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Mixed strategies in discriminatory divisible-good auctions*

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Abstract

Using the concept of market-distribution functions, we derive general optimality conditions for discriminatory divisible-good auctions, which are also applicable to Bertrand games and non-linear pricing. We introduce the concept of offer distribution function to analyze randomized offer curves, and characterize mixed-strategy Nash equilibria for pay-as-bid auctions where demand is uncertain and costs are common knowledge; a setting for which pure-strategy supply function equilibria typically do not exist. We generalize previous results on mixtures over horizontal offers as in Bertrand-Edgeworth games, and we also characterize novel mixtures over partly increasing supply functions.

Key words: Pay-as-bid auction, divisible good auction, mixed strategy equilibria, wholesale electricity markets

JEL Classification D43, D44, C72

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1 Introduction

Modelling auctions has been one of the major successes in the application of game theory, since in this setting the rules controlling the interactions between agents are particularly well defined. With auction theory it has been possible to predict bidding behaviour under different auction formats, and this has helped auction designers to choose efficient formats and to avoid disastrous ones. In this paper we focus on multi-unit auctions where the auctioneer buys or sells several homogeneous objects at once and bidders are free to choose a separate price for each object. When the number of traded objects is large, such auctions are called divisible-goods auctions [7] [27], auctions of shares [28] or supply function auctions [15] [20], and each bid consists of a curve. Important markets with this character are treasury auctions, electricity auctions and auctions of emission permits.

Divisible-good auctions have two typical mechanisms. In a pay-as-bid (or discriminatory) procurement auction the auctioneer pays each supplier according to its offer curve, whereas in a uniform-price auction all sellers are paid the clearing price for all of their accepted supply. The debate between proponents of the two formats has a long history and the issue is still largely unsettled. Ausubel and Cramton [6] show that it will depend on the character of the market which format is preferable to the auctioneer. A survey has found that 39 out of 42 countries used the discriminatory format in their treasury auctions [8]. On the other hand, the vast majority of electricity markets use the uniform-price format. But there are exceptions. The balancing market in Britain switched to a pay-as-bid format in 2001, and a similar move has been considered in California [18] and more recently in Italy. Moreover, balancing markets in several European zonal markets are a blend of the uniform-price and pay-as-bid format, because producers' post-clearing adjustments that are used to relax local system constraints (counter-trading) are compensated in a discriminatory way. Some of the power system reserves are also procured using this mechanism, e.g. in Germany [25].

Most studies of bidding behaviour in divisible good auctions are limited to characterizations of pure-strategy equilibria. But such equilibria are sometimes non-existent, and in electricity markets with the pay-as-bid format this seems to be the rule rather than the exception [17]. Thus when applying game-theoretical analysis to real divisible good auctions it is often necessary to consider mixed strategies. For restrictive market assumptions Anwar [2] shows the existence of a special type of mixed-strategy equilibrium in a uniform-price multi-unit auction with independent increasing offers.[1] With this exception, previous mod-

[1] Anwar analyses a uniform-price auction where an offer is submitted for each discrete production unit. For constant marginal costs and uniformly distributed demand, he shows that there exists a mixed strategy NE with increasing offers that are independently chosen for each production unit.

els of mixed strategy equilibria are limited to one-dimensional mixtures in discriminatory auctions where each producer offers its entire capacity at one price. These are essentially Bertrand-Edgeworth Nash equilibria [1] [10] [22] [23] with the added complexity that the auctioneer's demand [2], [12], [14] [24] or bidders' costs/valuations [7] are uncertain. In this paper we generalise these results by considering general cost functions and general distributions of the auctioneer's demand. We are the first to characterise equilibria with mixtures over increasing offer curves. Our focus is on discriminatory electricity auctions, where non-existing pure-strategy Nash equilibria are a major concern, but our novel approach is general enough to be applied to other types of auctions.

We use the concept of a market distribution function [4], which implicitly determines the contour of the residual demand for each probability level, to derive optimality conditions of discriminatory divisible good auctions. This allows us to derive more general conditions for discriminatory divisible-good auctions than in the past; ours are valid for any uncertainty in the seller's residual demand curve, i.e. for any combination of demand uncertainty and uncertainty in competitors' offers (e.g. when competitor costs are unknown or when they randomize their offer curves). These general conditions also have applications in Bertrand games and in the theory of non-linear pricing [26][29]. In the latter case, the seller faces a continuous distribution of consumers with different demand curves, which can be represented by the market distribution function. Ex-ante it is difficult to determine whether a particular discriminatory auction will have a mixed- or pure-strategy equilibrium, so in an empirical study it is useful that our optimality conditions work for both types of strategies. Given firms' cost functions, our methodology also enables one to test the hypothesis that firms are bidding to maximize expected profits. But a restriction is that bidders are assumed to know their own costs, so the results are not directly applicable to settings with common [28] [7] [27] or affiliated uncertain values/costs [6].

The optimality conditions we derive enable us to calculate Nash equilibria for auctions where supplier costs are common knowledge and the exogenous non-strategic demand is uncertain. Nash equilibria in such settings are referred to as supply function equilibria (SFE) [20]. A corresponding sales auction version has been used to analyze how strategic bidding in treasury auctions is influenced by an uncertain amount of non-competitive bids [27]. It has been shown that pure-strategy SFE in pay-as-bid auctions with positive mark-ups do not exist if there is any output level for which both marginal costs are sufficiently flat and the hazard rate of the demand shock is increasing [17]. In this paper, we generalise this second-order condition and make it more precise: the mark-up times the hazard rate of the demand shock must be non-increasing, otherwise pure-strategy SFE cannot exist in pay-as-bid auctions. In electricity markets, marginal costs are approximately stepped, i.e. locally constant, and demand shocks are approximately normally distributed, which have an increasing hazard rate, so the existence of pure-strategy SFE in discriminatory electricity auctions is very much in doubt.

While the optimality conditions are valid for any number of heterogeneous agents, we restrict attention to games with two players when calculating Nash equilibria. To work with mixed strategy equilibria, we introduce the concept of an offer-distribution function. Such a function implicitly defines the contour of a producer's supply function for each probability level. This enables us to define a market distribution function and characterize the best response of the supplier's rival, which allows the calculation of mixed-strategy SFE for cases when pure-strategy Nash equilibria can be ruled out.

The type of equilibria that are observed in these duopoly games depends on whether producers are pivotal or not. A *pivotal producer* is one for which the sum of capacities of rival firms is less than demand with positive probability. In a market with inelastic demand the removal of a pivotal producer from the market would create a supply shortage with positive probability. Hence, a pivotal producer has monopoly power when all its rivals are at capacity. We find that mixed-strategy equilibria essentially divide into two classes depending on the presence or absence of pivotal producers.

In markets with inelastic demand, no price cap and non-pivotal producers, equilibrium mixtures over strictly increasing supply functions can be found. They might be representative for other markets, but we do not expect them to occur in electricity markets, which typically have pivotal producers and price caps. We show that these more realistic circumstances lead to mixtures over supply functions that are horizontal and slope-constrained for low output levels. For low mark-ups, offer curves in the mixture may be upward sloping for high outputs, so that the offer curve gets a hockey-stick shape. Mixtures over such "hockey-stick" bids are a new feature of models representing equilibria in pay-as-bid markets. For high mark-ups, the whole curve is slope-constrained and we get mixtures over horizontal (one-dimensional) bid curves. The slope-constrained mixed-strategy equilibria are uniquely determined by the price cap.

The slope-constrained equilibria can be intuitively explained as follows. Ex-post, after the demand shock has been realized, it is always optimal to offer all accepted bids horizontally in a pay-as-bid auction, so that the maximum price is obtained for all the quantity supplied. Hence, unless the demand density is sufficiently decreasing or marginal costs are sufficiently steep relative to mark-ups (in which case pure-strategy SFE can be found), producers have incentives to offer the very first unit at the same price as some of the units with a higher marginal cost. Hence, the lowest part of the offer curve becomes horizontal and producers have incentives to slightly undercut each other's lowest offers down to the marginal cost, as in a Bertrand game. With constant marginal costs and non-pivotal producers there is a pure-strategy Bertrand Nash equilibrium [27] [12] (which may not be unique if the demand density is sufficiently decreasing). But similar to a Bertrand-Edgeworth game there will be profitable deviations from such an outcome if producers are pivotal [14] [17] or costs are increasing, so the equilibrium must be a mixed one. Increasing marginal costs may become steep relative to mark-ups for higher outputs so that the producer will have incentives to increase the offers of more expensive units. In this case the offer gets a hockey-stick shape.

The paper is laid out as follows. In the next section we define the market distribution function and derive optimality conditions for agents offering in a general discriminatory divisible-good setting. In Section 3, we restrict the analysis to pure-strategy supply function equilibria in pay-as-bid auctions. In Section 4, we study mixed-strategy equilibria over strictly increasing supply functions, and we rule out pivotal producers, price caps and elastic demand for this case. In Section 5 we analyze pivotal producers and a price cap, and we show that there can be slope-constrained mixed-strategy equilibria in this case. The highest bids in the mixture are horizontal. When the price cap is sufficiently low, we show that mixed strategies will contain hockey-stick bids that have both horizontal and increasing sections.

2 Optimality conditions for pay-as-bid auctions

The optimality conditions we derive here are valid for situations in which a supplier is facing an uncertain residual demand curve and is offering a divisible homogeneous good with a discriminatory price-schedule. We assume that the level curves of the residual demand distribution (the market distribution function) are smooth, but otherwise we do not impose any restrictive assumptions on the uncertainty of the residual demand curve: it can be caused by demand uncertainty and uncertainty in competitors' offer curves (e.g. when competitor costs are unknown or when they randomize their offer curves). The producer may be a monopolist in the market or in the most general case, face competition from other producers offering differentiated goods (as long as product differentiation does not introduce any non-smoothness in the residual demand distribution). The optimality conditions also consider cases where a monotonicity constraint in the price-schedule binds. This provides useful first-order conditions for cases when the discriminatory price schedule is partly horizontal or vertical, including a first-order condition that is valid for any Bertrand game where the level curves of the residual demand distribution are smooth. In Section 2.1 we show the applicability of the optimality conditions to non-linear pricing.

Having emphasized the generality of the conditions we now turn our focus back to the standard setting where producers offer homogeneous goods with a price-schedule to a discriminatory divisible-good auction. Each agent offers a supply curve that indicates the amount they are prepared to supply at any given price. The market then clears when supply equals demand and each agent is paid according to their supply function. In particular if an agent with cost function $C(q)$ offers quantity q at price $p(q)$, and the market clears the agent at quantity $\bar{q}$, then the agent is paid $\int_0^{\bar{q}} p(q)dq$, and achieves a profit of

$$\int_0^{\bar{q}} p(q)dq - C(\bar{q}).$$

This objective function is to be maximized over monotonic functions $p(q)$: p may not be continuous and it may not be strictly monotonic (so that there can be horizontal segments). We will also make use of the inverse of the offer curve which we call a supply function $q(p)$; again these are monotonic and may have discontinuities or horizontal sections. For convenience, we choose all the supply functions to be right-continuous with respect to the price, and we use notation like $q(p-)$ to denote the appropriate limit: $\lim_{\delta \searrow 0} q(p-\delta)$. Throughout this paper we will assume that each agent has some maximum capacity, which we write as q_m.

Following [4] we define the market distribution function $\psi(q,p)$ to be the probability that a supply offer of quantity q at price p is not fully cleared by the market. The expected payoff of a supplier offering a curve $p(t)$ into a pay-as-bid market can be written as

$$\Pi = \int_0^{q_m} \left(\int_0^{q} (p(t)dt - C(q) \right) d\psi(q, p(q)) \\ + (1 - \psi(q_m, p(q_m))) \left(\int_0^{q_m} (p(t)dt - C(q) \right). \tag{1}$$

The integral with respect to ψ could be interpreted in the Lebesgue-Stieltjes sense since this formulation would apply even if ψ was not continuous. However, we will assume that ψ is well-behaved and in fact differentiable at every point where $\psi(q,p) \in (0,1)$: we only allow the market distribution function to be non-smooth at the ends of this interval. We will assume that C is differentiable with $C(0) = 0$.

Then integrating by parts gives

$$\Pi = \left[\left(\int_0^q (p(t) - C'(t))dt\right)\psi(q,p(q))\right]_0^{q_m} - \int_0^{q_m} (p(q) - C'(q))\psi(q,p(q))dq$$
$$+ (1 - \psi(q_m, p(q_m)))\left(\int_0^{q_m} (p(t) - C'(t))dt\right)$$
$$= \psi(q_m, p(q_m))\int_0^{q_m} (p(t) - C'(t))dt - \int_0^{q_m} (p(q) - C'(q))\psi(q,p(q))dq$$
$$+ (1 - \psi(q_m, p(q_m)))\left(\int_0^{q_m} (p(t) - C'(t))dt\right)$$

whence

$$\Pi = \int_0^{q_m} (p(q) - C'(q))(1 - \psi(q,p(q)))dq. \qquad (2)$$

This formula has another interpretation. We may consider each increment of capacity dq offered to the market to earn a marginal profit of $(p(q) - C'(q))dq$. The probability of this increment being dispatched is $(1 - \psi(q, p(q)))$, and so (2) represents the expected profit.

Now consider the problem of choosing a curve $p(q)$, $q \in [0, q_m]$ to maximize Π. In some cases this will not have an optimal solution, but where there is a price cap in operation this existence question can be dealt with using the same approach as was used by Anderson and Hu [5] for the uniform price version of this problem. We can model a supply function using a continuous curve $s = \{(x(t), y(t)), 0 \leq t \leq T\}$, in which the components $x(t)$ and $y(t)$ are continuous monotonic increasing functions of a parameter t, and $x(t)$ and $y(t)$ trace, respectively, the quantity and price components. An agent will have an offer curve that starts at some point $(0, y(0))$ and finishes at $(q_m, y(T))$. The clearing price is determined as though the offer curve began with a vertical segment from the origin to $(0, y(0))$, thus without loss of generality we may take $x(0) = y(0) = 0$ and we write Ω for the set of feasible offer curves (continuous monotonic curves s starting at the origin and ending with $x(T) = q_m$).

Proposition 1 *If there is a price cap P and both $\psi(q,p)$ and $C'(q)$ are continuous for $q \in [0, q_m]$ and $p \in [0, P]$, then there exists an optimal solution for the problem of maximizing profit Π over offer curves in Ω.*

Proof. A result of Anderson and Hu [5] (with the roles of price and quantity reversed) demonstrates that if Ω is the set of monotonic continuous curves starting at the origin and ending on the closed line segment, L, from $(q_m, 0)$ to (q_m, P), then Ω is compact under the Hausdorff metric:

$$|s_1 - s_2|_H = \max_{(x_1,y_1)\in s_1} \min_{(x_2,y_2)\in s_2} \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

which measures the maximum Euclidean distance between these two curves.

The next step is to show that when ψ is continuous then Π defined from (2) is a continuous function of the offer curve using the Hausdorff metric. When $s = \{(x(t), y(t)), 0 \le t \le T\}$ we let

$$h(s) = \{(x(t), (y(t) - C'(x(t)))(1 - \psi(x(t), y(t)))), 0 \le t \le T\}.$$

Then $\Pi(s_1) - \Pi(s_2)$ is the area between $h(s_1)$ and $h(s_2)$ (taking account of the right-hand boundary where $x(t) = q_m$). This area is bounded by the length of the curve $h(s_1)$ multiplied by $|h(s_1) - h(s_2)|_H$. Since s_1 is monotonic the length of $h(s_1)$ is bounded by $P + q_m$, and so we can deduce that the profit from using an offer curve s is a continuous function of the curve $h(s)$. Now if $\psi(q, p)$ and $C'(q)$ are continuous then they are uniformly continuous with Euclidean metrics and this is enough to establish that h is a continuous function in the Hausdorff metric. Hence in this case we have the required continuity property for Π which together with compactness of Ω establishes the result. ■

In the absence of any constraints an optimal $p(q)$ must satisfy the Euler equation

$$\frac{\partial}{\partial p}(p(q) - C'(q))(1 - \psi(q, p(q))) = 0,$$

which may be rewritten

$$1 - \psi\,(q, p(q)) - \psi_p(q, p(q))(p(q) - C'(q)) = 0. \tag{3}$$

In our case we require $p(q)$ to be monotonically non-decreasing (i.e. a supply curve). In the next lemma we establish that the Euler curve formula applies whenever the supply curve is neither horizontal or vertical.

Lemma 2 *On any section of an optimal curve with* $0 < p'(q) < \infty$ *we have*

$$1 - \psi(q, p(q)) - \psi_p(q, p(q))(p(q) - C'(q)) = 0.$$

Proof. Consider a vertical perturbation of the curve by $\delta > 0$, between limits q_1 and q_2 with $0 < p'(q) < \infty$ throughout $[q_1, q_2]$. This gives a new curve

$$r(q) = \begin{cases} p(q) + \delta, & q_1 \le q \le q_2 \\ \max\{p(q), p(q_2) + \delta\}, & q_2 \le q \\ p(q), & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \Pi(r) - \Pi(p) &= \int_0^{q_m} (r(q) - C'(q))(1 - \psi(q, r(q))dq \\ &\quad - \int_0^{q_m} (p(q) - C'(q))(1 - \psi(q, p(q))dq \\ &= \delta \int_{q_1}^{q_2} (1 - \psi(q, p(q))dq - \delta \int_{q_1}^{q_2} \psi_p(q, p(q))(p(q) - C'(q))dq + o(\delta). \end{aligned}$$

Since $p(q)$ is optimal we must have $\Pi(r) - \Pi(p) \leq 0$, and so we obtain

$$\int_{q_1}^{q_2} (1 - \psi(q, p(q)) - \psi_p(q, p(q))(p(q) - C'(q))dq \leq 0.$$

A similar perturbation by $\delta < 0$ yields

$$\int_{q_1}^{q_2} (1 - \psi(q, p(q)) - \psi_p(q, p(q))(p(q) - C'(q)))dq \geq 0.$$

Since this holds on any section of the curve we have

$$1 - \psi(q, p(q)) - \psi_p(q, p(q))(p(q) - C'(q)) = 0,$$

which gives the result. ■

We write

$$\begin{aligned} Z(q,p) &= \frac{\partial}{\partial p}(p(q) - C'(q))(1 - \psi(q, p(q))) \\ &= 1 - \psi(q,p) - \psi_p(q,p)(p - C'(q)). \end{aligned} \tag{4}$$

We may deduce from the proof of Lemma 2 that for an optimal increasing supply curve it is necessary that $Z(q,p) \leq 0$ for $p > p(q)$ and sufficiently close to $p(q)$. Also $Z(q,p) \geq 0$ for $p < p(q)$ and sufficiently close to $p(q)$. Note that $Z(q,p)$ gives the marginal expected profit increase if we increase the price of unit q from p to $p + dp$. This follows from observing that $[1 - \psi(q,p)]\, dp$, the increase of mark-up times the probability that this bid is accepted, is the marginal revenue from this perturbation. But the probability that the bid is accepted is reduced by $\psi_p(q,p)dp$. Multiplying the reduction in the acceptance probability by the mark-up gives the expected loss of an increase in the mark-up. On an optimal increasing supply curve these two terms will be equal and $Z(q,p) = 0$. To ensure a local profit maximum $Z(q,p)$ needs to be decreasing in p for a fixed q. Thus $Z(q,p)$ is negative for (q,p) above and to the left of the increasing offer curve (and positive for (q,p) below and to the right of the offer curve). We can express this in terms of the partial derivative of Z. If we let $q_i(p)$ be the offer curve of the analyzed firm, then,

$$\left. \frac{\partial Z(q,p)}{\partial q} \right|_{q=q_i(p)} \geq 0. \tag{5}$$

Now if the Euler curve decreases at some point then it cannot be a candidate supply curve. In this case part of $p(q)$ will be horizontal. Similarly if the Euler curve bends back on itself then $p(q)$ will have a vertical segment. The following results characterize these situations.

Lemma 3 *Suppose an optimal curve is increasing at q_1 and horizontal at $q_2 > q_1$. Then*

$$\int_{q_1}^{q_2} (1 - \psi(q, p(q)) - \psi_p(q, p(q))(p(q) - C'(q)))dq \geq 0.$$

Proof. Consider a vertical perturbation of the curve downwards by $\delta > 0$, between limits q_1 and q_2, to give a new curve

$$r(q) = \begin{cases} p(q) - \delta, & q_1 \leq q \leq q_2 \\ \min\{p(q), p(q_1) - \delta\}, & q \leq q_1 \\ p(q), & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \Pi(r) - \Pi(p) &= \int_0^{q_m} (r(q) - C'(q))(1 - \psi(q, r(q))dq \\ &- \int_0^{q_m} (p(q) - C'(q))(1 - \psi(q, p(q))dq \\ &= -\delta \int_{q_1}^{q_2} (1 - \psi(q, r(q))dq + \delta \int_{q_1}^{q_2} \psi_p(q, p(q))(p(q) - C'(q))dq + o(\delta). \end{aligned}$$

This must give $\Pi(r) - \Pi(p) \leq 0$, and so we obtain

$$\int_{q_1}^{q_2} (1 - \psi(q, p(q)) - \psi_p(q, p(q))(p(q) - C'(q)))dq \geq 0,$$

which gives the result. ■

Similarly we can prove

Lemma 4 *Suppose an optimal curve is horizontal at q_1 and increasing at $q_2 > q_1$. Then*

$$\int_{q_1}^{q_2} (1 - \psi(q, p(q)) - \psi_p(q, p(q))(p(q) - C'(q)))dq \leq 0.$$

Combining these results gives the following lemma.

Lemma 5 *Suppose an optimal curve is horizontal between q_1 and q_2 and these quantities are the end points of the horizontal segment. Then*

$$\int_{q_1}^{q_2} (1 - \psi(q, p(q)) - \psi_p(q, p(q))(p(q) - C'(q)))dq = 0.$$

Proof. Choose $q_0 < q_1$ with $p(q)$ increasing in (q_0, q_1). Then Lemma 2 and Lemma 3 combine to show that

$$\int_{q_1}^{q_2} (1 - \psi(q, p(q)) - \psi_p(q, p(q))(p(q) - C'(q)))dq \geq 0.$$

But by choosing $q_0 > q_2$ with $p(q)$ increasing in (q_2, q_0) and using Lemma 2 and Lemma 4 we establish the reverse inequality, hence giving the result we require. ■

From Lemma 5 it is straightforward to show that a general first-order condition for Bertrand games, where suppliers are restricted to offer their capacity at one price, is given by:

$$\int_{0}^{q_m} (1 - \psi(q, p) - \psi_p(q, p)(p - C'(q)))dq = 0.$$

Finally we can establish the equivalent result for the case of a vertical segment.

