



# Doubling Times in Finance

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# Declaration

I declare that this thesis has not already been submitted for any degree and is not being submitted as part of candidature for any other degree.

I also declare that the thesis has been written by me and that any help that I have received in preparing this thesis, and all sources used, have been acknowledged in this thesis.

Signature of Candidate

.....

Richard Philip

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# Abstract

This dissertation proposes an alternative measure of performance, termed *doubling time*. Doubling time is defined as the time taken for an initial investment in an asset to double in value. This alternative performance metric has an intuitive appeal yet has received little attention in the academic literature to date.

This thesis provides the foundations required for the use of doubling times in finance. The work begins by examining the problem of computing the expected doubling time from a sample of doubling times. Analytical formulae and a simulation are proposed as alternative approaches to estimating the expected doubling time. Using these methods, expected doubling times are computed for the Australian equity market, using both price and accumulation indices. Expected doubling times are also computed for bonds.

The doubling time is then modelled as a first passage time problem. It is shown that if returns are normally distributed then the doubling times will be inverse Gaussian distributed. It is also shown that, regardless of the underlying distribution the returns follow, the simulated doubling time is very likely to be inverse Gaussian distributed.

Following this, portfolio formation using doubling times is investigated. It is shown that minimising either the skewness, or the inverse shape parameter, of the doubling times inverse Gaussian distribution, result in points on the efficient frontier for returns that are identical to those obtained under the classical Markowitz framework. In this

way a two parameter (mean and variance) portfolio optimisation problem is reduced to a one parameter problem (skewness or shape).

Measurement errors can result in the ex-post performance of optimised portfolios being no better than naive equally weighted portfolios. This is sometimes referred to as the problem of error maximisation in portfolio formation. A doubling time transformation is used to reduce the problem of error maximisation, and the results indicate an improvement in the ex-post performance of the optimised portfolios.

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# Chapter 1: Introduction

In finance the measure of performance for an investment asset is typically defined by its percentage return. Traditionally this performance measurement is the profit generated after some fixed time span relative to the investment initially made. This work proposes an alternative measure of performance, termed *doubling time*. The doubling time is defined as the time taken for an initial investment in an asset to double in value. This alternative performance metric has an intuitive appeal along with the strength of there being only one doubling time; irrespective of whether continuous or discrete returns are used and regardless of whether returns are expressed on the basis of simple interest or compound interest. Analysis of these doubling times may uncover new insights and possibly stimulate new theories which are currently unexplored in the finance literature.

## 1.1 Motivation

There are several possible reasons for the study of doubling times. Doubling times are an intuitively attractive way to express returns. This is evident from the development of rules of thumb for estimating doubling times, such as the rule of seventy-two<sup>1</sup>. Despite this intuitive appeal, there has been very little academic study of doubling times for securities. Furthermore, transforming returns into the time domain provides a different perspective on returns. Viewing returns from this fresh angle may stimulate new ideas

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<sup>1</sup> The rule of seventy-two states that 72 divided by the interest percentage per period will approximate the number of periods required for the investment to double.

and insights that might not otherwise be obtained. Four possible uses for doubling times are suggested here, although it is unlikely that this is an exhaustive list.

*Truth in Lending:* Interest rates may be expressed as simple or compound rates, nominal or effective rates with different compounding periods and may be discrete or continuous. Whichever way the interest rate is expressed, there is only one doubling time. Thus doubling times could supply a standard benchmark for comparing loans and would probably have an intuitive appeal to consumers.

*Performance Measurement:* If doubling times are useful in truth in lending, this also suggests that they might provide a useful way to report performance to investors. For example investment funds could be asked to report how long ago one dollar would need to have been invested with the fund in order to have doubled in value by the current date. Funds could also report the doubling time based on the current period's returns.

*Capital Budgeting:* The payback period continues to be very popular in capital budgeting despite its well known deficiencies, Truong, Partington and Peat (2008). Doubling times might be used to replace the payback period and could be theoretically more defensible. The doubling time would have the intuitive appeal of payback, and could be set up to match the properties of decisions based on the IRR. This would require computation of the project's IRR and its conversion to a doubling time. The doubling time computed could then be compared to a benchmark doubling time derived from the cost of capital for the project

*Portfolio Theory:* Doubling times are an alternative way of viewing percentage returns. As doubling times are viewed as the time period taken to reach a fixed return, whereas percentage returns are viewed as the return received for a fixed time period. Perhaps analysing returns in the time domain may yield new insights into forming portfolios, yielding results which differ from the classical Markowitz Framework.

While it appears there are several potential uses for doubling times, there has been little academic research on this area. This dissertation hopes to lay a foundation for work with doubling times. This thesis will highlight the problems that can arise with doubling times and provide analytical and simulation methods to overcome these problems. The thesis also examines the distribution of doubling times and applications in portfolio theory as described below.

A fundamental problem in finance is what distribution percentage returns follow. Despite investigation for over a century the results remain inconclusive. Perhaps this seems like an unimportant question; however it is of considerable significance. All of the classical financial theories such as Black-Scholes option pricing, the Capital Asset Pricing Model and Markowitz portfolio theory have the underlying assumption that returns are normally distributed, or alternatively for portfolio theory, that investors employ a quadratic form of utility. This thesis also attempts to identify the distribution of returns, however, not traditional percentage returns, rather the doubling times. Additionally, possible relationships between the parameters of the doubling time distribution and the moments of the equivalent percentage returns are also derived.

Lastly, Markowitz Portfolio Theory suffers from a problem known as error maximisation (Michaud, 1989). Michaud argues that mean-variance optimisation overweights (underweights) those assets with a large (low) estimated return to variance ratio, and that these are the assets likely to have large estimation errors. This dilemma can make practical implementation of Markowitz Portfolio Theory particularly difficult, often resulting in portfolio weights with substantial errors. This work addresses the problem of error maximisation and a new method is suggested to reduce estimation error. Additionally, when using doubling times, possible new risk metrics for returns are derived that provide alternative methods to obtain the efficient frontier.

## 1.2 Outline

This thesis outlines the problems that can arise when taking traditional descriptive statistics such as the arithmetic mean and variance of doubling times. To overcome these problems an analytical solution and a simulation methodology are provided. Following this, doubling times are modelled as the first passage time or first hitting time of a stochastic process. Under the assumption that percentage returns are normally distributed, the distribution of the corresponding doubling times is derived. Also provided are the equations which relate the parameters for the doubling time distribution with the first two moments of the corresponding percentage returns distribution. Following this, using the simulation methodology previously suggested, various tests are performed to determine the possible effects of violating the underlying assumption that percentage returns are normally distributed. Finally, portfolio work in the time domain is conducted. Initially, risk metrics in the time domain which are equivalent to the traditional risk metric of variance is derived. Following this, a new

possible risk metric is suggested. This new risk metric transforms parameters in such a way that the efficient frontier obtained will differ from the Markowitz frontier, it is shown that this new frontier will result in reducing the problem of error maximisation.

The rest of this thesis is organised as follows. Chapter 2 discusses the background literature required for this research. Chapter 3 highlights the fundamentals of doubling times. This shows where possible problems may arise when working with doubling times and solutions to overcome these problems. Chapter 4 models doubling times as a first passage time problem when prices follow Brownian motion. Additionally, tests are performed to determine any ill effects when the assumption of Brownian motion is relaxed. Chapter 5 investigates forming portfolios in the time domain and considers a new method for reducing estimation error. Finally, Chapter 6 summarises the results of this thesis and discusses some ideas for future work in this area.

# Chapter 2: Background

## 2.1 Percentage Returns

The discussion about what distribution returns follow, and if in fact they follow any distribution at all, has been taking place for over a century. Bachelier (1900) was the first to address this issue and proposed that returns followed a normal distribution. However, Mandelbrot (1963) raised serious doubts about the validity of this proposal. Inspired by Mandelbrot (1963), Fama (1965) conducted a thorough investigation about the distribution of stock returns. Similar to Mandelbrot (1963), Fama (1965) proposed that prices follow Stable Paretian Distributions. The Stable Paretian Distribution may be defined by its characteristic function. The characteristic function is necessary as the explicit expressions for the density function is only known for three cases. Accordingly, the distribution's character function is defined using four parameters as follows:

$$\ln f(t) = i\delta t - \gamma|t|^\alpha \{1 + i\beta(t/|t|)\omega(t, \alpha)\}$$

Where:  $\omega(t, \alpha)$  equals  $\tan(\pi\alpha/2)$  if  $\alpha \neq 1$

$\omega(t, \alpha)$  equals  $(2/\pi)(\tan |t|)$  if  $\alpha = 1$

$\delta$  is referred to as the location parameter and can take on any real value.  $\gamma$  is the scale parameter satisfying  $\gamma \geq 0$ ,  $\beta$  is an index of skewness and satisfies  $-1 \leq \beta \leq 1$ . Lastly,  $\alpha$

is a measure of the height of the extreme tail areas and is referred to as the characteristic exponent and satisfies  $0 \leq \alpha \leq 2$ .

A random variable is said to have a stable distribution if it has the property that a linear combination of two independent copies of the variable has the same distribution, up to location and scale parameters. Or mathematically, the stable distribution has the property that for any independent random variables  $X_1, X_2, \dots, X_n$  all having the distribution function  $\Phi$ , there exists constants  $a$  and  $b$  such that the random variable

$$X = b(X_1 + X_2 + \dots + X_n) + a$$

Also has the same distribution function  $\Phi$ .

Following Fama (1965), there have been numerous tests to determine if returns follow the Stable Paretian Distribution. Notably, both Officer (1972) and Rozelle and Feilitz (1980) found some supporting evidence that returns followed the Stable Paretian Distribution, but their results were not conclusive. However, damning evidence against this hypothesis was provided by Akgiray and Booth (1988) and Lau, Lau and Wingender (1990) among others, which left little ground for the Stable Paretian Hypothesis to stand on.

During the testing and development of the Stable Paretian Distribution Hypothesis there was another school of thought initiated by Press (1967). Rather than returns being defined by one distribution, it was suggested they were a mixture of several distributions. This made intuitive economic sense, where returns could be decomposed into “normal” and “abnormal” components. The normal returns were the day to day returns which followed a Gaussian distribution, whereas the abnormal components

reflected news or shocks to the price, which were modelled using a Poisson distribution. However, the implementation of this was problematical and Press (1967) frequently obtained negative estimates for the variance parameters. Following Press (1967) many different attempts have been made to fit returns to a mixture of distributions with varying levels of success examples include Ball and Torous (1983) and Kon (1984).

With mounting evidence against the Stable Paretian distribution and the problems that a distribution jump process faced, in recent times, attempts have been made to fit returns to density functions with finite moments. Peiro (1994) fitted returns to the Student- $t$  distribution and also rejected Stable Paretian Distributions on the evidence of convergence to normality when the interval of returns calculated was increased.

McDonald and Xu (1995) proposed an Exponential Generalised Beta Distribution, while Theodossiou (1998) suggested the Skewed Generalised  $t$  Distribution. Using the Exponential Generalised Beta and the Skewed Generalised  $t$  distributions, Harris and Kucukozmen (2001) showed that both fit returns in emerging markets better than the Student  $t$ , Logistic, Normal, Power Exponential and Laplace distributions.

The vast amount of literature on the distribution of returns is yet to provide any conclusive evidence. Despite this, most financial models such as Black-Scholes, the CAPM, Markowitz portfolio theory and Value at Risk all rely on some underlying assumption about the distribution of returns. Perhaps if returns were viewed in the time domain they would have a better fit to a known distribution.

## 2.2 Doubling Times

### **The Rule of 72**

The rule of 72 is the most widely recognized rule of thumb for estimating doubling times. Internet sources<sup>2</sup> often credit Albert Einstein for the rule of 72. However almost 500 years earlier this rule was mentioned by Luca Pacioli (1494), but he does not derive or explain the rule so it is thus assumed the rule predates Pacioli<sup>3</sup>. The rule of 72 is for discrete returns and the rule of 69 has been developed for use when returns are continuously compounded.

The rule of 69 suggests that if there is an investment receiving continuously compounding returns at a fixed rate then the time taken until the investment doubles can be easily approximated by dividing that fixed rate into 69. For example if one was to receive a fixed return of 10% continuously compounded it would take them approximately 6.9 years to double their money.

While the rule of 72 can be used for returns compounded at discrete intervals, this rule approximates well only for low values of returns. For example if one were to receive a return of 100% p.a compounded annually then it would take them a year to double their investment, yet the rule of 72 suggests this would only take approximately three quarters of a year. Derivations of the rule of 72 and 69 are provided below:

Future values under continuous compounding are given by:

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<sup>2</sup> <http://socyberty.com/issues/financial-rule-of-72/>

<sup>3</sup> [http://en.wikipedia.org/wiki/Rule\\_of\\_72](http://en.wikipedia.org/wiki/Rule_of_72)

$$FV = PVe^{rt}$$

where  $FV$  is the Future Value,  $PV$  is the Present Value,  $r$  is the rate of return, and  $t$  the time to maturity.

By setting  $FV$  to 2 and  $PV$  to 1 and solving for time,  $t$  the doubling time,  $\tau$ , is obtained:

$$\tau = \frac{\log(2)}{r} \tag{2.1}$$

A result which is approximated by the rule of 69 as the natural logarithm of 2 equals 0.6931.

Similarly, future values with discrete returns are given by:

$$FV = PV(1+r)^t$$

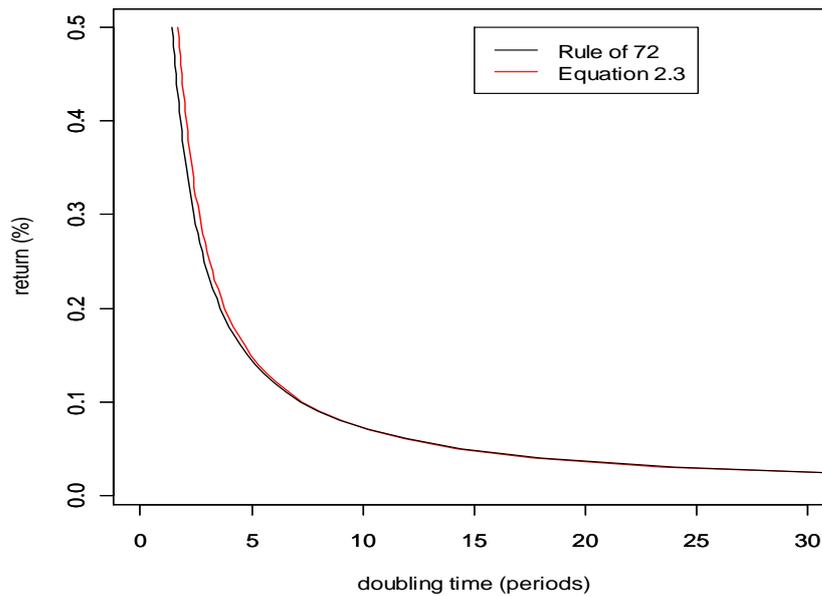
Again, by setting  $FV$  to 2 and  $PV$  to 1 and solving for  $t$  the following is obtained:

$$\tau = \frac{\log(2)}{\log(1+r)} \tag{2.2}$$

This equation is approximated by the rule of 72 for low values of  $r$ . This can be observed in Figure 2.1. Here, a plot of equation (2.2) and the rules of 72 are depicted on

the same axis. It is evident at low returns equation (2.2) is very well approximated by the rule of 72, but at high returns the two begin to deviate.

**Figure 2.1: Rule of 72 versus Equation 2.2**



**The Figure depicts the close approximation of equation (2.2) by the rule of 72. Particularly at low return values.**

For simple interest future values are given by:

$$FV = PV(1 + rt)$$

Setting  $FV$  equal to 2 and  $PV$  equal to 1 and solving for  $\tau$  equation (2.3) is obtained:

$$\tau = \frac{1}{r} \tag{2.3}$$

One advantage of expressing returns as doubling times is there is only one doubling time for a given asset, so it makes no difference to the doubling time whether the assets returns are expressed as a simple interest rate, or as a compound rate of return. If the returns are compounded, it makes no difference whether the compounding is discrete or continuous. Provided the returns are measured exactly and the computations are done correctly, then the same doubling time will result.

Although the expression of returns as doubling times has been known for over 500 years and that doubling times have intuitive appeal and the advantage of there being only one doubling time, there has been little academic work on equity market doubling times. Indeed, there is little written on doubling times in the financial literature, other than descriptions of the rules of 72 and 69. A shortcoming of the rules of 72 and 69 is that they do no account for stochastic returns and in relation to equity markets returns, due to their stochastic nature, these rules may have little added value.

While doubling time with uncertain returns has received little attention in the finance literature the doubling time concept has extensive use in several other fields. Doubling times are often applied to population growth as in Kendall (1949). Doubling times are also used in medicine; a common application is measuring the growth of a tumour, for example Hanks, D'Amico, Epstein and Schultheiss (1993) or the doubling time of cell growth (Zuk et al (2001)). The converse of doubling times, or half-lives, also has extensive use. Half lives are most commonly associated with radioactive decay, but can be applied to anything which decays. Possible medical applications involve nuclear medicine, Hendee (1979), or persistence of a drug or other substance in the body. The study of population extinctions also makes use of half lives for example Brooks, Pimm and Oyugi (1999). In the literature on doubling times/half lives which have a stochastic

return process, a common methodology is to model this behaviour as a first passage time or first hitting time problem (see Kendall (1949)).

## **First Passage Times**

The first passage time problem originally stems from the mathematics and physics literature where it is best described as the time taken for a particle following some random walk to reach a given barrier. However, this concept is now applied to many fields, such as biology, engineering, medicine and even finance.

Among the finance literature, the application to first passage times is most commonly found in pricing exotic options such as look-back, barrier and digital options, (see Kou and Wang (2003), Kou and Wang (2004)). A look-back option's value is path dependent, as the payoff depends on the maximum or minimum of the underlying assets price over the life of the option. Similarly, a barrier options value is also path dependent as the right to exercise depends on the underlying asset crossing or reaching a given barrier level. Because the value of these options is path dependent, first-passage times are often suggested as a method for valuing these options. In recent years Kou and Wang (2003)-(2004) have largely focused on this area, and they have extended the assumption of Brownian motion when valuing look-back and barrier options. In particular they have focused on a double exponential jump diffusion model (Kou and Wang (2003) and Kou and Wang (2004)). First passage times are not only used to value traded exotic options, but also options that may arise in other contexts. For example, Longstaff and Schwartz (1995) price digital credit-spread options with the logarithm of the credit spread assumed to follow a mean-reverting process. McDonald and Siegel (1986) determine the value of waiting to invest, using first passage times. Their work is

in regards to investment timing, and whether undertaking an irreversible project should be initiated now or deferred into the future.

The majority of the financial literature which uses first-passage times is predominately found in the options field, one notable exception is Cho and Frees (1988). In their work the volatility of discrete stock prices is estimated using an estimator of how quickly a price changes rather than how much the price changes.

While the foregoing financial literature has had applications for first passage time problems, none of this work has been closely related to doubling times. However, there has been some work where the focus was first passage times in returns. Simonsen, Jensin and Johansen (2002) estimate empirically the distribution of first passage times for the Dow Jones Industrial Average (DJIA) over a time period of 105 years. In their work barriers ranging from 1 to 20 percent above the starting point are used. Their results suggest that over these short return intervals the DJIA percentage returns are unlikely to be normally distributed. In this work, the empirical returns have had their trend removed using a wavelet filter. Consequently, the first passage time is no longer a function of the drift and volatility, but now only the volatility. When fitting these empirical first passage times the distribution that results has a well defined and pronounced maximum, followed by a long extended tail. This well defined maximum is referred to as the “optimal investment horizon” as it is the most likely first passage time. To my knowledge their work has had little follow up attention and the concepts they have suggested have not been further developed among the literature.

The use of doubling times provides an alternative performance metric to percentage returns. However, the idea of an alternative performance metric is by no means new. The time taken to reach a fixed level of wealth, or the first passage time, as a

performance metric can be found in MacLean and Ziemba (1999), MacLean, Ziemba and Blazenko (1992), Ethier (1987), Dohi, Tanaka, Kaio and Osaki (1994), and Browne (1997). This work is generally related to analysis of the Kelly criterion and betting proportions, where a trade-off of wealth growth and wealth security is required. The mentioned literature examines the probability of reaching wealth  $U$  and falling to wealth level  $L$ , the mean accumulated wealth at the end of period  $T$  and the mean exponential growth rate over period  $T$ . While these papers do not examine doubling times in the same manner as this research, they do support the notion that doubling times as a performance metric has merit.

## 2.3 Estimation Error

All models have their pitfalls and portfolio optimisation is no exception. In fact, one of the problems portfolio optimisation suffers from has been assigned its own unique name, error maximisation (see Michaud (1989)). This occurs because parameter estimates for the optimisation algorithm are approximated with some error. The optimiser tends to overweight the assets with high returns, low variance and low correlations and underweight the assets with low returns, high volatility and high correlations. The assets with these extremes are the cases in which estimation error is likely to be the most pronounced, as shown in Michaud (1989). A common way for estimating the expected returns and covariance matrix is through the use of historical data. However, when using historical data to estimate these parameters there are two areas of concern, the previously mentioned estimation errors, which can be somewhat reduced by increasing the sample size. The difficulty with increasing the sample size is this leads to the second problem, which is non-stationarity. The means, standard

deviations and correlations of security returns changeover time and this is a well documented effect (see Boness, Chen and Jatusipitak (1974), Christie (1982), Beaver (1968), Patell and Wolfson (1981) and Officer (1972)).

There has been a great deal of research in the area of estimation error (see Michaud (1989), Chopra and Ziemba (1993), Sherer (2002)) and various methods have been proposed to address these problems. The serious nature of estimation error was demonstrated in Jobson and Korkie (1981), who show that an equal weighted portfolio can outperform an optimal mean-variance portfolio which is computed using sample estimates. This result is due to errors in the estimates. Most of the error can be attributed to poor estimates of expected returns, rather than poor estimates of the standard deviations and correlations, as shown by Ceira and Stubbs (2006). They also point out that many portfolio managers have much more confidence in their risk estimates than in their expected return estimates.

The notion that volatility estimates are more accurate than expected return estimates suggests that when attempting to reduce estimation error adjusting the assets expected return such that the return to variance ratio is more uniform among all assets may be a valid approach. One approach is to use James-Stein estimators, as outlined in Jobson and Korkie (1981). This line of attack adjusts the expected returns towards the average expected return based on its volatility and distance from the average expected return. An alternative method has been proposed by Jorion (1985). This adjusts the estimates for the expected return towards the global minimum variance portfolio.

Another technique proposed for reducing estimation error is a Monte-Carlo procedure known as portfolio re-sampling introduced by Michaud (1999). This addresses estimation error by forming numerous optimal portfolios. These optimal portfolios result from optimising the parameter inputs obtained from different samples of the original data set. The average of all frontiers is then taken. In Sherer (2002) an example using this re-sampling technique is presented. In this work it is shown that the efficient frontier using estimates simply taken from historical data lies above the efficient frontier obtained using the re-sampling methodology, a positive result. The justification for this being a positive result is that the historical data frontier suffers from estimation error and accordingly is overly optimistic. It is the belief that as the re-sampled frontier lies beneath the historical data frontier it is less optimistic due to a reduction in the estimation error. However, Sherer (2002) also highlights some of the pitfalls of this methodology and its somewhat heuristic nature. One final method which has been proposed is that of Ceira and Stubbs (2006), who use robust optimisation, a methodology designed to explicitly consider parameter uncertainty in optimisation problems.

Broadie (1993) used a simulation procedure to determine the *estimated frontier* and the *true efficient frontier*. Where the *estimated frontier* was the frontier obtained using parameter estimates from a sample of a known distribution, while the *true efficient frontier* is the efficient frontier when the true parameters from the known distribution are used. From this study it is shown that the *true efficient frontier* lies beneath the *estimated frontier*. This highlights the fact that estimation error results in an overly optimistic frontier from what is really obtainable and is supportive of Sherers' (2002)

results which suggest the re-sampled frontier has less estimation error than the classical historical data frontier.

This thesis presents a new method based on doubling times for adjusting parameter estimates in an effort to reduce estimation error. By using a simulation method similar to that of Broadie (1993) it is possible to determine if the estimates from this new method provide an efficient frontier closer to the true efficient frontier than that estimated by a more traditional method.

## 2.4 Conclusion

In the first section of this chapter a review of the evolution of the various schools of thought regarding the distribution of returns was provided. It was revealed that in 1900 Bachelier suggested that returns were normally distributed; this notion was accepted until it was challenged and disproved in 1963 by Mandelbrot, who suggested the alternative hypothesis that returns followed a Stable Paretian Distribution. This hypothesis has continued to be tested until at least 1994 by Peiro, however the results over the years have provided more evidence against rather than for this hypothesis. An alternative hypothesis to the Stable Paretian Distribution was suggested by Press in 1967, who proposed that rather than returns being encompassed by one distribution, that they were a mixture of several distributions. This was an intuitive idea, but one which faced econometric difficulties to prove, despite this, many publications related to this notion can be found. Similarly, there have been many attempts to fit returns to various known distributions such as the Student- $t$ , the Exponential Generalised Beta,

the Skewed Generalised  $t$ , just to name a few. In spite of all this work, there is still no conclusive evidence as to the distribution returns' actually follow. With no conclusive evidence for the distribution of percentage returns perhaps a distribution of doubling times can be more easily determined.

Section 2 of this chapter investigated the use of doubling times among the finance literature. It was shown that despite the rule of 72 dating back as far as the late 1400's where it is mentioned by Luca Pacioli, doubling times have received little attention in the modern financial literature, particularly when stochastic returns are present.

Accordingly, an eye has been cast to other disciplines such as ecology, medicine and physics where doubling times with stochastic growth rates or drifts can be found. In these disciplines doubling times are modelled as first passage time or first hitting time problems. The use of first passage times among the finance literature is predominately found among the exotic options literature which has little relevance to the doubling times being investigated in this work. There is one notable exception among the literature, with the work by Simonsen, Jensen and Johansen (2002) stimulating some ideas developed in this thesis. With little literature on doubling times found among the finance literature, the theoretical foundations for doubling times must first be laid.

In the final section of this chapter, the problem of estimation error when optimising a portfolio under the Markowitz framework is addressed. From the literature it is shown that slight deviations in the mean and covariance estimates for the optimisation process can yield significantly different weights, a phenomenon which highlights the fact that hefty error to the weights can be caused by small errors in parameter estimates. This observable fact is referred to as error maximisation by Michaud (1989) and its serious

nature is reflected in Jobson and Korkie (1981), who show that an equal weighted portfolio can outperform an optimal mean-variance portfolio which is computed using sample estimates. To overcome error maximisation several methodologies have been suggested. Notably, the use of James-Stein estimators found in Jobson and Korkie (1981), the Monte Carlo method known as portfolio re-sampling suggested by Michaud (1999) and robust optimisation proposed by Ceira and Stubbs (2006) are reported. It is anticipated that viewing returns as doubling times may yield a new insight into portfolio optimisation and contribute to the body of literature which attempts to reduce estimation error. It is shown that this is indeed the case in Chapter 6.

# Chapter 3: Doubling Time Fundamentals

## 3.1 Introduction

The rules of 72 and 69 can easily be used to compute the time taken to double an investment if a constant return is given. However if returns are varying through time the problem is more difficult. It is a simple matter to compute doubling times period by period and so obtain a distribution of doubling times. However, the mean of that distribution (the mathematical expectation) does not give the time over which the investor should expect to double their money (the anticipated doubling time).

