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Second-Order Nonlinear Processes in Warm Unmagnetized Plasmas

A thesis submitted in fulfilment of the requirements for the degree of Doctor of Philosophy

by

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December, 2013
Declaration of originality

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Chapters 2 to 4 are based on the following published papers.

Chapter 2  **First-order thermal correction to the quadratic response tensor and rate for second harmonic plasma emission**  
B. Layden, D. J. Percival, I. H. Cairns, and P. A. Robinson,  
I was primarily responsible for this work.

Chapter 3  **Exact evaluation of the quadratic longitudinal response function for an unmagnetized Maxwellian plasma**  
B. Layden, I. H. Cairns, P. A. Robinson, and D. J. Percival,  
I was primarily responsible for this work.

Chapter 4  **Exact evaluation of the rates of electrostatic decay and scattering off thermal ions for an unmagnetized Maxwellian plasma**  
B. Layden, I. H. Cairns, and P. A. Robinson,  
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Abstract

Nonlinear processes are commonly invoked to describe a wide range of phenomena in both space plasmas and laboratory plasmas, allowing wave energy in a particular mode to be transferred to different modes. Relevant processes include three-wave interactions and nonlinear wave-particle scattering. The strength of the wave coupling—and hence the nonlinear rates—for these processes is determined by the quadratic response tensor. The general expression for this tensor involves a number of velocity-space integrals of the velocity distribution function and denominators related to the Cerenkov resonance between waves and particles. Due to the difficulty of evaluating these integrals they are typically approximated by making assumptions about the phase speeds of the waves; for an unmagnetized Maxwellian plasma, the phase speeds of Langmuir and transverse waves are assumed to be much greater than the electron thermal speed, and the phase speed of ion-sound waves is assumed to be much less than the electron thermal speed. However, the ranges of validity for these approximations are unclear, and the resulting approximate quadratic response tensors and nonlinear rates may be inaccurate when modeling the nonlinear processes in particular space plasmas.

Conversely, an exact expression for the quadratic response tensor of an unmagnetized Maxwellian plasma has been derived previously in terms of generalized plasma dispersion functions. This expression is valid for any phase speeds of the waves, but its length and complexity prevents its use in the calculation of nonlinear rates. What is lacking in the literature are more accurate explicit expressions for the quadratic response tensor that are appropriate for nonlinear rate calculations, from which also the accuracy of the typical approximations can be assessed.

This thesis presents new, more accurate analytical expressions for the quadratic response tensor for various second-order processes in unmagnetized plasmas, and analytical and numerical calculations of the corresponding nonlinear rates. Comparisons are then made between these new nonlinear rates and the previous approximate rates, allowing the accuracy of the previous rates to be
assessed and sharper bounds placed on their regimes of validity for the first time.

The structure of the thesis is as follows. Chapter 1 presents an overview of the relevant theory for nonlinear processes, a discussion of the previous work on the topic, and an outline of the work presented in the following chapters. In Chapter 2 a first-order thermal correction to the cold-plasma quadratic response tensor is derived, which is valid for interactions between three waves with phase speeds greater than the electron thermal speed in an unmagnetized plasma with an arbitrary isotropic velocity distribution. From this a thermal correction to the rate of second harmonic plasma emission (via Langmuir-wave coalescence) is calculated, and its importance is assessed for various space physics contexts. Chapter 3 presents an exact evaluation of the longitudinal part of the quadratic response tensor (known as the quadratic longitudinal response function) for an unmagnetized Maxwellian plasma, which gives an exact description of the wave coupling in second-order processes involving electrostatic waves only. The expression we derive involves a number of generalized plasma dispersion functions. Errors in certain aspects of previous expressions for the generalized plasma dispersion functions are corrected, a new set of expressions are derived that converge more rapidly, and the accuracy of various approximations to these functions are assessed. Chapter 4 contains an exact evaluation of the rates of electrostatic decay and scattering off thermal ions using the exact quadratic longitudinal response function derived in Chapter 3, and comparisons are made between these exact rates and the previous approximate rates in various contexts. A summary of the results in this thesis and directions for future work are presented in Chapter 5.
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Chapter 1

Introduction and literature review

In this chapter we introduce the necessary plasma theory for describing non-linear processes and discuss previous work on their rates. We begin in Sec. 1.1 with a basic description of plasmas and particle motions. In Sec. 1.2 we outline a derivation of the inhomogeneous wave equation from Maxwell’s equations. We then describe in Sec. 1.3 the Vlasov equation and show how it can be used to derive the equivalent dielectric tensor, and hence the wave modes, of an unmagnetized Maxwellian plasma. The general emission formula and its application to second-order nonlinear processes are presented in Sec. 1.4. In Sec. 1.5 previously derived expressions for the quadratic response tensor are presented and their limitations are discussed; we then describe the work presented in this thesis and how it addresses these limitations. Finally, in Sec. 1.6 we discuss the applications of this work to space plasma phenomena, including solar radio bursts and radio emission near planetary bow shocks.

1.1 Properties of plasmas

A plasma is a partially or fully ionized gas consisting of delocalized electrons, positively charged ions, and possibly negatively charged ions and neutral atoms, with a mean charge density that is equal to zero. There are a number of natural frequencies associated with particle motions in a plasma. The angular plasma frequency for the particle species \( a \) is defined by (e.g., Chen [1984])

\[
\omega_{pa} = \left( \frac{n_a q_a^2}{m_a \varepsilon_0} \right)^{1/2},
\]  

(1.1)

where \( n_a, q_a, \) and \( m_a \) are the number density, charge, and mass of particles of species \( a \), respectively, and \( \varepsilon_0 \) is the permittivity of free space. (Henceforth we omit the subscript \( e \) for the electron plasma frequency.) In the presence of an
ambient magnetic field $B_0$ the charged plasma particles undergo a spiralling motion along and about $B_0$, with an angular frequency (e.g., Chen [1984])

$$\Omega_a = \frac{q_a |B_0|}{m_a}, \quad (1.2)$$
called the cyclotron frequency. A plasma is said to be unmagnetized when this spiralling motion can be regarded as unimportant; this assumption is valid when $B_0$ is sufficiently weak (e.g., where $\Omega_a \ll \omega_{pa}$).

The plasma particles of species $a$ also undergo collisions with particles of the same or different species $a'$, with a frequency denoted $\nu_{a,a'}$. The frequency of electron-ion (Coulomb) collisions is given by (e.g., Chen [1984])

$$\nu_{e,i} = \frac{\omega_{pe}^2 q_i e}{4\pi \epsilon_0 m_e V_e^3} \ln \Lambda_c, \quad (1.3)$$

where $\ln \Lambda_c \sim 10$ is the Coulomb logarithm and $e$ is the elementary charge. A plasma can be considered collisionless when $\nu_{a,a'} \ll \omega_{pa}, \Omega_a$; this is satisfied if the plasma is sufficiently hot and diffuse. Collective electrostatic and electromagnetic interactions between the charged particles then determine the plasma dynamics.

Plasmas also have various characteristic length scales. The Debye length $\lambda_{Da}$ is the scale over which mobile charge carriers screen out electric fields in the plasma, given by (e.g., Chen [1984])

$$\lambda_{Da} = \frac{V_a}{\omega_{pa}}, \quad (1.4)$$

where $V_a = \sqrt{k_B T_a / m_a}$ is the thermal speed, with $T_a$ the particle temperature and $k_B$ the Boltzmann constant. The radius of gyration of a charged plasma particle in a magnetic field, called the gyroradius or Larmor radius, is (e.g., Chen [1984])

$$r_{ga} = \frac{m_a v_\perp}{|q_a||B_0|}, \quad (1.5)$$

where $v_\perp$ is the particle speed perpendicular to the magnetic field. The mean free path of a particle of species $a$ undergoing collisions with particles of species $a'$ is defined by (e.g., Melrose [1980a])

$$\lambda_a = \frac{V_a}{\nu_{a,a'}}, \quad (1.6)$$

In this thesis we consider unmagnetized collisionless electron-ion plasmas. Such a description is appropriate for the nonlinear processes relevant to space plasmas that we will be investigating.
1.2 The inhomogeneous wave equation

Plasmas support a variety of wave modes, which are obtained by deriving and solving a general equation for electromagnetic waves in a medium. Maxwell’s equations are the fundamental equations that govern the electric and magnetic fields in a plasma. They are given by

\[
\begin{align*}
\nabla \cdot \mathbf{E}(t, x) &= \frac{\rho(t, x)}{\epsilon_0}, \\
\nabla \cdot \mathbf{B}(t, x) &= 0, \\
\n\nabla \times \mathbf{E}(t, x) &= -\frac{\partial \mathbf{B}(t, x)}{\partial t}, \\
\n\nabla \times \mathbf{B}(t, x) &= \mu_0 \mathbf{J}(t, x) + \frac{1}{c^2} \frac{\partial \mathbf{E}(t, x)}{\partial t},
\end{align*}
\]

(1.7) (1.8) (1.9) (1.10)

where \(\rho\) is the charge density, \(\mathbf{J}\) is the current density, \(\mu_0\) is the permeability of free space, and \(c = 1/\sqrt{\epsilon_0 \mu_0}\) is the speed of light. It is useful to introduce the scalar and vector potentials \(\phi\) and \(\mathbf{A}\), which are related to the the electric and magnetic fields by

\[
\begin{align*}
\mathbf{E}(t, x) &= -\nabla \phi(t, x) - \frac{\partial \mathbf{A}(t, x)}{\partial t}, \\
\mathbf{B}(t, x) &= \nabla \times \mathbf{A}(t, x).
\end{align*}
\]

(1.11) (1.12)

The Fourier transforms of Eqs (1.10), (1.11), and (1.12) in the temporal gauge (where \(\phi = 0\) so that the electromagnetic field is described completely by \(\mathbf{A}\)) give (e.g., Melrose [1986a])

\[
\frac{\omega^2}{c^2} \mathbf{A}(\omega, \mathbf{k}) + \mathbf{k} \times [\mathbf{k} \times \mathbf{A}(\omega, \mathbf{k})] = -\mu_0 \mathbf{J}(\omega, \mathbf{k}).
\]

(1.13)

The current density can be separated into induced and external parts, i.e., \(\mathbf{J} = \mathbf{J}^{(\text{ind})} + \mathbf{J}^{(\text{ext})}\). An expansion of \(\mathbf{J}^{(\text{ind})}\) in powers of \(\mathbf{A}\), known as the weak-turbulence expansion, yields (e.g., Tsytovich [1970]; Melrose [1986a])

\[
J_i^{(\text{ind})}(k) = \sum_{n=1}^{\infty} J_i^{(n)}(k),
\]

(1.14)

where

\[
J_i^{(1)}(k) = \alpha_i^{(1)}(k) A_j(k),
\]

(1.15)
and

\[ J_i^{(n)}(k) = \int d\lambda^{(n)} \alpha_{ij_1 ... j_n}^{(n)}(k, k_1, \ldots, k_n) A_j(k_1) \cdots A_j(k_n), \quad n \geq 2, \]  

(1.16)

with \( i \) a free tensor index running over \( x, y, \) and \( z \). In Eqs (1.15) and (1.16), \( k_m \) collectively denotes \( \omega_m \) and \( k_m \) for the \( m \)th wave, and \( d\lambda^{(n)} \) is the \( n \)th-order convolution integral given by

\[ d\lambda^{(n)} = \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \cdots \frac{d^4k_n}{(2\pi)^4} \delta^4(k - k_1 - \ldots - k_n), \]  

(1.17)

with

\[ d^4k = d\omega \, d^3k, \]  

(1.18)

and

\[ \delta^4(k) = \delta(\omega)\delta^3(k). \]  

(1.19)

Equations (1.15) and (1.16) define the linear response tensor \( \alpha_{ij}^{(1)} \) and the nonlinear response tensors \( \alpha_{ij_1 ... j_n}^{(n)} \), respectively. Taking \( J^{(1)} \) to the left hand side of Eq. (1.13) and now defining \( J^{(\text{ext})} \) so that it includes \( J^{(n)} \) for \( n \geq 2 \) yields the inhomogeneous wave equation (e.g., Melrose [1986a])

\[ \Lambda_{ij}(k) A_j(k) = -\frac{\mu_0 c^2}{\omega^2} J_i^{(\text{ext})}(k). \]  

(1.20)

Here,

\[ \Lambda_{ij}(k) = \frac{|k|^2 c^2}{\omega^2} (\kappa_i \kappa_j - \delta_{ij}) + K_{ij}(k), \]  

(1.21)

with \( \kappa = k/|k| \), and where

\[ K_{ij}(k) = \delta_{ij} + \frac{1}{\omega^2 \epsilon_0} \sum_a \alpha_{ij}^{(1)(a)}(k) \]  

(1.22)

is the equivalent dielectric tensor. The sum in Eq. (1.22) is over contributions from the different particle species \( a \).

The source terms for Eq. (1.20) are \( J^{(\text{ext})} \) and the implicit source term from the antihermitian part of \( \Lambda_{ij} \) which describes the dissipative part of the response. Neglecting these source terms gives the homogeneous wave equation

\[ \Lambda_{ij}^H(k) A_j(k) = 0, \]  

(1.23)
where $H$ denotes the hermitian part. A nontrivial solution of this equation requires that
\[ \Lambda(k) := \det[\Lambda_{ij}(k)] = 0, \quad (1.24) \]
which is known as the dispersion equation. Equation (1.24) is typically solved for $\omega$ as a function of $k$; a particular solution $\omega = \omega_M(k)$ is called a dispersion relation, which defines a wave mode $M$. The solution $A_M(k) := A(k_M)$ of Eq. (1.23), where $k_M$ collectively denotes $\omega_M(k)$ and $k$, has an arbitrary amplitude $A_M(k)$ and phase $\varphi_M(k)$. The polarization vector $e_M(k)$ is introduced as the unit vector in the direction of $A_M(k)$, i.e.,
\[ e_M(k) = \frac{A_M(k)}{A_M(k)}. \quad (1.25) \]
We then define the wave field $A_M(k)$ as
\[ A_M(k) = e_M(k)A_M(k) \exp[i\varphi_M(k)]2\pi \left\{ \delta [\omega - \omega_M(k)] + \delta [\omega - \omega_M(-k)] \right\}. \quad (1.26) \]

An explicit expression for $K_{ij}$ is needed to find the wave modes for a plasma. Such an expression can be calculated using either a fluid or a kinetic approach. Here we will focus on the more general kinetic approach, with the starting point being the Vlasov equation.

### 1.3 The Vlasov equation and wave modes

Plasmas are collections of large numbers of charged particles, with the motion of each particle determined by external electromagnetic fields as well as fields arising from the microscopic distribution of all other charged particles. A statistical (or “kinetic”) description of the plasma is therefore necessary to describe the particle motions and the macroscopic plasma properties. The central quantity in kinetic theory is the distribution function $f(v, x, t)$, which is defined such that the total number of particles at time $t$ in a six-dimensional phase space element $d^3v \, d^3x$ centered at $(v, x)$ is equal to $f(v, x, t) \, d^3v \, d^3x$.

The Boltzmann equation is a general kinetic equation describing the evolution of the distribution function. It is given by (e.g., Klimontovich [1982])
\[ \frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial x} + \frac{q}{m} (E + v \times B) \cdot \frac{\partial f}{\partial v} = \left( \frac{\partial f}{\partial t} \right)_{\text{coll.}} + \left( \frac{\partial f}{\partial t} \right)_{\text{other}}. \quad (1.27) \]
The left hand side of Eq. (1.27) is the total time derivative of the distribution function, assuming that the only force acting on the particles is the Lorentz force $F = q(E + v \times B)$. The right hand side describes how the distribution function evolves due to collisions and other effects (e.g., charge-exchange collisions, ionization, chemical reactions, scattering by waves, etc.). For a collisionless plasma in the absence of any other processes, the right hand side can be set to zero and the equation is called the collisionless Boltzmann equation.

An approximation to the collisionless Boltzmann equation comes from assuming that particle interactions are manifested solely through the average self-consistent electric and magnetic fields, which are calculated by taking the source terms in Maxwell’s equations to be the average charge and current densities given by

$$
\rho(t, x) = \sum_a q_a n_a = \sum_a q_a \int d^3 v f_a(v, x, t),
$$

(1.28)

$$
J(t, x) = \sum_a q_a n_a \langle v_a \rangle = \sum_a q_a \int d^3 v v f_a(v, x, t). 
$$

(1.29)

The resulting equation is called the Vlasov equation [Vlasov, 1968].

There are two main approaches for solving the Vlasov equation. The first is the fluid approach, in which quantities averaged over velocity space are obtained by calculating moments of the Vlasov equation. The second approach is the kinetic approach, in which the Vlasov equation is solved by a perturbation expansion in powers of the electric field amplitude. This method retains information about the velocity distribution, and is thus more general than the fluid approach. Taking the Fourier transform of the Vlasov equation gives (e.g., Sitenko [1982]; Melrose [1986a])

$$
- i(\omega - k \cdot v)f(v, \omega, k) + \frac{q}{m} \int d\lambda^{(2)} \frac{1}{\omega_1} [(\omega_1 - k_1 \cdot v) \delta_{s_j} + k_1 s v_j]\n\times E_j(\omega_1, k_1) \frac{\partial f(v, \omega_2, k_2)}{\partial v_s} = 0.
$$

(1.30)

The distribution function is expanded according to

$$
f(v, \omega, k) = f^{(0)}(v)(2\pi)^4 \delta^4(k) + \sum_{n=1}^{\infty} f^{(n)}(v, \omega, k),
$$

(1.31)

with $f^{(n)} \propto E^n$. Solving Eq. (1.30) to first order gives

$$
f^{(1)}(v, \omega, k) = - \frac{iq}{m \omega(\omega - k \cdot v)} [(\omega - k \cdot v) \delta_{s_j} + k_s v_j] E_j(\omega, k) \frac{\partial f^{(0)}(v)}{\partial v_s}. 
$$

(1.32)
Comparing the two expressions for \( J^{(1)} \) in Eqs (1.15) and (1.29) (with \( f = f^{(1)} \)) and partially integrating with respect to \( \mathbf{v} \) leads to the identification

\[
\alpha^{(1)}_{ij}(\omega, \mathbf{k}) = \frac{q^2}{m} \int d^3 \mathbf{v} f(\mathbf{v}) \left[ \delta_{ij} + \frac{k_i v_j + k_j v_i}{\omega - \mathbf{k} \cdot \mathbf{v}} + \frac{(|\mathbf{k}|^2 - \omega^2/c^2) v_i v_j}{(\omega - \mathbf{k} \cdot \mathbf{v})^2} \right],
\]

(1.33)

with \( K_{ij} \) then following from Eq. (1.22). (In Eq. (1.33) and henceforth the superscript for \( f^{(0)}(\mathbf{v}) \) is omitted for brevity.)

For an unmagnetized plasma with an isotropic distribution function, \( K_{ij} \) can be separated into longitudinal and transverse parts via

\[
K_{ij}(\omega, \mathbf{k}) = K^L(\omega, \mathbf{k}) \kappa_i \kappa_j + K^T(\omega, \mathbf{k})(\delta_{ij} - \kappa_i \kappa_j).
\]

(1.34)

For a Maxwellian velocity distribution

\[
f(\mathbf{v}) = \frac{n}{(2\pi)^{3/2} V_a^3} \exp(-v^2/2V_a^2),
\]

(1.35)

the longitudinal and transverse parts can be written as

\[
K^L(\omega, \mathbf{k}) = 1 - \sum_a \frac{\omega^2_{pa}}{\omega^2 V_a^2} \int d^3 \mathbf{v} \frac{\omega}{\omega - \mathbf{k} \cdot \mathbf{v}} \left( \frac{\mathbf{k} \cdot \mathbf{v}}{|\mathbf{k}|} \right)^2 \exp(-v^2/2V_a^2) \frac{(2\pi)^{3/2} V_a^3}{(2\pi)^{3/2} V_a^3},
\]

(1.36)

\[
K^T(\omega, \mathbf{k}) = 1 - \sum_a \frac{\omega^2_{pa}}{2\omega V_a^2} \int d^3 \mathbf{v} \frac{\omega}{\omega - \mathbf{k} \cdot \mathbf{v}} \left[ v^2 - \left( \frac{\mathbf{k} \cdot \mathbf{v}}{|\mathbf{k}|} \right)^2 \right] \exp(-v^2/2V_a^2) \frac{(2\pi)^{3/2} V_a^3}{(2\pi)^{3/2} V_a^3}.
\]

(1.37)

The integrands in Eqs (1.36) and (1.37) have singularities at \( \omega = \mathbf{k} \cdot \mathbf{v} \), corresponding to a Cerenkov resonance between waves and particles. On rotating the coordinate system so that \( v_z \) is parallel to \( \mathbf{k} \), the correct integration contour for the \( v_z \) integral has an infinitesimal semicircular deformation into the lower-half plane around the singularity at \( v_z = \omega/|\mathbf{k}| \). Choosing this contour ensures that \( K_{ij} \) is a causal function [Landau, 1946]. An equivalent approach involves adding to \( \omega \) an infinitesimal imaginary part \( i\delta \) so that the contour along the real \( v_z \) axis passes below the singularity.

Both \( K^L \) and \( K^T \) can be evaluated in terms of the Fried-Conte plasma dispersion function [Fried and Conte, 1961], defined by

\[
Z(u) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dt \frac{e^{-t^2}}{t - u}, \quad \text{Im}(u) > 0,
\]

(1.38)
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and its analytic continuation for $\text{Im}(u) \leq 0$. This yields

$$K^L(\omega, \mathbf{k}) = 1 + \sum_a \frac{\omega^2 p_a}{|\mathbf{k}|^2 V_a^2} \left[ 1 + s_a Z(s_a) \right],$$  \hspace{1cm} (1.39)

$$K^T(\omega, \mathbf{k}) = 1 + \sum_a \frac{\omega^2 p_a}{\omega^2} s_a Z(s_a),$$  \hspace{1cm} (1.40)

where

$$s_a = \frac{\omega + i 0}{|\mathbf{k}| V_a \sqrt{2}}.$$  \hspace{1cm} (1.41)

Now having an explicit expression for $K_{ij}$ given by Eqs (1.39) and (1.40) in Eq. (1.34), Eq. (1.24) can be solved to give the wave modes for an unmagnetized isotropic Maxwellian plasma. The dispersion equation is then (e.g., Melrose [1986a])

$$\Lambda(\omega, \mathbf{k}) = \text{Re} \left[ K^L(\omega, \mathbf{k}) \right] \left\{ N^2 - \text{Re} \left[ K^T(\omega, \mathbf{k}) \right] \right\}^2 = 0,$$  \hspace{1cm} (1.42)

where $N = |\mathbf{k}| c / \omega$ is the refractive index. The first solution of Eq. (1.42) is

$$\text{Re} \left[ K^L(\omega, \mathbf{k}) \right] = 0.$$  \hspace{1cm} (1.43)

This solution gives the electrostatic (longitudinal) wave modes, i.e., those with $\mathbf{e}_M(\mathbf{k}) = \mathbf{k}$. The two wave modes of interest from Eq. (1.43) are the Langmuir ($L$) and ion-sound ($S$) modes. The Langmuir dispersion relation is derived by assuming that the wave’s phase speed $v_\phi = \omega / |\mathbf{k}|$ is much greater than $V_e$ so that $s_e \gg 1$; for $\omega \sim \omega_p$ this corresponds to the long wavelength limit $|\mathbf{k}| \lambda_D \ll 1$. Retaining the first three terms in the asymptotic expansion for $\text{Re}[Z(s_e)]$ and neglecting $K^L(i)$ gives the dispersion relation for Langmuir waves:

$$\omega_L(\mathbf{k}) = \left( \omega_p^2 + 3 |\mathbf{k}|^2 V_e^2 \right)^{1/2}.$$  \hspace{1cm} (1.44)

The dispersion relation for the ion-sound mode is obtained by assuming $V_i \ll v_\phi \ll V_e$ so that $s_e \ll 1 \ll s_i$. Keeping the first two terms of the asymptotic expansion for $\text{Re}[Z(s_i)]$ and neglecting $\text{Re}[Z(s_e)]$, since it is $O(s_e)$, gives

$$\omega_S(\mathbf{k}) = \frac{|\mathbf{k}| v_S}{\sqrt{1 + |\mathbf{k}|^2 \lambda_D^2}},$$  \hspace{1cm} (1.45)

which in the $|\mathbf{k}| \lambda_D \ll 1$ limit becomes

$$\omega_S(\mathbf{k}) = |\mathbf{k}| v_S.$$  \hspace{1cm} (1.46)
Here,

\[ v_S = V_e \left( \frac{\gamma m_e}{m_i} \right)^{1/2} \]  \hspace{1cm} (1.47)

is the ion-sound speed, with \( \gamma = 1 + \eta T_i/T_e \) and \( \eta = \left( \sqrt{1 + 12T_i/T_e} - 1 \right)T_e/2T_i \) \cite{Cairns et al., 1998}.

The transverse (T) mode results from the double solution of Eq. (1.42); i.e.,

\[ N^2 = K^T(\omega, \mathbf{k}). \]  \hspace{1cm} (1.48)

The waves are assumed to have \( v_\phi \gg V_e \). Taking the first term in the asymptotic expansion for Re\[Z(s_e)\] and neglecting \( K_L(\mathbf{k}) \) yields the dispersion relation

\[ \omega_T(\mathbf{k}) = \left( \omega_p^2 + |\mathbf{k}|^2 c^2 \right)^{1/2}. \]  \hspace{1cm} (1.49)

These waves have transverse polarization, i.e., \( \mathbf{e}_T(\mathbf{k}) \cdot \kappa = 0 \). The dispersion relations for the Langmuir, ion-sound, and transverse modes are shown in Fig. 1.1.

Plasma waves experience collisionless (“Landau”) damping when the resonance condition \( \omega = \mathbf{k} \cdot \mathbf{v} \) is satisfied for some waves and particles in the distribution \cite{Landau, 1946}. If \( \partial f(v)/\partial v < 0 \) at \( v = v_\phi \) the wave-particle interactions result in a net transfer of energy from the waves to the particles. Landau damping is formally treated by allowing \( \omega \) to have an imaginary part and considering the dissipative response of the plasma described by \( K^A_{ij} \) in the dispersion equation \cite{Landau, 1946; Melrose, 1986a}. The damping of Langmuir waves is weak for \( |\mathbf{k}| \lambda_D \ll 1 \) and strong for \( |\mathbf{k}| \lambda_D \gtrsim 1 \) where there are more particles that can satisfy the resonance condition. Ion-sound waves are weakly damped for \( T_i/T_e \ll 1 \) and strongly damped for \( T_i/T_e \gtrsim 1 \). There is strictly no Landau damping of transverse waves since \( v_{\phi T} > c \) from Eq. (1.49), so particles cannot resonate with them.

### 1.4 The emission formula

The continuity equation for electromagnetic energy is

\[ \frac{\partial}{\partial t} \left( \frac{\varepsilon_0 |\mathbf{E}|^2}{2} + \frac{|\mathbf{B}|^2}{2\mu_0} \right) + \nabla \cdot \left( \frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \right) = -\mathbf{J} \cdot \mathbf{E}. \]  \hspace{1cm} (1.50)

The terms on the left hand side of Eq. (1.50) are the rate of change of energy density in the electromagnetic field and the divergence of the flux of electromagnetic energy density. The right hand side of Eq. (1.50) is the rate
per unit volume at which work is done by a current on the electromagnetic field. The induced part of the current is part of the self-consistent field in the medium, therefore only the external current adds energy to the electromagnetic field [Melrose, 1986a]. Solving the inhomogeneous wave equation in Eq. (1.20) and substituting the solution for $A_M(k) \equiv -i E_M(k)/\omega$ into the source term $-J^{(\text{ext})} \cdot E$ leads to the power radiated in the mode $M$ per unit volume in the range $d^3k/(2\pi)^3$ of $k$ as (e.g., Melrose [1986a])

$$\frac{\partial W_M(k)}{\partial t} = \lim_{T \to \infty} \frac{R_M(k)}{TV\epsilon_0} \left| e_M^*(k) \cdot J^{(\text{ext})}(\omega_M(k), k) \right|^2.$$ \hspace{1cm} (1.51)

Here $W_M(k)$ is the energy density in the mode $M$ in the range $d^3k/(2\pi)^3$ of $k$, $V$ is the volume of the system, and $T$ is the normalization time. The quantity $R_M(k)$ in Eq. (1.51) is the ratio of electric to total energy in the mode $M$, given by (e.g., Melrose [1986a])

$$R_M(k) = \left. \left| \omega^{-1} e_M^*(k) e_M(k) \partial[\omega^2 K_{ij}(\omega, k)]/\partial \omega \right|^{-1} \right|_{\omega=\omega_M(k)}.$$ \hspace{1cm} (1.52)
For each emission process, the emission rate is calculated by substituting the corresponding external current into Eq. (1.51). We now describe two important second-order nonlinear processes that will be the focus of this thesis: three-wave interactions and nonlinear wave-particle scattering.

### 1.4.1 Three-wave interactions

Nonlinear processes occur due to currents generated by the nonlinear response of the plasma to electromagnetic fields. The second-order plasma response to the simultaneous beating of two wave fields (in arbitrary modes \( P \) and \( Q \)) gives rise to a third wave field (in the arbitrary mode \( M \)); this is called a three-wave interaction, denoted \( P + Q \rightarrow M \). The quadratic current \( J^{(2)} \) for this interaction is derived by substituting \( A = A_P + A_Q \) into Eq. (1.16) and keeping only the cross terms, giving

\[
J_i^{(2)}(k) = 2 \int d\lambda^{(2)} \alpha_{ijl}(k, k_1, k_2) A_{Pj}(k_1) A_{Ql}(k_2).
\]

To derive the above equation the symmetrization

\[
\alpha_{ijl}(k, k_1, k_2) = \alpha_{ilj}(k, k_2, k_1)
\]

has been imposed. The external current in Eq. (1.51) is then identified as \( J^{(2)} \). A simplification to the resulting emission formula comes from assuming that the phases of \( A_P \) and \( A_Q \) are random, so that an average over phases in Eq. (1.26) can be performed for each wave field. This “random phase approximation” is valid when the timescale over which the wave fields decohere is much shorter than the timescale of the nonlinear process [Sagdeev and Galeev, 1969; Tsytovich, 1970; Davidson, 1972; Zakharov et al., 1985; Melrose, 1986a,b; Robinson, 1997; Cairns, 2000]. This condition may be expressed as \( \Delta \omega \gg \Gamma \) where \( \Delta \omega \) is the bandwidth of the growing waves and \( \Gamma = (\partial W/\partial t)/W \) is the nonlinear growth rate [Tsytovich, 1970; Zakharov et al., 1985; Robinson, 1997].

In the random phase approximation the waves can be interpreted semiclassically as a collection of wave quanta with energy \( \hbar \omega_M(k) \) and momentum \( \hbar k \) [Tsytovich, 1970; Melrose, 1986a]. The occupation number \( N_M(k) \) is then introduced as the number density of wave quanta within the range \( d^3k \) of \( k \), so that

\[
N_M(k) = \frac{W_M(k)}{\hbar \omega_M(k)}.
\]
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In terms of $N_M(k)$ the rate of the three-wave interaction $P + Q \rightarrow M$ can be calculated from Eq. (1.51) as [Tsytovich, 1970; Melrose, 1986a]

$$\frac{\partial N_M(k)}{\partial t} = \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3} u_{MPQ}(k, k_1, k_2)N_P(k_1)N_Q(k_2),$$

where $u_{MPQ}$ is the interaction probability given by

$$u_{MPQ}(k, k_1, k_2) = \frac{\sigma \hbar R_M(k)R_P(k_1)R_Q(k_2)}{\omega_M(k)\omega_P(k_1)\omega_Q(k_2)}|\alpha_{MPQ}(k, k_1, k_2)|^2 \times (2\pi)^4 \delta^4(k_M - k_1 - k_2),$$

and

$$\alpha_{MPQ}(k_M, k_1, k_2) = \alpha_{ji}^{(2)}(k_M, k_1, k_2)e_M^*(k)e_P(k_1)e_Q(k_2).$$

If the symmetrized form of $\alpha_{ji}^{(2)}$ is used in Eq. (1.57) then $\sigma = 4$, whereas if the unsymmetrized form of $\alpha_{ji}^{(2)}$ is used then $\sigma = 1$. We will discuss these two different forms of the quadratic response tensor in Sec. 1.5. The delta function $\delta^4(k_M - k_1 - k_2)$ in Eq. (1.57) implies the wave matching conditions

$$k = k_1 + k_2,$$

$$\omega_M(k) = \omega_P(k_1) + \omega_Q(k_2),$$

which are interpreted as conservation of momentum and energy respectively in this semiclassical framework (with the common factor of $\hbar$ omitted). These wave matching conditions must be satisfied for a three-wave interaction to take place; three-wave interactions $P + Q \rightarrow M$ may be forbidden for a particular set of wave vectors, or in some cases for all sets of wave vectors.

