Raising operators, recurrences, and the Littlewood–Richardson polynomials

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This was made possible by the following people:

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Contents

Introduction

The aim of this thesis is to calculate two types of Littlewood–Richardson polynomials. These are structure coefficients in the ring of double symmetric functions $\Lambda(x\|a)$ which has a distinguished basis consisting of the double Schur functions $s_\lambda(x\|a)$. The first type of Littlewood–Richardson polynomials arises when we consider the product of two double Schur functions, the second when the comultiplication operation in the ring $\Lambda(x\|a)$ is applied to a double Schur function. When the ring $\Lambda(x\|a)$ is specialised to the ring of symmetric functions $\Lambda(x)$, we recover the Littlewood–Richardson coefficients. Apart from their applications in the combinatorics of symmetric functions, the Littlewood–Richardson polynomials are important for the following reasons. They are applied in geometry and representation theory. The first type of polynomials describe a multiplication rule for equivariant Schubert classes, and also a multiplication rule for virtual quantum immanants and higher Capelli operators. The second type is relevant to describing equivariant cohomology of infinite grassmanians.

The structure of this thesis is as follows. In Chapter 1, we introduce well known definitions associated with the ring of symmetric functions $\Lambda(x)$. Using the Pieri rule and Jacobi–Trudi identity, we then present a proof of a rule used to calculate the Littlewood–Richardson coefficients. This is Theorem 1.8. This proof we present is a simplified version of our main result in Chapter 3.

In Chapter 2, we introduce the ring of double symmetric functions $\Lambda(x\|a)$, which is a generalisation of the classical ring $\Lambda(x)$ depending on an extra set of infinite variables $a = (a_i)_{i\in\mathbb{Z}}$. We introduce the basis of double Schur functions, and then explain how the two types of Littlewood–Richardson polynomials arise as structure coefficients involving the double Schur functions. We also discuss the significance of these structure coefficients in combinatorics, representation theory, and geometry.

In Chapter 3, we present one of the main results of this thesis using raising operators. This is a new proof of Theorem 3.33, a known formula which calculates the Littlewood–Richardson polynomials arising between the product of two double Schur functions. Our proof relies on two things: first, we introduce a Jacobi–Trudi identity for the double Schur functions. Second, we derive a Pieri rule for the ring $\Lambda(x\|a)$. This Pieri rule is in turn a specialisation of a more general rule which we also introduce for the ring A generated by the indeterminates $h_{r,s}$ from the 9th Variation of Macdonald [14].

In Chapter 4, we discuss the dual Littlewood–Richardson polynomials which arise when comultiplication is applied to the double Schur functions. We also discuss the dual Schur functions and skew double Schur functions. The dual Littlewood–Richardson polynomials then give combinatorial identities involving these functions. In the conclusion of Chapter 4, we present another main result of this thesis. This is Theorem 4.3, which provides a stable formula to calculate the dual Littlewood–Richardson polynomials.

In Chapter 5, we introduce the ring of generalised supersymmetric functions $\Lambda(x/y\|a)$, which has a distinguished basis consisting of generalised Frobenius–Schur functions $s_\lambda(x/y\|a)$. Using a recurrence relation, we produce another main result of this thesis. This is a Pieri rule which gives the structure coefficients arising out of the product between the functions $s_\theta(x/y\|a)$ and $s_\lambda(x/y\|a)$, where λ is an arbitrary partition and θ is a skew partition not containg a 2×2 subdiagram; this is Theorem 5.34. A specialisation of this theorem then lets us evaluate some of the dual Littlewood–Richardson coefficients.

1 Littlewood–Richardson Coefficients

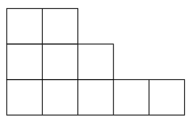
This thesis is about the combinatorics which arise when attempting to solve problems in the *ring of double symmetric functions*. We would like to give such an example in this chapter, using a simplified version of a rule we use in Chapter 3. First, we will state the *Pieri rule* and *Jacobi–Trudi identity* without proof. Then, the focus is to use both of them to give a version of the *Littlewood–Richardson* rule (Theorem 1.8). The proof of Theorem 1.8 illustrates the type of combinatorics of *tableaux* which we use throughout the thesis. This is our main focus and in this chapter we will try to keep definitions to a minimum; in the next chapter a proper introduction of the ring of double symmetric functions will be given.

1.1 Definitions

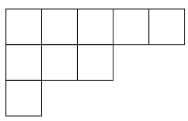
A *composition* α of length l is a sequence of nonnegative integers $(\alpha_1, \alpha_2, \dots, \alpha_l)$ such that α_l is positive. The *length* of α is denoted by $l(\alpha)$. We will also write α as the sequence $(\alpha_1, \dots, \alpha_l, 0, \dots)$; that is we will add as many zeroes as we like to the end of the sequence α. The *empty composition* $\varnothing$ is an infinite sequence of zeroes, with length equal to 0 by definition. We say a composition α *contains* the composition β if $\alpha_i \geqslant \beta_i$, for all $i > 0$, and denote this by $\alpha \supseteq \beta$. If α and β are compositions the *sum* $\alpha + \beta$ is the composition defined as the pointwise sum of α and β; that is $\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \dots)$, extending the sequence α or β with as many zeroes as necessary to make their lengths equal. A *partition* λ is a composition such that $\lambda_1 \geqslant \lambda_2 \geqslant \dots$.

The *diagram* of a composition α is a finite collection of boxes, left justified, with rows numbered 1 to l, starting from the top and ending at the bottom, such that there are α_i boxes in row i. Note that we will use the words diagram and composition interchangeably.

Example 1.1. For example, for the composition $\alpha = (2, 3, 5)$ we have the diagram

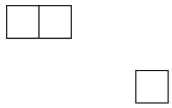
$\alpha =$ 

and for $\lambda = (5, 3, 1)$ we have the diagram

$\lambda =$ 

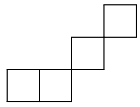
□

If α and β are both compositions and $\alpha \supseteq \beta$, we may define the skew composition α/β as the set theoretic difference consisting of boxes in α but not in β. For example, if $\alpha = (2,3,5)$ and $\beta = (0,3,4)$, we have the skew composition:

$\alpha/\beta =$ 

If $\mu \subseteq \nu$ are a pair of partitions, then the skew diagram ν/μ is called a *horizontal strip* if there is at most one box in every column of ν/μ.

For example, for the partitions $\nu = (5,4,3)$ and $\mu = (4,3,1)$ we have that

$\nu/\mu =$ 

is a horizontal strip.

Let A be the ring of polynomials in the variables $x = (x_1, \dots, x_n)$ with coefficients from $\mathbb{Z}$. The symmetric group on n elements $\mathfrak{S}_n$ acts on A by permuting the indices of the variables x_i. The *ring of symmetric polynomials* $\Lambda_n(x)$ is the subring of A consisting of all polynomials which are invariant under the action of all permutations in the group $\mathfrak{S}_n$. It is generated over $\mathbb{Z}$ by the *complete symmetric polynomials*, which we denote by h_p, for all $p = 1, 2, \dots,$ with

$$h_p = \sum_{n \geqslant i_1 \geqslant i_2 \geqslant \dots \geqslant i_p \geqslant 1} x_{i_1} x_{i_2} \dots x_{i_p}.$$

This means that any element in $\Lambda_n(x)$ is written as a polynomial $P(h_1, h_2, \dots)$ with coefficients from $\mathbb{Z}$. For convenience, define $h_0 = 1$ and $h_p = 0$ if $p < 0$. If κ is a composition, we denote by h_κ the product $h_{\kappa_1} h_{\kappa_2} \dots h_{\kappa_{l(\kappa)}}$.

The ring $\Lambda_n(x)$ has a basis over $\mathbb{Z}$ consisting of the family of *Schur polynomials* $\{s_\lambda\}$, indexed by all partitions $\Lambda(x)$ with length at most n. We can define these polynomials in terms of the complete symmetric polynomials using the *Jacobi–Trudi* identity. First we need to define the action of permutations on compositions, that is, for the transposition $\sigma = (i, i+1) \in \mathfrak{S}_l$, we have $\sigma(\alpha)$ is the composition equal to α with its i-th and $i+1$-th entry swapped. Let $\tau_l = (l-1, l-2, \dots, 0)$ be a composition. For a permutation $\omega \in \mathfrak{S}_l$, define the composition $\kappa^\omega = \omega(\lambda + \tau_l) - \tau_l$ and $\operatorname{sgn}(\lambda^\omega) = \operatorname{sgn}(\omega)$, the parity of the permutation ω. We now use these definitions to write the Schur polynomial s_λ in terms of the complete symmetric polynomials h_p.

Let $\Lambda(x)$ be a partition and $l = l(\lambda)$. The Schur polynomial s_λ may be expressed in terms of the complete symmetric polynomials via the *Jacobi–Trudi identity*:

$$\begin{aligned} s_\lambda &= \det(h_{\lambda_i + j - i})_{1 \leqslant i,j \leqslant l} \\ &= \sum_\kappa \operatorname{sgn}(\kappa) h_\kappa, \end{aligned} \tag{1.1}$$

summed over compositions $\kappa = \lambda^\omega$, for all $\omega \in \mathfrak{S}_l$. The second line of this equation follows by expanding the determinant; each summand under this expansion is of the form $\text{sgn}\,(\kappa)h_\kappa$, for a $\kappa = \lambda^\omega$.

Example 1.2. Let $\lambda = (3,1)$. Then for the transposition $(1,2) \in \mathfrak{S}_2$, we have that $\kappa = \lambda^{(1,2)}$ creates the composition $\kappa = (0,4)$. Thus, using the Jacobi–Trudi identity (1.1):

$$\begin{aligned} s_{(3,1)} &= h_{(3,1)} - h_{(0,4)} \\ &= \det \begin{pmatrix} h_3 & h_4 \\ h_0 & h_1 \end{pmatrix} \end{aligned}$$

□

We consider the product $s_\lambda s_\mu$. Since the Schur polynomials form a basis we may expand this product in terms of the Schur polynomials,

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda\mu}(n) s_\nu, \tag{1.2}$$

summed over partitions ν. The structure coefficients $c^\nu_{\lambda\mu}(n)$ arising when we decompose the product of two Schur polynomials (1.2) are integers which are called the *Littlewood–Richardson coefficients*, and the *Littlewood–Richardson rule* gives a calculation of $c^\nu_{\lambda\mu}$. These coefficients play an important role in the combinatorics of the symmetric polynomials [15], representation theory [24], and geometry [4]. In principle, the coefficient $c^\nu_{\lambda\mu}(n)$ depends on the number of variables $x = (x_1, \ldots, x_n)$. However, this coefficient stabilises as n goes towards infinity. An equivalent way of considering this is that the coefficients $c^\nu_{\lambda\mu}(n)$ do not depend on n when n is large enough; henceforth we will simply write $c^\nu_{\lambda\mu}$ to denote the Littlewood–Richardson coefficients.

1.2 Reformulating the Pieri rule

Our AIM: In this chapter, we wish to calculate the Littlewood–Richardson coefficient $c^\nu_{\lambda\mu}$, where λ, μ, ν are partitions. For a composition κ, define the *Kostka numbers* $K^{\ \nu}_{\kappa\mu}$ to be the structure coefficient occuring in the product between a complete symmetric polynomial and a Schur polynomial:

$$h_\kappa s_\mu = \sum_\nu K^{\ \nu}_{\kappa\mu} s_\nu, \tag{1.3}$$

summed over all partitions ν. Since the Jacobi–Trudi identity (1.1) gives the following expansion for s_λ:

$$s_\lambda = \sum_\kappa \text{sgn}\,(\kappa) h_\kappa,$$

we conclude that the Littlewood–Richardson coefficients are given by the following alternating sum:

$$c^{\nu}_{\lambda\mu} = \sum_{\kappa} \operatorname{sgn}(\kappa) K^{\nu}_{\kappa\mu}, \tag{1.4}$$

summed over compositions κ of the form λ^{ω}, where ω runs over the permutations of $\mathfrak{S}_l$. Thus, if we can furnish a formula for $K^{\nu}_{\kappa\mu}$, for all $\kappa = \lambda^{\omega}$ such that $\omega \in \mathfrak{S}_l$, we will have a formula for $c^{\nu}_{\lambda\mu}$.

Our starting point is the Pieri rule, which tells us the product between the complete symmetric polynomial h_p, for any $p \geqslant 0$, and a Schur polynomial s_{μ},

$$h_p s_{\mu} = \sum_{\rho} s_{\rho}, \tag{1.5}$$

summed over diagrams ρ containing μ such that ρ has p more boxes than μ, and ρ/μ is a horizontal strip. For a proof of this rule see Chapter 3, or [15].

We can use the Pieri rule (1.5) iteratively by calculating

$$h_{\kappa} s_{\mu} = h_{\kappa_1}(\dots(h_{\kappa_{l-1}}(h_{\kappa_l} s_{\mu}))),$$

starting with the first pair $h_{\kappa_l} s_{\mu}$. To do this calculation, we will use combinatorial objects known as *reverse tableaux*.

Let $\lambda = (p)$, which is the partition that corresponds to the diagram with p boxes in the first row. A reverse (p)-tableau T (plural tableaux) is obtained by inserting a positive integer into each box of the diagram (p), so that these integers weakly decrease when read from left to right. These integers are called the *entries* of T, and for a box $\alpha \in (p)$ the entry of T in box α is denoted $T(\alpha)$.

Example 1.3. For $p = 6$, the following is a (6)-tableau

$$U = \boxed{5\,|\,5\,|\,4\,|\,3\,|\,3\,|\,1}.$$

□

We say that each reverse (p)-tableau T has a (uniquely determined) *row word*, which is the sequence of positive integers $S = s_1 s_2 \dots s_p$ such that the s_i's, $i = 1, \dots, p$, are the entries of T reading left to right. We *apply* such a sequence to the composition μ by forming a sequence of compositions terminating at ν:

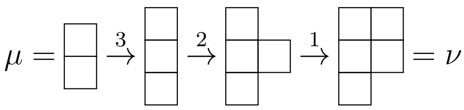
$$\mu = \rho^{(0)} \xrightarrow{s_1} \rho^{(1)} \xrightarrow{s_2} \dots \xrightarrow{s_p}, \rho^{(p)} = \nu$$

such that $\rho^{(i-1)} \xrightarrow{s_i} \rho^{(i)}$ means adding a box to the end of row s_i of $\rho^{(i-1)}$ to form $\rho^{(i)}$. We will say that the sequence S *takes* μ *to* ν, denote this by $S : \mu \to \nu$, and also define S to be *Yamanouchi when applied to* μ if $\rho^{(i)}$ is a partition for all $0 \leqslant i \leqslant p$.

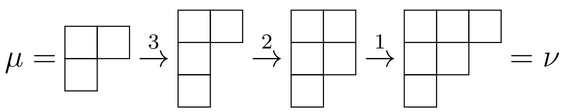
Example 1.4. Using the tableau U

$$U = \boxed{3}\boxed{2}\boxed{1},$$

we see that the row word of U is $S = 321$. When we apply S to the partition $\mu = (1^2)$ we obtain the following sequence of diagrams



so that the word S when applied to μ is not Yamanouchi, since the third diagram is not a partition. On the other hand, if we let $\mu = (2, 1)$, then we have the following sequence of diagrams



in which case S when applied to this μ is Yamanouchi, since all the diagrams in the sequence above are partitions. □

We reformulate the Pieri rule (1.5) with these reverse (p)-tableaux.

Proposition 1.5. *The coefficient* $c^{\nu}_{(p)\mu}$ *is the number of reverse* (p)*-tableaux* T *with row word* S *which takes* μ *to* ν *and is Yamanouchi.*

This theorem is equivalent to the following facts:

(1) If ν/μ is a horizontal strip and ν has p more boxes than μ, there is exactly ONE tableau T such that the sequence S takes μ to ν and is Yamanouchi.

(2) If ν/μ is not a horizontal strip or ν has does not have p more boxes than μ, there are no tableaux T such that the sequence S takes μ to ν and is Yamanouchi.

Proof. Firstly, since S contains p entries, when S is applied to μ we will obtain a composition with p more boxes than μ. Thus ν must have p more entries than μ, otherwise there will be no tableaux T such that the sequence S takes μ to ν. Suppose S takes μ to ν. Since the entries of T weakly decrease from left to right, we have that

$$S = l \dots l(l-1)\dots(l-1)\dots 1 \dots 1,$$

where the number of integers i appearing in this sequence is equal to $\nu_i - \mu_i$. Thus, this sequence is not Yamanouchi when applied to μ if ν/μ is not a horizontal strip. If ν/μ is a horizontal strip there is exactly one such S which takes μ to ν. □

We will use the reformulated Pieri rule (1.5) and reverse tableaux to give a formula for the Kostka numbers $K^{\nu}_{\kappa\mu}$ (1.3). First, we need to extend our definition of a reverse tableau.

Let κ be a composition. Then we obtain a reverse κ-tableau T by filling in the boxes of κ with positive integers, so that the entries in rows of T weakly decrease from left to right. We impose no conditions on the columns of T. To each T we associate the (uniquely determined) sequence of integers $S = s_1 s_2 \dots$, by reading the entries T from left to right in each row, starting with the bottom row. We will call S the *row word* of T.

For example, consider the composition $\kappa = (2, 3)$, such that the following is a reverse κ-tableau

3	3	
4	3	1

with row word $S = 43133$.

The following is a formula for the Kostka numbers $K^{\nu}_{\kappa\mu}$ using reverse κ-tableaux.

Proposition 1.6. *The coefficient $K^{\nu}_{\kappa\mu}$ is equal to the number of reverse κ-tableaux T with row word S which takes μ to ν and is Yamanouchi when applied to μ.*

Proof. We argue by induction. As the base case, the Pieri rule (Proposition 1.5) tells us how we decompose the product $h_{\kappa_l} s_\mu$. Let the composition $\kappa^* = (\kappa_2, \kappa_3, \dots, \kappa_l)$ be formed from the composition κ by excluding the entry κ_1. Then, as the induction hypothesis, we assume that $K^{\chi}_{\kappa^*\mu}$ is equal to the number of reverse κ^*-tableaux with row word $R : \mu \to \chi$. Then, we have

$$\begin{aligned} h_\kappa s_\mu &= h_{\kappa_1} h_{\kappa^*} s_\mu \\ &= h_{\kappa_1} \sum_{\chi} K^{\chi}_{\kappa^*\mu} s_\chi \\ &= \sum_{\chi} \sum_{\nu} K^{\chi}_{\kappa^*\mu} s_\nu, \end{aligned}$$

summed over all partitions χ and ν such that ν/χ is a horizontal strip with κ_1 boxes. The proposition then follows.

□

Earlier in this subsection, we said that by obtaining a formula for $K^{\nu}_{\kappa\mu}$, we would have a formula for $c^{\nu}_{\lambda\mu}$, namely:

$$c^{\nu}_{\lambda\mu} = \sum_{\kappa} \mathrm{sgn}\,(\kappa) K^{\nu}_{\kappa\mu},$$

with the coefficients $K^{\nu}_{\kappa\mu}$ equal to the number of reverse κ-tableaux T with a Yamanouchi row word which takes μ to ν.

However this formula turns out to be unsatisfactory, as we will discuss in the next subsection.

1.3 Calculating the Littlewood–Richardson coefficients

Recall equation (1.4)

$$c^{\nu}_{\lambda\mu} = \sum_{\kappa} \mathrm{sgn}\,(\kappa) K^{\nu}_{\kappa\mu},$$

which states that $c^{\nu}_{\lambda\mu}$ is equal to an alternating sum of the coefficients $K^{\nu}_{\kappa\mu}$. This is not a *combinatorial* formula, because the coefficients $c^{\nu}_{\lambda\mu}$ turn out to be nonnegative, for the following reasons in geometry and representation theory.

Firstly, the coefficients $c^{\nu}_{\lambda\mu}$ describe the intersection theory of geometrical objects called *Schubert cells*. The intersection theory of Schubert cells is analogous to the problem of solving for the number of intersection points of curves in the $\mathbb{R}^2$ plane. In particular, the number of intersections is a NONNEGATIVE number. Secondly, let $\Lambda(x)$ be a partition of n, i.e. $|\lambda| = n$. Then, the irreducible character χ^{λ} of the symmetric group $\mathfrak{S}_n$ is mapped to the Schur function s_{λ} by the characteristic map, see Macdonald [15, I, 7]. The induced product of the irreducible characters of the symmetric groups χ^{λ} and χ^{μ} is a character of the symmetric group and so it is a nonnegative integer linear combination of the irreducible characters. However, under the characteristic map this induced product is equivalent to taking the product of s_{λ} with s_{μ}, thus the Littlewood–Richardson coefficients are nonnegative. Thirdly, the Schur functions are the characters of irreducible representations of the general linear group GL_n, see Macdonald [15, IV, 6]. The product of the Schur functions corresponds to the character of the tensor product of two irreducibles of GL_n. Its decomposition into irreducibles contains nonnegative multiplicities, again showing that the Littlewood–Richardson coefficients are nonnegative.

So what do mean when we say that equation (1.4) is not a combinatorial formula? Perhaps the best way to describe this is to use the words of Berenstein and Zelevinsky [1]; *the obstacle to a combinatorial interpretation is the alternation* in equation (1.4). The alternating sum implies the possibility that the coefficients $c^{\nu}_{\lambda\mu}$ may in fact be negative. Thus, the KEY PROBLEM in finding a *combinatorial formula* is finding a formula that implies the nonnegativity of $c^{\nu}_{\lambda\mu}$. This is done in the rest of this section by eliminating unnecessary terms, such as the ones with negative coefficients, from the alternating sum in equation (1.4).

We first need the following definitions. Let κ be a composition. To every κ-tableau T we associate the (uniquely determined) *column word* $R = r_1 r_2 \ldots$, the sequence of entries of T, read from the first column to the last column, and from the bottom to the top of each column. Finally, call T *column strict* if the entries of T strictly decrease down each column.

Example 1.7. Let $\kappa = (3^2)$, then the following is a column strict reverse κ-tableau

$$\begin{array}{|c|c|c|}\hline 2&2&2\\ \hline 1&1&1\\ \hline\end{array}$$

with column word 121212. □

Theorem 1.8. *The Littlewood–Richardson coefficient $c^{\nu}_{\lambda\mu}$ is equal to the number of column strict reverse λ-tableaux T, with column word R such that R takes μ to ν and is Yamanouchi.*

This theorem appears as a special case of Molev's [18, Thm 2.1], where the more general Littlewood–Richardson polynomial is calculated. However in [18] a vanishing property of the Schur functions and a corresponding recurrence relation is used to obtain the result. Here, we rely on the Pieri rule and Jacobi–Trudi identity, which is a method used to calculate the coefficients $c^{\nu}_{\lambda\mu}$ previously in [5, 23], see also [26]. Although our rule looks different it is in fact equivalent to other rules used to calculate the Littlewood–Richardson coefficients, see for example the survey paper by van Leeuwen [27].

Proof. For κ a composition, call a κ-tableau T *bad* if it is not column strict or its word R is not Yamanouchi when applied to μ. Then we prove that Theorem 1.8 is true from equation (1.4) by cancelling out the weight of bad tableaux contributing to the sum $\sum_{\kappa} \operatorname{sgn}(\kappa) h_{\kappa}$ in (1.1). Throughout this proof we will refer to an example where $\lambda = (3^2)$ and $\mu = (3, 1)$ and $\nu = (6, 4)$. We remark that a similar argument is used in Chapter 3 to prove a more general result.

Let us consider our example. Let $\sigma = (2, 4)$. Since $\lambda = (3^2)$, by equation (1.4) we have that

$$c^{\nu}_{\lambda\mu} = K^{\nu}_{\lambda\mu} - K^{\nu}_{\sigma\mu}.$$

Using Proposition 1.6 we calculate the value of $K^{\nu}_{\lambda\mu}$ and $K^{\nu}_{\sigma\mu}$. First, we write down all of the λ-tableaux with a row word $S : \mu \to \nu$ which is Yamanouchi; they are

$$\begin{array}{|c|c|c|}\hline 2&2&2\\ \hline 1&1&1\\ \hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 2&2&1\\ \hline 2&1&1\\ \hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 2&1&1\\ \hline 2&2&1\\ \hline\end{array}$$

Second, we write down all of the σ-tableaux with a row word $S : \mu \to \nu$ which is Yamanouchi; they are

$$\begin{array}{|c|c|c|c|}\hline 2&1\\ \cline{1-2} 2&2&1&1\\ \hline\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 2&2\\ \cline{1-2} 2&1&1&1\\ \hline\end{array}$$

Therefore, by Proposition 1.6 the coefficient $c^{\nu}_{\lambda\mu} = 3 - 2 = 1$.

We return to the general argument. We start with some preliminary definitions. Suppose $\kappa = \lambda^{\omega}$, for a $\omega \in \mathfrak{S}_{l(\lambda)}$. Call a reverse $\Lambda(x)$-tableau T *good* if it is column strict and has a column word $R : \mu \to \nu$ which is Yamanouchi. On the other hand, a reverse κ-tableau T is *bad* if and only if it obeys one of the following two properties:

(P1) The tableau T is not column strict.

(P2) The composition κ is not a partition.

Thus, returning to our example, we see that we have one good tableau:

$$\begin{array}{|c|c|c|}\hline 2&2&2\\ \hline 1&1&1\\ \hline\end{array}$$

and four bad tableaux:

$$\begin{array}{|c|c|c|}\hline 2&2&1\\ \hline 2&1&1\\ \hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 2&1&1\\ \hline 2&2&1\\ \hline\end{array}$$

$$\begin{array}{|c|c|cc}\cline{1-2} 2&1\\ \hline 2&2&1&\multicolumn{1}{c|}{1}\\ \hline\end{array} \qquad \begin{array}{|c|c|cc}\cline{1-2} 2&2\\ \hline 2&1&1&\multicolumn{1}{c|}{1}\\ \hline\end{array}$$

To prove Theorem 1.8, we will introduce an involution on the set of bad κ-tableaux, such that a bad κ-tableau T is paired to a bad $\widetilde{\kappa}$-tableau $\widetilde{T}$, where $\widetilde{\kappa}$ is the composition formed from κ by swapping the boxes in two adjacent rows; that is, we apply the transposition $(i, i+1) \in \mathfrak{S}_{l(\lambda)}$ to κ. In this way, the number of bad κ-tableaux will cancel with the number of bad $\widetilde{\kappa}$-tableaux in the sum (1.4).

In particular, our involution will create the following pairs of tableaux from our example:

Pair 1: $\begin{array}{|c|c|c|}\hline 2&2&1\\ \hline 2&1&1\\ \hline\end{array}$ with $\begin{array}{|c|c|cc}\cline{1-2} 2&1\\ \hline 2&2&1&\multicolumn{1}{c|}{1}\\ \hline\end{array}$

Pair 2: $\begin{array}{|c|c|c|}\hline 2&1&1\\ \hline 2&2&1\\ \hline\end{array}$ with $\begin{array}{|c|c|cc}\cline{1-2} 2&2\\ \hline 2&1&1&\multicolumn{1}{c|}{1}\\ \hline\end{array}$

Returning to the general argument, let the integer i be the minimal row number of κ such that the subtableau formed from T consisting of rows i and $i+1$ is not column strict. If T is column strict, let i be the first row such that $\kappa_i < \kappa_{i+1}$. Therefore, by the definition that T is bad, such a row number i must always exist. For such an i and for each $2 \leqslant k \leqslant l(\nu)$ we define another integer n_k in the following way. For all integers k, let r_k be the number of barred k's in every row below and including row i, and let r'_k be the number of barred k's in every row strictly below row $i+1$. Then, for each $2 \leqslant k \leqslant l(\nu)$ the integer n_k is defined to be $n_k = \max(\mu_k + r_k - \mu_{k-1} - r'_{k-1}, 0)$. The row word of T is Yamanouchi. Thus, if n_k is positive, then there are at least n_k

boxes containing k in row i of T, and n_k boxes containing $k-1$ in row $i+1$ of T. For example, consider the tableau

$$\begin{array}{|c|c|c|}\hline 2&1&1\\\hline 2&2&1\\\hline\end{array}$$

and we have that $i=1$. Further, we have the integers $r_2=3$ and $r'_1=0$, thus $n_2=\max(\mu_2+r_2-\mu_1-r'_1,0)=1$. Thus, we claim that row 1 contains at least $n_2=1$ box containing a 2, and row 2 contains at least 1 box containing a 1, which is true.

We now describe the involution which pairs a bad κ-tableau T to a bad $\widetilde{\kappa}$-tableau $\widetilde{T}$. We do the following if T is not column strict. Consider the integer j, $1\leqslant j\leqslant\kappa_i$, to be minimal such that $T(i,j)\leqslant T(i+1,j)$. Let X denote the subtableau of entries of T in boxes (i,j) up to (i,κ_i) and Y the subtableau of entries of T in boxes $(i+1,j+1)$ up to (i,κ_{i+1}). Then, we apply two operations to T. The first operation is to swap the subtableaux X and Y to form an intermediate tableau T^1. Now, for each $2\leqslant k\leqslant T(i+1,j)$, we have n_k boxes containing k in row $i+1$ of T^1, and n_k boxes containing $k-1$ in row i of T^1, by definition of n_k. The second operation is to do the following independently for each $2\leqslant k\leqslant T(i+1,j)$: take the right most n_k entries with value k in row $i+1$ of T^1 and swap them with the left most n_k entries with value $k-1$ in row i of T^1. This forms the tableau $\widetilde{T}$, of shape $\widetilde{\kappa}$, paired with the tableau T under the involution. We demonstrate that this operation is indeed an involution by way of the following example.

Let us examine the bad tableau

$$T=\begin{array}{|c|c|c|}\hline 2&1&1\\\hline 2&2&1\\\hline\end{array}$$

Since the entries $T(1,1)\leqslant T(2,1)$, we note that the entries of T do not strictly decrease down the first column, and thus the column number $j=1$. Then, the subtableau X consists of the entries in boxes between $(1,1)$ and $(1,3)$, and the subtableau Y consists of the entries in boxes between $(2,2)$ and $(2,3)$; these subtableaux consist of the bold entries in row 1 and row 2 of the following tableau respectively:

$$\begin{array}{|c|c|c|}\hline \mathbf{2}&\mathbf{1}&\mathbf{1}\\\hline 2&\mathbf{2}&\mathbf{1}\\\hline\end{array}$$

We now swap X and Y, that is, we swap the bold entries in the first row with those in the second

$$T^1=\begin{array}{|c|c|cc}\cline{1-2} \mathbf{2}&\mathbf{1}&\multicolumn{1}{c}{}&\\\hline 2&\mathbf{2}&\multicolumn{1}{|c|}{\mathbf{1}}&\multicolumn{1}{c|}{\mathbf{1}}\\\hline\end{array}$$

Since $n_2=1$, we mark in bold the left most 1 in row 1 of T^1 and the rightmost 2 in row 2 of T^1

$$T^1=\begin{array}{|c|c|cc}\cline{1-2} 2&\mathbf{1}&\multicolumn{1}{c}{}&\\\hline 2&\mathbf{2}&\multicolumn{1}{|c|}{1}&\multicolumn{1}{c|}{1}\\\hline\end{array}$$

We swap these entries to create the tableau $\widetilde{T}$:

$$\widetilde{T} = \begin{array}{cccc} 2 & 2 & & \\ 2 & 1 & 1 & 1 \end{array}$$

We see that $\widetilde{T}$ is a bad tableau of shape $\sigma = (2, 4)$. We apply our operations on $\widetilde{T}$ again. Recall that in this case $\widetilde{T}(1, 1) \leqslant \widetilde{T}(1, 2)$, so we swap the bold entries in row 1 with those in row 2:

$$\widetilde{T} = \begin{array}{cccc} \mathbf{2} & \mathbf{2} & & \\ 2 & \mathbf{1} & \mathbf{1} & \mathbf{1} \end{array} \to \widetilde{T}^1 = \begin{array}{ccc} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 2 & \mathbf{2} & \mathbf{2} \end{array}$$

Again, note that there is at least one box containing a 1 in row 1, and one box containing a 2 in row 2; we now swap the left most 1 in row 1 with the right most 2 in row 2:

$$\widetilde{T}^1 = \begin{array}{ccc} \mathbf{1} & 1 & 1 \\ 2 & 2 & \mathbf{2} \end{array} \to \begin{array}{ccc} 2 & 1 & 1 \\ 2 & 2 & 1 \end{array}$$

Note that this final tableau formed is equal to T.

For completion of the proof, we describe the involution for the case where T is column strict but κ is not a partition. Recall that in this case $\kappa_i < \kappa_{i+1}$. Note that since $\kappa = \lambda^\omega$, for some $\omega \in \mathfrak{S}_{l(\lambda)}$, and $\lambda_i \geqslant \lambda_{i+1}$, we have that $\kappa_i \leqslant \kappa_{i+1} - 2$. Let Y be the subtableau of T formed from the entries in box $(i + 1, \kappa_i + 2)$ up to $(i + 1, \kappa_i + 2)$. Then we form $\widetilde{T}$ by moving the subtableau Y to the end of row i. $\square$

2 The symmetric functions

In this chapter we introduce the ring of double symmetric functions $\Lambda(x\|a)$, and three distinguished bases of this ring. We also introduce the generators consisting of the double power sums symmetric functions, and the structure coefficients of the ring $\Lambda(x\|a)$. This ring is a generalisation of the classical ring of symmetric functions $\Lambda(x)$, such that we have two (infinite) sets of variables, $x = (x_1, x_2, \dots)$ and $a = (a_i)_{i\in\mathbb{Z}}$. We would like to study the ring $\Lambda(x\|a)$ because the Littlewood–Richardson polynomials arise as structure coefficients in this ring, and many of the combinatorial questions dealt with in this thesis arise from attempts to describe these polynomials. We remark that under the specialisation $a_i = 0$, for all $i \in \mathbb{Z}$, the classical ring of symmetric functions $\Lambda(x)$ is recovered, for a detailed description of the ring $\Lambda(x)$ see Macdonald [15]. In this chapter, we will focus on the features of $\Lambda(x\|a)$, but many of our discussions will still hold for the classical case when we specialise a to the sequence of zeroes.

When we describe the bases of $\Lambda(x\|a)$ we will emphasize the usage of *tableaux*. There are other ways to describe these bases, for example, in the classical case one could define the bases of $\Lambda(x)$ algebraically, but since the combinatorics described in this thesis are combinatorics of tableaux, it seems sensible to approach the subject using tableaux as the main focus right from the outset.

2.1 The ring of double symmetric polynomials

Our introduction to the ring of double symmetric functions encapsulates ideas introduced by Molev [16, 18]. We first define the *ring of double symmetric polynomials* $\Lambda(x^{(n)}\|a)$, which depends on a FINITE set of variables $x^{(n)} = (x_1, x_2, \dots, x_n)$ and an infinite sequence of variables $a = (a_i)$, $i \in \mathbb{Z}$.

Consider the ring of polynomials $\mathbb{Q}[a]$ in the variables a_i with rational coefficients. For each $n \geqslant 0$, let $x^{(n)} = (x_1, x_2, \dots, x_n)$ denote another set of variables. Then, permutations from the symmetric group $\mathfrak{S}_n$ on n elements act naturally on polynomials in the variables $x^{(n)}$ by permuting the indices of x_i. For example, for the transposition $\sigma = (1, 2)$ we have that

$$\begin{aligned}(1,2)(x_1^2 - \frac{1}{2}a_1x_2) &= x_{\sigma(1)}^2 - \frac{1}{2}a_1x_{\sigma(2)} \\ &= x_2^2 - \frac{1}{2}a_1x_1\end{aligned}$$

We denote by $\Lambda(x^{(n)}\|a)$ the *ring of symmetric polynomials* in $x_1, \dots, x_n$ with coefficients in $\mathbb{Q}[a]$. This ring consists of all polynomials in the polynomial ring

$\mathbb{Q}[a][x_1, \dots, x_n]$ that are invariant under the action of all permutations in the symmetric group $\mathfrak{S}_n$. For example, the polynomial

$$p_2(x\|a) = \sum_{i=1}^{n} (x_i^2 - a_i^2)$$

is invariant under the tranpositions $(i, i+1)$, for all $1 \leqslant i < n$, and thus belongs to $\Lambda(x^{(n)}\|a)$.

We would like to define the ring of double symmetric functions, which depends on an INFINITE set of variables $x = (x_1, x_2, \dots)$ and the sequence a. First we define a filtration on the ring $\Lambda(x^{(n)}\|a)$. We do this by treating the elements of $\Lambda(x^{(n)}\|a)$ as polynomials in x_i with $\deg x_i = 1$ and $\deg a_i = 0$. Then, the ring $\Lambda(x^{(n)}\|a)$ has a filtration consisting of the $\mathbb{Q}[a]$-submodules $\Lambda^k(x^{(n)}\|a)$, consisting of elements from $\Lambda(x^{(n)}\|a)$ with degree at most k.

Let $P(x_1, \dots, x_n)$ denote a polynomial in $\Lambda(x^{(n)}\|a)$. For each n the evaluation map

$$\varphi_n : \Lambda(x^{(n)}\|a) \to \Lambda(x^{n-1}\|a) \qquad P(x_1, \dots, x_n) \mapsto P(x_1, \dots, x_{n-1}, a_n) \tag{2.1}$$

is a homomorphism of filtered rings so we can define the inverse limit ring $\Lambda(x\|a)$ by

$$\Lambda(x\|a) = \varprojlim \Lambda(x^{(n)}\|a), \qquad n \to \infty$$

in the category of filtered rings. Then, the elements of $\Lambda(x\|a)$ are sequences $L = (L_0, L_1, L_2 \dots)$ with $L_n \in \Lambda(x^{(n)}\|a)$, elements of bounded degree, for all $n \geqslant 0$, such that

$$\varphi_n(L_n) = L_{n-1}, \qquad \text{for } n = 1, 2, \dots,$$

with multiplication between L and another element $M = (M_0, M_1, \dots)$ in the ring $\Lambda(x\|a)$ defined by

$$LM = (L_0M_0, L_1M_1, \dots).$$

Remark 2.1. Under the specialisation $a_i \to 0$, the ring $\Lambda(x^{(n)}\|a)$ becomes the ring of symmetric polynomials in the variables $x_1, \dots, x_n$. Then, it is possible to define the ring of symmetric functions $\Lambda(x)$ as an inverse limit in the category of *graded* rings, see Macdonald [15, Chapter 1].

We will consider the elements of the ring $\Lambda(x\|a)$ as formal series in the variables $x = (x_1, x_2, \dots)$ with coefficients from $\mathbb{Q}[a]$. As such, we call them *double symmetric functions* and the ring $\Lambda(x\|a)$ is the *ring of double symmetric functions*. This is as distinct from the notion of a double symmetric polynomial, which is a polynomial in the finite variables $x^{(n)} = (x_1, \dots, x_n)$. The ring $\Lambda(x\|a)$ is also an algebra over the ring $\mathbb{Q}[a]$.

There are six distinguished bases of the classical ring of symmetric functions $\Lambda(x)$. Their 'double' analogues form six distinguished $\mathbb{Q}[a]$-bases of the ring $\Lambda(x\|a)$. These are

1. The double complete symmetric functions $h_\lambda(x\|a)$.
2. The double elementary symmetric functions $e_\lambda(x\|a)$.
3. The double Schur functions $s_\lambda(x\|a)$.
4. The double power sums symmetric functions $p_\lambda(x\|a)$.
5. The double monomial symmetric functions $m_\lambda(x\|a)$.
6. The double forgotten symmetric functions $f_\lambda(x\|a)$.

Out of these, the double complete, elementary, and Schur functions will be defined in terms of tableaux, which we do in this section, starting with the definition of the double Schur function. Also, we define the double power sums symmetric functions algebraically. These generators of the ring $\Lambda(x\|a)$ play an important role in the comultiplication on the ring $\Lambda(x\|a)$. The final two bases are not discussed in this thesis, the interested reader is referred to Molev [16, Equations 5.12, 5.13].

2.1.1 The double Schur functions

Fix an integer $n \geqslant 0$. We first define the *double Schur polynomial*, which is a symmetric polynomial in the (finite) variables $x^{(n)} = (x_1, \ldots, x_n)$. For a partition λ, recall that a column strict reverse λ-tableau has entries which weakly decrease along its rows and strictly decrease down its columns. For such a tableau T, we define the weight of one of its entries $T(\alpha)$ to be

$$\mathrm{ev}(T(\alpha)) = x_{T(\alpha)} - a_{T(\alpha)-c(\alpha)},$$

and the weight of T is defined to be

$$\mathrm{ev}(T) = \prod_{\alpha\in\lambda}(x_{T(\alpha)} - a_{T(\alpha)-c(\alpha)}).$$

Definition 2.2. The double Schur polynomial $s_\lambda(x^{(n)}\|a)$ is defined to be

$$\begin{aligned} s_\lambda(x^{(n)}\|a) &= \sum_T \mathrm{ev}(T) \\ &= \sum_T \prod_{\alpha\in\lambda}(x_{T(\alpha)} - a_{T(\alpha)-c(\alpha)}), \end{aligned}$$

summed over all column strict reverse λ-tableaux T with maximum entry n.

Definition 2.3. For each integer $p \geqslant 0$, the *double complete symmetric polynomial* $h_p(x^{(n)} \| a)$ is defined by

$$h_p(x^{(n)} \| a) = s_{(p)}(x^{(n)} \| a),$$

and the *double elementary symmetric polynomial* $e_p(x^{(n)} \| a)$ is defined by

$$e_p(x^{(n)} \| a) = s_{(1^p)}(x^{(n)} \| a),$$

For a partition λ with length l, define

$$h_\lambda(x^{(n)} \| a) = h_{\lambda_1}(x^{(n)} \| a) h_{\lambda_2}(x^{(n)} \| a) \ldots h_{\lambda_l}(x^{(n)} \| a)$$

which we also call a *double complete symmetric polynomial*, and also

$$e_\lambda(x^{(n)} \| a) = e_{\lambda_1}(x^{(n)} \| a) e_{\lambda_2}(x^{(n)} \| a) \ldots e_{\lambda_l}(x^{(n)} \| a)$$

which we will call a *double elementary symmetric polynomial.*

There are two alternate expressions for the double Schur polynomials; they may be expressed as a *ratio of alternants*, see Molev [16, Sec 2.1], or in terms of the Jacobi–Trudi identity, see Macdonald [14, 9th Var.]. To write $s_\lambda(x^{(n)} \| a)$ as a ratio of alternants, set

$$A_\alpha(x^{(n)} \| a) = \det\left((x_i \| a)^{\alpha_j}\right)_{i,j=1}^n, \tag{2.2}$$

for any composition α of length n. Here, the polynomial $(x_i \| a)^0 = 1$ and

$$(x_i \| a)^r = (x_i - a_n)(x_i - a_{n-1}) \ldots (x_i - a_{n-r+1}),$$

for all $r \geqslant 1$.

Let $\delta = (n-1, n-2, \ldots, 0)$. Then the polynomial $s_\lambda(x^{(n)} \| a)$ is the ratio

$$s_\lambda(x^{(n)} \| a) = \frac{A_{\lambda+\delta}(x^{(n)} \| a)}{A_\delta(x^{(n)} \| a)}, \tag{2.3}$$

where $A_\delta(x^{(n)} \| a)$ is equal to the Vandermonde determinant,

$$A_\delta(x^{(n)} \| a) = \prod_{1 \leqslant i < j \leqslant n} (x_i - x_j).$$

This definition shows that the double Schur polynomial is symmetric, since the numerator of equation (2.3) is a alternating polynomial which is a multiple of the denominator.

The double Schur polynomial may also be defined using the *Jacobi–Trudi* identity. Let the ring automorphism $\tau : \Lambda(x^{(n)} \| a) \to \Lambda(x^{(n)} \| a)$ take a_i to a_{i+1} for all integers

i. Let λ be a partition with length l at most equal to n, the number of variables. Then, the double Schur polynomial may be expressed in terms of the double complete symmetric polynomials,

$$s_\lambda(x^{(n)}\|a) = \det(h_{\lambda_i+j-i}(x^{(n)}\|\tau^{j-1}a))_{1\leqslant i,j\leqslant l}. \tag{2.4}$$

The Jacobi–Trudi identiy (2.4) plays a very important role in our calculations in Chapter 3, and we will explore it further there. In particular, we introduce an analogue of the Jacobi–Trudi identity which applies in the ring $\Lambda(x\|a)$ involving the infinite variables x; along the way we will also prove equation 2.4.

Proposition 2.4. *The double Schur polynomial $s_\lambda(x^{(n)}\|a)$ is consistent with the evaluation homomorphism φ_n (2.1),*

$$\varphi_n(s_\lambda(x^{(n)}\|a)) = s_\lambda(x^{(n-1)}\|a).$$

Proof. By definition of $s_\lambda(x^{(n)}\|a)$, we have that

$$\begin{aligned} s_\lambda(x^{(n)}\|a) &= \sum_T \mathrm{ev}(T) \\ &= \sum_T \prod_{\alpha\in\lambda}(x_{T(\alpha)} - a_{T(\alpha)-c(\alpha)}), \end{aligned}$$

summed over all column strict reverse λ-tableaux T with maximum entry n. If T is such a tableau with maximum entry not exceeding $n-1$, then $\varphi_n(\mathrm{ev}(T)) = \mathrm{ev}(T)$, since x_n does not appear in the product $\mathrm{ev}(T)$. The weight of such a T then contributes to the polynomial $s_\lambda(x^{(n-1)}\|a)$. On the other hand, if T contains an entry with integer value n, then the entry $T(1,1) = n$, since entries of T strictly decrease down columns and weakly decrease along rows. It follows then that

$$\varphi_n(\mathrm{ev}(T)) = 0,$$

since

$$\begin{aligned} \varphi_n(\mathrm{ev}(T(1,1))) &= \varphi_n(x_n - a_{n-c(1,1)}) \\ &= a_n - a_n \\ &= 0, \end{aligned}$$

which completes the proof. □

Since by Proposition 2.4 the double Schur polynomials are stable, we may define the *double Schur function* $s_\lambda(x\|a)$, depending on the infinite sequence of variables $x = (x_1, x_2, \dots)$.

Definition 2.5. For a partition λ, the double Schur function $s_\lambda(x\|a)$ is defined to be

$$\begin{aligned} s_\lambda(x\|a) &= \sum_T \operatorname{ev}(T) \\ &= \sum_T \prod_{\alpha\in\lambda} (x_{T(\alpha)} - a_{T(\alpha)-c(\alpha)}), \end{aligned}$$

summed over all column strict reverse λ-tableaux T.

