The Newton Polygon and the Puiseux Characteristic

Michael Barwick

Supervisor: A. Prof. Laurentiu Paunescu

A thesis submitted in partial fulfillment of
the requirements for the degree of
Master of Science

Pure Mathematics

THE UNIVERSITY OF SYDNEY

May 26, 2011

Contents

Acknowledgements

I would like to thank my supervisor Laurentiu Paunescu for his assistance throughout my candidature. Also, I would like to thank Tzee-Char Kuo for some invaluable discussions. Finally, I would like to thank my parents for their support.

Introduction

In this thesis, we will use the Newton polygon and the Puiseux characteristic to study complex analytic curves in $\mathbb{C}\{x,y\}$ and $\mathbb{C}[[x,y]]$. This allows us to topologically classify the plane curve singularities.

Chapter 1 will introduce the Newton polygon, the process of sliding towards a root and polar curve. The first section of chapter 2 contains the technical background to this topic. The second section introduces the Puiseux characteristic, and the third uses results from knot theory to classify the plane curve singularities as the cone over an iterated torus knot.

In the third chapter, we will look at the Kuo-Lu theorem, which is a generalisation of Rolle's theorem to complex curves. Finally, in the fourth chapter, we will give an application of the previous results to show a method of calculating the Lojasiewicz exponent.

Notation and Symbols

This section lists some of the commonly used notation and symbols of this thesis. Where appropriate, a reference to the page where the symbol was first used or defined is given.

θ_a - angle of E_a, 6
D_ϵ - disk of radius ϵ
E_a - the a-th edge, counted from right to left. 6
$\mathbb{C}(y)^*$ - field of fractional power series, 23
E_h - highest edge, 6
$\mathbb{C}[y^{1/n}]$ - integral domain of fractional power series in $y^{1/n}$, 22
$\mathcal{L}(f)$ - Lojasiewicz exponent, 39
$\ell(\lambda)$ - see page 39
mult f - multiplicity of f
$\mathbb{P}(f, \lambda)$ - Newton polygon of f relative to λ, 9
$O(f)$ - order of f, 10
f_x - partial derivative of f relative to x
$\mathbb{R}^+$ - set of positive real numbers $[0, \infty)$
$M(f)$ - tree model of f, 36
$M^*(f)$ - see page 36

Chapter 1

Newton Polygon

Throughout this chapter let $f(x, y) = \sum a_{ij} x^i y^{j/N} \in \mathbb{C}[x, y^{1/N}]$ be a power series with complex coefficients a_{ij}, with $(i, j) \in \mathbb{N} \times \mathbb{N}$ and $N \in \mathbb{N}$. In chapter 2, we will amend this to allow $(i, j) \in \mathbb{Z} \times \mathbb{Z}$. Note that ordinarily fractional (and negative) powers of complex numbers are not well defined. The justification for their use can be found in chapter 2. Negative powers will not be used until the second chapter, although some of the theorems will be proven in the more general case. The reader may either assume that fractional powers are well defined, or take $N = 1$ for most of the first chapter.

Definition 1.0.1 (Mini-Regular)**.** *Express $f(x, y)$ in the form of a Taylor series $f(x, y) = H_k(x, y) + H_{k+1}(x, y) + \dots$, where $H_i(x, y) = c_0 x^i + c_1 x^{i-1} y + \dots + c_i y^i$. We say that $f(x, y)$ is mini-regular in x if $H_k(1, 0) \neq 0$.*

Throughout, unless otherwise stated we will assume that f is mini-regular in x. Note that if $f(x, y)$ is not mini-regular in x, the function $f(x, y + cx)$ is mini-regular in x for almost all values of c (specifically all but finitely many), and so we can make this assumption without loss of generality.

1.1 The Newton Polygon

Definition 1.1.1 (Newton Diagram)**.** *The Newton diagram for $f(x, y)$, is obtained by plotting a dot in $\mathbb{R}^2$ at $(i, j/N)$ for each $a_{ij} \neq 0$.*

Recall that a set X is convex if it is a subset of a vector space such that the line segment joining any two points in X is contained in X, and that the convex hull of a set X is the smallest convex set containing X.

Consider the area constructed by translating the quadrant $(\mathbb{R}^+)^2$ to each Newton dot in the Newton diagram of f. For the example $f(x, y) = x^6 - x^3y^3 - x^2y^7 + y^8$, this area is shown in figure 1.1.

Hence for our example $f(x, y) = x^6 - x^2y^7 - x^3y^3 + y^8$, the convex hull of the area constructed by translating quadrants is shown in figure 1.2, and the boundary of the convex hull of this area is the bold line.

Definition 1.1.2 (Newton Polygon)**.** *The boundary of the convex hull of the area defined above is called the Newton polygon of f and is denoted $\mathbb{P}(f, 0)$. This consists of a* *horizontal half line joined to a vertical half line by a polygonal line. Each segment of the polygonal line is called an edge.*

We want to construct the boundary of the convex hull of this area. Let (a_1, b_1) and (a_2, b_2) be two Newton dots. From the definition of a convex set, the convex hull must contain all lines between any two points of the form (a_1+x_1, b_1+x_2) and (a_2+x_3, b_2+x_4), where $x_i \in \mathbb{R}^+$. Hence the convex hull will contain the line between (a_1, b_1) and (a_2, b_2) and all points above and to the right of this line.

For a given edge E, the *lowest dot* on E is the dot (x_1, y_1) with the smallest y value, and the *highest dot* is the dot (x_2, y_2) with the largest y value. We will also define the *length* of E to be $x_1 - x_2$. The *angle* of E is defined to be the acute angle between E and a line parallel to the x-axis. We will generally denote the angle of an edge E_a by θ_a. We will call the edge with the largest angle the *highest edge*, and will denote it E_h. Note that this edge is well defined by proposition 1.1.4. From the definition of convex hull, it will be the leftmost edge.

Proposition 1.1.3. *If $f(x, y)$ is mini-regular in x, all edges in the Newton polygon of f have angle θ_i with $tan(\theta_i) \geq 1$.*

Proof. First note that as $H_k(1, 0) \neq 0$, there is a Newton dot $(k, 0)$. This is clearly on the Newton polygon of f. As $H_k(x, y)$ is the first term in the Taylor series of f at the origin, all terms $a_{ij}x^iy^j$ in $f(x, y)$ will have $i + j \geq k$. Hence all Newton dots will lie on or above the line $y + x + k = 0$. Hence the edge $tan(\theta_1) \geq 1$. As $\mathbb{P}$ is the convex hull of the Newton dots, $tan(\theta_i) \geq 1$ for all i. □

Proposition 1.1.4. *The Newton polygon for infinite sums of the form*

$$\sum_{i,j \geq 0} a_{ij} x^i y^{j/N}$$

consists of finitely many edges.

Proof. Let $f = \Sigma a_{ij} X^i Y^{j/N}$ be an infinite power series, with $(i, j) \in \mathbb{N} \times \mathbb{N}$, and let (a, b) be a Newton dot which is on the Newton polygon of f.

Consider the set of dots which are higher than (a, b), and order them by height, so we have $(a_1, b_1), (a_2, b_2), \ldots$, where $b < b_1 < b_2 < \ldots$. As $\mathbb{P}(f)$ is the boundary of a convex hull, the gradient of the edges must be monotonically increasing, and so we get $a > a_1 > a_2 > \ldots$. As all dots are of the form $(i, j/N)$ with $i, j \in \mathbb{N}$, we have $a_i - a_{i+1} \geq 1$. Hence there are at most a dots on the Newton polygon of f above (a, b). A similar argument shows that there are at most $N \cdot b$ dots on the Newton polygon of f below (a, b). Hence there are only finitely many edges on the Newton polygon of f. □

Remark 1.1.5. *This remains true if we simply assume that the denominators are bounded. Specifically, if $f = \sum_{i,j} a_{ij} x^i y^{j/n_j}$, and the denominators are bounded by M, then for some N, we can write $f = \sum_{i,j} a'_{i,j} x^i y^{j/N}$ as a function in $x^i y^{j/N}$, where the a'_{ij} can be determined from the a_{ij}.*

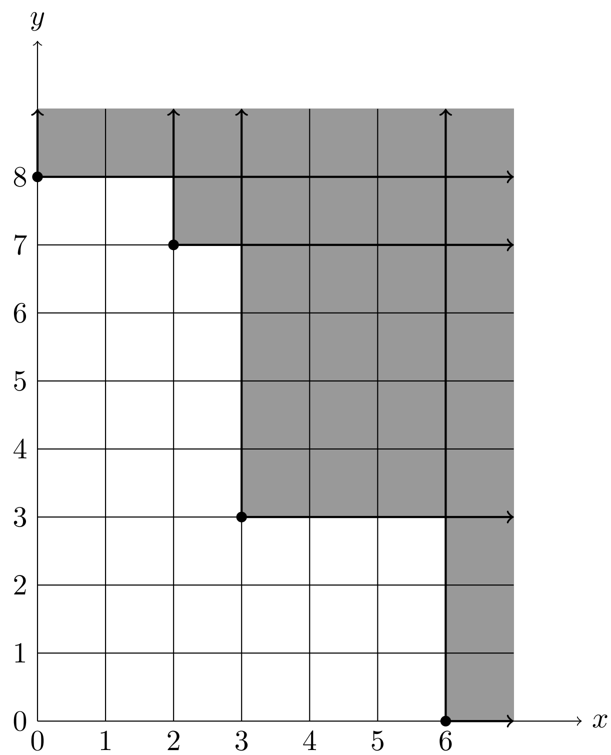

Figure 1.1: The area constructed by translating the quadrant $(\mathbb{R}^+)^2$ to each Newton dot in the Newton diagram of $f(x, y) = x^6 - x^3y^3 - x^2y^7 + y^8$

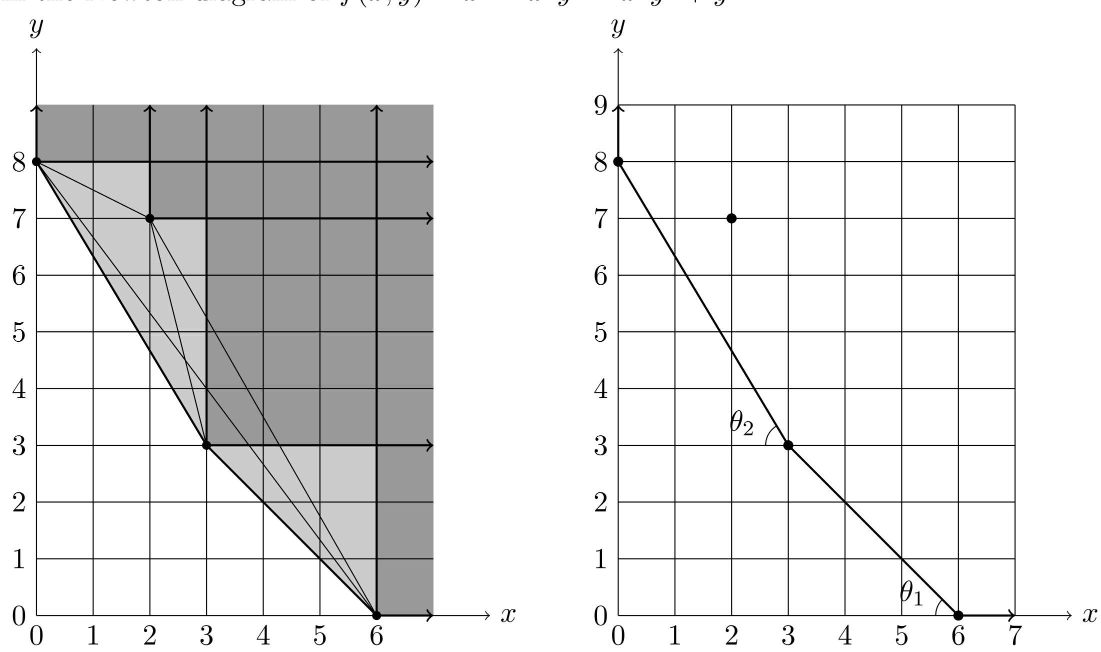

Figure 1.2: The convex hull (left) and Newton polygon of $f(x, y) = x^6 - x^2y^7 - x^3y^3 + y^8$

Note that this is not valid for infinite sums involving negative powers. For example, consider the function

$$f(x,y) = \sum_{i=0}^{\infty} x^{-i} y^{i^2} = 1 + x^{-1}y^1 + x^{-2}y^4 + x^{-3}y^9 + \ldots$$

Clearly this will have a Newton polygon consisting of infinitely many edges.

Similarly, if we allow the denominator of the powers to be unbounded, the proposition will no longer be true. For example, the Newton polygon of the function:

$$f(x,y) = \sum_i x^i y^{1/i}$$

has infinitely many edges.

Corollary 1.1.6. *The edges of this polygonal arc have increasing gradient, and so if we extend an edge E_a to intersect the x and y axis, the points of intersection will be on or below the Newton polygon of f, and on only in the case where E_a has a dot on the axis. See figure 1.3 for an example of this.*

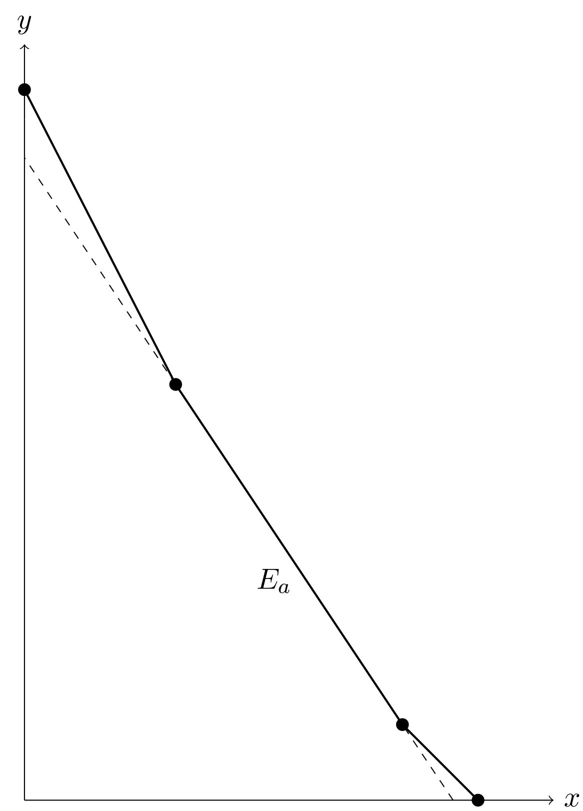

Figure 1.3: Example of increasing angle in the Newton polygon

Definition 1.1.7 (Analytic Arc)**.** *An analytic arc is the image set of an analytic mapping from $\mathbb{C}$ to the complex plane $\mathbb{C}^2$, $t \to (x(t), y(t))$.*

Definition 1.1.8 (Parametrisation of an Arc)**.** *In the above definition, the analytic mapping* $t \to (x(t), y(t))$ *is called a parametrisation of the resulting arc.*

Note that the curve defined by $x = \lambda(y)$ is an analytic arc if there is a parametrisation $t \to (x(t), y(t))$, with $x(t)$ and $y(t)$ analytic, such that $x(t) = \lambda(y(t))$ for all t.

Definition 1.1.9 (Newton Polygon Relative to an Arc)**.** *Let* $x = \lambda(y) = c_1 y^{\theta_1} + c_2 y^{\theta_2} + \dots$ *be an analytic arc. Define* $F(X, Y) = f(X + \lambda(Y), Y)$*. The Newton polygon of* f *relative to* λ *is the Newton polygon of* $F(X, Y)$ *and is denoted* $\mathbb{P}(f, \lambda)$*.*

As $f(X + c_1 Y^{\theta_1} + c_2 Y^{\theta_2} + \dots, Y) = f((X + c_2 Y^{\theta_2} + \dots) + c_1 Y^{\theta_1}, Y)$, we may recursively construct $\mathbb{P}(F(X, Y))$ by constructing the Newton polygons of $F_1, F_2, \dots$ where $F_1 = f(X + a_1 Y^{\theta_1}, Y)$, $F_2 = F_1(X + a_2 Y^{\theta_2}, Y), \dots$.

Now consider $\mathbb{P}(F_{i+1})$. As $F_{i+1}(X, Y) = F_i(X + c_{i+1} Y^{\theta_{i+1}}, Y)$, if (p, q) is a point on the Newton diagram of F_i (corresponding to $a_{pq} x^p y^q$), then in the expansion of F_{i+1} this point gives:

$$a_{pq}(X + c_{i+1} Y^{\theta_{i+1}})^p Y^q = a_{pq} X^p Y^q + a_{pq} \binom{p}{1} c_{i+1} X^{p-1} Y^{q+\theta_{i+1}} + a_{pq} \binom{p}{2} c_{i+1}^2 X^{p-2} Y^{q+2\theta_{i+1}} + \dots .$$

Hence the dots of $\mathbb{P}(F_{i+1})$ will consist of dots of $\mathbb{P}(F_i)$ and dots of the form $(p - k, q + k\theta_{i+1})$. Note that dots may be cancelled, as in the following example.

Example 1.1.10. *Let* $f(x, y) = x^6 - x^2 y^7 - 20x^3 y^3 + y^8$*. Consider the Newton polygon of* f *relative to the arc* λ *defined by* $x = y$*.*

$$f(x + \lambda(y), y) = (x + y)^6 - (x + y)^2 y^7 - 20(x + y)^3 y^3 + y^8$$

Expanding this will cancel the term in $x^3 y^3$*, which will cancel the corresponding dot.*

Definition 1.1.11 (Associated Form and Polynomial)**.** *The associated form* $\tilde{\mathcal{E}}_s(X, Y)$ *and associated polynomial* $\mathcal{E}_s(z)$ *of an edge* E_s *are defined to be the following:*

$$\tilde{\mathcal{E}}_s(X, Y) = \sum a_{ij} X^{i/N} Y^{j/N}, (i/N, j/N) \in E_s \ .$$
$$\mathcal{E}_s(z) = \tilde{\mathcal{E}}_s(z, 1)$$

Example 1.1.12. *For our previous example,* $f(x, y) = x^6 - x^2 y^7 - x^3 y^3 + y^8$ *(the Newton polygon of which is figure 1.2), the associated form and polynomial for each edge are as follows:*

$$\tilde{\mathcal{E}}_1(X, Y) = x^6 - x^3 y^3$$
$$\mathcal{E}_1(Z) = z^6 - z^3$$
$$\tilde{\mathcal{E}}_2(X, Y) = -x^3 y^3 + y^8$$
$$\mathcal{E}_2(Z) = -z^3 + 1$$

1.2 Newton-Puiseux Roots

Definition 1.2.1 (Newton-Puiseux Root)**.** *The arc $x = \lambda(y)$ is a Newton-Puiseux root of f if $f(\lambda(y), y) \equiv 0$.*

Theorem 1.2.2. *$x = \lambda(y)$ is a Newton-Puiseux root of f if and only if $\mathbb{P}(f, \lambda)$ has no Newton dots on the y-axis.*

Recall that the *order* of a power series is the smallest exponent of a non-zero term. For power series with multiple variables we will define the order of a term as the sum of its exponents, and the order of the power series is the smallest order of a non-zero term. For example, the order of $f(x, y) = x^i y^j$ is $i + j$. Let m be the multiplicity of f, and assume that the coefficient of x^m is non-zero. (Note that by applying the linear transform $y = y + cx$ for some generic c, we can assume this without loss of generality.)

Lemma 1.2.3. *Let $\phi_1 < \phi_2 < \dots$ be the order of the roots of f. Let α_i be the number of roots of order ϕ_i. Let $m = \sum \alpha_i$ be the total number of roots. We claim that the edges of the Newton polygon of f are between the vertices:*

$$(m - \sum_{i=1}^{j} \alpha_i, \sum_{i=1}^{j} \alpha_i \phi_i) \ ,$$

for $j = 0, 1, \dots,$. In particular, the angle of the edges will be $\phi_1, \phi_2, \dots$, and the length of the edge of angle ϕ_i will be number of roots with order ϕ_i. Note that there may be other dots on the Newton Polygon of f, but they will be on one of the lines between the above vertices.

Proof. First recall that we can without loss of generality assume that there is a dot at $x = m$, where m is the multiplicity of f, which will clearly be the leftmost dot on the $x-axis$. Let $\phi_1 < \phi_2 < \phi_3 < \dots$ be the orders of the roots of f. We will write the roots of f as β_{ij}, $j = 1, 2, \dots \alpha_i$, where ϕ_i is the order of β_{ij}. Now consider the expansion of $f(x, y)$:

$$\begin{aligned}\prod_{i,j}(x - \beta_{ij}) =& x^m + x^{m-1}\sum_{i,j}(\beta_{ij}) + x^{m-2}\sum_{i_1,j_1,i_2,j_2}\beta_{i_1j_1}\beta_{i_2j_2} + \\ &+ x^{m-3}\sum_{i_1,j_1,i_2,j_2,i_3,j_3}\beta_{i_1j_1}\beta_{i_2j_2}\beta_{i_3j_3} + \dots + \prod_{i,j}\beta_{ij}\end{aligned}$$

Now we construct the Newton diagram of f. Consider the dots on $x = m - 1$: these are given by $\sum_{i,j}(\beta_{ij})$. Clearly the lowest potential dot on this line is at $(m - 1, \phi_1)$. (Note that there is not necessarily a dot at this position). Similarly, the lowest potential dot on $x = \alpha_1$ is at $(m - \alpha_1, \alpha_1\phi_1)$. Consider the term in $x^{m-\alpha_1}y^{\alpha_1\phi_1}$: the coefficient of this term is the product $\prod_j c_{1,j}$. Clearly this is non-zero. Hence the lowest dot on $x = (m - \alpha_1)$ is $(m - \alpha_1, \alpha_1\phi_1)$.

Similarly the lowest potential dot on $x = (m - \alpha_1 - 1)$ is at $\alpha_1\phi_1 + \phi_2$, and the lowest potential dot on $x = (m - \alpha_1 - \alpha_2)$ is at $\alpha_1\phi_1 + \alpha_2\phi_2$. The coefficient of

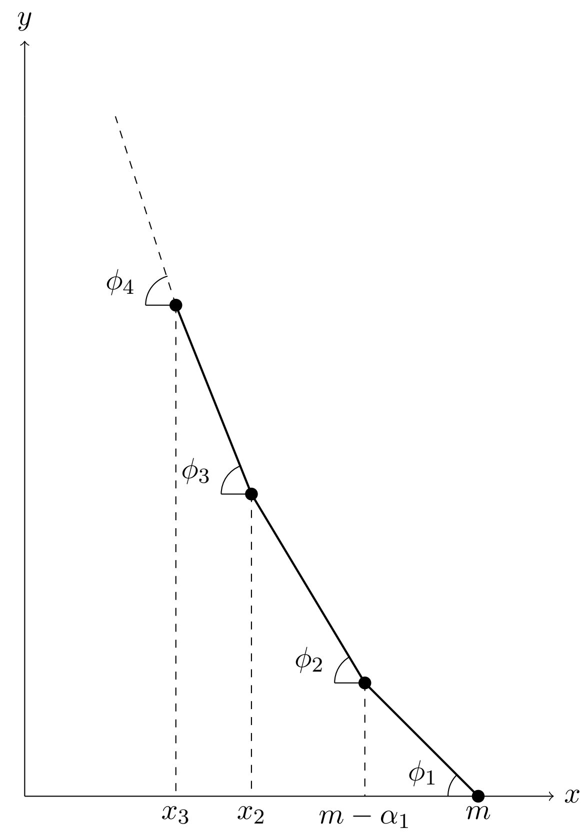

Figure 1.4: In the above diagram $x_2 = m - \alpha_1 - \alpha_2$, and $x_3 = m - \alpha_1 - \alpha_2 - \alpha_3$

$x^{m-\alpha_1-\alpha_2}y^{\alpha_1\phi_1+\alpha_2\phi_2}$ is given by $\prod_{j_1} c_{1,j_1} \prod_{j_2} c_{2,j_2}$, which is non-zero. Hence there is also a dot at $(m - \alpha_1 - \alpha_2, \alpha_1\phi_1 + \alpha_2\phi_2)$. Clearly this process can be continued and hence we have the required result. Note that each step from $m - \sum_i \alpha_i$ to $m - \sum_i \alpha_i - 1$ results in an increase in angle of the edge. □

Example 1.2.4. *Consider the function* $f(x, y) = x^3 + x^2y^3 - xy^2 - y^5$*. by factorising we get* $f(x, y) = (x+y)(x-y)(x+y^3)$ *Hence the Newton-Puiseux roots for* f *are* $x = \pm y$ *and* $x = -y^3$*.*

In many cases, the Newton-Puiseux roots of a function cannot be easily discovered. Often they will in fact be infinite power series. In this case, we can use a technique called sliding to find the roots.