Lemma 6 *Suppose an optimal curve is vertical at $\bar{q}$ between p_1 and p_2, and these prices are at the end points of the vertical segment. Then*

$$(p_2 - C'(\bar{q}))(1 - \psi(\bar{q}, p_2)) = (p_1 - C'(\bar{q}))(1 - \psi(\bar{q}, p_1))$$

Proof. Consider a horizontal perturbation of the curve by $\delta > 0$, between limits p_1 and p_2, to give a new curve

$$r(q) = \begin{cases} p_1, & \bar{q} \leq q \leq \bar{q} + \delta \\ p(q), & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \Pi(r) - \Pi(p) &= \int_0^{q_m} (r(q) - C'(q))(1 - \psi(q, r(q)))dq \\ &\quad - \int_0^{q_m} (p(q) - C'(q))(1 - \psi(q, p(q)))dq \\ &= -\delta((p_2 - C'(\bar{q}))(1 - \psi(\bar{q}, p_2)) - (p_1 - C'(\bar{q}))(1 - \psi(\bar{q}, p_1))) + o(\delta) \end{aligned}$$

This must give $\Pi(r) - \Pi(p) \leq 0$, and so we obtain

$$(p_2 - C'(\bar{q}))(1 - \psi(\bar{q}, p_2)) - (p_1 - C'(\bar{q}))(1 - \psi(\bar{q}, p_1)) \geq 0.$$

Similarly a horizontal perturbation of the curve by $\delta > 0$, between limits p_1 and p_2, to give a new curve

$$r(q) = \begin{cases} p_2, & \bar{q} - \delta \leq q \leq \bar{q} \\ p(q), & \text{otherwise,} \end{cases}$$

yields

$$(p_2 - C'(\bar{q}))(1 - \psi(\bar{q}, p_2)) - (p_1 - C'(\bar{q}))(1 - \psi(\bar{q}, p_1)) \leq 0,$$

which gives the result. ■

2.1 Non-linear pricing

In this Section we make a brief excursion from the main topic to illustrate that the optimality conditions are also of relevance for non-linear pricing. Consider a monopolist who uses non-linear pricing to discriminate between a continuum of types with individual demand curves [26][29]. Let θ be a taste parameter indicating the type of consumer. The distribution of the types is given by a density function $h(\theta) = H'(\theta)$ with the support $\left[\underline{\theta}, \overline{\theta}\right]$. For simplicity we normalize the problem so that the number of consumers integrates to 1, i.e. $H(\underline{\theta}) = 0$ and $H\left(\overline{\theta}\right) = 1$. The monopolist charges in total $T(q) = \int_0^q p(t)dt$ from any consumer buying q units. For simplicity, we assume that consumers have quasi-linear preferences. This is a good approximation when the consumers' expenditure on the good is small in comparison to their income [26]. Hence, a consumer's utility is given by $U = v(q,\theta) + Q$, where q is the good sold with a non-linear price-schedule $T(q)$ and Q is the numeraire, the price of which is normalized to 1. Let w be the income of the consumer. Hence, from the budget constraint it follows that

$$U = v(q,\theta) + w - \int_0^q p(t)\,dt,$$

so the consumer maximizes its utility when

$$\frac{\partial U}{\partial q} = \frac{\partial v(q,\theta)}{\partial q} - p(q) = 0.$$

Thus the consumer's demand is given by the condition that the marginal price $p(q) = T'(q)$ equals the consumer's marginal value of the good $\frac{\partial v(q,\theta)}{\partial q}$. We note that the demand is independent of income, so the demand curve is a function of the type and the marginal price, i.e. $D(p,\theta)$. We assume that the types are ordered such that $\frac{\partial D(p,\theta)}{\partial \theta} > 0$. Analogous to Section 2 we now define the market distribution function $\psi(q,p)$ to be the fraction of consumers buying less than q units when the marginal price $T'(q)$ equals p. Thus

$$\psi(q,p) = H(\theta_c(q,p)),$$

where $D(p,\theta_c) = q$. Analogous to (2) the profit of the monopolist is given by:

$$\Pi = \int_0^{\overline{q}} (p(q) - C'(q))(1 - \psi(q,p(q)))dq,$$

where $D\left(p(\overline{q}), \overline{\theta}\right) = \overline{q}$.

Many applications require that $T(q)$ is a concave function of the purchased quantity. This precludes customers from making arbitrage by opening multiple accounts and purchasing a small amount from each one [29]. Concavity of $T(q)$ is equivalent to the requirement that the price schedule $p(q)$ is non-increasing — allowing quantity discounts but never imposing quantity premia. It is straightforward to verify that the first-order condition in Lemma 2 is directly applicable to non-linear pricing, also when the price-schedule is strictly decreasing. The second-order condition in (5), however, would be reversed to $\left.\frac{\partial Z(q,p)}{\partial q}\right|_{q=q_i(p)} \leq 0$ when the price-schedule must be non-increasing. Similarly, the inequalities in Lemma 3 and 4 would be reversed. Lemma 5 also applies to situations where offer curves are slope-constrained due to a non-increasing constraint. Imposing such slope constraints in the non-linear pricing literature is referred to as "ironing" [29].

3 Supply function equilibrium

In this section we use the optimality conditions to derive necessary conditions for pure-strategy supply function equilibria. Now, we assume that costs are common knowledge and that demand is uncertain; a standard assumption for electricity auctions [15]. Supply function equilibria in the pay-as-bid auction, have been studied in the context of electricity markets by Holmberg [17]. He shows that symmetric supply function pure-strategy equilibria can be ruled out if the shock distribution has a locally increasing hazard rate when marginal costs are locally sufficiently flat. We re-examine this in the context of market distribution functions.

In the model we discuss, demand is elastic, being represented by a differentiable demand curve $D(p)$, and an additive demand shock ε with probability distribution F having a well-defined density function f with support $[\underline{\varepsilon}, \overline{\varepsilon}]$. At this point we should also point out that the pay-as-bid formulation creates some difficulties in circumstances in which there is elastic demand. Similar to Bertrand models [23], the question is how much of the excess demand at lower offered prices will remain at higher prices. As in Federico and Rahman [13], we assume that demand depends on the highest accepted offer in the market and we refer to this price as the clearing price - though much of the demand is met at lower prices, because of the pay-as-bid mechanism. The corresponding assumption in Bertrand models is called parallel rationing [23], which is equivalent to the assertion that demand of the good does not depend on income [22].[2] Moreover, as in other mod-

[2] The other common assumption in Bertrand models is called proportional rationing [23], and was used in the original work by Edgeworth on price competition between pivotal firms

els of electricity markets and Bertrand models we assume that the consumers' bid curves are simply determined by their marginal value of the commodity. We know from the derivation in Section 2.1 that this will be the case when consumers have quasi-linear preferences and pay the average of the producers' accepted offers. The latter assumption implies that consumers' total expenditures equal producers' total revenue, which would not be the case if consumers paid according to their bids and producers were paid according to their offers.

Now consider a producer i, who submits a supply curve $q_i(p)$. Suppose that its competitors' total quantity offered at price p is given by $q_j(p)$. It is easy to see that

$$\psi(q,p) = F(q + q_j(p) - D(p)).$$

Thus, when q_j is differentiable, the Euler curve (3) can be rewritten

$$1 - F(q + q_j(p) - D(p)) - (p - C'(q))f(q + q_j(p) - D(p))(q_j'(p) - D'(p)) = 0.$$

Then for $q = q_i(p)$ to be an optimal increasing offer in response to $q_j(p)$ we require that

$$\begin{aligned} &(1 - F(q + q_j(p) - D(p))) - \\ &(p - C'(q)) f(q + q_j(p) - D(p))(q_j'(p) - D'(p)) \begin{cases} \leq 0, & q < q_i(p) \\ = 0, & q = q_i(p) \\ \geq 0, & q > q_i(p) \end{cases} \end{aligned} \tag{6}$$

In the region where $F(q + q_j(p) - D(p)) < 1$ we can rewrite the equality in (6) as

$$1 - (p - C'(q))\, H(q + q_j(p) - D(p))(q_j'(p) - D'(p)) = 0$$

where $H(x) = f(x)/(1 - F(x))$ is the hazard rate of the demand shock. Moreover, the inequalities in (6) imply that

$$\left.\frac{\partial}{\partial q}\left[(p - C'(q))\, H(q + q_j(p) - D(p))\right]\right|_{q=q_i(p)} \leq 0, \tag{7}$$

provided that $q_j'(p) > 0$ (and hence $q_j'(p) - D'(p) > 0$).

This generalizes a previous result by Holmberg [17], who shows that if $H'(x) > 0$, and costs are close to linear, then there is no pure-strategy supply function equilibrium in the pay-as-bid auction. Note that both the first-order and second-order condition need to be satisfied at each level of output. Marginal costs in electricity markets are approximately stepped, i.e. locally constant, so that the second-order condition (7), which is required for pure-strategy SFE in such markets, is close to a requirement that $H'(x) \leq 0$. This is a very strong restriction

[11]. The two rationing assumptions are identical for perfectly inelastic demand.

on the form of F. It means that the density of the demand shock must decrease faster than e^{-x} throughout its range which rules out most demand shocks that one would encounter in practice.

We can gain some intuition for what is going on here by considering the effect of one player raising the price as much as possible for the initial quantity δ that it offers. So $p(q)$ becomes horizontal over a range $(0, \delta)$. If this player ends up supplying an amount more than δ then it improves its profit due to the higher price received for this first part of its output. The only loss occurs when demand is very low and the player ends up supplying less than δ. For a supply function that is not horizontal to occur, these considerations must balance. This can only happen when there is approximately equal probability of supplying an amount less than or greater than δ. This demonstrates that we will need a demand function weighting low demand values very highly, and hence the very steeply decreasing density functions implied by $H'(x) \leq 0$.

4 Mixtures with non-binding slope constraints

In the previous section we concluded that pure-strategy SFE can be ruled out for many pay-as-bid markets that we encounter in practice. Now we begin the main task of this paper which is an analysis of mixed-strategy equilibria for pay-as-bid markets. In this section we analyze cases where producers mix over a range of offer curves each of which satisfies the Euler condition $Z(q, p) = 0$. We call these supply functions with *non-binding slope constraints*. We will show that equilibria of this form may occur, but the conditions for such an equilibrium are very restrictive; they normally only exist when demand is inelastic and producers are non-pivotal (so that demand can still be met even if the largest firm exits the market). A second class of equilibrium mixture with slope-constrained bids is analyzed in Section 5.

We consider an equilibrium in which there is mixing over a whole range of solutions each of which falls into a region Γ for which

$$Z(q, p) \equiv 0, \quad (q, p) \in \Gamma. \tag{8}$$

Moreover we suppose that the mixing takes place over a continuum of solutions with no gaps, and thus we assume that $\{p : (q_0, p) \in \Gamma\}$ is an interval for any choice of $q_0 \in (0, q_m)$. We also assume that Γ^o, the interior of Γ, is a non-empty connected set.

Substituting for Z using (4) implies that the function $(p - C'(q))(1 - \psi(q, p))$ is independent of p in the region Γ. Hence

$$(p - C'(q))(1 - \psi(q, p)) = \theta(q)$$

for some arbitrary function θ. Thus the market distribution function of a mixing producer can be written

$$\psi(q,p) = 1 - \frac{\theta(q)}{p - C'(q)} = \frac{p - k(q)}{p - C'(q)}, \tag{9}$$

where $k(q)$ is some arbitrary function, such that $C'(q) \leq k(q) \leq p$.

In a mixed strategy equilibrium, each producer's strategy can be expressed by means of its offer distribution function $G(q,p)$, which is defined as the probability that the producer offers strictly more than q units at a price p or less. Implicitly this function determines the contours of the producer's supply for each probability level. In general there will be many different ways to produce the same offer distribution function by mixing over complex sets of offer curves. However the simplest way for a producer to generate a mixture of offer curves corresponding to G is to offer contours $G(q,p) = \gamma$ where γ has a uniform distribution on $(0,1)$. This is the only way to obtain G if the mixture is taken over supply functions that do not cross.

As supply functions are monotonic by assumption, we have that $\frac{\partial}{\partial p}G \geq 0$ and $\frac{\partial}{\partial q}G \leq 0$, where these derivatives exist. We will assume that all quantities supplied are non-negative. We write $q^L(p)$ for the infimum of quantities offered by any supply function at price p for producer i. If the mixture has a finite mass on the lowest supply function then G will be discontinuous at this point. But if G is continuous (in its first argument) then from the definition of G we will have $G\left(q^L(p), p\right) = 1$.

Without loss of generality we can assume that the lower boundary of Γ is included in Γ and corresponds to the highest supply function. We write $q^U(p)$ for this highest supply function in the mixture. Then by definition $G\left(q^U(p), p\right) = 0$ even if G has a discontinuity there.

4.1 Mixed supply function equilibria

For simplicity we limit our analysis to a symmetric equilibrium in a symmetric duopoly market. We consider the offer of producer i and we denote the offer distribution function of its competitor by $G_j(q,p)$. The accepted output of producer i at price p is given by the difference between two independent random variables: the shock outcome ε and $q_j(p)$, the supply of the competitor at the price p. Hence, the probability that an offer of q_i by producer i is not fully dispatched if offered at the price p is

$$\psi_i(q_i,p) = \int_{-\infty}^{\infty} f(\varepsilon)\, G_j(\varepsilon + D(p) - q_i, p)\, d\varepsilon.$$

Making the substitution $t = \varepsilon + D(p) - q_i$ yields

$$\psi_i(q,p) = F(q - D(p)) + \int_0^\infty f(t + q - D(p))\, G_j(t,p)\, dt, \tag{10}$$

since $G_j(t,p) = 1$ for $t < 0$. Here we have assumed that F is continuous and f is well-defined, but we could clearly write the equivalent formulae with sums instead of integrals in the case that the demand distribution is discrete.

We let $\underline{p}$ be the infimum of clearing prices with a positive output. For a given demand, the lowest clearing price will occur when the producers offer the largest amount i.e. when the offers are $q_i^U(p)$. Thus $\underline{p}$ can be defined explicitly as

$$\underline{p} = \inf\{p : q_1^U(p) + q_2^U(p) \geq D(p) + \varepsilon > 0 \text{ for some } \varepsilon \in [\underline{\varepsilon}, \overline{\varepsilon}]\}. \tag{11}$$

Analogously we introduce a highest clearing price $\overline{p}$, which is defined by

$$\overline{p} = \sup\{p : q_1^L(p) + q_2^L(p) \leq D(p) + \overline{\varepsilon}\}. \tag{12}$$

Our main purpose with this section is to show that mixed-strategy equilibria with non-binding slope constraints can only occur under very restricted conditions. Thus, our focus is on deriving necessary conditions for such equilibria. Sufficient conditions are only provided for some special cases. Our analysis will proceed in stages. The lowest clearing price with a positive output, i.e. $\underline{p}$, will be very important in this development. The first result in this section establishes that in a mixed-strategy equilibrium with non-binding slope constraints it is necessary for the most competitive supply curve $q_i^U(p)$ in the mixture to offer at least the minimum possible demand at this lowest price. In other words the first units of any producer cannot be accepted with certainty, because then it would be optimal to increase the price of these units, and this shows that the highest offered quantity in the mixture must be at least as large as the lowest demand at $\underline{p}$. This means that in the case with deterministic demand neither producer can be pivotal if there is to be a mixed-strategy equilibrium with non-binding slope constraints. In the case of uncertain demand, it will later be shown that both producers in a mixed-strategy equilibrium offer the maximum demand at $\underline{p}$, and so neither can be pivotal in this case as well. As established in Proposition 8, one implication of this is that demand must be inelastic for mixtures without slope-constraints, because if demand were elastic non-pivotal suppliers could increase their profits by postponing the crossing of the maximum demand curve, so they would offer all their supply at $\underline{p}$, i.e. they would not mix.

Lemma 7 *Any Nash equilibrium with mixtures over supply functions with non-binding slope constraints has* $q_i^U(p) = q_i^L(p) = 0$ *for* $p < \underline{p}$ *and* $q_i^U(\underline{p}) \geq \underline{\varepsilon} + D(\underline{p})$.

In addition, if $\underline{p} > C'(\max(0, \underline{\varepsilon} + D(\underline{p})))$*, then* $q_i^L(\underline{p}) = 0$ *and* $G_i(q, \underline{p}) = 0$*, for* $q > 0$.

Proof. In a pay-as-bid auction, it can never be best to offer a positive output below $\underline{p}$ (since raising all prices to $\underline{p}$ will increase revenue without altering any clearing prices that occur). Hence $q_i^U(p) = q_i^L(p) = 0$, for $p < \underline{p}$.

Next we prove the inequality for $q_i^U(\underline{p})$ by assuming that $\underline{\varepsilon} + D(\underline{p}) > q_j^U(\underline{p}) = q_i^U(\underline{p})$ and deriving a contradiction. To do this define $w_i = \underline{\varepsilon} + D(\underline{p}) - q_j^U(\underline{p})$, which is the minimum value of the residual demand faced by firm i at price $\underline{p}$, and suppose that $w_i > 0$. From (11) we have $q_i^U(\underline{p}) \geq w_i$, and so $q_i^U(\underline{p}) > 0$. Observe that since we use mixtures with non-binding slope constraints, we have $Z_i = 0$ in the interior of the section of the q_i^U offer curve which may be intersected by some realization of residual demand (this is true even if the region Γ is at a higher price than $\underline{p}$). Thus if $w_i < q_i^U(\underline{p})$ we have $Z_i(q, \underline{p}) = 0$ for $q \in (w_i, \min(q_i^U(\underline{p}), \overline{\varepsilon} + D(\underline{p})))$.

Now consider perturbing q_i^U upwards by an amount $\delta > 0$ along the entire horizontal section at $\underline{p}$: thus we change $q_i^U(p)$ to $r_i(p)$ with $r_i(p) = 0$ for $p \in [\underline{p}, \underline{p} + \delta)$, and $r_i(p) = q_i^U(p)$ otherwise. If $w_i < q_i^U(\underline{p})$, then there is a section where Z_i is zero and the associated change in profit for this section is of order δ^2. Otherwise $w_i = q_i^U(\underline{p})$ and this section vanishes. In either case, since $\psi_i(q, \underline{p}) = 0$ for $q \leq w_i$, we can deduce from (2) that the profit Π_i increases by

$$\delta w_i + O(\delta^2).$$

This is positive for δ chosen small enough, contradicting the optimality of q_i^U as part of the mixture played by firm i. Hence $w_i \leq 0$, $i = 1, 2$ and we have established the result for $q_j^U(\underline{p})$, and hence for $q_i^U(\underline{p})$ by symmetry.

To show that $G_i(q, \underline{p}) = 0$ we suppose otherwise and assume that $G_i(t, \underline{p}) \geq \gamma > 0$ for $t \in (0, T)$ with $T > 0$ and T chosen small enough that $\underline{p} > C'(T)$. In other words player i is offering a curve with a horizontal section at $\underline{p}$ with a nonzero probability, and from the definition of G, we have $q_i^U(\underline{p}) \geq T$. By symmetry $q_j^U(\underline{p}) \geq T$. Let $q_0 = \max(T, \underline{\varepsilon} + D(\underline{p}))$ and observe that from the first part of the Lemma that $q_i^U(\underline{p}) \geq q_0$.

In equilibrium all supply functions in the mixture return the same expected profit and are all optimal; we will show that $q_j^U(p)$ can be improved which gives a contradiction. Specifically consider the change from $q_j^U(p)$ to another supply function given by $r_j(p) = q_0$ for $p \in [\underline{p} - \delta, \underline{p})$ and $r_j(p) = q_j^U(p)$ otherwise. Note that r_j is non-decreasing. The deviation will never decrease the production of firm j, and for $\varepsilon + D(\underline{p})$ in the range $R = [\max(0, \underline{\varepsilon} + D(\underline{p})), \min(\overline{\varepsilon} + D(\underline{p}), 2q_0)]$ the firm produces at least an amount of $\min(q_0, \varepsilon + D(\underline{p}))$ rather than having to share demand. It follows that there is a fixed positive probability of demand $\varepsilon + D(\underline{p})$ exceeding q_0 (which is at least the probability of demand being in the upper half of the range R), and so there is a lower bound on the increase in production for j by moving from q_j^U to r_j. The exact amount of increase will depend on the sharing rule (which we assume is symmetric between the two players).

Since $\underline{p} > C'(\max(0, \underline{\varepsilon} + D(\underline{p})))$ by assumption and $\underline{p} > C'(T)$, we have $\underline{p} > C'(q_0)$. Hence if δ is sufficiently small this gives an improvement in profit. Any reduced profits made in other circumstances through offering at a price $\underline{p} - \delta$ rather than $\underline{p}$ are dependent on δ. Hence by taking δ small enough we will ensure that the profit for player j with r_j is higher than the profit for player j with q_j^U which contradicts the optimality of q_j^U for firm j. Hence, we can conclude that $G_i\left(q, \underline{p}\right) = 0$, for $q > 0$.

Now that we have shown $G_i\left(q, \underline{p}\right) = 0$, so that there is no accumulation of offers at any point $(q, \underline{p})$, then we have also established that $q_i^L\left(\underline{p}\right) = 0$, since otherwise we would have such an accumulation of horizontal offers at $\underline{p}$ in the output range $\left[0, q_i^L\left(\underline{p}\right)\right]$, giving a contradiction. ■

The statement in this Lemma that G_i is zero when $p = \underline{p}$ is equivalent to observing that, in a symmetric equilibrium, there cannot be an accumulation of horizontal offers at this lowest mixing price $\underline{p}$. The reason is that if this were true then one player would be better off by deviating to offer instead at a marginally lower price. In fact the same argument can be used to establish that in a symmetric equilibrium there can never be a positive probability of offering an offer curve with a horizontal section, if this is at a price where the market may clear. If one player does this at a price p_0 that is higher than its marginal cost then it can be shown that the other player can improve profits by moving the section of the offer curve at price p_0 to the lower price $p_0 - \varepsilon$ for ε chosen small enough. This result holds regardless of the sharing rule. Notice that in our discussion of expected payoff in section 2 we assumed a differentiable ψ which would not occur if one of the players had a positive probability of offering a curve with a horizontal section.

The condition $Z(q, p) = 0$ in the region $(q, p) \in \Gamma$ implies that a unit q will return the same expected profit even if the price at which it is offered varies. But it is not possible to maintain constant expected profit for a unit if demand is elastic and a supply curve in the mixture crosses $q = \overline{\varepsilon} + D(p)$, because the expected profit is surely zero when $q > \overline{\varepsilon} + D(p)$. The following proposition makes this precise.

Proposition 8 *In a Nash equilibrium with mixtures over supply functions with non-binding slope constraints, if* $q_i^U\left(\underline{p}\right) \geq \overline{\varepsilon} + D\left(\underline{p}\right)$ *and* $\underline{p} > C'(\max(0, \underline{\varepsilon} +$

$D\left(\underline{p}\right))$*, then* $\overline{p}=\infty$*,* $D(p)$ *is constant for* $p\geq\underline{p}$ *and* $q_i^L(p)=0$ *for all* p*.*

Proof. We consider the line of maximum residual demand for firm i given by $\overline{\varepsilon}+D(p)-q_j^L(p)$ and let p_U be a price at which this maximum residual demand line crosses a supply function in the interior of the mixture, so $\left(\overline{\varepsilon}+D(p_U)-q_j^L(p_U),p_U\right)\in\Gamma^o$ and one of the supply functions in the mixture has $q_i(p_U)=\overline{\varepsilon}+D(p_U)-q_j^L(p_U)$. Now the definition of $q_i(p_U)$ implies that $\psi(q_i(p_U),p_U)=1$. Inside the region Γ we have from (9)

$$\psi_i(q,p)=1-\frac{\theta(q)}{p-C'(q)},$$

and so $\theta(q_i(p_U))=0$. Hence $\psi(q_i(p_U),p)=1$ for all p with $(q_i(p_U),p)\in\Gamma$. Hence for small enough δ the maximum dispatch quantity for i at $p_U-\delta$ is the same as at p_U, i.e.

$$\overline{\varepsilon}+D(p_U-\delta)-q_j^L(p_U-\delta)=\overline{\varepsilon}+D(p_U)-q_j^L(p_U).$$

Since D is a non-increasing function and q_j^L is non-decreasing, $D(p_U-\delta)=D(p_U)$ and $q_j^L(p_U-\delta)=q_j^L(p_U)$. This establishes that both $D(p)$ and $q_j^L(p)$ are constant for any p where $\left(\overline{\varepsilon}+D(p)-q_j^L(p),p\right)\in\Gamma^o$.

We now show that $\left(\overline{\varepsilon}+D(p)-q_j^L(p),p\right)\in\Gamma^o$ at prices $p\in\left(\underline{p},\overline{p}\right)$ where $D(p)-q_j^L(p)$ is strictly decreasing. By assumption $q_i^U\left(\underline{p}\right)\geq\overline{\varepsilon}+D\left(\underline{p}\right)$, Lemma 7 implies $q_i^L\left(\underline{p}\right)=0$. Thus $q_i^U\left(\underline{p}\right)\geq\overline{\varepsilon}+D(\underline{p})-q_j^L\left(\underline{p}\right)$ and $\left(q,\underline{p}\right)\in\Gamma$ for all $q\in(0,D\left(\underline{p}\right)+\overline{\varepsilon}-q_j^L(\underline{p}))$. Consider any $p_Z<\overline{p}$ at which $D(p)-q_j^L(p)$ is strictly decreasing. Since by definition $\overline{p}$ is the supremum of prices at which a solution crosses the maximum residual demand curve, we can choose a solution $q_i(p)$ in the mixture crossing the maximum residual demand line at a price p_i strictly between p_Z and $\overline{p}$. The monotonicity of $q_i(p)$ implies that this solution remains above or on the horizontal line (q,p_i) for all $q\in(D(p_i)+\overline{\varepsilon}-q_j^L(p_i),q_m)$. So for all $q\in(D(p_i)+\overline{\varepsilon}-q_j^L(p_i),D\left(\underline{p}\right)+\overline{\varepsilon}-q_j^L(\underline{p}))$ the points (q,p) are in-between the curves $q_i(p)$ and $q_i^U\left(\underline{p}\right)$ when $p\in(\underline{p},p_i)$. Our assumption on the form of Γ now shows that $(q,p)\in\Gamma^o$ for all $q\in(D(p_i)+\overline{\varepsilon}-q_j^L(p_i),D\left(\underline{p}\right)+\overline{\varepsilon}-q_j^L(\underline{p}))$ and $p\in(\underline{p},p_i)$. Since $D(p_Z)-q_j^L(p_Z)>D(p_i)-q_j^L(p_i)$, we thus have $\left(D(p_Z)+\overline{\varepsilon}-q_j^L(p_Z),p_Z\right)\in\Gamma^o$.

