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Stochastic Stability In A Double Auction

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ABSTRACT

In a k-double auction, a buyer and a seller must simultaneously announce a bid and an ask price respectively. Exchange of the indivisible good takes place if and only if the bid is at least as high as the ask, the trading price being the bid price with probability k and the ask price with probability (1 - k). We show that the stable equilibria of a complete information k-double approximate an asymmetric Nash Bargaining solution with the seller's bargaining power decreasing in k. Note that ceteras paribus, the payoffs of the seller of the one-shot game increase in k. Nevertheless, as the stochastically stable equilibrium price decreases in k, choosing the seller's favourite price with a relatively higher probability in individual encounters makes him worse off in the long run.

JEL CLASSIFICATION: C78, D83
KEYWORDS: k-double auction, multiple equilibria, risk potential, stochastic stability, Nash Bargaining Solution.

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Contents

1 Introduction

This paper characterizes stochastically stable equilibria in a complete information version of the widely studied k-double auction of Satterthwaite and Williams (1989). In a k-double auction, a buyer and seller simultaneously choose a price. Trade takes place only if the buyer's price (bid) is at least as high as the seller's price (ask). In this case, the seller gets her preferred price (the bid) with probability k.

The interest in this problem is for two reasons. First, the double auction is used to stylize important trading institutions such as the call markets[1]. Call markets are for instance used to fix the daily opening price of every stock listed on the New York stock exchange. Twice a day, call markets are also used to fix the price of gold and copper in London. As these are markets in which traders participate repeatedly (and mostly anonymously), these are arguably environments where equilibrium play could be learned. Stochastically stable equilibria are equilibria that are robust to certain kinds of perturbations in the learning procedures.

The second reason for undertaking this study is a curiosity about the relationship between the stochastically stable outcome and the parameter k. To illustrate, suppose a risk neutral buyer and a seller seek to trade when the reservation values are one and zero respectively. Assume $0 < p < P < 1$ are the two feasible prices. The double auction is the following 2x2 game where $q = kP + (1-k)p$.

		Seller	
		p	P
Buyer	p	$1-p, p$	$0, 0$
	P	$1-q, q$	$1-P, P$

Figure 1: k-double auction with two prices

Note that (p, p) and (P, P) are the only strict Nash equilibria for all $k \in (0, 1)$. Although k does not affect the equilibrium set, on a heuristic basis, it is tempting to speculate that higher values of k

[1] See Cason and Friedman (1997), Rustichini and Satterthwaite (1994)

indicate a stronger position for the seller and a weaker position for the buyer. After all, the seller's payoffs are (weakly) increasing in k — whenever trade occurs, a higher k means the seller's favorite price (i.e. the bid) is more likely. Indeed, if a seller were to choose, on this heuristic basis, between two such one shot encounters that differ only in the size of k, she would probably pick the one with the higher k.

Now suppose we attempt to select between the two equilibria on the basis of perturbed learning dynamics. Note that higher the k, the out of equilibrium payoffs of the seller increase while those of the buyer decrease. In other words, it is less costly for the seller to ask for less than the buyer's bid and more costly for the buyer to bid more than the seller's asks. Therefore, when the bid is above the ask, there is more pressure for the bid to fall than for the ask to rise. Both these forces seem to suggest an increase in the size of the basin of attraction of the lower price. Consequently a lower price equilibrium may be more likely outcome for higher values of k. q It is easy to verify this intuition for the game in Figure 1. Suppose that the above game is being played according to the slightly noisy dynamics of Young (1993a). The players would then learn to coordinate on the *stochastically stable equilibrium*; for 2x2 games this is the same as the risk-dominant equilibrium. A direct calculation shows that for the above example there is a threshold $k^* \in (0,1)$ such that (p,p) is the risk dominant equilibrium if $k > k^*$ while (P,P) is the risk dominant equilibrium if $k < k^*$.

Clearly, k plays an important role in the *selection* of equilibria, even though it does not affect the equilibrium set. The curiosity is that *higher* values of k select the *lower* price. Thus, *higher* the probability that the seller's favorite price is selected in the one shot game, the *less* favorable is the long run outcome toward the seller. This is the opposite of what one might argue on an albeit informal examination of the one shot game.

In the above example, there are only two feasible prices. To show that these observations are not special, in Section 3, we study a model of price formation in a market where a population of buyers is matched at random and pairwise with a population of sellers. A k-double auction with *many* feasible prices is played within each encounter. If a trade for the indivisible object takes place then the pair exits the market and is replaced with a new one. Traders are myopic and play a best response to the distribution of prices across the matches from the previous period, unless they are matched for two consecutive periods. In the latter event they repeat the same strategy in some probability. Finally, there is a positive probability that they err and propose a price at random — allowing errors is only natural in an environment where players are learning to play the "right" strategies.

We seek those price distributions that are most likely to prevail in the long run when the probability of errors becomes negligible; such distributions are said to be stochastically stable distributions (SSPD)[2]. As in the case of the above example, a SSPD turns out to be a degenerate distribution with all traders coordinating on a strict equilibrium of the static double auction game. Such a strict equilibrium of the one shot game is said to be a *stochastically stable.*

In the two prices example, we could use the risk dominance criterion and identify the stochastically stable (SS) equilibrium. It is now well known that for games with more than two strategies, a pairwise ranking of strict equilibria based on the risk dominance criterion is not enough to identify the stochastically equilibria. In order to allow us to identify a stochastically stable equilibrium in terms of the payoffs of the underlying game, we introduce the notion of *risk-potential* of an equilibrium: Suppose (s_1^*, s_2^*) is a strategy profile with the property that s_i^* is the unique best response to a (mixed) strategy of player j if and only if j's strategy assigns a probability greater than γ to s_j^*. Then γ is said to be its risk potential[3].

Proposition 1, Section 2 contains a simple characterization of the risk potential of an equilibrium. In Theorem 1, we use this characterization to show that the equilibrium with the lowest risk-potential approximates the Nash bargaining solution of an asymmetric bargaining problem in which the bargaining power of the seller is $1-k$. Theorem 2 in Section 3.2 shows that in an SSPD, all traders coordinate on the equilibrium with the lowest risk-potential. The two theorems together yield the conclusion that the price that is most likely to prevail when the errors are increasingly unlikely is decreas-

[2]SSPD basically correspond to a stochastically stable states of the learning procedure. The notion of stochastic stability was first introduced by Foster and Young (1990). A precise definition is given in Section 3.2.

[3] The notion of risk potential is related to the the notion of p-dominance of Morris and Rob (1995) in the following way: Let $A((s_1^*, s_2^*)) = \{p \mid (s_1^*, s_2^*) \text{ is } p-\text{dominant.}\}$. Then γ is the infimum (but not the minimum) of the set $A((s_1^*, s_2^*))$.

ing in k.