1.3 Sliding Towards a Root

We will start this section with a simple example of sliding. Recall that the *Implicit Function Theorem* states that if $f(x, y)$ is an analytic function such that $f(0, 0) = 0$ and $\partial f/\partial x \neq 0$ on a neighbourhood of the origin, then there exists a unique analytic function $g(y)$ such that in a neighbourhood of $(0, 0)$, $f(x, y) = 0$ if and only if $x = g(y)$.

Example 1.3.1. *Let $f(x, y) = x - 2xy + y^4$. Hence $\frac{\partial f}{\partial x} = 1 - 2y$. Hence by applying the implicit function theorem to $f(x, y)$, there is a unique analytic function $\beta(y)$, defined on an open neighbourhood U of $(0, 0)$, such that $f(\beta(y), y) = 0$. By theorem 1.2.2, the Newton diagram of $F(X, Y) = f(x + \beta(y), y)$ will have no dots on the $y-axis$. Hence we can recursively construct the solution β:*
In order to cancel the lowest dot on $X = 0$, the first term in β must be $\beta_1 = -y^4$. Expanding $F_1(x, y) = f(x + \beta_1, y)$ gives us:

$$F_1 = x - 2xy + 2y^5 \ .$$

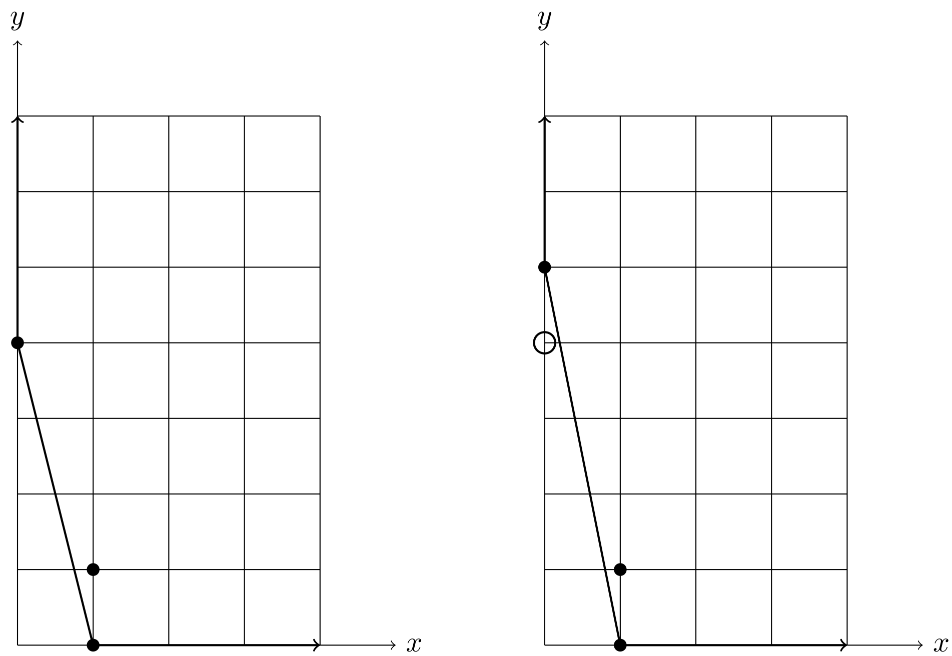

Figure 1.5: The Newton diagrams of $f(x, y) = x^2 - 2xy + y^4$ (left) and F_1 (right)

Now we will repeat the process from this: in order to cancel the dot at $(0, 5)$ the second term in the expansion of β must be $-2y^5$. So $\beta_2 = -y^4 - 2y^5$. Now

$$f(x + \beta_2, y) = x - 2xy + 4y^6 \ .$$

So the next term in β will be $4y^6$.

We can continue this process indefinitely to find the solution. As the root guaranteed by the implicit function theorem is unique,β must be this root, and hence is convergent. Note that as the dots at $(1, 0)$ and $(1, 1)$ are fixed, the exponents in β will always increase by 1.

Now let $f(x, y) = \sum a_{ij} x^i y^j$ be a (convergent) power series, and $\lambda = \sum_{i=1}^{k} c_i y^{\theta_i/N}$ be a polynomial in $y^{1/N}$. Let E_a be an edge of $\mathbb{P}(f, \lambda)$, and write θ_a for the angle of E_a. Consider the associated polynomial $\mathcal{E}_a$ of E_a. Let c be a (non-zero) solution of $\mathcal{E}_a$. We define $\lambda_1 = \lambda + cy^{\theta_a}$. λ_1 is a sliding of λ along E_a.

Now consider the actions of sliding on the Newton polygon $\mathbb{P}(f, \lambda)$. For each term $a_{ij}x^i y^j$, sliding gives us terms:

$$a_{ij}(x + cy^{\theta_a})^i y^j = \sum_{k=0}^{i} a_{ij} \binom{i}{k} c^{i-k} x^k y^{(i-k)\theta_a + j} \ .$$

Hence the Newton diagram of f relative to λ_1 will consist of dots of the form $(i-k, j+k\theta_a)$ where (i, j) is a dot on $\mathbb{P}(f, \lambda)$.

Remark 1.3.2. *A more visual way of expressing this is that all dots of* $\mathbb{P}(f, \lambda_1)$ *will be on the lines of gradient* $-tan(\theta_a)$ *starting at each dot of* $\mathbb{P}(f, \lambda)$.

Note that there may be cancellation of terms, but that this may only occur when there are at least two dots of $\mathbb{P}(f)$ on a line of gradient $-tan(\theta_a)$. In particular, the edges below E_a will be unchanged in $\mathbb{P}(f, \lambda_1)$, and the lowest dot on E_a will also remain.

Proposition 1.3.3. *All of the dots of* $\mathbb{P}(f, \lambda_1)$ *will be on or above the line extending* E_a*. (Where we take above to mean that we may write any dot in the form* $(p, q + c)$*, where* $c > 0$*, and* (p, q) *is a point on the line extending* E_a*.) Also, the only dots on this line will be the dots of* $\tilde{\mathcal{E}}_a(x + \lambda_1(y), y)$.

Proof. Recall that the Newton polygon is the boundary of a convex set. Hence we get trivially that if (i, j) is a Newton dot not on E_a, (i, j) is above the line extending E_a

Hence, as all dots of $\mathbb{P}(f, \lambda_1)$ will be on lines parallel to E_a, all dots in the Newton diagram of f relative to λ_1 will be on or above the line extending E_a, and the dots on this line will result from dots on E_a. □

Remark 1.3.4. *In addition, the edges with angle less that* θ *in the Newton polygon* $\mathbb{P}(f, \lambda_1)$ *will be the same as the edges with angle less that* θ *in the Newton polygon* $\mathbb{P}(f)$.

Remark 1.3.5. *In particular, as* $\theta_a \geq 1$*, this means that if* $f(x, y)$ *is mini-regular in* x*, then* $f(x + \lambda_1(y), y)$ *will be mini-regular in* x.

Proposition 1.3.6. *The lowest dot on* $X = 0$ *in the diagram of* $\mathbb{P}(f, \lambda_1)$ *is above the intersection of the line extending* E_a *and the* $Y - axis$.

Proof. Let $(0, h_0)$ be the intersection of the line extending E_a and the $Y - axis$.

From the previous proposition, we only need to consider dots on E_a. Consider the expansion of

$$\tilde{\mathcal{E}}_a(x + cy^{\theta_a}, y) = \sum_{i|(i,j)\in E_a} a_{ij}(x + cy^{\theta_a})^i y^j,$$

where $j = h_0 - i\theta_a$ (as (i, j) is a dot on the edge E_a). The term in X^0 in this expansion is given by:

$$
\begin{aligned}
a_{0,h_0} &= \sum_{(i,j)\in E_a} a_{ij}(cy^{\theta_a})^i y^j \\
&= \sum_{(i,j)\in E_a} a_{ij}c^i y^{i\theta_a} y^{h_\alpha+(\alpha-i)\theta_a} \\
&= \sum_{(i,j)\in E_a} a_{ij}c^i y^{i\theta+h_\alpha+(\alpha-i)\theta_a} \\
&= \sum_{(i,j)\in E_a} a_{ij}c^i y^{h_\alpha+\alpha\theta_a} \\
&= y^{h_0} \sum_{(i,j)\in E_a} a_{ij}c^i \qquad\qquad (\text{as } h_\alpha+\alpha\theta_a = h_0) \\
&= y^{h_0}\mathcal{E}_a(c) \\
&= 0
\end{aligned}
$$

Hence the lowest dot on $X = 0$ is above $(0, h_0)$. □

Definition 1.3.7 (Relevant Edge). *Let $\lambda = c_1 y^{\theta_1} + \cdots + d_i y^{\theta_i}$ be a fractional power series. An edge E of $\mathbb{P}(f, \lambda)$ is called relevant if the angle of E is greater than θ_i.*

Definition 1.3.8 (Lowest Relevant Dot). *The lowest relevant dot is defined to be the lowest dot which is on a relevant edge.*

Remark 1.3.9. *The above proposition can be restated as: $\mathbb{P}(f, \lambda_1)$ will have at least one relevant edge. In fact, It can be shown that $\mathbb{P}(f, \beta)$ will have a relevant edge if and only if β is equal to the first n terms of a Newton-Puiseux root λ for some n.*

Theorem 1.3.10. *Let $\beta = \sum_i c_i y^{\theta_i}$ be an arc found by sliding along relevant edges of a (convergent) power series $f(x, y)$, and let β_i be the sum of the first i terms of β. Write $\theta_i = \frac{n_i}{d_1 d_2 \ldots d_i}$, where d_i and n_i have no common factor. (Note that d_j may have a common factor with n_i for $i \neq j$.) We claim that all but finitely many of the d_i are equal to 1.*

Proof. Consider the Newton polygon $\mathbb{P}(f, \beta_i)$. Clearly the exponent of y in any term of the expansion of $f(x + \beta_i(y), y)$ may be written in the form $(a_1 n_1 + a_2 n_2 + \cdots + a_i n_i)/(d_1 d_2 \ldots d_i)$ for some $a_1, \ldots, a_i$. Hence we may write the dots of $\mathbb{P}(f, \beta_i)$ in the form $(j_1, j_2/(d_1 \ldots d_i)$. Now consider $\beta_{i+1} = \beta_i + c_{i+1} y^{\theta_{i+1}}$. Let E be the edge of $\mathbb{P}(f, \beta_i)$ of angle θ_{i+1}, and let (α_i, h_{α_i}) be the lowest dot on E. We want to show that the lowest relevant dot in $\mathbb{P}(f, \beta_{i+1})$ will have x-coordinate at most $\alpha_i - d_{i+1} + 1$.

As the edges of the Newton polygon below E are unchanged, the lowest relevant dot will be the highest dot on the line extending E. Now from proposition 1.3.3, all dots on the line extending E are dots of $\tilde{\mathcal{E}}(x + c_{i+1} y^{\theta_{i+1}}, y)$. For convenience we will write $a_{\alpha-k}$ for the coefficient of the dot $(\alpha - k, h_\alpha + k\theta_{i+1})$. So

$$
\tilde{\mathcal{E}}(x, y) = a_\alpha x^\alpha y^{h_\alpha} + a_{\alpha-1d_{i+1}} x^{\alpha-d_{i+1}} y^{h_\alpha+d_{i+1}\theta_{i+1}} + a_{\alpha-2d_{i+1}} x^{\alpha-2d_{i+1}} y^{h_\alpha+2d_{i+1}\theta_{i+1}} + \ldots \quad ,
$$

where $a_\alpha \neq 0$.

Now consider the expansion $\tilde{\mathcal{E}}(x + c_{i+1}y^{\theta_{i+1}}, y)$:

$$\tilde{\mathcal{E}}(x + c_{i+1}y^{\theta_{i+1}}, y) = a_\alpha(x + c_{i+1}y^{\theta_{i+1}})^\alpha y^{h_\alpha} + a_{\alpha-d_{i+1}}(x + c_{i+1}y^{\theta_{i+1}})^{\alpha-d_{i+1}} y^{h_\alpha+d_{i+1}\theta_{i+1}} + \dots$$

The term in $x^{\alpha-d_{i+1}+1}y^{h_\alpha+(d_{i+1}-1)\theta_{i+1}}$ in this expansion will be equal to the corresponding term in $a_\alpha(x + c_{i+1}y^{\theta_{i+1}})^\alpha$. This is: $a_\alpha\binom{\alpha}{d_{i+1}-1}c_{i+1}^{d_{i+1}-1}$, which is non-zero. Hence there is a dot at $(\alpha - d_{i+1} + 1, h_\alpha + (d_{i+1} - 1)\theta_{i+1})$. Hence the lowest relevant dot in $\mathbb{P}(f, \beta_{i+1})$ will have x-coordinate at most $\alpha_i - d_{i+1} + 1$.

Recall that we have assume that f is mini-regular in x. Hence the x-coordinate of the lowest relevant dot of $\mathbb{P}(f)$ will be $O(f)$. If $d_i > 1$, then $\alpha_i - d_{i+1} + 1 \leq \alpha_i - 1$. As the lowest relevant dot must have x-coordinate at least 1, at most $O(f)$ of the d_i will be greater than 1.

As there are only finitely many d_i not equal to 1, the product $\prod_i d_i$ is always finite. In particular, if we set $d = \prod_i d_i$, then we may write $\lambda = \sum_i c_i y^{m_i/d}$. □

Remark 1.3.11. *It is possible to strengthen this condition as follows:*
If a power series $f(x, y)$ is irreducible (as defined in chapter 2), the product of the d_i is equal to the multiplicity of f. $\prod_i d_i = ord(f)$.

If f is reducible, the the sum over the irreducible components of f of the product of the d_i will be equal to the multiplicity of f.

Proposition 1.3.12. *The above process (of sliding along relevant edges) will produce a Newton-Puiseux root for f, possibly after infinite iterations.*

Proof. Let β_i be the i-th iteration of the above process, and let $\beta_{i+1} = \beta_i + c_{i+1}y^{n_{i+1}/d}$ be the next iteration. Write $F(X, Y) = f(X + \beta_i(Y), Y)$, and call $(0, h_0)$ the lowest Newton dot on $X = 0$. From proposition 1.3.6 the lowest dot on $X = 0$ in $\mathbb{P}(f, \beta_{i+1})$ will be above $(0, h_0)$. Hence if there is a Newton dot on $X = 0$ it has height at least $h_0 + 1/d$ and so if the process of sliding is iterated indefinitely, the height of the lowest Newton dot will tend to ∞. Hence the process of sliding will eventually (although possibly after infinitely many iterations) produce an analytic arc β, such that there are no Newton dots on $X = 0$ in $\mathbb{P}(f, \beta)$. By theorem 1.2.2, β is a formal root of f. □

Theorem 1.3.13. *If all of the Newton-Puiseux roots of $f(x, y)$ are of multiplicity 1, the above algorithm will reach either a root or a highest edge of length 1 after a finite number of steps.*

Proof. First assume that the above algorithm requires sliding indefinitely (if not then the theorem is trivial). As there are finitely many possible lowest relevant dots, and the x-coordinate of the lowest relevant dot can only increase by sliding, we can find an integer N_0 such that (α, h_α) is the lowest relevant dot in $\mathbb{P}(f, \beta_i)$ for all $i > N_0$. By the same argument as in the proof of proposition 1.3.12, we can show that the lowest dots on $X = \alpha - 1, \alpha - 2, \dots$ will tend to infinity as i tends to infinity. Hence there will be no dots on $X = \alpha - 1, \alpha - 2, \dots$ in the Newton polygon $\mathbb{P}(f, \beta)$. But as we will see in section 1.5, if λ is a root of multiplicity 1, there will be a dot on $X = 1$ in the Newton polygon $\mathbb{P}(f, \lambda)$. Hence there must be a dot on $X = 1$ in $\mathbb{P}(f, \beta)$, and so $\alpha = 1$. □

Remark 1.3.14. *Note that so far we have only shown that there is a formal power series β which satisfies $f(\beta, y) = 0$. In fact, if $f(x, y)$ converges, it can be shown that β converges. We will do this in section 1.4.*

Proposition 1.3.15. *All the roots of f can be found by sliding.*

Proof. Let $\beta = c_1y^{\theta_1} + c_2y^{\theta_2} + \dots$ be an arbitrary root of f, and assume that β cannot be found by sliding. Consider the functions $F_1 = f(x + c_1y^{\theta_1}, y), F_2 = f(x + c_1y^{\theta_1} + c_2y^{\theta_2}, y), \dots$. Clearly $F_i = F_{i-1}(x + c_iy^{\theta_i})$, and

$$\lim_{i\to\infty} F_i = f(x + \beta(y), y) \quad .$$

As β cannot be found by sliding, there must be a coefficient c_i which is not a root of the associated polynomial of the edge with angle θ_i in the Newton polygon of F_{i-1} (where F_0 is taken to be f). Now let j be the lowest such indice. (So in particular $c_1y^{\theta_1} + \dots + c_{j-1}y^{\theta_{j-1}}$ can be found by sliding). Now clearly $x = \sum_{i\geq j} c_iy^{\theta_i}$ is a root of $F_{j-1}(x, y)$. Hence from lemma 1.2.3, there is an edge, which we will call E_j, of F_{j-1} with angle θ_j.

Let $(0, h_0)$ be the point at which the line extending E_j intersects the y-axis. As we have assumed that c_j is not a root of the associated polynomial $\mathcal{E}_j$ of E_j, the coefficient of y^{h_0} must be non-zero. The proposition now follows from a simple geometry argument based on proposition 1.3.3. From this proposition, all dots in F_j may be written as $(p, q+r)$, where (p, q) lies on the line extending E_j. In particular, we have that $p\theta_j + q = h_0$.

Now consider F_{j+1}. This will have additional terms in $x^{p-k}, y^{k\theta_{j+1}+q}$, for $1 \leq k \leq p$. Clearly as $\theta_{j+1} > \theta_j$, all of these will be above the line extending E_j, and in particular the term in y^{h_0} is unchanged. We may trivially extend this argument to an induction, which shows that the dot at $(0, h_0)$ is on the Newton polygon of F_j relative to the arc $\sum_{i\geq j} c_iy^{\theta_i}$, which is trivially equal to $\mathbb{P}(f, \lambda)$. Hence we can find β by sliding, and as β was an arbitrary root, we can find all roots of f by sliding. $\square$

Corollary 1.3.16. *Let λ be an analytic arc which is not a root of f, and consider $\mathbb{P}(f, \lambda)$. Define*

$$A = \{\beta_i | O(\beta_i - \lambda) \geq O(\beta_j - \lambda) \forall \beta_j\} \quad ,$$

where β_i is a root of f, and $O(P)$ is the order of P. We claim that $\beta_i \in A$ if and only if β_i can be obtained by sliding along the highest edge E_h of $\mathbb{P}(f, \lambda)$.

Proof. Let θ_h be the angle of E_h. If β can be obtained by sliding along E_h, then by sliding we can write $\beta = \lambda + c_1y^{\theta_h} + H.O.T.$. Hence $O(\beta - \lambda) = \theta_h$, and so $\beta \in A$.

Now, if $\beta \in A$, $O(\beta - \lambda) \geq \theta_h$, and so we can write $\beta = \lambda + cy^{\theta_h} + H.O.T.$. Hence β cannot be found by sliding along any other edge. But all roots can be found by sliding. Hence β can be found by sliding along E_h. $\square$

Proposition 1.3.17. *Again assume that the roots of f all have multiplicity* 1*. Let E_a be an edge of $\mathbb{P}(f)$. If we slide along E_a, then the number of roots that we find will be equal to the length of E_a.*

In order to prove this, we first need to prove a technical lemma:

Lemma 1.3.18. *If we slide along an edge, E_a, of length l, the sum of the lengths of the edges higher than E_a in the polygon $\mathbb{P}(f, cy^{\theta_a})$ plus the number of roots of f we find will be equal to l.*

Proof. If c is a root of multiplicity m of the associated polynomial $\tilde{\mathcal{E}}_a$ of an edge E_a then we want to show that by choosing c when we slide along E_a, the highest dot on E'_a will have x-coordinate m. As c is of multiplicity m, we may write $E_a = (x-cy^\theta)^m \prod(x-c_i y^\theta)$. Now consider the expansion of $E_a(x+cy^\theta, y)$:

$$\begin{aligned}\mathcal{E}'_a &= y^{h_a}((x+cy^\theta) - cy^\theta)^m \prod(x+cy^\theta - c_i y^\theta) \\ &= x^m y^{h_a} \prod(x+cy^\theta - c_i y^\theta)\end{aligned}$$

As the lowest exponent of x in this expansion is m, the m uppermost dots on this edge are zero. Also, as $c \neq c_i$, the coefficient of the term in $x^m y^{h_a+(h_a-m)\theta}$ is non-zero. Hence there is a dot at $(m, h_a + (h_a - m)\theta)$.

Now consider $\mathbb{P}(f, cy^\theta)$. As the only dots on the edge E'_a are as a result of sliding along E_a, we have that E'_a is an edge of $\mathbb{P}(f, cy^\theta)$. As we have assumed that there is no root of multiplicity greater than 1, we must have a dot somewhere on either $X = 0$ or $X = 1$. Now we consider the edges $E'_{a+1}, E'_{a+2}, \dots E_h$. The total length of these edges is m or $m-1$ and if it is $m-1$ then cy^θ is a solution. Note that if cy^θ is a solution then we can still slide along the edges $E'_{a+1}, E'_{a+2}, \dots E_h$. (See example B.1.1)

Now we recall that the sum of the multiplicities of the roots of $\tilde{\mathcal{E}}_a$ (which is simply a complex polynomial in z) is m. Hence we have the required number of edges for E_a. □

The proposition follows almost immediately:

Proof. From the technical lemma we have that each time we slide along the edges higher than E_a in the polygon of $f(x + cy^{\theta_a}, y)$ the number of solutions found plus edges left is equal to the length of E_a. But by theorem 1.3.13 eventually we will either be sliding along an edge of length 1, or we will have found a root. As each edge of length 1 corresponds to a single root of f, we have that the sum of the roots found by sliding along E_a is equal to the length of E_a. □

We will conclude this section with a simple example of sliding towards a root. More complex examples can be found in appendix B.1.

Example 1.3.19. *Let $f(x, y) = x^2 + y^3 - y^4$.*

f has one edge: E_1 with $tan\theta_1 = 3/2$. The associated form for this edge is $\tilde{\mathcal{E}}_1(X, Y) = X^2 + Y^3$, and so the associated polynomial $\mathcal{E}_1(z) = z^2 + 1$.

To find the Newton-Puiseux roots of f, we must first find the roots of $\mathcal{E}_1(z) = z^2+1$. These are $c_1 = i$ and $c_2 = -i$. So $\lambda_1 = \pm iy^{3/2}$

Now we consider $\mathbb{P}(f, \lambda_1)$.

$$\begin{aligned}F(X + \lambda_1(Y), Y) &= X^2 \pm 2iXY^{3/2} - Y^3 + Y^3 - Y^4 \\ &= X^2 \pm 2iXY^{3/2} - Y^4\end{aligned}$$

Hence we now have two edges E_1 and E_2 with $tan(\theta_1) = 3/2$ and $tan(\theta_2) = 5/2$. The associated form for E_2 is $\tilde{\mathcal{E}}_2(X, Y) = \pm 2iXY^{3/2} - Y^4$, and so the associated polynomial $\mathcal{E}_2(z) = \pm 2iz + 1$. As this is of degree 1, we may terminate by the IFT. This gives us two unique roots of the form:

$$\lambda(y) = \pm iy^{3/2} + H.O.T.$$

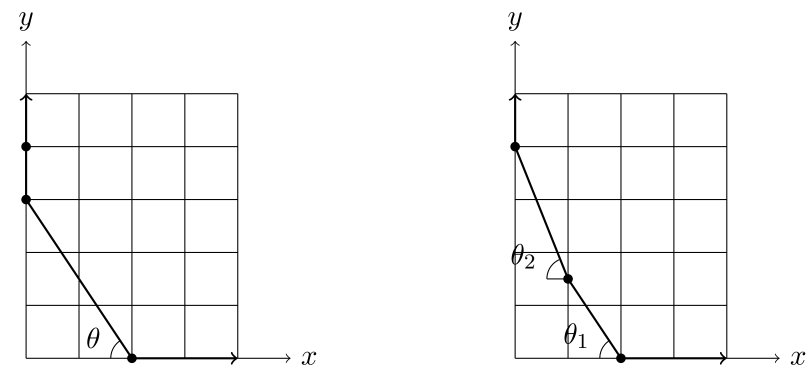

Figure 1.6: The Newton diagrams for f (left) and F_1

1.4 Convergence

In this section we will show that the roots found in section 1.3 converge. We will do this by manipulating the x terms in $f(x, y)$ in order to apply the implicit function theorem to show the existence of convergent roots (and then use an algebraic trick to show that the roots found by sliding must be the same as these roots and hence are convergent). Throughout we will say that a power series $f(x, y)$ is *convergent* if there exists $\delta > 0$ such that $f(x, y)$ converges for all $|(x, y)| < \delta$.