The argument above shows that $D(p)$ and $q_j^L(p)$ are constant for $p\in[\underline{p},\overline{p})$. Moreover from Lemma 7 and symmetry of the equilibrium $q_i^L(p)=0$, $p\in[\underline{p},\overline{p})$. It only remains to show that $\overline{p}=\infty$. Suppose that this is not the case and $\bar{p}<\infty$. If $q_i^L(\overline{p})=0$ then every offer in the mixture has zero profit. However, this contradicts the observation that the expected profit from the most competitive curve q_i^U is positive, because offers from the most competitive curve $q_i^U(p)$ are accepted with a positive probability and the mark-ups of the accepted offers are positive. Thus

$q_i^L(\overline{p}) > 0$, so this implies a horizontal section at $\overline{p}$ with $q \in [0, q_i^L(\overline{p})]$. From the discussion below Lemma 7 we also know that there cannot be a horizontal accumulation of offers at $\bar{p}$. Hence, offers at this price must be almost surely rejected, i.e. $\psi_i(q, \overline{p}) = 1$ for $q \in \left(0, q_i^L(\bar{p})\right)$. However, observe from (9) that $\psi_i(q,p)$ can only equal 1 when $\theta(q) = 0$ and hence can never approach 1 as p approaches $\bar{p}$ from below. So we have a contradiction if $\bar{p} < \infty$. ■

Now consider the special case with certain demand, so that $\overline{\varepsilon} = \underline{\varepsilon}$. Lemma 7 implies that in a mixed-strategy equilibrium with non-binding slope constraints $q_i^U\left(\underline{p}\right) \geq \overline{\varepsilon} + D\left(\underline{p}\right)$, so that each producer's capacity is sufficient to meet the highest demand at the lowest price. In other words producers must be non-pivotal for this to happen. From Proposition 8 it now follows that $D(p)$ is constant and $q_i^L(p) = 0$ for $p > \underline{p}$. Without loss of generality we assume that $D(p) = 0$. The form of the equilibrium with certain demand can now be derived from (10), which gives

$$\psi_i(u,p) = G_j(\overline{\varepsilon} - u, p) \text{ if } u \in (0, \overline{\varepsilon}).$$

Thus setting $q = \overline{\varepsilon} - u$, it follows from (9) that $G_j(q,p)$ must satisfy

$$G_j(q,p) = \frac{p - k(\overline{\varepsilon} - q)}{p - C'(\overline{\varepsilon} - q)} \text{ if } (p,q) \in \Gamma.$$

When $\underline{p} > C'(\overline{\varepsilon})$, we know from Lemma 7 that $G_j\left(q, \underline{p}\right) = 0$ if $q \in (0, \overline{\varepsilon})$, so $k(\overline{\varepsilon} - q) = \underline{p}$. Thus we have established that any symmetric Nash equilibrium has the form

$$G_j(q,p) = \frac{p - \underline{p}}{p - C'(\overline{\varepsilon} - q)} \text{ if } p \geq \underline{p} \text{ and } q \in (0, \overline{\varepsilon}). \tag{13}$$

Example 1: Symmetric duopoly with certain demand

Suppose $C(q) = q^2/2$ and $\overline{\varepsilon} = 1$ and assume both firms have capacity greater than 1. This gives

$$G(q,p) = \frac{p - \underline{p}}{p - (1-q)}, \tag{14}$$

the contours of which are plotted in Figure 1 for $\underline{p} = 2$.

It is easy to verify that G defines an offer distribution function since $0 \leq G \leq 1$, $\frac{\partial}{\partial p}G \geq 0$ and $\frac{\partial}{\partial q}G \leq 0$. By virtue of its derivation, every offer in the mixing region has the same profit (in this case $\frac{3}{2}$). Moreover, if the competitor follows the outlined strategy, offers at $\underline{p}$ are accepted with the same probability as offers at lower prices, so it is never profitable to undercut $\underline{p}$. Using (14) we can calculate a NE for any $\underline{p} > 1$, so we have a continuum of mixed-strategy NE. This is not surprising as similar results have been shown for Bertrand games with non-pivotal producers and unbounded prices [9][16][19].

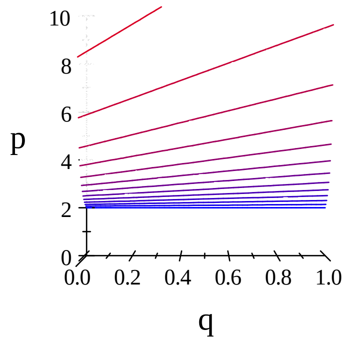

Figure 1: Contours of G for mixed strategy equilibrium, where $\underline{p} = 2$.

We now turn our attention to the question of whether an equilibrium with mixtures over supply functions with non-binding slope constraints can occur in a market with uncertain demand. Except for cases with constant marginal costs, we have not been able to find such an equilibrium when demand is uncertain and $\underline{\varepsilon} + D\left(\underline{p}\right) > 0$. The problem is that discontinuities in the shock density or in its derivatives gives discontinuities in $Z(q,p)$ and its derivatives, and this can be used to rule out the existence of such equilibria. Thus we will now focus on cases with $\underline{\varepsilon} + D\left(\underline{p}\right) \leq 0$. Outcomes with $\underline{\varepsilon} + D\left(\underline{p}\right) < 0$ are only possible if demand is elastic; in this case the minimum demand curve may intersect producers' vertical segment at zero output at a price below $\underline{p}$.

For certain demand we showed that $q_i^U\left(\underline{p}\right) \geq \overline{\varepsilon} + D\left(\underline{p}\right)$, so the lowest offer of the mixture starts with a horizontal section. Otherwise, the first unit would be accepted with certainty and the producer would have an incentive to increase the offer price of this unit relative to later units. When demand is uncertain, we expect mixed-strategy equilibria to arise in a similar way to those for certain demand, i.e. whenever lower demand outcomes combined with the competitor's offer do not provide sufficient elasticity in the residual demand distribution to counteract the incentive to markup the price on the first unit. We are able to show that $q_i^U\left(\underline{p}\right) \geq \varepsilon + D\left(\underline{p}\right)$ for all outcomes of the demand shock, under the following condition on its distribution:

$$\frac{\partial}{\partial q} \frac{\left(\underline{p} - C'\left(q\right)\right) f\left(q + \varepsilon\right)}{1 - F(q - D(\underline{p}))}\Bigg|_{q=0+} > 0, \quad \text{for each } \varepsilon \in \left[-D(\underline{p}), \overline{\varepsilon}\right]. \tag{15}$$

Here we use the notation $\frac{\partial}{\partial q}\Big|_{q=0+}$ for the right hand derivative at $q = 0$ and the assumption implies that these partial derivatives exist for each ε in the range $\left[-D(\underline{p}), \overline{\varepsilon}\right]$.

Proposition 9 *Suppose the distribution on the demand shock satisfies (15), and* $\underline{\varepsilon} + D\left(\underline{p}\right) \leq 0$*. Then any mixed-strategy equilibrium with non-binding slope constraints needs to satisfy* $q_i^U\left(\underline{p}\right) \geq \overline{\varepsilon} + D(\underline{p})$*.*

Proof. We calculate the value of Z when the price is $\underline{p}$. Observe that $f\left(t + q - D(p)\right) = 0$ for $t > \overline{\varepsilon} - q + D\left(\underline{p}\right)$, and so (10) becomes

$$\psi_i\left(q, p\right) = F(q - D(p)) + \int_0^{\overline{\varepsilon}-q+D(p)} f\left(t + q - D(p)\right) G\left(t, p\right) dt. \tag{16}$$

Differentiating (16) with respect to p and evaluating at $\underline{p}$ yields:

$$\psi_p\left(q, \underline{p}\right) = -f(q - D\left(\underline{p}\right))D'\left(\underline{p}\right) - \int_0^{\overline{\varepsilon}-q+D\left(\underline{p}\right)} f'\left(t + q - D\left(\underline{p}\right)\right) D'\left(\underline{p}\right) G\left(t, \underline{p}\right) dt$$
$$+ \int_0^{\overline{\varepsilon}-q+D\left(\underline{p}\right)} f\left(t + q - D\left(\underline{p}\right)\right) G_p\left(t, \underline{p}\right) dt + f\left(\overline{\varepsilon}\right) G\left(\overline{\varepsilon} - q + D\left(\underline{p}\right), \underline{p}\right) D'\left(\underline{p}\right), \tag{17}$$

where $G_p\left(t, p\right) = \partial G\left(t, p\right)/\partial p$. From Lemma 7 we know that $G\left(t, \underline{p}\right) = 0$ for any $t > 0$, so we obtain for $q < \overline{\varepsilon} + D\left(\underline{p}\right)$ that

$$\begin{aligned} Z(q, \underline{p}) &\equiv 1 - \psi\left(q, \underline{p}\right) - \psi_p(q, \underline{p})\left(\underline{p} - C'\left(q\right)\right) \\ &= 1 - F(q - D\left(\underline{p}\right)) + \left(\underline{p} - C'\left(q\right)\right) f(q - D\left(\underline{p}\right))D'\left(\underline{p}\right) \\ &\quad - \left(\underline{p} - C'\left(q\right)\right) \int_0^{\overline{\varepsilon}-q+D\left(\underline{p}\right)} f\left(t + q - D\left(\underline{p}\right)\right) G_p\left(t, \underline{p}\right) dt. \end{aligned}$$

Notice that for any $t > q_j^U\left(\underline{p}\right)$ we have $G(t, \underline{p} + \delta) = 0$ for δ chosen small enough and so $G_p\left(t, \underline{p}\right) = 0$. Now, suppose $q_j^U\left(\underline{p}\right) < \overline{\varepsilon} + D\left(\underline{p}\right)$. Then for any $q > 0$ and chosen close enough to 0, we will have $q_j^U\left(\underline{p}\right) < \overline{\varepsilon} - q + D\left(\underline{p}\right)$. For such a q, we can replace the upper limit of the integral in this expression for Z and write

$$\begin{aligned} Z(q, \underline{p}) &= \left[1 - F(q - D\left(\underline{p}\right))\right]\left[1 + E_1(q) + E_2(q)\right] \\ E_1(q) &= \left(\underline{p} - C'\left(q\right)\right) \frac{f(q - D\left(\underline{p}\right))}{1 - F(q - D\left(\underline{p}\right))} D'\left(\underline{p}\right) \\ E_2(q) &= -\left(\underline{p} - C'\left(q\right)\right) \int_0^{q_j^U\left(\underline{p}\right)} \frac{f\left(t + q - D\left(\underline{p}\right)\right)}{1 - F(q - D\left(\underline{p}\right))} G_p\left(t, \underline{p}\right) dt. \end{aligned}$$

We are interested in the (right-hand) derivative of $Z(q, \underline{p})$ with respect to q at $q = 0$. Since $Z(0, \underline{p}) = 0$, $D'\left(\underline{p}\right) \leq 0$, $\underline{p} > C'(0)$ (implied by Assumption 1) and $G_p\left(t, \underline{p}\right) \geq 0$, Assumption 1 implies that $Z(q, \underline{p})$ is decreasing with respect to q at $q = 0$. But this violates the condition that $Z = 0$ on the boundary of Γ. This contradiction establishes that $q_i^U\left(\underline{p}\right) \geq \overline{\varepsilon} + D(\underline{p})$. ■

Observe that by Proposition 8, if $\underline{p} > C'(\max(0, \underline{\varepsilon} + D\left(\underline{p}\right)))$ then $D(p)$ is constant for $(q, p) \in \Gamma$, $\overline{p} = \infty$ and $q_i^L\left(p\right) = 0$ for all p.

Given this observation, we will now restrict attention in the remainder of this section to the case of inelastic demand with $\underline{\varepsilon} + D\left(\underline{p}\right) \leq 0$, which becomes equivalent to taking $D\left(\underline{p}\right) = 0$ and $\underline{\varepsilon} = 0$. So when (15) holds, $q_i^U\left(\underline{p}\right) \geq \overline{\varepsilon}$ and the entire horizontal line $(0, \underline{p})$ to $(\overline{\varepsilon}, \underline{p})$ is the lower boundary of the region Γ where mixing takes place. Thus using the equations (9) and (10) as well as $G\left(q, \underline{p}\right) = 0$ from Lemma 7 we get

$$F(q) = \psi\left(q, \underline{p}\right) = \frac{\underline{p} - k(q)}{\underline{p} - C'(q)},$$

provided $\underline{p} > C'(\overline{\varepsilon})$. This determines the function

$$k(q) = \underline{p} - \left(\underline{p} - C'(q)\right) F(q) \tag{18}$$

and substitution back into (9) shows that

$$\psi_i(q, p) = \frac{p - \underline{p} + \left(\underline{p} - C'(q)\right) F(q)}{p - C'(q)} \text{ for } q \in (0, \overline{\varepsilon}) \text{ and } p > \underline{p}.$$

If $p \leq \underline{p}$, then any offer of (q, p) by producer i will be fully dispatched with probability $1 - F(q)$, so

$$\psi_i(q, p) = F(q) \text{ for } q \in (0, \overline{\varepsilon}) \text{ and } p \leq \underline{p}.$$

It is easy to verify by substituting the expression for $\psi_i(q, p)$ into (10) that

$$\int_0^{\overline{\varepsilon} - q} f(t + q)\, G_j(t, p)\, dt = \frac{p - \underline{p} + \left(\underline{p} - C'(q)\right) F(q)}{p - C'(q)} - F(q)$$

so

$$\int_0^{\overline{\varepsilon} - q} f(t + q)\, G_j(t, p)\, dt = \frac{(p - \underline{p})(1 - F(q))}{p - C'(q)}. \tag{19}$$

In principle, at least, equation (19) can be solved for G. This is straightforward when F has a uniform distribution. In this case we can characterize the form of the mixture analytically.

Proposition 10 *Suppose that demand is inelastic and has a uniform distribution with support* $[0, \overline{\varepsilon}]$. *If each producer has capacity* $q_m > \overline{\varepsilon}$, *and for some* $\underline{p}$ *and every* $q \in [0, \overline{\varepsilon}]$

$$(\underline{p} - C'(q)) \geq C''(q)(\overline{\varepsilon} - q), \tag{20}$$

and

$$2C''(q) \geq (\frac{2C''(q)^2}{\underline{p} - C'(q)} + C'''(q))(\overline{\varepsilon} - q), \tag{21}$$

then there exists a unique symmetric mixed-strategy equilibrium with non-binding slope constraints and lowest clearing price $\underline{p}$. *This is defined by the offer distribution function*

$$G_j(q,p) = \begin{cases} 0, & p < \underline{p},\ q \in (0, \overline{\varepsilon}), \\ (p - \underline{p})\frac{(p - C'(\overline{\varepsilon} - q)) - qC''(\overline{\varepsilon} - q)}{(p - C'(\overline{\varepsilon} - q))^2}, & p \geq \underline{p},\ q \in (0, \overline{\varepsilon}). \end{cases}$$

Proof. The formula for $G_j(q,p)$ can be obtained by setting $f(q) = \frac{1}{\overline{\varepsilon}}$, and $F(q) = \frac{q}{\overline{\varepsilon}}$ and differentiating (19). This gives

$$\begin{aligned} -\frac{1}{\overline{\varepsilon}} G_j(\overline{\varepsilon} - q, p) &= \frac{\partial}{\partial q}\left(\frac{(p - \underline{p})(1 - \frac{q}{\overline{\varepsilon}})}{p - C'(q)}\right) \\ &= (p - \underline{p})\frac{(1 - \frac{q}{\overline{\varepsilon}})C''(q) - (p - C'(q))\frac{1}{\overline{\varepsilon}}}{(p - C'(q))^2} \end{aligned}$$

so

$$G_j(\overline{\varepsilon} - q, p) = (p - \underline{p})\frac{(p - C'(q)) - (\overline{\varepsilon} - q)C''(q)}{(p - C'(q))^2}$$

as required.

Observe that

$$G(\overline{\varepsilon} - q, p) = \frac{p - \underline{p}}{p - C'(q)} \frac{(p - C'(q)) - (\overline{\varepsilon} - q)C''(q)}{p - C'(q)}$$

so (20) implies

$$0 \leq G(\overline{\varepsilon} - q, p) \leq 1, \quad \text{for every } q \in [0, \overline{\varepsilon}], \text{ and } p > \underline{p}.$$

We next check that $G_j(q,p)$ is monotonic.

$$\begin{aligned} \frac{\partial}{\partial p} G_j(\overline{\varepsilon} - q, p) &= \frac{\partial}{\partial p}\left((p - \underline{p})\frac{(p - C'(q)) - (\overline{\varepsilon} - q)C''(q)}{(p - C'(q))^2}\right) \\ &= \frac{(p - C'(q))(\underline{p} - C'(q)) + C''(q)(\overline{\varepsilon} - q)(p + C'(q) - 2\underline{p}))}{(p - C'(q))^3} \end{aligned}$$

which is nonnegative for every $q \in [0, \bar{\varepsilon}]$ when

$$(p - C'(q))(\underline{p} - C'(q)) + C''(q)(\bar{\varepsilon} - q)(p + C'(q) - 2\underline{p})) \geq 0.$$

The left-hand side of this inequality is increasing in p, and so this is equivalent to requiring

$$(\underline{p} - C'(q))(\underline{p} - C'(q)) + C''(q)(\bar{\varepsilon} - q)(\underline{p} + C'(q) - 2\underline{p})) \geq 0,$$

which follows from (20).

We also require

$$\frac{\partial}{\partial q} G_j(\bar{\varepsilon} - q, p) \geq 0$$

which is equivalent to

$$(p - C'(q))\, 2C''(q) \geq (2C''(q)^2 + (p - C'(q))C'''(q))(\bar{\varepsilon} - q)$$

or

$$2C''(q) \geq (\frac{2C''(q)^2}{p - C'(q)} + C'''(q))(\bar{\varepsilon} - q).$$

The right-hand side of this inequality is decreasing in p, so this is equivalent to (21).

It follows from the construction of G that every offer in the mixing region has the same profit, and offers at $\underline{p}$ are accepted with the same probability as offers at lower prices, so it is never profitable to undercut $\underline{p}$. ■

Example 2: Symmetric duopoly with uniform demand

As before we consider a symmetric duopoly market with $C(q) = q^2/2$, and ε uniformly distributed on $[0, 1]$. Proposition 10 gives a mixture defined by

$$G(q, p) = \frac{(p - 1)(p - \underline{p})}{(p + q - 1)^2},$$

as long as conditions (20) and (21) hold. It is easy to see that these correspond to $\underline{p} \geq 1$. Observe that condition (15) is equivalent to $\underline{p} > 1$, which shows that this is not necessary for a mixed strategy to exist. The contours of G correspond to the offer curves over which mixing takes place, and are shown in Figure 2 for $\underline{p} = 2$.

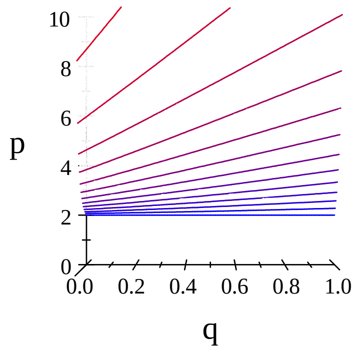

Figure 2: Contours of G for mixed strategy equilibrium for $\underline{p} = 2$.

5 Mixtures over slope-constrained bid curves

We saw in Section 3 that pure-strategy SFE can be ruled out in markets where $(p - C'(q))\, H(q + S_j(p) - D(p))$ is locally increasing. In Section 4, we were able to find equilibria with mixtures over supply functions with non-binding slope constraints under such circumstances when demand is inelastic and firms have sufficiently large capacities. But under an assumption on the demand distribution, we show such mixed-strategy equilibria do not exist for markets with pivotal producers, price caps, or with elastic demand. In this section we will show that mixtures over slope-constrained offer curves can exist under such circumstances. This takes us back to the discussion in section 2 in which we provided optimality conditions for solutions having this type of structure. We no longer require $Z = 0$ over a region Γ: the conditions now involve an integral of the Z function. We analyze two cases. We start with mixtures over horizontal bids, i.e. supply curves are slope-constrained along the whole output. We show that there exist equilibria where all offer curves in the mixture are of this type if the price cap is sufficiently high. For lower price caps, there is another mixed-strategy equilibrium, which we call a hockey-stick mixture. Offers in this mixture also start with horizontal segments. But in this case, the supply slopes upwards at high outputs for the lowest offer curves in the mixture.

5.1 Mixtures over horizontal bids

In this subsection we will consider mixtures over horizontal bid curves, i.e. the supply curves are slope constrained for the whole output. We consider the case with two players, with capacities q_i^m, $i = 1, 2$, where demand may exceed $\max(q_1^m, q_2^m)$ but not $q_1^m + q_2^m$. As before we let $\underline{p}$ be the lowest clearing price and $\overline{p}$ be the highest clearing price, where this exists. We consider a situation in which producer $j \neq i$ offers its capacity at a price p or below with probability $G_j(p)$. From (10) we have the resulting market distribution function of firm i :

$$\begin{aligned}\psi_i(q_i, p) &= F(q_i - D(p)) + \int_0^{q_j^m} f(t + q_i - D(p))\, G_j(p)\, dt \\ &= (1 - G_j(p))F(q_i - D(p)) + G_j(p)\, F(q_i + q_j^m - D(p)). \end{aligned} \tag{22}$$

Note that $F(\varepsilon) = 1$ for $\varepsilon > \overline{\varepsilon}$. The payoff of a horizontal offer at price p is given by (2)

$$\Pi_i(p) = \int_0^{q_i^m} (p - C_i'(q))(1 - \psi_i(q, p))dq.$$

In equilibrium we require the offer of q_i^m at any price p in the support of G_j to yield the same expected profit, and we let K_i be the value of $\Pi_i(p)$ in this region. After substituting for ψ_i this gives

$$K_i = \int_0^{q_i^m} (p - C_i'(q))(1 - (G_j(p)F(q + q_j^m - D(p)) + (1 - G_j(p))F(q - D(p)))dq.$$

After rearranging we get

$$G_j(p) = \frac{\int_0^{q_i^m} (p - C_i'(q))(1 - F(q - D(p)))dq - K_i}{\int_0^{q_i^m} (p - C_i'(q))(F(q + q_j^m - D(p)) - F(q - D(p)))dq}. \tag{23}$$

This generalizes the necessary first-order condition for mixed-strategy Nash equilibria in discriminatory auctions given by Anwar [2], Fabra et al. [12] and Son et al. [24], who consider cases with constant marginal costs and vertical demand.[3]

In Bertrand-Edgeworth games, demand is often assumed to equal $D(p)$ with certainty, i.e. $\underline{\varepsilon} = \overline{\varepsilon} = 0$. In this case, and under the assumption that $q_i^m \leq D(p)$ for prices that occur, (23) can be simplified to

$$G_j(p) = \frac{\int_0^{q_i^m} (p - C_i'(q))dq - K_i}{\int_{D(p)-q_j^m}^{q_i^m} (p - C_i'(q))dq} \tag{24}$$

$$= \frac{pq_i^m - C_i(q_i^m) - K_i}{p(q_i^m + q_j^m - D(p)) - C_i(q_i^m) + C_i(D(p) - q_j^m)}. \tag{25}$$

[3]Unlike us, Fabra et al. [12] consider a strategy space in which offers are constrained to be horizontal. Still our and their first-order condition are identical, because pay-offs for the horizontal offers are the same at any price p in the support of G_j in both models. Note that our first-order condition needs to be partially integrated to become similar to their condition.

