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Mr T Tran-Cong
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Dear Mr Tran-Cong,

I am pleased to advise you that you have been awarded the degree of Doctor of Philosophy. This follows consideration of the reports of the examiners of your thesis entitled "A Study on the Papkovich-Neuber Solution in Elasticity and its Applications".

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Yours sincerely,

Kenneth W Knight
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UNIVERSITY OF SYDNEY

DEPARTMENT OF AERONAUTICAL ENGINEERING

THESIS

presented for the degree of

DOCTOR OF PHILOSOPHY

A STUDY ON THE PAPKOVICH-NEUBER SOLUTION IN ELASTICITY AND ITS APPLICATIONS

by

Ton TRAN-CONG

(B.E. Honours, M.E.Sc.)

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CONTENTS

ABSTRACT

This thesis is aimed to be a detail study on the Papkovich-Neuber solution and is designed to bring forwards certain of its (novel) applications.

The solution is introduced in its chronological order of development in the first part of chapter 1. The remaining part of the chapter (and also part of chapter 9) examines, in detail, its applicability. Chapter 2 brings to light the underlying theorems of the applicability of the solution, and also certain interesting properties related to the solution. The studies in chapter 3 then go into the elimination of the scalar harmonic function from the solution. In that chapter, a "simple" proof is derived to replace the existing "difficult" proof, and also a counter example is given to clarify a (previously) unclear situation.

A few (novel) specific applications of the solution are presented in the following three chapters. Chapter 4 presents the transformation of the solution, in axi-symmetric deformations, to the forms used by Lur'e, Sadowsky, Boussinesq, Love, and Timpe. The equivalence between the Papkovich-Neuber solution and Airy stress function, for plane problems, is dealt with in Chapter 5. This equivalence is a stepping stone in extending plane problems to some three-dimensional problems; it also shows that the Papkovich-Neuber solution is complete in terms of two harmonic functions, for plane strain problems. Chapter 6 uses the Papkovich-Neuber solution to reveal some relationships between two-dimensional plane problems and axi-symmetric problems. This chapter also brings in the role of the Fourier and Hankel (mathematical) transformation in those relationships. The example at

the end of that chapter also serves to introduce the method of using complex harmonic functions in the Papkovich-Neuber solution.

The next two chapters of the thesis deal with some practical engineering problems. Chapter 7 is an extension of the plane-strain strip problems to the finite-thickness ones. This chapter carries further the concept of complex harmonic functions introduced in chapter 6. Improved assymptotic formulae for eigenvalues are also presented there. Chapter 8 is an application of the Papkovich-Neuber solution to the axi-symmetric problems. Improved algebraic results as well as a proposed "Generalised Least Square Method" are presented there. The latter is possible after overcoming some difficulties involved in certain kind of integrals of Bessel functions.

The last chapter 9 contains miscellaneous side issues that has been left outside the main theme of the thesis for the purpose of clarity. One of these issues involves the Galerkin's vector. An epilogue at the end of the chapter gives an overview of some (current) work being done by other authors that can be of relevance to the work presented in this thesis.

Except for the assistance provided by those people indicated in the acknowledgements, and except where expressly stated otherwise, all the work and ideas presented in this thesis is wholely due to the author.

CHAPTER 1

The Papkovich-Neuber solution to Lamé's equation in elasticity.

This chapter introduces to the readers certain fundamental ideas and equations of elasticity. Once the basis has been built, the Papkovich-Neuber solution is directly introduced, neglecting all other unrequired concepts and developments.

1.1 Introduction to elasticity and its fundamental equations.

1.1.1 - *Introduction to some fundamentals of elasticity.*

Elasticity is the branch of engineering that studies the very small deformations of, and the forces applied on, *elastic* materials, which possess the property of returning to their original forms once the applying forces are removed. (The word small is in the sense of small almost everywhere).

In all this thesis, each elastic body is assumed to be *homogeneous*, that is each small cut element of the material is indistinguishable from any other, and *isotropic*, that is the properties of each cut element do not depend on the orientation of the element. It is also assumed that there are enough constraints to prevent the body from moving as a rigid body, so that no displacements of the body are possible without a deformation of it.

Only small deformations such as commonly occur in engineering are considered. The small displacement $\underset{\sim}{V}$ of the point (x,y,z) is resolved into its cartesian components (along x,y,z) as u, v and w respectively.

The strains at a point (x,y,z) are the quantities which describe the deformations at that point. There are three direct strains (ε's), which describe the elongations of a cut element at that point, and six shear strains (γ's), which describe the angular deformations of the same cut element. For small deformations, strains are related to displacements by the Cauchy's equations, which are

$$\varepsilon_{xx} = \frac{\partial u}{\partial x},$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y},$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z}, \qquad (1.1.1)$$

$$\gamma_{xy} = \gamma_{yx} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},$$

$$\gamma_{xz} = \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x},$$

$$\gamma_{yz} = \gamma_{zy} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}.$$

The study of elasticity also requires the definitions of stresses, they are denoted by $\sigma_{x_ix_j}$ and $\tau_{x_ix_j}$ (σ for $i = j$, and τ for $i \neq j$), they are the x_j component of the force acting on a (small) surface normal to x_i of a cut element at the point (x,y,z). For economy reasons $\sigma_{x_ix_j}$ is sometimes written as σ_{x_i} as the notation $\sigma_{x_ix_j}$ is used only for $i = j$.

Consideration of the equilibrium of each (very small) element of the material leads to $\tau_{x_ix_j}$ being equal to $\tau_{x_jx_i}$ for each choice of i and j. Furthermore, the equilibrium conditions also give the following (Navier's) equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = -\rho F_x,$$

$$\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = -\rho F_y, \qquad (1.1.2)$$

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = -\rho F_z,$$

where ρ is the specific mass of the material at the point considered (ρ is a constant in a homogenous material), and F_x, F_y, F_z are the components of body force per unit mass at the point considered. It is noted that $\underset{\sim}{F} = (F_x, F_y, F_z)$ need not be the usual (conservative) gravity force, it can be any kind of body force or even a combination of them. Sometimes the body forces can be non-conservative such as electromagnetic forces in a varying electromagnetic field.

When displacements and strains at a point (x,y,z) are small, the stresses and strains are related by Hooke's Law, which gives

$$\varepsilon_{xx} = \frac{1}{E}\,[\sigma_{xx} - \nu\sigma_{yy} - \nu\sigma_{zz}] \tag{1.1.3a}$$

$$\varepsilon_{yy} = \frac{1}{E}\,[\sigma_{yy} - \nu\sigma_{zz} - \nu\sigma_{xx}] \tag{1.1.3b}$$

$$\varepsilon_{zz} = \frac{1}{E}\,[\sigma_{zz} - \nu\sigma_{xx} - \nu\sigma_{yy}] \tag{1.1.3c}$$

where E is called the Young's modulus of the material and ν, called the Poisson's ratio, expresses the lateral contraction due to each longitudinal elongation. For non-isotropic material, there are three different sets of E and ν for the three directions x, y and z. However this thesis only considers homogenous, isotropic materials, hence the Young's modulus and the Poisson's ratio, each is one single constant for any elastic body considered.

By a different orientation of direct stresses and elongations in an isotropic material, the following three relations can be derived from the last three,

$$\gamma_{xy} = \frac{2(1+\nu)}{E}\,\tau_{xy}, \tag{1.1.3d}$$

$$\gamma_{yz} = \frac{2(1+\nu)}{E}\,\tau_{yz}, \tag{1.1.3e}$$

$$\gamma_{zx} = \frac{2(1+\nu)}{E}\,\tau_{zx}. \tag{1.1.3f}$$

The six equations (1.1.3) will be called the equations relating stresses and strains.

1.1.2 - *Lamé's equation.*

Combining the above three sets of equations [(1.1.1), (1.1.2), and (1.1.3)], the Lamé's equation in elasticity is obtained, as

$$\nabla^2 \underset{\sim}{V} + \frac{1}{1-2\nu}\,\underset{\sim}{\nabla}(\underset{\sim}{\nabla}\cdot\underset{\sim}{V}) = -\,\frac{2(1+\nu)}{E}\,\rho\underset{\sim}{F}, \tag{1.1.4}$$

where $\underset{\sim}{\nabla}$ is the notation for the operator

$$\underset{\sim}{\nabla} = \underset{\sim}{i}\,\frac{\partial}{\partial x} + \underset{\sim}{j}\,\frac{\partial}{\partial y} + \underset{\sim}{k}\,\frac{\partial}{\partial z}.$$

Very often, the body force F is identically zero throughout the material region, in this case, the Lamé's equation reduces to its homogeneous form

$$\nabla^2 \underset{\sim}{V} + \frac{1}{1-2\nu}\,\underset{\sim}{\nabla}(\underset{\sim}{\nabla}\cdot\underset{\sim}{V}) = 0. \tag{1.1.4a}$$

The general solution to the above equations is named after its (independent) discoverers, Papkovich and Neuber. This solution and its applications will be the subject of study in this thesis.

1.1.3 - *A survey on alternative methods of solution to elasticity.*

Although the solution of Papkovich and Neuber to the equation (1.1.4) appears to be the most straightforward method to deal with problems in elasticity, this solution is only obtainable recently, due to the intricacy of Lamé's equation. The following is a (non-comprehensive!) survey that lists most of the methods which are independent of the Papkovich-Neuber solution. It should be realised that despite the neatness of the Papkovich-Neuber solution, the other methods (some of them are much older than the former) are in no way obsolete with the appearance of the former. Rather each method has its own role and position in elasticity.

Before the discovery of the general solution to (1.1.4), stresses and strains are often taken as the basic unknowns in elasticity. However, when using strains as basic unknowns, there is a problem of the integrability of displacements from strains. Displacements in a simply connected region are integrable from strains if and only if the following system of equations is satisfied.

$$\begin{aligned}
\frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} &= \frac{1}{2}\frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right), \\
\frac{\partial^2 \varepsilon_{yy}}{\partial z \partial x} &= \frac{1}{2}\frac{\partial}{\partial y}\left(-\frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x}\right), \\
\frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} &= \frac{1}{2}\frac{\partial}{\partial z}\left(-\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y}\right), \\
\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} &= \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2}, \\
\frac{\partial^2 \gamma_{yz}}{\partial y \partial z} &= \frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2}, \\
\frac{\partial^2 \gamma_{zx}}{\partial z \partial x} &= \frac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial z^2}.
\end{aligned} \tag{1.1.5}$$

The proof that (1.1.5)'s are the necessary and sufficient conditions for the integrability of displacements is due to Cesaro [47] and can be found in the present text book of Sokolnikoff [5], p. 28. The above six equations are called the compatibility equations, and were first obtained (in a different way) by Saint-Venant in 1860 (see [23], p. 237-238).

In this thesis, displacements are taken as the basic unknowns, hence there will be no question of the compatibility conditions being satisfied. The above equations (1.1.5) are mentioned only as a point of interest.

When the number of (required) independent coordinate variables is less than three (for example, in plane-strain or axi-symmetric problems), certain functions can be devised to give displacements or stresses, such examples are the use of Airy stress function (see [23], p. 32) or Love's function [18], p. 274. Complex variables can also be ingeneously used in plane strain (or equivalent plane stress) problems. This method was put forward by Kolosoff [45] and the book of Muskhelishvili [48] is a famous treatise on it.

The problems of twisted, infinite prisms also require only two coordinate variables. Prandtl [43] had used the membrane analogy for their solutions (see also [23], Chapter 10), his ideas were further pursued by Griffith and Taylor [44].

Numerical methods developed rapidly since the advance of digital computers. These methods either assist in solving for mathematical quantities upon which the physical quantities are based, or solve for the physical quantities directly. Such examples are the finite-difference method and the finite-element method (the latter was first named by Argyris [49]).

The finite-difference method replaces the defining, differential equation with equivalent difference equations. The results are numerical values of the functions at discrete points or nodes throughout the body. From those values, the physical quantities are derived.

The finite-element method depends upon access to a large digital computer. This method divides the elastic body into a number of standard shapes called elements. The elements are analysed approximately by elasticity theory and assembled to form the elastic body, using either the Principle of Virtual Work or the Principle of Complementary Virtual Work.

Of the theoretical methods, most (at least!) of them are somehow connected to the Papkovich-Neuber solution. The relationships between the Papkovich-Neuber solution and Airy's, Love's, Boussinesq's, Timpe's,Sadowsky's functions will be specially treated in Chapters 4 and 5 of this thesis.

Recently, Naghdi and Hsu [7] also put forward their own general solution to the Lamé's equation. This solution uses only three harmonic functions, but it involves more differential and integral operations than the Papkovich-Neuber solution. The Naghdi-Hsu solution has not as yet been commonly used in elasticity and whether this solution will be widely used is an open question.

1.1.4 *Some auxiliary equations.*

The following definitions and equations assist in the rapid conversions of the results presented in this thesis to and from those appearing in standard texts, and also between formulae in this thesis.

The shear modulus is defined as

$$G = \frac{E}{2(1+\nu)} \tag{1.1.6}$$

The relations between stresses and displacements can be alternatively written as

$$\frac{1+\nu}{E}\sigma_x = \frac{\partial u}{\partial x} + \frac{\nu}{1-2\nu}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right),$$

$$\frac{1+\nu}{E}\sigma_y = \frac{\partial v}{\partial y} + \frac{\nu}{1-2\nu}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right),$$

$$\frac{1+\nu}{E}\sigma_z = \frac{\partial w}{\partial z} + \frac{\nu}{1-2\nu}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right), \tag{1.1.7}$$

$$\tau_{xy} = G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \tag{1.1.7}$$

$$\tau_{yz} = G\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right),$$

$$\tau_{zx} = G\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right),$$

It should be stressed that all the equations (1.1.5), (1.1.6) and (1.1.7) are *non-essential* to the work in this thesis: (1.1.5) is irrelevant since there is no question of compatibility when displacements are taken as the basic unknowns; (1.1.6) is only a convenient definition; and finally (1.1.7) is just another form of (1.1.3).

1.2 Papkovich's method in deriving the general solution for Lamé's equation (1.1.4).

The Lamé's equation (1.1.4) is a synthesis of the three groups of equations (1.1.1), (1.1.2) and (1.1.3). It expresses the equilibrium of each element of the elastic body, and at the same time the characteristic deformation of the elastic body, given its Young's modulus, Poisson's ratio and the acting forces. From these considerations, the equation plays an enormous role in elasticity. However solutions for this equation have been successful only in very simple cases (see for example, [46] p. 316, and the reference [54] cited there).

Stress functions, such as those of Maxwell or Morera (see [22], p. 259) have been used as the general solution for (1.1.4)is unobtainable. On the other hand, Galerkin [50] extended Love's biharmonic-function idea and developed the strain functions named after him. (Westergaard [55] developed Galerkin's functions further, and at a latter date has them treated in his book [56]). These developments (except the works of Westergaard, they appeared after 1932) have nevertheless prepared a platform for Papkovich's work on Lamé's equation.

In 1932, Papkovich [1] presented his solution to the Lamé's equation (1.1.4). The following is a summary of his paper.

Assuming that the displacement vector field $\underset{\sim}{V}$ in (1.1.4) is the body-force field acting on another elastic body of the same shape, having the Young's modulus E, Poisson ratio σ and constant specific weight μ, then the Lamé equation of the new elastic body gives

$$\underset{\sim}{V} = \nabla^2\underset{\sim}{W} + \frac{1}{1-2\sigma}\ \underset{\sim}{\nabla}(\underset{\sim}{\nabla}\cdot\underset{\sim}{W}) \tag{1.2.1}$$

where $-\frac{2(1+\sigma)}{E}\mu\underset{\sim}{W}$ represents the displacement vector at each point of the new, elastic body.

Replacing $\underset{\sim}{V}$ from (1.2.1) into equation (1.1.4), one has

$$\nabla^2\underset{\sim}{W} + \frac{1}{1-2\nu}\left[1 + \frac{2(1-\nu)}{1-2\sigma}\right]\nabla^2\underset{\sim}{\nabla}(\underset{\sim}{\nabla}\cdot\underset{\sim}{W}) = -\frac{2(1+\nu)}{E}\rho\underset{\sim}{F}\ .$$

By equating σ to $\left(\frac{3}{2}-\nu\right)$, the above equation becomes quite simple. Hence

$$\underset{\sim}{V} = \nabla^2 W - \frac{1}{2(1-\nu)}\ \underset{\sim}{\nabla}(\underset{\sim}{\nabla}\cdot\underset{\sim}{W}), \tag{1.2.2}$$

where

$$\nabla^2\nabla^2\underset{\sim}{W} = -\frac{2(1+\nu)}{E}\ \rho\underset{\sim}{F}. \tag{1.2.3}$$

As the special solution $\underset{\sim}{W_0}$ of the equation (1.2.3) is quite simple to construct, one is only interested in its homogeneous solution.

$\underset{\sim}{W}$ of (1.2.3) can be expressed as

$$\underset{\sim}{W} = \underset{\sim}{W_0} + \underset{\sim}{A_1} + \underset{\sim}{A_2}, \tag{1.2.4}$$

where $\underset{\sim}{A_1}$ is an arbitary *special* solution of

$$\nabla^2\underset{\sim}{A_1} = \underset{\sim}{B}$$

with $\underset{\sim}{B}$, and $\underset{\sim}{A_2}$ being the *general* expression for harmonic vector field, satisfying

$$\nabla^2\underset{\sim}{B} = \underset{\sim}{0},$$

$$\nabla^2\underset{\sim}{A_2} = \underset{\sim}{0}.$$

Expressions (1.2.4) and (1.2.2) then give

$$\underset{\sim}{V} = \underset{\sim}{V_0} + \underset{\sim}{B} - \frac{1}{2(1-\nu)}\ \underset{\sim}{\nabla}[\underset{\sim}{\nabla}\cdot(A_1 + A_2)], \tag{1.2.5}$$

where $\underset{\sim}{V_0}$ is a *special* solution of (1.1.4) given by

$$\underset{\sim}{V}_0 = \nabla^2 \underset{\sim}{W}_0 - \frac{1}{2(1-\nu)} \underset{\sim}{\nabla}(\underset{\sim}{\nabla} \cdot \underset{\sim}{W}_0). \tag{1.2.6}$$

As $\nabla^2(\underset{\sim}{\nabla} \cdot \underset{\sim}{A}_1)$ is only a *special* solution of $\underset{\sim}{\nabla} \cdot \underset{\sim}{B}$, $(\underset{\sim}{\nabla} . \underset{\sim}{A}_1)$ can be chosen to be

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{A}_1 = \frac{1}{2} \underset{\sim}{r} \cdot \underset{\sim}{B},$$

where $\underset{\sim}{r} = (x,y,z)$ is a vector pointing from the origin to the field point considered.

Substitution of this equation into (1.2.5) gives

$$\underset{\sim}{V} = \underset{\sim}{V}_0 + \underset{\sim}{B} - \frac{1}{4(1-\nu)} \underset{\sim}{\nabla}(\underset{\sim}{r} \cdot \underset{\sim}{B} + 2\underset{\sim}{\nabla} \cdot \underset{\sim}{A}_2).$$

As $2(\underset{\sim}{\nabla} \cdot \underset{\sim}{A}_2)$ is a harmonic function, it is written as B_0, and the expression for $\underset{\sim}{V}$ becomes

$$\underset{\sim}{V} = \underset{\sim}{V}_0 + \underset{\sim}{B} - \frac{1}{4(1-\nu)} (\underset{\sim}{r} \cdot \underset{\sim}{B} + B_0), \tag{1.2.7}$$

with

$$\nabla^2 \underset{\sim}{B} = \underset{\sim}{0}, \ \nabla^2 B_0 = 0. \tag{1.2.8}$$

As pointed out by Papkovich, his solution given by [(1.2.7) and (1.2.8)], is the generalised form of solutions obtained by Galerkin [50], Hertz [51], Boussinesq [15], and of the generalised solution of Hertz and Boussinesq, given by Savin [53].

For some mysterious reason, Papkovich mixed up the meanings of *general* and *special* (in the ideas of general solution and special solution) and wrongly concluded that $\underset{\sim}{\nabla} \cdot \underset{\sim}{A}_2$ can be set to zero, resulting in B_0 being set to zero in (1.2.7). The case for this (wrongly concluded) omission will be considered in Chapter $\underset{\sim}{3}$ of this thesis.

It should be noted that if σ is set to $\left(\frac{3}{2} - \nu\right)$, the new elastic body with the Poisson's ratio σ may get a negative potential energy for an increase in external static pressure. This implies that as the external pressure increases, the new elastic body expands its volume! This point is a weakness of Papkovich's proof and it certainly needs a remedy.

1.3 Derivation of Papkovich's solution by Mindlin's method.

Realizing the weakness of Papkovich's proof, Mindlin [2] offered another proof to Papkovich's solution. This proof replaces the fictitious material with Poisson ratio σ by the use of the Helmholtz transformation. The new proof is much more rigorous and it is reproduced in the following.

Helmholtz (transformation) theorem states that for any vector function $\underset{\sim}{V}$ in a region D, the vector can be written as

$$\underset{\sim}{V} = \underset{\sim}{\nabla}\phi + \underset{\sim}{\nabla} \times \underset{\sim}{S}$$

where ϕ is a scalar potential function and $\underset{\sim}{S}$ is solenoidal $(\underset{\sim}{\nabla}\cdot\underset{\sim}{S} = 0)$. The region D must satisfy certain conditions, but these are not of our concern at this stage.

Since $\underset{\sim}{S}$ is solenoidal, $\underset{\sim}{S}$ can be written as

$$\underset{\sim}{S} = - \underset{\sim}{\nabla} \times \underset{\sim}{W}.$$

The function ϕ is independent of $\underset{\sim}{S}$ and can, therefore, be written as

$$\phi = \frac{1}{\beta} \underset{\sim}{\nabla}\cdot\underset{\sim}{W}.$$

with β being a constant.

With $\underset{\sim}{V}$ as a function of $\underset{\sim}{W}$, expressible by

$$\underset{\sim}{V} = \frac{1}{\beta} \underset{\sim}{\nabla}(\underset{\sim}{\nabla}\cdot\underset{\sim}{W}) - \underset{\sim}{\nabla} \times \underset{\sim}{\nabla} \times \underset{\sim}{W},$$

the function $\underset{\sim}{V}$ can now be written in the following form

$$\underset{\sim}{V} = \nabla^2\underset{\sim}{W} - \left(1 - \frac{1}{\beta}\right)\underset{\sim}{\nabla}(\underset{\sim}{\nabla}\cdot\underset{\sim}{W}). \tag{1.3.1}$$

where the identity $\nabla^2 = \underset{\sim}{\nabla}(\underset{\sim}{\nabla}\cdot) - \underset{\sim}{\nabla}\times(\underset{\sim}{\nabla}\times)$ has been used. It is noted here that Papkovich had to resort to an imaginary elastic body to arrive at equation (1.2.1) which is similar to the equation (1.3.1) derived by Helmholtz theorem.

By setting the constant β

$$\beta = \frac{2(1-\nu)}{1-2\nu},$$

the Lamé's equation (1.1.4) for the elastic body becomes a quite simple equation

$$\nabla^2\nabla^2 \underset{\sim}{W} = -\frac{2(1+\nu)}{E}\rho F. \tag{1.3.2}$$

With the above value of β, $\underset{\sim}{V}$ takes the form

$$\underset{\sim}{V} = \nabla^2 \underset{\sim}{W} - \frac{1}{2(1-\nu)}\underset{\sim}{\nabla}(\underset{\sim}{\nabla}\cdot\underset{\sim}{W}). \tag{1.3.3}$$

Expressions (1.3.2) and (1.3.3) are identical to (1.2.3) and (1.2.2) in Papkovich's proof. The rest of this proof can then follow the same argument as that of Papkovich.

As the weakness of Papkovich's proof is removed, some restrictions are instead placed on the shape of the material region D under study. One of the restrictions is that D must be *non-periphractic*. (The definition of this word will be found in section 2.2), this is because a solenoidal vector S is expressible as the curl of another vector W only in a non-periphractic region (see the theorem by Stevenson, in section 2.2). It will be proved later (section 2.3) that the proof is valid whenever the Helmholtz transformation is applicable. One sufficient condition for the applicability of Helmholtz transformation is that the region is finite, *simply connected* (The definition of this word will be found in subsection *1.5.1*) and has a smooth boundary (or boundaries).

Hence the expression (1.2.7), in conjunction with (1.2.8), is indeed the general solution to Lamé's equation in a material region D, where the Helmholtz theorem applies.

The conditions for the applicability of Helmholtz theorem will become clear to readers at the end of section 2.3.

1.4 Neuber's solution for the homogeneous Lamé's equation.

Independent of Papkovich's work, in 1934, Neuber [3,4] suggested that the solution to Lamé's equation can be worked out in the following way.

Neuber stated that the displacement vector $\underset{\sim}{V}$ can be written as

$$\frac{E}{1+\nu}\underset{\sim}{V} = -\underset{\sim}{\nabla}\phi + 2\beta\underset{\sim}{B}, \tag{1.4.1}$$

where β is a constant, ϕ a scalar function, and $\underset{\sim}{B}$ a harmonic vector function, satisfying

$$\nabla^2 \underset{\sim}{B} = \underset{\sim}{0}. \qquad (1.4.2)$$

Replacing (1.4.1) into the homogenous Lame's equation (1.1.4a), one has

$$-\nabla^2 \underset{\sim}{\nabla} \phi + \frac{E}{1+\nu} \frac{1}{1-2\nu} \underset{\sim}{\nabla}(\underset{\sim}{\nabla} \cdot \underset{\sim}{V}) = \underset{\sim}{0}. \qquad (1.4.3)$$

This gives

$$\underset{\sim}{\nabla} \left[\frac{E}{1+\nu} \frac{1}{1-2\nu} \underset{\sim}{\nabla} \cdot \underset{\sim}{V} - \nabla^2 \phi \right] = 0,$$

or

$$\frac{E}{1+\nu} \frac{1}{1-2\nu} \underset{\sim}{\nabla} \cdot \underset{\sim}{V} - \nabla^2 \phi = \text{constant}. \qquad (1.4.4)$$

Since the constant which appears on the right hand side is non-essential, it can be set to zero, and then

$$\frac{E}{1+\nu} \left(\underset{\sim}{\nabla} \cdot \underset{\sim}{V} \right) = (1-2\nu) \nabla^2 \phi \qquad (1.4.4a)$$

On the other hand, the divergence of equation (1.4.1) gives

$$\frac{E}{1+\nu} \underset{\sim}{\nabla} \cdot \underset{\sim}{V} = - \nabla^2 \phi + 2\beta \underset{\sim}{\nabla} \cdot \underset{\sim}{B}. \qquad (1.4.5)$$

Equations (1.4.4a) and (1.4.5) then give

$$2(1-\nu) \nabla^2 \phi = 2\beta \underset{\sim}{\nabla} \cdot \underset{\sim}{B}.$$

Thus the desired relationship between the stress function and the three harmonic functions has been found. *Whereas the corresponding equation could only be solved in integral form in its former proposition, it can here be satisfied in a simple way.*

By setting

$$\phi = \underset{\sim}{r} \cdot \underset{\sim}{B} + \phi_0,$$

and

$$\beta = 2(1-\nu),$$

the equation for displacement is satisfied.

Hence the solution $\underset{\sim}{V}$ to the homogeneous Lamé's equation (1.1.4a) is expressible in the following form

$$\frac{E}{1+\nu}\underset{\sim}{V} = 4(1-\nu)\underset{\sim}{B} - \underset{\sim}{\nabla}(\underset{\sim}{r}\cdot\underset{\sim}{B} + \phi_0), \tag{1.4.6}$$

where $\nabla^2\underset{\sim}{B} = \underset{\sim}{0}, \qquad \nabla^2\phi_0 = 0.$ (1.4.7)

Although the expression of $\underset{\sim}{V}$ [as in (1.4.1) and (1.4.2)] does not appear as restricted as the conditions for the Helmholtz transformation, it does require certain conditions to be satisfied. These conditions will be the subject of study in the next section of this thesis.

After the expression for $\underset{\sim}{V}$ has been derived, Neuber gave an inconclusive proof for the omission of ϕ_0. This omission will be further discussed in Chapter $\underset{\sim}{3}$ of this thesis.

1.5 The applicability of Neuber's method of solution.

This section presents some discussions and clarifications on certain points which have not been convincingly dealt with by Neuber.

The first obscure point in Neuber's proof is the setting to zero of the constant on the right hand side of (1.4.4). Neuber stated that the constant is non-essential but gave no reason. It is found that if the constant is retained, and the proof carried out to its end, the constant only represents uniform dilation of the elastic body. The constant can thus be easily incorporated into the harmonic vector $\underset{\sim}{B}$. Hence the constant on the right hand side of (1.4.4) is indeed non-essential.

The second, but major, point in Neuber's proof is the expression of a given displacement vector in the form given by (1.4.1) and (1.4.2). If it can not be proved that an arbitrary solution $\underset{\sim}{V}$ of the homogeneous Lamé's equation (1.1.4a) is expressible in this form, then the solution [(1.4.6) and (1.4.7)] is *not* the *general* solution to the homogeneous Lamé's equation.

The following subsections *1.5.1* and *1.5.2* present the proposed proof for the expression of $\underset{\sim}{V}$ in the form [(1.4.1) and (1.4.2)].

1.5.1 - *Proof on the expression of a vector function $\underset{\sim}{F}$ in the form of (1.5.1) and (1.5.2).*

First, a necessary and sufficient condition is stated.

A necessary and sufficient condition for the expression of a vector function $\underset{\sim}{F}$ in the following form

$$\underset{\sim}{F} = \underset{\sim}{\nabla}\phi + \underset{\sim}{B} \tag{1.5.1}$$

where ϕ is a scalar function and $\underset{\sim}{B}$ is a harmonic vector function, i.e.

$$\nabla^2\underset{\sim}{B} = \underset{\sim}{0}, \tag{1.5.2}$$

is that there is a function ψ such that

$$\underset{\sim}{\nabla}\, \nabla^2\psi = \nabla^2\underset{\sim}{F}, \tag{1.5.3}$$

for the given vector function $\underset{\sim}{F}$.

The proof for this necessary and sufficient condition is omitted due to its simplicity.

Next, another condition is considered.

A necessary and sufficient condition for the expression of the ∇^2 of an arbitrary vector function $\underset{\sim}{F}$, in a finite, *simply connected* region D, in the form

$$\nabla^2\underset{\sim}{F} = \underset{\sim}{\nabla}\, \nabla^2\psi, \tag{1.5.4}$$

is that the given function $\underset{\sim}{F}$ must satisfy

$$\underset{\sim}{\nabla} \times (\nabla^2\underset{\sim}{F}) = \underset{\sim}{0}. \tag{1.5.5}$$

(A region D is simply connected if every closed circuit in it can be reduced in size to a single point, without crossing the boundary of the material region).

<u>Proof</u>

If $\nabla^2\underset{\sim}{F}$ is equal to the gradient of $\nabla^2\psi$, then obviously (1.5.5) is necessary.

On the other hand, since the material region is simply connected, a function f can be constructed such that

$$\underset{\sim}{\nabla} f = \nabla^2\underset{\sim}{F}$$

as $\underset{\sim}{\nabla} \times (\nabla^2\underset{\sim}{F})$ is zero. The function $\psi(x,y,z)$ defined by

$$\psi(x,y,z) = -\frac{1}{4\pi}\int_D \frac{f(\xi,\eta,\xi)}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\xi)^2}}\,d\tau$$

certainly satisfies

$$\nabla^2\psi = f,$$

and thus

$$\underset{\sim}{\nabla}\,\nabla^2\psi = \nabla^2\underset{\sim}{F}.$$

The above two necessary and sufficient conditions lead to the following, necessary and sufficient condition.

In a finite, simply connected region, every vector function $\underset{\sim}{F}$ is expressible in the following form

$$\underset{\sim}{F} = \underset{\sim}{\nabla}\phi + \underset{\sim}{B} \tag{1.5.1}$$

where

$$\nabla^2\underset{\sim}{B} = 0, \tag{1.5.2}$$

if and only if

$$\underset{\sim}{\nabla}\mathrm{x}(\nabla^2\underset{\sim}{F}) = \underset{\sim}{0}. \tag{1.5.4}$$

We are now in a position to clarify Neuber's proof. This is done as in the following.

1.5.2 - *Proof on the expression of a solution V to the homogeneous Lamé's equation, in the form [(1.4.1) and (1.4.2)].*

The homogeneous Lamé's equation (1.1.4a) is rewritten as

$$\nabla^2\underset{\sim}{V} - \frac{1}{1-2\nu}\,\underset{\sim}{\nabla}(\underset{\sim}{\nabla}\cdot\underset{\sim}{V}) = 0,$$

where $\underset{\sim}{V}$ is the displacement vector in the elastic body. Taking curl of this equation, one has

$$\underset{\sim}{\nabla}\mathrm{x}(\nabla^2\underset{\sim}{V}) = 0.$$

Using the last necessary and sufficient condition of the preceeding subsection, $\underset{\sim}{V}$ is expressible in the following form

$$\underset{\sim}{V} = \underset{\sim}{\nabla}\phi + \underset{\sim}{B},$$

where

$$\nabla^2\underset{\sim}{B} = \underset{\sim}{0}$$

if the material region D is finite and simply connected.

A *special* solution to the non-homogeneous Lamé's equation (1.1.4) must be found if the *general* solution to the non-homogeneous Lamé's equation is to be found. The vector function $\underset{\sim}{V}_0$ given by the equation (1.2.6) in Papkovich's proof can serve this purpose.

It should be noted that a solution $\underset{\sim}{V}$ to the non-homogeneous Lamé's equation (1.1.4) is *not* expressible in the form [(1.4.1) and (1.4.2)] if the body force $\underset{\sim}{F}$ on the right hand side of (1.1.4) is non-conservative.

1.5.3 - *Generality of Neuber's solution.*

The preceeding subsection has shown that

In a finite, simply connected material region D, the *general* solution $\underset{\sim}{V}$ to the homogeneous Lamé's equation (1.1.4a) is given by

$$\underset{\sim}{V} = \underset{\sim}{B} - \frac{1}{4(1-\nu)} \underset{\sim}{\nabla}(\underset{\sim}{r}\cdot\underset{\sim}{B} + B_0), \tag{1.5.5}$$

where $\nabla^2\underset{\sim}{B} = \underset{\sim}{0}, \qquad \nabla^2 B_0 = 0,$ (1.5.6)

Equations (1.5.5) and (1.5.6) were obtained from (1.4.6) and (1.4.7) with only a slight change in notation.

1.6 - Comparison of the three approaches to the solution of Lame's equation.

This section will compare the three methods of solving the Lamé's equation (1.1.4) as put forward by Papkovich, Mindlin (he offered a modification of Papkovich's proof), and Neuber, respectively. The three will be compared on their rigorousness, achievements, foundations and finally on their applicability. But the comparisons will be brief and not in depth, as not to disproportionate the contents of the thesis.

The method of Papkovich (section 1.2) is obviously non-rigorous as it requires the exitence of some exotic elastic material with its Poisson's ratio σ of more than $\frac{1}{2}$ $(\sigma > \frac{1}{2})$. Furthermore, even if such a material exists, its equilibrium state for the required loading may be non-existent. However, a hint in Papkovich's proof has led

Mindlin in successfully developing a rigorous proof, using Helmholtz' theorem. Despite its weakness, the proof due to Papkovich has offered a special solution $\mathbf{V}_0$ [equation (1.2.6)] to the non-homogeneous Lame's equation. This solution $\mathbf{V}_0$ is valuable as it permits the construction of one special solution to (1.1.4)) for any given body force field $\mathbf{F}$ [on the right hand side of (1.1.4)]. (The property of *special* solutions is that it does not matter how the solutions were obtained, it is sufficient as long as we have them).

Mindlin's proof to the Papkovich solution is rigorous and elegant. It will be shown in Chapter **2** that the proof can be easily extended to cover periphractic region where the Helmholtz transformation applies. For Mindlin's proof to stand, it is *sufficient* that there exist a solution to the external Neumann's problem on the outer boundary of the material region, and an individual solution to the internal Neumann's problem on each of the inner boundaries, once the boundary values are correctly set. These conditions require that, at least, each of the boundary surface of the material region must be smooth enough for the corresponding Neumann's problem to be solvable (see section **2.2** for more details). In brief, if Stevenson's theorem (in section **2.2**) is used to validate the required Helmholtz transformation, then Mindlin's proof to Papkovich's solution is applicable to finite, simply connected, periphractic (or non-periphractic) regions with smooth boundaries. The foundation for Mindlin's proof evidently involves very extensive mathematical arguments.

Let us turn now to Neuber's method of solution. His method certainly can not given a special solution to the non-homogeneous Lamé's equation. But it can easily borrow the special solution $\mathbf{V}_0$ [in equation (1.2.6)] from Papkovich's proof and hence the method can give the general solution to the non-homogeneous Lamé's equation (1.1.4). On the other hand, the advantage of Neuber's method is overwhelming. It is much more "fundamental" than Mindlin's proof, is quite presentable in "simple" theory of elasticity, its applicability is quite clear (on any finite, simply connected material region), and finally it is better than the non-rigorous proof of Papkovich.

Neuber's and Mindlin's proofs are applicable to different classes of material shapes. The former is applicable to finite, simply connected shapes while the latter to "Helmholtz-theorem-applicable" shapes.

If the Helmholtz theorem can only be proved by the use of Stevenson's theorem then Mindlin's proof is less general than Neuber's, since it is applicable only to finite, simply connected (periphractic or non-periphractic) material regions with smooth boundaries (or boundary). On the other hand, if the Helmholtz theorem is applicable to more than the above class of regions then Mindlin's proof also has wider application. But a comparison between Mindlin's and Neuber's proofs in the latter case is impossible as there is still uncertainty on the applicability of Helmholtz' theorem.

On infinite size elastic bodies, one may be led into believing that Mindlin's proof is more applicable than Neuber's proof. It can only be said that such a belief is untenable, since Mindlin's proof places as much restriction on the decay behaviour of $\underset{\sim}{V}$ at infinity, as Neuber's proof. The reasoning for the last statement is very lengthy and is not included herein.

1.7 - Conclusions.

Elasticity is always a complex and diverse subject. A study of the general solution to Lamé's equation would appear to provide a unification of the different solution methods in this field. The (independent) discovery of the general solution is quite an achievement credited to Papkovich and Neuber. (The works of Mindlin on Papkovich's solution should also be mentioned as it gives the required rigour to the latter).

The Papkovich-Neuber general solution to Lame's equation can be written down as

$$\underset{\sim}{V} = \underset{\sim}{B} - \frac{1}{4(1-\nu)} \nabla(\underset{\sim}{r} \cdot \underset{\sim}{B} + B_0), \tag{1.7.1}$$

where $\nabla^2 \underset{\sim}{B} = \underset{\sim}{0}, \qquad \nabla^2 B_0 = 0.$ (1.7.2)

The (finite) material region must either be simply connected or be such that the Helmholtz transformation is applicable. The displacement vector $\underset{\sim}{V}$ must of course be continuous and differentiable for a certain number of times, for the general solution to be valid. For infinite-sized elastic bodies certain decay characteristics of $\underset{\sim}{V}$ are also required.

Note

The above conclusion only states the *sufficient* conditions for (1.7.1) and (1.7.2) to be the general solution to (1.1.4).

A region which is not simply connected can also have (1.7.1) and (1.7.2) as its general solution if it is z-convex with respect to one of its constant-z cross-sections; for a proof, see [56], p. 123.

CHAPTER 2

Some fundamental theorems relating to the Papkovich-Neuber solution.

2.1 Introduction

Except for the very simple case of a parallelepiped, the applicability of the Helmholtz transformation, such as the one used in section 1.3, is not simple to prove. Thus, the proof to the Papkovich-Neuber solution due to Mindlin (section 1.3) needs some precise descriptions of the topology of the material region. The theorem cited in the next section 2.2, of this chapter, is the one that can clarify this point. Using this theorem, it is even possible to extend Mindlin's proof to cover simply connected, periphractic material regions.

An interesting observation in elasticity is the deep involvement of biharmonic functions in solutions to problems of this discipline. The general form of biharmonic functions will be proved in an alternative way to the proof due to Fosdick [11]. The proposed proof is based solely on a theorem due to Stevenson [13], and is not explicitly dependent on some "Hölder continuous partial derivatives" of the biharmonic functions. In a remarkably similar way to the reduction of the Papkovich-Neuber solution (Chapter 3), the biharmonic functions do have their own reductions, these will be brought to light in section 2.6. Finally, still another method to derive the Papkovich-Neuber solution will be proposed. The method made use of the expression of a harmonic function in terms of divergence of another harmonic vector function. The results of this method lead to some observation on the role of B_0 in (1.7.1), and also pose certain questions.

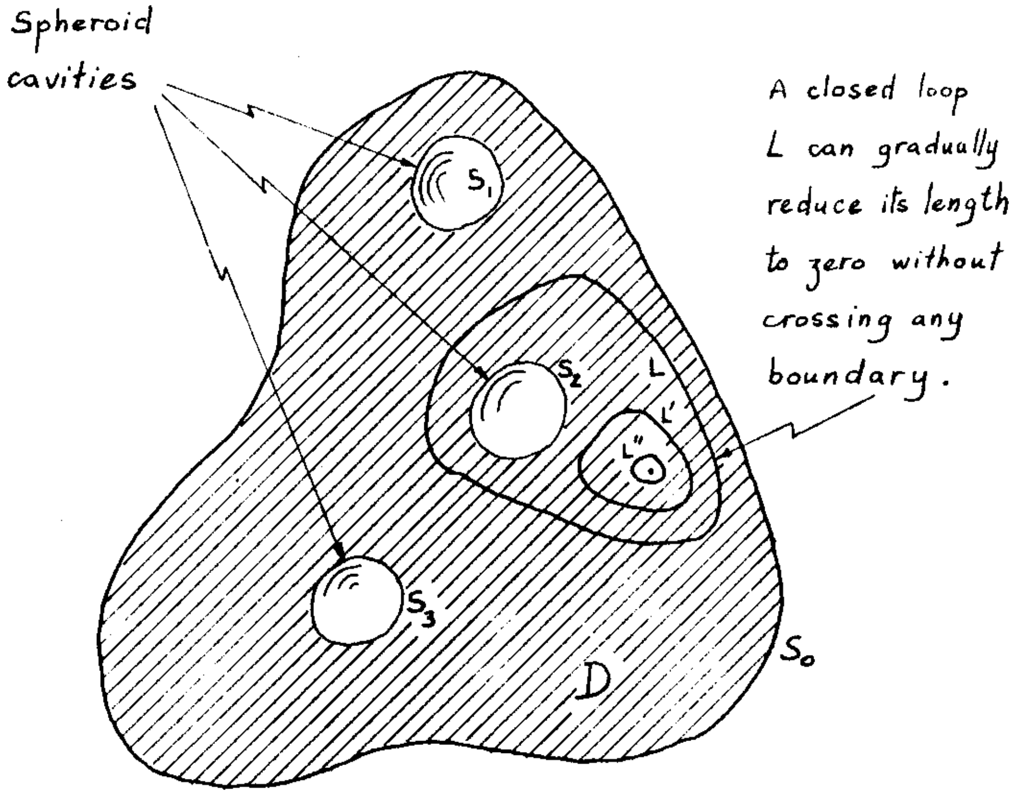


FIGURE 2.1:

A FINITE, SIMPLY CONNECTED, PERIPHRACTIC MATERIAL REGION D.

2.2 The expression of a divergence-free vector field in terms of curl ($\nabla\times$) of another vector field by the solutions of Neumann's problems.

In 1953, Stevenson [13] used the solution of the Neumann's problems to prove the following theorem:

Theorem.

For $\underset{\sim}{f}$ satisfying

$$\nabla\cdot\underset{\sim}{f} = 0 \text{ in a region } D, \tag{2.2.1}$$

$$\int_{S_i} \underset{\sim}{f}\cdot\underset{\sim}{n}\, d\sigma = 0, \quad i = 0,1,2 \ldots m, \tag{2.2.2}$$

where S_i is the smooth boundaries δD_i's of the finite, simply connected region D(S_o is the external boundary, all other S_i's are internal boundaries), $\underset{\sim}{f}$ can be expressed as

$$\underset{\sim}{f} = \underset{\sim}{\nabla}\times(\underset{\sim}{\nabla}\times\underset{\sim}{A}) \tag{2.2.3}$$

where $\underset{\sim}{A}$ is defined by

$$\underset{\sim}{A}(x,y,z) = \frac{1}{4\pi}\int_{\text{all space}} \frac{\underset{\sim}{f}_e}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}}\, d\tau(\xi,\eta,\zeta) \tag{2.2.4}$$

with $\underset{\sim}{f}_e$ being the extension of $\underset{\sim}{f}$ over the whole three dimensional space such that $\underset{\sim}{f}\cdot\underset{\sim}{n}$ and $\underset{\sim}{f}_e\cdot\underset{\sim}{n}$ are equal on each of the surfaces S_i's.