Theorem 1.4.1. *If $f(x, y)$ is a convergent power series such that all the roots of f have multiplicity 1, then the roots, of f as found in section 1.3 converge.*

Lemma 1.4.2. *If λ is an arc such that the highest edge of $\mathbb{P}(f, \lambda)$ is of length 1, then the root found by sliding along this edge is convergent.*

Proof. Write $F(x, y) = f(x + \lambda(y), y)$. Consider the change of variables given by $x = XY^{M\theta_{h-1}} y = Y^M$, where θ_{h-1} is the angle of the second highest edge, and M is defined such that all of the exponents in $F(XY^{M\theta_{h-1}}, Y^M)$ are integers. (Note that we can do this since the denominators of the exponents are bounded.)

Write $F(x, y) = \sum a_{ij} x^i y^j$. So:

$$\begin{aligned} F(XY^{M\theta_{h-1}}, Y^M) &= \sum a_{ij} (XY^{M\theta_{h-1}})^i Y^{Mj} \\ &= \sum a_{ij} X^i Y^{M(i\theta_{h-1}+j)} \quad . \end{aligned}$$

Now $i\theta_{h-1} + j$ is the y-value of the intersection of the line of gradient $-tan(\theta_{h_1})$ through (i, j), which is a dot on $x = 0$ on the newton polygon $\mathbb{P}(F, \epsilon y^{h_1}$ for generic ϵ. Now by applying Proposition 1.3.3 these dots will be on or above the dot $(0, \theta_{h-1} + h_1)$. Hence in particular $i\theta_{h-1} + j \geq \theta_{h-1} + h_1$, and so we may factorise to get $F(XY^{M\theta_{h-1}}, Y^M) = Y^{M\theta_{h-1}+h_1} g(X, Y)$.

Now from the term $a_{1,h_1} x^1 y^{h_1}$ of $F(x, y)$, we have a term in $F(XY^{M\theta_{h-1}}, Y^M)$ given by $a_{1,h_1} XY^{M\theta_{h-1}+Mh_1}$. So $g(X, Y) = a_{1,h_1} X + g_1(X, Y)$. Hence we may use the implicit function theorem on $g(X, Y)$ to find a root of $g(X, Y)$ within a neighbourhood of the origin. As this root is guaranteed to be both unique and convergent, and we can find another root by sliding, these must be the same root. In particular, the root found by sliding is convergent. ☐

Proof of Theorem 1.4.1. Let $f(x, y)$ be a convergent power series with only roots of multiplicity 1. Consider sliding towards a root β. By Theorem 1.3.13 after sliding finitely many times we will have an arc β_i such that the highest edge of $\mathbb{P}(f, \beta_i)$ is of length 1, and $\beta = \beta_i + \lambda$, where λ can be found by sliding along this edge. Now by lemma 1.4.2 λ converges. As β_i is a polynomial, $\beta(= \beta_i + \lambda)$ converges with the same radius of convergence as λ. From theorem 1.3.15 all roots can be found by sliding. Hence all roots of f are convergent. ☐

1.5 Polar Curves

Definition 1.5.1 (Polar Curve)**.** *The polar curves of a function $f(x, y)$ are the curves defined by $\partial f/\partial x = 0$.*

Throughout we will write f_x for $\frac{\partial f}{\partial x}$.

We will now compare the Newton polygon of f_x to the Newton polygon of f. Let $f(x + \lambda(y), y) = \Sigma a_{ij}(x + \lambda(y))^i y^{j/N}$.
Hence

$$f(x + \lambda(y), y) = \sum_{ijk} \binom{i}{k} a_{ij} x^k (\lambda(y))^{(i-k)} y^{j/N}$$

Then:

$$\begin{aligned} \partial/\partial x f(x + \lambda(y), y) &= \Sigma i a_{ij}(x + \lambda(y))^{i-1} y^{j/N} \\ &= \sum_{ijk} i \binom{i-N}{k} a_{ij} x^{k-1} (\lambda(y))^{(i-k)} y^{j/N} \end{aligned}$$

Hence the newton diagram of f_x can be obtained by shifting the Newton dots for f to the left by 1, and removing all dots to the left of the y-axis. From this we get that the edges of f_x include:
$E'_1, E'_2, \ldots, E'_{h-1}$, where E'_i is the i-th edge of f shifted one unit to the left, and $\mathbb{P}(f)$ has edges $E_1, \ldots, E_h$.
If E_h is of length 1, we have the Newton polygon of f_x.
Otherwise, the edges of f_x above E'_{h-1} must be obtained from the dots on the Newton diagram of f.

Remark 1.5.2. *Recall that in section 1.2, it was shown that if λ is a Newton-Puiseux root of f, $\mathbb{P}(f, \lambda)$ has no dots on the y-axis. Hence if γ is a Newton-Puiseux root of f_x, $\mathbb{P}(f_x, \gamma)$ has no dots on the y-axis, and so $\mathbb{P}(f, \gamma)$ has no Newton dots on the line $x = 1$.*

Remark 1.5.3. *If λ is a Newton-Puiseux root of multiplicity greater than 1 of f, then λ will be a polar curve of f, and in particular, $\mathbb{P}(f, \lambda)$ will not have any Newton dots on the lines $x = 0$ and $x = 1$. This can be easily generalised to: if λ is a root of multiplicity m,then $\mathbb{P}(f, \lambda)$ does not have any Newton dots on the lines $x = 0$, $x = 1 \ldots x = m - 1$*

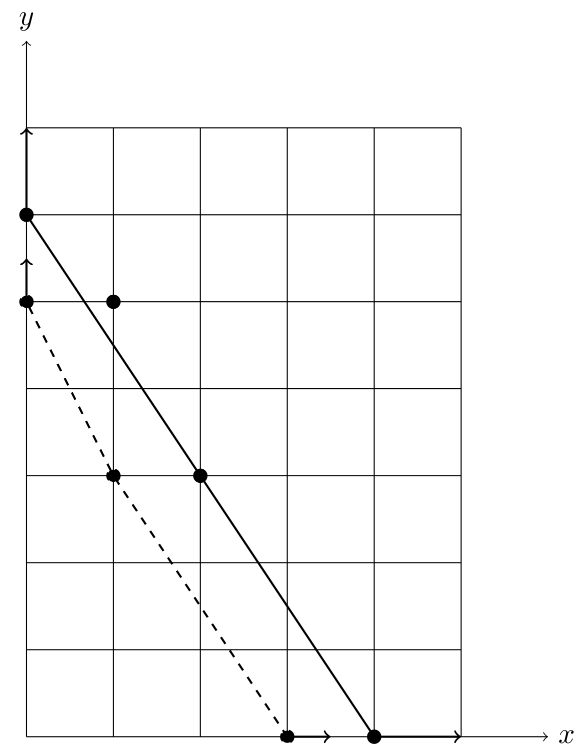

Figure 1.7: The broken line is f_x and the unbroken line is f

Example 1.5.4. *Let $f(x,y) = (x^2 - y^3)^2 - xy^5$. We will draw the Newton polygon of both f and f_x on the same diagram in figure 1.7.*

Note that in this example, the angle θ'_h of the highest edge of f_x is larger than the angle θ_h of the highest edge of f. Also, the lower edge of f_x is parallel to and one unit left of the section of the (only) edge of f between $(4,0)$ and $(2,2)$.

Chapter 2

Puiseux Theorem and Characteristic

2.1 Puiseux Theorem

First we will recall the definitions of algebraically closed and power series.

Definition 2.1.1 (Algebraically Closed Field)**.** *A field F is algebraically closed if every polynomial of one variable of degree at least one, with coefficients in F, has a root in F.*

Definition 2.1.2 (Power Series)**.** *A power series is an infinite sum*

$$c_0 + c_1 x^1 + c_2 x^2 + \dots \quad ,$$

with coefficients in F.

We will formally define $y^{m/n}$, with $m \in \mathbb{Z}, n \in \mathbb{N}$ by the relations:

$$y^{m/n} = (y^{1/n})^m,\ (y^{1/mn})^m = y^{1/n}, y^{1/1} = y\ .$$

Proposition 2.1.3. *From these relations we get the usual multiplication of fractional powers.*

Proof.

$$\begin{aligned} y^{a/c} \times y^{b/d} &= (y^{1/c})^a \times (y^{1/d})^b \\ &= \left((y^{1/cd})^d\right)^a \times \left((y^{1/cd})^c\right)^b \\ &= (y^{1/cd})^{ad} \times (y^{1/cd})^{bc} \\ &= (y^{1/cd})^{ad+bc} \\ &= y^{\frac{ad+bc}{cd}} \end{aligned}$$

□

This enables us to trivially extend the results of chapter 1 to fractional power series.

Definition 2.1.4 (The integral domain of fractional power series in $y^{1/n}$)**.** *For $n \in \mathbb{N}$, the integral domain $\mathbb{C}[y^{1/n}]$ is the set of all power series in $y^{1/n}$ with complex coefficients, using the operations defined below. So an element of this domain can be expressed as an infinite sum:*

$$\sum_{i=0}^{\infty} c_i y^{i/n} \quad .$$

For two fractional power series $\lambda_1 = \sum_i a_i y^{i/n}, \lambda_2 = \sum_j b_i y^{j/n}$ in $\mathbb{C}[y^{1/n}]$, we will define

$$\lambda_1 + \lambda_2 = \sum_i (a_i + b_i) y^{i/n}$$

and

$$\lambda_1 \cdot \lambda_2 = \sum_i (a_i b_0 + a_{i-1} b_1 + a_{i-2} b_2 + \dots + a_1 b_{i-1} + a_0 b_i) y^{i/n} \quad .$$

Clearly $\mathbb{C}[y^{1/n}]$ is closed under both addition and multiplication. It can also be easily checked that $\mathbb{C}[y^{1/n}]$ is an integral domain with these operations.

Remark 2.1.5. *If $k \geq 2$ is an integer, $\mathbb{C}[y^{1/n}] \subset \mathbb{C}[y^{1/kn}]$*

Proof. Let $\lambda = \sum_i a_i y^{i/n}$ be an element of $\mathbb{C}[y^{1/n}]$. Clearly we have:

$$\lambda = a_0 + 0y^{1/kn} + \dots + 0y^{(k-1)/kn} + a_1 y^{k/kn} + 0y^{(k+1)/kn} + \dots + 0y^{(2k-1)/kn} + a_2 y^{2k/kn} + \dots \quad ,$$

and as $k > 1$, $y^{1/kn}$ is not an element of $\mathbb{C}[y^{1/n}]$. (Hence the two sets are not equal.) □

Lemma 2.1.6. *Any element $\lambda = \sum_i a_i y^{i/n}$ of $\mathbb{C}[y^{1/n}]$ which has $c_0 \neq 0$ is a unit of $\mathbb{C}[y^{1/n}]$. (Recall that a unit is an element of a ring which has a multiplicative inverse.)*

Proof. Let $\lambda = c_0 + c_1 y^{\frac{1}{n}} + c_2 y^{\frac{2}{n}} + \dots$ where $c_0 \neq 0$. We will construct the inverse $\gamma = b_0 + b_1 y^{\frac{1}{n}} + b_2 y^{\frac{2}{n}} + \dots$ of λ:
First set $b_0 = \frac{1}{c_0}$. Now for $i > 0$, define

$$b_i = -\frac{1}{c_0} \sum_{j}^{i-1} a_{i-j} b_j \quad .$$

As $c_0 \neq 0$, b_i is well defined. Now

$$\begin{aligned}
\lambda \cdot \gamma &= b_0 c_0 + (b_1 c_0 + b_0 c_1) y^{\frac{1}{n}} + \dots + \sum_{j=0} b_j c_{i-j} y^{\frac{i}{n}} + \dots \\
&= 1 + \sum_{i=1}^{\infty} \sum_{j=0}^{i} b_j c_{i-j} y^{\frac{i}{n}} \\
&= 1 + \sum_{i=1}^{\infty} \Big(b_i c_0 + \sum_{j=0}^{i-1} a_{i-j} b_j \Big) y^{\frac{i}{n}} \\
&= 1 + \sum_{i=1}^{\infty} \Big(-\frac{1}{c_0} \sum_{j=0}^{i-1} a_{i-j} b_j \cdot c_0 + \sum_{j=0}^{i-1} a_{i-j} b_j \Big) y^{\frac{i}{n}} \\
&= 1
\end{aligned}$$

□

Definition 2.1.7 (The quotient field of fractional power series in $y^{\frac{1}{n}}$)**.** *The quotient field of fractional power series in $y^{1/n}$, $\mathbb{C}(y^{1/n})'$ is the quotient field of $\mathbb{C}[y^{1/n}]$.*

Remark 2.1.8. *If $k \geq 2$ is an integer, $\mathbb{C}(y^{1/n})' \subset \mathbb{C}(y^{1/kn})'$*

Proposition 2.1.9. *Any element of $\mathbb{C}(y^{1/n})'$ can be expressed in the form $\lambda = \frac{\lambda'}{y^h}$ where $\lambda' \in \mathbb{C}[y^{1/n}]$. Furthermore, we can also express λ in the form $\lambda = \sum_{i=-h_0}^{\infty} c_i y^{i/n}$.*

Proof. Let $\lambda = \frac{\gamma_1}{\gamma_2}$ be an element of $\mathbb{C}(y^{1/n})'$, where $\gamma_1, \gamma_2 \in \mathbb{C}[y^{1/n}]$. Write $\gamma_2 = c_0 y^{\frac{h_0}{n}} + c_1 y^{\frac{h_0+1}{n}} + \dots$, where h_0/n is the order of γ_2. Then

$$\gamma_2 = y^{\frac{h_0}{n}} \left(c_0 y^0 + c_1 y^{\frac{1}{n}} + c_2 y^{\frac{2}{n}} + \dots\right)$$

By lemma 2.1.6 there is an element γ_3 of $\mathbb{C}[y^{1/n}]$ such that

$$\gamma_3 \cdot \left(c_0 y^0 + c_1 y^{\frac{1}{n}} + c_2 y^{\frac{2}{n}} + \dots\right) = 1 \quad .$$

Hence

$$\begin{aligned}\lambda &= \frac{\gamma_1}{\gamma_2} \cdot \frac{\gamma_3}{\gamma_3} \\ &= \frac{\gamma_1 \cdot \gamma_3}{y^{\frac{h_0}{n}} \left(c_0 y^0 + c_1 y^{\frac{1}{n}} + c_2 y^{\frac{2}{n}} + \dots\right) \cdot \gamma_3} \\ &= \frac{\gamma_1 \cdot \gamma_3}{y^{\frac{h_0}{n}}}\end{aligned}$$

Write $\gamma_1 \cdot \gamma_3 = a_0 + a_1 y^{\frac{1}{n}} + a_2 y^{\frac{2}{n}} + \dots$ Then

$$\begin{aligned}\lambda &= \frac{a_0 + a_1 y^{\frac{1}{n}} + a_2 y^{\frac{2}{n}} + \dots}{y^{h_0/n}} \\ &= \frac{a_0}{y^{h_0/n}} + \frac{a_1 y^{\frac{1}{n}}}{y^{h_0/n}} + \frac{a_2 y^{\frac{2}{n}}}{y^{h_0/n}} + \dots \\ &= a_0 y^{-h_0/n} + a_1 y^{\frac{-h_0+1}{n}} + a_2 y^{\frac{-h_0+2}{n}} + \dots\end{aligned}$$

□

Definition 2.1.10 (Field of fractional power series)**.** *The field of fractional power series $\mathbb{C}(y)^*$ is defined to be the union of all of the fields $\mathbb{C}(y^{1/n})'$. So each element of $\mathbb{C}(y)^*$ is an element of $\mathbb{C}(y^{1/n})'$ for some n. Note that an element of this field will also be an element of $\mathbb{C}(y^{1/2n}), \mathbb{C}(y^{1/3n}), \dots$.*

Proposition 2.1.11. *This is indeed a field*

Proof. First, for all n, we have:

$$\begin{aligned}0 &= 0y^{0/1} \\ &= 0(y^{0/n})^n \\ &= 0y^{n \cdot 0/n} \\ &= 0y^{0/n}\end{aligned}$$

Similarly, $1 = 1y^{0/1} = 1y^{0/n}$.

In order to check the other conditions for $\mathbb{C}(y)^*$ to be a field, recall that for all n, $\mathbb{C}(y^{1/n})'$ is a field.

Let $\lambda_1 = \sum_i a_i y^{i/m} \in \mathbb{C}(y^{1/m})', \lambda_2 \sum_j b_j y^{j/n} \in \mathbb{C}(y^{1/n})', \lambda_3 = \sum_k c_k y^{k/l} \in \mathbb{C}(y^{1/l})'$ be three elements of $\mathbb{C}(y)^*$. By corollary 2.1.8 $\lambda_i \in \mathbb{C}(y^{1/lmn})'$, for $i = 1, 2, 3$. As $\mathbb{C}(y^{1/lmn})'$ is a field, the operations $+$ and $\cdot$ satisfy all the axioms required for $\mathbb{C}(y)^*$ to be a field. □

Theorem 2.1.12 (Puiseux Theorem)**.** *The field $\mathbb{C}(y)^*$ is algebraically closed.*

Proof. Let $f(x, y) = A_n(y)x^n + A_{n-1}(y)x^{n-1} + A_{n-2}(y)x^{n-2} + \cdots + A_0(y)$, with $n \geq 1$, $A_i(i) \in \mathbb{C}(y)^*$ and $A_n(y) \neq 0$, be a polynomial with coefficients in $\mathbb{C}(y)^*$. To prove that $\mathbb{C}(y)^*$ is algebraically closed, we must prove that f has a root.

By proposition 2.1.9, we can write: $A_i = y^{-h_i} B_i$, where $B_i = \sum_{j=0}^{\infty} b_{ij} y^{\frac{j}{N_i}}$ and $b_{i0} \neq 0$, for some $N_i \in \mathbb{N}$. Now:

$$\begin{aligned} f(x, y) &= \sum_{i=0}^{n} y^{-h_i} B_i x^i \\ &= y^{-h} \sum_{i=0}^{n} y^{h-h_i} B_i x^i \qquad \text{where } h = max(h_i) \\ &= y^{-h} g(x, y) \ , \end{aligned}$$

for some power series $g(x, y)$. As $h = max(h_i)$, all the exponents in $g(x, y)$ are non-negative, and there is at least one dot on $y = 0$ in the Newton diagram of $g(x, y)$. We now want to manipulate $g(x, y)$ into a form where we can apply proposition 1.3.12. First note that as there are finitely many integers N_i, we can find a number M (say $M = \prod_i N_i$) such that $g(x, y^M)$ is a power series. Let h_0 be the y-value of the lowest dot on $x = 0$. There are three cases to consider:

1. $h_0 > 0$.
   In this case, let $g_1(x, y) = g(x, y^{nM})$. Note that the exponents of y must be multiples of n. Consider the Newton polygon $\mathbb{P}(g_1)$. From above there will be at least one dot on $y = 0$. Specifically, there will be a term $c_i x^i$, for some $i \leq n$. Hence $g_1(x, y)$ must be mini-regular in x, and we may apply proposition 1.3.12 to get a root $x = \beta(y)$ of $g_1(x, y)$. Clearly $x = \beta(y^{1/nM})$ will be a root of $g(x, y)$, and hence also a root of $f(x, y)$.

2. $h_0 = 0$, no other dots on $y = 0$.
   In this case, let $g_1(x, y) = g(1/x, y)$.

$$\begin{aligned} g_1(x, y) &= \sum_{i=0}^{n} y^{h-h_i} B_i x^{-i} \\ &= x^{-n} \sum_{i=0}^{n} y^{h-h_i} B_i x^{n-i} \\ &= x^{-n} g_2(x, y) \ . \end{aligned}$$

Now the only dot on $y = 0$ in the Newton polygon $\mathbb{P}(g_2)$ will be at $x = n$. Hence we may use the same argument as in case 1 to find a root $x = \beta(y)$ of g_2. As $\mathbb{C}(y)^*$ is a field, β will have an inverse, β^{-1} which is also an element of $K(y)^*$. Clearly $x = \beta^{-1}$ is a root of $g(x, y)$ and hence is a root of $f(x, y)$.

3. $h_0 = 0$, at least one other dot on $y = 0$.
   To show this case we will slide once so that we can apply proposition 1.3.12. Let c be a root of the polynomial $g(z, 0)$. It can be seen by sliding that $g_1(x, y) = g(x + c, y)$ will not have a dot at $(0, 0)$. We may now use case 1 to find a root $x = \beta(y)$ of g_1. Hence $x = c + \beta(y)$ will be a root of $g(x, y)$ and hence also a root of $f(x, y)$.

Hence f has a root in $\mathbb{C}(y)^*$, and so the field $\mathbb{C}(y)^*$ is algebraically closed. □

2.2 Puiseux Characteristic

Let $x(t), y(t)$ be a parametrisation of an analytic arc, with $O(x) \leq O(y)$. As the field $\mathbb{C}(y)^*$ is algebraically closed, we can, by applying an appropriate algebraic transformation if necessary, write $x(t), y(t)$ in the form $x = t^m$, $y = \sum_{r=m}^{\infty} a_r t^r$.

Definition 2.2.1 (Irreducible Parametrisation)**.** *We will call a parametrisation $x = t^m$, $y = \sum_{r=m}^{\infty} a_r t^r$ of an arc $x = \lambda(y)$ irreducible if the integers $m, r (a_r \neq 0)$ have no common factor.*

Let $x = \lambda(y)$ be a root of f with an irreducible parametrisation:

$$x = t^m, \ y = \sum_{r=m}^{\infty} a_r t^r \ .$$

Define

$$\beta_1 = min\{k | a_k \neq 0, m \nmid k\} \text{ and } e_1 = gcd(m, \beta_1) \ .$$

Then we inductively define

$$\beta_{i+1} = min\{k | a_k \neq 0, e_i \nmid k\} \text{ and } e_{i+1} = gcd(e_i, \beta_{i+1}) \ ,$$

until we have β_g such that $e_g = 1$.

Definition 2.2.2 (Puiseux Characteristic)**.** *The series $(m; \beta_1, \beta_2, \ldots, \beta_g)$ is called the Puiseux characteristic of λ.*

Definition 2.2.3 (Puiseux Pairs)**.** *Again assume $x = \lambda(y)$ has an irreducible parametrisation, which we will write in the form*

$$x = t^m, \quad y = a_0 y^{n_0} + \cdots + a_{n_1} y^{\frac{n_1}{d_1}} + \cdots + a_{n_2} y^{\frac{n_2}{d_1 . d_2}} + \ldots \quad .$$

The pairs (n_i, d_i) are called Puiseux pairs.

Remark 2.2.4. *It is possible to obtain the Puiseux characteristic from the Puiseux pairs and vice-versa as follows:*
given $\{(n_i, d_i)\}$,

$$m = d_1 \dots d_g$$
$$\beta_i = n_i d_{i+1} d_{i+2} \dots d_g$$

Given $(m, \beta_1, \beta_2, \dots, \beta_g)$, *we have that*

$$\frac{\beta_i}{m} = \frac{n_i}{d_1 \dots d_i} \quad .$$

Example 2.2.5. *Let* $f(x, y) = (x^2 - y^3)^2 + xy^5$. *From example B.1.2, the roots of* f *are:*

$$\beta = \begin{array}{r} y^{3/2} + \frac{i}{2} y^{7/4} + \frac{1}{8} y^2 + H.O.T. \\ y^{3/2} - \frac{i}{2} y^{7/4} + \frac{1}{8} y^2 + H.O.T. \\ -y^{3/2} + \frac{1}{2} y^{7/4} - \frac{1}{8} y^2 + H.O.T. \\ -y^{3/2} - \frac{1}{2} y^{7/4} - \frac{1}{8} y^2 + H.O.T. \end{array}$$

Clearly these can be parametrised by $y = t^4, x = t^6 \pm \frac{i}{2} t^7 + \frac{1}{8} t^8 + \dots$, *and* $y = t^4, x = -t^6 \pm \frac{1}{2} t^7 - \frac{1}{8} t^8 + \dots$.

For both of these parametrisations we have $m = 4, \beta_1 = 6, e_1 = 2, \beta_2 = 7, e_2 = 1$. *As* $e_2 = 1$, *we may stop, and so the Puiseux characteristic for all the roots of* f *is* $(4; 6, 7)$.

It can also be easily seen that the Puiseux pairs of f *are* $(6, 2)$ *and* $(7, 2)$.