Since $G_j(\underline{p}) = 0$ we have

$$G_j(p) = \frac{(p - \underline{p})q_i^m}{p(q_i^m + q_j^m - D(p)) - C_i(q_i^m) + C_i(D(p) - q_j^m)}. \tag{26}$$

The choice of $\underline{p}$ gives different possible mixtures. This condition generalizes the previous conditions that have been used to calculate mixed-strategy Nash equilibria in Bertrand-Edgeworth games, which assume zero marginal costs and linear demand [10], [22]. Another difference is that we consider the case with parallel rationing rather than proportional rationing.

Now we give a simple example illustrating (26).

Example 3: Symmetric duopoly with elastic demand

To illustrate such an equilibrium consider a symmetric duopoly market with $C(q) = 0$. We assume that $D(p) = 1 - \frac{p}{10}$, and $\overline{\varepsilon} = \underline{\varepsilon} = 0$. Suppose $q_i^m = q_j^m = \frac{3}{4}$, and $\underline{p} = \frac{5}{24}$. Thus

$$G(p) = \frac{5}{16}\frac{24p - 5}{p(5 + p)}, \quad p \in [\frac{5}{24}, \frac{5}{4}].$$

This gives

$$\psi(q,p) = \begin{cases} 0, & q < (1 - \frac{p}{10} - \frac{3}{4}) \\ \frac{5}{16}\frac{24p-5}{p(5+p)}, & (1 - \frac{p}{10} - \frac{3}{4}) < q < (1 - \frac{p}{10}) \\ 1, & q > (1 - \frac{p}{10}) \end{cases}$$

and an expected payoff of $K = \frac{5}{32}$ for any horizontal offer of $\frac{3}{4}$ at a price $p \in [\frac{5}{24}, \frac{5}{4}]$. It is easy to see that there is no incentive to deviate outside this mixture. For example an offer of $\frac{3}{4}$ at a price $p = \frac{5}{4} + \epsilon$ will be dispatched a quantity $1 - \frac{\frac{5}{4}+\epsilon}{10} - \frac{3}{4}$ and earn

$$(1 - \frac{\frac{5}{4} + \epsilon}{10} - \frac{3}{4})(\frac{5}{4} + \epsilon) = \frac{5}{32} - \frac{1}{10}\epsilon^2 < K.$$

We note that this is the unique equilibrium without imposing a price cap. Any equilibrium mixture must have $\underline{p} \leq \frac{5}{24}$, because $G(p)$ is bounded away from 1 if $\underline{p}$ is chosen outside this range. Moreover any mixture over a price range with $\underline{p} < \frac{5}{24}$ will have $K < \frac{5}{32}$, and a price P less than $\frac{5}{4}$ where $G(P) = 1$, so we must impose a price cap at P to prevent a player deviating to an offer at $p = \frac{5}{4}$ which will yield a certain profit of $\frac{5}{32}$.

Finally we point out that this example is special in that deterministic elastic demand implies that $\psi(q,p)$ is discontinuous across the line $q = \frac{1}{4} - \frac{p}{10}$, and so the equilibrium that we have derived above does not satisfy the optimality condition

$$\int_0^{q^m} Z(q,p)dq = 0$$

as stated in Lemma 5. We must add a term to the optimality condition, because of the jump in $\psi(q,p)$ as the horizontal offer crosses the line $q = \frac{1}{4} - \frac{p}{10}$.

In the remainder of this Section we will focus on cases with inelastic demand. Without loss of generality we assume that $D(p) = 0$. In a discriminatory divisible-good auction, we can more or less eliminate the possibility of a horizontal mixture when the smallest producer is non-pivotal, i.e. when $\max(q_1^m, q_2^m) > \overline{\varepsilon}$ at prices that occur. The only exception is when marginal costs are constant.

Proposition 11 *In an equilibrium where a player i mixes over horizontal bid curves in some price interval (p_1, p_2) and $D(p) = 0$, it must be the case that $\overline{\varepsilon} \geq q_j^m$ for every $p \in (p_1, p_2)$, unless there are constant marginal costs.*

Proof. From Lemma 3 we know that a necessary characteristic of an optimal horizontal offer is that $Z(q,p)$ is non-negative for small values of q, otherwise we can obtain an improvement by reducing the first section of the offer. We will show that this property fails for player i when $\overline{\varepsilon}$ is less than the competitor's capacity. In this case $F(q_j^m) = 1$ and hence we can write (23) as

$$G_j(p) = 1 - \frac{K_i}{W(p)},$$

where

$$W(p) = \int_0^{q_i^m} (p - C_i'(q))(1 - F(q))dq,$$

and we have

$$W'(p) = \int_0^{q_i^m} (1 - F(q))dq. \tag{27}$$

Observe that if p is in the mixture then $W(p) > 0$ and so $p > C_i'(0)$. Now, from (22)

$$\psi_i(q,p) = (1 - G_j(p))F(q) + G_j(p)$$

and thus it follows from (4)

$$Z(q,p) = (1 - G_j(p))(1 - F(q)) - (p - C_i'(q))g_j(p)(1 - F(q)).$$

From the above relations we have

$$\begin{aligned} Z(0,p) &= (1 - F(0))[(1 - G_j(p)) - (p - C_i'(0))g_j(p)] \\ &= (1 - F(0))[\frac{K_i}{W(p)} - (p - C_i'(0))\frac{K_i W'(p)}{W(p)^2}] \\ &= \frac{K_i(1 - F(0))}{W(p)^2}[W(p) - (p - C_i'(0))W'(p)]. \end{aligned} \tag{28}$$

Substituting for $W(p)$ and $W'(p)$ we can show

$$Z(0,p) = \frac{K_i(1-F(0))}{W(p)^2} \int_0^{q_i^m} (C_i'(0) - C_i'(q))(1-F(q))dq \leq 0. \tag{29}$$

If marginal costs are not constant then $C_i'(0) < C_i'(q_i)$ and inequality (29) is strict, which gives the contradiction we need. ■

This proposition shows that the special case in which $C_i(x) = cx$ is the only case in which we can have a mixture over horizontal bids when $q_j^m > \bar{\varepsilon}$. In this case we can simplify (23) to

$$G_j(p) = 1 - \frac{K_i}{(p-c)\int_0^{q_i^m}(1-F(q))dq} = 1 - \frac{\underline{p}-c}{p-c},$$

because $G_j(\underline{p}) = 0$. Combined with (22) this gives

$$\begin{aligned}\psi_i(q_i,p) &= (1-G_j(p))F(q_i) + G_j(p) \\ &= 1 - \frac{(\underline{p}-c)(1-F(q))}{p-c}\end{aligned}$$

which satisfies the condition in (9) ensuring that this situation has $Z = 0$ throughout the region in which offers are made. Thus we are again in the non-slope-constrained case, even though the equilibrium offers are horizontal. As was the case for the non-slope-constrained mixtures in Section 4, there is a new horizontal mixture for every $\underline{p} > c$, and here it is identical to the mixed-strategy Bertrand NE in markets with non-pivotal producers and constant marginal costs [9]. Another similarity with Section 4 is that the highest offer needs to be unbounded, so the equilibrium over horizontal mixtures does not exist for finite price caps when producers are non-pivotal.

Although first-order conditions for mixtures over horizontal bids are similar in our framework and in the game analyzed by Fabra et al. and Bertrand-Edgeworth games, our strategy space is less constrained as it allows for strictly increasing supply functions. Thus one would expect sufficiency conditions to be different in our framework. For example, there would be profitable deviations from any potential equilibrium with $\underline{p} < C'(q_m)$. Genc [14] analyzes a supply function auction similar to ours, and shows that sufficiency conditions for mixtures over horizontal bids are satisfied if marginal costs are constant, and demand is inelastic and uniformly distributed. Anwar [2] provides sufficiency conditions that are more general than Genc, but they are also limited to constant marginal costs.

Our analysis is in a more general setting than [2] and [14], but we will restrict attention to the case where firms are symmetric, so that $C_1'(q) = C_2'(q) = C'(q)$

and $q_1^m = q_2^m = q_m$, and we look for an equilibrium in which both firms offer the same mixture of offers, $G_1(p) = G_2(p) = G(p)$.

It will be convenient to introduce some notation to shorten the expressions we deal with. We let

$$A = \int_0^{q_m} (1 - F(q))dq, \qquad B = \int_0^{q_m} (1 - F(q + q_m))dq, \tag{30}$$

and

$$L(p) = \int_0^{q_m} (p - C'(q))(F(q + q_m) - F(q))dq. \tag{31}$$

It is also convenient to define

$$J = -L(0) = \int_0^{q_m} C'(q)(F(q + q_m) - F(q))dq. \tag{32}$$

Under these assumptions we can rewrite (22) and (23) as

$$\psi(q, p) = F(q) + G(p)(F(q + q_m) - F(q)) \tag{33}$$

and

$$G(p) = \frac{\int_0^{q_m} (p - C'(q))(1 - F(q))dq - K}{L(p)}, \tag{34}$$

where

$$\begin{aligned} K &= \int_0^{q_m} (p - C'(q))(1 - (G(p)F(q + q_m) + (1 - G(p))F(q)))dq & (35) \\ &= \int_0^{q_m} (\underline{p} - C'(q))(1 - F(q))dq & (36) \\ &= \int_0^{q_m} (\overline{p} - C'(q))(1 - F(q + q_m))dq, & (37) \end{aligned}$$

because K is equal to the pay-off for all price levels, including the highest and lowest clearing price. Hence,

$$G(p) = \frac{(p - \underline{p})A}{L(p)}. \tag{38}$$

Writing $g(p) = G'(p)$ for the density function, we obtain

$$g(p) = \frac{L(\underline{p})A}{[L(p)]^2}, \tag{39}$$

which implies that $g(p) \geq 0$ as long as $p \geq \underline{p} \geq C'(q_m)$. Moreover,

$$g'(p) = \frac{-2L(\underline{p})A(A-B)}{[L(p)]^3} \leq 0 \tag{40}$$

if $p \geq \underline{p} \geq C'(q_m)$. Thus the density of the mixture is weighted towards lower prices.

A relation between the minimum and maximum prices in the horizontal mixture can be calculated from (36) and (37)

$$\underline{p} = (\overline{p}B + J)/A. \tag{41}$$

Observe that the existence of $\underline{p}$ defined by this expression guarantees that there is a finite maximum price (at which $G(p)$ reaches 1) if firms are pivotal so that $B > 0$.

We can establish the following general result for equilibria with mixtures over horizontal bids, when there is a price cap, P. As in Genc [14] and Fabra et al. [12] the existence of a price cap singles out a unique equilibrium.

Proposition 12 *When $D(p) = 0$ and $\overline{\varepsilon}/2 \leq q_m < \overline{\varepsilon}$ then an equilibrium where producers mix over horizontal bids can only exist if the price cap P satisfies*

$$P \geq \frac{1}{B^2}\left(A^2C'(q_m) - (A+B)J\right). \tag{42}$$

Moreover, the equilibrium is uniquely determined by (38), (41) and $\overline{p} = P$. The condition

$$\frac{\partial}{\partial q}\left[(\underline{p} - C'(q))\frac{(F(q+q_m) - F(q))}{1-F(q)}\right] > 0, \quad \text{for } q \in [0, \overline{\varepsilon} - q^m) \tag{43}$$

is sufficient to ensure that this is an equilibrium.

Proof. By means of (33) and (4) we can calculate

$$\begin{aligned} Z(q,p) &= 1 - F(q) - [G(p) + (p - C'(q))g(p)](F(q+q_m) - F(q)) \\ &= [1-F(q)]\,\Lambda(q,p), \end{aligned} \tag{44}$$

where

$$\Lambda(q,p) = 1 - [G(p) + (p - C'(q))g(p)]\frac{(F(q+q_m) - F(q))}{1-F(q)}. \tag{45}$$

We begin by establishing the first part of the proposition relating to the necessary conditions for an equilibrium. It follows from Lemma 5 that a necessary condition for an equilibrium with horizontal mixtures is:

$$\int_0^{q_m} Z(t,p)dt = 0, \tag{46}$$

so that the marginal profit from increasing the bid of the whole segment is zero. This implies that (38) is satisfied, since this is the condition to ensure the same payoff for horizontal bids at any price $p \in \left[\underline{p}, \overline{p}\right]$. Due to Lemma 3 we also require that

$$\textstyle\int_0^q Z(t,p)dt \geq 0, \quad q \leq q_m. \tag{47}$$

Otherwise the producer would find it profitable to deviate by reducing the price of the first part of the segment.

Now suppose $p > \overline{p}$ then from (44) we get

$$Z(q,p) = 1 - F(q + q_m), \tag{48}$$

because $g\left(p\right) = 0$ and $G\left(p\right) = 1$. Hence, $\int_0^{q_m} Z(t, \overline{p}+)dt > 0$ and producers have incentives to raise their highest bids unless it is prevented by the price cap: so we require that $\overline{p} = P$.

Using (41) we can show (after some algebra) that the condition (42) is equivalent to

$$\underline{p} \geq p^* = \frac{C'\left(q_m\right)A - J}{B},$$

which we will now prove. Note that since $2q_m \geq \overline{\varepsilon}$ we have from (45) that

$$\begin{aligned} \Lambda\left(q_m, \underline{p}\right) &= 1 - (\underline{p} - C'\left(q_m\right))g(\underline{p}) \\ &= 1 - \frac{(\underline{p} - C'\left(q_m\right))A}{\underline{p}(A-B) - J}. \end{aligned} \tag{49}$$

When $\underline{p} = p^*$ we obtain

$$\Lambda\left(q_m, \underline{p}\right) = 1 - \frac{\left(C'\left(q_m\right)\left(A-B\right) - J\right)A}{\left(C'\left(q_m\right)A - J\right)\left(A-B\right) - BJ} = 0.$$

Also it is easy to see from (49) that $\Lambda\left(q_m, \underline{p}\right)$ is decreasing in $\underline{p}$. Hence if $\underline{p} < p^*$ then $\Lambda\left(q_m, \underline{p}\right) > 0$ and hence $Z\left(q_m, \underline{p}\right) > 0$. But this would imply that $\int_0^q Z(t, \underline{p})dt$ is increasing in q as q approaches q_m from below, which leads to a contradiction from either (46) or (47). Thus we have established the condition we require that $\underline{p} \geq p^*$, which in turn leads to the condition (42).

Now we want to establish that under (43) this mixture over horizontal bids is a Nash equilibrium. Our assumption (43) implies that, for $q \in [0, \overline{\varepsilon} - q^m)$,

$$(\underline{p} - C'\left(q\right))\frac{\partial}{\partial q}\left\{\frac{(F(q+q_m) - F(q))}{1 - F(q)}\right\} > C''\left(q\right)\frac{(F(q+q_m) - F(q))}{1 - F(q)}.$$

As the right hand side is non-negative this establishes that $\underline{p} > C'(\overline{\varepsilon} - q^m)$ and that $\frac{\partial}{\partial q}\left\{\frac{(F(q+q_m)-F(q))}{1-F(q)}\right\} > 0$ in this interval. Observe that

$$\begin{aligned}
\Lambda_q(q,p) &= -\frac{\partial}{\partial q}\left\{[G(p) + (p - C'(q))g(p)]\frac{(F(q+q_m) - F(q))}{1-F(q)}\right\} \\
&= -G(p)\frac{\partial}{\partial q}\left\{\frac{(F(q+q_m) - F(q))}{1-F(q)}\right\} \\
&\quad - \frac{\partial}{\partial q}\left\{(p - C'(q))g(p)\frac{(F(q+q_m) - F(q))}{1-F(q)}\right\} \\
&= -\left(G(p) + g(p)(p - \underline{p})\right)\frac{\partial}{\partial q}\left\{\frac{(F(q+q_m) - F(q))}{1-F(q)}\right\} \\
&\quad - g(p)\frac{\partial}{\partial q}\left\{(\underline{p} - C'(q))\frac{(F(q+q_m) - F(q))}{1-F(q)}\right\}.
\end{aligned}$$

Thus $\Lambda_q(q,p) < 0$ for $p \geq \underline{p}$ and $q \in [0, \overline{\varepsilon} - q^m)$. On the other hand, from (45) we note that $\Lambda_q(q,p) = C''(q)\,g(p) \geq 0$ if $q > \overline{\varepsilon} - q_m$.

The results for $\Lambda_q(q,p)$ imply that $Z(q,p)$ cannot be always zero. We know that $Z(0,p) \geq 0$, $Z(q_m,p) \leq 0$ and $\int_0^{q_m} Z(q,p)dq = 0$ for $p \geq \underline{p}$. As the derivative of Λ changes from negative to positive at $\overline{\varepsilon} - q^m$ we can deduce that Λ (and therefore Z) has a single zero crossing moving from positive to negative at some point $q^*(p) \in (0, \overline{\varepsilon} - q^m)$ for $p \geq \underline{p}$.

Suppose that one player uses the mixture we have defined and consider the optimal choice of offer by the other player. It is clear that the lowest price used is $\underline{p}$. Now an optimal offer cannot contain a section with $0 < p'(q) < \infty$ since from Lemma 1 this only happens when $Z = 0$ and the optimal solution cannot follow $q^*(p)$ since this would contradict the second-order condition (5). Hence an optimal solution can only consist of horizontal and vertical segments. However we know that the integral of Z on the first horizontal segment will be positive if it finishes before q_m, and hence the solution can only be improved by raising this first horizontal section which is not admissible. Thus any optimal solution must consist of a single horizontal section, and thus is in the set of solutions already considered as part of our Nash equilibrium. ■

In the special case that there are constant marginal costs c and uniform demand we can deduce that there is always a solution of this form for pivotal producers and any price cap larger than c.

Corollary 13 *When $C(x) = cx$ and demand is inelastic and uniformly distributed on the interval $[0,1]$ then there is a mixed-strategy SFE with horizontal offers for any $P > c$ and $q_m \in (1/2, 1]$.*

Proof. In this case we can rewrite the condition (42) as

$$P \geq \frac{c}{B^2}\left(A^2 - (A+B)(A-B)\right) = c.$$

From (41) we have

$$\underline{p} = c + (P-c)\frac{B}{A} > c.$$

The sufficient condition is satisfied, since

$$\begin{aligned} &\frac{\partial}{\partial q}\left[(\underline{p} - C'(q))\frac{(F(q+q_m) - F(q))}{1-F(q)}\right] \\ &= \frac{\partial}{\partial q}\left[\frac{(\underline{p}-c)q_m}{1-q}\right] \\ &= \frac{(\underline{p}-c)q_m}{(1-q)^2} > 0, \text{ for } q \in [0, 1-q_m). \end{aligned}$$

Thus we have checked all the conditions of Proposition 12. ■

5.2 Mixtures with hockey stick offers

For increasing marginal costs, Proposition 12 shows that the price cap needs to be sufficiently high for mixtures over horizontal bids to exist. The problem for low price caps is that $Z(q,p)$ increases steeply in the interval $q \in (\bar{\varepsilon} - q_m, \bar{\varepsilon}]$, so that $Z(q,p)$ crosses zero at some point $q_A \in (\bar{\varepsilon} - q_m, \bar{\varepsilon}]$. Hence, we get $Z(q_m, p) > 0$, and the producer would have incentives to increase the price for the segment in the output interval $(q_A, q_m]$. The proposition below shows that in this case we can get another type of equilibrium where the lowest offer curves in the mixture are horizontal and slope-constrained in the interval $[0, q_A(p))$ and then strictly increasing and unconstrained along the curve $q_A(p)$ where $Z(q_A(p), p) = 0$. We call this a *hockey-stick* bid. The highest bids in the mixture are still horizontal along the whole output.

Proposition 14 *Assume that* $D(p) = 0$*,* $\bar{\varepsilon}/2 < q_m < \bar{\varepsilon}$ *and* $C'(q_m) \leq P$*. Then an equilibrium where producers mix over hockey stick bids has the following form:*

1. *There is some* p_m *such that for* $p \in [p_m, P]$ *producers mix over horizontal bids and the mixture is defined from*

$$G(p) = \frac{Ap - J - BP}{Ap - J - Bp}. \tag{50}$$

2. *There is some $\underline{p}$ such that for $p \in [\underline{p}, p_m]$ producers mix over hockey stick bids, where the individual offer, which can be parameterized by a price p, is defined by $p(q) = p$, for $q \in [0, q_A(p)]$ and $p(q) = q_A^{-1}(q)$ for $q \in [q_A(p), q_m]$. Moreover the functions $G(p)$ (defining the mixture) and $q_A(p)$ in the range $[\underline{p}, p_m]$ satisfy the linked differential equations:*

$$g(p) = G'(p) = \frac{1 - G(p)}{(p - C'(q_A(p)))}, \tag{51}$$

$$q_A'(p) = \frac{\int_0^{q_A(p)} 1 - F(q) - \frac{[p - C'(q) - G(p)(C'(q_A(p)) - C'(q))]}{(p - C'(q_A(p)))}(F(q + q_A(p)) - F(q))dq}{G(p)\int_0^{q_A(p)}(p - C'(q))f(q + q_A(p))dq}, \tag{52}$$

provided that $q_A(\underline{p}) > \bar{\varepsilon}/2$. The initial conditions for these differential equations are

$$\underline{p} = \frac{\int_0^{q_A(\underline{p})} \left[C'(q_A(\underline{p}))(1 - F(q)) - C'(q)(F(q + q_A(\underline{p})) - F(q))\right] dq}{\int_0^{q_A(\underline{p})}(1 - F(q + q_A(\underline{p})))dq}, \tag{53}$$

$$G(\underline{p}) = 0, \tag{54}$$

and, in addition,

$$q_A'(\underline{p}) = \frac{2\int_0^{q_A(\underline{p})}\left[C'(q_A(\underline{p})) - C'(q)\right](F(q + q_A(\underline{p})) - F(q))dq}{(\underline{p} - C'(q_A(\underline{p})))\int_0^{q_A(\underline{p})}\left[\begin{array}{c} 2(\underline{p} - C'(q))f(q + q_A(\underline{p})) \\ +(1 - F(q))C''(q_A(\underline{p}))\end{array}\right]dq}. \tag{55}$$

3. *The value of $\underline{p}$ is chosen so that a solution to the differential equations satisfies*

$$G(p_m) = \frac{Ap_m - J - BP}{Ap_m - J - Bp_m} \tag{56}$$

$$q_A(p_m) = q_{m.} \tag{57}$$

A sufficient condition for a mixture satisfying conditions 1-3 above to be a Nash equilibrium is that

$$\frac{\partial}{\partial q}\left[(\underline{p} - C'(q))\frac{f(q + u)}{1 - F(q)}\right] > 0, \tag{58}$$

for every pair $(u, q) \in \{u \geq 0, 0 \leq q \leq q_m : u + q < \bar{\varepsilon}\}$. Under this condition any mixed hockey stick equilibrium has $q_A(\underline{p}) > \bar{\varepsilon}/2$.

Proof. We begin by showing that the conditions of the Proposition statement are necessary for a hockey stick mixture to be an equilibrium. We start by considering the range above p_m (the first condition) where we have from (33)

$$\psi(q,p) = F(q) + G(p)(F(q+q_m) - F(q))$$

and from (44) and (45) that $Z(q,p) = [1 - F(q)]\,\Lambda\,(q,p)$ where

$$\Lambda\,(q,p) = 1 - [G(p) + (p - C'\,(q))g(p)]\,\frac{(F(q+q_m) - F(q))}{1 - F(q)}.$$

Now we note from our previous discussions that we can derive the equation for G in (50) from (34) and (37)

$$G(p) = \frac{\int_0^{q_m}(p - C'(q))(1 - F(q))dq - K}{\int_0^{q_m}(p - C'(q))(F(q+q_m) - F(q))dq}$$

where

$$K = \int_0^{q_m}(P - C'(q))(1 - F(q+q_m))dq,$$

which is the profit achieved by offering at the price cap. Thus the competitor mixing in this way is exactly what is required to ensure that all the horizontal offers with $p \geq p_m$ achieve the same profit for a producer, and hence, by construction, $\int_0^{q_m} Z(t,p)dt = 0$ for $p \in [p_m, P]$.

We also have P as the highest price offered, as in Proposition 12 . Otherwise producers would have incentives to increase their highest bid, because of the result in (48).