Example 2.6. We give some examples.

First, consider the partition $\lambda = (2,1)$. Then, the double Schur function $s_{(2,1)}(x\|a)$ is given *diagramatically* by

$$s_{(2,1)}(x\|a) = \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 1 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array} + \dots, \tag{2.5}$$

where each diagram represents its own weight in this equation. That is,

$$s_{(2,1)}(x\|a) = (x_2 - a_2)(x_1 - a_0)(x_1 - a_2) + (x_2 - a_2)(x_2 - a_1)(x_1 - a_2) + \dots,$$

where here we have calculated the weight of the two left most diagrams appearing in equation (2.5).

Second, for each integer $p \geqslant 0$, consider the polynomial

$$s_{(p)}(x\|a) = \prod_{i_1 \geqslant \dots \geqslant i_p \geqslant 1} (x_{i_1} - a_{i_1})(x_{i_2} - a_{i_2-1}) \dots (x_{i_p} - a_{i_p-p+1}),$$

which defines the *double complete symmetric function* $h_p(x\|a)$. In addition, the polynomial

$$s_{(1^p)}(x\|a) = \prod_{i_1 > \dots > i_p \geqslant 1} (x_{i_1} - a_{i_1})(x_{i_2} - a_{i_2+1}) \dots (x_{i_p} - a_{i_p+p-1})$$

defines the *double elementary symmetric function* $e_p(x\|a)$.

The corresponding double complete and elementary symmetric functions corresponding to a partition λ of length l are

$$h_\lambda(x\|a) = h_{\lambda_1}(x\|a) \dots h_{\lambda_l}(x\|a)$$

and

$$e_\lambda(x\|a) = e_{\lambda_1}(x\|a) \dots e_{\lambda_l}(x\|a)$$

respectively. □

We now show that the double Schur functions $s_\lambda(x\|a)$, indexed by the partitions, form a basis of the ring $\Lambda(x\|a)$. The proof also applies to the double complete, and double elementary symmetric functions. We only need to assume that the families $\{h_\lambda(x)\}$, $\{e_\lambda(x)\}$ and $\{s_\lambda(x)\}$, indexed by the partitions, form a basis of the classical ring of symmetric functions $\Lambda(x)$. For a proof of this see [15].

Proposition 2.7. *The family of double Schur functions $\{s_\lambda(x\|a)\}$, indexed by all the partitions, forms a basis of $\Lambda(x\|a)$.*

Proof. Fix a integer $n \geqslant 0$. Let the set $\{s_\lambda(x^{(n)}\|a)\}$, for all partitions λ with length not greater than n, denote the family of double Schur polynomials. We first argue that that this family $\{s_\lambda(x^{(n)}\|a)\}$ forms a basis of the ring of double symmetric polynomials $\Lambda(x^{(n)}\|a)$.

Recall that the variables x_i are considered to have degree equal to 1 and a_i are zero degree elements. The top order term of the polynomial $s_\lambda(x^{(n)}\|a)$ is equal to the classical symmetric polynomial $s_\lambda(x^{(n)})$. That is, we may write

$$s_\lambda(x^{(n)}\|a) = s_\lambda(x) + \sum \text{lower order terms in } x_i.$$

We argue by induction on the number of boxes in λ; the base case is when $\lambda = (1)$ and

$$s_{(1)}(x) = s_{(1)}(x^{(n)}\|a) - \sum_{i=1}^{n} a_i.$$

As the induction hypothesis, assume that

$$s_\lambda(x) = s_\lambda(x^{(n)}\|a) + \sum_\rho c_\rho(a) s_\rho(x^{(n)}\|a),$$

summed over partitions ρ with less boxes than λ, and coefficients $c_\rho(a)$ in $\mathbb{Q}[a]$. Then, for a partition λ^+ containing one more box than λ,

$$\begin{aligned} s_{\lambda^+}(x^{(n}) &= s_{\lambda^+}(x^{(n)}\|a) + \sum \text{lower order terms in } x_i \\ &= s_{\lambda^+}(x^{(n)}\|a) + \sum_\rho d_\rho(a) s_\rho(x^{(n)}\|a), \end{aligned}$$

summed over partitions ρ with at most $|\lambda|$ boxes. The second equality follows from the fact that the lower order terms in x_i are symmetric, and therefore can be written in terms of the polynomials $s_\rho(x^{(n)}$ with coefficients from $\mathbb{Q}[a]$. Thus, we have proven that $\{s_\lambda(x^{(n)}\|a)\}$, indexed by all partitions λ, forms a basis of the ring of double symmetric polynomials $\Lambda(x^{(n)}\|a)$.

Now we show that the family of double Schur functions $\{s_\lambda(x\|a)\}$ forms a basis of the ring $\Lambda(x\|a)$. Fix an integer k and let the integer n be much greater than k. Recall that $\Lambda^k(x^{(n)}\|a)$ is the $\mathbb{Q}[a]$-submodule of $\Lambda(x^{(n)}\|a)$ consisting of elements from $\Lambda(x^{(n)}\|a)$ with degree at most k. Consider an element $L \in \Lambda(x\|a)$ with degree at most k. Then L is an infinite sequence

$$L = (L_0, L_1, \dots)$$

such that the element $L_n \in \Lambda^k(x^{(n)}\|a)$ is stable under the evaluation homomorphism φ_n (2.1), i.e.

$$\varphi_n L_n = L_{n-1}$$

For any symmetric polynomial $L_n \in \Lambda^k(x^{(n)}\|a)$ we have

$$L_n = \sum_{\lambda} c_\lambda(a;n) s_\lambda(x^{(n)}\|a),$$

summed over partitions λ with $|\lambda| \leqslant k$ and coefficients $c_\lambda(a;n)$ in the ring $\mathbb{Q}[a]$. Apply φ_n to both sides of this equation to obtain

$$L_{n-1} = \sum_{\lambda} c_\lambda(a;n) s_\lambda(x^{(n-1)}\|a).$$

Since the family $\{s_\lambda(x^{(n-1)}\|a)\}$ form a basis of the ring $\Lambda(x^{(n-1)}\|a)$, we conclude that

$$c_\lambda(a;n) = c_\lambda(a;n-1)$$

and it follows that

$$L = \sum_{\lambda} c_\lambda(a) s_\lambda(x\|a),$$

where $c_\lambda(a) = c_\lambda(a;n)$ for any n which is large enough such that $s_\lambda(x^{(n)}\|a) \neq 0$. This concludes the proof of Proposition 2.7. □

Remark 2.8. The proof of Proposition 2.7 may be modified to show that the families $\{h_\lambda(x\|a)\}$ and $\{e_\lambda(x\|a)\}$, respectively consisting of the double complete and double elementary symmetric functions indexed by partitions λ, form a basis of the ring $\Lambda(x\|a)$.

2.2 Littlewood–Richardson polynomials

It is a well known fact that the classical ring of symmetric functions $\Lambda(x)$ is equipped with a coproduct, the $\mathbb{Q}$-linear ring homomorphism

$$\Delta : \Lambda(x) \to \Lambda(x) \otimes_{\mathbb{Q}} \Lambda(x)$$

defined on the power sums symmetric functions $p_k(x)$ by

$$\Delta(p_k(x)) = p_k(x) \otimes 1 + 1 \otimes p_k(x),$$

see for example Macdonald [15, Chapter 1].

The Littlewood–Richardson coefficients $c^{\nu}_{\lambda\mu}$, as defined in Chapter 1, play the role of structure coefficients in the decomposition of the product of two Schur functions,

$$s_\lambda(x)s_\mu(x) = \sum_\nu c^{\nu}_{\lambda\mu} s_\nu(x),$$

summed over partitions ν such that the number of boxes $|\nu| - |\mu| = |\lambda|$.

Further, these coefficients also occur in the decomposition

$$\Delta(s_\nu(x)) = \sum_{\lambda,\mu} c^{\nu}_{\lambda\mu} s_\lambda(x) \otimes s_\mu(x),$$

summed over partitions λ and ν such that the number of boxes $|\lambda| + |\mu| = |\nu|$.

The ring of double symmetric functions $\Lambda(x^{(n)}\|a)$ is also equipped with a comultiplication operation. In this generalised ring, there are TWO generalisations of the Littlewood–Richardson coefficients. One is a result of the prodct of two double Schur functions. The second results from applying the comultiplication operator to a double Schur function. We introduce both of these coefficients in the subsequent paragraphs.

Let λ, μ and ν be partitions. Define the coefficients $c^{\nu}_{\lambda\mu}(a)$, polynomials in $\mathbb{Q}[a]$, by the following product between two double Schur functions

$$s_\lambda(x\|a)s_\mu(x\|a) = \sum_\nu c^{\nu}_{\lambda\mu}(a) s_\nu(x\|a).$$

The coefficient $c^{\nu}_{\lambda\mu}(a)$ is the *Littlewood–Richardson polynomial*. We describe some properties of the coefficients $c^{\nu}_{\lambda\mu}(a)$.

First, if the coefficient $c^{\nu}_{\lambda\mu}(a)$ is not zero, then it is a polynomial in the ring $\mathbb{Q}[a]$ with degree equal to $|\nu/\mu|$, counting the variables a_i with degree 1. Further, if the partition ν/μ has the same number of boxes as λ, then the coefficient $c^{\nu}_{\lambda\mu}(a)$ is in fact equal to the Littlewood–Richardson coefficient $c^{\nu}_{\lambda\mu}$. This is because for any partition σ the 'top order' term of the double Schur polynomial $s_\lambda(x^{(n)}\|a)$ coincides with the Schur polynomial $s_\lambda(x^{(n)})$, i.e.

$$s_\sigma(x^{(n)}\|a) = s_\sigma(x^{(n)}) + \text{ lower orders terms in } x_i.$$

Second, the coefficients $c^{\nu}_{\lambda\mu}(a)$ obey an important condition known as *positivity.* This means that the coefficient $c^{\nu}_{\lambda\mu}(a)$ can be written as a polynomial in the differences $a_i - a_j$, with positive integer coefficients, such that $i < j$. This positivity condition on the Littlewood–Richardson polynomials was proved from the context of equivariant Schubert calculus by Graham [7]. In the classical case, this implies the Littlewood–Richardson coefficients $c^{\nu}_{\lambda\mu}$ from Section 1.3 are positive integers.

A summary of the applications of the polynomials $c^{\nu}_{\lambda\mu}(a)$ in combinatorics, geometry and representation theory can be found in [18]: under certain specialisations of the sequence a, the polynomials $c^{\nu}_{\lambda\mu}(a)$ arise from a multiplication rule for equivariant Schubert classes [9]. Moreover, under a different specialisation of a_i the polynomials $c^{\nu}_{\lambda\mu}(a)$ give a multiplication rule for virtual quantum immanants and higher Capelli operators [20], [21]. Furthermore, after a shift of variables, the Littlewood–Richardson polynomials become the structure coefficients for the symmetric polynomials in the basis of the generalised factorial Schur functions, first calculated in [19].

2.2.1 The dual Littlewood–Richardson polynomials

We now introduce *dual Littlewood–Richardson polynomials*. First, we have to introduce the double power sums symmetric functions which are generators of the ring $\Lambda(x\|a)$, and also the comultiplication on this ring.

Definition 2.9. For each integer $k \geqslant 0$, the *double power sum symmetric function* $p_k(x\|a)$ is

$$p_k(x\|a) = \sum_{i\geqslant 1}(x_i^k - a_i^k).$$

The ring $\Lambda(x\|a)$ is generated by these functions over the ring $\mathbb{Q}[a]$. Further, these functions play an integral role in the comultiplication structure of the ring $\Lambda(x\|a)$.

Definition 2.10. The *comultiplication* on the ring $\Lambda(x\|a)$ is the $\mathbb{Q}[a]$-linear ring homomorphism

$$\Delta : \Lambda(x\|a) \to \Lambda(x\|a) \otimes_{\mathbb{Q}[a]} \Lambda(x\|a)$$

defined on the double complete symmetric functions $p_k(x\|a)$ by

$$\Delta(p_k(x\|a)) = p_k(x\|a) \otimes 1 + 1 \otimes p_k(x\|a).$$

Then, the *dual Littlewood–Richardson polynomial* $\widehat{c}^{\nu}_{\lambda\mu}(a)$ occurs as the structure coefficient when comultiplication is applied to the double Schur function,

$$\Delta(s_\nu(x\|a)) = \sum_{\lambda,\mu} \widehat{c}^{\nu}_{\lambda\mu}(a) s_\lambda(x\|a) \otimes s_\mu(x\|a), \tag{2.6}$$

summed over partitions λ and μ, see Molev [16, Def 4.1].

We state some properties of the coefficients $\widehat{c}^{\nu}_{\lambda\mu}(a)$. First, when the number of boxes of ν equals $|\lambda| + |\mu|$, the coefficient $\widehat{c}^{\nu}_{\lambda\mu}(a)$ is the Littlewood–Richardson coefficient $c^{\nu}_{\lambda\mu}$. Further, the coefficients $\widehat{c}^{\nu}_{\lambda\mu}(a)$ are nonzero only if the inequality $|\nu| \geqslant |\lambda| + |\mu|$ holds. This fact is not immediately deducible from the definition (2.6), but it follows from one of the alternate definitions of the coefficient $\widehat{c}^{\nu}_{\lambda\mu}(a)$ in Chapter 4.

Understanding the dual Littlewood–Richardson coefficients will shed light on the comultiplication structure of the ring $\Lambda(x\|a)$ and also equivariant cohomology of infinite grassmanians, see recent work by Liou and Schwarz [8]. Further, the coefficients $\widehat{c}_{\lambda\mu}^{\nu}(a)$ describe a multiplication rule between *dual double Schur functions*, and also describe so called *skew double Schur functions* in terms of the usual double Schur functions corresponding to a partition. These applications will be discussed further in Chapter 4, or see [16].

2.2.2 The aims of this thesis

The first aim of this thesis is to provide a new proof of a combinatorial formula for the Littlewood–Richardson polynomials $c_{\lambda\mu}^{\nu}(a)$ using tableaux. We do this in Chapter 3 by first deriving a Pieri rule for the indeterminates $h_{r,s}$ defined in Macdonald's 9th Variation [14], which under specialisation to the ring $\Lambda(x\|a)$ allows us to calculate the product between a double complete symmetric function $h_p(x\|a)$ and a double Schur function $s_\lambda(x\|a)$. We then use the Pieri rule and the Jacobi–Trudi identity (2.4) to prove a rule used to calculate the polynomials $c_{\lambda\mu}^{\nu}(a)$.

The second aim of this thesis is to describe the dual Littlewood–Richardson polynomials $\widehat{c}_{\lambda\mu}^{\nu}(a)$. In [16], formulae are provided which calculate the dual Littlewood–Richardson polynomials. However, these formulae do not shed any light on stability or positivity properties of these coefficients. One of our main results is a refinement of a formula from [16], which we discuss in Chapter 4. The second main result which we discuss in Chapter 5 admits a special case which allows us to calculate $\widehat{c}_{\lambda\mu}^{\nu}(a)$ when the skew partition ν/μ does not contain a 2×2 subdiagram.

3 Raising operators and the Littlewood–Richardson polynomials

Compared to the Littlewood–Richardson coefficients $c_{\lambda\mu}^{\nu}$, there are fewer rules to calculate the Littlewood–Richardson polynomials $c_{\lambda\mu}^{\nu}(a)$ occuring in the decomposition of the product between two double Schur functions

$$s_{\lambda}(x\|a)s_{\mu}(x\|a) = \sum_{\nu} c_{\lambda\mu}^{\nu}(a)s_{\nu}(x\|a).$$

The first Graham positive [7] rule for the polynomial $c_{\lambda\mu}^{\nu}(a)$ was given in [9], from the context of equivariant Schubert calculus, whereas an earlier rule given in [19] lacks the positive property. The rule in [9] was expressed using the combinatorics of puzzles, see [29] for another rule expressed using puzzles.

The first positive rule given in terms of tableaux was independently derived in [10] and [18]. Although the rules given are equivalent, the methods used to derive the rules in [10] and [18] are quite different. In [18], a recurrence relation of [19] is used, whereas [10] generalises a concise proof of the classical Littlewood–Richardson rule by Stembridge [25], which relies on the definition of the Schur polynomial as a ratio of alternants (2.3).

The first main result of this thesis provided in this chapter is a new proof of the combinatorial formula for the polynomials $c_{\lambda\mu}^{\nu}(a)$ given by Molev [18, Theorem 2.1]. We define the action of *raising operators* on the indeterminates h_{rs}, appearing in the 9th Variation of Macdonald [14]. Here, the indeterminates h_{rs} are considered to be generalised complete symmetric functions. They then generate a ring A which we consider to be a general ring of symmetric functions. In this setting, we use the raising operators to produce a general Pieri rule, see Proposition 3.9. This work was inspired by Tamvakis [26]. We follow this by specialising the indeterminates h_{rs} to obtain the double complete symmetric polynomials $h_r(x^{(n)}\|\tau^s a)$, for a $n \geqslant 0$. This polynomial depends on the shift operator τ, where $(\tau a)_i = a_{i+1}$, for all i. From this, we recover the Pieri rule within the ring of double symmetric functions from the general Pieri rule.

To obtain a rule to calculate the polynomials $c_{\lambda\mu}^{\nu}(a)$, we utilise the *Jacobi–Trudi* identity (2.4). Recall that the coefficient $c_{\lambda\mu}^{\nu}(a)$ does not depend on the number of variables n. On the other hand, the Jacobi–Trudi identity (2.4) only defines the Schur polynomials $s_{\lambda}(x^{(n)}\|a)$, depending on n variables. This is because the automorphism τ only makes sense when there are a finite number of variables. We introduce an automorphism ϕ in the ring of double symmetric functions, an analogue of the automorphism τ, which allows us to introduce a version of the Jacobi–Trudi identity for the double Schur functions. This will enable us to provide a new proof of Molev's result [18, Theorem 2.1]. This proof is a more generalised version of the proof of Theorem 1.8. Further, under an appropriate specialisation the double Schur functions specialise to the shifted Schur functions from [20]; in this setting we find a generalisation of the Jacobi–Trudi identity for the shifted Schur functions provided by [20, Theorem 13.1].

The main results in this chapter are set out as follows. Proposition 3.4 establishes the Jacobi–Trudi identity for the Schur polynomials existing in the ring A generated by the indeterminates h_{rs} from Macdonald's 9th Variation. This then allows us to give a generalised Pieri rule for this ring, expressed by Proposition 3.9. The next main result is establishing a Jacobi–Trudi identity for the ring of double Schur functions $\Lambda(x\|a)$, which appears as Theorem 3.12. This relies on relation (3.9). Relation (3.9) also allows us to give a Pieri rule in the ring $\Lambda(x\|a)$, by specialisation of our Pieri rule for the ring A. In particular, we discover coefficients $d_{\sigma'}$, polynomials in the ring $\mathbb{Q}[a]$, which may be cancelled out; this is expressed by Proposition 3.16. To prove Proposition 3.16, we introduce tableaux which have weights which represent the coefficients $d_{\sigma'}$, this then allows us to prove Proposition 3.30, a stronger version of Proposition 3.16. Our Jacobi–Trudi identity, introduced in Theorem 3.12, then allows us to expand the double Schur function as an alternating sum of generalised double complete symmetric functions $h_{\kappa,\mathbf{1}}$. The structure coefficients $K^{\nu}_{\kappa\mu}(a)$ arising in the product between $h_{\kappa,\mathbf{1}}$ and $s_{\mu}(x\|a)$ is then given by Proposition 3.32. Finally, the proof of our main theorem, Theorem 3.33, simplifies the alternation arising from the Jacobi–Trudi identity and gives a new proof of a formula which calculates the coefficients $c^{\nu}_{\lambda\mu}(a)$.

3.1 Raising operators, Macdonald's 9th Variation, and the Pieri rule

We start by giving some preliminary definitions.

Definition 3.1. Let α be an *integer sequence*, which is the sequence $(\alpha_1, \alpha_2, \dots)$ of integers. The sum of a pair of integers sequences α and β is $\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \dots)$. An integer sequence α is called an *integer vector* if it only has a finite number of nonzero entries. Suppose there exist a positive integer l, such that α_l is the right most nonzero entry. Then, l is called the *length* of α, which is denoted by $l(\alpha)$. If α is the sequence of zeroes, then the length of α is set to be 0. We will identify an integer vector α with the finite sequence, $(\alpha_1, \alpha_2, \dots, \alpha_l)$. For an integer sequence α and $k > 0$, we define a *truncation* of α to be the integer vector $\alpha^k = (\alpha_1, \dots, \alpha_k)$. We say that an integer vector α *contains* another integer vector β if $\alpha_i \geqslant \beta_i$ for each i, and denote this by $\alpha \supseteq \beta$.

Thus, a composition is an integer vector such that every entry is nonnegative; a partition is a composition whose entries weakly decrease, reading left to right.

In the 9th Variation of Macdonald [14], a commutative ring A is defined as the ring generated by the independent indeterminates $h_{r,s}$, for $(r \geqslant 1, s \in \mathbb{Z})$, over $\mathbb{Z}$. For convenience, define $h_{0,s} = 1$ and $h_{r,s} = 0$ for all $r < 0$ and all $s \in \mathbb{Z}$. Define an automorphism τ of the ring A generated by the $h_{r,s}$ by $\tau(h_{r,s}) = h_{r,s+1}$ for all r, s. Let μ be an integer vector with $l = l(\mu)$ and β an integer sequence. We will define elements $h_{\mu,\beta}$ of A, corresponding to the pair (μ, β), to be

$$h_{\mu,\beta} = h_{\mu_1,\beta_1} h_{\mu_2,\beta_2} \dots h_{\mu_l,\beta_l}, \tag{3.1}$$

written in this order.

We define *raising operators* R_{st}, $1 \leqslant s < t$, acting on integer sequences $\sigma = (\sigma_1, \sigma_2, \dots)$ by raising the s-th component of σ by 1 and decreasing the t-th component of σ by 1. That is, $R_{st}\sigma = (\dots, \sigma_s + 1, \dots, \sigma_t - 1, \dots)$. These operators were first introduced by Young [28]. An explanation of how Young used these operators is given by Garsia [6].

Example 3.2. Let $\sigma = (0, -1, 2)$. Then the raising operator R_{13} acts on $(0, -1, 2)$ by

$$R_{13}(0, -1, 2) = (1, -1, 1).$$

Let R be a monomial in the raising operators R_{st}. We will also call R a raising operator and say that the operator R acts on the integer vector, integer sequence pair (μ, β) via $R(\mu, \beta) = (R\mu, R\beta)$.

Definition 3.3. The *Schur polynomial* $s_{\mu,\beta}$ corresponding to the pair (μ, β) is the following alternating sum:

$$s_{\mu,\beta} = \prod_{1 \leqslant s < t \leqslant l} (1 - R_{st}) h_{\mu,\beta}, \tag{3.2}$$

which has the following interpretation. Let R be a monomial in the R_{st}'s occurring in the expansion of the product in equation (3.2). Then we let the polynomial $Rh_{\mu,\beta}$ be the polynomial $h_{R\mu,R\beta}$, respecting the order in which the factors of $h_{\mu,\beta} = h_{\mu_1,\beta_1} \dots h_{\mu_l,\beta_l}$ are written in equation (3.1).

In other words, in equation (3.2) the action of the operator R on $h_{\mu,\beta}$ is induced by the action of R on the pair (μ, β). We may express the Schur polynomial in terms of a determinant using the Jacobi–Trudi identity.

Proposition 3.4. *Let* $R_l = \prod_{1\leqslant i<j\leqslant l}(1-R_{ij})$*. The Schur polynomial can be given by the* Jacobi–Trudi identity*:*

$$\begin{aligned} s_{\mu,\beta} &= R_l\,(h_{\mu,\beta}) \\ &= \det(h_{\mu_i+j-i,\beta_i+j-i})_{1\leqslant i,j\leqslant l}. \end{aligned} \tag{3.3}$$

Thus, the way we would like to interpret equation (3.2) is that it is a way of symbolically expanding the determinant in expression (3.3), such that there might be more terms than necessary. We give an example.

Example 3.5. Let $\mu = (2^3)$, and $\beta = (0,1,2)$. Then by equation (3.2) we have that

$$\begin{aligned} s_{\mu,\beta} &= (1-R_{12})(1-R_{13})(1-R_{23})h_{\mu,\beta} \\ &= (1-R_{12}-R_{13}-R_{23}+R_{12}R_{13}+R_{12}R_{23}+R_{13}R_{23}-R_{12}R_{13}R_{23})h_{(2,2,2),(0,1,2)} \\ &= h_{(2^3),(0,1,2)} - h_{(3,1,2),(1,0,2)} - h_{(3,2,1),(1,1,1)} - h_{(2,3,1),(0,2,1)} \\ &\qquad + h_{(4,1,1),(2,0,1)} + h_{(3,2,1),(1,1,1)} + h_{(3,3,0),(1,2,0)} - h_{(4,2,0),(2,1,0)}. \end{aligned}$$

Since the polynomial $-R_{13}h_{(2^3),(0,1,2)} = R_{12}R_{13}h_{(2^3),(0,1,2)}$ we have that these terms cancel in the above expansion, and thus

$$s_{\mu,\beta} = h_{(2^3),(0,1,2)} - h_{(3,1,2),(1,0,2)} - h_{(2,3,1),(0,2,1)} + h_{(4,1,1),(2,0,1)} + h_{(3,3,0),(1,2,0)} - h_{(4,2,0),(2,1,0)},$$

which is equivalent to the determinant of the following matrix:

$$\begin{pmatrix} h_{2,0} & h_{3,1} & h_{4,2} \\ h_{1,0} & h_{2,1} & h_{3,2} \\ h_{0,0} & h_{1,1} & h_{2,2} \end{pmatrix}.$$

□

When $\beta = (0,1,2,\dots)$ and μ is a partition, equation (3.3) is essentially equation (9.1') of Macdonald [14]. We now prove that Proposition 3.4 is true by suitably modifying an argument of Tamvakis [26].

Proof of Proposition 3.4. Consider the ring $\mathbb{B}$ of Laurent polynomials in the (non-commuting) variables $x_{i,k}$, for $i \in \mathbb{Z}, k = 1,2,\dots$, with coefficients in $\mathbb{Z}$. For an integer vector μ with $l = l(\mu)$, and integer sequence β, we let

$$x^{\mu}_{\beta} = x^{\mu_1}_{\beta_1,1}x^{\mu_2}_{\beta_2,2}\cdots x^{\mu_l}_{\beta_l,l}$$

be a monomial. Then a raising operator R acts on this monomial by $Rx^{\mu}_{\beta} = x^{R\mu}_{\beta}$. Note that the second subscript j in each $x_{i,j}$ makes the action of R on x^{μ}_{β} ordered in the same way as with the action of R on $h_{\mu,\beta}$.

Let $\psi_\mu : \mathbb{B} \to \mathbb{A}$ be the $\mathbb{Z}$-linear map which takes x_{nk}^m to $h_{m,m+n-\mu_k}$. For a pair of integers $1 \leqslant i < j$

$$\begin{aligned}\psi_\mu(x_\beta^{R_{ij}\mu}) &= \psi_\mu(x_{\beta_1,1}^{\mu_1} \dots x_{\beta_i,i}^{\mu_i+1} \dots x_{\beta_j,j}^{\mu_j-1} \dots x_{\beta_l,l}^{\mu_l}) \\ &= h_{\mu_1,\beta_1} \dots h_{\mu_i+1,\beta_i+\mu_i+1-\mu_i} \dots h_{\mu_j,\beta_j+\mu_j-1-\mu_j} \dots h_{\mu_l,\beta_l} \\ &= h_{\mu_1,\beta_1} \dots h_{\mu_i+1,\beta_i+1} \dots h_{\mu_j-1,\beta_j-1} \dots h_{\mu_l,\beta_l}.\end{aligned}$$

Since the raising operator R is a monomial in the operators R_{ij}, we may consider $Rh_{\mu,\beta}$ as the image of $x_\beta^{R\mu}$ under the map ψ_μ.

Let $R = R_{ij}$ for a pair (i, j) with $1 \leqslant i < j \leqslant l$. Consider the action of R on x_β^μ. It is equivalent to multiplying x_β^μ by $x_{\beta_i,i}x_{\beta_j,j}^{-1}$, thus we have:

$$\begin{aligned}R_{l(\mu)}x_\beta^\mu &= \prod_{1\leqslant i<j\leqslant l}(1 - R_{ij})x_\beta^\mu \\ &= \prod_{1\leqslant i<j\leqslant l}(1 - x_{\beta_i,i}x_{\beta_j,j}^{-1})x_\beta^\mu \\ &= \det(x_{\beta_i,i}^{\mu_i-i+j})_{1\leqslant i,j\leqslant l},\end{aligned} \tag{3.4}$$

with the last line following from the Vandermonde identity:

$$\prod_{1\leqslant i<j\leqslant l}(x_{\beta_j,j} - x_{\beta_i,i}) = \det(x_{\beta_i,i}^{j-1})_{1\leqslant i,j\leqslant l}.$$

Now apply ψ_μ to both sides of equation (3.4) and equation (3.3) is proven. □

Definition 3.6. For each pair of integers (i, j), $1 \leqslant i < j$ define the operator $\overline{R}_{ij}$ acting on an integer sequence α as follows:

$$\overline{R}_{ij}\alpha = (\alpha_1, \dots, \alpha_{i-1}, \alpha_j - 1, \alpha_{i+1}, \dots, \alpha_{j-1}, \alpha_i + 1, \alpha_{j+1}, \dots).$$

The operator $\overline{R}_{ij}$ swaps the entries in the i-th and j-th place of α and then decreases the i-th entry by 1, and increases the j-th entry by 1. Note that $\overline{R}_{ij}$ is not the inverse of R_{ij}; in fact, it is equivalent to applying R_{ij} and then swapping entries in the i-th and j-th place. Furthermore, $\overline{R}_{ij}$ is an involution on the set of integer sequences.

Example 3.7. Let $\sigma = (0, -1, 2)$. Then the operator $\overline{R}_{13}$ acts on $(3, 0, 2)$ by

$$\begin{aligned}\overline{R}_{23}(3, 0, 2) &= (2 - 1, 0, 3 + 1) \\ &= (1, 0, 4).\end{aligned}$$

□

Proposition 3.8 (Straightening law)**.** *Let μ be an integer vector and β an integer sequence. Let $(\mu', \beta') = (\overline{R}_{i,i+1}\mu, \overline{R}_{i,i+1}\beta)$, for a integer $i < l(\mu)$. Then, $s_{\mu,\beta} = -s_{\mu',\beta'}$.*

Proof. The Schur polynomial $s_{\mu,\beta} = \det A$ for some matrix A as defined from equation (3.3). Similarly, $s_{\mu',\beta'} = \det B$ for some matrix B. Then we may obtain B from A by swapping row i and $i+1$ of A. □

For the rest of this section, fix an integer sequence β. Let λ and μ be compositions. From now on in this chapter we define h_λ to be the polynomial $h_\lambda = h_{\lambda,\beta}$, and s_μ to be the polynomial $s_\mu = s_{\mu,\beta}$. Suppose $\lambda = (0, \dots, 0, p)$, a composition with length e and let $l = l(\mu)$. We wish to compute $h_\lambda s_\mu$, the product equal to $h_{p,\beta_e} s_{\mu,\beta}$ and write this product as a sum of the Schur polynomials. Recall the truncated integer vector $\beta^f = (\beta_1, \beta_2, \dots, \beta_f)$, for all $f > 0$. Denote by (μ, p) and (β^l, β_e) the concatenations $(\mu, p) = (\mu_1, \dots, \mu_l, p)$ and $(\beta^l, \beta_e) = (\beta_1, \dots, \beta_l, \beta_e)$. We have that

$$\begin{aligned} h_{p,\beta_e} s_\mu &= h_{p,\beta_e} R_l h_{\mu,\beta} \\ &= R_l\{h_{\mu,\beta} h_{p,\beta_e}\} \\ &= R_{l+1} \prod_{1 \leqslant i \leqslant l} (1 + R_{i,l+1} + R_{i,l+1}^2 + R_{i,l+1}^3 + \dots)\{h_{(\mu,p),(\beta^l,\beta_e)}\} \qquad (3.5) \\ &= R_{l+1} \sum_R R h_{(\mu,p),(\beta^l,\beta_e)}, \end{aligned}$$

summed over monomials R in the $R_{i,l+1}$'s, for all $1 \leqslant i \leqslant l$. The fourth equality holds because of two reasons. The first, because

$$(1 + R_{i,l+1} + R_{i,l+1}^2 + \dots)(1 - R_{i,l+1}) = 1.$$

The second reason is that the action of $R_{i,l+1}$ on the indeterminate h_{ij}, $1 \leqslant i \leqslant l$, is nilpotent for all j, that is for all integers $m > p$, the operator $R_{i,l+1}^m$ reduces the first index of the indeterminate h_{p,β_e} to a negative integer when $R_{i,l+1}^m$ is applied to the element $h_{(\mu,p),(\beta^l,\beta_e)}$. This leads to the Pieri rule.

Proposition 3.9 (Pieri rule)**.** *Let μ be a partition with $l = l(\mu)$, let p be a positive integer and e be an integer. Let $\beta' = (\beta^l, \beta_e - p)$. Then*

$$h_{p,\beta_e} s_\mu = \sum_\sigma s_{\mu+\sigma,\beta'+\sigma},$$

summed over compositions σ such that σ has p boxes and has length at most $l+1$.

Proof. From equation 3.5 we have that $h_{p,\beta_e} s_\mu = R_{l+1} \sum_R R h_{(\mu,p),(\beta^l,\beta_e)}$, summed over monomials R in the $R_{i,l+1}$'s, for all $1 \leqslant i \leqslant l$. For such a R, we have that

$$R h_{(\mu,p),(\beta^l,\beta_e)} = h_{\mu+\sigma,\beta'+\sigma},$$

for some σ such that σ has p boxes and has length at most $l+1$. That is, R acts on both the compositions (μ, p) and (β^l, β_e) by removing up to p boxes from row $l+1$ of each composition and adding them to the previous rows. The proposition follows. □

Example 3.10. Let $\mu = (1^2)$ and $\beta = (0, 1, 2, \dots)$. Then, the Pieri rule allows us to expand the product

$$h_{1,1} s_\mu = \sum_{\substack{\sigma, \\ |\sigma|=1}} s_{\mu+\sigma, \beta'+\sigma},$$

where $\beta' = (0, 1, 0)$. Thus,

$$h_{1,1} s_{(1,1),(0,1)} = s_{(2,1),(1,1)} + s_{(1,2),(0,2)} + s_{(1,1,1),(0,1,1)}.$$

□

3.2 Pieri rule: after specialisation

Define the integer vector $\mathbf{1} = (0, 1, 2, \dots)$. We will define the double Schur functions by fixing $\beta = \mathbf{1}$ and choosing a specialisation of the $h_{r,s}$ from the previous section. For any $n > 0$, we let $h^{(n)}_{r,s} = h_r(x^{(n)} \| \tau^s a)$, recalling the finite sequence of variables $x^{(n)} = (x_1, \dots, x_n)$ and the double complete symmetric polynomial

$$h_k(x^{(n)} \| a) = \sum_{n \geqslant i_1 \geqslant \dots \geqslant i_k \geqslant 1} (x_{i_1} - a_{i_1})(x_{i_2} - a_{i_2 - 1}) \dots (x_{i_k} - a_{i_k - k + 1}).$$

This means the ring A from the previous section specialises to $\Lambda(x^{(n)} \| a)$, when we set $h_{r,s} = h^{(n)}_{r,s}$ for a choice of $n > 0$. We have that the automorphism τ is the same as the operation on the sequence a such that $(\tau a)_i = a_{i+1}$ for all $i \in \mathbb{Z}$.

Proposition 3.11. *For all $r \geqslant 1$, we have the following relation between $h_{r,s}$ and $h_{r,s-1}$:*

$$h^{(n)}_{r,s} = h^{(n)}_{r,s-1} + (a_{s-r+1} - a_{n+s}) h^{(n)}_{r-1,s-1}. \tag{3.6}$$

Proof. This result can be directly calculated, or inferred from Molev [18, Lemma 2.4]. □

Let λ be a partition with $l = l(\lambda)$. Recall the integer vector $\mathbf{1}^l = (0, \dots, l-1)$ and one of the ways of defining the *double Schur polynomials* $s_\lambda(x_1, \dots, x_n \| a)$ is by specialising the Jacobi–Trudi identity (equation 3.3):

$$\begin{aligned} s_\lambda(x_1, \dots, x_n \| a) &= \det \left(h^{(n)}_{\lambda_i + j - i, j-1} \right)_{1 \leqslant i,j \leqslant l} \\ &= R_l h^{(n)}_{\lambda, \mathbf{1}^l} \end{aligned} \tag{3.7}$$

where here we mean that $h^{(n)}_{\lambda,\mathbf{1}^l} = h^{(n)}_{\lambda_1,0} \dots h^{(n)}_{\lambda_l,l-1}$. The last equality holds because we may use raising operators to rewrite the determinant. Equation (3.7) is precisely the Jacobi–Trudi identity given in Chapter 2, equation (2.4).

For partitions λ, μ and ν, the *Littlewood–Richardson polynomials* $c^{\nu}_{\lambda\mu}(a;n)$ are defined as the structure coefficients in the following expansion

$$s_\lambda(x^{(n)}\|a)s_\mu(x^{(n)}\|a) = \sum_{\nu} c^{\nu}_{\lambda\mu}(a;n)s_\nu(x^{(n)}\|a), \tag{3.8}$$

summed over partitions ν. Recall also that the polynomials $c^{\nu}_{\lambda\mu}(a)$ are defined to be the structure coefficients in the expansion between the product of two double Schur functions,

$$s_\lambda(x\|a)s_\mu(x\|a) = \sum_{\nu} c^{\nu}_{\lambda\mu}(a)s_\nu(x\|a),$$

summed over partitions ν.

The polynomials $c^{\nu}_{\lambda\mu}(a)$ and $c^{\nu}_{\lambda\mu}(a;n)$ are polynomials in $\mathbb{Z}[a]$, and in the case of $c^{\nu}_{\lambda\mu}(a;n)$, are dependent on n. However, since the polynomial $s_\lambda(x^{(n)}\|a)$ is stable under the homormorphisms (2.1), we have that $c^{\nu}_{\lambda\mu}(a;n)$ does not depend on n when n is big enough. This is a remarkable fact which means that for such a sufficiently big n the coefficient $c^{\nu}_{\lambda\mu}(a) = c^{\nu}_{\lambda\mu}(a;n)$. Our main aim is to use the Jacobi–Trudi identity (3.7) to calculate $c^{\nu}_{\lambda\mu}(a)$. We first sketch a way in which $c^{\nu}_{\lambda\mu}(a;n)$ from equation (3.8) may be calculated. First expand the determinant in equation (3.7) to obtain alternating summands consisting of products of double complete symmetric polynomials. Then calculate the product of each of these summands with $s_\mu(x^{(n)}\|a)$, and using the Pieri rule, decompose the result in terms of the double Schur polynomials. Hence each alternating summand from the expansion of (3.7) contributes to the polynomial $c^{\nu}_{\lambda\mu}(a;n)$.

In the classical case such a calculation for the Littlewood–Richardson coefficients has been explored; see [5, 23]. These authors used a 'sign reversing involution' to simplify the contribution of the alternating summands appearing in equation (3.7) to the Littlewood–Richardson coefficient. We will use this idea to calculate $c^{\nu}_{\lambda\mu}(a)$. This method motivates the following definition of ϕ, an automorphism of $\Lambda(x\|a)$, which applies to $h_p(x\|a) \in \Lambda(x\|a)$ in the following way:

$$\phi h_p(x\|a) = h_p(x\|a) + a_{2-p}h_{p-1}(x\|a), \tag{3.9}$$

for all $p \geqslant 1$. With this definition of ϕ, the Jacobi–Trudi identity (3.7) may be reinterpreted for $\Lambda(x\|a)$.

Theorem 3.12. *Recall the infinite sequence of variables $x = (x_1, x_2, \dots)$. Let λ be a partition with length l. Then the double Schur function may be given in terms of the* *double complete symmetric functions:*

$$s_\lambda(x\|a) = \det\left(\phi^{j-1}h_{\lambda_i+j-i}(x\|a)\right)_{1\leqslant i,j\leqslant l} \tag{3.10}$$

Proof. Fix a $n \geqslant 0$ and a partition λ. Using the Jacobi–Trudi equation (3.7) for the functions $s_\lambda(x^{(n)}\|a)$ and relation (3.6) allows us to write the double Schur function $s_\lambda(x^{(n)}\|a)$ completely in terms of the double complete symmetric functions,

$$s_\lambda(x^{(n)}\|a) = \sum_\sigma d_{\lambda,\sigma}(n)h_\sigma(x^{(n)}\|a),$$

summed over partitions σ and for some coefficients $d_{\lambda,\sigma}(n) \in \mathbb{Q}[a]$.

Now we rely on the fact that the double Schur functions and double complete symmetric functions form a basis of the ring $\Lambda(x\|a)$, see Proposition 2.7 and Remark 2.8. Since the polynomials $s_\lambda(x^{(n)}\|a)$ and and $h_\sigma(x^{(n)}\|a)$ are both consistent under the homomorphisms $\varphi_n : x_n \mapsto a_n$ (2.1), for any n large enough so that $h_\sigma(x^{(n)}\|a) \neq 0$, the coefficients $d_{\lambda,\sigma}(n)$ must be elements of the transition matrix between the bases of symmetric functions $\{s_\lambda(x\|a)\}$ and $\{h_\lambda(x\|a)\}$, indexed by partitions λ. Hence, these coefficients are polynomials in $\mathbb{Q}[a]$ which are independent of any n, and therefore the terms in relation (3.6) which contribute factors depending on n does not contribute to the expression for $s_\lambda(x\|a)$ and equation (3.10) must hold.

□

As a corollary of Theorem 3.12 we recover the Jacobi–Trudi identity for the shifted Schur functions defined in [20].

Corollary 3.13. *Let μ be a partition with length l. The following Jacobi–Trudi identity holds for the shifted Schur function s^*_μ from [20]*

$$s^*_\mu = \det\left(\phi^{j-1}h^*_{\mu_i-i+j}\right)_{1\leqslant i,j\leqslant l}$$

where ϕ is the automorphism on the generators defined by $\phi(h^*_k) = h^*_k + (k-t)h^*_{k-1}$, for all $k \geqslant 1$ and a constant $t \in \mathbb{Q}$.

Proof. In particular, when $t = 1$ this is Theorem 13.1 of [20]. Under the specialisiation $a_i = -i + t$, for all $i \in \mathbb{Z}$ and $x_i = y_i - i + t$, the double Schur function $s_\mu(x\|a)$ becomes the shifted Schur function s^*_μ from [20]. The corollary follows from this specialisation. □

We give an example which applies Theorem 3.12.

Example 3.14. Let λ be a partition with length 2. Then according to equation (3.7), we have

$$s_\lambda(x^{(n)}\|a) = \det \begin{pmatrix} h^{(n)}_{\lambda_1,0} & h^{(n)}_{\lambda_1+1,1} \\ h^{(n)}_{\lambda_2-1,0} & h^{(n)}_{\lambda_2,1} \end{pmatrix}.$$

The proof of Theorem 3.12 asserts that we may cancel the dependence on n. Indeed, using relation (3.6),

$$\det \begin{pmatrix} h^{(n)}_{\lambda_1,0} & h^{(n)}_{\lambda_1+1,1} \\ h^{(n)}_{\lambda_2-1,0} & h^{(n)}_{\lambda_2,1} \end{pmatrix} = \det \begin{pmatrix} h^{(n)}_{\lambda_1,0} & h^{(n)}_{\lambda_1+1,0} + a_{1-\lambda_1} h^{(n)}_{\lambda_1,0} \\ h^{(n)}_{\lambda_2-1,0} & h^{(n)}_{\lambda_2,0} + a_{2-\lambda_2} h^{(n)}_{\lambda_2-1,0} \end{pmatrix}$$
$$- \det \begin{pmatrix} h^{(n)}_{\lambda_1,0} & a_{n+1} h^{(n)}_{\lambda_1,0} \\ h^{(n)}_{\lambda_2-1,0} & a_{n+1} h^{(n)}_{\lambda_2-1,0} \end{pmatrix},$$

and we see that the second determinant on the right is equal to 0. □

From now onwards, we will work with the double Schur functions and double complete symmetric functions, and use relation (3.9) in our calculations; for the rest of this section let $h_{r,s}$ denote $\phi^s h_r(x\|a)$ and $s_\lambda = s_\lambda(x\|a)$, given by equation (3.10).