Furthermore, the variance of this distribution will not give the variance of the time taken for the investor to double their money. In this chapter two approaches are presented to computing the expected doubling time. An analytical approach, which shows that harmonic means can be used in estimating doubling times. Formulae using the harmonic mean are given for discrete and continuously compounded rates of return and also for simple interest rates. Secondly, a Monte Carlo simulation method is also used, which has the advantage of providing a distribution of the experienced doubling time.

The computations that underlie the results presented in this chapter were based on continuously compounded returns and hence doubling times were computed using equation (2.1). As previously noted, had the calculations used discrete returns and

equation (2.2) or simple interest rates and equation (2.3) identical doubling times would have been obtained.

## 3.2 Data

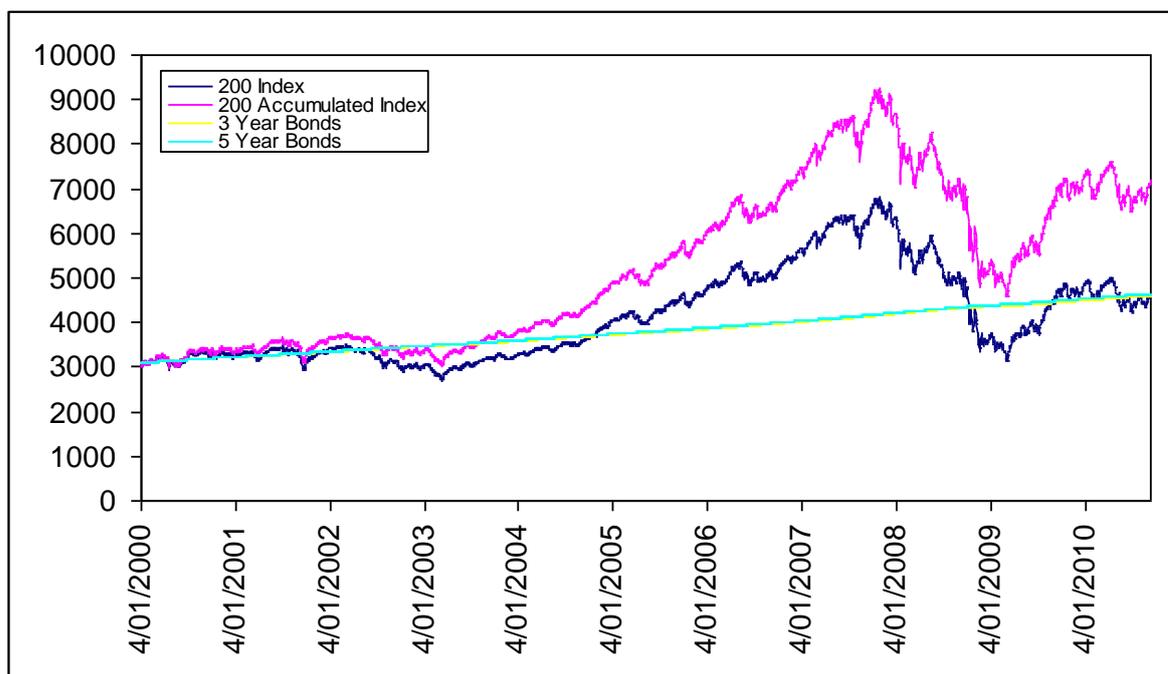
All data used for this chapter was obtained from Bloomberg where both equity and bond data was obtained. The equity data contained daily closing prices for the ASX S&P 200 Index and the ASX S&P Accumulation 200 Index. The bond data contained the yearly yields for the 3 year and 5 year bonds at their daily closing prices. All data sets ranged from the 4<sup>th</sup> of January 2000 to the 15<sup>th</sup> of September 2010 which gave a total of 2709 data points for each of the time series. The difference between the ASX S&P 200 Index and the ASX S&P Accumulation 200 Index is that the Accumulation Index assumes all dividends received have been fully reinvested whereas the ASX S&P 200 Index assumes no dividends have been reinvested.

Figure 3.1 plots the growth of capital spanning the life of the data set. Here it is evident that the Accumulation Index has outperformed the ASX S&P 200 Index. It is expected that the curves for equities will have a significantly higher endpoint than the bond curves. However, in this data sample it is evident that bonds perform no worse than the ASX S&P 200 index. This is a result of the global financial crisis causing the significant equity losses during 2008. To plot the capital growth of bonds is more difficult than an index as annual yields are recorded not index values. Using the annual yields the capital growth was determined by finding the daily yield, by assuming there

are 365 days in the year. Following this a day's capital level is simply computed as the previous days capital multiplied by that day's daily yield.

Investigations into any effect observation frequency may have for the ASX Indexes is achieved by additionally taking observations at a weekly or monthly interval. This is achieved by only observing the first price recorded each week, or the first price recorded every month. However, careful attention has been made such that daily, weekly or monthly observations all start and finish on the same day, so as to ensure regardless of observation frequency chosen, the investor's experience remains the same.

**Figure 3.1: Capital Growth for ASX Equities and Bonds**



**A time series plot of the capital growth over the last 10 years if it were invested in an index with no dividends reinvested (200 Index), with dividends reinvested (200 Accumulation Index) or placed in three or five year bonds. The three and five year bonds result in very similar performance and are overlaid on each other.**

Table 3.1 provides some descriptive statistics for the daily data sets used. Again it is evident that the Accumulation Index has a higher mean return than the bonds or un-Accumulation Index. Furthermore, it is evident that the returns offered on the bond market are significantly less volatile than those found on the equity market.

**Table 3.1: Descriptive Statistics of ASX Equity and Bond Data Used**

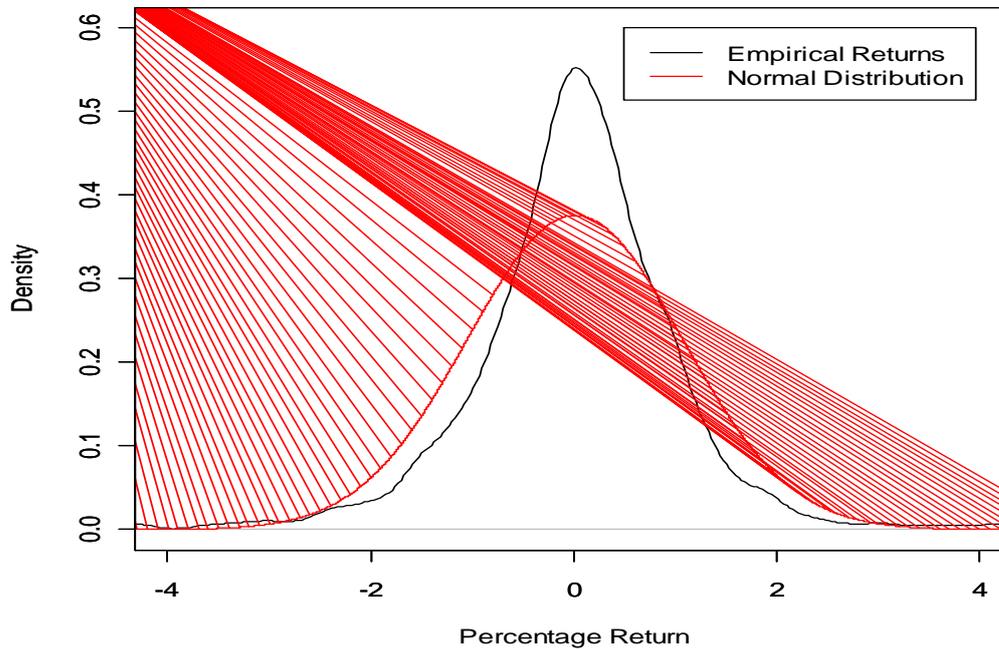
	S&P 200	S&P 200 Accum.	3 Year Bonds	5Year Bonds
mean	0.015%	0.031%	0.015%	0.015%
variance	0.011%	0.011%	0.000%	0.000%
stan Dev	1.061%	1.061%	0.002%	0.002%
min	-8.704%	-8.704%	0.008%	0.009%
max	5.628%	5.627%	0.019%	0.019%
median	0.039%	0.056%	0.015%	0.015%

**Descriptive statistics of the daily continuously compounded percentage returns used for this study. Here it is evident that the Accumulation index is offering a higher mean return than all other financial instruments. Additionally much less volatility in the bond market is evident.**

For illustrative purposes the empirical density of daily returns for the S&P 200 is plotted in Figure 3.2. Overlaid on this figure is the density plot of a normal distribution with mean and variance equal to those of the empirical returns. This figure highlights the commonly empirical regularity of stock returns having a higher kurtosis than would be the case if the returns were distributed normally.

In the next section the S&P 200 is used to illustrate the computation of doubling times and subsequently doubling time are compared across the three series discussed above.

**Figure 3.2: Density plot of empirical returns overlaid on a Gaussian distribution**

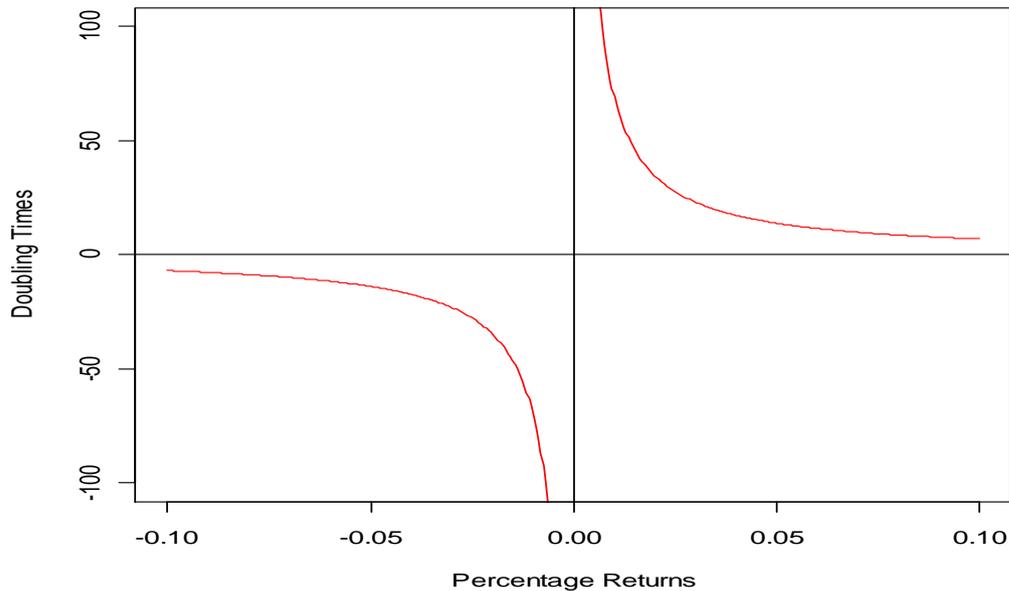


**A density plot of the empirical continuously compounded daily returns for the S&P 200 Index versus the density plot of a normal distribution with the same mean and variance of the Index's returns.**

### 3.3 Calculating Expected Doubling Time

A plot of doubling times against continuous returns was generated from equation (2.1) and is given in Figure 3.3. It is immediately evident from Figure 3.3 that negative doubling times (half lives) are a mirror image of positive doubling times. It is also evident that there is a discontinuity at zero. As returns approach zero, negative doubling times tend to minus infinity and positive doubling times tend to plus infinity.

**Figure 3.3: A plot of equation 2.1 for returns varying from -10% to +10%**



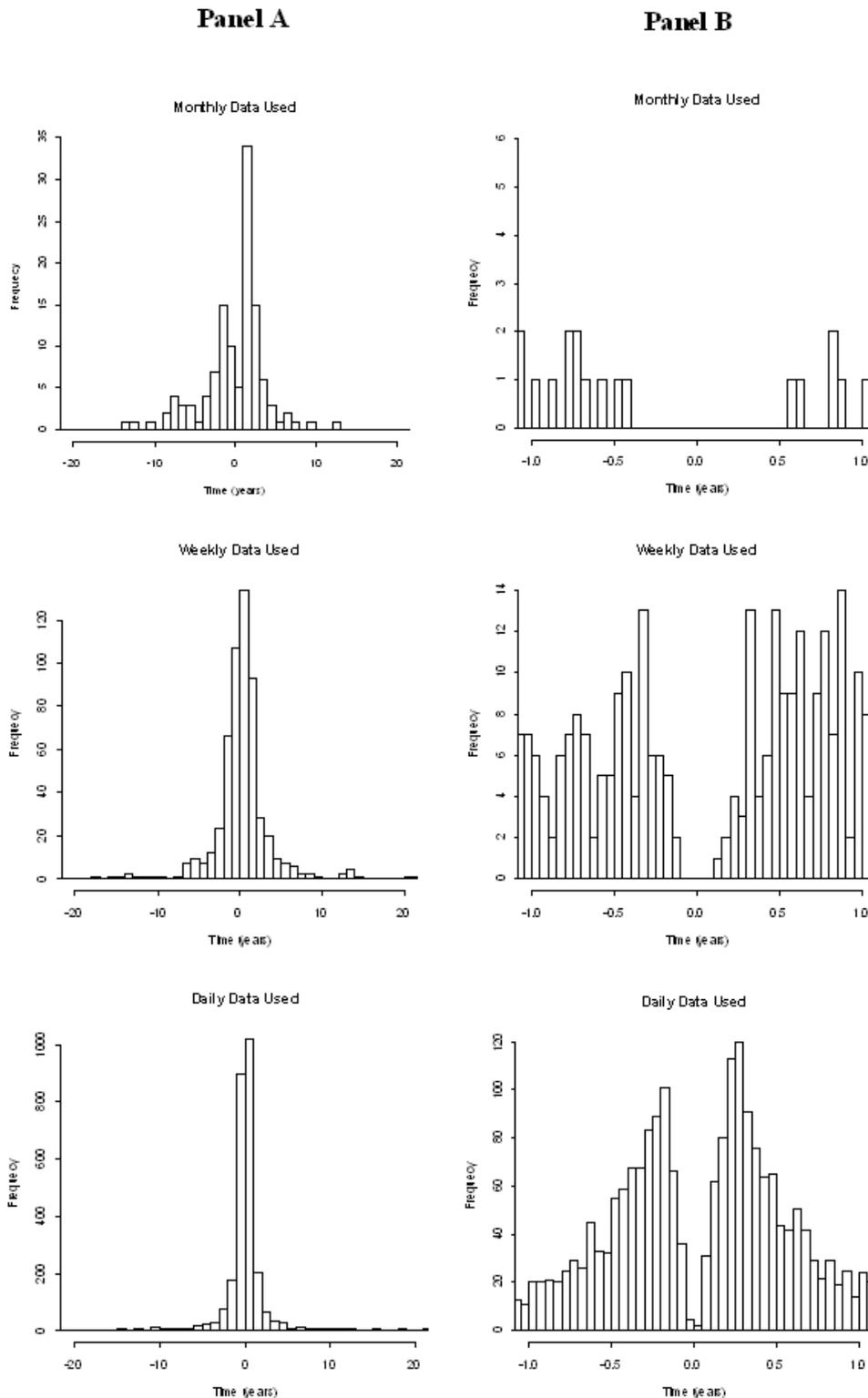
**Equation (2.1) is defined as  $\tau = \frac{\log(2)}{r}$  where  $\tau$  is doubling time and  $r$  is the percentage return**

Figure 3.3 suggests that in forming the parameters of doubling time distributions there will be a problem in handling cases with zero returns due to infinite doubling times. Care is also needed in combining half lives and doubling times. For example a half life represented by minus five years and a doubling time represented by plus five years have an arithmetic average of zero years. However an investor who experiences this combination of half-life and doubling time will not instantly double their money despite the arithmetic mean of the doubling time being zero. The point is more general than this, even for positive doubling times the arithmetic mean is generally not the same as the doubling time that investors experience. Investors are naturally interested in the doubling times they actually experience and a description of the computation of this parameter is provided in the following section.

Histograms for the combined distribution of half lives and doubling times for the ASX 200 Index are given in Panel A of Figure 3.4 for daily, weekly and monthly observations. It is clear that the great majority of doubling times and half lives are under ten years and it appears that a surprisingly large number of observations are close to zero. This observation, however, is somewhat misleading. Panel B of Figure 3.4 gives a plot drawn from the same distributions but scaled to magnify the observations centred on zero. It clearly illustrates that there is a discontinuity at 0 and suggests that the distributions of doubling times and half-lives are not identical. Perhaps, it might be better to analyse them as separate distributions.

Table 3.2 provides descriptive statistics based on the full data set, and for half lives and doubling times treated as separate distributions. Infinite doubling times are omitted in computing the descriptive statistics, but the number of such instances is reported as the number of zero returns removed.

**Figure 3.4: Histograms of Half Lives and Doubling Times combined for S&P 200 Index.**



**Histogram of combined doubling times and half lives for the S&P 200 Index using monthly, weekly and daily returns. Panel A shows the broad histogram, while Panel B shows a magnified version centred around 0 to highlight the discontinuity found around 0.**

**Table 3.2: Descriptive statistics for doubling times, half lives and both doubling times and half lives combined together computed from daily, weekly and monthly returns from the S&P 200 Index.**

Panel A: All values have been reported where periods are measured in years

	Daily Data		Weekly Data		Monthly Data	
	combined	doubling times half lives	combined	doubling times half lives	combined	doubling times half lives
mean	0.08459578	2.013169 -2.056091	-0.0324112	2.483745 -3.26457	8.032	17.249 -5.011
variance	103.02	131.5834 62.67773	71.28123	19.07258 120.0427	6214.554	10359.59 139.4085
standard deviation	10.14	11.4736 7.91692	8.4428	4.36721 10.9564	78.832	101.7821 11.8071
min	-169.01	0.047367 -169.0173	-146.7406	0.129703 -146.7406	-86.082	0.59799 -86.0828
max	355.1006	355.1006 -0.030628	33.5012	33.5012 -0.1136005	877.55	877.5532 -0.42177
median	0.13758	0.5117304 -0.520874	0.4555042	1.15171 -1.135758	1.27006	1.92002 -2.04198
zero returns removed	3		1		0	
Data sample size	2708		563		128	

Panel B: All values have been reported corresponding to the interval length of the data used

	Daily Data		Weekly Data		Monthly Data	
	combined	doubling times half lives	combined	doubling times half lives	combined	doubling times half lives
mean	21.99	523.424 -534.58	-1.685384	129.154 -169.75	96.384	206.9944 -60.14015
variance	6964231	8895035 4237015	192744	51572.27 324595	894895	1491781 20074.83
standard deviation	2638.98	2982.45 2058.401	439.027	227.095 569.7328	945.9893	1221.385 141.6857
min	-43944.49	12.31558 -43944.49	-7630.51	6.744558 -7630.51	-1032.994	7.17594 -1032.994
max	92326.16	92326.16 -7.963279	1742.065	1742.06 -5.907226	10530.64	10530.64 -5.061252
median	35.7712	133.049 -135.4273	23.6862	59.888 -59.05939	15.24072	23.0403 -24.50384
zero returns removed	3		1		0	

The problem of computing expected doubling times using the arithmetic mean is reflected in Table 3.2. Since all three data sets span the same period, they should all report the same mean doubling times. However this result does not arise. For example, the mean weekly doubling time for the combined data is -0.0324 years, while the mean monthly doubling time for the combined data is 8.032 years; two numbers that are considerably different from one another. A similar result is also obtained when considering the doubling times and half-lives separately. This shows that the mean doubling time depends on the frequency of observation and it appears that little useful information can be obtained by taking the arithmetic average of doubling times.

It is also evident that the distributions sample variances vary significantly across the different sampling periods chosen. For example, for the combined data, when daily data is used a variance of 103.02 years is computed, while monthly data has a variance of 6214.55 years. A noticeable feature of Table 3.2 is that the absolute values of the medians are rather low. This is not surprising for the cases where doubling times and half lives are mixed together in the combined data. However, even when the half lives and doubling times are considered separately the medians still appear small. It can be observed in the transition from daily to weekly to monthly data that the median doubling times increase in magnitude. The most likely explanation for this is as follows. In daily returns a move of one or two percent is not unusual, however a move of one or two percent consistently every day over a week is much less likely and even more improbable over a month. This is shown in the histograms depicted in Figure 3.4. When observing the histograms magnified around zero doubling time it is evident that many observations occur for the daily data, fewer for the weekly and very few are observed for the monthly data.

Investors are naturally interested in the doubling times they actually experience, yet the results obtained here show that simply taking the arithmetic average or variance of single period doubling times will not give metrics which reflect the overall experience of the investor, and the values reported appear to have little economic meaning.

### 3.4 Analytical Solutions

The equations provided here will give the expected time for an investment to double given a series of different single period doubling times. The analytical approach to computing the mean doubling time requires different formulae for discrete and continuous returns. The derivations are as follows,

*For discrete compounding:*

From equation (2.2):

$$(1+r)^\tau = 2 \quad \text{or} \quad \tau = \frac{\log(2)}{\log(1+r)}$$

Suppose the returns are  $(r_1, r_2, \dots, r_n)$  for n-periods. Then the equivalent compound rate  $R$  satisfies

$$(1+R)^n = \prod_{i=1}^n (1+r_i)$$

So  $(1+R)$  is the geometric mean of the one period terms  $(1+r_i)$ . Since  $(1+R)^\tau = 2$  and

$(1+r_i)^{t_i} = 2$ , the above equation can be rewritten as:

$$2^{n/\tau} = \prod_{i=1}^n 2^{1/\tau_i}$$

$$2^{n/\tau} = 2^{\sum_{i=1}^n 1/\tau_i}$$

which leads to

$$\tau = n \left( \sum_{i=1}^n \left( \frac{1}{\tau_i} \right) \right)^{-1} \quad (3.1)$$

This implies that the expected doubling-time is just the harmonic mean of the individual single period doubling times  $\tau_i$ .

*For continuous compounding:*

If the rate is continuously compounded at rate  $r$ , the doubling-time  $\tau$ , is defined by

$$e^{r\tau} = 2 \quad \text{or} \quad \tau = \frac{\log(2)}{r}$$

If the rate of return is  $r_i$  for a period  $t_i$ , the equivalent continuous rate  $R$  satisfies:

$$e^{RT} = e^{r_1 t_1 + r_2 t_2 + \dots + r_n t_n} \quad \text{where} \quad T = \sum_i t_i$$

Taking the natural logarithm of both sides of the above equation leads to

$$RT = r_1 t_1 + r_2 t_2 + \dots + r_n t_n$$

So  $R$  is the weighted arithmetic mean of the  $r_i$ . Since  $r = \log(2)/\tau$ , the above equation can be rewritten as:

$$T \log(2) / \tau = \sum_{i=1}^n t_i \log(2) / \tau_i$$

which simplifies to

$$\tau = T \left( \sum_{i=1}^n t_i / \tau_i \right)^{-1} \quad (3.2)$$

In this case the expected doubling time is the time weighted harmonic mean of the individual doubling times  $\tau_i$ .

*For simple interest rates:*

$$(1 + r\tau) = 2 \quad \text{or} \quad \tau = 1/r$$

Suppose the simple interest rates are  $r_i$  for a period  $t_i$ , then the equivalent flat rate  $R$  satisfies

$$R \sum_i^n t_i = \sum_i^n t_i r_i$$

Since  $r = 1/\tau$ , the above equation can be rewritten as:

$$\frac{1}{\tau} \sum_{i=1}^n t_i = \sum_{i=1}^n t_i / \tau_i$$

which leads to:

$$\tau = T \left( \sum_{i=1}^n \left( t_i / \tau_i \right) \right)^{-1} \quad \text{where } T = \sum_i t_i. \quad (3.3)$$

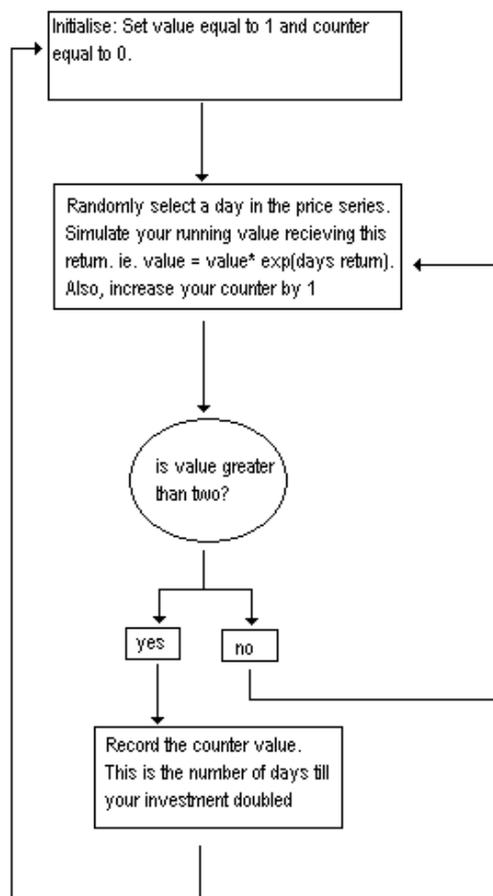
As for continuous compounding the expected doubling time is the time weighted harmonic mean of the individual doubling times  $\tau_i$ .

## 3.5 Simulation Method

The analytical results provided in section 3.4 give the expected doubling times for a series of single period doubling times. In this section a simulation is used to generate the doubling times an investor might experience from a given set of returns. The simulation will provide not only the expected doubling time, but also the distribution of doubling times.

In the simulation approach the returns are repeatedly re-sampled at random and a cumulative compound return computed. When this cumulative return sums to two, the doubling point has been reached. The number of iterations to reach this point gives the doubling time and many repetitions of this process give the sampling distribution for the mean doubling time. This method assumes that each single period return is equally likely and that these returns are stationary. A flow chart for the simulation is given in Figure 3.5 and the pseudo code is given in Figure 3.6.

**Figure 3.5: Simulation Flowchart**



The flowchart shows the logic behind simulating the time taken for an investment to double. This simulation can be repeated many times to form a distribution of doubling times

**Figure 3.6: Pseudo code for estimating the doubling time and its distribution**

**for**  $i=1$  **to** number of desired iterations

$runningValue = 1$   
 $periodCounter = 0$

**while** ( $runningValue < 2$ )  
     $singlePeriodReturn =$  random return drawn from return series  
     $runningValue = runningValue * \exp(singlePeriodReturn)$   
    increment  $periodCounter$   
**end while**

store  $periodCounter$  in a vector indexed by  $i$ .

**end for**

For the pseudo code in Figure 3.6 once all iterations are finished then the vector storing the values of *periodCounter* will contain different estimates of the time taken to double the investment and is the basis for forming the doubling time distribution.

## 3.6 Results

### 3.6.1 The investment experience

Table 3.3 provides descriptive statistics for the expected doubling time of the ASX S&P 200 Index consistent with an investor's experience. The expected doubling times are derived from the analytical approach and the simulation using daily, weekly and monthly observation frequencies. As these samples all started and finished on the same day, the experience faced by the investor is the same and the expected doubling times should be identical regardless of the frequency of measurement. In each case for the analytical calculation the expected time to double the investment is 18.175 years. For each data set the simulation process is performed over 5000 iterations, resulting in a sample of 5000 points to form the distribution of the doubling times. Density plots for these distributions are depicted in figure 3.7 for the daily, weekly and monthly data sets. The expected doubling time for each of these simulated distributions can be computed by simply taking the arithmetic average of the simulated distributions. Similarly the standard deviation of these distributions can also be computed.