For the learning procedure that we shall study, the notion of risk potential is closely related to the concepts of *radius* or *exit resistance* developed in Ellison (2000) and Maruta (1997) respectively [4]. Ellison (2000) in particular offers a variety of sufficient conditions that in principle can allow an easy characterization of the SS equilibria. Unfortunately, some of the simpler sufficient conditions, such as 1/2 dominance are not satisfied in our model. Here, one needs to construct explicitly the path of least resistance from one absorbing set to apply the sufficient condition of in Ellison (2000) that relates to the modified coradius and radius.

As a by product of the above method of proof, we are able to compute the expected waiting time to reach a stochastically stable state. An interesting insight from this exercise is the relationship between the absolute risk aversion of the traders and the expected waiting time. It turns out that the more risk averse the players are, faster is the convergence to the long run equilibrium. We discuss this in Section 4.3.

For the k-double auction, the equilibrium with the lowest risk potential is stochastically stable. We are however skeptical about the generality of the link between stochastic stability and risk potential even to particular classes of games. For, Maruta (1997) presents an example of 4x4, symmetric, supermodular game in which the equilibrium with the lowest risk potential is *not* the stochastically stable equilibrium.

We shall defer a discussion of the robustness of our results and the related literature (particularly Section 8.4, Young (1998)) to Section 4. Some of the formal arguments are in the Appendix.

2 The Double Auction And Risk-Potential

A buyer and a seller play a k-double auction in order to trade a single indivisible good. Normalize the utility of the no-trade outcome to zero and let $v(p)$ and $w(p)$ denote their respective VNM payoffs if trade occurs at a price p. Throughout the paper we maintain the following assumptions: i). $v(\cdot)$ and $w(\cdot)$ are common-knowledge, ii). $v(1) = w(0) = 0$, iii). $v(\cdot)$ and $w(\cdot)$ are strictly concave, twice differentiable and $v'(p) < 0$, $w'(p) > 0$, iii). $\lim_{p\to 1} v'(p) = -\infty$ and $\lim_{p\to 0} w'(p) = \infty$. Assumptions (iii) and (iv) are technical assumptions that do entail some loss of generality but substantially simplify the proofs. For instance, all of the properties are satisfied if we take $v(p) = (1-p)^\alpha$ and $w(p) = p^\alpha$ where $0 < \alpha < 1$ (where α is the coefficient of absolute risk aversion).

Let $\Sigma_n = \{i/n : i = 1 \ldots (n-1)\}$ be the set of pure strategies[5]. It is useful to think of $\delta_n = 1/n$ as a money unit and traders choose prices in multiples of this money unit. Some of our results concern the case when $n \to \infty$ and this corresponds to the case when the money unit is infinitesimal. A typical mixed strategy will be denoted by a vector $F = (f(p_1), \ldots, f(p_{n-1}))$ where $f(p)$ is the probability of playing the pure strategy p. The expected payoff of trader i upon choosing the price p_i when her opponent chooses F is

$$V_b(p_b, F) = \sum_{q \le p_b} f(q)\,[kv(p_b) + (1-k)v(q)] \tag{1}$$

$$V_s(p_s, F) = \sum_{q \ge p_s} f(q)\,[kw(q) + (1-k)w(p_s)] \tag{2}$$

A *best response* of trader i to F is a price p such that $V_i(p, F) \ge V_i(q, F)$ for all $q \in \Sigma_n$. As there are only finitely many strategies, the existence of a best response is not an issue. The following Lemma is stated without proof.

Lemma 1. *For every $p \in \Sigma_n$, the strategy profile (p, p) is a strict Nash equilibrium.*

As (p, p) is a strict Nash equilibrium, p continues to be the unique best response to F for either player even if F includes strategies other than p in its support – provided $f(p)$ is sufficiently large. If p can be the unique best response even for relatively smaller values of $f(p)$, then in an intuitive sense the equilibrium is more stable. This measure of riskiness is made precise below.

[4] The notion of risk-potential depends only on the payoffs of the underlying game. The radius depends both the underlying learning dynamic as well as the underlying payoffs. Risk potential is closely related to the idea of p-dominance described in Footnote 3 and plays a role in contexts other than best response learning dynamics. Kajii and Morris (1997) is an example. It is therefore useful to keep this concept distinct from the notion of radius.

[5] It is sufficient for our purposes to limit attention to prices in the range $(0, 1)$. For, these are the only prices that yield a positive utility for both players and other prices will not be played in any strict Nash equilibrium.

Definition 1. (Risk potential) An equilibrium (p,p) is said to have a risk potential of γ if for any mixed strategy F and $i = b, s$,

$$f(p) > \gamma \quad \Leftrightarrow \quad (V_i(p,F) > V_i(q,F), \ \ \forall q \in \Sigma_n, q \neq p)\,. \tag{3}$$

The risk potential of a game is related to the notion of p-dominance of Morris et. al (1995) as described in footnote 3.

For $i = s, b$ consider the functions $r_n^i : [\delta_n, 1-\delta_n] \to \Re$ and $r_n^b : [\delta_n, 1-\delta_n] \to \Re$ where

$$r_n^b(p) = \frac{k\,[v(p) - v(p+\delta_n)]}{kv(p) + (1-k)v(p+\delta_n)} \tag{4}$$

$$r_n^s(p) = \frac{(1-k)\,[w(p) - w(p-\delta_n)]}{(1-k)w(p) + kw(p-\delta_n)} \tag{5}$$

$$r_n(p) = \min\left\{r_n^b(p), r_n^s(p)\right\} \tag{6}$$

We alert the reader to the fact that $r_n^i(\cdot)$ and $r_n(\cdot)$ also depend on k. As we shall show presently in Proposition 1, the risk-potential of an equilibrium (p,p) can be characterized entirely in terms of $r_n(\cdot)$, provided the size of the money unit is small. But first in Lemma 2 below, we identify several key properties of $r_n^i(\cdot)$ and $r_n(\cdot)$ that are used in the proofs of various results to follow. First define,

$$q_{n,k} = \arg \max_{p\in[\delta_n, 1-\delta_n]} r_n(p) \tag{7}$$

Lemma 2. *Let $k \in (0,1)$. There exists an integer N_k such that for all $n \geq N_k$, the following hold:*

1. *$r_n^s(\cdot)$ is strictly decreasing and $r_n^b(\cdot)$ is strictly increasing.*
2. *$q_{n,k}$ is well defined and $r_n(p) = r_n^b(p)$ if $p \leq q_{n,k}$ and $r_n(p) = r_n^s(p)$ if $p \geq q_{n,k}$.*
3. *$\lim_{n\to\infty} q_{n,k} = q_k$ where q_k maximizes the asymmetric Nash product:*

$$q_k = \arg \max_{p\in[0,1]} v(p)^k w(p)^{(1-k)} \tag{8}$$

Proof. Divide and multiply the RHS $r_n^s(p)$ by δ_n to get

$$r_n^s(p) = \frac{(1-k)\delta_n}{(1-k)w(p) + kw(p-\delta_n)} \times \frac{w(p) - w(p-\delta_n)}{\delta_n}. \tag{9}$$

The first term on the RHS is strictly decreasing in p since $w(\cdot)$ is strictly increasing. The second term on the RHS is strictly decreasing in p since it is the slope of the concave function $w(\cdot)$ between $p - \delta_n$ and p. Therefore $r_n^s(\cdot)$ is strictly decreasing. A symmetric argument establishes that $r_n^b(\cdot)$ is strictly increasing. This proves Part 1 of the Lemma.