Let μ be a partition and let $l = l(\mu)$. Recall that $h_{\mu,\beta} = h_{\mu_1,\beta_1} \dots h_{\mu_l,\beta_l}$. Let $\mathbf{1}'$ denote the concatenation $(\mathbf{1}^l, e-p) = (0, 1, \dots, l-1, e-p)$. From Proposition 3.9, we have the corresponding Pieri rule for $\Lambda(x\|a)$:

$$\begin{aligned} h_{p,e} s_\mu &= \sum_\sigma s_{\mu+\sigma, \mathbf{1}'+\sigma} \\ &= \sum_\sigma R_{l+1} h_{\mu+\sigma,\mathbf{1}'+\sigma} \end{aligned}$$

summed over compositions σ with p boxes, and of at most length $l+1$. We use relation (3.9) to rewrite

$$\sum_\sigma h_{\mu+\sigma,\mathbf{1}'+\sigma} = \sum_{\sigma'} d_{\sigma'}(a) h_{\mu+\sigma',\mathbf{1}^{l+1}}, \tag{3.11}$$

summed over compositions $\sigma' \subseteq \sigma$, for all σ with p boxes and of at most length $l+1$. Here, the coefficient $d_{\sigma'}(a)$ is a polynomial in $\mathbb{Z}[a]$. Thus,

$$\begin{aligned} h_{p,e} s_\mu &= R_{l+1} \sum_{\sigma'} d_{\sigma'}(a) h_{\mu+\sigma',\mathbf{1}^{l+1}} \\ &= \sum_{\sigma'} d_{\sigma'}(a) s_{\mu+\sigma',\mathbf{1}^{l+1}} \end{aligned} \tag{3.12}$$

Example 3.15. We continue with Example 3.10, where we calculated the product

$$h_{1,1} s_{(1,1),(0,1)} = s_{(2,1),(1,1)} + s_{(1,2),(0,2)} + s_{(1,1,1),(0,1,1)}.$$

Let as examine the first double Schur function appearing in this product. We have

$$s_{(2,1),(1,1)} = R_2 h_{(2,1),(1,1)},$$

but

$$\begin{aligned} h_{(2,1),(1,1)} &= h_{2,1}h_{1,1} \\ &= h_{2,0}h_{1,1} + a_0 h_{1,0}h_{1,1} \\ &= h_{(2,1),(0,1)} + a_0 h_{(1,1),(0,1)}, \end{aligned}$$

where we obtain the second line of the equation from the first by applying relation (3.9) on the polynomial $h_{2,1}$; i.e.

$$h_{2,1} = h_{2,0} + a_0 h_{1,0}.$$

Thus,

$$\begin{aligned} s_{(2,1),(1,1)} &= R_2(h_{(2,1),(0,1)} + a_0 h_{(1,1),(0,1)}) \\ &= s_{(2,1)} + a_0 s_{(1,1)}. \end{aligned}$$

Similarly,

$$s_{(1,2),(0,2)} = s_{(1,2)} + a_1 s_{(1,1)},$$

and

$$s_{(1,1,1),(0,1,1)} = s_{(1^3)} - a_2 s_{(1^2)}.$$

In the last example we exploit the fact that

$$h_{1,2} = h_{1,1} + a_2 h_{0,1},$$

or equivalently

$$h_{1,1} = h_{1,2} - a_2 h_{0,1}.$$

The processes of using relation (3.9) to rewrite the Schur functions $s_{\lambda,\mathbf{1}}$ will be formalised by relations (3.15) and (3.18). □

We now simplify equation (3.12).

Proposition 3.16. *Let μ be a partition, and $\kappa = (0, \ldots, 0, p)$, a composition of length $e+1$; in this case, recall that $h_\kappa = h_{\kappa,\mathbf{1}} = h_{p,e}$. We claim that*

$$h_\kappa s_\mu = \sum_{\sigma'} d_{\sigma'}(a) s_{\mu+\sigma',\mathbf{1}^{l+1}} = \sum_{\nu} c^{\nu}_{\kappa\mu}(a) s_\nu(x\|a) \tag{3.13}$$

where the left is summed over compositions σ' with p boxes and the right is summed over partitions ν such that ν/μ is a horizontal strip with at most p boxes, with coefficients $c^{\nu}_{\kappa\mu}(a)$ in the ring $\mathbb{Q}[a]$.

The coefficients $c^{\nu}_{\kappa\mu}(a)$ appearing here are given in Proposition 3.30. The claim we make here is that when $\sum_{\sigma'} d_{\sigma'}(a)s_{\mu+\sigma',\mathbf{1}^{l+1}}$ is simplified the only nonzero coefficients which remain correspond to the functions $s_\nu(x\|a)$ for ν/μ a horizontal strip with at most p boxes. When a is specialised to the sequence of zeroes and κ is the partition $\kappa = (p)$, Proposition 3.16 is the usual Pieri rule in the classical ring of symmetric functions, which says that

$$h_p s_\mu(x) = \sum_\nu s_\nu(x),$$

summed over partitions ν such that ν/μ is a horizontal strip containing p-boxes.

We begin proving Proposition 3.16. Recall from equation (3.11) that when we consider the product $h_{p,e}s_\mu$ we obtain

$$\sum_\sigma h_{\mu+\sigma,\mathbf{1}^l+\sigma} = \sum_{\sigma'} d_{\sigma'}(a)h_{\mu+\sigma',\mathbf{1}^{l+1}}.$$

summed over compositions σ with p boxes and σ' contained in σ. If ν and μ are partitions and σ' a composition such that $\nu = \mu+\sigma'$, then call σ' a *good* composition if ν/μ is a horizontal strip, and *bad* otherwise. Then, we claim that $d_{\sigma'}(a) = c^{\nu}_{\kappa\mu}(a)$ when σ' is good; furthermore, if σ' is bad then $c^{\nu}_{\kappa\mu}(a) = 0$ and the contributions of $d_{\sigma'}(a)s_{\mu+\sigma',\mathbf{1}^{l+1}}$ for all bad σ''s will cancel on the left hand side of equation (3.13). The aim of the rest of this section is to prove these claims, and provide a way of calculating $d_{\sigma'}(a)$. We will do this using a sequence of lemmas.

We first calculate $d_{\sigma'}(a)$. It will be useful to think of our compositions as diagrams. Letting $l = l(\mu)$, for a $1 \leqslant j \leqslant l$, observe that if σ_j contains m boxes then m boxes are added to row j of μ when we form $\mu+\sigma$. Then, we will have the pair $(\mu_j+m, j-1+m)$ appearing as the j-th entry in the pair $(\mu+\sigma, \mathbf{1}^l+\sigma)$. This j-th entry corresponds to the polynomial $h_{\mu_j+m,j-1+m}$. To obtain the right hand side of equation (3.11) we wish to reduce the second index down to $j-1$.

Lemma 3.17. *We have that*

$$h_{\mu_j+m,j-1+m} = \sum_{d=0}^{m} K_d(a)h_{\mu_j+m-d,j-1}, \tag{3.14}$$

where $K_d(a)$ is the following degree d polynomial in $\mathbb{Z}[a]$:

$$K_d(a) = \sum_{b_1,\ldots,b_d} \prod_{i=1}^{d} a_{b_i-\mu_j-m}, \tag{3.15}$$

summed over integers b_i such that $j+d \leqslant b_d \leqslant \ldots \leqslant b_1 \leqslant j+m$.

Proof. Using relation (3.9),

$$\begin{aligned} h_{\mu_j+m,j-1+m} &= h_{\mu_j+m,j-1+m-1} + a_{j-\mu_j} h_{\mu_j+m-1,j-1+m-1} \\ &= h_{\mu_j+m,j-1} + \sum_{d=0}^{m-1} a_{j-m-\mu_j+d+1} h_{\mu_j+m-1,j-1+d}, \end{aligned} \tag{3.16}$$

and doing the same to $h_{\mu_j+m-1,j-1+d}$, and so on, we have

$$h_{\mu_j+m,j-1+m} = \sum_{d=0}^{m} K_d(a) h_{\mu_j+m-d,j-1},$$

where $K_d(a)$ is the following degree d polynomial in $\mathbb{Z}[a]$:

$$K_d(a) = \sum_{b_1,\ldots,b_d} \prod_{i=1}^{d} a_{b_i-\mu_j-m},$$

summed over integers b_i such that $j+d \leqslant b_d \leqslant \ldots \leqslant b_1 \leqslant j+m$. □

For the case where σ_{l+1} contains $p-k$ boxes, for a $0 \leqslant k \leqslant p$, we have the pair $(p-k, e-k)$ as the $l+1$-th entry in the pair $(\mu+\sigma, \mathbf{1}^l+\sigma)$. Since we assumed that e is smaller than l, the length of μ, we have that $e-k < l$. To obtain the right hand side of equation (3.11), we want this second index to be l.

Lemma 3.18. *We have that*

$$h_{p-k,e-k} = \sum_{d=0}^{p-k} G_d(a) h_{p-k-d,l} \tag{3.17}$$

where $G_d(a)$ is the following polynomial in $\mathbb{Z}[a]$ with degree d

$$G_d(a) = \sum_{b_1,\ldots,b_d} \prod_{i=1}^{d} (-1)^d a_{b_i-p+k+i+1}, \tag{3.18}$$

summed over integers b_i such that $e-k \leqslant b_1 \leqslant \ldots \leqslant b_d \leqslant l-1$.

Proof. We may use relation (3.9) to write

$$h_{p-k,e-k} = h_{p-k,l} - \sum_{d=e-k}^{l-1} a_{d-p+k+2} h_{p-k-1,d},$$

and doing the same to $h_{p-k-1,d}$ and so on, we can write

$$h_{p-k,e-k} = \sum_{d=0}^{p-k} G_d(a) h_{p-k-d,l}$$

where $G_d(a)$ is the following polynomial in $\mathbb{Z}[a]$ with degree d

$$G_d(a) = \sum_{b_1,\ldots,b_d} \prod_{i=1}^{d} (-1)^d a_{b_i - p + k + i + 1},$$

summed over integers b_i such that $e - k \leqslant b_1 \leqslant \ldots \leqslant b_d \leqslant l - 1$. $\square$

We will now eliminate bad compositions from the sum on the left hand side of equation (3.13), using tableaux which we define in the next section.

3.3 Using tableaux to calculate $d_{\sigma'}(a)$

We deduce from our calculations for $K_d(a)$ and $G_d(a)$ that if σ is a composition which contains the composition σ', then $h_{\nu,\mathbf{1}+\sigma}$ from equation (3.11) will contribute to a summand in $d_{\sigma'}(a)$. We will make the calculation of $d_{\sigma'}$ more precise using tableaux, which we now define. An example of all the following definitions exists at the end of this section.

Let κ be a composition. Recall that a *reverse κ-tableau* T of shape κ is obtained by filling each box of κ with a positive integer k which is unbarred, or a positive integer $\overline{k}$ which is barred; further, in each row of T, the entries weakly decrease from left to right. Note that we do not impose any conditions on the columns of T. The following definitions are all *associated* to the tableau T. If $\alpha = (i, j)$ is a box of κ, recall that we call $T(\alpha) = T(i, j)$ the *entry* of T in box α, and the *content* of box α is $c(\alpha) = c(i, j) = j - i$.

The *row order* is the ordering on the boxes of κ obtained by reading the boxes in rows from bottom to top, from left to right of each row. We extend this ordering to the entries of a tableau T of shape κ. Let α and β be two boxes of κ. Then $T(\alpha)$ is *before* $T(\beta)$ with respect to the row order if the box α is before β with respect to the row order.

Similarly, we define the *column order* on the boxes of κ by reading the boxes in columns left to right, from the bottom to the top of each column. Similarly, we will say $T(\alpha)$ is *before* $T(\beta)$ with respect to the column order if the box α is before β with respect to the column order.

Let μ be a composition and let $S = s_1 s_2 \ldots s_t$, be a sequence of positive integers. Recall that we *apply* S to μ by forming a sequence of compositions from μ which terminates in a composition π in the following way:

$$\mu = \rho^{(0)} \xrightarrow{s_1} \rho^{(1)} \xrightarrow{s_2} \ldots \xrightarrow{s_t} \rho^{(t)} = \pi,$$

such that $\rho^{(i)}$, $i = 1, \ldots, t$, are compositions and $\rho^{(i-1)} \xrightarrow{s_i} \rho^{(i)}$ means adding a box to the end of row s_i of $\rho^{(i-1)}$ to form $\rho^{(i)}$. We say that S *takes* μ to π (or π is *created* from

μ using S), and denote this as $S : \mu \to \pi$. We will say that S is *Yamanouchi* when applied to μ if $\rho^{(i)}$ is a partition for all $0 \leqslant i \leqslant t$, and *not Yamanouchi* otherwise.

If T is a reverse κ-tableau, we define the *row* and *column* word of T. The *row word S^r corresponding* to T is the sequence of barred entries in T listed left to right, from the first barred entry to the last with respect to the row order. Similarly, the *column word S^c corresponding* to T is the sequence of barred entries in T listed left to right, from the first to the last with respect to the column order. When writing the row and column words corresponding to T we will omit the bars. For every $\alpha = (i, j) \in \kappa$, let the (sub)word $S^r(\alpha) = S^r(i, j)$ be the subsequence of the word S^r consisting of the barred entries in T listed up to, and including box α, with respect to the row order. Let $S^c(\alpha) = S^c(i, j)$ be the subsequence of S consisting of the barred entries in T listed up to, and including box α, with respect to the column order. We will let $\rho^r(\alpha) = \rho^r(i, j)$ (resp. $\rho^c(\alpha) = \rho^c(i, j)$) be the composition created from μ using $S^r(\alpha)$ (resp. $S^c(\alpha)$).

For each $\alpha \in \kappa$, let $\rho(\alpha)$ be a composition. The set of compositions $\{\rho(\alpha)\}_{\alpha\in\lambda}$ is the *labelling* on the boxes of λ. We use the labelling of λ to define the *weight* of an entry $T(\alpha)$ denoted by $\mathrm{ev}(T(\alpha))$,

$$\mathrm{ev}(T(\alpha)) = a_{T(\alpha)-\rho(\alpha)_{T(\alpha)}} - a_{T(\alpha)-c(\alpha)}.$$

The *weight* of the tableau T is the weight of all unbarred entries of T multiplied together, denoted:

$$\mathrm{ev}(T) = \prod_{\substack{\alpha\in\kappa \\ \alpha \text{ unbarred}}} (a_{T(\alpha)-\rho(\alpha)_{T(\alpha)}} - a_{T(\alpha)-c(\alpha)}).$$

Example 3.19. Let $\kappa = (3, 5, 2)$, which corresponds to the diagram

.

We have the row and column ordering on the boxes of κ, which we illustrate in the following two diagrams by filling in the boxes of κ with integers so that the first box with respect to the ordering is labelled '1' and so on

7	8	9		
2	3	4	5	6
1				

row ordering

3	5	7		
2	4	6	8	9
1				

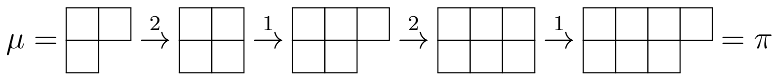
column ordering

Consider the following reverse κ-tableau T of shape κ:

$T =$

2	$\bar{2}$	$\bar{1}$			
2	2	$\bar{2}$	1	1	$\bar{1}$
1					

Then, the row word $S^r = 2121$ of T is the sequence of barred integers of T, listed with respect to the row order, and the column word $S^c = 2211$ of T is the sequence of barred integers of T, listed with respect to the column order. Let $\mu = (2, 1)$ be a partition. Then the row word S^r of T takes μ to $\pi = (4, 3)$ via the following sequence of compositions:



Since all of these are partitions, we say that S^r is Yamanouchi when applied to μ. For the box $(1, 2)$ we have the subword $S^r(1, 1) = 212$, the sequence of barred integers of T listed up to box $(2, 1)$ with respect to the row order. Correspondingly, the partition $\rho^r(1, 2)$ is the partition $(3, 3)$. We leave it to the reader to check that S^c takes $(2, 1)$ to $(4, 3)$ but is not Yamanouchi.

For the purposes of the weight of the entry $T(2, 4)$, let the labelling of the composition κ be $\rho(\alpha) = \rho^c(\alpha)$. Then, the weight of $T(2, 4)$ is

$$\begin{aligned} \mathrm{ev}(T(2,4)) &= a_{1-\rho^c(2,4)_1} - a_{1-c(2,4)} \\ &= a_{-2} - a_{-1}. \end{aligned}$$

□

3.4 Weights of tableaux express the coefficients $d_{\sigma'}(a)$

For the rest of this section let the composition $\kappa = (0, \ldots, 0, p)$ and have length $e + 1$ and assume that e is strictly less than l, the length of the partition μ. We will use reverse κ-tableaux to make sense of equation (3.11) and the coeffiecients $d_{\sigma'}(a)$ appearing in it. Recall that $d_{\sigma'}(a)$ is the coefficient of $h_{\mu+\sigma',\mathbf{1}^{l+1}}$ when we rewrite (3.11).

Lemma 3.20. *We claim that*

$$d_{\sigma'}(a) = \sum_T \mathrm{ev}(T), \tag{3.19}$$

summed over all reverse κ-tableaux T such that T has row word $S^r : \mu \to \mu + \sigma'$.

Example 3.21. For example, suppose $\sigma' = \sigma$ is a composition with p boxes, and is at most length $l+1$. Then, there is a unique reverse κ-tableau T with row word S^r such that $S^r : \mu \to \mu + \sigma$, so our claim in this instance is that $d_\sigma = 1$. When $\mu + \sigma'$ is a partition, then this is in fact just the classical Pieri rule, see for example Proposition 1.5. □

We will prove the claim in Lemma 3.20 for all compositions $\sigma' \subseteq \sigma$ in the following paragraphs.

Let T be a reverse κ-tableau, with row word $S^r : \mu \to \mu + \sigma'$, for a $\sigma' \subseteq \sigma$. Let us expand the weight of T in terms of monomials. We create a tableau U, *derived* from T, by doing the following: for every unbarred entry $x \in T$ we will either leave it unbarred, or put a prime on it, that is, we replace the entry x with x'. The tableau U inherits the definitions associated with T: for example, S^r, $\rho^r(\alpha)$ etc. The weight of an unbarred entry $U(\alpha)$ is then taken to be $a_{T(\alpha)-\rho^r(\alpha)_{T(\alpha)}}$, and the weight of a unbarred primed entry $U(\alpha)$ is taken to be $-a_{|T(\alpha)|-c(\alpha)}$, where $|T(\alpha)|$ means disregard the prime on the entry in box α. Thus we see that the weight of T may be expressed as

$$\mathrm{ev}(T) = \sum_{U} \prod_{\substack{\alpha\in\kappa,\\ T(\alpha)\text{ unbarred, unprimed}}} a_{T(\alpha)-\rho^r(\alpha)_{T(\alpha)}} \prod_{\substack{\alpha\in\kappa,\\ T(\alpha)\text{ unbarred, primed}}} -a_{|T(\alpha)|-c(\alpha)},$$

summed over all monomial tableaux U derived from T.

We now prove that Lemma 3.20 is true; that is, each monomial appearing in the coefficient $d_{\sigma'}(a)$ (3.12) may be given by the weight of a tableau U derived from a reverse κ-tableau T with row word $S^r : \mu \to \mu + \sigma'$. First, we provide a refinement of Lemma 3.20, by showing we only need to consider *good* monomial tableaux.

Definition 3.22. A tableau U is said to be *good* if it satisfies the following two conditions:

Cond. 1: The maximum unbarred, unprimed entry appearing in U is l.

Cond. 2: Let the integer y be the number of barred $l+1$'s appearing in U. Then there does not exist a primed entry $U(\alpha)$, for some $\alpha \in \kappa$, such that $|U(\alpha)| - c(\alpha) > l - y$.

Any tableau U which is not good is *bad.*

Let $\nu = \mu + \sigma$. Note that since $S^r : \mu \to \nu$ and $l(\nu) \leqslant l+1$ we have that the maximum barred integer appearing is $l+1$.

Example 3.23. Let $\kappa = (0,0,5)$, $\mu = (2,1)$, and $\nu = (2,1,3)$. Let T be the following reverse κ-tableau

$$T = \begin{array}{|c|c|c|c|c|}\hline \overline{3} & \overline{3} & 3 & \overline{3} & 2 \\ \hline\end{array}$$

which contains the row word 333 which takes μ to ν.

Then, the following is a bad monomial tableau U of shape κ derived from T which does not obey condition 1:

$$U = \begin{array}{|c|c|c|c|c|}\hline \overline{3} & \overline{3} & 3 & \overline{3} & 2 \\ \hline\end{array}$$

since its maximum unbarred entry is greater than the integer $l = l(\mu)$.

On the other hand, for the following reverse κ-tableau

$$\widetilde{T} = \boxed{\overline{3}}\boxed{\overline{3}}\boxed{\overline{3}}\boxed{2}\boxed{2}$$

with row word equal to 333. We have the following monomial tableau $\widetilde{U}$ derived from $\widetilde{T}$

$$\widetilde{U} = \boxed{\overline{3}}\boxed{\overline{3}}\boxed{\overline{3}}\boxed{2'}\boxed{2}.$$

We have the weight $\mathrm{ev}(\widetilde{U}(3,4)) = -a_{2-c(3,4)} = -a_1$. However, $2 - c(3,4) > l - y$, for the integer $y = 3$ which is the number of barred 3's in $\widetilde{U}$. Thus, the tableau $\widetilde{U}$ is bad since it violates condition 2. □

Let $\mathrm{ev}(U)$ denote the weight of a monomial tableau U,

$$\mathrm{ev}(U) = \prod_{\substack{\alpha\in\kappa,\\ T(\alpha)\text{ unbarred, unprimed}}} a_{T(\alpha)-\rho^r(\alpha)_{T(\alpha)}} \prod_{\substack{\alpha\in\kappa,\\ T(\alpha)\text{ unbarred, primed}}} -a_{|T(\alpha)|-c(\alpha)}.$$

The following is a refinement of Lemma 3.20.

Lemma 3.24. *We claim that*

$$d_{\sigma'}(a) = \sum_U \mathrm{ev}(U), \tag{3.20}$$

summed over all good monomial reverse κ-tableaux U derived from a reverse κ-tableau T such that T has row word $S^r : \mu \to \mu + \sigma'$.

Let $\mathbb{U}$ denote the set of bad tableaux U. We will show that Lemmas 3.20 and 3.24 are equivalent by showing

$$\sum_U \mathrm{ev}(U) = \sum_T \mathrm{ev}(T),$$

where the right hand side is summed over reverse κ-tableaux T such that T has row word $S^r : \mu \to \mu + \sigma'$, and the left over all good monomial tableaux U derivable from a tableau T appearing in the sum on the right. We do this by constructing a weight reversing involution on $\mathbb{U}$, such that a bad tableau U in $\mathbb{U}$ is paired to a tableau $\widetilde{U}$ in $\mathbb{U}$ with reverse weight to U. Since

$$\sum_T \mathrm{ev}(T) = \sum_U \mathrm{ev}(U),$$

summed over all monomial tableaux U derivable from T, the existence of such a weight reversing involution will cancel out the contributions of bad tableaux U on the right hand side.

If U is a bad tableau, for a box $\beta \in \lambda$ we will call an entry $U(\beta)$ *bad* if $U(\beta) > l$ is unprimed, or $U(\beta)$ is primed and $|U(\beta)| - c(\beta) > l - y$; i.e. the entry $U(\beta)$ violates Cond. 1 or Cond. 2 characterising good tableaux. Then, let $\alpha \in \kappa$ be the unique box satisfying both of the following conditions:

Cond. 1: $U(\alpha)$ is bad and the subscript of the weight of $U(\alpha)$ is the maximal subscript appearing in the weight of any bad entry of U. Call this subscript k.

Cond. 2: If there is more than one entry with weight equal to a_k or $-a_k$: let α be the box containing the primed entry with weight equal to $-a_k$ if it exists, otherwise, α is the leftmost box containing a unprimed, unbarred entry with weight equal to a_k.

Example 3.25. We continue with Example 3.23, where $\kappa = (0, 0, 5)$, $\mu = (2, 1)$, and $\nu = (2, 1, 3)$. Then, the monomial tableau

$$U = \boxed{\overline{3}}\boxed{\overline{3}}\boxed{3}\boxed{\overline{3}}\boxed{2}$$

is a bad tableau. The weight of $U(3, 3)$ is $\mathrm{ev}(U(3, 3)) = a_{3-2} = a_1$, and 1 is the highest subscript appearing as a weight of $U(\beta)$, for all $\beta \in (0, 0, 5)$, thus the box $\alpha = (3, 3)$.

For the monomial tableau

$$\widetilde{U} = \boxed{\overline{3}}\boxed{\overline{3}}\boxed{\overline{3}}\boxed{2'}\boxed{2}$$

the weight of $\widetilde{U}(3, 4) = a_1$ and thus the box $\alpha = (3, 4)$. □

Let the integer j be the column number of the box α, so that $\alpha = (e+1, j)$. There are two cases to consider in the construction of $\widetilde{U}$, the bad monomial tableau paired to U by our involution:

Case 1: Suppose that $U(e + 1, j)$ is unbarred and unprimed. Then, there exist a unique pair: a box $(e + 1, j')$, with $j' \geqslant j$ and a positive integer m such that $m \geqslant |U(e + 1, j' + 1)|$ and $m - c(e + 1, j') = k$. First, we argue that such an m exists. Let the integers $m(j') = k + c(e + 1, j')$ for all $j' \geqslant j$. Then, this is a strictly increasing sequence of integers, since $c(e + 1, j')$ strictly increases as j' increases. On the other hand $|U(e + 1, j')|$ weakly decreases as j' increases. Second, m is positive since $m - c(e + 1, j) \geqslant k$ and $e + 1 \leqslant l(\mu)$. Third, the pair is unique since the subscripts of the weights of any primed entries in U strictly decrease, reading left to right along the row. To form $\widetilde{U}$, we will remove the entry $U(e + 1, j)$, and move all entries from box $(e + 1, j + 1)$ to $(e + 1, j')$ inclusive, one box to the left. Now insert the entry m' into box $(e + 1, j')$. Note that there are no primed entries between box

$(e+1,j)$ and $(e+1,j')$, by assumption of maximality of k. Thus by construction, the tableau $\widetilde{U}$ formed this way has opposite weight to U.

Case 2: Suppose that $U(e+1,j)$ is primed and that $k = l+1-y'$, for some $0 \leqslant y' \leqslant y$. Since y is the number of barred $l+1$'s in U, there exist a box $(e+1,j')$, for some minimal $j' \leqslant j$, such that there are y' barred $l+1$'s in U strictly to the left the box $(e+1,j')$. We will remove the entry in box $(e+1,j)$, move all entries from box $(e+1,j')$ to $(e+1,j-1)$ inclusive one box to the right, and then insert an unbarred $l+1$ in box $(e+1,j')$. Suppose $k > l+1$; in this case we remove the entry in box $(e+1,j)$, move all entries from box $(e+1,1)$ to $(e+1,j-1)$ inclusive, one box to the right, and insert an unbarred k in box $(e+1,1)$. Again by construction, the tableaux formed in either of these ways has opposite weight to U.

We now give an example which illustrates this involution.

Example 3.26. We continue with Example 3.25. For the composition $\kappa = (0,0,5)$, and partitions $\mu = (2,1)$, and $\nu = (2,1,3)$, we have the following monomial tableaux U:

$U =$

$\overline{3}$	$\overline{3}$	3	$\overline{3}$	2

The entries $U(3,3)$, and $U(3,5)$ both have weight a_1. Since they are both unprimed, this is dealt with in Case 1. Thus, we pick α to be the leftmost box, i.e. $\alpha = (3,3)$. We have the tableau $\widetilde{U}$:

$\widetilde{U} =$

$\overline{3}$	$\overline{3}$	$\overline{3}$	$2'$	2

which is formed from U by removing the entry $U(3,3)$, moving the entry $U(3,4)$ one spot to the left, and inserting the entry $2'$ into the box $(3,4)$. This new entry has weight equal to $-a_{2-c(3,4)} = -a_1$. The entry $2'$ and the box $(3,4)$ are a unique pair because there is no other box of κ and a primed entry that can go in that box and have weight equal to $-a_1$.

Note that $\widetilde{U}$ has a primed entry which has weight $-a_1$, and '1' is in fact the maximum subscript of the weight of any entry in $\widetilde{U}$. Thus, this tableau is dealt with in Case 2. We restore U by noting that box $(3,3)$ is the leftmost box of $\widetilde{U}$ which has two barred 3's strictly to the left of it.

□

We now show that the weight of monomial tableaux U may be used to represent monomials appearing in $d_{\sigma'}$ (3.11). Let U be a good tableau such that for each $1 \leqslant j \leqslant l+1$, we have that there are σ'_j barred j's in U. This condition is necessary if U contains a row word which takes μ to $\mu+\sigma'$. Also, for any $1 \leqslant j \leqslant l$, let there be d unbarred, unprimed j's in U. Let let the product of these unprimed entries be the monomial V, written left to right with respect to the row order; thus, the subscripts of V weakly decrease. Let $\sigma_j = \sigma'_j + d$, and we will show that V is equal to a monomial in K_d, the coefficient of $h_{\mu_j+\sigma'_j, j-1}$ in equation (3.14) formed by applying the relation (3.9) to $h_{\mu_j+\sigma_j, j-1+\sigma}$. First note that V, like K_d, is a degree d monomial in $\mathbb{Z}[a]$. From equation (3.15), the maximum subscript appearing in a factor of V is $j - \mu_j$, while the minimum is $j - \mu_j - \sigma'_j$. Since the subscripts weakly decrease, reading left to right, the subscripts of V agree with the subscripts of a unique monomial appearing in the expansion of $K_d(a)$ in equation (3.15). Thus, the weight of the subtableau of U containing only j's is equal to a unique monomial in $K_d(a)$.

Example 3.27. We continue with example 3.15. Recall the product

$$h_{1,1}s_{(1,1),(0,1)} = s_{(2,1),(1,1)} + s_{(1,2),(0,2)} + s_{(1,1,1),(0,1,1)},$$

with corresponding decomposition of

$$s_{(2,1),(1,1)} = s_{(2,1)} + a_0 s_{(1,1)}.$$

We have that the coefficient of $s_{(1,1)}$ in this decomposition is

$$a_0 = \mathrm{ev}(U),$$

where U is the following monomial tableau of shape $(0, 1)$:

$$U = \boxed{1}$$

which has weight $\mathrm{ev}(U) = a_{1-\mu_1} = a_0$. □

Abusing notation, let there be d primed entries in U. Now consider the monomial W equal to the product of the weights of all primed entries in U, written left to right, with respect to the row order. Thus, the subscripts of W strictly decrease, reading left to right. We will show that W corresponds to a unique monomial appearing in the coefficient $G_d(a)$ in equation (3.17). We have that the maximum subscript appearing is $l - y$ where y is the number of barred $l+1$'s, since U is a good tableau. Noting that $y = p - k - d$ in equation (3.18), the integer $l - y$ agrees with the maximum subscript appearing in $G_d(a)$. The minimum subscript appearing in W is $1 - c(e+1, p) = 2 - p + e$. This agrees with the minimum subscript appearing in $G_d(a)$. Thus, we see that the subscripts of W agree with the subscripts of a unique monomial appearing in the expansion of $G_d(a)$.

Example 3.28. We continue with example 3.15. Recall the product

$$h_{1,1}s_{(1,1),(0,1)} = s_{(2,1),(1,1)} + s_{(1,2),(0,2)} + s_{(1,1,1),(0,1,1)},$$

with corresponding decomposition of

$$s_{(1,1,1),(0,1,1)} = s_{(1^3)} - a_2 s_{(1^2)}$$

We have that the coefficient of $s_{(1^2)}$ in this decomposition is

$$-a_2 = \mathrm{ev}(U),$$

where U is the following monomial tableau of shape $(0, 1)$:

$$U = \boxed{1'}$$

which has weight $\mathrm{ev}(U) = -a_{1-c(2,1)} = -a_2$. □

Let $\nu = \mu + \sigma'$ and $\kappa = (0, \ldots, 0, p)$ of length $e+1 \leqslant l(\mu)$. Thus we conclude that $d_{\sigma'}(a)$ from equation (3.12) may be expressed using the weights of reverse κ-tableaux;

$$d_{\sigma'} = \sum_T \left(a_{T(\alpha)-\rho^r(\alpha)_{T(\alpha)}} - a_{T(\alpha)-c(\alpha)}\right),$$

summed over tableaux T of shape κ, with row word $S^r : \mu \to \nu$, such that $\nu = \mu + \sigma'$.

Example 3.29. Continuing with example 3.15, we see that

$$h_{1,1} s_{(1,1)} = s_{(2,1)} + s_{(1,2)} + s_{(1^3)} + (a_0 + a_1 - a_2) s_{(1^2)}. \tag{3.21}$$

The coefficient of $s_{(1^2)}$ in equation 3.21 can be given *diagrammatically*,

$$a_0 + a_1 - a_2 = \boxed{1} + \boxed{2} + \boxed{1'},$$

where each diagram represents its own weight in this equation. Note that the tableaux given here are precisely the good monomial tableaux which obey conditions 1 and 2 in Definition 3.22. □

The main result of this section is the following proposition, which is a restatement of the claim made in (3.13).

3.5 Pieri rule: statement and proof

Recall that μ is a partition with length l and that $\kappa = (0, \ldots, 0, p)$ is a composition with length $e + 1$ less than l. We are now ready to prove Proposition 3.16, which describes the coefficients $c^{\nu}_{\kappa\mu}(a)$ occuring in the product between h_κ and s_λ,

$$h_\kappa s_\lambda = \sum_\nu c^{\nu}_{\kappa\mu}(a) s_\nu,$$

summed over compositions κ. We give the following proposition which is a stronger version of Proposition 3.16.

Proposition 3.30. *Let ν be a composition. If $\mu \not\subseteq \nu$, we have $c^{\nu}_{\kappa\mu}(a) = 0$. Otherwise,*

$$c^{\nu}_{\kappa\mu}(a) = \sum_{T} \prod_{\substack{\alpha\in\kappa,\\ T(\alpha)\ \mathit{unbarred}}} (a_{T(\alpha)-\rho(\alpha)_{T(\alpha)}} - a_{T(\alpha)-c(\alpha)}), \tag{3.22}$$

summed over reverse κ-tableaux T, with row word $S^r : \mu \to \nu$ that is Yamanouchi, with labelling $\rho(\alpha)$ defined by $S^r(\alpha) : \mu \to \rho(\alpha)$.

Proof. If $\mu \not\subseteq \nu$, we have $c^{\nu}_{\kappa\mu}(a) = 0$, since there is no row word that can create ν from μ. Let σ' be a composition with at most p boxes, and consider the composition $\nu = \mu+\sigma$. Recall σ' is bad if ν/μ is not a horizontal strip. If σ' is bad, we will call the composition $\nu = \mu + \sigma'$ bad as well. We will show that if S^r is not Yamanouchi, then the coefficient $c^{\nu}_{\kappa\mu}(a) = 0$. We do this by cancelling all contributions of $d_{\sigma'}(a)s_{\mu+\sigma',\mathbf{1}^{l+1}}$ on the left hand side of equation (3.13), for all bad σ''s.

Let the composition ν be bad, and S^r the row word, which is weakly decreasing, reading left to right, which takes μ to ν. Let $\mathbb{T}$ be the set of reverse κ-tableaux T with row word equal to S^r. Let the integer i be minimal such that $\nu_{i+1} - \mu_i > 0$. Such an i exists since ν is bad. Let $\widetilde{\nu} = \overline{R}_{i,i+1}\nu$, and $\widetilde{S}$ be the sequence of integers, weakly decreasing reading left to right, which takes μ to $\widetilde{\nu}$. Then, the composition $\widetilde{\nu}$ is bad. Let $\widetilde{\sigma'}$ be the composition equal to difference $\widetilde{\nu} - \mu$. Let $\widetilde{\mathbb{T}}$ be the set of reverse κ-tableaux T with row word equal to $\widetilde{S}$.

We claim the following:

$$\sum_{T\in\mathbb{T}} \mathrm{ev}(T) = \sum_{T\in\widetilde{\mathbb{T}}} \mathrm{ev}(T). \tag{3.23}$$

We will prove this by constructing a weight preserving bijection between monomial tableaux U derivable from $T \in \mathbb{T}$ and monomial tableaux $\widetilde{U}$ derivable from $\widetilde{T} \in \widetilde{\mathbb{T}}$. An example follows at the end. By proving that equation (3.23) is true, we will have shown Proposition 3.30. This is because by the straightening law (Proposition 3.8)

$$s_{\nu,\mathbf{1}^{l+1}} = -s_{\widetilde{\nu},\mathbf{1}^{l+1}},$$

since $\overline{R}_{i,i+1}\mathbf{1}^{l+1} = \mathbf{1}^{l+1}$.

Recall that we only need to consider monomial tableaux which obey conditions 1 and 2 from Definition 3.22. Let U be such a monomial tableau derived from $T \in \mathbb{T}$. Let $d = \nu_{i+1} - \nu_i - 1$. We construct $\widetilde{U}$ with equal weight to U; there are two cases depending on whether $d \geqslant 0$ or $d < 0$:

Case 1: Suppose the integer $d \geqslant 0$. This is the case where row $i+1$ of ν has more boxes than row i of ν. Let X be the subtableau of U containing d barred $i+1$'s, counting left from the right most $i+1$ in U. We will replace the d barred $i+1$'s in

X with barred i's to create a tableau with a new row word that will create $\widetilde{\nu}$ from μ. We do this in the following way. First, we do not do anything with the primed entries in U. Let X^{i+1} denote the sequence of unprimed $i+1$'s in X, reading left to right. Let Y denote the sequence of nonprimed entries equal to i in U, reading left to right. Delete the entries in X which are not primed, and also delete all unprimed entries equal to i, thus creating empty boxes. Let X^i be the sequence created from X^{i+1} by replacing all the unbarred $i+1$'s in X^{i+1} with unbarred i's, and all the barred $i+1$'s in X^{i+1} with barred i's. We will now fill in the newly created empty boxes as follows. Insert entries in the empty boxes of our tableau, from left to right, with the entries from Y, read left to right, and then the entries from X^i, read left to right.

Due to the previous processes applied to U, the entries of our current tableau, read left to right, may no longer weakly decrease. So the next step is to fix the order in which the barred entries equal to i and $i+1$ occur. Call a primed i (resp. $i+1$) *badly ordered* if it is to the left of a $i+1$ (resp. right of a i). We describe a process which fixes badly ordered primed i's. A very similar process will fix the badly ordered primed $i+1$'s; see the example below. Starting with the rightmost badly ordered primed i, do the following: Delete the badly ordered primed i. Move the $i+1$ to the right of it one box to the left. Now insert a primed $i+1$ in the blank box. Keep repeating this process on the next rightmost badly ordered primed i until none are left.

We end up with a tableau with entries that weakly decrease, read left to right, and this is the tableau $\widetilde{U}$ paired with U.

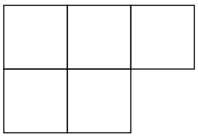
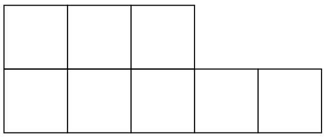
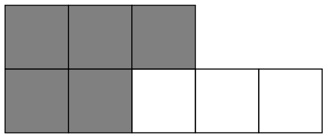
Case 2: Suppose $d < 0$. This is the case where row i of ν has at least the same number of boxes as row $i+1$ of ν. Then we can undo the processes described in Case 1. Let X be the subtableau of T containing $|d|$ barred i's, counting left from the rightmost i in T. Then we may reverse the process described in Case 1.

Note that the involution induces a natural pairing between the entries of U and those of $\widetilde{U}$.

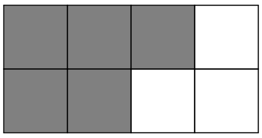
We check that the weight of U is the same as the weight of $\widetilde{U}$, for U a Case 1 tableau. Let the box α contain an entry of U that was changed from an i to $i+1$, or vice versa, in the creation of $\widetilde{U}$. Suppose α contains an unbarred $i+1$ that was changed into an unbarred i. Then by the definition of d, in the subtableau X there are d' barred $i+1$'s strictly to the left of α, for a $1 \leqslant d' \leqslant d$. This means the weight of $U(\alpha) = a_{i+1-(\nu_i+1+d')}$. Let the entry $\widetilde{U}(\beta)$ equal to an unbarred i in box β be the one paired to $U(\alpha)$ by the involution. The weight of the entry $U(\beta)$ is $a_{i-(\nu_i+d')}$, since the entry $U(\beta)$ occurs after d' barred i's in the sequence X^i. Thus, the weights of $U(\alpha)$ and $\widetilde{U}(\beta)$ are equal, that is $\mathrm{ev}(U(\alpha)) = \mathrm{ev}(\widetilde{U}(\beta))$. On the other hand, suppose α contained a badly ordered primed i that was changed into a primed $i+1$. Then

$\widetilde{U}(\beta) = (i+1)'$ is the entry paired with $U(\alpha)$, and β is one box to the right of α. Since $c(\beta) = c(\alpha)+1$, we conclude the weights of $U(\alpha)$ and $\widetilde{U}(\beta)$ are equal. A similar argument follows for badly ordered primed $i+1$'s that were changed into primed i's. □

Example 3.31. Let $\kappa = (8)$, $\mu = (3,2)$ be partitions and $\nu = (3,5)$ be a bad composition, since ν/μ is not a horizontal strip. We have the diagrams

$\mu =$  $\nu =$  $\nu/\mu =$ 

Then, the composition $\widetilde{\nu} = \overline{R}_{1,2}(\nu) = (4^2)$ is bad as well, since $\widetilde{\nu}/\mu$ is not a horizontal strip,

$\widetilde{\nu}/\mu =$ 

Let U be the following monomial tableau

$$U = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \overline{2} & \overline{2} & \overline{2} & 2' & 2 & 1' & 1 & 1' \\ \hline \end{array}$$

which has row word $S^r = 222$, which takes μ to ν. We have that $d = \nu_{i+1} - \nu_i - 1 = 1$. Then, the tableau X is the subtableau of entries from boxes $(1,3)$ up to $(1,5)$, since X contains $d = 1$ barred 2's, counting left from the rightmost 2. The sequence of unprimed entries in X, read left to right, is $X^2 = \overline{2}2$. The sequence $X^1 = \overline{1}1$ is formed from X^2 by replacing all 2's by 1's. The sequence of unprimed 1's in T is $Y = 1$. We now delete unprimed entries to form:

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline \overline{2} & \overline{2} & & 2' & & 1' & & 1' \\ \hline \end{array}$$

Now, fill in the blank boxes with entries from Y, then X^1, read left to right:

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline \overline{2} & \overline{2} & 1 & 2' & \overline{1} & 1' & 1 & 1' \\ \hline \end{array}$$

The entries do not weakly decrease, read left to right, so we must fix the badly ordered primed entries, by swapping the $2'$ with the 1 on its left, and then replacing $2'$ with $1'$. This forms the tableau $\widetilde{U}$:

$$\widetilde{U} = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \overline{2} & \overline{2} & 1' & 1 & \overline{1} & 1' & 1 & 1' \\ \hline \end{array}$$

We claim the weight of $U(1,5)$ is equal to the weight of $\widetilde{U}(1,7)$. This is because $2-\rho^r(1,5)_2 = 2-5$ and $1-\widetilde{\rho}^r(1,7)_1 = 1-4$. We claim the weight of $U(1,4)$ is equal to the weight of $\widetilde{U}(1,3)$. This is true since $c(1,3) = c(1,4)-1$, so $1-c(1,3) = 2-c(1,4)$. □

3.6 Littlewood–Richardson polynomials

Recall our notation that $h_\mu = h_{\mu,\mathbf{1}}$ and $s_\mu = s_{\mu,\mathbf{1}}$. We will keep this abbreviation to keep calculations neat. We come to the first of the main results of this thesis. We are ready to use the Pieri rule (Proposition 3.30) to calculate the Littlewood–Richardson polynomials $c^\nu_{\lambda\mu}(a)$. We first define some intermediate structure coefficients which arise out of the equation (3.10). Let μ and λ be partitions, and $l = l(\mu)$, and recall the integer vector $\mathbf{1} = (0, 1, \dots)$ and the Jacobi–Trudi identity (3.10):

$$s_{\mu,\mathbf{1}} = \det(h_{\mu_i+j-i,j-1})_{1\leqslant i,j\leqslant l}. \tag{3.24}$$

Let $\pi = (l-1, l-2, \dots, 0)$ and $\mathfrak{S}_l$ denote the symmetric group on l elements. For each $\omega \in \mathfrak{S}_l$, define the composition $\lambda^\omega = \omega(\lambda + \pi_l) - \pi_l$ and let $\operatorname{sgn}(\lambda^\omega) = \operatorname{sgn}(\omega)$, the parity of the permutation ω. We may write

$$s_{\mu,\mathbf{1}} = \sum_\kappa \operatorname{sgn}(\kappa) h_{\kappa,\mathbf{1}},$$

summed over $\kappa = \lambda^\omega$, for all $\omega \in \mathfrak{S}_l$. This is just an expansion of the determinant (3.24) into an alternating sum. For each κ, define $K^\nu_{\kappa\mu}(a)$ as the coefficients appearing in the expansion

$$h_{\kappa,\mathbf{1}} s_{\mu,\mathbf{1}} = \sum_\nu K^\nu_{\kappa\mu}(a) s_{\nu,\mathbf{1}}.$$

Then, we have that

$$c^\nu_{\lambda\mu}(a) = \sum_\kappa \operatorname{sgn}(\kappa) K^\nu_{\kappa\mu}(a). \tag{3.25}$$

summed over $\kappa = \lambda^\omega$, for all $\omega \in \mathfrak{S}_l$.

In the classical case, when κ is a partition and a is the sequence of zeroes, the coefficient $K^\nu_{\kappa\mu}(a)$ are the *Kostka numbers*. If κ is a diagram with only one row then this is the Pieri rule (3.22) and thus $K^\nu_{\kappa\mu}(a) = c^\nu_{\kappa\mu}(a)$. The aim of the rest of this section is to eliminate unwanted coefficients $K^\nu_{\kappa\mu}(a)$ from the above alternating sum; by doing this, we will be able to calculate the Littlewood–Richardson polynomials. First we give a formula for $K^\nu_{\kappa\mu}(a)$ using the Pieri rule.

Proposition 3.32. *Let $\kappa = \lambda^\omega$, for some $\omega \in \mathfrak{S}_l$. Then*

$$K^\nu_{\kappa\mu}(a) = \sum_T \prod_{\substack{\alpha\in\lambda,\\ T(\alpha)\ \text{unbarred.}}} \left(a_{T(\alpha)-\rho^r(\alpha)_{T(\alpha)}} - a_{T(\alpha)-c(\alpha)}\right), \tag{3.26}$$

summed over reverse κ-tableaux T, such that each T has row word $S^r : \mu \to \nu$ which is Yamanouchi.

Proof. The proof follows from repeated applications of the Pieri rule (3.22). Let $l' = l(\lambda)$.

We have that

$$\begin{aligned} h_\kappa s_\mu &= h_{\kappa_1} \dots h_{\kappa_{l'}} s_\mu \\ &= h_{\kappa_1,0}(\dots(h_{\kappa_{l'-1},l'-2}(h_{\kappa_{l'},l-1} s_\mu))\dots) \end{aligned} \tag{3.27}$$

where we evaluate each multiplicative pair using the Pieri rule, starting with $h_{\kappa_{l'}} s_\mu$. Each multiplication produces a tableau of shape $(0, \dots, 0, \kappa_i)$, of length i. We stack these tableau on top of each other to form a tableau T of shape κ; that is, the i-th row of T is equal to the tableau formed from the i-th multiplicative pair in expression 3.27. Furthermore, this tableau must contain a row word $S^r : \mu \to \nu$. □

One of our principle results in this thesis is a new proof of the following theorem, which expresses the Littlewood–Richardson polynomials $c^\nu_{\lambda\mu}(a)$ in terms of tableaux. Two alternative and different proofs were given by Molev [18] and Kreiman [10]. In [18], a recurrence relation is used, whereas [10] generalises a concise proof of the classical Littlewood–Richardson rule by Stembridge [25], which relies on the definition of the Schur polynomial as a ratio of alternants given by equation (2.3). Our proof, relying on the Jacobi–Trudi identity, is a generalisation of the method used previously by Gasharov [5], Remmel and Shimozono [23], and most recently Tamvakis [26].