Table 3.3 presents both the expected doubling times and standard deviation of these simulated doubling times and these are reported in both years and their respective measurement intervals. The simulated expected doubling times are found to be 18.463, 18.484 and 18.471 years when using the daily, weekly and monthly data sets respectively. While these values appear close to the analytical value of 18.175 years it is desirable to verify that they are not statistically different from the analytical value. The traditional *t-test* for the mean is not appropriate, since as depicted in Figure 3.6, and discussed later, the sample is not normally distributed. Rather it is assumed the sample follows the inverse Gaussian distribution (for reasons that are presented later). Using the test suggested by Chhikara and Folks (1989) the following nulls are formed:

$$\mathbf{H}_0: \mu = \mu_0 \text{ against } \mathbf{H}_A: \mu \neq \mu_0$$

where  $\mu$  is the sample mean and  $\mu_0$  is the analytical value.

When the shape parameter,  $\lambda$ , of the inverse Gaussian distribution is unknown then the critical region is defined as follows:

$$\left| \frac{\sqrt{(n-1)(\bar{X} - \mu_0)}}{\mu_0 \sqrt{\bar{X}V}} \right| \geq t_{1-\alpha/2}$$

where

$$\bar{X} = \frac{1}{n} \sum_1^n X_i, \quad V = \frac{1}{n-1} \sum_1^n \left( \frac{1}{X_i} - \frac{1}{\bar{X}} \right), \quad X_i \text{ is a data point and } n \text{ is the sample size.}$$

The critical *t* statistic at the 5% significance level is 1.9604, The computed *t*-values for the simulated doubling times when daily, weekly and monthly data sets are used are all reported in Table 3.3. It is evident that the largest computed *t*-value is 1.2354 and thus

the difference between the simulated and analytical value for the doubling time is not statistically significant.

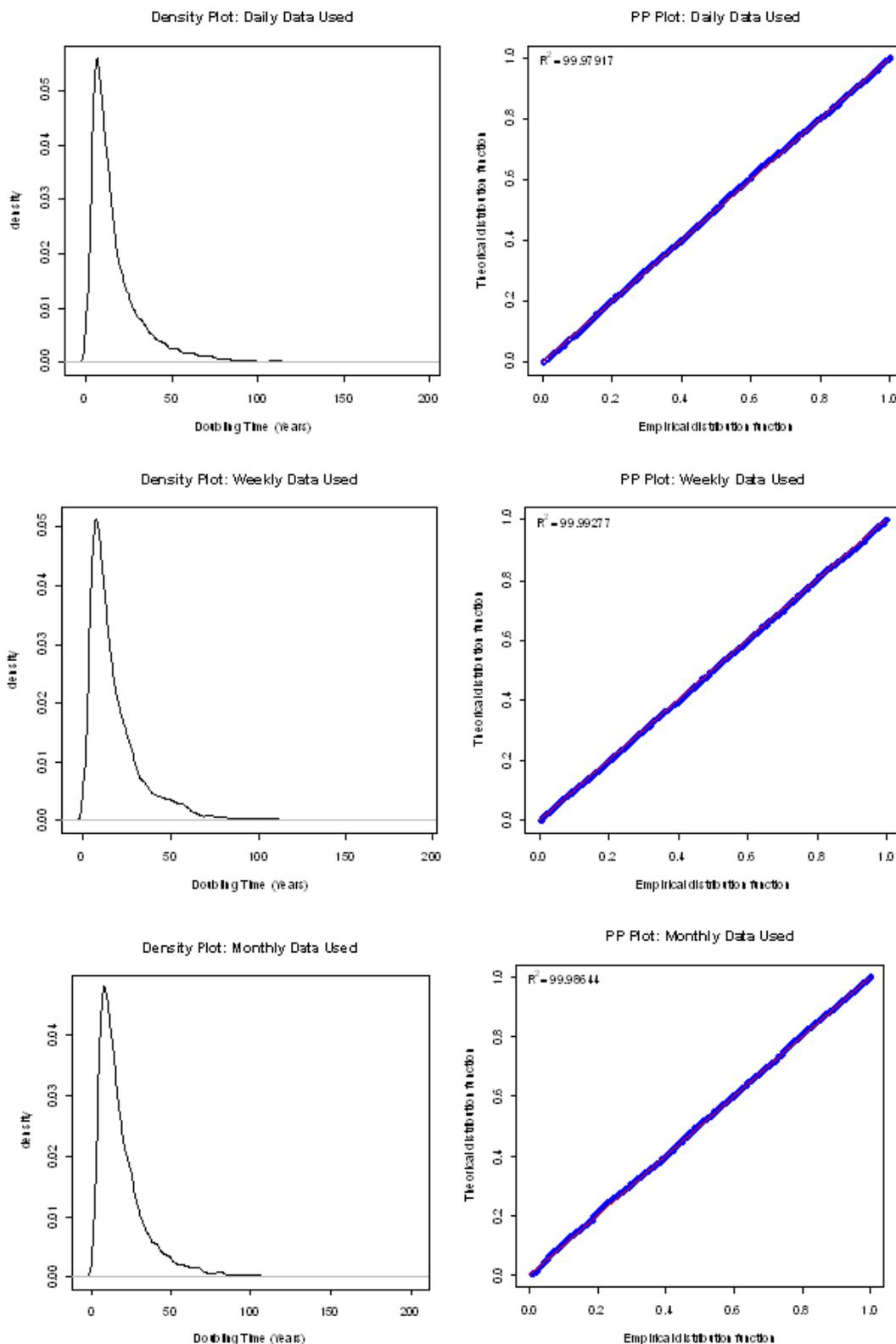
**Table 3.3: Descriptive Statistics for S&P 200 Index Doubling Times**

	Daily Data	Weekly Data	Monthly Data
Simulated expected doubling time (Periods)	4651.17	969.81	234.07
Analytical expected doubling time (Periods)	4578.49	953.57	237.38
Simulated standard deviation (Periods)	4674.36	982.37	229.20
Simulated expected doubling time (Years)	18.46	18.48	18.47
Analytical expected doubling time (Years)	18.18	18.18	18.18
Simulated standard deviation (Years)	18.56	18.72	17.92
Critical t statistic	1.96	1.96	1.96
Computed t Statistic	1.08	1.24	1.13

**Expected doubling time and variance of doubling times are reported for the simulated doubling times. The analytical doubling times are also reported along with t-statistics to infer if the analytical doubling time differs from the expected simulated doubling time. These values are reported in years and also their respective time periods.**

As previously mentioned, one advantage of the simulation methodology over the analytical method is that a distribution of doubling times can be formed as presented in Figure 3.7 where the density of the S&P 200 Index's simulated doubling times is depicted. The distribution is clearly non-normal and earlier in this section it was assumed that the distribution was inverse Gaussian. Accordingly the probability density function for the simulated distribution is plotted with a corresponding pp-plot for the inverse Gaussian distribution also depicted in Figure 3.7. The pp-plots strongly suggest that the simulated doubling times follows the inverse Gaussian distribution. The inverse Gaussian distribution describes the time a Brownian motion with positive drift takes to reach a fixed positive level. This is analogous to the time an investment subject to stochastic compounding takes to double.

**Figure 3.7: Doubling time density plot and corresponding pp-plot.**



**Density plots for the simulated doubling times (in years) using the S&P 200 Index data for daily, weekly and monthly observations. These simulated distributions are then compared to the inverse Gaussian distribution using a pp-plot which provide strong evidence these distributions are inverse Gaussian.**

### 3.6.2 Doubling times across asset classes

From Figure 3.1, it is reasonable to assume that the Accumulation Index will have the shortest doubling time and that the three year bonds and S&P 200 Index will have similar doubling times. This assumption is supported by Table 3.4 where the expected doubling time is reported for each asset class. Here it is evident that the Accumulation Index has a doubling time less than half that of the other two asset classes and the doubling times of the S&P 200 Index and three year Bonds are roughly the same.

**Table 3.4: Doubling time descriptive statistics for various asset classes**

	Analytical Doubling Time	Simulated Doubling Time	Standard Deviation
S&P 200	18.18	18.17533	18.5559
S&P200 Accum	8.83	8.772387	5.94062
3 Year Bonds	18.91	18.76942	0.04045

**The expected doubling time for the S&P 200 Index, S&P 200 Accumulation Index and three years bonds. The values provided were computed using the analytical method and simulation method. Furthermore, the standard deviation of simulated doubling times is also provided.**

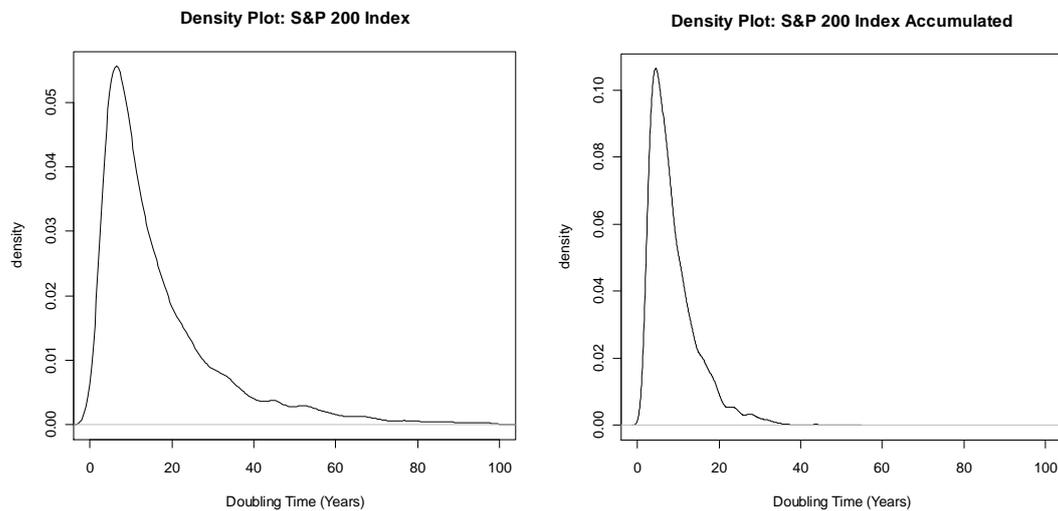
The S&P 200 Index and S&P 200 Accumulation are both invested in the top 200 stocks. Assuming that dividends are a relatively constant component of returns the two indices would have very similar variances in returns. This view is supported by Table 3.1 where the standard deviation of the S&P 200 Index is reported as 1.061% while the standard deviation of the S&P 200 Accumulation is also reported as 1.061%.

Interestingly, however, the standard deviation of doubling times is quite different for these two asset classes. As reported in Table 3.4 it can be seen that the S&P 200 Index has a standard deviation for doubling times approximately three times that of the

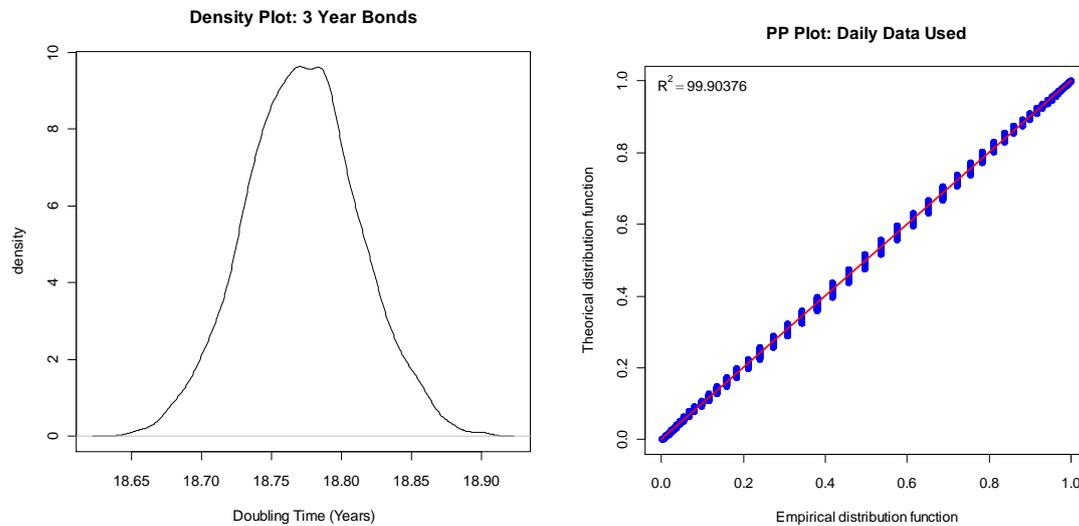
Accumulation Index. This was an unanticipated result and suggests that the variance of doubling times may provide an alternative risk metric. It appears that this risk measure is not just a function of the variance of returns but also the mean return.

The density plots in Figure 3.8 reflect the higher variance of the price index relative to the accumulation index, which is seen as increased weight and length in the right tail of the distribution. A density plot for the simulated doubling times for three year bonds is depicted in Figure 3.9. This shows the very tight dispersion around the mean and illustrates that with the reduction in dispersion the distribution becomes more bell shaped. However, a pp-plot confirms that the bonds doubling times still follow an inverse Gaussian distribution.

**Figure 3.8: Density plots of the simulated doubling times for the ASX S&P 200 Index and the ASX S&P 200 Accumulation Index**



**Figure 3.9: Density plot of the simulated doubling times for 3 year bonds plus its respective pp-plot with the inverse Gaussian distribution.**



### 3.7 Conclusion

This chapter examined the properties of doubling times both theoretically and empirically by analysis of returns on the ASX S&P 200 and Australian bond market. Computing doubling times period by period is comparatively simple, but combining those period by period estimates to compute the expected doubling time is shown to be more challenging.

Analytical formulae were presented to compute the expected doubling time for simple interest and for compound rates of return in the cases of discrete and continuous compounding. These formulae all give the same doubling time for a particular asset. Similarly, whether the doubling times are determined from monthly, weekly, or daily data makes no difference to the estimate of the expected doubling time. A Monte Carlo

simulation was also applied to the data and produced values consistent with the analytical formulae.

The simulation had the additional benefit of providing a distribution of the expected doubling times. The simulated distribution was shown to follow the inverse Gaussian distribution. This in turn suggests that the time until an asset doubles can be conveniently modelled as the first passage time for Brownian motion with drift, a result which is fully investigated in the following Chapter. Additionally, the simulated distributions suggested that the variance of expected doubling times could result in a new risk metric that is rather different from the traditional variance of returns.

# Chapter 4: Doubling Times as a First Passage Time Problem

## 4.1 Analogy to a First Passage Time Problem

### 4.1.1 Introduction

First passage times can be described as the time taken for an object which moves in random directions to reach a point, or absorbing barrier. Such a process is well studied in Cox and Miller (1965), and variations of the process such as moving absorbing barriers, partially reflecting barriers, multiple barriers or different underlying processes that the object follows are also well documented, for example Domine (1995), Atiya and Metwally (2004), Gut (1974), Tuckwell and Wan (1984) and Shepp (1967). Such processes have applications in biology, physics, and engineering and can also be found in finance, with regard to pricing exotic options.

The first passage time of a particle following Brownian motion with positive drift is known to follow the inverse Gaussian distribution (see Cox and Miller (1995)). This is a two-parameter distribution with probability density function given by:

$$f(x; \mu, \lambda) = \left[ \frac{\lambda}{2\pi x^3} \right]^{1/2} \exp \frac{-\lambda(x - \mu)^2}{2\mu^2 x} \quad (4.1)$$

For  $x > 0$ , where  $\mu > 0$  is the mean and  $\lambda > 0$  is the shape parameter.

In this chapter, the first passage time until an investment doubles in value is analysed assuming that the return on the asset follows Brownian motion with positive drift. A closed form solution to this problem is obtained, showing that the doubling time of the investment follows the inverse Gaussian distribution. Expressions for the two parameters that define the doubling time distribution, the mean and shape, are also derived. Violations of the Brownian motion with drift assumption will lead to deviations from the inverse Gaussian distribution and such deviations are examined using the simulation methodology defined in Section 3.5. Surprisingly, such deviations appear to have little effect on the doubling time distribution. A result attributed to the Central Limit Theorem.

#### 4.1.2 Analytical derivation

The simulation process suggested in Section 3.5 is analogous to the first time a stochastic process with drift travels from one to two. This is referred to as the first passage time or hitting time of a stochastic process with an absorbing barrier. In this section an analytical derivation is presented for the first passage time to doubling of capital undergoing continuous compounding. The derivation begins by using a transformation documented in Lax (1966), as follows:

The relationship from Lax (1966) shows that if:

$$\frac{dX(t)}{dt} = g(X(t))w(t)$$

Then the transformation

$$Y(X) = \int^X g(X')^{-1} dX'$$

Yields the process satisfying

$$\frac{dY(t)}{dt} = w(t)$$

Now, compounding returns following

$$X(t) = X(0)e^{\int_0^t r(s)ds}$$

give the stochastic differential equation:

$$\frac{dX(t)}{dt} = r(t)X(t)$$

Using the relationship from Lax (1966) defined above, the following is obtained

$$Y(X) = \int^X \frac{dX'}{X'}$$

$$\therefore Y(t) = \log X(t) \tag{4.2}$$

Which yields the random process satisfying

$$\frac{dY(t)}{dt} = r(t) \tag{4.3}$$

If  $E[r(t)dt] = \mu dt$  and  $Var[r(t)dt] = \sigma^2 dt$ , then this can be written as a stochastic differential equation defining a Weiner process with drift  $\mu$ .

$$dY(t) = \mu dt + \sigma Z(t)\sqrt{dt} \tag{4.4}$$

where  $Z(t)$  is a Gaussian process with  $E[Z(t)] = 0$  and  $Var[Z(t)] = 1$

The first passage time of this commonly known process is well documented; see Domine (1995), where the expected time to absorption and its variance are defined as:

$$E(T) = \frac{Y_B - Y_0}{\mu}, \quad V(T) = \frac{(Y_B - Y_0)\sigma^2}{\mu^3}$$

Where  $Y_B$  is the absorbing barrier and  $Y_0$  is the starting point for the process. This is also shown to be an inverse Gaussian distribution

Recalling equation (4.2), it is known that  $Y(t) = \log X(t)$

So if the starting point  $X_0 = 1$ , and the absorbing point  $X_B = 2$ , the expected doubling time and its corresponding variance can be computed as follows

$$E(DT) = \frac{\log(2) - \log(1)}{\mu_r} = \frac{\log(2)}{\mu_r} \quad (4.5)$$

$$V(DT) = \frac{(\log(2) - \log(1))\sigma_r^2}{\mu_r^3} = \frac{\log(2)\sigma_r^2}{\mu_r^3} \quad (4.6)$$

Where  $\mu_r$  and  $\sigma_r^2$  are the mean and variance of the underlying normally distributed percentage returns.

This derivation suggests that if returns are normally distributed, then doubling times should follow an inverse Gaussian distribution where the first two moments are defined by the normal distributions moments as in equation (4.5) and equation (4.6).

Traditionally, inverse Gaussian distributions are not defined by the mean and variance but by the mean and shape parameter, where the shape parameter is defined as

$$\lambda = \mu_{DT}^3 / \sigma_{DT}^2 \quad (4.7)$$

Where  $\mu_{DT}$  and  $\sigma_{DT}^2$  are the inverse Gaussian distributions mean and variance respectively.

By substituting in equations (4.5) and (4.6) the shape parameter, as a function of the first two moments of the underlying returns, is found to be

$$\lambda_{DT} = \frac{\log(2)^2}{\sigma_r^2} \quad (4.8)$$

The above results show that the doubling time of wealth, given that returns are normally distributed with a positive expected return, will be inverse Gaussian such that:

$$\text{Doubling Times} \sim IG\left(\frac{\log 2}{\mu_r}, \frac{\log(2)^2}{\sigma_r^2}\right) \quad (4.9)$$

## 4.2 Simulation versus Analytics

### 4.2.1 Method

In this section the analytical derivation of Section 4.1 will be directly compared to simulated results to verify that the simulation and analytics agree. The distribution parameters given by equations (4.5) and (4.6) will be compared to the distribution parameters obtained from the simulation.

To compare the simulation with the analytics it is assumed percentage returns are normally distributed (an assumption which was made in the analytics). Accordingly, the simulation daily returns are randomly drawn from the normal distribution and standardised to have a zero mean and standard deviation of one percent. By transformation, samples are then generated to have mean values varying from 0.1% to 1% in increments of 0.1%, while holding the standard deviation constant at 1%, resulting in ten distributions. Such values were chosen as they are not entirely unreasonable and covered a broad range. Next the mean is held constant and a transformation is used to generate samples with the standard deviations in the range from 1% to 4% in increments of 0.5%, while holding the mean constant at 0.4%. For each sample of returns a simulation is then performed to obtain a distribution of doubling times. Each of these simulations had 1000 iterations to generate simulated means and variances for doubling times. Kolmogorov Smirnov (K-S) tests were performed to determine whether the simulated distributions were significantly different

from the theoretical inverse Gaussian distribution derived in section 4.1<sup>4</sup>. The simulated doubling time distributions are compared with a distribution based on 1000 random draws from the inverse Gaussian distribution with theoretical mean and shape parameters given by equations (4.5) and (4.8)

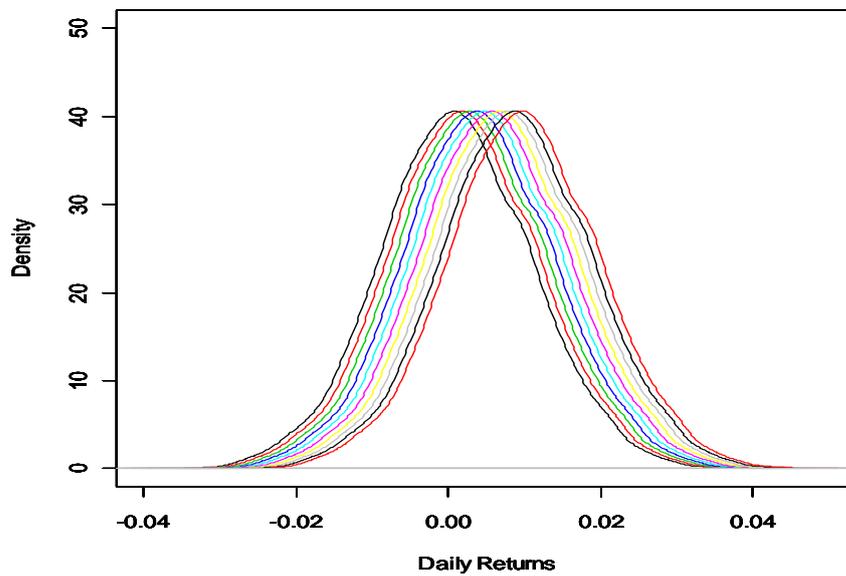
#### 4.2.2 Results

Figure 4.1 depicts the ten density functions of the underlying returns used for each simulation. This shows how the same underlying sample of returns has been used for each simulation; they have only had their means shifted by 0.1%. The corresponding distribution of simulated doubling times can be seen in Figure 4.2. Note that while the simulation was based on daily returns, the doubling times are expressed in years. The figure shows that as the underlying mean returns increases the simulated doubling times dispersion decreases, as does its mean. This outcome is consistent with the theoretical results derived earlier.

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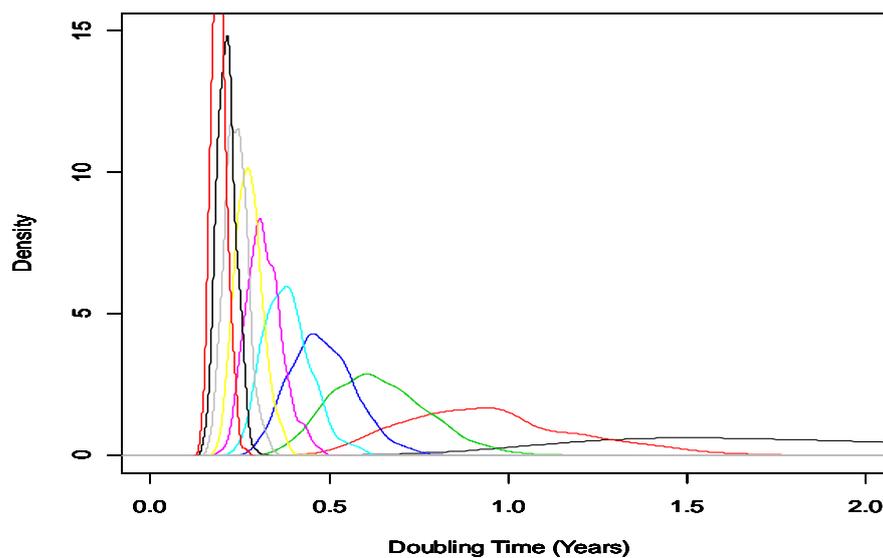
<sup>4</sup> The K-S test is chosen as it's a nonparametric test for the equality of continuous, one dimensional probability distributions.

**Figure 4.1: Density plots of underlying percentage returns**



**Plot of the ten density functions of the underlying returns used for each simulation**

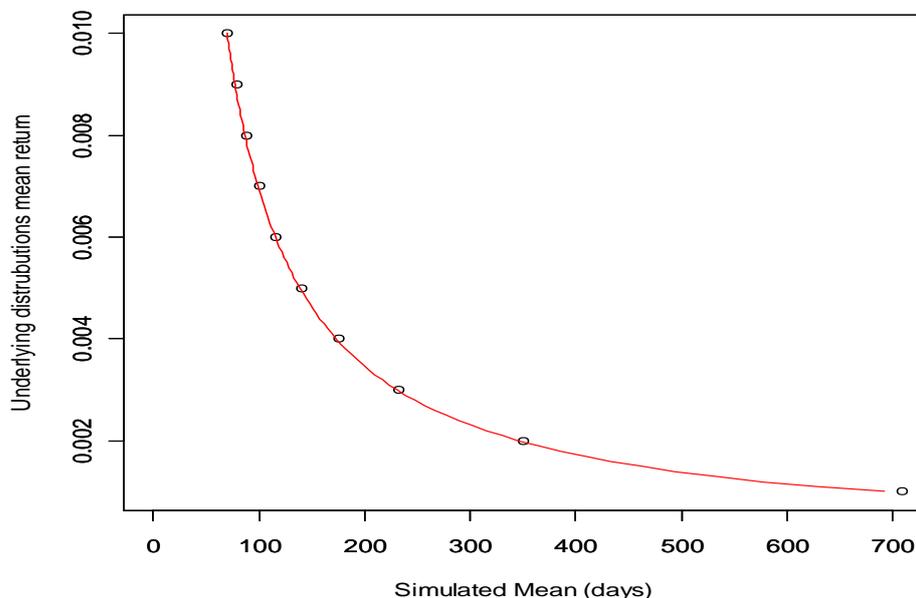
**Figure 4.2: Density plot of Simulated Doubling Times in years**



**Density plots of corresponding simulated doubling times (in years) as the underlying distribution's mean varies according to the distributions found in Figure 4.1, correspondence is matched by the colour of the lines.**

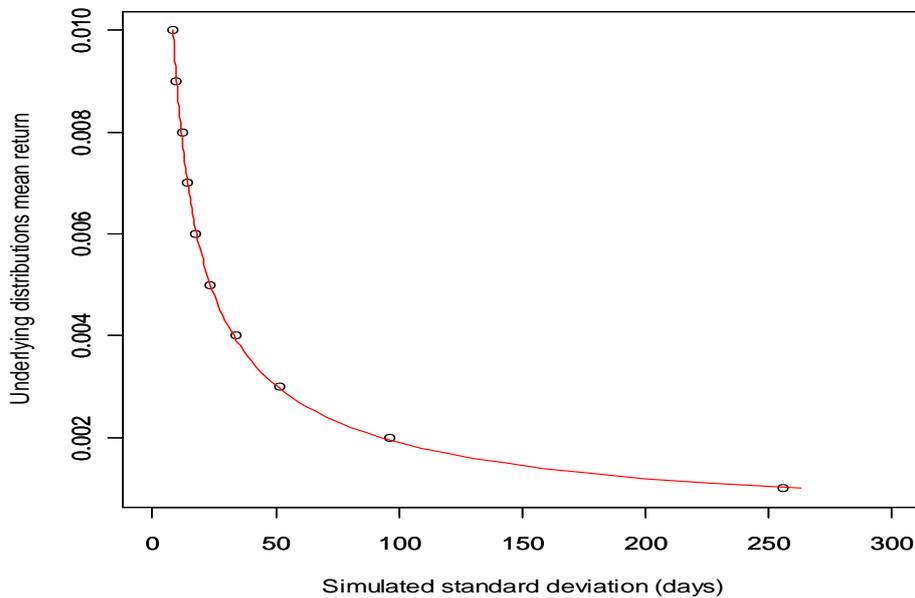
In order to examine whether the theoretical results agree with the simulation, the mean of the underlying returns is plotted against the corresponding mean doubling time from the simulation. This can be seen in Figure 4.3, where the circles represent the plotted points. Overlaid on Figure 4.3, is a continuous line which corresponds to the theoretical relationship as given by equation (4.5). This overlay fits the simulated plot very well, confirming that the means for simulated doubling times match theoretical results. Figure 4.4 plots the mean of the underlying returns against the corresponding simulated doubling times standard deviation. Similar, to Figure 4.3 a continuous line is plotted representing an overlay of the theoretical relationship given by equation (4.6). Figure 4.4 verifies that the variance of the simulated doubling time changes with the mean return in accordance with the theoretical model.