The theorem is a much stronger version of the commonly known theorem regarding the expression of a divergence-free vector such as those appearing in standard mathematics texts. The following is a summary of his proof.

Using the identity $\nabla^2 = -\underset{\sim}{\nabla}\times(\underset{\sim}{\nabla}\times) + \underset{\sim}{\nabla}(\underset{\sim}{\nabla}\cdot)$, the function $\underset{\sim}{f}(x,y,z)$ can be written as

$$4\pi\underset{\sim}{f}(x,y,z) = -\nabla^2 \int_D \frac{\underset{\sim}{f}(\xi,\eta,\zeta)}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}}\, d\tau$$

$$= \underset{\sim}{\nabla}\times\left[\underset{\sim}{\nabla}\times\int_D \frac{\underset{\sim}{f}(\xi,\eta,\zeta)}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}}\, d\tau\right]$$

$$-\underset{\sim}{\nabla}\left[\nabla\cdot\int_D \frac{\underset{\sim}{f}(\xi,\eta,\zeta)}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}}\, d\tau\right]$$

If the boundary surfaces are smooth enough, then the quantity inside the square bracket of the second term can be written as

$$\nabla\cdot\int_D \frac{\underset{\sim}{f}(\xi,\eta,\zeta)}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}}\,d\tau = -\int_D \frac{\underset{\sim}{\nabla}\cdot\underset{\sim}{f}(\xi,\eta,\zeta)}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}}\,d\tau - \int_S \frac{\underset{\sim}{f}\cdot\underset{\sim}{n}}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}}\,d\sigma.$$

Hence

$$4\pi\underset{\sim}{f}(x,y,z) = \underset{\sim}{\nabla}\times\left[\underset{\sim}{\nabla}\times\int_D \frac{\underset{\sim}{f}(\xi,\eta,\zeta)}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}}\,d\tau\right] + \underset{\sim}{\nabla}\int_S \frac{\underset{\sim}{f}\cdot\underset{\sim}{n}}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}}\,d\sigma,$$

since $\underset{\sim}{\nabla}\cdot\underset{\sim}{f}$ is zero throughout the material region D.

The solutions of the Neumann's problems are then used to cancel out the second term in the above equation, resulting in

$$\underset{\sim}{f}(x,y,z) = \underset{\sim}{\nabla}\times\left[\underset{\sim}{\nabla}\,\frac{1}{4\pi}\int_{\text{all space}} \frac{\underset{\sim}{f}_e}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}}\,d\tau\right],$$

where the function $\underset{\sim}{f}_e$ has been defined earlier.

Stevenson then reasoned that the function $\underset{\sim}{f}_e(x,y,z)$ must vanish at infinity to an order $\frac{1}{r^3}$ at least. Hence he concluded that his integral on the right hand size of the last equation converges.

The solutions of Neumann's problems are not of a simple nature, and they are rather put in the class of "advanced" mathematical problems. The existence of these solutions requires extensive discussion. Therefore the theorem by Stevenson is taken for granted in this thesis.

The importance of Stevenson's theorem is that it is applicable to finite simply connected, *non-periphractic* or *periphractic* regions. (A region is periphractic if it possesses a boundary surface which encloses at least one point not contained in the region) while the more commonly known version is (mostly) applicable only to parallelepipeds. The theorem will be used as a fundamental to the study of the applicability of the Papkovich-Neuber solution to be considered in this chapter. (The theorem can also be used to prove the applicability of the Helmholtz transformation to finite, simply connected, periphractic or non-periphractic regions).

2.3 Extension of Mindlin's method to cover periphractic regions.

The theorem of the previous section has established the validity of Mindlin's proof (section 1.3) for finite, simply connected, non-periphractic regions. Thus the Papkovich-Neuber general solution to elasticity, given by the equations (1.7.1) and (1.7.2), is applicable to any finite, simply connected, non-periphractic material region with a sufficiently smooth boundary.

It is clear that the original proof due to Mindlin is not applicable to periphractic regions. Thus we are in a position where assumptions have to be made about the suitability of the Papkovich-Neuber solution for a specific application (say, a hollow sphere) before the actual solution can be derived by matching the conditions on the boundary surfaces of the material region.

An alteration of Mindlin's proof is proposed here. It allows the Papkovich-Neuber solution to be applicable to finite, simply connected, periphractic material regions, such as the one in fig. 2.1.

The proposed modification is presented in the following:

Taking the divergence of $\underset{\sim}{V}$ over the whole (finite, simply connected and periphractic) material region D and integrating it over this region gives

$$\phi_1(x,y,z) = -\frac{1}{4\pi}\int_D (\underset{\sim}{\nabla}\cdot\underset{\sim}{V})\,\frac{d\tau(\xi,\eta,\zeta)}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}}\,.$$

The function $(\underset{\sim}{V}-\underset{\sim}{\nabla}\phi_1)$ is thus divergence-free in the material region D.

Further define

$$\phi_2(x,y,z) = -\frac{1}{4\pi}\,\frac{p_i}{|(x,y,z) - \underset{\sim}{a_i}|} \qquad i = 1, \ldots, m,$$

where $\underset{\sim}{a_i}$, i=1, ..., m are the vectors pointing from the origin of the coordinates to fixed points enclosed by the internal boundaries S_i, i=1, ...,m respectively; the constants, p_i, i=1, ..., m are defined by

$$p_i = \int_{S_i} (\underset{\sim}{V} - \underset{\sim}{\nabla}\phi_1)\cdot\underset{\sim}{n}\, d\sigma \qquad i = 1, \ldots, m.$$

Then the function $\underset{\sim}{V} - \underset{\sim}{\nabla}(\phi_1 + \phi_2)$ has the following properties

$$\underset{\sim}{\nabla}\cdot(\underset{\sim}{V} - \underset{\sim}{\nabla}(\phi_1 + \phi_2)) = 0 \text{ over } D$$

and

$$\int_{S_i} (\underset{\sim}{V} - \underset{\sim}{\nabla}(\phi_1 + \phi_2))\cdot\underset{\sim}{n}\, d\sigma = 0, \qquad i = 1, \ldots, m.$$

Hence $\underset{\sim}{V}$ is expressible as

$$\underset{\sim}{V} = \underset{\sim}{\nabla}\phi + \underset{\sim}{\nabla}\mathrm{x}\underset{\sim}{C},$$

where ϕ is the sum $(\phi_1 + \phi_2)$ and $\underset{\sim}{C}$ is a vector function, with its existence guaranteed by the theorem of section $\underset{\sim}{2}.\underset{\sim}{2}$.

For finite $\underset{\sim}{\nabla}\cdot\underset{\sim}{V}$, the functions ϕ and $\underset{\sim}{C}$ are finite in the material region D but they may be singular inside any internal boundary (ϕ certainly has at least one singularity point at $\underset{\sim}{a_i}$ inside each internal surface S_i).

The function $\underset{\sim}{C}$ is, in turn, written as

$$\underset{\sim}{C} = -\underset{\sim}{\nabla}\mathrm{x}\underset{\sim}{W} + \underset{\sim}{\nabla}\psi_1 - \underset{\sim}{\nabla}\sum_{i=1}^{m}\frac{1}{4\pi}\frac{q_i}{|\underset{\sim}{r} - \underset{\sim}{a_i}|}$$

where ψ and q_i's are defined by

$$\psi_1(x,y,z) = -\frac{1}{4\pi}\int_D (\underset{\sim}{\nabla}\cdot\underset{\sim}{C})\frac{d\tau(\xi,\eta,\zeta)}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}}$$

$$q_i = \int_{S_i} (\underset{\sim}{C} - \underset{\sim}{\nabla}\psi_1)\cdot\underset{\sim}{n}\, d\sigma, \qquad i = 1, \ldots, n.$$

Hence

$$\underset{\sim}{V} = \underset{\sim}{\nabla}\phi - \underset{\sim}{\nabla}\mathrm{x}\underset{\sim}{\nabla}\mathrm{x}\underset{\sim}{W}.$$

The above proof for $\underset{\sim}{C}$ implies that $\underset{\sim}{W}$ can be chosen such that it is divergence-free. Adding a term of

$$\underset{\sim}{W_0}(x,y,z) = -\frac{\beta}{4\pi}\underset{\sim}{\nabla}\int_D \phi\frac{d\tau(\xi,\eta,\zeta)}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}}$$

to the function $\underset{\sim}{W}$, which is divergence-free, leads to a relationship between the new value of $\underset{\sim}{W}$ and ϕ. The relationship is

$$\phi = \frac{1}{\beta}\underset{\sim}{\nabla}\cdot\underset{\sim}{W}.$$

Thus V is expressible by

$$V = \frac{1}{\beta}\underset{\sim}{\nabla}(\underset{\sim}{\nabla}\cdot\underset{\sim}{W}) - \underset{\sim}{\nabla}\mathrm{x}\underset{\sim}{\nabla}\mathrm{x}\underset{\sim}{W}. \qquad (2.3.1)$$

Equation (2.3.1) is identical to (1.3.1) and also (1.2.1). The remainder of the proof can be carried out as has been done previously in section 1.3.

Hence the Papkovich-Neuber solution, as given by (1.7.1) and (1.7.2), in the form

$$\underset{\sim}{V} = \underset{\sim}{B} - \frac{1}{4(1-\nu)} \nabla(\underset{\sim}{r}\cdot\underset{\sim}{B} + B_0), \qquad (1.7.1)$$

where $\nabla^2 \underset{\sim}{B} = 0,$ (1.7.2a)

$$\nabla^2 B_0 = 0, \qquad (1.7.2b)$$

is the general solution to the homogeneous Lamé's equation (1.1.4a) when the material region D is finite, simply connected and has smooth boundary surface(s). This statement holds whether the region D is non-periphractic or periphractic.

As a consequence, when using the Papkovich-Neuber solution to solve for problems involving finite, simply connected, and periphractic regions, the solution is known beforehand to cover those cases. Thus, when working with cases such as hollow spheres, or material with cavities, it is certain that the solution given by (1.7.1) and (1.7.2), does cover these cases, the only remaining problem is then to match the harmonic functions $\underset{\sim}{B}$, B_0 to the boundary conditions.

The proof in this section can also be carried out when D is *infinite*, simply connected (periphractic or non-periphractic) if certain restrictions are placed on the decay characteristics of $\underset{\sim}{V}$.

2.4 History of the general solution of biharmonic equations.

Let us turn now to biharmonic functions. These functions very often mingle with the solutions of elasticity. This is obvious in the case of plane strain (or equivalent plane stress), using Airy stress function, or in the case of axi-symmetric loadings, using Love's function. This section here will bring to light certain useful properties of biharmonic functions, some of these properties bear remarkable similarity to those of the Papkovich-Neuber general solution.

solution.

First, the general solution to biharmonic equations is under study. The first two theorems which express the general form of biharmonic functions are due to Almansi [14]. These theorems can be found in the present text book of Fung [10]; they are restated in the following.

Theorem

A biharmonic function f in a region D_1 can be expressed as

$$f = E + zC, \tag{2.4.1}$$

where E and C are harmonic functions in D_1.

Theorem

A biharmonic function f in a region D_2 can be expressed as

$$f = F + r^2 G \tag{2.4.2}$$

where F and G are harmonic functions in D_2.

However, the above theorems still leave something to be desired. It is because the first one is valid only when D_1 is z-convex and the second one valid only when D_2 is star-shaped with respect to the origin.

Recently, Fosdick [11] established the following theorem, which is

Theorem

Let D be a regular three-dimensional region with boundary $S = \bigcup_{i=0}^{m} S_i$. The outer boundary S_o as well as each inner boundary S_i, $i = 1, \ldots, m$ are assumed to be twice continuously differentiable. Let f have the following properties

$f \in C^{4+\alpha}(D) \cap C^4(\bar{D})$

f is biharmonic in D.

Then there exist constants k_i, $i = 1, \ldots, m$ and harmonic functions A,B,C,E in D such that

$$f = xA + yB + zC + E + \sum_{i=1}^{m} k_i |\underset{\sim}{r} - \underset{\sim}{a}_i| \tag{2.4.3}$$

where r is the vector (x,y,z) and $\underset{\sim}{a}_i$, i = 1, 2, ..., m represents a vector pointing to an arbitrary fixed point not in $\bar{D}$, but enclosed by the inner boundary S_i, i = 1, 2, ..., m, for each choice of index i.

In the above theorem, the notation $C^{N+\alpha}$ denotes the class of functions which are N times continuously differentiable and has N^{th} order Hölder continuous partial derivatives with exponent α.

The proof of this theorem is based on the existence of a vector field such that its curl is equal to another given field in the region D. The original proof of the latter (theorem) is due to Lichtenstein [12]. The proof of the general form of biharmonic function, due to Fosdick, is a straight application of Lichtenstein's theorem to the biharmonic f, using Green's theorem.

In the following section an alternative method of deriving the general form of biharmonic function is proposed. This method is based on Stevenson's theorem (which is similar to Lichtenstein's theorem but does not involve the Hölder continuous partial derivatives), and is *quite different* from the method used in Fosdick's proof.

2.5 The general solution of biharmonic equations based on the solutions of Neumann's problems.

The method proposed in this section is based on the theorem in section 2.2 due to Stevenson, and the latter is in turn based on the existence of the solutions of Neumann's problems.

First of all, a theorem is proposed, which will be of convenience to subsequent works in this chapter.

Proposed theorem

In a finite, simply connected region D, which has a smooth boundary $S = \bigcup_{i=0}^{m} S_i$. ($S_0$ is the external boundary, all other S_i's are internal boundaries), every scalar harmonic function A is expressible in the form

$$A = \underset{\sim}{\nabla} \cdot \underset{\sim}{B} + \sum_{i=1}^{m} \frac{q_i}{|\underset{\sim}{r} - \underset{\sim}{a}_i|} \tag{2.5.1}$$

with

$$\nabla^2 \underset{\sim}{B} = \underset{\sim}{0}, \tag{2.5.2}$$

where $\underset{\sim}{r}$ is the vector (x,y,z), q_i's, $i = 1, 2, \ldots, m$ are m constants, and $\underset{\sim}{a}_i$, $i = 1, \ldots, m$ represents a vector pointing to an arbitrary fixed point not in $\bar{D}$, but enclosed by the inner boundary S_i, $i = 1, 2, \ldots, m$, for each choice of index i.

Proof

Define the constants q_i's by

$$q_i = -\frac{1}{4\pi}\int_{S_i} \nabla A \cdot \underset{\sim}{n}\, d\sigma,$$

then, by the theorem in section 2.2, the function $\underset{\sim}{\nabla}\left(A - \sum_{i=1}^{m} \frac{q_i}{|\underset{\sim}{r} - \underset{\sim}{a}_i|}\right)$ is expressible as

$$\underset{\sim}{\nabla}\left(A - \sum_{i=1}^{m} \frac{q_i}{|\underset{\sim}{r}_i - \underset{\sim}{a}_i|}\right) = \underset{\sim}{\nabla}\mathrm{x}\underset{\sim}{G}.$$

Two new functions can now be defined:

$$\phi(x,y,z) = -\frac{1}{4\pi}\int_D \frac{A - \sum_{i=1}^{m} \frac{q_i}{|\underset{\sim}{r} - \underset{\sim}{a}_i|}}{\sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}}\, d\tau(x,y,z)$$

$$\underset{\sim}{H}(x,y,z) = -\frac{1}{4\pi}\int_D \frac{\underset{\sim}{G}}{\sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}}\, d\tau(x,y,z).$$

Then the function $\underset{\sim}{B}$ defined by

$$\underset{\sim}{B} = \underset{\sim}{\nabla}\phi - \underset{\sim}{\nabla}\mathrm{x}\underset{\sim}{H}$$

certainly satisfies the equations (2.5.1) and (2.5.2) in D. Hence the proposed theorem has been proved.

We are now ready to prove the following proposed theorem about the general form of a biharmonic function.

Proposed theorem

For a finite, simply connected region D with smooth boundary $S = \bigcup_{i=0}^{n} S_i$ (S_o is the external boundary, all other S_i's are internal boundaries), the general form for a biharmonic function f satisfying

$$\nabla^2(\nabla^2 f) = 0. \tag{2.5.3}$$

is

$$f = xA + yB + zC + E + \sum_{i=1}^{m} q_i\, |\underset{\sim}{r}_i - \underset{\sim}{a}_i|, \tag{2.5.4}$$

where A, B, C, E are four harmonic functions, q_i's, $i = 1, 2, \ldots, m$

are m vectors pointing from the origin to m points enclosed by m inner surfaces S_i's, $i = 1, 2, \ldots, m$.

Proof

A biharmonic function f satisfying

$$\nabla^2(\nabla^2 f) = 0$$

is the solution of

$$\nabla^2 f = G,$$

where G is a general harmonic function.

Using the preceeding proposed theorem, the function f is the solution of

$$\nabla^2 f = 2\nabla \cdot \mathbf{H} + \sum_{i=1}^{m} \frac{q_i}{|\mathbf{r}_i - \mathbf{a}_i|}$$

where $\mathbf{H}$ is a general harmonic vector function.

A *particular* solution f_o to the above equation can be easily found, which is

$$f_o = xA + yB + zC + \sum_{i=1}^{m} q_i |\mathbf{r}_i - \mathbf{a}_i|,$$

where A, B, C are the three scalar components along the x,y,z directions of vector $\mathbf{H}$.

Thus f is expressible in the form

$$f = xA + yB + zC + E + \sum_{i=1}^{m} q_i |\mathbf{t}_i - \mathbf{a}_i|,$$

where E, a harmonic function, is the *general* solution of the homogeneous equation

$$\nabla^2 f = 0.$$

Hence, the proposed theorem has been proved. Furthermore, it has been proved by the use of the theorem due to Stevenson (section 2.2), which does not involve the Hölder continuous partial derivatives of f.

It should be observed that f assumes the form (2.5.4) if it is possible to express every harmonic function as the divergence of another harmonic vector function. This requirement is certainly satisfied when the theorem due to Stevenson is applicable, but whether the requirement is equivalent to the applicability of Stevenson's theorem is unknown.

2.6 The reductions of the general form of biharmonic functions

When the region D, in the second proposed theorem of section 2.5, is z-convex or star-shaped with respect to the origin, then the general form of biharmonic functions, as given by (2.5.4), reduces to (2.4.1) and (2.4.2) respectively. It is so as the theorems due to Almansi (section 2.4) are applicable to these cases.

It is proposed in this section that there is another reduction theorem concerning the representation of a biharmonic function in a star-shaped region.

Proposed theorem

If the definition region of a biharmonic function f is star-shaped with respect to the origin then f can be represented as

$$f = xA + yB + zC + \text{constant} \tag{2.6.1}$$

where A, B, C are harmonic functions.

Proof

As the region is star-shaped, using the second proposed theorem of section 2.5, f can be written as

$$f = xH_x + yH_y + zH_z + E + k$$

where $\underset{\sim}{H}$ and E are harmonic functions, k a constant, and E satisfies

$$E(0,0,0) = 0.$$

Consider the function W defined by

$$W = \int_{r_0}^{r} \frac{E}{r}\, dr,$$

we have

$$r^2\nabla^2 W = \left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \nabla^{2*}\right]W$$

$$= \frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\int_{r_0}^{r}\frac{E}{r}\,dr\right) + \int_{r_0}^{r}\frac{\nabla^{2*}E}{r}\,dr$$

$$= r\,\frac{\partial E}{\partial r} + E + \int_{r_0}^{r}\frac{\nabla^{2*}E}{r}\,dr,$$

where the operator ∇^{2*} stands for

$$\nabla^{2*} = \nabla^2 - \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right).$$

Since

$$\int_{r_0}^{r} \frac{1}{r}\left(\frac{\partial}{\partial r} r^2 \frac{\partial E}{\partial r}\right) dr = \left[r\frac{\partial E}{\partial r} \right]_{r_0}^{r} + \left[E \right]_{r_0}^{r},$$

we can write

$$r^2 \nabla^2 W = \int_{r_0}^{r} \frac{1}{r} \nabla^2 E \, dr + \left[r\frac{\partial E}{\partial r} + E \right]_{r=r_0}.$$

As the region is star-shaped with respect to the origin, r_0 can be set to zero, leaving W harmonic, satisfying

$$\nabla^2 W = 0.$$

f can now be written as

$$f = x\left(H_x + \frac{\partial W}{\partial x}\right) + y\left(H_y + \frac{\partial W}{\partial y}\right) + z\left(H_z + \frac{\partial W}{\partial z}\right) + k.$$

By replacing the symbols, the desired form of (2.6.5) is arrived at.

The reductions of the general representation (2.5.4) for biharmonic functions are considered in this section because of the analogy between them and the reductions for the Papkovich-Neuber general solution (1.7.2) for elasticity (to be considered in Chapter 3).

2.7 The Papkovich-Neuber solution as derived from biharmonic functions.

It is observed that biharmonic functions appear very often in the solutions of the problems of elasticity. The reductions theorems in the previous sections are also remarkably similar to those of the Papkovich-Neuber solution. It is thus interesting to know how much interaction is there between the Papkovich-Neuber solution and biharmonic function. To throw some light on this question, this section proposes still another way of deriving the Papkovich-Neuber solution from the general form of biharmonic functions.

Consider a finite, simply connected, non-periphractic material region. Taking the divergence of the homogeneous Lamé's equation (1.1.4), we have

$$\nabla^2 (\nabla \cdot \underset{\sim}{V}) = 0. \qquad (2.7.1)$$

Hence $\underset{\sim}{\nabla}\cdot\underset{\sim}{V}$ is a harmonic function, the first proposed theorem of section 2.5 guarantees the existence of a vector $\underset{\sim}{A}$ such that

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{V} = \underset{\sim}{\nabla}\cdot\underset{\sim}{A}, \tag{2.7.2}$$

where $\underset{\sim}{A}$ is harmonic, satisfying

$$\nabla^2\underset{\sim}{A} = \underset{\sim}{0}. \tag{2.7.3}$$

Equations (2.7.2) and (2.7.3) show that $\underset{\sim}{V}$ can be split into two components, one is harmonic and the other is divergence-free

$$\underset{\sim}{V} = \underset{\sim}{A} + \underset{\sim}{C} \tag{2.7.4}$$

with $\underset{\sim}{A}$ satisfying (2.7.3) and $\underset{\sim}{C}$ satisfying

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{C} = 0. \tag{2.7.5}$$

Back substitution of $\underset{\sim}{A}$ and $\underset{\sim}{C}$ into (1.1.4) gives

$$\nabla^2\underset{\sim}{C} = -\frac{1}{1-2\nu}\underset{\sim}{\nabla}(\underset{\sim}{\nabla}\cdot\underset{\sim}{A}). \tag{2.7.6}$$

The application of another Laplacian operator to (2.7.6) shows that $\underset{\sim}{C}$ is biharmonic. This suggests that a special solution C_1 to the system (2.7.5) and (2.7.6) should be looked for in the form

$$\underset{\sim}{C_1} = (\underset{\sim}{r}\cdot\underset{\sim}{D} + G_x)\underset{\sim}{i} + (\underset{\sim}{r}\cdot\underset{\sim}{E} + G_y)\underset{\sim}{j} + (\underset{\sim}{r}\cdot\underset{\sim}{F} + G_z)\underset{\sim}{k}, \tag{2.7.7}$$

where

$$\nabla^2\underset{\sim}{D} = \nabla^2\underset{\sim}{E} = \nabla^2\underset{\sim}{F} = \nabla^2(G_x,G_y,G_z) = \underset{\sim}{0}. \tag{2.7.8}$$

It should be stressed that (2.7.7) and (2.7.8) are just suggestions to the form of C_1.

A special solution $\underset{\sim}{C_1}$ for (2.7.5) and (2.7.6) is

$$C_1 = -\frac{1}{2(1-2\nu)}\left[\left(r\cdot\frac{\partial}{\partial x}\underset{\sim}{A}\right)\underset{\sim}{i} + \left(\underset{\sim}{r}\cdot\frac{\partial}{\partial y}\underset{\sim}{A}\right)\underset{\sim}{j} + \left(r\cdot\frac{\partial}{\partial z}\underset{\sim}{A}\right)k\right] + \frac{1}{2(1-2\nu)}\underset{\sim}{A}.$$

This special solution $\underset{\sim}{C_1}$ is arrived at simply by equalizing the components of (2.7.6), giving

$$2\underset{\sim}{\nabla}\cdot\underset{\sim}{D} = -\frac{1}{1-2\nu}\underset{\sim}{\nabla}\cdot\left(\frac{\partial}{\partial x}\underset{\sim}{A}\right),$$

$$2\underset{\sim}{\nabla}\cdot\underset{\sim}{E} = -\frac{1}{1-2\nu}\nabla\cdot\left(\frac{\partial}{\partial y}\underset{\sim}{A}\right),$$

$$2\underset{\sim}{\nabla}\cdot\underset{\sim}{F} = -\frac{1}{1-2\nu}\underset{\sim}{\nabla}\cdot\left(\frac{\partial}{\partial z}\underset{\sim}{A}\right),$$

and then by adding $\frac{1}{2(1-2\nu)}\underset{\sim}{A}$ into $\underset{\sim}{C_1}$ such that (2.7.6) is satisfied.

$\underset{\sim}{C}_1$ can also be written in a more convenient form

$$\underset{\sim}{C}_1 = -\frac{1}{2(1-2\nu)}\underset{\sim}{\nabla}(\underset{\sim}{r}\cdot\underset{\sim}{A}) + \frac{1}{1-2\nu}\underset{\sim}{A} \tag{2.7.9}$$

Then $\underset{\sim}{C} = \underset{\sim}{C}_1 + \underset{\sim}{C}_o$ (2.7.10)

where $\underset{\sim}{C}_o$ is the general solution of the homogeneous equations corresponding to (2.7.5) and (2.7.6), i.e. $\underset{\sim}{C}_o$ satisfies

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{C}_o = 0, \tag{2.7.11}$$

and

$$\nabla^2\underset{\sim}{C}_o = \underset{\sim}{0}. \tag{2.7.12}$$

$\underset{\sim}{V}$ can now be written as

$$\underset{\sim}{V} = \frac{2(1-\nu)}{1-2\nu}\underset{\sim}{A} - \frac{1}{2(1-2\nu)}\underset{\sim}{\nabla}(\underset{\sim}{r}\cdot\underset{\sim}{A}) + \underset{\sim}{C}_o$$

with $\underset{\sim}{A}$ satisfying (2.7.3) and $\underset{\sim}{C}_o$ satisfying (2.7.11) and (2.7.12).

Since $\underset{\sim}{C}_o$ is harmonic, $\underset{\sim}{V}$ can also be expressed as

$$\underset{\sim}{V} = \underset{\sim}{B} - \frac{1}{4(1-\nu)}\underset{\sim}{\nabla}(\underset{\sim}{r}\cdot\underset{\sim}{B}) + \frac{1}{4(1-\nu)}\underset{\sim}{\nabla}(\underset{\sim}{r}\cdot\underset{\sim}{C}_o),$$

where $\underset{\sim}{B}$ is a general harmonic vector function, defined by

$$\underset{\sim}{B} = \frac{2(1-\nu)}{1-2\nu}\underset{\sim}{A} + \underset{\sim}{C}_o.$$

By noting that

$$\nabla^2(\underset{\sim}{r}\cdot\underset{\sim}{C}_o) = 2\underset{\sim}{\nabla}\cdot\underset{\sim}{C}_o + \underset{\sim}{r}\nabla^2\underset{\sim}{C}_o = 0,$$

$\underset{\sim}{V}$ can be rewritten in its final form

$$\underset{\sim}{V} = \underset{\sim}{B} - \frac{1}{4(1-\nu)}\underset{\sim}{\nabla}(\underset{\sim}{r}\cdot\underset{\sim}{B} + B_o), \tag{2.7.13}$$

where $\nabla^2\underset{\sim}{B} = 0,$ (2.7.14a)

$\nabla^2 B_o = 0.$ (2.7.14b).

Equations (2.7.13) and (2.7.14) are obviously the familiar Papkovich-Neuber general solution.

The proof presented above covers only finite, simply connected, non-periphractic regions. This proof is based on the expression of a harmonic function in terms of divergence of another harmonic (vector) function. It is certain that this requirement is satisfied whenever the theorem due to Stevenson is applicable, but the converse of this statement is not known to be true or false. (If the converse statement is false then new and interesting problems will probably surface in both elasticity and vector analysis). The proof in this section can not be simply extended to cover finite, simply connected, periphractic material regions. The reason for it is still another unanswered question.

In finite, simply connected, non-periphractic regions, the proof in this section also serves to demonstrate that the harmonic function B_0 required in the Papkovich-Neuber general expression (1.7.2) for displacements is (probably) not due to certain arbitrary chosen method. Rather it may be a real entity, appearing in every different method of solution. A confirmation of this (tentative) observation will be found in section 3.6.

2.8 Conclusions

This chapter has shown that

(a) The Papkovich-Neuber general expression for displacements [(1.7.1) and (1.7.2)] is applicable to finite, simply connected material regions with smooth boundaries, whether the region is non-periphractic or periphractic.

(b) Without any assumption regarding the "Holder continuous partial derivatives" of a biharmonic function f, the function has been proved to have the form (2.5.4) which is identical to the form (2.4.3) due to Fosdick. The only assumption used is that the definition region D of f is finite, simply connected and has a smooth boundary (or boundaries).

(c) When the definition region of a biharmonic function f is star-shaped with respect to the origin, the function f is expressible in the form (2.6.1).

(d) Finally, the Papkovich-Neuber solution for non-periphractic regions can be derived if every harmonic scalar function in the material region is expressible as the divergence of another harmonic vector function. The resulting solution is still identical with those given in Chapter 1.

CHAPTER 3

Completeness of the Papkovich-Neuber solution in terms of three harmonic functions.

3.1 Historical development of the three-harmonic function representation.

The Lamé equation for an elastic body, without any body force, is written as

$$\nabla^2 \underset{\sim}{V} + \frac{1}{1-2\nu}\, \nabla(\nabla . \underset{\sim}{V}) = 0 \,, \qquad (3.1.1)$$

where $\underset{\sim}{V}(x,y,z)$ is the displacement vector at the point (x,y,z) and ν is the Poisson ratio of the material.

As already known in chapter 1, both Papkovich [1] and Neuber [3], [4] independently established the general solution of the above equation as

$$\underset{\sim}{V} = \underset{\sim}{B} - \frac{1}{2(1-\nu)}\, \nabla(\tfrac{1}{2}\underset{\sim}{r}.\underset{\sim}{B} + B_0), \qquad (3.1.2)$$

where $\nabla^2 \underset{\sim}{B} = \underset{\sim}{0}, \qquad \nabla^2 B_0 = 0. \qquad (3.1.3)$

Here $\underset{\sim}{r}$ is the position vector of a field point, referring to an arbitrarily chosen (fixed) origin.

In this chapter, the validity of the solution given by the above system of (3.1.2) and (3.1.3) will neither be questioned nor investigated. Rather, they are accepted as starting points for a study which is completely different from those done in the preceding two chapters 1 and 2.

Having established (3.1.2), Papkovich [1] claimed that B_0 can be set to zero without affecting the generality of the solution. This unsupported claim has aroused much controversy over the generality of his (and also of Neuber's) final form for the displacement field $\underset{\sim}{V}$,

$$\underset{\sim}{V} = \underset{\sim}{B} - \frac{1}{2(1-\nu)}\, \nabla(\tfrac{1}{2}\underset{\sim}{r}.\underset{\sim}{B}). \qquad (3.1.4)$$

Neuber [3], [4] also asserted that any of the four harmonic functions B_0, B_x, B_y and B_z, where the last three are the scalar components of $\underset{\sim}{B}$, can be set to zero without altering the generality of (3.1.2). His statement regarding B_0 has also been shown to be unsupported and inconclusive [5], [6] since he assumed the existence of a function satisfying a pair of simultaneous partial differential equations.

Eubanks and Sternberg [6] showed that (a) if the region is convex with respect to one direction, the scalar component of $\underset{\sim}{B}$ in that direction can be set to zero without affecting the completeness of (3.1.2), or (b) the scalar function B_0 may be set to zero without loss of completeness if the region is star-shaped with respect to the origin and when 4ν is not an integer. When 4ν is an integer, they demonstrated, using a counter example, that B_0 can not be dropped from (3.1.2).

However, their proof for the omission of B_0 made use of its expansion into a series of solid spherical harmonic functions. This expansion gives rise to superficial difficulties when 4ν is an integer, also it does not allow the proof to be presented in "simple", "elementary" theory of elasticity. Stippes [8] also worked on the omission of B_0, but he used even more sophisticated mathematical methods, hence his work can neither be presented along "simple", "elementary" line of argument.

This chapter presents a method which uses only ordinary differentiation and integration rules to establish the necessary and sufficient conditions under which B_0 can be dropped from (3.1.2). This method also offers some insight into the significance of the starred-shape required by [6], the role of spherical harmonics, and also the significance of 4ν being an integer. At the end of the chapter, a counter example will show that the starred shape required by [6] is intrinsic to the omission of B_0, and is not a mathematical "over-condition".

3.2 The nullification of B_0 in (3.1.2)

Accepting the completeness of the system of (3.1.2) and (3.1.3), the term B_0 may be dropped *if and only if for any arbitrary harmonic* function ϕ, we can find a vector function $\underset{\sim}{C}$ satisfying

$$\alpha \underset{\sim}{C} - \underset{\sim}{\nabla}(\underset{\sim}{r}.\underset{\sim}{C}) = \underset{\sim}{\nabla}\phi, \qquad (3.2.1)$$

and

$$\nabla^2 \underset{\sim}{C} = \underset{\sim}{0}, \qquad (3.2.2)$$

where α is a notation denoting the commonly occuring quantity $4(1-\nu)$.

The proof for this condition is omitted due to its simplicity. *Note that, when α is not equal to 2, by taking the divergence of (3.2.1), it can be shown that any such vector $\underset{\sim}{C}$ satisfying both (3.2.1) and (3.2.2) will have its divergence equal to zero, i.e.*

$$\underset{\sim}{\nabla}.\underset{\sim}{C} = 0. \qquad (3.2.3)$$

Similarly, we also have $\underset{\sim}{C}$ irrotational for non zero α, i.e.

$$\underset{\sim}{\nabla} \times \underset{\sim}{C} = \underset{\sim}{0}. \qquad (3.2.4)$$

3.3 A necessary and sufficient condition for the existence of C

In this section, a necessary and sufficient condition for the existence of vector $\underset{\sim}{C}$ will be established. Assuming that α is a constant not equal to zero nor two $(0 \neq \alpha \neq 2)$ and ϕ is a given harmonic function, then there exists a vector function $\underset{\sim}{C}$ satisfying

$$\alpha \underset{\sim}{C} - \underset{\sim}{\nabla}(\underset{\sim}{r}.\underset{\sim}{C}) = \nabla\phi, \qquad (3.2.1)$$

and

$$\nabla^2 \underset{\sim}{C} = \underset{\sim}{0}, \qquad (3.2.2)$$

if and only if there is a function A satisfying

$$\alpha A - \underset{\sim}{r}.\underset{\sim}{\nabla} A = \phi, \qquad (3.3.1)$$

and

$$\nabla^2 A = 0 \qquad (3.3.2)$$

3.3.1. - *Sufficient condition*

If there is a function A satisfying (3.3.1) and (3.3.2), then the vector $\underset{\sim}{C}$ defined by $\underset{\sim}{C} = \underset{\sim}{\nabla} A$ will certainly satisfy (2.2.1) and (2.2.2).

3.3.2. - *Necessary condition*

If there is a vector function $\underset{\sim}{C}$ satisfying (3.2.1) and (3.2.2), then $\underset{\sim}{C}$ is irrotational, as pointed out in section 3.2, and can be expressed as

$$\underset{\sim}{C} = \underset{\sim}{\nabla} A'.$$

Substitution of this equation into (3.2.1) gives

$$\alpha A' - \underset{\sim}{r}.\underset{\sim}{\nabla} A' = \phi + k,$$

where k is an unknown constant.

Since α is non-zero, a function A may be defined, with A' already in existence, as

$$A = A' - \frac{k}{\alpha}.$$

The new function A satisfied (3.3.1), and since A differs from A' only by a constant, we also have

$$\underset{\sim}{C} = \underset{\sim}{\nabla} A;$$

and thus A satisfies (3.3.2) by virtue of (3.2.3).

Having the necessary and sufficient condition for the existence of $\underset{\sim}{C}$, it is not difficult to recognize the equivalence between the setting to zero of B_0 in [1], and of ϕ_0 in [4].

3.4 The existence of a function satisfying (3.3.1) and (3.3.2) for a given harmonic function $\phi(x,y,z)$.

Consider the transformation from the rectangular coordinates (x,y,z) into the coordinate (a,θ,γ) specified by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longleftrightarrow \begin{pmatrix} a \\ \theta \\ \gamma \end{pmatrix} = \begin{pmatrix} \log(x^2 + y^2 + z^2)^{\frac{1}{2}} \\ \theta \\ \gamma \end{pmatrix}$$

where θ and γ are the ordinary longitudinal and azimuthal angles in spherical coordinates.

For the function $P(a,\theta,\gamma)$ which is the "logarithmic-spherical" form of $A(x,y,z)$, we have

$$e^{\alpha a}\left[\frac{\partial}{\partial a}\left(e^{-\alpha a}P(a,\theta,\gamma)\right)\right]_{\theta,\gamma} = -\alpha A(x,y,z) + $$

$$+ x\left(\frac{\partial A}{\partial x}\right)_{y,z} + y\left(\frac{\partial A}{\partial y}\right)_{z,x} + z\left(\frac{\partial A}{\partial z}\right)_{x,y} \qquad (3.4.1)$$

here the subscripts denote the fixed variables in the partial differentiation processes.

Equation (3.3.1), with $\phi(x,y,z)$ replaced by its "logarithmic-spherical" form $\Omega(a,\theta,\gamma)$, gives

$$-\alpha A + \underset{\sim}{r}.\underset{\sim}{\nabla} A = \Omega(a,\theta,\gamma),$$

this equation is then written as

$$e^{\alpha a}\left[\frac{\partial}{\partial a}\left(e^{-\alpha a}P(a,\theta,\gamma)\right)\right]_{\theta'\gamma} = \Omega(a,\theta,\gamma). \qquad (3.4.2)$$

Equation (3.4.2) allows $P(a,\theta,\gamma)$ to be integrated and transformed to the cartesian form $A(x,y,z)$ as in the following

$$A(x,y,z) = P(a,\theta,\gamma) = -e^{+\alpha a}\left[\int_{a_0=\log r_0}^{a} e^{-\alpha a}\Omega(a,\theta,\gamma)\,da + s(\theta,\gamma)\right], \qquad (3.4.3)$$

where $s(\theta,\gamma)$ is a function of only θ and γ, and r_0 is an arbitrarily chosen reference radius.

If $W(r,\theta,\gamma)$ denotes the spherical form of $\phi(x,y,z)$, the function $A(x,y,z)$ can also be expressed as

$$A(x,y,z) = -r^{\alpha}\left[\int_{r_0}^{r} r^{-\alpha-1}W(r,\theta,\gamma)\,dr + s(\theta,\gamma)\right]. \qquad (3.4.4)$$

In general, (3.4.3), or its alternative form (3.4.4), allows $A(x,y,z)$ satisfying (3.3.1) to be determined.

Note that the term $r^{\alpha}s(\theta,\gamma)$ allows $s(\theta,\gamma)$ to assume different functional forms, differing only by surface harmonic function $S_{\alpha}(\theta,\gamma)$, without altering the harmonicity of A.

3.4.1. - *The harmonicity of A*

The Laplacian operator in spherical coordinates can be written as

$$\nabla^2 = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \nabla^{2*}\right], \tag{3.4.5}$$

where

$$\nabla^{2*} = \operatorname{cosec}\gamma\,\frac{\partial}{\partial\gamma}\left(\sin\gamma\,\frac{\partial}{\partial\gamma}\right) + \operatorname{cosec}^2\gamma\,\frac{\partial^2}{\partial\theta^2}\,.$$

Application of this operator to the terms on the right hand side of (3.4.4) gives

$$\nabla^2\left[r^{\alpha}\int_{r_0}^{r} r^{-\alpha-1}W(r,\theta,\gamma)\,dr\right] = r^{(\alpha-2)}\left[(\alpha+1)r_0^{-\alpha}W(r_0,\theta,\gamma) + r_0^{-\alpha+1}\left(\frac{\partial W}{\partial r}\right)_{\text{at}(r_0,\theta,\gamma)}\right] \tag{3.4.6}$$

and

$$\nabla^2\left[r^{\alpha}s(\theta,\gamma)\right] = r^{(\alpha-2)}\left[(\alpha+1)\alpha s(\theta,\gamma) + \nabla^{2*}s(\theta,\gamma)\right]. \tag{3.4.7}$$

The proof of (3.4.6) is deferred until section 3.5 due to its length.

Hence, for a given harmonic function ϕ, there is a function A satisfying (3.3.1) and (3.3.2) *if* both of the following conditions are satisfied.

a - The function

$$A^*(x,y,z) = -r^{\alpha}\int_{r_0}^{r} r^{-\alpha-1}W(r,\theta,\gamma)\,dr \tag{3.4.8a}$$

is differentiable at least to second degree over the material region.

b - The equation

$$(\alpha+1)\alpha s(\theta,\gamma) + \nabla^{2*}s(\theta,\gamma) = -r_0^{-\alpha}\left[(\alpha+1)W(r_0,\theta,\gamma) + r_0\left(\frac{\partial W}{\partial r}\right)_{\text{at}(r_0,\theta,\gamma)}\right] \tag{3.4.8b}$$

has a solution $s(\theta,\gamma)$. Several such solutions may exist but differ only by a surface harmonic of degree α.

It is worth remembering that r_0 is the lower limit for the integration in (3.4.4) and can be set at any value which facilitates the solving of (3.4.8b) for $s(\theta,\gamma)$.

The above conditions are also sufficient for the omission of B_0 in (3.1.2)

3.4.2. - *Some special cases*

a - If the material region is outside a closed surface and the straight line drawn through the origin and every point on that surface can extend to infinity without leaving the material region, and if W is differentiable over that region and remains *finite* when r tends to infinity, then the right hand side of (3.4.8b) can be shown to vanish for increasing r_0. Thus there is a solution $s(\theta,\gamma) = 0$ for (3.4.8b) when r_0 is infinite. Hence B_0 in (3.1.2) can be omitted in this case. This result is similar to one of Sokolnikoff's as discussed in [5], p. 333 *et seq*.

b - An argument similar to the case just mentioned results in the solution $A(x,y,z)$ of (3.3.1) to any harmonic polynomial of order less than α in x, y and z, *irrespective* of the form of the material.

c - If the material region is finite and star-shaped with respect to the origin, and α is not an integer then B_0 can also be omitted. The proof is to follow.

Since the region is finite, $\phi(x,y,z)$ can be expanded into a polynomial of x, y and z, using Taylor's theorem, with the remainder of order higher than α as in the following

$$\phi(x,y,z) = \sum_{\substack{k,\ell,m\geqslant 0\\ k+\ell+m\leqslant[\alpha]}} x^k y^\ell z^m E_{k\ell m} + \psi(x,y,z),$$

where $[\alpha]$ denotes the largest integer not exceeding α. It can be easily proved, using the limiting technique, for successive higher order, as r tends to zero, that each value of $(k+\ell+m)$ gives a harmonic polynomial. Hence the summation is harmonic.

Since the solution of

$$\alpha A_1 - \underset{\sim}{r}.\underset{\sim}{\nabla} A_1 = \sum_{\substack{k,\ell,m\geqslant 0\\ k+\ell+m\leqslant[\alpha]}} x^k y^\ell z^m E_{k\ell m}$$

can be easily found, and is certainly harmonic, it is only important to consider the solution corresponding to $\psi(x,y,z)$. The solution for

$$\alpha A_2 - \underset{\sim}{r}.\underset{\sim}{\nabla} A_2 = \psi(x,y,z)$$

can be found using (3.4.4), with $W(r,\theta,\gamma)$ replaced by $w(r,\theta,\gamma)$, the spherical form of $\psi(x,y,z)$.

Since $w(r,\theta,\gamma)$ is of order higher than α, the right hand side of (3.4.8b) tends to zero for r_0 tending to zero. Thus, there is a solution $s(\theta,\gamma) = 0$ for (3.4.8b) corresponding to A_2, and A_2 is now harmonic.

The sum A of A_1 and A_2 is harmonic and is the solution to

$$\alpha A - \underset{\sim}{r}.\underset{\sim}{\nabla} A = \phi(x,y,z).$$

Thus in this case, B_0 can also be omitted which corresponds to that given in [6].

d - If α is non-integer and B_0 is a harmonic polynomial in x,y and z then B_0 can be dropped from (3.1.2) *irrespective* of the form of the material region.

e - If there is a harmonic function B_0 such that

$$\phi(x,y.z) = W(r,\phi,\gamma) = B_0(x,y,z) = r^{\alpha} S_{\alpha}(\theta,\gamma) \tag{3.4.9}$$

then $A^*(x,y,z)$ defined by (3.4.8a) is not differentiable at the origin, and any function A satisfying (3.3.1), differeing from A^* by only a function $r^{\alpha} s(\theta,\gamma)$, is also not differentiable at the origin, hence B_0 can not be dropped from (3.1.2). This leads to the counter example cited in [6] , i.e. the representation (3.1.4) is incomplete when α is an integer (since there exist solid spherical harmonic functions of degree α).