Example 2.2.6. *Let* $f = 4x^3 - 4xy^3 - y^5$. *From example B.1.3, the roots of* f *are:*

$$\beta_1 = -1/4y^2 - 1/64y^3 + H.O.T.$$
$$\beta_2 = y^{3/2} + 1/8y^2 - 3/128y^{5/2} + H.O.T.$$
$$\beta_3 = -y^{3/2} + 1/8y^2 + 3/128y^{5/2} + H.O.T.$$

For β_2 *and* β_3, *we have a parametrisation* $y = t^2$, $x = \pm t^3 + 1/8t^4 - \pm 3/128t^5 + \dots$. *This has a Puiseux characteristic of* $(2; 3)$.

For β_1, $y = t$, $x = -1/4t^2 - 1/64t^3 + \dots$, *and so the characteristic of* β_1 *is undefined.*

Example 2.2.7. *Let* $f(x, y) = x - y^2 - xy^{1/2}$ *It can easily be seen that this has one root:* $x = y^2 + y^{5/2} + y^3 + y^{7/2} + \dots$. *Hence we have a parametrisation* $y = t^2$, $x = t^4 + t^5 + t^6 + \dots$. *Hence the Puiseux characteristic of* β *is* $(2; 5)$.

Now we will look at the characteristics of irreducible functions. First we will recall the definition of irreducibility.

Definition 2.2.8 (Irreducible)**.** *A power series* $f(x, y)$ *is irreducible if it cannot be expressed in the form* $f = g_1 g_2$, *where* g_1 *and* g_2 *are power series of order greater than* 0. *(Note that we are referring to power series with integer exponents, not fractional power series.) A power series is reducible if it is not irreducible.*

It is sometimes important to know if a given power series is irreducible. We will prove the theorems from this chapter in the appendix, and quote them here. If the first term in the Taylor expansion of f can be split, then it can be shown that f is reducible. Specifically we have the following:

Proposition 2.2.9. *Let $f(x, y)$ be a power series, and write it in the form of a Taylor series:*

$$f(x, y) = (a_1x+b_1y)^{m_1}(a_2x+b_2y)^{m_2} \dots (a_kx+b_ky)^{m_k}+H_{m+1}(x, y)+H_{m+2}(x, y)+\dots \quad ,$$

where $m = m_1 + m_2 + \cdots + m_k$. If there exists $i \neq j$, such that $a_ib_j - b_ia_j \neq 0$, then f is reducible.

The proof is in the appendix A.1. An obvious corollary is:

Corollary 2.2.10. *Let $f(x, y) = (a_1x+b_1y)^{m_1}(a_2x+b_2y)^{m_2} \dots (a_kx+b_ky)^{m_k}+H_{m+1}+H_{m+2}+\dots \quad$, as above. If for all $i, j \in \{1, \dots, k\}$ such that $i \neq j$, $a_ib_j - b_ia_j \neq 0$, then f is reducible in the form $f = f_1f_2 \dots f_k$.*

We will now look at Taylor series with irreducible first term. Note that by applying an appropriate linear transform we can write these as $f(x, y) = x^m + H_{m+1} + \dots$. We will rely on a technical lemma:

Lemma 2.2.11. *If λ is a Newton-Puiseux root of a power series f (with integer exponents), then the conjugates of λ (as defined below) will also be roots of f. Specifically if we write*

$$\lambda = c_1y^{n_1/d} + c_2y^{n_2/d} + \cdots + c_iy^{n_i/d} + \dots$$

(where d are minimal) then for any complex $d-th$ root of unit z_d, the arc

$$\lambda' = z_d^{n_1}c_1y^{n_1/d} + z_d^{n_2}c_2y^{n_2/d} + \cdots + z_d^{n_i}c_iy^{n_i/d} + \dots$$

is also a root of f.

Definition 2.2.12 (Conjugate Arc)**.** *Write $\lambda = c_1y^{m_1/d}+c_2y^{m_2/d}+\cdots+c_iy^{m_i/d}+\dots$. We will call an analytic arc λ' conjugate to λ if it is of the form*

$$\lambda' = z_d^{m_1}c_1y^{m_1/d} + z_d^{m_2}c_2y^{m_2/d} + \cdots + z_d^{m_i}c_iy^{m_i/d} + \dots$$

as above.

The proof of this lemma will be in the appendix. This has a number of important corollaries. Firstly from simple complex algebra we get:

Corollary 2.2.13. *The product $\prod_{\lambda'}(x - \lambda')$ over all arcs conjugate to λ has integer exponents.*

Corollary 2.2.14. *If f is irreducible, then all roots of f will be conjugate in the above sense.*

Proof. If λ is a root of f, then the product$\prod_{\lambda'}(x - \lambda')$ over all arcs conjugate to λ is a factor of f in the domain of power series with integer exponents. If f has a root β which is not conjugate to λ, then the product of all roots conjugate to β will also be a factor of f. Hence as f is irreducible, all roots of f are conjugate to λ. □

Recall that if $\lambda = c_1 y^{\theta_1} + \cdots + d_i y^{\theta_i}$ is a fractional power series, an edge E of $\mathbb{P}(f, \lambda)$ is called relevant if the angle of E is greater than θ_i.

Corollary 2.2.15. *Let β be a power series. If $\mathbb{P}(f, \beta)$ has more than one relevant edge, f is reducible.*

Proof. Let λ_1 and λ_2 be obtained from β by sliding along two different relevant edges. Clearly they are not conjugate to each other, and in particular any root conjugate to λ_1 will not be conjugate to λ_2 and vice-versa. From corollary 2.2.13 we have that the product over all roots conjugate to λ_1, $\prod_{\lambda_1'}(x - \lambda_1')$ and the product over all roots conjugate to λ_2, $\prod_{\lambda_2'}(x - \lambda_2')$ are both power series with only integer exponents. Hence we have found two different factors of f, and so f is reducible. □

Corollary 2.2.16. *If $\mathbb{P}(f, \beta)$ has a relevant edge of the form $E = \prod_i (x^m - w_i^m y^n)^d$, where $w_i \in \mathbb{C}$, such that there exists at least two w_i, w_j with $w_i^m \neq w_j^m$, then f is reducible.*

Proof. Assume w_i and w_j are different. Let $\lambda_i = \beta + w_i y^{\frac{n}{m}} + H.O.T.$ and $\lambda_j = \beta + w_j y^{\frac{n}{m}} + H.O.T.$ Clearly both of these will generate distinct factors of f. □

We now have a way to determine irreducibilty: for $f(x, y)$, we simply slide towards a root of f. If we find a relevant edge with more than one distinct factor over the field of power series, or more than one relevant edge, f is reducible. Otherwise, if we reach an arc such that $\mathbb{P}(f, \lambda)$ has a lowest relevant dot at $x = 1$, f is irreducible. (Although we have not proven this).

We will now state the original result of Kuo [3]:

Theorem 2.2.17 (Generalised Hensel's lemma)**.** *f is reducible if and only if there is an analytic arc $\lambda = c_1 y^{\theta_1} + \cdots + d_i y^{\theta_i}$ such that the Newton polygon of f relative to λ has a relevant edge E whose associated form is reducible with at least two distinct factors.*

In this case, given a factorisation $\mathcal{E}(z) = \eta_1(z)\eta_2(z)$ of the associated polynomial of E, we have a factorisation $f(x, y) = g(x, y)h(x, y)$ where η_1, η_2 are the associated polynomials for g and h respectively.

Example 2.2.18. *Let $f(x, y) = (x^2 - y^3)^2 + y^7$. The associated form for the edge E_1 is $(x^2 - y^3)^2$, which has only one distinct factor. Sliding along this edge gives us:*

$$f(x + y^{3/2}, y) = (x^2 - 2xy^{3/2})^2 + y^7 \quad .$$

This has edges E_1 and E_2. The associated form of E_2 is $4x^2 + y^4$, which is reducible. Hence f is reducible.

2.3 Iterated Torus Knots

In this section we will look at the topology of the singularity in a neighbourhood of a singular point. In particular, we will use some results from knot theory to show that the Puiseux characteristic is a topological invariant. We will not attempt to prove all of the theorems in this section, especially the ones which rely on knot theory. For a more detailed exposition, the reader is advised to look at Brieskorn and Knörrer

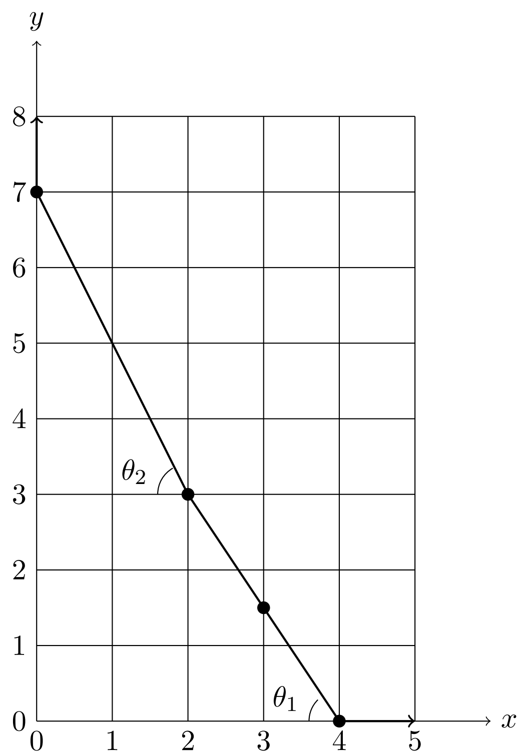

Figure 2.1: The Newton polygon of $f(x + y^{3/2}, y) = (x^2 - 2xy^{3/2})^2 + y^7$

[1] or Wall [11]. Alternatively, for a more knot-theoretical approach see Lickorish [7] or Eisenbud and Neumann [2]. Throughout let $f(x, y)$ be a complex valued analytic function expressed as a convergent power series.

We will start by quoting a theorem of Milnor, which is valid on algebraic sets. For the proof see chapter 2 of [8].

Definition 2.3.1 (Algebraic Set). *An algebraic set V is the set defined by the equations $P_1(x_1, x_2, \ldots x_n) = 0$, $P_2(x_1, x_2, \ldots x_n) = 0$, $\ldots$, $P_m(x_1, x_2, \ldots x_n) = 0$, where the P_i are polynomials in n variables.*

Definition 2.3.2 (Isolated Singularity). *z_0 is an isolated singularity of an algebraic set V if the disk of radius ϵ about the z_0 $D_\epsilon(z_0)$ in $\mathbb{C}^2$ contains no other singular points of V. This which is equivalent to the condition that V is transverse to the 3-sphere S_δ for all $\delta \leq \epsilon$.*

Remark 2.3.3. *In our case of mini-regular power series, $(0, 0)$ is an isolated singularity if and only if all Newton-Puiseux roots are of multiplicity 1.*

Theorem 2.3.4. *Let V be an algebraic set. For small ϵ the intersection of V with D_ϵ is homeomorphic to the cone over $K = V \cap S_\epsilon$.*

Note that this can be shown to hold for V defined using convergent power series (see Brieskorn and Knörrer [1]). From this theorem we have that in the neighbourhood of a singular point, we can determine the topology of the singularity by studying at the intersection of V with a sphere of sufficiently small radius.

As V is 2-dimensional it can easily be seen that the intersection of V with S_ϵ will be a 1-dimensional sub-manifold. As a 1-dimensional submanifold of a 3-sphere is not dense, we can use stereographic projection from a point not in $V \cap S_\delta$ to view it as a submanifold of $\mathbb{R}^3$, called a *link*. If $f(x, y)$ is irreducible, then this intersection will be a *knot*.

(Recall that a *knot* is defined to be an embedding of a circle S^1 in $\mathbb{R}^3$, and a *link* is the disjoint union of a finite number of knots.)

So we can restate theorem 2.3.4 as: if two curves have equivalent links they are topologically equivalent. From basic knot theory we have that the converse is true - topologically equivalent singularities have equivalent links. Let ϵ be sufficiently small. We will consider the intersection of V with S_ϵ.

Definition 2.3.5 (Torus Knot)**.** *A torus knot is an embedding of S^1 in the torus $S^1 \times S^1$.*

If m, n are co-prime, a (m, n) torus knot is an embedding of S^1 in a torus $\{(x, y) \in \mathbb{C}^2 | |x| = a, |y| = b\}$ (where $a, b > 0$) which winds m times around the circle $S^1 \times y_0$ (with $|x| = a$ and $|y_0| = b$) and n times around the circle $x_0 \times S^1$ (where $|x_0| = a$ and $|y| = b$). If $a > b$, we can embed this torus in $\mathbb{R}^3$ using the mapping

$$(x, y) = (ae^{i\theta}, be^{i\phi}) \to \big((a + bcos\phi)cos\theta, (a + bcos\phi)sin\theta, bsin\phi\big) \ .$$

In coordinates (u, v, w), this corresponds to the torus $(a - \sqrt{u^2 + v^2})^2 + w^2 = b^2$. Using this mapping, the circle $|x| = a$ maps to a latitude circle, and the circle $|y| = b$ maps to a longitude circle. (For example if $y_0 = (b, 0)$, $S^1 \times y_0$ maps to the circle $u^2 + v^2 = (a+b)^2$, and if $x_0 = (a, 0)$, $x_0 \times S^1$ maps to the circle $(u - a)^2 + w^2 = b^2$.) Note that it can be shown by elementary knot theory that two torus knots are topologically equivalent if and only if they are both of type (m, n) or (n, m).

Proposition 2.3.6. *If $f(x, y) = x^m - y^n$, the intersection of the set $\{(x, y)|f(x, y) = 0\}$ and a small sphere S_ϵ is a (m, n) torus knot.*

Proof. Let $f(x, y) = x^m - y^n$, where $n > m$, and define $V = \{(x, y)|f(x, y) = 0\}$. Consider the intersection of V with S_ϵ. Recall that this will be a 1-dimensional submanifold of S_ϵ.

From section 1.3, we may write $x = y^{n/m}$. This can then be parametrised by $y = t^m$, $x = t^n$. Now let $t = re^{i\theta}$. We have $y = (re^{i\theta})^m$, $x = (re^{i\theta})^n$. Now $|x|^2 + |y|^2 = r^{2m} + r^{2n}$. As this is a monotonically increasing function in r, there will only be one value of r for which $\big(x(re^{i\theta}), y(re^{i\theta})\big)$ will lie on S_ϵ. Hence if we look at this on the graph of $|x|$ relative to $|y|$, this intersection will correspond to a single point on the arc of radius ϵ. (See figure 2.2) In particular, the intersection of V with S_ϵ will be a subset of the set defined by $|x| = a$, $|y| = a^{m/n}$, where $a^2 + a^{2m/n} = \epsilon^2$. This set is in fact a torus, and so $V \cap S_\epsilon$ is a 1-dimensional submanifold of the torus.

Now consider the change in x and y as θ varies. Write $x = ae^{i\theta_x}$ and $y = a^{m/n}e^{i\theta_y}$. $\theta_x = n\theta$, $\theta_y = m\theta$. So as θ goes from 0 to 2π, the curve $V \cap S_\epsilon$ runs around the circle $|x| = a$, n times, and the circle $|y| = a^{m/n}$, m times. It can easily be seen that this is a (m, n) torus knot.

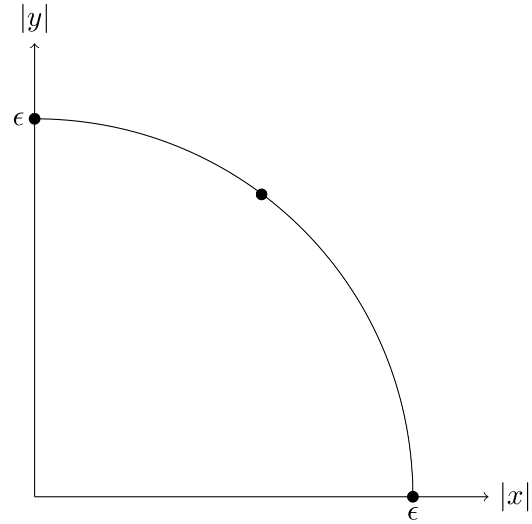

Figure 2.2: The graph of the sphere $|x|^2 + |y|^2 = \epsilon^2$ relative to $|x|$ and $|y|$. Note that each point on this arc corresponds to a torus in $\mathbb{C}^2$.

Now as f is irreducible, the other roots of f will be of the form $x = z_m^{2\pi k} y^{n/m}$. This can be parametrised by $x = ae^{in\theta}$ and $y = a^{m/n}e^{im\theta+2\pi k}$. Hence the other roots of f give the same knot. □

Definition 2.3.7 (Cable Knot)**.** *Let K be a knot in $\mathbb{S}^3$, and let T be a solid torus with boundary δT. Let τ be an embedding of the solid torus in a neighbourhood of K such that:*

1. *The intersection of a plane transverse to K with δT is a longitude circle $x_0 \times S^1$.*
2. *A latitude circle $S^1 \times y_0$ has linking number 0 with K.*

If T is a torus knot of type (m, n), we can consider T as a knot on the boundary of a solid torus. The (m, n) cable knot about K is the knot given by $\tau(T)$.

Definition 2.3.8 (Iterated Torus Knot)**.** *An iterated torus knot of order i is a knot constructed by the following procedure: let $(m_1, n_1), (m_2, n_2), \dots, (m_i, n_i)$ be pairs of co-prime integers. Define K_1 to be the (m_1, n_1) torus knot. We can now iteratively define the knot K_{j+1} to be the (m_{j+1}, n_{j+1}) cable knot about K_j. The knot K_i is an iterated torus knot of order i and type $(m_1, n_1), (m_2, n_2), \dots, (m_i, n_i)$.*

Proposition 2.3.9. *Let $f(x, y)$ be an irreducible analytic function with Puiseux pairs: $(n_1, d_1), (n_2, d_2), \dots, (n_g, d_g)$. Then the link of f is topologically equivalent to an iterated torus knot (cable knot) of order g and type $(m_1, d_1), (m_2, d_2), \dots, (m_g, d_g)$, where $m_1 = n_1$, and $m_i = n_i - n_{i-1}d_i + m_{i-1}d_{i-1}d_i$ for $i > 1$.*

Proof. We will only give the geometric ideas behind the proof of this proposition. First we note that the knot $(1, p)$ is trivial: specifically it is homeomorphic to the unknot (S^1). In particular, it can be shown that the $(1, p)$ cable knot about K is homeomorphic to K.

Now let $d = d_1 \cdot d_2 \cdot \dots \cdot d_g$, and parametrise f by $x(t) = t$, $y(t) = \dots + c_1 t^{n_1/d_1} + \dots + c_2 t^{n_2/d_1 d_2} + \dots + c_g t^{n_g/d} + \dots$, where $c_i \neq 0$. Now we can construct the link of f by inductively constructing the links of f_i, where the Puiseux roots of f_i are the first i terms of the Puiseux roots of f.

First note that from the above, if the first term is in y^k, then the link of f_1 will be the $(1, k)$ torus knot, which is the unknot. Hence we may assume that the first term is $c_1 t^{n_1/d_1}$. Now from proposition 2.3.6 the link of f_1 will be the (n_1, d_1) torus knot. Hence $m_1 = n_1$.

Now write $y = t^{n_1/d_1}(c_1 + \dots + c_2 t^{(n_2 - d_2 n_1)/d_2} + \dots + c_g t^{(n_g/d - n_1/d_1)} + \dots)$, and consider the second order approximation y_2:

$$y_2 = t^{n_1/d_1}(c_1 + c_2 t^{(n_2 - d_2 n_1)/d_2}) \quad .$$

Since $|t|$ is small, this will lie in a small tubular neighbourhood of K_1. By considering the parametrisation $t = re^{i\theta}$, it can be seen that this knot will rotate around the longitude of the tubular neighbourhood d_2 times, so K_2 is a (α, d_2) cable knot about K_1 (where α is defined below). If $d_2 = 1$, this is a trivial cable knot about K_1, which will be homeomorphic to K_1. So we may assume $d_2 > 1$.

Now we must evaluate α. K_2 will rotate $n_2 - d_2 n_1$ about a longitude on the tubular neighbourhood of K_1. However, as the embedding of the torus T into this tubular neighbourhood was twisted, K_2 will not be a $(d_2, n_2 - d_2 n_1)$ cable knot. The linking number of the longitude of T about the longitude of the tubular neighbourhood will be $m_1 d_1 d_2$. Hence K_2 is a iterated torus knot of order 2 and type $(m_1, d_1), (m_2, d_2)$.

For the general case, write $y_i = t^{n_1/d_1}(c_1 + t^{(n_2 - d_2 n_1)/d_2}(c_2 + \dots (c_{i-1} + c_i t^{n_i - n_{i-1} d_i / d_i})$, and assume K_i is an iterated torus knot of type $(m_1, d_1), (m_2, d_2), \dots, (m_i, n_i)$. Now y_{i+1} may be written as

$$y_{i+1} = t^{n_1/d_1}(c_1 + t^{(n_2 - d_2 n_1)/d_2}(c_2 + \dots (c_i + c_{i+1} t^{n_{i+1} - n_i d_{i+1}/d_{i+1}}))) \quad .$$

Since $|t|$ is small, this will lie in a small tubular neighbourhood of K_i. It will rotate d_{i+1} times about a longitude of the tubular neighbourhood (and hence by the same argument as above we may assume $d_{i+1} > 1$), and it will rotate $n_{i+1} - n_i d_{i+1}$ times about a latitude. It can be checked that the linking number of the longitude of the torus T about the longitude of the tubular neighbourhood will be $m_i d_i d_{i+1}$, and so K_{i+1} will be a (m_{i+1}, n_{i+1}) cable knot about K_i. Hence K_{i+1} is an iterated torus knot of order $i+1$ and type $(m_1, d_1), (m_2, d_2), \dots, (m_{i+1}, n_{i+1})$. The proposition follows by induction. □

Theorem 2.3.10. *The links of two irreducible curves will be topologically equivalent if and only if they have the same Puiseux characteristic.*

Proof. Proposition 2.3.9 gives us one direction. The other direction follows from the classification of iterated torus knots, which we will not attempt to do here. See Wall [11] or Brieskorn and Knörrer [1] for details. □

Corollary 2.3.11. *The Puiseux characteristic is a topological invariant of the singularity.*

Proof. Recall that from theorem 2.3.4 a neighbourhood of the singularity is homeomorphic to the cone over the link (and hence the link is a topological invariant of the singularity). By theorem 2.3.10 the link is completely determined by the characteristic, and so the Puiseux characteristic is a topological invariant. □

We will conclude this section with a short discussion of the reducible case. Let $f(x, y)$ be a reducible analytic curve with g irreducible components. In the intersection of f with a small sphere, each of these irreducible components will generate a cable knot, the type of which is determined by proposition 2.3.9. Hence this intersection will be the union of g linked cable knots. In fact it can be shown (see for example Reeve [9]) that the *intersection multiplicity* of the curve germs corresponding to the irreducible components is just the *linking number* of the corresponding knots. This leads to the topological classification of plane curve singularities:

Theorem 2.3.12. *Two singularities are topologically equivalent if and only if both the Puiseux pairs and the linking numbers of corresponding components are the same.*

Chapter 3

Kuo-Lu Theorem

3.1 The Kuo-Lu Theorem

In this section, we will give the original Kuo-Lu theorem, as stated in the paper [4].

This theorem is a generalisation of Rolle's theorem to the field of fractional power series. For the reader's convenience we will state Rolle's theorem, along with a basic extension to the complex field, which will be useful in the proof of the Kuo-Lu theorem.

Theorem 3.1.1 (Rolle's Theorem)**.** *If an analytic function $f(x) : \mathbb{R} \to \mathbb{R}$ has n solutions to $f(x) = c$ on the interval $[a, b]$, then there are at least $n - 1$ solutions to $f'(x) = 0$ on this interval.*

Lemma 3.1.2. *If $f(z)$ is a complex valued polynomial which has k different roots (all of which may have multiplicity greater than 1) then there will be $k - 1$ roots of $f'(z)$ (counted with multiplicity) which are not roots of $f(z)$.*

Proof. Write $f = \prod_{i=1}^{k}(z - c_i)^{m_i}$, where m_i is the multiplicity of the root c_i. Differentiating gives:

$$\begin{aligned} f'(z) &= \sum_{i=1}^{k} m_i(z - c_i)^{m_i-1} \prod_{j \neq i}(z - c_j)^{m_j} \\ &= \prod_{i}(z - c_i)^{m_i-1}(\sum_{j} m_j \prod_{l \neq j}(z - c_l)) \end{aligned}$$

As we are working over $\mathbb{C}$, the polynomial $\sum_j m_j \prod_{l \neq j}(z - c_l)$ has $k - 1$ roots, which are clearly not roots of f.

□

Now let $f(x, y)$ be a fractional power series. By Puiseux's theorem, f can be factorised into the product:

$$f(x, y) = \prod_{i}(x - \beta_i(y)),$$

where β_i are the Newton-Puiseux roots of f.

