APPENDIX D: A NON-LINEAR ELASTIC-PLASTIC MODEL FOR SAND

The model is isotropic, so it is convenient to formulate it in terms of mean and deviator stresses, which are also invariants.

D1. Elastic Behaviour

The stress-strain law may be written in terms of the tangent bulk modulus, $K_t$, and the tangent shear modulus, $G_t$, as follows:

\[
\begin{bmatrix}
 dp' \\
 dq
\end{bmatrix} = \begin{bmatrix}
 K_t & 0 \\
 0 & 3G_t
\end{bmatrix} \begin{bmatrix}
 de_p \\
 de_q
\end{bmatrix}
\]

where

\[
dp' = \frac{1}{3} [d\sigma'_1 + d\sigma'_2 + d\sigma'_3]
\]

\[
q = \frac{1}{\sqrt{2}} \sqrt{(d\sigma'_1 - d\sigma'_2)^2 + (d\sigma'_2 - d\sigma'_3)^2 + (d\sigma'_3 - d\sigma'_1)^2}
\]

\[
de_p = (de_1 + de_2 + de_3)
\]

\[
de_q = \sqrt{\frac{2}{3}} \sqrt{(de_1 - de_2)^2 + (de_2 - de_3)^2 + (de_3 - de_1)^2}
\]

and $d\sigma'_1$, $d\sigma'_2$ and $d\sigma'_3$ are increments of principal stresses, and $de_1$, $de_2$ and $de_3$ are increments of principal strain.

D2. Bulk Modulus, $K_t$

Following the suggestions of Naylor et al. (1981) and Lee and Salgado (2000), it is assumed that the tangent bulk modulus, $K_t$, is given by:

\[
K_t = D_s (p^\prime)^{n_k} (p_a)^{(1-n_k)}
\]

where \( p_a \) = atmospheric pressure

\( D_s \) and \( n_k \) are material constants, with \( n_k \) usually having a value of about 0.5

Therefore $K_t$ increases as a power law function of mean effective stress.
D3. Shear Modulus, \( G_t \)

D3.1 Small strain shear modulus, \( G_o \)

Following Hardin and Black (1966), it is assumed that:

\[
G_o = C_g \left( \frac{e_g - e_o}{1 + e_o} \right)^n \left( p_a \right)^{(i-n_g)(p')^n}, \tag{D-3}
\]

where \( C_g, n_g \) and \( e_g \) are material constants, and \( e_o = \text{initial void ratio} \)

D3.2 Secant Shear Modulus, \( G_s \)

Lee and Salgado (2000) extended the work of Fahey and Carter (1993), by proposing an expression for the secant shear modulus, as:

\[
\frac{G_s}{G_o} = \left( 1 - f \left( \frac{\sqrt{J_2} - \sqrt{J_{2o}}}{\sqrt{J_{2max}} - \sqrt{J_{2o}}} \right)^g \right)^{n_g} \left( \frac{I_1}{I_{1o}} \right)^{n_i}, \tag{D-4}
\]

where, the exponents, \( g \) and \( n_g \), and the factor, \( f \), are all material constants, \( I_1 = 3p' \)

\[
J_2 = 1/6 \left[ (\sigma'_1 - \sigma'_2)^2 + (\sigma'_2 - \sigma'_3)^2 + (\sigma'_1 - \sigma'_3)^2 \right]
\]

For triaxial conditions, \( J_2 = 1/3 \left( q \right)^2 \) \tag{D-5}

In equation D-4, \( I_{1o} \) and \( J_{2o} \) are the values of the stress invariants at the commencement of monotonic loading.

If \( J_{2max} \) is the maximum value of the second stress invariant which is attainable at the current mean stress, \( J_{2 max} \) can be found from the appropriate yield function, e.g. if \( q = Mp' \), then

\[
J_{2 max} = \frac{1}{3} q_{max}^2 = \frac{1}{3} (Mp')^2 \tag{D-6}
\]

where \( M = \text{current gradient of the failure envelope in } q-p' \text{ space} \)

\( p' = \text{current mean stress} \)
However it should be noted that other failure criteria (e.g. Mohr-Coulomb) could be used to determine $J_{2\text{ max}}$ for general stress conditions.

### D3.3 Tangent Shear Modulus, $G_t$

For conditions of simple shear ($\tau-\gamma$), Fahey and Carter (1993) demonstrated that

$$\frac{G_t}{G_o} = \frac{\left(\frac{G_t}{G_o}\right)^2}{1 - f(1-g)\left(\frac{\tau}{\tau_{\text{max}}}\right)^g}$$  \hspace{1cm} (D-7)

By analogy it is proposed that the generalised form of this equation should be

$$\frac{G_t}{G_o} = \frac{\left(\frac{G_t}{G_o}\right)^2}{1 - f(1-g)\left(\frac{\sqrt{J_{2\text{ max}}} - \sqrt{J_{2o}}}{\sqrt{J_{2\text{ max}}} - \sqrt{J_{2o}}_o}\right)^g}$$  \hspace{1cm} (D-8)

Equations D-2 and D-8 define the generalised forms of $K_t$ and $G_t$ to be substituted into the elastic stress-strain law, equation D-1.

#### Limit on $G_t$

It may also be necessary to place a limit on $G_t$ as it reduces with increasing deviator stress. If a limit is placed, the following condition must be satisfied:

$$\left(\frac{G_t}{G_o}\right) \geq r_g$$

where $r_g$ is a material parameter.

$r_g = 0$ corresponds to no limit on the reduction of $G_t$ with deviator stress

$r_g = 1$ corresponds to no reduction in $G_t$ with deviator stress

### D4. PLASTIC BEHAVIOUR

Refer to the basic model described in Appendix C.