Let $\phi_n : [\delta_n, 1-\delta_n] \to \Re$ as $\phi_n(p) = r_n^b(p) - r_n^s(p)$. Clearly $\phi_n(\cdot)$ is continuous. By Part 1, it is also strictly increasing. Note that $r^s(\delta_n) = 1$ while $\lim_{n\to\infty} r_n^s(1-\delta_n) = 0$. Therefore for n sufficiently large, $r_n^s(\cdot)$ is a function whose value is one when $p = \delta_n$ and is approximately zero when $p = (1-\delta_n)$. A symmetric argument shows that $r_n^b(\cdot)$ is an increasing function whose value is approximately equal to zero when $p = \delta_n$ but equals one when $p = (1-\delta_n)$. For all such n, $\phi_n(\delta_n) \approx -1$ and $\phi_n(1-\delta_n) \approx 1$. By the intermediate value theorem, a solution to $\phi_n(p) = 0$ where $p \in (\delta_n, 1-\delta_n)$ exists. That this solution is unique follows from the strict monotonicity of $\phi_n(\cdot)$. Let $q_{n,k}$ denote this unique solution. Part 2 is now immediate.

Assume, with no loss in generality[6] that $\lim_{n\to\infty} q_{n,k} = q_k$ is well defined. Then,

$$\begin{aligned}\lim_{n\to\infty} \frac{r_n^s(q_{n,k})}{\delta_n} &= \lim_{n\to\infty} \frac{(1-k)}{(1-k)w(q_{n,k}) + kw(q_{n,k}-\delta_n)} \\ &\quad \times \lim_{n\to\infty} \frac{w(q_{n,k}) - w(q_{n,k}-\delta_{n_i})}{\delta_n} \\ &= \frac{(1-k)}{w(q_k)} \times w'(q_k)\end{aligned} \tag{10}$$

Likewise

$$\lim_{n\to\infty} \frac{r_n^b(q_{n,k})}{\delta_n} = -\frac{k}{v(q_k)} \times v'(q_k). \tag{11}$$

From Part 2, we know that $r_n^s(q_{n,k}) = r_n^b(q_{n,k})$ for all n sufficiently large. Therefore we must have $\lim_{n\to\infty} \frac{r_n^s(q_{n,k})}{\delta_n} = \lim_{n\to\infty} \frac{r_n^b(q_{n,k})}{\delta_n}$. On using the above equations, we get

$$k\frac{v'(q_k)}{v(q_k)} = -(1-k)\frac{w'(q_k)}{w(q_k)} \tag{12}$$

[6] $\{q_{n,k}\}_n$ is a bounded sequence. One could apply the arguments to follow for each convergent subsequence and show that limit of each of them satisfies Eq. (12). Therefore, all of them converge to the same limit.

Part 3 of the Lemma now follows from the observation that Eq. (12) is in fact the first order condition that describes the unique interior solution to the concave programming problem $\max_{p\in(0,1)} v(p)^k w(p)^{1-k}$. ■

Proposition 1. *For each $k \in (0,1)$, there exists an integer N_k such that for all $n \geq N_k$, if $p \in \Sigma_n$ then the risk potential of the equilibrium (p,p) is given by $(1 - r_n(p))$.*

Proof. A sketch of the argument is as follows. Suppose $p \in \Sigma_n$ is such that $p > q_k$. Consider the mixed strategy F with p and $p - \delta_n$ as its support and $f(p) = 1 - r_n^s(p)$. A direct computation shows that $V_s(p,F) = V_s(p-\delta_n, F)$, i.e. both p and $p - \delta_n$ are the seller's best responses against the mixed strategy F. Therefore the risk potential of (p,p) is at least $(1 - r_n^s(p))$, which from Part 2, Lemma 2 equals $(1 - r_n(p))$.

Similarly when $p \leq q_k$, by considering the mixed strategy F with the support on p and $p + \delta_n$, where $f(p) = 1 - r_n^b(p)$, we note that both p and $p + \delta_n$ are the buyer's best responses to F. Therefore, the risk-potential is $(1 - r_n^b(p))$ which, again from Lemma 2 equals $(1 - r_n(p))$.

To complete the proof, we must show that p is the unique best response for each player to any mixed strategy F such that $f(p) > (1 - r_n(p))$, This is shown in the Appendix. ■

Define

$$K_{n,k} = \left\{p \in \Sigma_n : r_n(p) \geq r_n(\hat{p})\ \forall \hat{p} \in \Sigma_n\right\} \tag{13}$$

If n is sufficiently large, by Proposition 1, $K_{n,k}$ is the set of equilibrium prices in Σ_n with the lowest risk potential. A typical element of $K_{n,k}$ differs from $q_{n,k}$ in that the former maximizes $r_n(\cdot)$ on the smaller domain Σ_n. Even though $q_{n,k}$ is unique, as the following example illustrates $K_{n,k}$ need not be a singleton even when n is large. In other words, there can be more than one equilibrium with lowest risk potential even for large values of n.

Example 1. Take $v(p) = \sqrt{1-p}$ and $w(p) = \sqrt{p}$ and $k = 0.5$. Note that $r_n(p) = r_n(1-p)$ for all $p \in [\delta_n, 1-\delta_n]$ and achieves a maximum at $q_{n,k} = 0.5$. However, $q_{n,k} \in \Sigma_n$ only when n is an even number, in which case $K_{n,k} = \{0.5\}$. On the other hand when n is odd and sufficiently large, the maxima of $r_n(\cdot)$ on Σ_n are the neighboring prices of 0.5, which means $K_{n,k} = \{0.5 - 1/n, 0.5 + 1/n\}$.

We shall now proceed to discuss the comparative statics with respect to k of the equilibria with the lowest risk-potential as well as the limiting properties of $K_{n,k}$ as $n \to \infty$. As Theorem 1 below shows, there is a simple way to characterize this limit in general.