The use of the semiclassical formalism allows the rate for the process $M \rightarrow P + Q$ to be derived by appealing to the principle of detailed balance. The rate equations are then combined to give the rate for $P + Q \leftrightarrow M$ as (e.g., Tsytovich [1970]; Melrose [1986a])

$$\frac{\partial N_M(k)}{\partial t} = \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3} u_{MPQ}(k, k_1, k_2)$$

$$\times \{N_P(k_1)N_Q(k_2) - N_M(k)[N_P(k_1) + N_Q(k_2)]\},$$

with the nonlinear rates for the modes $P$ and $Q$ as

$$\frac{\partial N_P(k_1)}{\partial t} = -\int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3} u_{MPQ}(k, k_1, k_2)$$

$$\times \{N_P(k_1)N_Q(k_2) - N_M(k)[N_P(k_1) + N_Q(k_2)]\},$$

with the nonlinear rates for the modes $P$ and $Q$ as

$$\frac{\partial N_Q(k_2)}{\partial t} = -\int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k_1}{(2\pi)^3} u_{MPQ}(k, k_1, k_2)$$

$$\times \{N_P(k_1)N_Q(k_2) - N_M(k)[N_P(k_1) + N_Q(k_2)]\},$$

with the nonlinear rates for the modes $P$ and $Q$ as

$$\frac{\partial N_P(k_1)}{\partial t} = -\int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3} u_{MPQ}(k, k_1, k_2)$$

$$\times \{N_P(k_1)N_Q(k_2) - N_M(k)[N_P(k_1) + N_Q(k_2)]\},$$

with the nonlinear rates for the modes $P$ and $Q$ as

$$\frac{\partial N_Q(k_2)}{\partial t} = -\int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k_1}{(2\pi)^3} u_{MPQ}(k, k_1, k_2)$$

$$\times \{N_P(k_1)N_Q(k_2) - N_M(k)[N_P(k_1) + N_Q(k_2)]\},$$

with the nonlinear rates for the modes $P$ and $Q$ as

$$\frac{\partial N_P(k_1)}{\partial t} = -\int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3} u_{MPQ}(k, k_1, k_2)$$

$$\times \{N_P(k_1)N_Q(k_2) - N_M(k)[N_P(k_1) + N_Q(k_2)]\},$$

with the nonlinear rates for the modes $P$ and $Q$ as

$$\frac{\partial N_Q(k_2)}{\partial t} = -\int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k_1}{(2\pi)^3} u_{MPQ}(k, k_1, k_2)$$

$$\times \{N_P(k_1)N_Q(k_2) - N_M(k)[N_P(k_1) + N_Q(k_2)]\},$$
\[ \frac{\partial N_Q(k_2)}{\partial t} = -\int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k_1}{(2\pi)^3} u_{MPQ}(k, k_1, k_2) \times \{ N_P(k_1)N_Q(k_2) - N_M(k)[N_P(k_1) + N_Q(k_2)] \}. \] (1.63)

We note that these kinetic equations can also be derived from a multiple-timescale analysis of the Vlasov equation [Davidson, 1972; Sitenko, 1982; Yoon, 2000, 2006].

Nonlinear processes cannot transfer energy faster than the lowest frequency wave can respond; this imposes the constraint that the nonlinear growth rate must be less than the minimum frequency of the interacting waves [Zakharov et al., 1985; Cairns, 2000]. For example, for the electrostatic decay process \( L \leftrightarrow L' + S \), the nonlinear growth rate must be less than the ion-sound wave frequency. An additional requirement for this process to proceed is that the nonlinear growth rate for the \( L' \) waves exceed the linear damping rate so that the \( L' \) waves experience net growth [Cairns, 2000]. Analytical estimates of the nonlinear growth rate show that \( L' \) waves can undergo net growth for \( v_{\phi L'}/V_e \gtrsim 3 \), depending on plasma conditions [Mitchell et al., 2003]. Similarly, \( S \) waves can experience net growth for a range of plasma conditions, even when \( T_i/T_e \gtrsim 1 \) where their linear damping is strong [Mitchell et al., 2003]; however, net growth of \( S \) waves is not a necessary condition for the process to proceed [Cairns, 2000].

### 1.4.2 Nonlinear wave-particle scattering

The other second-order process in plasmas is nonlinear wave-particle scattering, denoted \( M + a \leftrightarrow P + a' \) where \( a \) is the species of scattering particle and \( a' \) denotes the recoiled particle. The quadratic current for this interaction is due to the beating between a wave field \( A_P \) and the Debye-shielding field \( A^{(q)} \) associated with a particle of charge \( q \); its expression is found by replacing \( A_Q(k_2) \) with \( A^{(q)}(k_2) \) in Eq. (1.53). The shielding field is derived by solving Eq. (1.20) with the external current identified as the single particle current

\[ J^{(sp)}(t, x) = qv \delta^3 [x - (x_0 + vt)], \] (1.64)

where \( x_0 \) is the initial position and \( v \) is the particle velocity. The solution of Eq. (1.20) is

\[ A_i^{(q)}(k) = -\frac{1}{\epsilon_0 \omega^2} \frac{\lambda_{ij}(k)}{\Lambda(k)} J_i^{(sp)}(k), \] (1.65)
where \( \lambda_{ij}(k) \) is the matrix of cofactors defined by

\[
\Lambda_{ij}(k) \lambda_{ij}(k) = \Lambda(k) \delta_{ij},
\]

and

\[
J^{(sp)}(k) = 2\pi q \mathbf{v} \exp(-i \mathbf{k} \cdot \mathbf{x}_0) \delta(\omega - \mathbf{k} \cdot \mathbf{v}).
\]

Equations (1.65) and (1.67) are then substituted into the emission formula in Eq. (1.51) and an average over random phases is performed to give the power radiated by a single particle. Scattering by a collection of particles is treated by replacing \( 1/\sqrt{N} \) with \( \int d^3 \mathbf{v} f(\mathbf{v}) \) in the emission formula. On rewriting in terms of the occupation number, then using the principle of detailed balance to include spontaneous emission, stimulated emission, and absorption, the kinetic equations for nonlinear wave-particle scattering are (e.g., Tsytovich [1970]; Melrose [1986a])

\[
\frac{\partial N_M(k)}{\partial t} = \int d^3 \mathbf{v} \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} w_{MP}(\mathbf{k}, \mathbf{k}_1, \mathbf{v}) \left\{ [N_M(k) - N_P(k_1)] f(\mathbf{v}) - N_M(k) N_P(k_1) \frac{\hbar (\mathbf{k} - \mathbf{k}_1)}{m} \cdot \frac{\partial f(\mathbf{v})}{\partial \mathbf{v}} \right\},
\]

\[
\frac{\partial N_P(k)}{\partial t} = \int d^3 \mathbf{v} \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} w_{MP}(\mathbf{k}, \mathbf{k}_1, \mathbf{v}) \left\{ [N_M(k) - N_P(k_1)] f(\mathbf{v}) - N_M(k) N_P(k_1) \frac{\hbar (\mathbf{k} - \mathbf{k}_1)}{m} \cdot \frac{\partial f(\mathbf{v})}{\partial \mathbf{v}} \right\},
\]

and

\[
\frac{\partial f(\mathbf{v})}{\partial t} = \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \frac{\hbar (\mathbf{k} - \mathbf{k}_1)}{m} \cdot \frac{\partial}{\partial \mathbf{v}} \left( w_{MP}(\mathbf{k}, \mathbf{k}_1, \mathbf{v}) \left\{ [N_M(k) - N_P(k_1)] f(\mathbf{v}) - N_M(k) N_P(k_1) \frac{\hbar (\mathbf{k} - \mathbf{k}_1)}{m} \cdot \frac{\partial f(\mathbf{v})}{\partial \mathbf{v}} \right\} \right),
\]

with

\[
w_{MP}(\mathbf{k}, \mathbf{k}_1, \mathbf{v}) = \frac{2\pi q^4}{e_0^2 m^2} \frac{R_M(k) R_P(k_1)}{[\omega_M(k) \omega_P(k_1)]^2} |A_{MP}(\mathbf{k}, \mathbf{k}_1, \mathbf{v})|^2 \times \delta[\omega_M(k) - \omega_P(k_1) - (\mathbf{k} - \mathbf{k}_1) \cdot \mathbf{v}],
\]

and

\[
A_{MP} = \frac{\sigma m e_{M} \alpha_P(\mathbf{k}_1) \alpha_{ijl}^{(2)}(k_M, k_{P1}, k_M - k_{P1}) \lambda_{im}(k_M - k_{P1}) v_m}{q e_0 \left[ \omega_M(k) - \omega_P(k_1) \right]^2 \Lambda(k_M - k_{P1})}.
\]
Here, $\sigma' = 2$ if the symmetrized $\alpha(2)_{ijl}$ is used, and $\sigma' = 1$ if the unsymmetrized $\alpha_{ijl}$ is used. In the case where the shielding field is electrostatic we have (e.g., Melrose [1986a])

$$\frac{\lambda_{ij}(k)}{\Lambda(k)} = \frac{\kappa_i \kappa_j}{K^L(k)},$$

and so Eq. (1.72) simplifies to

$$A_{MP} = \frac{\sigma'm}{q\epsilon_0}\frac{e_{Mi}^*(k) e_{Pj}(k_1)(k_1 - k_3l)\alpha_{ijl}^{(2)}(k_M, k_{P1}, k_M - k_{P1})(k - k_1) \cdot v}{[\omega_M(k) - \omega_P(k_1)]^2 |k - k_1|^2 K^L(k_M - k_{P1})},$$

From Eqs (1.57), (1.71), and (1.72), we see that the interaction probabilities—and hence the nonlinear rates—for both three-wave interactions and nonlinear wave-particle scattering depend on $\alpha_{ijl}^{(2)}$. To calculate the nonlinear rates for specific processes, an explicit expression for $\alpha_{ijl}^{(2)}$ is required. To derive a general expression for $\alpha_{ijl}^{(2)}$ we return to the Vlasov equation.

### 1.5 The quadratic response tensor

A general expression for the quadratic response tensor $\alpha_{ijl}^{(2)}$ is found by solving the Vlasov equation to second order in $E$. An analogous calculation to that for the linear response gives (e.g., Melrose [1986a])

$$\alpha_{ijl}^{(2)}(k, k_1, k_2) = \frac{q^3}{2m^2} \int d^3 v f(v) [a_{ij}(k, k_1, v)d_l(k_2, v) + a_{il}(k, k_2, v)d_j(k_1, v)] + a_{jl}(k_1, k_2, v)d_i(k, v),$$

where

$$a_{jl}(k_1, k_2, v) = \delta_{jl} + \frac{k_1v_j}{\omega_1 - k_1 \cdot v} + \frac{k_2v_l}{\omega_2 - k_2 \cdot v} + \frac{(k_1 \cdot k_2 - \omega_1\omega_2/c^2)v_jv_l}{(\omega_1 - k_1 \cdot v)(\omega_2 - k_2 \cdot v)},$$

$$d_i(k, v) = \frac{1}{\omega - k \cdot v} \left[ k_i + \frac{|k|^2 - \omega^2/c^2}{\omega - k \cdot v} v_i \right],$$

and where the symmetry in Eq. (1.54) has been imposed. The total quadratic response tensor is obtained by summing over the contributions from each particle species; however, the $m^{-2}$ dependence in Eq. (1.75) means that the
ion contribution is negligible. Writing Eq. (1.75) in terms of the individual velocity-space integrals yields (e.g., Percival and Robinson [1998a])

$$\alpha_{ijl}(k_1, k_2) = \frac{q^3}{2m^2} \left[ k_i + (|k|^2 - \omega^2/c^2) \frac{\partial}{\partial k_i} \right] [I(k)\delta_{jl} + k_{2j}J_l(k, k_2)$$

$$+ k_{1j}J_l(k, k_1) + (k_1 \cdot k_2 - \omega_1\omega_2/c^2)K_{jl}(k, k_1, k_2)]$$

$$+ (i, k) \leftrightarrow (j, k_1) + (i, k) \leftrightarrow (l, k_2) \right\}.$$  (1.78)

Here $(i, k) \leftrightarrow (j, k_1)$ represents the additional terms generated from those written explicitly by interchanging $k$ and $k_1$ and the associated tensor indices, and the integrals $I(k), J_i(k_1, k_2),$ and $K_{ij}(k_1, k_2)$ are given by

$$I(k) = \int d^3v f(v) \frac{1}{\omega - k \cdot v},$$  (1.79)

$$J_i(k_1, k_2) = \int d^3v f(v) \frac{v_i}{(\omega_1 - k_1 \cdot v)(\omega_2 - k_2 \cdot v)},$$  (1.80)

$$K_{ij}(k_1, k_2) = \int d^3v f(v) \frac{v_i v_j}{(\omega - k \cdot v)(\omega_1 - k_1 \cdot v)(\omega_2 - k_2 \cdot v)},$$  (1.81)

with each frequency containing an implicit additive term $i0$. (Henceforth $K_{ij}$ refers to the quantity defined by Eq. (1.81), which is unrelated to the equivalent dielectric tensor defined by Eq. (1.22).)

The quadratic response tensor is much more difficult to evaluate than the linear response tensor due to the multiple resonant denominators in Eqs (1.80) and (1.81), and so Eq. (1.78) is typically approximated. The approximations that are made depend on the assumed phase speeds of the waves participating in the nonlinear process. Langmuir and transverse waves are assumed to be fast, i.e., they are assumed to satisfy $v_\phi \gg V_e$; ion-sound waves and shielding fields associated with thermal ions are assumed to be slow ($v_\phi \ll V_e$).

For interactions between three fast waves the “cold-plasma approximation” is often made [Tsytovich, 1970; Sitenko, 1982; Melrose, 1986a]. Thermal effects in the wave coupling are assumed to be unimportant and so the limit $V_e \to 0$ is taken. In this limit, the Maxwellian distribution function becomes $f(v) = n\delta^3(v)$; substituting this expression into Eqs (1.79)–(1.81) gives $I(k_m) = n/\omega_m$ and $J_i(k_m) = K_{ij}(k_m, k_n) = 0$, for all $k_m$ and $k_n$. The cold-plasma quadratic response tensor is then (e.g., Tsytovich [1970]; Sitenko [1982]; Melrose [1986a])

$$\alpha_{ijl}^{(2)}(k, k_1, k_2) \approx \frac{q^3 n}{2m^2} \left( \frac{k_i \delta_{jl}}{\omega} + \frac{k_{1j} \delta_{il}}{\omega_1} + \frac{k_{2l} \delta_{ij}}{\omega_2} \right).$$  (1.82)

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This approximate form of the quadratic response tensor is commonly used for the processes \( L + L' \leftrightarrow T \) and \( L + T \leftrightarrow T' \).

A different approximation is required for interactions between two fast waves and one slow wave. Taking \( k_2 \) to be the frequency and wave vector of the slow wave, the dominant term in Eq. (1.75) is proportional to \( a_{ij}(k, k_1, v) d_l(k_2, v) \); Eq. (1.75) is then approximated by retaining this term and neglecting the other two terms in the integrand. The tensor \( a_{ij}(k, k_1, v) \) is also approximated by setting \( a_{ij}(k, k_1, v) = \delta_{ij} \). If the slow wave is electrostatic then the wave coupling is completely described by the contraction of \( \alpha_{ijl}^{(2)} \) with the polarization vector \( e_l(k_2) = \kappa_{2l} \), whose result we denote as \( \hat{\alpha}_{ij}^{(2)} \).

In the nonrelativistic limit \((c \to \infty)\), the reconstructed quadratic response tensor \( \alpha_{ijl}^{(2)} = \hat{\alpha}_{ij}^{(2)} \kappa_{2l} \) is (e.g., Melrose [1986a])

\[
\alpha_{ijl}^{(2)}(k, k_1, k_2) \approx \frac{q^3 n \omega_2 \delta_{ij} k_{2l}}{m^2} \int d^3 v \frac{f(v)}{(\omega_2 - k_2 \cdot v)^2}. \tag{1.83}
\]

The factor of 1/2 that appears in Eq. (1.75) is artificially omitted in Eq. (1.83) to give the correct unsymmetrized form; this is necessary because the slow wave is qualitatively different from the other two waves, and thus the symmetry in Eq. (1.54) is no longer valid (see, e.g., Melrose [1986a]).

The integral in Eq. (1.83) can be evaluated for a Maxwellian velocity distribution in terms of the plasma dispersion function defined by Eq. (1.38), giving (e.g., Melrose [1986a])

\[
\alpha_{ijl}^{(2)}(k, k_1, k_2) \approx \frac{q^3 n \omega_2 \delta_{ij} k_{2l}}{m^2} \frac{1}{|k_2|^2 V^2} \left[ 1 + s_{2e} Z(s_{2e}) \right], \tag{1.84}
\]

where \( s_{2e} = \omega_2 |k_2| V_e \sqrt{2} \). This expression is used for calculations relating to electrostatic decay \( L \leftrightarrow L' + S \), electromagnetic decay \( L \leftrightarrow T + S \), stimulated Brillouin scattering \( T \leftrightarrow T' + S \), and scattering off thermal ions \( L + i \leftrightarrow L' + i' \) and \( L + i \leftrightarrow T + i' \).

For interactions between three electrostatic waves, only the longitudinal part of \( \alpha_{ijl}^{(2)} \) given by

\[
\alpha^{(2)}(k, k_1, k_2) = \alpha_{ijl}^{(2)}(k, k_1, k_2) \kappa_i \kappa_{1j} \kappa_{2l}, \tag{1.85}
\]

called the quadratic longitudinal response function, is necessary to describe the wave coupling. On taking the longitudinal part of the general expression for \( \alpha_{ijl}^{(2)} \) in Eq. (1.75) various terms cancel, giving the quadratic longitudinal
response function as (e.g., Sitenko [1982])

\[
\alpha^{(2)}(k, k_1, k_2) = \frac{q^3}{m^2} \frac{\omega_1 \omega_2}{|k||k_1||k_2|} |\mathbf{k}_1 \cdot \mathbf{k}_2| |\mathbf{k}|^2 M(k; k_1, k_2) + (k \leftrightarrow k_1) + (k \leftrightarrow k_2),
\]

(1.86)

where

\[
M(k; k_1, k_2) = \int d^3 \mathbf{v} \frac{f(\mathbf{v})}{(\omega - \mathbf{k} \cdot \mathbf{v})^2(\omega_1 - \mathbf{k}_1 \cdot \mathbf{v})(\omega_2 - \mathbf{k}_2 \cdot \mathbf{v})}.
\]

(1.87)

Assuming that the wave fields \( k_1 \) and \( k_2 \) are fast and \( k_1 \) is slow, Eq. (1.86) is approximated by setting \( M(k; k_1, k_2) = M(k_1; k, k_2) = 0 \), and \( (\omega_m - \mathbf{k}_m \cdot \mathbf{v}) = \omega_m \) for the fast wave fields in \( M(k_2; k, k_1) \). This gives the same expression as the longitudinal part of Eq. (1.84).

Although the approximate expressions for the quadratic response tensor in Eqs (1.82) and (1.84) are commonly used, their ranges of validity are unclear. They are derived assuming the wave fields satisfy either \( v_\phi \gg V_e \) or \( v_\phi \ll V_e \), and therefore the approximations break down as the phase speed of one or more of the waves approaches \( V_e \). The nonlinear rates in Eqs (1.61)–(1.63) and (1.68)–(1.70) depend on the quadratic response tensor through Eqs (1.57) and (1.71), thus the rates also become inaccurate in this limit. It is therefore desirable to derive more accurate approximations or exact expressions for the quadratic response tensor, and apply these to the calculation of nonlinear rates.

A more accurate approximation for the quadratic longitudinal response function for interactions involving three electrostatic waves with \( v_\phi > V_e \) was derived by Sitenko [1982]. The expression was in the form of a thermal correction added to the cold-plasma quadratic longitudinal response function, which was calculated by performing a binomial expansion of the resonant denominators in Eq. (1.87) and then evaluating the integrals up to the next order in \( V/v_\phi \). However, this response function is only valid for the process \( L \leftrightarrow L' + L'' \), which does not occur since the wave matching conditions cannot be satisfied (see, e.g., Melrose and McPhedran [1991]).

In contrast to these approximate expressions, an exact expression for the quadratic response tensor was derived by Percival and Robinson [1998a] in terms of generalized plasma dispersion functions \( Y_{m,n} \) defined by

\[
Y_{m,n}(a, b, c) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx \ e^{-x^2} \frac{x^m}{(x-c)^n} Z(a+bx).
\]

(1.88)
This exact expression is given by

\[
\alpha_{ijl}^{(2)}(k, k_1, k_2) = \frac{q^3}{2m^2} \left\{ k_i + (|k|^2 - \omega^2/c^2) \frac{\partial}{\partial k_i} \right\} \left( -\frac{s}{\omega} Z(s) \delta_{jl} - \frac{b}{k^3 V \sqrt{2}} \right.
\times \left\{ \left[ a_1 Y_{0,1}^{(1)} + b_1 Y_{1,1}^{(1)} + Z(s) \right] k_{1l}(b_1 k_{2j} - b_2 k_{1j}) + \left[ a_2 Y_{0,1}^{(2)} + b_2 Y_{1,1}^{(2)} + Z(s) \right]
\times k_{2j}(b_2 k_{1l} - b_1 k_{2l}) + Y_{1,1}^{(1)} k_j k_{1l} + Y_{1,1}^{(2)} k_l k_{2j} \right\} + (k_1 \cdot k_2 - \omega_1 \omega_2/c^2)
\times \left[ k_i K_{jl}(k_1, k_2) + (|k|^2 - \omega^2/c^2) L_{ijl}(k_1, k_2) \right]
\left. \right\} + (i, k) \leftrightarrow (j, k_1) + (i, k) \leftrightarrow (l, k_2), \tag{1.89} \right.
\]

where

\[
a_j = \frac{\omega_j |k|}{|k \times k_1| V \sqrt{2}}, \tag{1.90} \]
\[
b_j = -\frac{k \cdot k_j}{|k \times k_1|}, \tag{1.91} \]
\[
s = \frac{\omega}{|k| V \sqrt{2}} = -\frac{a}{b}, \tag{1.92} \]

and where \(Y_{m,n}^{(j)}\) is an abbreviation for \(Y_{m,n}(a_j, b_j, s)\). The expressions for \(K_{ij}\) and \(L_{ijl}\) in Eq. (1.89) each contain 14 distinct generalized plasma dispersion functions, which are given in Eqs (59) and (60) of Percival and Robinson [1998a] but for brevity are not shown here. Series expressions for the generalized plasma dispersion functions were derived in Percival and Robinson [1998b]; we show these and subsequently correct certain aspects of them in Sec. 3.5.

Although it is desirable to use Eq. (1.89) for calculating nonlinear rates, there are difficulties involved in its practical implementation. The partial derivative \(\partial/\partial k_i\) in Eq. (1.89) needs to be evaluated analytically, which is impractical due to the numerous products of terms that are functions of \(k\). Problems also arise for the numerical calculation of the exact quadratic response tensor. Equation (1.89) contains many different generalized plasma dispersion functions, and the series expressions for these functions are prone to numerical instability as discussed later in Chapter 3. The slow convergence of these series expressions in particular regions of parameter space also means that the computation would be very time-consuming. The very large number of terms also increases the probability of catastrophic cancellation occurring.

What is lacking in the literature are expressions for the quadratic response tensor that are more accurate than the typical approximations in Eqs (1.82) and
(1.84), but which can be practically applied to the calculation of nonlinear rates. The work in this thesis addresses this problem. In Chapter 2 we derive a more accurate quadratic response tensor for interactions between three waves with $v_\phi > V_e$ and arbitrary polarization in an unmagnetized plasma with an arbitrary isotropic distribution function. This expression is given as a first-order thermal correction to the cold-plasma quadratic response tensor. Unlike the work by Sitenko [1982], this response tensor is general enough to be applied usefully to the rates of nonlinear processes. We use the derived thermal correction to the quadratic response tensor to calculate a first-order thermal correction to the rate of second harmonic plasma emission.

The calculation of nonlinear rates using the exact quadratic response function in Eq. (1.89) is infeasible. However, interactions between three electrostatic waves such as $L \leftrightarrow L' + S$ and $L + i \leftrightarrow L' + i'$ can be described completely by the quadratic longitudinal response function. In Chapter 3 we present an exact evaluation of the quadratic longitudinal response function in Eq. (1.86) for an unmagnetized Maxwellian plasma, in terms of the generalized plasma dispersion functions defined by Eq. (1.88). The resulting expression is much more compact than Eq. (1.89). We then apply the exact quadratic longitudinal response function to the rates of $L \leftrightarrow L' + S$ and $L + i \leftrightarrow L' + i'$ in Chapter 4.

1.6 Application to radio emissions in space plasmas

We now discuss these nonlinear processes in the context of radio emissions in space plasmas. There are various sources of radio emissions in the heliosphere. Radio emissions from the solar corona, interplanetary medium, and near planetary bow shocks can exhibit harmonic structure; i.e., emission at both a fundamental frequency and twice that frequency [Wild et al., 1953, 1954; Dunckel, 1974; Hoang et al., 1981]. These have long been interpreted as emission at the plasma frequency $\omega_p$ and the second harmonic $2\omega_p$, known as plasma emission. The mechanism for plasma emission therefore typically involves the generation of Langmuir waves with frequency $\omega_L \approx \omega_p$ and their subsequent conversion into transverse waves at $\omega_p$ and $2\omega_p$. (Other mechanisms not involving Langmuir waves have also been put forth, e.g., cyclotron maser emission [Wu et al., 2002; Pechhacker and Tsiklauri, 2012].)

Langmuir waves can be driven by a fast electron beam propagating through a plasma via the bump-on-tail instability [Vedenov et al., 1962; Drummond
Such beam-driven Langmuir waves have been observed in various space contexts [Scarf et al., 1971; Gurnett and Anderson, 1976; Filbert and Kellogg, 1979; Gurnett et al., 1981; Anderson et al., 1981; Lin et al., 1981; Greenstadt et al., 1995; Bale et al., 1999; Cairns and Robinson, 1999; Kasaba et al., 2000; Malaspina et al., 2009]. The different types of heliospheric radio emissions are generated by the same basic mechanism but with different processes that form the electron beam. Type II solar radio bursts are identified as radio emissions that slowly drift downward in frequency. They are typically associated with coronal mass ejections (CMEs) which drive shocks as they propagate through the solar corona and interplanetary medium [Wild et al., 1963; Cane et al., 1982; Nelson and Melrose, 1985; Cairns, 2011]; electron beams are then generated by acceleration at the shock front. The downward frequency drift is thus interpreted as the shock propagating outward into regions of lower plasma density and hence smaller $\omega_p$. Type III radio bursts have a much faster downward frequency drift. They are typically associated with flare-accelerated electrons that propagate along open magnetic field lines [Wild et al., 1963; Suzuki and Dulk, 1985; Cane et al., 2002]. Radio emissions are also observed from the foreshock regions of planetary bow shocks. The physics is qualitatively similar to type II solar radio bursts, the difference being that the electron beams are formed by electron acceleration at the standing collisionless shock resulting from the interaction of the solar wind plasma with the planet’s magnetic field [Filbert and Kellogg, 1979; Hoang et al., 1981; Cairns and Melrose, 1985; Cairns, 2011].

Numerous mode conversion mechanisms have been proposed for radio emission at $\omega_p$ and $2\omega_p$. Ginzburg and Zheleznyakov [1958] proposed the first detailed quantitative model of type III radio bursts in which the nonlinear processes $L + i \leftrightarrow T(\omega_p) + i'$ and $L + L' \leftrightarrow T(2\omega_p)$ were invoked for the emissions at $\omega_p$ and $2\omega_p$ respectively, where $L$ denotes a beam-driven Langmuir wave and $L'$ denotes a thermal Langmuir wave. This model was later revised since it was unable to account for the observed brightness temperatures of the radiation [Melrose, 1980b,c, 1982; Cairns and Melrose, 1985; Cairns, 1987a,b, 1988; Robinson et al., 1993]. The current version of the model involves, firstly, the electrostatic decay $L \leftrightarrow L' + S$ of beam-driven Langmuir waves $L$ into backscattered Langmuir waves $L'$ and ion-sound waves $S$. The product $S$ waves can then stimulate the electromagnetic decay $L \leftrightarrow T(\omega_p) + S$ to give fundamental emission, and the $L'$ waves can coalesce with the $L$ waves to give harmonic emission, i.e., $L + L' \leftrightarrow T(2\omega_p)$.

There is strong evidence for the electrostatic decay of beam-driven Lang-
muir waves in the source regions for the radio emission. Langmuir waves are often observed simultaneously with ion-sound waves during type II \cite{Lengyel-Frey et al., 1997} and type III \cite{Cairns, 1984; Lin et al., 1986; Robinson et al., 1993; Cairns and Robinson, 1995a} solar radio bursts and planetary foreshock emission \cite{Anderson et al., 1981; Cairns and Melrose, 1985; Cairns, 1986; Koons et al., 1987}. The ion-sound waves are not always directly observed since their levels are sometimes below the instrumental noise level, but their presence can be inferred from beating Langmuir waveforms with beat frequencies consistent with the frequency of ion-sound waves \cite{Cairns and Robinson, 1992a; Gurnett et al., 1993; Cairns, 1995; Hospodarsky and Gurnett, 1995}. Spectral analyses have also shown that the frequency difference between peaks in the high frequency (Langmuir wave) spectrum matches the peak in the low frequency (ion-sound wave) spectrum, required for the wave matching conditions to be satisfied \cite{Cairns and Robinson, 1995a; Henri et al., 2009; Graham and Cairns, 2013}.

There is also observational evidence for three-wave interactions involving transverse waves. \textit{Cairns and Robinson} \cite{1995a} found that the observed ion-sound wave frequencies were consistent with electromagnetic decay operating after electrostatic decay had produced observable levels of ion-sound waves. There was also good agreement between the timing of the radio emissions at $\omega_p$ and $2\omega_p$ and the growth of ion-sound waves up to these observable levels. \textit{Bale et al.} \cite{1996} calculated the bicoherence spectrum of waves in Earth’s electron foreshock and found phase coherence between Langmuir waves of two similar frequencies near $\omega_p$ and a transverse wave near $2\omega_p$, giving support to the $L + L' \leftrightarrow T$ process for these emissions.

Another plasma emission mechanism is linear mode conversion. In the presence of inhomogeneities, previously uncoupled wave modes become coupled in frequency-wave vector space. This allows wave energy to be converted linearly from one mode to another with constant frequency \cite{Field, 1956; Melrose, 1980c; Stix, 1992}. Linear mode conversion of Langmuir waves into transverse waves has been invoked for the radio emissions in type II and III radio bursts and planetary foreshock emissions \cite{Field, 1956; Melrose, 1980d; Lin et al., 1981; Yin et al., 1998; Kim et al., 2007, 2008}.

Langmuir eigenmodes trapped in trapped in density wells have recently been observed in the solar wind and Earth’s foreshock \cite{Ergun et al., 2008; Malaspina and Ergun, 2008; Graham et al., 2012a}. These eigenmodes can generate radio emission at $\omega_p$ \cite{Malaspina et al., 2012} and $2\omega_p$ \cite{Malaspina et al., 2010} due to radiation from nonlinear currents in the eigenmode structures, known as the
antenna mechanism. Strong evidence exists for Langmuir eigenmodes occurring frequently, and radio emission intensities from the eigenmodes were predicted to be high enough to account for a significant portion of the observed radiation.

Wave packet collapse is the process by which a localized Langmuir wave packet intensifies in field strength and decreases in size due to the ponderomotive force and nonlinear self-focusing; the ponderomotive force expels plasma from where the field is strongest and nonlinear self-focusing is the tendency of Langmuir waves to refract into and intensify in low density (high refractive index) regions of plasma [Zakharov, 1972; Zakharov et al., 1985; Robinson, 1997]. If the wave packet is intense enough the combined effects of the ponderomotive force and self-focusing will overcome linear dispersion and lead to collapse.

Antenna radiation resulting from Langmuir wave packet collapse has also been suggested for heliospheric radio emissions [Papadopoulos and Freund, 1978; Goldstein et al., 1979; Goldman et al., 1980]. However, numerous observations of localized Langmuir wave packets in planetary foreshocks and in type III sources show that the electric field strengths are typically an order of magnitude too small for wave packet collapse to occur [Cairns and Robinson, 1992b; Gurnett et al., 1993; Cairns and Robinson, 1995b; Bale et al., 1997; Graham et al., 2012a; Graham et al., 2012b].

Yoon [1995] proposed a nonlinear beam instability for the generation of plasma emission. The emission process involved the combined effects of the excitation of electrostatic waves and their concurrent conversion into electromagnetic waves by a nonlinear mode coupling process. This theory was applied to the radio emission from Earth’s foreshock. Several theoretical questions remain unanswered about this process [Cairns and Robinson, 2000].

To determine which of these mode conversion processes is dominant for particular observed radio emissions, it is crucial to have accurate calculations of the rates for each process. Currently it is not clear whether the approximate rates for three-wave interactions and nonlinear wave-particle scattering are sufficiently accurate for all the plasma conditions of interest to make a definite determination of the dominant process. The calculation of more accurate rates for these processes in this thesis will allow for better comparisons between data and theory for these heliospheric radio emissions.

We now briefly discuss the electron velocity distributions in space plasmas. For sufficiently high collision frequencies a plasma is in local thermodynamic equilibrium and the distribution function is Maxwellian. However, space plasmas are typically collisionless and hence non-Maxwellian distributions are often observed. The generalized Lorentzian (or “kappa”) distribution function de-
fined by

\[ f(v) = \frac{n}{(2\pi\kappa)^{3/2}V^3} \frac{\Gamma(\kappa + 1)}{\Gamma(\kappa - 1/2)} \left( 1 + \frac{v^2}{2\kappa V^2} \right)^{-(\kappa+1)}, \quad \kappa > \frac{3}{2} \]  

was introduced by Vasyliunas [1968] to model the electron distribution function in Earth’s magnetospheric plasma sheet. The parameter \( \kappa \) is called the spectral index, and \( \Gamma(x) \) is the gamma function [Olver et al., 2010]. The kappa distribution is Maxwellian-like for small \( v \) and power-law-like for large \( v \). As \( \kappa \to \infty \) the kappa distribution approaches the Maxwellian distribution, whereas for smaller \( \kappa \) the distribution has a longer tail. Kappa distributions of electrons have since been observed in the magnetospheres of Jupiter [Scudder et al., 1981] and Saturn [Schippers et al., 2008] and in the solar wind [Maksimovic et al., 1997a], and have been predicted to exist in the solar corona [Maksimovic et al., 1997b; Pierrard et al., 1999]. Their generation has been proposed in terms of particle acceleration by wave turbulence [Ma and Summers, 1999; Leubner, 2000; Yoon et al., 2005; Yoon, 2012], velocity-space Lévy flights [Collier, 1993], and as the equilibrium distribution in non-extensive statistics [Tsallis, 1988; Leubner, 2002].