Let $c_i = \underset{j \neq i}{Max}\ O(\beta_i - \beta_j)$.

Write $\beta_i = \sum_j a_j y^{\theta_j}$.
Define the polynomial g_{c_i} :

$$g_{c_i}(\beta_i) = \sum_{j|\theta_j < c_i} a_j y^{\theta_j} + cy^{c_i}$$

where c is a constant chosen from a dense subset of $\mathbb{C}$ or $\mathbb{R}$ depending on the context. We will call g_{c_i} the generic perturbation of β_i of degree c_i.

Assume that no two β_i's are identical.

Theorem 3.1.3 (Kuo-Lu Theorem)**.** *If β_i, β_j are roots of $f(x, y)$, then there is a root γ_k of f_x such that $O(\beta_i - \beta_j) = O(\beta_i - \gamma_k) = O(\beta_j - \gamma_k)$.*

Proof. Let g_{ij} be the generic perturbation of β_i at order $O(\beta_i - \beta_j)$. Consider the Newton diagram $\mathbb{P}(f, g_{ij})$. Let E_h be the highest edge of this diagram, and let θ be the angle of this edge. Let n be the length of the highest edge. Write $\mathcal{E}_h = \sum b_a x^a y^{h_0 - a\theta}$. As c is generic, it is not a root of f_x, and so the diagram $\mathbb{P}(f, g_{ij})$ will have a dot on $x = 1$. (In fact the same reasoning gives that there will be a dot on $x = k, 0 \leq k \leq mult(f)$.) From the remark 1.3.16 we have that all roots β_j of f with $O(\beta_i - \beta_j)$ can be found by sliding along E_h.

Now the associated polynomial $\tilde{\mathcal{E}}_h = \sum b_a z^a$. For each solution c_k of this polynomial, we find a root $\beta_k = g_{ij} + c_k y^{O(\beta_i - \beta_j)} + H.O.T.$ Note that unless β_i is maximal, the values c_k need not be distinct.

Now consider the graph $\mathbb{P}(f_x, g_{ij})$. From section 1.5 we have that

$$\mathbb{P}(f_x(x + g_{ij}, y) = \frac{\partial f}{\partial x} f(x + g_{ij}, y) \ ,$$

and so the diagram $\mathbb{P}(f_x, g_{ij})$ can be obtained from $\mathbb{P}(f, g_{ij})$ be shifting one space to the left.

We will now slide along the highest edge of this diagram $E_h{}'$ to find roots γ of f_x. The associated polynomial $\mathcal{E}_h{}' = \sum_a ab_a x^{a-1} y^{h_0 - a\theta}$. Hence we have $n - 1$ solutions c'_k to the associated form $\tilde{\mathcal{E}_h}{}' = \sum_a ab_a z^{a-1}$. By lemma 3.1.2, if there are m values c_k which solve $\tilde{\mathcal{E}_h}$, then $m - 1$ of the solutions c'_k of $\tilde{\mathcal{E}_h}{}'$ will not be solutions of $\tilde{\mathcal{E}_h}$. Let $\gamma_1, \ldots, \gamma_{m-1}$ be the roots of f_x such that $\gamma_k = g_{ij} + c'_k y^{O(\beta_i - \beta_j)} + H.O.T.$ As c'_k is not a root of $\tilde{\mathcal{E}_h}$, $O(\beta_i - \gamma_k) = O(\beta_j - \gamma_k) = O(\beta_i - \beta_j)$. Hence we have found $m - 1$ roots of f_x (counted with multiplicity) which satisfy the above relation. □

Example 3.1.4. *Consider $f(x, y) = (x^2 - y^3)^2 - xy^5$. Recall from example B.1.2 the roots of f are:*

$$\beta_1 = y^{3/2} + \frac{i}{2} y^{7/4} + H.O.T.$$
$$\beta_2 = y^{3/2} - \frac{i}{2} y^{7/4} + H.O.T.$$
$$\beta_3 = -y^{3/2} + \frac{1}{2} y^{7/4} + H.O.T.$$
$$\beta_4 = -y^{3/2} - \frac{1}{2} y^{7/4} + H.O.T.$$

and the roots of f_x are:

$$\gamma_1 = -1/4y^2 - 1/64y^3 + H.O.T.$$
$$\gamma_2 = y^{3/2} + 1/8y^2 - 3/128y^{5/2} + H.O.T.$$
$$\gamma_3 = -y^{3/2} + 1/8y^2 + 3/128y^{5/2} + H.O.T.$$

Now $O(\beta_1-\beta_3) = O(\beta_1-\beta_4) = O(\beta_2-\beta_3) = O(\beta_2-\beta_4) = 3/2$, and $O(\beta_i-\gamma_1) = 3/2$ for all i. Also $O(\beta_1 - \beta_2) = O(\beta_1 - \gamma_2) = O(\beta_2 - \gamma_2) = 7/4$ and $O(\beta_3 - \beta_4) = O(\beta_3 - \gamma_3) = O(\beta_4 - \gamma_3) = 7/4$. Hence for every pair of roots β_i, β_j, we have γ_k such that $O(\beta_i - \beta_j) = O(\beta_i - \gamma_k) = O(\beta_j - \gamma_k)$.

3.2 Tree Model

Let $f(x, y) = \prod(x - \beta_i(y))$ be a curve in $\mathbb{C}^2$. To construct the tree of f, we first draw a vertical line, called the *main trunk* and write next to it the multiplicity of f. Let $b_1 = min\{O(\beta_i - \beta_j\})$. We now draw a horizontal line, called a *bar*, touching the trunk and mark it with the number b_1, which is the *height* of the bar. Now we divide the roots into groups which have order of contact greater than b_1. For each such group, we draw a vertical line, called a *trunk*, and write next to it the number of members in the group. This is the *multiplicity* of the trunk. If the trunk is of multiplicity 1, it is called a *twig*, and we will omit its multiplicity. For each trunk T with multiplicity greater than 1, let $b_T = min\{O(\beta_i - \beta_j|\beta_i, \beta_j \in T\})$. We then draw a bar at the top of the trunk, and denote it b_T. This procedure is repeated until all trunks are of multiplicity 1.

The result of this is called the *tree model* of f and is denoted $M(f)$.

Remark 3.2.1. *Clearly we have a one-to-one correspondence between the bars of $M(f)$ and the trunks of multiplicity greater than 1 obtained by identifying the bar b_i with the trunk it is sitting on.*

Definition 3.2.2. *The diagram $M^*(f)$ is constructed from that of $M(f)$ by: for each bar of $M(f)$ with k trunks, adding $k - 1$ dashed twigs to $M(f)$.*

Proposition 3.2.3. *There is a one-to-one correspondence between the twigs of $M(f_x)$ and the dashed twigs of $M^*(f)$.*

Proof. First some notation:
we will label the bars $b_1 \dots b_m$, and the trunks under each bar b_i as t_i. We will call τ_i the number of trunks (including those of multiplicity 1 originating on b_i (see figure 3.1).

So we have a total of $1 + \sum \tau_i$ trunks and twigs (where the 1 is due to the bottom trunk).

Now from theorem 3.1.3 there are $\tau_i - 1$ roots of f_x on each bar b_i. So the total

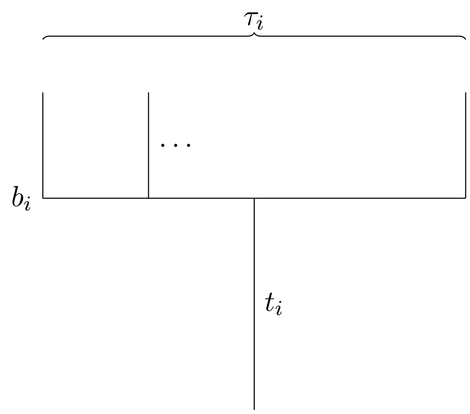

Figure 3.1: Tree model

number of roots of f_x is:

$$\begin{aligned}\sum_i(\tau_i-1) &= (\sum_i \tau_i) - O(\{b_i\})\\ &= \sum_{i\neq 1} t_i + \text{ number of twigs of M(f)} - O(\{b_i\})\\ &= mult(f) + \sum_i t_i - 1 - \sum_i t_i\\ &= mult(f) - 1\end{aligned}$$

From section 1.5, the Newton polygon of f_x is just the Newton polygon of f shifted to the left, and so $mult(f_x) = mult(f) - 1$.

□

Remark 3.2.4. *Note that the bars of f_x are not determined by the graph of $M^*(f)$. (See the third example in Appendix B.2).*

We can link the tree model to the Puiseux characteristic as follows:

Corollary 3.2.5. *If $f(x, y)$ is irreducible, then on the tree diagram of f, the height of the bars b_i is equal to $\frac{r_i}{m_1.m_2...m_i}$, and the number of trunks off this bar will be equal to m_i, where (r_i, m_i) are the Puiseux pairs of f.*

Proof. From the corollaries to lemma 2.2.11 we have that when sliding towards a root of f there will always be one relevant edge of the form $E = y^a(x^m + wy^n)^k$, and the roots are all conjugate to each other. Now consider sliding towards one of these roots: If $m = 1$, then we have only 1 root of the associated polynomial, and so the roots will not split. If $m \neq 1$, then we have m roots of the associated polynomial, and so the roots will split on the tree diagram. The number of additional roots will be equal to m. In addition, the new term in this root will have exponent $\frac{n}{m}$. This can be re-written to give the result. □

Note that this only holds if f is irreducible. For example, if $f(x, y) = (x^2 - y^3)^2 + y^7$, then the characteristic of all the roots of f is $(2; 3)$, which gives Puiseux pairs $(3, 2)$.

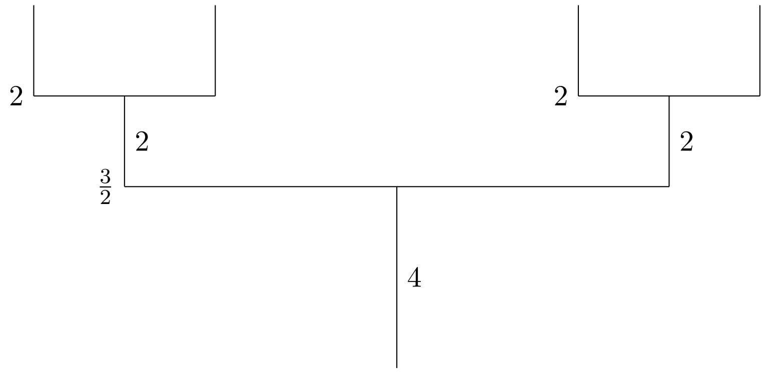

Figure 3.2: $f(x, y) = (x^2 - y^3)^2 + y^7$

Reading off the tree-diagram, the height of the bars b_i are: $b_1 = \frac{3}{2}, b_2 = 2$, and the number of trunks off the bars is $2, 2$. Hence $m_1 = m_2 = 2$, and $\frac{r_1}{2} = \frac{3}{2}$, $\frac{r_2}{4} = 2$. Hence we get $(3, 2), (8, 4)$. Note that this cannot be a Puiseux characteristic, as 4 divides 8.

Chapter 4

Lojasiewicz Exponent

Let $f(x,y) = \sum_{i,j} a_{ij}x^i y^j$ be a convergent power series with an isolated singularity at the origin. Let λ be an analytic arc:

$$\lambda(t) = (b_1 t^{m_1} + b_2 t^{m_2} + \dots, c_1 t^{n_1} + c_2 t^{n_2} + \dots).$$

Define $\ell(\lambda)$ by: for $|t|$ sufficiently small:

$$\|grad\ f(\lambda(t))\| \sim \|\lambda(t)\|^{\ell(\lambda)},$$

where $A \sim B$ means that there are positive constants $c_1 < c_2$ such that $c_1 < A/B < c_2$.

Definition 4.0.6 (Lojasiewicz Exponent)**.**

$$\mathcal{L}(f) := \underset{\lambda}{sup}\ \ell(\lambda)$$

4.1 The Lojasiewicz Exponent Using The Kuo-Lu Theorem

In this section, we will use the Kuo-Lu theorem to prove a result to the above about the Lojasiewicz exponent.

Write $f = \prod(x - \beta_i)$, the Puiseux factorisation of f. Let $c_i = \underset{j \neq i}{Max}\ O(\beta_i - \beta_j)$ and define g_{c_i} to be the generic perturbation of β_i of degree c_i.

Assume that no two β_i's are identical.

Theorem 4.1.1. *Let $l_i = O(f(g_{c_i}(\beta_i), y))$ and $\alpha = \underset{i}{Max}(l_i - 1)$. Then $\alpha = \mathcal{L}(f)$.*

Definition 4.1.2 (Minimal Root)**.** *β_i is a minimal root if for every root β_e such that $O(\beta_i - \beta_e) = c_i$ (recall $c_i = max_j O(\beta_i - \beta_j)$) there is no root β_h such that: $O(\beta_i - \beta_e) < O(\beta_h - \beta_e)$.*

Corollary 4.1.3. *In theorem 4.1.1, only indices where β_i are minimal need to be considered.*

We will now prove the main theorem of this section:

Proof. Let γ be a Newton Puiseux root of f_x, and let β_i be a Newton-Puiseux root of f such that $O(\beta_i - \gamma) = \max_j \big(O(\gamma - \beta_j)\big)$. Write $g_{c_i}(\beta_i)$ as the generic perturbation of β_i at $O(\beta_i - \gamma)$. Write $h_0(g_{c_i}(\beta_i))$ for the exponent of the lowest dot on $x = 0$ in the Newton polygon of f relative to $g_{c_i}(\beta_i)$. We want to show that $\ell(\gamma) = O(f(g_{c_i}(\beta_i), y)$.

Now $O(f(g_{c_i}(\beta_i), y) = h_0(g_{c_i}(\beta))$. We claim that $h_0(g_{c_i}(\beta)) \geq h_0(\gamma)$. Note that as γ is a Newton-Puiseux root of f_x, and f has no roots of multiplicity greater than 1, $h_0(\gamma)$ is non-zero.

To prove that $h_0(g_{c_i}(\beta_i)) \geq h_0(\gamma)$, assume otherwise. So there is a $\gamma_k = g_{c_i}(\beta_i)) + cy^\epsilon$ such that $h_0(\gamma_k) > h_0(g_{c_i}(\beta))$. Hence the term in $y^{h_0(g_{c_i}(\beta_i))}$ in $f(x + \gamma_k, y)$ is zero. As this term is zero, we get that c satisfies the associated polynomial for the highest edge, and that ϵ is the angle of the highest edge. Hence γ_k is actually a sliding towards a Newton-Puiseux root of f. Hence there is a root β_j which agrees with γ up to order ϵ. This is a contradiction. So $h_0(g_{c_i}(\beta_i)) \geq h_0(\gamma)$. As $h_0(\gamma) \neq 0$, by sliding we get that $h_0(g_{c_i}(\beta_i)) \leq h_0(\gamma)$. hence we have that $h_0(g_{c_i}(\beta_i)) = h_0(\gamma)$.

So for every root γ of f_x there is a root β_i of f such that $\ell(\gamma) = O(f(g_{c_i}(\beta_i), y)$. By theorem 4.2.1, $\mathcal{L} = max(\ell(\gamma))$. Hence

$$max_{\beta_i} O(f(g_{c_i}(\beta_i), y)) = \mathcal{L}(f)$$

as required. □

Proof of corollary. Let β_i be a root which is not minimal. So there exists β_e, β_h such that $O(\beta_i - \beta_e) = c_i < O(\beta_h - \beta_e)$. Hence we can write $\beta_e = g_{c_i}(\beta_i) + H.O.T.$

Now let β_j be such that $O(\beta_i - \beta_j) < c_i$. As $\beta_e = g_{c_i}(\beta_i) + H.O.T$, we get that $O(\beta_e - \beta_j) = O(\beta_i - \beta_j)$.

Let β_k be such that $O(\beta_i - \beta_k) = c_i$. As $\beta_e = g_{c_i}(\beta_i) + H.O.T$, we get that $O(\beta_e - \beta_k) \geq c_i = O(\beta_i - \beta_k)$. Hence for any root j:

$$O(\beta_e - \beta_j) \geq O(\beta_i - \beta_j).$$

Hence as $O(\beta_h - \beta_e) > O(\beta_i - \beta_e)$

$$\sum_j O(\beta_e - \beta_j) > \sum_j O(\beta_i - \beta_j)\ .$$

Therefore $l_e > l_i$ and so we only need to consider roots that are minimal. □

The tree model can be used to give an intuitive and visual method to evaluate L using the Kuo-Lu theorem (Theorem C of Kuo-Lu [4]).

Theorem 4.1.4. *Let τ_i be the multiplicity of a trunk of the tree model of f. Let b_i be the height of the bar on top of this trunk. Then if we trace up the tree-model to a Newton-Puiseux root of f,*

$$l_i = b_k \tau_k + b_{k-1}(\tau_{k-1} - \tau_k) + \cdots + b_1(\tau_1 - \tau_2)$$

Remark 4.1.5. *This can be easily rearranged to the following:*

$$l_i = \tau_k(b_k - b_{k-1}) + \tau_{k-1}(b_{k-1} - b_{k-2}) + \tau_{k-1}(b_{k-1} - b_{k-2}) + \ldots\ .$$

Proof of theorem 4.1.4. Consider

$$f(g_{c_i}(v), v) = \prod_j (g_{c_i}(\beta_i) - \beta_j(v))$$

Now $O(f(g_{c_i}(v), v)) = O(\prod_j (g_{c_i}(\beta_i) - \beta_j(v))) = \sigma_j O(g_{c_i}(\beta_i) - \beta_j(v))$.

By definition, $O(g_{c_i}(\beta_i) - \beta_j(v)) = b_h$, where b_h is the bar at which β_i and β_j split. Clearly the number of roots which split from β_i at height b_j is equal to $\tau_j - \tau_{j+1}$. Hence we have $O(f(g_{c_i}(v), v)) = \sum_j b_{j-1}(\tau_{j-1} - \tau_j)$. □

4.2 The Lojasiewicz Exponent Using Polar Curves

Theorem 4.2.1. *There is a Newton-Puiseux root γ of $f_x = 0$ such that $\mathcal{L}(f) = \ell(\gamma)$.*

The proof of this requires a technical lemma.

Lemma 4.2.2. *In the Newton diagram of f relative to an analytic arc λ, let $(0, h_0)$ and $(1, h_1)$ be the lowest Newton dots on $X = 0$ and $X = 1$. Then if $\theta_1 \geq 1$,*

$$\ell(\lambda) = min(h_0 - 1, h_1),$$

Proof. Let $f(x, y) = \sum a_{ij} x^i y^j$ be mini-regular in x with multiplicity k, and have an isolated singularity at 0. Let λ be an analytic arc parametrised by $\lambda(t) = \big(x(t), y(t)\big)$. Now

$$\|grad\ f(x(t), y(t))\| \sim \|(x(t), y(t))\|^{\ell(\lambda)}.$$

and hence

$$\ell(\lambda) = min\Big(O\big(\partial f/\partial x(x(t), y(t))\big), O\big(\partial f/\partial y(x(t), y(t))\big)\Big).$$

We will write

$$\lambda : x = \alpha(y) = a_1 y^{\theta_1} + a_1 y^{\theta_2} + \ldots, \quad y = y,$$

with $\theta_1 < \theta_2 < \ldots$.

Define $F(X, Y) = f(X + \alpha(Y), Y)$, and let h_0, h_1 be the lowest order terms in x^0 and x^1 respectively.

As we have assumed f to be mini-regular in x, we may ignore all curves with $\theta_1 < 1$, so assume $\theta_1 \geq 1$:

$$f_x = \sum_{i,j} i a_{ij} x^{i-1} y^j$$

$$\text{and so} \qquad f_x(\lambda) = \sum_{i,j} i a_{ij} \alpha^{i-1} y^j$$

Also: $\qquad F(X, Y) = f(X + \alpha(Y), Y) = \sum_{i,j} a_{ij}(X + \alpha(Y))^i Y^j$

Hence the terms in x^1 in $F(X, Y)$ are given by:

$$\sum_{i,j} a_{ij} \binom{i}{1} X(\alpha(Y))^{i-1} Y^j = f_x$$

Hence the order of f_x is equal to h_1

Now for f_y:

$$F_Y = \sum_{i,j} j a_{ij}(X+\alpha(Y))^i Y^{j-1} + \sum_{i,j} j a_{ij}(X+\alpha(Y))^{i-1} Y^j \alpha'(Y) \quad .$$

Along $X = 0$:

$$\begin{aligned} F_Y(0,Y) &= \sum_{i,j} j a_{ij}(\alpha(Y))^i Y^{j-1} + \sum_{i,j} j a_{ij}(\alpha(Y))^{i-1} Y^j \alpha'(Y) \\ &= \frac{d}{dY}\Big(\sum_{i,j} a_{ij}\alpha^i Y^j\Big) \end{aligned}$$

Hence along $x = 0$ the order of F_Y is $h_0 - 1$. Now

$$\begin{aligned} f_y(\lambda) &= \sum_{i,j} j a_{ij}(\alpha)^i y^{j-1} \\ &= F_Y(0,Y) - \sum_{i,j} j a_{ij}(\alpha(y))^{i-1} y^j \alpha'(y) \\ &= F_Y(0,Y) - \alpha'(y) f_x(\lambda) \end{aligned}$$

If $O(F_Y|_{(0,Y)} \neq O(\alpha'(y)f_x)$, then

$$\begin{aligned} O(f_y) &= min(O(F_Y), O(\alpha'(y)f_x)) \\ &= min(h_0 - 1, h_1 + \theta_1 - 1) \end{aligned}$$

Recall that $\theta_1 \geq 1$. Hence

$$\begin{aligned} \ell(\lambda) &= min(O(f_y), O(f_x)) \\ &= min(min(h_0 - 1, h_1 + \theta_1 - 1), h_1) \\ &= min(h_0 - 1, h_1). \end{aligned}$$

Otherwise, $O(F_Y) = O(\alpha'(y)f_x)$. Note that from this we get that $h_0 - 1 = O(F_Y) \geq O(f_x) = h_1$. In particular, $min(h_0 - 1, h_1) = h_1$.

Now $f_y(\lambda) = F_Y(x = 0) - \alpha'(y)f_x(\lambda)$, and so we either have $F_Y = \alpha'(y)f_x$, which gives us $f_y(\lambda) = 0$, or $O(f_y) \geq min(O(F_Y), O(\alpha'(y)f_x)) = O(\alpha'(y)f_x)$.

If $f_y(\lambda) = 0$:

$$\|grad\ f(x,y)\| = |f_x(\lambda)|^2 + |f_y(\lambda)|^2 = |f_x(\lambda)|^2 \quad ,$$

and so $\ell(\lambda) = O(f_x) = h_1$.

Otherwise, if $O(f_y) \geq O(\alpha'(y)f_x)$, we have that $O(f_y) \geq O(f_x)$ (recall that O $\alpha'(y) = \theta - 1 \geq 0$) and so

$$\ell(\lambda) = min(O(f_x, y_y)) = O(f_x) = h_1 \quad .$$

Hence if $O(F_Y) = O(\alpha'(y)f_x)$, $\ell(\lambda) = h_1 = min(h_0 - 1, h_1)$. Hence if $\theta_1 \geq 1$, $\ell(\lambda) = min(h_0 - 1, h_1)$. □

Proof of theorem 4.2.1. Let $\lambda = \sum_i c_i y_{\phi_i}$ be an analytic arc with $\phi_i \geq 1$. Write $F(X,Y) = f(X + \lambda(Y), Y)$. Let $(0, h_0)$ and $(1, h_1)$ be the lowest Newton dots on $X = 0$ and $X = 1$ respectively. Let $\lambda + cy^\theta$ be an arc found by sliding along a relevant edge of $\mathbb{P}(\partial f/\partial x, \lambda)$. (Note that this means that $\theta > \phi_i$ for all i.)

First recall that $F(X + cY^\theta, Y) = f(x + \lambda + cy^\theta, y)$ and hence the Newton diagram of F relative to cY^θ is equivalent to the Newton diagram of F relative to $\lambda + cY^\theta$.

In the Newton diagram of $F(x,y) = f(x + \lambda(y), y)$, let $(0, h_0)$ and $(1, h_1)$ be the lowest Newton dots on $x = 0$ and $x = 1$ respectively. Also, in the Newton diagram of F relative to cy^θ, let $(0, \eta_0)$ and $(1, \eta_1)$ be the lowest Newton dots on $x = 0$ and $x = 1$ respectively. We want to show that $min(\eta_0 - 1, \eta_1) \geq min(h_0 - 1, h_1)$.