**Figure 4.3: Simulated Doubling Time Mean versus Corresponding Underlying Mean Return**



**The mean of the underlying returns is plotted against the corresponding mean doubling time from the simulation represented by circles. The red line represents equation (4.5).**

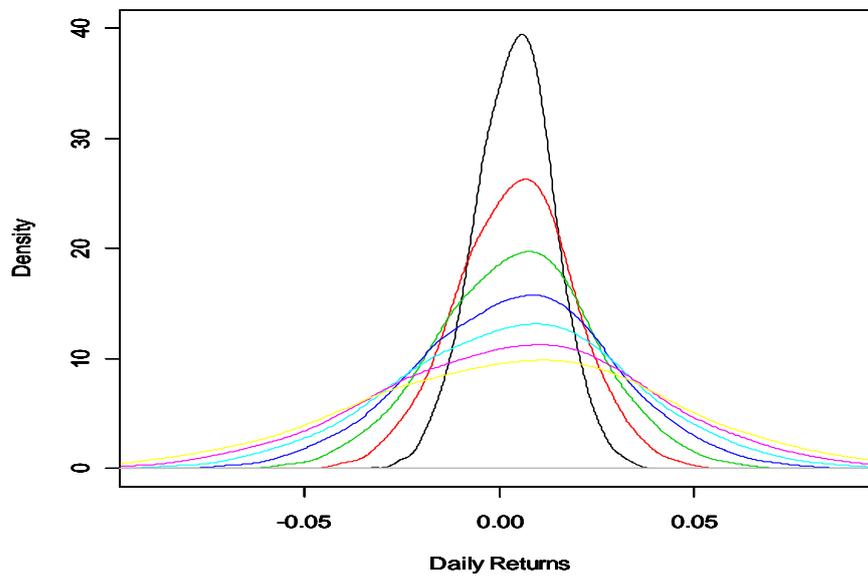
**Figure 4.4: Simulated Doubling Time Standard Deviation versus Corresponding Underlying Mean Return**



**The mean of the underlying returns is plotted against the corresponding simulated doubling time standard deviation represented by circles. The red line represents equation (4.6).**

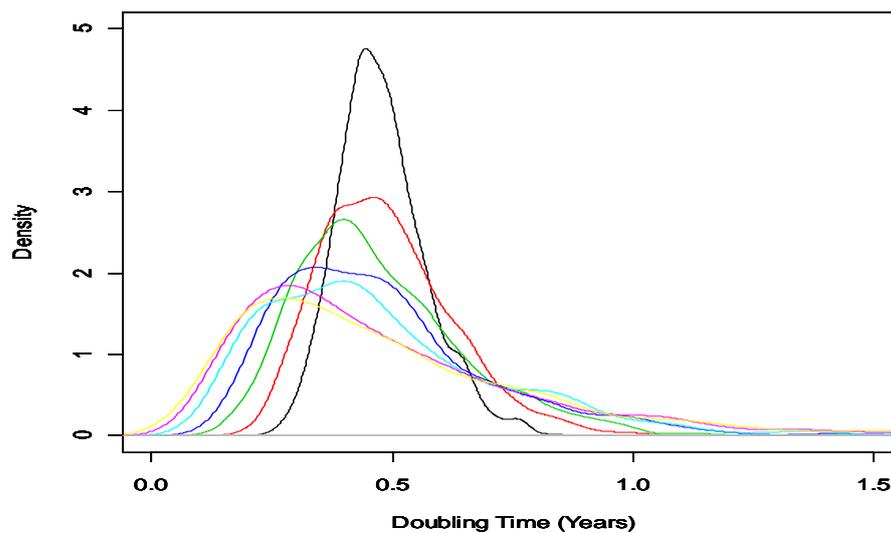
The density functions for underlying returns, created by changing the standard deviation, are displayed in Figure 4.5. The corresponding density function for the simulated doubling times are plotted in Figure 4.6. The density functions show that as the standard deviation of underlying returns increases so does the standard deviation of the doubling times, this is consistent with the analytical results. It is also noticeable that the skewness of simulated doubling times increases as the underlying standard deviation of returns increases. This is a natural consequence of doubling times being non-negative and hence being bounded below by zero.

**Figure 4.5: Density plots of underlying percentage return**



**Plot of the density functions of the underlying returns each with the same mean but different standard deviation used for each simulation**

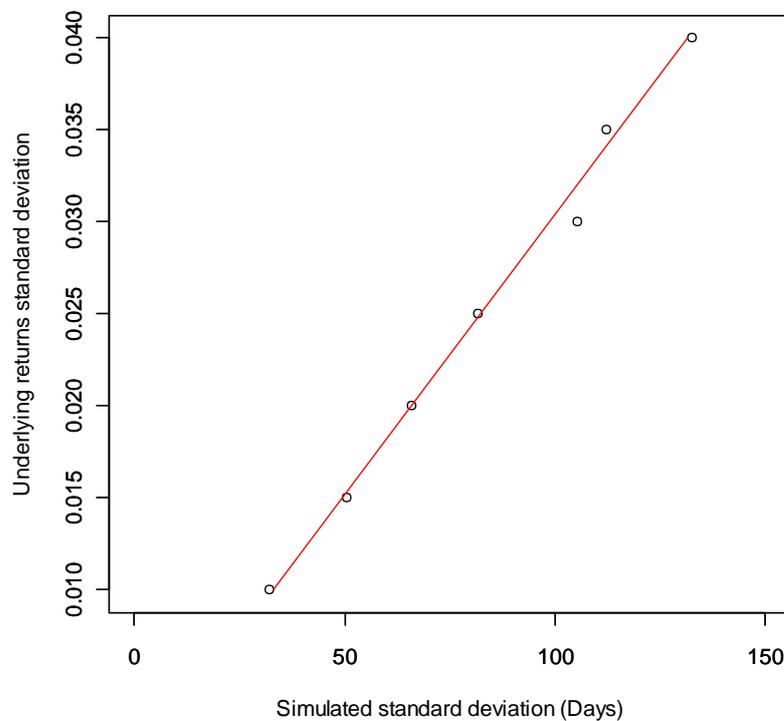
**Figure 4.6: Density plot of Simulated Doubling Times in years**



**Density plots of corresponding simulated doubling times (in years) as the underlying distribution's standard deviation varies according to the distributions found in Figure 4.5**

A plot of the simulated standard deviation versus the standard deviation of underlying returns is represented by the circles in Figure 4.7.<sup>5</sup> The continuous line represents an overlay of the theoretical relationship given by equation (4.6). Clearly there is a close correspondence between the simulation results and the theoretical results

**Figure 4.7: Simulated Doubling Time Standard Deviation versus Underlying Returns Standard Deviation**



**The standard deviation of the underlying returns is plotted against the corresponding simulated doubling time standard deviation represented by circles. The overlaid red line represents equation (4.6).**

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<sup>5</sup> No plot is presented for the means since there is no theoretical relation between the mean doubling time and the variance of returns.

Finally Kolomologrov-Smirnov (K-S) tests are performed to determine whether the simulated distributions and the corresponding theoretical distributions are significantly different. For brevity the results of these tests can be found in Appendix A.1 but it can be reported that all of the tests did not reject the null hypothesis that the two samples came from the same distribution (at the 5% significance level.)

## 4.3 What if Assumptions are Violated?

### 4.3.1 Analytical derivation

Given the results of Section 4.2, it is natural to ask how the doubling times distribution performs when the assumption of normality of returns is violated. To answer this question the Central Limit Theorem plays an important role.

In the simulation process; returns are randomly drawn with replacement until the compound value of these returns is equal to or slightly larger than 2, or algebraically:

$$\exp(r_1 + r_2 + \dots + r_n) \approx 2$$

By taking the natural log of both sides the following is obtained:

$$(r_1 + r_2 + \dots + r_n) \approx 0.69 \tag{4.10}$$

From equation (4.10) it is evident the sum of the  $n$  returns will approximately equal 0.69 or the natural log of 2.

Assume that the returns were uniformly distributed then:

$$\{r_1, r_2, \dots, r_n\} \sim \text{Uniform}(a, b)$$

However, if the sum of each return with its adjacent value is taken, thereby reducing the sample size by fifty percent and creating a new set of observations, the distribution of this new set of values would be entirely different. It would become the convolution of the original distribution with itself, provided all returns are independent of each other.

$$\{r_1 + r_2, r_3 + r_4, \dots, r_{n-1} + r_n\} \sim \text{conv}[\text{Uniform}(a, b), \text{Uniform}(a, b)]$$

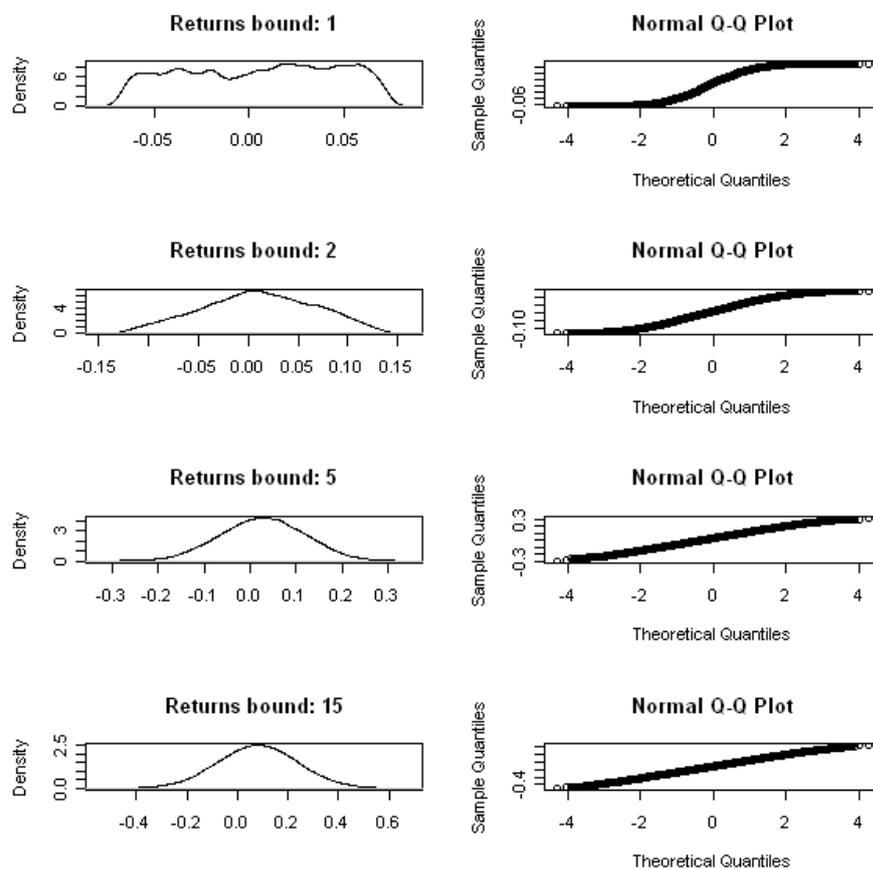
In the above example the convolution of the uniform distribution with itself will form a triangular distribution.

If the above step is repeated frequently the convolution of the same distribution with itself many times is created. Here the Central Limit Theorem comes into play; if enough variables are added to each other the result is a close approximation to the normal distribution.

An illustrative example of this process can be seen in Figure 4.8 below, again the uniform distribution has been chosen. Initially in the top left plot the density distribution of the returns are approximately uniformly distributed. The corresponding

normal qq-plot is also shown which reflects the poor fit to the normal distribution. When two returns are summed together the triangular distribution is generated, as expected. When five returns are summed it can be seen the density plot has the distinctive bell curve shape of the normal distribution, but the qq-plot suggests the tails are still not quite normally distributed. Finally, when 15 returns are summed together their resulting distribution appears to have a good fit to the normal distribution.

**Figure 4.8: Illustrative example of the Central Limit Theorem**



**This figure is an illustration of the Central Limit Theorem. Initially a uniform distribution is used. Its density plot can be found in the top left plot and its corresponding qq-plot found in the top right panel. It is evident that as more of this distribution is convolved with itself then the density plot converges to the normal distribution as verified by each distributions qq-plot.**

The foregoing analysis provides an explanation of why the simulation process, of sampling continuously compounded returns and summing them until the doubling point is reached, tends to normalize the returns as the doubling times are computed. Such a result would suggest that the doubling time distribution will be inverse Gaussian even if the underlying returns are not normally distributed.

#### 4.3.2 Empirical test of the Central Limit Theorem

The analysis in Section 4.2 maintained the underlying assumption that returns are normally distributed. In this section that assumption is relaxed in order to investigate the robustness of the simulation results to the assumption of normality. In particular the investigation considers whether the simulated inverse Gaussian distribution for doubling times is robust to divergence from normality in the higher moments of the return distribution. The effect of kurtosis is investigated by assuming returns come from the Student  $t$  Distribution. This allows one to sample returns from a distribution with varying degrees of freedom, thereby creating simulated distributions with different levels of kurtosis. These distributions are then standardised to have the same mean (0.1%) and standard deviation (1%), and a simulation of the doubling time is performed for each of the standardised distributions. The values for the mean and standard deviation are chosen as they are comparable to real world values. Furthermore, the degrees of freedom are chosen to range from 3 to 14, by increments of one. These degrees of freedom are chosen to allow for a wide range of kurtosis values, as reported in Table 4.1. If kurtosis does not play a role on the doubling time distribution then it would be expected that all the simulated doubling times would follow the inverse Gaussian distribution with the same parameters.

**Table 4.1: Varying degrees of Kurtosis**

d.o.f	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>	<b>7</b>	<b>8</b>
Kurtosis	21.34	14.59	8.12	5.1	4.84	4.42
d.o.f	<b>9</b>	<b>10</b>	<b>11</b>	<b>12</b>	<b>13</b>	<b>14</b>
Kurtosis	4.01	3.96	3.78	3.52	3.51	3.5

**This table provides the various levels of kurtosis used for the various simulations.**

Skewness is also investigated by performing simulations where returns are assumed to follow a variety of different distributions, some symmetrical such as a Uniform, Normal and Student  $t$  distribution, and some non-symmetrical such as the Lognormal, Weibull and Gamma distributions. Where the Gamma was positively skewed, the Weibull was negatively skewed and the Lognormal had a heavy positive skew. The sample skewness for each of the distributions is provided in Table 4.2. These values were chosen such that a range of skew is covered, from no skew (symmetric distributions), to mild positive skew (Gamma), mild negative skew (Weibull) and very large positive skew (Lognormal).

**Table 4.2: The sample skewness of the six distributions used**

	Student t	Gaussian	Uniform	Gamma	Weibull	Lognormal
Sample Skewness	<b>0.292</b>	<b>0.025</b>	<b>-0.008</b>	<b>1.263</b>	<b>-1.405</b>	<b>10.183</b>

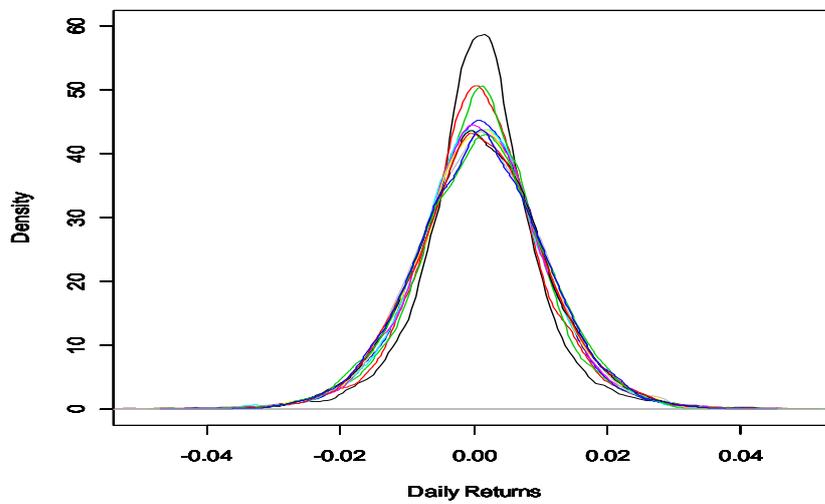
All these distributions were standardised to have the same first two moments, but obviously have varying levels of skewness and kurtosis. This potentially creates a problem in distinguishing between the effects of skewness and kurtosis. However, the results suggest that simulated doubling times are robust with respect to kurtosis, and a proof of this notion is provided later in Section 4.3.3. As such, deviations from the

inverse Gaussian distribution and its theoretical moments may reasonably be attributed to the effect of skewness.

### 4.3.3 Results

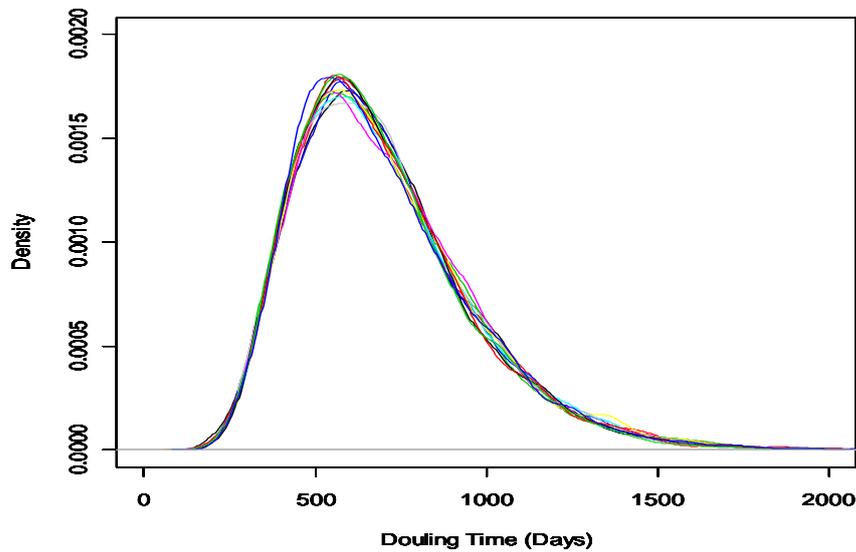
Sampling from the Student  $t$  distribution with different degrees of freedom, generates samples of returns with varying amounts of kurtosis, as depicted in Figure 4.9. It should be noted that these distributions have been re-scaled so they have the same sample mean and sample variance. The density functions for the corresponding simulated doubling times are depicted in Figure 4.10. Visually, all simulated doubling times appear similar; implying that kurtosis of the underlying returns has little or no effect on the doubling times distribution. This is confirmed by Kolmogorov-Smirnov tests for each distribution against every other distribution. There was no significant difference at the 5% level. (see Appendix A.2 for detailed results.). These results hold for a range of kurtosis values from over 20 down to that of a Gaussian distribution at 3. However, the above conclusion cannot be drawn for values outside this range. The conditions for which these results hold are provided later in Section 4.4.

**Figure 4.9: Density plots of underlying percentage returns**



Plot of the density functions of the underlying returns all with the same mean and standard deviation, but different levels of kurtosis.

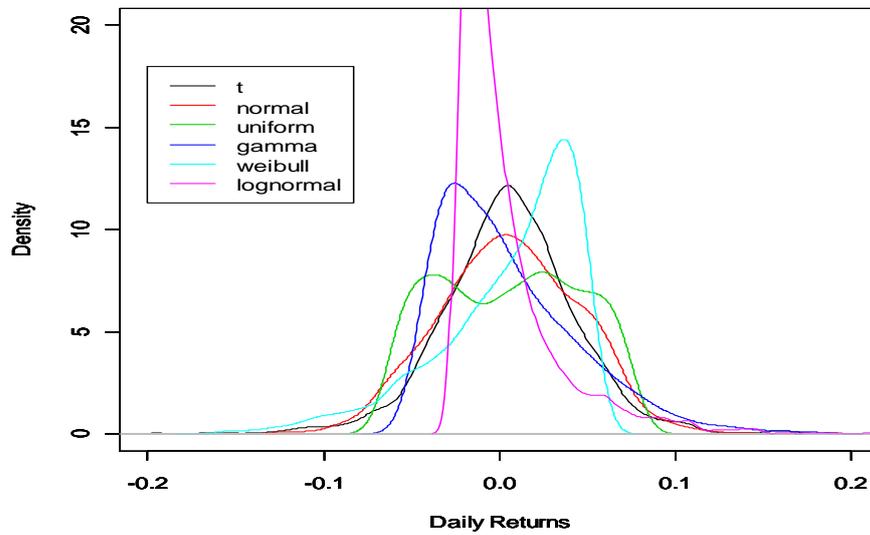
**Figure 4.10: Density plot of Simulated Doubling Times in days**



Density plots of corresponding simulated doubling times (in days) as the underlying distribution's kurtosis varies according to the distributions found in Figure 4.8

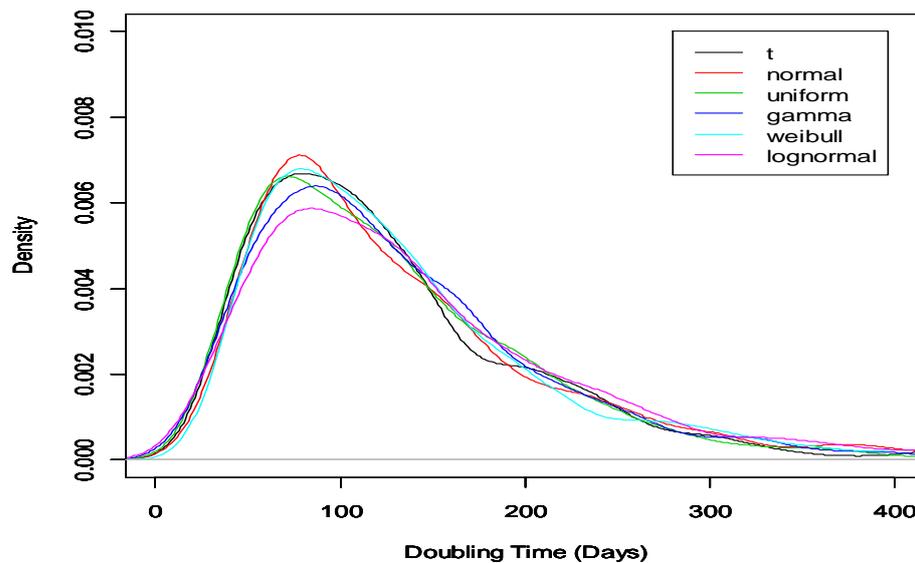
If the underlying returns are drawn from various distributions and standardised so they have the same mean and variance, then any departure of the simulated doubling times from the theoretical inverse Gaussian distribution may be attributed to the underlying returns skewness, as kurtosis has previously been shown to have little effect, and sample mean and sample variance are the same across all underlying distributions. The effect of skewness was examined by assuming the underlying returns come from three symmetric distributions, namely the Student  $t$ , Gaussian and Uniform distribution and also from three asymmetric distributions, namely the Gamma, Weibull and Lognormal. Density plots for these various underlying returns can be seen in Figure 4.11. The corresponding simulated doubling time distribution can be seen in Figure 4.12. On close inspection it appears that the simulated doubling times, whose underlying returns followed a Lognormal Distribution differs from the other simulated doubling times, and to lesser extent the same appears evident for the Gamma distribution. However, this claim is difficult to view visually and pp-plots and Kolmogorov-Smirnov tests can provide better conclusions.