The common presence of kappa distributions in space plasmas means that it is desirable to incorporate them in analyses of nonlinear processes. The first-order thermal correction to the quadratic response tensor that we derive in Chapter 2 is valid for any isotropic distribution function, and the subsequent analysis of the rate of second harmonic plasma emission is valid for a kappa distribution. However, our analyses of the quadratic longitudinal response function and the corresponding nonlinear rates in Chapters 3 and 4 are valid only for a Maxwellian distribution. Performing these analyses for a kappa distribution is a topic for future work.
Chapter 2

First-order thermal correction to the quadratic response tensor and rate for second harmonic plasma emission


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2.1 Abstract

Three-wave interactions in plasmas are described, in the framework of kinetic theory, by the quadratic response tensor (QRT). The cold-plasma QRT is a common approximation for interactions between three fast waves. Here, the first-order thermal correction (FOTC) to the cold-plasma QRT is derived for interactions between three waves with \( v_\phi > V_e \) in a warm unmagnetized collisionless plasma, whose particles have an arbitrary isotropic distribution function. The FOTC to the cold-plasma QRT is shown to depend on the second moment of the distribution function, the phase speeds of the waves, and the interaction geometry. Previous calculations of the rate for second harmonic plasma emission (via Langmuir-wave coalescence) assume the cold-plasma QRT. The FOTC to the cold-plasma QRT is used here to calculate the FOTC to the second harmonic emission rate, and its importance is assessed in various physical situations. The FOTC significantly increases the rate when the ratio of the Langmuir phase speed to the electron thermal speed is less than about 3.
2.2 Introduction

Plasma emission, which is the generation of radiation at multiples of the local electron plasma frequency $f_p$, is observed in various locations and phenomena in our solar system; these include type II [Wild and McCready, 1950; Wild et al., 1963; Cane et al., 1982] and type III [Wild and McCready, 1950; Wild et al., 1963; Suzuki and Dulk, 1985; Robinson and Cairns, 2000] solar radio bursts in the solar corona and interplanetary medium, and terrestrial foreshock emission [Dunckel, 1974; Gurnett, 1975; Hoang et al., 1981]. Although several mechanisms have been proposed for these emissions, such as linear mode conversion [Field, 1956; Melrose, 1980d; Yin et al., 1998] and cyclotron maser emission [Wu et al., 2002], they are generally attributed to three-wave interactions between Langmuir, transverse, and ion-sound waves [Ginzburg and Zheleznyakov, 1958; Wild et al., 1963; Melrose, 1980b; Nelson and Melrose, 1985; Cairns and Melrose, 1985]. Three-wave interactions include the coalescence of two waves to give a product wave, and the decay of one wave into two product waves. These processes occur due to the nonlinear response of the plasma medium to the wave fields. In kinetic theory, the response of a plasma to an electromagnetic disturbance is described by linear and nonlinear response tensors [Sagdeev and Galeev, 1969; Tsytovich, 1970; Sitenko, 1982; Melrose, 1980c]. Assuming the plasma response to be weakly nonlinear, induced plasma properties such as the induced charge and current densities can be expanded in powers of the amplitude of the electromagnetic field; this is termed the “weak-turbulence expansion” [Tsytovich, 1977; Melrose, 1980c; Sitenko, 1982]. The quadratic response tensor (QRT), defined as the coefficient of the second-order term in this expansion, describes the response of the plasma to two fields beating simultaneously to produce a third wave; therefore, the QRT is the relevant response tensor for three-wave interactions.

The general form of the QRT involves integrals over the velocity distribution function of the plasma particles. These integrals are difficult to evaluate exactly, and so the integrand is often approximated before performing the integrations, where the approximation that is made depends on the dispersion relations of the three wave modes involved. For interactions between three “fast” waves, that is, waves with a phase speed much greater than the thermal speed, the “cold-plasma approximation” to the QRT is often made, in which thermal effects are neglected in the description of wave coupling [Melrose, 1980c; Sitenko, 1982]. This approximation lends itself to a simple derivation of the response tensor, and the resulting expression can be readily used in practical applica-
tions. However, the cold-plasma approximation becomes less accurate as the phase speed of one or more of the interacting waves approaches the thermal speed, and its range of validity is poorly defined.

Conversely, the QRT has been evaluated exactly for a thermal plasma by Percival and Robinson [1998a], in terms of generalized plasma dispersion functions [Percival and Robinson, 1998b]. The resulting exact QRT accurately describes wave coupling in three-wave interactions between any wave modes. Although the use of the exact QRT is desirable, the expression is too cumbersome to apply analytically, and even its numerical evaluation presents difficulties as described in Sec. 2.3.

Due to the inaccuracy of the cold-plasma QRT at low phase speeds and the difficulty of applying the exact QRT, accurate approximations to the QRT are needed for a proper treatment of three-wave interactions. To this end, we derive here the first-order thermal correction (FOTC) to the cold-plasma QRT for a warm collisionless unmagnetized plasma, whose constituent particles have an arbitrary isotropic velocity distribution. The approximate response tensor derived in this chapter, which is the sum of the cold-plasma QRT and its FOTC, has the advantages that it is more accurate than the cold-plasma QRT and more tractable than the exact QRT. This approximation is valid for interactions between three waves with $v_\phi > V_e$, such as the Langmuir-wave coalescence process that generates second harmonic emission. We find that the FOTC depends on the second moment of the distribution function, the phase speeds of the waves, and the interaction geometry.

First-order thermal corrections to the quadratic response have been derived before in the literature, but the cases treated are not suitable for modeling the Langmuir-wave coalescence process in space plasmas. For example, a FOTC to the cold-plasma longitudinal quadratic susceptibility was previously derived by Sitenko [1982]; however, this quantity only describes interactions between three electrostatic waves with $v_\phi > V_e$ and is thus inadequate for treating second harmonic plasma emission, in which electromagnetic transverse waves are produced. The FOTC to the cold-plasma QRT was derived in Percival [1992] for a Maxwellian velocity distribution of the plasma particles, but space plasmas are commonly observed to have power-law tails which must be modeled by a non-Maxwellian distribution, often the generalized Lorentzian (or “kappa”) distribution [Vasyliunas, 1968; Maksimovic et al., 1997a]. The expression that we derive is equivalent to that in Percival [1992] when the velocity distribution is Maxwellian, but allows the treatment of three-wave processes in non-Maxwellian plasmas.
The rate of a three-wave interaction is dependent on the strength of the coupling between the waves, which is described by the QRT. Until now, the rate of second harmonic plasma emission via Langmuir-wave coalescence has been calculated assuming the cold-plasma approximation [Cairns, 1987b; Willes et al., 1996; Li et al., 2005]. We use the FOTC to the cold-plasma QRT to calculate the FOTC to the rate of Langmuir-wave coalescence, and assess its contribution to the total interaction rate in various situations. There is an increase in the rate by more than 100% when the ratio of the Langmuir phase speed to the electron thermal speed is less than about 3.

The chapter is structured as follows. In Sec. 2.3, the theory of response tensors and their derivation is described. In Sec. 2.4, we derive and discuss the FOTC to the cold-plasma QRT. The FOTC to the rate of second harmonic emission is derived in Sec. 2.5, where its importance in various situations is also analyzed.

2.3 Theoretical context

Nonlinear plasma response tensors are defined by expanding an induced plasma property, such as the induced current density, in powers of the amplitude of the Fourier transformed electromagnetic field; this is termed the weak-turbulence expansion. The QRT is the coefficient of the second-order term in this expansion. On choosing to describe the electromagnetic field by the vector potential $\mathbf{A}$ in the temporal gauge, the induced current density is given in Fourier space by (e.g., Melrose [1980c])

$$J_i(k) = \sum_{n=1}^{\infty} J_i^{(n)}(k),$$

(2.1)

where

$$J_i^{(n)}(k) = \int d\lambda^{(n)} \alpha_{ij_1j_2...j_n}(k, k_1, \ldots, k_n) A_{j_1}(k_1)A_{j_2}(k_2) \cdots A_{j_n}(k_n).$$

(2.2)

In Eqs (2.1) and (2.2), $k_m$ collectively denotes $\omega_m$ and $k_m$ for the $m$th wave, and $d\lambda^{(n)}$ is the $n$th-order convolution integral given by

$$d\lambda^{(n)} = \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \cdots \frac{d^4k_n}{(2\pi)^4} (2\pi)^4 \delta^4(k - k_1 - \ldots - k_n),$$

(2.3)

with

$$d^4k = d\omega d^3\mathbf{k},$$

(2.4)
and
\[ \delta^4(k) = \delta(\omega)\delta^3(k). \] (2.5)

The nonlinear response tensors are most commonly calculated via the Vlasov equation, which relates the distribution function to the wave fields for a collisionless plasma. Solving the Vlasov equation by employing a weak-turbulence expansion of the distribution function and expressing the induced current as a moment of the distribution yields the general form of the QRT (e.g., Percival and Robinson [1998a]),

\[ \alpha_{ijl}(k, k_1, k_2) = \frac{q^3}{2m^2} \left\{ k_i + \left( |k|^2 - \omega^2/c^2 \right) \frac{\partial}{\partial k_i} \right\} \left[ I(k)\delta_{jl} + k_{j1}J_{lj}(k, k_2) + k_{i1}J_{lj}(k, k_1) + (k_1 \cdot k_2 - \omega_1\omega_2/c^2)K_{lj}(k, k_1, k_2) \right] + (i, k) \leftrightarrow (j, k_1) + (i, k) \leftrightarrow (l, k_2). \] (2.6)

where \((i, k) \leftrightarrow (j, k_1)\) represents the additional terms generated from those written explicitly by interchanging \(k\) and \(k_1\) and the associated tensor indices, and the integrals \(I(k), J_i(k_1, k_2),\) and \(K_{ij}(k, k_1, k_2)\) are given by

\[ I(k) = \int d^3v f(v) \frac{1}{\omega - k \cdot v}, \] (2.7)

\[ J_i(k_1, k_2) = \int d^3v f(v) \frac{v_i}{(\omega_1 - k_1 \cdot v)(\omega_2 - k_2 \cdot v)}, \] (2.8)

\[ K_{ij}(k, k_1, k_2) = \int d^3v f(v) \frac{v_i v_j}{(\omega - k \cdot v)(\omega_1 - k_1 \cdot v)(\omega_2 - k_2 \cdot v)}. \] (2.9)

It was noted in Percival and Robinson [1998a] that the partial derivative \(\partial K_{ij}/\partial k_i\) cannot be taken without ambiguity since one of the wave vectors in the integrand is no longer independent of the other two. The integral

\[ L_{ijl} = \frac{\partial K_{ijl}}{\partial k_i} \]

\[ = \int d^3v f(v) \frac{v_i v_j v_l}{(\omega - k \cdot v)^2(\omega_1 - k_1 \cdot v)(\omega_2 - k_2 \cdot v)} \] (2.10)

must therefore be evaluated directly.

An alternative expression for the general QRT involves partial derivatives with respect to velocity, rather than wave vector as in Eq. (2.6). These partial derivatives in the alternative expression have been evaluated explicitly in
Eq. (5) of Yoon [2005]. Our detailed analyses, not reproduced here, show that this equation is identical to Eq. (2.6) above. Equation (5) of Yoon [2005] could also be used for analyses similar to those below.

Thermal effects in the wave coupling are often ignored for interactions between three fast waves; neglecting these effects is known as the cold-plasma approximation. The cold-plasma QRT may be calculated by substituting $f(v) = n\delta^3(v)$ into Eqs (2.7)–(2.10) then evaluating the integrals. The integrals $J$, $K$, and $L$ vanish, and $I(k)$ is replaced by $n/\omega$. This leads to

$$
\alpha_{ijl}^{(\text{cold})}(k, k_1, k_2) = \frac{q^3 n}{2m^2} \left( \frac{k_i \delta_{jl}}{\omega} + \frac{k_{1j} \delta_{il}}{\omega_1} + \frac{k_{2l} \delta_{ij}}{\omega_2} \right). \tag{2.11}
$$

The cold-plasma approximation is made in order to simplify the mathematical analysis, but the neglected thermal effects may become significant in some circumstances.

Percival and Robinson [1998a] have calculated the integrals in Eqs (2.7)–(2.10) exactly for a plasma in which the particles have a Maxwellian velocity distribution. This analysis yields an exact expression for the QRT shown in Eq. (1.89) in terms of generalized plasma dispersion functions [Percival and Robinson, 1998b], but calculating the interaction rate (as described below) is problematic because it involves integrals of the squared modulus of the QRT contracted with the relevant polarization tensors. Due to the large number of generalized plasma dispersion functions in the exact QRT and the numerical instability associated with their computation (described in Chapter 3), as well as the possible occurrence of catastrophic cancellation between the various terms, it is infeasible to use this response tensor directly in the calculation of rates.

Once the response tensors are known, the emission and absorption of waves can be studied. These processes are often treated semiclassically [Tsytovich, 1970; Melrose, 1980c]; the waves are interpreted as a collection of wave quanta with momentum $h\mathbf{k}$ and energy $h|\omega_M(k)|$. The occupation number $N_M(k)$ is introduced, being defined as the number density of wave quanta within the elemental range $d^3k$ of $\mathbf{k}$. This quantity is related to the energy density per unit volume of $k$-space, $W_M(k)$, by

$$
N_M(k) = \frac{W_M(k)}{h|\omega_M(k)|}. \tag{2.12}
$$

In a coalescence process the current $J^{(2)}(k)$ induced by the simultaneous response of the plasma to two wave fields $\mathbf{A}_P(k_1)$ and $\mathbf{A}_Q(k_2)$, given by Eq. (2.2),
is the source of a third wave field $A_M(k)$, where $k = k_1 + k_2$ as implied by the delta function. Assuming the random phase approximation, the kinetic equation for the wave mode $M$ in the three-wave interaction $P(k_1) + Q(k_2) \leftrightarrow M(k)$ is given by (e.g., Tsytovich [1970]; Melrose [1980c])

$$\frac{\partial N_M(k)}{\partial t} = \int \frac{d^3 k_1}{(2\pi)^3} \int \frac{d^3 k_2}{(2\pi)^3} u_{MPQ}(k, k_1, k_2) \times \{N_P(k_1)N_Q(k_2) - N_M(k)[N_P(k_1) + N_Q(k_2)]\}. \tag{2.13}$$

Alternatively the rate can be expressed in terms of $T_M$, the effective temperature for the wave mode $M$, as

$$\frac{\partial T_M(k)}{\partial t} = \int \frac{d^3 k_1}{(2\pi)^3} \int \frac{d^3 k_2}{(2\pi)^3} u_{MPQ}(k, k_1, k_2) \times \{T_P(k_1)T_Q(k_2) - T_M(k)[T_P(k_1) + T_Q(k_2)]\} \times \omega_M(k)/[\hbar \omega_P(k_1)\omega_Q(k_2)], \tag{2.14}$$

where the effective temperature is related to the occupation number by

$$T_M(k) = \hbar \omega_M(k) N_M(k). \tag{2.15}$$

The equation for the interaction probability $u_{MPQ}$ is (e.g., Tsytovich [1970] and Melrose [1980c])

$$u_{MPQ}(k, k_1, k_2) = \frac{4\hbar}{e_0^3} \frac{R_M(k)R_P(k_1)R_Q(k_2)}{[\omega_M(k)\omega_P(k_1)\omega_Q(k_2)]^2} |\alpha_{MPQ}(k_M, k_{P1}, k_{Q2})|^2 \times (2\pi)^4 \delta^4(k_M - k_{P1} - k_{Q2}), \tag{2.16}$$

where $R_M$ is the ratio of electric to total energy in the wave mode $M$, and

$$\alpha_{MPQ}(k_M, k_{P1}, k_{Q2}) = \alpha_{ijl}(k_M, k_{P1}, k_{Q2}) e_{M_i}^*(k)e_{P_j}(k_1)e_{Q_l}(k_2), \tag{2.17}$$

with $e_M(k)$ the polarization vector for the wave mode $M$. The quantity $k_M$ collectively denotes $\omega_M(k)$ and $k$, and similarly for $k_{P1}$ and $k_{Q2}$. In Eq. (2.16), the delta function in wave vector is interpreted in the semiclassical description as expressing conservation of momentum,

$$k = k_1 + k_2, \tag{2.18}$$

and the delta function in frequency expresses conservation of energy,

$$\omega_M(k) = \omega_P(k_1) + \omega_Q(k_2), \tag{2.19}$$
where the common factor of $\hbar$ is omitted.

We note that kinetic equations for three-wave interactions have also been derived using the theory of electromagnetic fluctuations, also known as the “statistical mechanical” approach [Gorbunov et al., 1965; Sitenko, 1982; Yoon, 2006]. The kinetic equation for the Langmuir-wave coalescence process in Yoon [2006] is given, in our notation, by

$$\frac{\partial N_T(k)}{\partial t} = \int \frac{d^3 k_1}{(2\pi)^3} \int \frac{d^3 k_2}{(2\pi)^3} u_{TLL'}(k, k_1, k_2) \times \left\{ N_L(k_1)N_L'(k_2) - N_T(k)[N_L(k_1) + N_L'(k_2)]/2 \right\}. \quad (2.20)$$

Thus, the semiclassical and statistical mechanical approaches lead to the same kinetic equation, except that the factor of $\frac{1}{2}$ multiplying $N_T$ in Eq. (2.20) does not appear in Eq. (2.13); this difference is not understood at present. It is shown in Sec. 2.5.1 below that this difference does not change the ratio of the FOTC rate to the cold-plasma rate.

### 2.4 First-order thermal correction to the cold-plasma quadratic response tensor

In this section, the general quadratic response tensor (QRT) in Eq. (2.6) is approximated by deriving the first-order thermal correction (FOTC) to the cold-plasma QRT given by Eq. (2.11). We discuss the expression obtained for the FOTC, including its importance and validity, for different interactions.

#### 2.4.1 Derivation

First the resonant denominators in Eqs (2.7)–(2.10) are binomially expanded in powers of $k_m \cdot \nu/\omega_m$ (which may be expressed as $v_\parallel /v_\phi$ where $\parallel$ is with respect to $k_m$) using the binomial expansion $(1 - x)^{-1} = \sum_{n=0}^{\infty} x^n$. The $v_\phi$ in the denominator of the expanded quantity ensures that this expansion of the integrals is convergent, as shown below. We show that this expansion recovers the cold-plasma QRT plus additional terms of order $(V/v_\phi)^{2n}$ relative to the cold-plasma terms, where $v_\phi$ is the phase speed, $V$ is the thermal speed of the particles, and $n$ is an integer. The $n = 1$ terms are called the FOTC to the cold-plasma QRT.

Two additional assumptions are made to derive the FOTC: the first is that the distribution function is isotropic, and the second is that the thermal effects
for each of the three wave fields can be treated equally, which will be discussed after the derivation.

A binomial expansion of the resonant denominators of the integrals in Eqs (2.7)–(2.10) gives

\[
I(k) = \frac{1}{\omega} \int d^3v f(v) \left[ 1 + \frac{k \cdot v}{\omega} + \left( \frac{k \cdot v}{\omega} \right)^2 + \mathcal{O} \left( \frac{k \cdot v}{\omega} \right)^3 \right], \quad (2.21)
\]

\[
\approx \frac{1}{\omega} \int d^3v f(v) + \frac{k_q}{\omega^2} \int d^3v f(v) v_q + \frac{k_r k_s}{\omega^3} \int d^3v f(v) v_r v_s, \quad (2.22)
\]

\[
J_i(k, k_1) = \frac{1}{\omega \omega_1} \int d^3v f(v) v_i \left[ 1 + \frac{k \cdot v}{\omega} + \frac{k_1 \cdot v}{\omega_1} + \mathcal{O} \left( \frac{k \cdot v}{\omega} \right)^2 \right], \quad (2.23)
\]

\[
\approx \frac{1}{\omega \omega_1} \int d^3v f(v) v_i + \frac{1}{\omega \omega_1} \left( \frac{k_s}{\omega} + \frac{k_{s_1}}{\omega_1} \right) \int d^3v f(v) v_i v_s, \quad (2.24)
\]

\[
K_{ij}(k, k_1, k_2) = \frac{1}{\omega \omega_1 \omega_2} \int d^3v f(v) v_i v_j \left[ 1 + \mathcal{O} \left( \frac{k \cdot v}{\omega} \right) \right], \quad (2.25)
\]

\[
\approx \frac{1}{\omega \omega_1 \omega_2} \int d^3v f(v) v_i v_j, \quad (2.26)
\]

\[
L_{ijl}(k, k_1, k_2) = \frac{1}{\omega \omega_1 \omega_2} \int d^3v f(v) v_i v_j v_l \left[ 1 + \mathcal{O} \left( \frac{k \cdot v}{\omega} \right) \right], \quad (2.27)
\]

\[
\approx \frac{1}{\omega \omega_1 \omega_2} \int d^3v f(v) v_i v_j v_l. \quad (2.28)
\]

We let \( v_i = \delta_{iq} v_q \) and \( v_i = \delta_{ir} v_r \) in the first and second integrals in Eq. (2.24) respectively. Then substituting the binomially approximated \( I, J, K, \) and \( L \) given by Eqs (2.21)–(2.28) respectively into the general QRT in Eq. (2.6) gives

\[
\alpha_{ijl} = \frac{q^3}{2m^2} \left\{ k_i + (|k|^2 - \omega^2/c^2) \frac{\partial}{\partial k_i} \right\} \left[ \frac{\delta_{jl}}{\omega} \int d^3v f(v) + A_q(v) \left( \frac{k_q \delta_{jl}}{\omega^2} \right) + B_{rs}(v) \left( \frac{k_r k_s \delta_{jl}}{\omega^3} + \frac{k_{r_1} \delta_{jl}}{\omega^2} + \frac{k_{s_1} \delta_{jl}}{\omega} + \frac{k_{s_2}}{\omega} + \frac{k_{s_3}}{\omega_1} + \frac{k_{s_4}}{\omega_2} \right) \times \left( \frac{k_s}{\omega} + \frac{k_{s_1}}{\omega_1} \right) \right\} + \frac{(k_1 \cdot k_2 - \omega_1 \omega_2/c^2) \frac{\partial}{\partial k_l}}{\omega \omega_1 \omega_2} \left[ \frac{k_i}{1 \cdot k_2} + \frac{\omega^2}{\omega_2} \right] C_{ijl}(v) + (i, k) \leftrightarrow (j, k_1) + (i, k) \leftrightarrow (l, k_2), \quad (2.29)
\]
where the arguments of the response tensor have been omitted for brevity. Here we have defined

\[ A_i(v) = \int d^3v f(v)v_i, \]  
\[ B_{ij}(v) = \int d^3v f(v)v_i v_j, \]  
\[ C_{ijl}(v) = \int d^3v f(v)v_i v_j v_l. \]

We now re-express the integrals over velocity space in terms of moments of the distribution function \( f(v) \). The distribution function is defined such that the particle number density \( n \) is given by

\[ n = \int d^3v f(v). \]

The moment of a quantity \( Q(v) \) is defined by

\[ \langle Q(v) \rangle = \frac{1}{n} \int d^3v f(v)Q(v). \]

We make the assumption that the distribution function is isotropic, hence \( f(v) = f(v) \) where \( v = (v_x^2 + v_y^2 + v_z^2)^{1/2} \) and so \( f \) is an even function of \( v_x, v_y, \) and \( v_z \). In Cartesian coordinates, the tensor indices run over \( x, y, \) and \( z \). The integrals \( A_i \) and \( C_{ijl} \) then vanish because every choice of indices gives an integrand that is odd in one or all of the variables \( v_x, v_y, \) and \( v_z \), and the integration limits are symmetric about the origin. By symmetry in \( v_x, v_y, \) and \( v_z \) all the diagonal components of \( B_{ij} \) are equal, as are the off-diagonal components. The off-diagonal components of the \( B_{ij} \) vanish due to oddness of the integrand, so \( B_{ij} = B\delta_{ij} \) where \( B = B_{xx} = B_{yy} = B_{zz} \). To calculate \( B \) we note that \( B = (B_{xx} + B_{yy} + B_{zz})/3 \) and so

\[ B_{ij} = \frac{\delta_{ij}}{3} \int d^3v f(v)(v_x^2 + v_y^2 + v_z^2), \]  
\[ = \frac{n\langle v^2 \rangle \delta_{ij}}{3} \]

from Eq. (2.34).

Substituting Eqs (2.33) and (2.36), and \( A_q = C_{ijl} = 0 \), into Eq. (2.29) and
simplifying gives
\[
\alpha_{ijl} = \frac{q^3 n}{2m^2} \left[ k_i + (|k|^2 - \omega^2 / c^2) \frac{\partial}{\partial k_i} \right] \left[ \frac{\delta_{ij} \omega}{3} \left\{ \frac{|k|^2 \delta_{jl}}{\omega^3} + \frac{k_{2j} \omega}{\omega_1 \omega_2} \right\} \right.
\]
\[
\times \left\{ \frac{k_1 + k_{2i}}{\omega} + \frac{k_{1l}}{\omega_1} \left( \frac{k_j + k_{1j}}{\omega + \frac{\omega_3}{\omega_1}} \right) \right\} + \frac{(v^2) \delta_{jl} k_1 (k_1 \cdot k_2 - \omega_1 \omega_2 / c^2)}{3 \omega_1 \omega_2} + (i, k) \leftrightarrow (j, k_1) + (i, k) \leftrightarrow (l, k_2) \right\}.
\]  
(2.37)

On performing the partial derivative in Eq. (2.37), using \( \partial |k|^2 / \partial k_i = 2k_i \) and \( \partial k_j / \partial k_i = \delta_{ij} \), we have
\[
\alpha_{ijl} = \frac{q^3 n}{2m^2} \left[ k_i + (|k|^2 - \omega^2 / c^2) \frac{\partial}{\partial k_i} \right] \left[ \frac{\delta_{ij} \omega}{3} \left\{ \frac{|k|^2 \delta_{jl}}{\omega^3} + \frac{k_{2j} \omega}{\omega_1 \omega_2} \right\} \right.
\]
\[
\times \left\{ \frac{k_1 + k_{2i}}{\omega} + \frac{k_{1l}}{\omega_1} \left( \frac{k_j + k_{1j}}{\omega + \frac{\omega_3}{\omega_1}} \right) \right\} + \frac{(v^2) \delta_{jl} k_1 (k_1 \cdot k_2 - \omega_1 \omega_2 / c^2)}{3 \omega_1 \omega_2} + (i, k) \leftrightarrow (j, k_1) + (i, k) \leftrightarrow (l, k_2) \right\}.
\]  
(2.38)

Rearrangement and factorization of Eq. (2.38) yields
\[
\alpha_{ijl} = \frac{q^3 n}{2m^2} \left[ k_i + (|k|^2 - \omega^2 / c^2) \frac{\partial}{\partial k_i} \right] \left[ \frac{\delta_{ij} \omega}{3} \left\{ \frac{|k|^2 + 2(|k|^2 - \omega^2 / c^2) + k_1 \cdot k_2 - \omega_1 \omega_2 / c^2}{\omega_1 \omega_2} \right\} \right.
\]
\[
\times \left\{ \frac{k_1 + k_{2i}}{\omega} + \frac{k_{1l}}{\omega_1} \left( \frac{k_j + k_{1j}}{\omega + \frac{\omega_3}{\omega_1}} \right) \right\} + \frac{(v^2) \delta_{jl} k_1 (k_1 \cdot k_2 - \omega_1 \omega_2 / c^2)}{3 \omega_1 \omega_2} + (i, k) \leftrightarrow (j, k_1) + (i, k) \leftrightarrow (l, k_2) \right\}.
\]  
(2.39)

The first term inside the braces and its interchanges are identified as the cold-plasma QRT given by Eq. (2.11); the remaining terms are the FOTC to Eq. (2.11), denoted by \( \Delta \alpha_{ijl} \). That is, \( \alpha_{ijl} = \alpha_{ijl}^{(\text{cold})} + \Delta \alpha_{ijl} \). In the nonrelativistic \((c \to \infty)\) limit, the FOTC to the cold-plasma QRT, for an isotropic particle velocity distribution, is
\[
\Delta \alpha_{ijl} = \frac{q^3 n}{2m^2} \frac{(v^2)}{3} \left\{ \left[ \frac{3|k|^2}{\omega^2} + \frac{k_1 \cdot k_2}{\omega_1 \omega_2} \right] \frac{k_i \delta_{jl}}{\omega} + \frac{|k|^2}{\omega^2} \left[ \frac{k_{1l} \delta_{ij}}{\omega_1} + \frac{k_{2j} \delta_{il}}{\omega_2} \right] + \frac{k_i}{\omega}
\]
\[
\times \left( \frac{k_{1l} k_j}{\omega_1^2} + \frac{k_{1l} k_{1j}}{\omega_2} + \frac{k_{2j} k_l}{\omega_2^2} + \frac{k_{2j} k_{2l}}{\omega_2^2} \right) \right\} + (i, k) \leftrightarrow (j, k_1) + (i, k) \leftrightarrow (l, k_2) \right\}.
\]  
(2.40)
2.4.2 Discussion

The particle temperature $T$ is related to the second moment of the distribution function by the definition $k_B T = m \langle v^2 \rangle / 3$, where $k_B$ is Boltzmann’s constant. For a Maxwellian distribution

$$f(v) = \frac{n}{(2\pi)^{3/2} V^3} e^{-v^2/2V^2}, \quad (2.41)$$

the thermal speed is given by $V = \sqrt{k_B T/m}$, whence $\langle v^2 \rangle / 3$ is replaced by $V^2$ in Eq. (2.40); this reproduces the expression for $\Delta \alpha_{ijl}$ in Percival [1992].

The terms in the cold-plasma QRT and FOTC are of order $1/v_{\phi}$ and $V^2/v_{\phi}^3$ respectively, from Eqs (2.11) and (2.40), where $v_{\phi}$ here represents the phase speeds of any of the three interacting waves. The ratio of these orders, $V^2/v_{\phi}^2$, therefore determines the significance of the FOTC: if it is much less than unity then the cold plasma approximation will be accurate, but as it approaches unity the cold plasma approximation begins to break down and the FOTC must be included for an accurate description of the plasma response.

From Eqs (2.6) and (2.21)–(2.28), it follows that higher-order thermal corrections to the cold-plasma QRT involve integrals of the form

$$S_{i_1i_2\cdots i_n}(v) = \int d^3v f(v)v_{i_1}v_{i_2}\cdots v_{i_n}, \quad (2.42)$$

where $i_m$ are tensor indices running over $x$, $y$, and $z$. For odd $n$, each choice of indices will give an odd power in at least one of the variables $v_x$, $v_y$, or $v_z$, hence in this case the integral will be zero. Therefore, the $n$th-order thermal correction will be $O \{ (V/v_{\phi})^{2n} \}$ relative to the cold-plasma terms. For the expansion of the general QRT by a binomial expansion of the resonant denominators to be convergent, one requires $v_{\phi} > V$ for each wave mode.

Thermal effects from one or two of the participating waves may be more important than those from the other wave or waves. This is the situation for the Langmuir-wave coalescence process: although both Langmuir and transverse waves have $v_{\phi} > V_e$, the thermal effects from the Langmuir waves will be more important since they have a significantly lower phase speed than the transverse wave, and so the terms involving the transverse wave phase speed may be neglected. Equation (2.40) may also be applied to the Raman scattering process $L + T \leftrightarrow T'$ which has been proposed for third and higher harmonic emission [Zlotnik, 1978; Cairns, 1987c; Rhee et al., 2009]. However, the FOTC will not be as important as in Langmuir-wave coalescence since Raman scattering involves two transverse waves and only one Langmuir wave.
2.5 First-order thermal correction to the second harmonic emission rate

In this section the FOTC to the cold-plasma QRT, given by Eq. (2.40), is applied to the rate calculation of second harmonic plasma emission via Langmuir-wave coalescence. The ratio of the FOTC rate to the cold-plasma rate is derived, and its range of values is calculated for different physical situations to assess the importance of the FOTC.

2.5.1 Derivation

The FOTC to quantities in this section will be denoted by the prefix ∆, such that

\[ x \approx x^{(\text{cold})} + \Delta x \]

for some quantity \( x \). We define the electron thermal speed by

\[ v_e = \sqrt{\langle v^2 \rangle / 3} \]

where the angle brackets denote the moment of the electron distribution function. Primary Langmuir waves \( L(k_1) \) are assumed to be generated by an electron beam via the bump-on-tail instability, such that the phase speed of the Langmuir waves is approximately equal to the speed of the electron beam, i.e., \( v_{\phi 1} \approx v_b \). Backscattered Langmuir waves \( L'(k_2) \), with which the primary waves coalesce, are assumed to be the product of the decay process \( L \leftrightarrow L' + S \) [Melrose, 1982; Cairns, 1987b; Robinson and Cairns, 1998a,b,c; Li et al., 2005]. Since the mass of the ions is much greater than that of the electrons and the QRT has a \( m^{-2} \) dependence for each particle species, the ionic contribution to the QRT is neglected.