It is interesting to know that the proof of part c of this subsection requires that the $[\alpha]+1$ partial derivatives of a harmonic function $\phi(x,y,z)$ be finite within a small open sphere centered on the origin of the coordinates (x,y,z). A simple proof for that will be reproduced in the following section $\underset{\sim}{3}.\underset{\sim}{5}$.

3.5 Proofs of some properties required in the preceding section

This section provides the proofs for (3.4.6) and for the differentiability of $\phi(x,y,z)$ as mentioned in the last paragraph of the preceding section. These proofs were brought out of the last section solely due to their lengths and their numerous details.

3.5.1 - *Proof for (3.4.6)*

Consider the quantity M, defined by

$$M = \nabla^2 * \left[r^{\alpha} \int_{r_0}^{r} r^{-\alpha-1} W(r,\theta,\gamma)\, dr \right].$$

Since $\nabla^2 *$ is defined in the spherical coordinates as

$$\nabla^2 * = \operatorname{cosec}\gamma \frac{\partial}{\partial \gamma}\left(\sin\gamma \frac{\partial}{\partial \gamma} \right) + \operatorname{cosec}^2\gamma \frac{\partial^2}{\partial \theta^2},$$

the quantity M is rewritten as

$$M = r^{\alpha} \int_{r_0}^{r} r^{-\alpha-1} \nabla^2 * W(r\theta,\gamma\ )\, dr.$$

Using the harmonicity of $W(r,\theta,\gamma)$,

$$\nabla^2 * W(r,\theta,\gamma) = -\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} \right) W(r,\theta,\gamma),$$

it is arrived at

$$M = -r^{\alpha} \int_{r_0}^{r} r^{-\alpha-1} \frac{\partial}{\partial r}\left(r^2 \frac{\partial W}{\partial r} \right) dr.$$

After two successive partial integrations, the quantity M becomes

$$M = -r^{\alpha}\left[r^{-\alpha+1} \frac{\partial W}{\partial_r} \right]_{r_0}^{r} - (\alpha+1) r^{\alpha} \left[r^{-\alpha} W \right]_{r_0}^{r} - \alpha(\alpha+1) r^{\alpha} \int_{r_0}^{r} r^{-\alpha-1} W dr.$$

On the other hand, consider the quantity N defined by

$$N = \left[\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} \right) \right] \left[r^{\alpha} \int_{r_0}^{r} r^{-\alpha-1} (r,\theta,\gamma)\, dr \right].$$

Applications of elementary rules for differentiation gives

$$N = \alpha(\alpha+1) r^{\alpha} \int_{r_0}^{r} r^{-\alpha-1} W(r,\theta,\gamma)\, dr + \alpha W(r,\theta,\gamma) + \frac{\partial}{\partial r}(rW).$$

By noting that

$$\nabla^2 \left[r^{\alpha} \int_{r_0}^{r} r^{-\alpha-1} W(r,\theta,\gamma)\, dr \right] = \frac{1}{r^2}(M+N);$$

the final result is arrived at

$$\nabla^2 \left[r^{\alpha} \int_{r_0}^{r} r^{-\alpha-1} W(r,\theta,\gamma)\, dr \right] = r^{\alpha-2} \Big[(\alpha+1) r_0^{-\alpha} W(r_0,\theta,\gamma) + $$
$$+ r_0^{-\alpha+1} \left(\frac{\partial W}{\partial r} \right)_{\text{at } (r_0,\theta,\gamma)} \Big].$$

3.5.2. - *Proof on the differentiability of a harmonic function* ϕ

The proof in this subsection is widely available in standard texts on electrostatics. It is reproduced here *solely for the purpose of comparison* between the method presented in this chapter and that of expanding $\phi(x,y,z)$ into series of spherical harmonic functions.

The harmonic function $\phi(x,y,z) \equiv W(r,\theta,\gamma)$ is defined on the material region, which encloses at least a small closed sphere of surface S and radius R, centered on the origin O. Hence $W(r,\theta,\gamma)$ is defined and finite for all values of θ and γ.

The solution of the Dirichlet problem (Appendix A) gives the value of $\phi(x,y,z)$ on this closed sphere of radius R as

$$\phi(x,y,z) = \frac{1}{4\pi} \int \phi(\xi,\eta,\zeta) \frac{R^2 - (x^2+y^2+z^2)}{R[(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2]^{3/2}} dS(\xi,\eta,\zeta),$$

where the value of $\phi(\xi,\eta,\zeta)$ is that of $W(R,\theta,\gamma)$ since $\xi^2 + \eta^2 + \zeta^2 = R^2$.

The above expression shows that the first derivative of $\phi(x,y,z)$ along any direction is defined on the open sphere of radius R, centered on 0.

Choosing R_1 slightly smaller than R, the same reasoning can be used on the sphere R_1 to prove that all second degree partial derivatives of ϕ is defined on the open sphere of radius R_1, centered on 0.

This process can be repeated for any number of times, and R_1 can be chosen as close to R as we like. Thus ϕ has all the partial derivatives up to degree m, where m is a finite positive integer, on the open sphere of radius R, centered on 0.

Hence, the omission of B_0 has been proved, in this chapter, by a method which uses less mathematical requisites than those required by [6].

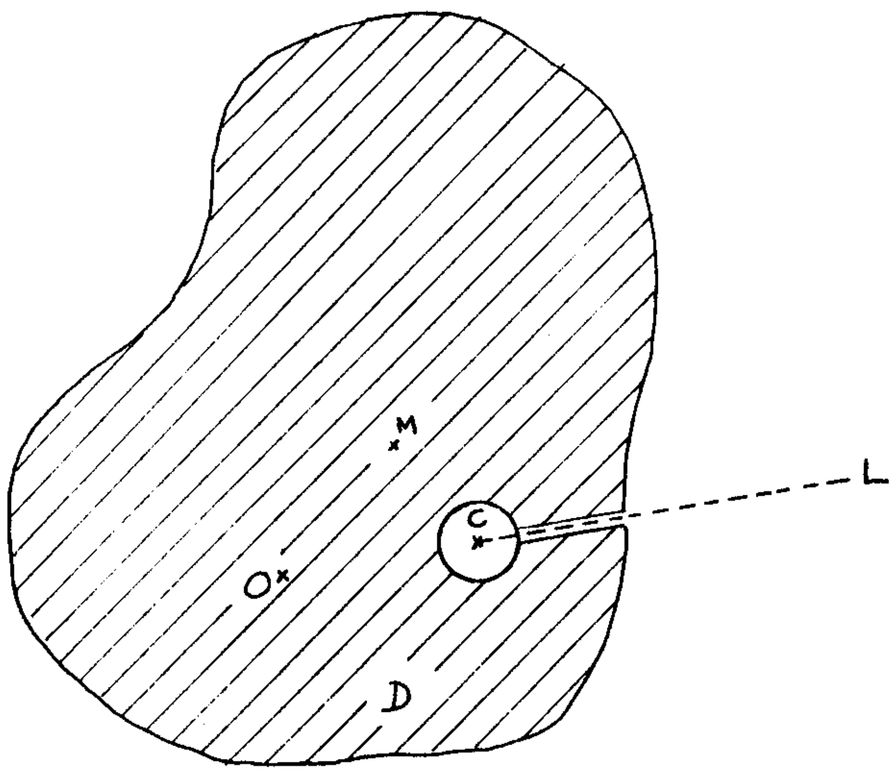


FIGURE 3.1:

AN EXAMPLE OF NON STAR-SHAPED MATERIAL REGIONS.

3.6 Counter example for the three harmonics representation in non-star-shaped regions

It may be thought that the star-shape required in subsection *3.4.2* for the omission of B_0 is artificial and is due to a particular method. This (wrong) observation may thus lead one to capitalise on the use of various different methods aiming at removing the star-shape required in *3.4.1*. This section will give a counter-example on the incompleteness of (3.1.4) in a non-star-shaped region. The counter-example thus serves to demonstrate that the star-shape required in *3.4.1* is inherent to the problem and is not due to any particular chosen method.

Consider now the function $A_1(x,y,z)$ defined by

$$A_1[x(r,\theta,\gamma),\ y(r,\theta,\gamma),\ z(r,\theta,\gamma)] = -r^{\alpha}\int_0^r r^{-\alpha-1}W(r,\theta,\gamma)\,dr, \tag{3.6.1}$$

where $W(r,\theta,\gamma)$ is the spherical form of the function $1/|MC|$, see fig. 3.1. The function so constructed is defined for the region D_L which is the region D minus a half line starting from C pointing away from O. The half line is named L. Close to L, the function A_1 will get infinitely large negative values. Thus, the function A_1 satisfies the system (3.3.1), (3.3.2) and is defined on D_L.

Consider a solution A (if it ever exists) of the system (3.3.1), (3.3.2) such that it is defined in the *whole* region D. The difference between this solution and A_1 in the region D_L is a function named A_2.

A_2 satisfies the homogeneous system of (3.3.1) and (3.3.2). Integration of the *homogeneous* equation (3.3.1) gives

$$A_2 = r^{\alpha}s(\theta,\gamma), \tag{3.6.2}$$

where $s(\theta,\gamma)$ is a function of only θ and γ.

As A_2 is harmonic and defined inside a sphere of radius $|OC|/2$, centered on O, its form, given by (3.6.2), assures that it is defined and harmonic *everywhere* in the three-dimensional space.

Since A_1 is the difference between two functions A and A_2, both of them are harmonic and defined in the *whole* D, A must be defined and harmonic on the *whole* D. This contradicts its singularity property on the half line L.

It is now clear that the star-shape required for the omission of B_0 in (3.1.2) is not a mathematical "over-condition", but is really intrinsic to the problem itself.

It is also noted that the function A_2 can be shown to be identically zero in the case of non-integer α, as in the following.

Since A_2 assumes the form

$$A_2 = r^{\alpha} s(\theta,\gamma),$$

the n^{th} partial derivative of A_2 with respect to r is

$$\frac{\partial^n A_2}{\partial r^n} = \frac{1}{\alpha(\alpha-1)\ldots(\alpha-n+1)}\, r^{+\alpha-n}\, s(\theta,\gamma).$$

It is known in *3.5.2* that A_2 has partial derivatives of any finite order inside a sphere centered on the origin 0. The last equation is contradictory to this property unless $s(\theta,\gamma)$ is identically zero.

Hence A_2 must be identically zero for non-integer α.

3.7 Conclusions

If the aim of mathematics is to find the simplest route to arrive at certain results, then the proofs in this chapter, regarding the omission of B_0, can be considered as having been done just that. The chapter does not present any new results but it does give a new insight to the problem of the omission of B_0. The results of the work done in this chapter can be summarised as in the following:

(a) When the material region in finite and star-shaped with respect to the origin and the value of 4ν is non-integer, then the representation (3.1.4) is complete.

(b) The representation (3.1.4) is incomplete when 4ν has an integer value.

(c) The representation (3.1.4) is incomplete when the material region is not star-shaped.

(d) When the material region is infinite and outside a closed surface such that every point on the surface can be joined to infinity, keeping the values of θ and γ unchanged, without leaving the material region and 4ν is not an integer then every harmonic function B_0 which is finite at infinity can be dropped from (3.1.2).

(e) The surface spherical harmonic functions $S_n(\theta,\gamma)$'s are inherent to the necessary and sufficient conditions for the omission of B_0 in (3.1.2), as manifested by the left hand side of (3.4.8b).

(f) There is an equivalence between the following two systems

$$\alpha \underset{\sim}{C} - \nabla(\underset{\sim}{r}.\underset{\sim}{C}) = \underset{\sim}{\nabla}\phi$$

$$\nabla^2 \underset{\sim}{C} = \underset{\sim}{0}$$

and

$$\alpha A - \underset{\sim}{r}.\underset{\sim}{\nabla} A = \phi,$$

$$\nabla^2 A = 0,$$

where ϕ is harmonic and α is neither 0 nor 2.

(g) The equation

$$\alpha A - x\frac{\partial A}{\partial x} - y\frac{\partial A}{\partial y} - z\frac{\partial A}{\partial z} = \phi(x,y,z)$$

can be integrated, in general, to give the function A explicitly, as in the following

$$A(x,y,z) = -r^{\alpha}\int_{r_0}^{r} r^{-\alpha-1}W(r,\theta,\gamma)\,dr - r^{\alpha}s(\theta,\gamma),$$

where $W(r,\theta,\gamma)$ is the spherical form of $\phi(x,y,z)$ and $s(\theta,\gamma)$ is a function independent of r.

(h) Depending on appropriate conditions being satisfied, the harmonicity of $\phi(x,y,z)$ will entail that of $A(x,y,z)$, in the above paragraph.

(i) When the definition region of ϕ contains a singularity point C, the semi-infinite straight line starting from C pointing away from O is a singularity line for the equation in paragraph (g). The equation does not have a solution such that it is defined wherever ϕ is defined.

CHAPTER 4

The various solutions to elastic problems in cylindrical coordinates

4.1 Introduction

As presented in [6], an application of the Papkovich-Neuber solution to the cylindrical coordinates gives rise to the already known Boussinesq solution [15]. Another application of the same Papkovich-Neuber solution to cylindrical coordinates can also give rise to another form of solution. The latter has already been used by Lur'e [16] although he had no formal proof for it.

This chapter supplies the required proof for the solution used by Lur'e. This solution is then proved to be equivalent to various forms of solution for torsion-free axi-symmetric problems. A short discussion on the choice of one form over the others will also be made at the end of the chapter.

4.2 The cylindrical form used by Lur'e and its origination from the Papkovich-Neuber form.

The solution given by Papkovich [1] to Lamé's equation for the equilibrium of a three dimensional elastic body in the Cartesian coordinate system (x,y,z) is written as

$$U_x = 4(1-\nu)A_x - \frac{\partial}{\partial x}(xA_a + yA_y + zA_z + A_o), \tag{4.2.1a}$$

$$U_y = 4(1-\nu)A_y - \frac{\partial}{\partial y}(xA_x + yA_y + zA_z + A_o), \tag{4.2.1b}$$

$$U_z = 4(1-\nu)A_z - \frac{\partial}{\partial z}(xA_x + yA_y + zA_z + A_o), \tag{4.2.1c}$$

where (U_x,U_y,U_z) is the displacement vector and A_x,A_y,A_z,A_o are four harmonic functions.

When this solution is applied to the case of torsion-free axisymmetric deformations, with z being the axis of symmetry, A_x and A_y can be omitted, as shown by [6], giving the well known Boussinesq solution [15], with A_o, A_z being functions of only r and z. This solution is widely used in elasticity theory due to its generality and simplicity.

On the other hand, when the conditions for the omission of B_z are satisfied (see [6]), the solution for torsion-free axi-symmetric deformations assumes the following form

$$u(r,z) = 4(1-\nu)D(r,z,\theta) - \frac{\partial}{\partial r}[rD(r,z,\theta) + B(r,z,\theta)], \quad (4.2.2a)$$

$$0 = v(r,z) = 4(1-\nu)C(r,z,\theta) - \frac{1}{r}\frac{\partial}{\partial\theta}[rD(r,z,\theta) + B(r,z,\theta)], \quad (4.2.2b)$$

$$w(r,z) = -\frac{\partial}{\partial z}[rD(r,z,\theta) + B(r,z,\theta)], \quad (4.2.2c)$$

where $B(r,z,\theta)$ is the cylindrical form of $A_o(x,y,z)$; $D(r,z,\theta)$ and $C(r,z,\theta)$ are given from $A_x(x,y,z)$ and $A_y(x,y,z)$ as

$$A_x(x,y,z) + iA_y(x,y,z) = [D(r,z,\theta) + iC(r,z,\theta)]e^{i\theta}, \quad (4.2.3)$$

and (u,v,w) is the displacement vector in cylindrical coordinates.

The difficulty involved in the use of this form of solution is the dependence of B, C and D on θ, a point not noted by previous authors on this subject.

In this chapter it will be proved that there exist other functions $d_o(r,z)$ and $b_o(r,z)$ such that

$$u(r,z) = 4(1-\nu)d_o(r,z) - \frac{\partial}{\partial r}[rd_o(r,z) + b_o(r,z)], \quad (4.2.4a)$$

$$v(r,z) = 0, \quad (4.2.4b)$$

$$w(r,z) = -\frac{\partial}{\partial z}[rd_o(r,z) + b_o(r,z)], \quad (4.2.4c)$$

and

$$\nabla^2 d_o(r,z) = \frac{d_o(r,z)}{r^2}, \quad (4.2.5a)$$

$$\nabla^2 b_o(r,z) = 0. \quad (4.2.5b)$$

This representation for displacements will be called the alternagive form to the Boussinesq form for torsion-free axi-symmetric problems. The representation has already been used by Lur'e in his book [16], but no convincing derivation of it has been offered, and thus it has not previously been systematically incorporated into the Papkovich-Neuber solution.

Once this representation is derived, it can be readily transformed to and from its various equivalents, namely those associated with Boussinesq [15], Sadowsky [17], Timpe [32] and Love [18].

4.3 Displacements in terms of two functions, independent of θ.

The application of Papkovich-Neuber solution to torsion-free axi-symmetric deformations give rise to (4.2.2), with $C(r,z,\theta)$ and $D(r,z,\theta)$ defined by (4.2.3).

Since A_x and A_y are harmonic, C and D must satisfy

$$\nabla^2\left[(D + iC)e^{i\theta} \right] = 0,$$

or

$$\nabla^2 D - \frac{D}{r^2} - \frac{2}{r^2}\frac{\partial C}{\partial \theta} = 0,$$

$$\nabla^2 C - \frac{C}{r^2} + \frac{2}{r^2}\frac{\partial D}{\partial \theta} = 0.$$

Using the equations (4.2.2) , the above two equations, together with the equation for $B(r,z,\theta)$, can be written as

$$\nabla^2 D(r,z,\theta) = \frac{D}{r^2} + \frac{1}{2(1-\nu)r^3}\frac{\partial^2}{\partial \theta^2}(rD + B), \tag{4.3.1a}$$

$$\nabla^2 B(r,z,\theta) = 0, \tag{4.3.1b}$$

$$\nabla^2 C(r,z,\theta) = -\frac{C}{r^2} - \frac{2}{r}\frac{\partial C}{\partial r}. \tag{4.3.1c}$$

Using the equations (4.2.2c), (4.2.2a) and (4.2.2b), in this order, it is straightforward to prove that

$$\frac{\partial}{\partial \theta}(rD + B) = \text{function of } (r,\theta),$$

$$\frac{\partial D}{\partial \theta} = \text{function of } (r,\theta),$$

$$rC = \text{function of } (r,\theta).$$

As a result,

$$\frac{\partial B}{\partial \theta} = \text{function of } (r,\theta).$$

Hence, B, C and D can be rewritten as

$$D(r,z,\theta) = d(r,\theta) + d_1(r,z), \tag{4.3.2a}$$

$$B(r,z,\theta) = b(r,\theta) + b_2(r,z), \tag{4.3.2b}$$

$$C(r,z,\theta) = \frac{1}{r}\, c(r,\theta). \tag{4.3.2c}$$

By the use of (4.2.2) and (4.3.2), it can be shown that

$$\frac{\partial c(r,\theta)}{\partial r} = \frac{\partial d(r,\theta)}{\partial \theta}.$$

This equation establishes the existence of a potential function $\phi(r,\theta)$ defined by

$$\phi(r,\theta) = \int_{(r_0,\theta_0)}^{(r,\theta)} [d(r,\theta)dr + c(r,\theta)d\theta],$$

where r_0,θ_0 are arbitrary reference values. $\phi(r,\theta)$ has the property that

$$\frac{\partial\phi\ (r,\theta)}{\partial r} = d(r,\theta),$$

and

$$\frac{\partial\phi(r,\theta)}{\partial\theta} = c(r,\theta).$$

With this newly defined function ϕ, the functions D and C can be rewritten as

$$D(r,z,\theta) = \frac{\partial\phi(r,\theta)}{\partial r} + d_1(r,z), \tag{4.3.3a}$$

$$C(r,z,\theta) = \frac{1}{r}\,\frac{\partial\phi(r,\theta)}{\partial\theta}. \tag{4.3.3c}$$

Using (4.3.3a), (4.3.3c) and (4.3.2b), $v(r,z)$ can now be written as

$$v(r,z) = \frac{4(1-\nu)}{r}\,\frac{\partial\phi(r,\theta)}{\partial\theta} - \frac{1}{r}\,\frac{\partial}{\partial\theta}\left[r\,\frac{\partial\phi(r,\theta)}{\partial r} + b(r,\theta) \right] \equiv 0.$$

This equation allows $b(r,\theta)$ to be related to $\phi(r,\theta)$ by

$$4(1-\nu)\phi - r\,\frac{\partial\phi}{\partial r} - b = f(r),$$

where $f(r)$ is a function of r only. Thus $B(r,z,\theta)$ can now be written as

$$B(r,z,\theta) = 4(1-\nu)\phi(r,\theta) - r\frac{\partial\phi(r,\theta)}{\partial r} - f(r) + b_2(r,z),$$

or

$$B(r,z,\theta) = 4(1-\nu)\phi(r,\theta) - r\,\frac{\partial\phi(r,\theta)}{\partial r} + b_1(r,z), \tag{4.3.3b}$$

with $b_1(r,z)$ equal to the sum of $-\ f(r)$ and $b_2(r,z)$.

Equations (4.3.3) allow displacements to be written as

$$u(r,z) = 4(1-\nu)d_1(r,z) - \frac{\partial}{\partial r}\,[rd_1(r,z)+b_1(r,z)], \tag{4.3.4a}$$

$$v(r,z) = 0, \tag{4.3.4b}$$

$$w(r,z) = -\frac{\partial}{\partial z}[rd_1(r,z) + b_1(r,z)]. \tag{4.3.4c}$$

It should be noted that no attempt has been made to demonstrate that $d_1(r,z)$ and $b_1(r,z)$ are harmonic.

4.4 Displacements in terms of one harmonic and another harmonic-related functions.

Equations (4.3.1c) and (4.3.3c) give

$$\nabla^2\left[\frac{1}{r}\frac{\partial\phi(r,\theta)}{\partial\theta}\right] + \frac{1}{r^3}\frac{\partial\phi(r,\theta)}{\partial\theta} + \frac{2}{r}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial\phi(r,\theta)}{\partial\theta}\right] = 0,$$

or $\quad \dfrac{\partial}{\partial\theta}\nabla^2\phi(r,\theta) = 0.$

This implies

$$\nabla^2\phi(r,\theta) = h(r),$$

where $h(r)$ is a function of only r.

Another function $\psi(r,\theta)$, which is harmonic, can be defined by

$$\psi(r,\theta) = \phi(r,\theta) - \int_{r_0}^{r}\frac{1}{r}\int_{r_0}^{r} rh(r)dr\, dr. \tag{4.4.1}$$

Hence, D,B and C can be rewritten as

$$D(r,z,\theta) = \frac{\partial\psi(r,\theta)}{\partial r} + d_o(r,z), \tag{4.4.2a}$$

$$B(r,z,\theta) = 4(1-\nu)\psi(r,\theta) - r\frac{\partial\psi(r,\theta)}{\partial r} + b_o(r,z), \tag{4.4.2b}$$

$$C(r,z,\theta) = \frac{1}{r}\frac{\partial\psi(r,\theta)}{\partial\theta}, \tag{4.4.2c}$$

where the functions $d_o(r,z)$, $b_o(r,z)$ are given by

$$d_o(r,z) = \frac{1}{r}\int_{r_0}^{r} rh(r)dr + d_1(r,z)$$

and

$$b_o(r,z) = 4(1-\nu)\int_{r_0}^{r}\frac{1}{r}\int_{r_0}^{r} rh(r)drdr - \int_{r_0}^{r} rh(r)dr + b_1(r,z).$$

Displacements can now be written in the desired form (4.2.4), which is

$$u(r,z) = 4(1-\nu)\, d_o(r,z) - \frac{\partial}{\partial r}[rd_o(r,z) + b_o(r,z)], \tag{4.2.4a}$$

$$v(r,z) = 0, \tag{4.2.4b}$$

$$W(r,z) = -\frac{\partial}{\partial z}\,[rd_o(r,z) + b_o(r,z)]. \tag{4.2.4c}$$

Using (4.3.1a), (4.3.1b) together with (4.4.2c), (4.4.2b) it can be shown that

$$\nabla^2 d_o - \frac{d_o}{r^2} + \left[\frac{\partial}{\partial r}\nabla^2\psi\right] = 0,$$

$$\nabla^2 b_o + \left[4(1-\nu)\nabla^2\psi - \frac{1}{r}\frac{\partial}{\partial r}(r^2\nabla^2\psi)\right] = 0.$$

Knowing that ψ is harmonic, the above equations can be rewritten in the form of (4.2.5), which is

$$\nabla^2 d_o(r,z) = \frac{d_o(r,z)}{r^2} \tag{4.2.5a}$$

$$\nabla^2 b_o(r,z) = 0. \tag{4.2.5b}$$

Thus, the displacements in torsion-free axi-symmetric deformation are expressible by (4.2.4) with $d_o(r,z)$ and $b_o(r,z)$ satisfying (4.2.5), when the conditions for the omission of A_z in the Papkovich-Neuber general solution are satisfied.

4.5 Alternative form of (4.2.4) and (4.2.5).

If displacements are given by the equations (4.2.4) and (4.2.5), then a function $a(r,z)$ can be defined by

$$a(r,z) = \int_{r}^{r} d_o(r,z)\,dr,$$

where r_1 is an arbitrary reference value. Hence

$$\frac{\partial}{\partial r}\nabla^2 a(r,z) = \nabla^2\frac{\partial a}{\partial r} - \frac{1}{r^2}\frac{\partial a}{\partial r}$$

$$= \nabla^2 d_o - \frac{d_o}{r^2} = 0.$$

This leads to

$$\nabla^2 a(r,z) = p(z),$$

where $p(z)$ is a function of only z.

Define the function $a_o(r,z)$ by

$$a_o(r,z) = a(r,z) - \int_{z_1}^{z}\int_{z_1}^{z} p(z)\,dz\,dz,$$

where z_1 is an arbitrary reference value.

Then

$$u = -\frac{\partial}{\partial r}\left[-4(1-\nu)a_o + r\frac{\partial a_o}{\partial r} + b_o\right], \tag{4.5.1a}$$

$$v = 0, \tag{4.5.1b}$$

$$w = -\frac{\partial}{\partial z}\left[r\frac{\partial a_o}{\partial r} + b_o\right], \tag{4.5.1c}$$

with

$$\nabla^2 a_o = 0, \tag{4.5.2a}$$

$$\nabla^2 b_o = 0. \tag{4.5.2b}$$

On the other hand, if displacements are given by (4.5.1) and (4.5.2), by defining

$$d_o(r,z) = \frac{\partial}{\partial r} a_o(r,z),$$

then displacements are also expressible by (4.2.4) and (4.2.5).

Thus (4.5.1) and (4.5.2) are an equivalent form of (4.2.4) and (4.2.5).

A trivial variation of (4.5.1) and (4.5.2) gives the following form

$$u = \frac{\partial}{\partial r}\left[r\frac{\partial \phi_1}{\partial r} + \psi_1\right], \tag{4.5.3a}$$

$$w = \frac{\partial}{\partial r}\left[4(1-\nu)\phi_1 + r\frac{\partial \phi_1}{\partial r} + \psi_1\right], \tag{4.5.3b}$$

with

$$\nabla^2 \psi_1 = 0, \tag{4.5.4a}$$

$$\nabla^2 \phi_1 = 0. \tag{4.5.4b}$$

This latter form is thought to have been used by Sadowsky in his unpublished lecture notes [17].

4.6 Relationship between (4.5.1), (4.5.2) and the Boussinesq solution.

As mentioned in section 4.2, the Boussinesq solution and the solution of the form (4.5.1), (4.5.2) can be obtained by zeroing different harmonic functions in the Papkovich-Neuber solution. However, these two forms, obtainable along two different paths, can be conveniently linked to each other as in the following.

Assuming that displacements are given by (4.5.1) and (4.5.2), define $E(r,z)$ and $F(r,z)$ by

$$E(r,z) = -\frac{\partial a_o}{\partial z},$$

$$F(r,z) = r\frac{\partial a_o}{\partial r} + z\frac{\partial a_o}{\partial z} + b_o - 4(1-\nu)a_o.$$

Then, by the use of the above equations together with (4.5.1), displacements can be written as

$$u(r,z) = -\frac{\partial}{\partial r}\left[zE + F\right], \tag{4.6.1a}$$

$$w(r,z) = 4(1-\nu)E - \frac{\partial}{\partial z}\left[zE + F\right] \tag{4.6.1b}$$

Using (4.5.2), it is only routine calculation to show that E and F are harmonic, i.e.,

$$\nabla^2 E(r,z) = 0, \tag{4.6.2a}$$

$$\nabla^2 F(r,z) = 0. \tag{4.6.2b}$$

On the other hand, if displacements are given by (4.6.1) and (4.6.2), then by defining

$$a(r,z) = -\int_{z_1}^{z} E(r,z)dz,$$

it can be shown that

$$\frac{\partial}{\partial z}\nabla^2 a(r,z) = 0, \text{ by virtue of (4.6.2a),}$$

or

$$\nabla^2 a(r,z) = q(r),$$

where $q(r)$ is a function of only r.

Define another function

$$a_o(r,z) = a(r,z) - \int_{r_1}^{r}\frac{1}{r}\int_{r_1}^{r} rq(r)drdr,$$

which is harmonic, and another one

$$b_o(r,z) = F(r,z) + 4(1-\nu)a_o + r\frac{\partial a_o}{\partial r} + z\frac{\partial a_o}{\partial z}$$

which is also harmonic by (4.6.2) and the harmonicity of a_o.

Using (4.6.1), it is trivial to show that displacements are also expressible by (4.5.1), and since a_o and b_o are harmonic, (4.5.2) is satisfied.

4.7 Relationship between the Boussinesq solution and the Love solution.

The Boussinesq solution and the Love solution were derived by two independent methods ([15], [18]). The relationship between them, as established in the following, enables one form of solution to be transformed to and from the other. Eventually each form of solution presented in this chapter can be transformed to and from another.

Assuming that displacements are given by (4.6.1) and (4.6.2), define χ_0 by

$$\chi_0(r,z) = \int_{z_1}^{z} (zE + F)dz,$$

then it can be shown that

$$\frac{\partial}{\partial z}\nabla^2\chi_0 = 2\frac{\partial E}{\partial z}.$$

The above equation implies

$$\nabla^2\chi_0 = 2E + s(r),$$

where $s(r)$ is a function of only r.

Define another function

$$\chi = \chi_0 - \int_{r_1}^{r} \frac{1}{r}\int_{r_1}^{r} rs(r)drdr.$$

Then χ has the property of

$$\frac{\partial\chi}{\partial z} = zE + F,$$

and $\nabla^2\chi = 2E.$

Hence displacements are given, using the above two equations in conjunction with (4.6.1), by

$$u = \frac{\partial^2\chi}{\partial r\partial z}, \tag{4.7.1a}$$

$$w = 2(1-\nu)\nabla^2\chi - \frac{\partial^2\chi}{\partial z^2}. \tag{4.7.1b}$$

Since E is harmonic, the equation relating χ and E gives

$$\nabla^4\chi = 0. \tag{4.7.2}$$

On the other hand, when displacements are given by (4.7.1) and (4.7.2), the functions E and F defined by

$$E = \frac{1}{2}\nabla^2\chi,$$

$$F = \frac{\partial \chi}{\partial z} - \frac{1}{2} z\nabla^2\chi,$$

can obviously satisfy (4.6.1). They are also harmonic, due to χ being biharmonic, satisfying (4.6.2).

4.8 Equivalence between (4.5.3), (4.5.4) and the Timpe solution.

The transformation from Timpe's two stress functions to Boussinesq's two stress functions has been established in [20] under appropriate conditions. However this is not sufficient for the equivalence between the two sets of stress functions. Thus it has not been shown that the Timpe form of solution can be derived from the Boussinesq form or to and from any of the forms appearing in the previous sections of this chapter.

This section will show that the Timpe solution is equivalent to the solution expressed by (4.5.3) and (4.5.4), and thus to all the solutions appearing in this chapter.

Displacements in terms of Timpe's two functions G and H are

$$u = \frac{\partial H}{\partial r} - 4(1-\nu)\frac{H}{r} + \frac{G}{r}, \tag{4.8.1a}$$

$$v = 0, \tag{4.8.1b}$$

$$w = \frac{\partial H}{\partial z} + \frac{\partial G}{\partial r}, \tag{4.8.1c}$$

with G and H satisfying the following requirements

$$\nabla^2 G(r,z) = 0, \tag{4.8.2a}$$

$$\nabla^2 H(r,z) = \frac{2}{r}\frac{\partial H}{\partial r}. \tag{4.8.2b}$$

It will be first proved that any displacements expressible by (4.8.1) and (4.8.2) are also expressible by (4.5.3) and (4.5.4). To this end, it is only necessary to find ϕ_1 and ψ_1 satisfying

$$\nabla^2\phi_1(r,z) = 0, \tag{4.8.3a}$$

$$\nabla^2\psi_1(r,z) = 0, \tag{4.8.3b}$$

and

$$\frac{\partial}{\partial r}\left(r\frac{\partial \phi_1}{\partial r} + \psi_1\right) = \frac{\partial H}{\partial r} - 4(1-\nu)\frac{H}{r}, \tag{4.8.4a}$$

$$\frac{\partial}{\partial z}\left[\;4(1-\nu)\phi_1 + r\,\frac{\partial \phi_1}{\partial r} + \psi_1\;\right] = \frac{\partial H}{\partial z}, \tag{4.8.5b}$$

as the resulting ψ_1 can be increased by G to satisfy (4.8.3) and (4.8.4).

The functions ϕ_2 and ψ_2 defined below will satisfy the above equations (4.8.5) if they take the places of ϕ_1 and ψ_1,

$$\phi_2 = \int_{r_o}^{r} \frac{H}{r}\,dr, \tag{4.8.6a}$$

$$\psi_2 = -\,4(1-\nu)\int_{r_o}^{r} \frac{H}{r}\,dr, \tag{4.8.6b}$$

where r_o is an arbitrary datum.

Using (4.8.6a) and (4.8.6b), it can be shown that

$$\frac{\partial}{\partial r}\left(\nabla^2\phi_2\right) = 0,$$

or $\quad \nabla^2\phi_2(r,z) = t(z),$

where $t(z)$ is a function of only z.

Now consider the functions ϕ_1 and ψ_1 defined in the following

$$\phi_1 = \phi_2 - \int_{z_o}^{z}\int_{z_o}^{z} t(z)\,dz\,dz, \tag{4.8.7a}$$

$$\psi_1 = -\,4(1-\nu)\phi_1, \tag{4.8.7b}$$

where z_o is an arbitrary datum. These functions obviously satisfy (4.8.5) and also (4.8.3).

Thus, any displacements expressible by (4.8.1) and (4.8.2) are also expressible by (4.5.3) and (4.5.4).

It will next be proved that any displacements expressible by (4.5.3) and (4.5.4) are also expressible by (4.8.1) and (4.8.2). For this aim, it is only necessary to find G and H satisfying (4.8.2) and

$$\frac{\partial H}{\partial r} - 4(1-\nu)\frac{H}{r} + \frac{\partial G}{\partial r} = \frac{\partial}{\partial r}\left(r\frac{\partial \phi_1}{\partial r} + \psi_1\right), \tag{4.8.8a}$$

$$\frac{\partial H}{\partial z} + \frac{\partial G}{\partial z} = \frac{\partial}{\partial z}\left[\;4(1-\nu)\phi_1 + r\,\frac{\partial \phi_1}{\partial r} + \psi_1\right]. \tag{4.8.8b}$$

Again, for the same reason as in the last proof, it is only necessary to find G and H satisfying (4.8.2) and

$$\frac{\partial H}{\partial r} - 4(1-\nu)\frac{H}{r} + \frac{\partial G}{\partial r} = \frac{\partial}{\partial r}\left(r\,\frac{\partial \phi_1}{\partial r}\right), \tag{4.8.9a}$$

$$\frac{\partial H}{\partial z} + \frac{\partial G}{\partial z} = \frac{\partial}{\partial z}\left[\,4(1-\nu)\phi_1 + r\,\frac{\partial \phi_1}{\partial r}\right]. \tag{4.8.9b}$$

The functions G and H defined by

$$G = 4(1-\nu)\phi_1, \tag{4.8.10a}$$

and

$$H = r\,\frac{\partial \phi_1}{\partial r}, \tag{4.8.10b}$$

obviously satisfy (4.8.9). Using the harmonicity of ϕ_1 and ψ_1, as given by (4.5.4), it is only straightforward calculation to show that G and H satisfy the equation (4.8.2).

Thus, any displacements expressible by (4.5.3) and (4.5.4) are also expressible by (4.8.1) and (4.8.2).

To this point, it has been established that any of the solution forms appearing in this and the previous sections of the chapter can be analytically transformed to and from any other. The actual transformations may be long and tedious, but for a solution given in any of the above forms, its equivalent in other forms can certainly be derived.

4.9 The relative values of different approaches.

The solution form (4.2.4) and (4.2.5), formally derived in this chapter has been successfully used by Lur'e [16] for the axi-symmetric problems of a cylinder. Boussinesq's solution has also been used in the same book [16], p. 391, this solution has an advantage of having z multiplying one harmonic function, an example where this is used to its advantage can be found in [22], p. 280. The form given by Sadowsky [(4.5.3) and (4.5.4)] has been fruitful in the variational approach to the end problem of a cylinder [21]. Timpe's solution is rarely used but Love's biharmonic function is unquestionably very familiar and is widely adopted in various works and texts, such as [23].

A simple guide in choosing a particular solution for a given problem can be expressed as:

a/Compatibility conditions being satisfied. (All the methods considered in this chapter do meet this requirement).

b/Simplicity of the form of the solution.

c/Separation of a dimension from the rest.

d/Ease of adopting the solution form to a particular method (as illustrated in [21]).

A few more points relating to the choice of the various forms can also be found in [41].

However, it is apparent that there is no rigid guide in evaluating a method for a particular problem. The choice is almost entirely dependent on the ingenuity of users.

4.10 Conclusions.

When the conditions for the omission of A_z in the Papkovich-Neuber solution [(4.2.1) and (4.2.2)] are satisfied, displacements in torsion-free axi-symmetric problems are expressible in terms of two functions $d_o(r,z)$ and $b_o(r,z)$ in the manner specified by (4.2.4) and (4.2.5). The conditions for this representation are thus more restricted than the corresponding ones for Boussinesq representation.

The various representations of [(4.2.4), (4.2.5)], of [(4.5.1), (4.5.2)], of Boussinesq [(4.6.1), (4.6.2)], of Love [(4.7.1), (4.7.2)], and of Timpe [(4.8.1), (4.8.2)] are all interconnected. When the transformations between any two of the above forms are possible, the two forms are equivalent and if one is complete, so is the other.

For displacements which are expressible in more than one of the equivalent representations, the choice of one over the others depends only on its convenience.

CHAPTER 5

The application of Papkovich-Neuber solution to plane strain problems.

5.1 Introduction.

For plane strain problems, it is always convenient to work with Airy biharmonic stress function, or the complex variable method. The transformation between these two methods is standard and can be found in standard texts, such as [23], p. 175-179.

This chapter will show that the Airy stress function can be transformed into, or derived from, a Papkovich-Neuber solution, as applied to plane strain cases. The two transformations presented in this chapter also allow the omission of one of the three harmonic functions in the Papkovich-Neuber solution for plane strain.

Compared to the Airy stress function, the Papkovich-Neuber form is less convenient to work with, however it can be extended to three dimensions while the other can not. The transformation from an Airy stress function to the Papkovich-Neuber functions is thus the first step toward extending two-dimensional problems to three-dimensional ones, as will be shown in Chapter 7.

5.2 Derivation of a Papkovich-Neuber solution from a given Airy stress function.

Given an Airy biharmonic stress function $\phi(x,y)$ satisfying

$$\nabla^2(\nabla^2\phi) = 0, \tag{5.2.1}$$

the gradients of displacements can be written as

$$\frac{E}{1+\nu}\frac{\partial u}{\partial x} = -\frac{\partial^2\phi}{\partial x^2} + (1-\nu)\nabla^2\phi, \tag{5.2.2a}$$

$$\frac{E}{1+\nu}\frac{\partial v}{\partial y} = -\frac{\partial^2\phi}{\partial y^2} + (1-\nu)\nabla^2\phi. \tag{5.2.2b}$$

It has been shown ([23], p. 173-174) that any biharmonic function $\phi(x,y)$ is expressible as

$$\phi(x,y) = y\xi(x,y) + \eta(x,y), \tag{5.2.3}$$

where

$$\nabla^2\xi(x,y) = 0 \tag{5.2.4a}$$

$$\nabla^2\eta(x,y) = 0. \tag{5.2.4b}$$

The equation (5.2.2) can now be written as

$$\frac{E}{1+\nu}\frac{\partial u}{\partial x} = -\frac{\partial^2}{\partial x^2}(y\xi+\eta) + 2(1-\nu)\frac{\partial \xi}{\partial y}, \tag{5.2.5a}$$

$$\frac{E}{1+\nu}\frac{\partial v}{\partial y} = -\frac{\partial^2}{\partial y^2}(y\xi+\eta) + 2(1-\nu)\frac{\partial \xi}{\partial y}. \tag{5.2.5b}$$

Since

$$\frac{\partial \xi}{\partial y} = \left(\frac{\partial \xi}{\partial y}\right)_{y=c} + \int_c^y \frac{\partial^2 \xi}{\partial y^2}\,dy = \left(\frac{\partial \xi}{\partial y}\right)_{y=c} - \int_c^y \frac{\partial^2 \xi}{\partial x^2}\,dy,$$

where c is a constant,

$\frac{\partial u}{\partial x}$ can be written as

$$\frac{E}{1+\nu}\frac{\partial u}{\partial x} = -\frac{\partial^2(y\xi+\eta)}{\partial x^2} - 2(1-\nu)\frac{\partial}{\partial x}\int_0^y \frac{\partial \xi}{\partial x}\,dy + 2(1-\nu)\left(\frac{\partial \xi}{\partial y}\right)_{y=c}.$$

Integration of the last equation, and of (5.2.5b) gives

$$\frac{E}{1+\nu}u = -\frac{\partial}{\partial x}(y\xi+\eta) - 2(1-\nu)\int_c^y \frac{\partial \xi}{\partial x}\,dy + 2(1-\nu)\int\left(\frac{\partial \xi}{\partial y}\right)_{y=c} dx + g(y), \tag{5.2.6a}$$

$$\frac{E}{1+\nu}v = -\frac{\partial}{\partial y}(y\xi+\eta) + 2(1-\nu)\xi + f(x). \tag{5.2.6b}$$

On the other hand, the relation between $\phi(x,y)$ and τ_{xy} gives

$$\frac{E}{2(1+\nu)}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = -\frac{\partial^2 \phi}{\partial x \partial y}. \tag{5.2.7}$$

Replacing the values of u, v, ϕ as given by (5.2.6) and (5.2.3), into equation (5.2.7), it is shown that

$$\frac{\partial g(y)}{\partial y} + \frac{\partial f(x)}{\partial x} = 0,$$

or

$$f(x) = ax + b,$$

$$g(y) = -ay + d.$$

The last two equations show that f(x) and g(y) represent only rigid body displacements, hence they can be safely omitted from (5.2.6).