**Figure 4.11: Density plots of the various underlying distributions used**



**Density plots of the six different underlying percentage return distributions used. Each distribution has been rescaled to have the same mean and variance.**

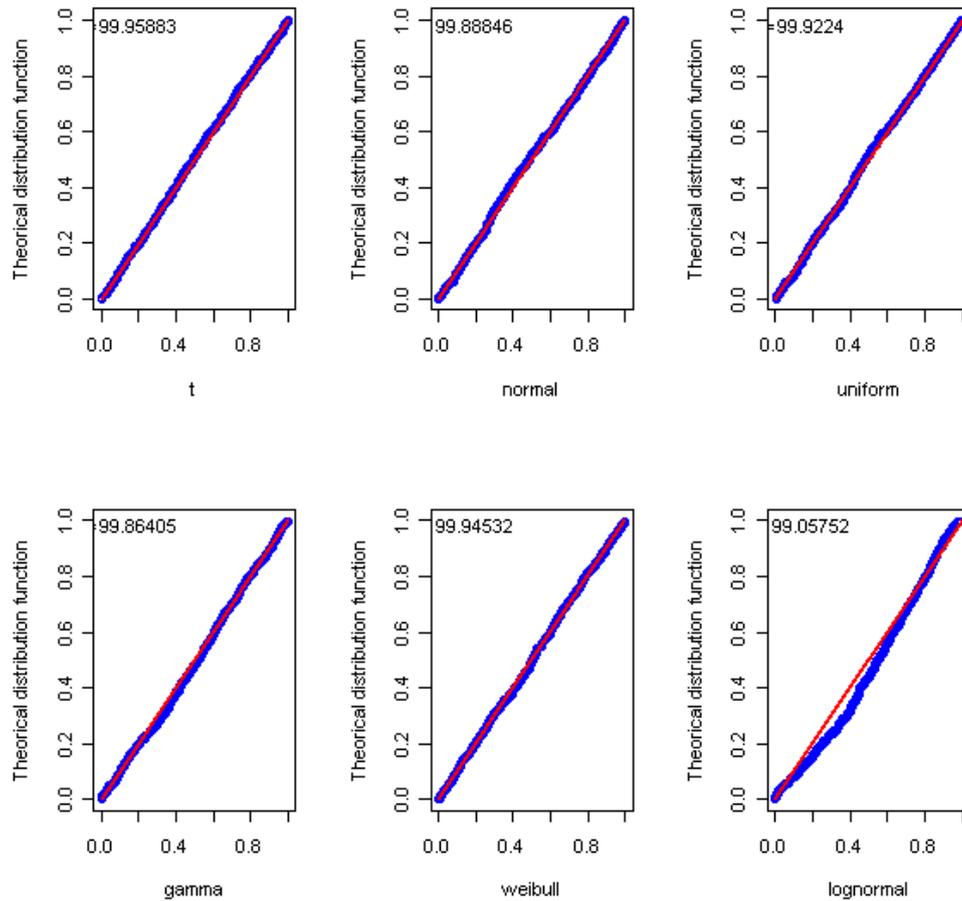
**Figure 4.12: Density plots of simulated doubling times**



**Density plots of corresponding simulated doubling times (in days) for each of the various underlying distributions. Notice the Gamma and Lognormal appear to diverge from the majority.**

The simulated doubling times should all be distributed following the inverse Gaussian distribution with mean  $\frac{\log 2}{\mu}$  and variance  $\frac{\sigma^2 \log 2}{\mu^3}$ . The fit of the simulated distributions to the theoretical one is investigated using two methods. Initially a pp-plot of the actual simulated doubling times against the theoretical distribution is performed, with the results depicted in Figure 4.13. Here the blue line represents the simulated distribution, while the red line represents the theoretical distribution. Visually, it appears that the only simulated doubling times not following the theoretical inverse Gaussian distribution are those whose underlying returns follow the Lognormal Distribution. The fit is also formally investigated by performing a Kolmogorov-Smirnov (K-S) test of the simulated doubling times against the theoretical inverse Gaussian distribution. The significance of these tests is reported in Tables 4.3 below.

**Figure 4.13: pp-plot of the six various doubling time distributions against the theoretical inverse Gaussian distribution**



**Table 4.3: Results of the Kolmogorov-Smirnov Tests**

	Student t	Gaussian	Uniform	Gamma	Weibull	Lognormal
Test Stat	<b>0.027</b>	<b>0.051</b>	<b>0.039</b>	<b>0.08</b>	<b>0.046</b>	<b>0.104</b>
pValue	<b>0.8592</b>	<b>0.1483</b>	<b>0.4324</b>	<b>0.0033*</b>	<b>0.241</b>	<b>0.000**</b>

\*\* < 0.001, \* < 0.01

**Results of the K-S tests for each simulated doubling time against the theoretical inverse Gaussian distribution. It is evident that the Lognormal and Gamma distributions reject the null.**

As the primary difference between the Lognormal distribution and the other distributions is its extremely large skew, it is arguable that the rejection of the null for the K-S test at the 0.1% level for the Lognormal Distribution suggests that if returns are extremely skewed, as is the case for the Lognormal Distribution, then the simulated doubling time may depart from the theoretical inverse Gaussian distribution. The K-S test also suggests that skew may have an effect in the case of the Gamma Distribution, but the result is weaker. A probable explanation is given in the next Section.

## 4.4 The Berry-Esseen Theorem

The Central Limit Theorem suggested that the results would be robust to violations of the normality assumption. However, it leaves unanswered why high levels of skewness appear to cause deviations from the inverse Gaussian distribution for doubling times.

The answer is provided by the Berry-Esseen Theorem.

The Berry-Esseen Theorem states the following:

Let  $X_1, X_2, \dots$ , be iid random variables with  $E(X_i) = 0$ ,  $E(X_i^2) = \sigma^2 > 0$ , and

$E(|X_1|^3) = \rho < \infty$ . With the sample mean defined as follows:

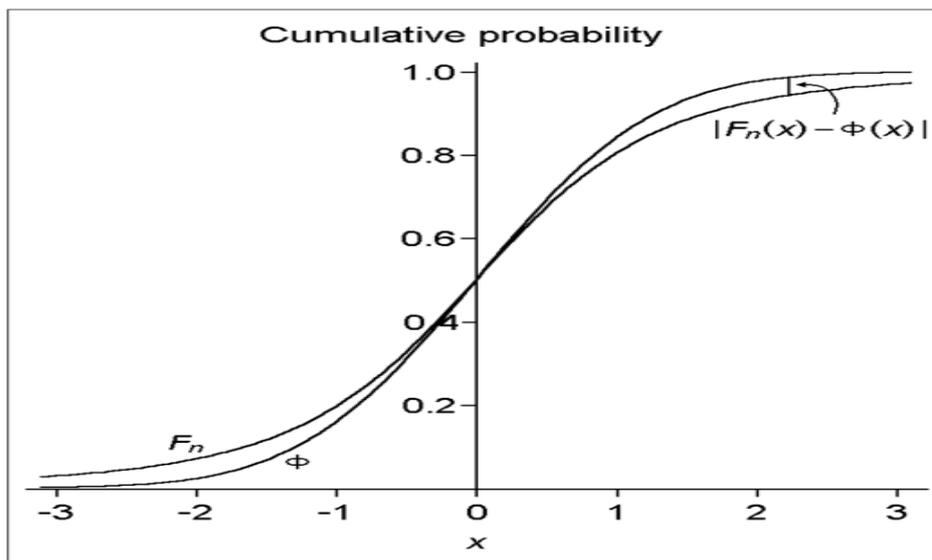
$$Y_n = \frac{X_1 + X_2 + \dots + X_n}{n}$$

And  $F_n$  being the cdf of  $Y_n \sqrt{n}/\sigma$  and  $\phi$  the cdf of the normal distribution, then there exists some constant  $c$  such that for all  $x$  and  $n$ ,

$$|F_n(x) - \phi(x)| \leq \frac{c\rho}{\sigma^3 \sqrt{n}} \quad (4.11)$$

The constant,  $c$  has been shown by Esseen (1956) to be greater than or equal to 0.40973, with the best current upper bound for  $c$  being 0.7056 shown in Shevtsova (2007). An illustration of the difference in cumulative distribution functions as given by the theorem can be found in Figure 4.14 below.

**Figure 4.14: Graphical Representation of Berry-Esseen Theorem**



**Difference of the divergence in cumulative distribution functions alluded to by the Berry-Esseen Theorem**

This theorem shows that the speed of convergence with normality is at least of the order of  $1/\sqrt{n}$ , with the convergence also being related to the underlying distributions rho,  $\rho$ .

Rho, is closely related to the distribution's skewness, as:

$$skewness = \rho / \sigma^3$$

Accordingly, larger skewness will result in larger values for rho. This implies, given a fixed sample size,  $n$ , the deviation from the normal distribution will be larger for skewed distributions than symmetric distributions. Accordingly, a skewed distribution will need a much larger sample size than its symmetric equivalent before convergence to the normal distribution occurs.

The following discussion illustrates these ideas showing convergence for the uniform distribution and incomplete convergence for the lognormal distribution.

From equation (4.10) the following is known:

$$(r_1 + r_2 + \dots + r_n) \approx 0.69$$

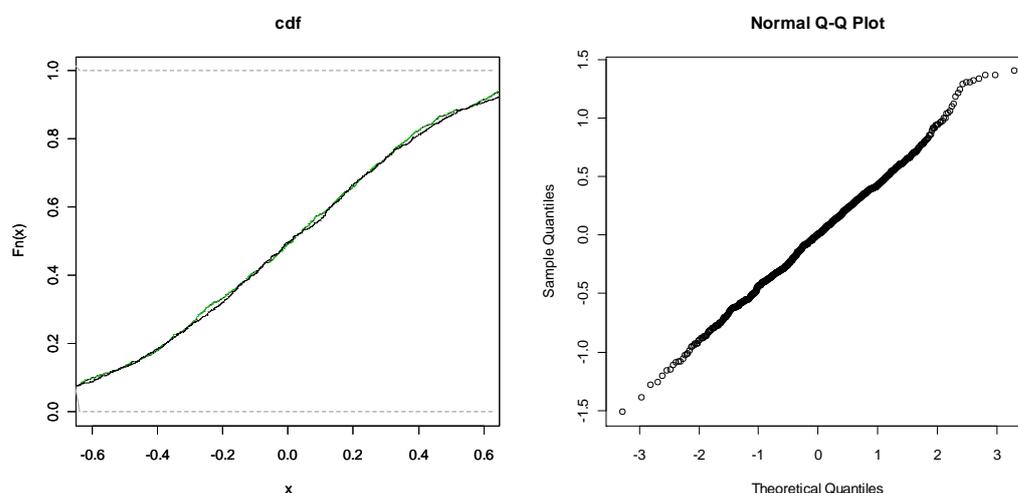
Accordingly, if the inverse Gaussian distribution for simulated doubling times is to be robust to non-normality in the return distribution, then the sum of the returns must converge to a normal distribution before their sum has an expected value greater than 0.69.

In the case of the Uniform Distribution, if two uniform distributions are added, both with mean of 0.0054 and standard deviation of 0.04<sup>6</sup> the new distribution formed will have a mean of 0.0108 and will be a Triangle Distribution. Similarly, if 10 distributions are added together this new distribution will have a mean of 0.0540. As explained above if the sum of the distributions has converged to the normal distribution before the mean of the summed distributions equals 0.69, then the simulated doubling time will be inverse Gaussian, as the assumption of the underlying distribution being normally distributed has been met, courtesy of the Central Limit Theorem. So in the case of the uniform distribution, if the sum of 127 ( $0.69/0.0054$ ) Uniform Distributions has converged to a normal distribution then the simulated doubling time will be inverse Gaussian distributed. This is shown in Figure 4.15 below. Here the green line is the cdf of the sum of 127 uniform distributions, while the black line is the cdf of the normal distribution. It is clear that there is very close correspondence between the two distributions. The normal qq-plot for the summed distributions, also suggests a close correspondence to normality, except in the extreme right tail.

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<sup>6</sup> This mean and standard deviation were the values used in the investigation of the effect of skewness in Section 4.3.

**Figure 4.15: Cumulative Distribution Function of the Gaussian distribution and summed uniform distributions along with corresponding qq-plot**

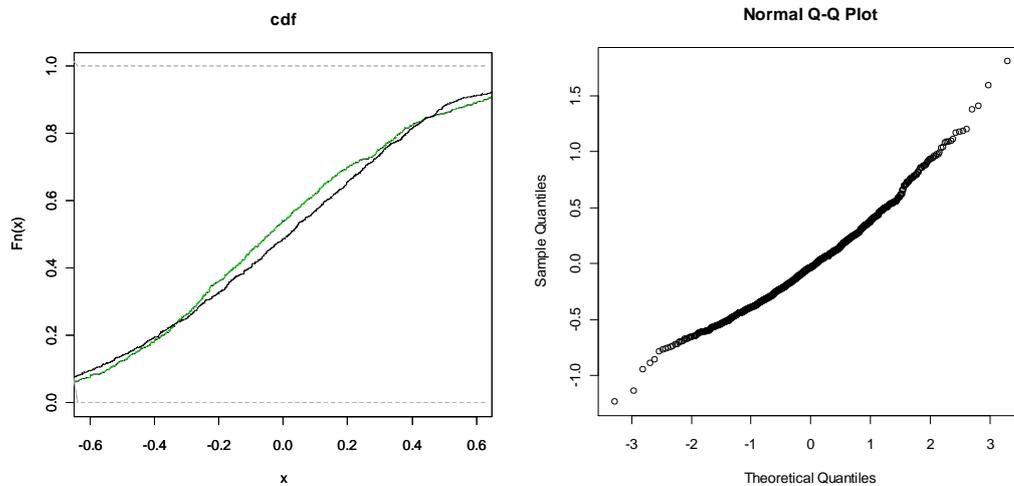


**The left plot has the cdf of a normal distribution (black line) the corresponding cdf of the summation of 127 uniform distributions is also plotted (green line). On the right panel a qq-plot of the summed uniform distributions is also depicted highlighting the close fit to the normal distribution**

Now consider the case where the underlying returns are assumed to be Lognormal distributed. Again the expected value of the daily returns is 0.0054. So if the sum of 127 ( $0.69/0.0054$ ) Lognormal Distributions has converged to the normal distribution then the simulated doubling time will be inverse Gaussian distributed.

From Figure 4.16 below it is evident that the sum of 127 lognormal distributions has a cdf which varies from its normal equivalent, a result verified by its corresponding normal qq-plot. This explains that as the sum of 127 lognormal distributions has yet to converge to a Gaussian distribution, then the corresponding simulated doubling time will not be inverse Gaussian distributed.

**Figure 4.16: Cumulative Distribution Function of the Gaussian distribution and summed lognormal distributions along with corresponding qq-plot**



The left plot has the cdf of a normal distribution (black line) the corresponding cdf of the summation of 127 lognormal distributions is also plotted (green line). On the right panel a qq-plot of the summed lognormal distributions is also depicted highlighting the poor fit to the normal distribution.

## 4.5 The Probability of Doubling After a Fixed Time.

This section analyses the probability of reaching a given wealth level after a fixed period of time, for example, the probability of doubling your wealth after two years.

The analytical and simulated probabilities are derived and compared in the following subsections.

### 4.5.1 Analytical Probability

From MacLean and Ziemba (1999) a suggested performance metric is the end of horizon wealth. This is defined as the probability of achieving a level of wealth after a given time period. MacLean and Ziemba provide the following equation for computing the probability an investors wealth is less than or equal to a give value,  $X$ , after a fixed time period,  $t$ :

$$P(P_t \leq X) = P \left( z \leq \frac{\log\left(\frac{X}{P_0}\right) - \mu_r t}{\sigma_r \sqrt{t}} \right) \quad (4.12)$$

Where  $P_0$  is the initial investment,  $\mu_r$  is the mean percentage return and  $\sigma_r$  is the standard deviation of percentage returns.

### 4.5.2 Simulated Probability

Alternatively, using a Monte Carlo simulation methodology similar to that outlined in Section 3.5 a simulated probability can also be estimated. This simulation has the variation that the loop is not exited once the investor's wealth has doubled, but rather after a fixed time period. The simulated probability that the investor's wealth is less than  $X$  can be simply calculated as the total number of iterations where final wealth is less than  $X$  divided by the total number of iterations used. The pseudo code for this simulation is provided in Figure 4.17.

**Figure 4.17: Pseudo code for estimating the probability that wealth is less than X.**

**for**  $i=1$  to number of desired iterations

*runningValue* = 1  
*periodCounter* = 1

**while** (*periodCounter* <= time Period To Investigate)  
    *singlePeriodReturn* = random return drawn from return series  
    *runningValue* = *runningValue*\*exp(*singlePeriodReturn*)  
    increment *periodCounter*  
**end while**

store *runningValue* in a vector indexed by  $i$ .

**end for**

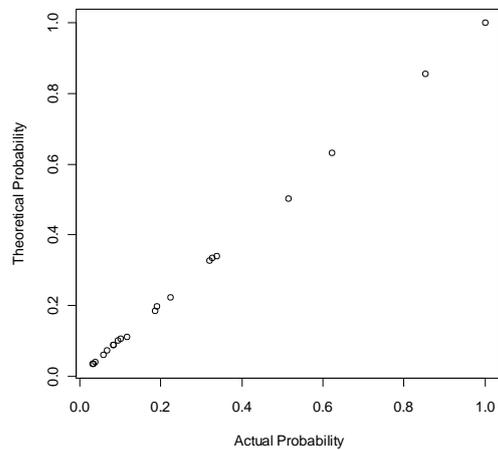
Probabilty = (number of *runningValues* <  $X$  in the vector) / number of desired iterations

### 4.5.3 Verification probabilities agree

Verification that the probabilities obtained using equation (4.12) and the simulation methodology are the same was achieved using the following process The daily close prices for the S&P Accumulation 200 Index from the 4<sup>th</sup> of January 2000 to the 15<sup>th</sup> of September 2010 was used to form a series of daily percentage returns, which gave a total of 2709 data points. This data was obtained from Bloomberg. The simulation methodology was run with the period fixed at 500 trading days (two years)and the number of iterations was set to 1,000,The probability that an investor's wealth is less than  $X$  was then determined from the proportion of iterations where the final wealth value was less than  $X$ . This probability was compared to the probability obtained using equation (4.12). This process was repeated for various values of  $X$ , where  $X$  was chosen randomly between values of 0 and 5. The simulated probability and the analytical

probability were plotted against each other for various values of  $X$ , as depicted in Figure 4.18, it is evident that the probabilities estimated using the simulation methodology are practically the same as those obtained from equation (4.12). Such a result is statistically verified by regressing the theoretical probability against the actual probability where no intercept term is used. The resulting slope coefficient is not statistically different from 1 at the 0.1% level.

**Figure 4.18: Theoretical Probability versus Actual Probability**



**Plot of the probability that and investors wealth is less than  $X$  after 500 trading days obtained using the simulation methodology (actual) versus the probability obtained using equation (4.12) (theoretical). These results were obtained using the ASX S&P Accumulation 200 Index.**

## 4.6 Conclusion

In this chapter doubling times were modelled as a first passage time process.

Analytically, it was shown that when an investment's returns are normally distributed, with a positive expected value, then its corresponding doubling time will be inverse

Gaussian distributed. Furthermore, it was shown that the inverse Gaussian distributions' two parameters, mean and shape, can both be derived as functions of the underlying return's mean and variance. These analytical results were consistent with Monte-Carlo simulations.

The impact on the simulated doubling time distribution of non-normality in returns was also investigated. It was shown that, as a result of the Central Limit Theorem, if the underlying returns deviate from normality, convergence of the simulated doubling times distribution to the inverse Gaussian distribution is still likely. Convergence is shown to be slower if the underlying returns are highly skewed, a direct result of the Berry-Esseen Theorem. The condition required for convergence is analysed.

Lastly, the probability of reaching a given wealth level after a fixed period of time, for example, the probability of doubling your wealth after two years was also analysed. This probability was shown to be a function of both the expected return and standard deviation of the returns. The analytical formula for the probability and a Monte Carlo simulation of the probability were shown to give consistent results.

# Chapter 5: Portfolios in the Time Domain

## 5.1 Introduction

This chapter analyses portfolio construction in the time domain. Firstly, various risk metrics are suggested along with possible new objective functions. The portfolios formed using these risk metrics are compared to those obtained using the traditional Markowitz portfolio optimisation method.

The results suggest that the same efficient frontier will arise when forming a portfolio in the time domain as when forming one in the traditional Markowitz framework.

## 5.2 Possible Risk Metrics

In the previous chapter it was shown that the distribution of doubling time could be represented by the inverse Gaussian distribution, where the expected time to double an investment is as follows:

$$\mu_{DT} = \frac{\log(2)}{\mu_r} \tag{5.1}$$

with variance defined by:

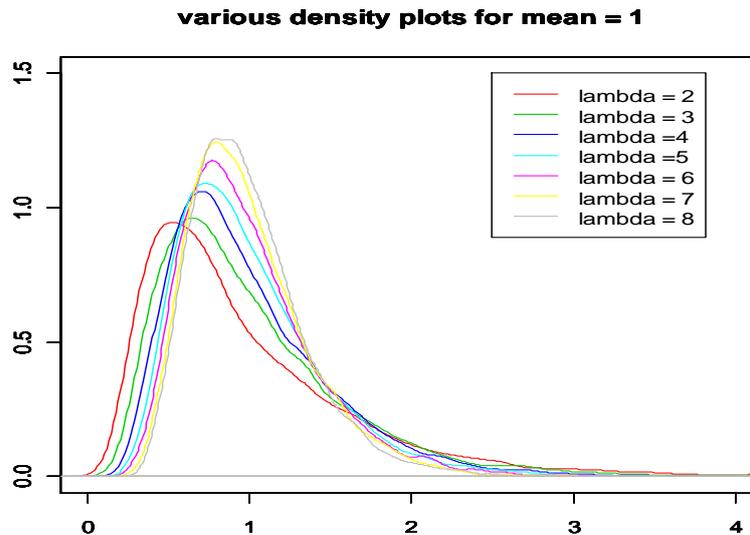
$$\sigma_{DT}^2 = \frac{\log(2)\sigma_r^2}{\mu_r^3} \quad (5.2)$$

Where  $\mu_r$  and  $\sigma_r^2$  are the mean and variance respectively of the underlying percentage returns.

Portfolio optimisation in the time domain becomes an almost inverse problem of traditional portfolio optimisation. Instead of maximising expected returns, expected doubling times are now minimised, on the assumption that investors wish to amass wealth as quickly as possible. While minimising doubling time is clearly analogous to maximising returns, the choice of risk metric is less obvious.

As the inverse Gaussian distribution is defined by two parameters, the mean,  $\mu_{DT}$  and shape,  $\lambda_{DT}$  it is plausible as a preliminary hypothesis to suggest that the shape parameter may be an appropriate risk metric. This hypothesis is further supported by observing Figure 5.1 where the mean is held constant but the shape parameter varies.

**Figure 5.1: Density plots as shape parameter changes with mean held constant**



**Plot of the inverse Gaussian distribution for various shape parameters, while the mean is held constant**

From Figure 5.1 it is evident that when the mean is held constant and the shape parameter increases, the dispersion of the distribution decreases. This plot suggests that the inverse of the shape parameter may be a possible time domain risk metric which is analogous to the traditional risk measure of variance.

An alternative hypothesised risk metric is the skewness of the doubling time. This risk metric has intuitive appeal as an investor would want to minimise the long positive tails of the distribution as these increase the risk of long waiting periods to double the investment.

Finally, a third hypothesised risk metric is the variance of doubling time, consistent with variance being the risk metric for percentage returns.

In the following subsections the inverse of the shape parameter, the skewness and the variance of doubling times will be analytically investigated as possible risk metrics. The relation between the first two moments of the underlying percentage returns distribution and the risk metrics will be explored.

### 5.2.1 The inverse of the shape parameter

From equation 4.7 the shape parameter is given by:

$$\lambda_{DT} = \frac{\mu_{DT}}{\sigma_{DT}^2} \quad (5.3)$$

Substituting equations (5.1) and (5.2) into equation (5.3) the following is obtained:

$$\lambda_{DT} = \frac{\log(2)^2}{\sigma_r^2} \quad (5.4)$$

$$\therefore (\lambda_{DT})^{-1} = \frac{\sigma_r^2}{c} \quad \text{where } c = \log(2)^2 \quad (5.5)$$

Equation (5.5) highlights that the inverse of the shape parameter is equivalent to the percentage returns variance, scaled by a constant. As such, when optimising a portfolio in doubling times, if the shape parameter,  $\lambda$ , is maximised then the portfolio weights

obtained should be equivalent to those estimated using traditional methods for the global minimum variance portfolio.

### 5.2.2 The skewness

An alternative hypothesised risk metric is skewness, as an investor may want to minimise the risk of long periods to double their investment. Skewness,  $\gamma$ , of the inverse Gaussian distribution is defined by its two parameters using equation (5.6):

$$\gamma = 3 \left( \frac{\mu_{DT}}{\lambda_{DT}} \right)^{1/2} \quad (5.6)$$

Substituting equations (5.1) and (5.2) into equation (5.6) the following is found:

$$\gamma = 3 \left( \frac{\sigma_r^2}{\mu_r \log(2)} \right)^{1/2} \quad (5.7)$$

$$\therefore \gamma = c \left( \frac{\sigma_r}{\sqrt{\mu_r}} \right) \quad \text{where } c \text{ is a constant} = \frac{3}{\sqrt{\log(2)}} \quad (5.8)$$

Accordingly, minimising skewness is the same as minimising a risk-return function in returns and their variance, or, equivalently, maximising a ratio of return to risk. This is comparable to maximising the return-to-risk ratio in a Markowitz framework which would result in a portfolio lying on the tangent of the origin and the efficient frontier.

### 5.2.3 The variance

The final suggested risk metric is the variance of doubling times. From equation (5.2):

$$\sigma_{DT}^2 = \frac{\log(2)\sigma_r^2}{\mu_r^3}$$

Accordingly, if the doubling time variance was minimised, then it would be equivalent to minimising a non-linear relationship in returns and the variance of returns. The non-linearity arises from the cubic term for the expected percentage return. An equivalent to minimising the doubling time variance would be to maximise its inverse, that is

$\max \mu_r^3 / \log(2)\sigma_r^2$ . For the combinations of mean return and variance commonly

encountered in securities markets, the numerator of the foregoing ratio will be very much greater than the denominator. Consequently, a portfolio which minimises the variance of doubling times will tend to have very heavy weights in high expected return stocks and strongly negative weights in low expected return stocks.

The above analysis has shown that optimising a portfolio in doubling times where either the shape or skewness is a risk metric, should result in portfolio weights equivalent to those of the global minimum variance portfolio and tangency portfolio when the Markowitz framework is used (a hypothesis which will be verified empirically in the following section).

## 5.3 Empirical Evidence that the Doubling Time Frontiers are Equivalent to the Markowitz Frontier

### 5.3.1 Method

In Section 5.2 three possible risk metrics were suggested when optimising portfolios in time domain. Firstly, the inverse shape parameter was suggested as a possible risk metric. Here it was shown that when minimised, the portfolio weights computed are equal to the portfolio weights for the global minimum variance portfolio when estimated using the Markowitz framework. The second suggested risk metric was the skewness. Here evidence was provided suggesting that minimising skewness should compute portfolio weights identical to those when maximising the return-to-risk ratio under the Markowitz framework. This is the tangency point which lies on the efficient frontier (which stems from the origin in the mean-variance dimension) and is the point on the frontier which has the highest return-to-risk ratio. Lastly, the doubling time variance was suggested as a possible risk metric. However, it was shown that when minimising the doubling time variance, a portfolio which gives significant positive weight to stocks with high expected returns and significant negative weight to low expected return stocks is likely to occur. This suggests that the doubling time variance appears to be a poor risk metric to use in the time domain.

To examine the above, several numerical experiments have been formulated. Using an identical data set to Broadie (1993), five assets are selected with means, variances and correlations provided in Tables 5.1 and 5.2 below.

**Table 5.1: The true means and standard deviation parameters and their equivalent doubling time parameters**

	Asset 1	Asset 2	Asset 3	Asset 4	Asset 5
Mean	0.006	0.01	0.014	0.018	0.022
Standard Deviation	0.085	0.08	0.095	0.09	0.1
Expected Doubling Time	50.17	30.10	21.50	16.72	13.68
Doubling Time Standard Deviation	100.35	43.89	31.47	20.45	16.81

**Table 5.2: The true correlation matrix**

	Asset 1	Asset 2	Asset 3	Asset 4	Asset 5
Asset 1	1	0.3	0.3	0.3	0.3
Asset 2	0.3	1	0.3	0.3	0.3
Asset 3	0.3	0.3	1	0.3	0.3
Asset 4	0.3	0.3	0.3	1	0.3
Asset 5	0.3	0.3	0.3	0.3	1

Given this data, solutions to the following constrained optimisations are obtained:

*The Inverse Shape Parameter.*

Minimise:  $\lambda^{-1}$  **5.3.1.1**

Subject to:  $\sum w_i = 1$

*The Skewness.*

Minimise:  $\gamma$  **5.3.1.2**

Subject to:  $\sum w_i = 1$

*The Doubling Time Variance.*

Minimise:  $\sigma_{DT}^2$

**5.3.1.3**

Subject to:  $\sum w_i = 1$

For these minimisation problems the objective functions will have only one local minimum which corresponds to the global minimum. Accordingly, the weights computed can be found using the Nelder-Mead method<sup>7</sup>. Once the asset weightings are computed for each of these objective functions, their corresponding portfolio mean and variance can be plotted. This plot can then be overlaid on the efficient frontier determined using the traditional Markowitz mean-variance framework.