We recall some definitions which are used. Let λ be a partition and T be a barred λ-tableau. First, recall the row and column words of T, which are $S^r(\alpha)$ and $S^c(\alpha)$ respectively consisting of the barred entries of T listed in row and column order. Second, we have the labelling $\rho^r(\alpha)$ which consists of compositions $\rho^r(\alpha)$ formed by applying the word $S^r(\alpha)$ to the partition μ. Lastly, recall that the column word S^c of the tableau T is said to be Yamanouchi if for all boxes $\alpha \in \lambda$, the compositions $\rho^c(\alpha)$ are partitions.

Theorem 3.33. *Let λ, μ, and ν be partitions. If $\nu \not\subseteq \mu$, then the coefficient $c^\nu_{\lambda\mu}(a) = 0$. If $\mu \subseteq \nu$, we have that*

$$c^\nu_{\lambda\mu}(a) = \sum_T \prod_{\substack{\alpha\in\lambda \\ T(\alpha)\ \text{unbarred}}} (a_{T(\alpha)-\rho^r(\alpha)_{T(\alpha)}} - a_{T(\alpha)-c(\alpha)}), \tag{3.28}$$

where the sum is taken over reverse λ-tableaux T obeying the following. First, the column word S^c of T is Yamanouchi and $S^c : \mu \to \nu$. Secondly, the entries in T strictly decrease down each column; that is, T is column strict.

We begin the proof of Theorem 3.33. The statement that $c^\nu_{\lambda\mu}(a) = 0$ if $\mu \not\subseteq \nu$ follows from Proposition 3.32, since if ν does not contain μ then there is no tableau with row or column word that can take μ to ν. We will split the proof up into sections but first we introduce some terminology. We will call a tableau T *good* if it appears in the sum (3.28). On the other hand, a *bad* tableau T is one appearing in the expansion of $K^{\nu}_{\kappa\mu}(a)$ for all $\kappa = \lambda^{\omega}$ in equation (3.26) which is not good. The following is an equivalent description of bad tableaux:

Let κ be the shape of a bad tableau T, and $l = l(\kappa)$. For each row i, let $T^{\geqslant i}$ denote the subtableau of T consisting of entries in row i and below. If a box α is in row i of κ, let $L(\alpha)$ be the column word corresponding to the barred entries of $T^{\geqslant i}$ in the boxes before and including α, with respect to the column order. Then a tableau T is bad if and only if T has one of the following properties:

(P1) There exist a row i and a box α in row i of κ such that the sequence $L(\alpha)$ is not Yamanouchi when applied to μ.

(P2) There exist a row i of κ such that the subtableau of T formed from rows i and $i+1$ of T is not column strict.

(P3) There is a row i of κ such that $\kappa_i < \kappa_{i+1}$.

We split the proof of Theorem 3.33 into two sections. In the first section we will describe an involution on the set of bad tableaux. Namely, we pair a bad tableau T to another bad tableau $\widetilde{T}$ of shape $\widetilde{\kappa} = R_{i,i+1}\kappa$ appearing in the sum (3.25), for some $1 \leqslant i \leqslant l$. This is not a weight preserving involution, however it is close to one, as we will discover later. In the second section, we will use a sequence of lemmas to show that it is possible to cancel out the weights of bad tableaux from the sum (3.25). An example which ties these two sections together will follow at the end.

If T is a bad tableau of shape κ, then $\kappa_m \neq \kappa_{m+1} - 1$ for all rows m of T. This is because $\lambda_i \geqslant \lambda_{i+1}$ which necessarily means that $\kappa_i \geqslant \kappa_{i+1}$ or $\kappa_i \leqslant \kappa_{i+1} - 2$. Furthermore, there exist a unique pair of integers (i, j), subject to both of the following conditions on T:

(C1) The row number i is maximal such that one of properties (P1), (P2) or (P3) hold for T.

(C2) The column number j is minimal so one of the following (mutually exclusive) conditions hold for T:

 (C2a) The property (P1) holds for T and $\alpha = (i, j)$, and $T(i, j) > T(i + 1, j)$.

 (C2b) The property (P2) holds for T and $\alpha = (i, j)$; by this we mean $T(i, j) \leqslant T(i + 1, j)$.

(C2c) The property (P3) holds for i, and there is no column j such that (P1) and (P2) hold for box $\alpha = (i, j)$ in row i of T.

In the case where condition (C2c) holds, let $j = \kappa_i + 1$. For all cases, we will call the tableau T *bad* in row i, column j.

We begin constructing our involution on the set of bad tableaux. Throughout this construction, if α and β are boxes in a row of κ, when we write "between boxes α and β" we mean that the boxes α and β are included in this range. If we want to exclude either of these boxes from the range we will specifically say so.

Let T be a bad tableau with shape κ. We will construct $\widetilde{T}$, a bad tableau paired to T, such that: 1) The shape $\widetilde{\kappa}$ of $\widetilde{T}$ is equal to $\overline{R}_{i,i+1}\kappa$. 2) The row word $\widetilde{S}^r$ corresponding to $\widetilde{T}$ takes μ to ν.

To construct $\widetilde{T}$ we use a sequence of processes, which are summed up by the following diagram:

$$T \xrightarrow{\psi_1} T^1 \xrightarrow{\psi_2} \widetilde{T},$$

where we start with the tableau T, and apply processes ψ_1 and ψ_2 in the order indicated by the arrows. This will create a sequence of tableaux involving the intermediate tableau T^1, and this sequence terminates at $\widetilde{T}$, which is the bad tableaux paired with T. The first process, ψ_1, is called a "tail swap" and the second, ψ_2, is called "reorder barred entries". The detailed description of each of these processes appear in the following subsections with the same name.

3.7 ψ_1: tail swap

In this process, we take the tableau T which is bad in row i, column j and create a tableau T^1, of shape $\widetilde{\kappa}$, which preserves the 'bad in row i, column j' condition. We first describe the process for tableaux which obey conditions (C2b), (C2c), and leave the argument for (C2a) till last, and then we will examine the properties of T^1.

Suppose T obeys condition (C2b): Let $T(i, j) = b$ and $T(i + 1, j) = c$, and from the definition of $\alpha = (i, j)$ in condition (C2b) we have $b \leqslant c$. We call the pair b and c a *bad column pair*. We do the following; let X be the subtableau of entries in boxes (i, j) to (i, κ_i) of T, and Y the subtableau of entries in boxes $(i+1, j+1)$ to (i, κ_{i+1}) of T. Form the tableau T^1 by swapping the subtableaux X and Y. Swapping means two things:

1. The shape of T^1 is $\widetilde{\kappa} = \overline{R}_{i,i+1}\kappa$.

2. The entries of T^1 in boxes $(i+1, j+1)$ up to $(i+1, \kappa_i+1)$ are equal to the entries of X, read left to right, and the entries of T^1 in boxes (i, j) up to $(i, \kappa_{i+1} - 1)$ are equal to the entries of Y, read left to right.

We will call this process of forming the tableau T^1 from the tableau T a *tail swap*. There is a natural *pairing* of boxes affected by the tail swap; e.g. if $\alpha = (m, n)$ is a box in X, then the box $\alpha' = (m+1, n+1)$ is the box paired with α by the tail swap. In a similar vein, the entry $T(\alpha)$ is paired with the entry $T^1(\alpha')$. This pairing holds for the other tail swaps described for cases (C2a) and (C2c) as well.

Suppose T obeys condition (C2c): Let Y be the subtableau of entries of T in boxes $(i+1, \kappa_i + 2)$ to $(i+1, \kappa_{i+1})$. Then, the tail swap is performed by moving Y to the end of row i of T to form T^1. We can think of the process for T obeying condition (C2c) as a special case of the process for tableaux obeying condition (C2b); that is we have that the subtableau X is empty.

The case where T obeys condition (C2a) involves an additional step, which we now describe.

Suppose T obeys condition (C2a): Let $T(i, j) = b$, and $T(i+1, j) = c$. We will also call the pair b and c a *bad column pair*. By definition of $\alpha = (i, j)$ in (C2a), we have that the sequence $L(\alpha)$ is not Yamanouchi when applied to μ. Therefore, we have that $b = c+1$ and the entry $T(i, j)$ is a barred $c+1$. Define the column number q to be the maximum integer so that the entry $T(i, q)$ is a b, barred or unbarred. Suppose that there are s barred b's in boxes (i, j) up to (i, q). We now examine the structure of the entries of the row i and $i+1$ of T. For any $k \geqslant 1$, a *block* of unbarred k's is an uninterrupted sequence, reading left to right, of unbarred k's in a row of T. Between boxes (i, j) and (i, q), we have s disjoint blocks of unbarred b's, with each block to the right of a barred b (starting with the one in box (i, j)). Note that some of these blocks may be empty. Let x_i, $i = 1, \ldots, s$, be the number of entries in each block respectively, reading the blocks left to right. Define the column number r to be the minimum integer such that there are s barred c's between box $(i+1, j+1)$ and $(i+1, r)$. This column number exists since the row word of T is Yamanouchi. We have that $T(i+1, r) = \overline{c}$ by definition of r. Between boxes $(i+1, j+1)$ and $(i+1, r)$ we have s disjoint blocks of unbarred c's, with each block to the left of a barred c. Let y_i, $i = 1, \ldots, s$, be the number of entries in each block respectively, reading left to right. There are two cases depending on whether $q \geqslant r$ or $q < r$:

Case 1: If $q \geqslant r$, we will form an intermediate tableau $T^{\frac{1}{2}}$ by doing the following process, which we call *fixing the column ordering*: Replace the entries in boxes (i, j) up to $(i, r-1)$ such that the new entries consist of s blocks, each containing y_i, $1 \leqslant i \leqslant s$, unbarred b's, with each block to the right of a barred b (starting with the one in box (i, j)). Let σ be the subdiagram of κ containing the boxes from box $(i+1, j+1)$ up to $(i+1, r)$, then (i, r) up to (i, q). Then, starting from box $(i+1, j+1)$ and ending in box (i, q), replace the entries in the boxes of σ with s blocks, each containing x_i unbarred c's, $1 \leqslant i \leqslant s$, such that each block is before a barred c, with respect to the row order imposed on σ. The tableau formed this way is $T^{\frac{1}{2}}$. Note that in the tableau $T^{\frac{1}{2}}$ we have that the entries in boxes (i,j) up to $(i,r-1)$ consist solely of b's, and the entries in the boxes of σ consist solely of c's. In particular $|T^{\frac{1}{2}}(i+1,r)| = |T^{\frac{1}{2}}(i,r)| = c$.

Let X be the subtableau of $T^{\frac{1}{2}}$ containing the entries from box (i,r) up to (i,κ_i), and Y the subtableau of $T^{\frac{1}{2}}$ containing entries from box $(i+1,r+1)$ up to $(i+1,\kappa_{i+1})$. Now swap X with Y to obtain the tableau T^1. Note that that the shape of T^1 is $\widetilde{\kappa} = R_{i,i+1}\kappa$ and the rows of T^1 weakly decrease.

Case 2: We now deal with the case where $q < r$. Define X (resp. Y) to be the subtableau of entries of T from box $(i,q+1)$ up to (i,κ_i) (resp. $(i+1,q+2)$ up to (i,κ_{i+1})). Swap the subtableaux X and Y and the tableau obtained is defined to be $T^{\frac{1}{2}}$. Again, the shape of $T^{\frac{1}{2}}$ is $\widetilde{\kappa} = R_{i,i+1}\kappa$ and the rows of $T^{\frac{1}{2}}$ weakly decrease.

If $q = r-1$, set $T^1 = T^{\frac{1}{2}}$. If $q < r-1$, in the tableau $T^{\frac{1}{2}}$ we have that the entries in boxes (i,j) up to (i,q) consist solely of b's. Let σ be the subdiagram of $\widetilde{\kappa}$ consisting of the boxes (i,j) up to $(i,q+1)$, then $(i,q+1)$ up to $(i,r-1)$. Then, the entries in σ consist solely of c's. Recall that the tableau $T^{\frac{1}{2}}$ formed in Case 1 has a very similar property. Now, we apply a process which we will also call fixing the column ordering: Replace the entries of $T^{\frac{1}{2}}$ in boxes (i,j) up to $(i,r-1)$ such that the new entries consist of s blocks, each containing y_i, $1 \leqslant i \leqslant s$, unbarred b's, with each block to the right of a barred b (starting with the one in box (i,j)). Then, replace the entries in boxes $(i+1,j+1)$ up to $(i+1,q+1)$ with s blocks, each containing x_i unbarred c's, $i = 1,\ldots,s$, with each block to the left of a barred c. This creates the tableau T^1.

We have just described a process of creating the tableau T^1 from a tableau T obeying (C2a), (C2b), or (C2c). The following properties of T^1 will be relevant when we want to show the processes described are an involution on the set of bad tableaux:

1. T^1 is bad in row i, column j.

2. The rows of T^1 weakly decrease, left to right.

3. T^1 is of shape $\widetilde{\kappa} = \overline{R}_{i,i+1}\kappa$.

4. If T obeys condition (C2a), and falls under Case 1 (with $q \geqslant r$), then $\widetilde{T}$ will obey condition (2a), and will fall under Case 2 with $q < r-1$. The converse holds as well: if T obeys condition (C2a), and falls under Case 2 with $q < r-1$, then $\widetilde{T}$ will obey condition (2a), and will fall under Case 1 (with $q \geqslant r$).

We will now describe the second process, to be applied to T^1.

3.8 ψ_2: reorder barred entries

We require this process because the row word formed from the barred entries of T^1 might not be Yamanouchi when applied to μ. To fix this we will rearrange barred entries equal to at most c in rows i and $i+1$ of T^1.

For each $2 \leqslant k \leqslant c$, let the integer r_k be the number of barred k's in every row below, and including row i. For each $1 \leqslant k < c$, let the integer r'_k be the number of barred k's in the rows strictly below row $i+1$. Let the integer $n_k = \max(\mu_k + r_k - \mu_{k-1} - r'_{k-1}, 0)$, for $k = 2, \ldots, c$. What is the significance of the integer n_k? Since the row word of T is Yamanouchi, in row $i+1$ of T there are at least n_k barred $k-1$'s, and in row i of T there are at least n_k barred k's. Thus, in row $i+1$ of T^1 there must be at least n_k barred k's, and in row i of T^1 there must be at least n_k barred $k-1$'s.

For each $2 \leqslant k \leqslant c$, define subtableaux P_k and Q_k of T^1, where P_k is the subtableau in row i of T^1 containing n_k barred $k-1$'s, counting right from the leftmost k in row i, and Q_k is the subtableau in row $i+1$ of T^1 containing n_k barred k's, counting left from the rightmost k in row $i+1$. Thus, the rightmost box of P_k contains a barred $k-1$ and the leftmost box of Q_k contains a barred k.

We will now form $\widetilde{T}$ from T^1. We slightly abuse notation to let us communicate the process without requiring messy subscripts. For each $k = 2, 3, \ldots, c$, do the following independently: Define the integers $n = n_k$, $P = P_k$, and $Q = Q_k$. The subtableau P consists of n blocks of unbarred $k-1$'s, so that each block is to the left of a barred $k-1$. Let v_i, $i = 1, \ldots, n$, be the number of entries in each block respectively, reading the blocks left to right. Similarly, the subtableau Q consist of n blocks of unbarred k's, so that each block is to the right of a barred k. Let the integer w_i, $i = 1, \ldots, n$, be the number of entries in each block respectively, reading the blocks left to right. Replace the entries in P with n blocks, each containing v_i unbarred k's, $i = 1, \ldots, n$, with each block to the right of a barred k. Call the subtableau of entries replacing P in this way $\widetilde{Q}_k$. Also, replace the entries in Q with n blocks, each containing w_i unbarred $k-1$'s, $i = 1, \ldots, n$, with each block to the left of a barred $k-1$. Call the subtableau of entries replacing Q in this way $\widetilde{P}_k$. The process works by exchanging n_k barred k's in row $i+1$ with the same amount of barred $k-1$'s in row i, while keeping the weights of the affected unbarred entries unchanged (we will check this later). The tableau formed after applying this process independently to each pair of P_k, and Q_k, $k = 2, \ldots, c$ is the tableau $\widetilde{T}$, and this completes the process of *reordering barred entries* and we have finished describing the involution on the set of barred tableaux.

We now show that applying the operations ψ_1, then ψ_2, is an involution on the set of bad tableaux.

Lemma 3.34. *Let T be a tableau bad in row i, column j. Denote by $\psi : T \to \widetilde{T}$ the process of applying ψ_1 then ψ_2 to T according to the diagram:*

$$T \xrightarrow{\psi_1} T^1 \xrightarrow{\psi_2} \widetilde{T}.$$

Then we claim ψ is an involution, that is, $\psi : \widetilde{T} \to T$.

Proof. As a result of the tail swap (process ψ_1), we have the following properties of $\widetilde{T}$:

1. $\widetilde{T}$ is bad in row i, column j.
2. The rows of $\widetilde{T}$ weakly decrease, left to right.
3. $\widetilde{T}$ is of shape $\widetilde{\kappa} = \overline{R}_{i,i+1}\kappa$.

Recall that if T obeys condition (C2a), then we had to apply the process of fixing the column ordering. Recall the integers q and r (page 52). There were two cases, Case 1 was for $q \geqslant r$ and Case 2 was for $q < r$. If T is a tableau that has the property $q \geqslant r$ then $\widetilde{T}$ has the property that $q < r$. Similarly, if T is a tableau that has the property $q < r$ then $\widetilde{T}$ has the property that $q \geqslant r$. Moreover, the process of fixing the column ordering is an involution.

For all T, the tail swap when applied to $\widetilde{T}$ of shape $\widetilde{\kappa}$ restores the shape κ. For each $2 \leqslant k \leqslant c$, the tail swap also sends the subtableaux $\widetilde{P_k}$ to row i and the subtableaux $\widetilde{Q_k}$ to row $i+1$. Then, the process of reordering barred entries restores the subtableaux P_k and Q_k to their original locations in T.

□

We now give a sequence of lemmas to check that the weights of T and $\widetilde{T}$ are almost equal.

3.9 Weights of T and $\widetilde{T}$ are almost equal

For the purposes of this subsection, let $\rho(\alpha)$ denote the labelling $\rho^r(\alpha)$ on T, and $\widetilde{\rho}(\alpha)$ denote the labelling $\widetilde{\rho}^r(\alpha)$ on $\widetilde{T}$.

Lemma 3.35. *Suppose T obeys condition (C2a) and $q \neq r-1$. Let $\beta = (i, q')$ be a box containing an unbarred entry of T between boxes (i, j) and (i, q), and $\gamma = (i+1, r')$ be a box containing an unbarred entry of T between boxes $(i+1, j+1)$ and $(i+1, r)$. Let β' and γ' be the boxes $(i+1, q')$ and (i, r') respectively; that is, β' and γ' are the boxes below β and above γ respectively. Then* $\mathrm{ev}(T(\beta)) = \mathrm{ev}(\widetilde{T}(\beta'))$ *and* $\mathrm{ev}(T(\gamma)) = \mathrm{ev}(\widetilde{T}(\gamma'))$.

Proof. Suppose $q \neq r-1r$. In this case, the process of fixing column ordering pairs the entry $T(\beta)$ to the entry $\widetilde{T}(\beta')$. We claim that the weight of $T(\beta)$ is the same as the weight of $\widetilde{T}(\beta')$. Suppose there are t barred b's between box (i,j) and (i,q') of T. Then there are $t-1$ barred c's between box $(i+1,j+1)$ and $(i+1,q')$ of $\widetilde{T}$. Since $b=c+1$, we have $a_{b-\rho(\beta)_b} = a_{c-\rho(\beta')_c}$ and $a_{b-c(\beta)} = a_{c-c(\beta')}$. Thus, the weight of $T(\beta)$ is the same as the weight of $\widetilde{T}(\beta')$. A similar argument shows that the weight of $T(\gamma)$ is the same as the weight of $\widetilde{T}(\gamma')$. □

Note that if T obeys condition (C2a) but $q=r-1$, then the entries in boxes (i,j) to (i,q) and $(i+1,j+1)$ to $(i+1,r)$ of T and $\widetilde{T}$ are the same.

Lemma 3.36. *For $2 \leqslant k \leqslant c$, the weight of P_k is equal to the weight of $\widetilde{Q}_k$, and the weight of Q_k is equal to the weight of $\widetilde{P}_k$.*

Proof. Note that for each $2 \leqslant k \leqslant c$ the barred subtableau P_k is paired to the subtableau of entries in T which contain the first n_k barred $k-1$'s in row $i+1$ of T, with respect to the row order. Similarly, the subtableau Q_k is paired to the subtableau of T which contains the last n_k barred k's in row i of T. After barred entries are reordered, the subtableau $\widetilde{P}_k$ (which occupies the boxes of Q_k) contains the first n_k barred $k-1$'s in row $i+1$ of $\widetilde{T}$ and $\widetilde{Q}_k$ (which occupies the boxes of P_k) contains the last n_k barred k's in row i of $\widetilde{T}$. Then, the proof follows in exactly the same manner as the previous proof. □

Lemma 3.37. *The weight of all barred tableaux T of shape κ, bad in row i, column j, is equal to the weight of all barred tableaux T of shape $\widetilde{\kappa}$, bad in row i, column j.*

Proof. The process of applying ψ_1 then ψ_2 is almost a weight preserving involution on barred tableaux bad in row i, column j. This is because the unbarred entries of T which are affected by the process of fixing column ordering (if T obeys condition (C2a)), and entries in P_k or Q_k for all $2 \leqslant k \leqslant c$ are paired to entries with corresponding weight in $\widetilde{T}$ by the previous two lemmas. However, the other entries might not have the same weight because of the tail swap. We were unable to find an involution on these entries that would preserve the weight, so we adopt the approach of cancelling paired monomials ocurring in the weight of T and $\widetilde{T}$.

We define the *unaffected entries* of $\widetilde{T}$ to be the entries of $\widetilde{T}$ which are unaffected by the process of fixing column ordering (if T obeys condition (C2a)), and entries strictly to the right of boxes $(i,j-1)$ and $(i+1,j)$ that are not in $\widetilde{P}_k$ or $\widetilde{Q}_k$ for all $2 \leqslant k \leqslant c$. Furthermore, we define the *unaffected entries* of T which are the entries of T which are paired with the unaffected entries of $\widetilde{T}$ by the tail swap.

For a $1 \leqslant k \leqslant c-1$, let δ denote the subdiagram of κ which contains the unaffected entries equal to k in row i of T. Similarly, let ϵ denote the subdiagram of κ which contains the unaffected entries equal to k in row $i+1$ of T. In fact, note that these unaffected entries occur after n_{k+1} barred k's in row $i+1$, and before n_k barred k's in row i of T, with respect to the row order. Let $\widetilde{\delta}$ and $\widetilde{\epsilon}$ denote the respective subdiagrams of $\widetilde{\kappa}$ that are paired to δ and ϵ by the tail swap.

Let M be the subtableaux containing barred and unbarred k's of T in the subdiagrams δ and ϵ. Then the weight of M is

$$\prod_{\substack{\alpha\in\delta\cup\epsilon \\ T(\alpha)=k \text{ unbarred}}} \left(a_{T(\alpha)-\rho^r(\alpha)_{T(\alpha)}} - a_{T(\alpha)-c(\alpha)}\right).$$

We wish to split this weight into monomials, so recall our definition of monomial tableaux from Section 2. A *monomial subtableau* N is *derived* from M by doing the following: for each unbarred entry in M, either add a prime as a superscript of that unbarred entry or do nothing. Then, the weight of M can be expanded as:

$$\begin{aligned}\operatorname{ev}(M) &= \sum_N \operatorname{ev}(N) \\ &= \sum_N \prod_{\substack{\alpha\in\delta\cup\epsilon \\ T(\alpha)=k \text{ unbarred,} \\ \text{unprimed}}} \left(a_{T(\alpha)-\rho^r(\alpha)_{T(\alpha)}}\right) \prod_{\substack{\alpha\in\delta\cup\epsilon \\ T(\alpha)=k \text{ unbarred,} \\ \text{primed}}} \left(-a_{|T(\alpha)|-c(\alpha)}\right)\end{aligned}$$

summed over all monomial subtableaux N derived from M.

Let N be a monomial tableaux derived from M. We will find a monomial subtableau $\widetilde{N}$ of $\widetilde{T}$ in the subdiagrams $\widetilde{\delta}\cup\widetilde{\epsilon}$ such that the weight of N and $\widetilde{N}$ are equal. Let N' denote the sequence of unprimed k's in M, listed left to right, first in the subdiagram δ, then in the subdiagram ϵ. Let χ be the subset of boxes of the subdiagram $\delta\cup\epsilon$ containing the primed entries of M. Let $\widetilde{\chi}$ be the subset of the subdiagram $\widetilde{\delta}\cup\widetilde{\epsilon}$ paired to χ by the process of tail swapping. To form $\widetilde{N}$ first fill in the boxes of $\widetilde{\chi}$ with primed k's. Then, fill in the boxes of $\widetilde{\delta}\cup\widetilde{\epsilon}$ not in $\widetilde{\chi}$ with unprimed k's such that the sequence of replaced entries, read left to right, first from row $i+1$ and then from row i, is equal to N'. This forms the monomial tableaux $\widetilde{N}$. Note that these replaced entries occur after n_{k+1} barred k's in row $i+1$ of $\widetilde{T}$, and before n_k barred k's in row i of $\widetilde{T}$, thus the weight of N and $\widetilde{N}$ are equal. □

This completes the proof of Theorem 3.33 since we have cancelled out all unwanted summands from (3.25).

We may express the Littlewood–Richardson polymonials using the following alternative form. Recall that $\rho^c(\alpha)$ is the labelling on the boxes of the partition λ formed by applying the column word $S^c(\alpha)$ to the partition μ.

Corollary 3.38. *Let λ, μ, ν be partitions. If $\nu \not\subseteq \mu$, then $c^{\nu}_{\lambda\mu}(a) = 0$. If $\mu \subseteq \nu$, we have that*

$$c^{\nu}_{\lambda\mu}(a) = \sum_{T} \prod_{\substack{\alpha \in \lambda \\ T(\alpha) \text{ unbarred}}} \left(a_{T(\alpha)-\rho^c(\alpha)_{T(\alpha)}} - a_{T(\alpha)-c(\alpha)}\right), \tag{3.29}$$

where the sum is taken over reverse λ-tableaux T obeying the following. First, the column word S^c of T is Yamanouchi and $S^c : \mu \to \nu$. Secondly, the entries in T strictly decrease down each column.

Proof. The corollary follows from the fact that the entries in T strictly decrease down each column. This means that for each $i \geqslant 1$, the subtableau of T containing entries with integer value i is a horizontal strip. Thus, the labelling $\rho^c(\alpha)$ is equivalent to the labelling $\rho^r(\alpha)$ when applied to this subtableau. □

We make one comment on the corollary, and also give an example, before we proceed with our example of the involution. For any partition ν, a λ-tableau T is said to be *ν-bounded* if $T(1,j) \leqslant \nu'_j$, for each $1 \leqslant j \leqslant \lambda_1$. Then, the formula given by equation 3.29 is Graham positive [7] if we further impose the ν-bounded condition on the tableaux T appearing in the sum. This makes the formula equivalent to [18, Theorem 2.1].

Example 3.39. Consider partitions $\lambda = (3,2,1,1)$, $\mu = (2^2)$, and $(\nu = 4,3,1,1)$. Then, for the calculation of $c^{\nu}_{\lambda\mu}(a)$, we need to consider the weight of column strict λ-tableaux that contain a column word that is Yamanouchi and takes μ to ν. The following table gives all such tableaux, along with their weights.

Tableaux	Word	Weight
$\begin{array}{ccc} \bar{4} & 2 & 1 \\ \bar{3} & \bar{1} & \\ \bar{2} & & \\ \bar{1} & & \end{array}$	12341	$(a_{-1} - a_3)(a_{-3} - a_{-1})$
$\begin{array}{ccc} \bar{4} & 2 & \bar{1} \\ \bar{3} & 1 & \\ \bar{2} & & \\ \bar{1} & & \end{array}$	12341	$(a_{-1} - a_3)(a_{-2} - a_1)$

Tableaux	Word	Weight
$\overline{4}\ \overline{2}\ 1$ $\overline{3}\ \overline{1}$ 2 $\overline{1}$	13412	$(a_0 - a_4)(a_{-3} - a_{-1})$
$\overline{4}\ \overline{2}\ \overline{1}$ $\overline{3}\ 1$ 2 $\overline{1}$	13421	$(a_0 - a_4)(a_{-2} - a_1)$
$\overline{4}\ \overline{2}\ \overline{1}$ $\overline{3}\ \overline{1}$ 2 1	34121	$(a_0 - a_4)(a_{-1} - a_4)$
$\overline{4}\ \overline{2}\ 2$ $\overline{3}\ \overline{1}$ 2 $\overline{1}$	13412	$(a_0 - a_4)(a_{-1} - a_0)$

Tableaux	Word	Weight
$\begin{array}{\|c\|c\|c\|}\hline \overline{4} & 2 & \overline{2} \\ \hline \overline{3} & \overline{1} & \\ \cline{1-2} 2 & & \\ \cline{1-1} \overline{1} & & \\ \cline{1-1}\end{array}$	13412	$(a_0 - a_4)(a_0 - a_1)$
$\begin{array}{\|c\|c\|c\|}\hline \overline{4} & 2 & 2 \\ \hline \overline{3} & \overline{1} & \\ \cline{1-2} \overline{2} & & \\ \cline{1-1} \overline{1} & & \\ \cline{1-1}\end{array}$	12341	$(a_{-1} - a_1)(a_{-1} - a_0)$

Table 1: Table of all tableaux that have weight contributing to the polynomial $c^{\nu}_{\lambda\mu}(a)$

Thus, for the partitions $\lambda = (4, 2, 1, 1)$, $\mu = (2^2)$, and $\nu = (4, 3, 1, 1)$, the coefficient $c^{\nu}_{\lambda\mu}(a)$ is given by summing up all the weights in Table 1.

3.10 Example of the involution

This example illustrates the involution used in the proof of Theorem 3.33.

Example 3.40. We give an example of a bad tableau T which falls in Subcase 1a of the proof. Let $\mu = (2^2)$, $\kappa = (9^2)$, and $\nu = (4, 3, 2, 1)$. Then the following is a bad tableau appearing in $K^{\nu}_{\kappa\mu}(a)$:

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline \overline{4} & 4 & 4 & 4 & 3 & 2 & \overline{2} & 2 & 2 \\ \hline 3 & 3 & \overline{3} & 3 & \overline{3} & 3 & 1 & \overline{1} & \overline{1} \\ \hline \end{array}$$

Let $\alpha = (i, j) = (1, 1)$. Since $L(\alpha) = 4$, we have that $L(\alpha)$ takes μ to $(2, 2, 0, 1)$, which is not a partition. Thus, the word $L(\alpha)$ is not Yamanouchi, (P1) holds, and T is bad. We have the entries $b = \overline{4}$, and $c = 3$, in boxes $(1, 1)$ and $(2, 1)$ respectively. In row 1, box $(1, q) = (1, 4)$ is the rightmost box containing a 4, barred or unbarred, and there are $s = 1$ barred 4's between box $(1, 1)$ and box $(1, 4)$. Counting right from box $(2,2)$, we see there is 1 barred 3 up until box $(2,r) = (2,3)$. There are $x_1 = 3$ unbarred 4's between box α and box $(1,4)$ and $y_1 = 1$ unbarred 3's between box $(2,2)$ and $(2,3)$. We have $r \leqslant q$, so T obeys condition (C2a), Case 1. We obtain the following tableau $T^{\frac{1}{2}}$ after we fix the column ordering:

$T^{\frac{1}{2}} =$

$\overline{\mathbf{4}}$	**4**	**3**	$\overline{\mathbf{3}}$	3	2	$\overline{2}$	2	2
3	**3**	**3**	3	$\overline{3}$	3	1	$\overline{1}$	$\overline{1}$

where the bold entries are the ones which have been affected. Now, we see that we have $y_1 = 1$ unbarred 4's, and $x_1 = 3$ unbarred 3's. The entries weakly decrease down column 3. Thus, let X be the subtableau of entries from box $(1,3)$ to $(1,9)$, and Y the subtableau of entries from box $(2,4)$ to $(2,9)$; X and Y are the bold entries in row 1 and 2 respectively of the tableau:

$T^{\frac{1}{2}} =$

$\overline{4}$	4	**3**	$\overline{\mathbf{3}}$	**3**	**2**	$\overline{\mathbf{2}}$	**2**	**2**
3	3	3	**3**	$\overline{\mathbf{3}}$	**3**	**1**	$\overline{\mathbf{1}}$	$\overline{\mathbf{1}}$

We perform a tailswap on X and Y, producing the tableau T^1:

$T^1 =$

$\overline{4}$	4	**3**	$\overline{\mathbf{3}}$	**3**	**1**	$\overline{\mathbf{1}}$	**1**		
3	3	3	**3**	$\overline{\mathbf{3}}$	**3**	**2**	$\overline{\mathbf{2}}$	**2**	**2**

T^1 has shape $\overline{R}_{1,2}(9,9) = (8,10)$ and each row of T^1 weakly decreases from left to right. We claim that the weight of $\mathrm{ev}(T^1(2,3))$ is equal to the weight of $\mathrm{ev}(T(1,3))$. This is because $3 - \mu_3 = 4 - \rho^r(1,3)$, and $c(2,3) = c(1,3) - 1$ We now apply the second process: reordering the barred entries. We calculate n_2 and n_3. Since there are no entries below row 2, $r'_1 = r'_2 = 0$. The number of barred 2's and 3's in rows 1 and 2 are $r_2 = 1$ and $r_3 = 2$ respectively. Then, $n_2 = \max(\mu_2 + r_2 - \mu_1 - r'_1, 0) = 1$ and $n_3 = \max(\mu_3 + r_3 - \mu_2 - r'_2, 0) = 0$. Thus, we find subtableau P_2 and Q_2 of T containing one barred 1 and one barred 2 respectively. The subtableau P_2 and Q_2 consists of the bold entries in row 1 and row 2 respectively of the tableau:

$T^1 =$

$\overline{4}$	4	3	$\overline{3}$	3	**1**	$\overline{\mathbf{1}}$	$\overline{1}$		
3	3	3	3	$\overline{3}$	3	2	$\overline{\mathbf{2}}$	**2**	**2**

,

It is a fact that the row word of T^1 is not Yamanouchi but we will fix this shortly. We have that $v_1 = 1$ and $w_1 = 2$; these are the numbers of unbarred 1's and 2's in P_2

and Q_2 respectively. We obtain $\widetilde{P_2}$ and $\widetilde{Q_2}$, which are the bold entries in row 2 and row 1 respectively of the tableau:

$\widetilde{T} =$

$\bar{4}$	4	3	$\bar{3}$	3	$\bar{\mathbf{2}}$	$\mathbf{2}$	$\bar{1}$		
3	3	3	3	$\bar{3}$	3	2	$\mathbf{1}$	$\mathbf{1}$	$\bar{\mathbf{1}}$

,

which completes the process of reordering barred entries and thus we have formed $\widetilde{T}$ from T. We claim that the weight of $\widetilde{T}(1,7)$ is equal to the weight of $T(2,7)$. The entry $T(2,7)$ has no barred 1's before it. The entry $\widetilde{T}(1,7)$ has one barred 2 before it. The weights are equal since $1 - \rho^r(2,7)_1 = 2 - \widetilde{\rho}^r(1,7)_2$ and $c(2,7) = c(1,7) - 1$.

The entries of $\widetilde{T}$ not in $\widetilde{P}_2$ and $\widetilde{Q}_2$ which are also unaffected by fixing the column ordering are marked in bold:

$\bar{4}$	4	$\mathbf{3}$	$\bar{\mathbf{3}}$	$\mathbf{3}$	$\bar{2}$	2	$\bar{\mathbf{1}}$		
3	3	3	3	$\bar{3}$	$\mathbf{3}$	$\mathbf{2}$	1	1	$\bar{1}$

We will check that the weights of the unaffected entries can be cancelled out later, but first we check that applying the involution to $\widetilde{T}$ will restore T (throughout bold entries denote affected entries). We do the tail swap, first obtaining

$\widetilde{T}^{\frac{1}{2}} =$

$\bar{4}$	4	$\mathbf{3}$	$\bar{\mathbf{3}}$	$\mathbf{3}$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$	$\bar{\mathbf{1}}$
3	3	3	$\mathbf{3}$	$\bar{\mathbf{3}}$	$\mathbf{3}$	$\bar{\mathbf{2}}$	$\mathbf{2}$	$\bar{\mathbf{1}}$

from which we fix the column ordering and obtain:

$\widetilde{T}^{\frac{1}{2}} =$

$\bar{4}$	$\mathbf{4}$	$\mathbf{4}$	$\mathbf{4}$	3	2	1	1	$\bar{1}$
3	$\mathbf{3}$	$\bar{\mathbf{3}}$	3	$\bar{3}$	3	$\bar{2}$	2	$\bar{1}$

Then, we reorder the barred entries, obtaining:

$T =$

$\bar{4}$	4	4	4	3	2	$\bar{\mathbf{2}}$	$\mathbf{2}$	$\mathbf{2}$
3	3	$\bar{3}$	3	$\bar{3}$	3	$\mathbf{1}$	$\bar{\mathbf{1}}$	$\bar{1}$

Thus, the process restores T from $\widetilde{T}$.

Consider the subtableaux of unaffected entries in $\widetilde{T}$, marked in bold, which are the entries not affected by fixing the column ordering and not in $\widetilde{P}_2$ or $\widetilde{Q}_2$ of $\widetilde{T}$:

$\bar{4}$	4	$\mathbf{3}$	$\bar{\mathbf{3}}$	$\mathbf{3}$	$\bar{2}$	2	$\bar{\mathbf{1}}$		
3	3	3	3	$\bar{3}$	$\mathbf{3}$	$\mathbf{2}$	1	1	$\bar{1}$

(3.30)

This tableau originated from the bad tableau T of shape (9,9):

$\overline{4}$	4	4	4	**3**	**2**	$\overline{2}$	2	2
3	3	$\overline{3}$	**3**	$\overline{\mathbf{3}}$	**3**	1	$\overline{1}$	$\overline{\mathbf{1}}$

(3.31)

with the bold entries of tableau (3.31) paired to the bold entries of tableau (3.30) by the tail swap. By the previous arguments the weight of the nonbold entries in tableau (3.31) and (3.30) are equal.

From the bold entries in tableau (3.31) we form the following monomial subtableaux, we are only concerned with the bold entries of tableau (3.31) equal to 3 so we omit the rest:

$N =$

				3				
			$3'$	$\overline{3}$	3			

From this tableau, we have the integer sequence $N' = \overline{3}33$ formed by listing the 3's left to right and omitting any primed 3's, first from row 1, then row 2. We form the following subtableaux:

$\widetilde{N} =$

		$3'$	3	3					
					$\overline{3}$				

which again has the sequence $N' = \overline{3}33$ when the unprimed entries of $\widetilde{N}$ are listed, omitting primed entries, left to right, starting from row 2, then row 1. Since the boxes containing the primed 3 in N and $\widetilde{N}$ have the same content, the weight of the primed entries are equal. We claim that the weight of the unprimed entries are equal. This follows when we consider that N is the monomial subtableau marked in bold inside the following tableau

$\overline{4}$	4	4	4	**3**	2	$\overline{2}$	2	2
3	3	$\overline{3}$	$\mathbf{3'}$	$\overline{\mathbf{3}}$	**3**	1	$\overline{1}$	$\overline{1}$

and $\widetilde{N}$ is the monomial subtableau marked in bold inside the tableau

$\overline{4}$	4	$\mathbf{3'}$	**3**	**3**	$\overline{2}$	2	1		
3	3	3	3	$\overline{3}$	$\overline{\mathbf{3}}$	2	1	1	$\overline{1}$

where the nonbold entries of the above two tableau have equal weight, by previous arguments. Note that the bold entries occur after one barred 3 in both tableaux, with respect to the row order. Thus, the weights of the unprimed bold entries in both tableaux are equal, and we conclude that the weights of N and $\widetilde{N}$ are equal. $\square$

In summary, in this chapter we calculated the Littlewood–Richardson polynomials $c_{\lambda\mu}^{\nu}(a)$. These structure coefficients occur in the decomposition of PRODUCTS between double Schur functions. In the rest of this thesis, we explore the COMULTIPLICATION structure of the ring of double symmetric functions. Here, the *dual Littlewood–Richardson polynomials* $\widehat{c}_{\lambda\mu}^{\nu}(a)$ arise as the structure coefficients when the COPRODUCT is applied to a double Schur function. The rest of this thesis is focused on investigating the coefficients $\widehat{c}_{\lambda\mu}^{\nu}(a)$.

4 The dual Littlewood–Richardson polynomials

Recall the definition of the double power sums symmetric functions

$$p_k(x\|a) = \sum_{i\geqslant 1}(x_i^k - a_i^k), \qquad \text{for } k = 1, 2, \dots,$$

which generate the ring of double symmetric functions $\Lambda(x\|a)$ over the ring $\mathbb{Q}[a]$. The comultiplication on the ring $\Lambda(x\|a)$ is the $\mathbb{Q}[a]$-linear ring homomorphism

$$\Delta : \Lambda(x\|a) \to \Lambda(x\|a) \otimes_{\mathbb{Q}[a]} \Lambda(x\|a)$$

defined on the double complete symmetric functions $p_k(x\|a)$ by

$$\Delta(p_k(x\|a)) = p_k(x\|a) \otimes 1 + 1 \otimes p_k(x\|a).$$

When the operation Δ is applied to the basis element $s_\nu(x\|a)$, for ν a partition, we obtain the dual Littlewood–Richardson polynomials $\widehat{c}_{\lambda\mu}^{\nu}(a)$:

$$\Delta(s_\nu(x\|a)) = \sum_{\lambda,\nu} \widehat{c}_{\lambda\mu}^{\nu}(a) s_\lambda(x\|a) \otimes s_\mu(x\|a),$$

summed over partitions λ and ν such that $|\lambda| + |\mu| \geqslant |\nu|$, see Section 2.2.1 and Molev [16]. When the number of boxes $|\lambda| + |\mu| = |\nu|$, the polynomial $\widehat{c}_{\lambda\mu}^{\nu}(a)$ is the Littlewood–Richardson coefficient $c_{\lambda\mu}^{\nu}$ defined by the product of two classical Schur functions,

$$s_\lambda(x)s_\mu(x) = \sum_{\nu} c_{\lambda\mu}^{\nu} s_\nu(x).$$

In contrast, the double Littlewood–Richardson polynomials $c_{\lambda\mu}^{\nu}(a)$ defined by the product

$$s_\lambda(x\|a)s_\mu(x\|a) = \sum_{\nu} c_{\lambda\mu}^{\nu}(a) s_\nu(x\|a)$$

between two double Schur functions has the property that $c_{\lambda\mu}^{\nu}(a) = c_{\lambda\mu}^{\nu}$ when the number of boxes $|\lambda| + |\mu| = |\nu|$, and is nonzero only when $|\lambda| + |\mu| \leqslant |\nu|$.

The coefficients $\widehat{c}_{\lambda\mu}^{\nu}(a)$ are important because they describe the comultiplication structure of the ring $\Lambda(x\|a)$ and also the equivariant cohomology of infinite grassmanian [8]. In addition, there are several nice combinatorial identities involving them, see equations (4.1) and (4.2) below, first described by Molev [16]. However, not much is known about the polynomials $\widehat{c}_{\lambda\mu}^{\nu}(a)$. In [16] Molev provides a rule which calculates the polynomials $\widehat{c}_{\lambda\mu}^{\nu}(a)$ using tableaux. However, this rule does not shed any light on any positivity (in the sense of Graham [7]) or stability properties of the coefficients $\widehat{c}_{\lambda\mu}^{\nu}(a)$, recalling that

1. Stability means that the coefficients do not depend on the number of variables n.

2. Positive means that the coefficients may be expressed in terms of differences $a_i - a_j$, with positive integer coefficients, where $i < j$.

We have two main results in the rest of this thesis. The first is a refinement of the rule provided by Molev which explicitly expresses the coefficients $\widehat{c}_{\lambda\mu}^{\nu}(a)$ in a stable way. The second, given in the next chapter, is a rule to calculate $\widehat{c}_{\lambda\mu}^{\nu}(a)$ for the special case where ν/μ does not contain a subdiagram consisting of 2×2 boxes. This will be in fact a special case of a more general rule involving generalised Frobenius–Schur functions which we provide and prove.

In addition to describing the comultiplication structure on the ring $\Lambda(x\|a)$, the dual Littlewood–Richardson polynomials $\widehat{c}_{\lambda\mu}^{\nu}(a)$ also describe a multiplication rule for the *dual Schur functions*, see equation (4.1). They also express skew double Schur functions in terms of nonskew double Schur functions, see equation (4.2). Finally, they also arise as a special case of the structure coefficients for the ring of *double supersymmetric functions* $\Lambda(x/y\|a)$; namely they occur in the multiplication rule for the distinguished basis elements called the *generalised Frobenius–Schur functions*. The calculation of the dual Littlewood–Richardson polynomials will occupy our efforts for the rest of this thesis.

In [16, Sec 3.1], Molev defines the ring $\hat{\Lambda}(x\|a)$ to be the ring of formal series of the symmetric functions in the set of indeterminates $x = (x_1, x_2, \dots)$ with coefficients in $\mathbb{Q}[a]$. An element $q(x)$ in $\hat{\Lambda}(x\|a)$ may be considered as

$$q(x) = \sum_{\lambda} b_\lambda(a) s_\lambda(x),$$

summed over all partitions λ and with coefficients $b_\lambda(a) \in \mathbb{Q}[a]$.