Now we turn to the part of the solution below p_m. We will show that the differential equations (51) and (52) arise from the requirement that $\int_0^{q_A(p)} Z(t,p)dt = 0$ and $Z(q_A(p),p) = 0$ for $p \in \left[\underline{p}, p_m\right]$. We have

$$\psi(q,p) = F(q) + G(p)(F(q + q_A\,(p)) - F(q))$$

for $p \in \left[\underline{p}, p_m\right]$. This follows from the fact that with probability $G(p)$ the other player offers one of the hockey stick offers with price below p, and each of these offer curves coincides at the quantity $q_A(p)$ at price p. By means of (4), we can now calculate

$$\begin{aligned} Z(q,p) &= 1 - \psi(q,p) - (p - C'\,(q))\,\psi_p(q,p) \\ &= 1 - F(q) - [G(p) + (p - C'\,(q))g(p)]\,(F(q + q_A\,(p)) - F(q)) \qquad (59) \\ &- (p - C'\,(q))q_A'\,(p)\,f(q + q_A\,(p))G(p) \\ &= [1 - F(q)]\,\Lambda\,(q,p)\,, \end{aligned}$$

where

$$\Lambda(q,p) = 1 - [G(p) + (p - C'(q))g(p)] \frac{(F(q + q_A(p)) - F(q))}{1 - F(q)} \\ - (p - C'(q))q'_A(p) \frac{f(q + q_A(p))}{1 - F(q)} G(p). \tag{60}$$

As we assume that $q_A(\underline{p}) > \bar{\varepsilon}/2$, we also have $q_A(p) > \bar{\varepsilon}/2$ and we can simplify (59) at $q = q_A(p)$ to obtain

$$Z(q_A(p), p) = 1 - F(q_A(p)) - [G(p) + (p - C'(q_A(p)))g(p)](1 - F(q_A(p))). \tag{61}$$

Since increasing supply functions must follow a $Z(q,p) = 0$ curve, we require $Z(q_A(p), p) = 0$. This implies (51), as required for $p \in [\underline{p}, p_m]$.

From (51) and (59) we also have

$$Z(q,p) = 1 - F(q) - \left(\frac{p - C'(q) - G(p)(C'(q_A(p)) - C'(q))}{p - C'(q_A(p))} \right) (F(q + q_A(p)) - F(q)) \\ - (p - C'(q))q'_A(p) f(q + q_A(p))G(p). \tag{62}$$

Since $\int_0^{q_A(p)} Z(t,p)dt = 0$ we can deduce (52) as required.

By definition the lowest price $\underline{p}$ has $G(\underline{p}) = 0$. So, we can deduce from (51) that

$$g(\underline{p}) = \frac{1}{(\underline{p} - C'(q_A(\underline{p})))}. \tag{63}$$

Thus, (59) can be simplified to

$$Z(q,\underline{p}) = 1 - F(q) - \frac{(\underline{p} - C'(q))}{\underline{p} - C'(q_A(\underline{p}))} (F(q + q_A(\underline{p})) - F(q)).$$

Hence, the condition $\int_0^{q_A(p)} Z(t,p)dt = 0$ yields:

$$0 = \int_0^{q_A(\underline{p})} \left[1 - F(q) - \frac{(\underline{p} - C'(q))}{\underline{p} - C'(q_A(\underline{p}))} (F(q + q_A(\underline{p})) - F(q)) \right] dq. \tag{64}$$

from which we obtain (53).

Now consider the value of $q'_A(\underline{p})$ as given by the expression (52) evaluated at $\underline{p}$. Using (64) and the fact that $G(\underline{p}) = 0$ shows that both the numerator and denominator are equal to zero. Hence, we must calculate $q_A'(\underline{p})$ from (55) by means of l'Hôpital's rule. Let $q_A'(p) = N(p)/M(p)$ where

$$N(p) = (p - C'(q_A(p))) \int_0^{q_A(p)} [1 - F(q)]\, dq$$
$$- \int_0^{q_A(p)} (p - C'(q) - G(p)[C'(q_A(p)) - C'(q)])(F(q + q_A(p)) - F(q))dq$$

and

$$M(p) = (p - C'(q_A(p)))G(p) \int_0^{q_A(p)} (p - C'(q)) f(q + q_A(p))dq.$$

In order to apply l'Hôpital's rule we calculate $N'(\underline{p})$ and $M'(\underline{p})$ using $G(\underline{p}) = 0$, $Z(q_A(\underline{p}), \underline{p}) = 0$ and Leibniz' rule. Note that the contribution from differentiating the integration limits are zero in both cases.

$$N'(\underline{p}) = (1 - C''(q_A(\underline{p}))\, q_A'(\underline{p})) \int_0^{q_A(\underline{p})} (1 - F(q))\, dq$$
$$- \int_0^{q_A(\underline{p})} \left(1 - g(\underline{p})\left[C'(q_A(\underline{p})) - C'(q)\right]\right)(F(q + q_A(p)) - F(q))dq$$
$$- q_A'(\underline{p}) \int_0^{q_A(\underline{p})} (\underline{p} - C'(q)) f(q + q_A(\underline{p}))dq$$

and

$$M'(\underline{p}) = (\underline{p} - C'(q_A(\underline{p})))g(\underline{p}) \int_0^{q_A(\underline{p})} (\underline{p} - C'(q)) f(q + q_A(\underline{p}))dq.$$

Since $q_A'(\underline{p}) = N'(\underline{p})/M'(\underline{p})$ and using (63) we have

$$q_A'(\underline{p}) \int_0^{q_A(\underline{p})} (\underline{p} - C'(q)) f(q + q_A(\underline{p}))dq = N'(\underline{p}).$$

We can now collect all terms with $q_A'(p))$ and use (63), so that

$$q_A'(\underline{p}) = \frac{\int_0^{q_A(\underline{p})} \left\{[1 - F(q)] - \left(1 - \frac{C'(q_A(\underline{p})) - C'(q)}{\underline{p} - C'(q_A(p))}\right)(F(q + q_A(\underline{p})) - F(q))\right\} dq}{\int_0^{q_A(\underline{p})} \left\{2(\underline{p} - C'(q)) f(q + q_A(\underline{p})) + [1 - F(q)]\, C''(q_A(\underline{p}))\right\} dq}. \tag{65}$$

The relationship (55) now follows from (64).

The condition (56) follows from (50) and the fact that $G(p)$ is continuous at p_m which is the price at which the curve $q_A(p)$ hits the right-hand boundary q_m.

Now we consider the second part of the Proposition. We begin by showing that $q_A(\underline{p}) > \overline{\varepsilon}/2$ follows from the assumption $\frac{\partial}{\partial q}\left[(\underline{p} - C'(q))f(q+u)/(1-F(q))\right] > 0$. Under this assumption we can deduce

$$\begin{aligned}\frac{\partial}{\partial q}\left[(\underline{p} - C'(q))\frac{(F(q+q_A(p)) - F(q))}{1-F(q)}\right] &= \frac{\partial}{\partial q}\left[\frac{(\underline{p} - C'(q))}{1-F(q)}\int_0^{q_A(p)} f(q+u)du\right] \\ &= \int_0^{q_A(p)} \frac{\partial}{\partial q}\left[(\underline{p} - C'(q))\frac{f(q+u)}{1-F(q)}\right] du > 0\end{aligned} \tag{66}$$

if $q + q_A(\underline{p}) \leq \overline{\varepsilon}$. Hence, if $2q_A(\underline{p}) \leq \overline{\varepsilon}$ then (60) and $G(\underline{p}) = 0$ implies

$$\Lambda_q(q, \underline{p}) = -g(\underline{p})\frac{\partial}{\partial q}\left\{(\underline{p} - C'(q))\frac{(F(q+q_A(\underline{p})) - F(q))}{1-F(q)}\right\} < 0$$

if $q < q_A(\underline{p})$. This would imply that $Z(q, \underline{p})$ reaches 0 from a positive Z-value as $q \to q_A(\underline{p})$. But it would violate the necessary conditions from Lemma 5 and Lemma 3, respectively, that $\int_0^{q_A} Z(t,p)dt = 0$ and $\int_0^q Z(t,p)dt \geq 0$ for $q \leq q_A(p)$. Hence, we have established $2q_A(p) \geq 2q_A(\underline{p}) > \overline{\varepsilon}$, as required.

In order to show sufficiency we will need to establish a number of different things, and we start with the case $p \in (\underline{p}, p_m)$. First we show that the conditions are enough to guarantee that $G(p)$ and $q_A(p)$ are non-decreasing as functions of p. First observe from (53) that

$$\underline{p} \geq \frac{C'(q_A(\underline{p}))\int_0^{q_A(\underline{p})}\left[(1-F(q)) - (F(q+q_A(\underline{p})) - F(q))\right]dq}{\int_0^{q_A(\underline{p})}(1-F(q+q_A(\underline{p})))dq} = C'(q_A(\underline{p})).$$

Note that the condition (58) of the Proposition statement can be written more explicitly as

$$(\underline{p} - C'(q))\frac{\partial}{\partial q}\left[f(q+u)/(1-F(q))\right] - f(q+u)/(1-F(q))\,C''(q) > 0$$

for $u \geq 0$, $0 \leq q \leq q_m$ with $u + q < \overline{\varepsilon}$. This assumption would be violated for $q = q_A(\underline{p}) \leq q_m < \overline{\varepsilon}$ if $\underline{p} = C'(q_A(\underline{p}))$. Hence, we can establish that $\underline{p} > C'(q_A(\underline{p}))$, so that the above inequality is strict. So $G'(p) > 0$, because of (51), and from (55) we observe that $q_A'(\underline{p}) \geq 0$. Now consider differentiating the identity $\int_0^{q_A(p)} Z(t,p)dt = 0$ with respect to p under the assumption that $q_A'(p) =$

0 and we make use of (51) and (62).

$$\int_0^{q_A(p)} (p - C'(q)) q_A''(p) f(q + q_A(p)) G(p) dq$$
$$= -\int_0^{q_A(p)} \frac{\partial}{\partial p} \left[\frac{(p - C'(q)) + G(p)(C'(q) - C'(q_A(p)))}{(p - C'(q_A(p)))} \right] (F(q + q_A(p)) - F(q)) dq$$
$$= \int_0^{q_A(p)} \left[\frac{2[1 - G(p)](C'(q_A(p)) - C'(q))}{(p - C'(q_A(p)))^2} \right] (F(q + q_A(p)) - F(q)) dq \geq 0.$$

Hence, $q_A''(p) \geq 0$ whenever $q_A'(p) = 0$. Thus the derivative $q_A'(p)$ can move from negative to positive but not the other way around as p increases. Since $q_A'(\underline{p}) \geq 0$ and this value is defined by continuity from above, there can be no changes of sign in q_A' between $\underline{p}$ and p_m and so q_A is increasing throughout this range.

We know that $Z(q_A(p), p) = 0$. Now we will analyze other potential zero-crossings where $Z(q, p) = 0$ in the interval $q \in (0, q_A(p))$. From (60) we note that $\Lambda_q(q, p) = C''(q) g(p) \geq 0$ if $q > \overline{\varepsilon} - q_A(p)$, and so Λ is non-decreasing in this range and zero at $q = q_A(p)$, which implies that $\int_{\overline{\varepsilon} - q_A(p)}^{q_A(p)} Z(q, p) dq \leq 0$ and $Z(\overline{\varepsilon} - q_A(p)+, p) \leq 0$. Together with the condition $\int_0^{q_A(p)} Z(q, p) dq = 0$ this implies that there must be at least one $q^* \in [0, \overline{\varepsilon} - q_A(p)]$ with $Z(q^*, p) = 0$.

From (60) we have

$$\Lambda_q(q, p) = -\frac{\partial}{\partial q} \left[[G(p) + (p - C'(q)) g(p)] \frac{(F(q + q_A(p)) - F(q))}{1 - F(q)} \right]$$
$$- q_A'(p) G(p) \frac{\partial}{\partial q} \left[(p - C'(q)) \frac{f(q + q_A(p))}{1 - F(q)} \right].$$

Notice that the first term can be rewritten

$$\frac{\partial}{\partial q} \left[[G(p) + (p - C'(q)) g(p)] \frac{(F(q + q_A(p)) - F(q))}{1 - F(q)} \right] \tag{67}$$
$$= \left(\frac{G(p)}{p - C'(q)} + g(p) \right) \frac{\partial}{\partial q} \left[(p - C'(q)) \frac{(F(q + q_A(p)) - F(q))}{1 - F(q)} \right]$$
$$+ \frac{G(p)}{p - C'(q)} C''(q) \frac{(F(q + q_A(p)) - F(q))}{1 - F(q)}.$$

It is straightforward to show that condition (58) is satisfied for all prices $p > \underline{p} > C'(q)$ if it is satisfied for $p = \underline{p}$. Thus we can conclude from (66) and (67) that the first term in $\Lambda_q(q, p)$ is negative for $q < \overline{\varepsilon} - q_A(p)$. As $q_A'(p) \geq 0$ the second term in $\Lambda_q(q, p)$ is non-positive from (58) and so we have established that $\Lambda_q(q, p) < 0$ for $q < \overline{\varepsilon} - q_A(p)$. This implies that there is exactly one point at which $\Lambda(q,p) = 0$ in this range. Since Λ changes sign at q^* the same must be true for Z.

But there might be a range of points where $Z(q,p) = 0$ for $q > \bar{\varepsilon} - q_A(p)$. The relations $Z(q_A(p), p) = 0$ and $\Lambda_q(q,p) = C''(q)\, g(p)$ if $q > \bar{\varepsilon} - q_A(p)$ imply that if $C''(q) = 0$ for q in some range (q_B, q_C) and $q_A(p) \in [q_B, q_C]$ then $Z(q,p) = 0$ for $q \in (\max[q_B, \bar{\varepsilon} - q_A(p)], q_C)$. In cases where marginal costs are non-constant around $q_A(p)$ we let $q_B = q_C = q_A(p)$, and we can sum up the situation as follows:

$$
\begin{aligned}
&Z(q,p) > 0, \text{ for } q \in (0, q^*) \text{ and } q \in (q_C, q_m),\\
&Z(q,p) < 0, \text{ for } q \in (q^*, \max[q_B, \bar{\varepsilon} - q_A(p)]),\\
&Z(q,p) = 0, \text{ for } q = q^* \text{ and } q \in (\max[q_B, \bar{\varepsilon} - q_A(p)], q_C).
\end{aligned}
$$

Now we want to establish that this mixture over hockey stick bids is a Nash equilibrium. Suppose that one player uses the mixture we have defined and consider the optimal choice of offer by the other player. As in Proposition 12 it is clear that the other player has no incentives to bid below $\underline{p}$. We know that Z has a single zero crossing in the interval $[0, \bar{\varepsilon} - q_A(p)]$ moving from positive to negative at some point $q^*(p) \in (0, \bar{\varepsilon} - q_A(p))$ for $p \in (\underline{p}, p_m)$. Now an optimal offer cannot follow the $q^*(p)$ curve since this would contradict the condition (5). So the only place where an optimal offer can have $0 < p'(q) < \infty$ is along the curve $q_A(p)$ (or in a region surrounding $q_A(p)$ where $Z = 0$ in the case when marginal costs are constant). We can change the offer curves within the region where $Z = 0$ without changing their optimality. Hence we may suppose that an optimal reponse is adjusted to lie along the curve $q_A(p)$ as much as possible. Thus where $0 < p'(q) < \infty$ it follows the curve $q_A(p)$ and it does not begin a horizontal segment or end a vertical segment within this $Z = 0$ region. Apart from the section along the curve $q_A(p)$, an optimal offer can only consist of horizontal and vertical sections. Consider the final horizontal section, say from q_X to q_Y at price p. Since there are no more horizontal segments we must have either $q_Y = q_m$ if $p \geq p_m$, or $q_Y \geq q_A(p)$ if $p \in (\underline{p}, p_m)$. Suppose that this horizontal section does not start at zero, so $q_X > 0$.

Now, consider the case $p \in (\underline{p}, p_m)$ and suppose that $q_Y = q_A(p)$. Then $\int_{q_X}^{q_A(p)} Z(t,p)dt < 0$ since $\int_0^{q_A(p)} Z(q,p)dq = 0$ and either $Z(t,p) < 0$ throughout the interval $t \in (q_X, q_D)$ or $Z(t,p) > 0$ for $t \in (0, q_X)$. Hence this solution can be improved by moving the horizontal section slightly downwards (as in Lemma 3) contradicting the claimed optimality. Thus we must have $q_Y > q_A(p)$. But we know that $Z(q,p) \geq 0$ below the $q_A(p)$ curve, so without lost profit we can increase the price of the units $q \in (q_A(p), q_Y)$ up to this curve. Thus it can never be a profitable deviation to have $q_Y > q_A(p)$.

The argument for the case when $p \geq p_m$ is easier. As in Proposition 12 we simply establish that $\int_{q_X}^{q_m} Z(t,p)dt < 0$ when $q_X > 0$ and hence use Lemma 3 to show the deviation is not optimal.

Thus we have established that there is exactly one horizontal section starting at zero and finishing on the q_A curve or at q_m. So any optimal response is already represented in the mixture and this is enough to show that the solution is a Nash equilibrium. ■

The proposition below provides sufficient conditions for a hockey-stick mixture to exist.

Proposition 15 *Assume that* $D(p) = 0$, $\bar{\varepsilon}/2 < q_m < \bar{\varepsilon}$ *and* $\underline{p} \in \left[\underline{p}^{\min}, \underline{p}^{\max}\right]$, *where*

$$\underline{p}^{\min} = \frac{\int_0^{\bar{\varepsilon}/2} \left[C'\left(\bar{\varepsilon}/2\right)(1 - F(q)) - C'(q)(F(q + \bar{\varepsilon}/2) - F(q))\right] dq}{\int_0^{\bar{\varepsilon}/2} (1 - F(q + \bar{\varepsilon}/2))dq} \tag{68}$$

and

$$\underline{p}^{\max} = \frac{C'\left(q_m\right) A - J}{B}. \tag{69}$$

Then there is a price cap value $P > p_m$ *for which a mixed hockey stick equilibrium exists if for some* $\delta > 0$

$$\frac{\partial}{\partial q}\left[\left(\underline{p} - C'\left(q\right)\right)\frac{f(q+u)}{1 - F(q)}\right] > \delta$$
$$\text{for } (u,q) \in \left\{u \geq 0, 0 \leq q \leq q_m : u + q < \bar{\varepsilon}\right\}.$$

Proof. From the result of Proposition 14 it is enough to find a set of price cap values P such that there will be choices of $\underline{p}$ and p_m and a solution of the differential equations (51) and (52) with initial conditions (53), (54) and (55) satisfying the conditions (56) and (57). We will generate these solutions by showing that each of a range of possible starting points is matched to a final price cap value P. We do this by starting with one of the initial points given by (53), (54) and constructing a solution to the differential equations (51) and (52) from this point (which will automatically satisfy (55)).

We need to establish that the set of possible starting points is non-empty. First recall the defining relationship for $q_A\left(\underline{p}\right)$ in (64)

$$0 = \int_0^{q_A\left(\underline{p}\right)} \left[1 - F(q) - \frac{\left(\underline{p} - C'\left(q\right)\right)}{\underline{p} - C'\left(q_A\left(\underline{p}\right)\right)}\left(F(q + q_A\left(\underline{p}\right)) - F(q)\right)\right] dq. \tag{70}$$

We can differentiate both sides of (70) with respect to $\underline{p}$. The calculations are simplified since the integrand is $Z\left(q,\underline{p}\right)$ and $Z\left(q_A\left(\underline{p}\right),\underline{p}\right)=0$.

$$\begin{aligned}
0 &= -\int_0^{q_A(\underline{p})} \frac{(F(q+q_A\left(\underline{p}\right))-F(q))}{\underline{p}-C'\left(q_A\left(\underline{p}\right)\right)}dq \\
&+ \int_0^{q_A(\underline{p})} \left[\frac{(\underline{p}-C'(q))(F(q+q_A\left(\underline{p}\right))-F(q))}{\left[\underline{p}-C'\left(q_A\left(\underline{p}\right)\right)\right]^2}\right]dq \\
&- \int_0^{q_A(\underline{p})} \frac{(\underline{p}-C'(q))f(q+q_A\left(\underline{p}\right))\frac{\partial q_A(\underline{p})}{\partial \underline{p}}}{\underline{p}-C'\left(q_A\left(\underline{p}\right)\right)}dq \\
&= \int_0^{q_A(\underline{p})} \left[\frac{(C'\left(q_A\left(\underline{p}\right)\right)-C'(q))(F(q+q_A\left(\underline{p}\right))-F(q))}{\left[\underline{p}-C'\left(q_A\left(\underline{p}\right)\right)\right]^2}\right]dq \\
&- \int_0^{q_A(\underline{p})} \frac{(\underline{p}-C'(q))f(q+q_A\left(\underline{p}\right))\frac{\partial q_A(\underline{p})}{\partial \underline{p}}}{\underline{p}-C'\left(q_A\left(\underline{p}\right)\right)}dq,
\end{aligned}$$

which implies that $\frac{\partial q_A(\underline{p})}{\partial \underline{p}} \geq 0$, because $C'\left(q_A\left(\underline{p}\right)\right)-C'(q)\geq 0$. So the highest value of $\underline{p}$ occurs when $q_A\left(\underline{p}\right)$ is q_m. With this value we get (69) from (53). Moreover, the lowest value of $\underline{p}$ occurs when $q_A\left(\underline{p}\right)=\bar{\varepsilon}/2$, which gives us (68).

We show that the capacity constraint q_m must bind at some price p_m where $G\left(p_m\right)<1$. We know from (51) that

$$\frac{G'(p)}{1-G(p)}=\frac{1}{(p-C'\left(q_A\left(p\right)\right))}.$$

Since $G(\underline{p})=0$, integration gives

$$-\ln(1-G(p))=\int_{\underline{p}}^{p}\frac{dp}{p-C'\left(q_A\left(p\right)\right)}=\alpha(p),$$

so

$$G(p)=1-e^{-\alpha(p)}.$$

The assumption that the inequality (58) holds for

$$(u,q)\in\{u\geq 0, 0\leq q\leq q_m : u+q<\bar{\varepsilon}\}$$

shows that $\underline{p}-C'(q)$ is bounded away from zero as q approaches q_m. Hence there exists some δ such that

$$p-C'\left(q_A\left(p\right)\right)\geq \underline{p}-C'\left(q_m\right)\geq\delta>0$$

for every $p \in [\underline{p}, p_m]$. Thus $\alpha(p)$ is finite for finite p, whence $G(p) < 1$, i.e. a hockey stick mixture never reaches $G(p) = 1$ at a finite price.

Next we will show that $q_A(p) > q_m$ as $p \to \infty$. We make the contradictory assumption that $q_A(p) \leq q_m$ in this limit. For $p > \underline{p}$ we then have from (52) that

$$q'_A(p) = \frac{\int_0^{q_A(p)} 1 - F(q) - \left[G(p) + \frac{(p-C'(q))[1-G(p)]}{(p-C'(q_A(p)))}\right](F(q+q_A(p)) - F(q))dq}{\int_0^{q_A(p)} (p - C'(q)) f(q + q_A(p)) G(p) dq}$$
$$> \frac{\int_0^{1-q_m} [1 - F(q + q_m)]\, dq}{p} + O\left(\frac{1}{p^2}\right) = \frac{k}{p} + O\left(\frac{1}{p^2}\right),$$

where k is some positive constant. Thus

$$q_A(p) = q_A(\underline{p}) + \int_{\underline{p}}^{p} q'_A(p)\, dp > q_A(\underline{p}) + k \ln p + O\left(\frac{1}{p}\right).$$

Hence, $q_A(p) > q_m$ for sufficiently large p, which is a contradiction. Hence, the capacity constraint q_m must bind at some finite price p_m where $G(p_m) < 1$.