Theorem 1. *Let $k \in (0,1)$. Let $\{p_{n,k}\}_n$ be an infinite sequence of prices such that $p_{n,k}$ is an equilibrium price with the lowest risk-potential. Then $\lim_{n\to\infty} p_{n,k} = q_k$ where q_k is as defined in Eq. (8).*

Moreover for every $k > \hat{k} \in (0,1)$, there exists an integer N such that if $p_{N,k}$ and $p_{N,\hat{k}}$ are any corresponding equilibrium prices with the lowest risk potential, then $p_{n,k} < p_{n,\hat{k}}$.

In particular, note that $\lim_{k\to 1} q_k \to 0$. Therefore for small values of k the equilibrium prices with the lowest risk-potential are almost zero.

Proof. Let $k \in (0,1)$. By Proposition 1, for all n sufficiently large, $p_{n,k} \in K_{n,k}$. Due to the monotonicity properties of $r_n(\cdot)$ that follow from Lemma 2,

$$\mid p_{n,k} - q_{n,k} \mid < \delta_n, \qquad \text{for all } n \text{ sufficiently large.} \tag{14}$$

This together with Part 3, Lemma 2 implies that $\lim_{n\to} p_{n,k} = q_k$. That $p_{n,k} < p_{n,\hat{k}}$ if $k > \hat{k}$ now follows from the fact that $q_k < q_{\hat{k}}$. ■

3 A Matching Model of Price Formation

At each date, a finite number of buyers and sellers are matched in pairs and play the k-double auction described in the previous sections. At the end of each period, successful pairs of traders exit the market and are replaced by identical pairs. Unsuccessful traders return to the pool and await to be matched again. Every trader in this pool has a positive probability of being matched. Let m denote the constant number of matches in each period.

Within a match, strategies are chosen as follows: If the trader is subject to a certain independent, random, idiosyncratic shock which occurs with probability ϵ, she chooses each of the pure strategies with equal probability. These are perturbations to an underlying price formation process and can be thought of in different ways - as mutations, as errors by the agents in choosing otherwise optimal responses or as uncertainty on the part of the modeler regarding the specification of the dynamic learning procedure. For concreteness, we shall use the interpretation of Young (1993a) and think of the perturbations as mistakes committed by the traders. We reiterate however, that other interpretations are equally valid.

When a trader does not make errors, she plays a best response to the distribution of prices of the opposing population from the previous period – except when matched for two consecutive periods. In the latter case, there is a small but positive probability that she repeats her strategy from the previous match (perhaps being subject to inertia). If there are several best responses, then each of them is played with an equal probability.

Let Δ_n denote the set of all mixed strategies on Σ_n which can be generated from m observations. From the above description of the decision rules, it is clear that a trader's choices at date t depend only on the distribution of the bid and ask prices realized at date $t-1$. Consequently, we can think of the state of the economy as being an element of $\Omega_n \equiv \Delta_n \times \Delta_n$. Using the above decision rules and standard probability calculus, we can compute the probability of being in a state $(\tilde{F}_b, \tilde{F}_s)$ at date $t+1$ conditional on being in a state (F_b, F_s) at date t. Therefore the (probabilistic) evolution of the bid-ask distributions over time can be described by means of a finite state Markov Process $\langle \Omega_n, \mathbf{P}_n^\epsilon \rangle$, where the state space is Ω_n and $\mathbf{P}_n^\epsilon$ is the matrix of transition probabilities. We shall refer to $\langle \Omega_n, \mathbf{P}_n^\epsilon \rangle$ as the *perturbed price-formation process* (PPFP).

Ultimately, we are interested in the long run behavior of the PPFP in the limiting case when $\epsilon \to 0$. Before we do this, we shall consider the case when $\epsilon = 0$ in the following subsection.

Further general comments on this learning procedure are deferred to Section 4.

3.1 The Price Formation Process

When $\epsilon = 0$, we shall refer to evolution of prices described in the previous section as the *price-formation process* or PFP.

Definition 2. $(F_b, F_s) \in \Delta_n$ is an *absorbing state* of the PFP if the probability of transition from (F_b, F_s) to *any other state is zero.*

Often, one needs a precise computation of the matrix of transition probabilities in order to identify the absorbing states and the global convergence properties of a Markov Chain. Fortunately, in this case, we can provide more direct arguments to show that the absorbing states are isomorphic to the set of strict Nash equilibria of the one shot game.

Let $\mathbf{p}$ denote a bid (or ask) distribution in which all the buyers (or sellers) propose the pure strategy p. Suppose that the state at date t is (F_b, F_s). Let p_i be a best response to F_j, where $j \neq i, i, j = s, b$. In subsequent periods, there is a positive probability that all the traders play a best response to the prices in the previous period. Therefore, the sequence of transitions $(F_b, F_s) \rightarrow (\mathbf{p}_b, \mathbf{p}_s) \rightarrow (\mathbf{p}_s, \mathbf{p}_b)$ can occur with a positive probability. From this it is clear that for (F_b, F_s) to be an absorbing state, it must be that $F_b = F_s = \mathbf{p}$ for some $p \in \Sigma_n$.

Conversely, take an arbitrary $p \in \Sigma_n$ and let $(\mathbf{p}, \mathbf{p})$ be the state at date t. As all the bids equal the asks in each of the matches at date $t-1$, all of those matches would have been successful. Therefore, in each of the matches at date t, a trader *must* play a best response to $\mathbf{p}$ which is p. Therefore, the state at date $t+1$ is necessarily $(\mathbf{p}, \mathbf{p})$. Therefore, $(\mathbf{p}, \mathbf{p})$ is an absorbing state.

A similar set of arguments can also be used to show that the PFP must globally converge to an absorbing state. Lemma 3 provides the requisite intermediate step for showing this.

Lemma 3. *Let p_b and p_s be best responses to F_s and F_b respectively. From an initial state (F_b, F_s), there is a positive probability that PFP enters the state $(\mathbf{p}_s, \mathbf{p}_s)$ or the state $(\mathbf{p}_b, \mathbf{p}_b)$ in at most four periods.*

Proof. Let (F_b, F_s) be the current state at date t and assume that p_b and p_s are as in the hypothesis of the Lemma. There is a positive probability that all the traders who are matched at date t play a best response to the distribution of prices from the previous period. Therefore, there is a positive probability that the state at date $t+1$ is $(\mathbf{p}_b, \mathbf{p}_s)$.

If $p_b = p_s$, the proof is complete.