After the omission of f(x) and g(y), equations (5.2.6) can be written as

$$\frac{E}{1+\nu}\,u = -\frac{\partial}{\partial x}\left[y\xi + \eta + 2(1-\nu)\int_c^y \xi dy - 2(1-\nu)\iint \left(\frac{\partial \xi}{\partial y}\right)_{y=c} dxdx\right] \tag{5.2.8a}$$

$$\frac{E}{1+\nu}\,v = 4(1-\nu)\xi - \frac{\partial}{\partial y}\left[y\xi + \eta + 2(1-\nu)\int_c^y \xi dy - 2(1-\nu)\iint \left(\frac{\partial \xi}{\partial y}\right)_{y=c} dxdx\right]. \tag{5.2.8b}$$

Since

$$\nabla^2\left[\int_c^y \xi dy - \iint\left(\frac{\partial \xi}{\partial y}\right)_{y=c} dxdx\right] = \frac{\partial \xi}{\partial y} + \int_c^y \frac{\partial^2 \xi}{\partial x^2}\,dy - \left(\frac{\partial \xi}{\partial y}\right)_{y=c}$$

$$= \left(\frac{\partial \xi}{\partial y}\right)_{y=c} + \int_c^y \nabla^2 \xi\,dy - \left(\frac{\partial \xi}{\partial y}\right)_{y=c} = 0,$$

a new harmonic function ψ can be defined as

$$\psi(x,y) = \eta + 2(1-\nu)\int_c^y \xi dy - 2(1-\nu)\iint\left(\frac{\partial \xi}{\partial y}\right)_{y=c} dxdx. \tag{5.2.9}$$

Hence, displacements are expressible by the Papkovich-Neuber form as

$$u(x,y) = -\frac{\partial}{\partial x}\left[y\left(\frac{1+\nu}{E}\xi\right) + \left(\frac{1+\nu}{E}\psi\right)\right],$$

$$v(x,y) = 4(1-\nu)\left(\frac{1+\nu}{E}\xi\right) - \frac{\partial}{\partial y}\left[y\left(\frac{1+\nu}{E}\xi\right) + \left(\frac{1+\nu}{E}\psi\right)\right].$$

With the definitions of

$$A_y = \frac{1+\nu}{E}\,\xi,$$

and

$$A_0 = \frac{1+\nu}{E}\,\psi,$$

displacements can be written as

$$u(x,y) = -\frac{\partial}{\partial x}(yA_y + A_0), \tag{5.2.10a}$$

$$v(x,y) = 4(1-\nu)A_y - \frac{\partial}{\partial y}(yA_y + A_0), \tag{5.2.10b}$$

where A_y, A_0 satisfy

$$\nabla^2 A_y(x,y) = 0, \tag{5.2.11a}$$

and

$$\nabla^2 A_0(x,y) = 0. \tag{5.2.11b}$$

It is noted that simple (harmonic) terms can be added to A_0, A_y to take up any solid body displacements in the plane (x,y). Thus the Papkovich-Neuber form (5.2.10) and (5.2.11) can express any displacements of the given Airy stress function (5.2.1) after the some slight modification of the functions A_y, A_0.

5.3 Derivation of an Airy stress function from a given Papkovich-Neuber solution.

If a plane strain problem has its displacements given as

$$u(x,y) = -\frac{\partial}{\partial x}\left[yA_y + A_0\right], \tag{5.3.1a}$$

$$v(x,y) = 4(1-\nu)A_y - \frac{\partial}{\partial y}\left[yA_y + A_0\right], \tag{5.3.1b}$$

where

$$\nabla^2 A_0(x,y) = 0,$$

$$\nabla^2 A_y(x,y) = 0,$$

then by defining a function B(x,y),

$$B(x,y) = A_0(x,y) - 2(1-\nu)\int_c^y A_y dy + 2(1-\nu)\iint\left(\frac{\partial A_y}{\partial y}\right)_{y=c} dxdx \tag{5.3.2}$$

which is harmonic, u and v can be written as

$$u = -2(1-\nu)\int_c^y \frac{\partial A_y}{\partial x} dy + 2(1-\nu)\int\left(\frac{\partial A_y}{\partial y}\right)_{y=c} dx - \frac{\partial}{\partial x}(yA_y + B)$$

$$v = 2(1-\nu)A_y - \frac{\partial}{\partial y}(yA_y + B).$$

The last two equations and the harmonicity of A_y and B give

$$\frac{\partial u}{\partial x} = 2(1-\nu)\frac{\partial A_y}{\partial y} - \frac{\partial^2}{\partial x^2}(yA_y + B), \tag{5.3.3a}$$

$$\frac{\partial v}{\partial y} = 2(1-\nu)\frac{\partial A_y}{\partial y} - \frac{\partial^2}{\partial y^2}(yA_y + B), \tag{5.3.3b}$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2\frac{\partial^2}{\partial x \partial y}(yA_y + B). \tag{5.3.3c}$$

If the quantity $yA_y + B$ is named $\phi(x,y)$, it is obvious that

$$\nabla^2\left[\nabla^2\phi(x,y)\right] = 0,$$

and

$$\frac{\partial u}{\partial x} = -\frac{\partial^2\phi}{\partial x^2} + (1-\nu)\nabla^2\phi,$$

$$\frac{\partial v}{\partial y} = -\frac{\partial^2\phi}{\partial y^2} + (1-\nu)\nabla^2\phi,$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2\frac{\partial^2\phi}{\partial x\partial y} .$$

For plane strain problem, the relations between stresses and strains are

$$\frac{1+\nu}{E}\sigma_x = \frac{\partial u}{\partial x} + \frac{\nu}{1-2\nu}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) ,$$

$$\frac{1+\nu}{E}\sigma_y = \frac{\partial v}{\partial y} + \frac{\nu}{1-2\nu}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) ,$$

$$\frac{2(1+\nu)}{E}\tau_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} .$$

The last six equations give

$$\sigma_{xx} = \frac{\partial^2}{\partial y^2}\left(\frac{E}{1+\nu}\phi\right) , \tag{5.3.4a}$$

$$\sigma_{yy} = \frac{\partial^2}{\partial x^2}\left(\frac{E}{1+\nu}\phi\right) , \tag{5.3.4b}$$

$$\tau_{xy} = -\frac{\partial^2}{\partial x\partial y}\left(\frac{E}{1+\nu}\phi\right) . \tag{5.3.4c}$$

The function $\frac{E}{1+\nu}\phi$ is also biharmonic, i.e.

$$\nabla^2\left\{\nabla^2\left[\frac{E}{1+\nu}\phi(x,y)\right]\right\} = 0, \tag{5.3.5}$$

hence it is the required Airy stress function.

It should be noted that the accepted Papkovich-Neuber form for plane strain problem is

$$u = 4(1-\nu)A_x - \frac{\partial}{\partial x}(xA_x + yA_y + A_0), \tag{5.3.6a}$$

$$v = 4(1-\nu)A_y - \frac{\partial}{\partial y}(xA_x + yA_y + A_0), \tag{5.3.6b}$$

with

$$\nabla^2 A_x(x,y) = 0, \tag{5.3.7a}$$

$$\nabla^2 A_y(x,y) = 0, \tag{5.3.7b}$$

$$\nabla^2 A_0(x,y) = 0. \tag{5.3.7c}$$

The treatment in this section gives an individual Airy stress function for each of the functions A_x, A_y, A_0; the sum of all the three stress functions is the Airy stress function for the displacements described by (5.3.6).

Generally, for three given harmonic Papkovich-Neuber functions A_x, A_y, A_0, an Airy stress function is generated, from this function two new Papkovich-Neuber harmonic functions A^*_y, A^*_0 are derived. The last two functions give the same displacements as the given three, except for solid body displacements of the form

$$u = ay + b, \tag{5.3.8a}$$

$$v = -ax + d. \tag{5.3.8b}$$

As the two functions A^*_y, A^*_0 can be easily modified into two harmonic functions $A^*_y{}^*$, $A^*_0{}^*$ to absorb the solid body displacements given by (5.3.8), these final functions $A^*_y{}^*$, $A^*_0{}^*$ then give the displacements described by (5.3.6).

The above observations prove that in general, one of the functions A_x, A_y in the Papkovich-Neuber form (5.3.6) for plane strain can be set to zero without losing the completeness of the form.

5.4 An example on the derivation of the Papkovich-Neuber solution from a given Airy stress function.

In this section, a derivation of the Papkovich-Neuber solution from a known Airy stress function is made. The derived functions will be of good help in comparing the results in section 7.5, which is an extension of a plane strain problem into a three-dimensional one.

If the known Airy stress function $\phi(x,y)$ is given as

$$\phi = \sin\delta x(k_1\cosh\delta y + k_2\sinh\delta y + k_3 y\cosh\delta y + k_4 y\sinh\delta y)$$

where k_1, k_2, k_3, k_4, δ are constants, then the function $\phi(x,y)$ can be written as

$$\phi = yA(x,y) + B(x,y),$$

where

$$\nabla^2 A = 0,$$

$$\nabla^2 B = 0.$$

In this case, the functions A and B are easily found, they are

$$A(x,y) = \sin\delta x\ (k_3 \cosh\delta y + k_4 \sinh\delta y),$$

$$B(x,y) = \sin\delta x\ (k_1 \cosh\delta y + k_2 \sinh\delta y).$$

A new harmonic function $C(x,y)$ can then be defined, in the manner specified by (5.2.9), as

$$C(x,y) = B(x,y) + 2(1-\nu)\int_0^y A dy - 2(1-\nu)\iint \left(\frac{\partial A}{\partial y}\right)_{y=0} dx dx$$

$$= B + 2(1-\nu)\ \frac{\sin\delta x}{\delta}\left[k_3 \sinh\delta y + k_4 \cosh\delta y\right]$$

$$= \sin\delta x \left\{\left[k_2 + \frac{2(1-\nu)}{\delta}\ k_3\right]\sinh\delta y + \left[k_1 + \frac{2(1-\nu)}{\delta} k_4\right]\cosh\delta y\right\}.$$

The displacements u, v are now expressible by

$$\frac{E}{1+\nu}\ u = -\ \frac{\partial}{\partial x}\ (yA + C),$$

$$\frac{E}{1+\nu}\ v = 4(1-\nu)A - \frac{\partial}{\partial y}\ (yA + C),$$

where A and C are harmonic functions satisfying

$$\nabla^2 A = \nabla^2 C = 0.$$

After some interchanging of notation, the results from this section will be able assist in comparing the results from section 7.5 with those in plane strain cases.

5.5 Conclusions

The Airy stress function approach and the Papkovich-Neuber approach to plane strain problems are equivalent, except for some solid body displacement. Comparing the convenience between the Papkovich-Neuber approach, the Airy stress function approach and the complex variable approach, the first one is certainly more cumbersome to use, and also redundant in its pure form, i.e. in the form containing three harmonic functions A_x, A_y and A_0 of (5.3.6).

The transformations in this chapter constitute the first step toward extending a plane strain solution to a three dimensional solution as will be shown in chapter 7.

CHAPTER 6

Some relationships between plane strain and torsion-free axi-symmetric problems.

6.1 Introduction.

Symmetric plane strain (or equivalent symmetric plane stress) and torsion-free axi-symmetric problems are both two-dimensional. The existence of relationships between these two types of problems are thus expected both from the theoretical and economical points of views.

The forward and reverse transformations from a torsion-free axi-symmetric state to a symmetric plane state can be carried out according to the following plans:

(i) Rotation of the symmetric plane state around its axis of symmetry to give a torsion-free axi-symmetric state.

(ii) Linear translation of a torsion-free axi-symmetric state along a direction perpendicular to its axis of rotation to give a symmetric plane state.

Using a stress function approach for axi-symmetric problems, C. Weber [24], [42] pointed out the analogy between his approach and the Airy stress function approach. Transformations between the two types of stress functions were carried out according to plans (i) and (ii). However, only their shear stresses and direct stresses along the direction of the axis of symmetry can be related; the relationships for their other stress components and for displacements between the two states remains unclear.

A. Ia. Aleksandrov [26], on the other hand, worked exclusively with stresses and displacements, for the transformations between the two states according to plans (i) and (ii). Although his paper gives much physical meaning to the transformations, the relationships for the stress functions were left out.

Therefore, although stress functions can be transformed according to (i) and (ii) in a similar manner to the physical quantities (displacements, strains, stresses, etc.), there is no theory dealing with all of them. A theory which deals with the functions and their physical quantities as a whole, is of practical importance as it permits better understanding of the relationships and lends to a more effective use of them.

This chapter applies the transformations (i) and (ii) to the Papkovich-Neuber expressions for displacements in the symmtric plane strain and torsion-free axi-symmetric problems. The functions of the two states are thus related and they are shown to be the same functions which generate the corresponding physical displacements. Then, using the uniqueness theorem for stresses, these are shown to be similarly related. The harmonic functions, their displacements, strains, stresses are thus related according to the appropriate (tensorial) transformation rules.

The plan (ii) is also shown to correspond to successive applications of zeroth order Hankel and inverse Fourier cosine transformations. The two mathematical transformations, in turn, offer some more means for relating the harmonic functions of the two geometries.

Finally, the method of transformation is illustrated in section 6.6, through the problems of side loading of an infinite strip and of an infinite cylinder; this shows that the two problems are analogous in every aspect.

6.2 Displacements in terms of harmonic functions.

For plane strain problems, displacements are given by

$$u_1(x,z) = -\frac{\partial}{\partial x}(zA + B), \tag{6.2.1a}$$

$$w_1(x,z) = 4(1-\nu)A - \frac{\partial}{\partial z}(zA + B), \tag{6.2.1b}$$

where $u_1(x,z)$, $w_1(x,z)$ are displacements along the x and z directions, ν is the Poisson ratio, and $A(x,z)$, $B(x,z)$ are two harmonic functions in the Cartesian coordinates (x,z).

On the other hand, the Boussinesq solution for a torsion-free axi-symmetric state gives

$$u_2(r,z) = -\frac{\partial}{\partial r}(z\mathcal{C} + \mathcal{D}), \qquad (6.2.2a)$$

$$w_2(r,z) = 4(1-\nu)\mathcal{C} - \frac{\partial}{\partial z}(z\mathcal{C} + \mathcal{D}), \qquad (6.2.2b)$$

where $u_2(r,z)$, $w_2(r,z)$ are displacements along the r and z directions, $\mathcal{C}(r,z)$ and $\mathcal{D}(r,z)$ are two axi-symmetric harmonic functions in the cylindrical coordinates (r,θ,z).

This chapter will relate the symmetric plane strain state and the torsion-free axi-symmetric state by the use of the above four equations. These equations allow a simple physical interpretation of the generation of a torsion-free axi-symmetric (or symmetric plane strain) state from a symmetric plane strain (or torsion-free axi-symmetric) one.

It is interesting to note that although the procedures (i) and (ii) have been used in both [24] and [26], without the use of (6.2.1) and (6.2.2) neither paper dealt with the functions and the physical quantities as a whole.

It is also worth noting that torsion-free axi-symmetric states are quite difficult to relate to symmetric plane stress states (although the latter are sometimes considered to be equivalent to symmetric plane strain states) because of the following considerations.

a - Plane stress problems require one more equation to account for displacement in the third Cartesian direction. As a result, expressions like (6.2.1) may need more than two harmonic functions for completeness of the solutions.

b - Plane stress and plane strain problems are known to be equivalent only approximately. It therefore requires more than the proof in sections 6.3 and 6.4 to relate symmetric plane stress and torsion-free axi-symmetric states.

A proof which relates a torsion-free axi-symmetric state to a plane state, where the latter state is not exactly specified, is therefore imprecise and prone to misinterpretation.

6.3 Rotation of a symmetric plane strain state to give a torsion-free axi-symmetric state.

The given symmetric plane strain state has its displacements satisfying the equations (6.2.1). A super-imposition of finely and equally spaced of an infinite number of this state, each rotated around its axis of symmetry, gives a torsion-free axi-symmetric state.

The Cartesian harmonic functions A(x,z) and B(x,z) then give two cylindrical harmonic functions $\mathcal{A}(r,z)$ and $\mathcal{B}(r,z)$,

$$\mathcal{A}(r,z) = \int_0^{\pi} A(r\cos\theta, z)\,d\theta, \tag{6.3.1a}$$

$$\mathcal{B}(r,z) = \int_0^{\pi} B(r\cos\theta, z)\,d\theta. \tag{6.3.1b}$$

Displacements $u_1(x,z)$ and $w_1(x,z)$ of the same equations (6.2.1) give

$$u_4(r,z) = \int_0^{\pi} u_1(r\cos\theta, z)\cos\theta\,d\theta, \tag{6.3.2a}$$

$$w_4(r,z) = \int_0^{\pi} w_1(r\cos\theta, z)\,d\theta. \tag{6.3.2b}$$

As

$$\begin{aligned} u_4(r,z) &= -\int_0^{\pi} \frac{\partial}{\partial(r\cos\theta)}\,[zA(r\cos\theta,z) + B(r\cos\theta,z)]\cos\theta\,d\theta \\ &= -\int_0^{\pi} \frac{\partial}{\partial r}\,[zA(r\cos\theta,z) + B(r\cos\theta,z)]\,d\theta \\ &= -\frac{\partial}{\partial r}\,[z\mathcal{A}(r,z) + \mathcal{B}(r,z)], \end{aligned}$$

and

$$\begin{aligned} w_4(r,z) &= \int_0^{\pi}\left\{4(1-\nu)A(r\cos\theta,z) - \frac{\partial}{\partial z}\,[zA(r\cos\theta,z) + B(r\cos\theta,z)]\right\}d\theta \\ &= 4(1-\nu)\mathcal{A}(r,z) - \frac{\partial}{\partial z}\,[z\mathcal{A}(r,z) + \mathcal{B}(r,z)], \end{aligned}$$

it is clear that $u_4(r,z)$ and $w_4(r,z)$ are the displacements generated by the axi-symmetric harmonic functions $\mathcal{A}(r,z)$ and $\mathcal{B}(r,z)$. Thus, the rotation has trnasformed (6.2.1a) and (6.2.1b) into

$$u_4(r,z) = -\frac{\partial}{\partial r}(zA + B), \tag{6.3.3a}$$

$$w_4(r,z) = 4(1-\nu)A - \frac{\partial}{\partial z}(zA + B). \tag{6.3.3b}$$

Hence, the rotation of a symmetric plane state gives a torsion-free axi-symmetric state whose harmonic functions, displacements and displacement derived quantities (to follow) are the transformed variables from the plane state.

Since the displacements of the transformed state (as specified by (6.3.2a) and (6.3.2b) are generated according to plan (i) of section 6.2, all other physical quantities, which are derived from displacements, such as stresses, strains, dilation ... are also generated according to plan (i). The quantities at ($r\cos\theta,z$) are thus projected (tensorially) to the required (tensorial) direction, and then integrated for the range of θ from 0 to π.

For example, the stresses in the axi-symmetric state are the transformed variables of those in the plane state:

$$\sigma_{rr4} = \int_0^{\pi} [\sigma_{xx1}(r\cos\theta,z)\cos^2\theta + \sigma_{yy1}(r\cos\theta,z)\sin^2\theta]d\theta, \tag{6.3.4a}$$

$$\sigma_{\theta\theta4} = \int_0^{\pi} [\sigma_{xx1}(r\cos\theta,z)\sin^2\theta + \sigma_{yy1}(r\cos\theta,z)\cos^2\theta]d\theta, \tag{6.3.4b}$$

$$\sigma_{zz4} = \int_0^{\pi} \sigma_{zz1}(r\cos\theta,z)d\theta, \tag{6.3.4c}$$

$$\tau_{rz4} = \int_0^{\pi} \tau_{xz1}(r\cos\theta,z)\cos\theta\, d\theta, \tag{6.3.4d}$$

where the subscripts 1 and 4 denote the original plane state and its transformed axi-symmetric state, respectively.

Concerning the inverse of this rotation, it is sufficient to point out that all the equations (6.3.1) to (6.3.3) (for (6.3.4a) and (6.3.4b), their sum and difference) are of the Abel type; therefore their inverses are given by

$$A(x,z) = \frac{1}{\pi}\frac{\partial}{\partial z}\int_0^{x} A(r,z)\frac{rdr}{\sqrt{x^2-r^2}}, \tag{6.3.5a}$$

$$B(x,z) = \frac{1}{\pi}\frac{\partial}{\partial x}\int_0^x \mathcal{B}(r,z)\frac{r\,dr}{\sqrt{x^2-r^2}}, \tag{6.3.5b}$$

$$u_1(x,z) = \frac{1}{\pi x}\frac{\partial}{\partial x}\int_0^x u_4(r,z)\frac{r^2\,dr}{\sqrt{x^2-r^2}}, \tag{6.3.6a}$$

$$w_1(x,z) = \frac{1}{\pi}\frac{\partial}{\partial x}\int_0^x w_4(r,z)\frac{r\,dr}{\sqrt{x^2-r^2}}. \tag{6.3.6b}$$

Stresses in the plane state can also be worked out from stresses of the transformed state. For a detailed treatment of this inverse transformation, readers are referred to [26].

6.4 Linear translation of a torsion-free axi-symmetric state to give a symmetric plane state.

The superimposition of an infinite number of finely and equally spaced axi-symmetric states, each has been translated along a direction which is perpendicular to the axis of rotation of the initial state, gives a symmetric plane state.

The cylindrical harmonic functions $\mathcal{C}(r,z)$ and $\mathcal{D}(r,z)$ of equations (6.2.2) give the following cartesian harmonic functions

$$C(x,z) = \int_{-\infty}^{\infty} \mathcal{C}(\sqrt{x^2+y^2},z)\,dy, \tag{6.4.1a}$$

$$D(x,z) = \int_{-\infty}^{\infty} \mathcal{D}(\sqrt{x^2+y^2},z)\,dy. \tag{6.4.1b}$$

Displacements $u_2(r,z)$ and $w_2(r,z)$ of the same equations (6.2.2) give

$$u_3(x,z) = \int_{-\infty}^{\infty} u_2(\sqrt{x^2+y^2},z)\,\frac{x}{\sqrt{x^2+y^2}}\,dy, \tag{6.4.2a}$$

$$w_3(x,z) = \int_{-\infty}^{\infty} w_2(\sqrt{x^2+y^2},z)\,dy. \tag{6.4.2b}$$

As

$$\begin{aligned} u_3(x,z) &= -\int_{-\infty}^{\infty}\frac{\partial}{\partial(\sqrt{x^2+y^2})}\,[z\mathcal{C}(\sqrt{x^2+y^2},z) + \mathcal{D}(\sqrt{x^2+y^2},z)]\,dy, \\ &= -\int_{-\infty}^{\infty}\frac{\partial}{\partial x}\,[z\mathcal{C}(\sqrt{x^2+y^2},z) + \mathcal{D}(\sqrt{x^2+y^2},z)]\,\frac{x}{\sqrt{x^2+y^2}}\,dy, \\ &= -\frac{\partial}{\partial x}\,[zC(x,z) + D(x,z)], \end{aligned}$$

and

$$w_3(x,z) = \int_{-\infty}^{\infty}\left\{4(1-\nu)\mathcal{C}(\sqrt{x^2+y^2},z) - \frac{\partial}{\partial z}\left[z\mathcal{C}(\sqrt{x^2+y^2},z) + \mathcal{D}(\sqrt{x^2+y^2},z)\right]\right\}dy,$$

$$= 4(1-\nu)C(x,z) - \frac{\partial}{\partial z}\left[zC(x,z) + D(x,z)\right],$$

it is also clear that $u_3(x,z)$ and $w_3(x,z)$ are the displacements generated by the cartesian harmonic functions $C(x,z)$ and $D(x,z)$. Thus, the translation has effectively transformed (6.2.2a) and (6.2.2b) into

$$u_3(x,z) = -\frac{\partial}{\partial x}(zC + D), \qquad (6.4.3a)$$

$$w_3(x,z) = 4(1-\nu)C - \frac{\partial}{\partial z}(zC + D). \qquad (6.4.3b)$$

Similar to section 6.3, a symmetric plane strain state has been derived from a torsion-free axi-symmetric state according to plan (ii). The harmonic functions, displacements and all other derived quantities are generated according to plan (ii). The quantities at $(\sqrt{x^2+y^2},z)$ are projected (tensorially) to the required (tensorial) direction and then integrated for the range of y from $-\infty$ to ∞.

Stresses of the derived plane state are related to those of the original axi-symmetric state by

$$\sigma_{xx_3}(x,z) = \int_{-\infty}^{\infty}\left[\sigma_{rr_2}(\sqrt{x^2+y^2},z)\frac{x^2}{x^2+y^2} + \sigma_{\theta\theta_2}(\sqrt{x^2+y^2},z)\frac{y^2}{x^2+y^2}\right]dy, \qquad (6.4.4a)$$

$$\sigma_{yy_3}(x,z) = \int_{-\infty}^{\infty}\left[\sigma_{rr_2}(\sqrt{x^2+y^2},z)\frac{y^2}{x^2+y^2} + \sigma_{\theta\theta_2}(\sqrt{x^2+y^2},z)\frac{x^2}{x^2+y^2}\right]dy, \qquad (6.4.4b)$$

$$\sigma_{zz_3}(x,z) = \int_{-\infty}^{\infty}\sigma_{zz_2}(\sqrt{x^2+y^2},z)\,dy, \qquad (6.4.4c)$$

$$\tau_{rz_3}(x,z) = \int_{-\infty}^{\infty}\tau_{rz_2}(\sqrt{x^2+y^2},z)\frac{x}{\sqrt{x^2+y^2}}\,dy, \qquad (6.4.4d)$$

where the subscripts 2 and 3 denote the original axi-symmetric state and its transformed plane state respectively.

The inverse of equation (6.4.1a) is worked out as in the following.

$$C(x,z) = \int_{-\infty}^{\infty}\mathcal{C}(\sqrt{x^2+y^2},z)\,dy$$

$$= \int_{-\infty}^{\infty}\mathcal{C}(\sqrt{x^2+t_1^2},z)\,dt_1,$$

so

$$\int_{-\infty}^{\infty} c(\sqrt{x^2+t_2{}^2},z)\,dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{C}(\sqrt{x^2+t_1{}^2+t_2{}^2},z)\,dt_1dt_2$$

$$= \int_0^{2\pi} d\theta \int_0^{\infty} \mathcal{C}(\sqrt{x^2+t_3{}^2},z)\,t_3dt_3$$

$$= 2\pi \int_x^{\infty} \mathcal{C}(t_4,z)\,t_4dt_4.$$

Hence

$$-\frac{\partial}{\partial x}\int_{-\infty}^{\infty} c(\sqrt{x^2+t_2{}^2},z)\,dt_2 = 2\pi\mathcal{C}(x,z)\,x,$$

or

$$\mathcal{C}(x,z) = -\frac{1}{2\pi x}\frac{\partial}{\partial x}\int_{-\infty}^{\infty} c(\sqrt{x^2+t_2{}^2},z)\,dt_2.$$

Then

$$\mathcal{C}(r,z) = -\frac{1}{2\pi r}\frac{\partial}{\partial r}\int_{-\infty}^{\infty} c(\sqrt{r^2+t^2},z)\,dt,$$

or

$$\mathcal{C}(r,z) = -\frac{1}{\pi}\int_r^{\infty} \frac{\partial C(x,z)}{\partial x}\,\frac{dx}{\sqrt{x^2-r^2}}\,. \tag{6.4.5a}$$

Similarly,

$$\mathcal{D}(r,z) = -\frac{1}{\pi}\int_r^{\infty} \frac{\partial D(x,z)}{\partial x}\,\frac{dx}{\sqrt{x^2-r^2}}\,. \tag{6.4.5b}$$

Following essentially the same argument, displacements are given by

$$u_2(r,z) = -\frac{1}{\pi}\int_r^{\infty} \frac{\partial u_3(x,z)}{\partial x}\,\frac{x\,dx}{r\sqrt{x^2-r^2}}, \tag{6.4.6a}$$

$$w_2(r,z) = -\frac{1}{\pi}\int_r^{\infty} \frac{\partial w_3(x,z)}{\partial x}\,\frac{dx}{\sqrt{x^2-r^2}}\,. \tag{6.4.6b}$$

(6.4.4c) and (6.4.4d) can also be dealt with in the same manner. For (6.4.4a) and (6.4.4b) their sum and difference are used instead. For a detailed treatment of the inverse transformation in this section, readers are referred to [26].

6.5 The role of Fourier-cosine and zero-order Hankel transformations in the relationship between plane and axi-symmetric states.

The Fourier-cosine transformation on a symmetric plane harmonic function $f(x,z)$ which satisfies

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial z^2} = 0,$$

gives a function $\phi(s,z)$ which satisfies the corresponding transformed equation

$$-s^2\phi(s,z) + \frac{\partial^2\phi}{\partial z^2} = 0.$$

On the other hand, the Hankel transformation of order zero of an axi-symmetric harmonic function $F(r,z)$, which satisfies

$$\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{\partial^2 F}{\partial z^2} = 0,$$

is a function $\Phi(s,z)$ which satisfies

$$-s^2\Phi(s,z) + \frac{\partial^2\Phi}{\partial z^2} = 0.$$

By equating $\phi(s,z)$ and $\Phi(s,z)$, transformations between symmetric plane and axi-symmetric harmonic functions can be carried out as in the following sub-sections.

6.5.1 *Cylindrical to plane.*

With

$$\Phi(s,z) = \int_0^\infty rF(r,z)J_o(sr)dr, \tag{6.5.1}$$

and

$$f(x,z) = \int_0^\infty \Phi(s,z)\cos(sx)ds, \tag{6.5.2}$$

it is deduced that

$$f(x,z) = \sqrt{\frac{2}{\pi}}\int_{s=o}^\infty \cos(sx)\int_{r=o}^\infty rF(r,z)J_o(sr)drds,$$

or

$$f(x,z) = \sqrt{\frac{2}{\pi}}\int_{r=o}^\infty rF(r,z)\left[\int_{s=o}^\infty \cos(sx)J_o(sr)ds\right]dr. \tag{6.5.3}$$

Since

$$\int_o^\infty \cos(sx)J_o(sr)ds = \begin{cases} \dfrac{1}{\sqrt{r^2-x^2}} & r > x \\ 0 & r < x, \end{cases} \tag{6.5.4}$$

$f(x,z)$ is written in its final form as

$$f(x,z) = -\sqrt{\frac{2}{\pi}}\int_0^\infty F(\sqrt{h^2+x^2},z)dh. \tag{6.5.5}$$

Apart from the constant $-(2\pi)^{-\frac{1}{2}}$, the above relationship is identical to (6.4.1a) and (6.4.1b).

6.5.2 *Plane to cylindrical.*

With

$$\phi(s,z) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x,z) \cos(sx) dx, \tag{6.5.6}$$

and

$$F(r,z) = \int_0^\infty s\phi(s,z) J_o(sr) ds, \tag{6.5.7}$$

it is deduced that

$$F(r,z) = \sqrt{\frac{2}{\pi}} \int_{s=o}^\infty sJ_o(sr) \int_{x=o}^\infty f(x,z) \cos(sx) dx ds.$$

It should be noted that interchanging the order of integration of the above double integral is not permitted. Instead, additional assumptions are needed to reduce the above integral to an amenable form.

Partial integration gives

$$\int_0^\infty f(x,z)\cos(sx)dx = \left[\frac{1}{s} f(x,z)\sin(sx)\right]_{x=o}^\infty - \int_{x=o}^\infty \frac{\partial f}{\partial x} \frac{\sin(sx)}{s} dx.$$

With the assumption that $f(x,z)$ vanishes at infinities along the x direction, the square bracket in the above equation is eliminated, leaving

$$\int_0^\infty f(x,z) \cos(sx) dx = - \int_0^\infty \frac{\partial f}{\partial x} \frac{\sin(sx)}{s} dx. \tag{6.5.8}$$

Thus

$$F(r,z) = - \sqrt{\frac{2}{\pi}} \int_{s=o}^\infty sJ_o(sr) \left[\int_{x=o}^\infty \frac{\partial f}{\partial x} \frac{\sin(sx)}{s} dx \right] ds.$$

$$= - \sqrt{\frac{2}{\pi}} \int_0^\infty J_o(sr) \left[\int_{x=o}^\infty \frac{\partial f}{\partial x} \sin(sx) dx \right] ds$$

$$= - \sqrt{\frac{2}{\pi}} \int_{x=o}^\infty \frac{\partial f}{\partial x} \left[\int_{s=o}^\infty J_o(sr)\sin(sx) ds \right] dx. \tag{6.5.9}$$

A standard result is that

$$\int_{s=o}^\infty J_o(rs)\sin(xs)ds = \begin{cases} 0 & r > x \\ -\dfrac{1}{\sqrt{x^2-r^2}} & r < x. \end{cases} \tag{6.5.10}$$

Hence,

$$F(r,z) = \sqrt{\frac{2}{\pi}} \int_{x=r}^\infty \frac{\partial f}{\partial x} \frac{dx}{\sqrt{x^2-r^2}}. \tag{6.5.11}$$

Apart from the constant $-(2\pi)^{-\frac{1}{2}}$ in front of $F(r,z)$, this equation is identical to the equation (6.4.5a).

The mathematical processes carried out in sub-sections *6.5.1* and *6.5.2* thus correspond to the forward and reverse linear translation of a torsion-free axi-symmetric state to give a symmetric plane state.

If the cosine function in (6.5.2) is replaced by a sine function, then another kind of transformation is obtained

$$f_6(x,z) = \int_0^x F_5(\sqrt{x^2-t^2},z)\,dt \qquad (6.5.12)$$

where the subscripts 5 and 6 denote the original cylindrical and the transformed cartesian states, respectively.

When certain additional conditions are imposed on $F_5(r,z)$ then its harmonicity induces that of $f_6(x,z)$.

The formulation of (6.5.12) also suggests another kind of transformation, from a symmetric plane state to a torsion-free axi-symmetric state. It is

$$F_7(r,z) = \int_0^\infty f_8(r\cosh\theta,z)\,d\theta, \qquad (6.5.13)$$

where the subscripts 8 and 7 denote the original cartesian and the transformed cylindrical states, respectively.

As for (6.5.12), when certain additional conditions are satisfied, the harmonicity of $f_8(x,z)$ induces that of $F_7(r,z)$.

The transformations for harmonic functions as specified by (6.5.12) and (6.5.13) are purely mathematical and their physical meanings are still lacking. As their physical meaning (if any) is not simple, these transformations may not have any practical value.

6.6 Example of the relationship between a symmetric two dimensional state and an axi-symmetric state.

As an example of the relationship between a symmetric plane strain problem and an axi-symmetric one, this section considers the problem of an infinite strip and of an infinite cylinder loaded on their sides. The section also introduces the technique of using complex harmonic functions in the Papkovich-Neuber solution. This technique will be the one in use in the next two chapters 7 and 8.

The infinite strip along the z direction, with sinusoidal loadings on the sides of $x = \pm 1$ (fig. 6.1), has its complex harmonic functions as

$$A(x,z) = -\frac{i\delta}{2\pi\gamma} e^{i\delta z} \cosh(\gamma x), \tag{6.6.1a}$$

and

$$B(x,z) = \frac{e^{i\delta z}}{2\pi}\left[x \sinh(\gamma x) + \frac{i\delta}{\gamma} z \cosh(\gamma x) - \frac{K+2(1-\nu)}{\gamma}\cosh(\gamma x)\right], \tag{6.6.1b}$$

where γ, δ and K are three constant. The constants γ and δ are further required to satisfy

$$\gamma^2 = \delta^2. \tag{6.6.2}$$

Actually, it is possible to use only one of these two constants. However, the following arguments are simpler with both of them retained.

Displacements, according to (6.2.1), are given by

$$u_1(x,z) = \frac{e^{i\delta z}}{2\pi}\left[(K+1-2\nu)\sinh(\gamma x) - \gamma x \cosh(\gamma x)\right], \tag{6.6.3a}$$

$$w_1(x,z) = i\delta\, \frac{e^{i\delta z}}{2\pi}\left[-x \sinh(\gamma x) + \frac{K-2(1-\nu)}{\gamma}\cosh(\gamma x)\right]. \tag{6.6.3b}$$

Stresses in plane strain problems are related to displacements by the following equations

$$\frac{1+\nu}{E}\sigma_{xx} = \frac{\partial u}{\partial x} + \frac{\nu}{1-2\nu}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right),$$

$$\frac{1+\nu}{E}\sigma_{zz} = \frac{\partial w}{\partial z} + \frac{\nu}{1-2\nu}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right),$$

$$2\frac{1+\nu}{E}\tau_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}.$$

These three equations give the values of σ_{xx_1}, σ_{zz_1} and τ_{xz_1} as

$$\frac{1+\nu}{E}\sigma_{xx_1} = \frac{e^{i\delta z}}{2\pi}\gamma\left[-\gamma x \sinh(\gamma x) + K\cosh(\gamma x)\right], \tag{6.6.4a}$$

$$\frac{1+\nu}{E}\sigma_{zz_1} = \frac{e^{i\delta z}}{2\pi}\gamma\left[\gamma x \sinh(\gamma x) + (2-K)\cosh(\gamma x)\right], \tag{6.6.4b}$$

$$\frac{1+\nu}{E}\tau_{xz_1} = \frac{i\delta e^{i\delta z}}{2\pi}\left[(K-1)\sinh(\gamma x) - \gamma x\cosh(\gamma x)\right]. \tag{6.6.4c}$$

As (γ,δ) are replaced by $(\bar{\gamma},-\bar{\delta})$, the quantities A, B, u_1, w_1, σ_{xx_1}, σ_{zz_1}, τ_{xz_1} all change into their complex conjugates. Thus, their real and imaginary parts each form a system of real loading.

For real K and γ(and so δ), the above system can represent any x-symmetric loading which varies sinusoidally along the z direction with wavelength $1/\gamma$. Since any x-symmetric loading on the sides $x = \pm 1$ of the strip can be Fourier analysed into constituent modes such as the two modes (one from the real, the other from the imaginary parts of A, B, u_1, w_1, σ_{xx_1}, σ_{zz_1}, τ_{xz_1}) considered above, it is thus sufficient to investigate only these two.

It is also noted that if all the functions $\sinh(\gamma x)$ and $\cosh(\gamma x)$ in the previous calculations of this section are replaced by $\cosh(\gamma x)$ and $\sinh(\gamma x)$ respectively, then a x-anti-symmetric loading of the strip is obtained. However, this anti-symmetric loading is of no interest to this chapter and will not be mentioned any further within the chapter.

A rotation of the above plane state around its z-axis gives an axi-symmetric state (fig. 6.2), the harmonic functions of which are given by (6.3.1) as

$$\mathcal{A}(r,z) = \int_0^{\pi} A(r\cos\theta, z)d\theta,$$

and

$$\mathcal{B}(r,z) = \int_0^{\pi} B(r\cos\theta, z)d\theta.$$

With the values of A and B given in (6.6.1), and the help of the following formulae

$$I_o(c) = \frac{1}{\pi}\int_0^{\pi}\cosh(c\cos\theta)d\theta, \qquad \text{(any complex c)}$$

$$I_1(c) = \frac{1}{\pi}\int_0^{\pi}\sinh(c\cos\theta)\cos\theta d\theta, \qquad \text{(any complex c)}$$

the functions $\mathcal{A}$ and $\mathcal{B}$ are evaluated as

$$\mathcal{A}(r,z) = -\frac{i\delta}{2\gamma}e^{i\delta z}I_o(\gamma r), \tag{6.6.5a}$$

and

$$\mathcal{B}(r,z) = \tfrac{1}{2}e^{i\delta z}[rI_1(\gamma r) + \frac{i\delta}{\gamma} zI_o(\gamma r) - \frac{K+2(1-\nu)}{\gamma} I_o(\gamma r)], \quad (6.6.5b)$$

where γ, δ and K are the three constants defined earlier at the end of equation (6.6.1). The constants γ and δ still have to satisfy the equation (6.6.2), which is

$$\gamma^2 = \delta^2. \quad (6.6.2)$$

Similarly to the last case, only one of the two constants γ and δ is really needed. However, for the following arguments to be simple, both of them are retained.

Displacements, which satisfy (6.2.2) as well as (6.3.2), for this case of axi-symmetric deformation are written as

$$u_4(r,z) = \frac{e^{i\delta z}}{2}\left[(K+2-2\nu)I_1(\gamma r) - \gamma r I_o(\gamma r) \right], \quad (6.6.6a)$$

and

$$w_4(r,z) = i\delta\frac{e^{i\delta z}}{2}\left[-rI_1(\gamma r) + \frac{K-2+2\nu}{\gamma} I_o(\gamma r)\right]. \quad (6.6.6b)$$

Stresses as deduced from the above displacements, also satisfying (6.3.4), are given by

$$\frac{1+\nu}{E}\sigma_{rr_4} = \frac{e^{i\delta z}}{2}\left[\left(\frac{-K-2+2\nu}{r} - \gamma^2 r\right) I_1(\gamma r) + (K+1)\gamma I_o(\gamma r) \right], \quad (6.6.7a)$$

$$\frac{1+\nu}{E}\sigma_{zz_4} = \frac{e^{i\delta z}}{2}\gamma\left[\gamma r I_1(\gamma r) + (2-K)I_o(\gamma r) \right], \quad (6.6.7b)$$

and

$$\frac{1+\nu}{E}\tau_{rz_4} = \frac{i\delta e^{i\delta z}}{2}\left[KI_1(\gamma r) - \gamma r I_o(\gamma r) \right]. \quad (6.6.7c)$$

As (γ,δ) are replaced by $(\overline{\gamma},-\overline{\delta})$, the quantities $\mathcal{A}$, $\mathcal{B}$, u_4, w_4, σ_{rr_4}, σ_{zz_4}, τ_{rz_4} all change into their complex conjugates. Thus, their real and imaginary parts each forms a system of real loading.

For real K and γ(and so δ), the above derived system can represent any torsion-free axi-symmetric loading (on the cylindrical surface r=1 of the cylinder) which varies sinusoidally along the z direction with wavelength $1/\gamma$. Since any torsion-free axi-symmetric loading on the surface r=1 can be Fourier analysed into constituent modes such as the two modes just considered, it can be said that any torsion-free axi-symmetric loading on the surface r=1 can be treated by the method derived above.

To this point, the solution for a torsion-free axi-symmetric loading on the cylindrical surface r=1 of an infinite cylinder has been derived from the solution for an infinite (plane strain) strip, symmetrically loaded on its sides $x = \pm 1$.

Moreover, the homogeneous solutions of these two problems are actually related, as in the following.

Applying the homogeneous conditions ($\sigma_{xx_1} = \tau_{xz_1} = 0$ on $x = \pm 1$) to the equations (6.6.4a) and (6.6.4b), the values of K and γ are required to satisfy

$$-\gamma \sinh(\gamma) + K\cosh(\gamma) = 0, \tag{6.6.8a}$$

and $$(K-1)\sinh(\gamma) - \gamma\cosh(\gamma) = 0. \tag{6.6.8b}$$

As the plane state is rotated around its axis, the equations (6.6.4a) and (6.6.4b) have their counterparts in the axi-symmetric state the equations (6.6.7a) and (6.6.7b). The homogeneous conditions ($\sigma_{rr_4} = \tau_{rz_4} = 0$ on r=1) of a torsion-free axi-symmetrically loaded cylinder require that the right hand sides of (6.6.7a) and (6.6.7c) are equal to zero for r=1. Hence, for the axi-symmetric case, K and δ are required to satisfy

$$(-K-2+2\nu - \gamma^2)I_1(\gamma) + (K+1)\gamma I_o(\gamma) = 0, \tag{6.6.9a}$$

and

$$KI_1(\gamma) - \gamma I_o(\gamma) = 0. \tag{6.6.9b}$$

The relationship between (6.6.7) from (6.6.4) has thus given some insight to the similarity between the eigenvalues of the problems of a semi-infinite strip and of a semi-infinite cylinder. These eigenvalues will be encountered in chapters 7 and 8 respectively.

It is finally observed that as δ is replaced by γ in all the equations of this section, the real and the imaginary parts of the so obtained harmonic functions, displacements, strains and stresses form two system of real loading, one corresponds to all the real parts, the other to all the imaginary parts. This observation will prove to be useful in the next two chapters.

6.7 Conclusions.

A translation of a torsion-free axi-symmetric state gives a symmetric plane strain state, which has its harmonic functions, displacements and derived quantities as the transformation of those in the former state. Conversely, a rotation of a symmetric plane strain state by an angle π around its axis of symmetry gives a torsion-free axi-symmetric state which has its stress-functions, displacements and derived quantities as the transformation of those in the plane state.

The translation of a torsion-free axi-symmetric state, described in the above paragraphs, corresponds to the successive applications of Hankel (of zeroth order) and inverse Fourier cosine transforms. The inverse of this translation (which is *not* the rotation described in the above paragraph) then corresponds to the successive applications of Fourier cosine and inverse (zeroth order) Hankel transformations.

The two methods of transformation presented in the first paragraph can be applied to symmetric plane strain and torsion-free axi-symmetric problems of corresponding shapes to reveal their relationships. Section 6.6 is such an example. It is also easy to see that when the boundary conditions are given on constant-z cross-sections, and the solutions are known to vanish from the axis of symmetry, each symmetric plane strain (or torsion-free axi-symmetric) problem corresponds to another torsion-free axi-symmetric (or symmetric plane strain) one with the transformed boundary conditions.

Some interesting problems in elasticity.

In the following chapters, two interesting problems will be examined. The clamped prism problem and the end loading problem for a cylinder.

The clamped prism problem is an extension of the two-dimensional strip problem; the extension is made by the use of the Papkovich-Neuber solution to the strip problem. The end loading problem is considered along the method of least square error as proposed by Lur'e, also some numerical and algebraic results are presented to advance the method from the state appearing in Lur'e's book [16].

These problems serve to demonstrate the usefulness of the Papkovich-Neuber solution, as well as the difficulties which may be encountered when dealing with a particular problem.

CHAPTER 7

The finite thickness, frictionlessly clamped prism.

7.1 Introduction.

The problems of rectangular strips have been treated by a number of authors, such as J. Fadle [29], P. F. Papkovich [30], and others (see [23], p. 62 footnotes). However, all the problems having been considered are in two dimensions, with the simplification that all the strains related to the third dimension are zero. This simplification limits those treatments to infinitely thick strips (plane strain) or their equivalent infinitely thin strips (plane stress).