Consider the Newton diagram of f_x relative to λ. From section 1.5 the newton diagram of f_x can be obtained by shifting the Newton dots for f to the left by 1, and removing all dots to the left of the y-axis. In particular, the lowest dot on $x = 0$ in $\mathbb{P}(f_x)$ will have the same y-value as the lowest dot on $x = 1$ in $\mathbb{P}(f)$. Now by sliding, the lowest dot on $x = 0$ in $\mathbb{P}(f_x, \lambda + cy^\theta)$ will be higher than the lowest dot on $x = 0$ in $\mathbb{P}(f_x, \lambda)$, and hence we have either $\eta_1 > h_1$, or η_1 is not defined (in which case $\lambda + cy^\theta$ is a root of f_x from theorem 1.2.2).

Let E_a be the edge of $\mathbb{P}(f, \lambda)$ which contains the point $(1, h_1)$. From proposition 1.3.3, all of the dots of $\mathbb{P}(f_x, \lambda + cy^\theta)$ will be on or above the line extending E_a, or specifically the line through $(1, h_1)$ with gradient $-tan(\theta)$. It can be easily seen that the dots of $\mathbb{P}(f, \lambda + cy^\theta)$ which are generated by sliding will be on or above the line through $(1, h_1)$ with gradient $-tan(\theta)$. Hence we either have $\eta_0 \geq h_1 + \theta$, $\eta_0 = h_0$ or there are no dots on $x = 0$. As $\theta \geq 1$, we have either $\eta_0 \geq min(h_0, h_1 + 1)$, or η_0 is not defined. If η_0 is undefined, then $\lambda + cy^\theta$ is a root of f.

As f has no roots of multiplicity greater than 1, one of η_0 and η_1 must be defined, and so we have two cases to consider:
If η_1 is undefined, $\eta_0 \geq min(h_0, h_1 + 1)$. Hence $min(\eta_0 - 1, \eta_1) \geq min(h_0 - 1, h_1)$, and as $\lambda + cy^\theta$ is a root of f_x we have finished.

Now assume η_0 is undefined. In this case $\eta_1 > h_1$. Consider h_0: If h_0 is undefined then $min(\eta_0 - 1, \eta_1) \geq min(h_0 - 1, h_1)$. Otherwise, assume h_0 is defined. Then the dot at h_0 is cancelled by sliding from $\mathbb{P}(f, \lambda)$ along cy^θ. Hence we must have $h_0 \geq h_1 + \theta$, and so $min(h_0 - 1, h_1) = h_1 < \eta_1$. Hence $min(\eta_0 - 1, \eta_1) \geq min(h_0 - 1, h_1)$.

Hence $\ell(\lambda + cy^\theta) \geq \ell(\lambda)$. Hence sliding along $\partial f/\partial x$ does not decrease $\ell(\lambda)$. Hence $\mathcal{L}$ is obtained along a polar curve. □

Remark 4.2.3. *Recall that in section 1.5 it was shown that if γ is a polar curve of f, the Newton diagram of f relative to γ will not have a dot at h_1, and so if we know γ, we only have to compute $h_0(\mathbb{P}(f, \gamma))$ to find the value of $\ell(\gamma)$.*

Remark 4.2.4. *The Lojasiewicz exponent need not be only obtained along a polar curve. Also, the Lojasiewicz exponent will not necessarily be obtained on every polar curve.*

Example 4.2.5. *Consider the function $f(x,y) = x^3 - x^2y + y^5$. Differentiating gives $f_x = 3x^2 - 2xy$. This has two Newton-Puiseux roots: $\gamma_1 : x = 0$ and $\gamma_2 : x = \frac{2}{3}y$.*

To evaluate $\ell(\gamma)$, consider $f_y(\gamma, y)$. $f_y = -x^2 + 5y^4$. Hence along $x = 0$, $\|gradf\|^2 = |f_x|^2 + |f_y|^2 = 25y^8$.
Hence $\ell(\gamma_1) = O(\|gradf\|) = 4$.

Similarly, along $x = y$, $\|gradf\|^2 = |f_x|^2 + |f_y|^2 = (-(2/3y)^2 + 5y^4)^2$. Hence $\ell(\gamma_2) = O(\|gradf\|) = 2$.

So the Lojasiewicz exponent of f is only obtained along one polar curve of f.

Note that this method requires sliding towards a root of $\partial f/\partial x = 0$, possibly indefinitely. We can however ignore terms of high degree in the expansion of λ.

Proposition 4.2.6. *Let λ_i be the sum of the first i terms of λ. Let h_0 and h_1 be the height of x^0 and x^1 respectively on the Newton diagram $\mathbb{P}(f, \lambda_i)$. If $h_1 + \theta_i > h_0$ then $\ell(\lambda_i) = \ell(\lambda)$.*

Proof. Let λ_i be a polynomial in $y^{1/N}$ such that $h_1 + \theta_{i+1} > h_0$. We want to show that the lowest dot on $x = 0$ in $\mathbb{P}(f, \lambda_{i+1})$ is at h_0. Recall from section 1.5, that as there is a dot on $x = 1$ in $\mathbb{P}(f, \lambda_i)$, the Newton polygon $\mathbb{P}(f, \lambda_i)$ will be equal to the Newton polygon $\mathbb{P}(f_x, \lambda_i)$ shifted one unit to the right. Now from proposition 1.3.3, all dots in $\mathbb{P}(f_x, \lambda_i)$ are on or above the line of gradient $-tan(\theta_i)$ which passes through $(0, h_1)$. As the exponents of x are integers, we have that all of the dots of $\mathbb{P}(f, \lambda_i)$ not on $x = 0$ are on or above the line of gradient $-tan(\theta_i)$ which passes through $(1, h_1)$.

Now consider the dots on $x = 0$ in $\mathbb{P}(f, \lambda_{i+1})$. These will be of the form $(0, q+p\theta_{i+1})$, where (p, q) is a dot of $\mathbb{P}(f, \lambda_i)$. If $p > 0$, (p, q) will be on or above the line of gradient $-tan(\theta_i)$ which passes through $(1, h_1)$. In particular, $q + (p-1)\theta_i \geq h_1$. As $\theta_{i+1} > \theta_i$, we have

$$\begin{aligned} q + p\theta_{i+1} &\geq q + (p-1)\theta_i + \theta_{i+1} \\ &\geq h_1 + \theta_{i+1} \\ &> h_0 \ . \end{aligned}$$

Now when $p = 0$, there is a dot $(0, h_0)$. Hence this is the lowest dot on $x = 0$ in $\mathbb{P}(f, \lambda_{i+1})$.

So, by induction, in the Newton polygon $\mathbb{P}(f, \lambda)$, the lowest dot on $x = 0$ will be at h_0. But λ is a Newton-Puiseux root of f_x. Hence $\mathbb{P}(f, \lambda)$ will not have a dot on $x = 1$, and so $\ell(\lambda)$ will be h_0. (Note that h_0 must exist, as otherwise λ would be a root of f of multiplicity greater than one.) □

Remark 4.2.7. *If β is a Newton-Puiseux root of f of multiplicity greater than 1, then $\ell(\beta) = 0$.*

Proof. Let $f(x, y) = (x - \beta(y))^2 g(x, y)$. Then:

$$\begin{aligned} \frac{\partial f}{\partial x} &= 2(x - \beta(y))g(x, y) + (x - \beta(y))^2 \frac{\partial f}{\partial x} g(x, y) \\ \frac{\partial f}{\partial y} &= 2\beta'(y)(x - \beta(y))g(x, y) + (x - \beta(y))^2 \frac{\partial f}{\partial y} g(x, y) \end{aligned}$$

Hence along the curve $x = \beta(y)$, $\partial f/\partial x = 0 = \partial f/\partial y$. □

For example: $f(x, y) = (x^2 - y^3)^2$.

$$\frac{\partial f}{\partial x} = 4x(x^3 - y^3)$$

This has three Newton-Puiseux roots: $x = 0, \ \pm y^{3/2}$. For $\lambda = \pm y^{3/2}$, consider $\mathbb{P}(f, \lambda)$:

$$\begin{aligned} f(x+\lambda(y), y) &= (x^2 \pm 2xy^{3/2} + y^3 - y^3)^2 \\ &= x^4 \pm 4x^3y^{3/2} + 4x^2y^3. \end{aligned}$$

This has no Newton dots on the line $x = 1$.

Example 4.2.8. *Let* $f(x, y) = (x - y + y^2)^2 = x^2 - 2xy + y^2 + 2xy^2 - 2y^3 + y^4$.

$$\frac{\partial f}{\partial x} = 2x - 2y + 2y^2.$$

The Newton diagram for this is shown in figure 4.1

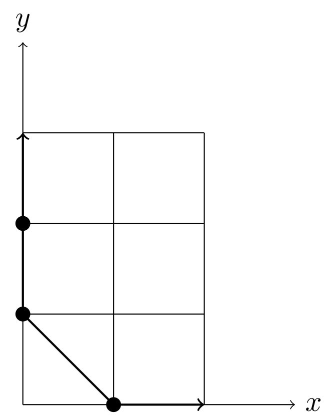

Figure 4.1: Newton polygon of $f_x = 2x - 2y + 2y^2$

The only edge for this is E, *with* $tan\theta = 1$, *and associated polynomial:* $\mathcal{E}(z) = 2z - 2$. *Hence* $\beta_1 = y$. *Clearly the Newton-Puiseux root for* $\partial f/\partial x$ *is* $\beta = y - y^2$.

To calculate $\ell(\beta_1)$:

$$\begin{aligned} f(x + \beta_1, y) &= (x + y - y + y^2)^2 \\ &= x^2 + 2xy^2 + y^4 \end{aligned}$$

Hence $h_0 = 4$ *and* $h_1 = 2$. *Hence* $\ell(\beta_1) = 2$.

Note that for $\mathbb{P}(f, \beta)$ *both* h_0 *and* h_1 *are undefined. To calculate* $\ell(\beta)$:

$$\frac{\partial f}{\partial x} = 2(x - y + y^2)$$
$$\frac{\partial f}{\partial y} = 2(2y - 1)(x - y + y^2)$$

Substituting $x = y - y^2$ *gives* $\partial f/\partial x = 0 = \partial f/\partial y$.

Example 4.2.9. *We will give an example where sliding towards a polar curve reduces the value of* h_0.

Let $f(x,y) = (x+y^3)(x^2-3y^2) = x^3 - 3xy^2 + x^2y^3 - 3y^5$, *and let* $\lambda = y$. *So* $h_0 = 5, h_1 = 2$.

Consider $\mathbb{P}(f,0)$ *and* $\mathbb{P}(f,\lambda)$*:*

$$f(x+\lambda, y) = x^3 + 3x^2y + y^3 + x^2y^3 + 2xy^4 \quad .$$

Hence $h'_0 = 3, h'_1 = 4$. *Note that* $min(h_0 - 1, h_1) = 2 = min(h'_0 - 1, h'_1)$, *as stated by theorem 4.2.1*

Appendix A

Irreducibility

A.1 Hensel's Lemma

Note that although we will not show this, the following proofs can be modified to also work for analytic functions.

Definition A.1.1 (Irreducible)**.** *A power series $f(x, y)$ is irreducible if it cannot be expressed in the form $f = g_1 g_2$, where g_1 and g_2 are power series of multiplicity greater than 0. (Note that we are referring to power series with integer exponents, not fractional power series.) A function is reducible if it can be expressed in the form $f = g_1 g_2$.*

Definition A.1.2 (Unit)**.** *A unit is an element of a ring which has a multiplicative inverse.*

Remark A.1.3. *It can be easily shown that an element of the domain of power series is a unit if and only if it is of the form $a_0 + a_1 x + \dots$, where a_0 is non-zero.*

Remark A.1.4. *All power series of order greater than 1 are reducible over the field of fractional power series.*

It is sometimes important to know whether a given power series is reducible. In this section, we give some criterion of increasing strength for the reducibility of a power series $f(x, y)$.

Throughout we will write f in the form $f = H_i + H_{i+1} + H_{i+2} + \dots$, where H_j are the terms of order j. Specifically,

$$H_i = \sum_{j=0}^{i} c_{j,i-j} x^j y^{i-j} \quad .$$

Example A.1.5. $f(x, y) = x^2 - y^3 \quad \big(= (x - y^{3/2})(x + y^{3/2}) \big)$ *is irreducible over the field of power series, but is reducible over the field of fractional power series.*

We will first give a simple example of the process which will be used to prove the theorems in this section.

Example A.1.6. *Consider a function f of the form $f(x,y) = xy + H_3(x,y) + H_4(x,y) + \dots$. We want to show that f is reducible. To do this we must find two power series: $g_1 = x + L_2(x,y) + L_3(x,y) + \dots$ and $g_2 = y + K_2(x,y) + K_3(x,y) + \dots$ (L_i and K_i are the terms of order i) such that $g_1 g_2 = f$. Expanding gives:*

$$\begin{aligned}(x + L_2 + L_3 + \dots)(y + K_2 + K_3 + \dots) &= xy + \big(yL_2 + xK_2\big) + \big(yL_3 + L_2K_2 + xK_3\big) + \dots \\ &= xy + \sum_{i=2} \big(yL_i + xK_i + \sum_{j=2}^{i-2} L_jK_{i-j}\big) + \dots\end{aligned}$$

Let $H_3 = a_{3,0}x^3 + a_{2,1}x^2y + a_{1,2}xy^2 + a_{0,3}y^3$. We can define $L_2 = a_{0,3}y$, $K_2 = a_{3,0}x^2 + a_{2,1}xy + a_{1,2}y$, which gives $H_3 = yL_2 + xK_2$. Note that there are infinitely many other possibilities for K_2 and L_2.

Now consider $H_4' = H_4 - K_2L_2$. Write $H_4' = a_{3,0}x^3 + a_{2,1}x^2y + a_{1,2}xy^2 + a_{0,3}y^3$. We can define $L_2 = a_{0,3}y^2$, $K_2 = a_{3,0}x^2 + a_{2,1}xy + a_{1,2}y^2$, which gives $H_4' = yL_3 + xK_3$, and hence:

$$H_4 = H_4' + K_2L_2 = yL_3 + L_2K_2 + xK_3 \quad .$$

Note again that there are infinitely many other possibilities for K_3 and L_3.

We can inductively define L_i and K_i as follows: Let $H_i' = H_i - \sum_{j=2}^{i-2} L_jK_{i-j}$, and write $H_i' = \sum_{j=0}^{i} a_{j,i-j}x^jy^{i-j}$. Now we will define $L_i = a_{0,i}y^{i-1}$ and $K_i = \sum_{j=1}^{i} a_{j,i-j}x^{j-1}y^{i-j}$. Clearly $yL_i + xK_i = H_i'$, and so:

$$H_i = H_i' + \sum_{j=2}^{i-2} L_jK_{i-j} = yL_i + xK_i + \sum_{j=2}^{i-2} L_jK_{i-j} \quad .$$

This process can be easily extended to power series such as $x^2y + H_4 + H_5 + \dots$, and indeed to all series of the form $f(x,y) = x^iy^j + H_{i+j+1} + H_{i+j+2} + \dots$, $i,j \geq 1$. (Morally because every element of H_{i+j+1} must be divisible by either x^i or y^j.)

Example A.1.7. *Let $f(x,y) = (x+y)(x-y) + H_3 + H_4 + \dots$, and define $g_1 = (x+y) + L_2 + L_3 + \dots$ and $g_2 = (x-y) + K_2 + K_3 + \dots$.*

$$\begin{aligned}g_1g_2 &= \big((x+y) + L_2 + L_3 \dots\ \big)\big((x-y) + K_2 + K_3 + \dots\ \big) \\ &= (x+y)(x-y) + \big((x+y)K_2 + (x-y)L_2\big) + \dots + \big(yL_i + xK_i + \sum_{j=2}^{i-2} L_jK_{i-j}\big) + \dots\end{aligned}$$

Now let $H_i' = H_i - \sum_{j=2}^{i-2} L_jK_{i-j}$, and write $H_i' = \sum_{j=0}^{i} a_jx^jy^{i-j}$. We want to find L_{i-1} and K_{i-1} such that $H_i = (x-y)L_{i-1} + (x+y)K_{i-1}$. Write

$$L_{i-1} = \sum_{j=0}^{i-1} b_jx^jy^{i-j-1}$$

$$K_{i-1} = \sum_{j=0}^{i-1} c_jx^jy^{i-j-1}$$

Expanding $(x-y)L_{i-1}+(x+y)K_{i-1}$ and equating coefficients give us a set of linear equations to solve:

$$\begin{aligned}
a_0 &= b_0 + c_0 \\
a_1 &= c_0 - b_0 + b_1 + c_1 \\
a_2 &= c_1 - b_1 + b_2 + c_2 \\
\vdots &\ \ \vdots \qquad \vdots \\
a_{i-1} &= c_{i-2} - b_{i-2} + c_{i-1} + b_{i-1} \\
a_i &= c_{i-1} - b_{i-1}
\end{aligned}$$

There are $i+1$ equations in $2(i-1)$ unknowns, with $i \geq 3$, and so we have infinitely many solutions. For example, if we set $b_j = 0$ and $c_j = \sum_{n=1}^{i-j}(-1)^{n+1}a_{n+j}$ for $j \geq 1$, then for $k \geq 2$, $a_j = c_{j-1} - b_{j-1} + c_j + b_j$, and b_0 and c_0 must satisfy $b_0 + c_0 = a_0$, and:

$$\begin{aligned}
c_0 - b_0 + c_1 + b_1 &= a_1 \\
c_0 - b_0 &= a_1 - c_1 \\
&= a_1 - \big(\sum_{n=1}^{i-1}(-1)^{n+1}a_{n+j}\big) \\
&= a_1 - a_2 + a_3 - a_4 + \dots \\
&= \sum_{n=0}^{i-1}(-1)^n a_{n+1}
\end{aligned}$$

Clearly the following values satisfy this:

$$\begin{aligned}
b_0 &= \frac{1}{2}\big(a_0 - \sum_{n=0}^{i-1}(-1)^n a_{n+1}\big) \\
c_0 &= \frac{1}{2}\big(a_0 + \sum_{n=0}^{i-1}(-1)^n a_{n+1}\big)
\end{aligned}$$

Hence, by induction, f is reducible.

If $f(x,y)$ has a first term which does not split into powers of x and y, but is reducible into the product of powers of two linearly independent functions of the form $ax+by$, we can apply a change of variables to use the above theorem. Specifically we have the following proposition:

Proposition A.1.8. *Let $f(x,y) = (a_1x+b_1y)^{m_1}(a_2x+b_2y)^{m_2} + H_m + H_{m+1} + \dots$. If $a_1b_2 - a_2b_1 \neq 0$, f is reducible.*

Proof. Let $X = a_1x+b_1y$, and $Y = a_2x+b_2y$. As the matrix defined by $(a_1,a_2;b_1,b_2)$ has non-zero determinant, it is invertible and so we can find $(c_1,c_2;d_1,d_2)$ with $c_1d_2-c_2d_1 \neq 0$ such that $x = c_1X+d_1Y$, and $y = c_2X+d_2Y$. Hence we can write:

$$H_i(x,y) = H_i(c_1X+d_1Y, c_2X+d_2Y) = H_i'(X,Y)$$

Note that the order of H'_i is i. Hence $f = X^{m_1}Y^{m_2} + H'_m(X,Y) + H'_{m+1}(X,Y) + \dots$. From the above example, we can reduce this if $m_1, m_2 \geq 1$. So

$$f = \Big(X^{m_1} + L'_{m_1+1}(X,Y) + \dots\Big)\Big(Y^{m_2} + K'_{m_2+1}(X,Y) + \dots\Big)$$

We can now use the substitution $X = a_1x + b_1y$, $Y = a_2x + b_2y$ to get:

$$f = \Big((a_1x+b_1y)^{m_1}+L'_{m_1+1}(x,y)+L'_{m_1+2}(x,y)+\dots\Big)\Big((a_2x+b_2y)^{m_2}++K'_{m_2+1}(x,y)+\dots\Big)$$

as required. □

In fact, we can generalise this to the case when there are finitely many linearly independent functions of the form $ax + by$.

Proposition A.1.9. *Let $f(x,y) = (a_1x+b_1y)^{m_1}(a_2x+b_2y)^{m_2}\dots(a_kx+b_ky)^{m_k}+H_{m+1}+H_{m+2}+\dots$. If there exists $i \neq j$, such that $a_ib_j - b_ia_j \neq 0$, then f is reducible.*

Proof. First, by applying an appropriate linear transformation if neccessary, we can write:

$$f(x,y) = x^{m_1}y^{m_2}(a_3x + b_3y)^{m_3}\dots(a_kx + b_ky)^{m_k} + H_{m+1} + H_{m+2} + \dots \quad .$$

In order to show that this is reducible, we have to find a series of polynomials $L_1, L_2, \dots$ and $K_1, K_2, \dots$ such that

$$f(x,y) = \Big(x^{m_1}+L_1(x,y)+L_2(x,y)+\dots\Big)\Big(y^{m_2}(a_3x+b_3y)^{m_3}\dots(a_kx+b_ky)^{m_k}+K_1(x,y)+\dots\Big) \quad .$$

Consider the terms of order $m + i$ in the expansion of $\Big(x^{m_1} + L_1(x,y) + L_2(x,y) + \dots\Big)\Big(y^{m_2}(a_3x+b_3y)^{m_3}\dots(a_kx+b_ky)^{m_k} + K_1(x,y) + K_2(x,y) + \dots\Big)$: we want to define these so that this sum is H_i. Write $f_i(x,y) = \Big(x^{m_1}+L_1(x,y)++\dots+L_i(x,y)\Big)\Big(y^{m_2}(a_3x+b_3y)^{m_3}\dots(a_kx+b_ky)^{m_k}+K_1(x,y)+\dots+K_i(x,y)\Big)$. We will work by induction: assume we have defined $K_1, K_2, \dots, K_{i-1}$ and $L_1, L_2, \dots L_{i-1}$ such that

$$f_{i-1}(x,y) = x^{m_1}y^{m_2}(a_3x+b_3y)^{m_3}\dots(a_kx+b_ky)^{m_k}+H_{m+1}+H_{m+2}+\dots H_{m+i-1}+H.O.T.$$

Now

$$H_{m+i} = x^{m_1}K_i(x,y)+y^{m_2}(a_3x+b_3y)^{m_3}\dots(a_kx+b_ky)^{m_k}L_i+R(x,y,L_1,\dots L_{i-1},K_1,\dots K_{i-1}) \quad .$$

Where $R(x,y,L_1,\dots L_{i-1},K_1,\dots K_{i-1})$ is a polynomial. Note that from the induction assumption we have already defined the terms in $L_1, L_2, \dots L_{i-1}$ and $K_1, K_2, \dots K_{i-1}$. Define H'_i to be $H_{m+i} - R(x,y,L_1,\dots L_{i-1},K_1,\dots K_{i-1})$.