We define the dual Schur function as a ratio of alternants. We work with the finite set of variables $x^{(n)} = (x_1, \dots, x_n)$, for a $n \geqslant 0$. For any n-tuple $\alpha = (\alpha_1, \dots, \alpha_n)$ define the skew-symmetric polynomial

$$A_\alpha(x^{(n)}, a) = \det\big((x_i, a)^{\alpha_j}(1 - a_{n-\alpha_j-1}x_i)(1 - a_{n-\alpha_j-2}x_i)\dots(1 - a_{1-\alpha_j}x_i)\big)_{i,j=1}^{n}$$

where $(x_i, a)^0 = 1$ and

$$(x_i, a)^r = \frac{x_i^r}{(1 - a_0x_i)(1 - a_{-1}x_i)\dots(1 - a_{1-r}x_i)},$$

for all $r \geqslant 1$.

For a partition λ, define the *dual Schur polynomial* $\hat{s}_\lambda(x^{(n)}\|a)$ to be the ratio

$$\hat{s}_\lambda(x^{(n)}\|a) = \frac{A_\lambda(x^{(n)}, a)}{A_\delta(x^{(n)}, a)},$$

where δ is the partition $\delta = (n-1, n-2, \ldots, 0)$. Since $A_\delta(x^{(n)}, a)$ is the Vandermonde determinant, this is a ratio of two alternants and hence the polynomial $\hat{s}_\lambda(x^{(n)}\|a)$ belongs to $\hat{\Lambda}(x\|a)$.

It turns out by Theorem 4.3 in Molev [16] that the dual Littlewood–Richardson polynomials describe a multiplication rule for the functions $\hat{s}_\lambda(x\|a)$. Let μ be a partition. Then, we can decompose the product

$$\hat{s}_\lambda(x\|a)\hat{s}_\mu(x\|a) = \sum_\nu \widehat{c}_{\lambda\mu}^{\nu}(a)\hat{s}_\nu(x\|a), \tag{4.1}$$

summed over partitions ν, by using the dual Littlewood–Richardson polynomials $\widehat{c}_{\lambda\mu}^{\nu}(a)$.

Let θ be a skew diagram. The definition of a double Schur polynomial in the variables $x^{(n)} = (x_1, \ldots, x_n)$ can be extended to skew diagrams λ as well. For example, we could define a polynomial $\widetilde{s}_\theta(x^{(n)}\|a)$ by

$$\widetilde{s}_\theta(x^{(n)}\|a) = \sum_T \prod_{\alpha\in\theta}(x_{T(\alpha)} - a_{T(\alpha)-c(\alpha)})$$

summed over column strict reverse θ-tableau T with entry at most n. This polynomial which results from such a definition is not stable with respect to the evaluation homomorphism $\varphi_n : x_n \to a_n$, equation (2.1), and thus it does not permit us to define a skew double Schur function $\widetilde{s}_\lambda(x\|a)$ where $x = (x_1, x_2, \ldots)$ is an infinite sequence of variables. We will now investigate a function $s_\theta(x\|a)$ which we will call the *skew double Schur function*.

To define the function $s_\theta(x\|a)$, we discuss the so called *generalised Frobenius–Schur functions* which are defined in the next chapter. These functions are denoted by $s_\theta(x/y\|a)$, where $y = (y_1, y_2, \ldots)$ is another infinite sequence of variables. They were first introduced in [13] and form a basis of the ring of generalised supersymmetric functions $\Lambda(x/y\|a)$. Let $(a')_i = -a_{1-i}$, for all $i \in \mathbb{Z}$, be an infinite sequence of variables related to the sequence a. It was noted in Molev [16] that the ring isomorphism between $\Lambda(x/y\|a) \to \Lambda(x\|a)$ given by the map $y_i \mapsto -a_i$ maps the generalised Frobenius–Schur functions to the double Schur functions, that is, for partitions λ,

$$s_\lambda(x, y\| - a') \mapsto s_\lambda(x\|a).$$

Thus, we define the *skew double Schur function* $s_{\nu/\mu}(x\|a)$, by setting it to be the image of $s_{\nu/\mu}(x, y\| - a')$ under the evaluation map $y_i \mapsto -a_i$.

Suppose $\theta = \nu/\mu$, for some partition ν which contains another partition μ. Then, the function $s_\theta(x\|a)$ admits the following decomposition

$$s_\theta(x\|a) = s_{\nu/\mu}(x\|a) = \sum_\lambda \widehat{c}_{\lambda\mu}^{\nu}(a) s_\lambda(x\|a), \tag{4.2}$$

summed over partitions λ, see Molev [16, Section 2.4]. In other words, the dual Littlewood–Richardson polynomials $\widehat{c}_{\lambda\mu}^{\nu}(a)$ can be used to expand the skew double Schur function in terms of the basis of non-skew double Schur functions.

The decomposition given by equation (4.2) gives rise to a formula that we can use to calculate the coefficients $\widehat{c}_{\lambda\mu}^{\nu}(a)$. This formula relies on *ν/μ-supertableaux T* which are defined as follows. Fix a $n > 0$. We fill in the boxes of the skew diagram ν/μ with the symbols $1, 1', 2, 2', \dots, n, n'$ such that the following two conditions are obeyed on the rows and columns of ν/μ:

1. In each row (resp. column) each primed index is to the left (resp. above) of each unprimed index.

2. Unprimed indices weakly decrease along the rows and strictly decrease down the columns, while primed indices strictly increase along the rows and weakly increase down the column.

The *column order* is the ordering on boxes of ν/μ by reading in columns from left to right and from bottom to top in each column. We write $\alpha < \beta$ if α is strictly before β with respect to this column order.

We define *barring* on a ν/μ supertableaux T as follows, let $\alpha_1 < \alpha_2 < \cdots < \alpha_n$ be a sequence of n boxes of ν/μ containing unprimed entries. We will put a bar over the entries in these boxes, and call the tableau formed this way a barred ν/μ-supertableau.

Let T be a barred ν/μ-supertableau. The sequence of barred entries $r_1 r_2 \dots r_t = T(\alpha_1)T(\alpha_2)\dots T(\alpha_t)$, written with respect to column order, is the (barred) *word* R of T. As in Chapter 2, we say that a word R is *Yamanouchi* when applied to the empty partition $\varnothing$ if the following is a sequence of partitions

$$\varnothing = \rho^{(0)} \xrightarrow{r_1} \rho^{(1)} \xrightarrow{r_2} \dots \xrightarrow{r_t} \rho^{(t)} = \nu,$$

recalling that by $\rho^{(i-1)} \xrightarrow{r_i} \rho^{(i)}$ we mean we form the composition $\rho^{(i)}$ from $\rho^{(i-1)}$ by adding a box to the end of row r_i of the composition $\rho^{(i-1)}$.

For each box $\alpha \in \nu/\mu$ which is occupied by an unprimed, unbarred entry, we define the *labelling* $\rho(\alpha)$ on the box α to be $\rho(\alpha) = \rho^{(i)}$, for $0 \leqslant i \leqslant i$ such that $\alpha_i < \alpha < \alpha_{i+1}$.

Example 4.1. Let $\mu = (3, 3, 2)$ and $\nu = (6, 5, 4, 3, 3)$. Then we illustrate the column order on the diagram ν/μ by filling in the boxes with integers 1 up to 13, such that the first box is filled with 1, and so on.

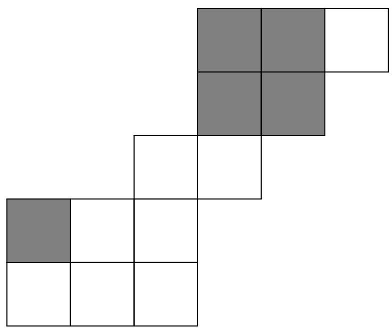
			10	12	13
			9	11	
		7	8		
2	4	6			
1	3	5			

The following is a barred ν/μ-supertableau T

$T =$

			$1'$	$2'$	$\overline{8}$
			$1'$	$5'$	
		4	4		
$7'$	$\overline{5}$	3			
$\overline{4}$	$\overline{4}$	$\overline{2}$			

As an illustration we grey out the boxes of T containing primed entries, and leave the boxes containing unprimed entries white



We remark that the condition that primed indices of T in each row (resp. column) should be to the left (resp. above) of unprimed indices of T is equivalent to the fact that there exist a partition ρ such that $\mu \subseteq \rho \subseteq \nu$ with all primed indices within the skew diagram ρ/μ and all unprimed indices within the skew diagram ν/ρ. In the case of this example, $\rho = (5^2, 2, 1)$, and the skew diagram ρ/μ consists of the boxes marked in grey.

The word of T is the sequence of integers $R = 44528$, which are the barred entries of T from first to last, listed left to right with respect to the column order. $\square$

The following is a formula for the dual Littlewood–Richardson polynomials $\widehat{c}_{\lambda\mu}^{\nu}(a)$, given as Theorem 4.6 from [16]. Let λ, μ and ν be partitions such that ν contains μ. Also, fix an integer $n > 0$ such that $\nu_i' - \mu_i' \leqslant n$, for all i; that is, the columns of ν/μ contain at most n boxes. Then, we have

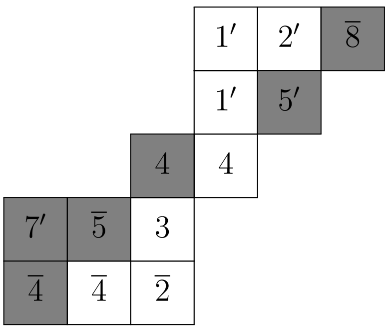
$$\widehat{c}_{\lambda\mu}^{\nu}(a) = \sum_{T} \prod_{\substack{\alpha\in\nu/\mu,\\ T(\alpha)\text{ unprimed, unbarred}}} \left(a_{T(\alpha)-\rho(\alpha)_{T(\alpha)}} - a_{T(\alpha)-c(\alpha)}\right) \prod_{\substack{\alpha\in\nu/\mu,\\ T(\alpha)\text{ primed}}} \left(a_{T(\alpha)-c(\alpha)} - a_{T(\alpha)}\right), \tag{4.3}$$

summed over ν/μ-supertableaux T with maximum entry n such that the tableau T contains the word $R : \varnothing \to \lambda$ which is Yamanouchi.

Molev makes the following remark [16, Rmk 4.7] about equation (4.3): the formula provided by this equation involves terms which cancel pairwise, and it would be interesting to find a combinatorial presentation of $\widehat{c}_{\lambda\mu}^{\nu}(a)$ analagous to Theorem 3.33 and understand the positivity properties. As a conclusion to this chapter, we give a refinement which introduces the stability property to $\widehat{c}_{\lambda\mu}^{\nu}(a)$, that is, we eliminate the condition that the entries of T appearing in the sum (4.3) should be at most n. Note that this stability property is implied by the fact that the skew double Schur functions are specialisations of the generalised Frobenius–Schur functions (Chapter 5). The Pieri rule we derive for these generalised Frobenius–Schur functions in Chapter 5 will have the stability property as well.

We start with the following definition of *border entries* of T. For a box $\alpha \in \nu/\mu$, if $T(\alpha)$ is primed, then $T(\alpha)$ is said to be a *border entry* if it does not have a primed entry to the south or east of it. If $T(\alpha)$ is unprimed, then $T(\alpha)$ is also said to be a *border entry* if it does not have an unprimed entry to the north or west of it.

Example 4.2. We use our tableau T from the previous example. In the following illustration, we grey out the boxes which contain the border entries of T.



We can think of the border entries as entries which can either be primed or unprimed, without changing the requirement that primed indices of T in each row (resp. column) should be to the left (resp. above) of unprimed indices of T. □

Theorem 4.3. *Formula (4.3) still holds if we impose two further conditions on tableaux T contributing to the sum (4.3):*

Cond. 1: *The maximum entry occuring in T, primed or not, takes value at most $l(\lambda)$.*

Cond. 2: *For each $i \geqslant 1$, the most southwest border entry of T taking value i is a barred unprimed entry.*

Condition 1 of Theorem 4.3 provides the stability property for the coefficients $\widehat{c}^{\nu}_{\lambda\mu}(a)$; that is, this coefficient does not depend on the number of variables $(x_1, \ldots, x_n)$, and only the length of λ. We will now prove this theorem; an example of the definitions and involutions used within the proof follow after.

Proof. First, we will show that we may impose Condition 1 on the tableaux appearing in the sum (4.3) without changing the sum. We will construct an involution on the set of tableaux that do not obey Condition 1. We first make the following definition. Suppose a border entry meets one of the following two exclusive conditions:

1. $T(\alpha)$ is a primed border entry, and the tableau formed by removing the prime from $T(\alpha)$ is still a valid ν/μ supertableau.
2. Otherwise, suppose $T(\alpha)$ is a unprimed border entry, and the tableau formed by adding a prime to the entry $T(\alpha)$ is still a valid ν/μ-supertableau.

Condition 1. only happens if the entry to the east of $T(\alpha)$ is at most $T(\alpha)$, and the entry south of $T(\alpha)$ is strictly smaller than $T(\alpha)$. Condition 2. only happens if the entry west of $T(\alpha)$ is strictly less than $T(\alpha)$, and the entry north of $T(\alpha)$ is at most $T(\alpha)$. If either of these two conditions is met, we say that the border entry $T(\alpha)$ is *movable.*

Now, we construct an involution on the set of tableaux that do not obey Condition 1. The main claim is that if T is such a tableau, then there exist a most southwest movable border entry $T(\alpha)$ greater than $l(\lambda)$. Note that since $T(\alpha) > l(\lambda)$ the entry $T(\alpha)$ must be unbarred, since the barred entries of T provide a word $R : \varnothing \to \lambda$. First, note that since primed entries increase along rows and down columns, each primed border entry is the largest entry in its row and column. Similarly, note that since unprimed entries decrease along rows and down columns, each unprimed border entry is the largest entry in its row and column. Then, our claim follows if we let $T(\alpha)$ be the largest border entry. In case there are multiple border entries with the same maximum value, we let $T(\alpha)$ be the most southwest such entry. We justify our claim as follows. There are two cases, either $T(\alpha)$ is primed, or it is not.

Suppose $T(\alpha)$ is primed. If they exist, let α^s and α^e be the boxes south and east of α respectively. Then, we assert that $T(\alpha)$ is movable because $T(\alpha^s) < T(\alpha)$ and $T(\alpha^e) \leqslant T(\alpha)$. This is because $T(\alpha^s)$ is at most equal to the border entry to the west of it, which is smaller than $T(\alpha)$, by our assumption that the box α is most southwest. Furthermore, $T(\alpha^e)$ is less than the border entry to the north of it, which takes value at most $T(\alpha)$.

On the other hand, suppose $T(\alpha)$ is unprimed. If they exist, let α^n and α^w be the boxes north and west of α respectively. Then, we assert that $T(\alpha)$ is movable because $T(\alpha^n) \leqslant T(\alpha)$ and $T(\alpha^w) < T(\alpha)$. This is because $T(\alpha^n)$ is at most equal to the border entry to the east of it, which is at most equal to $T(\alpha)$. Furthermore, $T(\alpha^w)$ is at most equal to the border entry south of it, which is smaller than $T(\alpha)$, by our assumption that α is most southwest.

Thus, if T is a tableau that does not obey Condition 1, there exists a most southwest movable border entry $T(\alpha)$. Then, under the involution the tableau T is paired to the tableau $\widetilde{T}$ formed by removing the prime from $T(\alpha)$ if $T(\alpha)$ is primed, or by adding a prime to $T(\alpha)$ if $T(\alpha)$ is not primed. We check that the weights of T and $\widetilde{T}$ are opposite. In fact, all the entries of T and $\widetilde{T}$ are equal except for $T(\alpha)$ and $\widetilde{T}(\alpha)$. Suppose $T(\alpha)$ is unprimed, then we have

$$\begin{aligned} \mathrm{ev}(T(\alpha)) &= a_{T(\alpha)-\rho(\alpha)_{T(\alpha)}} - a_{T(\alpha)-c(\alpha)} \\ &= a_{T(\alpha)} - a_{T(\alpha)-c(\alpha)} \\ &= -\mathrm{ev}(\widetilde{T}(\alpha)), \end{aligned}$$

where the second equality holds because $\rho(\alpha) \subseteq \lambda$ which has less rows than $T(\alpha)$. The same argument applies if $T(\alpha)$ is primed instead. This demonstrates that if T has an entry greater than $l(\lambda)$ then there is a paired tableau $\widetilde{T}$ with opposite weight, thus we may cancel their contributions from equation (4.3).

We now do the same for the set of tableaux T which does not obey Condition 2. We may assume that Condition 1 holds for these tableaux. We claim that there exist a most southwest box α of T which obeys the following condition: the entry $T(\alpha)$ is the largest movable, unbarred, border entry with integer value i which is also the most southwest occurence of i in T. Suppose this $T(\alpha)$ is primed. Then it is movable because $T(\alpha^s)$ is less than $T(\alpha)$, since it must be at most equal to the border entry west of it, which is less than $T(\alpha)$, by our assumption that $T(\alpha)$ is largest and most southwest. Furthermore, $T(\alpha^e)$ is at most equal to $T(\alpha)$, since $T(\alpha^e)$ is at most equal to the border entry north of it, which is at most equal to $T(\alpha)$. The same arguments hold if $T(\alpha)$ is unprimed. This demonstrates that we may cancel out tableaux violating Condition 2 from the sum (4.3). □

Example 4.4. We start by giving examples of movable border entries occurring in skew tableaux containing 3 boxes. First, the primed entry in the following tableau is a movable border entry:

$3'$	3
2	

since if we remove the prime

3	3
2	

the unprimed entries weakly decrease along the row and strictly decrease down the column. Thus, the tableau formed by removing the prime is a valid supertableaux.

Second, the unprimed entry in the following tableau is a movable border entry:

	$4'$
$3'$	4

since if we add a prime to the unprimed entry

	$4'$
$3'$	$4'$

the primed entries weakly decrease down the column, and weakly increase along the row. Thus, the tableau formed by adding a prime to the unprimed entry is a valid supertableaux.

For a general ν/μ-supertableaux T, we can think of all movable border entries of T as belonging to a subtableaux of T, consisting of 3 entries, similar to one of the above tableaux.

Suppose that there is a tableau T which does not obey Condition 1. or Condition 2. of Theorem 4.3, but contributes to the sum in equation 4.3. Suppose that the tableau T does not obey Cond. 1. Then there is an entry of T which is greater than $l(\lambda)$. We let α be the box which contains the greatest and most southwest such entry. It belongs in one of the following trios of boxes, either

$T(\alpha)'$	$T(\alpha^e)$
$T(\alpha^s)$	

(4.4)

if $T(\alpha)$ is primed, or

$$\begin{array}{cc} \cline{2-2} & \multicolumn{1}{|c|}{T(\alpha^n)'} \\ \hline \multicolumn{1}{|c|}{T(\alpha^w)'} & \multicolumn{1}{|c|}{T(\alpha)} \\ \hline \end{array} \tag{4.5}$$

if $T(\alpha)$ is unprimed. Note in both these sets of subtableaux we slightly abuse notation by inserting a prime to distinguish between the primed and unprimed entries.

In the subtableau in (4.4), we have that $|T(\alpha)| \geqslant |T(\alpha^e)|$ and $|T(\alpha)| > |T(\alpha^s)|$. This is because $T(\alpha^s)$ must be at most equal to the unprimed border entry to the west of it, which is smaller than $|T(\alpha)|$ by assumption on the maximality of $|T(\alpha)|$. Similarly, $T(\alpha^e)$ is smaller than the unprimed border entry to the north of it, which again is at most $|T(\alpha)|$ because α is the most southwest box with integer value $|T(\alpha)|$. Thus the prime may be removed from the entry in box $T(\alpha)$ and the trio of entries created will still be a valid subtableau. A similar argument holds for the trio of entries in equation (4.5).

$\square$

5 A Pieri rule for the generalised Frobenius–Schur functions

Introduce the following finite sequence of variables $y^{(n)} = (y_1, y_2, \dots, y_n)$, and recall the finite sequence $x^{(n)} = (x_1, x_2, \dots, x_n)$. Also, introduce the infinite sequence of variables $y = (y_1, y_2, \dots)$ which is a counterpart to the sequence $x = (x_1, x_2, \dots)$. Let $\Lambda(x^{(n)}/y^{(n)})$ denote the ring of supersymmetric functions, which is generated over $\mathbb{Q}[a]$ by the power sums supersymmetric polynomials

$$p_k(x^{(n)}, y^{(n)}) = \sum_{i=1}^{n} \left(x_i^k + (-1)^{k-1} y_i^k \right),$$

for integers $k \geqslant 0$. These polynomials are stable with respect to the simultaneous specialisations $x_n = 0$ and $y_n = 0$. Therefore, we may define the ring of supersymmetric functions $\Lambda(x/y\|a)$ generated over $\mathbb{Q}[a]$ by the power sums supersymmetric functions

$$p_k(x, y) = \sum_{i=1}^{\infty} \left(x_i^k + (-1)^{k-1} y_i^k \right),$$

see for example Macdonald [15].

In this chapter, we discuss a multiparameter generalisation of $\Lambda(x/y)$ depending on the extra sequence of variables $a = (a_i)_{i \in \mathbb{Z}}$. This is the ring $\Lambda(x/y\|a)$, which has a distinguished $\mathbb{Q}[a]$-basis consisting of the generalised Frobenius-Schur functions $s_\lambda(x/y\|a)$, over all partitions λ. The elements $s_\lambda(x/y\|a)$ may be considered as the super-analogue of the double Schur function. Factorial supersymmetric Schur polynomials were first introduced in [17], whereas the definition of their stable version, the functions $s_\lambda(x/y\|a)$ we give in this chapter, is due to [13].

We are interested in the functions $s_\lambda(x/y\|a)$ for the following reasons. First, the definition of the generalised Frobenius-Schur function holds if $\lambda = \theta$, a skew diagram. Let a' be another infinite sequence related to a via the following

$$(a')_i = -a_{1-i},$$

for all integers i. If λ is a partition, the evaluation map $y_i \mapsto -a_i$ for all integers i sends the function $s_\lambda(x/y\|-a')$ to the double Schur function $s_\lambda(x\|a)$, which is a consequence of Proposition 2.5 and Remark 2.4 of [16]. In this way, for the skew partition θ we may define the double Schur function $s_\theta(x\|a)$ as the image of $s_\theta(x/y\| - a')$ under the map $y_i \mapsto -a_i$.

Second, let $k_{\theta\pi}^{\lambda}(a)$ denote the structure coefficients occuring between the product of two generalised Frobenius–Schur functions,

$$s_\theta(x/y\|a) s_\pi(x/y\|a) = \sum_{\lambda} k_{\theta\pi}^{\lambda}(a) s_\lambda(x/y\|a),$$

such that θ is a skew partition, and π and λ are both partitions such that λ has at least the number of boxes of π.

As a special case, when $\pi = \varnothing$ and for the sequence $-a'$, we have

$$s_\theta(x/y\| - a') = \sum_\lambda k^\nu_{\theta\varnothing}(-a')s_\lambda(x/y\| - a'). \tag{5.1}$$

Let $\theta = \nu/\mu$ for a partition ν containing another partition μ. Recall from equation (4.2) we can use the dual Littlewood–Richardson polynomials $\widehat{c}^{\,\nu}_{\lambda\mu}(a)$ to decompose the skew double Schur functions

$$s_\theta(x\|a) = s_{\nu/\mu}(x\|a) = \sum_\lambda \widehat{c}^{\,\nu}_{\lambda\mu}(a)s_\lambda(x\|a)$$

in terms of the double Schur functions $s_\lambda(x\|a)$ corresponding to partitions λ. Under the map $y_i \mapsto -a_i$, equation (5.1) identifies the coefficient $k^\nu_{\theta\varnothing}(-a')$ with $\widehat{c}^{\,\nu}_{\lambda\mu}(a)$. Thus, by giving a rule to calculate the coefficients $k^\lambda_{\theta\pi}(a)$, we will also be able to calculate the dual Littlewood–Richardson polynomials $c^\nu_{\lambda\mu}(a)$.

The aim of this chapter is to use a recurrence relation to calculate the coefficients $k^\lambda_{\theta\pi}(a)$ where θ is a disjoint union of skew hooks; that is, θ is a skew diagram that does not contain a 2×2 subdiagram. As a special case of this, we will then derive a rule to calculate $\widehat{c}^{\,\nu}_{\lambda\mu}(a)$ when ν/μ is a skew hook. This is the last major result of this thesis.

The main results of this chapter are set out as follows. First we give necessary definitions so that we may introduce the recurrence relation, difference formula, and the barred tableaux which we will use in our calculations. The main statement of our result is Theorem 5.34, but leading up to this theorem, we first calculate (in Section 5.9) the coefficients $k^\lambda_{\theta\pi}(a)$ for when the skew partition λ/π has one box. Then, we calculate the coefficients $k^\lambda_{\theta\pi}(a)$ for a particular case when the skew partition λ/π has two boxes. This calculation involves Lemma 5.21, which is essential in proving Theorem 5.34. We then discuss Theorem 5.34 and its proof. Finally, we conclude with a discussion of how the general coefficients $k^\lambda_{\theta\pi}(a)$ for arbitrary partitions λ and π may be calculated.

We begin by introducing a tableaux representation for the generalised Frobenius–Schur functions $s_\lambda(x/y\|a)$.

5.1 Generalised Frobenius–Schur functions

We give a combinatorial formula for $s_\lambda(x/y\|a)$ in terms of tableaux, which we define shortly. This is equivalent to two given in [13, Section 4]. An equivalent definition is also given in [16, Section 2.4].

Define the alphabet $\mathbb{A} = \{1, 1', 1^*, 2, 2', 2^* \dots\}$ of unprimed, primed, and starred integers. If $a \in \mathbb{A}$, we denote by $|a|$ the *integer value* of a, without any primes or stars. Let θ be a skew partition. Then the $\mathbb{A}$-*tableau* T of shape θ is the diagram of θ filled in with entries from the alphabet $\mathbb{A}$ subjected to further additional conditions on the rows and columns of T:

1. First, each row weakly increases reading left to right and each column weakly increases, from top to bottom. By weakly increase we mean that if a and b are entries of a row (resp. column) of T, then $|a| \leqslant |b|$ if the entry b is to the right of (resp. below) the entry a.

2. Second, the most northeastern entry in each connected component of T containing entries equal to i, primed or unprimed, is an i^*.

3. Third, for each $i \geqslant 1$ there is to be only one occurrence of an unprimed i or i^* in each column, and one occurrence of a primed i or i^* in each row in the tableau T.

This last condition means that for each $i \geqslant 1$, the subtableaux of T consisting of entries with integer value i is a disjoint union of *skew hooks*, where a skew hook is a connected skew diagram which does not contain a 2×2 block of squares.

The *weight* of a tableau T is defined as follows. First we define the weight for unprimed, primed, and starred entries:

Unprimed: If $T(\alpha)$ is unprimed, then the weight of $T(\alpha)$ is

$$\mathrm{ev}(T(\alpha)) = x_{T(\alpha)} - a_{c(\alpha)+1}$$

Primed: If $T(\alpha)$ is primed, then the weight of $T(\alpha)$ is

$$\mathrm{ev}(T(\alpha)) = y_{T(\alpha)} + a_{c(\alpha)+1}$$

Starred: If $T(\alpha)$ is starred, then the weight of $T(\alpha)$ is

$$\mathrm{ev}(T(\alpha)) = x_{T(\alpha)} + y_{T(\alpha)}$$

The weight of a tableau T is then defined to be the product of the weights of the entries of T:

$$\mathrm{ev}(T) = \prod_{\alpha \in \theta} \mathrm{ev}(T(\alpha)),$$

and the generalised Frobenius–Schur function $s_\theta(x/y\|a)$ is equal to

$$s_\theta(x/y\|a) = \sum_T \mathrm{ev}(T),$$

summed over all $\mathbb{A}$-tableau T of shape θ.

Example 5.1. Consider the following $\mathbb{A}$-tableau of shape $(3^3)/(1)$.

	1	1^*
1	$1'$	2^*
$1'$	2	$2'$

If we consider the subtableaux T_1 and T_2 containing entries with integer value 1 and 2 respectively we have the following:

In both subtableaux we see that the entries occupy a skew diagram known as a skew hook.

The following are the weights of each entry in the second subtableau. First, the starred entry in box $(2,3)$ has weight equal to x_2+y_2. The primed entry in box $(3,3)$ has weight equal to $y_2 + a_{c(3,3)+1} = y_2 + a_1$. The unprimed entry in box $(3,2)$ has weight equal to $x_2 - a_{c(3,2)+1} = x_2 - a_0$. Thus the weight of the second subtableau is

$$\mathrm{ev}(T_2) = (x_2 + y_2)(y_2 + a_1)(x_2 - a_0).$$

□

Example 5.2. We provide another example which demonstrates that under the evaluation map $y_i \mapsto -a_i$ we recover the double Schur function $s_\lambda(x\|a)$ from $s_\lambda(x/y\| - a')$. Let $\lambda = (2)$, and $\rho = (1^3)$. We have that

$$s_\lambda(x/y\| - a') = \sum_{1\leqslant j}(x_j - a_0)(x_j + y_j) + \sum_{1\leqslant i<j}(x_i + y_i)(x_j + y_j).$$

The map $y_i \mapsto -a_i$ maps $s_\lambda(x/y\| - a')$ to

$$\begin{aligned}\sum_{1\leqslant j}(x_j - a_0)(x_j - a_j) + &\sum_{1\leqslant i<j}(x_i - a_i)(x_j - a_j)\\ &= \sum_{1\leqslant j}(x_1 - a_0 + x_2 - a_1 + \cdots + x_j - a_{j-1})(x_j - a_j)\\ &= s_\lambda(x\|a)\end{aligned}$$

as claimed.

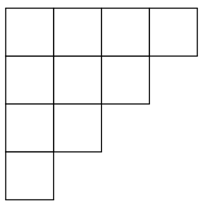
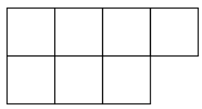
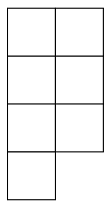
□

We come to the second combinatorial method referenced by the title of this thesis. This method is due to the work of Okounkov on interpolation formulae for the double Schur functions, see [21], [22] and also [20]. We will describe the recurrence relation which applies to the generalised Frobenius–Schur functions, starting with the vanishing theorem found by [13]. Then, to calculate the polynomials $k_{\theta\pi}^{\lambda}(a)$ we apply methods used by Molev [18] to calculate the Littlewood–Richardson polynomials.

5.2 Vanishing property for the generalised Frobenius–Schur functions

Let ν be a partition. Define the *depth* of ν, denoted $d(\nu)$, to be the number of boxes of ν lying on the diagonal of ν, which then consists of the boxes $(1,1),\ldots,(d(\nu),d(\nu))$. The *rows of* ν *above the diagonal* are the first $d(\nu)$ rows of ν, and the *columns of* ν *below the diagonal* are the first $d(\nu)$ columns of ν. Essentially what we are doing is emphasizing that the partition ν is split into two halves by its diagonal; the upper half consisting of the rows above the diagonal and the lower half consisting of the columns below the diagonal. We will define $\nu^{\wedge}$ to be the partition formed by taking the first $d(\nu)$ rows of ν, and $\nu^{\vee}$ to be the partition formed by taking the first $d(\nu)$ columns of ν.

Example 5.3. Let $\mu = (4,3,2,1)$. Then the boxes lying on the diagonal of ν consist of the boxes $(1,1)$ and $(2,2)$, and the depth of ν is 2. Then we have the following diagrams

$\mu =$  $\mu^{\wedge} =$  $\mu^{\vee} =$ 

□

The generalised Frobenius Schur functions obey a vanishing property, subject to the following specialisation of the sequences x and y. Define the sequence of variables $x(\nu)$ and $y(\nu)$ as follows:

$$\begin{aligned} x(\nu)_i = a_{\nu_i-i+1}, \qquad y(\nu)_i &= a'_{\nu'_i-i+1}, && \text{if } 1 \leqslant i \leqslant d(\nu) \\ x(\nu)_i = y(\nu)_i &= 0, && \text{if } i > d(\nu). \end{aligned}$$

A key feature of the definition of the weight of an entry of T is the dependence on whether the entry is unprimed, primed or starred. These correspond to a weight depending on the sequence x, or y, or both x and y respectively. Under the specialisation $x \to x(\nu)$ and $y \to y(\nu)$, the weights of unprimed, primed or starred entries then depend on rows of ν above the diagonal, and columns of ν below the diagonal, or both, respectively.

Example 5.4. Let $\rho = (2^3)$ be a partition. Then we have the specialisation

$$\begin{aligned} x(\rho) &= (a_{2-1+1}, a_{2-2+1}, 0, \dots) \\ &= (a_2, a_1, 0, \dots), \end{aligned}$$

and the specialisation

$$\begin{aligned} y(\rho) &= (a'_{3-1+1}, a'_{3-2+1}, 0, \dots) \\ &= (a'_3, a'_2, 0, \dots). \end{aligned}$$

□

Taking the notation $s_\lambda(x(\nu), y(\nu)\|a)$ to mean evaluating the function $s_\lambda(x/y\|a)$ at $x = x(\nu), y = y(\nu)$ we have the following result from [13, Theorem 5.1]:

Theorem 5.5 (Vanishing Theorem)**.** *For partitions λ and ν we have the following 2 results.*

1. *Suppose that $\lambda \not\subseteq \nu$. Then $s_\lambda(x(\nu), y(\nu)\|a) = 0$.*
2. *Suppose that $\lambda = \nu$. Then, $s_\lambda(x(\lambda), y(\lambda)\|a) \neq 0$.*

In [13, Theorem 5.1] the Frobenius coordinates of ν are used in the statement of the theorem, which is equivalent to our specialisations $x(\nu)$ and $y(\nu)$ defined here. Further, the polynomial $s_\lambda(x(\lambda), y(\lambda)\|a)$ is explicitly calculated.

Example 5.6. Let $\lambda = (2)$, and $\rho = (1^3)$. Then the partition λ is not contained in the partition ρ. We have that

$$s_\lambda(x/y\|a) = \sum_{1\leqslant j}(x_j - a_1)(x_j + y_j) + \sum_{1\leqslant i\leqslant j}(x_i + y_i)(x_j + y_j),$$

and thus

$$\begin{aligned} s_\lambda(x(\rho)/y(\rho)\|a) &= (x(\rho)_1 - a_1)(x(\rho)_1 + y(\rho)_1) \\ &= (a_{1-1+1} - a_1)(a_{1-1+1} + a'_{3-1+1}) \\ &= 0 \end{aligned}$$

as predicted by the Vanishing theorem (Theorem 5.5). □

Let $P(x/y\|a)$ be an element in $\Lambda(x/y\|a)$. Then, let the coefficient $k^{\lambda}_{P\pi}(a)$ be defined by the following expansion

$$P(x/y\|a)s_{\pi}(x/y\|a) = \sum_{\lambda} k^{\lambda}_{P\pi}(a)s_{\lambda}(x/y\|a). \tag{5.2}$$

The following is a corollary of the vanishing theorem.

Corollary 5.7. *If $\pi \not\subseteq \lambda$, we have that $k^{\lambda}_{P\pi}(a) = 0$. Furthermore, when $\pi = \lambda$*

$$k^{\lambda}_{P\lambda}(a) = P(x(\lambda), y(\lambda)\|a),$$

where $P(x(\lambda)/y(\lambda)\|a)$ is the polynomial in $\mathbb{Q}[a]$ formed by evaluating $P(x/y\|a)$ at $x = x(\lambda), y = y(\lambda)$.

Proof. We calculate the coefficient $k^{\zeta}_{P\pi}$ for a partition $\zeta \not\supseteq \pi$. Without loss of generality, we may assume that ζ has a minimal number of boxes. Then, we make the substitution $x = x(\zeta), y = y(\zeta)$ in equation (5.2), giving

$$P(x(\zeta)/y(\zeta)\|a)s_{\pi}(x(\zeta)/y(\zeta)\|a) = \sum_{\lambda} k^{\lambda}_{P\pi}(a)s_{\lambda}(x(\zeta)/y(\zeta)\|a).$$

By the vanishing theorem, the left hand side of this equation is equal to zero, since π is not contained in ζ. By our assumption on the minimality of $|\zeta|$, on the right hand side, all terms are equal to 0 except when $\lambda = \zeta$. The polynomial $s_{\zeta}(x(\zeta)/y(\zeta))$ is nonzero, again by the vanishing theorem. Thus the coefficient $k^{\zeta}_{P\pi}(a) = 0$.

To prove the second part of the corollary, we make the substitution $x = x(\pi), y = y(\pi)$ in equation (5.2),

$$P(x(\pi)/y(\pi)\|a)s_{\pi}(x(\pi)/y(\pi)\|a) = k^{\pi}_{P\pi}(a)s_{\pi}(x(\pi)/y(\pi)\|a) + \sum_{\lambda \supset \pi} k^{\lambda}_{P\pi}(a)s_{\lambda}(x(\pi)/y(\pi)\|a),$$

but all terms $s_{\lambda}(x(\pi)/y(\pi)\|a)$ vanish, for $\lambda \supset \pi$, since $\lambda \not\subseteq \pi$. Thus we are left with

$$P(x(\pi)/y(\pi)\|a)s_{\mu}(x(\pi)/y(\pi)\|a) = k^{\mu}_{P\mu}(a)s_{\mu}(x(\mu)/y(\pi)\|a),$$

and since $s_{\mu}(x(\mu)/y(\pi)\|a)$ is nonzero (again by the vanishing theorem) we may divide both sides of this equation by $s_{\mu}(x(\pi)/y(\pi)\|a)$ to obtain the required result. □

This same idea is used to derive the recurrence relation in the next section.

5.3 The recurrence relation

Let $\pi \subseteq \lambda$ be a pair of partitions, and $P = P(x/y\|a)$ an element of $\Lambda(x/y\|a)$. Recall that the coefficients $k^{\lambda}_{P\pi}(a)$ are given by the expansion of

$$P(x/y\|a)s_{\pi}(x/y\|a) = \sum_{\lambda} k^{\lambda}_{P\pi}(a)s_{\lambda}(x/y\|a),$$

summed over partitions λ which contain π.

Theorem 5.8. *We have the following formula for the coefficients $k^{\lambda}_{P\pi}(a)$:*

$$k^{\lambda}_{P\pi}(a) = \frac{1}{k^{\lambda}_{(1)\lambda}(a) - k^{\pi}_{(1)\pi}(a)}\left(\sum_{\pi^+} k^{\lambda}_{P\pi^+}(a) - \sum_{\lambda^-} k^{\lambda^-}_{P\pi}(a)\right), \tag{5.3}$$

summed over all possible diagrams π^+ formed by adding a box to the end of a row of π, and diagrams λ^- formed by deleting a box from the end of a row of λ.

Proof. Consider the product

$$s_{(1)}(x/y\|a)\left(P(x/y\|a)s_{\pi}(x/y\|a)\right) \tag{5.4}$$

which we expand in two ways. First

$$\begin{aligned} s_{(1)}(x/y\|a)(P(x/y\|a)s_{\pi}(x/y\|a)) &= s_{(1)}(x/y\|a)\sum_{\rho} k^{\rho}_{P\pi}(a)s_{\rho}(x/y\|a) \\ &= \sum_{\rho} k^{\rho}_{P\pi}(a)\sum_{\chi} k^{\chi}_{(1)\rho}(a)s_{\chi}(x/y\|a), \end{aligned} \tag{5.5}$$

summed over all partitions ρ which contain π, and χ with at most one box more than ρ.

The second way to expand the product (5.4) is

$$\begin{aligned} P(x/y\|a)(s_{(1)}(x/y\|a)s_{\pi}(x/y\|a)) &= P(x/y\|a)\sum_{\rho} k^{\rho}_{(1)\pi}(a)s_{\rho}(x/y\|a) \\ &= \sum_{\rho} k^{\rho}_{(1)\pi}(a)\sum_{\chi} k^{\chi}_{P\rho}(a)s_{\chi}(x/y\|a), \end{aligned} \tag{5.6}$$

summed over all partitions ρ with at most one box more than π, and partitions χ which contain ρ.

Let λ be a partition such that $\pi^+ \subseteq \lambda \subseteq \chi^-$, that is, λ is contained between a partition formed by adding a box to π and another formed by deleting a box from χ. Since the generalised Frobenius–Schur functions are a basis of $\Lambda(x/y\|a)$, the coefficient of $s_\lambda(x/y\|a)$ appearing in equations (5.5) and (5.6) must be equal. Respectively, they are

$$k^{\pi}_{(1)\pi}(a)k^{\lambda}_{P\pi}(a) + \sum_{\pi^+} k^{\pi^+}_{(1)\pi}(a)k^{\lambda}_{P\pi^+}(a)$$

and

$$k^{\lambda}_{P\pi}(a)k^{\lambda}_{(1)\lambda}(a) + \sum_{\lambda^-} k^{\lambda^-}_{P\pi}(a)k^{\lambda}_{(1)\lambda^-}(a),$$

where the respective sums are over partitions π^+ formed by adding a box to the end of a row of π, and λ^- formed by deleting a box from the end of a row of λ. Thus when equating these coefficients, we obtain

$$k^{\pi}_{(1)\pi}(a)k^{\lambda}_{P\pi}(a) + \sum_{\pi^+} k^{\pi^+}_{1\pi}(a)k^{\lambda}_{P\pi^+}(a) = k^{\lambda}_{P\pi}(a)k^{\lambda}_{(1)\lambda}(a) + \sum_{\lambda^-} k^{\lambda^-}_{P\pi}(a)k^{\lambda}_{(1)\lambda^-}(a)$$

$$k^{\lambda}_{P\pi}(a) = \frac{1}{k^{\lambda}_{(1)\lambda}(a) - k^{\pi}_{(1)\pi}(a)} \left(\sum_{\substack{\pi^+, \\ \pi\to\pi^+}} k^{\lambda}_{P\pi^+}(a) - \sum_{\substack{\lambda^-, \\ \lambda^-\to\lambda}} k^{\lambda^-}_{P\pi}(a) \right),$$

The second equation follows from the first because $k^{\pi^+}_{1\pi}(a) = k^{\lambda}_{(1)\lambda^-}(a) = 1$ which is a direct result of the Pieri rule in the classical case.

□

By specialising $P(x/y\|a)$ to the element $s_\theta(x/y\|a)$, for a skew partition θ, Theorem 5.8 allows the coefficients $k^{\lambda}_{\theta\pi}(a)$ appearing in the expansion

$$s_\theta(x/y\|a)s_\pi(x/y\|a) = \sum_{\lambda} k^{\lambda}_{\theta\pi}(a)s_\lambda(x/y\|a),$$

to be calculated by induction on the number of boxes of λ/π. For a pair of partitions $\pi \subseteq \lambda$, we start with the base case where $k^{\pi}_{\theta\pi}(a)$ and $k^{\lambda}_{\theta\lambda}(a)$ are both known as a result of Corollary 5.7. Then the recurrence relation will allow us to express $k^{\lambda}_{\theta\pi}(a)$ in terms of $k^{\lambda}_{\theta\pi^+}(a)$ and $k^{\lambda^-}_{\theta\pi}(a)$, where the partitions π^+ and λ^- are formed from π and λ by adding a box to π and removing a box from λ respectively. This is the method used in [18] to calculate the Littlewood–Richardson polynomials $c^{\nu}_{\lambda\mu}(a)$ defined in Section 2.2. A key technique which will be used in our calculations is defined in the following section.

5.4 The difference formula

Let $(v_1, v_2, \dots, v_n)$ and $(u_1, u_2, \dots, u_n)$ be a sequence of n formal commuting variables. Consider the difference between the products

$$\begin{aligned} v_1 v_2 \dots v_n - u_1 u_2 \dots u_n =& (v_1 - u_1) u_2 u_3 \dots u_n \\ &+ v_1 (v_2 - u_2) u_3 u_4 \dots u_n \\ &\vdots \\ &+ v_1 v_2 \dots v_{n-1} (v_n - u_n) \\ =& \sum_{i=1}^{n} (v_i - u_i) v_1 \dots v_{i-1} \hat{u}_i u_{i+1} \dots u_n, \end{aligned} \tag{5.7}$$

where in this context the hat over an entry v_i in a summand means we omit the entry from the product.

Since the variables v_i and u_i commute, we may instead choose to rewrite the difference in the following way:

$$\begin{aligned} v_n v_{n-1} \dots v_1 - u_n u_{n-1} \dots u_1 =& (v_n - u_n) u_{n-1} u_{n-2} \dots u_1 \\ &+ v_n (v_{n-1} - u_{n-1}) u_{n-2} u_{n-3} \dots u_1 \\ &\vdots \\ &+ v_n v_{n-1} \dots v_2 (v_1 - u_1) \\ =& \sum_{i=1}^{n} (v_i - u_i) u_1 \dots u_{i-1} \hat{u}_i v_{i+1} \dots v_n. \end{aligned} \tag{5.8}$$

Clearly, the sum we end up with on the right hand side of equations (5.7) and (5.8) depends on the chosen ordering on the product $v_1 \dots v_n$ and $u_1 \dots u_n$. We formalise this as follows.

Definition 5.9. Let I be a totally ordered finite set with n elements, and $\{v_i\}_{i \in I}$ and $\{u_i\}_{i \in I}$ be two families of formal variables. Order the elements of I using the total ordering, that is, we write $i_1 \leqslant i_2 \leqslant \dots \leqslant i_n$, for all $i_k \in I$. Then the difference

$$v_{i_1} v_{i_2} \dots v_{i_n} - u_{i_1} u_{i_2} \dots u_{i_n},$$

taken with respect to the the total order on I is the difference

$$\sum_{j=1}^{n} v_{i_1} \dots v_{j_{i-1}} \widehat{u_{j_i}} u_{j_{i+1}} \dots u_{i_n},$$

where the hatted factor is omitted from each summand.

Thus, in the case of equation (5.7), the difference formula is applied with respect to the ordering where the indices of the variables v_i increase when read left to right, and vice versa for equation (5.8). Throughout this chapter we will state the ordering we use whenever we apply the difference formula, unless the ordering is evident from the context.

What is the relevance of the difference formula to the calculation of the coefficients $k^{\lambda}_{\theta\pi}(a)$? We illustrate this with an example.