It is difficult to extend those two-dimensional treatments to cover three-dimensional cases because of their use of the Airy stress function. As have been pointed out before, an Airy stress function can be used to deduce its corresponding Papkovich-Neuber displacement functions. This chapter will use the Papkovich-Neuber equivalents of the Airy stress-functions to extend the treatments to three-dimensional cases, where the strains related to the third dimension do not need to be identically zero. The results given in this chapter reduce to those of two-dimensional cases if variations along the third dimension are set to zero.

A clarification should be made in advance that the treatment in this chapter is incomplete in the sense that it can not deal with some given shear stresses (τ_{xy} and τ_{zy}) on the unclamped sides of the prism.

7.2 The problem of a frictionlessly clamped, finite thickness y-symmetrically loaded prism, solved by the Papkovich-Neuber functions.

In this chapter, the (x,z) plane corresponds to the two-dimensional plane for other two-dimensional treatments. Thus the direction y is the one for infinite thickness in two-dimensional treatments. Also,

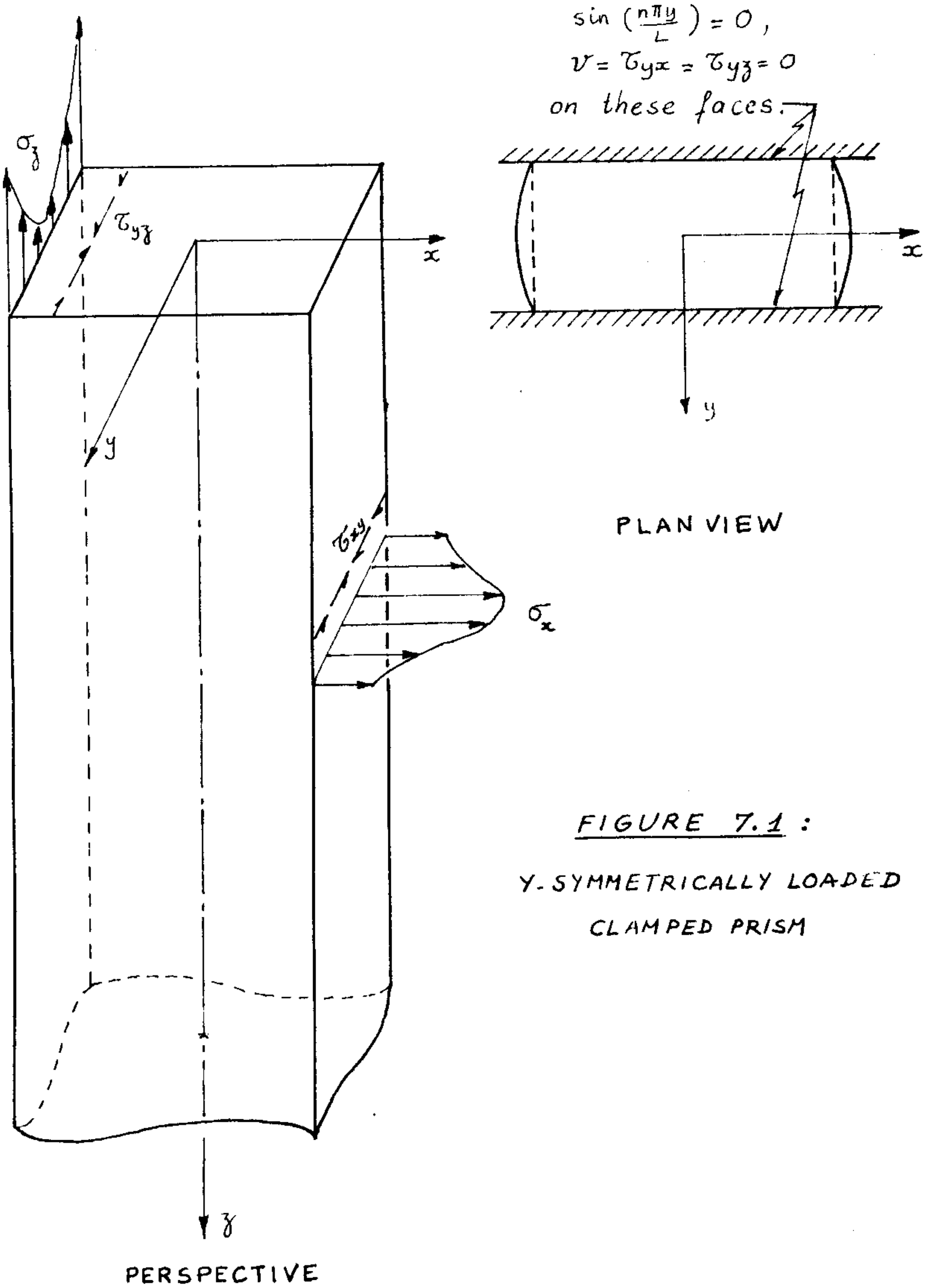


FIGURE 7.1 :

Y-SYMMETRICALLY LOADED CLAMPED PRISM

in this chapter, only the stresses σ_x, σ_z, τ_{xz} are considered in boundary conditions on the unclamped sides. Generally, the shear stresses τ_{xy} and τ_{zy} can not be given in advance in this chapter but they depend on the other stresses σ_x, σ_y, σ_z and τ_{xz}.

Consider now the Papkovich-Neuber form for displacement

$$u = 4(1-\nu)A_x - \frac{\partial}{\partial x}(xA_x + yA_y + zA_z + A_o), \tag{7.2.1a}$$

$$v = 4(1-\nu)A_y - \frac{\partial}{\partial y}(xA_x + yA_y + zA_z + A_o), \tag{7.2.1b}$$

$$w = 4(1-\nu)A_z - \frac{\partial}{\partial z}(xA_x + yA_y + zA_z + A_o). \tag{7.2.1c}$$

For a prism as in figures 7.1, the representation is still complete when either A_y or A_z is set to zero. In the treatment here, both of them will be set to zero, and the remaining functions A_x, A_o are assumed to vary sinusoidally as

$$A_x = \cos\left(\frac{n\pi y}{L}\right) C(x,z), \tag{7.2.2a}$$

$$A_o = \cos\left(\frac{n\pi y}{L}\right) D(x,z), \tag{7.2.2b}$$

$$A_y = 0, \tag{7.2.2c}$$

$$A_z = 0. \tag{7.2.2d}$$

Hence the functions given in (7.2.2) can not be expected to represent all loading on the prism.

When the forms given in (7.2.2) for the Papkovich-Neuber functions are substituted into the expression (7.2.1) for displacements, the results are

$$u = \cos\left(\frac{n\pi y}{L}\right) \left\{4(1-\nu)C - \frac{\partial}{\partial x}(xC+D) \right\}, \tag{7.2.3a}$$

$$v = \frac{n\pi}{L} \sin\left(\frac{n\pi y}{L}\right) \{xC+D\}, \tag{7.2.3b}$$

$$w = \cos\left(\frac{n\pi y}{L}\right) \left\{ -\frac{\partial}{\partial z}(xC+D)\right\}, \tag{7.2.3c}$$

with C(x,z) and D(x,z) satisfying

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) C(x,z) = \frac{n^2\pi^2}{L^2} C(x,z), \tag{7.2.4a}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) D(x,z) = \frac{n^2\pi^2}{L^2} D(x,z). \tag{7.2.4b}$$

where n is a non-negative integer.

The gradients of displacements are then given as

$$\frac{\partial u}{\partial x} = \cos\left(\frac{n\pi y}{L}\right)\left\{ 4(1-\nu)\ \frac{\partial C}{\partial x} - \frac{\partial^2}{\partial x^2}\ (xC+D)\right\}, \tag{7.2.5a}$$

$$\frac{\partial v}{\partial y} = \cos\left(\frac{n\pi y}{L}\right)\left\{ \frac{n^2\pi^2}{L^2}\ (xC+D)\right\}, \tag{7.2.5b}$$

$$\frac{\partial w}{\partial z} = \cos\left(\frac{n\pi y}{L}\right)\left\{ -\frac{\partial^2}{\partial z^2}\ (xC+D)\right\}, \tag{7.2.5c}$$

$$\frac{\partial u}{\partial z} = \cos\left(\frac{n\pi y}{L}\right)\left\{ 4(1-\nu)\ \frac{\partial C}{\partial z} - \frac{\partial^2}{\partial x\partial z}\ (xC+D)\right\}, \tag{7.2.5d}$$

$$\frac{\partial w}{\partial x} = \cos\left(\frac{n\pi y}{L}\right)\left\{ -\frac{\partial^2}{\partial x\partial z}\ (xC+D)\right\}, \tag{7.2.5e}$$

$$\frac{\partial u}{\partial y} = -\sin\left(\frac{n\pi y}{L}\right)\frac{n\pi}{L}\left\{ 4(1-\nu)C - \frac{\partial}{\partial x}\ (xC+D)\right\}, \tag{7.2.5f}$$

$$\frac{\partial v}{\partial x} = -\sin\left(\frac{n\pi y}{L}\right)\frac{n\pi}{L}\left\{ -\frac{\partial}{\partial x}\ (xC+D)\right\}, \tag{7.2.5g}$$

$$\frac{\partial w}{\partial y} = -\sin\left(\frac{n\pi y}{L}\right)\frac{n\pi}{L}\left\{ -\frac{\partial}{\partial z}\ (xC+D)\right\}, \tag{7.2.5h}$$

$$\frac{\partial v}{\partial z} = -\sin\left(\frac{n\pi y}{L}\right)\frac{n\pi}{L}\left\{ -\frac{\partial}{\partial z}\ (xC+D)\right\}. \tag{7.2.5i}$$

The equations (1.1.1) and (1.1.3) give the values of shear stresses on a surface of constant y as

$$\tau_{xy} = \frac{E}{2(1+\nu)}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

and

$$\tau_{zy} = \frac{E}{2(1+\nu)}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right).$$

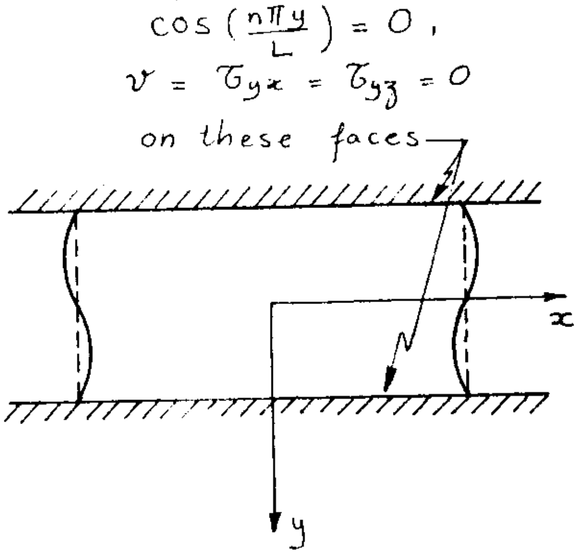
With the values of the gradients of displacements given by (7.2.5), it is clear that τ_{xy} and τ_{zy} all vanish along with v at the zeros of $\sin\left(\frac{n\pi y}{L}\right)$. The given boundary conditions of "frictionlessly clamped on the surfaces $y = \pm L/2$" can be written as

$$v = \tau_{xy} = \tau_{yz} = 0 \qquad \text{for } y = L/2, \tag{7.2.6a}$$

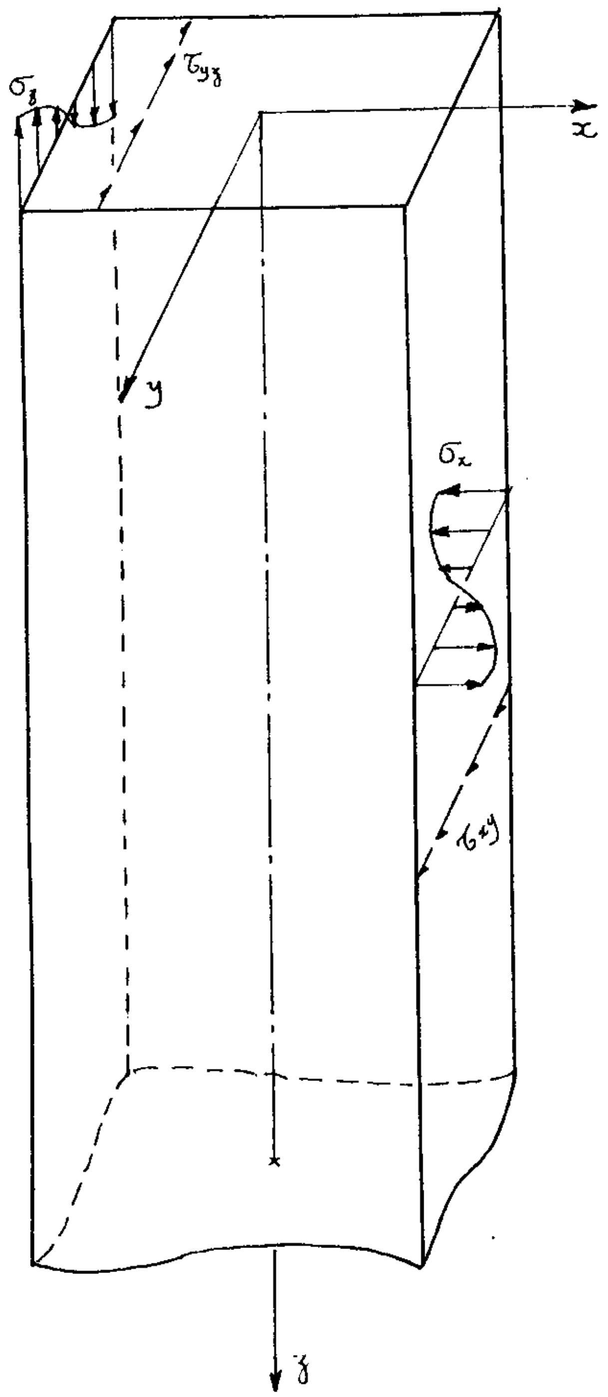
and

$$v = \tau_{xy} = \tau_{yz} = 0 \qquad \text{for } y = -L/2. \tag{7.2.6b}$$

This situation, as illustrated in figures 7.1, can arise realistically when a small rectangular prism is clamped between two smooth, well oiled jaws of a giant vice. Naturally, the displacement v is not zero on the two clamped surfaces $y = \pm L/2$ in this case, but after taking away the simple uniform compression mode of $(u = -\nu x,\ v = y,$



PLAN VIEW



PERSPECTIVE

FIGURE 7.2 :

Y-ANTISYMMETRICALLY LOADED CLAMPED PRISM

$w = -\nu z$), the remaining displacements will have zero v on the two clamped surfaces.

As pointed out previously, $\sin\left(\frac{n\pi y}{L}\right)$ must be zero on the two boundary surfaces $y = \pm L/2$, so the integer n must satisfy

$$\sin\left(\frac{n\pi}{2}\right) = 0,$$

or

$$n = 0,\ 2,\ 4,\ 6,\ \ldots$$

Hence, n can take only even integer values for the case illustrated by figures 7.1 . For zero n, the problem becomes the standard plane strain problem in the (x,z) plane.

7.3 The problem of a frictionlessly clamped, finite thickness, y-antisymmetrically loaded prism, solved by the Papkovich-Neuber functions.

An argument similar to the one in the last section will be used here, and the differences between the two sections are only subtle details.

It is assumed that

$$A_x = \sin\left(\frac{n\pi y}{L}\right) C(x,z), \tag{7.3.1a}$$

$$A_o = \sin\left(\frac{n\pi y}{L}\right) D(x,z), \tag{7.3.1b}$$

$$A_y = 0, \tag{7.3.1c}$$

and

$$A_z = 0, \tag{7.3.1d}$$

then displacements will be

$$u = \sin\left(\frac{n\pi y}{L}\right) \{4(1-\nu)C - \frac{\partial}{\partial x}(xC+D)\}\ , \tag{7.3.2a}$$

$$v = -\frac{n\pi}{L}\cos\left(\frac{n\pi y}{L}\right) \{xC+D\}, \tag{7.3.2b}$$

$$w = \sin\left(\frac{n\pi y}{L}\right) \{ -\frac{\partial}{\partial z}(zC+D) \}, \tag{7.3.2c}$$

with $C(x,z)$ and $D(x,z)$ still satisfying the equations (7.2.4).

The gradients of displacements are given in this case as

$$\frac{\partial u}{\partial x} = \sin\left(\frac{n\pi y}{L}\right)\left\{4(1-\nu)\frac{\partial C}{\partial x} - \frac{\partial^2}{\partial x^2}(xC+D)\right\}, \qquad (7.3.3a)$$

$$\frac{\partial v}{\partial y} = \sin\left(\frac{n\pi y}{L}\right)\left\{\frac{n^2\pi^2}{L^2}(xC+D)\right\}, \qquad (7.3.3b)$$

$$\frac{\partial w}{\partial z} = \sin\left(\frac{n\pi y}{L}\right)\left\{-\frac{\partial^2}{\partial z^2}(xC+D)\right\}, \qquad (7.3.3c)$$

$$\frac{\partial u}{\partial z} = \sin\left(\frac{n\pi y}{L}\right)\left\{4(1-\nu)\frac{\partial C}{\partial z} - \frac{\partial^2}{\partial x\partial z}(xC+D)\right\}, \qquad (7.3.3d)$$

$$\frac{\partial w}{\partial x} = \sin\left(\frac{n\pi y}{L}\right)\left\{-\frac{\partial^2}{\partial x\partial z}(xC+D)\right\}, \qquad (7.3.3e)$$

$$\frac{\partial u}{\partial y} = \cos\left(\frac{n\pi y}{L}\right)\frac{n\pi}{L}\left\{4(1-\nu)C - \frac{\partial}{\partial x}(xC+D)\right\}, \qquad (7.3.3f)$$

$$\frac{\partial v}{\partial x} = \cos\left(\frac{n\pi y}{L}\right)\frac{n\pi}{L}\left\{-\frac{\partial}{\partial x}(xC+D)\right\}, \qquad (7.3.3g)$$

$$\frac{\partial w}{\partial y} = \cos\left(\frac{n\pi y}{L}\right)\frac{n\pi}{L}\left\{-\frac{\partial}{\partial z}(xC+D)\right\}, \qquad (7.3.3h)$$

$$\frac{\partial v}{\partial z} = \cos\left(\frac{n\pi y}{L}\right)\frac{n\pi}{L}\left\{-\frac{\partial}{\partial z}(xC+D)\right\}. \qquad (7.3.3i)$$

Shear stresses τ_{xy} and τ_{zy} vanish along with v at the zeros of $\cos\left(\frac{n\pi y}{L}\right)$, similarly to the last section.

The situation, illustrated by figures 7.2 , is different from that of figures 7.1 as a compression on the surface $y = L/2$ requires a tension on $y = -L/2$. However this can also arise after the simple uniform compression mode ($u = -\nu x$, $v = y$, $w = -\nu z$) has been taken out of the actual loading.

Similarly to the last case, the boundary conditions of "frictionlessly clamped" on the sides $y = \pm L/2$ require that

$$v = \tau_{xy} = \tau_{yz} = 0 \qquad \text{for } y = \pm L/2.$$

The above, in turn, require that

$$\cos\left(\frac{n\pi}{2}\right) = 0,$$

or

$$n = 1, 3, 5, 7, \ldots,$$

for the case illustrated by figures 7.2 .

For problems with the following boundary conditions

$$v = c_1 \, , \tau_{xy} = \tau_{zy} = 0 \quad \text{on the surface } y = L/2,$$

$$v = c_2 \, , \tau_{xy} = \tau_{zy} = 0 \quad \text{on the surface } y = -L/2,$$

it is only a simple step, using translation and uniform compression or tension, to reduce the given conditions to

$$v = \tau_{xy} = \tau_{zy} = 0 \qquad \text{on } \; y = L/2 \, , \tag{7.3.4a}$$

$$v = \tau_{xy} = \tau_{zy} = 0 \qquad \text{on } \; y = -L/2 \, . \tag{7.3.4b}$$

The boundary conditions expressed in (7.3.4) allow the prism and its loading to be extended from the range ($y = -L/2$, $y = L/2$) to the range ($y = -L/2$, $y = 3L/2$). The new double thickness prism then has a periodic loading in the y dimension with a period of 2L. Hence, if only σ_x, σ_z, τ_{xz} are specified on the remaining four sides of the original prism, the treatment in this and the previous sections will *completely* analyse the loading since the loadings considered in these two sections have periods of 2L, 2L/2, 2L/3, 2L/4, ...etc.

It should be emphasised again that if all the stresses σ_x, σ_z, τ_{xz}, τ_{xy} and τ_{zy} are specified on the four unclamped sides of the prism then the treatment in these two sections will *not* be able to analyse the given loading. This is easily seen as all the modes of these two sections can not give certain non-trivial τ_{xy} and τ_{yz} loading while σ_x, σ_z, τ_{xz}, are kept zero on the same four unclamped sides.

7.4 Combined representation for y-symmetric and y-antisymmetric loadings of a frictionlessly clamped, finite thickness prism.

The two preceding sections laid the foundations for this section. For reasons of economy, some properties which would have been investigated in the preceding two sections have been delayed until this section. To be self-contained and also for subsequent references, this section has been made independent of the preceding two sections.

The Papkovich-Neuber form for displacements is written again as

$$u = 4(1-\nu)A_x - \frac{\partial}{\partial x}(xA_x + yA_y + zA_z + A_o) \ , \tag{7.4.1a}$$

$$v = 4(1-\nu)A_y - \frac{\partial}{\partial y}(xA_x + yA_y + zA_z + A_o) \ , \tag{7.4.1b}$$

$$w = 4(1-\nu)A_z - \frac{\partial}{\partial z}(xA_x + yA_y + zA_z + A_o) \ . \tag{7.4.1c}$$

For a finite thickness, frictionlessly clamped prism, the two previous sections showed that the following Papkovich-Neuber functions can describe *completely* all the stresses σ_x, σ_z, τ_{xz} on the four unclamped faces of the prism (except for simple uniform compression or tension):

$$A_x = \frac{1}{\sqrt{2}}\left[\sin\left(\frac{n\pi y}{L} + \frac{\pi}{4}\right) - (-1)^n \sin\left(\frac{n\pi y}{L} - \frac{\pi}{4}\right)\right]C(x,z) \ , \tag{7.4.2a}$$

$$A_o = \frac{1}{\sqrt{2}}\left[\sin\left(\frac{n\pi y}{L} + \frac{\pi}{4}\right) - (-1)^n \sin\left(\frac{n\pi y}{L} - \frac{\pi}{4}\right)\right]D(x,z) \ , \tag{7.4.2b}$$

$$A_y = 0 \ , \tag{7.4.2c}$$

$$A_z = 0 \ . \tag{7.4.2d}$$

The above Papkovich-Neuber functions are equal to those of equations (7.3.1) for odd and positive n; and to those of equations (7.2.2) for even and non-negative n. For zero n, the problem becomes a plane strain one.

The function C(x,z) and D(x,z) are required to satisfy

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) C(x,z) = \frac{n^2\pi^2}{L^2} C(x,z) , \tag{7.4.3a}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) D(x,z) = \frac{n^2\pi^2}{L^2} D(x,z) . \tag{7.4.3b}$$

Displacements are then given as

$$u = \frac{1}{\sqrt{2}}\left[\sin\left(\frac{n\pi y}{L} + \frac{\pi}{4}\right) - (-1)^n \sin\left(\frac{n\pi y}{L} - \frac{\pi}{4}\right)\right]\left\{4(1-\nu)C - \frac{\partial}{\partial x}(xC+D)\right\},$$

$$v = -\frac{n\pi}{L}\frac{1}{\sqrt{2}}\left[\cos\left(\frac{n\pi y}{L} + \frac{\pi}{4}\right) - (-1)^n \cos\left(\frac{n\pi y}{L} - \frac{\pi}{4}\right)\right]\left\{ xC + D \right\},$$

$$w = \frac{1}{\sqrt{2}}\left[\sin\left(\frac{n\pi y}{L} + \frac{\pi}{4}\right) - (-1)^n \sin\left(\frac{n\pi y}{L} - \frac{\pi}{4}\right)\right]\left\{-\frac{\partial}{\partial z}(xC + D)\right\},$$

(7.4.4a, b, c)

and their gradients as

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{2}}\left[\sin\left(\frac{n\pi y}{L} + \frac{\pi}{4}\right) - (-1)^n \sin\left(\frac{n\pi y}{L} - \frac{\pi}{4}\right)\right]\left\{4(1-\nu)\frac{\partial C}{\partial x} - \frac{\partial^2}{\partial x^2}(xC+D)\right\},$$

$$\frac{\partial v}{\partial y} = \frac{1}{\sqrt{2}}\left[\sin\left(\frac{n\pi y}{L} + \frac{\pi}{4}\right) - (-1)^n \sin\left(\frac{n\pi y}{L} - \frac{\pi}{4}\right)\right]\left\{\frac{n^2\pi^2}{L^2}(xC + D)\right\},$$

$$\frac{\partial w}{\partial z} = \frac{1}{\sqrt{2}}\left[\sin\left(\frac{n\pi y}{L} + \frac{\pi}{4}\right) - (-1)^n \sin\left(\frac{n\pi y}{L} - \frac{\pi}{4}\right)\right]\left\{-\frac{\partial^2}{\partial z^2}(xC + D)\right\},$$

(7.4.5a, b, c)

$$\frac{\partial u}{\partial z} = \frac{1}{\sqrt{2}}\left[\sin\left(\frac{n\pi y}{L} + \frac{\pi}{4}\right) - (-1)^n \sin\left(\frac{n\pi y}{L} - \frac{\pi}{4}\right)\right]\left\{4(1-\nu)\frac{\partial C}{\partial z} - \frac{\partial^2}{\partial x\partial z}(xC+D)\right\},$$

$$\frac{\partial w}{\partial x} = \frac{1}{\sqrt{2}}\left[\sin\left(\frac{n\pi y}{L} + \frac{\pi}{4}\right) - (-1)^n \sin\left(\frac{n\pi y}{L} - \frac{\pi}{4}\right)\right]\left\{-\frac{\partial^2}{\partial x\partial z}(xC + D)\right\},$$

(7.4.5d, e)

$$\frac{\partial u}{\partial y} = \frac{n\pi}{L}\frac{1}{\sqrt{2}}\left[\cos\left(\frac{n\pi y}{L} + \frac{\pi}{4}\right) - (-1)^n \cos\left(\frac{n\pi y}{L} - \frac{\pi}{4}\right)\right]\left\{4(1-\nu)C - \frac{\partial}{\partial x}(xC+D)\right\},$$

$$\frac{\partial v}{\partial x} = \frac{n\pi}{L}\frac{1}{\sqrt{2}}\left[\cos\left(\frac{n\pi y}{L} + \frac{\pi}{4}\right) - (-1)^n \cos\left(\frac{n\pi y}{L} - \frac{\pi}{4}\right)\right]\left\{-\frac{\partial}{\partial x}(xC+D)\right\},$$

(7.4.5f, g)

$$\frac{\partial w}{\partial y} = \frac{n\pi}{L}\frac{1}{\sqrt{2}}\left[\cos\left(\frac{n\pi y}{L} + \frac{\pi}{4}\right) - (-1)^n \cos\left(\frac{n\pi y}{L} - \frac{\pi}{4}\right)\right]\left\{-\frac{\partial}{\partial z}(xC + D)\right\},$$

$$\frac{\partial v}{\partial z} = \frac{n\pi}{L}\frac{1}{\sqrt{2}}\left[\cos\left(\frac{n\pi y}{L} + \frac{\pi}{4}\right) - (-1)^n \cos\left(\frac{n\pi y}{L} - \frac{\pi}{4}\right)\right]\left\{-\frac{\partial}{\partial z}(xC + D)\right\}.$$

(7.4.5h, i)

The formula for σ_z gives

$$\frac{1+\nu}{E}\ \sigma_x = \frac{\partial u}{\partial x} + \frac{\nu}{1-2\nu}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right).$$

With the values of the gradients of displacements as given by (7.4.5), the stress σ_x becomes

$$\frac{1+\nu}{E}\,\sigma_x = \frac{1}{\sqrt{2}}\left[\sin(\frac{n\pi y}{L}+\frac{\pi}{4})-(-1)^n\sin(\frac{n\pi y}{L}-\frac{\pi}{4})\right]\left\{2(1-\nu)\frac{\partial C}{\partial x} - x\frac{\partial^2 C}{\partial x^2} - \frac{\partial^2 D}{\partial x^2}\right\}. \tag{7.4.6a}$$

Similarly, the other two direct stresses are given by

$$\frac{1+\nu}{E}\,\sigma_y = \frac{1}{\sqrt{2}}\left[\sin(\frac{n\pi y}{L}+\frac{\pi}{4})-(-1)^n\sin(\frac{n\pi y}{L}-\frac{\pi}{4})\right]\left\{\frac{n^2\pi^2}{L^2}(xC+D) + 2\nu\,\frac{\partial C}{\partial x}\right\}, \tag{7.4.6b}$$

and

$$\frac{1+\nu}{E}\,\sigma_z = \frac{1}{\sqrt{2}}\left[\sin(\frac{n\pi y}{L}+\frac{\pi}{4})-(-1)^n\sin(\frac{n\pi y}{L}-\frac{\pi}{4})\right]\left\{2\nu\,\frac{\partial C}{\partial x} - \frac{\partial^2}{\partial z^2}(xC+D)\right\}. \tag{7.4.6c}$$

The shear stresses are given by equations (1.1.1) and (1.1.3). For τ_{xz}, the formula is

$$\frac{2(1+\nu)}{E}\,\tau_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\ .$$

With the values of the terms on the right hand side given in (7.4.5), the value of τ_{xz} is

$$\frac{1+\nu}{E}\,\tau_{xz} = \frac{1}{\sqrt{2}}\left[\sin(\frac{n\pi y}{L}+\frac{\pi}{4})-(-1)^n\sin(\frac{n\pi y}{L}-\frac{\pi}{4})\right]\left\{2(1-\nu)\frac{\partial C}{\partial z} - \frac{\partial^2}{\partial x\partial z}(xC+D)\right\}. \tag{7.4.6d}$$

Similarly, the value for τ_{xy} and τ_{yz} are given by

$$\frac{1+\nu}{E}\,\tau_{xy} = \frac{n\pi}{L}\frac{1}{\sqrt{2}}\left[\cos(\frac{n\pi y}{L}+\frac{\pi}{4})-(-1)^n\cos(\frac{n\pi y}{L}-\frac{\pi}{4})\right]\left\{2(1-\nu)C - \frac{\partial}{\partial x}(xC+D)\right\}, \tag{7.4.6e}$$

and

$$\frac{1+\nu}{E}\,\tau_{yz} = \frac{n\pi}{L}\frac{1}{\sqrt{2}}\left[\cos(\frac{n\pi y}{L}+\frac{\pi}{4})-(-1)^n\cos(\frac{n\pi y}{L}-\frac{\pi}{4})\right]\left\{-\frac{\partial}{\partial z}(xC + D)\right\}. \tag{7.4.6f}$$

From the equations (7.4.4) and (7.4.6), it is clear that τ_{xy}, τ_{zy} and v all vanish at the zeros of $\left[\cos(\frac{n\pi y}{L}+\frac{\pi}{4})-(-1)^n\cos(\frac{n\pi y}{L}-\frac{\pi}{4})\right]$. It is also quite simple to show that the quantity in the square brackets vanishes for all non-negative, integer value of n on the surfaces $y = \pm L/2$. Hence the functions given in the equations (7.4.2) are the ones for the problem of a frictionlessly clamped prism.

As have been mentioned earlier, after the simple uniform compression mode has been taken out, the loading of σ_x, σ_z, τ_{xz} on the remaining four sides of the clamped prism can be decomposed into y-symmetric and y-antisymmetric components. Each component can then be extended from the range (-L/2, L/2) to the range (-L/2, 3L/2). The loading on the new double thickness prism can be Fourier analysed *completely* with all the modes of loading presented in this section, with n = 1, 2, 3, ...(This is possible since simple uniform compression has been taken out).

Equations (7.4.6d) and (7.4.6e) show that τ_{xz} and τ_{xy} are related by

$$\frac{\partial}{\partial z}\tau_{xy} = \frac{n\pi}{L}\ \frac{\cos(\frac{n\pi y}{L}+\frac{\pi}{4}) - (-1)^n\cos(\frac{n\pi y}{L}-\frac{\pi}{4})}{\sin(\frac{n\pi y}{L}+\frac{\pi}{4}) - (-1)^n\sin(\frac{n\pi y}{L}-\frac{\pi}{4})}\ \tau_{xz}\ . \qquad (7.4.7)$$

On the surfaces of constant x, the equation (7.4.7) prevent τ_{xy} from being specified independently of τ_{xz}. Hence, up to this point, it is only certain that all the Papkovich-Neuber functions presented here can completely analyse σ_x, σ_z, and τ_{xz} when τ_{xy} and τ_{yz} are not specified on the four unclamped surfaces of the prism.

A further observation reveals that there is no constraint at all on the values of σ_z, τ_{xz} and τ_{yz} on a surface of constant z. Hence, it is only a simple proof to show that the loading on a clamped prism can be completely described by the three Papkovich-Neuber functions A_o, A_x and A_z, which are

$$A_x = \frac{1}{\sqrt{2}}\left[\sin(\frac{n\pi y}{L}+\frac{\pi}{4})-(-1)^n\sin(\frac{n\pi y}{L}-\frac{\pi}{4})\right]C(x,z)\,, \qquad (7.4.8a)$$

$$A_o = \frac{1}{\sqrt{2}}\left[\sin(\frac{n\pi y}{L}+\frac{\pi}{4})-(-1)^n\sin(\frac{n\pi y}{L}-\frac{\pi}{4})\right]D(x,z)\,, \qquad (7.4.8b)$$

$$A_z = \frac{1}{\sqrt{2}}\left[\sin(\frac{n\pi y}{L}+\frac{\pi}{4})-(-1)^n\sin(\frac{n\pi y}{L}-\frac{\pi}{4})\right]E(x,z)\,, \qquad (7.4.8c)$$

$$A_y = 0\ . \qquad (7.4.8d)$$

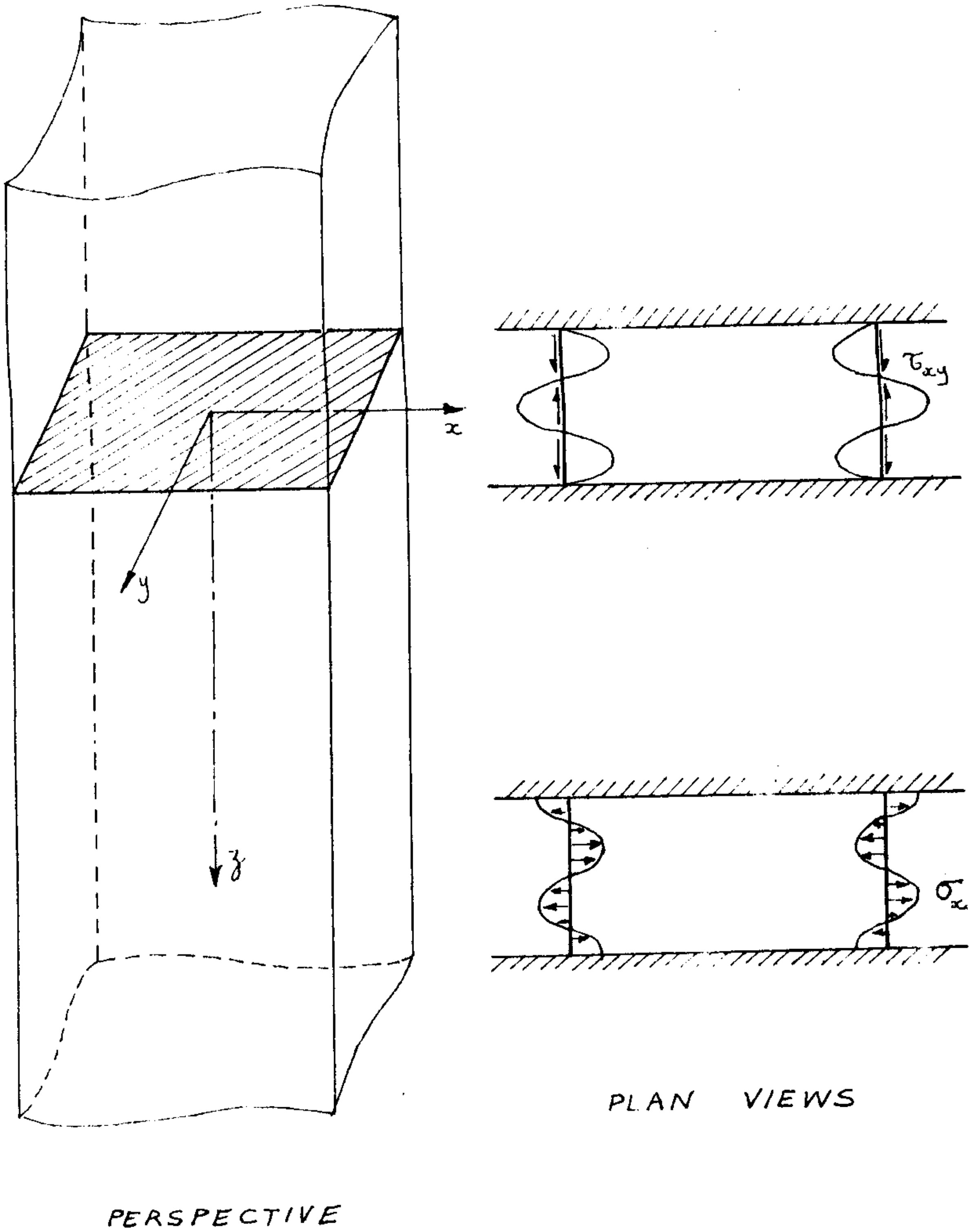


FIGURE 7.3: SIDE LOADING OF A CLAMPED PRISM

The functions C, D and E are required to satisfy

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) C(x,z) = \frac{n^2\pi^2}{L^2} C(x,z) , \tag{7.4.9a}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) D(x,z) = \frac{n^2\pi^2}{L} D(x,z) , \tag{7.4.9b}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) E(x,z) = \frac{n^2\pi^2}{L} E(x,z) . \tag{7.4.9c}$$

However, for the rest of the chapter, only the forms (7.4.2) and (7.4.3) are used and any loading of τ_{xy} and τ_{yz} is considered as of only secondary importance.

7.5 Side loading of a frictionlessly clamped, finite thickness, infinitely long, rectangular prism.

The problems of an infinitely thick (or thin) rectangular strip of infinite length have been previously treated, as mentioned in section 7.1 . In this section, the problems will be generalised, by the use of constituent modes for a clamped, finite thickness, infinitely long prism. The problem is defined as in figures 7.3 for the case of n = 3.

The Papkovich-Neuber functions for a frictionlessly clamped prism is given by the last section as

$$A_x = \frac{1}{\sqrt{2}} \left[\sin\left(\frac{n\pi y}{L} + \frac{\pi}{4}\right) - (-1)^n \sin\left(\frac{n\pi y}{L} - \frac{\pi}{4}\right)\right] C(x,z), \tag{7.5.1a}$$

$$A_o = \frac{1}{\sqrt{2}} \left[\sin\left(\frac{n\pi y}{L} + \frac{\pi}{4}\right) - (-1)^n \sin\left(\frac{n\pi y}{L} - \frac{\pi}{4}\right)\right] D(x,z), \tag{7.5.1b}$$

$$A_y = 0, \tag{7.5.1c}$$

$$A_z = 0. \tag{7.5.1d}$$

Take the functions C(x,z) and D(x,z) as

$$C(x,z) = \left[B_1\cosh(\gamma x) + B_2\sinh(\gamma x)\right]\sin(\delta z) , \tag{7.5.2a}$$

$$D(x,z) = \left[B_3\cosh(\gamma x) + B_4\sinh(\gamma x)\right]\sin(\delta z) , \tag{7.5.2b}$$

with B_1, B_2, B_3, B_4, γ, δ being six constants; γ and δ are further required to satisfy

$$\gamma^2 - \delta^2 = \frac{n^2\pi^2}{L^2} \quad . \tag{7.5.3}$$

The conditions on the clamped surfaces, namely

$$v = \tau_{xy} = \tau_{zy} = 0 \qquad \text{for } y = \pm L/2 \ , \tag{7.5.4}$$

are automatically satisfied, due to the proof in the last section. It remains to see what kind of loading can be applied to the remaining sides ($x = \pm 1$) of the prism.

The equation for direct stress σ_x is

$$\frac{1+\nu}{E}\sigma_x = \frac{\partial u}{x} + \frac{\nu}{1-2\nu}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) .$$

From this and the first three of equations (7.4.5), σ_x is given as

$$\frac{1+\nu}{E}\sigma_x = \frac{1}{\sqrt{2}}\left[\sin\left(\frac{n\pi y}{L} + \frac{\pi}{4}\right) - (-1)^n \sin\left(\frac{n\pi y}{L} - \frac{\pi}{4}\right)\right]\left\{2(1-\nu)\frac{\partial C}{\partial x} - \left(x\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 D}{\partial x^2}\right)\right\},$$

or

$$\begin{aligned}\frac{1+\nu}{E}\sigma_x = {} & \frac{1}{\sqrt{2}}\left[\sin\left(\frac{n\pi y}{L} + \frac{\pi}{4}\right) - (-1)^n \sin\left(\frac{n\pi y}{L} - \frac{\pi}{4}\right)\right] \sin(\delta z) \times \\ & \times \{\left[2(1-\nu)\sinh(\gamma x) - x\gamma\cosh(\gamma x)\right]\gamma B_1 + \\ & + \left[2(1-\nu)\cosh(\gamma x) - x\gamma\sinh(\gamma x)\right]\gamma B_2 - \gamma^2\cosh(\gamma x) B_3 \\ & - \gamma^2 \sinh(\gamma x) B_4\} \ .\end{aligned} \tag{7.5.5}$$

Similarly, from the equations for shear stresses

$$\frac{2(1+\nu)}{E}\tau_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} ,$$

it is deduced that

$$\frac{2(1+\nu)}{E}\tau_{xz} = \frac{1}{\sqrt{2}}\left[\sin\left(\frac{n\pi y}{L} + \frac{\pi}{4}\right) - (-1)^n \sin\left(\frac{n\pi y}{L} - \frac{\pi}{4}\right)\right]\left\{4(1-\nu)\frac{\partial C}{\partial z} - 2\frac{\partial^2}{\partial x \partial z}(xC+D)\right\},$$

or

$$\frac{1+\nu}{E}\tau_{xz}=\frac{1}{\sqrt{2}}\left[\sin\left(\frac{n\pi y}{L}+\frac{\pi}{4}\right)-(-1)^n\sin\left(\frac{n\pi y}{L}-\frac{\pi}{4}\right)\right]\cos(\delta z)\times$$

$$\times\{\left[(1-2\nu)\cosh(\gamma x)-x\gamma\sinh(\gamma x)\right]\delta B_1+$$

$$+\left[(1-2\nu)\sinh(\gamma x)-x\gamma\cosh(\gamma x)\right]\delta B_2-\gamma\delta\sinh(\gamma x)B_3$$

$$-\gamma\delta\cosh(\gamma x)B_4\}\ . \qquad (7.5.6)$$

If the loadings on the sides of $x = \pm 1$ are given as

$$\frac{1+\nu}{E}\sigma_x=\frac{1}{\sqrt{2}}\left[\sin\left(\frac{n\pi y}{L}+\frac{\pi}{4}\right)-(-1)^n\sin\left(\frac{n\pi y}{L}-\frac{\pi}{4}\right)\right]\sin(\delta z)K_1 \text{ at } x=1,$$

$$\frac{1+\nu}{E}\sigma_x=\frac{1}{\sqrt{2}}\left[\sin\left(\frac{n\pi y}{L}+\frac{\pi}{4}\right)-(-1)^n\sin\left(\frac{n\pi y}{L}-\frac{\pi}{4}\right)\right]\sin(\delta z)K_2 \text{ at } x=-1,$$

$$\frac{1+\nu}{E}\tau_{xz}=\frac{1}{\sqrt{2}}\left[\sin\left(\frac{n\pi y}{L}+\frac{\pi}{4}\right)-(-1)^n\sin\left(\frac{n\pi y}{L}-\frac{\pi}{4}\right)\right]\cos(\delta z)L_1 \text{ at } x=1,$$

$$\frac{1+\nu}{E}\tau_{xz}=\frac{1}{\sqrt{2}}\left[\sin\left(\frac{n\pi y}{L}+\frac{\pi}{4}\right)-(-1)^n\sin\left(\frac{n\pi y}{L}-\frac{\pi}{4}\right)\right]\cos(\delta z)L_2 \text{ at } x=-1,$$

where K_1, K_2, L_1, L_2 are constants, then the constants B_1, B_2, B_3 and B_4 can be determined from the following system of equations

$$\left[2(1-\nu)\sinh(\gamma)-\gamma\cosh(\gamma)\right]\gamma B_1+\left[2(1-\nu)\cosh(\gamma)-\gamma\sinh(\gamma)\right]\gamma B_2$$

$$-\gamma^2\cosh(\gamma)B_3-\gamma^2\sinh(\gamma)B_4=K_1\ , \qquad (7.5.7a)$$

$$-\left[2(1-\nu)\sinh(\gamma)-\gamma\cosh(\gamma)\right]\gamma B_1+\left[2(1-\nu)\cosh(\gamma)-\gamma\sinh(\gamma)\right]\gamma B_2$$

$$-\gamma^2\cosh(\gamma)B_3+\gamma^2\sinh(\gamma)B_4=K_2\ , \qquad (7.5.7b)$$

$$\left[(1-2\nu)\cosh(\gamma)-\gamma\sinh(\gamma)\right]\delta B_1+\left[(1-2\nu)\sinh(\gamma)-\gamma\cosh(\gamma)\right]\delta B_2$$

$$-\gamma\delta\sinh(\gamma)B_3-\gamma\delta\cosh(\gamma)B_4=L_1\ , \qquad (7.5.7c)$$

$$\left[(1-2\nu)\cosh(\gamma)-\gamma\sinh(\gamma)\right]\delta B_1-\left[(1-2\nu)\sinh(\gamma)-\gamma\cosh(\gamma)\right]\delta B_2$$

$$+\gamma\delta\sinh(\gamma)B_3-\gamma\delta\cosh(\gamma)B_4=L_2\ . \qquad (7.5.7d)$$

The natural coupling between B_1 and B_4, also between B_2 and B_3 enables the above system to be split into two systems of two equations in two unknowns. One of the new system contains only B_1 and B_4, the other contains B_2 and B_3. The solutions for (7.5.7) are therefore very simple.