We first outline the derivation of the cold-plasma interaction probability for Langmuir-wave coalescence (see, e.g., Melrose [1980c]). Transverse waves have \( e_T \cdot \kappa_T = 0 \) and hence \( e_T^* \cdot \kappa_T^* = 0 \). In the case of no spatial damping, \( \kappa_T \) is real and so \( e_T^* \cdot \kappa_T = 0 \). On contracting the cold-plasma QRT in Eq. (2.11) with \( e_{T i}^*(k_T) e_{Lj}(k_1) e_{Ll}(k_2) \), i.e., with \( e_{T i}^* \kappa_{1j} \kappa_{2l} \), where \( \kappa = k / |k| \), we have

\[ \alpha_{TLL'}^{(\text{cold})}(k_T, k_1, k_2) = -\frac{e^3 n_e}{2 m_e^2} \left( \frac{e_{T i}^* \cdot \kappa_1}{v_{\phi 2}} + \frac{e_{T j}^* \cdot \kappa_2}{v_{\phi 1}} \right) ; \] (2.43)

hence, using \( |x + y|^2 = |x|^2 + |y|^2 + 2 \text{Re}\{x^*y\} \), we find

\[ \left| \alpha_{TLL'}^{(\text{cold})} \right|^2 = \frac{e^6 n_e^2}{4 m_e^4} \left( \frac{|e_{T i}^* \cdot \kappa_1|^2}{v_{\phi 2}^2} + \frac{|e_{T j}^* \cdot \kappa_2|^2}{v_{\phi 1}^2} + 2 \text{Re} \left( \frac{(e_{T i}^* \cdot \kappa_1)(e_{T j}^* \cdot \kappa_2)}{v_{\phi 1} v_{\phi 2}} \right) \right) . \] (2.44)

When the polarization of the transverse waves \( T \) is of no interest, an average over the two initial states of polarization and a sum over the two final states...
of polarization is performed. This leads to the replacement
\[ |e_T^* \cdot \kappa_1|^2 = |\kappa_T \times \kappa_1|^2 / 2 \]

(2.45)

and similarly for \(|e_T^* \cdot \kappa_2|^2\). Making the approximations that \(R_L \approx \frac{1}{2}\), \(\omega_L \approx \omega_p\), and \(\omega_T \approx 2\omega_p\), and since \(R_T = \frac{1}{2}\), we have the interaction probability for a cold plasma as
\[ u_{\text{TLL'}}^{(\text{cold})} \approx \frac{\hbar}{4e^3 \omega_B^3} \left| \alpha_{\text{TLL'}}^{(\text{cold})} \right|^2 (2\pi)^4 \delta^4 (k_T - k_1 - k_2). \]

(2.46)

To obtain \(\Delta \alpha_{\text{TLL'}}\), Eq. (2.40) is contracted with \(e_{T,ij}\). Grouping \(\Delta \alpha_{\text{TLL'}}\) by order in \(v_{\phi T}\) gives
\[ \Delta \alpha_{\text{TLL'}} = \Delta \alpha_{\text{TLL'}}^{(0)} + \Delta \alpha_{\text{TLL'}}^{(1)} + \Delta \alpha_{\text{TLL'}}^{(2)} \]

(2.47)

where
\[ \Delta \alpha_{\text{TLL'}}^{(0)} = -\frac{e^3}{2m_e^2 v_e^2} \left[ \frac{3 e_T^* \cdot \kappa_2}{v_{\phi 1}^3} + \frac{2(e_T^* \cdot \kappa_2)(\kappa_1 \cdot \kappa_2) + e_T^* \cdot \kappa_1}{v_{\phi 1}^2 v_{\phi 2}} \right] + (\kappa_1, v_{\phi 1}) \leftrightarrow (\kappa_2, v_{\phi 2}), \]

(2.48)

\[ \Delta \alpha_{\text{TLL'}}^{(1)} = -\frac{e^3}{2m_e^2 v_e v_{\phi T}} \left[ \frac{2(\kappa_T \cdot \kappa_2)(e_T^* \cdot \kappa_1)}{v_{\phi 1}^2} + \frac{(\kappa_T \cdot \kappa_2)(e_T^* \cdot \kappa_2)}{v_{\phi 1} v_{\phi 2}} \right] + (\kappa_1, v_{\phi 1}) \leftrightarrow (\kappa_2, v_{\phi 2}), \]

(2.49)

\[ \Delta \alpha_{\text{TLL'}}^{(2)} = -\frac{e^3}{2m_e^2 v_e v_{\phi T}} \left[ \frac{(\kappa_1 \cdot \kappa_2)(e_T^* \cdot \kappa_1)}{v_{\phi 1}} + (\kappa_1, v_{\phi 1}) \leftrightarrow (\kappa_2, v_{\phi 2}) \right]. \]

(2.50)

From these equations, \(\Delta \alpha_{\text{TLL'}}^{(0)} \approx \mathcal{O}(V_e^2/v_b^3)\), \(\Delta \alpha_{\text{TLL'}}^{(1)} \approx \mathcal{O}(V_e^2/v_b^2 v_{\phi T})\), and \(\Delta \alpha_{\text{TLL'}}^{(2)} \approx \mathcal{O}(V_e^2/v_b v_{\phi T}^2)\). The electron beam speed is typically less than a few tenths of the speed of light and \(v_{\phi T} > c\), so to first order \(\Delta \alpha_{\text{TLL'}} \approx \Delta \alpha_{\text{TLL'}}^{(0)}\).

Including the first order thermal correction in the nonlinear response implies \(\alpha_{\text{TLL'}} = \alpha_{\text{TLL'}}^{(\text{cold})} + \Delta \alpha_{\text{TLL'}}\). Hence, to the next order after the cold-plasma term,
\[ |\alpha_{\text{TLL'}}|^2 = \left| \alpha_{\text{TLL'}}^{(\text{cold})} \right|^2 + 2 \text{Re} \left[ (\alpha_{\text{TLL'}}^{(\text{cold})})^* \Delta \alpha_{\text{TLL'}}^{(0)} \right]. \]

(2.51)

So, the second term on the right hand side of Eq. (2.51) is the first order...
correction to $|\alpha_T LL'|^2$, which is then

$$\Delta \left( |\alpha_T LL'|^2 \right) = \frac{e^4 n_e V_e^2}{2 m_e^2} \Re \left\{ \frac{3|\mathbf{e}_T^* \cdot \mathbf{\kappa}_2|^2}{v_{\phi_1}^4} + \frac{2(\mathbf{e}_T^* \cdot \mathbf{\kappa}_2)(\mathbf{\kappa}_1 \cdot \mathbf{\kappa}_2) + \mathbf{e}_T^* \cdot \mathbf{\kappa}_1}{v_{\phi_1}^4 v_{\phi_2}} (\mathbf{e}_T^* \cdot \mathbf{\kappa}_2)^* \right\}$$

$$+ \frac{3(\mathbf{e}_T^* \cdot \mathbf{\kappa}_2)(\mathbf{e}_T^* \cdot \mathbf{\kappa}_1)^*}{v_{\phi_1}^4 v_{\phi_2}} + \frac{2(\mathbf{e}_T^* \cdot \mathbf{\kappa}_2)(\mathbf{\kappa}_1 \cdot \mathbf{\kappa}_2) + \mathbf{e}_T^* \cdot \mathbf{\kappa}_1}{v_{\phi_1}^2 v_{\phi_2}^2} (\mathbf{e}_T^* \cdot \mathbf{\kappa}_1)^*$$

$$+ (\kappa_1, v_{\phi_1}) \rightarrow (\kappa_2, v_{\phi_2}) \right\}. \quad (2.52)$$

From Eqs (2.16), (2.51), and (2.52), the FOTC to the interaction probability is

$$\Delta u_{T LL'} \approx \frac{\hbar}{4 e_0^3 \omega_p^3} \Delta \left( |\alpha_T LL'|^2 \right) (2\pi)^4 \delta^4(k_T - k_1 - k_2). \quad (2.53)$$

The FOTC to the interaction rate is next calculated in terms of the effective temperature using a modified system of spherical coordinates (as in Li et al. [2005]), where the Langmuir wave numbers $k_{1,2}$ take on positive and negative values, and the polar angle $\theta$ ranges from 0 to $\pi/2$. The angle between the primary and backscattered Langmuir waves is assumed to be greater than $\pi/2$, hence $\text{sgn}[k_2] = -\text{sgn}[k_1]$. From Eq. (2.14), the FOTC to the rate of Langmuir-wave coalescence, neglecting the back-reaction $T \rightarrow L + L'$, is

$$\Delta \frac{\partial T_T(k_T)}{\partial t} \approx \frac{2}{\hbar \omega_p} \int \frac{d^3 k_1}{(2\pi)^3} \int \frac{d^3 k_2}{(2\pi)^3} \Delta u_{T LL'}(k_T, k_1, k_2) T_L(k_1) T_{L'}(k_2). \quad (2.54)$$

We assume that $\omega_{1,2} \approx \omega_p$, whence $v_{\phi_1,2} = \omega_p/k_{1,2}$, to simplify the integrand. The delta function $\delta^3(k_T - k_1 - k_2)$ is used to integrate over $d^3 k_2$. Thus on integration,

$$k_2 = k_T - k_1, \quad (2.55)$$

$$\mathbf{e}_T^* \cdot \mathbf{\kappa}_2 = \frac{-k_1}{k_2(k_1, k_T)} \mathbf{e}_T^* \cdot \mathbf{\kappa}_1, \quad (2.56)$$

$$\mathbf{\kappa}_1 \cdot \mathbf{\kappa}_2 = \frac{k_T \cos \psi - k_1}{k_2(k_1, k_T)}, \quad (2.57)$$

where $\psi$ is the angle between the $T$ and $L$ wave vectors, and

$$k_2(k_1, k_T) = -\text{sgn}[k_1] \left( k_1^2 + k_T^2 - 2k_1 k_T \cos \psi \right)^{1/2}. \quad (2.58)$$
The rate then becomes
\[
\frac{\Delta \partial T_T(k_T)}{\partial t} = \frac{e^2 V_e^2}{16\pi^2 \epsilon_0 m_e^2 \omega_p^2} \int d^3 k_1 \, g(k_1, k_T) |e_T^* \cdot \kappa_1|^2 \\
\times \delta [\omega_T(k_T) - \omega_L(k_1) - \omega_L(k_T - k_1)] T_L(k_1) T_L'(k_T - k_1),
\]
with
\[
g(k_1, k_T) = \frac{3k_1^6}{[k_2(k_1, k_T)]^2} + 2k_1^4 \left[ \frac{k_1 (k_T \cos \psi - k_1)}{[k_2(k_1, k_T)]^2} - 2 \right] + k_1^2 \frac{[k_2(k_1, k_T)]^2}{[k_2(k_1, k_T)]^2} \\
\times \left[ 1 - \frac{4k_1 (k_T \cos \psi - k_1)}{[k_2(k_1, k_T)]^2} + \frac{k_1^2}{[k_2(k_1, k_T)]^2} \right] + 2k_1[k_2(k_1, k_T)]^3 \\
\times \frac{[(k_T \cos \psi - k_1) - 2k_1]}{k_2(k_1, k_T)} + 3[k_2(k_1, k_T)]^4.
\]

Expanding Eq. (2.60), simplifying, and factorizing leads to
\[
g(k_1, k_T) = \left\{ k_1^2 - [k_2(k_1, k_T)]^2 \right\} \left\{ 3k_1^2 + 3[k_2(k_1, k_T)]^2 + 2k_1(k_T \cos \psi - k_1) \right\} \\
- \frac{k_1^2 (2k_1 \cos \psi - k_T)^2}{[k_2(k_1, k_T)]^2} (4k_1^2 + 3k_T^2 - 4k_1 k_T \cos \psi).
\]

On summing the states of polarization of the $T$ waves in the final state and averaging over the initial states of polarization, $|e_T^* \cdot \kappa_1|^2$ is replaced by $|\kappa_T \times \kappa_1|^2/2 = (\sin^2 \psi)/2$ in Eq. (2.59).

To simplify the delta function, the assumption
\[
\omega_L(k) \approx \omega_p + 3k^2 V_e^2 / 2\omega_p
\]
is made for both Langmuir waves, which is valid for $k \ll \lambda_D^{-1} = (V_e/\omega_p)^{-1}$. The dispersion relation for Langmuir waves in a generalized-Lorentzian plasma (i.e., one in which the electrons have a kappa distribution) is also given by Eq. (2.63) in the limit $k \ll \lambda_D^{-1}$ [Thorne and Summers, 1991]. So, on substituting Eq. (2.63) into the delta function in Eq. (2.59), we find
\[
\delta[\omega_T(k_T) - \omega_L(k_1) - \omega_L(k_T - k_1)] \\
= \delta \left\{ [\omega_T(k_T) - 2\omega_p] - \frac{3V_e^2}{2\omega_p} \left( 2k_1^2 + k_T^2 - 2k_1 k_T \cos \psi \right) \right\}.
\]
Using $\delta(ax) = \delta(x)/|a|$ and rearranging gives
\[
\delta[\omega_T(k_T) - \omega_L(k_1) - \omega_L'(k_T - k_1)]
= \frac{\omega_p}{3V_e^2|k_1|k_T}\delta\left\{ \cos \psi - \frac{1}{2k_1k_T}\left[ 2k_1^2 + k_T^2 - \frac{2\omega_p}{3V_e^2} [\omega_T(k_T) - 2\omega_p]\right] \right\}.
\]
(2.65)

We assume the effective temperatures have arc distributions [Willes et al., 1996; Edney and Robinson, 2001]
\[
T_L(k_1) = T_L(k_1) \exp(\beta \cos \theta_1),
\]
(2.66)
\[
T_L'(k_2) = T_L'(k_2) \exp(-\beta \cos \theta_2).
\]
(2.67)

These spectra are good approximations to the functional form obtained through numerical solutions of the Zakharov equations [Robinson and Newman, 1989; Robinson et al., 1992]. Here $\cos \theta_j = \kappa_j \cdot k_z$ for both spectra, where $k_z$ is the unit vector parallel to $k_b = \omega_p v_b/v_0^2$.

The coordinate system is then rotated such that the new $z$ axis is parallel to $k_T$. Hence, in the new system
\[
\cos \theta_1 = \cos \chi \cos \psi - \sin \chi \sin \psi \cos \phi,
\]
(2.68)
\[
\cos \theta_2 = (k_T \cos \chi - |k_1| \cos \theta_1)/|k_2|,
\]
(2.69)
where $\cos \chi = \kappa_T \cdot k_z$. Evaluating the integrals over $\cos \theta$ and $\phi$ leads to the rate equation
\[
\Delta \frac{\partial T_T(k_T, \chi)}{\partial t} = \frac{e^2}{48\pi e_0 m_e^2 v_p^3} \int dk_1 g(k_1, k_T) \exp\left\{ \beta \cos \chi [\cos \psi(k_1, k_T)
\times \left( 1 + \frac{|k_1|}{|k_2(k_1, k_T)|} \right) + \frac{k_T}{k_2(k_1, k_T)} \right]\right\} I_0 \left[ \beta \sin \chi \sqrt{1 - \cos^2 \psi(k_1, k_T)}
\times \left( 1 + \frac{|k_1|}{|k_2(k_1, k_T)|} \right) T_L(k_1)T_L'[k_2(k_1, k_T)]. \right\}
\]
(2.70)

where $I_0$ is the zeroth-order modified Bessel function [Olver et al., 2010] and $\cos \psi$ satisfies
\[
\cos \psi(k_1, k_T) = \frac{1}{2k_1k_T}\left( 2k_1^2 + k_T^2 - \frac{2\omega_p}{3V_e^2} [\omega_T(k_T) - 2\omega_p] \right).
\]
(2.71)

Substituting Eq. (2.71) into Eq. (2.62) yields
\[
g(k_1, k_T) = \frac{|k_1|k_T[2k_1 \cos \psi(k_1, k_T) - k_T]^2}{|k_2(k_1, k_T)|^2} \left( k_T^2 + \frac{4\omega_p}{3V_e^2} [\omega_T(k_T) - 2\omega_p] \right)
\times \left[ 1 - \cos^2 \psi(k_1, k_T) \right].
\]
(2.72)
The cold-plasma interaction rate derived by Li et al. [2005] is given by Eq. (2.70) on replacing $g(k_1, k_T)$ with

$$h(k_1, k_T) = \frac{\omega_p^2}{2V_e^2} \frac{|k_1| k_T [2k_1 \cos \psi(k_1, k_T) - k_T]^2}{[k_2(k_1, k_T)]^2 [1 - \cos^2 \psi(k_1, k_T)]}.$$  

(2.73)

The ratio $R = g(k_1, k_T)/h(k_1, k_T)$ is then given by

$$R(k_T) = \frac{2k^2 T V_e^2 \omega_p^2}{3\omega_p} + 8 \left[ \frac{\omega_T(k_T) - 2\omega_p}{\omega_p} \right].$$  

(2.74)

Since $R$ is not a function of $k_1$, Eq. (2.74) gives the ratio of the FOTC rate to the cold-plasma rate, independent of the integral over $k_1$. At this point we emphasize that $R$ is therefore independent of whether Eq. (2.13) or Eq. (2.20) is used as the kinetic equation for the interaction (even if the back reaction were included). On substituting the minimum transverse wave number $k_{T0} = \omega_p \sqrt{3/c}$ into Eq. (2.74), the ratio can be expressed as

$$R(k_T) = 6 \left( \frac{V_e}{c} \right)^2 \left( \frac{k_T}{k_{T0}} \right)^2 + \frac{8}{3} \left\{ 1 + 3 \left( \frac{k_T}{k_{T0}} \right)^2 \right\}^{1/2} - 2.$$  

(2.75)

To obtain the emission rate, a particular Langmuir spectrum $T_L(k)$ must first be assumed, after which the integral in Eq. (2.70) can be evaluated. For a Gaussian Langmuir spectrum,

$$T_L(k) = \exp \left[ -\frac{(k - k_f)^2}{K^2_f} \right] + \exp \left[ -\frac{(k - k_b)^2}{K^2_b} \right],$$  

(2.76)

the emission rate peaks at a transverse wave number

$$k_{T_{\text{max}}} = k_{T0}(1 + \epsilon)$$  

(2.77)

where $\epsilon = (k_f^2 + k_b^2)/k_D^2$ with $k_D = \lambda_e^{-1}$. Since the $L'$ waves are produced by the electrostatic decay process $L \leftrightarrow L' + S$, $k_b \approx k_f - k_0$ [Cairns, 1987b; Willes et al., 1996], where $k_f = \omega_p/v_b$ and $k_0 = 2\omega_p v_s/3V_e^2$ with $v_s$ the ion acoustic speed. This leads to

$$\epsilon \approx 2 \left( \frac{V_e}{v_b} \right)^2 \left( 1 - \frac{2v_s v_b}{3V_e^2} + \frac{2v_s^2 v_b^2}{9V_e^4} \right)$$  

(2.78)

in Eq. (2.77).
2.5.2 Discussion

The FOTC to the cold-plasma rate of second harmonic emission has been derived by applying the FOTC to the cold-plasma QRT. This derivation is valid for both Maxwellian and generalized Lorentzian distributions of electrons; this is due to the FOTC to the cold-plasma QRT being valid for arbitrary isotropic velocity distributions, and to Langmuir waves having the same dispersion relation for both distributions in the long wavelength ($k\lambda_D \ll 1$) limit. The resulting ratio $R(k_T)$ of the FOTC to the cold-plasma emission rate, given by Eq. (2.75), does not depend on the integration over $k_1$, so it is the same for all Langmuir wave spectra.

We define the dimensionless quantity $R_{\text{max}}$ to be the ratio $R(k_T)$ evaluated at $k_T = k_{T\text{max}}$ in Eq. (2.75). We choose $R_{\text{max}}$ to quantify the importance of the FOTC to the second harmonic emission rate. For most applications $V_e/c \ll 1$, and so the first term in Eq. (2.75) is small. Importantly, Eqs (2.77) and (2.78) then imply that $R_{\text{max}}$ depends mainly on the ratios $v_b/V_e$ and $v_s/V_e$, not on the individual speeds. As $v_b/V_e$ decreases, $k_{T\text{max}}$ increases, and hence $R_{\text{max}}$ increases. Since $v_s/V_e \ll 1$ unless $T_i \gg T_e$, the final term on the right hand side of Eq. (2.78) can be neglected, so $R_{\text{max}}$ decreases with increasing $v_s/V_e$. The dependence of $R_{\text{max}}$ on $v_b/V_e$ is stronger than on $v_s/V_e$, which can be seen in Fig. 2.1.

The significance of the FOTC to the cold-plasma rate of second harmonic emission is now assessed for different applications. In coronal type III solar radio bursts, typical parameters are $v_b/c \approx 0.2 - 0.5$, $V_e/c \approx 0.02$ and $v_s/c \approx 1.5 \times 10^{-4}$. This gives $k_{T\text{max}}/k_{T0} = 1.02 - 1.003$ from Eqs (2.77) and (2.78), and hence $R_{\text{max}} = 0.08 - 0.01$ from Eq. (2.75). However, Dulk et al. [1987] determined much lower electron beam speeds from their observations, ranging from $v_b/c = 0.07 - 0.25$, with an average of 0.14$c$. These values lead to the range $k_{T\text{max}}/k_{T0} = 1.16 - 1.04$ and $R_{\text{max}} = 0.66 - 0.05$, with an average of $k_{T\text{max}}/k_{T0} = 1.04$ and $R_{\text{max}} = 0.16$. Thus, the second harmonic emission rate may be well approximated by assuming a cold plasma for faster beams, but the FOTC becomes important for the slower electron beams measured by Dulk et al..

The electron beams responsible for significant radio emission are typically much slower in the “foreshock” regions upstream of shocks. Examples are Earth’s foreshock radio emissions, produced upstream of Earth’s bow shock, and type II solar radio bursts associated with traveling shocks. At Earth’s foreshock, $V_e/c \approx 3 \times 10^{-3}$, $v_s/c \approx 3 \times 10^{-4}$, and $v_b/V_e \approx 2 - 10$ (Kuncic et al.
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[2004]), which gives \( k_{T_{\text{max}}}/k_{T0} = 1.44 - 1.01 \) and \( R_{\text{max}} = 1.82 - 0.04 \). Knock et al. [2001] calculated, for interplanetary type II bursts, a maximum in the emissivity of second harmonic radiation where \( v_b/V_e \approx 3.5 \) for a thermal speed \( V_e/c = 0.005 \). In this case, taking \( v_s/c = 1.3 \times 10^{-4} \) leads to \( k_{T_{\text{max}}}/k_{T0} = 1.15 \) and \( R_{\text{max}} = 0.63 \). These values of \( R_{\text{max}} \) indicate that the FOTC may be a significant contribution to the total rate in foreshock emissions, and can even exceed the cold-plasma contribution. Figure 2.2 shows the emission rate versus \( k_T \) for typical coronal type III burst and Earth’s foreshock parameters. Notably, the peak wavenumber \( k_{T_{\text{max}}} \) stays almost constant when the FOTC is added to the emission rate.

Assuming the first term in \( R(k_T) \) in Eq. (2.75) to be negligible, and also that \( v_s/V_e \ll 1 \), we obtain \( R > 1 \) for \( v_b/V_e < 2.9 \). Thus, for sufficiently slow electron beams, the contribution from the FOTC exceeds the cold-plasma contribution to the emission rate. However, the assumption made in Eq. (2.63) that \( k_L \ll \lambda_D^{-1} \), which corresponds approximately to \( v_b/V_e \gg 1 \), is not satisfied very well for these slow foreshock parameters. Thus for small \( v_b/V_e \) the expression for \( \cos \psi(k_1, k_T) \) given in Eq. (2.71), and hence the rate in Eq. (2.70), will be less accurate.

2.6 Summary and conclusion

Both the cold-plasma quadratic response tensor (QRT) and the exact QRT describe three-wave interactions in which each wave has a phase speed that is greater than the electron thermal speed. However, neither is ideal for the calculation of interaction rates: the cold-plasma QRT is readily calculable, but is only accurate where all phase speeds are much greater than the thermal speed; conversely, the exact QRT provides an accurate description of three-wave interactions between any waves, but its direct application to the calculation of rates is infeasible. The approximate QRT that we have derived here, which is the sum of the cold-plasma QRT and its first-order thermal correction (FOTC), overcomes these disadvantages since it is more accurate than the cold-plasma QRT alone but still permits a calculation of the interaction rate. It is also valid for arbitrary isotropic velocity distributions. This approximate QRT is therefore suitable for modeling three-wave interactions in space plasmas, in which thermal effects are important for the interacting waves, and the velocity distributions are commonly non-Maxwellian.

The rate of second harmonic plasma emission via Langmuir-wave coales-
cence has previously been treated with the cold-plasma QRT. Therefore, the resulting expression is inaccurate where the phase speed of one or more of the waves is similar to the thermal speed. Using our result for the approximate QRT, we have derived the FOTC to the rate of second harmonic plasma emission. The ratio of the FOTC rate to the cold-plasma rate is easily calculated using Eq. (2.75); it is only a function of the transverse wave number $k_T$, and does not require an integral over the Langmuir wave number. It is therefore independent of whether the semiclassical or the statistical mechanical kinetic equation is used. The importance of the FOTC to the emission rate is determined by the ratios $v_b/V_e$ and $v_s/V_e$: the FOTC to the emission rate becomes larger compared to the cold-plasma emission rate as both $v_b/V_e$ and $v_s/V_e$ decrease. The FOTC to the cold-plasma emission rate is therefore important in foreshock emission, where the electron beam speed is not much larger than the electron thermal speed (within a factor of $\sim 2 - 10$). In the case where $v_s/V_e \ll 1$, the FOTC to the cold-plasma emission rate is greater than the cold-plasma emission rate for $v_b/V_e \lesssim 3$. 
Figure 2.1: $R_{\text{max}}$ versus (a) $v_b/V_e$, where $v_s/V_e = 0.1$; and (b) $v_s/V_e$, where $v_b/V_e = 4$. For both (a) and (b), $V_e/c = 0.003$. 
Figure 2.2: Second harmonic emission rate $\partial T/\partial t$ in units of $Js^{-1}$ for (a) coronal type III burst parameters: $V_e/c = 0.02$, $v_b/V_e = 10$, and $v_s/V_e = 0.03$; and (b) Earth’s foreshock parameters: $V_e/c = 0.003$, $v_b/V_e = 3$, and $v_s/V_e = 0.1$. Dashed lines are for a cold plasma while solid lines include the FOTC.
Chapter 3

Exact evaluation of the quadratic longitudinal response function for an unmagnetized Maxwellian plasma


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3.1 Abstract

The quadratic longitudinal response function describes the second-order non-linear response of a plasma to electrostatic wave fields. An explicit expression for this function in the weak-turbulence regime requires the evaluation of velocity-space integrals involving the velocity distribution function and various resonant denominators. Previous calculations of the quadratic longitudinal response function were performed by approximating the resonant denominators to facilitate the integration. Here we evaluate these integrals exactly for a non-relativistic collisionless unmagnetized isotropic Maxwellian plasma in terms of generalized plasma dispersion functions, and correct certain aspects of expressions previously derived for these functions. We show that in the appropriate limits the exact expression reduces to the approximate form used for interactions between two fast waves and one slow wave, such as the electrostatic decay of Langmuir waves into Langmuir waves and ion-sound waves, and the scattering of Langmuir waves off thermal ions.
3.2 Introduction

Plasmas are highly nonlinear media, with nonlinear effects commonly invoked to describe the behavior of both space and laboratory plasmas. In kinetic theory, the response of a plasma to electromagnetic wave fields is described by the linear response tensor and a hierarchy of nonlinear response tensors (or equivalently, conductivity or susceptibility tensors) [Tsytovich, 1970; Davidson, 1972; Tsytovich, 1977; Melrose, 1980c; Sitenko, 1982; Melrose, 1986a]. If only the response to electrostatic (i.e., longitudinal) wave fields is being considered, then the longitudinal parts of these response tensors, termed the linear and nonlinear longitudinal response functions, are sufficient to model the plasma response [Tsytovich, 1970; Davidson, 1972; Tsytovich, 1977; Melrose, 1980c; Sitenko, 1982; Melrose, 1986a].

In the case that the nonlinear response to the electromagnetic disturbance is weak, induced plasma properties such as the induced charge and current densities can be expanded in powers of the wave-field amplitude; this is called the “weak-turbulence expansion”. The quadratic (i.e., second-order) response tensor is defined as the coefficient of the second-order term in an expansion of the induced current density in powers of the vector potential [Melrose, 1980c, 1986a]. This quantity describes wave coupling in three-wave interactions, which include the coalescence of two waves to give a product wave and the decay of a wave into two product waves, as well as the coupling in nonlinear scattering of waves by particles. The third-order term in the weak-turbulence expansion, called the cubic response tensor, describes the following four-wave interactions: the decay of one wave into three waves, the coalescence of three waves into one wave, and interactions between couples of waves. Four-wave interactions may also result from the product of a three-wave interaction taking part in another three-wave interaction, giving rise to an “effective cubic response” of the plasma. The effective cubic response tensor is then the sum of the cubic response tensor and three combinations of the product of two quadratic response tensors [Sitenko, 1982; Melrose, 1986c].

The Vlasov equation may be used to derive general expressions for the linear and nonlinear response tensors, for instance for a nonrelativistic unmagnetized collisionless plasma, in terms of velocity-space integrals involving the velocity distribution function and various resonant denominators [Davidson, 1972; Tsytovich, 1977; Melrose, 1980c; Sitenko, 1982; Melrose, 1986a]. These integrals must be evaluated to obtain explicit expressions for the response tensors, which are needed for calculating the rates of the corresponding linear and nonlinear
processes. The most common approach is to approximate the integrands by making assumptions about the frequencies of the waves before evaluating the integrals [Suramlishvili, 1963; Akhiezer et al., 1964; Akhiezer, 1965; Liperovskii and Tsytovich, 1965; Kovrizhnykh, 1966; Melrose, 1980c; Sitekno, 1982; Melrose, 1986a]. The terms in the integrand that are neglected depend on whether the wave is fast or slow (i.e., whether $\omega/|k| \gg V$ or $\omega/|k| \ll V$, where $V$ is the relevant thermal speed). In a warm unmagnetized plasma, Langmuir ($L$) and transverse ($T$) waves are assumed fast and ion-sound ($S$) waves are assumed slow. Performing these approximations leads to a simplified calculation of the integrals [Suramlishvili, 1963; Akhiezer et al., 1964; Akhiezer, 1965; Liperovskii and Tsytovich, 1965; Kovrizhnykh, 1966; Melrose, 1980c; Sitekno, 1982; Melrose, 1986a]; however, the ranges of validity of the resulting expressions are not clear, and they may be inaccurate in certain physical applications. In Layden et al. [2011] (Chapter 2) a thermal correction was derived for the cold-plasma quadratic response tensor, which is the approximate expression often used where the three waves involved are assumed to be fast. The thermal correction was predicted to possibly be important when treating the rate of Langmuir-wave coalescence $L + L' \leftrightarrow T$ at Earth’s foreshock and in type II radio bursts.

On the other hand, an exact evaluation of the quadratic response tensor was carried out by Percival and Robinson [1998a], where the resulting expression was given in terms of generalized plasma dispersion functions [Percival and Robinson, 1998b]. The exact expression accurately describes the wave coupling in any three-wave interaction or nonlinear scattering process. Despite this advantage over the approximate forms, the large number of terms in the exact expression presents difficulties in applying it to specific interactions. The full quadratic response tensor, however, is not needed when studying the wave coupling between three electrostatic waves. Such processes include electrostatic (Langmuir-wave) decay $L \leftrightarrow L' + S$ [Tsytovich, 1970; Davidson, 1972; Tsytovich, 1977; Melrose, 1980c; Sitekno, 1982; Melrose, 1986a; Cairns, 2000; Yoon, 2000], where a Langmuir wave decays into a backscattered Langmuir wave and an ion-sound wave, and scattering of Langmuir waves off thermal ions (or “nonlinear Landau damping”) $L + i \leftrightarrow L' + i'$ [Sturrock, 1957; Ginzburg and Zheleznyakov, 1958; Tsytovich, 1970; Davidson, 1972; Tsytovich, 1977; Melrose, 1980c; Sitekno, 1982; Melrose, 1986a; Cairns, 2000; Yoon, 2000], in which Langmuir waves are scattered off the Debye shielding (or “electron polarization”) clouds around thermal ions. For these processes only the longitudinal part (i.e., the quadratic longitudinal response function) contributes to
the plasma response. An explicit expression for the exact quadratic longitudinal response function has yet to be derived in the literature. This would permit a more accurate calculation of the rates of electrostatic decay and scattering off thermal ions for all plasma parameters.

In this chapter, we derive an exact expression for the quadratic longitudinal response function of a nonrelativistic collisionless unmagnetized isotropic plasma with Maxwellian velocity distributions of particles. Rather than contracting the cumbersome exact quadratic response tensor \cite{percival1998a} with the polarization vectors to obtain its longitudinal part, we instead begin our analysis from a general expression for the quadratic longitudinal response function. In Sec. 3.3 we outline the relevant nonlinear plasma theory, such as the general and approximate expressions for the quadratic longitudinal response function and how they are derived. In Sec. 3.4 we evaluate exactly the velocity-space integrals that arise in the general expression. We thus obtain an exact expression in terms of generalized plasma dispersion functions, and we discuss the properties of this expression and the generalized plasma dispersion functions. In Sec. 3.5 we identify and correct some errors in the derivation of expressions for the generalized plasma dispersion functions \cite{percival1998b}. To curb the numerical instability associated with calculating these functions, in Sec. 3.6 we derive alternative series expressions for the generalized plasma dispersion functions which converge faster than those in Sec. 3.5. We show in Sec. 3.7 that the exact expression for the quadratic longitudinal response function reduces to the approximate form used for interactions between two fast waves and one slow wave in the appropriate limits. In Sec. 3.8 we assess the accuracy of various approximations for the generalized plasma dispersions. Sec. 3.9 contains a summary and the conclusions. These results will be applied to the rates of electrostatic decay and scattering off thermal ions in Chapter 4.

3.3 Theoretical context

The linear and nonlinear plasma response tensors may be defined by expanding an induced plasma property in powers of the amplitude of the Fourier-transformed electromagnetic field, which is called the weak-turbulence expansion. General expressions for the linear and nonlinear plasma responses to the wave fields may then be calculated in a variety of ways. These include the forward-scattering \cite{melrose1986a, melrose1987} and Lagrangian \cite{whitham1965}.
Galloway and Kim, 1971] methods. The linear and quadratic responses have also been constructed using symmetry and dimensionality arguments and the relations of charge continuity and gauge invariance [Percival, 1997]. The responses are most commonly calculated using the Vlasov equation [Davidson, 1972; Tsytovich, 1977; Melrose, 1980c; Sitenko, 1982; Melrose, 1986a], which relates the distribution function to the wave fields; this is the method that we outline here.

The first assumption of the Vlasov equation is that the electric and magnetic fields and particle motions in the plasma are self-consistent; i.e., the fields are determined by statistically averaged charge and current densities in the plasma expressed as moments of the distribution function. The plasma is also assumed collisionless, which is valid where the electron collision frequency is much lower than each of the relevant wave frequencies. The collisionless assumption is appropriate for most space and fusion plasmas. An ambient magnetic field $B_0$ may be included in the Vlasov equation; however, the analysis of the quadratic plasma response becomes much more complicated (see, e.g., Sitenko [1982] for general expressions for the quadratic longitudinal response function and the quadratic response tensor). In this chapter we assume that the plasma is unmagnetized, i.e. $B_0 = 0$, which is valid where $\Omega_e/\omega_p \ll 1$ with $\Omega_e = eB_0/m_e$ the electron cyclotron frequency and $\omega_p = (n_e e^2/\epsilon_0 m_e)^{1/2}$ the electron plasma frequency. The final assumption made here is that the thermal speed of each particle species satisfies $V/c \ll 1$, so that the plasma is nonrelativistic.