Assume then, that the current state is $(\mathbf{p}_b, \mathbf{p}_s)$ and $p_b < p_s$. Trade would not have occurred in any of the matches in the last period. So there is a positive probability that all the matches in the current period are repetitions from the last period. If this event occurs, then there is in turn a positive probability that each of the buyers (subject to inertia) repeats her strategy p_b of the previous period while each of the sellers plays the best response p_b to the distribution $\mathbf{p}_b$. Hence, there is a positive probability that the state in the next period[7] is $(\mathbf{p}_b, \mathbf{p}_b)$.

Now consider the case when the current state is $(\mathbf{p}_b, \mathbf{p}_s)$ and $p_b > p_s$. In each of the matches, there is a probability that a trader of type i plays the best response p_j to the distribution $\mathbf{p}_i$ where $i \neq j$, $i, j = s, b$. Therefore, there is a positive probability that the PFP enters the state $(\mathbf{p}_s, \mathbf{p}_b)$ in the next period. As the bid price is now greater than the ask price, we are in the case considered in the previous paragraph. Repeat those arguments to conclude that the transition, $(\mathbf{p}_s, \mathbf{p}_b) \rightarrow (\mathbf{p}_s, \mathbf{p}_s)$ can occur with a positive probability. ■

The above Lemma shows that regardless of the initial state, the PFP enters a state in which players coordinate on a common price in at most four periods. Since $\langle \Omega_n, \mathbf{P}_n^0 \rangle$ is a stationary process, the overall probability that the PFP *does not* enter an absorbing state must converge to zero over the infinite horizon. We collect these observations in the form of Proposition 2.

Proposition 2. *For every $p \in \Sigma_n$, $(\mathbf{p}, \mathbf{p})$ is an absorbing state. Conversely, if (F_b, F_s) is an absorbing state, then $F_b = F_s = \mathbf{p}$ for some $p \in \Sigma_n$. Moreover, the PFP globally converges (with probability one) to an absorbing state.*

3.2 Stochastically Stable Price Distributions

Now suppose that $\epsilon > 0$ so that there is a positive probability that traders choose arbitrary strategies at random. In this section we shall characterize those stable price distributions which are robust to such perturbations of the PFP. This procedure is in the spirit of Kandori and Mailath (1993) and Young (1993a) among others.

Let $\mu_T^\epsilon(F_b, F_s)$ denote the relative frequency of observing the state (F_b, F_s) in the first T periods. Naturally, this frequency would depend on the initial state. The PPFP is however ergodic and the impact of the initial state wears off as time goes on and therefore $\mu^\epsilon(F_b, F_s) = \lim_{T\to\infty} \mu_T^\epsilon(F_b, F_s)$ does not depend on the initial state. Essentially, $\mu^\epsilon(F_b, F_s)$ describes the long run probability of observing the state (F_b, F_s) for a given ϵ. Stochastic stability requires that this long run probability is positive even when mistakes are unlikely, i.e in the limit as $\epsilon \to 0$.

Definition 3. (F_b, F_s) is said to be a *stochastically stable state* (SSS), if $\lim_{\epsilon\to 0} \mu^\epsilon(F_b, F_s) > 0$. $p \in \Sigma_n$ is said to be a *stochastically stable price* if $(\mathbf{p}, \mathbf{p})$ is SSS.

Define

$$K_{n,k}^m = \left\{p \in \Sigma_n : [mr_n(p)] \geq [mr_n(\hat{p})],\ \hat{p} \in \Sigma_n\right\} \tag{15a}$$

where $[x] = x$ if x is an integer and equals one plus the integer part of x if x is not an integer. We shall see that $K_{n,k}^m$ plays a crucial role in relating the SSS to equilibria with the lowest risk potential. From Lemma 2, we know that $r_n(\cdot)$ is strictly increasing to the left of $q_{n,k}$ and strictly decreasing to the right of $q_{n,k}$. Therefore,

$$K_{n,k}^m = \left\{i_1\delta_n, \ldots, i\delta_n, (i+1)\delta_n, \ldots i_2\delta_n\right\} \tag{15b}$$

for some integers i_1, i_2 such that $1 \leq i_1 \leq q_{n,k} \leq i_2 \leq (n-1)$. That is to say, the maxima of $[mr_n(\cdot)]$ are adjacent to each other and lie on either side of $q_{n,k}$.

Theorem 2. *Let $k \in (0,1)$. A stochastically stable state is an absorbing state of the PFP. Moreover, there exists an integer N_k such that for each $n \geq N_k$, the following hold:*

1. *If $p \in \Sigma_n$ is a stochastically stable price, then $p \in K_{n,k}^m$.*
2. *If the number of matches is sufficiently large, then a stochastically stable price has the lowest risk potential.*

Proof. Choose n sufficiently large so that Proposition 1 applies and $K_{n,k}$ is the set of all equilibrium prices with the lowest risk potential. Clearly, $K_{n,k} \subseteq K_{n,k}^m$ for all m but the inclusion may be strict for smaller values of m due to integer problems. These integer problems do not matter if m is sufficiently large and the maxima of $[mr_n(\cdot)]$ are the same as the maxima of $r_n(\cdot)$ on the domain Σ_n. Consequently, $K_{n,k} = \cap_{m\geq 1} K_{n,k}^m$. It is now immediate that Part 1 implies Part 2.

[7] Similarly, if instead the buyers play a best responses and the sellers repeat their strategy, the PFP would enter the state $(\mathbf{p}_s, \mathbf{p}_s)$.

Now consider the following union of limit sets:

$$K^* = \cup_{p \in K^m_{n,k}} \{(\mathbf{p}, \mathbf{p})\} \tag{16}$$

Proof of Part 1 now relies on the techniques described in Ellison (2000). Given two states $z = (F_b, F_s)$ and $\hat{z} = (\hat{F}_b, \hat{F}_s)$, define the cost $c(z, \hat{z})$ as the *minimum* number of mistakes that are required for the PPFP to move from z to $\hat{z}$. In the rest or this paper, we shall routinely use terms such as *path*, *modified cost*, *radius* and *coradius*. All of these terms are as defined in Ellison (2000) and we refer the reader to this work for formal details. Roughly speaking, the radius of a state z is the smallest cost that must be incurred for the PPFP to leave the basin of attraction of z. If the PPFP begins in a state z_1 and ends in a state z_{n+1} after transiting through a sequence $z_2, \ldots, z_i, \ldots, z_n$ of states, the cost of this path is simply $\sum_{i=1}^{n} c(z_i, z_{i+1})$. The modified cost of the above path is its cost net of the radii of all the interim states z_i, $i = 1, \ldots, n-1$. Ultimately, what we shall argue is that the radius of K^* is strictly greater than its modified coradius. Through an application of Theorem 2, Ellison (2000), we shall then conclude that K^* must contain an SSS.

Let $z = (\mathbf{p}, \mathbf{p})$.