Finally we define the price cap P from (56). ■

Now we are ready to establish the following:

Theorem 16 *Assume that $D(p) = 0$, $\bar{\varepsilon}/2 < q_m < \bar{\varepsilon}$ and that for some $\delta > 0$*

$$\frac{\partial}{\partial q}\left[(\underline{p}^{\min} - C'(q))\frac{f(q+u)}{1-F(q)}\right] > \delta$$
$$\text{for } (u, q) \in \{u \geq 0, 0 \leq q \leq q_m : u + q < \bar{\varepsilon}\}.$$

Then a unique hockey stick mixture exists for each price cap $P \in [P_{\min}, P_{\max}]$ where $P_{\min}$ is determined from the mixture starting at $\underline{p}^{\min}$ and

$$P_{\max} = \frac{1}{B^2}\left(A^2 C'(q_m) - (A + B) J\right).$$

Proof. It follows from Proposition 15 that a hockey-stick mixture will exist for some price cap for any $\underline{p} \in [\underline{p}^{\min}, \underline{p}^{\max}]$. The necessary equations outlined in Proposition 14 ensures that the hockey-stick solution $(G(p), q_A(p))$ is differentiable, and hence continuous, with respect to p. Proposition 24 in Appendix 2 proves that there is a unique continuous hockey-stick solution for each $\underline{p}$. Lemma 28 and Theorem 30 in Appendix 3 prove that the price cap is continuous and strictly increasing with respect to $\underline{p}$. Thus the lowest price cap for which a hockey-stick mixture occurs can be calculated from the initial value $\underline{p}^{\min}$. Moreover, the highest price cap occurs when $\underline{p} = \underline{p}^{\max}$, i.e. when $q_A(\underline{p}) = q_m$. Using $G(\underline{p}) = 0$, (56), and (69) we see that this value corresponds to a price cap at

$$P_{\max} = \frac{1}{B^2}\left(A^2 C'(q_m) - (A+B)J\right).$$

■

Note that according to Proposition 12, $P = \frac{1}{B^2}\left(A^2 C'(q_m) - (A+B)J\right)$ is the lowest price cap for which a mixture over horizontal offers can exist. Thus there is no price cap for which both types of slope-constrained mixtures exist.

It can also be shown that:

Lemma 17 *For a hockey-stick mixture satisfying* $\underline{p} > C'\left(q_A(\underline{p})\right)$ *we must have* $C'\left(q_A(\underline{p})\right) > C'(0)$.

Proof. This follows directly from (53). ■

Genc [14] proves that mixed-strategy equilibria over partly increasing supply functions do not exist for pivotal producers and constant marginal costs when demand is uniformly distributed. Our findings do not contradict this result. Lemma 17 implies that the sufficiency condition for hockey-stick mixtures in (58) is never satisfied for constant marginal costs. Hence, non-constant marginal costs are necessary to satisfy this sufficiency condition. We also know from Proposition 15 that hockey-stick mixtures satisfying (58) have a horizontal mixture at the top, i.e. $P > p_m$. By means of Proposition 11 we can therefore rule out hockey-stick mixtures satisfying (58) for non-pivotal producers.

5.3 Examples

We conclude this section with two examples of equilibria with mixed strategies. In both examples we assume a symmetric duopoly with each player having capacity $q_m \in (\frac{1}{2}, 1)$ and $C(q) = \frac{1}{2}q^2$, and inelastic demand that is uniformly distributed on $[0, 1]$.

Example 4: Horizontal bids

With a uniform demand shock and quadratic costs we get

$$A = \frac{1}{2}q_m(2 - q_m),$$

$$B = \frac{1}{2}(1 - q_m)^2,$$

and

$$J = \frac{1}{6}(-1 + 3q_m - q_m^3).$$

This gives

$$\begin{aligned} P &\geq \frac{1}{B^2}\left(A^2 C'(q_m) - (A+B)J\right) \\ &= \frac{13q_m^3 - 12q_m^4 + 3q_m^5 + 1 - 3q_m}{3(1-q_m)^4} \end{aligned}$$

for an equilibrium. We can see that this is sufficient because using this value of P gives

$$\begin{aligned} \underline{p} &= \frac{\int_0^{q_m} C'(q)(1-F(q))dq + \int_0^{q_m}(P - C'(q))(1 - F(q+q_m))dq}{\int_0^{q_m}(1-F(q))dq} \\ &= \frac{\int_0^{q_m} q(1-q)dq + \int_0^{1-q_m}(P-q)(1-(q+q_m))dq}{\int_0^{q_m}(1-q)dq} \\ &= \frac{-2q_m^3 + 6q_m^2 - 3q_m + 1}{3(1-q_m)^2} \end{aligned}$$

which is easily seen to be strictly greater than 1 for $q_m \in (\frac{1}{2}, 1)$. This means that

$$\frac{\partial}{\partial q}\left[(\underline{p} - C'(q))\frac{(F(q+q_m) - F(q))}{1-F(q)}\right] = q_m \frac{\underline{p}-1}{(q-1)^2} > 0, \quad \text{for } q \in [0, 1-q_m)$$

which is sufficient for an equilibrium by Proposition 12. So for every value of $q_m \in (\frac{1}{2}, 1)$ and price cap $P \geq \frac{1}{3}\frac{13q_m^3 - 12q_m^4 + 3q_m^5 + 1 - 3q_m}{(1-q_m)^4}$ there is a mixed-strategy equilibrium with horizontal bids. The equilibrium can be uniquely determined by

$$\underline{p} = \frac{-\frac{1}{6}q_m^3 - \frac{1}{6} + \frac{1}{2}q_m + \frac{1}{2}P - Pq_m + \frac{1}{2}Pq_m^2}{q_m - \frac{1}{2}q_m^2}$$

and

$$G(p) = \frac{3q_m(2-q_m)\left(p - \underline{p}\right)}{-3p - 3q_m + 12pq_m + q_m^3 - 6pq_m^2 + 1}.$$

In this example we have assumed that demand is inelastic. Using the same approach, we are able to construct similar examples in which $D(p) \neq 0$, as long as $q_m < D(P) + \bar{\epsilon}$.

Example 5: Hockey-stick bids

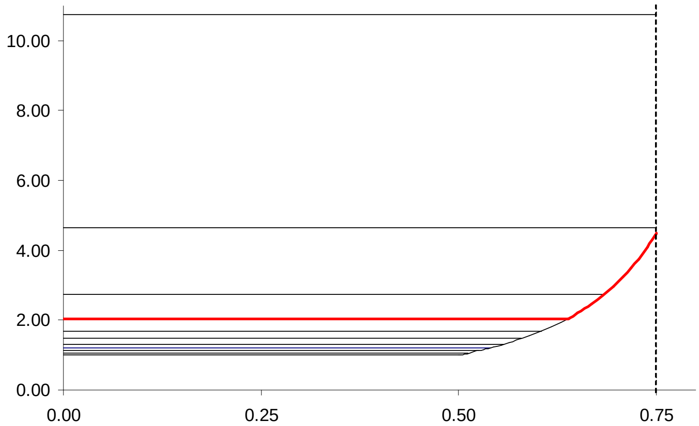

Figure 3: Mixed-strategy equilibrium for $\underline{p} = 1$ and $P = 10.734$, showing hockey-stick bids.

The choice $q_m = \frac{3}{4}$ gives

$$\frac{1}{3}\frac{13q_m^3 - 12q_m^4 + 3q_m^5 + 1 - 3q_m}{(1-q_m)^4} = 98\frac{1}{12}.$$

If we choose a price cap $P < 98\frac{1}{12}$, then by Proposition 12 there does not exist a mixed-strategy equilibrium in horizontal bids. However for every $P \in (10.734, 98\frac{1}{12})$ there are equilibria with mixtures over hockey-stick bids and horizontal bids. The most competitive equilibrium where $q_A(\underline{p}) = \frac{1}{2}$ is plotted in Figure 3. The lines shown are at contours of G taking values 0, 0.1, 0.2, up to 1.0. It has $q_A(p)$ starting at $\underline{p} = 1$, and shows a typical hockey-stick offer using a solid line. All the hockey-stick curves in the mixture meet $q_m = 0.75$ at $p_m = 4.5$ (approximately), and correspond to a price cap $P = 10.734$. Observe that the offers in the mixtures are horizontal for $p > p_m = 4.5$. As a comparison, the most competitive mixture is compared with another mixture in Figure 4. The higher priced equilibrium starts at $\underline{p} = \frac{29}{15}$, and gives $q_A(\underline{p}) = \frac{6}{10}$ and p_m approximately 5.54. The lowest priced hockey stick offer in this mixture is shown as a dashed line in Figure 4, along with the lowest priced horizontal offer in this mixture (at p_m =5.54). The highest priced offer in this mixture (at $P = 24.6$) is not shown.

6 Conclusions

In this paper we derive general optimality conditions for pay-as-bid procurement auctions that are valid for any uncertainty in a producer's residual demand curve,

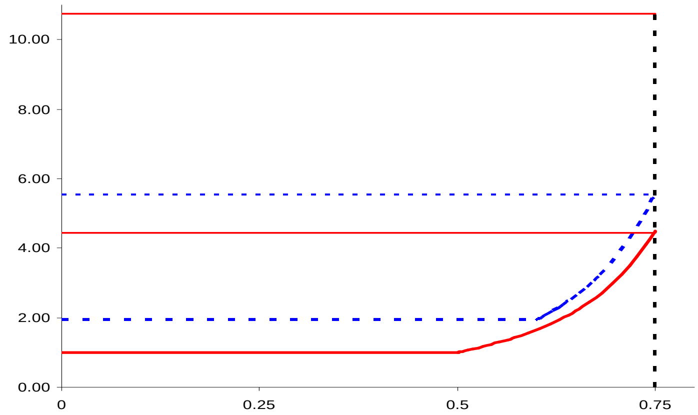

Figure 4: Comparison of two hockey-stick mixtures with $\underline{p} = 1$ (solid) and $\underline{p} = \frac{29}{15}$ (dashed). The price cap (24.6) for the dashed mixture is not shown.

i.e. for any combination of demand uncertainty, uncertainty in competitors' costs or randomization of competitors' offer curves. It is our belief that these conditions can also be useful in the theory of Bertrand games and non-linear pricing. We use the conditions to derive necessary conditions for pure-strategy equilibria in electricity auctions, i.e. when costs are common knowledge and demand uncertain. We show that they fail to exist whenever the market clears at a point where a producer's mark-up times the hazard rate of the demand shock is increasing. Hence, it is of great interest to analyze mixed strategy equilibria under those circumstances. As far as we know we are the first to analyze mixed-strategy equilibria in multi-unit/divisible good auctions with offer curves that are not necessarily horizontal. We consider a symmetric duopoly market. It is shown that mixtures over strictly increasing supply functions can occur in markets with non-pivotal producers, inelastic demand and no price cap. With pivotal producers we get mixtures over slope-constrained supply curves with horizontal segments, and the equilibria are uniquely determined by the price cap when demand is inelastic. When price caps are sufficiently high all offer curves of the producers are slope-constrained along the whole output. These one-dimensional mixtures correspond to mixed-strategy equilibria previously analyzed by Anwar [2], Fabra et al. [12], Genc [14] and Son et al. [24], and they are also Nash equilibria in corresponding Bertrand-Edgeworth games with uncertain demand. For lower price caps, so that the mark-up for the lowest offers becomes small relative to the curvature of the cost curve, we get a new type of mixture where the lowest bids are hockey-stick shaped. They are slope-constrained and horizontal for low outputs and strictly increasing for high outputs. The highest bids in the mixture are still horizontal along the whole output. We show that there are no mixed-strategy equilibria of these types when the market has a price cap and producers are non-pivotal.

Mixed-strategy equilibria are a nuisance for agents in the market, as they cause additional uncertainty. Another disadvantage is that mixed strategies will lead to welfare losses due to inefficient production. With symmetric producers and convex increasing costs, production is most efficient if all producers have the same output. But with randomized offer curves, the realized production will typically be asymmetric. On the other hand, mixed-strategies reduce the problem with multiple equilibria, because all bids are accepted with a positive probability. So there are no out-of-equilibrium bids which could otherwise support very non-competitive equilibria. Similarly, collusion is harder in pay-as-bid auctions compared to uniform-price auctions, because bids can not be used as costless threats and signals [21]. Hence, in spite of the potential problems with mixed-strategy equilibria, the pay-as-bid auction may still be attractive for market designers.

With our model it becomes possible to quantitatively compare strategic bidding in uniform-price and pay-as-bid electricity auctions for previously unexplored but still very relevant cases, for example when producers are pivotal, marginal costs are stepped and demand shocks are normally distributed. Moreover, the use of market distribution functions and offer distribution functions should be of general interest, since they can be applied to characterize mixed-strategy equilibria in any multi-unit or divisible-good auction, including uniform-price auctions. We believe that the optimality conditions that we derive will be useful in empirical work, for example to test whether producers maximize their expected profit in discriminatory divisible-good auctions. As it is very difficult to tell beforehand what type of equilibria will occur in such auctions, it is helpful that the conditions can be applied in general circumstances, including both mixed and pure-strategies.

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7 Appendix 1: Some technical lemmas

In this appendix we define some notation and prove some technical lemmas that are used in Appendix 2 and Appendix 3. We begin by defining

$$\Phi(q,p) = 1 - F(q) - \frac{[p - C'(q) - G(p)(C'(q_A(p)) - C'(q))]}{(p - C'(q_A(p)))}(F(q + q_A(p)) - F(q)), \tag{71}$$

which is the integrand in the numerator of the expression for $q'_A(p)$ in Proposition 14. We also set

$$\xi(q,p) = (1 - F(q))(p - C'(q_A(p))) - (p - C'(q))(F(q + q_A(p)) - F(q)). \tag{72}$$

Observe that

$$\begin{aligned}(p - C'(q_A(p)))\,\Phi(q,p) &= (1 - F(q))(p - C'(q_A(p))) \\ &\quad - (p - C'(q))(F(q + q_A(p)) - F(q)) \\ &\quad + G(p)(C'(q_A(p)) - C'(q))(F(q + q_A(p)) - F(q)) \\ &= \xi(q,p) + G(p)(C'(q_A(p)) - C'(q))(F(q + q_A(p)) - F(q))\end{aligned}$$

so it follows from (59) and (63) that

$$\left(\underline{p} - C'\left(q_A\left(\underline{p}\right)\right)\right)\Phi(q,\underline{p}) = \xi(q,\underline{p}) = \left(\underline{p} - C'\left(q_A\left(\underline{p}\right)\right)\right)Z(q,\underline{p}) \tag{73}$$

since $G(\underline{p}) = 0$. Thus (64) yields

$$\int_0^{q_A(\underline{p})} \xi(q,\underline{p})dq = 0. \tag{74}$$

It follows directly from the definitions of $\xi(q,p)$ and $\Phi(q,p)$ that

$$\xi(q_A(p), p) = 0 \tag{75}$$

and

$$\Phi(q_A(p), p) = 0 \tag{76}$$

whenever $q_A(p) > \overline{\varepsilon}/2$. Moreover, $Z(q_A(p), p) = 0$, (59) and (51) implies that

Inside the proof of Proposition 14 we showed that $q'_A(p) \geq 0$, but as we show below the inequality is actually strict.

Lemma 18 *Assume that $D(p) = 0$, $\overline{\varepsilon}/2 < q_m < \overline{\varepsilon}$ and $C'(q_m) \leq P$ then $q'_A(p) > 0$ if $p > \underline{p}$.*

Proof. We have shown in Proposition 14 that $q_A'(p) \geq 0$, for $p \geq \underline{p}$. If $C'(q_A(p)) = C'(0))$ then $C'(q_A(p)) = C'(q))$ for every $q \in [0, q_A(p)]$, so

$$\Phi(q,p) = (1 - F(q + q_A(p)),$$

giving

$$\begin{aligned} q_A'(p) &= \frac{\int_0^{q_A(p)} \Phi(q,p)dq}{G(p)\int_0^{q_A(p)}(p - C'(q))f(q + q_A(p))dq} \\ &= \frac{\int_0^{q_A(p)}(1 - F(q + q_A(p)))dq}{G(p)\int_0^{q_A(p)}(p - C'(q))f(q + q_A(p))dq} \\ &> 0, \end{aligned}$$

since $G(p) > 0$ if $p > \underline{p}$ and $q_A(p) \in (\bar{\varepsilon}/2, \bar{\varepsilon})$.

If for some $p > \underline{p}$, $C'(q_A(p)) > C'(0))$ and $q_A'(p) = 0$, then we can use the following relation (derived in Proposition 14)

$$\begin{aligned} &\int_0^{q_A(p)} (p - C'(q))q_A''(p) f(q + q_A(p))G(p)dq \\ &= \int_0^{q_A(p)} \left[\frac{2[1 - G(p)](C'(q_A(p)) - C'(q))}{(p - C'(q_A(p)))^2}\right](F(q + q_A(p)) - F(q))dq \end{aligned}$$

to show that $q_A''(p) > 0$, which rules out $q_A'(p) = 0$ if $p > \underline{p}$. ■

Lemma 19 $\int_0^{q_A(\underline{p})} \Phi(q,\underline{p})dq = 0$ *and* $\int_0^{q_A(p)} \Phi(q,p)dq > 0$ *if* $p > \underline{p}$.

Proof. When $p = \underline{p}$, the result follows directly from (73) and (74). When $p > \underline{p}$ we have from Proposition 14 that

$$q_A'(p) = \frac{\int_0^{q_A(p)} \Phi(q,p)dq}{G(p)\int_0^{q_A(p)}(p - C'(q))f(q + q_A(p))dq}$$

so the result follows directly from $q_A'(p) > 0$, and $G(p) > 0$. ■

Lemma 20 *For hockey-stick mixtures satisfying* $\bar{\varepsilon}/2 < q_A(\underline{p}) \leq q_m < \bar{\varepsilon}$ *we have for every* $p \geq \underline{p}$

$$\int_0^{q_A(p)} \xi(q,p)dq \leq 0.$$

The inequality is strict when $p > \underline{p} > C'(q_A(\underline{p}))$.

Proof. It follows from (74) that $\int_0^{q_A(\underline{p})} \xi(q,\underline{p})dq = 0$. So to prove the statement for $p > \underline{p}$, it is enough to show that

$$\frac{\partial}{\partial p}\int_0^{q_A(p)} \xi(q,p)dq \leq 0$$

for $p \geq \underline{p}$ whenever

$$\int_0^{q_A(p)} \xi(q,p)dq = 0. \tag{77}$$

Since $q_A(p) > \bar{\varepsilon}/2$ we have $\xi(q_A(p), p) = 0$ so

$$\begin{aligned}
\frac{\partial}{\partial p}\int_0^{q_A(p)} \xi(q,p)dq &= \int_0^{q_A(p)} \frac{\partial}{\partial p}\xi(q,p)dq \qquad (78)\\
&= \int_0^{q_A(p)} [1 - F(q + q_A(p))]\, dq \\
&- q_A'(p)\int_0^{q_A(p)} [(1-F(q))\, C''(q_A(p)) + (p - C'(q)) f(q + q_A(p))]\, dq \\
&\leq \int_0^{q_A(p)} [1 - F(q + q_A(p))]\, dq \\
&- q_A'(p)\int_0^{q_A(p)} (p - C'(q)) f(q + q_A(p))\, dq
\end{aligned}$$

since $C''(q_A(p)) \geq 0$ and $q_A'(p) \geq 0$. We note that $\frac{\partial}{\partial p}\int_0^{q_A(p)} \xi(q,p)dq$ is continuous at $\underline{p}$ since $q_A'(p)$ and $q_A(p)$ are continuous at that point. It has been shown in Lemma 17 that $\underline{p} > C'\left(q_A\left(\underline{p}\right)\right)$ implies $C'\left(q_A\left(\underline{p}\right)\right) > C'(0)$. Hence, the inequality above is strict when $\underline{p} > C'\left(q_A\left(\underline{p}\right)\right)$. We know from Proposition 14 that when $G(p) > 0$,

$$q_A'(p) = \frac{\int_0^{q_A(p)} \Phi(q,p)dq}{G(p)\int_0^{q_A(p)} (p - C'(q)) f(q + q_A(p))dq},$$

so

$$\begin{aligned}
\frac{\partial}{\partial p}\int_0^{q_A(p)} \xi(q,p)dq &\leq \int_0^{q_A(p)} (1 - F(q + q_A(p)))\, dq - \frac{\int_0^{q_A(p)} \Phi(q,p)dq}{G(p)} \\
&= \int_0^{q_A(p)} \left(1 - F(q + q_A(p)) - \frac{\Phi(q,p)}{G(p)}\right) dq.
\end{aligned}$$

Recall

$$(p - C'(q_A(p)))\Phi(q,p) = \xi(q,p) + G(p)(C'(q_A(p)) - C'(q))(F(q + q_A(p)) - F(q)),$$

which gives

$$
\begin{aligned}
&(p - C'(q_A(p)))\left((1 - F(q + q_A(p))) - \frac{\Phi(q,p)}{G(p)}\right) \\
&= (p - C'(q_A(p)))(1 - F(q + q_A(p))) - \frac{\xi(q,p)}{G(p)} \\
&- (C'(q_A(p)) - C'(q))(F(q + q_A(p)) - F(q)) \\
&= (p - C'(q_A(p)))(1 - F(q)) - (p - C'(q))(F(q + q_A(p)) - F(q)) - \frac{\xi(q,p)}{G(p)} \\
&= \xi(q,p) - \frac{\xi(q,p)}{G(p)}.
\end{aligned}
$$

So for every $p > \underline{p}$ for which (77) holds

$$
\begin{aligned}
(p - C'(q_A(p)))\frac{\partial}{\partial p}\int_0^{q_A(p)} \xi(q,p)dq \leq \int_0^{q_A(p)} &\left(\xi(q,p) - \frac{\xi(q,p)}{G(p)}\right) dq \\
= 0&
\end{aligned}
$$

demonstrating that $\frac{\partial}{\partial p}\int_0^{q_A(p)} \xi(q,p)dq \leq 0$ for every $p > \underline{p}$ for which (77) holds. Continuity of $\frac{\partial}{\partial p}\int_0^{q_A(p)} \xi(q,p)dq$ at $\underline{p}$ ensures that the inequality is satisfied also in the limit. The inequality is strict when $\underline{p} > C'\left(q_A\left(\underline{p}\right)\right)$. ■

8 Appendix 2: Hockey-stick uniqueness result

In this appendix we prove that the set of differential equations defined in Proposition 14 has at most one solution $q_A(p), G(p)$ given initial conditions $q_A\left(\underline{p}\right), G\left(\underline{p}\right)$. This is not self-evident, because of a singularity at $\underline{p}$. We assume that $(q_A(p), G(p))$ is a valid solution and study perturbations of this solution. Let $(q_A(p) + u(p), G(p) + v(p))$ be another solution with both u and v approaching zero as $p \to \underline{p}$. We will show that u and v must be identically zero, so that the original solution is unique.