The system (7.5.7) is only slightly more complicated than the one for plane strain case. It does reduce to the plane strain system when n is equal to zero. A comparison to the plane strain system such as the one in [23], pp. 53-60, can be made with the use of equations from section 5.4.

Similar results are also obtained when $\sin(\delta z)$ and $\cos(\delta z)$ are replaced by $\cos(\delta z)$ and $-\sin(\delta z)$.

After all the coefficients B_1, B_2, B_3 and B_4 are determined, the shear stress τ_{xy} is given by (7.4.6e) as

$$\frac{1+\nu}{E}\tau_{xy} = \frac{n\pi}{L}\frac{1}{\sqrt{2}}\left[\cos\left(\frac{n\pi y}{L}+\frac{\pi}{4}\right)-(-1)^n\cos\left(\frac{n\pi y}{L}-\frac{\pi}{4}\right)\right]\sin(\delta z)\times$$

$$\times\{[(1-2\nu)\cosh(\gamma x)-x\gamma\sinh(\gamma x)]B_1$$

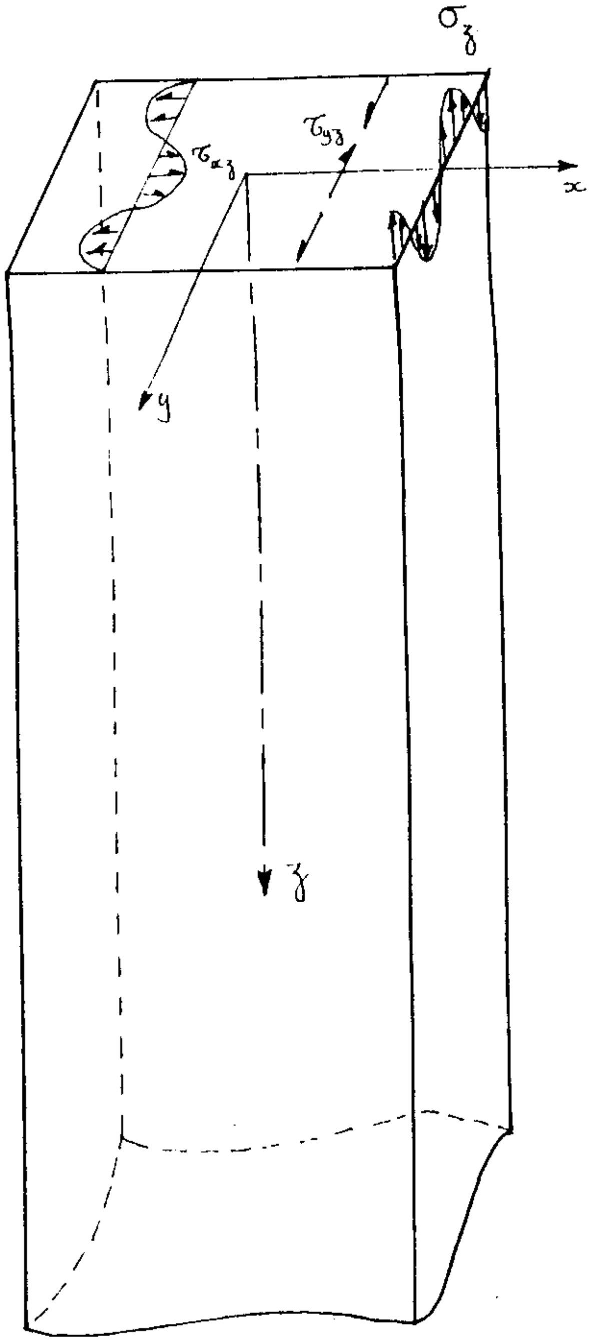
$$+[(1-2\nu)\sinh(\gamma x)-x\gamma\cosh(\gamma x)]B_2-[\gamma\sinh(\gamma x)]B_3$$

$$-[\gamma\cosh(\gamma x)]B_4\}\ .$$

On the boundary surfaces of $x = \pm 1$, the two stresses τ_{xy} and τ_{xz} are obviously connected by equation (7.4.7). In order that the solution in this section be applicable, the boundary shear stress must be set to the above calculated stress.

As far as the method here is concerned, all applications of Fourier series to the plane-strain strip problem apply equally to this clamped, rectangular prism problem (provided τ_{yx} and τ_{yz} are left to be determined by the solution). The problem of a point force applied to the side of a plane strain strip (see [31], [32], [33], [34], [35]) has its equivalent as forces applied to a line ($x = 1$ and z = constant) on a clamped prism. The latter problem can be solved analogously to the former if τ_{yx} and τ_{yz} are left to be determined by the solution.

Similarly to the point force problem, the side loading of a plane strain strip also has its equivalent. If τ_{yx} and τ_{yz} are left to be determined by the solution then the problem of side



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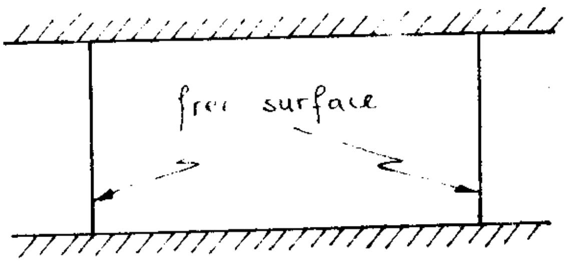


PLAN VIEW

FIGURE 7.4:

END LOADING OF A CLAMPED PRISM

loading of a prism will be easily solved by the use of the constituent modes presented in this section.

7.6 End loading of a finite thickness, frictionlessly clamped, semi-infinite, rectangular prism.

As in section 7.5, the problem of end loading of a semi-infinite rectangular strip has been previously investigated, but only for plane strain case. In this section, the problem will be generalised to the end loading of a finite thickness, clamped semi-infinitely long rectangular prism. The problem is defined as in figures 7.4.

Again, the Papkovich-Neuber functions are given by (7.4.2)

$$A_x = \frac{1}{\sqrt{2}}\left[\sin\left(\frac{n\pi y}{L}+\frac{\pi}{4}\right)-(-1)^n\sin\left(\frac{n\pi y}{L}-\frac{\pi}{4}\right)\right]C(x,z)\ , \qquad (7.6.1a)$$

$$A_o = \frac{1}{\sqrt{2}}\left[\sin\left(\frac{n\pi y}{L}+\frac{\pi}{4}\right)-(-1)\ \sin\left(\frac{n\pi y}{L}-\frac{\pi}{4}\right)\right]D(x,z)\ , \qquad (7.6.1b)$$

$$A_y = 0\ , \qquad (7.6.1c)$$

$$A_z = 0\ . \qquad (7.6.1d)$$

In this case, the function C(x,z) and D(x,z) are complex (the reason for this will become clear at the end of the section), being

$$C(x,z) = B_1\cosh(\gamma x)e^{i\delta z}\ , \qquad (7.6.2a)$$

$$D(x,z) = B_4\sinh(\gamma x)e^{i\delta z}\ , \qquad (7.6.2b)$$

where B_1, B_4, γ, δ are complex constants, with γ, δ satisfying

$$\gamma^2 - \delta^2 = \frac{n^2\pi^2}{L^2}\ . \qquad (7.6.3)$$

The conditions on the clamped surfaces, namely

$$v = \tau_{xy} = \tau_{zy} = 0 \quad \text{for} \quad y = \pm L/2 \qquad (7.6.4)$$

are automatically satisfied.

Manipulations similar to those in the last section lead to

$$\frac{1+\nu}{E}\sigma_x = \frac{1}{\sqrt{2}}\left[\sin\left(\frac{n\pi y}{L}+\frac{\pi}{4}\right)-(-1)^n\sin\left(\frac{n\pi y}{L}-\frac{\pi}{4}\right)\right]e^{i\delta z}\times$$

$$\times\ \{\left[2(1-\nu)\sinh(\gamma x)-x\gamma\cosh(\gamma x)\right]\gamma B_1-\gamma^2\sinh(\gamma x)B_4\},\tag{7.6.5a}$$

$$\frac{1+\nu}{E}\tau_{xz}= \frac{1}{\sqrt{2}}\left[\sin\left(\frac{n\pi y}{L}+\frac{\pi}{4}\right)-(-1)^n\sin\left(\frac{n\pi y}{L}-\frac{\pi}{4}\right)\right]i\delta e^{i\delta z}\times$$

$$\times\ \{\left[(1-2\nu)\cosh(\gamma x)-x\gamma\sinh(\gamma x)\right]B_1-\gamma\cosh(\gamma x)B_4\},\tag{7.6.6a}$$

and

$$\frac{1+\nu}{E}\tau_{xy}= \frac{n\pi}{L}\frac{1}{\sqrt{2}}\left[\cos\left(\frac{n\pi y}{L}+\frac{\pi}{4}\right)-(-1)^n\cos\left(\frac{n\pi y}{L}-\frac{\pi}{4}\right)\right]e^{i\delta z}\times$$

$$\times\ \{\left[(1-2\nu)\cosh(\gamma x)-x\gamma\sinh(\gamma x)\right]B_1-\gamma\cosh(\gamma x)B_4\}.\tag{7.6.7a}$$

The conditions of no loading on the two sides $x = \pm 1$, namely

$$\sigma_{xx} = \tau_{xy} = \tau_{xz} = 0 \qquad \text{for} \quad x = \pm 1,$$

require that the determinant of the following system

$$\left[2(1-\nu)\sinh(\gamma) - \gamma\cosh(\gamma)\right]\gamma B_1 - \gamma^2\sinh(\gamma)B_4 = 0\ ,$$

$$\left[(1-2\nu)\cosh(\gamma) - \gamma\sinh(\gamma)\right]\delta B_1 - \gamma\delta\cosh(\gamma)B_4 = 0$$

to be zero. This leads to the following eigenvalue equation

$$0 = 2\gamma - \sinh(2\gamma)\ .\tag{7.6.8a}$$

Obviously, $\gamma = 0$ is one root of the above equation, but this value of γ correspond to no loading at all on the prism, hence only the non-zero roots of (7.6.8a) are of interest. The first ten (complex) roots of (7.6.8a) are given by table 7.1 .

If the functions $C(x,z)$, $D(x,z)$ are instead taken as

$$C(x,z) = B_2\sinh(\gamma x)e^{i\delta z}\ ,\tag{7.6.2c}$$

$$D(x,z) = B_3\cosh(\gamma x)e^{i\delta z}\ ,\tag{7.6.2d}$$

where B_2, B_3, γ, δ are complex constants with γ, δ still satisfying

(7.6.3), then a similar treatment gives

$$\frac{1+\nu}{E}\,\sigma_x = \frac{1}{\sqrt{2}}\left[\sin\left(\frac{n\pi y}{L}+\frac{\pi}{4}\right)-(-1)^n\sin\left(\frac{n\pi y}{L}-\frac{\pi}{4}\right)\right]e^{i\delta z}\times$$

$$\times\ \{\left[2(1-\nu)\cosh(\gamma x)-x\gamma\sinh(\gamma x)\right]\gamma B_2-\gamma^2\cosh(\gamma x)B_3\}, \quad (7.6.5b)$$

$$\frac{1+\nu}{E}\,\tau_{xz} = \frac{1}{\sqrt{2}}\left[\sin\left(\frac{n\pi y}{L}+\frac{\pi}{4}\right)-(-1)^n\sin\left(\frac{n\pi y}{L}-\frac{\pi}{4}\right)\right]i\delta e^{i\delta z}\times$$

$$\times\ \{\left[(1-2\nu)\sinh(\gamma x)-x\gamma\cosh(\gamma x)\right]B_2-\gamma\sinh(\gamma x)B_3\}, \quad (7.6.6b)$$

$$\frac{1+\nu}{E}\,\tau_{xy} = \frac{n\pi}{L}\frac{1}{\sqrt{2}}\left[\cos\left(\frac{n\pi y}{L}+\frac{\pi}{4}\right)-(-1)^n\cos\left(\frac{n\pi y}{L}-\frac{\pi}{4}\right)\right]e^{i\delta z}\times$$

$$\times\ \{\left[(1-2\nu)\sinh(\gamma x)-x\gamma\cosh(\gamma x)\right]B_2-\gamma\sinh(\gamma x)B_3\}. \quad (7.6.7b)$$

The conditions on the clamped surfaces, being

$$v = \tau_{xy} = \tau_{yz} = 0 \quad \text{for} \quad y = \pm L/2 \qquad (7.6.4)$$

are automatically satisfied by the choice of the Papkovich-Neuber functions. The conditions on the remaining sides of $x = \pm 1$ lead to the following equations

$$\left[2(1-\nu)\cosh(\gamma) - \gamma\sinh(\gamma)\right]\gamma B_2 - \gamma^2\cosh(\gamma)B_3 = 0,$$

and

$$\left[(1-2\nu)\sinh(\gamma) - \gamma\cosh(\gamma)\right]\delta B_2 - \gamma\delta\sinh(\gamma)B_3 = 0.$$

The eigenvalue equation for the above system is

$$0 = 2\gamma + \sinh(2\gamma). \qquad (7.6.8b)$$

Table 7.1 gives the first ten non-zero roots of (7.6.8b).

The roots of (7.6.8) appear in quadruple, i.e. if γ is a root to either (7.6.8a) or (7.6.8b) so are $(-\gamma)$, $\bar{\gamma}$, $(-\bar{\gamma})$ to the same equation.

The modulus of the roots of the equations (7.6.8) increases steadily as their order increases. The assymptotic formula for large roots of (7.6.8) presented here is

$$\gamma_m = \frac{\ln[(2m+1)\pi]}{2} + \frac{2\{\ln[(2m+1)\pi]\}^2-4\{\ln[(2m+1)\pi]\} -1}{2[(2m+1)\pi]^2}$$

$$+ i \left\{ \frac{(2m+1)\pi}{4} - \frac{\ln[(2m+1)\pi]}{(2m+1)\pi} \right\} , \qquad (7.6.9)$$

with $m = 1, 3, 5, \ldots$ for roots of $\sinh(2\gamma_m) + 2\gamma_m = 0$,

$m = 2, 4, 6, \ldots$ for roots of $\sinh(2\gamma_m) - 2\gamma_m = 0$.

The assymptotic expression (7.6.9) for γ_m is essentially in agreement with that appearing in [36] except that the latter contains an error of $\frac{1}{2}(2m+1)^{-2}$, and also it applies only to the case of odd m.

The order of the remainder of the above assymptotic formula is

$$O\left(\frac{\text{remainder } R(\gamma_m)}{R(\gamma_m)}\right) = O\left(\frac{\text{remainder } I(\gamma_m)}{I(\gamma_m)}\right) = \frac{\{\ln[(2m+1)\pi]\}^3}{[(2m+1)\pi]^4} . \qquad (7.6.10)$$

For m = 19, that is for the tenth non-zero root of (7.6.8b), the order of the remainder is

$$O\left(\frac{\text{remainder } R(\gamma_m)}{R(\gamma_m)}\right) = O\left(\frac{\text{remainder } I(\gamma_m)}{I(\gamma_m)}\right) \simeq 5\times10^{-7},$$

and the precise and assymptotic values of the root γ_{19} are listed for comparison as

$\gamma_{19} = 2.405013 + 30.591295$ (precise value),

and $\gamma_{19} = 2.40501(2) + 30.5912(84)$ (assymptotic value).

It may be asked why only certain combinations of B_1, B_2, B_3 and B_4 are used in the equations (7.6.2). The answer is that those combinations are the ones involved in the natural coupling of the equations in the system (7.5.7).

As mentioned earlier, the roots γ_m of (7.6.8) occur in quadruples. This fact, together with (7.6.3) lead to the values of δ occur also in quadruples of $(\delta, -\delta, \bar{\delta}, -\bar{\delta})$. However, for each value of γ_m, the corresponding value of δ (or $-\delta$) must share the same quadrant of the complex plane. By choosing suitable combinations of the complex modes of loading, it is possible to sum them up giving real modes of loading.

For example, with L = 1, n = 2, m = 4, the values of γ_m and $\delta_{m,n}$ are given by table 7.2 as

$$\gamma = 1.6761 + i6.9500 ,$$

$$\delta = 1.2522 + i9.3027 .$$

The functions A_x, A_o are then

$$A_x = \cos(2\pi y)\cosh(\gamma x)e^{i\delta z} ,$$
$$A_o = \frac{B_4}{B_1}\cos(2\pi y)\sinh(\gamma x)e^{i\delta z} , \qquad (7.6.11)$$

with (B_1/B_4) given by

$$\frac{B_1}{B_4} = \frac{2(1-\nu)\sinh(\gamma)-\gamma\cosh(\gamma)}{\gamma\sinh(\gamma)} .$$

As δ can be replaced by $(-\delta)$ without upsetting any equation derived in this section, there is another solution

$$A_x = \cos(2\pi y)\cosh(\bar{\gamma}x)e^{-i\bar{\delta}z},$$
$$A_o = \overline{\left(\frac{B_4}{B_1}\right)}\cos(2\pi y)\sinh(\bar{\gamma}x)e^{-i\bar{\delta}z}. \qquad (7.6.11a)$$

The right hand side members of (7.6.11) and (7.6.11a) are just complex conjugates of one another. Hence the real and the imaginary parts of A_x and A_o in (7.6.11) are each a solution to the problem.

Corresponding to (7.6.11), displacements and stresses under consideration are

$$u = \cos(2\pi y)e^{i\delta z}\{(3-4\nu-\frac{B_4}{B_1}\gamma)\cosh(\gamma x) - x\gamma\sinh(\gamma x)\} ,$$

$$v = 2\pi\sin(2\pi y)e^{i\delta z}\{x\cosh(\gamma x) + \frac{B_4}{B_1}\sinh(\gamma x)\} , \qquad (7.6.12)$$

$$w = -\cos(2\pi y)i\delta e^{i\delta z}\{x\cosh(\gamma x) + \frac{B_4}{B_1}\sinh(\gamma x)\} ,$$

and

$$\frac{1+\nu}{E}\sigma_x = \cos(2\pi y)e^{i\delta z}\{\left[2(1-\nu)\sinh(\gamma x)-x\gamma\cosh(\gamma x)\right]\gamma-\gamma^2\sinh(\gamma x)\frac{B_4}{B_1}\} ,$$

$$\frac{1+\nu}{E}\sigma_z = \cos(2\pi y)e^{i\delta z}\{(2\nu\gamma+\delta^2\frac{B_4}{B_1})\sinh(\gamma x)+\delta^2 x\cosh(\gamma x)\} ,$$

$$\frac{1+\nu}{E}\tau_{xz} = \cos(2\pi y)i\delta e^{i\delta z}\{\left[(1-2\nu)\cosh(\gamma x)-x\gamma\sinh(\gamma x)\right]-\gamma\cosh(\gamma x)\frac{B_4}{B_1}\} ,$$

$$\frac{1+\nu}{E}\sigma_y = \cos(2\pi y)e^{i\delta z}\{\left[2\nu\gamma+4\pi^2\frac{B_4}{B_1}\right]\sinh(\gamma x)+4\pi^2 x\cosh(\gamma x)\} ,$$

$$\frac{1+\nu}{E}\tau_{xy} = -\sin(2\pi y)2\pi e^{i\delta z}\{-x\gamma\sinh(\gamma x)+(1-2\nu-\frac{B_4}{B_1}\gamma)\cosh(\gamma x)\} ,$$

$$\frac{1+\nu}{E}\tau_{yz} = -\sin(2\pi y)2\pi i\delta e^{i\delta z}\{-\frac{B_4}{B_1}\sinh(\gamma x)-x\cosh(\gamma x)\} .$$

(7.6.13a, b, c, d, e, f)

When (γ, δ) are replaced by $(\overline{\gamma},-\overline{\delta})$ the right hand side members of (7.6.12) and (7.6.13) change into their complex conjugates. Hence the real (or imaginary) parts of equations (7.6.11), (7.6.12), (7.6.13) form a real solution for the problem of end loading of a clamped prism.

Similar results are also obtained for different values of L, n and m.

From table 7.2 , or alternatively by a simple proof, the argument of δ is known to be closer to the value of $\pi/2$ than the argument of γ. Also the modulus of δ is always greater than that of γ. Hence the eigenfunctions of this problem decay faster with z as n/L increases (The plane strain case corresponds to n equal to zero).

For any loading σ_z, τ_{xz} on the end of a clamped semi-infinite rectangular prism, the loading can be analysed into a Fourier series in the y direction, each term of the series is a constituent mode considered in this section. The sum of these constituent modes gives the required end loading. Similarly to the previous sections, although τ_{yx} is zero on the sides $x = \pm 1$, the shear stress τ_{yz} must be left to be determined by the solution.

7.7 Conclusions.

The problem of a frictionlessly clamped rectangular prism is not much more difficult than the problem of a plane strain rectangular strip (provided that τ_{xy} and τ_{yz} are left to be determined by the solutions). The difference is only in the modulation of displacements and stresses by sine and cosine functions across the thickness y, and the more rapid modulation along the z direction, compared to the plane strain case.

If τ_{xy} and τ_{yz} are included in the boundary conditions for this problem then the complete Papkovich-Neuber solution (7.4.8) must be used. Although this Papkovich-Neuber solution is complete, the matching of all the stresses σ_x, σ_z, τ_{xz}, τ_{xy}, τ_{yz} on the four unclamped surfaces is not simple, and is beyond the scope of this chapter.

Table 7.1: Roots of equations (7.6.8), as appeared in [37] (see also [38] and [39]).

Order of root	Roots of $\sinh(2\gamma)+2\gamma = 0$		Roots of $\sinh(2\gamma)-2\gamma = 0$	
	Real	Imaginary	Real	Imaginary
1	1.125 365	2.106 196	1.384 339	3.748 838
2	1.551 575	5.356 269	1.676 105	6.949 980
3	1.775 544	8.536 683	1.858 384	10.119 259
4	1.929 405	11.699 178	1.991 571	13.277 274
5	2.046 853	14.854 060	2.096 626	16.429 872
6	2.141 891	18.004 933	2.183 398	19.579 409
7	2.221 723	21.153 414	2.257 320	22.727 036
8	2.290 553	24.300 342	2.321 714	25.873 384
9	2.351 048	27.446 203	2.378 758	29.018 831
10	2.405 013	30.591 295	2.429 959	32.163 617

Table 7.2: Typical values of δ (L = 1 in this table).

m \ n	0	1	2	3	4
1	1.1254 +2.1061i	0.6461 +3.6683i	0.3624 +6.5406i	0.2470 +9.5946i	0.1867 +12.6932i
2	1.3843 +3.7488i	1.0781 +4.8134i	0.7187 +7.2203i	0.5158 +10.0613i	0.3978 +13.0464i
3	1.5516 +5.3563i	1.3484 +6.1621i	1.0169 +8.1728i	0.7726 +10.7567i	0.6117 +13.5857i
4	1.6761 +6.9500i	1.5334 +7.5970i	1.2522 +9.3027i	1.0014 +11.6328i	0.8154 +14.2854i
5	1.7755 +8.5366i	1.6699 +9.0763i	1.4369 +10.5482i	1.1983 +12.6485i	1.0024 +15.1208i
6	1.8583 +10.1192i	1.7771 +10.5817i	1.5840 +11.8715i	1.3655 +13.7708i	1.1702 +16.0695i
7	1.9294 +11.6991i	1.8649 +12.1035i	1.7037 +13.2487i	1.5073 +14.9748i	1.3191 +17.1114i
8	1.9915 +13.2772i	1.9391 +13.6363i	1.8031 +14.6645i	1.6280 +16.2418i	1.4504 +18.2301i
9	2.0468 +14.8540i	2.0033 +15.1768i	1.8874 +16.1088i	1.7316 +17.5578i	1.5662 +19.4118i
10	2.0966 +16.4298i	2.0599 +16.7229i	1.9600 +17.5745i	1.8214 +18.9126i	1.6685 +20.6456i

$$\left(\delta^2 = \gamma^2 - \frac{n^2\pi^2}{L^2} \right)$$

CHAPTER 8

The end problems of a cylinder and their application to an annular crack in an infinite cylinder.

8.1 Introduction.

This chapter follows the general line of argument presented by Lur'e ([16],p.380-439), the reader is assumed to be familiar with that reference.

The chapter will first present some improvements on the accuracy of the eigenvalues of the end problems. A "generalised least square method" will be presented in section 8.4 . This method is an extension of Lur'e's to cover some "mixed known-conditions" at the end of the cylinder. Application of this method is made, in section 8.5, to the problem of an annular crack in an infinite solid cylinder. The results give a fair description of stresses and displacements caused by the crack. Finally, some "difficult to integrate" integrals, which are frequently involved in this method, will be calculated and listed in section 8.6.

8.2 Lur'e method for the end problems of a cylinder.

Lur'e method uses (without proof) the two harmonic function form derived in chapter 4 of this thesis, namely the form

$$u(r,z) = 4(1-\nu)a_0 - \frac{\partial}{\partial r}(ra_0 + b_0) \tag{8.2.1a}$$

$$v(r,z) \equiv 0 \tag{8.2.1c}$$

$$w(r,z) = -\frac{\partial}{\partial z}(ra_0 + b_0) \tag{8.2.1b}$$

where u,v,w are the radial, tangential and axial displacements in the cylindrical coordinates, and a_0, b_0 satisfy

$$\nabla^2 a_0(r,z) = 0 \tag{8.2.2a}$$

$$(\nabla^2 - \frac{1}{r^2})b_0(r,z) = 0 \tag{8.2.2b}$$

over the material region.

The particular points in Lur'e's treatment are his assumptions that

$$a_0 = f(r)e^{i\beta z} , \tag{8.2.3a}$$

$$b_0 = g(r)e^{i\beta z} , \tag{8.2.3b}$$

with the functions f, g being complex. This may seem strange at first sight, but due to the appearance of eigenvalues in quadrubles, the real and imaginary parts of displacements can be separated and each part constitutes a valid solution to the problem, as already illustrated in section 7.5.

The harmonic equations for a_0 and b_0 gives the following equations for f and g

$$f''(r) + \frac{1}{r} f'(r) - (\beta^2 + \frac{1}{r^2}) f(r) = 0 , \tag{8.2.4a}$$

$$g''(r) + \frac{1}{r} g'(r) - \beta^2 g(r) = 0 . \tag{8.2.4b}$$

From the above equations, the functions f and g are known as

$$f(r) = C_1 I_1(\beta r) + C_2 K_1(\beta r) , \tag{8.2.5a}$$

$$g(r) = C_3 I_0(\beta r) + C_4 K_0(\beta r) . \tag{8.2.5b}$$

From (8.2.3) and (8.2.4), together with the properties of the modified-Bessel functions I_n, K_n, the displacements and stresses are written as

$$u = \left[4(1-\nu)(C_1 I_1(\beta r) + C_2 K_1(\beta r)) - \beta(C_3 I_1(\beta r) - C_4 K_1(\beta r)) - \beta r(C_1 I_0(\beta r) - C_2 K_0(\beta r))\right] e^{i\beta z} \tag{8.2.6a}$$

$$w = \left[C_3 I_0(\beta r) + C_4 K_0(\beta r) + r(C_1 I_1(\beta r) + C_2 K_1(\beta r))\right] \cdot (-i\beta) e^{i\beta z} \tag{8.2.6b}$$

$$\begin{aligned} \frac{1}{2G}\sigma_{rr} = \Big\{ & C_1\left[(3-2\nu)\beta I_0(\beta r) - \left(\frac{4(1-\nu)}{r} + \beta^2 r\right) I_1(\beta r)\right] \\ & + C_2\left[-(3-2\nu)\beta K_0(\beta r) - \left(\frac{4(1-\nu)}{r} + \beta^2 r\right) K_1(\beta r)\right] \\ & + C_3\left[\frac{\beta}{r} I_1(\beta r) - \beta^2 I_0(\beta r)\right] \\ & + C_4\left[-\frac{\beta}{r} K_1(\beta r) - \beta^2 K_0(\beta r)\right] \Big\} e^{i\beta z} \end{aligned} \tag{8.2.6c}$$

$$\begin{aligned} \frac{1}{2G}\sigma_{\theta\theta} = \Big\{ & C_1\left[\frac{4(1-\nu)}{r} I_1(\beta r) - (1-2\nu)\beta I_0(\beta r)\right] \\ & + C_2\left[\frac{4(1-\nu)}{r} K_1(\beta r) + (1-2\nu)\beta K_0(\beta r)\right] \\ & + C_3\left[-\frac{\beta}{r} I_1(\beta r)\right] \\ & + C_4\left[\frac{\beta}{r} K_1(\beta r)\right] \Big\} e^{i\beta z} \end{aligned} \tag{8.2.6d}$$

$$\frac{1}{2G}\sigma_{zz} = \Big\{C_1\big[2\nu\beta I_0(\beta r) + \beta^2 r I_1(\beta r)\big] + C_2\big[-2\nu\beta K_0(\beta r) + \beta^2 r K_1(\beta r)\big] + C_3\big[\beta^2 I_0(\beta r)\big] + C_4\big[\beta^2 K_0(\beta r)\big]\Big\}\, e^{i\beta z} \qquad (8.2.6e)$$

$$\frac{1}{2G}\tau_{rz} = \Big\{C_1\big[\beta r I_0(\beta r) - 2(1-\nu) I_1(\beta r)\big] + C_2\big[-\beta r K_0(\beta r) - 2(1-\nu) K_1(\beta r)\big] + C_3\big[\beta I_1(\beta r)\big] + C_4\big[-\beta K_1(\beta r)\big]\Big\}\,(-i\beta) e^{i\beta z} \qquad (8.2.6f)$$

For a solid cylinder, if the stresses and displacements at its central axis are to be finite, the coefficients C_2 and C_4 must be zero, as required by (8.2.6). Hence, there are only two coefficients C_1 and C_3 to be determined in the formulae (8.2.6) for a solid cylinder.

The infinite solid cylinder with free lateral surface has the boundary conditions as

$$\sigma_{rr} = 0 \qquad \text{at} \qquad r = 1 , \qquad (8.2.7a)$$

and

$$\tau_{rz} = 0 \qquad \text{at} \qquad r = 1 . \qquad (8.2.7b)$$

The above conditions lead to the equation for possible values of β, which is

$$\Psi(\beta) \equiv \beta^2\big[I_0^2(\beta) - I_1^2(\beta)\big] - 2(1-\nu) I_1^2(\beta) = 0 . \qquad (8.2.8)$$

There are an infinite number of possible values of β, they are called the eigenvalues for an axi-symmetrically end loaded cylinder. Lur'e has proved that the roots of (8.2.8) occur in quadrubles so that if β is a root, so are $\overline{\beta}$, $(-\beta)$, and $(-\overline{\beta})$. Of course, $\beta = 0$ is a solution for (8.2.8) but this value of β gives no loading at all on the cylinder. The non-zero roots of (8.2.8) which lie in the first quadrant of the complex plane are ordered by their absolute values. Lur'e has given the values of the first three roots in the first quadrant of the complex plane. Other authors have then supplied the values of more roots and at the same time improved their accuracy. Tables 8.1a and 8.1b give further improved values of β_n, which are much more accurate than those supplied by [16], [28], and [25].

For large values of β_n, the assymptotic formula given by Lur'e is

$$\beta_n \sim n\pi i + \frac{1}{2}\log(4\pi n) - i\left[\frac{\log(4\pi n)}{4\pi n} - \frac{1}{2\pi n}(2\nu-\tfrac{7}{4})\right], \quad (8.2.9)$$

the formula was said to satisfy the eigenvalue equation (8.2.8) up to the term of order (log n)/n , but there is no mention of the order of the remainder for the assymptotic formula (8.2.9).

With the values of β_n known, and by setting $C_1 = \left[\beta I_1(\beta)\right]^{-1}$, the values of displacements and stresses are given by (8.2.6) as

$$u(r,z,\beta_n) = \frac{1}{I_1(\beta_n)}\left[-rI_0(\beta_n r) + \left(\frac{2(1-\nu)}{\beta_n} + \frac{I_0(\beta_n)}{I_1(\beta_n)}\right)I_1(\beta_n r)\right]\cdot \cdot\exp(i\beta_n z), \quad (8.2.10a)$$

$$w(r,z,\beta_n) = \frac{-i}{I_1(\beta_n)}\left[rI_1(\beta_n r) + \left(\frac{2(1-\nu)}{\beta_n} - \frac{I_0(\beta_n)}{I_1(\beta_n)}\right)I_0(\beta_n r)\right]\cdot \cdot\exp(i\beta_n z), \quad (8.2.10b)$$

$$\frac{1}{2G}\sigma_{rr}(r,z,\beta\ ) = \frac{1}{I_1(\beta_n)}\left[I_0(\beta_n r)\left(1 + \beta_n\frac{I_0(\beta_n)}{I_1(\beta_n)}\right) - \frac{I_1(\beta_n r)}{r}\left(\frac{2(1-\nu)}{\beta_n} + \frac{I_0(\beta_n)}{I_1(\beta_n)} + \beta_n r^2\right)\right]\cdot \cdot\exp(i\beta_n z), \quad (8.2.10c)$$

$$\frac{1}{2G}\sigma_{\theta\theta}(r,z,\beta_n) = \frac{1}{I_1(\beta_n)}\left[-(1-2\nu)I_0(\beta_n r) + \frac{I_1(\beta_n r)}{r}\cdot \cdot\left(\frac{2(1-\nu)}{\beta_n} + \frac{I_0(\beta_n)}{I_1(\beta_n)}\right)\right]\exp(i\beta_n z), \quad (8.2.10d)$$

$$\frac{1}{2G}\sigma_{zz}(r,z,\beta_n) = \frac{1}{I_1(\beta_n)}\left[\left(2 - \beta_n\frac{I_0(\beta_n)}{I_1(\beta_n)}\right)I_0(\beta_n r) + + r\beta_n I_1(\beta_n r)\right]\exp(i\beta_n z), \quad (8.2.10e)$$

$$\frac{1}{2G}\tau_{rz}(r,z,\beta_n) = \frac{-i}{I_1(\beta_n)}\beta_n\left[rI_0(\beta_n r) - \frac{I_0(\beta_n)}{I_1(\beta_n)}I_1(\beta_n r)\right]\cdot \cdot\exp(i\beta_n z). \quad (8.2.10f)$$

As various use will be made to the real and imaginary parts of the above displacements and stresses, the following abbreviation will be used in the subsequent sections of the chapter:

$$
\begin{aligned}
u(\ n,r,z) &= \mathrm{Real}\left[u(r,z,\beta_n)\right] \quad , \\
u(in,r,z) &= \mathrm{Im}\left[u(r,z,\beta_n)\right] \quad , \\
w(\ n,r,z) &= \mathrm{Real}\left[w(r,z,\beta_n)\right] \quad , \\
w(in,r,z) &= \mathrm{Im}\left[w(r,z,\beta_n)\right] \quad , \\
\sigma_r(\ n,r,z) &= \mathrm{Real}\left[\sigma_{rr}(r,z,\beta_n)\right] \quad , \\
\sigma_r(in,r,z) &= \mathrm{Im}\left[\sigma_{rr}(r,z,\beta_n)\right] \quad , \\
\sigma_\theta(\ n,r,z) &= \mathrm{Real}\left[\sigma_{\theta\theta}(r,z,\beta_n)\right] \quad , \\
\sigma_\theta(in,r,z) &= \mathrm{Im}\left[\sigma_{\theta\theta}(r,z,\beta_n)\right] \quad , \\
\sigma_z(\ n,r,z) &= \mathrm{Real}\left[\sigma_{zz}(r,z,\beta_n)\right] \quad , \\
\sigma_z(in,r,z) &= \mathrm{Im}\left[\sigma_{zz}(r,z,\beta_n)\right] \quad ,
\end{aligned}
\tag{8.2.11}
$$

and

$$
\begin{aligned}
\tau_{rz}(\ n,r,z) &= \mathrm{Real}\left[\tau_{rz}(r,z,\beta_n)\right] \quad , \\
\tau_{rz}(in,r,z) &= \mathrm{Im}\left[\tau_{rz}(r,z,\beta_n)\right] \quad .
\end{aligned}
$$

8.3 Improvements on the accuracy of eigenvalues.

The values of β_n, as given by Lur'e and subsequent authors (e.g., [28], [41]), have been further improved in tables 8.1a and 8.1b. These tables are obtained with the use of a digital computer and careful programming for the Newton-Raphson's method of root finding.

The technique used here consists of expanding $\Psi(\beta)$ and $\Psi'(\beta)$ into series, instead of using the Bessel functions. As n increases, the number of terms required for the valuation of $\Psi(\beta)$ and $\Psi'(\beta)$ increases much faster; for value of $n \geq 25$ this method is unapplicable to the computer available due to accumulative errors (The computer in use is a PDP 11-40 with an available relative accuracy of better than 10^{-16}). Although it is possible to devise special programs to improve the relative accuracy, it is too cumbersome to use such technique, as it is only possible to go to $n \simeq 35$ for a relative accuracy of 10^{-32}. The tables are

Table 8.1a: Eigenvalues for a solid cylider ($\nu = 0.25$).

n	Real(β_n)	Imaginary(β_n)
1	1.367 356 999	2.697 651 804
2	1.638 147 085	6.051 222 300
3	1.828 534 160	9.261 273 455
4	1.967 428 297	12.438 444 451
5	2.076 421 142	15.602 204 445
6	2.166 039 326	18.759 055 655
7	2.242 108 030	21.911 845 435
8	2.308 175 643	25.062 031 889
9	2.366 560 685	28.210 443 716
10	2.418 860 400	31.357 588 567
11	2.466 221 463	34.503 795 629
12	2.509 494 714	37.649 288 138
13	2.549 328 779	40.794 223 059
14	2.586 229 395	43.938 714 111
15	2.620 598 480	47.082 845 781
16	2.652 760 712	50.226 682 205
17	2.682 982 119	53.370 272 973
18	2.711 483 391	56.513 657 060
19	2.738 449 628	59.656 865 536
20	2.764 037 595	62.799 923 483

Table 8.1b: Eigenvalues for a solid cylinder ($\nu = 0.3$).

n	Real(β_n)	Imaginary(β_n)
1	1.362 197 076	2.722 175 533
2	1.637 624 407	6.060 083 224
3	1.828 255 822	9.266 835 313
4	1.967 241 172	12.442 529 146
5	2.076 283 753	15.605 440 657
6	2.165 933 153	18.761 738 397
7	2.242 023 045	21.914 137 705
8	2.308 105 814	25.064 033 571
9	2.366 502 128	28.212 220 544
10	2.418 810 481	31.359 186 155
11	2.466 178 328	34.505 246 947
12	2.509 457 014	37.650 617 799
13	2.549 295 509	40.795 449 930
14	2.586 199 787	43.939 852 978
15	2.620 571 938	47.083 908 449
16	2.652 736 765	50.227 678 247
17	2.682 920 389	53.371 210 263
18	2.711 463 573	56.514 542 153
19	2.738 431 470	59.657 703 945
20	2.764 020 890	62.800 719 891

thus left to stop at $n = 20$.

The assymptotic formula (8.2.9) for large values of β_n's, as given by Lur'e, is not of sufficient accuracy for today's numerical computations. Also Lur'e has not specified the accuracy of his formula. Further investigation into the assymptotic form of β_n is therefore desirable.

Following Lur'e's method for developping the assymptotic formula for β_n, but with more terms, the following formula is arrived at

$$\beta_n \sim i(\pi n) + \frac{1}{2}\log(4\pi n) - \frac{i}{4\pi n}\left[\log(4\pi n) - (4\nu - \frac{7}{2})\right]$$
$$+ \frac{1}{(4\pi n)^2}\left[\left(\log(4\pi n)\right)^2 + (5-8\nu)\log(4\pi n) + \frac{7}{2} - 16(1-\nu)^2\right], \tag{8.3.1}$$

with the order of the remainder to be

$$0\left(\frac{\text{remainder } R(\beta_n)}{R(\beta_n)}\right) = 0\left(\frac{\text{remainder } I(\beta_n)}{I(\beta_n)}\right) = \frac{\left(\log(4\pi n)\right)^3}{(4\pi n)^4} .$$

For $n = 20$, the relative accuracy of β_n is thus predicted to be

$$\left(\frac{\delta R(\beta_n)}{R(\beta_n)}\right) \text{ and } \left(\frac{\delta I(\beta_n)}{I(\beta_n)}\right) < \simeq \frac{\left(\log(4\pi n)\right)^3}{(4\pi n)^4} = 0.3 * 10^{-6} .$$

To examine the remainder of β_n as given by (8.3.1) it is only necessary to consider the expansion of $\Psi(\beta)$ in terms of β, it is

$$0 = \Psi(\beta) \equiv \beta^2 \sum_{m=0}^{\infty} \left[1 - (1-\nu)\frac{2m+1}{(m+1)(m+2)}\right] \frac{(2m)!}{(m+1)!(m!)^3}\left(\frac{\beta}{2}\right)^{2m} .$$

With the first few terms for large β_n given by (8.3.1), as

$$\beta_n \sim iAn + B + iC\frac{1}{n} + D\frac{1}{n^2} + X\frac{1}{n^3} + \dots ,$$

where the real constants A, B, C, D may contain $\log(4\pi n)$, $\left(\log(4\pi n)\right)^2$, but not any power of n, we can expand $\Psi(\beta)$ to the power of $(1/n)^3$ to prove that X must be purely imaginary.

Hence β_n can now be written as

$$\beta_n \sim iAn + B + iC\frac{1}{n} + D\frac{1}{n^2} + iE\frac{1}{n^3} + F\frac{1}{n^4} + \ldots ,$$

where the coefficients A, B, C, D, E, F, ... are all real. Furthermore, each coefficient, from C, D onward, can contain only powers of $\log(4\pi n)$ not exceeding that of n, i.e. C can contain only $\log(4\pi n)$, D can contain $\log(4\pi n)$, $(\log(4\pi n))^2$, E can contain only $\log(4\pi n)$, $(\log(4\pi n))^2$, $(\log(4\pi n))^3$, and so on.

For $\nu = 0.25$, $n = 20$, the precise value of β_n is

$$\beta_{20} = 2.764037595 + i\ 62.799923483 .$$

This value can be used to judge the value given by the new assymptotic formula (8.3.1), which is

$$\beta_{20} = 2.764037(25) + i\ 62.7999(15) ,$$

and also the value given by Lur'e assymptotic formula (8.2.9), as

$$\beta_{20} = 2.76(337) + i\ 62.7999(15) .$$

The digits in brackets are those given by the corresponding assymptotic formula but is not correct when compared to the precise values.

8.4 Least square method for the end problems of a cylinder.

The least square method presented by Lur'e in [16] is carried a little bit further in this section, as the loading is known for only part of the end surface of the cylinder and other quantities are known on the remaining of the surface. The least square method here has to minimise the difference between the calculated quantities and the given quantities, whether they are stresses or strains or a combination of them.

The functions which measure the difference between the given and the calculated quantities are called the measuring functions, and are evaluated only from the center of the end surface to certain radius R. For a problem where the measuring function must change in type as one goes from one zone to another zone of the end surface, the sum of these measuring functions can be used (provided that they are compatible) to make a single measuring function for the problem. This will be done in section 8.5.