Example 5.10. Let $\pi \subseteq \lambda$ be a pair of partitions such that λ is formed from π by adding 1 box to row r of the partition π above the diagonal, for a $1 \leqslant r \leqslant l(\pi)+1$. For each p, we have the generalised Frobenius–Schur function $s_{(p)}(x/y\|a)$ expressed as

$$s_{(p)}(x/y\|a) = \sum_{1\leqslant i_1\leqslant\ldots\leqslant i_p} (x_{i_1}+b_1)(x_{i_2}+b_2)\ldots(x_{i_p}+b_p),$$

where for each particular sequence $(i_1,\ldots,i_p)$ occuring in the sum the coefficients b_k either represent y_{i_k} or $-a_k$. By Corollary 5.7 of the vanishing theorem we have that

$$k^{\pi}_{(p)\pi}(a) = \sum_{i_1\geqslant\ldots\geqslant i_p\geqslant 1} (x(\pi)_{i_1}+b_1)\ldots(x(\pi)_{i_p}+b_p),$$

and

$$k^{\lambda}_{(n)\lambda}(a) = \sum_{i_1\geqslant\ldots\geqslant i_p\geqslant 1} (x(\lambda)_{i_1}+b_1)\ldots(x(\lambda)_{i_p}+b_p).$$

However, by the recurrence relation (5.3) we have that

$$k^{\lambda}_{(p)\pi}(a) = \frac{1}{k^{\lambda}_{(1)\lambda}(a)-k^{\pi}_{(1)\pi}(a)}(k^{\lambda}_{(p)\lambda}(a)-k^{\pi}_{(p)\pi}(a)). \tag{5.9}$$

Since λ differs from π by the addition of a box to row r of π, we have the difference

$$\begin{aligned} k^{\lambda}_{(1)\lambda}(a)-k^{\pi}_{(1)\pi}(a) &= \sum_{j=1}^{l(\lambda)} (x(\lambda)_j - x(\pi)_j + y(\lambda)_j - y(\pi)_j) \\ &= x(\lambda)_r - x(\pi)_r \end{aligned} \tag{5.10}$$

since $x(\lambda)_i = x(\pi)_i$, for all $i \neq r$, and $y(\lambda)_i = y(\pi)_i$ for all integers i.

Now consider the difference

$$\begin{aligned} k^{\lambda}_{(p)\lambda}(a)-k^{\pi}_{(p)\pi}(a) = \sum_{1\leqslant i_1\leqslant\ldots\leqslant i_p} \big((x(\lambda)_{i_1}+b_1)\ldots(x(\lambda)_{i_p}+b_p) \\ - (x(\pi)_{i_1}+b_1)\ldots(x(\pi)_{i_p}+b_p)\big)\,. \end{aligned} \tag{5.11}$$

Since $x(\lambda)_i = x(\pi)_i$ for all $i \neq r$, and $y(\lambda)_i = y(\pi)_i$, for all integers i, the only difference between terms occurs for all $1 \leqslant j \leqslant p$ such that the subscript $i_j = r$.

For the purposes of this example, we consider the case where all the subscripts i_j take value r. Then, we have the difference

$$(x(\lambda)_r + b_1)\dots(x(\lambda)_r + b_p) - (x(\pi)_r + b_1)\dots(x(\pi)_r + b_p)$$

occuring as a summand in equation (5.11). Thus we may apply the difference formula (5.7) by specialising $v_i \mapsto (x(\lambda)_i + b_i)$ and $u_i \mapsto (x(\pi)_i + b_i)$, for each $1 \leqslant i \leqslant p$. We see in fact that

$$\begin{aligned}(x(\lambda)_r + b_1)&\dots(x(\lambda)_r + b_p) - (x(\pi)_r + b_1)\dots(x(\pi)_r + b_p) \\ &= (x(\lambda)_r - x(\pi)_r)\sum_{j=1}^{n}\Big((x(\lambda)_r + b_1)\dots(x(\lambda)_r + b_{j-1})(\widehat{x(\lambda)_r + b_j}) \\ &\qquad \times (x(\pi)_r + b_{j+1})\dots(x(\pi)_r + b_p)\Big),\end{aligned} \tag{5.12}$$

where the hat over the factor $x(\lambda)_r + b_j$ means omit this factor from the product. Then, the factor $(x(\lambda)_r - x(\pi)_r)$ appearing in the right hand side cancels with the denominator from equation (5.10). □

Manipulations of differences such as in equation (5.11) are rather unwieldy, as demonstrated in the example. This motivates the definition of *tableaux* in the next section as these tableaux encode the polynomials in the indeterminates a appearing in equations such as equation (5.11), and make calculating their differences more streamlined.

5.5 Barred tableaux

Suppose T is a $\mathbb{A}$-tableaux of shape θ, where θ is a disjoint union of skew hooks. We order the boxes of θ in the following way, a box α is said to be *before* β if α is to the northeast of beta, whereby northeast we mean either north, or east. (Similarly southwest means either south or west, and so on; however if we say north, for example, we mean to exclude boxes which are both north and east). Thus the first box of θ under this ordering is the most northeastern box of θ; on the other hand, the last box of θ under this ordering is the most southwestern box of θ. We call this ordering the *normal ordering* on boxes in θ, and if α is before β then we will write $\alpha < \beta$. This ordering of the boxes induces the *normal ordering* on entries of T; namely, $T(\alpha) < T(\beta)$ if $\alpha < \beta$.

We will create a *barred* $\mathbb{A}$-tableaux of shape θ from T, which are tableaux like T with the addition of two features, *barring* of entries of T, and *labelling* on entries of T. We first discuss barring. For some integer $n \geqslant 0$, let $\{\alpha_1, \dots, \alpha_n\}$ be a collection of n boxes of θ, such that $\alpha_1 < \cdots < \alpha_n$. We bar the entries in the boxes $\alpha_1, \ldots, \alpha_n$, where barring means the following:

$T(\alpha_j)$ **unprimed:** If $T(\alpha_j) = i$ is unprimed for some $1 \leqslant j \leqslant n$, then we may bar $T(\alpha)$ by putting a left arrow on top of the i, that is $T(\alpha) = \overleftarrow{i}$.

$T(\alpha_j)$ **primed:** If $T(\alpha) = i$ is primed for some $1 \leqslant j \leqslant n$, then we may bar $T(\alpha)$ by putting a right arrow on top of the i, that is $T(\alpha) = \overrightarrow{i'}$.

$T(\alpha_j)$ **starred:** If $T(\alpha) = i$ is starred for some $1 \leqslant j \leqslant n$, then we may bar $T(\alpha)$ by putting a left arrow, or right arrow, or a double arrow on top of the i, that is $T(\alpha) = \overleftarrow{i^*}, \overrightarrow{i^*}$, or $\overleftrightarrow{i^*}$.

If a starred entry has a double arrow on top of it, then it counts as having both a left, and a right arrow on top of it.

Example 5.11. Consider the following barred $\mathbb{A}$-tableau

	$\overleftarrow{1}$	1	$\overleftrightarrow{1^*}$
	$1'$		
1	$\overrightarrow{1'}$		
$1'$			

There are 3 barred entries, with the 1^* in the northeast corner counting as being barred by both a left and a right arrow. □

What role do barred tableaux play in simplifying calculations involving the recurrence relation (5.3)? We revisit the previous example.

Example 5.12. From Example 5.10 we had the difference

$$(x(\lambda)_r + b_1) \ldots (x(\lambda)_r + b_p) - (x(\pi)_r + b_1) \ldots (x(\pi)_r + b_p). \tag{5.13}$$

We can use two tableaux of shape (p) to represent this difference using a *diagrammatic relation*:

r	$\ldots$	r^*	$-$	r	$\ldots$	r^*
	λ				π	

where these tableau represent their own weights, with the first weight evaluated at $x = x(\lambda), y = y(\lambda)$ and the second weight evaluated at $x = x(\pi), y = y(\pi)$. The result

$$(x(\lambda)_r - x(\pi)_r)\sum_{j=1}^{n}\Big((x(\lambda)_r + b_1)\dots(x(\lambda)_r + b_{j-1})(x(\widehat{\lambda)_r +}\, b_j)$$
$$\times\,(x(\pi)_r + b_{j+1})\dots(x(\pi)_r + b_p)\Big)$$

from applying the difference formula to equation (5.13) may be represented using a diagrammatic relation as well:

$$(x(\lambda)_r - x(\pi)_r)\textstyle\sum_{j=1}^{n} \begin{array}{|c|c|c|c|c|c|c|} \hline r & \dots & r & \overleftarrow{r} & r & \dots & r^* \\ \hline \multicolumn{1}{c}{} & \multicolumn{1}{c}{\lambda} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\pi} & \multicolumn{1}{c}{} \end{array}$$

Here, the sum is over $\mathbb{A}$-tableaux of shape (p) with the j-th entry barred by a left arrow. Furthermore, each entry to left of the barred entry of a tableau appearing in this sum represents its own weight evaluated at $x = x(\theta), y = y(\theta)$, and each entry to right of the barred entry represents its weight evaluated at $x = x(\pi), y = y(\pi)$. □

5.6 Word of a tableau

Let θ be a disjoint union of skew hooks, that is, it does not contain a 2×2 subdiagram. In a similar vein to the construction in Chapter 2, we associate a *word* to the barred entries of a $\mathbb{A}$-tableau T of shape θ. There will in fact be two words, one corresponding to entries barred by left arrows, and one to entries barred with right arrows. We define the *unprimed (barred) word* of T, and the *primed (barred) word* of T. First we define the *unprimed* and *primed* ordering. The *unprimed ordering* on T is equal to the normal ordering of T, that is, entries are read from the northeast to the southwest. Suppose $T(\alpha)$ and $T(\beta)$ are barred unprimed entries, or starred entries barred by a left arrow. Then, we say that $T(\alpha)$ is *before* $T(\beta)$ with respect to the unprimed ordering if $\alpha < \beta$. We will denote this relationship as $T(\alpha) \sqsubset T(\beta)$. The unprimed ordering on barred entries in T induces the *unprimed ordering* on the boxes of θ containing barred unprimed entries, or starred entries barred by a left arrow, which we will again denote by $\alpha \sqsubset \beta$.

The *primed ordering* on entries of T is defined as follows. First, order the entries of T from southwest to northeast (i.e. opposite to the unprimed ordering). Then, we introduce the following modification: for each $r > 1$, the most northeast r^* is defined to be before the most southwest entry taking integer value r. To expand on this point, consider the following algorithm for each r. Let α contain the most southwest entry with integer value r. Let α^- be the box directly southwest of α. There exist a box $\beta \leqslant \alpha$ (i.e. northeast of α) such that β contains the most northeast r^*. Then if the entries are ordered from southwest to northeast, we have the sequence $T(\alpha^-)$, $T(\alpha)$, then $T(\beta)$. Under the primed ordering the entry $T(\beta)$ occurs before $T(\alpha)$, i.e. we have the sequence $T(\alpha^-)$, $T(\beta)$, then $T(\alpha)$, with respect to the primed ordering. The *primed ordering* on the boxes of θ containing barred primed entries, or starred entries barred by a right arrow naturally follows from the this ordering. We will denote the primed ordering with $\prec$, that is, we write $\alpha \prec \beta$ and correspondingly $T(\alpha) \prec T(\beta)$ if $T(\alpha)$ is before $T(\beta)$ with respect to the primed ordering.

We define the *unprimed (barred) word* of T, and the *primed (barred) word* of T as follows:

Unprimed word: The *unprimed word* of T or the word R is the sequence of barred unprimed entries or starred entries barred by a left arrow of T, listed left to right, starting with the first and ending with the last, with respect to the unprimed ordering on T.

Primed word: The *primed word* of T or the word R is the sequence of barred primed entries or starred entries barred by a right arrow of T, listed left to right, starting with the first and ending with the last, with respect to the primed ordering on T.

To keep things neat when we write the word of T we will usually write the entries of each word without stars or primes. Denote the unprimed and primed words of T by R^+ and R^- respectively, and we will also use the pair (R^+, R^-) to represent the words of T. Since starred entries with a double arrow on top are considered to have both a left, and a right arrow on top, the total number of entries in R^+ and R^- might exceed the number of barred entries of T.

Example 5.13. We continue with the tableau from Example 5.11:

$$T = \begin{array}{|c|c|c|c|} \hline & \overleftarrow{1} & 1 & \overleftrightarrow{1^*} \\ \hline & 1' & & \\ \hline 1 & \overrightarrow{1'} & & \\ \hline 1' & & & \\ \hline \end{array}$$

First let us label the boxes of T according to how they would be ordered by the unprimed ordering:

$$T = \begin{array}{cccc} & \overleftarrow{1} & 1 & \overleftrightarrow{1}^* \\ & 1' & & \\ 1 & \overrightarrow{1}' & & \\ 1' & & & \end{array} \quad \leftrightarrow \quad \begin{array}{cccc} & 3 & 2 & 1 \\ & 4 & & \\ 6 & 5 & & \\ 7 & & & \end{array}$$

Thus, on the right the most northeast box is the first box, and the most southwest box is the last box.

Next, we label the boxes of T according to how they would be ordered by the primed ordering:

$$T = \begin{array}{cccc} & \overleftarrow{1} & 1 & \overleftrightarrow{1}^* \\ & 1' & & \\ 1 & \overrightarrow{1}' & & \\ 1' & & & \end{array} \quad \leftrightarrow \quad \begin{array}{cccc} & 6 & 7 & 1 \\ & 5 & & \\ 3 & 4 & & \\ 2 & & & \end{array}$$

The most northeast box is still the first box, because this box contains the most northeast 1^* in T. Thereafter, the rest of the boxes are read from southwest to northeast. We give another example to illustrate the last point.

Consider the following tableaux S, with its boxes labelled according to the primed ordering on the right:

$$S = \begin{array}{cc} & 1^* \\ & 2^* \\ & 2' \\ 1^* & \\ 1' & \\ 2^* & \\ 2' & \end{array} \quad \leftrightarrow \quad \begin{array}{cc} & 4 \\ & 1 \\ & 7 \\ 6 & \\ 5 & \\ 3 & \\ 2 & \end{array}$$

that is, the entries are arranged in southwest to northeast order, except the 2^* in box $(2, 2)$ is before the $2'$ in box $(7, 1)$, and the 1^* in box $(1, 2)$ is before the $1'$ in box $(5, 1)$.

We now return to the previous tableau T, and write down the words (R^+, R^-) of this tableau. The unprimed word R^+ consists of listing the entries of T barred by a left arrow, with respect to the unprimed order. Thus, $R^+ = 1^*1$. The primed word R^- consists of listing the entries of T barred by a right arrow, with respect to the primed order. Thus, $R^- = 1^*1'$. In the future, we will drop any primes or stars when writing the words.

$\square$

Fix a pair of partitions λ which contains π. Recall that the depth of λ is the number of boxes on the diagonal of λ. Then, the partitions $\lambda^\wedge$ and $\lambda^\vee$ denote the first $d(\lambda)$ rows and columns of λ respectively, and the same holds for the partitions $\pi^\wedge$ and $\pi^\vee$. Consider a sequence R^+ of partitions $(\rho_+^{(i)})_{0\leqslant i\leqslant p}$ with $\rho_+^{(0)} = \pi^\wedge$ and $\rho_+^{(p)} = \lambda^\wedge$ such that

$$\pi^\wedge = \rho_+^{(0)} \subset \rho_+^{(1)} \subset \cdots \subset \rho_+^{(p)} = \lambda^\wedge,$$

that is, $\rho_+^{(i)}$ is contained in and has less boxes than $\rho_+^{(i+1)}$. In fact, we will write this sequence of partitions in the following way

$$\pi^\wedge = \rho_+^{(0)} \xrightarrow{r_1} \rho_+^{(1)} \xrightarrow{r_2} \dots \xrightarrow{r_p} \rho_+^{(p)} = \lambda^\wedge,$$

such that $\rho_+^{(i-1)} \xrightarrow{r_i} \rho_+^{(i)}$ means that $\rho_+^{(i)}$ is formed from $\rho_+^{(i-1)}$ by doing one of the following mutually exclusive actions. Let the integer $d_i = d(\rho_+^{(i)})$ denote the depth of the partition $d(\rho_+^{(i)})$:

1. If $r_i \leqslant d_{(i-1)}$, then adding a box to the end of row r_i of $\rho_+^{(i-1)}$ forms $\rho_+^{(i)}$.
2. If $r_i = d+1$, where $d = d_{(i-1)}$, adding $d+1$ boxes to row $d+1$ of $\rho_+^{(i-1)}$ forms $\rho_+^{(i)}$.

In the last case when $r_i = d(\rho_+^{(i-1)}) + 1$, we will call the process *adding a box onto the diagonal*. We will abuse notation and write $R^+ = r_1 \dots r_p$ and associate the sequence of integers $r_1 \dots r_p$ to the sequence of partitions R^+, and say that the sequence $r_1 \dots r_p$ takes $\pi^\wedge$ to $\lambda^\wedge$ and denote this as $R^+ : \pi^\wedge \to \lambda^\wedge$.

Similarly, consider a sequence of partitions $(\rho_-^{(i)})_{0\leqslant i\leqslant q}$ with $\rho_-^{(0)} = \pi^\vee$ and $\rho_-^{(q)} = \lambda^\vee$ such that

$$\pi^\vee = \rho_-^{(0)} \subset \rho_-^{(1)} \subset \cdots \subset \rho_-^{(q)} = \lambda^\vee,$$

that is, $\rho_-^{(i-1)}$ is contained in and has less boxes than $\rho_-^{(i)}$. Again, we will write this sequence of partitions in the following way

$$\pi^\vee = \rho_-^{(0)} \xrightarrow{s_1} \rho_-^{(1)} \xrightarrow{s_2} \dots \xrightarrow{s_q} \rho_-^{(q)} = \lambda^\vee,$$

such that $\rho_-^{(i-1)} \xrightarrow{s_i} \rho_-^{(i)}$ means that $\rho_-^{(i)}$ is formed from $\rho_-^{(i-1)}$ by doing one of the following mutually exclusive actions. Let the integer $d_i = d(\rho_-^{(i)})$ denote the depth of the partition $d(\rho_-^{(i)})$:

1. If $s_i \leqslant d_{(i-1)}$, then adding a box to the end of column s_i of $\rho_-^{(i-1)}$ forms $\rho_-^{(i)}$.

2. If $s_i = d+1$, where $d = d_{(i-1)}$, adding $d+1$ boxes to column $d+1$ of $\rho_-^{(i-1)}$ forms $\rho_-^{(i)}$.

In the last case when $s_i = d(\rho_-^{(i-1)})+1$, we will call the process *adding a box onto the diagonal*. We will abuse notation and write $R^- = s_1 \dots s_q$ and associate the sequence of integers $s_1 \dots s_q$ to the sequence of partitions R^-, and say that the sequence $s_1 \dots s_q$ takes $\pi^\vee$ to $\lambda^\vee$ and denote this as $R^- : \pi^\vee \to \lambda^\vee$.

When both the sequences $R^+ : \pi^\wedge \to \lambda^\wedge$ and $R^- : \pi^\vee \to \lambda^\vee$, we will say the pair of sequences (R^+, R^-) takes π to λ and denote this with $(R^+, R^-) : \pi \to \lambda$. Let T be a $\mathbb{A}$-tableau with unprimed and primed words (S^+, S^-). Fix a partition π contained in another partition λ. We will say that the words of T are *Yamanouchi* if there exist a sequence of partitions $R^+ : \pi^\wedge \to \lambda^\wedge$ and another sequence of partitions $R^- : \pi^\vee \to \lambda^\vee$ such that the words (S^+, S^-) coincide with the sequence of integers (R^+, R^-). In this case, we will say the words (R^+, R^-) of T takes π to λ and again denote this by $(R^+, R^-) : \pi \to \lambda$.

Fix a sequence $R^+ : \pi^\wedge \to \lambda^\wedge$ and a sequence $R^- : \pi^\vee \to \lambda^\vee$. Let $\mathcal{T}(R^+, R^-)$ be the set of all tableaux which obey the following two conditions.

1. If the tableau T belongs to $\mathcal{T}(R^+, R^-)$ then it contains unprimed and primed words R^+ and R^- respectively.

2. Furthermore, the number of barred entries in such a T is equal to the number of boxes in λ/π.

Condition 1. means that the tableau T has words $(R^+, R^-) : \pi \to \lambda$ which are Yamanouchi.

Example 5.14. Let $\pi = \varnothing$, a partition, such that $\pi^\wedge = \pi^\vee = \varnothing$. Let $\lambda = (2,1)$, such that $\lambda^\wedge = (2)$ and $\lambda^\vee = (1^2)$. We have that R^+ is the following sequence of partitions which takes $\pi^\wedge$ to $\lambda^\wedge$:

$$\pi^\wedge = \varnothing \xrightarrow{1} (1) \xrightarrow{1} (2) = \lambda^\wedge.$$

Similarly, R^- is the following sequence of partitions which takes the partition $\pi^\vee$ to the partition $\lambda^\vee$:

$$\pi^\vee = \varnothing \xrightarrow{1} (1) \xrightarrow{1} (1^2) = \pi^\vee.$$

Recall the tableau T from the previous examples

$$T = \begin{array}{|c|c|c|c|} \cline{2-4} \multicolumn{1}{c|}{} & \overleftarrow{1} & 1 & \overleftrightarrow{1}^{*} \\ \cline{2-4} \multicolumn{1}{c|}{} & 1' & \multicolumn{2}{c}{} \\ \cline{1-2} 1 & \overrightarrow{1'} & \multicolumn{2}{c}{} \\ \cline{1-2} 1' & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

has unprimed word $R^+ = 11$ and primed word $R^- = 11$, which consist respectively of entries barred with left, and right arrows respectively. Thus, the words (R^+, R^-) of the tableau T are Yamanouchi and take π to λ. □

5.7 Weight of tableaux

Continue fixing partitions π and λ such that there are two sequences of partitions $R^+ : \pi^\wedge \to \lambda^\wedge$ and $R^- : \pi^\vee \to \lambda^\vee$, with $R^+ = r_1 \dots r_p$ and $R^- = s_1 \dots s_q$. Let T be a $\mathbb{A}$-tableau of shape θ in the set $\mathcal{T}(R^+, R^-)$. The second feature of a barred $\mathbb{A}$-tableau T in this set is the labelling of its entries, which will be used to define the weight of the tableau T. For each box $\alpha \in \theta$, let $\sigma(\alpha)$ be a partition. The set of partitions $\{\sigma(\alpha)\}$, for all boxes $\alpha \in \theta$, forms the *labelling* of the entries of T. For example, two distinguished labellings of entries are the partitions $\rho^\wedge(\alpha)$ and $\rho^\vee(\alpha)$, which we define as follows. Let the entries of T barred by a left arrow occupy the boxes $\alpha_1^+, \dots, \alpha_p^+$, listed with respect to the unprimed ordering, and the entries of T barred by a right arrow occupy the boxes $\alpha_1^-, \dots, \alpha_q^-$, listed with respect to the primed ordering. Then the partition $\rho^\wedge(\alpha)$ is defined to be

$$\rho^\wedge(\alpha) = \rho_+^{(i)}, \qquad \text{if } \alpha_i \sqsubseteq \alpha \sqsubset \alpha_{i+1}.$$

Similarly, the partition $\rho^\vee(\alpha)$ is defined to be

$$\rho^\vee(\alpha) = \rho_-^{(i)}, \qquad \text{if } \alpha_i \preccurlyeq \alpha \prec \alpha_{i+1}.$$

We will also define the labelling $\rho(\alpha) = \rho^\wedge(\alpha) \cup \rho^\vee(\alpha)$, a (non-disjoint) union of partitions, to be another labelling of entries of T. We will encounter these labellings later when we define the weight of tableaux.

We define the *weight* of barred $\mathbb{A}$-tableaux T, by defining the *weight* of entries of T. Let $\sigma(\alpha)$ be a labelling on the entries of T and let $T(\alpha)$ be an entry of T. Then,

Unprimed: If $T(\alpha)$ is unprimed as well as unbarred, then the weight of $T(\alpha)$ is

$$\mathrm{ev}(T(\alpha)) = x(\sigma(\alpha))_{T(\alpha)} - a_{c(\alpha)+1}.$$

Primed: If $T(\alpha)$ is primed as well as unbarred, then the weight of $T(\alpha)$ is

$$\mathrm{ev}(T(\alpha)) = y(\sigma(\alpha))_{T(\alpha)} + a_{c(\alpha)+1}.$$

Starred: If $T(\alpha)$ is unprimed as well as unbarred, then the weight of $T(\alpha)$ is

$$\mathrm{ev}(T(\alpha)) = x(\sigma(\alpha))_{T(\alpha)} + y(\sigma(\alpha))_{T(\alpha)}.$$

Then, the weight of T is the product of the weight of all unbarred entries in T, which we again denote by $\mathrm{ev}(T)$:

$$\mathrm{ev}(T) = \prod_{\substack{\alpha\in\lambda,\\ T(\alpha)\text{ unbarred}}} \mathrm{ev}(T(\alpha)).$$

Example 5.15. Previously, we had the tableau T

$$T = \begin{array}{cccc} \cline{2-4} & \multicolumn{1}{|c|}{\overleftarrow{1}} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{\overleftrightarrow{1}^*} \\ \cline{2-4} & \multicolumn{1}{|c|}{1'} & & \\ \cline{1-2} \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{\overrightarrow{1}'} & & \\ \cline{1-2} \multicolumn{1}{|c|}{1'} & & & \\ \cline{1-1} \end{array}$$

with word $(R^+, R^-) = (11, 11)$ which takes $\pi = \varnothing$ to $\lambda = (2, 1)$, with the two sequences of partitions:

$$\pi^\wedge = \varnothing \xrightarrow{1} (1) \xrightarrow{1} (2) = \lambda^\wedge.$$

and

$$\pi^\vee = \varnothing \xrightarrow{1} (1) \xrightarrow{1} (1^2) = \lambda^\vee.$$

We now show the labelling $\rho^\wedge(\alpha)$ and $\rho^\vee(\alpha)$ on the entries of T by displaying the labels in the boxes of the following tableaux

$$T = \begin{array}{cccc} \cline{2-4} & \multicolumn{1}{|c|}{\overleftarrow{1}} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{\overleftrightarrow{1}^*} \\ \cline{2-4} & \multicolumn{1}{|c|}{1'} & & \\ \cline{1-2} \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{\overrightarrow{1}'} & & \\ \cline{1-2} \multicolumn{1}{|c|}{1'} & & & \\ \cline{1-1} \end{array} \qquad \begin{array}{cccc} \cline{2-4} & \multicolumn{1}{|c|}{(1)} & \multicolumn{1}{c|}{(1)} & \multicolumn{1}{c|}{(1)} \\ \cline{2-4} & \multicolumn{1}{|c|}{(2)} & & \\ \cline{1-2} \multicolumn{1}{|c|}{(2)} & \multicolumn{1}{c|}{(2)} & & \\ \cline{1-2} \multicolumn{1}{|c|}{(2)} & & & \\ \cline{1-1} \end{array} \qquad \begin{array}{cccc} \cline{2-4} & \multicolumn{1}{|c|}{(1^2)} & \multicolumn{1}{c|}{(1^2)} & \multicolumn{1}{c|}{(1)} \\ \cline{2-4} & \multicolumn{1}{|c|}{(1^2)} & & \\ \cline{1-2} \multicolumn{1}{|c|}{(1)} & \multicolumn{1}{c|}{(1^2)} & & \\ \cline{1-2} \multicolumn{1}{|c|}{(1)} & & & \\ \cline{1-1} \end{array}$$

$\rho^\wedge(\alpha)$ $\qquad$ $\rho^\vee(\alpha)$

where the second tableau represents the labelling $\rho^\wedge(\alpha)$ and the third tableau $\rho^\vee(\alpha)$.

The labelling $\rho(\alpha)$ is then the non-disjoint union $\rho(\alpha) = \rho^{\wedge}(\alpha) \cup \rho^{\vee}(\alpha)$. Recall that for a partition σ we have the specialisation $x(\sigma)_i = a_{\sigma_i - i+1}$ and $y(\sigma)_i = a'_{\sigma'_i - i+1}$. The weight of the unbarred entries of T follows. First, the unprimed entry in box $(1,3)$ has weight $x(\rho(1,3))_1 - a_{c(1,3)+1} = a_1 - a_3$, and the unprimed entry in box $(3,1)$ has weight $x(\rho(3,1))_1 - a_{c(3,1)+1} = a_2 - a_{-1}$. On the other hand, the primed entry in box $(2,2)$ has weight $y(\rho(2,2))_1 + a_{c(2,2)+1} = a'_2 + a_1$ and the primed entry in box $(4,1)$ has weight $y(\rho(4,1))_1 + a_{c(4,1)+1} = a'_1 + a_{-2}$. Under the labelling $\rho(\alpha)$, the weight of the tableau T is then:

$$\mathrm{ev}(T) = (a'_1 + a_{-2})(a'_2 + a_1)(a_2 - a_{-1})(a_1 - a_3),$$

which is equal to the product of the weights of all the unbarred entries of T.

We remark that only the labelling $\rho^{\wedge}(\alpha)$ is relevant to the weight of unprimed entries, whereas only the labelling $\rho^{\vee}(\alpha)$ is relevant to the weight of primed entries.

□

5.8 Dual tableaux

Recall the variables $(a')_i = -a_{-i+1}$, for all $i \in \mathbb{Z}$. The generalised Frobenius–Schur functions obey the following duality property:

$$s_\pi(x, y\|a) = s_{\pi'}(y, x\|a'),$$

which is a result of Proposition 4.1 and equation (4.6) of [13]. This motivates the definition of a *dual* barred $\mathbb{A}$-tableau $\breve{T}$ of shape θ which is dual to a $\mathbb{A}$-tableau T. We set up the following definitions so that a tableau T has the same weight as its dual. Before we define it though, we emphasize that all the properties of the usual barred $\mathbb{A}$-tableaux have their dual equivalents. For example, $\breve{T}$ has dual words $\breve{R}^+$, $\breve{R}^-$, dual primed and unprimed orderings, and labellings $\breve{\rho}(\alpha)$, $\breve{\rho}^{\wedge}(\alpha)$, $\breve{\rho}^{\vee}(\alpha)$.

Given a tableau T, its dual $\breve{T}$ is constructed as follows. For every $i \geqslant 1$, do the following: Let β contain the most northeast i^* in T, and let α be the box containing the most southwest entry of T with integer value i. Then, in the subtableau of T containing entries with integer value i, delete the most northeast entry i^* in box β. Then, move the remaining entries in this subtableau to the box directly to their northeast in the subtableau. This means that box α is empty; fill in this box with i^* and if neccessary bar it so that the new entry $\breve{T}(\alpha)$ is equal to $T(\beta)$. This process induces a natural pair between entries $T(\gamma)$ of T and entries $\breve{T}(\breve{\gamma})$ of $\breve{T}$. Note that $\breve{T}$ obeys the following conditions on its rows and columns: each row weakly increases reading left to right and each column weakly increases, from top to bottom, and for each $i \geqslant 1$ there is only one occurrence of an unprimed i or i^* in each column, and one occurrence of a primed i or i^* in each row.

The labelling on the entries of $\breve{T}$ is set as follows. Let γ be a box in θ. If $\breve{T}(\breve{\gamma})$ is an entry of $\breve{T}$ paired with the entry $T(\gamma)$, then the labelling on the entry $\breve{T}(\breve{\gamma})$ is set to be $\breve{\rho}^{\wedge}(\breve{\gamma}) = \rho^{\wedge}(\gamma)$ and $\breve{\rho}^{\vee}(\breve{\gamma}) = \rho^{\vee}(\gamma)$, and $\breve{\rho}(\breve{\gamma}) = \breve{\rho}^{\wedge}(\breve{\gamma}) \cup \breve{\rho}^{\vee}(\breve{\gamma})$.

Example 5.16. Consider the following tableaux T, and its dual $\breve{T}$:

$T =$

			1	1	$\overrightarrow{1}^*$
			$1'$		
			$\overleftrightarrow{2}^*$		
			$2'$		
1	$\overleftarrow{1}^*$	2	$2'$		

$\leftrightarrow$ $\breve{T} =$

			$1'$	1	1
			$\overleftarrow{1}^*$		
			$2'$		
			$2'$		
$\overrightarrow{1}^*$	1	$\overleftrightarrow{2}^*$	2		

□

The now define a new ordering on the entries of a dual tableaux. *dual normal ordering* on $\breve{T}$ is given by reading the entries of $\breve{T}$ starting from the southwest and ending at the northeast. If α and β are two boxes of θ, denote by $T(\alpha)\breve{>}T(\beta)$ if $T(\alpha)$ is after $T(\beta)$ under this ordering. The *dual primed ordering* on $\breve{T}$ is the same as the dual normal ordering. Denote by $T(\alpha)\breve{>}^{-}T(\beta)$ if $T(\alpha)$ is after $T(\beta)$ under this ordering. The *dual unprimed ordering* on $\breve{T}$, denoted by $\breve{>}^{+}$, is defined analagously to the primed ordering on T, see Section 5.6. First order the entries of $\breve{T}$ from northeast to southwest. Then, for each $r \geqslant 1$, the most southwest entry equal to r^* is taken to be before the most northeast entry with integer value r.

Example 5.17. Consider the unprimed and primed ordering on the entries of T:

			3	2	1
			4		
			5		
			6		
10	9	8	7		

unprimed

			9	10	1
			8		
			4		
			7		
2	3	5	6		

primed

On the other hand, we have the dual primed and unprimed ordering on the entries of $\breve{T}$:

			8	9	10
			7		
			6		
			5		
1	2	3	4		

unprimed

			4	3	2
			5		
			7		
			8		
1	10	6	9		

primed

□

The dual labellings $\breve{\rho}(\alpha)$, $\breve{\rho}^\wedge(\alpha)$, $\breve{\rho}^\vee(\alpha)$ of $\breve{T}$ are equivalent to the labellings $\rho(\alpha)$, $\rho^+(\alpha)$, $\rho^-(\alpha)$ of T, in the sense that $T(\alpha)$ and its pair $\breve{T}(\breve{\alpha})$ are labelled by the same partition, even though $R^\pm$ may not be the same as $\breve{R}^\pm$.

Example 5.18. Continuing with the previous example, we write down the primed word of T. This is the sequence of entries barred with a left arrow, listed left to right with respect to the primed order of T (which we can see from Example 5.14). We have $R^- = 12$. Now we refer to the dual tableau $\breve{T}$ and the corresponding dual primed ordering, and write down the dual primed word $\breve{R}^- = 12$, which we see agrees with R^-. Furthermore, if we list the entries with integer value 1 in T and $\breve{T}$ with respect to the primed order we obtain the same sequence $\overrightarrow{1}^*\ 1\ \overrightarrow{1}^*\ 1'11$, thus the labelling $\rho^\vee(\alpha)$ and $\breve{\rho}^\vee(\alpha)$ label the same entries in this sequence and hence are equivalent in this sense. Note however that the sequence of entries of T and $\breve{T}$ listed in primed and dual primed order respectively are not equivalent; we have the sequences $\overrightarrow{1}^*\ 1\ \overleftarrow{1}^*\overleftrightarrow{2}^*\ 22'2'1'11$ and $\overrightarrow{1}^*\ 1\ \overleftrightarrow{2}^*\ 22'2'\ \overrightarrow{1}^*\ 1'11$ respectively. What will become important later when we discuss the weight of dual tableaux is that for each r the subsequence of entries of $R^\pm$ and $\breve{R}^\pm$ equal to r is the same. □

The *weight* of entries of $\breve{T}$ is defined in exactly the same way as the weight of entries of T, with a small modification; for a labelling $\sigma(\alpha)$ on $\breve{T}$:

Unprimed: If $\breve{T}(\alpha)$ is unprimed as well as unbarred, then the weight of $\breve{T}(\alpha)$ is

$$\mathrm{ev}(\breve{T}(\alpha)) = x(\sigma(\alpha))_{\breve{T}(\alpha)} - a_{c(\alpha)}.$$

Primed: If $\breve{T}(\alpha)$ is primed as well as unbarred, then the weight of $\breve{T}(\alpha)$ is

$$\mathrm{ev}(\breve{T}(\alpha)) = y(\sigma(\alpha))_{\breve{T}(\alpha)} + a_{c(\alpha)}.$$

Starred: If $\breve{T}(\alpha)$ is starred as well as unbarred, then the weight of $\breve{T}(\alpha)$ is

$$\mathrm{ev}(\breve{T}(\alpha)) = x(\sigma(\alpha))_{\breve{T}(\alpha)} + y(\sigma(\alpha))_{\breve{T}(\alpha)}.$$

In other words, if an entry is unbarred, and unprimed or primed, we make the modification $c(\alpha)+1 \to c(\alpha)$ in the second term appearing in the weight, c.f. Section 5.7. This modification is necessary to compensate for the process in which we moved all entries of T except the most northeast starred occurence of each r to the box directly northeast of them. Furthermore, we note that the content of a box does not affect the weight of a starred entry. Thus, since the labellings of $\breve{T}$ are equivalent to those on T, the weight of $T(\alpha)$, for some $\alpha \in \lambda$, is the same as the weight of the entry $\breve{T}(\breve{\alpha})$ paired to it.

We define the weight of $\breve{T}$ to be the weight of all its entries multiplied together. Thus, we have achieved our goal of defining a dual tableau $\breve{T}$ with the same weight as T. What will these dual tableaux be used for? It turns out that in the following sections the combinatorics involved in simplifying equation (5.3) become a lot easier. This is because the primed ordering on $\mathbb{A}$-tableaux T is a lot more complicated than the dual primed ordering on dual $\mathbb{A}$-tableaux $\breve{T}$. It will be convenient to refer to dual tableaux and the corresponding dual primed ordering when making calculations involving the labelling $\rho^{\vee}(\alpha)$.

5.9 Calculating $k^{\lambda}_{\theta\pi}(a)$ when λ/π contains one box

Fix a skew partition θ, a disjoint union of skew hooks, that is, the skew partition θ does not contain a 2×2 subdiagram. We now prove the formula for $k^{\lambda}_{\theta\pi}(a)$ when the skew partition λ/π is one box. There are 3 cases, depending on whether we have to add a box to a row of π above the diagonal, a column below the diagonal, or onto the diagonal of π. We have, by the recurrence relation (5.3)

$$k^{\lambda}_{\theta\pi}(a) = \frac{1}{k^{\lambda}_{(1)\lambda}(a) - k^{\pi}_{(1)\pi}(a)}\Big(\sum_{\substack{\pi^+,\\ \pi\to\pi^+}} k^{\lambda}_{\lambda\pi^+}(a) - \sum_{\substack{\lambda^-,\\ \lambda^-\to\lambda}} k^{\lambda}_{\lambda\pi^-}(a)\Big).$$

In each of the following cases, we calculate the factor $k^{\lambda}_{(1)\lambda}(a) - k^{\pi}_{(1)\pi}(a)$ and also the factor

$$\sum_{\substack{\pi^+,\\ \pi\to\pi^+}} k^{\lambda}_{\lambda\pi^+}(a) - \sum_{\substack{\lambda^-,\\ \lambda^-\to\lambda}} k^{\lambda^-}_{\lambda\pi}(a)$$

5.9.1 Case 1:

Suppose that $\pi \xrightarrow{r} \lambda$ where a box is added to row r of π above the diagonal to form λ. In this case, we have

$$k^{\lambda}_{(1)\lambda}(a) - k^{\pi}_{(1)\pi}(a) = x(\lambda)_r - x(\pi)_r, \tag{5.14}$$

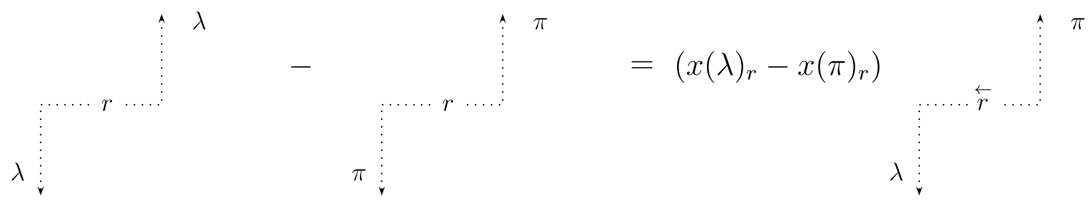
since the rows and columns of π and λ are of equal length except in row r. Suppose T is a $\mathbb{A}$-tableau with its entries labelled by λ, and $\widetilde{T}$ is a $\mathbb{A}$-tableau with entries equal to T but is labelled by π. Then, the difference

$$\mathrm{ev}(T) - \mathrm{ev}(\widetilde{T})$$

only differs in entries of T and $\widetilde{T}$ equal to unprimed r's, since the rows and columns of π and λ are of equal length except in row r. These entries contribute the difference

$$\sum_{\substack{\alpha,\\ T(\alpha)=r}} (x(\lambda)_r - a_{c(\alpha)+1}) - \sum_{\substack{\alpha,\\ T(\alpha)=r}} (x(\pi)_r - a_{c(\alpha)+1}),$$

thus, we invoke the difference formula (5.7) while ordering the unprimed r's in T and $\widetilde{T}$ with respect to the unprimed ordering on T. The formula follows immediately; the diagrammatic relation below, first introduced in [18], illustrates this point:



Here on the right hand side, we take the diagram to mean that we sum over all possible $\mathbb{A}$-tableaux of shape λ with a r barred by a left arrow. Furthermore, the entries northeast of the barred r (i.e. those which are before the barred r with respect to the normal ordering) are labelled by π, while those southwest are labelled by λ. We see that the factor $x(\lambda)_r - x(\pi)_r$ appearing on the right hand side cancels with $k^{\lambda}_{(1)\lambda}(a) - k^{\pi}_{(1)\pi}(a)$ from equation (5.14) to give the required result:

$$k^{\lambda}_{\theta\pi}(a) = \sum_{T} \mathrm{ev}(T),$$

summed over tableaux T with a r barred by a left arrow. We now briefly cover the other cases.

5.9.2 Case 2:

Suppose that $\pi \xrightarrow{r} \lambda$ where a box is added to column r of π below the diagonal to form λ. In this case, we have

$$k^{\lambda}_{(1)\lambda}(a) - k^{\pi}_{(1)\pi}(a) = y(\lambda)_r - y(\pi)_r, \tag{5.15}$$

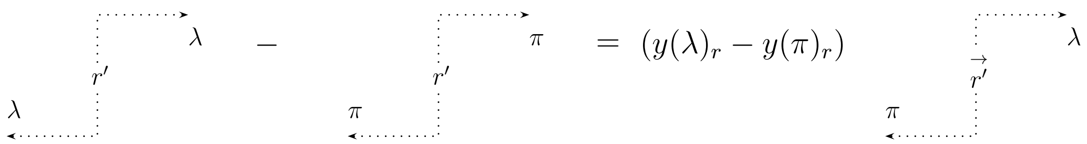
since the rows and columns of π and λ are of equal length except in column r. Suppose T is a $\mathbb{A}$-tableau with its entries labelled by λ, and $\widetilde{T}$ is a $\mathbb{A}$-tableau with entries equal to T but is labelled by π. Then, the difference

$$\mathrm{ev}(T) - \mathrm{ev}(\widetilde{T})$$

only differs in entries of T and $\widetilde{T}$ equal to primed r's, since the rows and columns of π and λ are of equal length except in column r. These entries contribute the difference:

$$\sum_{\substack{\alpha,\\ T(\alpha)=r}} (y(\lambda)_r - a_{c(\alpha)+1}) - \sum_{\substack{\alpha,\\ T(\alpha)=r}} (y(\pi)_r - a_{c(\alpha)+1}),$$

thus, we invoke the difference formula (5.7) while ordering the primed r's in T and $\widetilde{T}$ with respect to to the primed ordering on T. The formula follows immediately; the diagrammatic relation below illustrates this point:



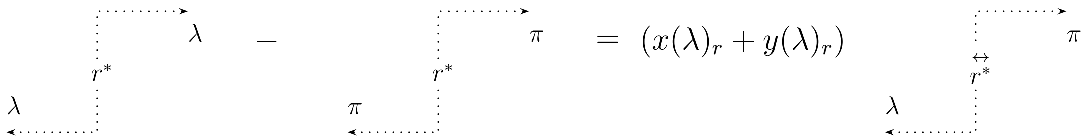
where again the diagram on the right hand side represents a summation of all tableaux of shape λ that have a r' barred by a right arrow. With respect to the primed ordering, all entries in the tableau before the barred r' are labelled with π, and those after are labelled with λ.

5.9.3 Case 3:

Suppose that $\pi \xrightarrow{r} \lambda$ where $r = d + 1$, with $d = d(\pi)$. Thus a box is added to the diagonal of π to form λ. In this case, we have

$$k^{\lambda}_{(1)\lambda}(a) - k^{\pi}_{(1)\pi}(a) = (x(\lambda)_{d+1} + y(\lambda)_{d+1}), \tag{5.16}$$

since the rows and columns of π and λ are of equal length except that the diagonal of λ is longer than the diagonal of π by one box. We have the diagrammatic relation



which we explain as follows. On the lefthand side we have the difference between tableaux T labelled by λ and tableaux $\widetilde{T}$, with the same entries as T, but labelled by π. As before, the only thing that will contribute to the difference in weights are entries equal to $r = d + 1$. However, if $\widetilde{T}$ contains such an entry r, then it must contain a most northeast box containing a starred r. The weight of this box is 0 since $x(\pi)_{d+1} = y(\pi)_{d+1} = 0$. Thus the righthand side is summed over tableaux where the most northeast starred r is barred by a double arrow. The labelling of λ to the southwest, and π to the northeast is consistent with both the primed and unprimed ordering on tableaux.

This completes our 3 cases, we now move on to the case where λ/π contains two boxes.

5.10 Calculating $k^{\lambda}_{\theta\pi}(a)$ when λ/π contains two boxes

Continue fixing θ, a skew partition with no 2×2 subdiagram. In this section we describe the case of calculating $k^{\lambda}_{\theta\pi}(a)$ where the skew partition λ/π has two boxes, either added in the same column in consecutive rows above the diagonal, or added in the same row in consecutive columns below the diagonal. The calculations we provide here are crucial in proving Theorem 5.34 later.

We will have some examples at the end of the section as well. There are 2 other cases where λ/π has two boxes; the first is when two boxes are added consecutively to different rows and columns, and the second is when two boxes are added consecutively to the same row above the diagonal, or column below the diagonal. However, these two cases are relatively easy to understand so we leave the proof and description of them till later, when we tackle the general case.

5.10.1 Adding two boxes to the same column above the diagonal

Let $d = d(\pi)$. We consider the calculation for $k^{\lambda}_{\theta\pi}(a)$ for the case where $rs : \pi \to \lambda$ with $s = r + 1$, where λ is formed from π by adding two boxes, in row r and then s, for a $r \leqslant d$, such that the added boxes lie in the same column.