Using the differential equations defined in Proposition 14 we have

$$
\begin{aligned}
G'(p) + v'(p) &= \frac{1 - (G(p) + v(p))}{p - C'(q_A(p) + u(p))} \\
&= \frac{1 - G(p)}{p - C'(q_A(p))} + \frac{\partial}{\partial q_A}\left\{\frac{1 - G(p)}{p - C'(q_A(p))}\right\} u(p) \\
&+ \frac{\partial}{\partial G}\left\{\frac{1 - G(p)}{p - C'(q_A(p))}\right\} v(p) + o(u) + o(v).
\end{aligned}
$$

So

$$v'(p) = A(p)\,u(p) - B(p)\,v(p) + o(u) + o(v)$$

where

$$A(p) = \frac{(1 - G(p))\,C''(q_A(p))}{(p - C'(q_A(p)))^2}, \tag{79}$$

$$B(p) = \frac{1}{p - C'(q_A(p))}. \tag{80}$$

Next we establish a similar identity for $u'(p)$. We have

$$u'(p) = -D(p)\,u(p) + E(p)\,v(p) + o(u) + o(v),$$

where

$$D(p) = -\frac{\partial}{\partial q_A}\left\{\frac{\int_0^{q_A(p)}\left[1 - F(q) - \frac{[p - C'(q) - G(p)(C'(q_A(p)) - C'(q))]}{(p - C'(q_A(p)))}(F(q + q_A(p)) - F(q))\right]dq}{G(p)\int_0^{q_A(p)}(p - C'(q))f(q + q_A(p))dq}\right\},$$

$$E(p) = \frac{\partial}{\partial G}\left\{\frac{\int_0^{q_A(p)}\left[1 - F(q) - \frac{[p - C'(q) - G(p)(C'(q_A(p)) - C'(q))]}{(p - C'(q_A(p)))}(F(q + q_A(p)) - F(q))\right]dq}{G(p)\int_0^{q_A(p)}(p - C'(q))f(q + q_A(p))dq}\right\}.$$

The calculations are simplified because $\bar{\varepsilon}/2 < q_A(p)$, so that $F(2q_A(p)) = 1$ and $f(2q_A(p)) = 0$. We also note that because of (76) the integrands are zero at the upper limit and we have no contributions from differentiating the integration limits. This gives

$$\begin{aligned}
D(p) = {} & \frac{\int_0^{q_A(p)}\frac{\partial}{\partial q_A}\left\{\frac{[p - C'(q) - G(p)(C'(q_A(p)) - C'(q))]}{(p - C'(q_A(p)))}(F(q + q_A(p)) - F(q))\right\}dq}{G(p)\int_0^{q_A(p)}(p - C'(q))f(q + q_A(p))dq} \\
& + \int_0^{q_A(p)}(p - C'(q))f'(q + q_A(p))dq \times \\
& \frac{\int_0^{q_A(p)}\left[1 - F(q) - \frac{[p - C'(q) - G(p)(C'(q_A(p)) - C'(q))]}{(p - C'(q_A(p)))}(F(q + q_A(p)) - F(q))\right]dq}{G(p)\left(\int_0^{q_A(p)}(p - C'(q))f(q + q_A(p))dq\right)^2} \\
= {} & \frac{\int_0^{q_A(p)}\frac{\partial}{\partial q_A}\left\{\frac{[p - C'(q) - G(p)(C'(q_A(p)) - C'(q))]}{(p - C'(q_A(p)))}(F(q + q_A(p)) - F(q))\right\}dq}{G(p)\int_0^{q_A(p)}(p - C'(q))f(q + q_A(p))dq} \\
& + \frac{q_A'(p)\int_0^{q_A(p)}(p - C'(q))f'(q + q_A(p))dq}{\int_0^{q_A(p)}(p - C'(q))f(q + q_A(p))dq}.
\end{aligned}$$

Observe that

$$\lim_{p\to\underline{p}}\frac{\partial}{\partial q_A}\left\{\frac{[p-C'(q)-G(p)(C'(q_A(p))-C'(q))]}{(p-C'(q_A(p)))}(F(q+q_A(p))-F(q))\right\}>0 \tag{81}$$

because

$$\frac{\partial}{\partial q_A}\left\{\frac{[p-C'(q)-G(p)(C'(q_A(p))-C'(q))]}{(p-C'(q_A(p)))}(F(q+q_A(p))-F(q))\right\} \tag{82}$$

$$\begin{aligned}
&=\frac{p-C'(q)}{p-C'(q_A(p))}f(q+q_A(p))\\
&-\frac{G(p)(C'(q_A(p))-C'(q))}{p-C'(q_A(p))}f(q+q_A(p))\\
&+C''(q_A(p))\frac{p-C'(q)}{(p-C'(q_A(p)))^2}(F(q+q_A(p))-F(q))\\
&-C''(q_A(p))\frac{G(p)(C'(q_A(p))-C'(q))}{(p-C'(q_A(p)))^2}(F(q+q_A(p))-F(q))\\
&-\frac{G(p)C''(q_A(p))}{(p-C'(q_A(p)))}(F(q+q_A(p))-F(q))\\
&\to\frac{\underline{p}-C'(q)}{\underline{p}-C'(q_A(\underline{p}))}f(q+q_A(\underline{p}))\\
&+C''(q_A(\underline{p}))\frac{\underline{p}-C'(q)}{(\underline{p}-C'(q_A(\underline{p})))^2}(F(q+q_A(\underline{p}))-F(q)) \qquad (83)\\
&>0
\end{aligned}$$

as $p\longrightarrow\underline{p}$ because $G(\underline{p})=0$. We can then expect the first term of $D(p)$ to become large and positive as $p\to\underline{p}$, while the second term remains bounded which gives $D(p)>0$ for p sufficiently close to $\underline{p}$.

Similarly using the definitions of $\Phi(q,p)$ and $\xi(q,p)$ introduced in Appendix 1

$$\begin{aligned}
E(p)&=\frac{\int_0^{q_A(p)}(C'(q_A(p))-C'(q))(F(q+q_A(p))-F(q))dq}{(p-C'(q_A(p)))G(p)\int_0^{q_A(p)}(p-C'(q))f(q+q_A(p))dq}\\
&-\frac{\int_0^{q_A(p)}(p-C'(q_A(p)))\Phi(q,p)dq}{(p-C'(q_A(p)))G^2(p)\int_0^{q_A(p)}(p-C'(q))f(q+q_A(p))dq}\\
&=-\frac{\int_0^{q_A(p)}\xi(q,p)dq}{G^2(p)(p-C'(q_A(p)))\int_0^{q_A(p)}(p-C'(q))f(q+q_A(p))dq}\\
&\geq 0
\end{aligned}$$

for $p \geq \underline{p}$ by virtue of Lemma 20. By means of l'Hôpital's rule, (74) and (78), it straightforward to show that

$$\underset{p\to\underline{p}}{lim}\frac{\int_0^{q_A(p)}\xi(q,p)dq}{G(p)}=\frac{\frac{\partial}{\partial p}\int_0^{q_A(p)}\xi(q,p)dq\Big|_{p=\underline{p}}}{g(\underline{p})}$$

is finite, so like $D(p)$, $E(p)$ is unbounded when $p \to \underline{p}$. From (83) we know that $\lim_{p\to\underline{p}} D(p)\, G(p) > 0$ and we have just shown that $\lim_{p\to\underline{p}} E(p)\, G(p)$ is finite. We can deduce from this that $\lim_{p\to\underline{p}} \frac{E(p)}{D(p)}$ is finite.

In summary, the system of differential equations for the perturbations $v(p)$ and $u(p)$ is

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} -D(p) & E(p) \\ A(p) & -B(p) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \tag{84}$$

where $D(p) > 0$ and $E(p) \geq 0$ are unbounded near the singularity at $\underline{p}$, while $A(p) \geq 0$ and $B(p) \geq 0$ are bounded. The following result proves that diagonal terms dominate the off-diagonal terms at $p = \underline{p}$.

Lemma 21 *$B(\underline{p}) - \frac{E(\underline{p})A(\underline{p})}{D(\underline{p})} > 0$ for hockey-stick mixtures satisfying $\bar{\varepsilon}/2 < q_A(\underline{p}) \leq q_m < \bar{\varepsilon}$.*

Proof. We write

$$E_{rel} = \lim_{p\to\underline{p}} \frac{E(p)}{D(p)}.$$

In the limit we can ignore the second term in the expression for $D(p)$ and hence

$$E_{rel} = \lim_{p\to\underline{p}} \frac{-\int_0^{q_A(p)} [(1-F(q))(p-C'(q_A)) - [p-C'(q)](F(q+q_A(p))-F(q))]\, dq}{(p-C'(q_A))G(p)\int_0^{q_A(p)} \frac{\partial}{\partial q_A}\left\{\left[G(p)+\frac{(p-C'(q))[1-G(p)]}{(p-C'(q_A(\underline{p})))}\right](F(q+q_A(p))-F(q))\right\} dq}.$$

Using the expressions for A, B and $\xi(q,p)$ in (79), (80) and (72) gives

$$B(\underline{p}) - A(\underline{p})\, E_{rel} = \lim_{p\to\underline{p}} \left\{ \frac{1}{(p-C'(q_A(p)))} + \right.$$

$$\left. \frac{(1-G(p))\, C''(q_A(p)) \int_0^{q_A(p)} \xi(q,p)\, dq}{(p-C'(q_A(\underline{p})))^3 G(p)\int_0^{q_A(p)} \frac{\partial}{\partial q_A}\left\{\left[G(p)+\frac{(p-C'(q))[1-G(p)]}{(p-C'(q_A(\underline{p})))}\right](F(q+q_A(p))-F(q))\right\} dq} \right\}.$$

After rearrangement, and using the fact that $\lim_{p\to\underline{p}}\left[\frac{G(p)}{1-G(p)}\right]=0$, we can establish that the inequality we want to prove is

$$\Psi\left(\underline{p}\right)=\left(\underline{p}-C'\left(q_A\left(\underline{p}\right)\right)\right)^2\int_0^{q_A\left(\underline{p}\right)}\frac{\partial}{\partial q_A}\left\{\frac{\left(\underline{p}-C'\left(q\right)\right)}{\underline{p}-C'\left(q_A\left(\underline{p}\right)\right)}\left(F(q+q_A\left(p\right))-F(q)\right)\right\}dq \tag{85}$$
$$+\,C''\left(q_A\left(\underline{p}\right)\right)\lim_{p\to\underline{p}}\frac{\int_0^{q_A(p)}\xi\left(q,p\right)dq}{G(p)}$$
$$>0.$$

Now

$$\frac{\partial}{\partial q_A}\left\{\frac{\left(\underline{p}-C'\left(q\right)\right)}{\left(\underline{p}-C'\left(q_A\left(\underline{p}\right)\right)\right)}\left(F(q+q_A\left(p\right))-F(q)\right)\right\} \tag{86}$$
$$=\frac{C''\left(q_A\left(\underline{p}\right)\right)\left(\underline{p}-C'\left(q\right)\right)}{\left(\underline{p}-C'\left(q_A\left(\underline{p}\right)\right)\right)^2}\left(F(q+q_A\left(\underline{p}\right))-F(q)\right)+\frac{\left(\underline{p}-C'\left(q\right)\right)}{\left(\underline{p}-C'\left(q_A\left(\underline{p}\right)\right)\right)}f(q+q_A\left(\underline{p}\right)).$$

From (74) we see that the limit in the second term of $\Psi\left(\underline{p}\right)$ is of the type $\frac{0}{0}$ and it can be calculated using l'Hôpital's rule. Thus we have from $\int_0^{q_A\left(\underline{p}\right)}\xi(q,\underline{p})dq=0$ and (78) that

$$\lim_{p\to\underline{p}}\frac{\partial}{\partial p}\int_0^{q_A(p)}\xi(q,p)dq$$
$$=\int_0^{q_A\left(\underline{p}\right)}\left[1-F(q+q_A\left(\underline{p}\right))\right]dq$$
$$-\,q_A'(\underline{p})\int_0^{q_A\left(\underline{p}\right)}\left[\left(1-F(q)\right)C''\left(q_A(\underline{p})\right)+\left(\underline{p}-C'\left(q\right)\right)f\left(q+q_A\left(\underline{p}\right)\right)\right]dq.$$

Using l'Hôpital's rule, and combining the result above with the expressions for

$q'_A(\underline{p})$ and $g(p)$ in Proposition 14 we now get

$$\lim_{p\to\underline{p}}\frac{\int_0^{q_A(p)}\xi(q,p)\,dq}{G(p)}=\frac{\lim_{p\to\underline{p}}\frac{\partial}{\partial p}\int_0^{q_A(p)}\xi(q,p)dq}{g(\underline{p})}$$

$$=\left(\underline{p}-C'\left(q_A\left(\underline{p}\right)\right)\right)\int_0^{q_A\left(\underline{p}\right)}\left[1-F(q+q_A\left(\underline{p}\right))\right]dq$$

$$-\frac{2\int_0^{q_A\left(\underline{p}\right)}\left[C'\left(q_A\left(\underline{p}\right)\right)-C'(q)\right](F(q+q_A\left(\underline{p}\right))-F(q))dq}{\int_0^{q_A\left(\underline{p}\right)}\left[2(\underline{p}-C'(q))f(q+q_A\left(\underline{p}\right))+(1-F(q))\,C''\left(q_A\left(\underline{p}\right)\right)\right]dq}$$

$$\times\int_0^{q_A\left(\underline{p}\right)}\left[(1-F(q))\,C''\left(q_A(\underline{p})\right)+(\underline{p}-C'(q))f\left(q+q_A\left(\underline{p}\right)\right)\right]dq$$

$$\geq\left(\underline{p}-C'\left(q_A\left(\underline{p}\right)\right)\right)\int_0^{q_A\left(\underline{p}\right)}\left[1-F(q+q_A\left(\underline{p}\right))\right]dq$$

$$-2\int_0^{q_A\left(\underline{p}\right)}\left[C'\left(q_A\left(\underline{p}\right)\right)-C'(q)\right](F(q+q_A\left(\underline{p}\right))-F(q))dq.$$

Combining the result above with (85) and (86) yields:

$$\Psi\left(\underline{p}\right)\geq\int_0^{q_A\left(\underline{p}\right)}(\underline{p}-C'(q))(\underline{p}-C'\left(q_A\left(\underline{p}\right)\right))f(q+q_A\left(\underline{p}\right))dq$$

$$+C''\left(q_A\left(\underline{p}\right)\right)\int_0^{q_A\left(\underline{p}\right)}(\underline{p}-C'(q))(F(q+q_A\left(\underline{p}\right))-F(q))dq$$

$$+C''\left(q_A\left(\underline{p}\right)\right)\int_0^{q_A\left(\underline{p}\right)}\left(\underline{p}-C'\left(q_A\left(\underline{p}\right)\right)\right)\left[1-F(q+q_A\left(\underline{p}\right))\right]dq$$

$$-2C''\left(q_A\left(\underline{p}\right)\right)\int_0^{q_A\left(\underline{p}\right)}\left[C'\left(q_A\left(\underline{p}\right)\right)-C'(q)\right](F(q+q_A\left(\underline{p}\right))-F(q))dq.$$

Dropping the first term and rearranging the others we have

$$\Psi\left(\underline{p}\right)>C''\left(q_A\left(\underline{p}\right)\right)\int_0^{q_A\left(\underline{p}\right)}(\underline{p}-C'\left(q_A\left(\underline{p}\right)\right))(1-F(q))dq$$

$$-C''\left(q_A\left(\underline{p}\right)\right)\int_0^{q_A\left(\underline{p}\right)}\left[C'\left(q_A\left(\underline{p}\right)\right)-C'(q)\right](F(q+q_A\left(\underline{p}\right))-F(q))dq.$$

Now using (64) and that $\underline{p} > C'\left(q_A\left(\underline{p}\right)\right)$ we get

$$\begin{aligned}\Psi\left(\underline{p}\right) &> C''\left(q_A\left(\underline{p}\right)\right)\int_0^{q_A\left(\underline{p}\right)}\left[\underline{p} - C'(q)\right](F(q + q_A\left(\underline{p}\right)) - F(q))dq \\ &\quad - C''\left(q_A\left(\underline{p}\right)\right)\int_0^{q_A\left(\underline{p}\right)}\left[C'\left(q_A\left(\underline{p}\right)\right) - C'(q)\right](F(q + q_A\left(\underline{p}\right)) - F(q))dq \\ &> 0,\end{aligned}$$

which implies that $B\left(\underline{p}\right) - A\left(\underline{p}\right)E_{rel} > 0$. ■

Now, consider a price interval $\left(\underline{p}, p_0\right]$. Our proof will show that for sufficiently small $p_0 > \underline{p}$, the only solution to (84) over $\left(\underline{p}, p_0\right]$ is $u = 0$, $v = 0$. This will establish that the set of differential equations defined in Proposition 14 has at most one solution

Let $\underline{D}$ be the smallest $D(p)$ in the interval $\left(\underline{p}, p_0\right]$. Similarly, $\overline{E}_{rel}$ and $\underline{E}_{rel}$ are the largest and smallest values of $E(p)/D(p)$, respectively, in the interval. We let $\overline{A}$ and $\overline{B}$ be the largest values of $A(p)$ and $B(p)$, respectively, and $\underline{B}$ is the smallest $B(p)$. For small enough p_0 all these bounds, which are positive, are close to their values at $\underline{p}$ and we can use continuity to show that $\underline{B} - \overline{E}_{rel}\overline{A} > 0$ from Lemma 21.

We divide the uv-plane into four regions as follows. In region 1 we have $u > \overline{E}_{rel}v$ and $u > \underline{E}_{rel}v$, so $u > E_{rel}v$. In region 2 we have $\underline{E}_{rel}v < u < \overline{E}_{rel}v$, which implies that $v \geq 0$. In region 3 we have $u < \overline{E}_{rel}v$ and $u < \underline{E}_{rel}v$, so $u < E_{rel}v$. In region 4 we have $\overline{E}_{rel}v < u < \underline{E}_{rel}v$, which implies that $v \leq 0$. The four regions are illustrated in Figure 5.

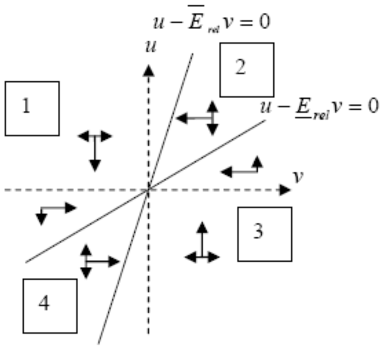

Figure 5: The four regions referred to in the proof of Lemma 22. The directions of the arrows show the changes in u and v as p increases in each region.

Lemma 22 *If* $\underline{B} - \overline{E}_{rel}\overline{A} > 0$, *then we can make the following claims:*
i) Either $u'u < 0$ *or* $v'v < 0$.
ii) $u' < 0$ *in region 1.*
iii) $v' < 0$ *in region 2.*
iv) $u' > 0$ *in region 3.*
v) $v' > 0$ *in region 4.*

Proof. First we note that the assumption $\underline{B} - \overline{E}_{rel}\overline{A} > 0$ implies that $\frac{\underline{B}}{\overline{A}} > \overline{E}_{rel}$.
Claim i). Make the contradictory assumption that $u'u \geq 0$ and $v'v \geq 0$. Thus it follows from (84) that $-u^2 + E_{rel}uv \geq 0$ and $Auv - Bv^2 \geq 0$, respectively. To satisfy both inequalities we need $uv \geq 0$. Thus together the two inequalities imply $\frac{B}{A} \leq \frac{u}{v} \leq E_{rel}$, which would violate the assumption that $\underline{B} - \overline{E}_{rel}\overline{A} > 0$.
Claim ii) The first equation in (84) implies that $u' = -Du+Ev = -D\left(u - E_{rel}v\right) < 0$, because $u > E_{rel}v$ in region 1.
Claim iii) Make the contradictory assumption that $v' \geq 0$ in region 2, so that $Au - Bv \geq 0$. In this region we have $u < \overline{E}_{rel}v$ and $v \geq 0$. Thus $A\overline{E}_{rel}v - Bv \geq 0$, so $A\overline{E}_{rel} - B \geq 0$. But this would contradict our assumption that $\underline{B} - \overline{E}_{rel}\overline{A} > 0$.
Claim iv) $u\prime = -Du + Ev = -D\left(u - E_{rel}v\right) > 0$, because $u < E_{rel}v$ in region 3.
Claim v) Make the contradictory assumption that $v' \leq 0$ in region 4, so that $Au - Bv \leq 0$. In this region we have $u > \overline{E}_{rel}v$ and $v \leq 0$. Thus $A\overline{E}_{rel}v - Bv \leq 0$, so $A\overline{E}_{rel} - B \geq 0$, because $v \leq 0$. But this would contradict our assumption that $\underline{B} - \overline{E}_{rel}\overline{A} > 0$.
■

Claim i) of the Lemma implies that whenever $|u|$ is increasing at some p then $|v|$ must be decreasing at that point. We now consider the interval $\left(\underline{p}, p_0\right]$, where p_0 is chosen sufficiently close to $\underline{p}$, so that we can find a finite $k > \overline{E}_{rel}$, such that

$$\underline{D}\left(k - \overline{E}_{rel}\right) > k\left(k\overline{A} + \overline{B}\right) \geq 0. \quad (87)$$

Such a p_0 can always be found, because D is unbounded at $\underline{p}$. We proceed to show that the initial value problem $q_A\left(\underline{p}\right) = q_0$ and $G\left(\underline{p}\right) = 0$ of hockey-stick mixtures has at most one continuous solution over the interval $\left(\underline{p}, p_0\right]$.

Lemma 23 *Suppose that there are at least two different solutions to (84) with the initial condition* $u(\underline{p}) = v(\underline{p}) = 0$ *over the price interval* $\left(\underline{p}, p_0\right]$. *Then for every* $p \in \left(\underline{p}, p_0\right]$ *either* $u(p) \geq 0$, $v(p) > 0$ *or* $u(p) \leq 0$, $v(p) < 0$.

Proof. One obvious solution is $u(p) = v(p) = 0$ for all $p \in \left(\underline{p}, p_0\right]$, but suppose that there is at least one additional solution over the price interval $\left(\underline{p}, p_0\right]$.

The system of differential equations is Lipschitz continuous at every price in this interval, and so any two solutions must differ at this price, otherwise they would be identical throughout the interval $\left(\underline{p}, p_0\right]$. This means that there is no $p^* \in \left(\underline{p}, p_0\right]$ for which $u(p^*) = v(p^*) = 0$.

First we show that there is no $p^* \in \left(\underline{p}, p_0\right]$, for which $|u(p^*)| > 0$ and $v(p^*) = 0$, by proving a stronger result, namely that there is no $p^* \in \left(\underline{p}, p_0\right]$, with $|u(p^*)| > k|v(p^*)|$. Suppose that such a p^* exists. Then it follows from (87) that

$$\begin{aligned} D|u(p^*)| - Ev(p^*) &\geq D|u(p^*)| - E|v(p^*)| > D\left(k - \overline{E}_{rel}\right)|v(p^*)| \\ &\geq \underline{D}\left(k - \overline{E}_{rel}\right)|v(p^*)| > 0. \end{aligned}$$

Hence (84) implies that

$$-u'(p^*)\,u(p^*) = D|u(p^*)|^2 - Ev(p^*)\,u(p^*) > 0,$$

which shows that $u^2(p)$ is strictly decreasing at p^*. Since this deduction follows for any p^* with $|u(p^*)| > k|v(p^*)|$ we cannot have $|u(p)| > k|v(p)|$ near $\underline{p}$ for this would contradict the fact that $u^2(\underline{p}) = 0$. So, by continuity, there is some $p_X \in \left(\underline{p}, p^*\right]$ with $|u(p_X)| = k|v(p_X)| > 0$. We take the smallest such p_X. Now (84) implies that

$$\begin{aligned} |u'(p_X)| &= |Du(p_X) - Ev(p_X)| \\ &\geq D|u(p_X)| - E|v(p_X)| \\ &= D\left(|u(p_X)| - E_{rel}|v(p_X)|\right) \\ &\geq \underline{D}\left(|u(p_X)| - \overline{E}_{rel}|v(p_X)|\right) \\ &= \underline{D}\left(k - \overline{E}_{rel}\right)|v(p_X)| \\ &> k\left(k\overline{A} + \overline{B}\right)|v(p_X)| \\ &= k\left(\overline{A}|u(p_X)| + \overline{B}|v(p_X)|\right) \\ &\geq k|v'(p_X)|. \end{aligned}$$

As $|u(p)| > k|v(p)|$ on some interval $(p_X, p_Y]$, $|u(p)|$ is strictly decreasing on this interval, and so we have $\frac{d}{dp}|u(p)| \leq 0$ at $p = p_X$ and

$$\begin{aligned} \frac{d}{dp}|u(p_X)| &= -|u'(p_X)| < -k|v'(p_X)| \\ &\leq k\frac{d}{dp}|v(p_X)|. \end{aligned}$$

Thus $|u(p)| - k|v(p)|$ is decreasing at p_X which contradicts $|u(p_X)| = k|v(p_X)|$ and $|u(p)| > k|v(p)|$, $p \in (p_X, p_Y]$.