A weak-turbulence expansion of the induced current density $J(k)$ in powers of the vector potential $A(k_m)$ in the temporal gauge yields (e.g., Melrose [1986a])

$$J_i(k) = \sum_{n=1}^{\infty} J_i^{(n)}(k),$$

(3.1)

where

$$J_i^{(1)}(k) = \alpha_{ij}^{(1)}(k) A_j(k),$$

(3.2)

and

$$J_i^{(n)}(k) = \int d\lambda^{(n)} \alpha_{ij_1...j_n}^{(n)}(k, k_1, \ldots, k_n) A_{j_1}(k_1) \cdots A_{j_n}(k_n), \quad n \geq 2.$$  

(3.3)

In Eqs (3.1)–(3.3), $k_m$ collectively denotes $\omega_m$ and $k_m$ for the $m$th wave, and $d\lambda^{(n)}$ is the $n$th-order convolution integral given by

$$d\lambda^{(n)} = d^4k_1 d^4k_2 \cdots d^4k_n (2\pi)^4 \delta^4(k - k_1 - \ldots - k_n),$$

(3.4)
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with

\[ d^4k = d\omega \, d^3k, \quad (3.5) \]

and

\[ \delta^4(k) = \delta(\omega) \, \delta^3(k). \quad (3.6) \]

The quantity \( \alpha_{ij}^{(1)} \) is called the linear response tensor, which is the coefficient of the first-order term in the weak-turbulence expansion of the induced current. The coefficients of the higher-order terms are the nonlinear response tensors denoted \( \alpha_{ij \ldots jn}^{(n)} \), where \( n \geq 2 \).

For interactions between electrostatic waves, the strength of the wave coupling is described by the longitudinal part of these response tensors [Tsytovich, 1977; Melrose, 1986a], given by

\[ \alpha^{(n)}(k, k_1, \ldots, k_n) = \frac{\alpha_{ij \ldots jn}^{(n)}(k; k_1, \ldots, k_n)}{|k||k_1| \cdots |k_n|} k_i k_{i_1} \cdots k_{j_n} j_n. \quad (3.7) \]

Taking the longitudinal part of the general quadratic response tensor in Eq. (1.75) and taking the nonrelativistic limit \( c \to \infty \) gives the general form of the quadratic longitudinal response function for a nonrelativistic unmagnetized plasma (e.g., Sitenko [1982]):

\[ \alpha^{(2)}(k, k_1, k_2) = \frac{q^3}{m^2} \frac{\omega \omega_1 \omega_2}{|k||k_1||k_2|} [k_1 \cdot k_2 |k|^2 M(k; k_1, k_2) + (k \leftrightarrow k_1) + (k \leftrightarrow k_2)], \quad (3.8) \]

where

\[ M(k; k_1, k_2) = \int d^3v \frac{f(v)}{(\omega - k \cdot v)^2(\omega_1 - k_1 \cdot v)(\omega_2 - k_2 \cdot v)}, \quad (3.9) \]

and where \( (k_m \leftrightarrow k_n) \) denotes additional terms generated by the interchange of \( k_m \) and \( k_n \). The contribution from each particle species is summed to give the total plasma response, but for an electron-ion plasma for instance, the ionic contribution is almost always neglected in view of the \( m^{-2} \) dependence. The expression in Eq. (3.8) is suitable for interactions in which one of the waves is slow; an additional factor of \( 1/2 \) must be introduced if all of the waves are fast, due to symmetry considerations (see, e.g., Melrose [1986a]).

The delta-function in the convolution integral in Eq. (3.3) implies the relation

\[ k = k_1 + \ldots + k_n, \quad (3.10) \]
which expresses conservation of frequency and wave vector, also known as the wave-matching conditions. For the quadratic response these state that

\[ \omega = \omega_1 + \omega_2, \tag{3.11} \]

and

\[ k = k_1 + k_2. \tag{3.12} \]

Equations (3.11) and (3.12) may be used to simplify \( \alpha^{(2)} \) after the interchanges in Eq. (3.8) are performed.

The integrals \( M \) given by Eq. (3.9) are difficult to evaluate, and so the integrands are often approximated prior to carrying out the integrations. The approximate calculation for the quadratic longitudinal response function where two of the waves are fast and one is slow proceeds as follows (see Sitenko [1982] and Melrose [1986a]): the general expression in Eq. (3.8) is first approximated by keeping only the highest power of the resonant denominator \( (\omega_2 - k_2 \cdot v) \) where \( k_2 \) denotes the slow wave field; this corresponds to assuming \( M(k; k_1, k_2) = M(k_1; k, k_2) = 0 \). It is further approximated by neglecting thermal effects for the fast wave fields, denoted \( k \) and \( k_1 \), which is achieved by assuming \( \omega_j - k_j \cdot v \approx \omega_j \) for these waves. This yields

\[ \alpha^{(2)}(k, k_1, k_2) \approx -\frac{q^3 n}{m^2} \frac{\omega_2 k \cdot k_1}{|k| |k_1|} \int d^3v f(v) \frac{1}{(\omega_2 - k_2 \cdot v)^2}. \tag{3.13} \]

For the Maxwellian distribution function, given by

\[ f(v) = Ae^{-v^2/2V^2}, \tag{3.14} \]

where the thermal speed \( V = \sqrt{k_B T/m} \) and the normalization constant is

\[ A = \frac{n}{(V \sqrt{2})^3 \pi^{3/2}}, \tag{3.15} \]

the integral in Eq. (3.13) can be evaluated in terms of the Fried-Conte plasma dispersion function [Fried and Conte, 1961], defined by

\[ Z(u) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dt \frac{e^{-t^2}}{t - u}, \quad \text{Im}(u) > 0, \tag{3.16} \]

and its analytic continuation for \( \text{Im}(u) \leq 0 \). This leads to

\[ \alpha^{(2)}(k, k_1, k_2) = -\frac{q^3 n}{m^2} \frac{\omega_2 k \cdot k_1}{|k||k_1||k_2|V^2} [1 + s_2 Z(s_2)], \tag{3.17} \]
where

\[ s_j = \frac{\omega_j}{|k_j|V\sqrt{2}}, \]

(3.18)

with \( j \) labeling the wave fields \( k, k_1, \) and \( k_2 \). This expression is often approximated further; assuming \( s_2 \ll 1 \) and using that the lowest order term in the power series expansion of \( uZ(u) \) about \( u = 0 \) is \( \mathcal{O}(u) \), the final term in Eq. (3.17) is neglected, giving

\[ \alpha^{(2)}(k, k_1, k_2) = -\frac{q^2 n}{m^2} \frac{\omega_2 k \cdot k_1}{V^2 |k||k_1||k_2|}. \]

(3.19)

Equations (3.17) and (3.19) have been used by many authors for the wave coupling in electrostatic decay [Caponi and Davidson, 1971; Cairns, 1987d; Robinson et al., 1993; Cairns, 2000; Yoon, 2000; Ziebell et al., 2001; Edney and Robinson, 2001; Kontar and Pécseli, 2002; Li et al., 2003; Vásquez and Gómez, 2004; Ziebell et al., 2008] and scattering off thermal ions [Kaplan and Tsytovich, 1968; Zheleznyakov and Zaitsev, 1970; Muschietti and Dum, 1991; Cairns, 2000; Yoon, 2000; Ziebell et al., 2001; Kontar and Pécseli, 2002; Ziebell et al., 2008].

The approximation for the quadratic longitudinal response function in Eq. (3.19) is valid where the phase speed approximations \( \omega/|k| \gg V, \omega_1/|k_1| \gg V, \) and \( \omega_2/|k_2| \ll V \) (i.e., \( s, s_1 \gg 1 \) and \( s_2 \ll 1 \)) are satisfied. However, as \( s \) and \( s_1 \) approach unity from above and \( s_2 \) approaches unity from below, this approximation must break down, and it is not clear when this occurs. An exact expression for the quadratic longitudinal response function is therefore desirable, as it would accurately describe the wave coupling strength between three electrostatic waves of any phase speed. Also, the accuracy of the commonly used approximations could then be assessed.

An exact evaluation of the quadratic response tensor has been performed by Percival and Robinson [1998a]. The expression for the exact response tensor shown in Eq. (1.89) involves generalized plasma dispersion functions \( Y_{m,n} \) defined by

\[ Y_{m,n}(a, b, c) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx e^{-x^2} \frac{x^m}{(x-c)^n} Z(a + bx), \]

(3.20)

with \( Z \) given by Eq. (3.16). Various properties of these functions were derived in Percival and Robinson [1998b], including series expressions which may be employed for their numerical evaluation. To obtain the quadratic longitudinal
response function, one could apply Eq. (3.7) to the exact quadratic response tensor; however, the cumbersome expression means that such an approach is impractical. In this chapter we therefore derive an exact expression for the quadratic longitudinal response function by working from the general expression in Eq. (3.8), and evaluating the velocity-space integrals exactly in terms of the $Y_{m,n}$.

Once an explicit expression for the quadratic longitudinal response function is derived, the rate of three-wave interactions or nonlinear scattering processes involving electrostatic waves can be calculated. The kinetic equation for the power radiated per unit volume in the mode $M$ due to the three-wave coalescences and decays $P + Q \to M$ and $M \to P + Q$, in the random-phase approximation, is \cite{Tsytovich, Davidson, Tsytovich, Melrose, Sitenko, Sitenko, Melrose, Yoon}

$$\frac{\partial N_M(k)}{\partial t} = \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3} u_{MPQ}(k, k_1, k_2) \times \{N_P(k_1)N_Q(k_2) - N_M(k)[N_P(k_1) + N_Q(k_2)]\}, \quad (3.21)$$

with similar kinetic equations for the modes $P$ and $Q$. Here, $N_M(k)$ is the occupation number for the wave mode $M$, which is related to the wave energy density $W$ by $N_M(k) = W_M(k)/\hbar\omega_M(k)$. The equation for the interaction probability $u_{MPQ}$ is (in the notation of \textit{Melrose} [1986a])

$$u_{MPQ}(k, k_1, k_2) = \frac{4\hbar R_M(k)R_P(k_1)R_Q(k_2)}{\epsilon_0^2 |\omega_M(k)|\omega_P(k_1)|\omega_Q(k_2)|} |\alpha^{(2)}(k_M, k_1, k_2)|^2 \times (2\pi)^4 \delta^4(k_M - k_P - k_Q), \quad (3.22)$$

where $R_M$ is the ratio of electric to total energy in the wave mode $M$, and the quantity $k_M$ collectively denotes $\omega_M(k)$ and $k$ (similarly for $k_P$ and $k_Q$). For scattering off thermal ions, the kinetic equation is \cite{Tsytovich, Davidson, Tsytovich, Melrose, Sitenko, Sitenko, Melrose, Yoon}

$$\frac{\partial N_M(k)}{\partial t} = \int d^3v \int \frac{d^3k_1}{(2\pi)^3} w_{MP}(k, k_1, v) \left\{ [N_P(k_1) - N_M(k)]f_i(v) - N_P(k_1)N_M(k) \frac{\hbar(k_1 - k)}{m_i} \cdot \frac{\partial f_i(v)}{\partial v} \right\}, \quad (3.23)$$
with
\[
\omega_{MP} = \frac{8\pi q^2}{\epsilon_0} \left. R_M(k) R_P(k_1) \right| \frac{\alpha^{(2)}(k_M, k_{P1}, k_M - k_{P1})(k - k_1) \cdot v}{[\omega_M(k) - \omega_P(k_1)]^2 |k - k_1| K^L(k_M - k_{P1})} \big| \delta[\omega_M(k) - \omega_P(k_1) - (k - k_1) \cdot v],
\]
and where \( f_i(v) \) is the ion distribution function and \( K^L \) is the longitudinal part of the equivalent dielectric tensor.

Previous calculations of the rates in Eqs (3.21) and (3.23) have assumed the approximate quadratic longitudinal response function given by Eq. (3.17) for substitution in the interaction probabilities in Eqs (3.22) and (3.24). In Chapter 4 we will use the results derived here for the exact quadratic longitudinal response function to determine the rates of electrostatic decay and scattering off thermal ions, and assess the accuracy of the previous calculations.

We note that the nonlinear longitudinal response functions used in this chapter are closely related to the widely used nonlinear longitudinal susceptibilities \( \chi^{(n)} \), which are the coefficients in an expansion of the induced polarization \( P/\epsilon_0 \) in powers of the electric field strength \( E \) (e.g., Sitenko [1982] and Melrose and McPhedran [1991]), through
\[
\chi^{(n)}(k, k_1, \ldots, k_n) = \frac{(-i)^{n-1}}{\omega_1 \cdots \omega_n} \alpha^{(n)}(k, k_1, \ldots, k_n).
\]
An equivalent analysis to that below could be performed in terms of \( \chi^{(2)} \).

### 3.4 Derivation of the exact quadratic longitudinal response function

We first evaluate \( M(k; k_1, k_2) \) in Eq. (3.9) for the Maxwellian velocity distribution function given by Eq. (3.14). We choose the same coordinate system used in Percival and Robinson [1998a], where the \( z \)-axis is orientated parallel to \( k \) and the \( x \)-axis is directed so that \( k \) and \( k_1 \) lie in the \( xz \)-plane with a component of \( k_1 \) along the positive \( x \)-axis. The wave vectors and velocity vector may then be written as
\[
\begin{align*}
k &= (0, 0, k_\parallel), \\
k_1 &= (k_{1\perp}, 0, k_{1\parallel}), \\
k_2 &= (k_{2\perp}, 0, k_{2\parallel}), \\
v &= (v_\perp \cos \phi, v_\perp \sin \phi, v_\parallel),
\end{align*}
\]
with \( k_\parallel = k_1\parallel + k_2\parallel, \ k_\perp = -k_2\perp = k_\perp, \) \( \phi \) the azimuthal angle of the velocity vector, and where \( \parallel \) and \( \perp \) denote vector components parallel and perpendicular to the \( z \)-axis respectively. On substituting Eq. (3.14) and Eqs (3.26)–(3.29) into Eq. (3.9), and expressing velocity components in units of \( V\sqrt{2} \), we have

\[
M(k; k_1, k_2) = \frac{A (V\sqrt{2})^3}{(k_\perp V\sqrt{2})^4} \int_{-\infty}^{\infty} dv_\parallel e^{-v_\parallel^2} \int_{0}^{\infty} dv_\perp e^{-v_\perp^2} \times \int_{-\pi}^{\pi} d\phi \frac{1}{d^2 (d_1 - v_\perp \cos \phi) (d_2 + v_\perp \cos \phi)},
\]

where

\[
d_j = d_j(v_\parallel) = \frac{\omega_j - k_j \parallel V\sqrt{2}}{k_\perp V\sqrt{2}}.
\]

We first evaluate the integral over \( \phi \). A partial fraction decomposition yields

\[
\frac{1}{(d_1 - v_\perp \cos \phi) (d_2 + v_\perp \cos \phi)} = \frac{1}{d_1 + d_2} \left( \frac{1}{d_1 - v_\perp \cos \phi} + \frac{1}{d_2 + v_\perp \cos \phi} \right).
\]

We note that a sign ambiguity arises when evaluating the integral over \( \phi \), i.e.,

\[
\int_{-\pi}^{\pi} d\phi \frac{1}{S - T \cos \phi} = \pm \frac{2\pi}{\sqrt{S^2 - T^2}}
\]

for \( |S| > |T| \), where the upper sign corresponds to \( \text{Im}(S) > 0 \) and the lower one to \( \text{Im}(S) < 0 \). The correct choice of sign requires an application of the Landau prescription [Landau, 1946]: the causality of the response is imposed by adding infinitesimal imaginary parts \( i0 \) to the frequencies \( \omega_j \), and hence to the quantities \( d_j \) by Eq. (3.31). Therefore the upper sign in Eq. (3.33) is chosen, giving

\[
M(k; k_1, k_2) = \frac{A (2\pi)}{k_\perp^4 V\sqrt{2}} \int_{-\infty}^{\infty} dv_\parallel e^{-v_\parallel^2} \int_{0}^{\infty} dv_\perp e^{-v_\perp^2} \times \left( \frac{1}{\sqrt{d_1^2 - v_\perp^2}} + \frac{1}{\sqrt{d_2^2 - v_\perp^2}} \right).
\]

Next we integrate over \( v_\perp \) using the result

\[
\int_{0}^{\infty} dx e^{-x^2} \frac{x}{\sqrt{\xi^2 - x^2}} = -\frac{Z(\xi)}{2}, \quad \text{Im}(\xi) > 0,
\]

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derived in *Percival and Robinson [1998b]*, where $Z$ is the plasma dispersion function in Eq. (3.16). Hence

$$M(k; k_1, k_2) = -\frac{A\pi}{k_\perp V^2} \int_{-\infty}^{\infty} dv_\parallel \frac{e^{-v_\parallel^2}}{d^2(d_1 + d_2)} [Z(d_1) + Z(d_2)]. \quad (3.36)$$

In order to express Eq. (3.36) in terms of generalized plasma dispersion functions, we write

$$d_j = a_j + b_j v_\parallel, \quad (3.37)$$

where

$$a_j = \frac{\omega_j}{k_\perp V\sqrt{2}} = \frac{\omega_j |k|}{|k \times k_1| V \sqrt{2}}, \quad (3.38)$$

and

$$b_j = -\frac{k_\parallel}{k_\perp} = -\frac{k \cdot k_j}{|k \times k_1|}. \quad (3.39)$$

Performing a partial fraction decomposition gives

$$M(k; k_1, k_2) = -\frac{A\pi}{k_\perp V^2} \int_{-\infty}^{\infty} dv_\parallel \frac{1}{r - s} \left\{ \int_{-\infty}^{\infty} dv_\parallel \frac{e^{-v_\parallel^2}}{(v_\parallel - r)^2} [Z(a_1 + b_1 v_\parallel)] + Z(a_2 + b_2 v_\parallel) \right\}$$

$$+ \left[ Z(a_2 + b_2 v_\parallel) - \int_{-\infty}^{\infty} dv_\parallel \frac{e^{-v_\parallel^2}}{(v_\parallel - s)^2} [Z(a_1 + b_1 v_\parallel) + Z(a_2 + b_2 v_\parallel)] \right], \quad (3.40)$$

where we define

$$r = -\frac{a_1 + a_2}{b_1 + b_2}, \quad (3.41)$$

and where $s_j$ is given by Eq. (3.18) but may be alternatively expressed as

$$s_j = -\frac{a_j}{b_j}. \quad (3.42)$$

Using Eqs (3.15) and (3.20) in (3.40) gives, in a coordinate-independent (i.e., vector) representation,

$$M(k; k_1, k_2) = \frac{n}{4V^4 |k \times k_1||(k \cdot k_1 + k \cdot k_2)|} \int_{-\infty}^{\infty} dv_\parallel \frac{1}{r - s} \left\{ \frac{1}{r - s} [Y_{0,1}(a_1, b_1, r)$$

$$+ Y_{0,1}(a_2, b_2, r) - Y_{0,1}(a_1, b_1, s) - Y_{0,1}(a_2, b_2, s)] - Y_{0,2}(a_1, b_1, s)$$

$$- Y_{0,2}(a_2, b_2, s) \right\}. \quad (3.43)$$
Substituting Eq. (3.43) into Eq. (3.8) gives the exact quadratic longitudinal response function of an unmagnetized Maxwellian plasma. Once the interchanges in Eq. (3.8) are performed, the wave-matching condition in Eq. (3.11) may be used to simplify the expression. Specifically, the wave-matching condition can be used to simplify the expression for $M(k; k_1, k_2)$ given in Eq. (3.36) by noting that it implies $d = d_1 + d_2$. Hence, a more compact expression is

$$M(k; k_1, k_2) = \frac{n}{4V^4|k \times k_1||k_2|^2} \left[ Y_{0,3}(a_1, b_1, s) + Y_{0,3}(a_2, b_2, s) \right]. \quad (3.44)$$

Therefore, the exact quadratic longitudinal response function is

$$\alpha^{(2)} = \frac{q^2 n}{m^2} \frac{\omega_1 \omega_2}{4V^4|k||k_1||k_2||k_1 \times k_2|} \left( \frac{k_1 \cdot k_2}{|k_1 + k_2|} \right)^2 \left\{ \frac{1}{r - s} \left[ Y_{0,1}(\bar{a}_1, \bar{b}_1, \bar{r}) + Y_{0,1}(\bar{a}_2, \bar{b}_2, \bar{r}) ight] ight\}$$

$$+ \frac{k \cdot k_1 |k_2|^2}{(k \cdot k_2 + k_1 \cdot k_2)} \left\{ \frac{1}{r - s} \left[ Y_{0,1}(\bar{a}_1, \bar{b}_1, \bar{r}) + Y_{0,1}(\bar{a}_2, \bar{b}_2, \bar{r}) ight] ight\}$$

$$- Y_{0,1}(\bar{a}_1, \bar{b}_1, s_1) + Y_{0,1}(\bar{a}_2, \bar{b}_2, s_1) - Y_{0,2}(\bar{a}_1, \bar{b}_1, s_1) - Y_{0,2}(\bar{a}_2, \bar{b}_2, s_1) \right\}, \quad (3.45)$$

where the arguments of $\alpha^{(2)}$ are omitted for brevity. The tilde over a variable indicates that the interchange $k \leftrightarrow k_1$ has been done, and the bar denotes the interchange $k \leftrightarrow k_2$. The interchange quantities for $a_{1,2}$ and $b_{1,2}$ are given by Eqs (3.96)–(3.103) in Sec. 3.A. In Eqs (3.45) and (3.96)–(3.103) we have used the wave-matching condition Eq. (3.12) to write $|k \times k_2|$ and $|k_1 \times k_2|$ as $|k \times k_1|$.

The magnitudes of the terms in Eq. (3.45) depend on both the interaction geometry and the phase speeds of the waves. This dependence occurs through the factors multiplying the generalized plasma dispersion functions, and their arguments. For example, the parameters $a_1$, $b_1$, and $s$ which are the arguments of the first $Y_{0,3}$ function may be written as $v_{\phi 1} \csc \psi / V \sqrt{2}$, $- \cot \psi$, and $v_{\phi} / V \sqrt{2}$, respectively, where $v_{\phi j} = \omega_j / |k_j|$ and $\psi$ is the angle between $k$ and $k_1$. We see that $b_1$ and $s$ depend only on the interaction geometry and phase speeds respectively, whereas $a_1$ depends on both of these.

We note that the non-resonant part (i.e., real part for real $\omega$ and $k$) of the quadratic longitudinal response function is the important quantity when
modeling three-wave coalescences and decays [Melrose, 1972]. The resonant part (i.e., imaginary part for real \( \omega \) and \( k \)) may be important in the treatment of nonlinear frequency shifts and other such phenomena associated with the cubic plasma response [Melrose and Kuijpers, 1984]; there is a contribution from the resonant part of the quadratic response function to the real part of the effective cubic response function, resulting from the multiplication of two quadratic response functions.

### 3.5 Corrections to the generalized plasma dispersion function expressions

Here we correct some results derived in Percival and Robinson [1998b] concerning the generalized plasma dispersion functions \( Y_{m,n} \), and then summarize the correct expressions for the real and imaginary parts of the generalized plasma dispersion functions at the end of the section. Where the arguments of \( Y_{m,n} \) are omitted below it is understood to refer to \( Y_{m,n}(a,b,c) \). Series expansions for the generalized plasma dispersion function \( Y_{0,1} \) were derived in Eqs (21) and (32) of Percival and Robinson [1998b]:

\[
Y_{0,1} = Z(c)Z(a + bc) + \sum_{n=1}^{\infty} \frac{b^n}{n!(2i)^{n-1}} H_{n-1}(-ic)Z^n(a + bc), \quad |b| < 1, \tag{3.46}
\]

and

\[
Y_{0,1} = -\sum_{n=1}^{\infty} \frac{b^{-n}}{n!(2i)^{n-1}} H_{n-1}[-i(a + bc)]Z^n(c), \quad |b| > 1. \tag{3.47}
\]

We show in this section that Eq. (3.47) is only valid for \( b < -1 \). Here \( H_\alpha(z) \) refers to the Hermite polynomial of degree \( \alpha \) [Olver et al., 2010]. Then using Eq. (13) of Percival and Robinson [1998b], viz.,

\[
\frac{\partial Y_{m,n}}{\partial c} = nY_{m,n+1}, \tag{3.48}
\]

series expressions for \( Y_{0,p} \) were derived:

\[
Y_{0,p} = \sum_{n=0}^{p-1} \frac{b^n}{n!(p-n-1)!} Z^{(p-n-1)}(c)Z^n(a + bc)
+ \sum_{n=p}^{\infty} \frac{b^n}{n!(2i)^{n-p}} H_{n-p}(-ic)Z^n(a + bc), \quad |b| < 1, \tag{3.49}
\]
\[
Y_{0,p} = -b^{p-1} \sum_{n=p}^{\infty} \frac{b^{-n}}{n!(2i)^{n-p}} H_{n-p} \left[-i(a + bc)\right] Z^{(n)}(c), \quad |b| > 1. \tag{3.50}
\]

The series expressions in Eqs (3.49) and (3.50) have the same range of validity as their counterparts in Eqs (3.46) and (3.47). As such, we find the range of validity for Eq. (3.50) to be only \( b < -1 \). In this section we derive new expressions for \( Y_{0,p} \) that are valid for \( b > 1 \).

A closed-form expression was found for the imaginary part of \( Y_{0,1} \) given in Percival and Robinson [1998b] Eq. (38), viz,

\[
\text{Im } Y_{0,1} = \text{Im } Z(c) \text{Re } Z(a + bc) + \text{Im } Z \left( \frac{a}{\sqrt{1 + b^2}} \right) \text{Re } Z \left( \frac{b(a + bc) + c}{\sqrt{1 + b^2}} \right).	ag{3.51}
\]

Using Eq. (3.48) we derive the imaginary part of \( Y_{0,p} \):

\[
\text{Im } Y_{0,p} = \frac{1}{(p-1)!} \left\{ \sum_{n=0}^{p-1} \binom{p-1}{n} b^n \text{Re } Z^{(n)}(a + bc) \text{Im } Z^{(p-n-1)}(c) \right. \\
+ \left. \left( \sqrt{1 + b^2} \right)^{p-1} \text{Im } Z \left( \frac{a}{\sqrt{1 + b^2}} \right) \text{Re } Z^{(p-1)} \left[ \frac{b(a + bc) + c}{\sqrt{1 + b^2}} \right] \right\}.\tag{3.52}
\]

Equation (3.52) differs from the expression for \( \text{Im } Y_{0,p} \) in Percival and Robinson [1998b] Eq. (40); this is because the partial derivative of the first term in Eq. (3.51) was performed incorrectly in Percival and Robinson [1998b], by only retaining the \( n = 0 \) and \( n = p - 1 \) terms in the summation. We graph \( \text{Im } Y_{0,p} \) in Fig. 3.1. By virtue of closed-form expressions for \( \text{Im } Y_{0,p} \) in Eqs (3.51) and (3.52), series expressions for the \( Y_{0,p} \) are needed only for the calculation of their real parts.

We now turn our attention to the derivation of Eq. (3.47). To derive a convergent series expression for \( Y_{0,1} \) where \( |b| > 1 \), the function \( Y_{0,1} \) was written in Percival and Robinson [1998b] as

\[
Y_{0,1} = \frac{1}{\pi} \int_{-\infty}^{\infty} dy \frac{e^{-y^2}}{y - (a + bc)} \int_{-\infty}^{\infty} dx \frac{e^{-x^2}}{x - c - \frac{1}{x - (y - a)/b}}.\tag{3.53}
\]

From Eq. (3.38), application of the Landau prescription leads to the replacement \( a_j \rightarrow a_j + i0 \) in \( Y_{0,1}(a_j, b, c) \). For \( b < -1 \) the calculation proceeds as
Figure 3.1: Contour plots of \( \text{Im} Y_{0,p}(a, b, c) \) versus \( a \) and \( b \) for different values of \( c \) and \( p \). Contour intervals are logarithmic, with contour lines separated by factors of \( 10^{1/2} \). Positive regions are colored red, while negative regions are colored blue. Regions where the absolute value of the function is less than \( 10^{-4} \) are colored white.

in Percival and Robinson [1998b]. However, for \( b > 1 \) the imaginary part of \( (y - a)/b \) flips sign, leading to an additional term for \( Y_{0,1} \). On performing the \( x \) integration in Eq. (3.53) where \( b > 1 \) in terms of plasma dispersion functions, and from the definition of the generalized plasma dispersion function in
Eq. (3.20), we have

\[ Y_{0,1}(a, b, c) = Z(c)Z(a + bc) - Y_{0,1}(-a/b, 1/b, a + bc) + 2i\sqrt{\pi}I(a, b, c), \]

(3.54)

where

\[ I(a, b, c) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dy \frac{e^{-y^2}e^{-(y-a)/b}}{y - (a + bc)}. \]

(3.55)

The final term in Eq. (3.54) does not appear in the expression for \( Y_{0,1} \) where \( b < -1 \). It arises when the additional \( 2i\sqrt{\pi}e^{-u^2} \) term from the analytic continuation of the plasma dispersion function,

\[ Z(u) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dt e^{-t^2} + 2i\sqrt{\pi}e^{-u^2}, \quad \text{Im}(u) < 0, \]

(3.56)

is taken into account when integrating the second term in square brackets in Eq. (3.53) over \( x \). This term was not taken into account in Percival and Robinson [1998b]. The substitution \( u = (y - a)/b \) in Eq. (3.55) yields

\[ I(a, b, c) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} du \frac{e^{-(a+bu)^2}e^{-u^2}}{u - c}. \]

(3.57)

Using the Rodrigues formula for Hermite polynomials,

\[ H_n(u) = (-1)^n e^{a^2} \frac{d^n}{du^n} e^{-u^2}, \]

(3.58)

the power series expansion

\[ e^{-u^2} = e^{-c^2} \sum_{n=0}^{\infty} (-1)^n H_n(c)(u - c)^n \]

(3.59)

may be derived. Substituting Eq. (3.59) into Eq. (3.57), taking the summation outside the integral, and changing variables using \( y = a + bu \) gives

\[ I(a, b, c) = e^{-c^2} \left\{ Z(a + bc) + \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{(-b)^{-n}H_n(c)}{n!} \int_{-\infty}^{\infty} dy e^{-y^2} \right\} \times [y - (a + bc)]^{n-1}, \]

(3.60)

where the \( n = 0 \) term has been explicitly written in terms of the plasma dispersion function. On using the integral in Eq. (3.462.4) of Gradshteyn and Ryzhik [2007], i.e.,

\[ \int_{-\infty}^{\infty} dx (x - \beta)^n e^{-x^2} = \sqrt{\pi}(2i)^{-n}H_n(-i\beta), \quad n \geq 0, \]

(3.61)
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we have

\[ I(a, b, c) = e^{-c^2} \left[ Z(a + bc) + S(a, b, c) \right], \]  
(3.62)

where

\[ S(a, b, c) = \sum_{n=1}^{\infty} \frac{(-b)^n}{n!(2i)^{n-1}} H_{n-1} \left[ -i(a + bc) \right] H_n(c). \]  
(3.63)

Substituting Eq. (3.62) and the series expression for \( Y_{0,1} \) in Eq. (3.46) into the right hand side of Eq. (3.54) gives the convergent series expression

\[ Y_{0,1} = -\sum_{n=1}^{\infty} \frac{b^{-n}}{n!(2i)^{n-1}} H_{n-1} \left[ -i(a + bc) \right] Z^{(n)}(c) + 2i\sqrt{\pi}e^{-c^2} \left[ Z(a + bc) + S(a, b, c) \right], \quad b > 1. \]  
(3.64)

Then from Eq. (3.48) we derive

\[ Y_{0,p} = -b^{p-1} \sum_{n=p}^{\infty} \frac{b^{-n}}{n!(2i)^{n-p}} H_{n-p} \left[ -i(a + bc) \right] Z^{(n)}(c) + 2i\sqrt{\pi}e^{-c^2} \times \left[ b^{p-1} \sum_{n=0}^{p-1} \frac{(-b)^n}{n!(p-n-1)!} H_n(c) Z^{(p-n-1)}(a + bc) + \sum_{n=0}^{p-1} \frac{(-1)^n}{n!(p-n-1)!} \right] \times H_n(c) \frac{\partial^{p-n-1} S}{\partial c^{p-n-1}}, \quad b > 1. \]  
(3.65)

We see that the expressions in Eqs (3.64) and (3.65) for \( Y_{0,1} \) and \( Y_{0,p} \) have additional terms and series compared to the expressions given by Eqs (3.47) and (3.50) derived in Percival and Robinson [1998b], which are only valid for \( b < -1 \).

We now determine which parts of the expression for \( Y_{0,p} \) in Eq. (3.65) are real and which are imaginary, as only its real part requires evaluation. We first note that \( H_n(u) \) contains only even powers of \( u \) where \( n \) is even, and only odd powers of \( u \) where \( n \) is odd. Therefore where \( u \) is real, \( H_n(iu) \) is real for even \( n \) and imaginary for odd \( n \); hence, \( i^{-n}H_n(iu) \) is real for all \( n \). The contribution to Re\( Y_{0,p} \) is therefore from Re \( Z^{(n)}(c) \) in the first series and from Im \( Z^{(p-n-1)}(a + bc) \) in the second series. Next we observe that the function \( S \), given by Eq. (3.63), and its partial derivatives are real. When the third series in Eq. (3.65) is multiplied by the imaginary quantity outside the square brackets it contributes only to Im \( Y_{0,p} \), and as such it need not be evaluated.
In summary, Eqs (3.49), (3.50), and (3.65) may be used to calculate the real part of \( Y_{0,p} \) for \(|b| < 1\), \( b < -1\), and \( b > 1\), respectively, and Eq. (3.52) permits calculation of the imaginary part of \( Y_{0,p} \). The differences between these results and those in Percival and Robinson [1998b] are significant. The corrected series expression for \( Y_{0,p} \) where \( b > 1 \) in Eq. (3.65) has several additional terms resulting from the analytic continuation of the plasma dispersion function; this implies that \( Y_{0,p} \) is not even (nor odd) in \( b \) as the previous expression implies. The corrected closed-form expression for \( \text{Im} Y_{0,p} \) in Eq. (3.52) also has extra terms.