If the known quantities on the end face of the cylinder are stresses, as

$$\begin{aligned} \sigma_z(r,z) &= S(r) \qquad \text{for } z = 0\ ,\ 0 \le r \le R \le 1\ , \\ \tau_{rz}(r,z) &= T(r) \qquad \text{for } z = 0\ ,\ 0 \le r \le R \le 1\ , \end{aligned} \tag{8.4.1}$$

then the measuring function is defined as

$$\Psi(R,M_1,M_2,\ldots,M_n,N_1,N_2,\ldots,N_n) =$$

$$= \int_0^R \left\{ \left[S(r) - \sum_{k=1}^{n}\left(M_k\sigma_z(k,r,0) + N_k\sigma_z(ik,r,0)\right)\right]^2 + \left[T(r) - \sum_{k=1}^{n}\left(M_k\tau_{rz}(k,r,0) + N_k\tau_{rz}(ik,r,0)\right)\right]^2 \right\} r\,dr\ , \tag{8.4.2}$$

where the notations $\sigma(k,r,z)$ and $\tau(k,r,z)$ follow the rules established by (8.2.11).

The conditions for minimum Ψ are derived from (8.4.2) as

$$\frac{\partial\Psi}{\partial M_k} = 0\ ,\quad \frac{\partial\Psi}{\partial N_k} = 0\ , \qquad\qquad k = 1,\ 2,\ldots,\ n.$$

Using the notations

$$\begin{aligned}
A_{km}(R) &= \int_0^R [\sigma_z(k,r,0)\sigma_z(m,r,0) + \tau_{rz}(k,r,0)\tau_{rz}(m,r,0)]r\,dr\ , \\
B_{km}(R) &= \int_0^R [\sigma_z(ik,r,0)\sigma_z(m,r,0) + \tau_{rz}(ik,r,0)\tau_{rz}(m,r,0)]r\,dr\ , \\
C_{km}(R) &= \int_0^R [\sigma_z(ik,r,0)\sigma_z(im,r,0) + \tau_{rz}(ik,r,0)\tau_{rz}(im,r,0)]r\,dr\ , \\
\Phi_k(R) &= \int_0^R [S(r)\sigma_z(k,r,0) + T(r)\tau_{rz}(k,r,0)]r\,dr\ , \\
\Gamma_k(R) &= \int_0^R [S(r)\sigma_z(ik,r,0) + T(r)\tau_{rz}(ik,r,0)]r\,dr\ ,
\end{aligned} \tag{8.4.3}$$

the conditions of minimum Ψ become

$$\sum_{k=1}^{n} (A_{km}M_k + B_{km}N_k) = \Phi_m\ , \qquad m = 1,\ 2,\ldots,\ n\ ,$$

and

$$\sum_{k=1}^{n} (B_{mk}M_k + C_{km}N_k) = \Gamma_m\ , \qquad m = 1,\ 2,\ldots,\ n\ . \tag{8.4.4}$$

In order to facillitate the computation of A_{km}, B_{km}, C_{km}, the following integrals are introduced

$$J_{\pm}(R,\beta_k,\beta_m) \equiv \int_o^R [\sigma_z(r,0,\beta_k)\sigma_z(r,0,\beta_m) \pm \tau_{rz}(r,0,\beta_k)\tau_{rz}(r,0,\beta_m)]\; r\,dr\ ,$$

where the functions $\sigma_z(r,z,\beta_k)$ and $\tau_{rz}(r,z,\beta_k)$ are those given by the equation (8.2.10).

The coefficients A_{km}, B_{km}, C_{km} for M_k, N_k can now be written as

$$A_{km}(R) = \frac{1}{4}\,[J_+(R,\beta_k,\beta_m) + J_+(R,\overline{\beta}_k,\overline{\beta}_m) + J_-(R,\beta_k,\overline{\beta}_m) + J_-(R,\overline{\beta}_k,\beta_m)],$$

$$B_{km}(R) = \frac{-i}{4}\,[J_+(R,\beta_k,\beta_m) - J_+(R,\overline{\beta}_k,\overline{\beta}_m) + J_-(R,\beta_k,\overline{\beta}_m) - J_-(R,\overline{\beta}_k,\beta_m)],$$

$$C_{km}(R) = \frac{1}{4}[-J_+(R,\beta_k,\beta_m) - J_+(R,\overline{\beta}_k,\overline{\beta}_m) + J_-(R,\beta_k,\overline{\beta}_m) + J_-(R,\overline{\beta}_k,\beta_m)].$$

(8.4.5)

The values of $J_{\pm}(R,\beta_k,\beta_m)$ are given in tables 8.3, their evaluations are to be found in section 8.6.

It is also noted that

$$\Phi_k(R) + i\Gamma_k(R) = \int_o^R [S(r)\sigma_z(r,0,\beta_k) + T(r)\tau_{rz}(r,0,\beta_k)]\; r\,dr\ .$$

(8.4.6)

The coefficients $A_{km}(1)$, $B_{km}(1)$, $C_{km}(1)$ are evaluated once and for all, their values are given in the following system of equations

$$1.075M_1 - .401N_1 - .265M_2 + .049N_2 - .123M_3 + .092N_3 - .052M_4 + .065N_4 = \Phi_1,$$

$$-.401M_1 + .290N_1 + .526M_2 - .236N_2 + .083M_3 - .124N_3 + .020M_4 - .057N_4 = \Gamma_1,$$

$$-.265M_1 + .526N_1 + 2.694M_2 - 1.670N_2 - .674M_3 - .172N_3 - .388M_4 + .183N_4 = \Phi_2,$$

$$+.049M_1 - .236N_1 - 1.670M_2 + 1.434N_2 + 1.530M_3 - .579N_3 + .203M_4 - .371N_4 = \Gamma_2,$$

$$-.123M_1 + .083N_1 - .674M_2 + 1.530N_2 + 4.152M_3 - 2.840N_3 - .955M_4 - .638N_4 = \Phi_3,$$

$$+.092M_1 - .124N_1 - .172M_2 - .579N_2 - 2.840M_3 + 2.859N_3 + 2.697M_4 - .866N_4 = \Gamma_3,$$

$$-.052M_1 + .020N_1 - .388M_2 + .203N_2 - .955M_3 + 2.697N_3 + 5.718M_4 - 3.955N_4 = \Phi_4,$$

$$+.065M_1 - .057N_1 + .183M_2 - .371N_2 - .638M_3 - .866N_3 - 3.955M_4 + 4.455N_4 = \Gamma_4,$$

(8.4.7)

where the coefficients Φ's, Γ's are evaluated at $R = 1$.

By solving systems like the above, the quantity Ψ can be set to its minimum.

On the other hand, if displacements are known on the end surface of the cylinder, as

$$\begin{aligned} u(r,z) &= U(r) \quad \text{for} \quad z = 0 \;,\; 0 \leq r \leq R \leq 1 \;, \\ w(r,z) &= W(r) \quad \text{for} \quad z = 0 \;,\; 0 \leq r \leq R \leq 1 \;, \end{aligned} \tag{8.4.8}$$

then the measuring function is defined as

$$\begin{aligned} &\psi(R,M_1,M_2,\ldots,M_n,N_1,N_2,\ldots,N_n) = \\ &= \int_0^R \Bigg\{\Big[U(r) - \sum_{k=1}^{n}\Big(M_k u(k,r,0) + N_k u(ik,r,0)\Big)\Big]^2 + \\ &\qquad \Big[W(r) - \sum_{k=1}^{n}\Big(M_k w(k,r,0) + N_k w(ik,r,0)\Big)\Big]^2\Bigg\}\, r\, dr \;, \end{aligned} \tag{8.4.9}$$

where the notations $u(l,r,z)$ and $w(l,r,z)$ follow the rules established by (8.2.11).

The conditions for minimum ψ are then

$$\frac{\partial\psi}{\partial M_k} = 0 \;,\quad \frac{\partial\psi}{\partial N_k} = 0 \;, \qquad k = 1,\, 2,\ldots,\, n.$$

Using the notations

$$\begin{aligned} a_{km}(R) &= \int_0^R [u(k,r,0)u(m,r,0) + w(k,r,0)w(k,r,0)]r\, dr \;, \\ b_{km}(R) &= \int_0^R [u(ik,r,0)u(m,r,0) + w(ik,r,0)w(m,r,0)]r\, dr \;, \\ c_{km}(R) &= \int_0^R [u(ik,r,0)u(im,r,0) + w(ik,r,0)w(im,r,0)]r\, dr \;, \\ \phi_k(R) &= \int_0^R [U(r)u(k,r,0) + W(r)w(k,r,0)]r\, dr \;, \\ \gamma_k(R) &= \int_0^R [U(r)u(ik,r,0) + W(r)w(ik,r,0)]r\, dr \;, \end{aligned} \tag{8.4.10}$$

the conditions for minimum ψ become

$$\sum_{k=1}^{n} (a_{km}M_k + b_{km}N_k) = \phi_m \;, \qquad m = 1,\, 2,\ldots,\, n \;,$$

and (8.4.11)

$$\sum_{k=1}^{n} (b_{mk}M_k + c_{km}N_k) = \gamma_m \;, \qquad m = 1,\, 2,\ldots,\, n \;.$$

Similarly to the previous case, the following integrals are introduced

$$L_{\pm}(R,\beta_k,\beta_m) \equiv \int_0^R [u(r,0,\beta_k)u(r,0,\beta_m) \pm w(r,0,\beta_k)w(r,0,\beta_m)]\ r\,dr\ ,$$

where the functions $u(r,z,\beta_k)$ and $w(r,z,\beta_k)$ are those given by the equations (8.2.10) and $L_{\pm}(R,\beta_k,\beta_m)$ are given in table 8.4.

With the introduced functions $L_{\pm}(R,\beta_k,\beta_m)$, the coefficients a_{km}, b_{km}, c_{km} for M_k and N_k can be written as

$$a_{km}(R) = \frac{1}{4}\ [L_+(R,\beta_k,\beta_m) + L_+(R,\overline{\beta}_k,\overline{\beta}_m) + L_-(R,\beta_k,\overline{\beta}_m) + L_-(R,\overline{\beta}_k,\beta_m)],$$

$$b_{km}(R) = \frac{-i}{4}\ [L_+(R,\beta_k,\beta_m) - L_+(R,\overline{\beta}_k,\overline{\beta}_m) + L_-(R,\beta_k,\overline{\beta}_m) - L_-(R,\overline{\beta}_k,\beta_m)],$$

$$c_{km}(R) = \frac{1}{4}[-L_+(R,\beta_k,\beta_m) - L_+(R,\overline{\beta}_k,\overline{\beta}_m) + L_-(R,\beta_k,\overline{\beta}_m) + L_-(R,\overline{\beta}_k,\beta_m)].$$

(8.4.12)

In this case, the functions ϕ_k and γ_k are evaluated by

$$\phi_k(R) + i\gamma_k(R) = \int_0^R [U(r)u(r,0,\beta_k) + W(r)w(r,0,\beta_k)]\ r\,dr\ .$$

(8.4.13)

Similarly to the last case, the coefficients $a_{km}(1)$, $b_{km}(1)$, $c_{km}(1)$ are evaluated once and for all, and their first few values are given in the following system

$$0.1068M_1-.0590N_1-.0287M_2-.0033N_2-.0085M_3+.0021N_3-.0035M_4+.0013N_4=\phi_1\ ,$$

$$-.0590M_1+.1043N_1+.0312M_2+.0078N_2+.0069M_3+.0021N_3+.0034M_4+.0015N_4=\gamma_1\ ,$$

$$-.0287M_1+.0312N_1+.0734M_2-.0413N_2-.0165M_3-.0044N_3-.0059M_4+.0021N_4=\phi_2\ ,$$

$$-.0033M_1+.0078N_1-.0413M_2+.0434N_2+.0266M_3-.0027N_3+.0037M_4-.0025N_4=\gamma_2\ ,$$

$$-.0085M_1+.0069N_1-.0165M_2+.0266N_2+.0505M_3-.0311N_3-.0108M_4-.0060N_4=\phi_3\ ,$$

$$+.0021M_1+.0021N_1-.0044M_2-.0027N_2-.0311M_3+.0327N_3+.0230M_4-.0035N_4=\gamma_3\ ,$$

$$-.0035M_1+.0034N_1-.0059M_2+.0037N_2-.0108M_3+.0230N_3+.0389M_4-.0245N_4=\phi_4\ ,$$

$$+.0013M_1+.0015N_1+.0021M_2-.0025N_2-.0060M_3-.0035N_3-.0245M_4+.0276N_4=\gamma_4\ ,$$

(8.4.14)

where the coefficients ϕ's, γ's are evaluated at $R = 1$.

Solving for the coefficients M_k, N_k from systems as (8.4.4) or (8.4.11) then gives the minimum possible value with 2n real eigenmodes (n modes with coefficients M's, n with N's) for the measuring function. Hence, the more modes are used the smaller is the minimum for the measuring function. The next logical thing is thus to establish a quantity which indicates the variation of the minimum of the measuring function with increasing n, this quantity will be named the "standardised deviation" function. The purpose of this "standardised deviation" function is to indicate how well the calculated quantities can fit the given ones. An example on this newly defined function is given below.

Suppose that the function Ψ defined by (8.4.2) is the measuring function for a given case, then the standardised deviation function s is defined as

$$s(R,n) = \frac{\Psi(R, M_1, M_2, \ldots, M_n, N_1, N_2, \ldots, N_n)}{\Psi(R,\ 0,\ 0, \ldots,\ 0,\ 0,\ 0, \ldots,\ 0)} ,$$

where M's, N's are the solution of (8.4.4).

If no mode is used ($2n = 0$), then $s(R,n)$ is equal to unity, and if the given boundary conditions can be expressed by a linear combination of 2n real modes corresponding to $M_1, M_2, \ldots, M_n$, $N_1, N_2, \ldots, N_n$ then $s(R,n)$ is equal to zero. Hence as n increases, $s(R,n)$ decreases, and if all the eigenmodes entering the measuring function can express any arbitrary given loading, then the value of $s(R,\infty)$ is zero. For any non-zero value of n, a value of $s(R,n)$ greater than unity is certainly the indication of some erroneous calculation(s) !

The standardised deviation is generally defined for any measuring function as

$$s(n) = \frac{\left\{ \begin{array}{c} \text{Minimum possible of measuring function} \\ \text{with } M_1, M_2, \ldots, M_n, N_1, N_2, \ldots, N_n \end{array} \right\}}{\left\{ \begin{array}{c} \text{Measuring function with all} \\ \text{M's, N's equal to zero} \end{array} \right\}} . \tag{8.4.15}$$

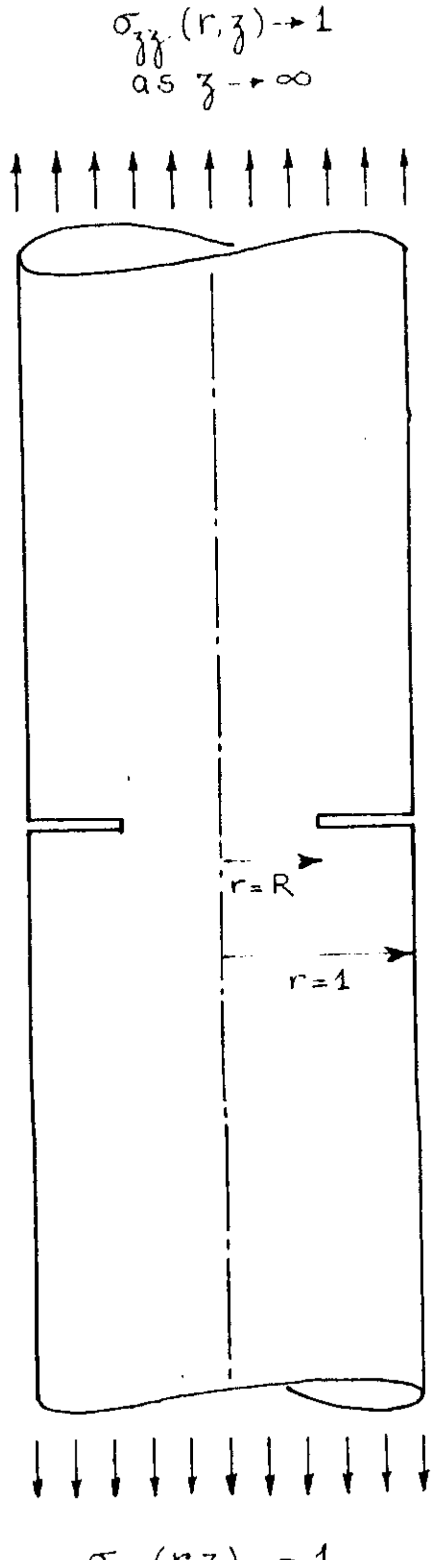


FIGURE 8.1: EXTERNAL CRACK IN AN INFINITE SOLID CYLIDER

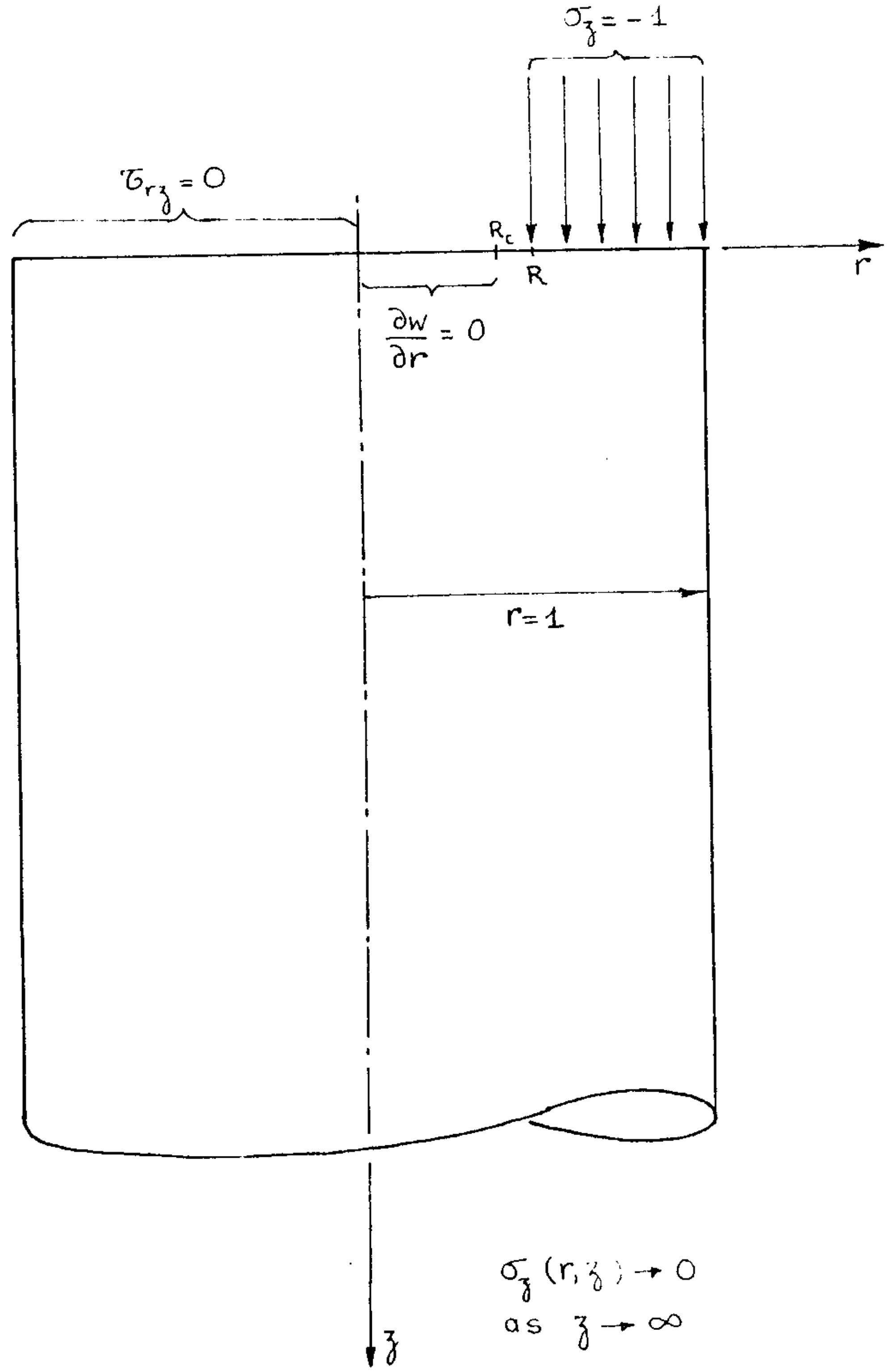


FIGURE 8.2 :

END LOADING OF A SEMI-INFINITE CYLINDER ACCORDING TO EQUATION (8.5.2)

A standardised deviation function defined as above will be used in the next section

8.5 Least square method applied to the external crack in an infinite cylinder.

In this section, the least square method proposed in the last section is used to investigate the problem of an external crack in an infinite cylinder.

The crack starts at the radius $R < 1$ and goes right through to the lateral surface of the cylinder. This crack not only reduces the cross section of the cylinder, but also increases drastically, stresses at its tip, that is at the radius $r = R-\varepsilon$, where ε is a very small length. It is thus interesting to investigate the behaviours of the physical quantities like displacements and stresses around this crack.

The problem is equivalent to the problem of a semi-infinite cylinder under the following conditions on the end surface

$$\sigma_z(r,z) = 0 \qquad \text{for} \quad z = 0 \;,\; R < r \leq 1 \;, \qquad (8.5.1a)$$

$$\tau_{rz}(r,z) = 0 \qquad \text{for} \quad z = 0 \;,\; 0 \leq r \leq 1 \;, \qquad (8.5.1b)$$

$$\frac{E}{1+\nu}\,\frac{\partial w(r,z)}{\partial r} = 0 \qquad \text{for} \quad z = 0 \;,\; 0 \leq r < R \;, \qquad (8.5.1c)$$

and for very large value of z,

$$\sigma_z(r,z) \to 1 \qquad \text{for} \quad z \to \infty \;, \qquad (8.5.1d)$$

$$\tau_{rz}(r,z) \to 0 \qquad \text{for} \quad z \to \infty \;, \qquad (8.5.1e)$$

$$\frac{E}{1+\nu}\,\frac{\partial w(r,z)}{\partial r} \to 0 \qquad \text{for} \quad z \to \infty \;. \qquad (8.5.1f)$$

Superimposing a uniform compression of $\sigma_z(r,z) = -1$ on the loading of the semi-infinite cylinder, the following system of boundary conditions is arrived at

$$\sigma_z(r,z) = -1 \qquad \text{for} \quad z = 0 \;,\; R < r \leq 1 \;, \qquad (8.5.2a)$$

$$\tau_{rz}(r,z) = 0 \qquad \text{for} \quad z = 0 \;,\; 0 \leq r \leq 1 \;, \qquad (8.5.2b)$$

$$\frac{E}{1+\nu}\frac{\partial w(r,z)}{\partial r} = 0 \qquad \text{for} \quad z = 0 \;,\; 0 \le r < R \;, \qquad (8.5.2c)$$

and

$$\sigma_z(r,z) \to 0 \qquad \text{for} \quad z \to \infty \;, \qquad (8.5.2d)$$

$$\tau_{rz}(r,z) \to 0 \qquad \text{for} \quad z \to \infty \;, \qquad (8.5.2e)$$

$$\frac{E}{1+\nu}\frac{\partial w(r,z)}{\partial r} \to 0 \qquad \text{for} \quad z \to \infty \;. \qquad (8.5.2f)$$

The eigenfunctions for end-loading of a semi-infinite cylinder as given by (8.2.10) will well suit this problem. However, the determination of the coefficients for the eigenmodes is not a simple task.

It is noted that as the actual value of $w(r,z)$ in the region of ($z = 0$, $r < R$) is unknown, the elegant "biorthogonality method" developed by Little and Childs [28] is not applicable to this problem. Even if a new set of biorthogonal functions involving σ_z, τ_{rz}, and $\frac{\partial w}{\partial z}$ is developed, the method is still unsuitable as there is no free lateral surface at the radius R. (The biorthogonality method is based on the Sturm-Liouville system of equations which, in turn, depends on the known conditions at its two end points, i.e. at the radii $r = 0$ and $r = R$.)

The least square method as proposed in the last section is thus the only reasonable choice left, and it will be used to investigate the problem.

As for the proposed least square method, the known quantities must be first defined, they are

$$\sigma_z(r,z) = S(r) = -1 \qquad \text{for} \quad z = 0 \;,\; R < r \le 1 \;, \qquad (8.5.3a)$$

$$\tau_{rz}(r,z) = T(r) = 0 \qquad \text{for} \quad z = 0 \;,\; 0 \le r \le 1 \;, \qquad (8.5.3b)$$

$$\frac{E}{1+\nu}\frac{\partial w(r,z)}{\partial r} = V(r) = 0 \quad \text{for} \quad z = 0 \;,\; 0 \le r \le R_c . \qquad (8.5.3c)$$

Since all the eigenmodes die away from the end $z = 0$, their use ensures that the conditions (8.5.2d) to (8.5.2f) are automatically satisfied. The value of $\frac{\partial w}{\partial r}$ is fitted to the known value $V(r) = 0$ for only $0 \le r \le R_c$. The reason is that initial attempts at letting R_c (named here the "continuity radius") equal to R (named here the "crack radius") led to some non-convergence on the coefficients of the eigenmodes.

The measuring function is now defined as

$$\Omega(R,R_c,M_1,M_2,\ldots,M_n,N_1,N_2,\ldots,N_n) =$$

$$= \int_R^1 \left\{[S(r) - \sum_{k=1}^{n}\left(M_k\sigma_z(k,r,z) + N_k\sigma_z(ik,r,z)\right)]^2\right\} r\,dr \; +$$

$$+ \int_o^1 \left\{[T(r) - \sum_{k=1}^{n}\left(M_k\tau_{rz}(k,r,z) + N_k\tau_{rz}(ik,r,z)\right)]^2\right\} r\,dr \; +$$

$$+ \int_o^{R_c} \left\{[V(r) - \frac{E}{1+\nu}\sum_{k=1}^{n}\left(M_k\frac{\partial w(k,r,z)}{\partial r} + N_k\frac{\partial w(ik,r,z)}{\partial r}\right)]^2\right\} r\,dr. \tag{8.5.4}$$

The factor $\frac{E}{1+\nu}$ is placed in front of $\frac{\partial w}{\partial r}$ to make this term of the same dimension and magnitude to stresses. Without this factor, $\frac{\partial w}{\partial r}$ will be either too large or too small for stresses, and hence the measuring function may neglect the smaller term(s). The meaning of the word "compatible" used in the second paragraph of section 8.4 is thus illuminated here. In this section, to simplify the calculations, the value of $E/(1+\nu)$ is assumed to be unity.

The conditions for minimum Ω are now

$$\frac{\partial\Omega}{\partial M_k} = 0\ , \quad \frac{\partial\Omega}{\partial N_k} = 0\ , \qquad\qquad k = 1, 2,\ldots, n.$$

Using the notations

$$D_{km}(R,R_c) = \int_R^1 [\sigma_z(k,r,0)\sigma_z(m,r,0)]r\,dr \; +$$

$$+ \int_o^1 [\tau_{rz}(k,r,0)\tau_{rz}(m,r,0)]r\,dr \; +$$

$$+ \int_o^{R_c} [\frac{\partial w(k,r,0)}{\partial r}\;\frac{\partial w(m,r,0)}{\partial r}]r\,dr\ , \tag{8.5.5a}$$

$$E_{km}(R,R_c) = \int_R^1 [\sigma_z(ik,r,0)\sigma_z(m,r,0)]r\,dr \; +$$

$$+ \int_o^1 [\tau_{rz}(ik,r,0)\tau_{rz}(m,r,0)]r\,dr \; +$$

$$+ \int_o^{R_c} [\frac{\partial w(ik,r,0)}{\partial r}\;\frac{\partial w(m,r,0)}{\partial r}]r\,dr\ , \tag{8.5.5b}$$

$$F_{km}(R,R_c) = \int_R^1 [\sigma_z(ik,r,0)\sigma_z(im,r,0)]r\,dr + $$
$$+ \int_o^1 [\tau_{rz}(ik,r,0)\tau_{rz}(im,r,0)]r\,dr +$$
$$+ \int_o^{R_c} [\frac{\partial w(ik,r,0)}{\partial r} \ \frac{\partial w(im,r,0)}{\partial r}]r\,dr \ , \qquad (8.5.5c)$$

and

$$\xi_k(R,R_c) = \int_R^1 [S(r)\sigma_z(k,r,0)]r\,dr +$$
$$+ \int_o^1 [T(r)\tau_{rz}(k,r,0)]r\,dr +$$
$$+ \int_o^{R_c} [V(r) \ \frac{\partial w(k,r,0)}{\partial r}]r\,dr \ , \qquad (8.5.5d)$$

$$\eta_k(R,R_c) = \int_R^1 [S(r)\sigma_z(ik,r,0)]r\,dr +$$
$$+ \int_o^1 [T(r)\tau_{rz}(ik,r,0)]r\,dr +$$
$$+ \int_o^{R_c} [V(r) \ \frac{\partial w(ik,r,0)}{\partial r}]r\,dr \ , \qquad (8.5.5e)$$

the conditions for minimum Ω become

$$\sum_{k=1}^{n} (D_{km}M_k + E_{km}N_k) = \xi_m \qquad m = 1, 2, \ldots, n \ ,$$

and (8.5.6)

$$\sum_{k=1}^{n} (E_{mk}M_k + F_{km}N_k) = \eta_m \qquad m = 1, 2, \ldots, n \ .$$

To evaluate D's, E's, F's, the following formulae are are needed

$$\int_o^R \sigma_z(k,r,0)\sigma_z(m,r,0) \ r\,dr =$$
$$= \frac{1}{4} [J_1(R,\beta_k,\beta_m) + J_1(R,\overline{\beta}_k,\overline{\beta}_m) + J_1(R,\beta_k,\overline{\beta}_m) + J_1(R,\overline{\beta}_k,\beta_m)], \qquad (8.5.7a)$$

$$\int_o^R \sigma_z(ik,r,0)\sigma_z(m,r,0) \ r\,dr =$$
$$= \frac{-i}{4} [J_1(R,\beta_k,\beta_m) - J_1(R,\overline{\beta}_k,\overline{\beta}_m) + J_1(R,\beta_k,\overline{\beta}_m) - J_1(R,\overline{\beta}_k,\beta_m)], \qquad (8.5.7b)$$

$$\int_0^R \sigma_z(ik,r,0)\sigma_z(im,r,0)\ r\ dr =$$

$$= \frac{1}{4}\ [-J_1(R,\beta_k,\beta_m) - J_1(R,\overline{\beta}_k,\overline{\beta}_m) + J_1(R,\beta_k,\overline{\beta}_m) + J_1(R,\overline{\beta}_k,\beta_m)], \tag{8.5.7c}$$

where the function $J_1(R,a,b)$ is defined by

$$J_1(R,a,b) = \frac{1}{2}\ [\ J_+(R,a,b) + J_-(R,a,b)\ ]\ . \tag{8.5.8}$$

With the help of the above formulae, the coefficients D's, E's, F's can be evaluated as

$$\begin{aligned} D_{km} = \frac{1}{4}\ [\ & J_+(1,\beta_k,\beta_m) + J_+(1,\overline{\beta}_k,\overline{\beta}_m) + J_-(1,\beta_k,\overline{\beta}_m) + J_-(1,\overline{\beta}_k,\beta_m)] \\ & - \frac{1}{4}\ [\ J_1(R,\beta_k,\beta_m) + J_1(R,\overline{\beta}_k,\overline{\beta}_m) + J_1(R,\beta_k,\overline{\beta}_m) + J_1(R,\overline{\beta}_k,\beta_m)] \\ & + \frac{1}{4}\ [\ K(R_c,\beta_k,\beta_m) + K(R_c,\overline{\beta}_k,\overline{\beta}_m) - K(R_c,\beta_k,\overline{\beta}_m) - K(R_c,\overline{\beta}_k,\beta_m)], \end{aligned} \tag{8.5.9a}$$

$$\begin{aligned} E_{km} = \frac{-i}{4}\ [\ & J_+(1,\beta_k,\beta_m) - J_+(1,\overline{\beta}_k,\overline{\beta}_m) + J_-(1,\beta_k,\overline{\beta}_m) - J_-(1,\overline{\beta}_k,\beta_m)] \\ & + \frac{i}{4}\ [\ J_1(R,\beta_k,\beta_m) - J_1(R,\overline{\beta}_k,\overline{\beta}_m) + J_1(R,\beta_k,\overline{\beta}_m) - J_1(R,\overline{\beta}_k,\beta_m)] \\ & - \frac{i}{4}\ [\ K(R_c,\beta_k,\beta_m) - K(R_c,\overline{\beta}_k,\overline{\beta}_m) - K(R_c,\beta_k,\overline{\beta}_m) + K(R_c,\overline{\beta}_k,\beta_m)], \end{aligned} \tag{8.5.9b}$$

$$\begin{aligned} F_{km} = \frac{1}{4}\ [& -J_+(1,\beta_k,\beta_m) - J_+(1,\overline{\beta}_k,\overline{\beta}_m) + J_-(1,\beta_k,\overline{\beta}_m) + J_-(1,\overline{\beta}_k,\beta_m)] \\ & - \frac{1}{4}\ [-J_1(R,\beta_k,\beta_m) - J_1(R,\overline{\beta}_k,\overline{\beta}_m) + J_1(R,\beta_k,\overline{\beta}_m) + J_1(R,\overline{\beta}_k,\beta_m)] \\ & + \frac{1}{4}\ [-K(R_c,\beta_k,\beta_m) - K(R_c,\overline{\beta}_k,\overline{\beta}_m) - K(R_c,\beta_k,\overline{\beta}_m) - K(R_c,\overline{\beta}_k,\beta_m)], \end{aligned} \tag{8.5.9c}$$

where the function $K(R_c,\beta_k,\beta_m)$ is defined by

$$K(R_c,\beta_k,\beta_m) = \int_0^{R_c} [\ \frac{\partial w(r,0,\beta_k)}{\partial r}\ \frac{\partial w(r,0,\beta_m)}{\partial r}\ ]r\ dr\ , \tag{8.5.10}$$

and its value is given in table 8.5.

Since $T(r)$ and $V(r)$ vanish on their known ranges, the coefficients ξ_k and η_k involve only the integrals containing $S(r)$. The

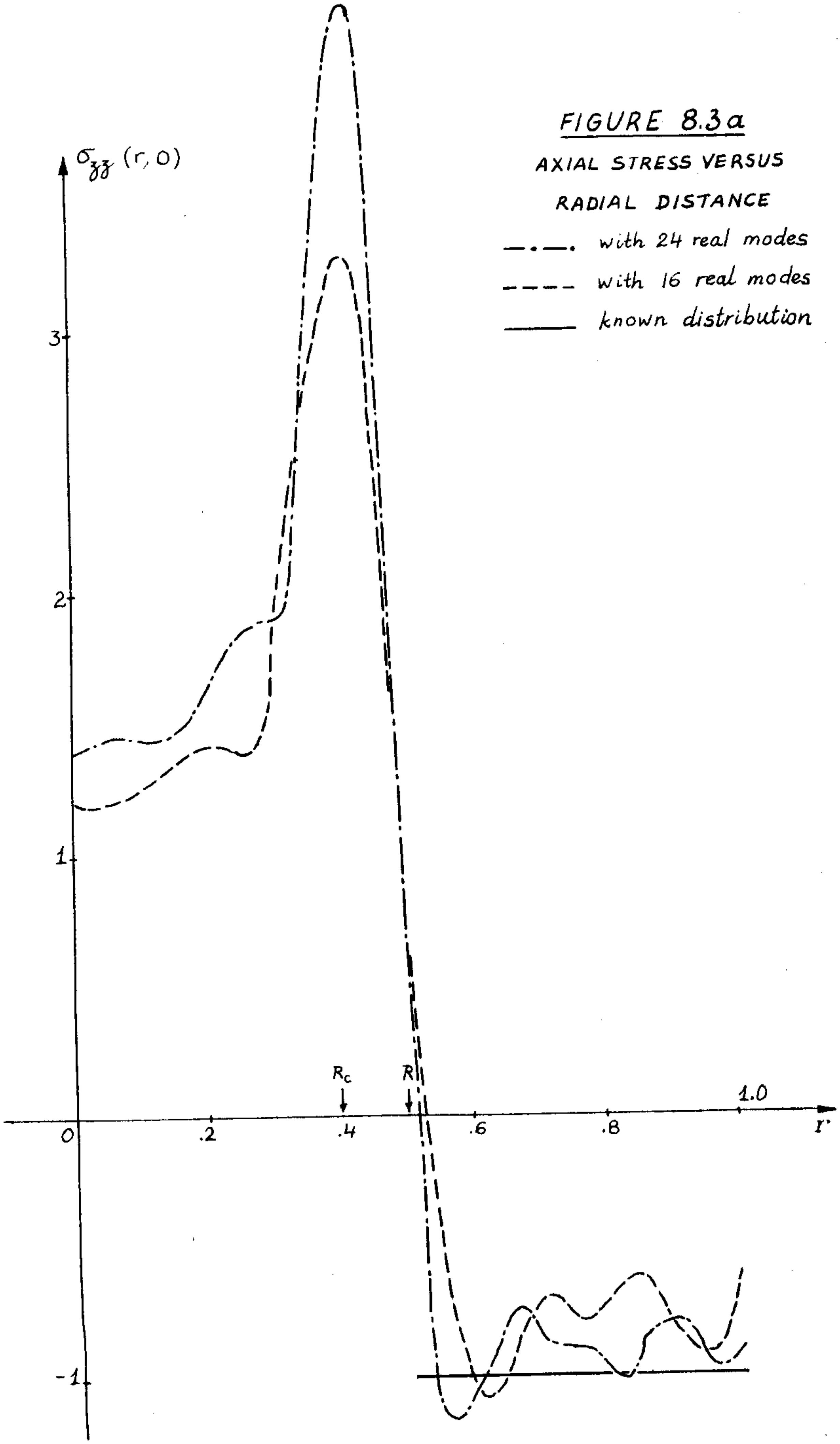


FIGURE 8.3a

AXIAL STRESS VERSUS RADIAL DISTANCE

—·—· with 24 real modes

- - - - with 16 real modes

—— known distribution

FIGURE 8.3b

SHEAR STRESS VERSUS RADIAL DISTANCE

—.—. with 24 real modes
– – – – – with 16 real modes
———— known distribution

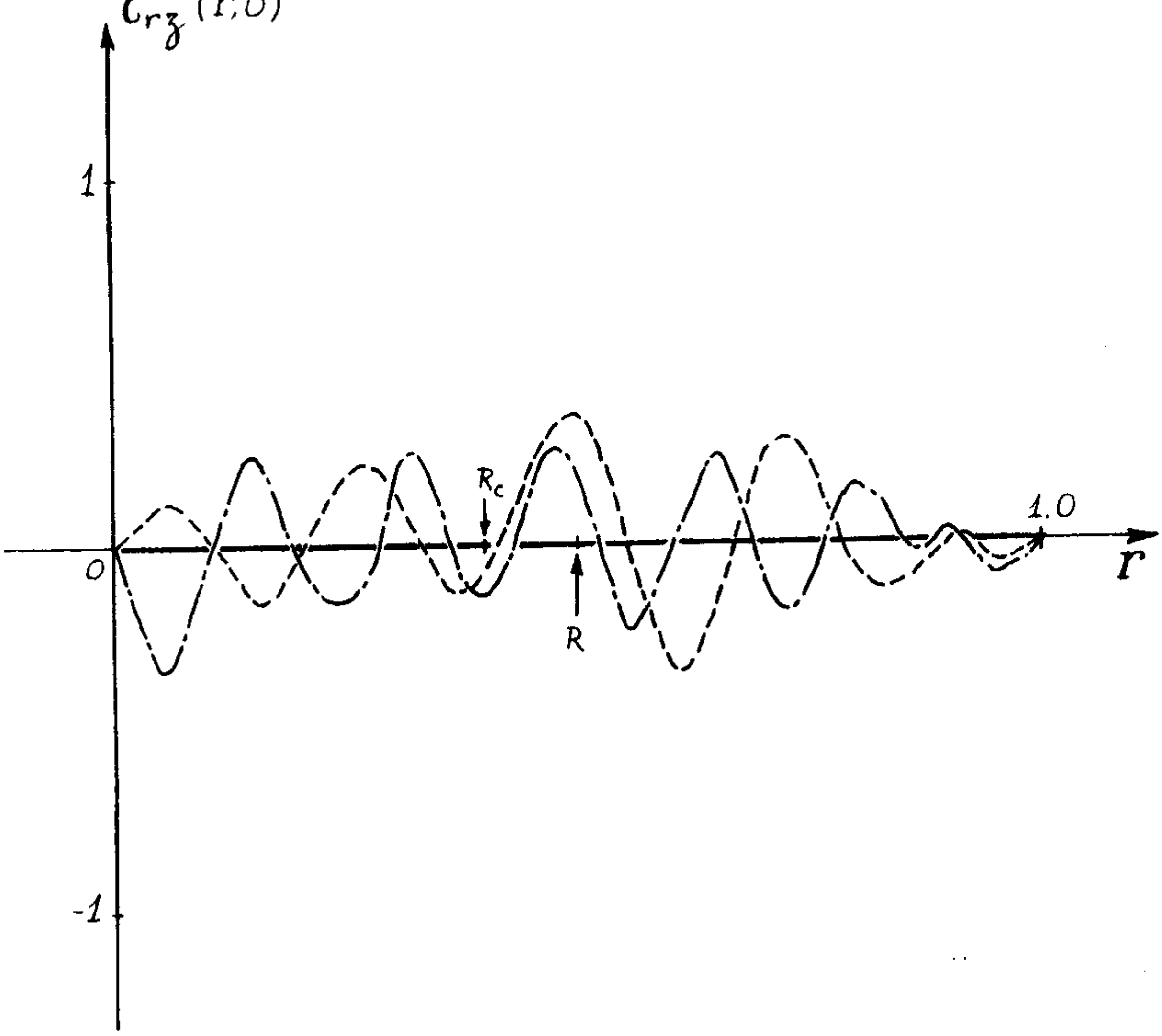


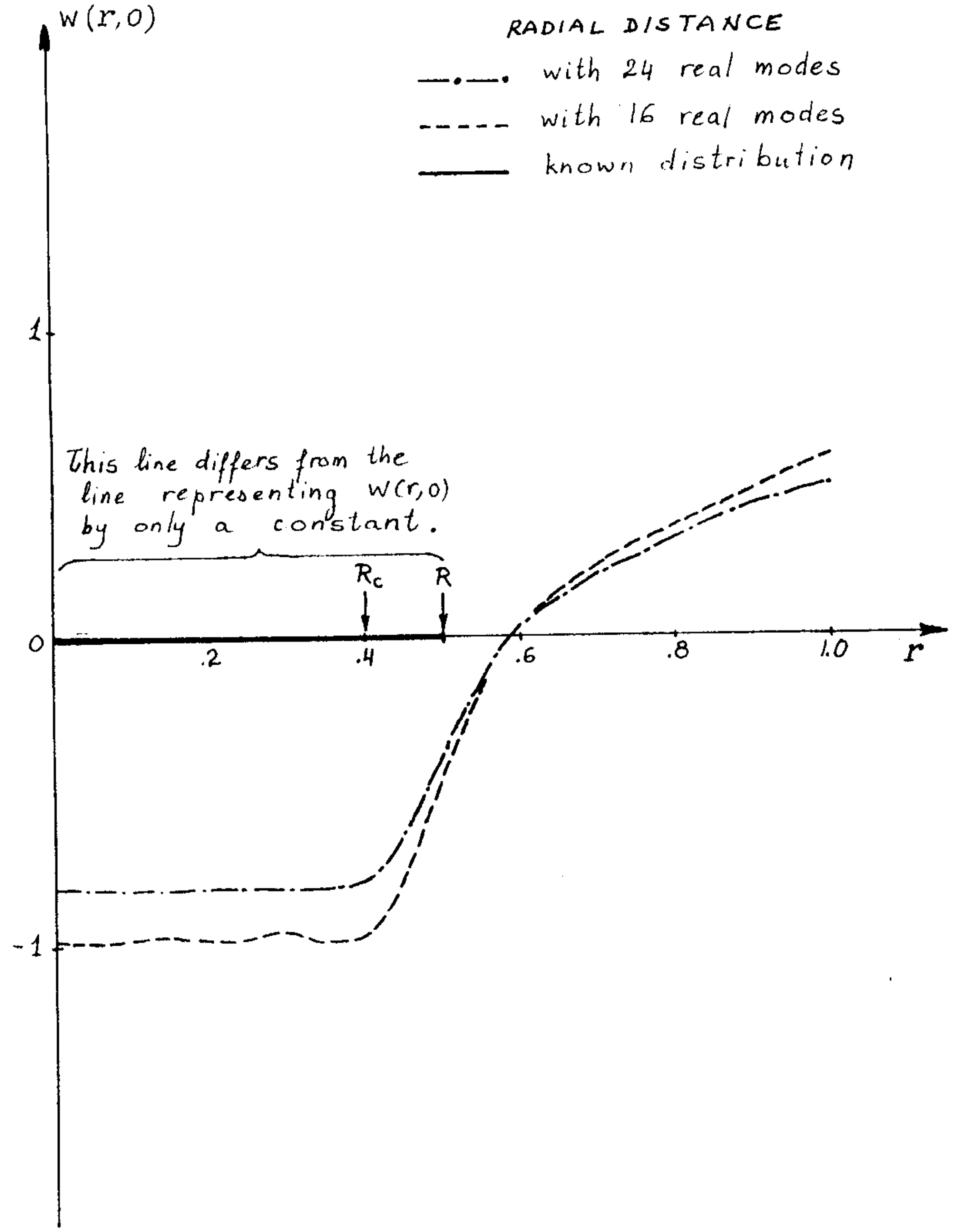
FIGURE 8.3c

AXIAL DISPLACEMENT VERSUS RADIAL DISTANCE

—·—· with 24 real modes

- - - - - with 16 real modes

——— known distribution



values of ξ_k and η_k are thus given by

$$\xi_k + i\eta_k \;=\; \int_{R_c}^{1} [\, S(r)\sigma_z(r,0,\beta_k)]\; r\, dr\; . \qquad (8.5.11)$$

According to the defining equation (8.4.15), the standardised deviation function is

$$s(R,R_c,n) = \frac{\Omega(R,R_c,M_1,M_2,\ldots,M_n,N_1,N_2,\ldots,N_n)}{\Omega(R,R_c,\;0,\;0,\ldots,\;0,\;0,\;0,\ldots,\;0)}\;, \qquad (8.5.12)$$

where $M_1, M_2, \ldots, M_n, N_1, N_2, \ldots, N_n$ are the solution of the system (8.5.6).