As previously mentioned, it is expected that the minimum skewness portfolio will lie on the tangency point between the efficient frontier and the origin, while the portfolio for the minimum inverse shape parameter will lie on the global minimum variance of the efficient frontier. Conversely, the minimum doubling time variance portfolio is expected to not lie on the efficient frontier, but instead assign large weights for high expected return stocks and large negative weights for low expected return stocks.

Once these individual portfolios have been formed the next step is to determine whether the whole Markowitz efficient frontier can be formed using the doubling time risk

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<sup>7</sup> The Nelder-Mead is a downhill Simplex method which although not as efficient as some gradient methods, is more robust and less likely to be trapped by local minima.

metrics. The variance of the doubling time is not included in this analysis, since as explained above it clearly will not give Markowitz efficient portfolios, a result confirmed in the following section. To construct the efficient frontiers the objective functions of the optimisations are modified as follows:

$$\text{Minimise: } \tau_{\lambda} \times \mu_{DT} + \lambda_{DT}^{-1} \quad \mathbf{5.3.1.4}$$

$$\text{Subject to: } \sum w_i = 1$$

Where  $\tau$  is a risk tolerance parameter and  $w$  is the weight assigned to asset  $i$ .

To trace the efficient frontier the objective function can be solved for a range of risk tolerance parameters.

To find the efficient frontier when using skewness as a risk metric, a comparable approach can be taken by solving the below objective function across various risk tolerance levels:

$$\text{Minimise: } \tau_{\gamma} \times \mu_{DT} + \gamma \quad \mathbf{5.3.1.5}$$

$$\text{Subject to: } \sum w_i = 1$$

These two frontiers should lie on the Markowitz efficient frontier in the mean-variance plane, defined by the following objective function:

$$\text{Maximise: } \tau_{\sigma} \times \mu_r - \sigma_r^2$$

**5.3.1.6**

$$\text{Subject to: } \sum w_i = 1$$

It is suggested that all frontiers obtained using objective function 5.3.1.4, 5.3.1.5 and 5.3.1.6 will be the same as it has been shown when minimising the two risk metrics (inverse shape and skewness), portfolios which lie on some point of the Markowitz frontier will be obtained. Accordingly, with the extension that accounts for doubling times in the objective function, any portfolios obtained will still lie on the Markowitz efficient frontier as doubling times are simply a function of the corresponding percentage returns, as demonstrated in equation (5.1).

While the same efficient frontier will be obtained regardless of the risk metrics used, what may be of interest is that the risk tolerance parameters may all differ on any given point of the efficient frontier. For example, the tangency portfolio computed using skewness as a risk metric may have a different risk tolerance parameter for the exact same tangency portfolio using the shape parameter as a risk metric. Accordingly, analysis will be performed to determine whether the risk tolerance parameter for one objective function can be functionally related to the risk tolerance parameter of an alternate objective function.

The examination of risk tolerances will be performed by firstly determining two points on the efficient frontier by maximising objective function 5.3.1.6 (Markowitz framework) for two different risk tolerance parameters,  $\tau_1$  and  $\tau_2$ , using the Nelder-Mead Algorithm. These will be called *Portfolio1* and *Portfolio2* and have known means

defined as  $\mu_1$  and  $\mu_2$ . Using the two fund theorem, any portfolio with a desired return can be formed using the following:

$$\mu_{desired} = \alpha \times \mu_1 + (1 - \alpha) \times \mu_2 \quad (5.9)$$

By rearranging equation (5.9), a value for alpha can be found such that the desired return is achieved:

$$\alpha = \frac{(\mu_{desired} - \mu_2)}{(\mu_1 - \mu_2)} \quad (5.10)$$

Given a risk tolerance parameter of  $\tau_{optim}$ , the objective function, where inverse shape is the risk metric (objective function 5.3.1.4,) can be minimised using the Nelder-Mead algorithm. This will result in a portfolio for the given risk tolerance parameter with an expected return,  $\mu_{optim}$ . The two fund theorem can be used to find a portfolio which is a combination of *Portfolio1* and *Portfolio2* with a return equal to  $\mu_{optim}$ . In other words alpha is found using equation (5.10) and letting  $\mu_{desired} = \mu_{optim}$ . Risk tolerance parameters have a linear sum similar to expected returns. Thus, the equivalent risk tolerance parameter for the Markowitz framework (objective function 5.3.1.6) is computed using the following:

$$\tau_{equivalent} = \alpha \times \tau_1 + (1 - \alpha) \times \tau_2 \quad (5.11)$$

The Markowitz risk tolerance parameter  $\tau_{equivalent}$  applied to objective function 5.3.1.6 gives an identical portfolio to that obtained when solving objective function 5.3.1.4 for

risk tolerance parameter  $\tau_{\text{optim}}$ . This process can be repeated for various values of  $\tau_{\text{optim}}$  to trace out a relationship between the risk tolerance parameters.

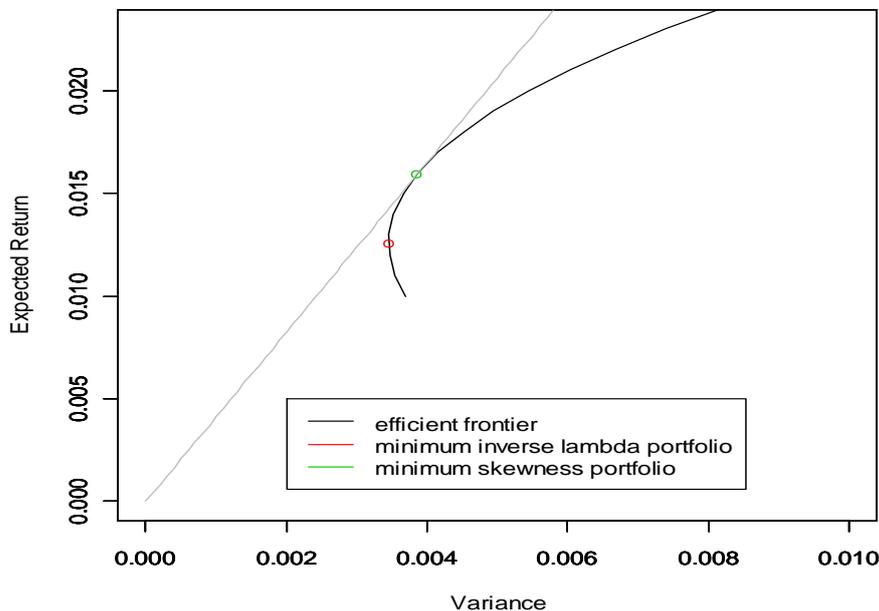
A similar approach is also taken when comparing risk tolerance parameter between the minimum skewness objective function (5.3.1.5) and the Markowitz framework objective function (5.3.1.6).

### 5.3.3 Results

In this Chapter, doubling time portfolios were formed with two possible risk metrics. The first of these was the inverse of the shape parameter for the inverse Gaussian distribution, while the second was the skew parameter for the inverse Gaussian distribution.

Theoretically, it was shown that a portfolio optimised to minimise the inverse of the shape parameter should result in the same weight allocation as those of the global minimum variance portfolio when the traditional mean variance framework is used. Additionally, when the skewness parameter is minimised it is shown that this should correspond to a tangency portfolio in the traditional mean-variance framework. To verify this empirically, the weights for a doubling time portfolio of 5 assets which minimise the inverse shape parameter were calculated and the weights which minimise skewness were also estimated. These two estimated portfolios were then plotted on the efficient frontier, at their respective points, as depicted in Figure 5.2 below.

**Figure 5.2: Efficient frontier and respective points**



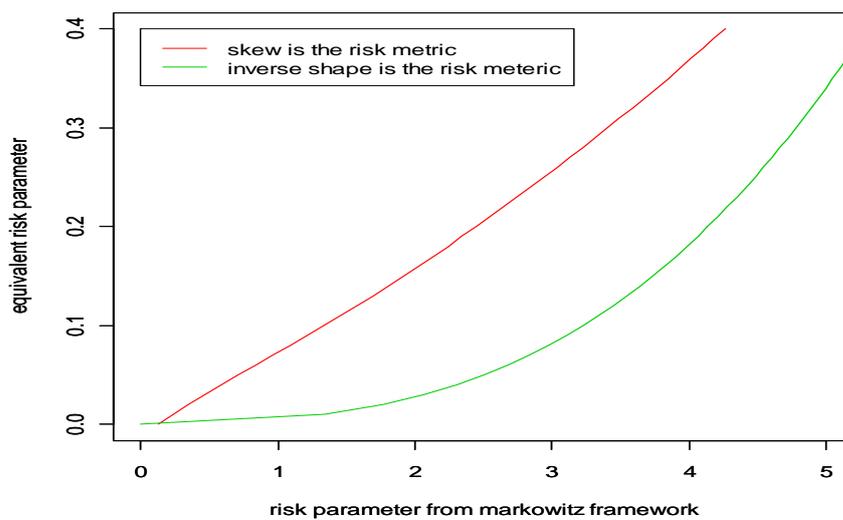
**This figure depicts the efficient frontier using mean, variance and correlation parameters found in Tables 5.1 and 5.2. The efficient frontier was computed using the traditional mean-variance framework and is shown as the black line. The red dot represents the minimum inverse shape parameter which lies at the global minimum variance portfolio. The green dot is the minimum skewness portfolio which is shown to lie on the tangency point between the origin and the frontier (a grey line is superimposed across these points).**

Figure 5.2 highlights that, consistent with theory, the portfolio estimated in doubling times which minimises the inverse shape parameter corresponds to the equivalent global minimum variance portfolio estimated under a traditional mean-variance framework. Similarly, a portfolio which minimises the skewness of doubling times is shown to correspond to the portfolio which occurs when a tangent is drawn from the point where expected return and variance both equal zero. These results confirm that optimising portfolios in doubling times or mean-variance framework result in the same portfolio. The portfolio obtained when minimising the doubling times variance is not depicted in Figure 5.2 as the asset weightings obtained approached unbounded values,

as hypothesised, resulting in almost infinite expected returns, with correspondingly infinite variances.

The objective functions given by equations 5.3.1.4 (for inverse shape) and 5.3.1.5 (for skew) give the utility for different levels of risk tolerance defined by  $\tau$ . While both these objective functions result in the same efficient frontier, equivalent points on the frontier will not necessarily have equivalent risk tolerance parameters. Using the method described in section 5.3.1, the relationship between the risk tolerance parameters of objective function 5.3.1.4 and objective function 5.3.1.6 was found. This relationship is depicted in Figure 5.3 below. Figure 5.3 also depicts the relationship between risk tolerance parameters when the risk metric used is skewness (objective function 5.3.1.5) and it is referenced against the Markowitz framework (objective function 5.3.1.6).

**Figure 5.3: Equivalent risk tolerance parameters**



**Plot of the relationship between risk tolerance parameters for utility functions, 5.3.1.4 and 5.3.1.5, which equate the same portfolio found under the Markowitz framework defined by utility function 5.3.1.6**

Figure 5.3 highlights the fact that to obtain the same portfolio under the Markowitz framework as when using skew or the inverse shape parameter, a much higher risk tolerance parameter must be used. For example, using the data set chosen for these experiments under the Markowitz framework, if objective function 5.3.1.6 is optimised using 3 as the risk tolerance parameter, the portfolio obtained lies on the efficient frontier with an expected return of 9.4% and standard deviation of 49.2%. To obtain a portfolio on the efficient frontier with the same expected return and standard deviation by optimising objective function 5.3.1.4, where inverse shape is now the risk metric, then a risk tolerance parameter of 3 would no longer be used, but instead a risk tolerance parameter of 0.08 is required. Similarly, to again find a portfolio on the efficient frontier with expected return of 9.4% and standard deviation of 49.2% by optimising objective function 5.3.1.5, where skew is the risk metric, then a risk tolerance parameter of 0.25 would be needed.

## 5.4 Conclusion

Chapter 5 has analytically derived two possible risk metrics which when minimised will result in critical points on the efficient frontier. Firstly, when minimising the inverse of the shape parameter it was shown that this will result in the equivalent global minimum variance portfolio which would be obtained using the classical Markowitz framework. Secondly, it was shown that when minimising the skewness of doubling times that the tangency portfolio with the highest reward-to-risk ratio is obtained. The skewness result has particular intuitive appeal. This appeal arises as investors would

want to minimise any positive skew in their investment doubling times. Long waiting periods until their investments double would naturally be undesirable. Furthermore, this doubling time optimisation problem only requires one parameter (skewness) to obtain the tangency portfolio, yet the Markowitz framework requires two parameters (mean and variance), thereby resulting in a simplified yet equivalent optimisation problem. These results provide a new perspective from which to view portfolio theory and an alternative calculus for generating the efficient frontier

The analytical derivation suggested that minimising the skewness, or the inverse shape, of the doubling time distribution would each give a portfolio that lay on the Markowitz efficient frontier. This was then demonstrated empirically.

By optimising utility functions, while varying the risk tolerance parameter, an optimal set of doubling time portfolios was generated. This was done for both the skewness and the inverse shape measures of risk. These optimal sets of doubling time portfolios were both shown to reproduce the Markowitz efficient frontier when transformed from doubling times to returns. Analysis was then undertaken to determine the equivalent risk tolerance parameters. That is the parameters for risk tolerance were such that the same efficient portfolios were formed using the risk metrics of skewness or inverse shape in the time domain, or variance in the return domain. The results showed that the risk tolerance parameters using inverse shape or skew as risk metrics were about an order of magnitude ten times less than the equivalent portfolios using the traditional Markowitz framework.

# Chapter 6: Reducing Estimation Error

## 6.1 Introduction

As discussed in Section 2.3, when optimising under the Markowitz framework error maximisation can be a serious problem (see Michaud (1989)). The weights estimated from the optimisation process are sensitive to errors in the estimates of the mean and covariance parameters used. For example, if an expected return for an asset is overestimated it will get a higher weight than it otherwise should have, with the biggest errors receiving the highest weights. Several methods have been proposed to overcome this problem (see Jobson and Korkie (1981), Jorion (1985) and Michaud (1999)). This chapter proposes a new method which is computationally quick.

## 6.2 Method

In Chapter 5 it was shown that when a portfolio is formed in doubling times the same asset weighting to those obtained using the traditional Markowitz framework will arise for the global minimum variance portfolio, and the tangency portfolio. Thus, forming a portfolio in doubling times, or using traditional techniques will result in the same asset weighting. Consequently it might be asked: where is the advantage of optimising a portfolio using doubling times over optimising the same portfolio in a mean-variance framework? The answer is that, one can take advantage of an interesting property of the

inverse Gaussian distribution, which may result in better estimation of the input parameters for the optimisation process. In turn this should yield better estimates for the asset allocation weights. The inverse Gaussian distribution has both positive and negative moments, and this can be utilised to transform the raw parameters and obtain improved estimates. The transformation arises due to Jensen's inequality and is explained as follows. From Chapter 4, equation (4.5) gives:

$$E[r] = \log(2) / E[\tau]$$

Where  $\tau$  is the doubling time and  $r$  is the underlying percentage return.

What may be of interest is to instead determine

$$E[r_{trans}] = E\left[\log(2) / \tau\right]$$

Where  $r_{trans}$  is the transformed expected return. The above simplifies down to

$$E[r_{trans}] = \log(2) E[\tau^{-1}]$$

Now  $E[\tau^{-1}]$  is the first negative moment of the inverse Gaussian distributed doubling times,  $\tau$ , as the expectation of  $\tau^{-1}$  is taken as opposed to the expectation of  $\tau$ . This is why the name negative moment transform has been chosen. A similar notion is also used for the variance.

An alternative transformation involving doubling times is also investigated in this chapter. Simonsen, Jensen and Johansen (2002) suggest that the peak of the doubling time distribution is the *optimal investment horizon* as it is the most likely time horizon over which an investment's value will double. Accordingly, rather than using the expected doubling time in transforming returns, the most likely doubling time is used instead. The negative moment transform described above and the optimal investment horizon transform described in this paragraph are rigorously derived in the sub sections below.

### 6.2.1 The negative moment transform

The negative moment transform can be derived as follows:

Using the continuous compounding returns equation:

$$FV = PVe^{rt} \tag{6.1}$$

Letting  $FV = 2$  and  $PV = 1$ , the transformed returns can be computed as:

$$r_{trans} = \frac{\log(2)}{\tau} \tag{6.2}$$

Where  $\tau$  is the doubling time

The equation used to achieve the first negative moment of an inverse Gaussian distribution is the following (see Seshadri (1999)):

$$E[X^{-1}] = \frac{1}{\mu} + \frac{1}{\lambda} \quad (6.3)$$

Applying (6.2) to (6.3), the percentage return, corresponding to the expected doubling time, can be written in terms of doubling times as follows:

$$E\left[\frac{\log(2)}{\tau}\right] = \frac{\log(2)}{\mu_{DT}} + \frac{\log(2)}{\lambda_{DT}} \quad (6.4)$$

If equation (5.1) and (5.4), for the mean and variance of doubling time in terms of the underlying returns first two moments, are substituted into equation (6.4) the following transformation is obtained:

$$\mu_{trans} = \mu_r + \frac{\sigma_r^2}{\log(2)} \quad (6.5)$$

This highlights the notion that the mean percentage return to double an investment (the transformed mean) is different to the mean of underlying percentage return distribution.

This arises due to Jensen's inequality as  $E\left[\frac{1}{x}\right] \neq \frac{1}{E[x]}$

Analysis similar to that above can be used to determine the transformed returns variance.

From Seshadri (1999) the equation for the inverse Gaussians second moment is defined as the following:

$$E[X^{-2}] - E[X]^{-2} = \frac{1}{\mu\lambda} + \frac{2}{\lambda^2} \quad (6.6)$$

Using (6.2) to rewrite (6.6) the variance of returns can be estimated from doubling times as:

$$\text{var} \left[ \frac{\log(2)}{\tau} \right] = \frac{\log(2)^2}{\mu_{DT} \lambda_{DT}} + \frac{2\log(2)^2}{\lambda_{DT}^2} \quad (6.7)$$

If equations (5.1) and (5.4), for the mean and shape parameter of doubling time in terms of underlying returns, are substituted into equation (6.7) the resulting variance transformation is:

$$\sigma_{trans}^2 = \sigma_r^2 \left( \frac{\mu_r}{\log(2)} + \frac{2\sigma_r^2}{\log(2)^2} \right) \quad (6.8)$$

Equations (6.5) and (6.8) give the mean and variance of percentage returns, transformed from doubling times, as a function of the original percentage return's first two moments.

Jobson and Korkie (1981) and Jorion (1985), attempted to reduce the error maximisation problem by transforming risk return parameters to form a stronger risk-return relation. The negative moment transformation above achieves this objective, giving a stronger and more linear risk return relation, as will be empirically demonstrated later in the chapter. Consider, for example, an asset which is estimated to have a high expected return and a low variance. It is evident that when the return

transformation of (6.5) is applied the increase in expected return will be relatively less for low variance assets relative to high variance assets. Furthermore, when the variance transformation of (6.8) is applied it is evident that a higher return will lead to a higher variance relative to assets with a low return. With the transformations applied, therefore, the asset is no longer a high expected return and low volatility asset relative to the other assets, as its mean will have had less increase relative to the other assets and its volatility will have had a larger increase relative to the other assets. Intuitively, such transformations may reduce the problem of error maximisation. A stock that, due to estimation error, has an extremely high expected return and a low variance relative to other assets is likely to get a heavy portfolio weight in the classical Markowitz optimisation process. After transformation however, its return is relatively smaller and its variance relatively higher, its portfolio weight is likely to be reduced.

### 6.2.2 The optimal investment transform

An alternative transformation comes from the work of Simonsen, Jensen and Johansen (2002), where the peak of the doubling time distribution is referred to as the *optimal investment horizon*. Their work motivates the use of the most likely doubling time; rather than the expected doubling time. The most likely doubling time is simply the mode of the doubling time distribution. The transformation is derived as follows.

The mode of an inverse Gaussian distribution can be determined by its two parameters using the following (see Seshadri (1999)):

$$MODE = \mu \left[ \left( 1 + \frac{9\mu^2}{4\lambda^2} \right)^{1/2} - \frac{3\mu}{2\lambda} \right] \quad (6.9)$$

If equations (5.1) and (5.4) are substituted into equation (6.9), the optimal investment horizon (the mode) as a function of the underlying percentage returns mean and variance is given as:

$$MODE = \frac{\log(2)}{\mu_r} \left[ \left( 1 + \frac{9\sigma_r^4}{4(\log(2))^2 \mu_r^2} \right)^{1/2} - \frac{3\sigma_r^2}{2\log(2)\mu_r} \right] \quad (6.10)$$

The optimal investment horizon, can now be transformed into a percentage return. This is done by simply dividing  $\log(2)$  by the mode giving:

$$\mu_{trans} = \log(2) / \frac{\log(2)}{\mu_r} \left[ \left( 1 + \frac{9\sigma_r^4}{4(\log(2))^2 \mu_r^2} \right)^{1/2} - \frac{3\sigma_r^2}{2\log(2)\mu_r} \right] \quad (6.11)$$

This simplifies to:

$$\mu_{trans} = \mu_r \left[ \left( 1 + \frac{9\sigma_r^4}{4(\log(2))^2 \mu_r^2} \right)^{1/2} - \frac{3\sigma_r^2}{2\log(2)\mu_r} \right]^{-1} \quad (6.12)$$

Equation (6.12) provides an alternative transformation for the expected returns. This transformation for expected returns is paired with the transformation for the variance given by equation (6.8). Thus, it is only the different transformation for expected

returns that distinguishes the negative moment transformation from the optimal investment horizon transformation. As discussed in the following section, transformations of the variance are of lesser importance in addressing the error maximisation problem.

### 6.2.3 Testing the suggested transformations

The analysis commences with a graphical representation of the underlying estimated returns versus underlying estimated variance, as compared to the transformed returns versus transformed variance. This permits examination of whether the transformed parameters yield a more linear risk-return relationship.

In Chapter 2, it was mentioned that the estimation error is most pronounced in the means, and less evident in the standard deviations. As such, a desirable outcome of the transformation is that adjustments are more pronounced in the mean rather than the variance. Whether this is the case is determined by drawing several samples of returns and plotting the underlying sample means against the transformed means. Similarly, the underlying sample standard deviations are plotted against the transformed standard deviations.

The impact of the transformation on portfolio formation is analysed by the same method as Broadie (1993). This involves using simulated data, to evaluate whether the efficient frontier is more accurately estimated using transformed returns. This simulated data involves five assets, each with a known mean, known standard deviation and

known correlations. The parameters used are the same as those used in Broadie (1993), and are provided in Tables 5.1 and 5.2 in Section 5.2.

Using the known parameters, returns consistent with the five distributions were randomly generated from a multivariate normal distribution. This allows the comparison of the *True Efficient Frontier*, computed using the known true parameters, with the *Estimated Frontier* obtained using the sample mean and covariance. *Estimated frontiers* are obtained using the raw and transformed means and variances. It is also possible to examine the *Actual Frontier* that would be obtained by using the *Estimated Frontier's* portfolio weights applied to the true parameters.

An example of the various frontiers can be seen in Figure 6.1 below, which is taken from Broadie (1993). Here it can be seen that due to estimation error the *Estimated Frontier* is overly optimistic lying above the *True Efficient Frontier*. The result that actually arises by using the weights obtained on the *Estimated Frontier* can be seen on the *Actual Frontier*, which lies below the *True Efficient Frontier*, reflecting the loss of performance due to estimation error. The closer the *Actual Frontier* is to the *True Efficient Frontier*, the less estimation error is a problem.

## Figure 6.1: Broadie (1993) three different frontiers

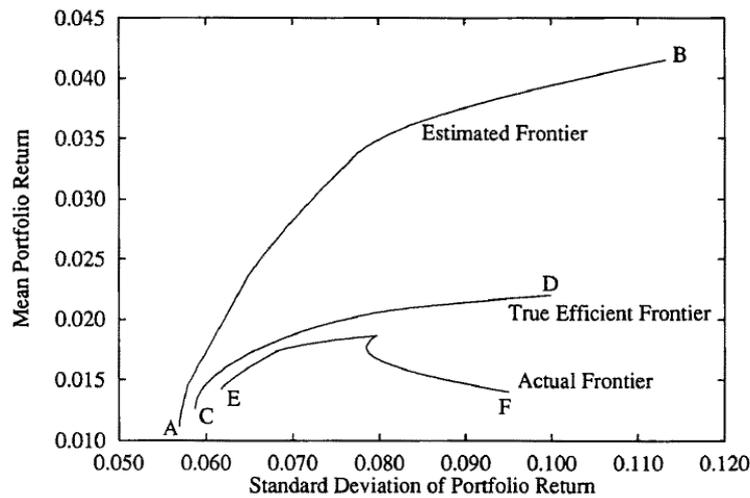


Fig. 1. Mean-variance frontiers using 24 months of simulated data.

**The three different frontiers discussed in Broadie (1993). The *Estimated Frontier* is the frontier obtained using parameters estimated from simulated data. The *True Efficient Frontier* is the frontier which arises using the true parameters, which are unknown in real life. The *Actual Frontier* is the frontier which arises when the weights obtained on the *Estimated Frontier* are applied to the true parameters.**

The *Actual Frontiers* to be estimated in the following sections will be formed using parameters from the negative moment transformation, the optimal investment horizon transformation and untransformed parameters. If the transformations reduce estimation error, they will give an *Actual Frontier* that lies closer to the *True Efficient Frontier* than the *Actual Frontier* found using untransformed parameters. Additionally, as the sample size of the draw increases, it is expected that the *Estimated Frontier* and *Actual Frontier* will converge towards the *True Efficient Frontier*, because the estimation error is reduced as the sample size increases.

The following outlines how the *Actual Frontiers* were determined. For the *Actual Frontier* where no transformations are made, the assets expected returns and covariance

matrix are estimated from the data drawn from the multivariate normal distribution. The expected returns are defined as  $\mu_{est}$  which is a vector of length 5, as there are 5 assets used. Similarly, the covariance matrix will be a 5x5 matrix and will be defined as  $\Sigma_{est}$ . The weights for two efficient portfolios can be found by computing the following:

$$w_1 = \frac{\Sigma_{est}^{-1} \varepsilon}{\varepsilon \Sigma_{est}^{-1} \varepsilon} \quad (6.13)$$

$$w_2 = \frac{\Sigma_{est}^{-1} \mu_{est}}{\varepsilon \Sigma_{est}^{-1} \mu_{est}} \quad (6.14)$$

Where  $\varepsilon$  is a vector of ones of length 5

The asset weights given by  $w_1$  will be the weights for the *Estimated Frontier's* global minimum variance portfolio, while the asset weights given by  $w_2$  will be the weights for a tangency portfolio on the *Estimated Frontier*. With these two efficient portfolios found, it is a simple matter now to find efficient portfolios which trace out the *Estimated Frontier* via the two fund theorem. The asset weights for any efficient portfolio can be determined as a combination of these two efficient portfolio such that

$$w_p = \alpha w_1 + (1 - \alpha) w_2 \quad (6.15)$$

Where  $w_p$  is a vector of asset weightings for any efficient portfolio dependant on  $\alpha$ .