Now suppose that the PPFP is initially in state z and because of the occurrence of a series of mistakes, it wanders to a state $\hat{z} = (F_b, F_s)$. Suppose $\hat{z}$ is in fact the first state such that some trader has a best response that differs from p. Let $1 - k_i/m$ be the frequency with which p is observed in F_i, $i = s, b$. Then by Proposition 1, it must necessarily be the case that $(1 - k_i/m) \leq r_n(p)$ and each of the k_i observations that differ from p constitute a mistake. Therefore, the total number of mistakes involved in the transition from z to $\hat{z}$ is bounded below by $[mr_n(p)]$. In other words,

$$c(z, \hat{z}) \geq [mr_n(p)] \tag{17}$$

With a slight abuse in notation, we shall write the cost of reaching a state $\hat{z} = (\hat{\mathbf{p}}, \hat{\mathbf{p}})$ from z as $c(p, \hat{p})$. We show in the Appendix, that

$$c(p, p + \delta_n) = [mr_n(p)] \qquad \text{if } p < q_{n,k}, \tag{18a}$$
$$c(p, p - \delta_n) = [mr_n(p)] \qquad \text{if } p \geq q_{n,k}. \tag{18b}$$

Two important conclusions can be drawn from Eq. (17) and Eq. (18). First, the radius of z is in fact $[mr_n(p)]$ for every $p \in \Sigma_n$.

Now suppose p lies to the left of $K^m_{n,k}$. In fact suppose that $p = i_1\delta_n - i_0\delta_n$ so that it is a distance $d = i_0\delta_n$ from the smallest element of $K^m_{n,k}$. From Eq. (18a), it is possible for the PPFP to sequentially transit through $(\mathbf{p}_i, \mathbf{p}_i)$ where $p_i = p + i\delta_n$ and $i = 1 \ldots i_0$ and reach the set K^* with the total cost of the path being

$$c(p, p_1) + \sum_{i=1}^{i_0-1} c(p_i, p_{i+1}) = [mr_n(p)] + \sum_{i=1}^{i_0-1} [mr_n(p_i)] \tag{19}$$

Subtracting the radii of the intermediate states, we conclude that the modified cost of reaching K^* from z is $[mr_n(p)]$. A symmetric argument for the case when p lies to the right of $K^m_{n,k}$ shows that the modified cost of reaching K^* from z is again $[mr_n(p)]$.

Thus for each $p \notin K^m_{n,k}$, we have exhibited a path from $z = (\mathbf{p}, \mathbf{p})$ to K^* whose modified cost is exactly equal to the radius of z. Therefore this is the path from z to K^* with the *lowest* modified cost. The modified coradius of K^* is therefore the maximum value of the modified cost of the above path from $(\mathbf{p}, \mathbf{p})$, which is $[mr_n(p)]$ as p varies over the set $\Sigma_n \setminus K^m_{n,k}$. In otherwords, the modified coradius is the second highest value taken by $[mr_n(\cdot)]$ on the set $K^m_{n,k}$. The radius of the set K^* is however $[mr_n(p)]$ for $p \in K^m_{n,k}$ and is therefore the highest value of $[mr_n(\cdot)]$ on $K^m_{n,k}$. This leads to the second conclusion that the radius of K^* is greater than the modified coradius of K^*. Apply Theorem 2, Ellison (2000) to conclude that K^* contains a stochastically stable state.

It remains to show that a price that is not in $K^m_{n,k}$ cannot be stochastically stable. As discussed above, the modified cost of reaching a state in K^* from a state $(\mathbf{p}, \mathbf{p})$ where $p \notin K^m_{n,k}$ is $[mr_n(p)]$ which is less than the radius of every state in K^*. By Theorem 3, Ellison (2000) a price that is not in $K^m_{n,k}$ cannot be stochastically stable. ■

4 Discussion

4.1 Of related Literature.

Recently, there have been a number of papers that seek to characterize the stochastically stable equilibria in various economic models.

Troger (2002) studies dynamics involving bargaining model the size of the surplus is determined by a prior stage investment decision. He shows that the equilibria that favor forward induction and lead to equitable outcomes are stochastically stable. Agastya (1999) characterizes stochastically stable equilibria in coalition form games. He shows that for convex coalition form games, an allocation that maximizes the product of players' utilities among all core allocations is stochastically stable. In an evolutionary bargaining model, Ellingsen (1997) studies the interaction between sophisticated agents (who play adaptively) and "obstinate agents" who stick to particular demands. Noldeke and Samuelson (1997) for instance show that the Riley equilibrium is stochastically stable in a model of market signaling. Young (1993b) studies the Nash demand game. Our study of the k-double auction adds to this growing literature.

In fact, our work is most closely related to the study of stochastic stability in a variant of the Nash demand game described in Section 8.4, Young (1998). Young studies a matching model with two populations. At each date, one representative is chosen from the two populations. The pair play a modified Nash demand game – they simultaneously demand shares, x and y of a unit surplus. If these are compatible, i.e. $x+y \leq 1$ each gets what she demands and share half of the remaining surplus $(1-x-y)$. On the other hand, if demands are not compatible, which is the case when $x+y > 1$ they get nothing. To decide how much to ask, an agent samples a part of the recent history to forecast her opponent's strategy and plays a best response. Building on some of his earlier results[8], Young outlines an argument by which the stochastically stable allocation is shown to be the Nash bargaining solution. Moreover, if the two populations differ in the size of the sample they choose making their best response, one gets an asymmetric Nash bargaining solution.

There is a conceptual difference between Young's work and ours: Young can be interpreted as saying that asymmetry in the *way players learn to play* explains relative bargaining strengths. We were deliberate in excluding such asymmetries in the learning procedure. (See below however.) Our point is that the fundamentals of the underlying game that are *considered irrelevant under standard equilibrium analysis* can determine relative strengths even with a symmetric learning process.

[8]Such as those in Young (1993a,b) and Chapter 8, Young (1998)

From a technical viewpoint, our work differs from Young in terms of the underlying game. The payoffs of a mixed strategy F in the modified Nash demand game are

$$\begin{aligned}\hat{V}_b(p_b, F) &= \sum_{q \leq p_b} f(q) v\left(\frac{p_b + q}{2}\right) \\ \hat{V}_s(p_s, F) &= \sum_{q \geq p_s} f(q) w\left(\frac{p_b + q}{2}\right)\end{aligned}$$

Clearly $\hat{V}_i \equiv V_i$ for $i = s, b$ if and only if $v(\cdot)$ and $w(\cdot)$ are linear and $k = 1/2$. Binmore et. el (2002) study the more general version of this problem for other values of k.