Proposition 5.19. *The coefficient* $k^{\lambda}_{\theta\pi}(a)$ *is given by*

$$k^{\lambda}_{\theta\pi}(a) = \sum_{T} \mathrm{ev}(T),$$

where the sum is over barred $\mathbb{A}$*-tableaux* T *in* $\mathcal{T}(R^+, R^-)$*, where* $(R^+, R^-) : \pi \to \lambda$ *and is Yamanouchi.*

We make some preliminary remarks before proving this proposition. Recall that the words (R^+, R^-) are said to be Yamanouchi if there exist sequences of partitions $R^+ : \pi^\wedge \to \lambda^\wedge$ and $R^- : \pi^\vee \to \lambda^\vee$. The set $\mathcal{T}(R^+, R^-)$ is the set of tableaux that have words (R^+, R^-), and also the number of barred entries in each tableau is equal to the number of boxes in λ/π. The last condition is necessary because the degree of $k^{\lambda}_{\theta\pi}(a)$ is equal to $|\lambda| - |\lambda/\pi|$. Without the last condition it is possible to produce barred tableaux with words which take π to λ but do not have the right degree. For example, if we take $\pi = \varnothing$ and $\lambda = (1)$, then the following barred tableau of shape λ

$$\begin{array}{|c|c|}\hline \overleftarrow{1} & \overrightarrow{1}^{*} \\ \hline \end{array}$$

has words $(R^+, R^-) = (1, 1)$ which respectively take $\pi^\wedge \to \lambda^\wedge$ and $\pi^\vee \to \lambda^\vee$, but the weight of this tableau does not have the right degree.

There are two cases to consider. Either the second box is added to the diagonal, or it is not. Let the partition ρ be defined by $\pi \xrightarrow{r} \rho$. Denote the first case by Case 1, and the second by Case 2. Since the arguments are similar, we will consider both cases simultaneously by pointing out where the argument diverges for each case, and then returning to the general argument for both cases.

We apply the recurrence relation (5.3) and obtain

$$k^{\lambda}_{\theta\pi}(a) = \frac{1}{k^{\lambda}_{(1)\lambda}(a) - k^{\pi}_{(1)\pi}(a)} \left(k^{\lambda}_{\theta\rho}(a) - k^{\rho}_{\theta\pi}(a) \right).$$

We use a sequence of lemmas to prove Proposition 5.19.

Lemma 5.20. *The following equation holds:*

$$\left(k^{\lambda}_{(1)\lambda}(a) - k^{\pi}_{(1)\pi}(a) \right) \sum_{T \in \mathcal{T}(R^+, R^-)} \mathrm{ev}(T) = k^{\lambda}_{\theta\rho}(a) - k^{\rho}_{\theta\pi}(a),$$

summed over the set $\mathcal{T}(R^+, R^-)$ *of tableaux with words* $(R^+, R^-) : \pi \to \lambda$ *which are Yamanouchi.*

We prove Lemma 5.20 with the following arguments. By the previous section, we know that

$$k^{\lambda}_{\theta\rho}(a) = \sum_{T} \mathrm{ev}(T),$$

summed over tableaux T with the most northeast $(d+1)^*$ barred by a double arrow in Case 1, and tableaux T with a s or s^* barred by a left arrow in Case 2, with the appropriate labelling. Also, we know

$$k^{\rho}_{\theta\pi}(a) = \sum_{T} \mathrm{ev}(T),$$

summed over tableaux T with a r or r^* barred by a left arrow, with the appropriate labelling, in both Case 1, and 2. First we start by calculating $k^{\lambda}_{(1)\lambda}(a) - k^{\pi}_{(1)\pi}(a)$, and then we will calculate $k^{\lambda}_{\theta\rho}(a) - k^{\rho}_{\theta\pi}(a)$. We note that $x(\lambda)_s = x(\pi)_r$ in the below calculation. This is because $x(\lambda)_s = \lambda_s - s + 1 = \pi_r - r + 1 = x(\pi)_r$, since row r of π has one less box than row s of λ.

Case 1: Since λ differs from π through the addition of a box in row d, and a box on the diagonal, we have that

$$\begin{aligned} k^{\lambda}_{(1)\lambda}(a) - k^{\pi}_{(1)\pi}(a) &= x(\lambda)_d + y(\lambda)_d + x(\lambda)_{d+1} + y(\lambda)_{d+1} - x(\pi)_d - y(\pi)_d \\ &= x(\lambda)_d + y(\lambda)_{d+1}, \end{aligned}$$

where the second equality follows because the columns of λ are of equal length to the columns of π, except λ has a $(d+1)$-th column and π does not, and $x(\lambda)_{d+1} = x(\pi)_d$.

Case 2: Since λ differs from π through the addition of a box in row r, and a box not on the diagonal, we have that

$$\begin{aligned} k^{\lambda}_{(1)\lambda}(a) - k^{\pi}_{(1)\pi}(a) &= x(\lambda)_r + y(\lambda)_r + x(\lambda)_s + y(\lambda)_s - x(\pi)_r - y(\pi)_r - x(\pi)_s - y(\pi)_s \\ &= x(\lambda)_r - x(\pi)_s, \end{aligned}$$

where again terms cancel because $x(\lambda)_s = x(\pi)_r$ and the length of columns below the diagonal of π and λ are the same.

Fix a box $\alpha \in \theta$. *Case 1:* Let T be a $\mathbb{A}$-tableau such that the most northeast $(d+1)^*$ of T barred by a double arrow is in a box α. The entries to the northeast of α are labelled ρ and the entries to the southeast of α are labelled λ. *Case 2:* Let T be a $\mathbb{A}$-tableau with a s barred by a left arrow in the box α, such that the entries to the northeast of α are labelled ρ and the entries to the southeast of α are labelled λ. Note that an equivalent way of setting the labelling in both cases is to say that entries before α are labelled ρ and those after are labelled λ, with respect to the unprimed ordering. However, we believe that it is easier to understand compass directions so we will continue to stick with compass directions in this section. Returning to the general argument, in both cases, T is a tableau which contributes to the coefficient $k^{\lambda}_{\theta\rho}(a)$. Let T_1 be the tableau with exactly the same entries as T, except the entries to the northeast of α are labelled π and the entries to the southeast of α are labelled

λ. Then the difference between $\mathrm{ev}(T)$ and $\mathrm{ev}(T_1)$ is

$$\mathrm{ev}(T) - \mathrm{ev}(T_1) = \sum_{\widetilde{T}} (x(\lambda)_r - x(\pi)_r)\,\mathrm{ev}(\widetilde{T}),$$

where the sum is over $\mathbb{A}$-tableaux $\widetilde{T}$ in $\mathcal{T}(R^+, R^-)$ with entries equal to T, except there exist a barred r to the northeast of α in $\widetilde{T}$.

Fix a box $\beta \in \theta$. In both cases, let T' be a $\mathbb{A}$-tableau with a r barred by a left arrow in box β, such that the entries to the northeast of β are labelled π and those to the southeast of β are labelled ρ.

Let T_2 be the tableau with exactly the same entries as T', except the entries to the northeast of β are labelled π and the entries to the southeast of β are labelled λ. Then the difference between $\mathrm{ev}(T_2)$ and $\mathrm{ev}(T')$ is different depending on the case.

Case 1: In this case,

$$\mathrm{ev}(T_2) - \mathrm{ev}(T') = \sum_{\widetilde{T}} (x(\lambda)_{d+1} + y(\lambda)_{d+1})\,\mathrm{ev}(\widetilde{T}),$$

where the sum is over $\mathbb{A}$-tableaux $\widetilde{T}$ in $\mathcal{T}(R^+, R^-)$ with entries equal to T', except the first starred s southwest of β is barred by a double arrow. This barred starred s is the most northeast starred s in $\widetilde{T}$, since if it were not $\widetilde{T}$ would have zero weight.

Case 2: In this case,

$$\mathrm{ev}(T_2) - \mathrm{ev}(T') = \sum_{\widetilde{T}} (x(\lambda)_s - x(\pi)_s)\,\mathrm{ev}(\widetilde{T}),$$

where the sum is over $\mathbb{A}$-tableaux $\widetilde{T}$ with entries equal to T', except there exist an s barred with a left arrow to the southeast of α in $\widetilde{T}$.

We note that in Case 1

$$x(\lambda)_d - x(\pi)_d + x(\lambda)_{d+1} + y(\lambda)_{d+1} = x(\lambda)_d + y(\lambda)_{d+1}, \tag{5.17}$$

since $x(\lambda)_{d+1} = x(\pi)_d$. In Case 2,

$$x(\lambda)_r - x(\pi)_r + x(\lambda)_s - x(\pi)_s = x(\lambda)_r - x(\pi)_s, \tag{5.18}$$

since $x(\lambda)_s = x(\pi)_r$.

Therefore in both cases, the differences (5.17) and (5.18) are equal to the factor $k^{\lambda}_{(1)\lambda}(a) - k^{\pi}_{(1)\pi}(a)$. Thus, if we can cancel the weight of tableaux like T_1 with the weight of tableaux like T_2, the desired formula for $k^{\lambda}_{\theta\pi}(a)$ will follow. We do this in the subsequent paragraphs, starting by defining two sets of $\mathbb{A}$-tableaux, $\mathcal{T}$ and $\mathcal{T}'$, the first of which depends on the case.

Case 1: let $\mathcal{T}$ denote the set of $\mathbb{A}$-tableaux T such that the most northeast starred s in the tableau T is barred by a double arrow. For a tableau $T \in \mathcal{T}$, the entries to the northeast of the barred entry are labelled π and the entries southeast of the barred entry are labelled λ.

Case 2: let $\mathcal{T}$ denote the set of $\mathbb{A}$-tableaux containing a s barred by a left arrow. For a tableau $T \in \mathcal{T}$, the entries to the northeast of the barred entry are labelled π and the entries southeast of the barred entry are labelled λ.

In both cases, let $\mathcal{T}'$ denote the set of $\mathbb{A}$-tableaux containing a r barred by a left arrow. For a tableau $T \in \mathbb{T}'$, the entries to the northeast of the barred entry are labelled π and the entries southeast of the barred entry are labelled λ.

We wish to prove the following lemma.

Lemma 5.21. *For the sets $\mathcal{T}$ and $\mathcal{T}'$ defined above, we have*

$$\sum_{T \in \mathcal{T}} \mathrm{ev}(T) = \sum_{T \in \mathcal{T}'} \mathrm{ev}(T).$$

This lemma is essential in Section 5.14 when we wish to prove Theorem 5.34. Also, the proof of Lemma 5.20 follows since the following statement is a consequence of Lemma 5.21 and equations (5.17) and (5.18). We have that

$$k^{\lambda}_{\theta\rho}(a) - \sum_{T \in \mathcal{T}} \mathrm{ev}(T) + \sum_{T \in \mathcal{T}'} \mathrm{ev}(T) - k^{\rho}_{\theta\pi}(a) = (k^{\lambda}_{(1)\lambda}(a) - k^{\pi}_{(1)\pi}(a)) \left(\sum_{T \in \mathcal{T}(R^+, R^-)} \mathrm{ev}(T) \right),$$

where the last sum is over tableaux in the set $\mathcal{T}(R^+, R^-)$ consisting of tableaux with $|\lambda/\pi|$ barred entries and containing words $(R^+, R^-) : \pi \to \lambda$ which are Yamanouchi.

To make the proof of Lemma 5.21 easier, we define a set $\mathcal{Q}'$, which will have equal weight to $\mathcal{T}'$, in the sense that

$$\sum_{T \in \mathcal{Q}'} \mathrm{ev}(T) = \sum_{T \in \mathcal{T}'} \mathrm{ev}(T)$$

The tableaux in $\mathcal{Q}'$ consist of tableaux Q formed from tableaux T in $\mathcal{T}'$ by doing the following: First, replace each connected component of θ containing integer values r with its dual subtableaux. Second, entries of the new tableau $Q \in \mathcal{Q}'$ to the northeast of the barred entry are labelled by the partition π and those southwest are labelled with the partition λ.

Example 5.22. Consider the following tableau T in $\mathcal{T}'$

$\overleftarrow{2}$	2	2^*	3	3	3^*
$2'$					
$2'$					
3^*					
$3'$					
$3'$					

Let $r = 2$. Then, the above process forms the following tableau Q in $\mathcal{Q}'$:

$2'$	$\overleftarrow{2}$	2	3	3	3^*
$2'$					
2^*					
3^*					
$3'$					
$3'$					

In both the tableaux, entries northeast of the barred entry are labelled π and those southwest are labelled λ. Note that the entries that do not take integer value 2 are unaffected. Also, all the entries of both tableaux have the same weight, except for 2^*, since the entry $T(1,3) = 2^*$ is labelled λ and the entry $Q(3,1) = 2^*$ is labelled π. $\square$

Lemma 5.23. *The weight of $\mathcal{Q}'$ and $\mathcal{T}'$ are equal, that is:*

$$\sum_{T \in \mathcal{T}'} \mathrm{ev}(T) = \sum_{T \in \mathcal{Q}'} \mathrm{ev}(T).$$

Proof. Without loss of generality, we may assume that the tableau T is connected and only contains entries with integer value r. We have that

$$\begin{aligned}
LHS &= \sum_{T\in\mathcal{T}'} \mathrm{ev}(T) && (5.19)\\
&= \frac{1}{x(\lambda)_r - x(\pi)_r}\left(\sum_{T_\lambda} \mathrm{ev}(T_\lambda) - \sum_{T_\pi} \mathrm{ev}(T_\pi)\right) && (5.20)\\
&= \frac{1}{x(\lambda)_r - x(\pi)_r}\left(\sum_{Q_\lambda} \mathrm{ev}(Q_\lambda) - \sum_{Q_\pi} \mathrm{ev}(Q_\pi)\right) && (5.21)\\
&= \sum_{Q\in\mathcal{Q}'} \mathrm{ev}(Q) && (5.22)\\
&= RHS, && (5.23)
\end{aligned}$$

where in expression (5.20) the first sum is over all $\mathbb{A}$-tableaux T_λ with no barred entries and all entries labelled by λ, and the second sum is over all $\mathbb{A}$-tableaux T_π with no barred entries and all entries labelled by π. In expression (5.21), the sums are over tableaux Q_λ and Q_π formed from T_λ and T_π respectively by replacing the tableaux T_λ and T_π by their dual subtableaux respectively. The second and fourth equality follows from applying the difference formula (5.7), with respect to the dual primed ordering, and noting that the weights of the tableau in the sums in (5.20) and (5.21) only differ in boxes containing an unprimed, or starred r. □

Furthermore, we define the set $\mathcal{Q}$ in a very similar way; for each T in $\mathcal{T}$, we create a tableau in $\mathcal{Q}$ formed from T by replacing the subtableau of T with entries taking integer value r by its dual subtableau. The labelling of the entries are unchanged; those northeast of the barred s are still labelled by π and those southwest are labelled by λ. It is obvious that doing this replacement does not change the weight of the tableau.

Example 5.24. Letting $s = 3$, consider the following tableau which belongs to $\mathcal{T}$

2	2	2^*	3	$\overleftarrow{3}$	3^*
$2'$					
$2'$					
3^*					
$3'$					
$3'$					

We replace it with the following tableau, which is in $\mathcal{Q}$:

$2'$	2	2	3	$\overleftarrow{3}$	3^*
$2'$					
2^*					
3^*					
$3'$					
$3'$					

□

We return to the proof of Lemma 5.21, which reduces to proving that the weight of $\mathcal{Q}$ and $\mathcal{Q}'$ are equal,

$$\sum_{T\in\mathcal{Q}} \mathrm{ev}(T) = \sum_{T\in\mathcal{Q}'} \mathrm{ev}(T).$$

In both cases, we distinguish between two types of tableaux in $\mathcal{Q}$:

Type (1a) The box α containing a barred s does not have an entry with integer value r in the box directly northeast of it.

Type (1b) The box α containing a barred s has an entry with a r^* in the box directly northeast of it.

Similarly, there are two types of tableaux in $\mathcal{Q}'$:

Type (2a) The box β containing a barred r does not have a s^* in the box directly southwest of it.

Type (2b) The box β containing a barred r has a s^* in the box directly southwest of it.

Example 5.25. Letting $r = 2$ and $s = 3$, the following is an example of a type (2a) tableau:

$2'$	$\overleftarrow{2}$	2	3	3	3^*
$2'$					
2^*					
3^*					
$3'$					
$3'$					

since the barred 2 does not have a 3^* directly southwest of it.

Continuing with the same values of r and s, the following is an example of a type (2b) tableau:

$2'$	2	2	3	3	3^*
$2'$					
$\overleftarrow{2^*}$					
3^*					
$3'$					
$3'$					

since the barred 2^* has a 3^* directly southwest of it. □

Let Q be a tableau of type (1a). Let β be the most southwest box of the connected component containing α which contains entries with integer value s; this means $Q(\beta) = s$, and the box directly southwest of β does not contain an entry with integer value s. Define a tableau R: except in boxes between β and α, R is the tableau with entries equal to Q; we replace the entries in boxes between β and α with unbarred entries with integer value r. The replaced entries are primed or left unprimed so that Condition 1 of the definition of a $\mathbb{A}$-tableau is preserved (see Section 5.1). Furthermore, if the box directly southwest of β contains an entry with integer value r, we bar the r in box β with a left arrow. Otherwise, if the box directly southwest of β does not contain an entry with integer value r, we replace the entry with a r^* and bar it with a left arrow. The labelling on R is set as follows: for entries southwest of the barred entry are labelled λ and those northeast are labelled π. In both cases, R is a type (2a) tableau. Furthermore, the most northeast box of the connected component containing the box β and entries with integer value r is the box α. Hence Q and R are a unique pair.

Example 5.26. Let Q be the following tableau of type (1a), which only has nonzero weight in Case 2.

$$Q = \begin{array}{|c|c|c|c|c|c|}
\hline
2' & 2 & 2 & 3 & \overleftarrow{3} & 3^* \\
\hline
2' \\
\cline{1-1}
2^* \\
\cline{1-1}
3^* \\
\cline{1-1}
3' \\
\cline{1-1}
3' \\
\cline{1-1}
\end{array}$$

Here, the entries southwest of (1,5) are labelled λ, and those northeast are labelled π. The most southwest entry with integer value 3 connected to the barred entry in box $\alpha = (1,5)$ is in box $\beta = (1,4)$. We make the following replacement to form R:

$$R = \begin{array}{|c|c|c|c|c|c|}
\hline
2' & 2 & 2 & \overleftarrow{2} & 2 & 3^* \\
\hline
2' \\
\cline{1-1}
2^* \\
\cline{1-1}
3^* \\
\cline{1-1}
3' \\
\cline{1-1}
3' \\
\cline{1-1}
\end{array}$$

such that the entries to the southwest of (1,4) are labelled λ and those northeast are labelled π. The weight of $Q(1,4)$ is equal to the weight of $R(1,5)$, since

$$\begin{aligned}
\mathrm{ev}(Q(1,4)) &= a_{\lambda_3-2} - a_{c(1,4)+1} \\
&= a_{\pi_2-1} - a_{c(1,5)} \\
&= \mathrm{ev}(R(1,5)),
\end{aligned}$$

recalling that the third equality holds because the weight of the entry $R(\beta)$ in the dual subtableau containing entries with integer value 2 is $a_{\pi_{R(\beta)}-R(\beta)+1} - a_{c(\beta)}$.

Consider another pair of examples. Let Q be the following tableau of type (1a):

$$Q = \begin{array}{|c|c|c|c|c|c|}
\hline
3 & 3 & 3 & 3 & \overleftarrow{3} & 3^* \\
\hline
3' \\
\cline{1-1}
3' \\
\cline{1-1}
3' \\
\cline{1-1}
3' \\
\cline{1-1}
3' \\
\cline{1-1}
\end{array}$$

from which we form R:

$$R = \begin{array}{|c|c|c|c|c|c|}
\hline
2' & 2 & 2 & 2 & 2 & 3^* \\
\hline
2' \\
\cline{1-1}
2' \\
\cline{1-1}
2' \\
\cline{1-1}
2' \\
\cline{1-1}
\overleftarrow{2}^* \\
\cline{1-1}
\end{array}$$

The weight of the entries of Q in boxes $(1,1)$ up to $(1,4)$ are equal to the weight of entries of R in boxes $(1,2)$ to $(1,5)$. However, the primed entries of Q in boxes $(2,1)$ to $(6,1)$ have different weights to those of R in boxes $(1,1)$ to $(5,1)$. We examine this in the following argument. $\square$

Let γ be a box between β and α, such that there is at least one box north of γ. We will call γ a box in a *vertical section between β and α*. For each such γ let γ' be the box above γ. Thus, γ' has at least one box to the south of it. For each γ, define tableaux Q_γ and $R_{\gamma'}$, which both have entries, except between boxes β and α, equal to Q and R respectively: In Q_γ, the boxes between β and γ are filled with entries taking integer value s's, with the entry in box γ barred and the rest unbarred. Here, we mean that the boxes are filled with appropriate unprimed, primed or starred entries so that this still results in a $\mathbb{A}$-tableau being formed. Furthermore, the boxes between γ' and α are filled in with entries with integer value r's, all unbarred. In $R_{\gamma'}$, the boxes between β and γ are filled in with entries with integer value s's, all unbarred. Furthermore, the boxes between γ' and α are filled in with entries with integer value r's, with the entry in box γ' barred and the rest unbarred.

In both the tableaux Q_γ and $R_{\gamma'}$, the entries to the northeast of the barred entry are labelled by π and those southwest are labelled by λ. Note that the tableaux Q_γ and $R_{\gamma'}$ are tableau of type (1b) and (2b) respectively.

Example 5.27. Continuing with the pair of tableaux Q and R:

$$Q = \begin{array}{|c|c|c|c|c|c|} \hline 3 & 3 & 3 & 3 & \overleftarrow{3} & 3^* \\ \hline 3' \\ \cline{1-1} 3' \\ \cline{1-1} 3' \\ \cline{1-1} 3' \\ \cline{1-1} 3' \\ \cline{1-1} \end{array} \qquad R = \begin{array}{|c|c|c|c|c|c|} \hline 2' & 2 & 2 & 2 & 2 & 3^* \\ \hline 2' \\ \cline{1-1} 2' \\ \cline{1-1} 2' \\ \cline{1-1} 2' \\ \cline{1-1} \overleftarrow{2^*} \\ \cline{1-1} \end{array}$$

we have that $\gamma = (4, 1)$ is in the vertical section between $\alpha = (1, 5)$ and $\beta = (6, 1)$, since γ has at least one box north of it. Therefore, the box north of γ is set to be $\gamma' = (3, 1)$. We have the following tableau Q_γ and $R_{\gamma'}$ respectively:

$$Q_\gamma = \begin{array}{|c|c|c|c|c|c|} \hline 2' & 2 & 2 & 2 & 2 & 3^* \\ \hline 2' \\ \cline{1-1} 2^* \\ \cline{1-1} \overleftarrow{3^*} \\ \cline{1-1} 3' \\ \cline{1-1} 3' \\ \cline{1-1} \end{array} \qquad R_{\gamma'} = \begin{array}{|c|c|c|c|c|c|} \hline 2' & 2 & 2 & 2 & 2 & 3^* \\ \hline 2' \\ \cline{1-1} \overleftarrow{2^*} \\ \cline{1-1} 3^* \\ \cline{1-1} 3' \\ \cline{1-1} 3' \\ \cline{1-1} \end{array}$$

In both of these tableaux, the entries to the northeast of the barred entry are labelled π and those southwest are labelled λ. Note that in the tableau Q_γ, the barred 3^* has a 2^* directly northeast of it, therefore is a tableau of type (1b). Also, in the tableau $R_{\gamma'}$, the barred 2^* has a 3^* directly southwest of it, therefore this tableau is one of type (2b). Finally, the entries in boxes $(1, 2)$ to $(1, 6)$ in both Q_γ and $R_{\gamma'}$ have the same weight.

□

We will now cancel the weight of tableaux in $\mathcal{T}$ with those in $\widetilde{\mathcal{T}}$. We claim that the following equation holds:

$$\operatorname{ev}(Q) - \operatorname{ev}(R) - \sum_{\gamma} (\operatorname{ev}(R_{\gamma'}) - \operatorname{ev}(Q_\gamma)) = 0, \tag{5.24}$$

summed over all γ in a vertical section between β and α. The claim follows in parts. In the first part we evaluate $\text{ev}(Q_\gamma) - \text{ev}(R_{\gamma'})$ for a given γ, and in the second part we will use the difference formula 5.7 to simplify $\sum\limits_{\gamma} \text{ev}(Q_\gamma) - \text{ev}(R_{\gamma'})$, and compare it to $\text{ev}(Q) - \text{ev}(R)$.

First, the weight of unprimed entries are equal between boxes β and α of tableaux Q, R and all tableaux Q_γ, $R_{\gamma'}$, for any box γ in a vertical section between β and α. Furthermore, in Q_γ and $R_{\gamma'}$, every unbarred entry has the same weight except for $Q_\gamma(\gamma')$ and $R_{\gamma'}(\gamma)$; the weight of $Q_\gamma(\gamma)$ is $x(\pi)_r + y(\pi)_r$ and the weight of $R_{\gamma'}(\gamma')$ is $x(\lambda)_s + y(\lambda)_s$. Therefore, $\text{ev}(Q_\gamma) - \text{ev}(R_{\gamma'})$ only differs in weight in those two entries. Recalling that $x(\pi)_r = x(\lambda)_s$, we may write this difference as

$$\text{ev}(Q_\gamma) - \text{ev}(R_{\gamma'}) = (y(\pi)_r - y(\lambda)_s)\text{ev}(P_\gamma),$$

such that the tableau P_γ has entries equal to Q_γ, except we will also bar the entry in box γ'. Furthermore, the entries of P_γ northeast of γ' are labelled π and those southwest of γ are labelled λ.

We may then write $\sum_\gamma \text{ev}(Q_\gamma) - \text{ev}(R_{\gamma'})$ as

$$\sum_{\gamma} \text{ev}(Q_\gamma) - \text{ev}(R_{\gamma'}) = (y(\pi)_r - y(\lambda)_s) \sum_{\gamma} \text{ev}(P_\gamma), \tag{5.25}$$

summed over γ in a vertical section between β and α. We now illustrate this with an example.

Example 5.28. Using Q_γ and $R_{\gamma'}$ from previously, we have the diagrammatic relation:

$$Q_\gamma - R_{\gamma'} = \begin{array}{|c|c|c|c|c|c|}\hline 2' & 2 & 2 & 2 & 2 & 3^* \\ \hline 2' \\ \cline{1-1} 2^* \\ \cline{1-1} \overleftarrow{3^*} \\ \cline{1-1} 3' \\ \cline{1-1} 3' \\ \cline{1-1}\end{array} \quad - \quad \begin{array}{|c|c|c|c|c|c|}\hline 2' & 2 & 2 & 2 & 2 & 3^* \\ \hline 2' \\ \cline{1-1} \overleftarrow{2^*} \\ \cline{1-1} 3^* \\ \cline{1-1} 3' \\ \cline{1-1} 3' \\ \cline{1-1}\end{array}$$

$$= (y(\pi)_2 - y(\lambda)_3) \begin{array}{|c|c|c|c|c|c|}\hline 2' & 2 & 2 & 2 & 2 & 3^* \\ \hline 2' \\ \cline{1-1} \overleftarrow{2^*} \\ \cline{1-1} \overleftarrow{3^*} \\ \cline{1-1} 3' \\ \cline{1-1} 3' \\ \cline{1-1}\end{array}$$

$$= (y(\pi)_2 - y(\lambda)_3)P_\gamma$$

□

We return to the general argument. Consider $\mathrm{ev}(Q) - \mathrm{ev}(R)$: the only difference occurs in the weights of primed entries between β and α of Q and R. We apply the difference formula (5.7) to $\mathrm{ev}(Q) - \mathrm{ev}(R)$ and conclude that

$$\mathrm{ev}(Q) - \mathrm{ev}(R) = (y(\lambda)_s - y(\pi)_r) \sum_\gamma \mathrm{ev}(P_\gamma),$$

summed over all boxes γ in a vertical section between β and α. This concludes the proof of Lemma 5.21.

5.10.2 Adding two boxes to the same row below the diagonal

Fix a skew partition θ, not containing a 2×2 subdiagram. Let π be a partition with depth $d = d(\pi)$. We consider the calculation for the coefficient $k^\lambda_{\theta\pi}(a)$ for the case where $rs : \pi \to \lambda$, and $s = r + 1$, where λ is formed from π by adding two boxes in column r and then s, for a $r \leqslant d$ such that the added boxes lie in the same row. Let ρ be the partition formed from π by adding a box to the end of column r of π. We claim that

Proposition 5.29. *The coefficient $k^{\lambda}_{\theta\pi}(a)$ is given by*

$$k^{\lambda}_{\theta\pi}(a) = \sum_{T} \operatorname{ev}(T),$$

where the sum is over barred $\mathbb{A}$-tableaux T in $\mathcal{T}(R^+, R^-)$, where $(R^+, R^-) : \pi \to \lambda$ and is Yamanouchi.

Recall that a tableau T in the set $\mathcal{T}(R^+, R^-)$ has $|\lambda/\pi|$ barred entries. The statement and proof of this proposition is very similar to the one in the previous section; the difference arises due to the primed ordering on tableaux. We argue using dual tableaux defined in Section 5.8 and sketch the proof, recalling that an $\mathbb{A}$-tableau T and its dual $\breve{T}$ have the same weight.

The recurrence relation (5.3) gives

$$k^{\lambda}_{\theta\pi}(a) = \frac{1}{k^{\lambda}_{(1)\lambda}(a) - k^{\pi}_{(1)\pi}(a)} \left(k^{\lambda}_{\theta\rho}(a) - k^{\rho}_{\theta\pi}(a) \right). \tag{5.26}$$

We know that

$$k^{\lambda}_{\theta\rho}(a) = \sum_{T} \operatorname{ev}(T), \tag{5.27}$$

summed over dual tableaux T with a s' or s^* barred with a right arrow or a double arrow. Entries of T southwest of the barred entry are labelled ρ and those northeast are labelled λ. Also, we know

$$k^{\rho}_{\theta\pi}(a) = \sum_{T} \operatorname{ev}(T), \tag{5.28}$$

summed over tableaux T with a r' or r^* barred by a right arrow, with entries to the southwest labelled π and those to the northeast labelled ρ.

Lemma 5.30. *The following equation holds:*

$$(k^{\lambda}_{(1)\lambda}(a) - k^{\pi}_{(1)\pi}(a)) \sum_{T \in \mathcal{T}(R^+, R^-)} \operatorname{ev}(T) = k^{\lambda}_{\theta\rho}(a) - k^{\rho}_{\theta\pi}(a),$$

summed over the set $\mathcal{T}(R^+, R^-)$ of tableaux with words $(R^+, R^-) : \pi \to \lambda$ which are Yamanouchi.

The proof of this lemma follows in the same way as the proof of Lemma 5.20. Let T be a tableau with word which takes ρ to λ and is Yamanouchi, and T' a tableau with word which takes π to ρ and is Yamanouchi. Analagous to equations (5.17) and (5.18), define sets of tableau $\mathcal{T}$ and $\mathcal{T}'$ such that for $T_1 \in \mathcal{T}$ and $T_2 \in \mathcal{T}'$ we have

$$\operatorname{ev}(T) - \operatorname{ev}(T_1) + \operatorname{ev}(T_2) - \operatorname{ev}(T') = (k^{\lambda}_{(1)\lambda}(a) - k^{\pi}_{(1)\pi}(a))P(a),$$

where $P(a)$ is a polynomial in the variables a with integer coefficients.

Then, to prove Proposition 5.29, it suffices to show that the following lemma analagous to Lemma 5.21 is true.

Lemma 5.31. *For the sets $\mathcal{T}$ and $\mathcal{T}'$ defined above, we have*

$$\sum_{T\in\mathcal{T}} \mathrm{ev}(T) = \sum_{T\in\mathcal{T}'} \mathrm{ev}(T).$$

Lemma 5.31 is proved in the same way as Lemma 5.21, by defining sets $\mathcal{Q}$ and $\mathcal{Q}'$ of tableaux formed from tableaux T in $\mathcal{T}$ and $\mathcal{T}'$ respectively by replacing the subtableau of T consisting of entries with integer value s with its dual. We will then have the following two statements:

$$\sum_{T\in\mathcal{Q}} \mathrm{ev}(T) = \sum_{T\in\mathcal{T}} \mathrm{ev}(T),$$

and

$$\sum_{T\in\mathcal{Q}'} \mathrm{ev}(T) = \sum_{T\in\mathcal{T}'} \mathrm{ev}(T),$$

and proving Lemma 5.31 reduces to proving the following lemma

Lemma 5.32. *For the sets $\mathcal{Q}$ and $\mathcal{Q}'$ defined above, we have*

$$\sum_{T\in\mathcal{Q}} \mathrm{ev}(T) = \sum_{T\in\mathcal{Q}'} \mathrm{ev}(T).$$

We illustrate the proofs of all the above statements by providing an example.

Example 5.33. Suppose that $\pi = (3^4, 1)$ and $\lambda = (3^5)$. Thus, the words $(R^+, R^-) = (\varnothing, 23) : \pi \to \lambda$ and is Yamanouchi. Let $\rho = (3^4, 2)$ be the partition formed from π by adding a box to column 2 of π. Let $\theta = (5^2)/(4^2)$. Consider the following dual tableau T which contributes its weight to $k^{\lambda}_{\theta\rho}(a)$:

$$T = \begin{array}{ccccc}
\cline{5-5} & & & & \multicolumn{1}{|c|}{2'} \\
\cline{5-5} & & & & \multicolumn{1}{|c|}{2^*} \\
\cline{5-5} & & & & \multicolumn{1}{|c|}{3'} \\
\cline{5-5} & & & & \multicolumn{1}{|c|}{\vec{3}'} \\
\hline \multicolumn{1}{|c|}{2^*} & \multicolumn{1}{c|}{3^*} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{3} \\
\hline
\end{array}$$

which has entries to the northeast of the barred entry in box $(4, 5)$ labelled by λ, and those southwest labelled by ρ. Define the tableau T_1 of shape θ to have the same entries as T except the entries northeast of box $(4,5)$ are labelled λ and those southwest are labelled π.

Now consider the dual tableau T' which contributes its weight to the coefficient $k^{\rho}_{\theta\pi}(a)$:

$$T' = \begin{array}{|c|c|c|c|c|}
\cline{5-5} \multicolumn{4}{c|}{} & 2' \\
\cline{5-5} \multicolumn{4}{c|}{} & 2^* \\
\cline{5-5} \multicolumn{4}{c|}{} & 3' \\
\cline{5-5} \multicolumn{4}{c|}{} & 3' \\
\hline \overrightarrow{2^*} & 3^* & 3 & 3 & 3 \\
\hline
\end{array}$$

with entries northeast of box $(5,1)$ labelled by ρ. Consider the tableau T_2 with the same shape and entries as T', except entries northeast of box $(5,1)$ are labelled λ.

The tableaux T_1 and T_2 belong to sets $\mathcal{T}$ and $\mathcal{T}'$ respectively. These sets respectively contain tableaux with a $3'$ or 3^* barred by a right arrow, or tableaux with a $2'$ or 2^* barred by a right arrow. Entries of tableaux in both sets are labelled λ if they are northeast of the barred entry, or π if they are southwest. Then, the following equation holds

$$k^{\lambda}_{\theta\rho}(a) - \sum_{T\in\mathcal{T}} \mathrm{ev}(T) + \sum_{T\in\mathcal{T}'} \mathrm{ev}(T) - k^{\rho}_{\theta\pi}(a) = (k^{\nu}_{(1)\nu} - k^{\mu}_{(1)\mu}) \left(\sum_{T\in\mathcal{T}(R^+,R^-)} \mathrm{ev}(T) \right),$$

where the last sum is over tableaux with $|\lambda/\pi|$ barred entries and containing the words $(\varnothing, 23) : \pi \to \lambda$. A tableau T contains such a word if there is a $2'$ or 2^* barred with a right arrow southwest of a $3'$ or 3^* barred by a right arrow in T.

Thus, our formula for $k^{\lambda}_{\theta\pi}(a)$ follows if we can prove

$$\sum_{T\in\mathcal{T}} \mathrm{ev}(T) - \sum_{T\in\mathcal{T}'} \mathrm{ev}(T) = 0.$$

First, we construct the tableau Q from T_1 by replacing the subtableau containing 2's with its dual:

$$T_1 = \begin{array}{|c|c|c|c|c|}
\cline{5-5} \multicolumn{4}{c|}{} & 2' \\
\cline{5-5} \multicolumn{4}{c|}{} & 2^* \\
\cline{5-5} \multicolumn{4}{c|}{} & 3' \\
\cline{5-5} \multicolumn{4}{c|}{} & \overrightarrow{3'} \\
\hline 2^* & 3^* & 3 & 3 & 3 \\
\hline
\end{array}
\quad\longrightarrow\quad
\begin{array}{|c|c|c|c|c|}
\cline{5-5} \multicolumn{4}{c|}{} & 2^* \\
\cline{5-5} \multicolumn{4}{c|}{} & 2' \\
\cline{5-5} \multicolumn{4}{c|}{} & 3' \\
\cline{5-5} \multicolumn{4}{c|}{} & \overrightarrow{3'} \\
\hline 2^* & 3^* & 3 & 3 & 3 \\
\hline
\end{array} = Q$$

The entries to the northeast of the barred entry are labelled by λ and those southwest are labelled by π. Let the set $\mathcal{Q}$ consist of tableaux formed from tableaux in $\mathcal{T}$ by doing this replacement. We have that

$$\sum_{T \in \mathcal{T}} \mathrm{ev}(T) = \sum_{T \in \mathcal{Q}} \mathrm{ev}(T).$$

Second, the tableau Q' is constructed from T_2 by replacing the connected component to box $(5,1)$ of entries with integer value 2 with its dual:

$T_2 =$

				$2'$
				2^*
				$3'$
				$3'$
$\overrightarrow{2^*}$	3^*	3	3	3

$\longrightarrow$

				2^*
				$2'$
				$3'$
				$3'$
$\overrightarrow{2^*}$	3^*	3	3	3

$= Q'$

The entries northeast of the barred entry are labelled by λ and those southwest are labelled by π. Let the set $\mathcal{Q}'$ consist of tableaux formed from tableaux in $\mathcal{T}'$ by doing this replacement. We have that

$$\sum_{T \in \mathcal{T}'} \mathrm{ev}(T) = \sum_{T \in \mathcal{Q}'} \mathrm{ev}(T),$$

with this equation following in exactly the same manner as in the proof of Lemma 5.23.

We argue that

$$\sum_{T \in \mathcal{Q}} \mathrm{ev}(T) = \sum_{T \in \mathcal{Q}'} \mathrm{ev}(T)$$

using an example.

Consider the following tableau $Q \in \mathcal{Q}$

$Q =$

				2^*
				$2'$
				$3'$
				$3'$
$\overrightarrow{3^*}$	3	3	3	3

from which we form the tableau $R \in \mathcal{Q}'$

$$R = \begin{array}{|c|c|c|c|c|}
\cline{5-5}
\multicolumn{4}{c|}{} & 2^* \\ \cline{5-5}
\multicolumn{4}{c|}{} & 2' \\ \cline{5-5}
\multicolumn{4}{c|}{} & \vec{2'} \\ \cline{5-5}
\multicolumn{4}{c|}{} & 2' \\ \hline
2 & 2 & 2 & 2 & 2' \\ \hline
\end{array}$$

such that entries to the northeast of box $(5, 1)$ of R are labelled λ. The entries of Q in boxes $(1, 5)$ to $(4, 5)$ have the same weight as the unbarred entries of R in boxes $(1, 5)$ to $(5, 5)$.

Let γ be a box between boxes $(5, 2)$ and $(5, 5)$, and let γ' be the box to the left of γ. Define tableau Q_γ and $R_{\gamma'}$ in the following way: In both tableaux, boxes southwest of γ' are filled with entries taking integer value 2 and boxes northeast of γ are filled with entries taking integer value 3. In boxes γ' and γ we have a 2^* and 3^* respectively. In Q_γ the 3^* in box γ is barred with a right arrow, and in $R_{\gamma'}$ the 2^* in box γ' is barred with a right arrow. All entries to the northeast of the barred entry in both tableaux are labelled λ and those to the southwest are labelled π.

For example, consider the boxes $\gamma = (5, 3)$ and $\gamma' = (5, 2)$. Then, we have the following pair of tableaux

$$Q_{(5,3)} = \begin{array}{|c|c|c|c|c|}
\cline{5-5}
\multicolumn{4}{c|}{} & 2^* \\ \cline{5-5}
\multicolumn{4}{c|}{} & 2' \\ \cline{5-5}
\multicolumn{4}{c|}{} & 2' \\ \cline{5-5}
\multicolumn{4}{c|}{} & 2' \\ \hline
2 & 2^* & \vec{3^*} & 3 & 3 \\ \hline
\end{array} \quad \text{and} \quad R_{(5,2)} = \begin{array}{|c|c|c|c|c|}
\cline{5-5}
\multicolumn{4}{c|}{} & 2^* \\ \cline{5-5}
\multicolumn{4}{c|}{} & 2' \\ \cline{5-5}
\multicolumn{4}{c|}{} & 2' \\ \cline{5-5}
\multicolumn{4}{c|}{} & 2' \\ \hline
2 & \vec{2^*} & 3^* & 3 & 3 \\ \hline
\end{array}$$

such that entries to the northeast of the barred entry of both tableaux are labelled λ and those southwest are labelled π. Then, the entries of $Q_{(5,3)}$ and $R_{(5,2)}$ in all boxes other than $(5, 3)$ and $(5, 4)$ have equal weight to those in Q and Q'.

It follows from arguments used to prove equation (5.24) that

$$\mathrm{ev}(Q) + \sum_\gamma \mathrm{ev}(Q_\gamma) - \mathrm{ev}(R) - \sum_\gamma \mathrm{ev}(R_{\gamma'}) = 0,$$

summed over boxes γ between boxes $(5, 2)$ and $(5, 5)$. $\square$

5.11 The general case

We come to the main result of this chapter. Let θ be a skew diagram without a 2×2 subdiagram and let π be a partition. Recall that the coefficient $k_{\theta\pi}^{\lambda}(a)$ occurs in the decomposition

$$s_{\theta}(x/y\|a)s_{\pi}(x/y\|a) = \sum_{\lambda} k_{\theta\pi}^{\lambda}(a)s_{\lambda}(x/y\|a),$$

summed over partitions λ which contain π. Our main result here is a rule to calculate the coefficient $k_{\theta\pi}^{\lambda}(a)$, this is presented as Theorem 5.34. As a main corollary of this theorem, we are able to calculate the dual Littlewood–Richardson polynomials $\widehat{c}_{\lambda\mu}^{\nu}(a)$ defined in Section 2.2.

If T is a $\mathbb{A}$-tableaux of shape θ, define 3 subsets, θ^u, θ^p and θ^s, consisting of boxes of θ such that:

θ^u: The subset θ^u is the set of boxes $\alpha \in \theta$ such that $T(\alpha)$ is unbarred, unstarred and unprimed.

θ^p: The subset θ^p is the set of boxes $\alpha \in \theta$ such that $T(\alpha)$ is unbarred, unstarred and primed.

θ^s: The subset θ^s is the set of boxes $\alpha \in \theta$ such that $T(\alpha)$ is unbarred and starred.

Theorem 5.34. *Let θ be a disjoint union of skew hooks, and let π be a partition contained in another partition λ. Then*

$$k_{\theta\pi}^{\lambda}(a) = \sum_{T} \prod_{\alpha \in \theta^u} (a_{\rho(\alpha)_{T(\alpha)}-T(\alpha)+1} - a_{c(\alpha)+1}) \prod_{\alpha \in \theta^p} (a'_{\rho'(\alpha)_{T(\alpha)}-T(\alpha)+1} + a_{c(\alpha)+1})$$
$$\prod_{\alpha \in \theta^*} (a_{\rho(\alpha)_{T(\alpha)}-T(\alpha)+1} + a'_{\rho'(\alpha)_{T(\alpha)}-T(\alpha)+1}),$$

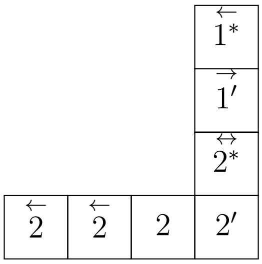
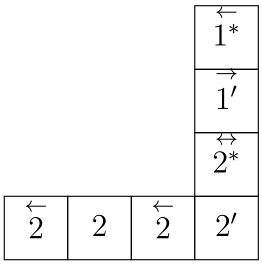
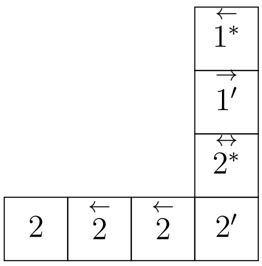
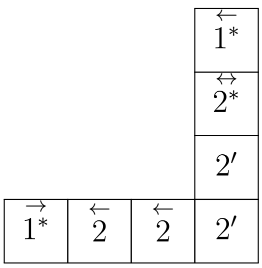
summed over $\mathbb{A}$-tableaux T of shape θ, such that the word (R^+, R^-) of T is Yamanouchi and takes π to λ.

First, we give an example where we calculate the coefficient $k_{\theta\pi}^{\lambda}(a)$.