We note that there are difficulties in computing \( \text{Re} Y_{0,p} \) in particular regions of parameter space, especially near \(|b| = 1\), using the series expressions in Eqs (3.49), (3.50), and (3.65). This is because the number of terms required for convergence can be large, and approaches infinity as \(|b| \to 1\), so that the computation of \( H_n(u) \) and \( Z^{(n)}(u) \) for larger \( n \) is necessary near these regions. Numerical algorithms for calculating these functions are typically based on their three-term recurrence relations [i.e., \( H_{n+1}(u) = 2uH_n(u) - 2nH_{n-1}(u) \) (e.g., Olver et al. [2010]) and \( Z^{(n+1)}(u) = -2uZ^{(n)}(u) - 2nZ^{(n-1)}(u) \) (Percival and Robinson [1998b]) for \( n \geq 2 \)] which become unstable after a large number of iterations (see, e.g., Gautschi [1967]). In our calculations the algorithms become unstable where \( n \sim 120 - 150 \) (depending on the size of the function arguments), thus restricting the range of parameters for which we can compute \( \text{Re} Y_{0,p} \) accurately. The onset of instability when calculating \( H_n(u) \) and \( Z^{(n)}(u) \) is sudden and drastic, so we can readily identify empirically when the calculations become inaccurate. We derive a new expression for \( Z^{(n)}(u) \) in Sec. 3.B which permits its computation using numerical routines for the Hermite polynomials of negative order and complex argument (e.g., as provided by Mathematica) but which are also susceptible to instability for large \( n \).

3.6 Alternative series expressions for the generalized plasma dispersion functions

The slow convergence of the set of series expressions for \( Y_{0,p} \) given by Eqs (3.49), (3.50), and (3.65) in particular regions of parameter space, leading to numerical instabilities, motivates the derivation of alternative series expressions. In this section we derive a set of series expressions for \( Y_{0,p} \) which converge more quickly than the previously derived set, and thus confine the regions of numerical instability to smaller regions about \(|b| = 1\).
To derive the new series expressions we expand $Z(a + bx)$ about $x = 0$ in $Y_{0,1}$ [where $Y_{m,n}$ is defined in Eq. (3.20)] instead of expanding about $x = c$. This yields

$$Y_{0,1} = \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{b^n Z^{(n)}(a)}{n!} \int_{-\infty}^{\infty} dx^n e^{-x^2} \frac{x^n}{x - c}.$$  \hspace{1cm} (3.66)

We now present two different methods of evaluating the integral in Eq. (3.66). To derive the first new expression we use polynomial division to write

$$\frac{x^n}{x - c} = \sum_{j=0}^{n-1} c^{n-j-1} x^j + \frac{c^n}{x - c}. \hspace{1cm} (3.67)$$

The integrals that arise from the summation on the right hand side of Eq. (3.67) may be evaluated using the integral identity in Eq. (3.461.2) of Gradshteyn and Ryzhik [2007], which can be written as

$$\int_{-\infty}^{\infty} dx x^j e^{-x^2} = \frac{[1 + (-1)^j]}{2} \Gamma \left( \frac{j + 1}{2} \right), \quad j \geq 0, \hspace{1cm} (3.68)$$

where $\Gamma(x)$ is the gamma function [Olver et al., 2010]. The integral from the other term can be evaluated directly from the definition of the plasma dispersion function $Z(u)$ for $\text{Im}(u) > 0$ in Eq. (3.16). On rearranging the resulting expression we have the first alternative series expression

$$Y_{0,1} = \sum_{n=0}^{\infty} \frac{(bc)^n Z^{(n)}(a)}{n!} \left[ Z(c) + T(n,0,c) \right], \hspace{1cm} (3.69)$$

where we define

$$T(n,m,u) = \frac{(-u)^{-m}}{\sqrt{\pi}} \sum_{j=0}^{n-1} \frac{[1 + (-1)^j]}{2} \Gamma \left( \frac{j + 1}{2} \right) \frac{(j + m)!}{j!} u^{-(j+1)}, \quad n > 1, \hspace{1cm} (3.70)$$

and $T(0,m,u) = 0$. In the limit as $c \to 0$ the expression for $Y_{0,1}$ must be reevaluated by setting $(x - c + i0)^{-1} = (x + i0)^{-1}$ in Eq. (3.66), then performing the integration. This gives

$$Y_{0,1}(a,b,0) = i \sqrt{\pi} Z(a) + \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{b^n Z^{(n)}(a)}{n!} \frac{[1 + (-1)^{n-1}]}{2} \Gamma \left( \frac{n}{2} \right). \hspace{1cm} (3.71)$$
Alternatively, the integral in Eq. (3.66) may be evaluated by first projecting $x^n$ onto the basis of Hermite polynomials; i.e.,

$$x^n = \sum_{j=0}^{n} \beta_j H_j(x). \quad (3.72)$$

The coefficients $\beta_j$ are given by

$$\beta_j = \frac{1}{2 j! \sqrt{\pi}} \int_{-\infty}^{\infty} dx \ x^n e^{-x^2} H_j(x), \quad (3.73)$$

where the integral may be evaluated to give

$$\beta_j = \begin{cases} 0, & j + n \text{ odd;} \\ \frac{1}{\sqrt{\pi}} \Gamma \left( \frac{n + 1}{2} \right) \frac{(n/2)!}{[(n-j)/2]!}, & j, n \text{ both even;} \\ \frac{1}{\sqrt{\pi}} \Gamma \left( \frac{n + 2}{2} \right) \frac{[(n-1)/2]!}{[(n-j)/2]!}, & j, n \text{ both odd.} \end{cases} \quad (3.74)$$

We then interchange summation and integration in Eq. (3.66) and use the alternative expression for $Z^{(n)}(u)$ in Robinson [1989], i.e.,

$$Z^{(n)}(u) = \frac{(-1)^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx \ H_n(x)e^{-x^2} \frac{1}{x-u}, \quad (3.75)$$

to derive the second alternative series expression

$$Y_{0,1} = \sum_{n=0}^{\infty} \frac{b^n Z^{(n)}(a)}{n!} \sum_{j=0}^{n} (-1)^j \beta_j Z^{(j)}(c). \quad (3.76)$$

We note that both of the new series expressions in Eqs (3.69) and (3.76) have the same convergence properties since we have evaluated the integral as a finite sum in both cases. We find that the expression for $Y_{0,1}$ given by Eq. (3.69) is more appropriate for numerical evaluation of $\text{Re}Y_{0,1}$ than the expression in Eq. (3.76); this is because it requires fewer calculations of $Z^{(n)}(u)$, for which the algorithms are slower and more prone to instability than those for the gamma function. Therefore, we will show only the derivation of a set of series expressions for the generalized plasma dispersion functions based on Eq. (3.69).

We find empirically via numerical computation that the series expression in Eq. (3.69) is convergent for $|b| < 1$. We therefore use the same techniques as in Sec. 3.5 to derive convergent expressions for $b < -1$ and $b > 1$. For $b < -1$ we use Eq. (31) of Percival and Robinson [1998b]; i.e.,

$$Y_{0,1}(a, b, c) = Z(c)Z(a + bc) - Y_{0,1}(-a/b, 1/b, a + bc), \quad (3.77)$$
which leads to

\[ Y_{0,1} = Z(c)Z(a + bc) - \sum_{n=0}^{\infty} \frac{b^{-n}(a + bc)^n Z^{(n)}(-a/b)}{n!} \]

\[ \times \left[ Z(a + bc) + T(n, 0, a + bc) \right]. \]  \hspace{1cm} (3.78)

Where \( b > 1 \), Eq. (3.54) may be used to derive convergent series expressions. We now evaluate the integral \( I \) in Eq. (3.55) via a similar approach to that used earlier in this section, such that the resulting expression converges more quickly than that given in Eq. (3.62). We first expand \( e^{-\left[(y-a)/b\right]^2} \) about \( y = 0 \) in Eq. (3.55), giving

\[ I(a, b, c) = \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-b)^{-n} H_n(-a/b)e^{-(a/b)^2}}{n!} \int_{-\infty}^{\infty} dy \frac{y^n e^{-y^2}}{y - (a + bc)}. \]  \hspace{1cm} (3.79)

The same method of evaluating the integral in Eq. (3.66) may be used here, giving

\[ I(a, b, c) = \sum_{n=0}^{\infty} \frac{(-b)^{-n}(a + bc)^n H_n(-a/b)e^{-(a/b)^2}}{n!} \left[ Z(a + bc) + T(n, 0, a + bc) \right]. \]  \hspace{1cm} (3.80)

Using Eq. (3.54) then gives

\[ Y_{0,1} = Z(c)Z(a + bc) + \sum_{n=0}^{\infty} \frac{b^{-n}(a + bc)^n}{n!} \left[ 2i\sqrt{\pi}(-1)^n H_n(-a/b)e^{-(a/b)^2} ight.\]

\[ - \left. Z^{(n)}(-a/b) \right] \left[ Z(a + bc) + T(n, 0, a + bc) \right]. \]  \hspace{1cm} (3.81)

Where \( a + bc = 0 \), i.e. where \( c = -a/b \), we have, for \( b < -1 \) and \( b > 1 \) respectively,

\[ Y_{0,1}(a, b, -a/b) = -\frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{b^{-n}Z^{(n)}(-a/b)[1 + (-1)^{n-1}]}{n!} \Gamma \left( \frac{n}{2} \right) \]  \hspace{1cm} (3.82)

and

\[ Y_{0,1}(a, b, -a/b) = \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{b^{-n}}{n!} \left[ 2i\sqrt{\pi}(-1)^n H_n(-a/b)e^{-(a/b)^2} - Z^{(n)}(-a/b) \right] \]

\[ \times \frac{[1 + (-1)^{n-1}]}{2} \Gamma \left( \frac{n}{2} \right). \]  \hspace{1cm} (3.83)
The $Y_{0,p}$ functions may then be derived using Eq. (3.48) with Eqs (3.69), (3.78), and (3.81), respectively. This gives

$$Y_{0,p} = \begin{cases} \sum_{k=0}^{p-1} \frac{c-k}{k!(p-k-1)!} \sum_{n=k}^{\infty} \frac{(bc)^n Z^{(n)}(a)}{(n-k)!} \left[ Z^{(p-k-1)}(c) + T(n, p - k - 1, c) \right], & |b| < 1; \\ b^{p-1} \sum_{k=0}^{p-1} \frac{1}{k!(p-k-1)!} \left\{ b^{-k} Z^{(p-k-1)}(a + bc) Z^{(k)}(c) - (a + bc)^{-k} \\ \times \sum_{n=k}^{\infty} \frac{b^{-n}(a + bc)^n Z^{(n)}(-a/b)}{(n-k)!} \left[ Z^{(p-k-1)}(a + bc) \\ + T(n, p - k - 1, a + bc) \right] \right\}, & b < -1; \\ b^{p-1} \sum_{k=0}^{\infty} \frac{1}{n!} \left\{ b^{-n} Z^{(n)}(a + bc) Z^{(n)}(c) + (a + bc)^{-k} \\ \times \sum_{n=k}^{\infty} \frac{b^{-n}(a + bc)^n Z^{(n)}(-a/b)}{(n-k)!} \left[ 2i\sqrt{\pi}(-1)^{n} H_n(-a/b)e^{-(a/b)^2} - Z^{(n)}(-a/b) \right] \\ \times \left[ Z^{(p-k-1)}(a + bc) + T(n, p - k - 1, a + bc) \right] \right\}, & b > 1. \end{cases}$$

Special cases for $Y_{0,p}$ are derived from Eqs (3.71), (3.82), and (3.83), giving

$$Y_{0,p} = \begin{cases} \sum_{n=0}^{p-1} \frac{b^n Z^{(n)}(a) Z^{(p-n-1)}(0)}{n!} + \frac{1}{\sqrt{\pi}} \sum_{n=p}^{\infty} \frac{b^n Z^{(n)}(a)}{n!} \left[ 1 + (-1)^{n-p} \right] \\ \times \Gamma \left( \frac{n-p+1}{2} \right), & |b| < 1, c = 0; \\ b^{p-1} \sum_{n=p}^{\infty} \frac{b^{-n} Z^{(n)}(-a/b)}{n!} \left[ 1 + (-1)^{n-p} \right] \Gamma \left( \frac{n-p+1}{2} \right), & b < -1, a + bc = 0; \\ b^{p-1} \sum_{n=p}^{\infty} \frac{b^{-n}}{n!} \left[ 2i\sqrt{\pi}(-1)^{n} H_n(-a/b)e^{-(a/b)^2} - Z^{(n)}(-a/b) \right] \\ \times \left[ 1 + (-1)^{n-p} \right] \Gamma \left( \frac{n-p+1}{2} \right), & b > 1, a + bc = 0, \end{cases}$$

(3.85)
Figure 3.2: Contour plots of $\text{Re} Y_{0,p}(a, b, c)$ versus $a$ and $b$ for different values of $c$ and $p$. Contour intervals are logarithmic, with contour lines separated by factors of $10^{1/2}$. Positive regions are colored red, while negative regions are colored blue. Regions where the absolute value of the function is less than $10^{-4}$ are colored white.

where [Percival and Robinson, 1998b]

$$Z^{(n)}(0) = \begin{cases} \frac{i\sqrt{\pi}(-1)^{2k+1}(2k)!}{k!}, & n = 2k; \\ (-1)^{k+1}2^{2k+1}k!, & n = 2k + 1. \end{cases} \tag{3.86}$$

We graph $\text{Re} Y_{0,p}$ in Fig. 3.2 using Eqs (3.84) and (3.85), where we have calculated $\text{Re} Y_{0,p}$ to 3 significant figures by truncating the series when its $n$th partial sum $S_n$ satisfies $|(S_{n+1} - S_n)/S_n| < 10^{-3}$. In plotting $\text{Re} Y_{0,p}$ we have
linearly interpolated over regions of numerical instability; these regions are shown in Fig. 3.3 for $Y_{0,1}$ along with the rates of convergence for the different series expressions. We find that the alternative set of series expressions in Eqs (3.69), (3.78), and (3.81) [with special cases given by Eqs (3.71), (3.82), and (3.83)] converges much faster than the series expressions given by Eqs (3.46), (3.47), and (3.64). As a result, the regions of numerical instability are greatly diminished to $0.98 \lesssim |b| \lesssim 1.02$ for $c = 0.1$, $c = 1$, and $c = 5$, with an additional region of instability where $2.4 \lesssim a \lesssim 3.2$ and $1.02 \lesssim b \lesssim 1.25$ for $c = 5$. We find similar results for the higher order generalized plasma dispersion functions of the same parameters. The smallness of the unstable regions for these parameters permits accurate linear interpolation of the functions over these regions. For larger arguments of $Y_{0,p}$ the numerical evaluation becomes unstable for a larger region of parameter space, in which case interpolation over these regions may not be particularly accurate (see Fig. 3.4 for unstable regions where $c = 10$ and $c = 15$). However, the dominant generalized plasma dispersion functions in Eq. (3.45) are those with smaller $a$ and $c$ arguments, as shown in Fig. 3.2, and thus the instability for large arguments does not prevent an accurate calculation of the quadratic longitudinal response function.

### 3.7 Approximation for two fast waves and one slow wave

To recover the approximate form of the quadratic longitudinal response function in Eq. (3.17) we take the limit of the exact expression in Eq. (3.45) where two of the waves have infinite phase speeds; i.e., $s, s_1 \to \infty$, and the other wave has a low phase speed; i.e., $s_2 \ll 1$. In these limits, $\tilde{a}_1, \tilde{a}_1, \tilde{a}_2 \to \infty$ and hence $|\tilde{r}|, |\tilde{r}| \to \infty$. The functions $Y_{0,p}$ may be approximated for large $c$ by [Percival and Robinson, 1998b]

$$Y_{0,p} \approx \frac{1}{(-c)p^{\frac{3}{2}} \sqrt{1 + b^2}} Z \left( \frac{a}{\sqrt{1 + b^2}} \right), \quad |c| \gg 1,$$

whence

$$Y_{0,p} \to 0 \quad \text{as} \quad |c| \to \infty.$$

However, a different approximation to $Y_{0,p}$ is required for $|c| \lesssim 1$ and $|a| \gg 1$. The main contribution to the integral in Eq. (3.20) for the generalized plasma
Figure 3.3: Number of terms $n$ (color coded) required for convergence of Re $Y_{0,1}(a, b, c)$ to 3 significant figures versus $a$ and $b$ for different values of $c$. Figures (a)–(c) are for the set of expansions in Sec. 3.5 while (d)–(f) are for the set of expansions in Sec. 3.6, where $c = 0.1$, 1, and 5, respectively. Regions of numerical instability are colored white, and regions where $n > 100$ are colored dark red.
dispersion function is then from \( x \approx c \), so we can approximate \( Z(a + bx) \) by \( Z(a) \approx -1/a \) in the integrand, provided \( |a|/|b| \gg 1 \) also holds. We then have

\[
Y_{0,p} \approx -\frac{1}{a} \int_{-\infty}^{\infty} dx \frac{e^{-x^2}}{(x - c)^p},
\]

\[
= -\frac{Z^{(p-1)}(c)}{a}.
\]

(3.89)

On substituting Eq. (3.88) into Eq. (3.45) the terms multiplied by \( k_1 \cdot k_2 \) and \( k \cdot k_2 \) vanish, as does \( Y_{0,1}(\bar{a}_j, \bar{b}_j, \bar{r}) \), since each of the relevant generalized plasma dispersion functions has \( |c| \to \infty \). From Eq. (3.90) we have \( Y_{0,1}(\bar{a}_j, \bar{b}_j, s_2)/\bar{r}Y_{0,2}(\bar{a}_j, \bar{b}_j, s_2) \approx Z(s_2)/\bar{r}Z'(s_2) \approx s_2/\bar{r} \ll 1 \), so the \( Y_{0,2} \) functions are dominant. The approximations Eqs (3.88) and (3.90) thus yield

\[
\alpha^{(2)} = \frac{q^3n}{m^2 4V^4|k||k_1||k_2||k \times k_1|} \left[ \frac{k \cdot k_1 |k_2|^2}{(k \cdot k_2 + k_1 \cdot k_2)} \frac{1}{\bar{r}} \left( \frac{1}{\bar{a}_1} + \frac{1}{\bar{a}_2} \right) Z'(s_2) \right].
\]

(3.91)

Writing \( \bar{r} \) and \( \bar{a}_j \) explicitly using Eqs (3.41) and (3.100)–(3.103) gives

\[
\alpha^{(2)} = \frac{q^3n}{m^2 4V^4|k||k_1||k_2||k \times k_1|} \left[ \frac{k \cdot k_1 |k_2|^2}{(k \cdot k_2 + k_1 \cdot k_2)} \frac{(k \cdot k_2 + k_1 \cdot k_2)}{|k \times k_1|} \right]
\]

\[
\times \frac{(V\sqrt{2})^2 |k \times k_1|^2}{\omega_1 |k_2|^2} Z'(s_2).
\]

(3.92)

On using the differential equation \( Z'(u) = -2 - 2uZ(u) \) [Fried and Conte, 1961] and after various cancellations we recover the approximate quadratic longitudinal response function in Eq. (3.17).

### 3.8 Accuracy of the approximations to the generalized plasma dispersion functions

In this section we assess the accuracy of the various approximations that have been derived for the generalized plasma dispersion functions; these include the approximations in Eqs (3.87) and (3.90), as well as another approximation for \( a \gg 1 \) [Percival and Robinson, 1998b],

\[
Y_{0,1} \approx Z(a + bc) \left[ Z(c) - Z(-a/b) \right],
\]

(3.93)
Figure 3.4: Graphs of $D = \log_{10}|(Y_{0,1}^{(exact)} - Y_{0,1}^{(approx)})/Y_{0,1}^{(exact)}|$ for the different approximations. (i)–(iii): $|c| \gg 1$ approximation [Eq. (3.87)], $c = 5, 10, \text{ and } 15$ respectively; (iv)–(vi): Padé approximant [Eq. (3.94)], $c = 0.1, 1, \text{ and } 5$ respectively; (vii)–(ix): $|a| \gg 1$ approximation [Eq. (3.93)], $a = 5, 10, \text{ and } 15$ respectively; (x)–(xii): $|c| \gg 1$ approximation [Eq. (3.90)], $a = 5, 10, \text{ and } 15$ respectively. Regions where $D > -1$ are colored white, and regions of numerical instability are colored gray.
and a Padé approximant [Percival and Robinson, 1998b],

\[
Y_{0,1} \approx \tilde{Z}(c)\tilde{Z}(a + bc) - \left[ \frac{\gamma}{a + bc - \delta} \tilde{Z} \left( \frac{\delta - a}{b} \right) + \frac{\gamma^*}{a + bc + \delta^*} \tilde{Z} \left( -\frac{\delta^* + a}{b} \right) \right],
\]

(3.94)

where \(\gamma = -0.5 - 0.7425i\), \(\delta = 0.5228 - 0.7763i\), and

\[\tilde{Z}(u) = \frac{\gamma}{u - \delta} + \frac{\gamma^*}{u + \delta^*}\]

(3.95)
is a two-pole Padé approximant to the Fried-Conte plasma dispersion function [Robinson and Newman, 1988]. Approximations to the higher order generalized plasma dispersion functions based on Eqs (3.93) and (3.94) may be derived using Eq. (3.48). We show a sample of the accuracy of these approximations for \(Y_{0,1}\) in Fig. 3.4.

From Fig. 3.4 we see that the approximation for \(|c| \gg 1\) in Eq. (3.87) is accurate only near \(a = |b|\) or \(b = 0\) for each value of \(c\). As \(c\) increases, the intervals around \(a = |b|\) and \(b = 0\) for which the approximation is accurate become larger. In contrast, the approximation for \(|a| \gg 1\) in Eq. (3.93) is accurate in a much larger region of parameter space; it fails mainly near \(a + bc = 0\), as stated in Percival and Robinson [1998b]. The other approximation for \(|a| \gg 1\) in Eq. (3.90) is accurate in smaller regions than for Eq. (3.93). The approximation in Eq. (3.90) is useful predominantly for recovering the approximate quadratic longitudinal response function in Eq. (3.17); it does not become accurate for much of the \(c < 1\) parameter space until \(|a| \gg 1\) due to the requirement that \(|a|/|b| \gg 1\). The Padé approximant fails in a large region of parameter space. It only becomes accurate for an appreciable region of parameter space when \(c\) is large, in which Eq. (3.87) is a more accurate approximation of \(Y_{0,1}\) anyway. The Padé approximant is therefore not very useful.

### 3.9 Summary and conclusions

We have derived an exact expression for the quadratic longitudinal response function of a nonrelativistic collisionless unmagnetized isotropic Maxwellian plasma, which we have written in Eq. (3.45) in terms of generalized plasma dispersion functions. This response function accurately describes the wave coupling between three electrostatic waves. The expression in Eq. (3.45) depends on both the phase speeds of the waves and the interaction geometry. We
have corrected expressions previously derived for the real and imaginary parts of the generalized plasma dispersion functions $Y_{0,p}(a, b, c)$ so that they are valid for all values of the parameters $b$ and $p$.

For specific numerical calculations, the accuracy of the quadratic longitudinal response function is determined by the accuracy to which the generalized plasma dispersion functions are computed. Series expressions for the generalized plasma dispersion functions included here require calculation of $Z^{(n)}(u)$, and $H_n(u)$ for some expressions. Numerical instability in iterating the three-term recurrence relations for these functions for large $n$ limits the number of calculable terms, and hence restricts the number of significant figures that can be obtained for the generalized plasma dispersion functions. We have derived a new set of series expressions for the generalized plasma dispersion functions in Eqs (3.84) and (3.85) which converge more rapidly than those derived in Percival and Robinson [1998b], thus reducing the regions of instability to small regions about $|b| = 1$. This allows computation of the functions to at least 3 significant figures for a large range of parameter space.

In the limit where two of the waves are fast and one is slow, which has been previously assumed for the wave coupling in electrostatic decay and scattering off thermal ions, we have shown by approximating the generalized plasma dispersion functions that the exact quadratic longitudinal response function reduces to the commonly used approximation in Eq. (3.17). The exact expression for the quadratic longitudinal response function that we have derived is relatively compact compared to the exact quadratic response tensor. It is therefore feasible to use Eq. (3.45) in calculating the rate for nonlinear wave-wave and wave-particle interactions involving electrostatic waves, which is the subject of Chapter 4.
3.A Interchanges of $a_j$ and $b_j$

\[ \tilde{a}_1 = \frac{\omega |k_1|}{|k \times k_1| V \sqrt{2}}, \]  
(3.96)

\[ \tilde{a}_2 = \frac{\omega_2 |k_1|}{|k \times k_1| V \sqrt{2}}, \]  
(3.97)

\[ \tilde{b}_1 = b_1 = - \frac{k \cdot k_1}{|k \times k_1|}; \]  
(3.98)

\[ \tilde{b}_2 = - \frac{k_1 \cdot k_2}{|k \times k_1|}; \]  
(3.99)

\[ \bar{a}_1 = \frac{\omega_1 |k_2|}{|k \times k_1| V \sqrt{2}}, \]  
(3.100)

\[ \bar{a}_2 = \frac{\omega_2 |k_2|}{|k \times k_1| V \sqrt{2}}, \]  
(3.101)

\[ \bar{b}_1 = \bar{b}_2 = - \frac{k_1 \cdot k_2}{|k \times k_1|}; \]  
(3.102)

\[ \bar{b}_2 = b_2 = - \frac{k \cdot k_2}{|k \times k_1|}. \]  
(3.103)

3.B Numerical computation of $Z^{(n)}(u)$

The derivatives of the plasma dispersion function $Z^{(n)}(u)$ may be calculated via the recursion relation $Z^{(n+1)}(u) = -2u Z^{(n)}(u) - 2n Z^{(n-1)}(u)$, using upward recursion for $0 \leq u \leq \sqrt{n}$ and downward recursion for $u \geq \sqrt{n}$ [Percival and Robinson, 1998b]. The derivatives may also be expressed in terms of repeated integrals of the complementary error function [Robinson, 1989, 1990]; however, this form is inappropriate for computation where $u$ is on or near the real axis [Percival and Robinson, 1998b]. We now derive an alternative expression for $Z^{(n)}(u)$ which permits computation using a numerical algorithm for Hermite polynomials of negative order and complex argument. Given the integral representation of the Hermite polynomials as [Gradshteyn and Ryzhik, 2007]

\[ H_n(u) = \frac{2^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} dt e^{-t^2} (u + it)^n, \]  
(3.104)

the plasma dispersion function in Eq. (3.16) can be expressed simply as

\[ Z(u) = 2i H_{-1}(-iu). \]  
(3.105)
Then using

\[ H_n'(-iu) = 2iH_{n-1}(-iu), \]  

(3.106)

the derivatives of the plasma dispersion function are given by

\[ Z^{(n)}(u) = n!(2i)^{n+1}H_{(n+1)}(-iu). \]  

(3.107)

An expression for the imaginary part of \( Z^{(n)}(u) \) derived in Percival and Robinson [1998b] is

\[ \text{Im } Z^{(n)}(u) = (-1)^n \sqrt{\pi} e^{-u^2} H_n(u). \]  

(3.108)

Thus Eq. (3.107) is needed only for computation of the real part of \( Z^{(n)}(u) \). We note that Eqs (3.107) and (3.108) show that the real and imaginary parts of \( Y_{0,p} \) in Eqs (3.49), (3.50), (3.65), and (3.52) can be expressed solely in terms of Hermite polynomials.
Chapter 4

Exact evaluation of the rates of electrostatic decay and scattering off thermal ions for an unmagnetized Maxwellian plasma

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4.1 Abstract

Electrostatic decay of Langmuir waves into Langmuir and ion-sound waves ($L \leftrightarrow L + S$) and scattering of Langmuir waves off thermal ions ($L + i \leftrightarrow L' + i'$, also called “nonlinear Landau damping”) are important nonlinear weak-turbulence processes. The rates for these processes depend on the quadratic longitudinal response function $\alpha^{(2)}$ (or, equivalently, the quadratic longitudinal susceptibility $\chi^{(2)}$), which describes the second-order response of a plasma to electrostatic wave fields. Previous calculations of these rates for an unmagnetized Maxwellian plasma have relied upon an approximate form for $\alpha^{(2)}$ that is valid where two of the wave fields are fast (i.e., $v_\phi = \omega/k \gg V_e$ where $\omega$ is the angular frequency, $k$ is the wavenumber, and $V_e$ is the electron thermal speed) and one is slow ($v_\phi \ll V_e$). Recently an exact expression was derived for $\alpha^{(2)}$ that is valid for any phase speeds of the three waves in an unmagnetized Maxwellian plasma. Here, this exact $\alpha^{(2)}$ is applied to the calculation of the three-dimensional rates for electrostatic decay and scattering off thermal ions, and the resulting exact rates are compared with the approximate rates. The
calculations are performed using previously derived three-dimensional rates for electrostatic decay given in terms of a general $\alpha^{(2)}$, and newly derived three-dimensional rates for scattering off thermal ions; the scattering rate is derived assuming a Maxwellian ion distribution, and both rates are derived assuming arc distributions for the wave spectra. For most space plasma conditions the approximate rate is found to be accurate to better than 20%; however, for sufficiently low Langmuir phase speeds ($v_\phi/V_e \approx 3$) appropriate to some spatial domains of the foreshock regions of planetary bow shocks and type II solar radio bursts, the use of the exact rate may be necessary for accurate calculations. The relative rates of electrostatic decay and scattering off thermal ions are calculated for a range of parameters using the exact expressions for the rates; electrostatic decay is found to have the larger growth rate over the whole range of parameters, consistent with previous approximate calculations.
4.2 Introduction

The linear and nonlinear evolution of electron plasma oscillations, known as Langmuir waves, is a fundamental area of study in plasma physics. Langmuir waves at the electron plasma frequency $\omega_p$ can be driven via the bump-on-tail instability by a fast electron beam propagating through a plasma [Vedenov et al., 1962; Drummond and Pines, 1962; Davidson, 1972; Melrose, 1986a; Stix, 1992; Gary, 1993]. Such beam-driven Langmuir waves are observed in a wide range of contexts in both laboratory [Roberson et al., 1971; Wong and Cheung, 1984] and space plasmas [Scarf et al., 1971; Gurnett and Anderson, 1976; Libbert and Kellogg, 1979; Gurnett et al., 1981; Anderson et al., 1981; Lin et al., 1981; Greenstadt et al., 1995; Bale et al., 1999; Cairns and Robinson, 1999; Kasaba et al., 2000; Malaspina et al., 2009]. These waves can participate in resonant linear and nonlinear processes, namely: linear wave-particle interactions, satisfying the resonance condition $\omega - k \cdot v = 0$, where $\omega$ is the angular frequency, $k$ is the wave vector, and $v$ is the particle velocity; nonlinear wave-particle interactions, satisfying $\sum_{i=1}^{n}(\omega_i - k_i \cdot v) = 0$, $n \geq 2$; and nonlinear wave-wave interactions, with $\sum_{i=1}^{n} \omega_i = 0$, $\sum_{i=1}^{n} k_i = 0$, and $n \geq 3$ [Sagdeev and Galeev, 1969; Tsytovich, 1970; Davidson, 1972; Tsytovich, 1977; Sitenko, 1982; Melrose, 1986a].

Such nonlinear processes involving Langmuir waves include scattering off thermal ions (also called “nonlinear Landau damping”, and hereafter abbreviated as STI), denoted $L + i \leftrightarrow L' + i'$, in which Langmuir waves $L$ scatter off the electron polarization (“Debye shielding”) clouds around thermal ions $i$, and the electrostatic decay (ESD) of Langmuir waves into product Langmuir waves $L'$ and ion-sound waves $S$, denoted $L \leftrightarrow L' + S$ [Sagdeev and Galeev, 1969; Tsytovich, 1970; Davidson, 1972; Tsytovich, 1977; Sitenko, 1982; Melrose, 1986a,b]. These two processes are closely related; the single-particle ion response in STI is replaced by a collective ion response (i.e., an ion-sound wave) in ESD. The relative importance of ESD or STI in the nonlinear evolution of Langmuir waves depends on their relative growth rates, in addition to “collective” and “timescale” constraints on whether these processes can proceed [Zakharov et al., 1985; Cairns, 2000]. The ESD process was predicted in Cairns [2000] and Mitchell et al. [2003] to dominate over STI in type III sources and high beam speed regions of Earth’s foreshock for a specific range of wave amplitudes, while for higher wave amplitudes only STI can proceed. The predicted dominance of ESD over STI has also been found in numerical solutions of the weak-turbulence kinetic equations by Kontar and Pécseli [2002].
Both STI and ESD have weak-turbulence (also called “random-phase” or “resistive”) and strong-turbulence (also called “parametric” or “reactive”) versions [Melrose, 1986a,b], corresponding respectively to the phases of the wave packets either decohering or remaining coherent over the timescale of the nonlinear interaction [Sagdeev and Galeev, 1969; Tsytovich, 1970; Davidson, 1972; Tsytovich, 1977; Zakharov et al., 1985; Melrose, 1986a,b; Robinson, 1997; Cairns, 2000]. These conditions may be expressed as \( \Delta \omega \gtrsim \Gamma \) and \( \Delta \omega \lesssim \Gamma \), respectively, where \( \Delta \omega \) is the bandwidth of the growing waves and \( \Gamma \) is the nonlinear growth rate [Tsytovich, 1970; Zakharov et al., 1985; Robinson, 1997]. It has been argued from analyses of bandwidths and nonlinear growth rates that the weak-turbulence versions of these processes are usually the relevant ones in space plasmas [Cairns, 2000], such as in Earth’s foreshock [Cairns et al., 1998] and other planetary foreshocks [Cairns and Robinson, 1992a], and in type III radio emissions from the solar corona to 1 AU [Cairns and Robinson, 1998].