Write $H'_{m+i} = \sum_j h_jx^jy^{m+i-j}$. We want to define $l_j, k_j \in \mathbb{C}$ such that $L_i = \sum_j l_jx^jy^{m+i-j}$ and $K_i = \sum_j k_jx^jy^{m-m_1+i-j}$ satisfy

$$x^{m_1}K_i + y^{m_2}(a_3x + b_3y)^{m_3}\dots(a_kx + b_ky)^{m_k}L_i = H'_i \quad .$$

We will only sketch the proof of this, as it is largely similar to the above: First, we define:

$$\begin{aligned}
l_0 &= h_0 \\
l_1 &= h_0 - \sum_{j=3}^{k} \binom{m_j}{1} a_j b_j^{m_j-1} \prod_{\alpha \neq j} b_\alpha^{m_\alpha} \\
l_2 &= h_0 - \Big(\sum_{j}^{k} \binom{m_j}{2} a_j^2 b_j^{m_j-2} \Big) \sum_{j_1 \neq j_2}^{k} \binom{m_{j_1}}{1} \binom{m_{j_2}}{1} a_{j_1} b_{j_1}^{m_{j_1}-1} a_{j_2} b_{j_2}^{m_{j_2}-1} \prod_{\alpha \neq j_1, j_2} b_\alpha^{m_\alpha} \\
\vdots &= \vdots
\end{aligned}$$

Specifically, we can define l_j in terms of h_j and $l_1, l_2, \dots, l_{j-1}$. Now we want to define k_j:

$$\begin{aligned}
k_0 &= h_{m-m_1+i} - \alpha_0(l_j) \\
k_1 &= h_{m-m_1+i+1} - \alpha_1(l_j, k_0) \\
k_2 &= h_{m-m_1+i+2} - \alpha_2(l_j, k_0, k_1) \\
\vdots &= \vdots
\end{aligned}$$

Where α_j is equal to the terms in $x^{m-m_1-i+j} y^{m_1+i-j}$ in the expansion of $x^{m_1} K_i + y^{m_2}(a_3x + b_3y)^{m_3} \dots (a_kx + b_ky)^{m_k} L_i$ generated by $l_0, l_1, \dots, l_{m-m_1+i}, k_0, \dots, k_{j-1}$. Note that the terms k_j can be 0 (and generally the first few will be zero). Now consider this expansion: We have defined l_j and k_j such that the first $m - m_1 + i$ and the last $m_1 + i$ terms will sum to h_j. As $m - m_1 + i + m_1 + i > m + i$, we have defined L_i and K_i such that $x^{m_1} K_i + y^{m_2}(a_3x + b_3y)^{m_3} \dots (a_kx + b_ky)^{m_k} L_i = H_i'$. Hence we have that the sum of the terms of order i in $f_I(x, y)$ is equal to H_i. Hence we have inductively defined L_i and K_i such that

$$f(x,y) = \Big(x^{m_1} + L_1(x,y) + L_2(x,y) + \dots\Big)\Big(y^{m_2}(a_3x+b_3y)^{m_3} \dots (a_kx+b_ky)^{m_k} + K_1(x,y) + \dots\Big)$$

as required. ☐

Remark A.1.10. *Note that if $f(x, y)$ is convergent, then we can in fact choose L_i and K_i to also be convergent.*

Corollary A.1.11. *Let $f(x, y) = (a_1x+b_1y)^{m_1}(a_2x+b_2y)^{m_2} \dots (a_kx+b_ky)^{m_k} + H_{m+1} + H_{m+2} + \dots$. If for all $i \neq j$, $a_ib_j - b_ia_j \neq 0$, then f is reducible in the form $f = f_1 f_2 \dots f_k$.*

Proof. From the previous proposition, we may factorise

$$f(x,y) = \Big((a_1x+b_1y)^{m_1} + L_1(x,y) + \dots\Big)\Big((a_2x+b_2y)^{m_2} \dots (a_kx+b_ky)^{m_k} + K_1(x,y) + \dots\Big)$$

We can then apply the previous proposition to the function $(a_2x + b_2y)^{m_2} \dots (a_kx + b_ky)^{m_k} + K_1(x, y) + K_2(x, y) + \dots$ and so on. Hence we have the result by induction. ☐

Remark A.1.12. *In particular we have that all power series with a reducible initial term are reducible. There are two cases left: power series with initial term of order* 1*, i.e.* $f(x,y) = (x+by)+H_2+\dots$ *, (which are trivially irreducible) and power series which have an initial term which is a complete power, i.e.* $f(x,y) = (ax+by)^n + H_{n+1} + \dots$ *. Note that by applying a change of variables if necessary, we can assume that* $a = 1, b = 0$*.*

A.2 Generalised Hensel's Lemma

Now we will consider power series which are of the form $f(x,y) = x^n + H_{n+1} + \dots$.

Recall the definitions of the relevant edge and dot:

Definition A.2.1 (Relevant Edge)**.** *Let* $\lambda = c_1 y^{\theta_1} + \dots + d_i y^{\theta_i}$ *be a polynonmial. An edge* E *of* $\mathbb{P}(f, \lambda)$ *is called relevant if the angle of* E *is greater than* θ_i*.*

Definition A.2.2 (Lowest Relevant Dot)**.** *The lowest relevant dot is defined to be the lowest dot which is on a relevant edge.*

We will now prove lemma 2.2.11. Recall that it states:
If λ is a Newton-Puiseux root of an integer valued power series f, then the conjugates of λ will also be roots of f. Specifically if we write

$$\lambda = c_1 y^{n_1/d} + c_2 y^{n_2/d} + \dots + c_i y^{n_i/d} + \dots$$

(where d are minimal) then for any complex $d-th$ root of unit z_d, the arc

$$\lambda' = z_d^{n_1} c_1 y^{n_1/d} + z_d^{n_2} c_2 y^{n_2/d} + \dots + z_d^{n_i} c_i y^{n_i/d} + \dots$$

is also a root of f.

Remark A.2.3. *If*

$$\lambda = c_{1,1} y^{m_{1,1}/d_1} + c_{1,2} y^{m_{1,2}/d_1} + \dots + c_{2,1} y^{m_{2,1}/d_1 d_2} + \dots + c_{j,1} y^{m_{j,1}/d_1 d_2 \dots d_j} + \dots \quad ,$$

then by setting $d = lcm(d_1, d_2, \dots, d_i)$ *and* $n_1 = m_{1,1} d/d_1, n_2 = m_{1,2} d/d_1, \dots$ *, we can write* $\lambda = c_1 y^{n_1/d} + c_2 y^{n_2/d} + \dots + c_i y^{n_i/d} + \dots$ *.*

First we will prove some technical lemmas:

Lemma A.2.4. *Let* λ *be a Newton-Puiseux root of* f*, and* λ_i *the arc constructed by taking the first* i *terms of* λ*. Write* $\lambda_i = c_1 y^{m_1/d'} + c_2 y^{m_2/d'} + \dots + c_i y^{m_i/d'}$*, where* d' *is minimal, then we have the following:*
if the $i+1-th$ *term in* λ *is equal to* $c_{i+1} y^{\frac{n}{d'm}}$*, then for any complex* $m-th$ *root of unity* z_m*, then there is a Newton-Puiseux root of* f *which is equal to* $\lambda_i + z_m c_{i+1} y^{\frac{n}{m}} + H.O.T.$

Proof. Let E be the edge of angle θ_{i+1} in the Newton polygon of f relative to λ_i, and let (x^{a_1}, y^{b_1}) and (x^{a_2}, y^{b_2}) be the endpoints of E. The associated form of E may be written: $\mathcal{E} = x^{a_2} y^{b_1} (d_0 x_1^a + \dots + d_{a_2 - a_1} y^{b_2})$. We may write this as

$$\mathcal{E} = x^{a_2} y^{b_1} (x^m + w_1 y^n)^{k_1} (x^m + w_2 y^n)^{k_2} \dots ,$$

where $hcf(m,n)=1$, $\frac{n}{m}=\theta_{i+1}$ and $w_i \neq w_j$ for $i \neq j$. We can order the w_i such that the coefficient of $y^{\theta_{i+1}}$ in λ (which we will denote c_i is a m-th root of w_1. Clearly $z_m c_i$ is also a m-th root of w_1, and so is a solution of the associated polynomial of E. Hence by sliding along E we get $\lambda_{i+1} = \lambda_i + z_m c_i y^{\theta_i}$.

So in particular, if the $i+1-th$ term in λ is equal to $c_{i+1} y^{\frac{n}{m}}$, where (n,m) are minimal, then $z_m c_{i+1}$ also satisfies the associated polynomial of a relevant edge of $\mathbb{P}(f,\lambda_i)$, and so there is a Newton-Puiseux root of f which is equal to $\lambda_i + z_m c_{i+1} y^{\frac{n}{m}} + H.O.T.$ $\square$

Lemma A.2.5. *If $\lambda = c_1 y^{n_1/d} + c_2 y^{n_2/d} + \dots$ and $\lambda' = z_d^{n_1} c_1 y^{n_1/d} + z_d^{n_2} c_2 y^{n_2/d} + \dots$, then the Newton polygon $\mathbb{P}(f,\lambda')$ of f relative to λ' will be the same as the Newton polygon $\mathbb{P}(f,\lambda)$ of f relative to λ. In addition, if $(p, q/d)$ is a dot on $\mathbb{P}(f,\lambda)$ corresponding to the term $a_{pq} x^p y^{q/d}$, the coefficients of the corresponding dot in $\mathbb{P}(f,\lambda')$ will be $z_d^{q/d} a_{pq}$.*

Proof. Let $g(y^{1/d}) = \lambda(y)$ and $g'(y^{1/d}) = \lambda'(y)$. So $g(z_d y^{1/d}) = g'(y^{1/d})$. The lemma follows trivially from writing $f(x+\lambda'(y), y)$ as $f(x + g(z_d y^{1/d}), (z_d y^{1/d})^d)$. $\square$

We will now prove lemma 2.2.11. Recall that this lemma states:
if λ is a Newton-Puiseux root of an integer valued power series f, then the conjugates of λ will also be roots of f

Proof of lemma 2.2.11. Let λ' be an arc of the form

$$\lambda' = z_d^{m_1} c_1 y^{m_1/d} + z_d^{m_2} c_2 y^{m_2/d} + \dots + z_d^{m_i} c_i y^{m_i/d} + \dots$$

We will prove that λ' is a root of f by sliding towards λ' on the Newton diagram of f.

First, by lemma A.2.4, we have that for any complex $d_1 - th$ root of unity, $z_{d_1} c_1$ is a root of the associated polynomial of a relevant edge of the Newton polygon of f. Hence for any choice of d, $z_d^{n_1/d_1} c_1$ will be a root of the associated polynomial, and so $z_d^{m_1/d} c_1$ will be a root of the associated polynomial. Hence there is a Newton-Puiseux root of f with first term $z_d^{m_1} c_1 y^{m_1/d}$.

Now assume that the first i terms $\lambda'_i = z_d^{m_1} c_1 y^{m_1/d} + z_d^{m_2} c_2 y^{m_2/d} + \dots + z_d^{m_i} c_i y^{m_i/d}$ are the first i terms of a root. We want to show that λ_{i+1} is the first $i+1$ terms of a root of f. Write the expansion of $f(x+\lambda_i, y)$ as $F_i = \sum_{p,q} a_{pq} x^p y^{q/d}$. Now consider the expansion of $f(x+\lambda'_i, y)$. From lemma A.2.5, this may be written as $F'_i = \sum_{p,q} z_d^{q/d} a_{pq} x^p y^{q/d}$, and the Newton polygons are equal. Now consider the edge E_{i+1} with angle θ_{i+1}. Assume that in the Newton polygon $\mathbb{P}(f,\lambda_i)$ the associated form of this edge is given by

$$\tilde{\mathcal{E}}_{i+1} = (x^m - w_1 y^{\frac{n}{d}})^{k_1} \cdot (x^m - w_2 y^{\frac{n}{d}})^{k_2} \cdot (x^m - w_3 y^{\frac{n}{d}})^{k_3} \cdot \dots \quad .$$

We claim that the associated form of the edge E'_{i+1} is:

$$\tilde{\mathcal{E}}'_{i+1} = (x^m - w_1 (z_d)^n y^{\frac{n}{d}})^{k_1} \cdot (x^m - w_2 (z_d)^n y^{\frac{n}{d}})^{k_2} \cdot (x^m - w_3 (z_d)^n y^{\frac{n}{d}})^{k_3} \cdot \dots \quad .$$

This follows from writing the associated form of the edge as $\tilde{\mathcal{E}}_{i+1} = x^p y^{q/d}(b_0 x^l + b_1 x^{l-1} y^{\theta_{i+1}} + b_2 x^{l-2} y^{2\theta_{i+1}} + \dots + b_l y^{l\theta_{i+1}})$. Now $\theta_{i+1} = \frac{m_{i+1}}{d}$, for some integer m_{i+1}. So we have $\tilde{\mathcal{E}}_{i+1} = x^p y^{q/d}(b_0 x^l + b_1 x^{l-1} y^{m_{i+1}/d} + b_2 x^{l-2} y^{2m_{i+1}/d} + \dots + b_l y^{l m_{i+1}/d})$.

Similarly $\tilde{\mathcal{E}}'_{i+1} = z_d^q x^p y^{q/d}(b_0 x^l + z_d^{m_{i+1}} b_1 x^{l-1} y^{m_{i+1}/d} + z_d^{2m_{i+1}} b_2 x^{l-2} y^{2m_{i+1}/d}) + \cdots + z_d^{lm_{i+1}} b_l y^{lm_{i+1}/d})$. Now let $t = z_d y^{1/d}$. Clearly

$$\begin{aligned}
\tilde{\mathcal{E}}'_{i+1} &= x^p t^q (b_0 x^l + b_1 x^{l-1} t^{m_{i+1}} + b_2 x^{l-2} t^{2m_{i+1}} + \cdots + b_l t^{lm_{i+1}}) \\
&= (x^m - w_1 t^n)^{k_1} \cdot (x^m - w_2 t^n)^{k_2} \cdot (x^m - w_3 t^n)^{k_3} \cdot \ldots \\
&= (x^m - w_1 (z_d)^n y^{\frac{n}{d}})^{k_1} \cdot (x^m - w_2 (z_d)^n y^{\frac{n}{d}})^{k_2} \cdot (x^m - w_3 (z_d)^n y^{\frac{n}{d}})^{k_3} \cdot \ldots \quad .
\end{aligned}$$

We also have the associated polynomials:

$$\mathcal{E}_{i+1}(s) = (s^m - w_1)^{k_1} \cdot (s^m - w_2)^{k_2} \cdot (s^m - w_3)^{k_3} \cdot \ldots \quad .$$

$$\mathcal{E}'_{i+1}(s) = (s^m - w_1 (z_d)^n)^{k_1} \cdot (s^m - w_2 (z_d)^n)^{k_2} \cdot (s^m - w_3 (z_d)^n)^{k_3} \cdot \ldots \quad .$$

As λ is a root, c_{i+1} is a root of $\mathcal{E}_{\rangle+\infty}$, and hence c_{i+1} is a m-th root of one of the w_j, say w_1. Now we will substitute $s = z_d^{m_{i+1}} c_{i+1}$ into $\mathcal{E}'_{i+1}$.

$$\begin{aligned}
\mathcal{E}'_{i+1}(s) &= ((z_d^{m_{i+1}} c_{i+1})^m - w_1 (z_d)^n)^{k_1} \cdot (s^m - w_2 (z_d)^n)^{k_2} \cdot \ldots \\
&= ((z_d^{mm_{i+1}} w_1) - w_1 (z_d)^n)^{k_1} \cdot (s^m - w_2 (z_d)^n)^{k_2} \cdot \ldots \\
&= w_1 (z_d^{mm_{i+1}} - z_d^n)^{k_1} \cdot \ldots
\end{aligned}$$

Now m_{i+1}/d is the exponent of the $i+1$-th term of λ. Also, by sliding, the $i+1$-th term is n/m. Hence $mm_{i+1} = n$, and so the associated polynomial $\mathcal{E}'_{i+1}(z_d^{m_{i+1}} c_{i+1}) = 0$. Hence by sliding there is a root of f which has first $i+1$ terms λ'_{i+1}. Hence by induction λ' is a root of f for any choice z_d. □

Appendix B

Examples

B.1 Examples of Sliding

Example B.1.1. $f(x, y) = (x - y)(x - y - y^2) = x^2 - 2xy + y^2 - xy^2 - y^3$.

The only edge of f has one root (of multiplicity 2) to its associated polynomial: $z = 1$ Hence we get $\lambda_1 = y$.

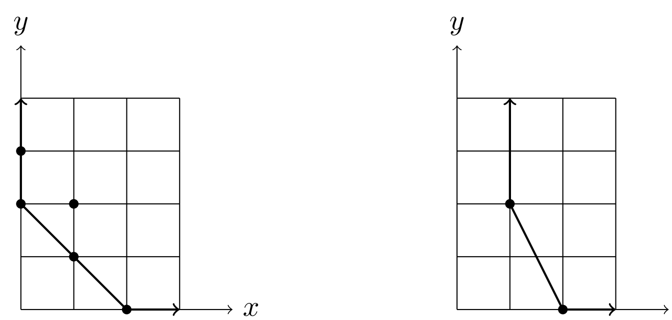

Figure B.1: The newton polygons of $f(x, y) = (x - y)(x - y - y^2)$ (left) and $f(x + y, y)$ (right)

Now $f(x + y, y) = x(x - y^2)$, and so $x = y$ is a root. We will now slide along the edge of $\mathbb{P}(f, y)$: the associated form for E is $x^2 - xy^2$, and so the associated form is $z^2 - z$. Hence we have two roots of the associated polynomial: $z = 0, 1$. This gives us the roots $x = y + y^2$ and $x = y$ (which we already had).

Example B.1.2. $f(x, y) = (x^2 - y^3)^2 - xy^5$

This has one edge: E_1, with $tan(\theta_1) = 3/2$. The associated form of E_1 is $\tilde{\mathcal{E}}_1(X, Y) = X^4 - 2X^2 + 1$, and so the associated polynomial $\mathcal{E}_2(z) = z^4 - 2z^2 + 1$. This has two roots, $z = \pm 1$, both of multiplicity 2, and so we get $\lambda_1(y) = \pm y^{3/2}$.

Now consider $\mathbb{P}(f, \lambda_1)$.

$$\begin{aligned} F(X + \lambda_1(Y), Y) &= (X^2 \pm 2XY^{3/2} + Y^3 - Y^3)^2 + XY^5 \pm Y^{13/2} \\ &= (X^2 \pm 2XY^{3/2})^2 + XY^5 \pm Y^{13/2} \\ &= X^4 \pm 4X^3Y^{3/2} + 4X^2Y^3 + XY^5 \pm Y^{13/2} \end{aligned}$$

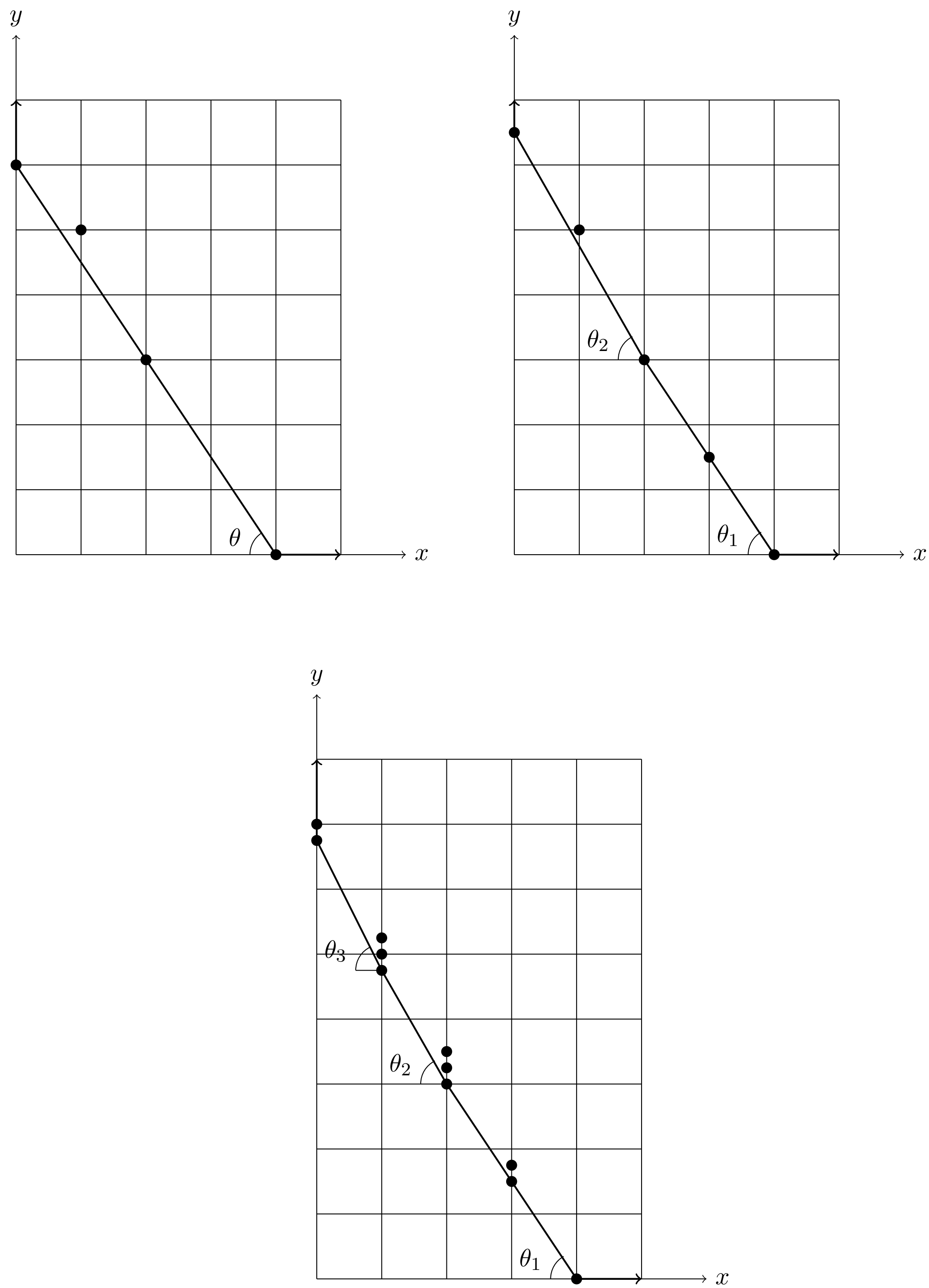

Figure B.2: Newton poygon of f, F_1 and F_2 for $f(x,y) = (x^2 - y^3)^2 - xy^5$.

We now have two edges E_1 and E_2 with $tan(\theta_1) = 3/2$ and $tan(\theta_2) = 7/4$. The associated form of E_2 is $4X^2Y^3 \pm Y^{13/2}$, and so the associated polynomial is: $\mathcal{E}_2(z) = 4z^2 \pm +1$. This gives four possible choices for λ_2 :

$$
\begin{aligned}
\lambda_2 &= y^{3/2} + \frac{i}{2}y^{7/4} \\
& \quad y^{3/2} - \frac{i}{2}y^{7/4} \\
& \quad -y^{3/2} + \frac{1}{2}y^{7/4} \\
& \quad -y^{3/2} - \frac{1}{2}y^{7/4}
\end{aligned}
$$

Now we must consider $\mathbb{P}(f, \lambda_2)$. We will use $\lambda_2 = y^{3/2} \pm \frac{i}{2}y^{7/4}$

$$
\begin{aligned}
F(X+\lambda_2(Y), Y) &= \big((X \pm \frac{i}{2}Y^{7/4})^2 + 2(X \pm \frac{i}{2}Y^{7/4})Y^{3/2}\big)^2 + (X \pm \frac{i}{2}Y^{7/4})Y^5 + Y^{13/2} \\
&= (X^2 \pm iXY^{7/4} - \frac{1}{4}Y^{7/2} + 2XY^{3/2} \pm iY^{13/4})^2 + XY^5 \pm \frac{i}{2}Y^{27/4} + Y^{13/2} \\
&= X^4 \pm 2\frac{i}{2}X^3Y^{7/4} - \frac{1}{2}X^2Y^{7/2} + 4X^3Y^{3/2} \pm 2iX^2Y^{13/4} - X^2Y^{7/2} \\
&\quad - \pm \frac{i}{2}XY^{21/4} \pm 4iX^2Y^{13/4} - 2iXY^5 + \frac{1}{16}Y^7 - XY^5 \\
&\quad - \pm \frac{i}{2}Y^{27/4} + 4X^2Y^3 \pm 4iXY^{19/4} - Y^{13/2} + XY^5 \pm \frac{i}{2}Y^{27/4} + Y^{13/2}
\end{aligned}
$$

Hence we now have three edges E_1, E_2 and E_3 with $tan(\theta_1) = 1.5$, $tan(\theta_2) = 1.75$ and $tan(\theta_3) = 2$. The associated form of E_3 is $\tilde{\mathcal{E}}_2(X,Y) = \pm 4iXY^{19/4} - \pm\frac{i}{2}Y^{27/4}$, and so the associated polynomial $\mathcal{E}_3(z) = \pm 4iz - \pm\frac{i}{2}$. As this is of degree 1, we may terminate by the IFT.

Using $\lambda_2 = -y^{3/2} \pm \frac{1}{2}y^{7/4}$ gives a similar result, and so the Newton-Puiseux roots of f are:

$$
\begin{aligned}
\beta = \quad & y^{3/2} + \frac{i}{2}y^{7/4} + H.O.T. \\
& y^{3/2} - \frac{i}{2}y^{7/4} + H.O.T. \\
& -y^{3/2} + \frac{1}{2}y^{7/4} + H.O.T. \\
& -y^{3/2} - \frac{1}{2}y^{7/4} + H.O.T.
\end{aligned}
$$

The next example is the partial derivative with respect to x of the function $g(x,y) = (x^2 - y^3)^2 - xy^5$.