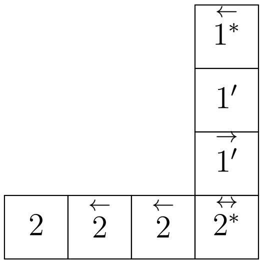
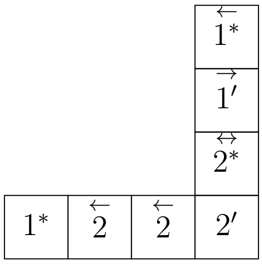
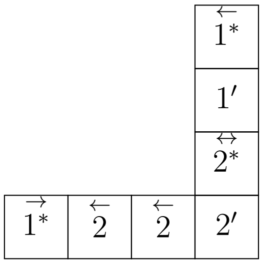
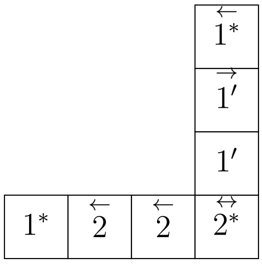
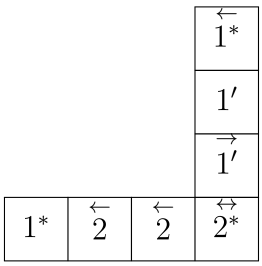
Example 5.35. Consider a skew diagram $\theta = (4^2)/(3^2)$, and partitions $\pi = (3,1)$ and $\lambda = (4,4,1)$. We wish to calculate the polynomial $k_{\theta\pi}^{\lambda}(a)$, which is given by the weights of $\mathbb{A}$-tableaux with words (R^+, R^-), such that the words R^+ and R^- are chosen from the following table:

R^+ :	1222	2122	2212
R^- :	12	21	

These words are Yamanouchi and take $\pi^{\pm}$ to $\lambda^{\pm}$. Tableaux which contain these words must have at least three 2's and at most five 2's. In addition, due to the shape of θ, no tableau contains the unprimed word $R^{+} = 2122$ or $R^{+} = 2212$. The following table gives all such possible $\mathbb{A}$-tableaux, along with their weight and words.

Tableaux	Words	Weight
	$(1222, 21)$	$(a_1 - a_0)^2$
	$(1222, 21)$	$(a_1 - a_0)(a_2 - a_{-1})$
	$(1222, 21)$	$(a_1 - a_0)(a_3 - a_{-2})$
	$(1222, 12)$	$(a_2 - a_0)(a_1 - a_0)$

Tableaux	Words	Weight
$\begin{array}{cccc} & & & \overleftarrow{1}^* \\ & & & \overrightarrow{1}' \\ & & & 1' \\ \overleftarrow{2} & \overleftarrow{2} & 2 & \overleftrightarrow{2}^* \end{array}$	$(1222, 21)$	$(a_2 - a_{-1})(a_1 - a_0)$
$\begin{array}{cccc} & & & \overleftarrow{1}^* \\ & & & \overrightarrow{1}' \\ & & & 1' \\ \overleftarrow{2} & 2 & \overleftarrow{2} & \overleftrightarrow{2}^* \end{array}$	$(1222, 21)$	$(a_2 - a_{-1})^2$
$\begin{array}{cccc} & & & \overleftarrow{1}^* \\ & & & \overrightarrow{1}' \\ & & & 1' \\ 2 & \overleftarrow{2} & \overleftarrow{2} & \overleftrightarrow{2}^* \end{array}$	$(1222, 21)$	$(a_2 - a_1)(a_3 - a_{-2})$
$\begin{array}{cccc} & & & \overleftarrow{1}^* \\ & & & 1' \\ & & & \overrightarrow{1}' \\ \overleftarrow{2} & \overleftarrow{2} & 2 & \overleftrightarrow{2}^* \end{array}$	$(1222, 21)$	$(a_3 - a_{-2})(a_1 - a_0)$
$\begin{array}{cccc} & & & \overleftarrow{1}^* \\ & & & 1' \\ & & & \overrightarrow{1}' \\ \overleftarrow{2} & 2 & \overleftarrow{2} & \overleftrightarrow{2}^* \end{array}$	$(1222, 21)$	$(a_3 - a_{-2})(a_2 - a_{-1})$

Tableaux	Words	Weight
	$(1222, 21)$	$(a_3 - a_{-2})^2$
	$(1222, 21)$	$(a_4 - a_{-1})(a_1 - a_0)$
	$(1222, 12)$	$(a_3 - a_{-2})(a_1 - a_0)$
	$(1222, 21)$	$(a_4 - a_{-2})(a_2 - a_{-1})$
	$(1222, 21)$	$(a_4 - a_{-1})(a_3 - a_{-2})$

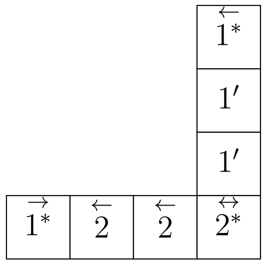
Tableaux	Words	Weight
	$(1222, 12)$	$(a_3 - a_{-2})(a_2 - a_{-2})$

Table 2: Table of all tableaux that have weight contributing to the polynomial $k^{\lambda}_{\theta\pi}(a)$.

Thus, for skew diagram $(4^2)/(3^2)$, partitions $\pi = (3,1)$ and $\lambda = (4,4,1)$, the coefficient $k^{\lambda}_{\theta\pi}(a)$ is given by summing up all the weights in Table 2.

□

Before we prove Theorem 5.34, consider the corollaries of this theorem. The first is a way to calculate the dual Littlewood–Richardson polynomials, and the second allows us to recover a Pieri rule for the ring of double symmetric functions $\Lambda(x\|a)$.

Let θ be the skew diagram ν/μ. Recall that one of the ways the coefficient $\widehat{c}^{\nu}_{\lambda\mu}(a)$ is defined is by decomposing the skew double Schur function

$$s_{\theta}(x\|a) = \sum_{\lambda} \widehat{c}^{\nu}_{\lambda\mu}(a) s_{\lambda}(x\|a), \tag{5.29}$$

summed over partitions λ.

Corollary 5.36. *Recall the sequence of variables $(a')_i = -a_{1-i}$ for all integers i. We have that*

$$\widehat{c}^{\nu}_{\lambda\mu}(a) = k^{\lambda}_{\theta\varnothing}(-a').$$

For a reasoning of why this equation holds, see the paragraphs which follow equation (5.1). Corollary 5.36 allows us to calculate the dual Littlewood–Richardson polynomials $\widehat{c}^{\nu}_{\lambda\mu}(a)$ for the special case where ν/μ is a disjoint union of skew hooks, i.e. does not contain a 2×2 subdiagram. As emphasized in Chapter 4, the Littlewood–Richardson polynomials are important because they describe the comultiplication structure of the ring $\Lambda(x\|a)$, equivariant cohomology of infinite grassmanians, and also combinatorial identities such as equation (5.29) which involves skew double Schur functions, and equation (4.1) which involves a product of dual Schur functions.

Corollary 5.37. *Let θ be the partition (p). Then,*

$$k^{\lambda}_{\theta\pi}(a) = \sum_{T} \prod_{\alpha\in\theta^u} (a_{\rho(\alpha)_{T(\alpha)}-T(\alpha)+1} - a_{c(\alpha)+1}) \prod_{\alpha\in\theta^*} (a_{\rho(\alpha)_{T(\alpha)}-T(\alpha)+1} + a'_{\rho'(\alpha)_{T(\alpha)}-T(\alpha)+1})$$

where θ^u is the subset of θ consisting of boxes which contain unprimed entries, and θ^s is the subset of θ consisting of boxes which contain starred entries.

In particular,

$$k^{\lambda}_{\theta\pi}(-a') = c^{\lambda}_{\theta\pi}(a).$$

This gives a new way of writing the Littlewood–Richardson polynomials $c^{\lambda}_{\theta\pi}(a)$ in terms of the Frobenius coordinates. A similar corollary holds when $\theta = (1^p)$, the partition corresponding to one column of boxes.

We prove Theorem 5.34 using a sequence of lemmas. Our proof will closely follow that of Molev and Sagan [19]. Fix a pair of partitions $\pi \subseteq \lambda$. For each sequence $(R^+, R^-) : \pi \to \lambda$ with corresponding sequences of partitions

$$\pi^{\wedge} = \rho_+^{(0)} \xrightarrow{r_1} \rho_+^{(1)} \xrightarrow{r_2} \dots \xrightarrow{r_p} \rho_+^{(p)} = \lambda^{\wedge},$$

and

$$\pi^{\vee} = \rho_-^{(0)} \xrightarrow{s_1} \rho_-^{(1)} \xrightarrow{s_2} \dots \xrightarrow{s_q} \rho_-^{(q)} = \lambda^{\vee},$$

we define two sets of tableaux, $\mathcal{T}_{k^+}(R^+, R^-)$, for a $1 \leqslant k \leqslant p$ and $\mathcal{T}_{k^-}(R^+, R^-)$, for a $1 \leqslant k \leqslant q$ as follows.

For a $1 \leqslant k \leqslant p$, define $R^+_k = r_1 \dots r_{k-1} r_{k+1} \dots r_p$ to be the integer sequence equal to R^+ except we omit the k-th entry. Then, the set $\mathcal{T}_{k^+}(R^+, R^-)$ is the set of all $\mathbb{A}$-tableaux with word (R^+_k, R^-). Similarly, for a $1 \leqslant k \leqslant q$, define $R^-_k = s_1 \dots s_{k-1} s_{k+1} \dots s_q$ to be the integer sequence equal to R^- except we omit the k-th entry. Then, the set $\mathcal{T}_{k^-}(R^+, R^-)$ is the set of all $\mathbb{A}$-tableaux with word (R^+, R^-_k).

For each $1 \leqslant k \leqslant p$, we associate two ways of labelling to the tableaux in $\mathcal{T}_{k^+}(R^+, R^-)$. Suppose $\alpha^+_1, \dots, \alpha^+_{k-1}, \alpha^+_{k+1}, \dots, \alpha^+_p$ are the boxes of θ with respect to the unprimed order which contain the barred entries $r_1, \dots r_{k-1}, r_{k+1}, \dots, r_p$ respectively. The first labelling, $\rho^u_{k^+}(a)$ is defined to be

$$\rho^u_{k^+}(\alpha) = \begin{cases} \rho_+^{(k)} \cup \rho^{\vee}(\alpha), & \text{if } \alpha^+_{k-1} \sqsubseteq \alpha \sqsubset \alpha^+_{k+1}. \\ \rho(\alpha), & \text{otherwise.} \end{cases}$$

and the second, $\rho^l_{k^+}(a)$ is defined to be

$$\rho^l_{k^+}(\alpha) = \begin{cases} \rho_+^{(k-1)} \cup \rho^{\vee}(\alpha), & \text{if } \alpha^+_{k-1} \preccurlyeq \alpha \prec \alpha^+_{k+1}. \\ \rho(\alpha), & \text{otherwise.} \end{cases}$$

Note that in either case if $\alpha^{\pm}_{k-1}$ or $\alpha^{\pm}_{k+1}$ do not exist we simply exclude them from the bounds on α. Let the set of all tableaux in $\mathcal{T}_{k^+}(R^+, R^-)$ labelled by $\rho^u_{k^+}(a)$ be the set denoted $\mathcal{T}^u_{k^+}(R^+, R^-)$, and let the set of all tableaux in $\mathcal{T}_{k^+}(R^+, R^-)$ labelled by $\rho^l_{k^+}(a)$ be the set denoted $\mathcal{T}^l_{k^+}(R^+, R^-)$.

For each $1 \leqslant k \leqslant q$, we associate two ways of labelling to the tableaux in $\mathcal{T}_{k^-}(R^+, R^-)$. Suppose $\alpha^-_1, \ldots, \alpha^-_{k-1}, \alpha^-_{k+1}, \ldots, \alpha^-_q$ are the boxes of θ with respect to the primed order which contain the barred entries $s_1, \ldots s_{k-1}, s_{k+1}, \ldots, s_q$ respectively. The first labelling, $\rho^u_{k^-}(a)$ is defined to be

$$\rho^u_{k^-}(\alpha) = \begin{cases} \rho^\wedge(\alpha) \cup \rho^{(k)}_-, & \text{if } \alpha^-_{k-1} < \alpha < \alpha^-_{k+1}. \\ \rho(\alpha), & \text{otherwise.} \end{cases}$$

and the second, $\rho^l_{k^-}(a)$ is defined to be

$$\rho^l_{k^-}(\alpha) = \begin{cases} \rho^\wedge(\alpha) \cup \rho^{(k-1)}_-, & \text{if } \alpha^-_{k-1} < \alpha < \alpha^-_{k+1}. \\ \rho(\alpha), & \text{otherwise.} \end{cases}$$

Let the set of all tableaux in $\mathcal{T}_{k^-}(R^+, R^-)$ labelled by $\rho^u_{k^-}(a)$ be the set denoted $\mathcal{T}^u_{k^-}(R^+, R^-)$, and let the set of all tableaux in $\mathcal{T}_{k^-}(R^+, R^-)$ labelled by $\rho^l_{k^-}(a)$ be the set denoted $\mathcal{T}^l_{k^-}(R^+, R^-)$.

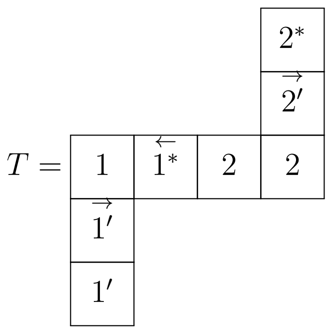
Example 5.38. Let $\pi = (2^2)$ and $\lambda = (3, 2, 2)$. Then the words $(R^+, R^-) = (1, 12)$ take π to λ and is Yamanouchi.We have the following sequences of partitions, corresponding to R^+:

$$\pi^\wedge = (2^2) \xrightarrow{1} (3, 2) = \lambda^\wedge,$$

and corresponding to R^-:

$$\pi^\vee = (2^2) \xrightarrow{1} (2^2, 1) \xrightarrow{2} (2^3) = \lambda^\vee.$$

The following is a tableau T of shape $\theta = (4^3, 1^2)/(3^2)$



which has words (R^+, R^-). We form 3 tableaux:

$$T_{1^+} = \begin{array}{cccc} & & & 2^* \\ & & & \overrightarrow{2'} \\ 1 & 1^* & 2 & 2 \\ \overrightarrow{1'} & & & \\ 1' & & & \end{array}$$

$$T_{1^-} = \begin{array}{cccc} & & & 2^* \\ & & & \overrightarrow{2'} \\ 1 & \overleftarrow{1^*} & 2 & 2 \\ 1' & & & \\ 1' & & & \end{array} \qquad T_{2^-} = \begin{array}{cccc} & & & 2^* \\ & & & 2' \\ 1 & \overleftarrow{1^*} & 2 & 2 \\ \overrightarrow{1'} & & & \\ 1' & & & \end{array}$$

which belong to $\mathcal{T}_{1^+}(R^+, R^-)$, $\mathcal{T}_{1^-}(R^+, R^-)$, and $\mathcal{T}_{2^-}(R^+, R^-)$ respectively. Each of these tableaux admit two different labellings, $\rho^u_{1^+}(\alpha)$ and $\rho^u_{1^-}(\alpha)$; the entries in boxes $(3,1)$, and $(3,2)$ are the same in both the tableaux $T^u_{1^+}$ and T_{1^+} but are labelled by $\lambda^\wedge \cup (2^2)$. On the other hand, the entries in boxes $(3,1)$ and $(3,2)$ of the tableaux $T^l_{1^+}$ and T_{1^+} are the same, but are labelled by $\pi^\wedge \cup (2^2)$.

□

For a partition ν, define the sequence $z(\nu) = x(\nu) + y(\nu)$. We will see the following relations used in the proofs below. If μ is another partition contained in ν, we have 3 cases:

1. ν/μ has one box in the k-th row, for a $k \leqslant d(\mu)$. Then $z(\nu)_r - z(\mu)_r = x(\nu)_k - x(\mu)_k$ when $r = k$, and 0 otherwise.

2. ν/μ has one box in the k-th column, for a $k \leqslant d(\mu)$. Then $z(\nu)_r - z(\mu)_r = y(\nu)_k - y(\mu)_k$ when $r = k$, and 0 otherwise.

3. ν/μ has one box in the $(d+1)$-th row and column, for $d = d(\mu)$. Then $z(\nu)_r - z(\mu)_r = x(\nu)_{d+1} + y(\nu)_{d+1}$ when $r = d+1$, and 0 otherwise.

Proposition 5.39. *Recall the sequences $(R^+, R^-) : \pi \to \lambda$. For a tableau $T \in \mathcal{T}_{k^\pm}(R^+, R^-)$ let $T^u_{k^\pm}$ and $T^l_{k^\pm}$ be the corresponding tableaux labelled by $\rho^u_{k^\pm}$ and $\rho^l_{k^\pm}$*

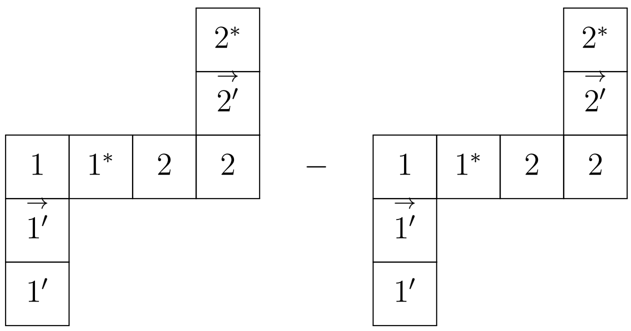
respectively. Then, the difference between the weights of $T^u_{k\pm}$ *and* $T^l_{k\pm}$ *is*

$$\mathrm{ev}(T^u_{k\pm}) - \mathrm{ev}(T^l_{k\pm}) = (z(\rho^{(k)}_{\pm}) - z(\rho^{(k-1)}_{\pm})) \sum_T \mathrm{ev}(T),$$

summed over tableaux $T \in \mathcal{T}(R^+, R^-)$ *labelled by* $\rho(\alpha)$ *with a* r_k *or* s_k *in a box strictly between* $\alpha^{\pm}_{k-1}$ *and* $\alpha^{\pm}_{k+1}$*, with respect to the unprimed and primed ordering respectively, barred by an appropriately directed arrow.*

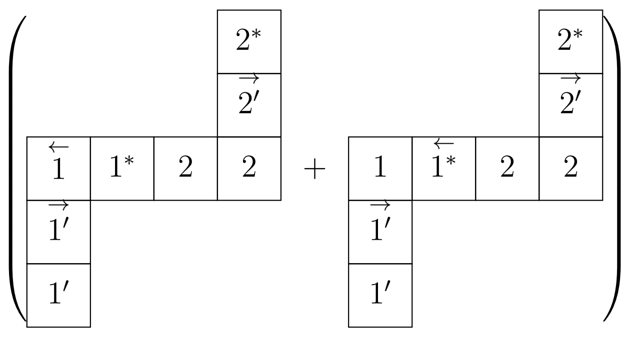
Proof. The difference in the labelling of $T^u_{k\pm}$ and $T^l_{k\pm}$ only lies in boxes strictly between $\alpha^{\pm}_{k-1}$ and $\alpha^{\pm}_{k+1}$ which take value r_k (in the case of T^u_{k+}) and s_k (in the case of T^u_{k-}). In $T^u_{k\pm}$ and $T^l_{k\pm}$ the only difference in labelling is between $\alpha^{\pm}_{k-1}$ and $\alpha^{\pm}_{k+1}$, since $T^u_{k\pm}$ is labelled by $\rho^u_{\pm}(\alpha)$ and $T^l_{k\pm}$ by $\rho^l_{\pm}(\alpha)$. The only weights of entries that differ under this labelling are those which take integer value equal to r_k and s_k; in fact, for T^u_{k-} and T^l_{k-} only the primed and starred entries differ, on the other hand for T^u_{k+} and T^l_{k+} only the unprimed and starred entries differ. The proposition follows from the definition of $\rho^u_{\pm}(\alpha)$ and $\rho^l_{\pm}(\alpha)$, and the difference formula 5.7. □

Example 5.40. Continuing with the previous example, consider the difference between $\mathrm{ev}(T^u_{1+}) - \mathrm{ev}(T^l_{1+})$, represented diagrammatically as follows:



where the first tableau is labelled by $\rho^u_{1+}(\alpha)$ and the second by $\rho^l_{1+}(\alpha)$. Thus, the weights only differ in the entries equal to unprimed or starred 1's:

$$\mathrm{ev}(T^u_{1+}) - \mathrm{ev}(T^l_{1+}) = (x(\lambda^\wedge)_1 - x(\pi^\wedge)_1)$$



We see that the two resultant tableaux obtained by taking the difference $\mathrm{ev}(T^u_{1^+}) - \mathrm{ev}(T^l_{1^+})$ contain the words (R^+, R^-). This also applies to the difference

$$\mathrm{ev}(T^u_{1^-}) - \mathrm{ev}(T^l_{1^-}) = (y(\rho^{(1)}_-)_1 - y(\pi^\vee)_1) \sum_T \mathrm{ev}(T),$$

summed over tableaux T with words (R^+, R^-), and also the difference

$$\mathrm{ev}(T^u_{2^-}) - \mathrm{ev}(T^l_{2^-}) = (y(\lambda^\vee)_2 - y(\rho^{(1)}_-)_2) \sum_T \mathrm{ev}(T),$$

summed over tableaux T with words (R^+, R^-). □

Proposition 5.41. *Let T be a tableau in $\mathcal{T}(R^+, R^-)$. Recall the sequence $z(\nu) = x(\nu) + y(\nu)$, for a partition ν. We have that*

$$\sum_{k=1}^{p}(z(\rho^{(k)}_+) - z(\rho^{(k-1)}_+)) + \sum_{k=1}^{q}(z(\rho^{(k)}_-) - z(\rho^{(k-1)}_-)) = k^\lambda_{(1)\lambda}(a) - k^\pi_{(1)\pi}(a).$$

Proof. First, we have the relation

$$k^\lambda_{(1)\lambda}(a) - k^\pi_{(1)\pi}(a) = \sum_{i=1}^{d(\lambda)} z(\lambda)_i - z(\pi)_i$$

Consider a $1 \leqslant t \leqslant d(\lambda)$. Suppose that m boxes are added to π in row t above the diagonal to form λ. Then, there are entries $r_{i_1}, \ldots, r_{i_m}$ of R^+ such that each of these entries are equal to t and represent entries t or t^* of T barred by left arrows. But we have that

$$\sum_{k=1}^{m}(x(\rho^{(i_k)}_+)_t - x(\rho^{(i_k-1)}_+)_t) = x(\lambda)_t - x(\pi)_t,$$

since $x(\rho^{(i_m)}_+)_t = x(\lambda)_t$ and $x(\rho^{(i_1-1)}_+)_t = x(\pi)_t$.

On the other hand if m' boxes are added to π in column t below the diagonal to form λ, there are entries $r_{j_1}, \ldots, r_{j_{m'}}$ of R^- such that each of these entries are equal to t and represent entries t' or t^* of T barred by right arrows. But we have that

$$\sum_{k=1}^{m'}(y(\rho^{(j_k)}_-)_t - y(\rho^{(j_k-1)}_-)_t) = y(\lambda)_t - y(\pi)_t,$$

since $y(\rho^{(j_m)}_-)_t = y(\lambda)_t$ and $y(\rho^{(j_1-1)}_-)_t = y(\pi)_t$.

Combining the two equations for boxes added to row t and column t of π gives:

$$x(\lambda)_i - x(\pi)_i + y(\lambda)_i - y(\pi)_i = z(\lambda)_i - z(\pi)_i,$$

and applying the argument to all rows and columns $1 \leqslant i \leqslant d(\lambda)$ completes the proof. □

Example 5.42. We continue with the previous example, where we had the partitions $\pi = (2^2)$ and $\lambda = (3, 2^2)$, with sequences $(1, 12) : \pi \to \lambda$. We see that

$$x(\lambda^\wedge)_1 - x(\pi^\wedge)_1 + y(\rho_-^{(1)})_1 - y(\pi^\vee)_1 + y(\lambda^\vee)_2 - y(\rho_-^{(1)})_2 = k^\lambda_{(1)\lambda}(a) - k^\pi_{(1)\pi}(a).$$

□

We are now ready to prove Theorem 5.34 by induction on the number of boxes in λ/π. The case where $|\lambda/\pi| = 0$ is known already, this is the coefficient $k^\lambda_{\theta\lambda}(a)$, and this forms the base case. We start with some preliminary definitions. For a $1 \leqslant k \leqslant p$, we define the two sums, $S^u_{k^+}(R^+, R^-)$ and $S^l_{k^+}(R^+, R^-)$ to be

$$S^u_{k^+}(R^+, R^-) = \sum_{T \in \mathcal{T}^u_{k^+}(R^+, R^-)} \mathrm{ev}(T)$$

and

$$S^l_{k^+}(R^+, R^-) = \sum_{T \in \mathcal{T}^l_{k^+}(R^+, R^-)} \mathrm{ev}(T)$$

respectively.

For a $1 \leqslant k \leqslant q$, we define the two sums, $S^u_{k^-}(R^+, R^-)$ and $S^l_{k^-}(R^+, R^-)$ to be

$$S^u_{k^-}(R^+, R^-) = \sum_{T \in \mathcal{T}^u_{k^-}(R^+, R^-)} \mathrm{ev}(T)$$

and

$$S^l_{k^-}(R^+, R^-) = \sum_{T \in \mathcal{T}^l_{k^-}(R^+, R^-)} \mathrm{ev}(T)$$

respectively.

As the induction hypotheses, we assume

$$\sum_{\pi^+} k^\lambda_{\theta\pi^+}(a) = \sum_{(R^+, R^-)} S^u_{1^\pm}(R^+, R^-) \tag{5.30}$$

and

$$\sum_{\lambda^-} k^{\lambda^-}_{\theta\pi}(a) = \sum_{(R^+, R^-)} \left(S^l_{p^+}(R^+, R^-) + S^l_{q^-}(R^+, R^-)\right), \tag{5.31}$$

where the sum on the left hand side of equation (5.30) is over partitions π^+ formed by adding a box to the end of a row or column of π, and the sum on the left hand side of equation (5.31) is over partitions λ^- formed by removing a box from the end of a row or column of λ. The right hand side of both equations is summed over all sequences (R^+, R^-) which take π to λ and are Yamanouchi. Note that if (R^+, R^-)

are sequences that take π to λ, then the sequences (R_1^+, R^-) will take π^+ to λ, where π^+ is formed from π by adding a box to row r_1 of π. The same argument applies for (R^+, R_1^-). Similarly, the sequences (R_p^+, R^-) and (R^+, R_q^-) take π to some partition λ^-, where λ^- is formed from λ by deleting a box from row r_p.

We will prove Theorem 5.34 by using Proposition 5.39 and Proposition 5.41. We are able to do this if we prove

$$k_{\theta\pi}^{\lambda}(a) = \frac{1}{k_{(1)\lambda}^{\lambda}(a) - k_{(1)\pi}^{\pi}(a)} \sum_{(R^+,R^-)} \left(\sum_{k=1}^{p} (S_{k^+}^u(R^+,R^-) - S_{k^+}^l(R^+,R^-)) \right. \\ \left. + \sum_{k=1}^{q} (S_{k^-}^u(R^+,R^-) - S_{k^-}^l(R^+,R^-)) \right). \tag{5.32}$$

To do this, all we need to show is that

$$\sum_{k=2}^{p} S_{k^+}^u(R^+,R^-) = \sum_{k=1}^{p-1} S_{k^+}^l(R^+,R^-) \tag{5.33}$$

and

$$\sum_{k=2}^{p} S_{k^-}^u(R^+,R^-) = \sum_{k=1}^{p-1} S_{k^-}^l(R^+,R^-) \tag{5.34}$$

and then we will have proved Theorem 5.34. Before proceeding with the general argument we provide an example to illustrate the idea:

Example 5.43. Recall from the previous example the partitions $\pi = (2^2)$ and $\lambda = (3, 2^2)$. From before, we had three tableaux:

$$T_{1^+} = \begin{array}{|c|c|c|c|} \cline{4-4} \multicolumn{3}{c|}{} & 2^* \\ \cline{4-4} \multicolumn{3}{c|}{} & \overrightarrow{2'} \\ \hline 1 & 1^* & 2 & 2 \\ \hline \overrightarrow{1'} & \multicolumn{3}{c}{} \\ \cline{1-1} 1' & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

$$T_{1^-} = \begin{array}{|c|c|c|c|} \cline{4-4} \multicolumn{3}{c|}{} & 2^* \\ \cline{4-4} \multicolumn{3}{c|}{} & \overrightarrow{2'} \\ \hline 1 & \overleftarrow{1^*} & 2 & 2 \\ \hline 1' & \multicolumn{3}{c}{} \\ \cline{1-1} 1' & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \qquad T_{2^-} = \begin{array}{|c|c|c|c|} \cline{4-4} \multicolumn{3}{c|}{} & 2^* \\ \cline{4-4} \multicolumn{3}{c|}{} & 2' \\ \hline 1 & \overleftarrow{1^*} & 2 & 2 \\ \hline \overrightarrow{1'} & \multicolumn{3}{c}{} \\ \cline{1-1} 1' & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

which admitted 3 pairs of labellings respectively: $\rho^u_{1+}(\alpha)$ and $\rho^l_{1+}(\alpha)$, $\rho^u_{1-}(\alpha)$ and $\rho^l_{1-}(\alpha)$, and $\rho^u_{2-}(\alpha)$ and $\rho^l_{2-}(\alpha)$. The weights of these tableaux with their respective labellings contribute to $S^u_{1+}(R^+, R^-)$ and $S^l_{1+}(R^+, R^-)$, $S^u_{1-}(R^+, R^-)$ and $S^l_{1-}(R^+, R^-)$, and $S^u_{2-}(R^+, R^-)$ and $S^l_{2-}(R^+, R^-)$ respectively.

We have that

$$\sum_{\rho} k^{\lambda}_{\theta\rho}(a) = S^l_{1+}(R^+, R^-) + S^l_{2-}(R^+, R^-),$$

summed over all partitions ρ formed from π by adding a box to π, and

$$\sum_{\rho} k^{\lambda}_{\theta\rho}(a) = S^u_{1+}(R^+, R^-) + S^u_{1-}(R^+, R^-),$$

again summed over all partitions ρ formed from π by adding a box to π.

Thus, we wish to demonstrate that

$$S^u_{2-}(R^+, R^-) = S^l_{1-}(R^+, R^-).$$

We show this by way of example.

Consider the following tableau T which belongs to $\mathcal{T}^u_{2-}(R^+, R^-)$:

$$T = \begin{array}{|c|c|c|c|}
\cline{4-4}
\multicolumn{3}{c|}{} & 2^* \\ \cline{4-4}
\multicolumn{3}{c|}{} & 2' \\ \hline
1 & \overleftarrow{1^*} & 2 & 2 \\ \hline
\overrightarrow{1'} & \multicolumn{3}{l}{} \\ \cline{1-1}
1' & \multicolumn{3}{l}{} \\ \cline{1-1}
\end{array}$$

where we label the primed 1's to the northeast of box $(4,1)$ with the partition λ and those to the southwest with the partition π (for these entries, this labelling is equivalent to the one defined on T^u_{2-}).

Now consider the following tableau $\widetilde{T}$:

$$\widetilde{T} = \begin{array}{|c|c|c|c|}
\cline{4-4}
\multicolumn{3}{c|}{} & 2' \\ \cline{4-4}
\multicolumn{3}{c|}{} & 2' \\ \hline
1 & \overleftarrow{1^*} & 2^* & 2 \\ \hline
2' & \multicolumn{3}{l}{} \\ \cline{1-1}
\overrightarrow{2^*} & \multicolumn{3}{l}{} \\ \cline{1-1}
\end{array}$$

which has the same weight as T. This tableaux is "dual" to one in $T^l_{1^-}(R^+, R-)$ as described in the proof of Lemma 5.23. □

Returning to the general argument, we will show that equation (5.33) is true by considering 3 different types of tableau which contribute to the sums on both sides of (5.33). Modifying the same arguments slightly will also show that equation (5.34) is true, so we will not prove equation (5.34) here. Consider a tableau T whose weight contributes to $S^l_{k^+}(R^+, R^-)$, for some $k = 1, \ldots, p-1$. There are 3 cases, since the skew diagram $\rho_+^{(k+1)}/\rho_+^{(k-1)}$ arising from the labelling on T obeys one of the following 3 mutually exclusive conditions:

Cond. 1: The skew diagram $\rho_+^{(k+1)}/\rho_+^{(k-1)}$ is not connected.

Cond. 2: The skew diagram $\rho_+^{(k+1)}/\rho_+^{(k-1)}$ is connected and has two boxes in the same row above the diagonal.

Cond. 3: The skew diagram $\rho_+^{(k+1)}/\rho_+^{(k-1)}$ is connected and has two boxes in the same column in consecutive rows above the diagonal.

If $\rho_+^{(k+1)}/\rho_+^{(k-1)}$ obeys Cond. 1 or 2, we construct a bijection between T and a tableau $\widetilde{T}$, whose weight contributes to $S^u_{(k+1)^+}(\widetilde{R}^+, \widetilde{R}^-)$, where the words $(\widetilde{R}^+, \widetilde{R}^-) : \pi \to \lambda$ and is Yamanouchi. The skew diagram $\widetilde{\rho}^{(k+1)}/\widetilde{\rho}^{(k-1)}$ arising from the labelling on $\widetilde{T}$ obeys two conditions which respectively correspond to Cond. 1 and Cond. 2 for T:

Cond. 1': The skew diagram $\widetilde{\rho}^{(k+1)}/\widetilde{\rho}^{(k-1)}$ has two boxes in separate rows and columns above the diagonal.

Cond. 2': The skew diagram $\widetilde{\rho}^{(k+1)}/\widetilde{\rho}^{(k-1)}$ has two boxes in the same row above the diagonal.

We construct the bijection separately for Cond. 1 and 2:

5.12 If $\rho_+^{(k+1)}/\rho_+^{(k-1)}$ obeys Cond. 1

We have that R_k^+ is the unprimed word of T. We have the following sequence of partitions appearing arising in the labelling $\rho^l_{k+}(\alpha)$ of T:

$$\ldots \rho_+^{(k-2)} \overset{r_{k-1}}{\to} \rho_+^{(k-1)} \overset{r_{k+1}}{\to} \rho_+^{(k+1)} \overset{r_{k+2}}{\to} \rho_+^{(k+2)} \ldots \qquad (5.35)$$

Strictly speaking, each of these partitions are of the form $\rho^{(i)} \cup \rho_-(\alpha)$, but for brevity we drop the second partition in this sequence now, and also in the future. Let $\widetilde{R}^+$ be the sequence equal to R^+, except we swap r_k and r_{k+1}, and set $\widetilde{R}^- = R^-$. Then

$\widetilde{R}^+ : \pi^\wedge \to \lambda^\wedge$ and is Yamanouchi, since $\rho^{(k+1)}/\rho^{(k-1)}$ obeys Cond. 1, which means that two boxes may be added to the end of row r_k and r_{k+1} of the partition $\rho_+^{(k-1)}$ in any order. The intermediate partitions $\widetilde{\rho}_+^{(j)}$, for all $1 \leqslant j \leqslant n$ and $j \neq k$ are in fact equal to $\rho_+^{(j)}$. Consider the tableau $\widetilde{T}$, which has the same entries as T and the same labelling as T. We claim that $\widetilde{T}$ belongs to $\mathcal{T}_{k+1}^u(\widetilde{R}^+, \widetilde{R}^-)$, and thus contributes to $S_{(k+1)^+}^u(R')$. This is true since the sequence

$$\dots \rho_+^{(k-2)} \overset{r_{k-1}}{\to} \rho_+^{(k-1)} \overset{\widetilde{r}_k}{\to} \rho_+^{(k+1)} \overset{r_{k+2}}{\to} \rho_+^{(k+2)} \dots \tag{5.36}$$

with $\widetilde{r}_k = r_{k+1}$ corresponds to the labelling $\rho_{(k+1)^+}^u(\alpha)$ of $\widetilde{T}$. It follows that the weights of T and $\widetilde{T}$ are equal.

5.13 If $\rho_+^{(k+1)}/\rho_+^{(k-1)}$ obeys Cond. 2

If $\rho_+^{(k+1)}/\rho_+^{(k-1)}$ obeys Cond. 2, then consecutive boxes are added in the same row above the diagonal to form $\rho_+^{(k+1)}$ from $\rho_+^{(k-1)}$. The following sequence of partitions appears in the labelling of T:

$$\dots \rho^{(k-2)} \overset{r_{k-1}}{\to} \rho^{(k-1)} \overset{r_{k+1}}{\to} \rho^{(k+1)} \overset{r_{k+2}}{\to} \rho^{(k+2)} \dots \tag{5.37}$$

with $r_{k+1} = r_k$. Let $\widetilde{R}^\pm = R^\pm$, and let $\widetilde{T} = T$. However, we consider $\widetilde{T}$ to be a tableau contributing to $S_{k+1}^u(\widetilde{R}^+, \widetilde{R}^-)$, which we may do since the following sequence

$$\dots \rho^{(k-2)} \overset{r_{k-1}}{\to} \rho^{(k-1)} \overset{r_k}{\to} \rho^{(k+1)} \overset{r_{k+2}}{\to} \rho^{(k+2)} \dots \tag{5.38}$$

is consistent with the labelling $\rho_{(k+1)^+}^u(\alpha)$.

5.14 If $\rho_+^{(k+1)}/\rho_+^{(k-1)}$ obeys Cond. 3

If $\rho_+^{(k+1)}/\rho_+^{(k-1)}$ obeys Cond. 3 then $r_{k+1} = r_k + 1$, that is we have to add consecutive boxes in the same columns in rows r_k and r_k+1 above the diagonal to form $\rho^{(k+1)}$ from $\rho^{(k-1)}$. Here we do not construct a bijection. Instead we argue that the followings weights are equal:

$$\sum_{T \in \mathcal{T}_{(k+1)^+}^u(R^+,R^-)} \mathrm{ev}(T) = \sum_{T \in \mathcal{T}_{k^+}^l(R^+,R^-)} \mathrm{ev}(T). \tag{5.39}$$

We will prove this statement using Lemma 5.21:

Fix a pair of boxes γ southwest of δ in the skew diagram θ. Recall that a tableau T in the set $\mathcal{T}_{k^+}^l(R^+, R^-)$ has an entry r_{k+1} which is barred in box α_{k+1}. Let $\mathcal{P}_{\gamma,\delta}$ denote the subset of $\mathcal{T}_{k^+}^l(R^+, R^-)$ which contains tableaux T obeying the following two conditions:

1. The first barred entry southwest of the entry $T(\alpha_{k+1})$ is in box γ.

2. The first barred entry northeast of the entry $T(\alpha_{k+1})$ is in box δ.

Now, recall that a tableau T in the set $\mathcal{T}^l_{(k+1)^+}(R^+, R^-)$ has an entry r_k which is barred in box α_k. Similarly, let $\widetilde{\mathcal{P}}_{\gamma,\delta}$ denote the subset of $\mathcal{T}^l_{(k+1)^+}(R^+, R^-)$ which contains tableaux T obeying the following two conditions:

1. The first barred entry southwest of the entry $T(\alpha_k)$ is in box γ.

2. The first barred entry northeast of the entry $T(\alpha_{k+1})$ is in box δ.

Proving that equation (5.39) is true reduces to showing that

$$\sum_{T \in \mathcal{P}_{\gamma,\delta}} \mathrm{ev}(T) = \sum_{T \in \widetilde{\mathcal{P}}_{\gamma,\delta}} \mathrm{ev}(T),$$

but this follows from Lemma 5.21.

5.15 Conclusion

As mentioned previously, the following are applications of Theorem 5.34. Firstly, we recover the 'traditional' Pieri rule when we set $\theta = (p)$ or $\theta = (1^p)$, for some integer $p \geqslant 0$, see Corollary 5.37. Now, consider skew diagrams θ and λ such that $\theta = \nu/\mu$ is a disjoint union of skew hooks. Then, from Corollary 5.36 the dual Littlewood–Richardson polynomial $\widehat{c}^{\,\nu}_{\lambda\mu}(a)$ is equal to

$$\widehat{c}^{\,\nu}_{\lambda\mu}(a) = k^{\lambda}_{\theta\varnothing}(-a'),$$

a special case of our rule for the coefficients $k^{\lambda}_{\theta\pi}(a)$.

We discuss some further possible work. Deriving a rule for the coefficients $k^{\lambda}_{\theta\pi}(a)$ for all skew diagrams θ will give a rule to calculate the coefficients $\widehat{c}^{\,\nu}_{\lambda\mu}(a)$ for any partition λ. We now discuss possible ways of doing this.

The first method of approach is to calculate examples of $k^{\lambda}_{\theta\pi}(a)$ for when θ is not a skew hook, using the recurrence relation (5.3), and using simple pairs of partitions π and λ. We explicitly did this calculation for $k^{\lambda}_{\theta\pi}(a)$ when $\theta = (2^2)$ and $\theta = (2^3)$. Unfortunately, in the latter case there does not seem to be a way of expressing the coefficient $k^{\lambda}_{2^3\pi}$ in terms of products of differences of the form $a_i - a_j$, as in the expression given by Theorem 5.34.

The second method of approach is inspired by Chapter 3, where we calculated the Littlewood–Richardson coefficients $c^{\nu}_{\lambda\mu}(a)$ using the Jacobi–Trudi identity. Due to

Ivanov [13, Appendix], the Jacobi–Trudi identity holds for the generalised Frobenius–Schur functions,

$$s_{\nu/\mu}(x^{(n)}/y^{(n)}\|a) = \det\left(s_{(\nu_i-\mu_j+j-i)}(x^{(n)}/y^{(n)}\|\tau^{\pi_j-j+1}a)\right)_{1\leqslant i,j\leqslant l(\nu)}. \tag{5.40}$$

This determinant may be expanded into an alternating sum of products involving the generalised Frobenius–Schur function $s_{(p)}(x^{(n)}/y^{(n)}\|\tau^j a)$ corresponding to a partition with one row with some shift τ^j applied. However, our Pieri rule (Theorem 5.34) can be adapted to calculate $s_{(p)}(x^{(n)}/y^{(n)}\|\tau^j a)s_\pi(x^{(n)}/y^{(n)}\|a)$, where π is a partition. This will give a formula for $k^\lambda_{\theta\pi}(a)$ which involves an alternating sum of polynomials in $\mathbb{Q}[a]$. It seems fitting to conclude with the words of Berenstein and Zevelinsky: the obstacle to a combinatorial interpretation of $\widehat{c}^{\,\nu}_{\lambda\mu}(a)$ is then the alternation arising from equation (5.40).

6 Conclusion

We conclude by summarising the three main results of this thesis, which describe how to calculate the Littlewood–Richardson polynomials $c^{\nu}_{\lambda\mu}(a)$, and the dual Littlewood–Richardson polynomials $\widehat{c}^{\nu}_{\lambda\mu}(a)$.

Result 1:

The first main result is a new proof of the following theorem, which calculates the coefficient $c^{\nu}_{\lambda\mu}(a)$.

Theorem 6.1. *Let λ, μ, and ν be partitions. If $\nu \not\subseteq \mu$, the coefficient $c^{\nu}_{\lambda\mu}(a) = 0$. If $\mu \subseteq \nu$, we have that*

$$c^{\nu}_{\lambda\mu}(a) = \sum_{T} \prod_{\substack{\alpha\in\lambda \\ T(\alpha)\ \textit{unbarred}}} (a_{T(\alpha)-\rho^r(\alpha)_{T(\alpha)}} - a_{T(\alpha)-c(\alpha)}),$$

where the sum is taken over reverse λ-tableaux T obeying the following. First, the column word S^c of T is Yamanouchi and $S^c : \mu \to \nu$. Secondly, the entries in T strictly decrease down each column; that is, T is column strict.

Result 2:

The second main result is a formula for the coefficients $\widehat{c}^{\nu}_{\lambda\mu}(a)$ which is stable; i.e. does not depend on the number of variables n.

Theorem 6.2. *Let λ, μ and ν be partitions. Then,*

$$\widehat{c}^{\nu}_{\lambda\mu}(a) = \sum_{T} \prod_{\substack{\alpha\in\nu/\mu, \\ T(\alpha)\ \textit{unprimed, unbarred}}} (a_{T(\alpha)-\rho(\alpha)_{T(\alpha)}} - a_{T(\alpha)-c(\alpha)}) \prod_{\substack{\alpha\in\nu/\mu, \\ T(\alpha)\ \textit{primed}}} (a_{T(\alpha)-c(\alpha)} - a_{T(\alpha)}),$$

summed over ν/μ-supertableaux T such that the tableau T contains the word $R : \varnothing \to \lambda$ which is Yamanouchi and obeys the following two conditions:

Cond. 1: *The maximum entry occuring in T, primed or not, takes value at most $l(\lambda)$.*

Cond. 2: *For each $i \geqslant 1$, the most southwest border entry of T taking value i is a barred unprimed entry.*

Result 3:

Let θ be a disjont union of skew hooks, and π and λ be partitions. The third main result provides a formula for the coefficients $k^{\lambda}_{\theta\pi}(a)$ occuring in the decomposition of the product of two generalised Frobenius–Schur functions:

$$s_\theta(x/y\|a)s_\pi(x/y\|a) = \sum_\lambda s_\lambda(x/y\|a).$$

If T is a $\mathbb{A}$-tableau of shape θ, recall the 3 sets, θ^u, θ^p and θ^s, are subsets consisting of boxes of θ such that:

θ^u: The subset θ^u is the set of boxes $\alpha \in \theta$ such that $T(\alpha)$ is unbarred, unstarred and unprimed.

θ^p: The subset θ^p is the set of boxes $\alpha \in \theta$ such that $T(\alpha)$ is unbarred, unstarred and primed.

θ^s: The subset θ^s is the set of boxes $\alpha \in \theta$ such that $T(\alpha)$ is unbarred and starred.

Theorem 6.3. *The coefficient $k^{\lambda}_{\theta\pi}(a)$ is given by the formula*

$$k^{\lambda}_{\theta\pi}(a) = \sum_T \prod_{\alpha\in\theta^u} (a_{\rho(\alpha)_{T(\alpha)}-T(\alpha)+1} - a_{c(\alpha)+1}) \prod_{\alpha\in\theta^p} (a'_{\rho'(\alpha)_{T(\alpha)}-T(\alpha)+1} + a_{c(\alpha)+1})$$
$$\prod_{\alpha\in\theta^*} (a_{\rho(\alpha)_{T(\alpha)}-T(\alpha)+1} + a'_{\rho'(\alpha)_{T(\alpha)}-T(\alpha)+1}),$$

summed over $\mathbb{A}$-tableaux T of shape θ, such that the word R of T is Yamanouchi and takes π to λ.

As a corollary of this result, when the skew diagram θ is equal to ν/μ, for a pair of partitions ν and μ, the dual Littlewood–Richardson polynomial $\widehat{c}^{\nu}_{\lambda\mu}(a)$ is given by

$$\widehat{c}^{\nu}_{\lambda\mu}(a) = k^{\lambda}_{\theta\pi}(-a'),$$

where a' is the infinite sequence of variables given by the rule $(a')_i = -a_{1-i}$ for all $i \in \mathbb{Z}$.

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