In kinetic theory the strength of the coupling between three electromagnetic waves, and hence the nonlinear interaction rate, is described by the quadratic response tensor [Melrose, 1986a; Percival and Robinson, 1998a] (or, equivalently, the second-order nonlinear susceptibility tensor [Sitenko, 1982; Yoon, 2005]). The general expression for the quadratic response tensor involves velocity-space integrals over the distribution function and various resonant denominators [Tsytovich, 1970; Davidson, 1972; Sitenko, 1982; Melrose, 1986a]. Typically these integrals are approximated by assuming that each wave field has a phase speed \( v_\phi = \omega / k \) that is either much greater or much less than the electron thermal speed \( V_e \); the former approximation is made for Langmuir and transverse \( (T) \) waves, and the latter for ion-sound waves and shielding fields associated with thermal ions [Suramlishvili, 1963; Akhiezer et al., 1964; Akhiezer, 1965; Lipervskii and Tsytovich, 1965; Kovrizhnykh, 1966; Tsytovich, 1970; Sitenko, 1982; Zakharov et al., 1985; Melrose, 1986a; Muschietti and Dum, 1991; Cairns, 2000; Ziebell et al., 2001; Kontar and Pécseli, 2002; Ziebell et al., 2008, 2011]. The ranges of validity for these approximate response tensors—and hence the nonlinear rates—are not clear, and the approximations become inaccurate as these phase speed assumptions break down. Conversely the quadratic response tensor for an unmagnetized Maxwellian plasma has been evaluated exactly in Percival and Robinson [1998a] (with the expression shown in Eq. (1.89)) in terms of generalized plasma dispersion functions [Percival and Robinson, 1998b], but the cumbersome expression prevents a feasible calculation of nonlinear rates.
In order to improve the cold-plasma description of the wave coupling where $v_\phi \sim V_e$, and due to the difficulty of applying the exact quadratic response tensor, a thermal correction to the cold-plasma quadratic response tensor was derived in Layden et al. [2011] (Chapter 2) for interactions between three waves with $v_\phi > V_e$. This was used to calculate a thermal correction to the rate of second harmonic plasma emission $L + L' \leftrightarrow T$. This correction was predicted to be important in regions of Earth’s foreshock and in foreshock sources of type II radio emissions where the speed of the electron beam driving the Langmuir waves is a few $V_e$. However, the accuracy of other nonlinear rates has yet to be determined.

Although it is infeasible to apply the full quadratic response tensor to nonlinear rate calculations, the longitudinal part of this response tensor, called the quadratic longitudinal response function, is sufficient to describe interactions between three electrostatic waves such as in ESD and STI. An exact expression for the quadratic longitudinal response function for an unmagnetized Maxwellian plasma was recently derived in Layden et al. [2012] (Chapter 3), also in terms of generalized plasma dispersion functions. This is much more compact than the full quadratic response tensor, thus allowing the corresponding nonlinear rates to be calculated.

In this chapter, we apply the exact quadratic longitudinal response function of Layden et al. [2012] (Chapter 3) to the calculation of the rates for ESD and STI; we thus derive expressions for these rates that are valid for any phase speeds of the three interacting waves. We compare these rates with the previously derived rates for ESD and STI, and assess the ranges of validity for the previous rates.

The chapter is structured as follows. In Sec. 4.3 we outline the theory for ESD and STI. We present the expressions for the approximate and exact quadratic longitudinal response functions and general expressions for the rates of ESD and STI. In Secs 4.4 and 4.5 we derive the rates of ESD and STI respectively for wave spectra with arc distributions, calculate the rates numerically for both the approximate and exact quadratic longitudinal response functions, and assess the accuracy of the approximate rates by comparing them with the exact rates. We calculate in Sec. 4.6 the relative rates of ESD and STI using the exact rates derived in Secs 4.4 and 4.5, and summarize the results of the chapter in Sec. 4.7.
4.3 Theory for nonlinear rates

The strength of wave coupling in second-order nonlinear processes (i.e., three-wave interactions and nonlinear wave-particle interactions) involving electrostatic wave fields is described in kinetic theory by the quadratic longitudinal response function \( \alpha^{(2)}(k, k_1, k_2) \). (Here, \( k_m \) collectively denotes \( \omega_m \) and \( k_m \) for the \( m \)th wave.) The quadratic longitudinal response function is defined to be the longitudinal part of the quadratic response tensor, which is the second-order coefficient in a weak-turbulence expansion of the induced current density in powers of the vector potential [Melrose, 1986a]. The general form of this function involves velocity-space integrals of the velocity distribution function and resonant denominators (see, e.g., Sitenko [1982])

\[
\alpha^{(2)}(k, k_1, k_2) = \frac{q^3}{m^2} \frac{\omega_1 \omega_2}{|k||k_1||k_2|} \left[ k_1 \cdot k_2 |k|^2 M(k; k_1, k_2) + (k \leftrightarrow k_1) + (k \leftrightarrow k_2) \right],
\]

where

\[
M(k; k_1, k_2) = \int d^3v \frac{f(v)}{(\omega - k \cdot v)^2(\omega_1 - k_1 \cdot v)(\omega_2 - k_2 \cdot v)};
\]

and where \((k_m \leftrightarrow k_n)\) denotes additional terms generated by the interchange of \( k_m \) and \( k_n \).

To obtain an explicit expression for \( \alpha^{(2)} \) for ESD and STI, the integrals \( M \) are typically approximated [Suramlishvili, 1963; Akhiezer et al., 1964; Akhiezer, 1965; Liperovskii and Tsytovich, 1965; Kovrizhnykh, 1966; Tsytovich, 1970; Sitenko, 1982; Zakharov et al., 1985; Melrose, 1986a; Muschietti and Dum, 1991; Cairns, 2000; Ziebell et al., 2001; Kontar and Pécsei, 2002; Ziebell et al., 2008, 2011] by assuming that the two Langmuir wave fields are fast (i.e., \( \omega/k \gg V_e \)) and the ion-sound/thermal-ion disturbance is slow (i.e., \( \omega/k \ll V_e \)), as well as assuming a Maxwellian velocity distribution (see, e.g., Melrose [1986a] and Layden et al. [2012] (Chapter 3) for an outline of the derivation). This leads to the approximate quadratic longitudinal response function [Suramlishvili, 1963; Liperovskii and Tsytovich, 1965; Tsytovich, 1970; Melrose, 1986a; Layden et al., 2012]

\[
\alpha^{(2)}(k, k_1, k_2) \approx -\frac{q^3 n}{m^2} \frac{\omega_2 k \cdot k_1}{|k||k_1||k_2|V_e^2} \left[ 1 + s_2 Z(s_2) \right],
\]

where

\[
s_j = \frac{\omega_j}{|k_j||V_e\sqrt{2}|},
\]

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and where

\[ Z(u) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dt \frac{e^{-t^2}}{t - u}, \quad \text{Im}(u) > 0 \]  \hspace{1cm} (4.5)

is the Fried-Conte plasma dispersion function [Fried and Conte, 1961] (defined by analytic continuation for \( \text{Im}(u) \leq 0 \)), with \( j \) labeling the wave fields \( k, k_1, \) and \( k_2. \)

Recently an exact expression for the quadratic longitudinal response function of a Maxwellian plasma was derived [Layden et al., 2012] (Chapter 3) by evaluating the integrals \( M \) in terms of generalized plasma dispersion functions \( Y_{m,n} \) [Percival and Robinson, 1998a,b], giving

\[ \alpha^{(2)} = q^3 n \frac{\omega \omega_1 \omega_2}{m^2 4 V^4 |k||k_1||k_2||k \times k_1|} \left( k \cdot k_2 \left[ Y_{0,3}(a_1, b_1, s) + Y_{0,3}(a_2, b_2, s) \right] \right. \\
+ \frac{k \cdot k_2 |k_1|^2}{(k \cdot k_1 + k_1 \cdot k_2)} \left. \frac{1}{\vec{r} - s_1} \left\{ \left[ Y_{0,1}(a_1, b_1, \tilde{r}) + Y_{0,1}(a_2, b_2, \tilde{r}) \right. \right. \\
- Y_{0,1}(\tilde{a}_1, b_1, s_1) - Y_{0,1}(\tilde{a}_2, b_2, s_1) \left] - Y_{0,2}(\tilde{a}_1, b_1, s_1) - Y_{0,2}(\tilde{a}_2, b_2, s_1) \right. \right\} + \frac{k \cdot k_1 |k_2|^2}{(k \cdot k_2 + k_1 \cdot k_2)} \left. \right. \\
\left. \frac{1}{\vec{r} - s_2} \left[ Y_{0,1}(a_1, b_1, \tilde{r}) + Y_{0,1}(a_2, b_2, \tilde{r}) \right. \right. \\
- Y_{0,1}(\tilde{a}_1, b_1, s_2) - Y_{0,1}(\tilde{a}_2, b_2, s_2) \left] - Y_{0,2}(\tilde{a}_1, b_1, s_2) - Y_{0,2}(\tilde{a}_2, b_2, s_2) \right. \right\} \right), \]  \hspace{1cm} (4.6)

where

\[ Y_{m,n}(a, b, c) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx \frac{e^{-x^2}}{(x - c)^n} Z(a + bx), \]  \hspace{1cm} (4.7)

\[ a_j = \frac{\omega_j |k|}{|k \times k_1|V \sqrt{2}}, \]  \hspace{1cm} (4.8)

\[ b_j = -\frac{k \cdot k_j}{|k \times k_1|}, \]  \hspace{1cm} (4.9)

and

\[ r = \frac{a_1 + a_2}{b_1 + b_2}. \]  \hspace{1cm} (4.10)

The variables \( \tilde{a}_j, \tilde{b}_j, \) and \( \tilde{r} \) in Eq. (4.6) are new variables defined by interchanging \( k \) and \( k_1 \) in Eqs (4.8), (4.9), and (4.10), respectively. Likewise, the variables
$a_j$, $b_j$, and $\bar{r}$ are new variables defined by interchanging $k$ and $k_2$ in Eqs (4.8), (4.9), and (4.10), respectively. In addition to deriving the exact quadratic longitudinal response function [Eq. (4.6)] in Layden et al. [2012] (Chapter 3), a more rapidly convergent set of expressions for the generalized plasma dispersion functions was derived and previous expressions were corrected (see Eq. (3.52) for $\text{Im} \ Y_{0,p}$ and Eqs (3.84) and (3.85) for $\text{Re} \ Y_{0,p}$).

Once an explicit expression for the quadratic longitudinal response function has been determined, the corresponding nonlinear rates can be calculated. For random-phase nonlinear processes the rates are often derived using a semiclassical formalism in which the waves are interpreted as a collection of wave quanta with momentum $\hbar k$ and energy $\hbar|\omega_M(k)|$. The occupation number $N_M(k)$ is introduced, being defined as the number density of wave quanta within the elemental range $d^3k$ of $k$. This quantity is related to the energy density per unit volume of $k$-space, $W_M(k)$, by

$$N_M(k) = \frac{W_M(k)}{\hbar \omega_M(k)}.$$  \hfill (4.11)

The kinetic equations are then derived via the principle of detailed balance. For ESD, the kinetic equation for $N_L$ is [Tsytovich, 1970; Davidson, 1972; Tsytovich, 1977; Melrose, 1980c; Sitenko, 1982; Melrose, 1986a; Yoon, 2000]

$$\left[ \frac{\partial N_L(k)}{\partial t} \right]_{\text{ESD}} = - \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3} u_{LL'S}(k, k_1, k_2) \times \{ N_L(k)[N_{L'}(k_1) + N_S(k_2)] - N_{L'}(k_1)N_S(k_2) \},$$ \hfill (4.12)

with similar kinetic equations for the modes $L'$ and $S$. The equation for the interaction probability $u_{LL'S}$ is

$$u_{LL'S}(k, k_1, k_2) = \frac{\hbar}{\epsilon_0} \left| \frac{R_L(k)R_{L'}(k_1)R_S(k_2)}{\omega_L(k)\omega_{L'}(k_1)\omega_S(k_2)} \right|^2 |x(2)(k, k_L, k_L', k_S)|^2 \times (2\pi)^4 \delta^3(k - k_1 - k_2)\delta[\omega_L(k) - \omega_{L'}(k_1) - \omega_S(k_2)],$$ \hfill (4.13)

where $R_M$ is the ratio of electric to total energy in the wave mode $M$, and the quantity $k_{Mj}$ collectively denotes $\omega_M(k_j)$ and $k_j$.

For STI the kinetic equation is [Tsytovich, 1970; Davidson, 1972; Tsytovich,
\[
\left[ \frac{\partial N_L(k)}{\partial t} \right]_{\text{STI}} = - \int d^3v_i \int \frac{d^3k_1}{(2\pi)^3} w_{LL'}(k, k_1, v_i) \left\{ [N_L(k) - N_{L'}(k_1)] \times f_i(v_i) - N_L(k) N_{L'}(k_1) \frac{\hbar(k - k_1)}{m_i} \frac{\partial f_i(v_i)}{\partial v_i} \right\},
\]
(4.14)

with
\[
w_{LL'}(k, k_1, v_i) = \frac{2\pi q_i^2 R_L(k) R_{L'}(k_1)}{\epsilon_0 m_i^2 |\omega_L(k)\omega_{L'}(k_1)|} \left| A_{LL'}(k, k_1, v_i) \right|^2 \\
\times \delta[\omega_L(k) - \omega_{L'}(k_1) - (k - k_1) \cdot v_i],
\]
(4.15)
\[
A_{LL'} = \frac{m_i}{q_i \epsilon_0} \frac{\alpha^{(2)}(k_L, k_L', k_L - k_{L'})}{|k_L\cdot v_i|} \frac{k - k_1}{|k_L - k_{L'}|},
\]
(4.16)

and where \( f_i(v_i) \) is the ion velocity distribution function. The terms proportional to \( N_L N_{L'} \) describe induced scattering by ions, while the terms proportional to \( N_L \) and \( N_{L'} \) describe spontaneous scattering. Equation (4.16) is valid where the shielding field is predominantly electrostatic, and \( |k| \lambda_D \ll 1 \) such that Thomson scattering is negligible [Melrose, 1986a]. The longitudinal part of the equivalent dielectric tensor, \( K^L \), is given by [Melrose, 1986a]
\[
K^L(\omega, k) = 1 + \sum_a \frac{\omega_a^2}{|k|^2 V_a^2} \left[ 1 + s_{a}Z(s_{a}) \right],
\]
(4.17)

where \( a \) denotes the particle species (i.e., electrons or ions). The approximate quadratic longitudinal response function in Eq. (4.3) then may be written as
\[
\alpha^{(2)}(k, k_1, k_2) \approx \frac{\epsilon e \omega_2 k \cdot k_1 |k_2|}{m_e |k||k_1|} K^{L(e)}(\omega_2, k_2),
\]
(4.18)

with \( K^{L(e)} \) the electronic contribution to \( K^L \).

Using the result
\[
\frac{\partial f(v)}{\partial v} = -\frac{v}{V^2} f(v)
\]
(4.19)

for a Maxwellian distribution, and averaging the probability \( w_{LL'} \) over a Maxwellian distribution of ions, the kinetic equation in Eq. (4.14) becomes
\[
\left[ \frac{\partial N_L(k)}{\partial t} \right]_{\text{STI}} = - \int d^3k_1 \frac{d^3k}{(2\pi)^3} \tilde{w}_{LL'}(k, k_1) \left\{ N_L(k) - N_{L'}(k_1) \right\} \\
+ \frac{\hbar [\omega_L(k) - \omega_{L'}(k_1)]}{m_i V_i^2} N_L(k) N_{L'}(k_1),
\]
(4.20)

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with the average probability \( \bar{w}_{LL'} \) (for \( q_i = e \)) given by

\[
\bar{w}_{LL'}(k, k_1) = \int d^3v_i f_i(v_i) w_{LL'}(k, k_1, v_i)
\]

\[
= \frac{D \exp\left(-s_{2i}^2\right)}{[\omega_L(k) - \omega_{L'}(k_1)]^2|k - k_1|^3 |K^L(k_L - k_{L'})|^2} \left|\alpha^{(2)}(k_L, k_{L'}, k_L - k_{L'})\right|^2, \tag{4.22}
\]

where

\[
s_{2i} = \frac{\omega_2}{|k_2| v_i \sqrt{2}}, \tag{4.23}
\]

and

\[
D = \frac{(2\pi)^{1/2} m_e}{4\varepsilon_0^3 V_i}. \tag{4.24}
\]

The approximate average scattering probability is typically written with \( \alpha^{(2)} \) in the form of Eq. (4.18), so Eq. (4.22) becomes

\[
\bar{w}_{LL'}(k, k_1) \approx \frac{D e^2 \varepsilon_0^2 |k \cdot k_1|^2 \exp\left(-s_{2i}^2\right) |K^{L}(e)(k_L - k_{L'})|^2}{m_e^2 |k|^2 |k_1|^2 |k - k_1| |K^L(k_L - k_{L'})|^2}. \tag{4.25}
\]

The interaction probability for both ESD and STI is proportional to \( |\alpha^{(2)}|^2 \), as expressed in Eqs (4.13) and (4.22), and the approximation in Eq. (4.3) is thus made for both processes. The analyses of the interaction probabilities and the nonlinear rates for these processes are therefore very similar. The difference between these processes relates to the frequency \( \omega_2 \). For ESD \( \omega_2 \) corresponds to the ion-sound wave frequency

\[
\omega_S(k) = |k| v_S, \quad |k| \lambda_D \ll 1, \tag{4.26}
\]

where

\[
v_S = V_e (\gamma m_e / m_i)^{1/2} \tag{4.27}
\]

with \( \gamma = 1 + \eta T_i / T_e \) and \( \eta = (\sqrt{1 + 12 T_i / T_e} - 1) T_e / 2 T_i \) (see Fig. 4.1) [Cairns et al., 1998]. For STI, \( \omega_2 \) corresponds to \( \omega_i = (k - k_1) \cdot v_i \).

In this chapter we perform numerical calculations of the nonlinear rates for both ESD and STI using the exact quadratic longitudinal response function in Eq. (4.6), and compare these with previous calculations based on the approximate response function in Eq. (4.3).
Figure 4.1: Dependence of the parameter $s_2$ for ESD [from Eqs (4.4) and (4.26)] on the temperature ratio $T_i/T_e$.

### 4.4 Electrostatic decay rate

To assess the importance of the exact quadratic longitudinal response function in the calculation of the nonlinear rates we need to evaluate the integrals in Eqs (4.12) and (4.20). We first perform the analysis for the ESD process, and then for STI.

We follow the method described in Edney and Robinson [2001] to reduce the six-dimensional integral in Eq. (4.12) analytically to a one-dimensional integral which can be evaluated numerically. The nonlinear rate for the evolution of primary Langmuir waves $L$ was not discussed in Edney and Robinson [2001] so we present the derivation here. Our derivation differs from that of Edney and Robinson [2001] in a few respects: we present the derivation in terms of occupation number rather than wave temperature; we introduce non-dimensional variables to make the dependences of the rate more explicit; and we include more detail about the permitted values for the wave vectors.

We assume the occupation number spectra have arc distributions [Willes
N_L(k) = N_L(k) \exp(\beta_L \cos \theta), \quad \text{(4.28)}
N_L'(k_1) = N_L'(k_1) \exp(-\beta_L' \cos \theta_1), \quad \text{(4.29)}
N_S(k_2) = N_S(k_2) \exp(\beta_S \cos \theta_2). \quad \text{(4.30)}

These spectra are good approximations to the functional form obtained through numerical solutions of the Zakharov equations [Robinson and Newman, 1989; Robinson et al., 1992]. Here \( \cos \theta_j = \hat{\mathbf{k}}_j \cdot \mathbf{k}_z \) for each spectrum, where \( \mathbf{k}_z \) is the unit vector parallel to \( \mathbf{k}_b = \omega_p v_b / v^2_b \), and where from henceforth \( k_j \) refers to \( |k_j| \). The parameters \( \beta_M \) may be estimated as

\[
\beta_M = \frac{\mu_M - 1}{\sqrt{1 + \sigma_M}} \quad \text{(4.31)}
\]

which we have derived from Eqs (A2) and (A3) of Li et al. [2005], where \( \mu_M > 0 \) is a constant (\( \sim 1-10 \)) related to the characteristic angular extent of the spectrum. We assume that each spectrum has a Gaussian radial distribution given by

\[
N_M(k) = n_M \exp \left[ -\frac{(k/k_{Mc} - 1)^2}{2\sigma^2_M} \right], \quad \text{(4.32)}
\]

where \( k_{Mc} \) is the peak wavenumber of the distribution and \( n_M \) is a constant [Robinson et al., 1993; Willes et al., 1996; Edney and Robinson, 2001]. The primary (beam-driven) Langmuir wave spectrum has \( k_{Lc} = k_b \) and \( \sigma_L = \Delta v_b / v_b \), where \( \Delta v_b \) is the beam spread; the decay process leads to \( L' \) and \( S \) wave distributions with \( k_{L'S} \approx k_b - k_0, k_{Sc} \approx 2k_b - k_0 \), and \( (\mu_{L'}, \sigma_{L'}) \approx (\mu_S, \sigma_S) \approx (\mu_L, \sigma_L) \) [Melrose, 1986a; Cairns, 1987b; Robinson et al., 1993], where

\[
k_0 = \frac{2\omega_p v_S}{3V_e^2}. \quad \text{(4.33)}
\]

From Eqs (4.31) and (4.32) a larger \( \sigma_M \) results in broader spreads for both the radial and angular distributions of each of the occupation number spectra \( N_M \).

We first integrate over \( k_2 \) using \( \delta^3(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2) \) to give

\[
\left[ \frac{\partial N_L(k)}{\partial t} \right]_{\text{ESD}} = -\frac{C}{(2\pi)^2} \int \delta^3 \mathbf{k}_1 \omega_S(\mathbf{K}_2) |\alpha^{(2)}(\mathbf{k}_1, \mathbf{K}_2)|^2 \{N_L(k) \times [N_L'(k_1) + N_S(\mathbf{K}_2)] - N_L'(k_1)N_S(\mathbf{K}_2) \} \times \delta[\omega_L(k) - \omega_{L'}(k_1) - \omega_S(\mathbf{K}_2)], \quad \text{(4.34)}
\]
with
\[ C = \frac{\hbar m_e/m_i}{8\epsilon_0^3\omega_p^4}, \]  
(4.35)

and where
\[ K_2 = k - k_1. \]  
(4.36)

The coordinate system is then rotated such that the new z axis is parallel to \( k \). Hence, in the new system
\[
\cos \theta_1 = \cos \theta \cos \psi - \sin \theta \sin \psi \cos \phi, 
\]
(4.37)
\[
\cos \theta_2 = (k \cos \theta - k_1 \cos \theta_1)/K_2, 
\]
(4.38)
where \( \cos \psi = \hat{k} \cdot \hat{k}_1 \). Transforming the integral over \( k_1 \) in Eq. (4.34) to spherical polar coordinates such that \( d^3k_1 = d(\cos \psi)\, dk_1\, k_1^2\, d\phi \) and integrating over the azimuthal angle \( \phi \) yields
\[
\left[ \frac{\partial N_L(k)}{\partial t} \right]_{\text{ESD}} = -\frac{C}{2\pi} \int_{-1}^{1} d(\cos \psi) \int_{0}^{\infty} dk_1\, k_1^2\, \omega_S(K_2)|\alpha^{(2)}(k, k_1, K_2)|^2 
\times N^2(k, k_1, K_2)\delta[\omega_L(k) - \omega_{L'}(k_1) - \omega_S(K_2)], 
\]
(4.39)

where
\[ N^2(k, k_1, K_2) = N_L(k) \exp(\beta_L \cos \theta) [N_{L'}(k_1) \exp(-\beta_{L'} \cos \theta \cos \psi)
\times I_0(\beta_{L'} \sin \theta \sin \psi) + N_S(k_2) \exp(-\beta_S k_1 \cos \theta \cos \psi/k_2) I_0(\beta_S k_1 \sin \theta \sin \psi/k_2)
\times \exp(\beta_S k_1 \cos \theta/k_2)] - N_{L'}(k_1) N_S(k_2) \exp[-(\beta_{L'} + \beta_S k_1/k_2) \cos \theta \cos \psi]
\times I_0[(\beta_{L'} + \beta_S k_1/k_2) \sin \theta \sin \psi] \exp(\beta_S k_1 \cos \theta/k_2) 
\]
(4.40)

with \( I_0 \) the zeroth-order modified Bessel function of the first kind [Olver et al., 2010]. We now substitute Eq. (4.26) and
\[
\omega_L(k) = \omega_p + \frac{3k^2V_e^2}{2\omega_p}, \quad k\lambda_D \ll 1, 
\]
(4.41)
into the frequency delta function in Eq. (4.39), so that
\[
\delta[\omega_L(k) - \omega_{L'}(k_1) - \omega_S(K_2)] = \delta[g(k_1; k, \cos \psi)], 
\]
(4.42)
where
\[
g(k_1; k, \cos \psi) = 3V_e^2(k^2 - k_1^2)/2\omega_p - K_2v_s, 
\]
(4.43)
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With
\[ K_2(k, k_1, \cos \psi) = (k^2 + k_1^2 - 2kk_1 \cos \psi)^{1/2}. \] (4.44)

We then use
\[ \delta[g(k_1; k, \cos \psi)] = \sum_i \frac{\delta[k_1 - K_{1i}(k, \cos \psi)]}{|g'[K_{1i}(k, \cos \psi)]|} \] (4.45)

to simplify the delta function, where \( K_{1i} \) are the zeros of \( g(k_1; k, \cos \psi) \), and \( g' = \partial g/\partial k_1 \). Introducing the rescaled variables
\[ \rho = k_1/k \] (4.46)
and
\[ \tau_0 = k/k_0, \] (4.47)
the equation \( g(k_1; k, \cos \psi) = 0 \) is equivalent to the quartic equation
\[ \rho^4 + E\rho^2 + F\rho + G = 0, \] (4.48)
with \( E = -(1 + 2\tau_0^2)/\tau_0^2 \), \( F = 2 \cos \psi/\tau_0^2 \), and \( G = (\tau_0^2 - 1)/\tau_0^2 \). Equation (4.48) has a discriminant \( \Delta > 0 \) and seminvariants \( H < 0 \) and \( Q > 0 \) (as defined by Eqs (19) and (21) in Cremona [1999]), so that Eq. (4.48) has four real roots (see, e.g., Cremona [1999]). The “resolvent cubic” of Eq. (4.48) is \[ z^3 + 2Ez^2 + (E^2 - 4G)z - F^2 = 0. \] (4.49)

The four roots of Eq. (4.48) may be expressed as [Uspensky, 1948]
\[ \rho_{\sigma,\sigma'} = \frac{\sigma \sqrt{z_0} + \sigma' \sqrt{2 \sqrt{z_0} - E - \sigma F/\sqrt{z_0}}}{2}, \] (4.50)
where \( \sigma, \sigma' = \pm \), and \( z_0 \) is any root of Eq. (4.49). If \( z_0 \) is the largest root of Eq. (4.49) then \( \rho_{-,\sigma'} < 0 \) and \( \rho_{+,\sigma'} > 0 \), with \( 1 \geq \rho_{+,-} \geq 1 - 1/\tau_0 \) and \( 1 + 1/\tau_0 \geq \rho_{+,+} \geq 1 \). The required solution of Eq. (4.48) must satisfy \( k_1 + k_0 \geq k \geq k_1 \) [Cairns, 1987b], i.e., \( 1 \geq \rho \geq 1 - 1/\tau_0 \); thus \( \rho_{+,+} \) is the required solution which we now denote as \( R(\cos \psi) \), and which we graph in Fig. 4.2. The wave vector \( k_1 \) that satisfies Eq. (4.48) is given by \( k_1 = K_1(k, \cos \psi) \), where \( K_1(k, \cos \psi) = kR(\cos \psi) \) from Eq. (4.46). Assuming (without loss of generality) that \( K_1(k, \cos \psi) \) lies in the \( x-z \) plane and has a positive \( x \)-component,
\[ K_1(k, \cos \psi) = kR(\cos \psi) \left( \sqrt{1 - \cos^2 \psi}, 0, \cos \psi \right). \] (4.51)
Figure 4.2: Graphs of $R(\cos \psi)$ given by Eq. (4.50) with $\sigma, \sigma' = +, -$. Solid line: $\tau_0 = 4$, long-dashed line: $\tau_0 = 8$, dash-dot line: $\tau_0 = 12$, and short-dashed line: $\tau_0 = 16$.

Thus, Eq. (4.45) may be written as

$$
\delta [\omega_L(k) - \omega_{L'}(K_1) - \omega_S(K_2)] = \frac{k_0 K_2 \delta[k_1 - K_1(k, \cos \psi)]}{v_S[2K_1(k, \cos \psi)K_2 + k_0[K_1(k, \cos \psi) - k \cos \psi]]}.
$$

Integrating over $k_1$ using the delta function in Eq. (4.52) thus yields

$$
\left[ \frac{\partial N_L(k)}{\partial t} \right]_{\text{ESD}} = -\frac{Ck_0}{2\pi} \int_{-1}^{1} \frac{d(\cos \psi) K_1^2 K_2^2}{|2K_1 K_2 + k_0[K_1(k, \cos \psi) - k \cos \psi]|} |\alpha^{(2)}(k, K_1, K_2)|^2 
\times N^2(k, K_1, K_2),
$$

Equation (4.53) may be written in terms of the non-dimensional variables $\tau_0$ and $R(\cos \psi)$ as

$$
\left[ \frac{\partial N_L(\tau_0, \theta)}{\partial t} \right]_{\text{ESD}} = -C' \tau_0^3 \int_{-1}^{1} \frac{d(\cos \psi) R^2 S^2}{|2\tau_0 RS + R - \cos \psi|} |\alpha^{(2)}(k, K_1, K_2)|^2 
\times N^2(k, K_1, K_2),
$$
where
\[
C' = \frac{\hbar k_0^3 m_s/m_e}{16\pi e_0^2 \omega_p^4}
\] (4.55)
and
\[
S(\cos \psi) = \left\{ 1 + [R(\cos \psi)]^2 - 2R(\cos \psi) \right\}^{1/2},
\] (4.56)
with the arguments of \(R(\cos \psi)\) and \(S(\cos \psi)\) omitted in Eq. (4.54) for brevity.

We now summarize the results for the \(\partial N_L/\partial t\) calculation. Frequency and wave vector conservation imply that \(\rho' \equiv k/k_1\) satisfies Eq. (4.48) with the replacements \((\rho, \tau_0) \rightarrow (\rho', \tau_0')\), where \(\tau_0' \equiv k_1/k_0\). The kinematically allowed solution to this equation satisfies \(1 + 1/\tau_0' \geq \rho' \geq 1\) and is thus given by \(R' \equiv \rho'_{+,+}\) defined by Eq. (4.50) with the aforementioned replacements. The nonlinear rate is then
\[
\left[ \frac{\partial N_L(\tau_0', \theta_1)}{\partial t} \right]_{\text{ESD}} = C' \tau_0'^3 \int_{-1}^{1} \frac{d(\cos \psi)}{|2R_0'R'S' - R' + \cos \psi|} |\alpha^{(2)}(K, k_1, K_2)|^2
\]
\[
\times N^2(K, k_1, K_2),
\] (4.57)
with \(S' = (1 + R'^2 - 2R' \cos \psi)^{1/2}\), and
\[
N^2(k, k_1, k_2) = N_L(k_1) \exp(-\beta_L' \cos \theta_1) [N_L(k) \exp(\beta_L \cos \theta_1 \cos \psi)]
\]
\[
\times I_0(\beta_L \cos \theta_1 \sin \psi) - N_S(k_2) \exp(\beta_S k \cos \theta_1 \cos \psi/k_2)I_0(\beta_S k \sin \theta_1 \sin \psi/k_2)
\]
\[
\times \exp(-\beta_S k_1 \cos \theta_1/k_1) + N_L(k)N_S(k_2) \exp((\beta_L + \beta_S k/k_2) \cos \theta_1 \cos \psi)
\]
\[
\times I_0(\beta_L + \beta_S k/k_2 \sin \theta_1 \sin \psi)] \exp(-\beta_S k_1 \cos \theta_1/k_2).
\] (4.58)

We note an error in Eq. (33) of Edney and Robinson [2001]: the denominator in the integrand of that equation was proportional to \(2\tau_0' R'S' - R' - \cos \psi\), whereas it should be proportional to \(2\tau_0' R'S' - R' + \cos \psi\), as in our Eq. (4.57).

We note that not all of the 9 vector component arguments of \(\alpha^{(2)}\) in Eq. (4.54) [nor in Eq. (4.57)] are required for its calculation: from the dependence of \(\alpha^{(2)}\) on its arguments in Eq. (4.6), and from Eqs (4.44) and (4.51), \(\alpha^{(2)}\) may be specified by only the arguments \(k, \cos \psi, \text{ and } R(\cos \psi)\). Similarly, \(N^2\) may expressed in terms of only \(k, \cos \psi, R(\cos \psi)\), and \(\cos \theta, \text{ and } \partial N_L/\partial t\) depends only on \(k\) and \(\cos \theta\). In terms of these arguments the approximate quadratic longitudinal response function in Eq. (4.3) can be expressed as
\[
\alpha^{(2)} = -\frac{\gamma^3 n}{m^2 V_e} \left( \frac{\gamma m_e}{m_i} \right)^{1/2} \cos \psi \left[ 1 + s_Z(s_Z) \right],
\] (4.59)
and the exact quadratic longitudinal response function in Eq. (4.6) may be written as

\[
\alpha^{(2)} = \frac{q^3 n}{4m^2 V_c} \frac{\omega_L(k)\omega_L'(k_1)\omega_S(k_2)}{(k\lambda_D)^3 RS \sin \psi} \left( (\cos \psi - R) \left[ Y_{0,3}(a_1, b_1, s) + Y_{0,3}(a_2, b_2, s) \right] 
+ \frac{1 - R \cos \psi}{2 \cos \psi - R \tilde{r} - s_1} \left\{ \frac{1}{\tilde{r} - s_1} \left[ Y_{0,1}(\bar{a}_1, \bar{b}_1, \tilde{r}) + Y_{0,1}(\bar{a}_2, \bar{b}_2, \tilde{r}) - Y_{0,1}(\bar{a}_1, \bar{b}_1, s_1) \right] 
- Y_{0,1}(\tilde{a}_2, \tilde{b}_2, s_1) \right\} \right) + \frac{S^2 \cos \psi}{1 - R^2} \frac{1}{\tilde{r} - s_2}
\times \left\{ \frac{1}{\tilde{r} - s_2} \left[ Y_{0,1}(\bar{a}_1, \bar{b}_1, \tilde{r}) + Y_{0,1}(\bar{a}_2, \bar{b}_2, \tilde{r}) - Y_{0,1}(\bar{a}_1, \bar{b}_1, s_1) \right] 
- Y_{0,2}(\tilde{a}_1, \tilde{b}_1, s_2) \right\}, \tag{4.60}\]

where \( \omega_M = \omega_M/\omega_p \). The terms defined by Eqs (4.4) and (4.8)–(4.10) as functions of \( k\lambda_D, \cos \psi \), and \( R(\cos \psi) \) are presented in Eqs (4.77)–(4.91).