Example B.1.3. $f(x,y) = 4x^3 - 4xy^3 - y^5$

The Newton diagram of f is shown in figure B.3

This has two edges: E_1 and E_2 with $tan(\theta_1) = 3/2$ and $tan(\theta_2) = 2$. E_2 is of length 1 and so we get one root from it: The associated polynomial for the second

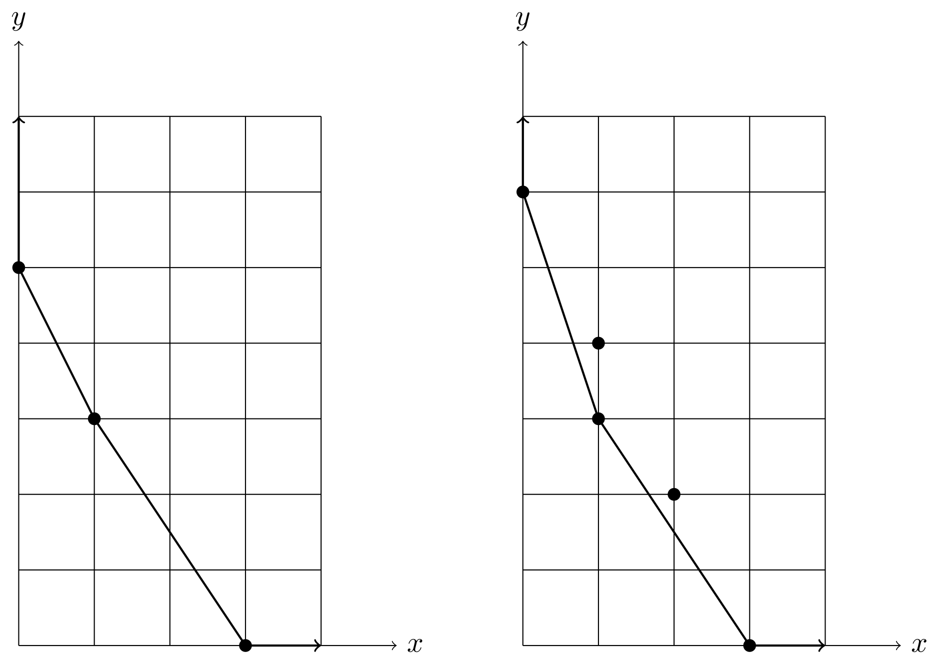

Figure B.3: Newton polygon of $f(x,y) = 4x^3 - 4xy^3 - y^5$ and $F_1 = f(x + -1/4y^2, y)$

edge is $\mathcal{E}_2(z) = -4z - 1$. This has one root: $z = -1/4$. So $\lambda_1 = -1/4y^2$. Consider $F_1 = f(x + \lambda_1, y)$:

$$F_1(x,y) = 4(x - \frac{1}{4}y^2)^3 - 4xy^3 + y^5 - y^5$$
$$= 4x^3 - 3x^2y^2 + \frac{3}{4}xy^4 - \frac{1}{16}y^6 - 4xy^3$$

So we have $\beta_1 = -1/4y^2 - 1/64y^3 + H.O.T.$.

The associated form for E_1 is $\tilde{\mathcal{E}}_2(x,y) = 4x^3 - 4xy^3$, and so the associated polynomial $\mathcal{E}_1(z) = 4z^3 - 4z$. This has three roots: $\pm 1, 0$. We will ignore $x = 0$ (this leads to the root we have already found).

Now consider β_2 and β_3: $\lambda_1 = \pm y^{3/2}$.

$$\begin{aligned} f_x(x \pm \lambda_1(y), y) &= 4x^3 \pm 12x^2y^{3/2} + 12xy^3 \pm 4y^{9/2} - 4xy^3 - \pm 4y^{9/2} - y^5 \\ &= 4x^3 \pm 12x^2y^{3/2} + 8xy^3 - y^5 \end{aligned}$$

Now consider the associated form for the highest edge: $E_h = 8xy^3 - y^5$, and so the associated polynomial $\mathcal{E}_h(z) = 8z - 1$. This has one root $z = 1/8$. Now consider $f(x + \lambda_2(y), y)$ where $\lambda_2 = \pm y^3/2 + 1/8y^2$.

$$f(x + \lambda_2(y), y) = 4(x + \frac{1}{8}y^2)^3 \pm 12(x + \frac{1}{8}y^2)^2y^{3/2} + 8(x + \frac{1}{8}y^2)y^3 - y^5$$
$$= 4(x + \frac{1}{8}y^2)^3 \pm 12(x + \frac{1}{8}y^2)^2y^{3/2} + 8xy^3 + y^5 - y^5$$

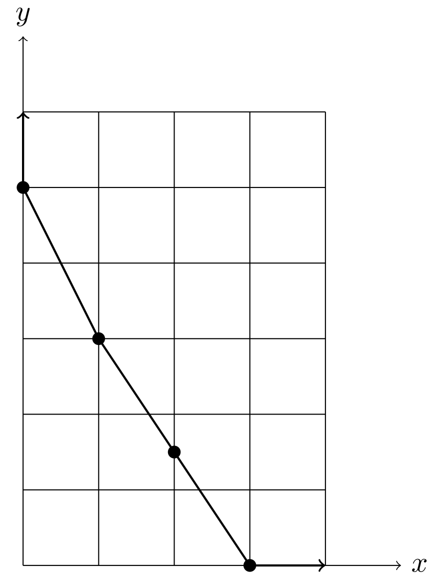

Figure B.4: Newton polygon of $f(x \pm y^{3/2}, y)$

The new highest edge has associated form $\tilde{\mathcal{E}}_h = 8xy^3 \pm 3/16y^{5.5}$, *and so the third term in* β_2, β_3 *is* $\pm 3/128y^{5/2}$. *Hence we have* $\beta_2 = y^{3/2} + 1/8y^2 - 3/128y^{5/2} + H.O.T., \beta_3 = -y^{3/2} + 1/8y^2 + 3/128y^{5/2} + H.O.T.$. *So the roots of* f *are:*

$$\beta_1 = -1/4y^2 - 1/64y^3 + H.O.T.$$
$$\beta_2 = y^{3/2} + 1/8y^2 - 3/128y^{5/2} + H.O.T.$$
$$\beta_3 = -y^{3/2} + 1/8y^2 + 3/128y^{5/2} + H.O.T.$$

Example B.1.4. $f(x, y) = x^4 + x^2y^2 + y^5$

For E_1 *we have* $\tilde{\mathcal{E}}_1(x, y) = x^4 + x^2y^2$, *and so* $\mathcal{E} = z^4 + z^2$. *The solutions of this are* $z = \pm i$ *and* $z = 0$. *Clearly we may ignore the* $z = 0$ *solution (this corresponds to the roots found by sliding along* E_2*). So we have* $\lambda_1 = \pm iy$. *Hence*

$$\begin{aligned} F_1(x, y) = f(x \pm iy, y) &= x^4 + 4ix^3y - 6x^2y^2 - 4ixy^3 + y^4 + x^2y^2 + 2ixy^3 - y^4 + y^5 \\ &= x^4 + 4ix^3y - 5x^2y^2 - 2ixy^3 + y^5 \end{aligned}$$

As the highest edge is of order 1, we may hence use the I.F.T to find a pair of unique roots given by: $\beta_1 = iy + H.O.T.$, $\beta_2 = -iy + H.O.T.$

Now for E_2 *we have* $\tilde{\mathcal{E}}_2(x, y) = x^2y^2 + y^5$, *and so* $\mathcal{E} = z^2 + 1$. *The solutions of this are* $z = \pm i$. *So we have* $\lambda_1 = \pm iy^{3/2}$. *Hence*

$$\begin{aligned} G_1(x, y) = f(x \pm iy^{3/2}, y) &= x^4 + 4ix^3y^{3/2} - 6x^2y^3 - 4ixy^{9/2} + y^6 + x^2y^2 + 2ixy^{7/2} - y^5 + y^5 \\ &= x^4 + 4ix^3y^{3/2} - 6x^2y^3 + x^2y^2 + 2ixy^{7/2} - 4ixy^{9/2} + y^6 \end{aligned}$$

We may hence use the I.F.T to find a pair of unique roots given by:
$\beta_3 = iy^{3/2} + H.O.T. \qquad \beta_4 = -iy^{3/2} + H.O.T.$

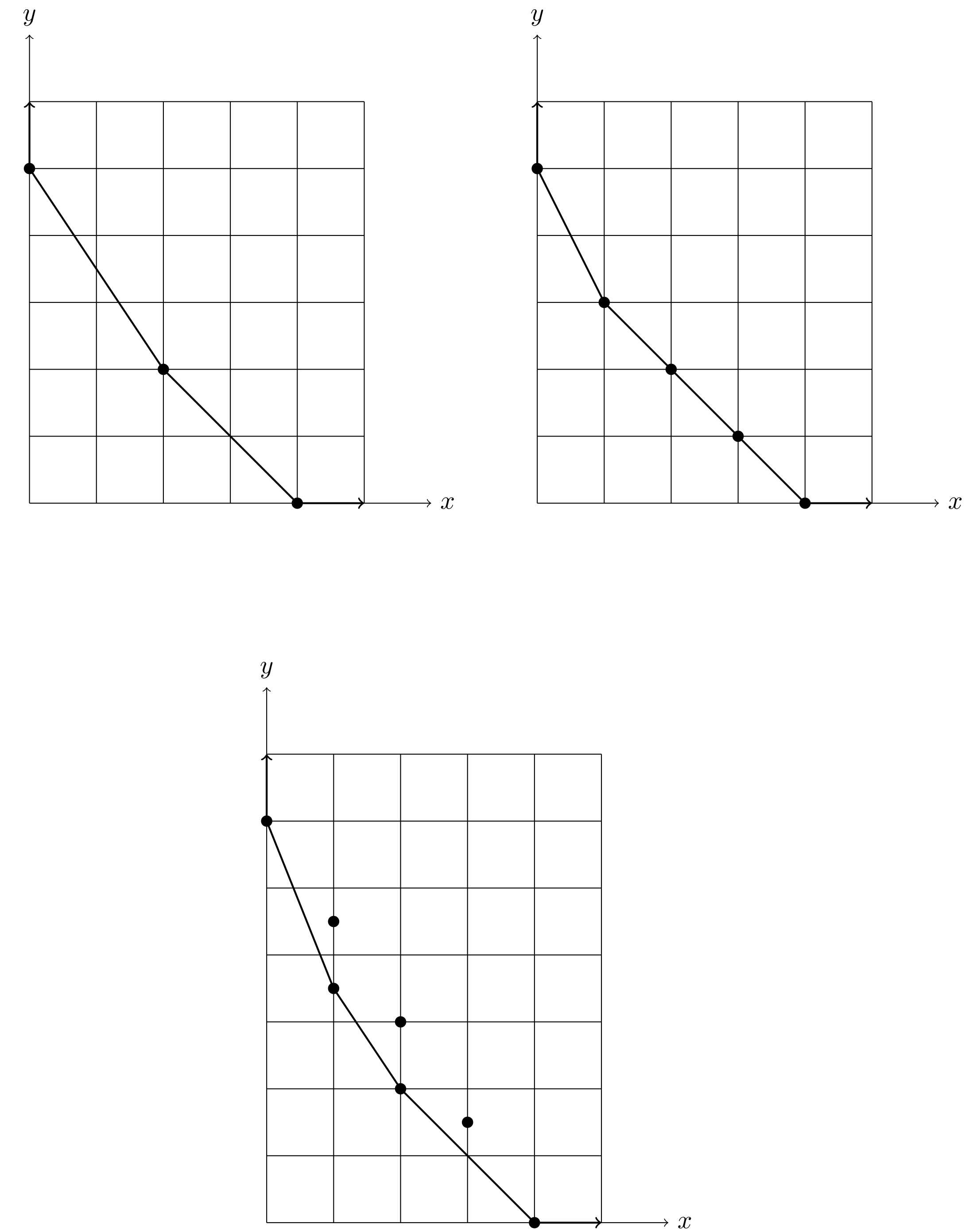

Figure B.5: Newton polygons of f (top left), F_1 (top right) and G_1 (below)

Hence the Newton-Puiseux roots for f are:

$$\beta_1 = iy + H.O.T.$$
$$\beta_2 = -iy + H.O.T.$$
$$\beta_3 = iy^{3/2} + H.O.T.$$
$$\beta_4 = -iy^{3/2} + H.O.T. \quad .$$

In this example each edge was of order two, and gave two roots.

B.2 Tree Model Examples

Example B.2.1. $f(x,y) = x^2 + y^3 - y^4$

The Newton-Puiseux roots for f are: $\lambda(y) = \pm iy^{3/2} + H.O.T.$ Hence the bars for these roots split at height 1.5. Hence the tree-model for f consists of a single bar with two twigs. The diagram $M^*(f)$ is constructed by adding a dashed twig to $M(f)$, as shown in the right diagram of figure B.6

Also, $\partial f/\partial x = 2x$, and so the tree model of f_x consists of a single twig.

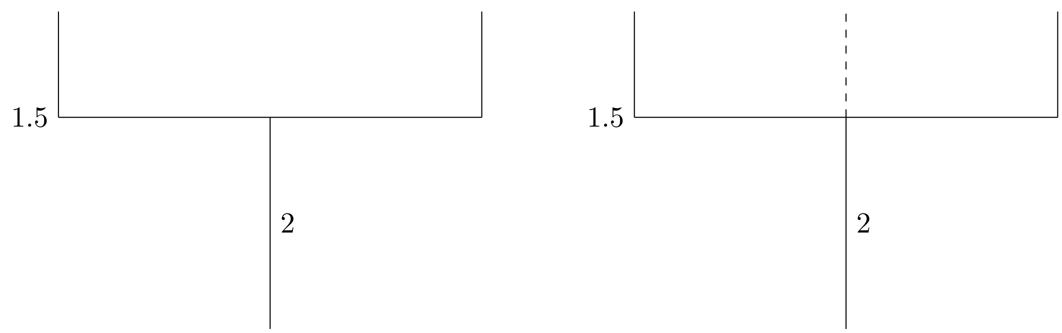

Figure B.6: Tree diagrams $M(f)$ (left) and $M^*(f)$ (right) for $f(x,y) = x^2 + y^3 - y^4$.

Example B.2.2. $f(x,y) = (x^2 - y^3)^2 - xy^5$

Recall from example B.1.2 that the Newton-Puiseux roots of f are:

$$\beta_1 = y^{3/2} + \frac{i}{2}y^{7/4} + H.O.T.$$
$$\beta_2 = y^{3/2} - \frac{i}{2}y^{7/4} + H.O.T.$$
$$\beta_3 = -y^{3/2} + \frac{1}{2}y^{7/4} + H.O.T.$$
$$\beta_4 = -y^{3/2} - \frac{1}{2}y^{7/4} + H.O.T.$$

Hence the tree diagram for f has a bar at $3/2$ with two trunks, and each of these trunks has a bar at $7/4$ with two twigs.

We will now draw the tree model of $f_x = 4x^3 + 4xy^3 + y^5$. Recall that the roots of f_x are:

$$\beta_1 = -1/4y^2 - 1/64y^3 + H.O.T.$$
$$\beta_2 = y^{3/2} + 1/8y^2 - 3/128y^{5/2} + H.O.T.$$
$$\beta_3 = -y^{3/2} + 1/8y^2 + 3/128y^{5/2} + H.O.T.$$

These are equal up to order $3/2$, and so the tree diagram $M(f_x)$ has one bar at $3/2$, with three twigs.

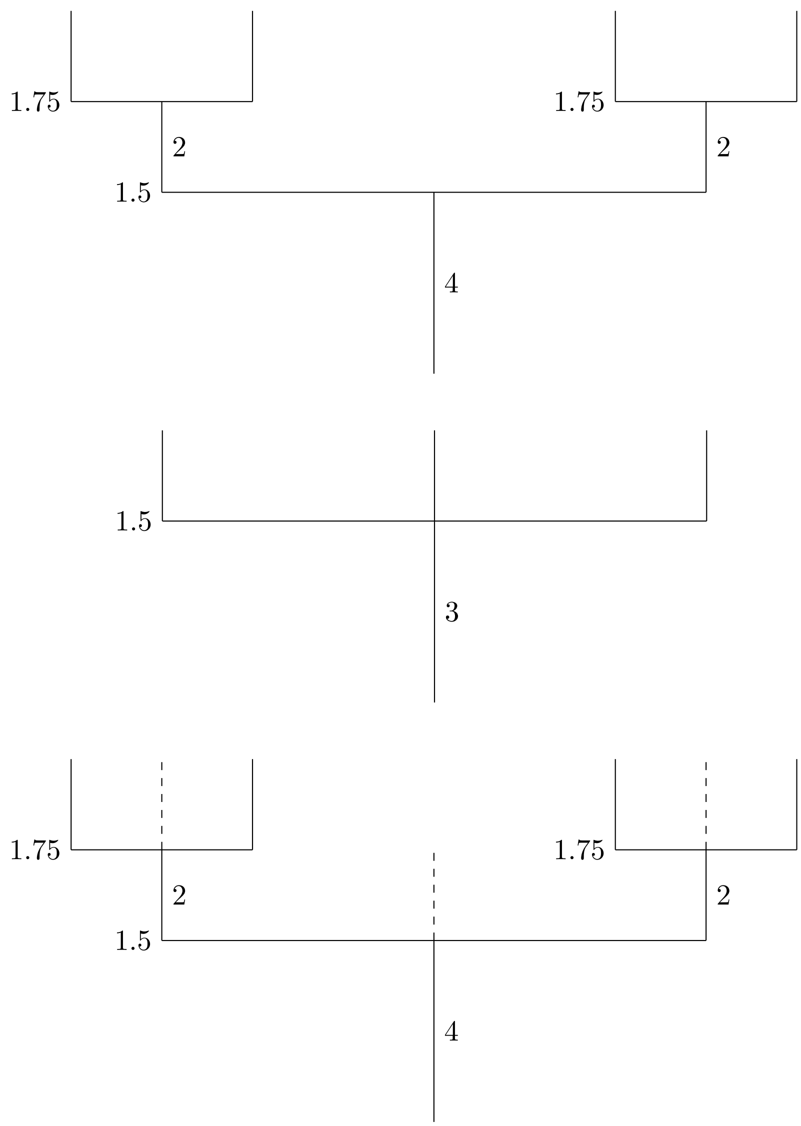

Figure B.7: Tree diagrams $M(f)$ (top), $M(f_x)$ (middle) and $M^*(f)$ (bottom) for $f(x,y) = (x^2 - y^3)^2 - xy^5$.

Example B.2.3 (The bars of f_x are not determined by the graph of $M^*(f)$)**.** *Let* $f(x,y) = x^3 - y^3 + xy^8$

The Newton-Puiseux roots for f *are:* $\lambda(y) = \sqrt[3]{1}y + H.O.T.$ *for each of the cube roots of* 1*. Hence the bars for these roots split at height* 1*. Hence the tree-model for* f *has one bar at* 1 *with three twigs, as shown on the top left of figure B.8.*

$\partial f/\partial x = 3x^2 + y^8$*. The Newton-Puiseux roots for* f_x *are:* $\gamma(y) = \pm iy^4$*. Hence the tree diagram for* f_x *has one bar at* 4 *with two twigs, as shown in the top right diagram of figure B.8.*

Now we will draw the tree model of $M^*(f)$*. Note that in this example, the bar of* $M(f_x)$ *is different to the bars of* $M^*(f)$*. Finally we will draw the tree model of* f *and* f_x *on the same diagram (figure B.9).*

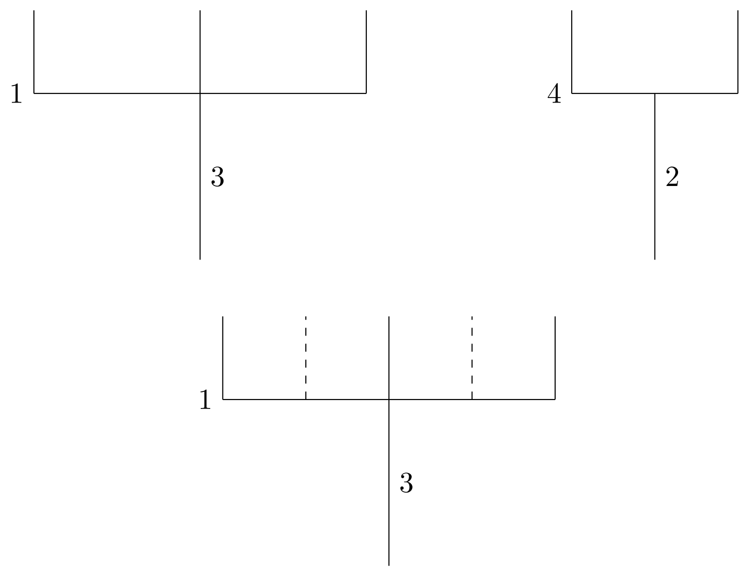

Figure B.8: Tree diagrams $M(f)$ (top left), $M(f_x)$ (top right) and $M^*(f)$ (bottom) for $f(x,y) = x^3 - y^3 + xy^8$.

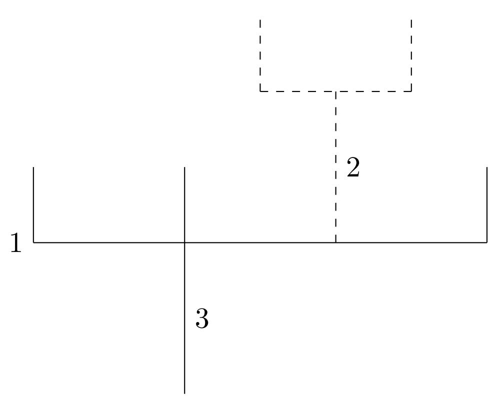

Figure B.9: Tree diagrams of $f(x,y) = x^3 - y^3 + xy^8$ and f_x on the same diagram.

If fact it is possible to have a function f such that the f_x has a root of multiplicity greater than 1. In this case, the tree diagram of f_x will have a trunk which has no bar above it.

Example B.2.4 (f_x has a root of mult. 2)**.** *Let* $f(x,y) = (x-y)^3 + y^3 = x(x^2 - 3xy + 3y^2)$.

Clearly f *has three Newton-Puiseux roots given by:* $x = 0, c_1y + H.O.T.\ , c_2y + H.O.T.$ *where* c_1, c_2 *are the roots of the associated equation* $z^2 - 3z + 3 = 0$.

The tree diagram for f *is: shown in figure B.10*

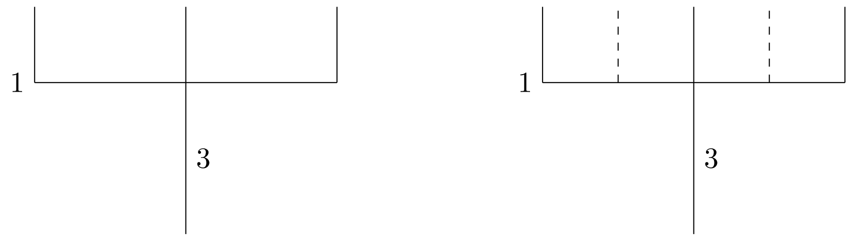

Figure B.10: Tree diagrams $M(f)$ (left) and $M^*(f)$ for $f(x,y) = (x-y)^3 + y^3$.

Now $f_x = 3(x-y)^2$*, and so the roots of* f_x *are* $x = y$ *with multiplicity 2. So the tree diagram for* f_x *consists of a single trunk. Note that the bars of* f_x *are different to the dashed bars of* $M^*(f)$ *(which is shown on the right in figure B.10).*

Bibliography

[1] Brieskorn E and Knörrer H. *Plane Algebraic Curves*, Birkhäuser Verlag, Basel, 1986. Translated by J. Stillwell.

[2] Eisenbud, S and Neumann, W. *Three-Dimensional Link Theory and Invariants of Plane Curve Singularities. Annals of Mathematics 110*, Princetown University Press, Princetown N.Y., 1985.

[3] Kuo, T-C. *Generalised Newton-Puiseux theory and Hensel's lemma in* $\mathbb{C}[x, y]$, Canadian journal of Mathematics XLI (1989), 1101-1116.

[4] Kuo, T-C and Lu, Y-C. *On analytic function germs of two complex variables*, Topology 16 (1977), 299-310.

[5] Kuo TC, Parusiński A. *Newton polygon relative to an arc, Real and complex singularities*, 5th Workshop on Real and Complex Singularities, H Brezis, R G Douglas, A Jeffrey (eds.), CRC Research Notes in Mathematics, Chapman and Hall; CRC Press, London, UK, (2000), 7693

[6] Kuo TC, Parusiński A. *Newton-Puiseux roots of Jacobian determinants, Journal of Algebraic Geometry*, 13 (2004), 579601

[7] Lickorish, W.B.R. *An Introduction to Knot Theory*, Springer-Verlag, New York, 1997.

[8] Milnor, J. *Singular Points of Complex Hypersurfaces. Annals of Mathematics 61*, Princetown University Press, Princetown N.Y., 1968.

[9] Reeve, J.E. *A summary of results in the topological classification of plane algebroid singularities*, Universitá e Polytecnico di Torino, Rendiconti del Seminaro Matematico, 1954/1955.

[10] Walker, R.J. *Algebraic Curves*, Dover Publucations, New York, 1950.

[11] Wall, C.T.C. *Singular Points of Plane Curves*, Cambridge University Press, Cambridge, 2004.

[12] Zariski, O. *Algebraic Surfaces. Second Supplemented Edition. Ergebnisse der Mathematik und Ihrer Grenzgebiete Band 61*, Springer-Verlag, Berlin Heidelberg New York, 1971.