For $R = 0.5$, $R_c = 0.4$, the coefficients M's and N's are given in table 8.6 for n equal to 8 and 12 . Their stresses σ_z, τ_{rz} and axial displacement w are plotted in figures 8.3a, 8.3b, and 8.3c. The standardised deviation function s takes the values of 0.287 and 0.183 for the cases n = 8 and n = 12 respectively. This fact shows that the least square method used here did give some relatively good results.

Regarding the gap left between R_c and R, it is possible that R_c can be made very close to R if computational errors can be made extremely small. For the accuracy in use ($10^{-7}$for addition, substraction, multiplication and division), the chosen value of $R_c = 0.4$ is not very close to the value of $R = 0.5$, however this chosen value gives consistently smaller s for larger n.

8.6 Evaluations of some integrals developed for this chapter.

Section 8.4 introduced the integrals $J_{\pm}(r,a,b)$, $J_1(r,a,b)$ etc..., whose evaluations are not simple and are deferred until this section.

For the sake of clarity, the following integrals will be first evaluated

$$i_{100}(r,a,b) \;\equiv\; \frac{1}{I_1(a)I_1(b)} \int_0^r I_0(ar)I_0(br)\; r\; dr\;,$$

$$i_{111}(r,a,b) \equiv \frac{1}{I_1(a)I_1(b)} \int_0^r I_1(ar)I_1(br)\ r\ dr\ ,$$

$$i_{201}(r,a,b) \equiv \frac{1}{I_1(a)I_1(b)} \int_0^r I_0(ar)I_1(br)\ r^2dr\ ,$$

$$i_{210}(r,a,b) \equiv \frac{1}{I_1(a)I_1(b)} \int_0^r I_1(ar)I_0(br)\ r^2dr\ ,$$

$$i_{300}(r,a,b) \equiv \frac{1}{I_1(a)I_1(b)} \int_0^r I_0(ar)I_0(br)\ r^3dr\ ,$$

$$i_{311}(r,a,b) \equiv \frac{1}{I_1(a)I_1(b)} \int_0^r I_1(ar)I_1(br)\ r^3dr\ .$$

The first two integrals can be calculated with the aid of tables of indefinite integrals such as those on p. 484 of [40]. The next four integrals can be calculated using the following relations, as is done in Lur'e's book,

$$b\int I_0(ax)I_1(bx)\ x^2dx + a\int I_1(ax)I_0(bx)\ x^2dx =$$

$$= x^2I_0(ax)I_0(bx) - 2\int I_0(ax)I_0(bx)\ x\ dx\ ,$$

$$a\int I_0(ax)I_1(bx)\ x^2dx + b\int I_1(ax)I_0(bx)\ x^2dx =$$

$$= x^2I_1(ax)I_1(bx)$$, for the middle equations, and

$$a\int I_0(ax)I_0(bx)\ x^3dx + b\int I_1(ax)I_1(bx)\ x^3dx =$$

$$= x^3I_1(ax)I_0(bx) - 2\int I_1(ax)I_0(bx)\ x^2dx\ ,$$

$$b\int I_0(ax)I_0(bx)\ x^3dx + a\int I_1(ax)I_1(bx)\ x^3dx =$$

$$= x^3I_0(ax)I_1(bx) - 2\int I_0(ax)I_1(bx)\ x^2dx\ ,$$

for the last two equations.

Thus one obtains the results presented in table 8.2a for $a \neq b$.

The integrals $i(r,a,a)$'s are evaluated in the following

$$I_1^2(a)\,i_{100}(r,a,a) = \lim_{b\to a} [I_1(a)I_1(b)\,i_{100}(r,a,b)]$$

$$= \lim_{b\to a} \frac{r}{a^2-b^2} [aI_1(ar)I_0(br)-bI_0(ar)I_1(br)]$$

$$= \frac{r^2}{2} [I_0^2(ar) - I_1^2(ar)] \ ,$$

$$I_1^2(a)\,i_{111}(r,a,a) = \lim_{b\to a} [I_1(a)I_1(b)\,i_{111}(r,a,b)]$$

$$= \lim_{b\to a} \frac{r}{a^2-b^2} [aI_0(ar)I_1(br)-bI_1(ar)I_0(br)]$$

$$= \frac{r^2}{2} [I_1^2(ar) - I_0^2(ar)] + \frac{r}{a} I_0(ar)I_1(ar) \ .$$

The first equation for each integral made use of Arzelà's theorem to bring the limit outside the integration sign. However this should not bother readers since those equations can be considered only assumptions in deriving the (correct) final results; as the final results can be verified by direct differentiation, there is no need for the proof of the first equations evaluating each integral.

Using the following equation

$$2a\int_0^r I_0(ar)I_1(ar)\ r^2dr = r^2I_0(ar)I_0(ar) - 2\int_0^r I_0(ar)I_0(ar)\ r\ dr \ ,$$

the integral $i_{201}(r,a,a)$ can be evaluated as

$$i_{201}(r,a,a) = \frac{1}{I_1^2(a)} \frac{r^2}{2a} I_1(ar)I_1(ar) \ .$$

For the last two integrals, consider the relation

$$\frac{\partial}{\partial r} \left[\frac{r^4}{2} \left(I_0^2(ar) - I_1^2(ar) \right) \right] = 2r^3I_0^2(ar) - r^3I_1^2(ar) \ ,$$

or

$$2\int_0^r I_0^2(ar)r^3dr - \int_0^r I_1^2(ar)r^3dr = \frac{r^4}{2} [I_0^2(ar) - I_1^2(ar)] \ ,$$

and the relation

$$\int_0^r I_0^2(ar)r^3dr + \int_0^r I_1^2(ar)r^3dr = \frac{r^3}{a} I_1(ar)I_0(ar) - \frac{2}{a}\int_0^r I_1(ar)I_0(ar)r^2dr.$$

They give

$$\int_0^r I_0(ar)I_0(ar)r^3dr = \frac{r^4}{6}[I_0^2(ar)-I_1^2(ar)] + \frac{r^3}{3a}I_1(ar)I_0(ar) - \frac{2}{3a}\int_0^r I_0(ar)I_1(ar)r^2dr \ .$$

The integral $i_{311}(r,a,a)$ is determined from

$$\int_0^r I_1(ar)I_1(ar)r^3dr = -\frac{r^4}{6}[I_0^2(ar)-I_1^2(ar)] + \frac{2}{3a}r^2I_0(ar)I_1(ar) - \frac{4}{3a}\int_0^r I_0(ar)I_1(ar)r^2dr \ .$$

Table 8.2b lists the results of the above derivations.

The functions $J_\pm(r,a,b)$ are then given as

$$\begin{aligned} J_\pm(r,a,b) = {} & [4 - 2(a\lambda_a + b\lambda_b) + ab\lambda_a\lambda_b]i_{100}(r,a,b) \\ & \mp ab\lambda_a\lambda_b i_{111}(r,a,b) \\ & + (2b - \lambda_a ab \pm \lambda_b ab)i_{201}(r,a,b) \\ & + (2a - \lambda_b ab \pm \lambda_a ab)i_{210}(r,a,b) \\ & \mp abi_{300}(r,a,b) + abi_{311}(r,a,b) \ , \end{aligned}$$

and $J_\pm(r,a,a)$ are obtained by replacing b by a in the above formula.

It should be noted that the above derived formula is correct only when a and b are eigenvalues derived from the equation (8.2.8).

Substitution of the values of the i(r,a,b) integrals into the above formula gives the value of $J_\pm(r,a,b)$ as in table 8.3.

The function $J_1(r,a,b)$ defined as

$$J_1(r,a,b) = \frac{1}{2}[J_+(r,a,b) + J_-(r,a,b)]$$

has its value easily worked out from table 8.3.

By definition $L_\pm(r,a,b)$ is

$$L_\pm(r,a,b) = \int_o^r [u(x,0,a)u(x,0,b) \pm w(x,0,a)w(x,0,b)]x\,dx.$$

Substitution of the values for u and w corresponding to the eigenvalues a and b gives

$$\begin{aligned}L_\pm(r,a,b) = \mp\ & (a\lambda_a^2 - a - \lambda_a)(b\lambda_b^2 - b - \lambda_b)i_{100}(r,a,b)\\ &+ (a\lambda_a^2 - a + \lambda_a)(b\lambda_b^2 - b + \lambda_b)i_{111}(r,a,b)\\ &- [2(1-\nu)(\tfrac{1}{b} \pm \tfrac{1}{a}) + (\lambda_b \mp \lambda_a)]i_{201}(r,a,b)\\ &- [2(1-\nu)(\tfrac{1}{a} \pm \tfrac{1}{b}) + (\lambda_a \mp \lambda_b)]i_{210}(r,a,b)\\ &+ i_{300}(r,a,b) \mp i_{311}(r,a,b)\ .\end{aligned}$$

The formula for $L_\pm(r,a,a)$ is obtained from the above by replacing b by a.

Substitution of the values of the i integrals into the formulae for L gives the results in table 8.4.

Similarly, the function K(r,a,b) is defined as

$$K(r,a,b) = \int_o^r \left(\frac{\partial w(x,0,a)}{\partial x}\right)\left(\frac{\partial w(x,0,b)}{\partial x}\right) x\,dx\ .$$

Substitution of the values for $\frac{\partial w}{\partial x}$ corresponding to the eigenvalues a and b gives

$$\begin{aligned}K(r,a,b) = -\ & ab(a\lambda_a^2 - a - \lambda_a)(b\lambda_b^2 - b - \lambda_b)i_{111}(r,a,b)\\ &- ab(b\lambda_b^2 - b - \lambda_b)i_{201}(r,a,b)\\ &- ab(a\lambda_a^2 - a - \lambda_a)i_{210}(r,a,b)\\ &- ab\,i_{300}(r,a,b)\ .\end{aligned}$$

The function K(r,a,a) is obtained by replacing b by a in the above formula.

Simple substitution for the values of the i integrals into the above formula gives the results in table 8.5.

8.7 Conclusions.

(a) The application of the Papkovich-Neuber solution to the end problem of an axi-symmetrically loaded semi-infinite cylinder is a nice demonstration of typical techniques and also of some difficulties involved in the use of this kind of solution.

(b) The improved values for β_n presented in section 8.3 enable more accurate numerical evaluations of eigenmodes of the cylinder than would be possible with previous literatures. However, even more accurate values are desirable if more than the first twenty eigenmodes are used in a particular problem.

(c) The generalised least square method presented in section 8.4 and its relative success in dealing with the annular crack give a fair idea on its applicability and capacity. The crack problem solved by this generalised least square method can have only fair describtion of its stresses and displacements. However, this is a typical characteristic of problems solved with the use of non-orthogonal eigenmodes.

Table 8.2a: Indefinite integrals of products of some Bessel functions.

For any complex numbers a and b ($0 \neq a \neq b \neq 0$):

$$i_{100}(r,a,b) \equiv \frac{1}{I_1(a)I_1(b)} \int_0^r I_0(ar)I_0(br)\ r\,dr$$

$$= \frac{1}{I_1(a)I_1(b)} \frac{r}{a^2-b^2} \left[aI_1(ar)I_0(br) - bI_0(ar)I_1(br)\right]$$

$$i_{111}(r,a,b) \equiv \frac{1}{I_1(a)I_1(b)} \int_0^r I_1(ar)I_1(br)\ r\,dr$$

$$= \frac{1}{I_1(a)I_1(b)} \frac{r}{a^2-b^2} \left[aI_0(ar)I_1(br) - bI_1(ar)I_0(br)\right]$$

$$i_{201}(r,a,b) \equiv \frac{1}{I_1(a)I_1(b)} \int_0^r I_0(ar)I_1(br)\ r^2dr$$

$$= \frac{1}{I_1(a)I_1(b)} \frac{r^2}{a^2-b^2} \left[aI_1(ar)I_1(br) - bI_0(ar)I_0(br)\right]$$

$$+ \frac{2}{a^2-b^2}\, b\ i_{100}(r,a,b)$$

$$i_{210}(r,a,b) \equiv \frac{1}{I_1(a)I_1(b)} \int_0^r I_1(ar)I_0(br)\ r^2dr$$

$$= \frac{1}{I_1(a)I_1(b)} \frac{r^2}{a^2-b^2} \left[aI_0(ar)I_0(br) - bI_1(ar)I_1(br)\right]$$

$$- \frac{2}{a^2-b^2}\, a\ i_{100}(r,a,b)$$

$$i_{300}(r,a,b) \equiv \frac{1}{I_1(a)I_1(b)} \int_0^r I_0(ar)I_0(br)\ r^3dr$$

$$= \frac{1}{I_1(a)I_1(b)} \frac{r^3}{a^2-b^2} \left[aI_1(ar)I_0(br) - bI_0(ar)I_1(br)\right]$$

$$- \frac{2}{a^2-b^2} \left[a\ i_{210}(r,a,b) - b\ i_{201}(r,a,b)\right]$$

$$i_{311}(r,a,b) \equiv \frac{1}{I_1(a)I_1(b)} \int_0^r I_1(ar)I_1(br)\ r^3dr$$

$$= \frac{1}{I_1(a)I_1(b)} \frac{r^3}{a^2-b^2} \left[aI_0(ar)I_1(br) - bI_1(ar)I_0(br)\right]$$

$$+ \frac{2}{a^2-b^2} \left[b\ i_{210}(r,a,b) - a\ i_{201}(r,a,b)\right]$$

Table 8.2b: Indefinite integrals of the squares of some Bessel functions.

For any complex number a ($a \neq 0$):

$$i_{100}(r,a,a) \equiv \frac{1}{I_1(a)I_1(a)} \int_0^r I_0(ar)I_0(ar)\ r\,dr$$

$$= \frac{1}{I_1^2(a)} \frac{r^2}{2} \left[I_0^2(ar) - I_1^2(ar)\right]$$

$$i_{111}(r,a,a) \equiv \frac{1}{I_1(a)I_1(a)} \int_0^r I_1(ar)I_1(ar)\ r\,dr$$

$$= \frac{1}{I_1^2(a)} \frac{r^2}{2} \left[I_1^2(ar) - I_0^2(ar)\right]$$

$$+ \frac{1}{I_1^2(a)} \frac{r}{a} I_0(ar)I_1(ar)$$

$$i_{201}(r,a,a) \equiv \frac{1}{I_1(a)I_1(a)} \int_0^r I_0(ar)I_1(ar)\ r^2 dr$$

$$= \frac{1}{I_1^2(a)} \frac{r^2}{2a} I_1^2(ar)$$

$$i_{300}(r,a,a) \equiv \frac{1}{I_1(a)I_1(a)} \int_0^r I_0(ar)I_0(ar)\ r^3 dr$$

$$= \frac{1}{I_1^2(a)} \frac{r^4}{6} \left[I_0^2(ar) - I_1^2(ar)\right]$$

$$+ \frac{1}{I_1^2(a)} \frac{r^3}{3a} I_0(ar)I_1(ar)$$

$$- \frac{2}{3a} i_{201}(r,a,a)$$

$$i_{311}(r,a,a) \equiv \frac{1}{I_1(a)I_1(a)} \int_0^r I_1(ar)I_1(ar)\ r^3 dr$$

$$= \frac{1}{I_1^2(a)} \frac{r^4}{6} \left[I_1^2(ar) - I_0^2(ar)\right]$$

$$+ \frac{1}{I^2(a)} \frac{2r^3}{3a} I_0(ar)I_1(ar)$$

$$- \frac{4}{3a} i_{201}(r,a,a)$$

Table 8.3: Integrals of eigenvalued stresses.

For any two different eigenvalues a and b ($a \neq b$ and $\Psi(a) = \Psi(b) = 0$):

$$
\begin{aligned}
J_{\pm}(r,a,b) \equiv{}& \int_0^r [\sigma_z(x,0,a)\sigma_z(x,0,b) \pm \tau_{rz}(x,0,a)\tau_{rz}(x,0,b)]\, x\, dx \\
={}& \frac{1}{I_1(a)I_1(b)} \frac{r}{a^2-b^2} I_0(ar)I_1(br)\Big\{ab(a\pm b)(r^2 \mp \lambda_a\lambda_b) \\
&+ 2b(a\lambda_a + b\lambda_b) + \frac{2ab^2}{a \mp b}\Big[(\pm\lambda_a - \lambda_b) \pm \frac{2}{a \mp b}\Big]\Big\} \\
&+ \frac{1}{I_1(a)I_1(b)} \frac{r}{a^2-b^2} I_1(ar)I_0(br)\Big\{ab(a\pm b)(\mp r^2 + \lambda_a\lambda_b) \\
&- 2a(a\lambda_a + b\lambda_b) + \frac{2a^2b}{a \mp b}\Big[(\mp\lambda_a + \lambda_b) \mp \frac{2}{a \mp b}\Big]\Big\} \\
&+ \frac{1}{I_1(a)I_1(b)} \frac{r^2}{a \mp b} I_0(ar)I_0(br)\Big\{2(a \mp b) \pm \frac{2ab}{a \mp b} + ab(\pm\lambda_a - \lambda_b)\Big\} \\
&+ \frac{1}{I_1(a)I_1(b)} \frac{r^2 ab}{a \mp b} I_1(ar)I_1(br)\Big\{-\frac{2}{a \mp b} + (-\lambda_a \pm \lambda_b)\Big\}
\end{aligned}
$$

For any eigenvalue a ($\Psi(a) = 0$):

$$
\begin{aligned}
J_{\pm}(r,a,a) \equiv{}& \int_0^r [\sigma_z(x,0,a)\sigma_z(x,0,a) \pm \tau_{rz}(x,0,a)\tau_{rz}(x,0,a)]\, x\, dx \\
={}& \frac{r^2}{I_1^2(a)} I_0^2(ar)\Big\{-\frac{1}{6} a^2r^2(1\pm 1) + \frac{1}{2}\big[4 - 4a\lambda_a + a^2\lambda_a^2(1\pm 1)\big]\Big\} \\
&+ \frac{r^2}{I_1^2(a)} I_1^2(ar)\Big\{\frac{1}{6} a^2r^2(1\pm 1) + \frac{1}{2}\Big[\frac{2}{3}(\pm 1 - 2) \\
&\qquad + 2(1\pm 1)a\lambda_a - (1\pm 1)a^2\lambda_a^2\Big]\Big\} \\
&+ \frac{ar}{I_1^2(a)} I_0(ar)I_1(ar)\Big\{\frac{1}{3} r^2(\mp 1 + 2) \mp \lambda_a^2\Big\}
\end{aligned}
$$

Table 8.4: Integrals of eigenvalued displacements.

For any two different eigenvalues a and b ($a \neq b$ and $\Psi(a) = \Psi(b) = 0$):

$$L_{\pm}(r,a,b) \equiv \int_{0}^{r} [u(x,0,a)u(x,0,b) \pm w(x,0,a)w(x,0,b)]x\,dx$$

$$= \frac{1}{I_1(a)I_1(b)} \frac{r}{a^2-b^2} I_0(ar)I_1(br)\Big\{-(b\pm a)r^2$$

$$\pm b(a\lambda_a^2-a-\lambda_a)(b\lambda_b^2-b-\lambda_b) + a(a\lambda_a^2-a+\lambda_a)(b\lambda_b^2-b+\lambda_b)$$

$$- \frac{b}{a^2-b^2}\Big[4\frac{a\pm b}{a\mp b} + 2(a\pm b)(\lambda_a\mp\lambda_b) \pm 2ab(\lambda_a^2-\lambda_b^2)\Big]\Big\}$$

$$+ \frac{1}{I_1(a)I_1(b)} \frac{r}{a^2-b^2} I_1(ar)I_0(br)\Big\{(a\pm b)r^2$$

$$\mp a(a\lambda_a^2-a-\lambda_a)(b\lambda_b^2-b-\lambda_b) - b(a\lambda_a^2-a+\lambda_a)(b\lambda_b^2-b+\lambda_b)$$

$$+ \frac{a}{a^2-b^2}\Big[4\frac{a\pm b}{a\mp b} + 2(a\pm b)(\lambda_a\mp\lambda_b) \pm 2ab(\lambda_b^2-\lambda_a^2)\Big]\Big\}$$

$$+ \frac{1}{I_1(a)I_1(b)} \frac{r^2}{a^2-b^2} I_0(ar)I_0(br)\Big\{-\frac{2(a\pm b)}{a\mp b}$$

$$+ (b\pm a)(\lambda_b\mp\lambda_a) \mp ab(\lambda_b^2-\lambda_a^2)\Big\}$$

$$+ \frac{1}{I_1(a)I_1(b)} \frac{r^2}{a^2-b^2} I_1(ar)I_1(br)\Big\{\frac{2(b\pm a)}{a\mp b}$$

$$+ (b\pm a)(\lambda_a\mp\lambda_b) - ab(\lambda_b^2-\lambda_a^2)\Big\}$$

For any eigenvalue a ($\Psi(a) = 0$):

$$L_{\pm}(r,a,a) \equiv \int_{0}^{r} [u(x,0,a)u(x,0,a) \pm w(x,0,a)w(x,0,a)]x\,dx$$

$$= \frac{r^2}{I_1^2(a)} I_0^2(ar)\Big\{\frac{r^2}{6}(1\pm1) - \frac{1}{2}(a\lambda_a^2-a+\lambda_a)^2 \mp \frac{1}{2}(a\lambda_a^2-a-\lambda_a)^2\Big\}$$

$$+ \frac{r^2}{I_1^2(a)} I_1^2(ar)\Big\{-\frac{r^2}{6}(1\pm1) + \frac{1}{2}(a\lambda_a^2-a+\lambda_a)^2 \pm \frac{1}{2}(a\lambda_a^2-a-\lambda_a)^2$$

$$- \frac{1\mp2}{3a^2} - (1\pm1)(\lambda_a^2-1) - \frac{\lambda_a}{a}(1\mp1)\Big\}$$

$$+ \frac{r}{aI_1^2(a)} I_0(ar)I_1(ar)\Big\{\frac{r^2}{3}(1\mp2) + (a\lambda_a^2-a+\lambda_a)^2\Big\}$$

Table 8.5: Integrals of eigenvalued $\frac{\partial w}{\partial r}$.

For any two different eigenvalues a and b ($a \neq b$ and $\Psi(a) = \Psi(b) = 0$):

$$K(r,a,b) \equiv \int_0^r \left(\frac{\partial w(x,0,a)}{\partial x}\right)\left(\frac{\partial w(x,0,b)}{\partial x}\right) x\,dx$$

$$= \frac{1}{I_1(a)I_1(b)} \frac{rab}{a^2-b^2} I_0(ar)I_1(br)\Big\{ br^2$$

$$- a(a\lambda_a^2-a-\lambda_a)(b\lambda_b^2-b-\lambda_b) + \frac{2b}{a^2-b^2}(a\lambda_a-b\lambda_b+2\frac{a^2+b^2}{a^2-b^2})\Big\}$$

$$+ \frac{1}{I_1(a)I_1(b)} \frac{rab}{a^2-b^2} I_1(ar)I_0(br)\Big\{-ar^2$$

$$+ b(a\lambda_a^2-a-\lambda_a)(b\lambda_b^2-b-\lambda_b) - \frac{2a}{a^2-b^2}(a\lambda_a-b\lambda_b+2\frac{a^2+b^2}{a^2-b^2})\Big\}$$

$$+ \frac{1}{I_1(a)I_1(b)} \frac{r^2ab}{a^2-b^2} I_0(ar)I_0(br)\Big\{a\lambda_a-b\lambda_b+2\frac{a^2+b^2}{a^2-b^2}\Big\}$$

$$+ \frac{1}{I_1(a)I_1(b)} \frac{r^2ab}{a^2-b^2} I_1(ar)I_1(br)\Big\{ b(a\lambda_a^2-\lambda_a)$$

$$- a(b\lambda_b^2-\lambda_b) - 4\frac{ab}{a^2-b^2}\Big\}$$

For any eigenvalue a ($\Psi(a) = 0$):

$$K(r,a,a) \equiv \int_0^r \left(\frac{\partial w(x,0,a)}{\partial x}\right)\left(\frac{\partial w(x,0,a)}{\partial x}\right) x\,dx$$

$$= \frac{1}{I_1^2(a)} \frac{r^2a^2}{2} I_0^2(ar)\Big\{-\frac{r^2}{3}+(a\lambda_a^2-a-\lambda_a)^2\Big\}$$

$$+ \frac{1}{I_1^2(a)} \frac{r^2a^2}{2} I_1^2(ar)\Big\{\frac{r^2}{3}-(a\lambda_a^2-a-\lambda_a)^2$$

$$- \frac{2}{a}(a\lambda_a^2-a-\lambda_a) + \frac{2}{3a^2}\Big\}$$

$$+ \frac{1}{I_1^2(a)} raI_0(ar)I_1(ar)\Big\{-\frac{r^2}{3}-(a\lambda_a^2-a-\lambda_a)^2\Big\}$$

Table 8.6: Values of coefficients M's and N's in figures 8.3.

($R = 0.5$, $R_c = 0.4$)

Coef.	2n = 16	2n = 24
M_1	-1.026 254 4	-1.208 941 2
N_1	-1.497 217 8	-1.790 214 3
M_2	0.727 703 0	0.905 629 8
N_2	0.352 294 5	0.486 759 2
M_3	0.773 838 7	0.909 433 7
N_3	1.419 885 2	1.710 143 2
M_4	-1.132 698 5	-1.369 013 9
N_4	-0.357 506 2	-0.528 785 1
M_5	-0.696 338 5	-0.604 603 7
N_5	-1.414 991 3	-1.562 642 0
M_6	1.053 299 5	1.467 353 3
N_6	0.161 524 4	0.722 913 1
M_7	0.467 118 5	0.100 369 3
N_7	0.772 639 5	1.296 839 7
M_8	-0.403 237 3	-1.529 171 3
N_8	-0.013 086 3	-0.913 577 3
M_9	0	0.042 876 5
N_9	0	-1.133 646 4
M_{10}	0	1.086 646 4
N_{10}	0	0.559 093 8
M_{11}	0	-0.110 057 8
N_{11}	0	0.282 338 0
M_{12}	0	-0.122 844 2
N_{12}	0	0.010 038 5
$s(R,R_c,n)$	0.287	0.182

CHAPTER 9

Miscellaneous issues and Epilogue.

This chapter presents some of the side issues which have been left aside in order to permit free flow of thought during the previous chapters. The epilogue will give some concluding notes for the thesis.

9.1 Notes on the Galerkin's vector.

During the study in chapter 1, there has been only one reference to the Galerkin's [50] vector. In the book of Westergaard [56], he states that the special solution (1.2.6) to the homogeneous Lame's equation is due to Galerkin (but not in the vector form). Westergaard also worked ingeniously with this vector. He proved that the Galerkin's vector can *completely* describe all displacements satisfying the Lame's equation in a z-convex material region ([56], p. 123). His proof requires neither simple connectivity nor the Helmholtz transformation.

The solution using Galerkin's vector and the Papkovich-Neuber solution are really equivalent in a finite region. Precisely, their equivalence can be expressed in the following proposition.

Proposition.

In a region D, if $\underset{\sim}{V}$ is expressible by

$$\underset{\sim}{V} = \alpha\nabla^2\underset{\sim}{F} - 2\underset{\sim}{\nabla}(\underset{\sim}{\nabla}\cdot\underset{\sim}{F}) , \qquad (9.1.1)$$

where α is a constant, and $\underset{\sim}{F}$ satisfies

$$\nabla^2(\nabla^2\underset{\sim}{F}) = \underset{\sim}{0} , \qquad (9.1.2)$$

then $\underset{\sim}{V}$ is also expressible as

$$\underset{\sim}{V} = \alpha\underset{\sim}{B} - \underset{\sim}{\nabla}(\underset{\sim}{r}\cdot\underset{\sim}{B} + B_0) \qquad (9.1.3)$$

where

$$\nabla^2 \underset{\sim}{B} = \underset{\sim}{0} \quad , \quad \nabla^2 B_0 = 0 \ ; \tag{9.1.4}$$

and vice-versa if D is finite and the constant α is different from 2 .

Proof:

If (9.1.1) and (9.1.2) hold, define

$$\underset{\sim}{B} = \nabla^2 \underset{\sim}{F} \ ,$$

which is harmonic, then one has

$$\nabla^2 (\underset{\sim}{\nabla} \cdot \underset{\sim}{F}) = \underset{\sim}{\nabla} \cdot \underset{\sim}{B} \ .$$

Thus $\underset{\sim}{\nabla} \cdot \underset{\sim}{F}$ is given as

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{F} = \tfrac{1}{2} \, \underset{\sim}{r} \cdot \underset{\sim}{B} + B_0 \ ,$$

where B_0 is harmonic. Hence (9.1.3) and (9.1.4) hold.

This proof is essentially the same as the one due to Papkovich (section $\underset{\sim}{1}.\underset{\sim}{2}$).

Conversely, let (9.1.3) and (9.1.4) hold. It will be proved that (9.1.1) and (9.1.2) also hold.

Define $\underset{\sim}{E}$ by

$$\underset{\sim}{E}(P) = -\frac{1}{4\pi} \int_D \frac{\underset{\sim}{B}(Q)}{r_{PQ}} \, d\tau(Q) \quad ,$$

then one has

$$\nabla^2 \underset{\sim}{E} = \underset{\sim}{B} \ ,$$

and $\quad \nabla^2 (\nabla^2 \underset{\sim}{E}) = \underset{\sim}{0} \ .$

Since

$$\nabla^2 \left[\underset{\sim}{\nabla} \cdot \underset{\sim}{E} - \tfrac{1}{2} (\underset{\sim}{r} \cdot \underset{\sim}{B} + B_0) \right] = 0 \ ,$$

one has

$$\underset{\sim}{r}\cdot\underset{\sim}{B} + B_0 = 2\underset{\sim}{\nabla}\cdot\underset{\sim}{E} + \psi$$

where ψ is harmonic function.

Define another function η by

$$\eta(P) = -\frac{1}{4\pi}\int_D \frac{\psi(Q)}{r_{PQ}}\, d\tau(Q) \quad ,$$

which satisfies $\nabla^2\eta = \psi$, and is biharmonic.

The function F defined by

$$\underset{\sim}{F} = \underset{\sim}{E} - \frac{1}{\alpha-2}\underset{\sim}{\nabla}\eta$$

is obviously biharmonic, satisfying (9.1.2). Simple substitution shows that $\underset{\sim}{V}$ is a function of $\underset{\sim}{F}$ as given by (9.1.1).

The two pairs of equations [(9.1.1) and (9.1.2)] and [(9.1.3) and (9.1.4)] are thus really equivalent.

Westergaard has shown that any one (say the z) component of the Galerkin's vector $\underset{\sim}{F}$ can be set to zero if the region D is convex in that (z) direction and one more direction (orthogonal to z). If Almansi's theorems (see section 2.4) are then applied to the remaining biharmonic components of $\underset{\sim}{F}$, the Papkovich-Neuber solution in terms of three harmonic functions will result. This is also the result obtained if B_z is omitted (see [6]) in the region D. (This point was brought to the author's attention by Dr. J.R. Barber of the University of Newcastle upon Tyne, U.K.)

If the region D does not allow any reduction neither in the number of components of $\underset{\sim}{F}$, nor in the form of each component, then the Galerkin's vector form [(9.1.1) and (9.1.2)] will look terribly awkward in terms of its twelve harmonic functions !

9.2 The use of Helmholtz transformation.

The authors of [6] are probably convinced that proofs which avoid using the Helmholtz transformation to deduce [(1.7.1) and

(1.7.2)] would be untenable. This thesis has shown that it is not so for Neuber's or Westergaard's proofs.

It should be pointed out that the necessary representation [(9.1.1) and (9.1.2)] for displacement $\underset{\sim}{V}$ needs only *one single* value of α, the value is $4(1-\nu)$. On the other hand, the Helmholtz transformation requires that (9.1.1) must be possible for any *arbitrary* given value of α. It is this point which yields the "over-requirement" in Mindlin's proof.

It should be noted that the Helmholtz transformation is only *superficially* simple. Indeed, it involves the theories of harmonic functions, of function spaces, of Fredholm's integrals, and many more. Its use, therefore, should be avoided as much as possible.

The proof due to Westergaard ([56], p. 123) consists of finding a harmonic (vector) function whose divergence is equal to another given harmonic (scalar) function. This point can be readily proved with the use of [(2.5.1) and (2.5.2)], which is a consequence of the Helmholtz transformation, for some shapes of material regions. But Westergaard's proof has a wider applicability than to just those shapes.

In [6], there is also mention of a paper by M.G. Solobodyansky [*General forms of solution in terms of harmonic functions, to the equation of elasticity for simply and multiply connected regions* (in Russian), Prikl. Mat. Mekh. Akad. Nauk. SSSR, $\underset{\sim\sim}{18}$, 1954-55], which "seemingly avoids" the use of the Helmholtz transformation. In the opinion of the authors of [6], his proof is "untenable". Also, at a later stage, he made some "sweeping claims" (on the completeness in terms of three harmonic functions) which are not proved, and even erroneous (when compared to the results of section $\underset{\sim}{3}.\underset{\sim}{6}$ of this thesis). It was considered unwarranted for the author of this thesis to investigate that paper during the time span allowed for this thesis (The paper is not readily obtainable, not easy to be translated into English and it was also published considerably after the pioneering works of Galerkin, Papkovich, Neuber, Mindlin and Westergaard).

9.3 Notes on the completeness of the Papkovich-Neuber solution in terms of only three harmonic functions.

It needs to be kept in mind that the Papkovich-Neuber solution requires only three harmonic functions in any sphere totally contained in the material region. This is a *local* property, it does not necessarily entail the corresponding *global* property (i.e. property which applies to the *whole* region under study). This is analoguous to the the following cases: the expression of a solenoidal vector in terms of the curl of another vector, or the expansion of a complex function in terms of power series ... etc. The counter-example in section 3.6 is a strong enough dissuation for any attempt aiming at the omission of B_0 for a *general* shape of the material region D.

9.4 The completeness of the eigenmode expansion for end loadings.

The authors of [21] have pointed out that there has been no proof for the completeness of the eigenmode expansion for end loadings of a strip or a cylinder. The completeness has been taken for granted in literatures and in chapters 7 and 8 of this thesis.

A sketch of the proofs for that completeness is presented in the following:

As the order m of the eigenvalues increases, the assymptotic formulae (7.6.9) and (8.3.1) show that they assymptote to the values $\{i(2m+1)\pi/4 + \log[(2m+1)\pi]/2\}$ for a strip, and $\{i\pi m + \log(4\pi m)/2\}$ for a solid cylinder. Substitution of the latter values into the formulae for eigenmodes, it is easily proved that these values give a *complete base* for all loadings. It can be shown that a non-linearly-dependent system of bases which assymptotes another *complete* system of bases (the number of elements in both partial systems are equal) is also complete. Hence, the eigenmodes are complete.

The actual proofs may contain a lot of algebraic manipulations, but their essence will be that given above.

9.5 Epilogue.

Elasticity is never a simple or a dead subject. It involves as much and as diverse mathematical arguments as one likes.

The applicability of the Papkovich-Neuber solution alone can carry one as far as to Neumann's problems, elliptic operators, functional spaces, Fredholm's integrals, and even to very fundamental mathematics. This could be carried further by including arguments and mathematics associated with Papkovich's fictitious materials (section 1.2). The elimination of one harmonic function in the Papkovich-Neuber solution may also lead to as many fundamental problems as the applicability of the solution.

Chapter 4 of the thesis brought in the relationship between a number of approaches to axi-symmetric problems, but their *comprehensive* conditions of equivalence are still to be investigated. Chapter 5 gave the equivalence between the Papkovich-Neuber solution and the Airy stress function. But the relationship between the former and the complex variable approach has not been entered as it is not justifiable in the time span allowed for this thesis.

Chapter 6 threw some light on the transformations between plane and axi-symmetric states. This chapter also gave some insight into other methods of transformation. There should be more investigation along this direction although the subject is not an easy one. Aleksandrov, Papkovich, Polozii (as cited by [26] and its references) are the few authors who have worked on that subject.

Chapter 7 is just a natural extension of a two-dimensional problem to a three-dimensional one. The method can be considered as a cross wise application of Fourier transform in the directions of x and z. The method is somehow similar to that used for asymmetric problems of a cylinder in a cylindrical coordinate system; the latter problems are now the current subject of interest.

Chapter 8 is rather an exercise on the eigenmodes of end loadings. There should be mention of a method [21] which sacrifies the equilibrium conditions in order to achieve at certain "close enough eigenmodes"; but this method is still far from full development. Another method, the biorthogonal method [28], which is based on the theory of eigenfunctions, does give some impressive results to certain problems; but one would question the amount of time spent in developing the biorthogonal series for a specific application. The use of complex-variable method in axi-symmetric problems has also been investigated; a few author on that approach are Aleksandrov and Polozii.

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Appendix A

Some fundamental theorems.

A.1 Uniqueness of the solution to a Dirichlet Problem.

The Uniqueness Theorem states that:

If $f(x,y,z)$ and $g(x,y,z)$ are harmonic functions in a region R (of the three dimensional space) with a smooth boundary Σ and $f = g$ on Σ then $f \equiv g$ in R .

Proof:

Consider the equality

$$\int_R [\,\underset{\sim}{\nabla}(f-g)\,]^2 d\tau + \int_R (f-g)\nabla^2(f-g)\, d\tau =$$

$$= \int_R \underset{\sim}{\nabla}\cdot\left[(f-g)\underset{\sim}{\nabla}(f-g)\right] d\tau = \int_\Sigma (f-g)\underset{\sim}{\nabla}(f-g)\cdot\underset{\sim}{n}\, d\sigma \quad .$$

Taking note that $f = g$ on Σ, and $\nabla^2(f-g) = 0$ in R, it is arrived at

$$\underset{\sim}{\nabla}(f-g) = 0 \qquad \text{in R} .$$

Hence $f \equiv g$ in R, as $f = g$ on Σ .

Similarly, two solutions to a Neumann's problem can be proved to differ by only a constant.

A.2 Explicit solution of the internal Dirichlet Problem for a sphere of radius ρ centered on the origin.

The internal Dirichlet Problem for a sphere Ω, of radius ρ centered at O, is to find a harmonic function $f(x,y,z)$, defined on the inside of the sphere, which has its value on the surface Σ of the sphere as

$$f\left[x(r,\theta,\gamma), y(r,\theta,\gamma), z(r,\theta,\gamma)\right] = g(\theta,\gamma) \quad ,$$

where $g(\theta,\gamma)$ is a given function on the surface of the sphere, and (r,θ,γ) are the spherical coordinates corresponding to (x,y,z).

Consider a *fixed* point M at (x_M,y_M,z_M) and its associated *fixed* point N with coordinates (x_N,y_N,z_N) defined by

$$\overline{OM}.\overline{ON} = \rho^2 .$$

The Green's function G(P) at the *variable* point P situated at (x,y,z), with respect to the surface Σ and its *fixed* pole M is defined by

$$G(P) = \frac{1}{|PM|} - \frac{\rho}{|OM|} \times \frac{1}{PN} .$$

This Green's function is harmonic for all P(x,y,z) inside the sphere Ω except at the point M. Furthermore, it vanishes on the surface Σ of the sphere Ω.

Application of the identity

$$\int_R (u\nabla^2 v - v\nabla^2 u)\, d\tau = \int_\Sigma (v\nabla u - u\nabla v)\cdot \underset{\sim}{n}\, d\sigma$$

to the region inside the sphere Ω and outside the sphere ω centered on M, of radius δ ($\delta << \rho$), gives

$$\int_S (f\nabla G - G\nabla f)\cdot \underset{\sim}{n}\, d\sigma = \int_s (f\nabla G - G\nabla f)\cdot \underset{\sim}{n}\, d\sigma ,$$

where S and s are the surfaces of the sphere Ω and ω respectively.

Letting δ tend to zero, the following identity is arrived at

$$f(x_M,y_M,z_M) = - \frac{1}{4\pi} \int_S f(x,y,z) \left(\frac{\partial G}{\partial n}\right)_{\text{at}(x,y,z)} d\sigma(x,y,z) .$$

Hence, the explicit solution to the Dirichlet problem for the sphere Ω is

$$f(x_M,y_M,z_M) = - \frac{1}{4\pi} \int_S g(\theta,\gamma) \left(\frac{\partial G}{\partial n}\right)_{\text{at}(\rho,\theta,\gamma)} d\sigma(\theta,\gamma)$$

since f(x,y,z) has its value on the surface of the sphere equal to $g(\theta,\gamma)$.

Finally, the solution is written in Cartesian coordinates as

$$f(x,y,z) \;=\; \frac{1}{4\pi}\int_S f(\xi,\eta,\zeta)\,\frac{\rho^2-(x^2+y^2+z^2)}{\rho\left[(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2\right]^{3/2}}\,d\sigma(\xi,\eta,\zeta)$$

where the value of $f(\xi,\eta,\zeta)$ is that of $g(\theta,\gamma)$ since $\xi^2+\eta^2+\zeta^2=\rho^2$.

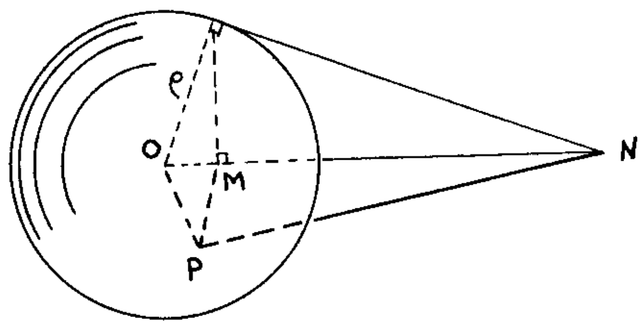

Figure A.1: The internal Dirichlet's problem for a sphere.

Appendix B

Some useful formulae for Bessel functions.

For an arbitray ν, the Bessel function $I_\nu(z)$ is given by

$$I_\nu(z) \quad = \quad (\tfrac{1}{2}z)^\nu \sum_{k=0}^{\infty} \frac{(\tfrac{1}{4}z^2)^k}{k!\,\Gamma(\nu+k+1)} \quad .$$

For a fixed ν and a large value of $|z|$, let $\mu = 4\nu^2$, the assymptotic formula for $I_\nu(z)$ is then

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}} \left\{ 1 - \frac{\mu-1}{8z} + \frac{(\mu-1)(\mu-9)}{2!(8z)^2} - \frac{(\mu-1)(\mu-9)(\mu-25)}{3!(8z)^3} + \ldots. \right\} ,$$

$$\text{if} \quad |\arg(z)| < \tfrac{1}{2}\pi \ .$$

The derivatives of $I_\nu(z)$ are given by

$$\left(\frac{1}{z}\frac{d}{dz}\right)^k \left\{ z^\nu \, I_\nu(z) \right\} \quad = \quad z^{\nu-k} \, I_{\nu-k}(z) \quad ,$$

$$\text{for} \quad k = 0,\ 1,\ 2,\ 3,\ \ldots.,$$

in particular

$$I_0'(z) \quad = \quad I_1(z) \ ,$$

$$I_1'(z) \quad = \quad I_0(z) - \frac{1}{z} I_1(z) \ .$$

The functions $I_0(z)$ and $I_1(z)$ are also expressible in the following forms

$$I_0(z) \quad = \quad \frac{1}{\pi} \int_0^\pi \exp(\pm z \cos\theta) \; d\theta \ ,$$

$$I_1(z) \quad = \quad \frac{1}{\pi} \int_0^\pi \exp(z \cos\theta) \, \cos\theta \; d\theta \ ,$$

or

$$I_1(z) \quad = \quad \frac{1}{\pi} \int_0^\pi \sinh(z \cos\theta) \, \cos\theta \; d\theta \ ,$$

the last three formulae are valid for all z.