The price formation process that we have studied here is also different from Young's model of adaptive play. The dynamic we consider makes the analysis somewhat simpler and hopefully more transparent than the the one considered by Young. Further, some inertia is all that one requires for the price formation process to converge. Young's dynamic of adaptive play requires a relatively stringent condition on the amount of information that players gather (and then forget) in order to ensure convergence. We also allow for the possibility that there are several matches in a period. Arguably, this is more descriptive of a market.

The inverse relationship between k and the equilibrium payoff of the seller has also been observed by Williams (1987) in an entirely different context. Williams studies bilateral trading mechanisms that are ex-ante efficient, interim incentive compatible and individually rational between privately informed traders. He shows that in this class of mechanisms, the seller's expected gains from trade are greatest when playing the k-double auction with $k = 1$, i.e. in which the buyer's bid is chosen with probability one. At this stage, it is unclear as to how the intuition/results in this paper[9] relate to those of Williams (1987).

[9]It is my conjecture (and work in progress) that the result of Williams (1987) and risk-potential can be related by studying equilibrium selection involving global games. This is however, beyond the scope of this paper.

4.2 Robustness of results to the specified learning procedure

It is well recognized that some kind of "dampening" of best response dynamics is needed for convergence. The assumption of inertia plays this role. An alternative would be to substitute this with the assumption that a trader samples a price distribution (of her opponents) that was realized in one of the K most recent periods, where $K \geq 2$. Our results remain unaffected.

Certain other specifications of the unperturbed dynamics do not affect our result in any essential way. For example, one could allow traders to gather more information about past prices. In the spirit of Young (1993b), suppose we said that buyers and sellers respectively choose a random sample of k_b and k_s prices from the most recent K periods. They then use an empirical distribution of these prices as their forecast. It can be shown that all our results hold by redefining $r_n(p) = \min\left\{k_b r_n^b(p), k_s r_n^b(p)\right\}$ – other things being the same, more information (i.e. a higher k_i) leads to a higher share of the surplus for trader i. (Convergence however requires that $k_i/2 \leq K$ for $i = b, s$, just as in Young (1993b)). We believe that other variations of the best response dynamics do not alter the qualitative relationship between stochastically stable price and k.

On the other hand, the result is sensitive to more general specifications of error processes. For instance, Theorem 2 is no longer true if one allows for state-dependent perturbations as in Bergin and Lipman (1996). (See Binmore and Samuelson (1999)for an informative account of the implications of alternative error processes.)

4.3 Risk Aversion & The Rate Of Convergence

For most of the paper we have concerned ourselves with the limiting invariant distributions of the perturbed process. It is of interest to know the expected waiting time to reach a stochastically stable state. Theorem 2, Ellison (2000) shows that the expected wait is in fact of the order ϵ^{-C^*} where C^* is the coradius of the stochastically stable state.

As we have seen in the proof of Theorem 2, in the present model, the coradius of the union of limit sets containing the stochastically stable set is the second highest value of $[mr_n(\cdot)]$. For n sufficiently large, this is approximately the same as its highest value, i.e. $r_n(p_{n,k})$. From Eq. (14) and Lemma 2, $\lim_{n\to\infty} nr_n(p_{n,k})$ is given by Eq. (12).

Now consider the special case where $v(p) = (1-p)^\alpha$ and $w(p) = p^\alpha$. This is the case when both traders have a common absolute risk aversion of α. Applying Eq. (12), we get $\lim_{n\to\infty} nr_n(p_{n,k}) = \alpha$. Therefore, for sufficiently large values of m and n, we have $C^* \approx [\frac{m}{n}\alpha]$. It is then evident that the less risk averse the players are, the longer is the expected wait, *ceteras paribus*.

4.4 Local Interaction & The Rate of Convergence

Ellison (2000) and Blume (1995) among others have shown that local interaction can improve the waiting time. While these papers for the most part deal with coordination games, the intuition they convey is of course general. In both global and local interaction, one requires a critical *fraction* of mistakes for the process to leave the basin of attraction of a steady state. With local interaction even if this critical number is achieved in a neighbourhood, and if there is sufficient overlap among the neighbourhoods, it sets off a contagion leading other neighbourhoods to follow suit without further mistakes. The discussion to follow illustrates the validity of this insight in the context of k-double auction game. The discussion is illustrative and we leave it to the reader to fill in many of the formal details that we gloss over.

Imagine that the m matches occur at m nodes on a circle. Think of m as being very large but finite integer. At each node the traders play the k-double auction with Σ_n as the set of strategies. Prior to making a choice, a trader chooses a random sample of size ℓ from[10] $\hat{m} = \eta m$ of the prices played in the previous period at adjacent locations on either side of her own. The parameter $0 < \eta \leq 1$ captures the extent of local interaction. To simplify, assume $\ell = \hat{m}/2$ and each sample of this size from the $\hat{m}$ observations is chosen with a positive probability. Given her sample, a trader chooses a price as per the behavioral rules described in Section 3.

Without perturbations, it is not hard to see that the process must converge with probability one to a configuration where all traders coordinate on a common price. For, starting from an initial configuration, there is a positive probability that all the traders located at

[10] I will ignore the integer problems in this discussion. It should be clear that they do not affect the main argument.

nodes $i = 1, \ldots \hat{m}$ choose the same sample of prices from the middle locations $\ell/4$ to $3\ell/4$. Consequently there is positive probabilty that all the buyers in locations 1 through $\hat{m}$ choose the same price p_b and all the corresponding buyers choose a price p_s. By continuing to look at the event where all of these traders sample from the middle locations and by following the arguments in the proof of Lemma 3, it is easy to see that within four periods all the traders on locations 1 through $\hat{m}$ coordinate on a common price, say p. In the next period, there is a positive probability that all of these traders continue to sample from the middle location *and* the traders located at $\hat{m}+1 \ldots \hat{m}+\ell$ sample the the common price p from the locations $\ell/2+1$ through $\hat{m}$. The unique best response is p. Consequently in the following period there is a positive probability that all the traders located at 1 through $\hat{m}+1$ trade at p. Continuing in this way, we note that starting from an arbitrary configuration, there is a positive probability of reaching a configuration where all traders trade at price p in finitely many periods. The probability of not converging to such a state over the infinite horizon is zero, which in turn implies that absorbing sets of this process are isomorphic to the set of strict Nash equilibria of the k-double auction.