The *Actual Frontier* can be computed by varying alpha ( $\alpha$ ) over a range of values with small increments each time. This will give estimated portfolio weights defined above as  $w_p$ . The *actual* expected return and variance using these weights can be determined using the true parameters which are provided in Tables 5.1 and 5.2. Letting the true expected returns and variance be defined as the vector  $\mu_{true}$  and matrix  $\Sigma_{true}$  respectively, the actual mean and variance for a given portfolio weight  $w_p$  is found using the following two equations:

$$\mu_{actual} = w_p' \mu_{true} \quad (6.16)$$

$$\sigma_{actual}^2 = w_p' \Sigma_{true}^{-1} w_p \quad (6.17)$$

The actual means and variances are computed for many different portfolios on the *Estimated Frontier* by varying alpha across a range of values resulting in enough data points to plot out the *Actual Frontier* over a reasonable region.

The only adjustments made to the above method when using the transformed parameters are to equations (6.13) and (6.14). Rather than  $\mu_{est}$  and  $\Sigma_{est}$  being estimated from the simulated data, the suggested transformations are applied. For example, using the negative moment transformation  $\mu_{est}$  for asset 1 will require transformations given by equation (6.5) and  $\Sigma_{est}$  will be found by using transformations suggested by equation (6.8). When applying these equations,  $\mu_r$  and  $\sigma_r$  defined in equations (6.5) and (6.8), will be the estimated mean and standard deviation from the simulated data.  $\Sigma_{est}$  is slightly more complicated to estimate. First, correlations are estimated from the simulated historical data set, and second the variance for each asset is transformed using equation (6.8). Third covariance is computed as:

$$\sigma_{AB} = \rho_{AB} \sigma_A \sigma_B$$

Where  $\rho_{AB}$  is the correlation coefficient for assets A and B and  $\sigma_A$  is the standard deviation of asset.

Once the covariance matrix is complete the *Actual Frontier* is obtained using the method described above. The *Actual Frontiers* for each method are estimated from 100 independent simulated data sets and the averages of these *Actual Frontiers* are reported. The averages of these *Actual Frontiers* are then compared against the *True Efficient Frontier*. If the transformations do reduce estimation error on average, then it is expected that the *Actual Frontier* estimated using transformed parameters will plot closer to the *True Efficient Frontier* than the *Actual Frontier* estimated using the original raw sample estimates.

## 6.3 Results

### 6.3.1 Preliminary Examination

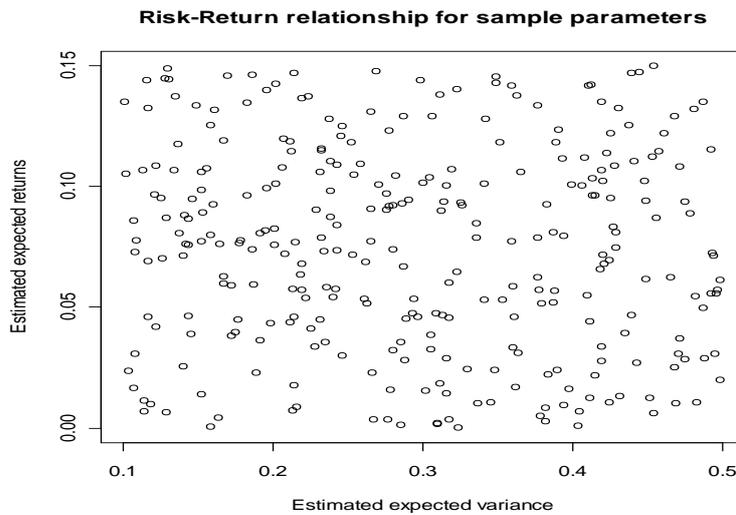
The analysis first involves plotting the risk-return relationship for a raw sample, and also plotting the risk-return relationship for the equivalent transformed parameters.

These plots were designed to determine if the transformed data did in fact have a stronger risk-return relationship than the raw data as hypothesised. For this analysis the

expected return and variance were simulated for 300 hypothetical assets. The expected returns were randomly drawn from a uniform distribution between 0 and 15%, while the variances were randomly drawn from a uniform distribution between 10% and 40%.

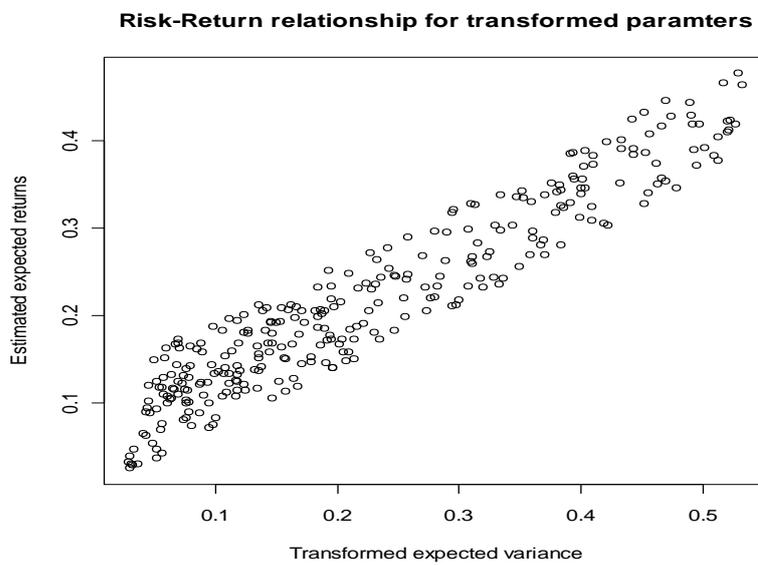
Figure 6.2 depicts the risk-return relationship for the sample parameters in the mean variance plane. For this plot there is no expected relationship as the data has been randomly drawn from the uniform distribution. Figure 6.3 plots the risk-return relationship for the parameters depicted in Figure 6.2 after being transformed by the negative moment transformation method. Lastly, Figure 6.4 plots the relationship for the transformed parameters using the optimal investment horizon transformation. These results highlight the fact that the negative moment transformation process results in a stronger risk-return relationship than that obtained from the raw parameters. The transformation using the optimal investment horizon method changes the data such that an extremely strong linear risk-reward relationship is formed.

**Figure 6.2: Raw Mean versus Raw Variance**



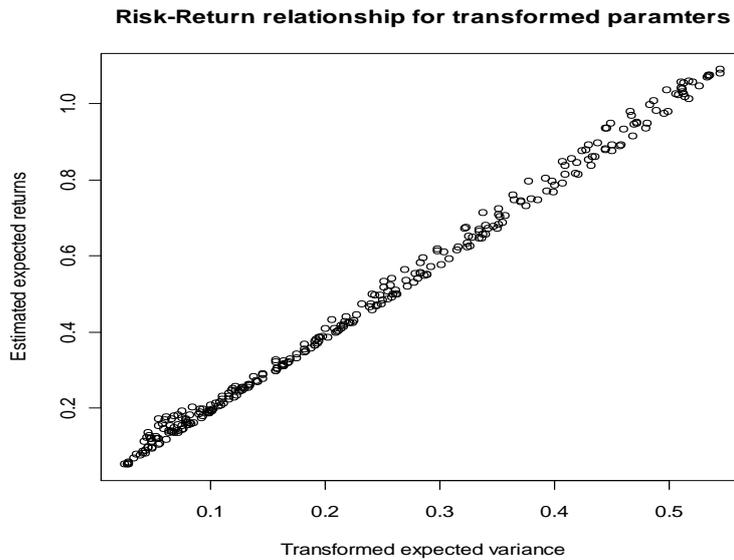
**Plot of randomly drawn data for mean ranging from 0 to 15% and variance ranging from 10% to 40%**

**Figure 6.3: Negative moment transformed mean versus Transformed Variance**



**Plot of transformed mean versus transformed variance using the negative moment transformations given by equations (6.5) and (6.8). The raw parameters transformed are those depicted in Figure 6.2**

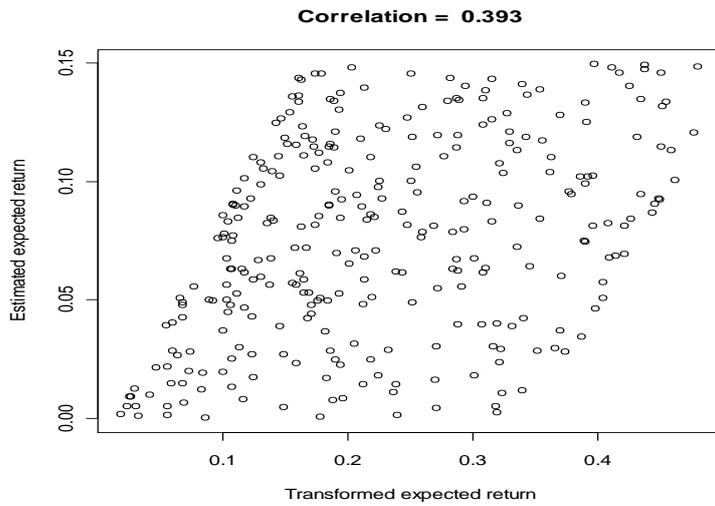
**Figure 6.4: Optimal investment horizon transformed mean versus Transformed Variance**



**Plot of transformed mean versus transformed variance using the optimal investment transformations given by equations (6.12) and (6.8). The raw parameters transformed are those depicted in Figure 6.2**

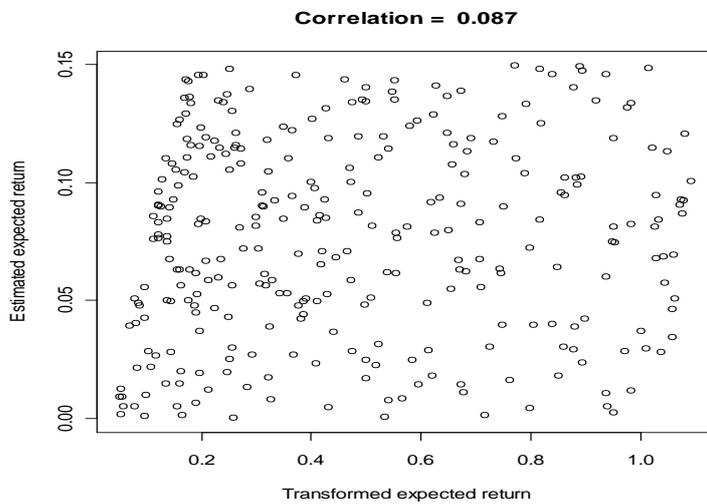
Figure 6.5 depicts the raw expected returns versus the corresponding transformed expected returns when using the negative moment method, along with the correlation coefficient between the raw and transformed first moments. A similar plot for transformed expected returns using the optimal investment horizon transformation is depicted in Figure 6.6. For both transformation methods the transformation to the standard deviation is the same as defined by equation (6.8). Figure 6.7 depicts the raw estimated variance plotted against the transformed variance, along with the correlation between the two.

**Figure 6.5: Expected return versus negative moment transformed return**



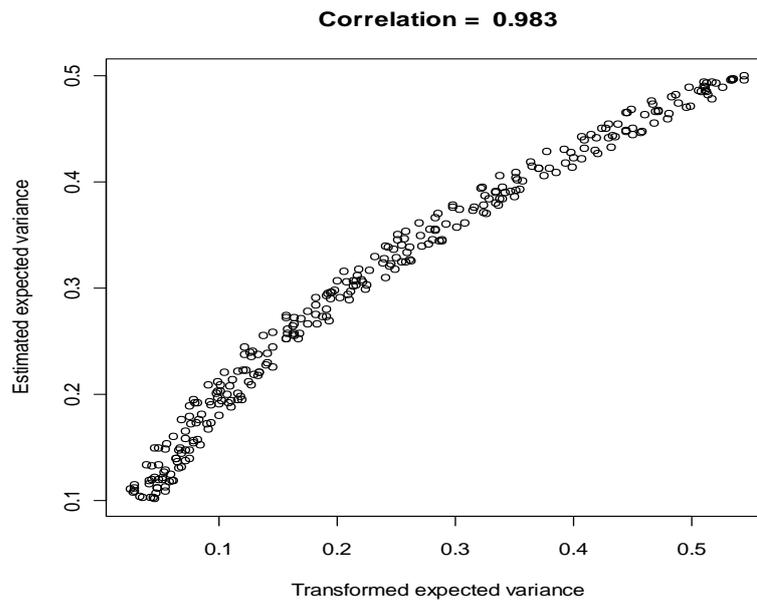
**Plot of raw expected returns against transformed expected returns using the negative moments transformation defined by equation (6.5)**

**Figure 6.6: Expected return versus optimal investment horizon transformed return**



**Plot of raw expected returns against transformed expected returns using the optimal investment transformation, defined by equation (6.12)**

**Figure 6.7: Variance versus negative moment transformed variance**



**Plot of raw variance against transformed variance using the negative moment transformation, defined by equation (6.8)**

The correlation between the raw estimated variance and its corresponding transformed estimate is significantly higher than the correlation between the raw estimated first moment and its corresponding transformed parameter for both transformation methods. This result suggests that more transformation is occurring to the first moment than to the second. This is a desired outcome, as more adjustment to the estimated expected returns is consistent with results of Ceira and Stubbs (2006), who show that most of the estimation error is due to errors in estimates of expected returns, and not in estimates of variance.

When comparing the transformed parameters using the negative moment transformations, as depicted in Figure 6.5, with the transformed parameters using the optimal investment horizon transformations, as shown in Figure 6.6, it is observed there

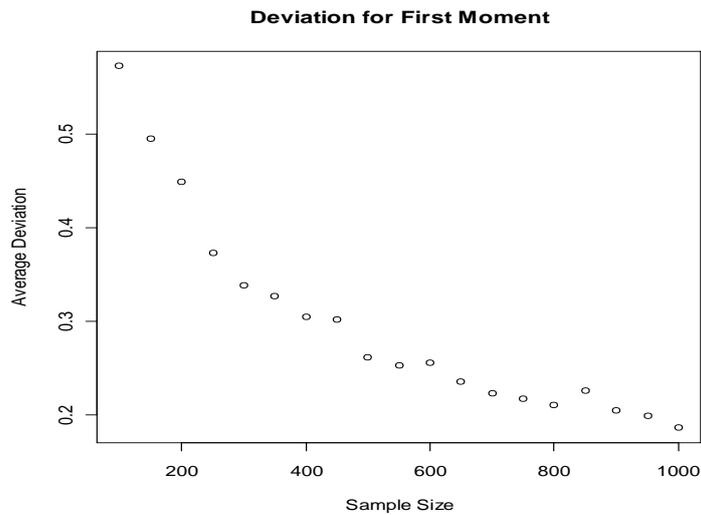
is a correlation of around 0.4 for the negative moment method; yet the correlation is approximately 0.09 for the optimal investment horizon method. This suggests that the negative moment methods transformed expected returns still contain some of the structure of the raw expected return estimates; however the optimal investment horizon transformed expected returns reflect much less of the structure originally contained in the raw parameter estimates.

### 6.3.2 Convergence rate for sample size.

Various sample sizes are used to illustrate how the effect of estimation error is reduced as sample size increases. The analysis shows that a large sample size is required to obtain an *Actual Frontier* close to the *True Efficient Frontier*. The mean absolute deviation (MAD) is used to measure the proximity of the *Actual Frontier* to the *True Efficient Frontier*. However, percentage deviations from their true parameters were taken rather than raw deviations in order to provide a metric which is comparable between the mean and variance.

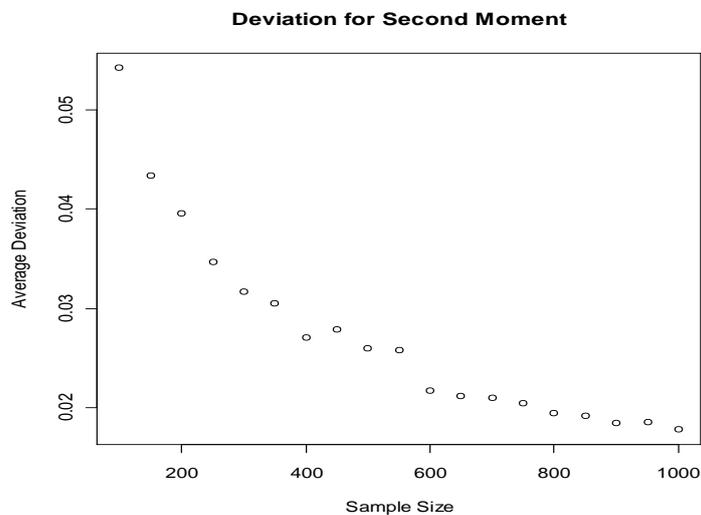
For this section the data used is Broadies (1993) which is presented in Tables 5.1 and 5.2. For each sample size the *Actual Frontier* is based on 100 random draws from the multivariate return distribution defined by Broadie's data, as described in Section 6.2.3. The averages for the mean absolute deviations are shown in Figures 6.8 and 6.9 below; where the deviation is plotted against sample size.

**Figure 6.8: Deviation from the Mean**



**Plot of a random draws average percentage deviation from its true mean for different sample sizes**

**Figure 6.9: Deviation from the Variance**



**Plot of a random draws average percentage deviation from its true variance for different sample sizes**

These results are consistent with the hypothesis, that as the sample size increases, the estimated parameters are closer to the true parameters. Interestingly, the percentage

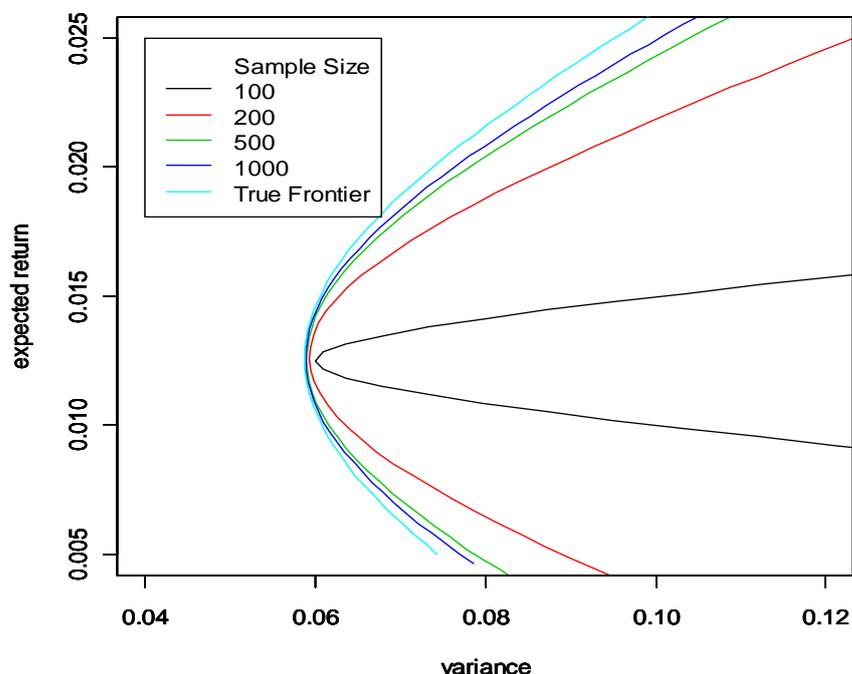
errors in the variances are an order of magnitude less than the errors in the mean<sup>8</sup>. This is consistent with Ceira and Stubbs (2006), who show that most estimation error occurs in estimates of the expected returns.

Using Broadie's (1993) data, *Actual Frontiers* were computed using parameter estimates based on different sample sizes. These *Actual Frontiers* were compared with the *True Efficient Frontier*. As the sample size increases, thereby reducing estimation error, it is expected that the *Actual Frontier* will converge to the *True Efficient Frontier*. The *Actual Frontiers* were estimated for samples of size 100, 200, 500, and 1000 observations. The frontiers plotted were the average of 100 experiments to eliminate any sample bias. The results are depicted in Figure 6.10 below.

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<sup>8</sup> This occurs regardless of whether one is using Eigenvalue decomposition or Choleski decomposition for the sample draw procedure.

**Figure 6.10: Actual Frontier versus True Efficient Frontier for Different Sample Sizes**



**Actual Frontiers for different sample sizes are plotted against the True Efficient Frontier (light blue line). Note that as the sample size increases the Actual Frontier lies closer to the True Efficient Frontier.**

Figure 6.10 highlights how dramatically the actual portfolio can differ from the true portfolio when estimation errors are present. For example, a sample size of 100, or roughly 8 years of monthly data, results in an *Actual Frontier* that is a long way from the *True Efficient Frontier*. A sample of 1000 observations or 83 years of stationary monthly data lies close to the *True Efficient Frontier*, but it is not a perfect approximation and the divergence from the *True Efficient Frontier* increases as the expected portfolio return increases. Thus even with estimates from a stationary set of data lasting 83 years, there is no guarantee of forming a set of portfolios that match the *True Efficient Frontier*. Noticeably, the global minimum variance portfolio is close to the *True Efficient Frontier* for all of the *Actual Frontiers*. This shows that the

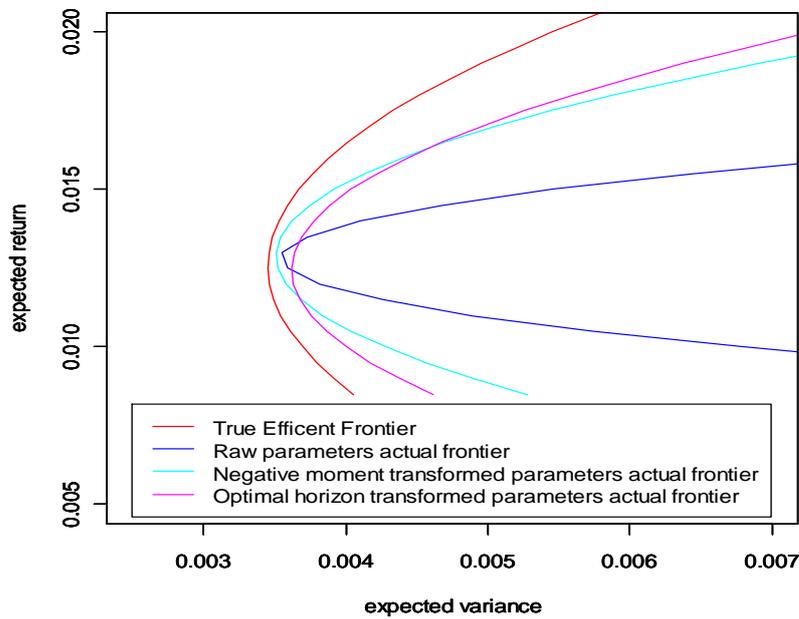
estimation error is less pronounced in the covariance matrix, as the expected returns are not used for the computation of the minimum variance portfolio. Overall, however, it is clear that estimation error can be a substantial problem for portfolio optimisation.

### 6.3.3 Empirical tests for the transforms

The effectiveness of the transforms will be judged by the proximity of the *Actual Frontier* to the *True Efficient Frontier* as outlined in Section 6.2.3, using data from Broadie (1993). A sample size of 200 points was chosen. This equates to over 16 years of monthly data, but as shown in the examples above it is a sample size which still allows substantial errors in the *Estimated Frontiers*.

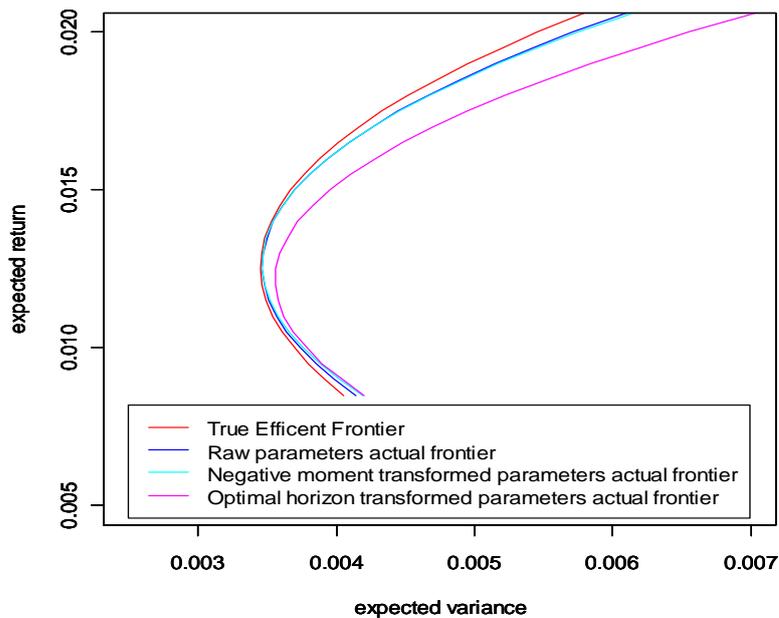
Figure 6.11 plots the *Actual Frontiers* obtained when using the raw parameters and the transformed parameters from the negative moments and optimal investment horizon method. Figure 6.12 is a replication of Figure 6.11, except that a sample size of 1000 data points is used (roughly 83 years of monthly data). This allows an investigation into whether these transformations are still beneficial if there is limited estimation error.

**Figure 6.11: Actual Frontier versus True Frontier**



**Actual Frontier determined using raw parameters or parameters obtained using either of the suggested transformation techniques for a sample size of 200.**

**Figure 6.12: Actual Frontier versus True Frontier**



**Actual Frontier determined using raw parameters or parameters obtained using either of the suggested transformation techniques for a sample size of 1000.**

Figure 6.11 shows that the transformed parameters using the negative moments method estimates the efficient frontier better than the raw estimated parameters, and accordingly results in an *Actual Frontier* closer to the *True Efficient Frontier*. The *Actual Frontier* using the negative moment transformed parameters always dominates the *Actual Frontier* using the raw parameters. The *Actual Frontier* using the optimal investment horizon transformation performs the worst in the region near the global minimum variance portfolio. However, performance improves as expected portfolio returns increase such that the optimal investment horizon transformation is the best performer at the higher end of expected returns.

In Figure 6.12, consistent with the results of Figure 6.10, the *Actual Frontier* using the raw parameters is very close to the *True Efficient Frontier*. This is because the estimation error is small due to the large sample size (1000). The *Actual Frontier* using parameters transformed by the negative moment technique is nearly identical to that obtained using the raw parameters. This result suggests that the negative moment transform is not detrimental when estimation error is small and from Figure 6.11 it appears to be beneficial when estimation error is larger. However, it is clear from Figure 6.12 that the optimal investment horizon transformation is the worst performer when estimation error is small.

In summary, if the negative moment transformation method is applied, then the *Actual Frontier* obtained is, on average, no worse than the *Actual Frontier* obtained using raw parameters when the estimation error is extremely small. However, when estimation error becomes a more significant problem, the *Actual Frontier* obtained from transformed parameters outperforms the *Actual Frontier* using raw parameter estimates

across all regions of the frontier. For the optimal investment horizon, when the estimation error is large, then the transformed parameters are only likely to dominate for the higher expected return areas on the frontier. When the estimation error is small, the optimal investment horizon transformation appears to be detrimental and the *Actual Frontier* is likely to lie substantially below the *Actual Frontier* obtained using raw parameters.