We find that the quadratic longitudinal response functions in Eqs (4.59) and (4.60) for a given \( \cos \psi \) depend on \( V_c, T_i/T_e \), and \( k\lambda_D \). Now the \( k\lambda_D \) dependence may be equivalently expressed in terms of \( v_\phi/V_c \), or \( \tau_0 \) for a given \( T_i/T_e \), which we compare in Fig. 4.3. Then, assuming that \( T_i/T_e \) and \( k\lambda_D \) are constant, both these response functions scale as \( V_c^{-1} \) and so the ratio of these functions is independent of \( V_c \).

Figure 4.4 shows \( |\alpha^{(2)}_{\text{approx}}|^2 \) and \( |\alpha^{(2)}_{\text{exact}}|^2 \) as functions of \( \cos \psi \) for different values of \( \tau_0 \), where \( \alpha^{(2)}_{\text{approx}} \) and \( \alpha^{(2)}_{\text{exact}} \) are defined by Eqs (4.59) and (4.60) respectively. The assumptions \( s, s_1 \gg 1 \) used in deriving Eq. (4.3) correspond to the condition \( v_\phi/V_c \gg 1 \). Unlike \( |\alpha^{(2)}_{\text{approx}}|^2 \), \( |\alpha^{(2)}_{\text{exact}}|^2 \) is not symmetric about \( \cos \psi = 0 \); instead, for larger \( \tau_0 \) the minimum of \( |\alpha^{(2)}_{\text{exact}}|^2 \) occurs at larger \( \cos \psi \). For \( \tau_0 = 18 \), \( |\alpha^{(2)}_{\text{exact}}|^2 \) is approximately 35% larger than \( |\alpha^{(2)}_{\text{approx}}|^2 \) at \( \cos \psi = -1 \) and 43% larger at \( \cos \psi = 1 \). For smaller values of \( \tau_0 \), corresponding to smaller \( k \), \( |\alpha^{(2)}_{\text{approx}}|^2 \) becomes more accurate: it is within about 5% of \( |\alpha^{(2)}_{\text{exact}}|^2 \) at \( \cos \psi = \pm 1 \) for \( \tau_0 \leq 8 \).

We find that the ratio of the exact to the approximate quadratic longitudinal response function varies negligibly with \( T_i/T_e \) for \( T_i/T_e \lesssim 10 \). This is because \( s_2 \) [derived using Eqs (4.4) and (4.26)] varies very weakly with \( T_i/T_e \) and remains much less than one, as shown in Fig. 4.1. In this parameter range the dominant term in the exact quadratic longitudinal response function is the one proportional to \( M(k_2; k, k_1) \) [defined by Eq. (4.2)], and the standard \( s_2 \ll 1 \) approximation for \( M(k_2; k, k_1) \) is accurate [Layden et al., 2012] (Chapter 3). Therefore, when assessing the accuracy of the approximate response
Figure 4.3: Comparison of \( \frac{v_0}{V_e} \), \( k \lambda D \), and \( \tau_0 \) (assuming \( T_i/T_e = 1/3 \)) using Eqs (4.33) and (4.41).

function and interaction probability, only the dependence on \( k \lambda D \) (or, equivalently, \( \frac{v_0}{V_e} \) or \( \tau_0 \)) needs to be considered.

We now numerically calculate the ESD rate in Eq. (4.54) for both \( |\alpha_{\text{approx}}(2)|^2 \) and \( |\alpha_{\text{exact}}(2)|^2 \). It is infeasible to calculate \( |\alpha_{\text{exact}}(2)|^2 \) directly for a large range of \( \tau_0 \) owing to the time-consuming computation of the various generalized plasma dispersion functions. Therefore, we calculate \( |\alpha_{\text{exact}}(2)|^2 \) for \(-1 \leq \cos \psi \leq 1 \) and \( 4 \leq \tau_0 \leq 18 \) in increments of \( 10^{-3} \) and 1 respectively, then interpolate \( |\alpha_{\text{exact}}(2)|^2 \) using a two-dimensional cubic spline with a \( 10^{-3} \) increment in \( \cos \psi \) and a \( 10^{-2} \) increment in \( \tau_0 \). These increments were chosen by reducing the increment size until a good level of accuracy was attained.

In Fig. 4.5 we graph the (three-dimensional) ESD rates using both \( |\alpha_{\text{approx}}(2)|^2 \) and \( |\alpha_{\text{exact}}(2)|^2 \) for nominal parameters. These graphs show that \( \partial N_L / \partial t \) is negative, corresponding to wave energy being transferred from the \( L \) wave distribution to the \( L' \) and \( S \) wave distributions. Both rates are maximal at \( \theta = 0 \) and near \( \tau_0 = \tau_{0Lc} \equiv k_{Lc}/k_0 \) (as discussed in, e.g., Tsytovich [1970], Cairns [1987b], and Edney and Robinson [2001] for the approximate rate). Thus the ratio of the maxima of the exact and approximate rates may be used to assess the accuracy of the approximate rate over \( \tau_0 \) and \( \theta \). We denote this ratio by
Figure 4.4: Graphs of $|\alpha_{\text{approx}}^{(2)}|^2$ and $|\alpha_{\text{exact}}^{(2)}|^2$, normalized by the value of $|\alpha_{\text{approx}}^{(2)}|^2$ at $\psi = \pi$ (perfect backscatter) for the same $\tau_0$, as a function $\cos \psi$. The black, light blue, red, dark blue, green, and purple solid lines correspond to the normalized $|\alpha_{\text{exact}}^{(2)}|^2$ for $\tau_0 = 16, 14, 12, 10, 8$, respectively. The dashed line shows the normalized $|\alpha_{\text{approx}}^{(2)}|^2$; i.e., $\cos^2 \psi$ [from Eq. (4.59)].

We graph $\xi_{\text{ESD}}$ which can be expressed as

$$\xi_{\text{ESD}} = \frac{\max \left\{ \left[ \frac{\partial N_L(\tau_0, \theta)}{\partial t} \right]_{\text{ESD,exact}} \right\}}{\max \left\{ \left[ \frac{\partial N_L(\tau_0, \theta)}{\partial t} \right]_{\text{ESD,approx}} \right\}}.$$  

(4.61)

We graph $\xi_{\text{ESD}}$ as a function of $\tau_0 L_c$ for different values of $\mu$ and $\sigma$ in Fig. 4.6. We see in Fig. 4.6 that $\xi_{\text{ESD}}$ increases with $\tau_0 L_c$, since $|\alpha_{\text{exact}}^{(2)}|^2/|\alpha_{\text{approx}}^{(2)}|^2$ increases with $\tau_0$ as shown in Fig. 4.4. The figure also shows that $\xi_{\text{ESD}}$ increases as $\sigma$ increases and $\mu$ decreases. This is because both these changes lead to smaller $\beta_M$ via Eq. (4.31); thus, the integrand in Eq. (4.54) decreases more slowly as $\cos \psi$ increases from $-1$ where it is maximal, and hence the contribution to the integral for larger $\cos \psi$ becomes more significant. The ratio $|\alpha_{\text{exact}}^{(2)}|^2/|\alpha_{\text{approx}}^{(2)}|^2$ becomes larger as $\cos \psi$ increases from $-1$ which may be seen from Fig. 4.4, leading to a larger $\xi_{\text{ESD}}.$

The values of $\xi_{\text{ESD}}$ in Fig. 4.6 imply that the approximate rate of ESD is accurate in comparison with the exact rate for a large range of parameters.
Ch. 4 EXACT RATES OF DECAY AND SCATTERING

Figure 4.5: Contour plots of the exact ESD rate (upper panel) and approximate ESD rate (lower panel) for $\tau_{0Lc} = 12$, $\mu = 3$, $\sigma = 0.2$, and $T_i/T_e = 1/3$. Color indicates the value of the rate normalized by $|\max \{[\partial N_L(\tau_0, \theta) / \partial t]_{ESD,approx} \}|$, with contour lines separated by increments of 0.1.

Specifically for $\tau_{0Lc} = k_{Lc}/k_0 < 15$, corresponding to $v_\phi/V_e > 3$, the exact rate is less than a factor of 1.4 larger than the approximate rate. These results suggest that the use of the exact quadratic response function is not necessary for modeling type III radio sources, where the electron beams driving the Langmuir waves arise from impulsive acceleration at the Sun (often associated with solar flares) and have speeds $v_b/V_e \approx 10 - 100$ [Lin et al., 1986; Dulk et al., 1987; Malaspina et al., 2011]. The use of the exact ESD rate may be important in foreshock regions of type II radio sources and planetary bow shocks where slow electron beams ($v_b/V_e \leq 3$) are found [Fitzenreiter et al., 1990, 1996]. The strongest levels of Langmuir waves, however, are generated by beams with $3 \lesssim v_b/V_e \lesssim 10$ [Knock et al., 2001; Malaspina et al., 2009], so that the use of the exact ESD rate is only important where the levels of Langmuir waves are relatively weak.

We also note that the approximate rate for ESD is accurate over a larger range of parameters than the approximate rate for the coalescence of Langmuir
waves $L + L' \leftrightarrow T$ is accurate, as derived in Layden et al. [2011] (Chapter 2). This is because the dominant terms in the quadratic response tensor $\alpha_{ijl}^{(2)}$ for Langmuir-wave coalescence arise from integrals involving resonant denominators of Langmuir wave fields; these integrals are not calculated as accurately using the standard approximations as the dominant integral $M(k_2 ; k, k_1)$ is here.

4.5 Scattering off thermal ions rate

We now calculate the rate of STI assuming the occupation number spectra are given by Eqs (4.29) and (4.30). Starting with Eqs (4.20) and (4.22) the
coordinate system is then rotated as in Sec. 4.4, giving
\[
\left[ \frac{\partial N_L(k)}{\partial t} \right]_{\text{STI}} = -\frac{D}{m_i V_i^3 (2\pi)^3 \sqrt{2}} \int_0^\infty d\rho \rho^2 \int_{-1}^1 d(\cos \psi) \int_{-\pi}^\pi d\phi \exp(-s_{2i}^2) \exp(-s_{2i}^2) \frac{\alpha^{(2)}(k_L, k_{L'1}, k_L - k_{L'1})}{|K^L(k_L - k_{L'1})|^2} \left\{ N_L(k) \exp(\beta_L \cos \theta) - N_L(k_1) \times \exp(-\beta_{L'} \cos \theta \cos \psi) \exp(\beta_L' \sin \theta \sin \psi \cos \phi) \right\} + N_L(k)N_L'(k_1),
\]
where
\[
s_{2i} \approx \frac{\tau_*(1 - \rho^2)}{(1 + \rho^2 - 2\rho \cos \psi)^{1/2} \sqrt{2}},
\]
and
\[
\tau_* = \frac{k}{k_*}.
\]
We then integrate over \( \phi \) and write the rate in terms of the non-dimensional parameter \( \tau_* \), giving
\[
\left[ \frac{\partial N_L(\tau_*, \theta)}{\partial t} \right]_{\text{STI}} = -\frac{D'}{\tau_*} \int_0^\infty d\rho \int_{-1}^1 d(\cos \psi) I(\rho, \cos \psi)
\]
where
\[
D' = \frac{\hbar m_e / m_i}{16\pi^{3/2} c_0^3 V_i^4 k_*},
\]
and
\[
I(\rho, \cos \psi) = \frac{\rho^2 \exp(-s_{2i}^2) \frac{\alpha^{(2)}(k_L, k_{L'1}, k_L - k_{L'1})}{|K^L(k_L - k_{L'1})|^2}}{s_{2i}(1 + \rho^2 - 2\rho \cos \psi)^{1/2}} \times \left\{ \frac{d}{\tau_* s_{2i}(1 + \rho^2 - 2\rho \cos \psi)^{1/2}} \left[ N_L(\tau_*) \exp(\beta_L \cos \theta) - N_L(\rho; \tau_*) \times \exp(-\beta_{L'} \cos \theta \cos \psi) I_0(\beta_L' \sin \theta \sin \psi) \right] + N_L(\tau_*)N_L'(\rho; \tau_*) \times \exp[(\beta_L - \beta_{L'} \cos \psi) \cos \theta] I_0(\beta_L' \sin \theta \sin \psi) \right\}
\]
with

\[ d = \frac{m_i V_i}{hk_a\sqrt{2}}. \]  

(4.69)

An analogous derivation for \( \partial N_L'/\partial t \) gives

\[
\left[ \frac{\partial N_{L'}(\tau'_s, \theta_1)}{\partial t} \right]_{\text{STI}} = \frac{D'}{\tau'_s} \int_0^\infty d\rho' \int_{-1}^1 d(\cos \psi) I'(\rho', \cos \psi),
\]

(4.70)

where \( \tau'_s = k_1/k_s \) and

\[
I'(\rho', \cos \psi) = \frac{\rho^2 \exp(-s_{2i}^2)}{s_{2i}(1 + \rho^2 - 2\rho' \cos \psi)^2} \left| \frac{\alpha^{(2)}(k_L, k_{L'1}, k_L - k_{L'1})}{|K'(k_L - k_{L'1})|^2} \right|^2 \\
\times \left\{ \frac{d}{\tau'_s s_{2i}(1 + \rho^2 - 2\rho' \cos \psi)^{1/2}} \left[ N_L'(\rho'; \tau'_s) \exp(\beta_L \cos \theta_1 \cos \psi) \right. \right.
\]
\[
\left. \times I_0(\beta_L \sin \theta_1 \sin \psi) - N_L'(\tau'_s) \exp(-\beta_L \cos \theta_1) \right] + N_L'(\rho'; \tau'_s) N_L'(\tau'_s) \right. \]
\[
\times \exp \left[ (\beta_L \cos \psi - \beta'_L) \cos \theta_1 \right] I_0(\beta_L \sin \theta_1 \sin \psi) \right\}. \]

(4.71)

For the following calculations we assume that the backscattered Langmuir wave spectrum has a peak wavenumber \( k_{Lc} = k_{Lc} - k_s \) [Melrose, 1986a; Muschietti and Dum, 1991; Cairns, 2000]. Unlike for the ESD process, \( \rho \) is not a function of \( \cos \psi \) for STI; instead the range of ion velocities in a Maxwellian distribution allows scattering to take place for any \( k_1 \). However, various factors in the integrand of Eq. (4.66) cause scattering to be efficient only for a narrow region of \((\rho, \cos \psi)\)-space. The factors in Eq. (4.66) that are related to the angular distribution of the \( L' \) occupation numbers, i.e., \( \exp(-\beta_L \cos \theta \cos \psi) \) and \( I_0(\beta_L \sin \theta \sin \psi) \), cause the integrand to be strongly peaked at \( \cos \psi = -1 \) and negligible outside the region \(-1 \leq \cos \psi \leq -0.9\). The factor \( N'_L(\rho; \tau'_s) \) causes the integrand to be significant only near its peak at \( \rho \approx (\tau_{Lc} - 1)/\tau'_s \), where we define \( \tau_{Lc} = k_{Lc}/k_s \).

The exact and approximate rates of STI are shown in Fig. 4.7. These rates have a similar \( \theta \) dependence to ESD; however, at \( \tau_0 \approx 11 \) there is a peak with \( \partial N_L/\partial t > 0 \), whereas there is no such peak for ESD. This is because the ion velocity distribution is isotropic, so the \( L' \) waves can couple with ions that have velocities in the \(-z\) direction, transferring Langmuir wave energy into the \(+z\) direction. For ESD, however, we have assumed only an \( S \) wave distribution centered on the \(+z\) axis.

We compare the exact and approximate rates for STI by graphing \( \xi_{\text{STI}} \) [defined analogously to \( \xi_{\text{ESD}} \) in Eq. (4.61)] as a function of \( \tau_{0Lc} \) for different
values of $\mu$, $\sigma$, and $T_i/T_e$ in Fig. 4.8. We find similar results to those for the ESD rate in Sec. 4.4. The quantity $\xi_{\text{STI}}$ increases as $\tau_{0Lc}$ increases [Figs 4.8(a)–(c)], $\mu$ decreases [Fig. 4.8(a)], and $\sigma$ increases [Fig. 4.8(b)], for the same reasons described in Sec. 4.4. However, unlike $\xi_{\text{ESD}}$, $\xi_{\text{STI}}$ decreases as $T_i/T_e$ increases. This is not due to the $T_i/T_e$ dependence of $\alpha^{(2)}$, since $s_2 \ll 1$ for $0.1 \leq T_i/T_e \leq 10$ and thus its $T_i/T_e$ dependence is negligible, as for ESD. As $T_i/T_e$ increases, the scattering rate peaks at smaller $\tau_0$ for which $|\alpha^{(2)}_{\text{exact}}|^2/|\alpha^{(2)}_{\text{approx}}|^2$ is smaller, resulting in a smaller value of $\xi_{\text{STI}}$. The values of $\xi_{\text{STI}}$ shown in Fig. 4.8 are similar to those for $\xi_{\text{ESD}}$ in Fig. 4.6, thus the approximate rate of STI has the same range of validity as the approximate ESD rate.

We now determine the velocity of the ions that most efficiently scatter the Langmuir waves. The kinematic constraint for STI expressed by the delta function in Eq. (4.15) may be written in the form

$$\omega_L(\mathbf{k}) - \omega_L'(\mathbf{k}_1) - |\mathbf{k} - \mathbf{k}_1|v_\parallel = 0,$$

(4.72)
Substituting Eq. (4.41) into Eq. (4.72) and rearranging yields
\[
\frac{v_{\parallel}}{V_i} = \frac{\tau_\ast (1 - \rho^2)}{(1 + \rho^2 - 2\rho \cos \psi)^{1/2}}. 
\] (4.73)

The most efficient ion velocities for the 3D calculation are well approximated by those for a 1D interaction geometry in which \(\theta = 0\), \(\cos \psi = -1\), and thus \(v_i\) is parallel to \(k - k_1\). This is because the rate is maximal at \(\theta = 0\) and decays quickly as \(\theta\) increases (see Fig. 4.7), and similarly the integrand of Eq. (4.66) is maximal at \(\cos \psi = -1\) and decays quickly as \(\cos \psi\) increases, as is well known [Tsytovich, 1970; Melrose, 1980c, 1986a]. For a 1D geometry Eq. (4.73)
becomes
\[ \frac{v_{\parallel}}{V_i} = \tau_s (1 - \rho). \]

(4.74)

Scattering is most efficient where \( I(\rho, -1) \) is maximal. The ion velocity is then found by substituting the value of \( \rho \) that maximizes \( I(\rho, -1) \) into Eq. (4.74).

We graph in Fig. 4.9 the velocity of the ions that most efficiently scatter the Langmuir waves. We find that the velocities \( v_{\parallel}/V_i \) lie approximately in the range \( 1 \lesssim v_{\parallel}/V_i \lesssim 4 \) for \( 0.1 \leq T_i/T_e \leq 10 \), where \( V_e \) is kept constant. The quantity \( v_{\parallel}/V_i \) increases with \( \tau_{0Lc} \) and decreases with \( T_i/T_e \). The dependence of \( v_{\parallel}/V_i \) on \( \sigma \) is very weak except where \( \tau_{0Lc} = 4 \), in which case \( v_{\parallel}/V_i \) is noticeably smaller for \( \sigma = 0.1 \) than \( \sigma = 0.3 \) near \( T_i/T_e = 1 \). The velocities calculated here are similar to the velocity \( v_{\parallel}/V_i = 1.9 \) calculated in Muschietti and Dum [1991] for \( k\lambda_D(V_e/V_i) = 1.8 \) (i.e., \( \tau_0 = 1.1 \) where \( T_i/T_e = 1/3 \)) using a 1D formalism for the scattering rate.

### 4.6 Relative growth rates of ESD and STI

Both ESD and STI produce backscattered Langmuir waves, transferring wave energy from a primary Langmuir wave distribution to a product Langmuir wave distribution at lower \( k \). The relative importance of these processes depends on their respective rates, as well as constraints that determine whether the process can proceed. These include “collective” constraints, e.g., that the timescale of the nonlinear process must not exceed the period of the lowest frequency wave (i.e., \( \omega_S \) for ESD and \( \omega_L \) for STI), and “timescale” constraints, e.g., that multiple wave periods fit within the available time and that the growth rate be large enough for at least several e-foldings to occur during the time available [Zakharov et al., 1985; Cairns, 2000]. In Secs 4.4 and 4.5 we derived expressions for the rates of ESD and STI for the exact quadratic response function, and compared these exact rates with the corresponding approximate rates for each process. We now compare the exact rate of ESD with the exact rate of STI.

Previous analytical work in Tsytovich [1970], Cairns [2000], and Mitchell et al. [2003] compared the rates of ESD and STI assuming the occupation numbers are constant within a solid angle of \( k \)-space and zero elsewhere, as well as approximating \( \alpha^{(2)} \) and \( K^{(a)} \) in the integrands. The resulting expression in Mitchell et al. [2003] is

\[ \frac{\Gamma_{ESD}^L}{\Gamma_{STI}^L} = \frac{2\gamma}{(2\pi)^{1/2}} \left( \frac{\gamma T_i}{T_e} \right)^{1/2} \left( 1 + \frac{T_e}{T_i} \right)^2, \]

(4.75)

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Figure 4.9: Graphs of $v_{\parallel}/V_i$ versus $T_i/T_e$ for (a) $\sigma = 0.1$ and (b) $\sigma = 0.3$. The dark blue, green, red, and light blue lines correspond to $\tau_{0Lc} = 4, 8, 12,$ and 16, respectively.

where

$$\Gamma_M^Q(k) = \frac{1}{N_M(k)} \left[ \frac{\partial N_M(k)}{\partial t} \right]_Q$$ (4.76)

is the nonlinear growth rate for the nonlinear process $Q$ (i.e., ESD or STI) and wave mode $M$. Here we improve the accuracy of this comparison by numerically calculating the rates of ESD and STI in Eqs (4.54) and (4.66), which are exact expressions for the nonlinear rates for arc wave spectra.

For the following calculations we assume that the nonlinear processes have proceeded for a sufficient time that the amplitudes of the product wave distributions $n_L$ and $n_S$ [defined by Eq. (4.32)] are approximately equal, but much less than the amplitude of the primary Langmuir wave distribution $n_L$; i.e.,
In this case the spontaneous terms—i.e., those arising from $N_L N_S$ in Eq. (4.12) for ESD and from $N_L$ and $N_L'$ appearing individually in Eq. (4.20) for STI—are negligible.

We graph $\Gamma_{\text{ESD}}^L / \Gamma_{\text{STI}}^L$ in Fig. 4.10 using Eqs (4.54) and (4.66). We find that $7 \lesssim \Gamma_{\text{ESD}}^L / \Gamma_{\text{STI}}^L \lesssim 320$ over the range of parameters considered. That is, ESD always dominates STI when the collective and timescale constraints for the decay are satisfied. This is in semiquantitative agreement with the previous analytical results in Tsytovich [1970], Cairns [2000], and Mitchell et al. [2003]. For instance, Eq. (4.75) (Mitchell et al. [2003]) gives $10 \lesssim \Gamma_{\text{ESD}}^L / \Gamma_{\text{STI}}^L \lesssim 40$ for $0.1 \leq T_i / T_e \leq 10$, as shown in Fig. 4.10(a).

The ratio $\Gamma_{\text{ESD}}^L / \Gamma_{\text{STI}}^L$ has a strong dependence on $\tau_{0Lc}$, which is not captured by the approximation in Eq. (4.75). For small $T_i / T_e$, $\Gamma_{\text{ESD}}^L / \Gamma_{\text{STI}}^L$ increases with $\tau_{0Lc}$; however for large $T_i / T_e$, $\Gamma_{\text{ESD}}^L / \Gamma_{\text{STI}}^L$ decreases with $\tau_{0Lc}$, as shown in Fig. 4.10(a). Figure 4.10(b) shows that increasing $\mu$ results in $\Gamma_{\text{ESD}}^L / \Gamma_{\text{STI}}^L$ increasing by a constant multiplicative factor over the full range of $T_i / T_e$. The dependence of $\Gamma_{\text{ESD}}^L / \Gamma_{\text{STI}}^L$ on $\sigma$ shown in Fig. 4.10(c) is similar to the dependence on $\tau_{0Lc}$.

### 4.7 Summary and conclusions

In this chapter we have calculated the three-dimensional rates of electrostatic decay (ESD) and scattering off thermal ions (STI) using for the first time an exact expression for the quadratic longitudinal response function $\alpha^{(2)}$ for an unmagnetized Maxwellian plasma derived in Layden et al. [2012] (Chapter 3). Unlike previous calculations of the rates which have used an approximate expression for $\alpha^{(2)}$, the exact rates calculated here are valid for any phase speeds of the three waves.

We find that the ratio of the exact rate to the approximate rate for both ESD and STI approaches unity as $v_\phi / V_e \to \infty$, and increases above unity as $v_\phi / V_e$ decreases. For most space plasma conditions $v_\phi / V_e$ is large enough ($\gtrsim 3$) for the approximate rates of both ESD and STI to be accurate. However, for $v_\phi / V_e \approx 3$ the exact rates can be up to 40% larger than the approximate rates. Thus the exact rates may be necessary for accurately modeling these nonlinear processes in some type II radio bursts and foreshock regions of planetary bow shocks. We have also calculated the velocities of the ions that most efficiently scatter the Langmuir waves, which we find are between 1 and 4 ion thermal speeds for $0.1 \leq T_i / T_e \leq 10$.  

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Figure 4.10: Graphs of $\Gamma_{\text{ESD}}^L / \Gamma_{\text{STI}}^L$ versus $T_i / T_e$. Black, red, and blue lines show: (a) $\tau_{0Lc} = 5, 10, \text{ and } 15$, respectively, for $\mu = 3$ and $\sigma = 0.1$; (b) $\mu = 3, 5, \text{ and } 10$, respectively, for $\tau_{0Lc} = 5$ and $\sigma = 0.1$; and (c) $\sigma = 0.1, 0.2, \text{ and } 0.3$, respectively, for $\tau_{0Lc} = 5$ and $\mu = 3$. The green line in (a) shows the approximation given by Eq. (4.75).
The relative (exact) rates of ESD and STI are calculated for a range of parameters; ESD is found to have the larger growth rate over the whole range of parameters considered, by at least a factor of 7 and up to a factor of about 320, in semiquantitative agreement with previous approximate calculations. Therefore, ESD is the dominant nonlinear process in space plasmas so long as the timescale and collective constraints for the process are satisfied.

A possible direction for future work is to calculate the quadratic longitudinal response function and rates of ESD and STI exactly for a non-Maxwellian electron distribution function such as a generalized Lorentzian (or “kappa”) distribution, since these distributions are commonly observed in space plasmas and may lead to greater enhancements of the nonlinear rates. Another direction involves calculating the quadratic response tensor and rate for electromagnetic decay $L \leftrightarrow T + S$ more accurately in a similar manner to the work presented here.
4.A Variables $a_j$, $b_j$, $r$, and their interchanges

\[
a_1 = \frac{\omega_{L'}}{Rk\lambda_D\sqrt{2}\sin\psi}, \tag{4.77}
\]
\[
\tilde{a}_1 = \frac{\omega_L}{k\lambda_D\sqrt{2}\sin\psi}, \tag{4.78}
\]
\[
\bar{a}_1 = \frac{\omega_{L'S}}{Rk\lambda_D\sqrt{2}\sin\psi}, \tag{4.79}
\]
\[
a_2 = \frac{\omega_S}{Rk\lambda_D\sqrt{2}\sin\psi}, \tag{4.80}
\]
\[
\tilde{a}_2 = \frac{\omega_S}{k\lambda_D\sqrt{2}\sin\psi}, \tag{4.81}
\]
\[
\bar{a}_2 = \frac{\omega_{LS}}{Rk\lambda_D\sqrt{2}\sin\psi}, \tag{4.82}
\]
\[
b_1 = -\frac{\cos\psi}{\sin\psi}, \tag{4.83}
\]
\[
\tilde{b}_1 = b_1, \tag{4.84}
\]
\[
\bar{b}_1 = \frac{R - \cos\psi}{\sin\psi}, \tag{4.85}
\]
\[
b_2 = \frac{\cos\psi - 1/R}{\sin\psi}, \tag{4.86}
\]
\[
\tilde{b}_2 = b_1, \tag{4.87}
\]
\[
\bar{b}_2 = b_2, \tag{4.88}
\]
\[
r = s, \tag{4.89}
\]
\[
\tilde{r} = \frac{\omega_L + \omega_S}{(2\cos\psi - R)k\lambda_D\sqrt{2}}, \tag{4.90}
\]
\[
\bar{r} = \frac{(\omega_L + \omega_{L'})S}{(1 - R^2)k\lambda_D\sqrt{2}}. \tag{4.91}
\]
Chapter 5

Concluding remarks and future directions

Previous calculations of the rates for second-order nonlinear processes have relied on approximate expressions for the quadratic response tensor. These approximate expressions were derived assuming that the phase speed of each wave is either much greater or much less than the electron thermal speed. However, the ranges of validity for these approximations are unclear, and more accurate expressions may be required for particular applications. This thesis presents new, more accurate analytical expressions for the quadratic response tensor for second-order nonlinear processes (i.e., three-wave interactions and nonlinear wave-particle scattering) in unmagnetized plasmas, and analytical and numerical calculations of the corresponding nonlinear rates.

In Chapter 2 a first-order thermal correction to the cold-plasma quadratic response tensor was derived, which is valid for three-wave interactions involving three fast waves in an unmagnetized plasma with an arbitrary isotropic velocity distribution. A first-order thermal correction to the rate of second harmonic plasma emission $L + L' \leftrightarrow T$ was then calculated using this new quadratic response tensor.

Chapter 3 presented an exact evaluation of the quadratic longitudinal response function for an unmagnetized Maxwellian plasma in terms of generalized plasma dispersion functions, whose expressions for both the real and imaginary parts were here corrected from previous work [Percival and Robinson, 1998b]. New, more rapidly convergent series expressions for the real part of the generalized plasma dispersion functions were derived, reducing the regions of numerical instability in their calculation so that they can be evaluated accurately over a large region of parameter space. This is important since we have shown that previously derived approximations to these functions are not accurate in appreciable regions of parameter space.
Unlike the full exact quadratic response tensor [Percival and Robinson, 1998a], the exact quadratic longitudinal response function derived in Chapter 3 is relatively compact, allowing us to calculate the rates of electrostatic decay \( L \leftrightarrow L' + S \) and scattering off thermal ions \( L + i \leftrightarrow L' + i' \) exactly in Chapter 4.

For all of the nonlinear rates considered in Chapters 2 to 4, the ratio of the beam speed to the electron thermal speed \( v_b/V_e \) is the crucial parameter that determines the accuracy of the approximate rates. By comparisons with the newly derived rates, the approximate rates were found to be accurate when \( v_b/V_e \gtrsim 3 \), which is well satisfied for electron beams in type III radio sources. However, for slow beams with \( v_b/V_e \lesssim 3 \) appropriate to some spatial domains of the foreshock regions of planetary bow shocks and type II radio sources, the approximate rates become less accurate. The exact rates of electrostatic decay and scattering off thermal ions can be more than a factor of 1.4 larger than the corresponding approximate rates for \( v_b/V_e < 3 \). For second harmonic plasma emission, the rate derived in Chapter 2 that includes a first-order thermal correction can be more than a factor of 2 larger than the cold-plasma rate when \( v_b/V_e < 3 \). The use of these new nonlinear rates derived in this thesis can therefore be important for accurate quantitative modeling of the nonlinear processes in space plasmas.

There are several avenues for future work. In this thesis we have calculated more accurate rates for the processes \( L + L' \leftrightarrow T \), \( L \leftrightarrow L' + S \), and \( L + i \leftrightarrow L' + i' \). Future work could involve calculating the rates of electromagnetic decay \( L \leftrightarrow T + S \) and scattering off thermal ions into transverse waves \( L + i \leftrightarrow T + i' \) more accurately using analogous methods. However, we predict that the approximate rates for these processes will be more accurate than those for \( L \leftrightarrow L' + S \) and \( L + i \leftrightarrow L' + i' \), since transverse waves better satisfy \( v_\phi \gg V_e \) than Langmuir waves. Put another way, the thermal corrections for these processes will be smaller than those for the processes \( L + L' \leftrightarrow T \), \( L \leftrightarrow L' + S \), and \( L + i \leftrightarrow L' + i' \). Another direction is to calculate the quadratic longitudinal response function and rates of electrostatic decay and scattering off thermal ions exactly for a non-Maxwellian electron distribution function such as a generalized Lorentzian (or “kappa”) distribution. We predict that the response function for a kappa distribution could be evaluated in terms of integrals involving the modified plasma dispersion function introduced by Summers and Thorne [1991]. Future work should also include incorporating our newly derived rates into existing simulations of type II (e.g., Schmidt and Cairns [2012]) and III radio bursts (e.g., Li and Cairns [2013]), which would lead to greater accuracy in data-theory comparisons.
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