It remains to show that there is no $p^* \in \left(\underline{p}, p_0\right]$, for which $u\left(p^*\right) v\left(p^*\right) < 0$. If this inequality holds then, from (84),

$$u(p^*)u'(p^*) = -Du\left(p^*\right)^2 + Ev\left(p^*\right)u\left(p^*\right) \\ < 0$$

so $|u|$ is strictly decreasing at p^*. So we cannot have $u\left(p^*\right) v\left(p^*\right) < 0$ near $\underline{p}$ for this would contradict the fact that $u(\underline{p}) = 0$. Thus, by continuity, there is some $p_X \in \left(\underline{p}, p^*\right]$ with $u\left(p_X\right) v\left(p_X\right) = 0$, and we take the smallest such p_X. Since, as argued above, we cannot have $u\left(p_X\right) = v\left(p_X\right) = 0$ nor $v\left(p_X\right) = 0$, this establishes that $u\left(p_X\right) = 0$ which contradicts $|u|$ strictly decreasing for p just above p_X. ■

Proposition 24 *The initial value problem* $q_A\left(\underline{p}\right) = q_0$ *and* $G\left(\underline{p}\right) = 0$ *of hockey-stick mixtures has at most one continuous solution.*

Proof. Suppose that there are at least two different solutions over the price interval $\left(\underline{p}, p_0\right]$. From Lemma 23 we know that $(u(p), v(p))$ is either in the positive or negative orthant and cannot move between them. We demonstrate the result when $u(p) \geq 0$ and $v(p) > 0$. In this case from Lemma 22 when $u(p) > \overline{E}_{rel}v(p)$ (in region 1) $u' < 0$ and when $\overline{E}_{rel}v(p) > u(p)$ (in region 2 or 3) $v'(p) < 0$. When $u(p) = \overline{E}_{rel}v(p)$ we have $u'(p) \leq 0$ and $v'(p) \leq 0$. Thus $\max(u(p), \overline{E}_{rel}v(p))$ is non-increasing throughout $\left(\underline{p}, p_0\right]$ and, since this is zero at $\underline{p}$, we have established that $u(p)$ and $v(p)$ are identically zero throughout $\left(\underline{p}, p_0\right]$. The argument when $(u(p), v(p))$ is in the negative orthant is exactly similar with $\min(u(p), \underline{E}_{rel}v(p))$ increasing throughout $\left(\underline{p}, p_0\right]$. ■

9 Appendix 3: P is strictly increasing with $\underline{\mathbf{p}}$

We now consider a family of hockey-stick solutions with different lowest prices $\underline{p}$. We define $G(p, \underline{p})$ to be the mixing distribution and $q_A\left(p, \underline{p}\right)$ to be the curved part of the hockey stick curve corresponding to the solution with lowest price $\underline{p}$, and thus we write $q_A(\underline{p}, \underline{p})$ for $q_A(\underline{p})$. We proceed to show that at any fixed $p > \underline{p}$, $q_A\left(p, \underline{p}\right)$ and $G\left(p, \underline{p}\right)$ are decreasing in $\underline{p}$. Throughout the Section we consider hockey-stick mixtures satisfying $\underline{p} > C'\left(q_A(\underline{p})\right)$.

The first result establishes that the curve describing the possible initial points, $q_A(\underline{p}, \underline{p})$, increases in $\underline{p}$ more slowly than the individual $q_A(p, \underline{p})$ curves at their starting points.

Lemma 25 $\frac{d}{d\underline{p}} q_A(\underline{p}, \underline{p}) < \left.\frac{\partial}{\partial p} q_A(p, \underline{p})\right|_{p=\underline{p}}$

Proof. We write $\zeta(\underline{p})$ for $\frac{d}{d\underline{p}}q_A(\underline{p},\underline{p})$. From (53)

$$\underline{p}\int_0^{q_A(\underline{p},\underline{p})}(1-F(q+q_A(\underline{p},\underline{p})))dq$$
$$=\int_0^{q_A(\underline{p},\underline{p})}\left[C'\left(q_A\left(\underline{p},\underline{p}\right)\right)(1-F(q))-C'(q)(F(q+q_A\left(\underline{p},\underline{p}\right))-F(q))\right]dq.$$

Taking derivatives with respect to $\underline{p}$ gives

$$\underline{p}\int_0^{q_A(\underline{p},\underline{p})}-f(q+q_A\left(\underline{p},\underline{p}\right))\zeta(\underline{p})dq+\int_0^{q_A(\underline{p},\underline{p})}(1-F(q+q_A\left(\underline{p},\underline{p}\right)))dq$$
$$=\int_0^{q_A(\underline{p},\underline{p})}\left[C''\left(q_A\left(\underline{p},\underline{p}\right)\right)\zeta(\underline{p})(1-F(q))-C'(q)f(q+q_A\left(\underline{p},\underline{p}\right)\zeta(\underline{p})\right]dq$$

since the integrands are zero at the upper limit when $\overline{\varepsilon}/2<q_A\left(p\right)$. Thus

$$\zeta(\underline{p})=\frac{\int_0^{q_A(\underline{p},\underline{p})}(1-F(q+q_A\left(\underline{p},\underline{p}\right)))dq}{\int_0^{q_A(\underline{p},\underline{p})}\left[C''\left(q_A\left(\underline{p},\underline{p}\right)\right)(1-F(q))+(\underline{p}-C'(q))f(q+q_A\left(\underline{p},\underline{p}\right)\right]dq}.$$

We want to compare this with $\left.\frac{\partial}{\partial p}q_A(p,\underline{p})\right|_{p=\underline{p}}$ given by (55), but first we make some derivations which are useful for this comparison. We have the following relationship from (64)

$$\left(\underline{p}-C'\left(q_A\left(\underline{p},\underline{p}\right)\right)\right)\int_0^{q_A(\underline{p},\underline{p})}(1-F(q))dq=\int_0^{q_A(\underline{p},\underline{p})}(\underline{p}-C'\left(q\right))(F(q+q_A\left(\underline{p},\underline{p}\right))-F(q))dq.$$

So

$$\int_0^{q_A(\underline{p},\underline{p})}\left[C'\left(q_A\left(\underline{p},\underline{p}\right)\right)-C'\left(q\right)\right](F(q+q_A\left(\underline{p},\underline{p}\right))-F(q))dq$$
$$=\int_0^{q_A(\underline{p},\underline{p})}(\underline{p}-C'\left(q\right))(F(q+q_A\left(\underline{p},\underline{p}\right))-F(q))dq$$
$$-\left(\underline{p}-C'\left(q_A\left(\underline{p},\underline{p}\right)\right)\right)\int_0^{q_A(\underline{p},\underline{p})}(F(q+q_A\left(\underline{p},\underline{p}\right))-F(q))dq$$
$$=\left(\underline{p}-C'\left(q_A\left(\underline{p},\underline{p}\right)\right)\right)\int_0^{q_A(\underline{p},\underline{p})}(1-F(q+q_A\left(\underline{p}\right)))dq$$

Thus (55) can be simplified to

$$\left.\frac{\partial}{\partial p} q_A(p, \underline{p})\right|_{p=\underline{p}} = \frac{\int_0^{q_A(\underline{p},\underline{p})} \left(1 - F(q + q_A\left(\underline{p}, \underline{p}\right))\right) dq}{\int_0^{q_A(\underline{p},\underline{p})} \left[(\underline{p} - C'(q)) f(q + q_A\left(\underline{p}, \underline{p}\right)) + (1/2)\left(1 - F(q)\right) C''\left(q_A\left(\underline{p}, \underline{p}\right)\right)\right] dq}$$

$$> \frac{\int_0^{q_A(\underline{p},\underline{p})} (1 - F(q + q_A\left(\underline{p}, \underline{p}\right)))dq}{\int_0^{q_A(\underline{p},\underline{p})} \left[(\underline{p} - C'(q)) f(q + q_A\left(\underline{p}, \underline{p}\right) + (1 - F(q)) C''\left(q_A\left(\underline{p}, \underline{p}\right)\right)\right] dq} = \zeta(\underline{p}).$$

■

The next two lemmas are related. Roughly speaking they show that as we vary the starting price $\underline{p}$ we have monotonicity in G and in q_A provided we also have monotonicity in the other.

Lemma 26 *Suppose $\underline{p}_1 < \underline{p}_2$. If $q_A(p, \underline{p}_2) \leq q_A(p, \underline{p}_1)$ throughout the interval $p \in (\underline{p}_2, p_0)$ and $p - C'\left(q_A(p, \underline{p}_1)\right) > 0$ for $p \in [\underline{p}_2, p_0]$, then $G\left(p, \underline{p}_1\right) > G\left(p, \underline{p}_2\right)$ for $p \in [\underline{p}_2, p_0]$.*

Proof. We write $G_\Delta(p) = G\left(p, \underline{p}_2\right) - G\left(p, \underline{p}_1\right)$. Observe that $G_\Delta(\underline{p}_2) = -G\left(\underline{p}_2, \underline{p}_1\right) < 0$. Now since C' is increasing, we have from (51) and the inequality on q_A

$$\begin{aligned}
G'_\Delta(p) &= g\left(p, \underline{p}_2\right) - g\left(p, \underline{p}_1\right) \\
&= \frac{1 - G(p, \underline{p}_2)}{(p - C'\left(q_A(p, \underline{p}_2)\right))} - \frac{1 - G(p, \underline{p}_1)}{(p - C'\left(q_A(p, \underline{p}_1)\right))} \\
&\leq -\frac{1}{(p - C'\left(q_A(p, \underline{p}_1)\right))} G_\Delta(p),
\end{aligned}$$

for $p \in (\underline{p}_2, p_0)$. Using Gronwall's lemma we obtain

$$G_\Delta(p) \leq G_\Delta(\underline{p}_2) \exp\left(\int_{\underline{p}_2}^{p} -\frac{1}{(s - C'\left(q_A(s, \underline{p}_1)\right))} ds\right),$$

for $p \in [\underline{p}_2, p_0]$. Since $p - C'\left(q_A(p, \underline{p}_1)\right) > 0$ throughout the closed interval $[\underline{p}_2, p_0]$ and it is continuous, it is also bounded below by some constant. Thus the integral is bounded and hence $G_\Delta(p) < 0$ and the result is established. ■

Lemma 27 *Suppose* $\underline{p}_1 < \underline{p}_2$. *If* $G\left(p, \underline{p}_1\right) > G\left(p, \underline{p}_2\right)$ *throughout the interval* $p \in [\underline{p}_2, p_0)$, *then* $q_A(p, \underline{p}_1) > q_A(p, \underline{p}_2)$ *for* $p \in [\underline{p}_2, p_0)$.

Proof. Suppose that for a given $\underline{p}_1$ the claim of the lemma is not true for some pair $(p, \underline{p}_2)$ then we fix $\underline{p}_2$ and choose the lowest $p \in [\underline{p}_2, p_0)$ for which $q_A(p, \underline{p}_2) \geq q_A(p, \underline{p}_1)$. First consider the case when the lowest p is larger than $\underline{p}_2$. Since curves $q_A(p, \underline{p})$ are continuous with respect to p this occurs when $q_A(p, \underline{p}_2) = q_A(p, \underline{p}_1)$. We suppose this occurs at p_Z with $q_A(p_Z, \underline{p}_2) = q_A(p_Z, \underline{p}_1) = T$. Then from (52)

$$\begin{aligned}
\left[\frac{\partial}{\partial p} q_A\left(p, \underline{p}_i\right)\right]_{p_Z} &= \frac{\int_0^T 1 - F(q) - \frac{\left[p_Z - C'(q) - G(p_Z, \underline{p}_i)(C'(T) - C'(q))\right]}{(p_Z - C'(T))}(F(q+T) - F(q))dq}{G(p_Z, \underline{p}_i)\int_0^T (p_Z - C'(q))f(q+T)dq} \\
&= \frac{\int_0^T 1 - F(q) - \frac{(p_Z - C'(q))}{(p_Z - C'(T))}(F(q+T) - F(q))dq}{G(p_Z, \underline{p}_i)\int_0^T (p_Z - C'(q))f(q+T)dq} \\
&\quad + \frac{\int_0^T \frac{(C'(T) - C'(q))}{(p_Z - C'(T))}(F(q+T) - F(q))dq}{\int_0^T (p_Z - C'(q))f(q+T)dq} \qquad (88)
\end{aligned}$$

for $i = 1, 2$. We know from Lemma 20 that

$$\int_0^T 1 - F(q) - \frac{(p_Z - C'(q))}{(p_Z - C'(T))}(F(q+T) - F(q))dq < 0,$$

because $p_Z > \underline{p}_2 > C'\left(q_A(\underline{p}_2)\right)$ so $G\left(p_Z, \underline{p}_1\right) > G\left(p_Z, \underline{p}_2\right)$ implies

$$\left[\frac{\partial}{\partial p} q_A\left(p, \underline{p}_1\right)\right]_{p_Z} > \left[\frac{\partial}{\partial p} q_A\left(p, \underline{p}_2\right)\right]_{p_Z}.$$

This gives a contradiction, since this implies that $q_A(p, \underline{p}_2) > q_A(p, \underline{p}_1)$ for p approaching p_Z from below.

Now we consider the case that $q_A(\underline{p}_2, \underline{p}_2) = q_A(\underline{p}_2, \underline{p}_1)$ (corresponding to $p_z = \underline{p}_2$) and we take the lowest value $\underline{p}_2$ for which this is true. From Lemma 25 we know that $q_A(p, \underline{p}_1) > q_A(p, p)$ for p sufficiently close to $\underline{p}_1$. Since $\underline{p}_2$ is the lowest value at which this inequality fails to hold, we have $q_A(p, \underline{p}_1) > q_A(p, p)$ as p approaches $\underline{p}_2$ from below. Hence for small $\delta > 0$ we have $q_A(\underline{p}_2 - \delta, \underline{p}_1) > q_A(\underline{p}_2 - \delta, \underline{p}_2 - \delta)$ and from Lemma 25 applied at $\underline{p}_2 - \delta$ we also have $q_A(\underline{p}_2, \underline{p}_2 - \delta) > q_A(\underline{p}_2, \underline{p}_2) = q_A(\underline{p}_2, \underline{p}_1)$. Thus the curve $q_A(p, \underline{p}_2 - \delta)$ must cross the curve $q_A(p, \underline{p}_1)$ at some price $p \in (\underline{p}_2 - \delta, \underline{p}_2)$. But this case has already been ruled out by our discussion above. ∎

Next we establish that the monotonicity results we want apply throughout the curved part of the hockey stick bids.

Lemma 28 *For fixed p, both $G(p, \underline{p})$ and $q_A(p, \underline{p})$ are continuous decreasing functions of $\underline{p}$ over the range where $q_A(p, \underline{p}) \leq q_m$.*

Proof. We begin by establishing that the functions are decreasing. Take $\underline{p}_2 > \underline{p}_1$. Since $G(p, \underline{p})$ and $q_A(p, \underline{p})$ are only defined for $p > \underline{p}$, it is enough to show that $G(p, \underline{p}_1) > G(p, \underline{p}_2)$ and $q_A(p, \underline{p}_1) > q_A(p, \underline{p}_2)$ for $p > \underline{p}_2$. Since $g(p, \underline{p}_1) > 0$ for $p > \underline{p}_1$ and $G(\underline{p}_2, \underline{p}_2) = 0$ we have $G(p, \underline{p}_1) > G(p, \underline{p}_2)$ for $p \in [\underline{p}_2, \underline{p}_2 + \delta)$ for $\delta > 0$ and small enough. Thus from Lemma 27 $q_A(p, \underline{p}_1) > q_A(p, \underline{p}_2)$ for $p \in [\underline{p}_2, \underline{p}_2 + \delta)$. We can now use Lemmas 26 and 27 to show that $G(p, \underline{p}_1) > G(p, \underline{p}_2)$ and $q_A(p, \underline{p}_1) > q_A(p, \underline{p}_2)$ for any p (provided $q_A(p, \underline{p}_1) < q_m$). Let p_W be the first value of p for which one of the inequalities fails; then from Lemma 26 we have $G(p, \underline{p}_1) > G(p, \underline{p}_2)$ for $p \in [\underline{p}_2, p_W]$. Thus we may choose an $\varepsilon > 0$ with $G(p, \underline{p}_1) > G(p, \underline{p}_2)$ for $p \in [\underline{p}_2, p_W + \varepsilon)$. But we can deduce from Lemma 27 that $q_A(p_W, \underline{p}_1) > q_A(p_W, \underline{p}_2)$, which contradicts the definition of p_W. Hence the required inequalities hold throughout the range.

Now suppose that $q_A(p, \underline{p})$ is not continuous in $\underline{p}$ for some $p = p_* > \underline{p}$. Then we have some $\underline{p}_0$ and $\delta_n \to 0$ with $q_A(p_*, \underline{p}_0)$ not equal to $\lim_{n\to\infty} q_A(p_*, \underline{p}_0 + \delta_n)$. We may choose the sequence $\{\delta_n\}$ to be either increasing to zero or decreasing to zero. We suppose the latter (the argument in the other case is similar). By the monotonicity result we have just established we have $q_A(p, \underline{p}_0 + \delta_n)$ and $G(p, \underline{p}_0 + \delta_n)$ are increasing sequences for all $p > \underline{p}_0$, which therefore have limits which we call $q_A^*(p, \underline{p}_0)$ and $G^*(p, \underline{p}_0)$ with $q_A(p_*, \underline{p}_0) \neq q_A^*(p_*, \underline{p}_0)$. Now observe that for each n the functions $(\partial/\partial p) q_A(p, \underline{p}_0 + \delta_n)$ and $g(p, \underline{p}_0 + \delta_n)$ satisfy the differential equations (52) and (51) and will be bounded. Moreover the derivatives $(\partial^2/\partial p^2) q_A(p, \underline{p}_0 + \delta_n)$ and $(\partial/\partial p) g(p, \underline{p}_0 + \delta_n)$ can also be obtained from these equations. It is not hard to see that these second derivatives will also be bounded. Choose K independent of n so that $\left|(\partial/\partial p) g(p, \underline{p}_0 + \delta_n)\right| < K$. Applying the mean value theorem twice we have

$$
\begin{aligned}
G(p+\delta, \underline{p}_0 + \delta_n) &= G(p, \underline{p}_0 + \delta_n) + g(p + x, \underline{p}_0 + \delta_n)\delta \\
&= G(p, \underline{p}_0 + \delta_n) + \left(g(p, \underline{p}_0 + \delta_n) + x \left[(\partial/\partial p) g(p, \underline{p}_0 + \delta_n) \right]_{p+y} \right) \delta
\end{aligned}
$$

for some $y < x < \delta$ Thus

$$G(p+\delta, \underline{p}_0 + \delta_n) - G(p, \underline{p}_0 + \delta_n) - \delta g(p, \underline{p}_0 + \delta_n) \in (-K\delta^2, K\delta^2).$$

We can choose a subsequence in which $g(p, \underline{p}_0 + \delta_n)$ approaches a limit which we write as $g^*(p, \underline{p}_0)$. Then taking limits in this subsequence we have

$$G^*(p+\delta, \underline{p}_0) - G^*(p, \underline{p}_0) - \delta g^*(p, \underline{p}_0) \in [-K\delta^2, K\delta^2].$$

Thus G^* has a derivative at p which is equal to g^*. Moreover since each $g(p, \underline{p}_0+\delta_n)$ satisfies (51) this will also be satisfied by g^* in the limit. The same argument can also be used for $q_A^*(p, \underline{p}_0)$ and so we have shown that $q_A^*(p, \underline{p}_0)$ and $G^*(p, \underline{p}_0)$. satisfy the differential equations (51) and (52). So $q_A^*(p, \underline{p}_0)$ and $G^*(p, \underline{p}_0)$ is another solution to these differential equations starting at $\underline{p}_0$, but having $q_A(p_*, \underline{p}_0) \neq q_A^*(p_*, \underline{p}_0)$ which contradicts the uniqueness result of Appendix 2. Thus we have established continuity. ■

The next lemma establishes that the derivative $g(p, \underline{p})$ of $G(p, \underline{p})$ has a positive jump at the price p_m satisfying $q_A(p_m) = q_m$. Observe that the definition of g changes at p_m, since the mixtures at higher prices are over horizontal bids rather than hockey-stick bids. Thus we write $g(p_m+, \underline{p})$ to represent the value that satisfies

$$\begin{aligned} 0 &= \int_0^{q_m} Z(q, p_m+)\, dq \\ &= \int_0^{q_m} \left[1 - F(q) - \left[G(p_m, \underline{p}) + (p_m - C'(q)) g(p_m+, \underline{p})\right] (F(q+q_m) - F(q))\right] dq, \end{aligned} \tag{89}$$

which follows from (44), and $g(p_m, \underline{p})$ to represent the value that satisfies

$$\begin{aligned} 0 &= \int_0^{q_m} Z(q, p_m)\, dq \\ &= \int_0^{q_m} \left[1 - F(q) - \left[G(p_m, \underline{p}) + (p_m - C'(q)) g(p_m, \underline{p})\right] (F(q+q_m) - F(q))\right] dq \\ &- \int_0^{q_m} (p_m - C'(q)) q_A'\left(p_m, \underline{p}\right) f(q+q_m) G\left(p_m, \underline{p}\right) dq, \end{aligned} \tag{90}$$

which follows from (59).

Lemma 29 *At the price p_m satisfying $q_A(p_m) = q_m$, $g(p_m+, \underline{p}) > g(p_m, \underline{p})$*

Proof. Observe that

$$\int_0^{q_m} (p_m - C'(q)) q_A'\left(p_m, \underline{p}\right) f(q+q_m) G\left(p_m, \underline{p}\right) dq > 0$$

by Lemma 18. Thus combining (89) and (90) gives

$$\begin{aligned} &\int_0^{q_m} \left[1 - F(q) - \left[G(p_m, \underline{p}) + (p_m - C'(q)) g(p_m, \underline{p})\right] (F(q+q_m) - F(q))\right] dq \\ &> \int_0^{q_m} \left[1 - F(q) - \left[G(p_m, \underline{p}) + (p_m - C'(q)) g(p_m+, \underline{p})\right] (F(q+q_m) - F(q))\right] dq \end{aligned}$$

yielding

$$g(p_m, \underline{p}) \int_0^{q_m} (p_m - C'(q))(F(q + q_m) - F(q))dq$$
$$< g(p_m+, \underline{p}) \int_0^{q_m} (p_m - C'(q))(F(q + q_m) - F(q))dq$$

which gives the result. ■

Theorem 30 *Hockey stick mixtures with a strictly higher* $\underline{p}$ *will have a strictly higher* $\overline{p}$ *if* $\overline{\varepsilon}/2 < q_A(\underline{p}) \le q_m < \overline{\varepsilon}$ *and* $p - C'(q_A(p)) > 0$.

Proof. We start by showing that $G(p, \underline{p})$ is decreasing in $\underline{p}$ for fixed $p = p^*$ at the point $\underline{p} = \underline{p}_0$ where $q_A(p^*, \underline{p}_0) = q_m$ (so $\underline{p}$ is chosen so $p_m = p^*$ in our previous notation). Suppose not, and

$$G(p^*, \underline{p}_0) \ge G(p^*, \underline{p}_0 - \varepsilon) \tag{91}$$

for some $\varepsilon > 0$. Let $p^*(\varepsilon)$ be the price at which the $q_A(p, \underline{p}_0 - \varepsilon)$ curve hits q_m so $q_A(p^*(\varepsilon), \underline{p}_0 - \varepsilon) = q_m$. By continuity (Lemma 28) $p^*(\varepsilon) \to p^*$ as $\varepsilon \to 0$. From the same lemma we also have $G(p^*(\varepsilon), \underline{p}_0 - \varepsilon) > G(p^*(\varepsilon), \underline{p}_0)$. But the only way this can happen in combination with (91) is for $g(p, \underline{p}_0 - \varepsilon) < g(p, \underline{p}_0)$ for some set of $p \in (p^*(\varepsilon), p^*)$. This contradicts Lemma 29 for ε chosen small enough. Thus for small enough $\varepsilon > 0$ we have

$$G(p^*, \underline{p}_0) < G(p^*, \underline{p}_0 - \varepsilon). \tag{92}$$

We have already shown in Lemma 28 that $G(p, \underline{p}_0) < G(p, \underline{p}_0 - \varepsilon)$ for $p < p^*$ and ε chosen small enough.

In (35) we have the following identity for mixtures over horizontal bids:

$$K \equiv \int_0^{q_m} (p - C'(q))(1 - (G(p)F(q + q_m) + (1 - G(p))F(q)))dq \tag{93}$$
$$\equiv \int_0^{q_m} (\overline{p} - C'(q))(1 - F(q + q_m))dq. \tag{94}$$

Thus if we write $K(\underline{p})$ for the constant associated with the solution starting at $\underline{p}$, then from (92) we have

$$K(\underline{p}_0) = \int_0^{q_m} (p^* - C'(q))(1 - (G(p^*, \underline{p}_0)F(q + q_m) + (1 - G(p^*, \underline{p}_0))F(q)))dq$$
$$> K(\underline{p}_0 - \varepsilon).$$

Hence applying (93) at any point $p > p^*$ establishes that $G(p, \underline{p}_0) < G(p, \underline{p}_0 - \varepsilon)$.

Thus we have shown that $G(p, \underline{p})$ is decreasing in $\underline{p}$ for all values of p. And hence $K(\underline{p})$ is increasing in $\underline{p}$. Thus from (94) the value of $\overline{p}$ is also increasing in $\underline{p}$. ■