#### 6.3.4 Empirical tests for the transforms when a strong risk-return relationship exists

As Figures 6.3 and 6.4 show, one effect of the transformations is to strengthen the relation between risk and return. The analysis that follows investigates whether the negative moment transformation is still beneficial<sup>9</sup> when the true parameters already exhibit a strong risk-return relationship. Accordingly, analysis similar to that of Figures 6.11 and 6.12 will be undertaken for data where there is a strong risk-return relation. The means and variances of the five assets chosen are provided in Table 6.1 and consistent with Broadie (1993), a correlation of 0.3 for each asset with all other assets was used (as in Table 5.2). The means were chosen to be the same as those in Broadie's work. However, the standard deviations have been slightly modified so a strong risk-return relation is created, as is shown in Figure 6.13.

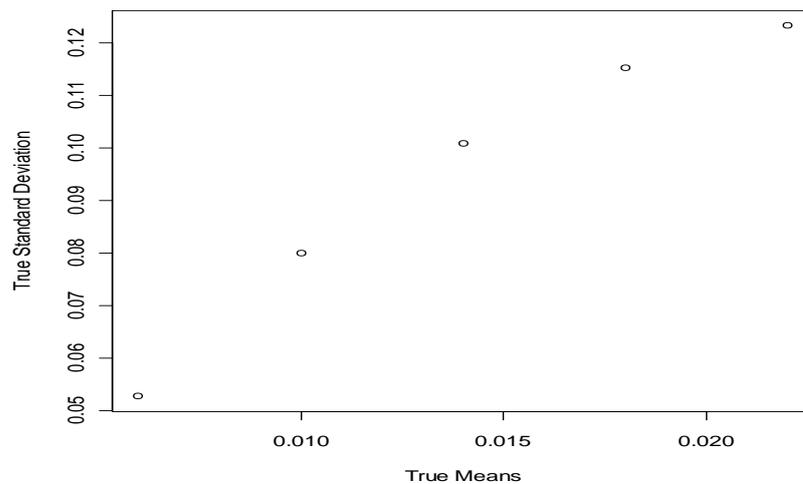
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<sup>9</sup> In the interests of clarity and brevity the optimal investment horizon transformation is not considered in this section, given that the prior results showed that it was not always beneficial.

**Table 6.1: The True Expected Return and Standard Deviation of 5 Assets**

	Asset 1	Asset 2	Asset 3	Asset 4	Asset 5
Mean	0.006	0.01	0.014	0.018	0.022
Standard Deviation	0.053	0.080	0.100	0.115	0.123

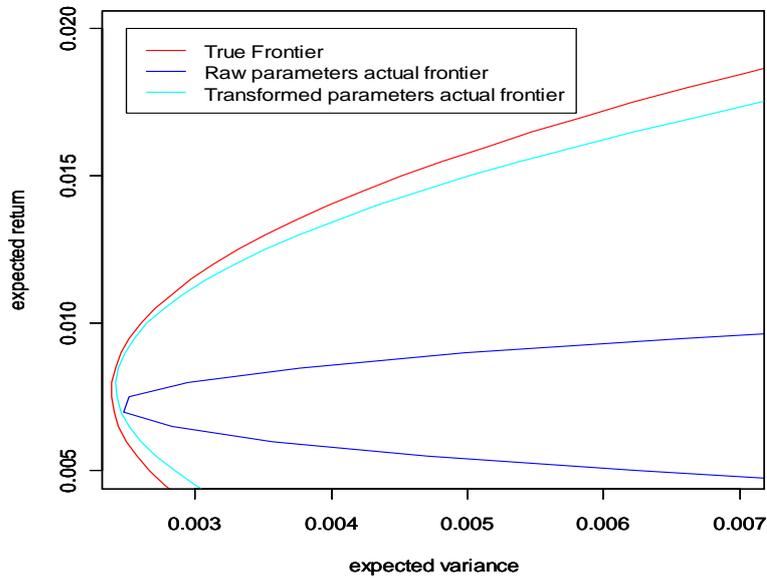
**Figure 6.13: Mean return versus Standard Deviation**



**Plot of the True Means versus the True Standard Deviations from Table 6.1. These points show a strong risk-return relation.**

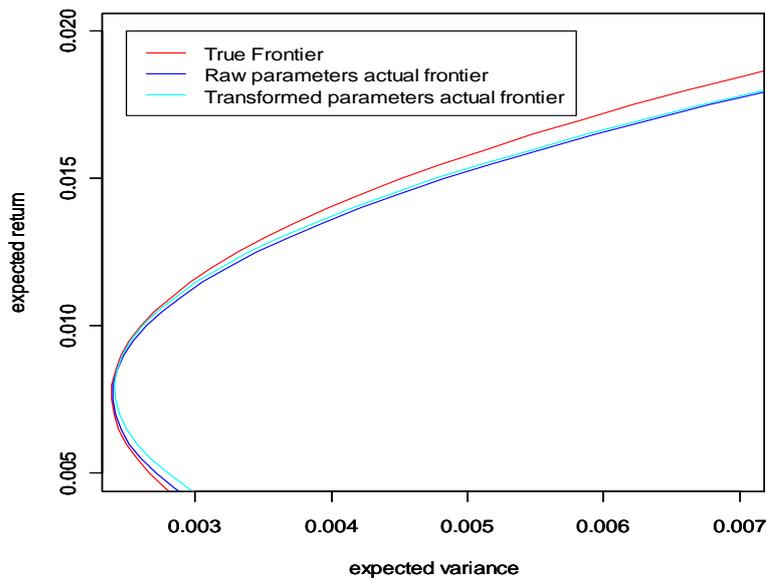
Figure 6.14 shows the *True Efficient Frontier* and the *Actual Frontiers* using the negative moment transformed parameters and the raw parameters, for a sample size of 200. The same frontiers when a sample size of 1000 is used can be found in Figure 6.15.

**Figure 6.14: Actual Frontier versus True Efficient Frontier**



**Actual Frontier determined using raw parameters or parameters obtained using the negative moment transform for a sample size of 200.**

**Figure 6.15: Actual Frontier versus True Frontier**



**Actual Frontier determined using raw parameters or parameters obtained using the negative moment transform for a sample size of 1000.**

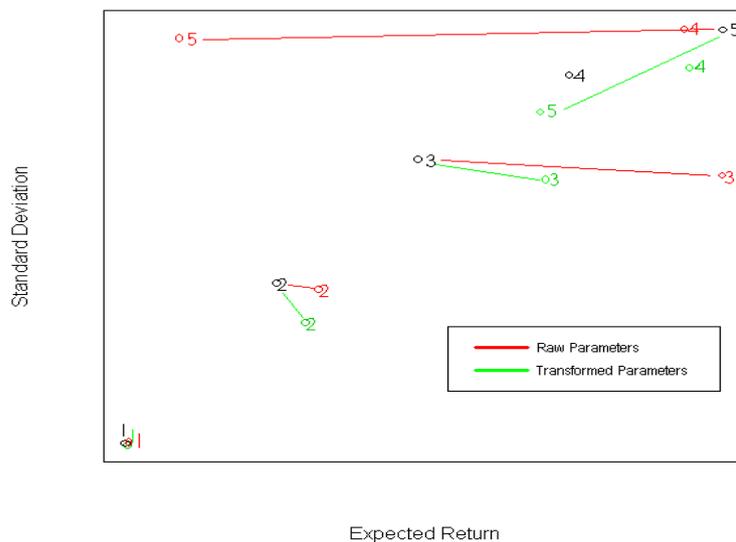
Figure 6.14 shows that even when a strong risk-return relationship is present in the true parameters, but error remains in the estimated parameters, the *Actual Frontier* obtained using the negative moment transformed parameters lies much closer to the *True Efficient Frontier* than the *Actual Frontier* obtained using the raw parameters. Figure 6.15 shows that when the estimation error is very small, the negative moment transformation just outperforms the raw parameters, but there is very little in it.

### 6.3.5 Intuition behind the results

Two factors are suggested as contributing to the beneficial effects of the negative moment transformation. A reduction in estimation error and also a reduction in the extent of over and under weighting. The transformation creates a stronger risk-return relation in the estimated parameters, as depicted previously in Figures 6.2 and 6.3. Even when the true parameters have a strong risk-return relation, the estimated parameters have some error and that risk-return relation is thus somewhat diminished. When the transformation is applied, the relationship is re-strengthened thereby reducing estimation error. This notion is depicted in Figure 6.16 below. The true parameters for five assets are plotted in black in the mean-standard deviation plane. Here it is evident that a strong positive mean-standard deviation relationship exists. In red, the plotted estimated mean and standard deviation for the five assets are also depicted, where the estimates have been made from a draw of 200 points from a multivariate normal distribution. Here it is evident that estimation error is present and part of the risk-return relationship is lost. Lastly, in green, the five assets are plotted using transformed

parameters. Lines have been overlaid extending from the true assets parameters to the estimated assets parameters. Essentially these lines are a graphical representation of estimation error. For asset two it is evident that the estimation error using the raw parameter estimates (red line) is approximately the same as the estimation error for the transformed parameter estimates (green line) as both lines are roughly the same length. However, when observing asset three and asset five it is clear that the estimation error for the raw parameter estimates is substantively larger than the corresponding transformed parameter's estimation error. This figure highlights how the transformation equations move the raw parameters estimates closer to the true parameters thereby reducing estimation error. It should be noted that for clarity the estimation error lines have only been plotted for assets two, three and five.

**Figure 6.16: Plot of estimated parameters versus true parameters**



**This figure depicts the proximity of estimated assets means and standard deviations (coloured points) to their true means and standard deviations (black points). The coloured lines represent errors in the estimates. For clarity these have only been plotted for assets 2, 3 and 5.**

Since the transformation strengthens the risk-return relation it will reduce the extent to which particular assets are given very large or very small weights. For example, from Figure 6.16 it is expected that when using raw parameters in the optimisation process, asset five will get a very heavy weighting due to its relatively high expected return for its given level of variance. However, when using transformed parameters, the expected return for asset five no longer seems as high relative to its corresponding level of variance. Such reweighting towards a more uniform set of weights can be beneficial, even if there is little or no relation between mean and variance for the true parameters. This idea is related to the work of Jobson and Korkie (1981), who show that an equal weighted portfolio can outperform an optimal mean-variance portfolio that has been computed using sample estimates. Making the weightings more uniform reduces the importance of individual parameter estimates.

The effect of the transformation on the portfolio weights is analysed by solving the following optimisation problem and comparing the dispersion in portfolio weights obtained when using transformed parameters versus raw parameters.

$$\begin{aligned} \text{Minimise:} \quad & w^t \Omega w \\ \text{Subject to:} \quad & R^t w = \mu \\ & \mathbf{1}^t w = 1 \end{aligned}$$

Where  $w$  is a vector of asset weights,  $\Omega$  is the covariance matrix,  $R$  is the vector of assets expected returns,  $\mu$  is the desired return for the portfolio and  $\mathbf{1}$  a vector of ones.

The difficulty in solving the above optimisation problem for a given  $\mu$  for both transformed parameters and raw parameters is that the transformations will change the scale. For example, assuming  $R_E$  is the vector of raw estimated returns and  $R_T$  is the vector of transformed expected returns, then it is evident that  $R_T^t w \neq R_E^t w$  despite having the same asset weighting,  $w$ . As such, comparing two portfolios with the same expected return,  $\mu$ , may be futile as the scale of the two portfolios has significantly changed. Accordingly, Broadies (1993) method, as outlined in Subsection 6.3.3, will be used. Again the *Actual Frontiers* will be calculated using raw parameters and transformed parameters. The reason for calculating the *Actual Frontiers* is that the problem of different scales can be removed. For example, assuming  $R_{true}$  is the true returns, then if  $w_e$  is the portfolio weights using raw estimated returns and  $w_t$  is the portfolio weights estimated using transformed parameters. It is evident that if  $w_t$  equals  $w_e$  then  $R_{true}^t w_t = R_{true}^t w_e$  and the same applies for the variance of the two portfolios. Thus the same portfolios have the same mean and variance. The scale problem has been removed as the portfolios have been rescaled back to the same dimension.

To conduct this experiment the true, estimated and transformed parameters must be known. These are provided in Tables 6.2, 6.3 and 6.4 below, and for simplicity it will be assumed that the true correlation between each asset is 0.3, and that this correlation is estimated without error.

**Table 6.2: True Parameters**

	Asset 1	Asset 2	Asset 3	Asset 4	Asset 5
Mean	0.006	0.01	0.014	0.018	0.028
Stan. Dev	0.08	0.08	0.095	0.09	0.08

**Table 6.3: Estimated Parameters**

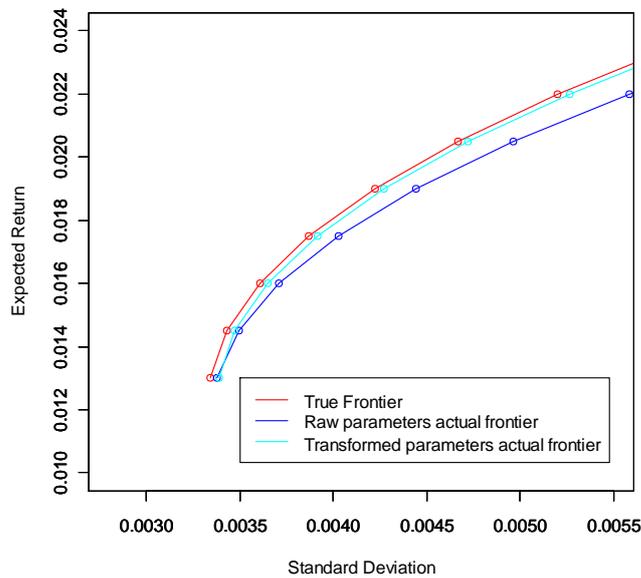
	Asset 1	Asset 2	Asset 3	Asset 4	Asset 5
Mean	0.0024	0.0114	0.0153	0.016	0.0252
Stan. Dev	0.0811	0.0802	0.0955	0.0893	0.0951

**Table 6.4: Transformed Parameters**

	Asset 1	Asset 2	Asset 3	Asset 4	Asset 5
Mean	0.0152	0.0192	0.027	0.0296	0.0372
Stan. Dev	0.0107	0.0123	0.0173	0.0175	0.0178

Using the true parameters, nine efficient portfolios which lie on the *True Efficient Frontier* are computed. These portfolios have an expected return ranging from 1.3% to 2.5% increasing in increments of 0.15% and are depicted as red points in Figure 6.17 below. Furthermore, nine portfolios with the same actual expected returns as the true portfolios previously found are also computed using estimated parameters and are plotted as dark blue points in Figure 6.17. Lastly, nine portfolios with the same actual expected returns are estimated using transformed parameters and are depicted as light blue points. Consistent with previous results, it is evident that the actual portfolios estimated using transformed parameters dominate the actual portfolios using estimated parameters.

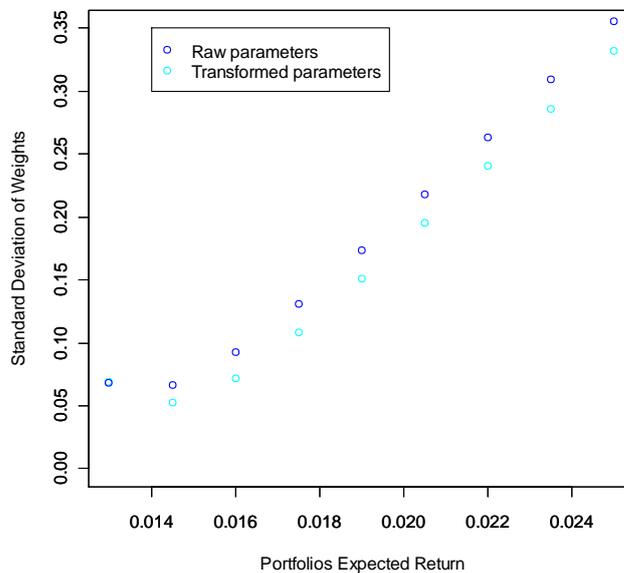
**Figure 6.17: Various Estimated Portfolios**



**This figure depicts the actual portfolios obtained for a specified level of expected return.**

From Figure 6.17, it is evident that for each portfolio estimated using raw parameters there is a corresponding portfolio with the same actual expected return which has been estimated using transformed parameters. By comparing these portfolios, it is possible to determine if the portfolios estimated using transformed parameters have more uniform weighting across assets than portfolios estimated using raw parameters. This is done by comparing the standard deviation of asset weights for a portfolio based on raw parameters and its corresponding portfolio based on transformed parameters. The results are depicted in Figure 6.18

**Figure 6.18: Plot of dispersion of asset weightings**



**This figure depicts the dispersion of weights assigned to assets for a portfolio with a given expected return**

From Figure 6.18 it is evident that, with the exception of the portfolio with the minimum variance,<sup>10</sup> all transformed parameter portfolios have less deviation among the asset weighting than the equivalent portfolio estimated using raw estimated parameters. These results are consistent with the stronger risk-return relation, created by the transformation, leading to a more uniform weighting and improved portfolio performance.

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<sup>10</sup> This portfolio is coincident with the global minimum variance portfolio and since the expected return is not used to compute this portfolio, its weights are unaffected by the strengthening of the risk-return relation.

## 6.4 Conclusion

This Chapter has suggested two new transformation techniques, based on doubling times, as a means to reduce estimation error in the portfolio formulation process. The first was a method which takes advantage of the fact that doubling times are inverse Gaussian distributed and such a distribution contains negative moments. This transformation was called the negative moment transformation. The alternative transformation was motivated by Simonsen, Jensen and Johansen (2002) who suggest that the mode of a doubling time's distribution is the *optimal investment horizon*. To test the effectiveness of both transformation techniques, a method similar to that of Broadie (1993) was employed. This involved comparing the *True Efficient Frontier*, based on the true parameters for mean and variance, with the *Actual Frontier* obtained by applying estimated portfolio weights to the true parameters. The estimated portfolio weights were obtained using raw estimates and transformed estimates for the mean and variance of the assets in the portfolio. The closer the *Actual Frontier* to the *True Efficient Frontier*, the better the performance.

The results suggested that while under some conditions the optimal investment horizon transformation outperforms the raw parameters, the out performance is not consistent under all conditions. However, the results for the negative moment transformation were more promising. It was shown that when there is very little estimation error the transformation does not yield any worse results than those obtained using raw parameters. However, when estimation error is present the transformed parameters appear to substantively outperform the raw parameters. This is particularly noticeable if there is a strong risk-return relationship between the assets used. Two possible

explanations for the improved performance were analysed. First, a reduction of estimation error due to a stronger relation between risk and return in the transformed data. Second a re-weighting towards a more uniform distribution reducing the impact of the more heavily overweight or underweight assets.

# Chapter 7: Conclusion and Future Work

This dissertation has investigated the notion of using doubling times as an alternative performance metric in finance.

Initially, it was shown that taking the arithmetic mean of individual doubling times does not give the expected time to double an investment. Analytically, it was shown that the time weighted harmonic mean of individual doubling times should be used to find the expected doubling time. However, no analytical formula for the variance of the doubling time was derived. Instead the variance of doubling times was obtained using a Monte Carlo simulation. This simulation enabled the estimation of the whole distribution of doubling times, so that traditional metrics such as variance and skewness could then be obtained.

Doubling times were then analytically modelled as a first passage time problem. It was shown that if percentage returns followed the normal distribution, then the corresponding doubling times were inverse Gaussian distributed. The inverse Gaussian distribution is described by two parameters, mean and shape. Functions were derived to express the mean and shape for the doubling time distribution in terms of the underlying percentage return's first two moments, mean and variance. It was then verified that the simulation of the doubling time distribution returned the same values as the analytical expression.

Through various simulation experiments it was shown that even if returns are not normally distributed the simulated doubling time can still be inverse Gaussian distributed, a result which is explained by the Central Limit Theorem, with a particular emphasis on the Berry-Esseen Theorem. Over a century of research has still yet to conclusively determine what distribution percentage returns follow. However, the distribution of doubling times has been shown to follow the inverse Gaussian distribution. In particular it was shown that regardless of the percentage returns distribution (within reason) that this result will hold.

The potential application of doubling times to portfolio formation was also investigated using various time domain risk metrics. It was proved analytically and verified empirically that when minimising one suggested risk metric (the inverse of the shape parameter), the portfolio obtained is equivalent to the global minimum variance portfolio found under the Markowitz framework. Similarly, when minimising the skewness of doubling times, it was shown that the tangency portfolio with the highest risk-return ratio is obtained. This approach has particular appeal due to the natural intuition that investors would want to minimise long waiting periods until their investments double. Additionally, the optimal portfolio problem has now been reduced from a two parameter problem (mean and variance) under the Markowitz framework to a one parameter problem (skewness) in the doubling time framework. This is a refined way to model portfolio optimisation which appears to make intuitive sense and may spark a different way to think about the efficient frontier.

Finally, an investigation was undertaken into a new method for reducing the problems created by estimation error in the portfolio optimisation process. Two doubling time transformations of the expected return and variance were suggested for this purpose. .

One transformation, motivated by Simonsen, Jensen and Johansen (2002) was based on the most likely doubling time (the mode of the doubling time distribution). The alternative transformation was the negative moment transform based on the expected doubling time and it was this transformation which yielded the most promising results. Through a simulation methodology similar to Broadie(1993) it was shown that, on average, the portfolios obtained using transformed parameters for the mean and variance will dominate the portfolios obtained using raw parameters. For all simulations completed, the negative moment transformation portfolios were, at worst, equivalent to the raw parameter portfolios. When the estimation error was substantive, as is likely in practice, portfolios formed using the negative moment transformations significantly outperformed portfolios formed using the raw parameters.

One considerable advantage of this transformation method, when compared to alternatives such as portfolio re-sampling, introduced by Michaud (1999), is that it is far more computationally efficient. As opposed to large simulations with many iterative steps a simple one step transformation is applied to the parameters to be input into the optimisation problem.

This work leaves scope for several further research topics. Continuing with the portfolio work, perhaps an investigation into forming portfolios with time varying parameters could be examined. Generally, in portfolio optimisation when the input parameters such as mean, variance or correlation change over time then the portfolio weights must be re-estimated and the portfolio rebalanced accordingly. However, by taking advantage of two ideas from this research perhaps a methodology which allows formation of a portfolio that requires no rebalancing can be developed. This can be

achieved by taking advantage of, firstly, the doubling time simulation methodology provided in Chapter 3 and, secondly, the proof that the minimum skewness portfolio is equivalent to Markowitz's tangency portfolio with maximum return-risk ratio in Chapter 5. For example, assume expected returns and the covariance matrix for a number of assets are known. However, these expected returns last for only one year, and following that year a series of new expected returns and covariance matrix for these assets exists. Traditionally, a portfolio will be estimated and the weights obtained for year one and then the weights are re-estimated and the portfolio rebalanced as required in the second year. The possible alternative method involves simulating the distribution of a portfolio's doubling time for a given set of portfolio weights. The trick here is that when using the simulation methodology, once a certain number of iterations have been reached, which equate to a simulated year, returns are no longer sampled using the first years estimates, but instead sampled from the following years estimates. Then using some form of optimisation process the weights are varied accordingly such that a portfolio is eventually formed which has the minimum doubling time skewness. Such a technique may prove useful when rebalancing costs are high as it eliminates the need of portfolio rebalancing.

As previously mentioned the use of doubling times could have some application in capital budgeting. The payback period continues to be very popular in capital budgeting despite its well known deficiencies (see Truong, Partington and Peat (2008)). Doubling times might be used to replace the payback period and could be theoretically more defensible. A common argument for the use of the payback period is its easy interpretation. Perhaps the IRR or alternatively the true rate of return could be re-

expressed as doubling times which would have a similarly simple interpretation as the payback period.

Lastly, in Chapter 5 it was shown that the risk tolerance parameter for a point on the efficient frontier is vastly different to the risk tolerance parameter for an equivalent point on the efficient frontier under the Markowitz framework. While various utility functions have been formulated for the Markowitz framework it may be interesting to investigate equivalent utility functions in the time domain.

These are three possible directions for future work using doubling times; however they are by no means collectively exhaustive. It is the author's hope that this dissertation may serve as a catalyst for further creative concepts to provide a novel prism through which to view, evaluate and quantify our constantly evolving financial world.

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# Appendix

## A.1. Kolmogorov-Smirnov tests for Section 4.2

**Table 1:** P-values for Kolmogorov-Smirnov tests that the simulated distribution is inverse Gaussian distributed with parameters defined by equation (4.9). The tests are performed for varying the levels of the expected return for the underlying distribution, with a range extending from 0.001 to 0.01. Here not one test rejects the null that the two distributions were drawn from the same continuous distribution. This suggests the simulated distributions are in agreement with its theoretical distribution defined by equation (4.9)

Underlying Percentage Mean	0.001	0.002	0.003	0.004	0.005
K-S test p value	0.71005	0.849736	0.507247	0.385711	0.666977
Underlying Percentage Mean	0.006	0.007	0.008	0.009	0.01
K-S test p value	0.620326	0.492434	0.229575	0.852519	0.961262

**Table 2:** P-values for Kolmogorov-Smirnov tests that the simulated distribution is inverse Gaussian distributed with parameters defined by equation (4.9). The tests are performed by varying the levels of variance for the underlying distribution, ranging from 0.01 to 0.04. Here not one test rejects the null that the two distributions were drawn from the same continuous distribution. This suggests the simulated distributions are in agreement with its theoretical distribution defined by equation (4.9)

Underlying Percentage Variance	0.01	0.015	0.02	0.025	0.03	0.035	0.04
K-S test p value	0.62169	0.9234	0.98725	0.894142	0.885041	0.7245	0.43264

## A.2. Kolmogorov-Smirnov tests for Section 4.3

**Table 3:** The p-values reported for numerous Kolmogorov-Smirnov tests. Each test involves comparing one simulated distribution with another simulated distribution. Each simulated distribution was formed using  $t$ -distributed underlying returns with mean equal to 0.0054 and standard deviation equal to 0.04 where the degrees of freedom used are provided in the table. This meant all simulated distributions had an underlying distribution with the same first two moments but a varying level of kurtosis. Of the 55 tests performed only one rejects the null hypothesis that the two distributions were drawn from the same continuous distribution. This one rejection is attributed to being a type 1 error.

	Degrees of Freedom											
	3	4	5	6	7	8	9	10	11	12	13	14
3	-	0.993	0.151	0.538	0.751	0.788	0.096*	0.310	0.357	0.483	0.329	0.406
4	-	-	0.923	0.967	0.431	0.803	0.229	0.313	0.123	0.428	0.982	0.932
5	-	-	-	0.565	0.496	0.154	0.167	0.418	0.178	0.814	0.497	0.743
6	-	-	-	-	0.772	0.463	0.986	0.897	0.485	0.642	0.700	0.767
7	-	-	-	-	-	0.790	0.489	0.366	0.410	0.963	0.798	0.655
8	-	-	-	-	-	-	0.655	0.441	0.820	0.138	0.815	0.248
9	-	-	-	-	-	-	-	0.147	0.189	0.441	0.162	0.817
10	-	-	-	-	-	-	-	-	0.508	0.390	0.329	0.741
11	-	-	-	-	-	-	-	-	-	0.675	0.197	0.603
12	-	-	-	-	-	-	-	-	-	-	0.563	0.450
13	-	-	-	-	-	-	-	-	-	-	-	0.486
14	-	-	-	-	-	-	-	-	-	-	-	-

\* Significance at the 10% level.