Let us now turn to a uniform perturbation of this process as done in Section 3.2. Now suppose that the process is in a state where all the traders are trading at the price p where $p < q_{n,k}$. Suppose then through a series of mistakes the process wanders to a state where some $\ell r^b_{n,k}(p)$ adjacent sellers, located say at conseceutive nodes starting at 1 ask for $p+\delta_n$ instead of p. All the remaining traders play p. Clearly, there are $\ell r^b_{n,k}(p)$ mistakes. Since the risk potential of p is $r^b_{n,k}(p)$, it follows that $p+\delta_n$ is a best response to any buyer who chooses a sample that includes the prices from these locations. As there is a positive probability that every buyer located between 1 and $\hat{m}$ chooses this sample, and the sellers at locations 1 through $\ell r^b_{n,k}(p)$ would not have traded, there is a positive probabiliy that the state in the next period is one where all the buyers located at 1 through $\hat{m}$ choose $p+\delta_n$ while the sellers located between 1 and $\ell r^b_{n,k}(p)$ choose $p+\delta$. Straighforward arguments as in the previous paragraph will show that the process converges to a state in which all the traders coordinate on the price $p+\delta_n$ with no further mistakes. The total number of mistakes involved in this transition is therefore $\ell r^b_{n,k}(p)$. In fact, only slight alterations to the part of the proof of Theorem 2 found in the Appendix will show that the equations corresponding to Eq.(18a) are

$$c(p, p+\delta_n) = \ell r_n(p) \qquad \text{if } p < q_{n,k}, \tag{20a}$$

$$c(p, p-\delta_n) = \ell r_n(p) \qquad \text{if } p \geq q_{n,k}. \tag{20b}$$

Repeat all the subsequent arguments in the proof of Theorem 2 to argue that a stochastically stable price must necessarily have the lowest risk potential. The coradius of the union of limit sets containing the stochastically stable state is the second highest value of $\ell r_n(\cdot)$.

Proceeding as in Section 4.3, when $v(\cdot)$ and $w(\cdot)$ exhibit constant absolute risk aversion of α , the rate of convergence is now approximately of order $\ell\alpha/n = \eta\alpha m/2n$. That smaller values of η mean faster convergence is now clear.

4.5 Method of Proof for Theorem 2

In this paper, we have used the methods of Ellison (2000) in the proof of Theorem 2. It is also possible to characterize the stochastically stable states[11] using the method described in Young (1993a). The chief advantage of the current approach is that it provides a sense of the expected waiting time for the PPFP to reach a stochastically stable state. This yields useful insights – the relationship between risk-aversion of the speed of convergence discussed in the Section 4.3 is an example.

On the other hand, the results of Ellison (2000) allow one only to conclude that the stochastically stable state is an element of K^* (See Eq. (16)). In fact, it is not hard to construct a $p-$tree for each $p \in K^m_{n,k}$ – by following the arguments used to construct a path from an arbitrary absorbing state $(\hat{\mathbf{p}}, \hat{\mathbf{p}})$ in the proof of Theorem 2 (in the Appendix) – and then showing that this has the highest stochastic potential. The tree surgery arguments of Young would then give us the stronger result that *every* $p \in K^m_{n,k}$ is stochastically stable.

4.6 A modified double auction

Our result is also robust to the following generalization of the double auction game: Suppose that $0 < k_i < 1$ is the probability that p_i is the trading price when $p_b \geq p_s$. There is no trade if $p_b < p_s$.

[11] Indeed, this was the method that was used in an earlier version.

Suppose that we only required that $k_b + k_s \leq 1$, so that we allow for the possibility that no agreement is reached even when the proposed prices are compatible. All our results remain unchanged by taking $k = k_b/(k_b + k_s)$.

To see this, note that the VNM utility for a buyer, resulting from the strategy profile (p_b, p_s) is

$$\tilde{u}(p_s, p_b) = \begin{cases} (k_b + k_s)[kv(p_b) + (1-k)v(p_s) & \text{if } p_b \geq p_s \\ 0 & \text{otherwise} \end{cases} \tag{21}$$

Likewise for the seller. Clearly, the VNM utility of an arbitrary strategy profile in this revised formulation is (k_b+k_s) times the VNM utility in our original formulation. As VNM utility is unique only up to a positive linear transformation, these are the same games.

Appendix

Proofs of Results in Section 2

Lemma 4 and Lemma 5 provide the necessary intermediate steps for the proof of Proposition 1.

Lemma 4. *Let F be such that $V_s(\tilde{p}, F) \geq V_s(p, F)$ for some $\tilde{p} \neq p$. Then,* $(1 - f(p)) \geq \min\left\{r_n^s(p), \frac{w(\delta_n)}{w(1)}\right\}$

Proof. First consider the case when $p < \tilde{p}$. Then,

$$\begin{aligned} V_s(\tilde{p}, F) - V_s(p, F) &= (1-k)\,(1 - F(\tilde{p} - \delta_n))\,[w(\tilde{p}) - w(p)] + \\ &\quad - \sum_{p \leq q < \tilde{p}} f(q)\,[(1-k)w(p) + kw(q)] \\ &\leq (1-k)\,(1 - F(\tilde{p} - \delta_n))\,[w(\tilde{p}) - w(p)] \\ &\quad - (F(\tilde{p} - \delta_n) - F(p - \delta_n))\,w(p). \end{aligned}$$

The above inequality is due to the monotonicity of $w(\cdot)$. Simplify the above RHS and use the hypothesis that $0 \leq V_s(\tilde{p}, F) - V_s(p, F)$ to conclude that

$$(1 - F(\tilde{p} - \delta_n)) \geq \frac{w(p)}{(1-k)w(\tilde{p}) + kw(p)}\,(1 - F(p - \delta_n))\,.$$

Therefore,

$$\begin{aligned} 1 - f(p) &= [1 - F(p)] + F(p - \delta_n) \\ &\geq [1 - F(\tilde{p} - \delta_n)] + F(p - \delta_n) \qquad (22) \\ &\geq \frac{w(p)}{(1-k)w(\tilde{p}) + kw(p)}\,(1 - F(p - \delta_n)) \\ &\qquad + F(p - \delta_n) \\ &\geq \frac{w(p)}{(1-k)w(\tilde{p}) + kw(p)} \geq \frac{w(\delta_n)}{w(1)} \end{aligned}$$

Now consider the case where $\tilde{p} < p$.

$$\begin{aligned} V_s(\tilde{p}, F) - V_s(p, F) &= (1 - F(p - \delta_n))\,(1-k)\,[w(\tilde{p}) - w(p)] + \\ &\quad \sum_{\tilde{p} \leq q < p} [(1-k)w(\tilde{p}) + kw(q)]\,f(q) \\ &\leq (1 - F(p - \delta_n))\,(1-k)\,[w(p - \delta_n) - w(p)] \\ &\quad F(p - \delta_n)w(p - \delta_n). \end{aligned}$$

Again, the inequality is an immediate consequence of the fact that $w(\cdot)$ is increasing. By hypothesis $V_s(\tilde{p}, F) - V_s(p, F) \geq 0$ and hence the RHS above must be non-negative, i.e.

$$\begin{aligned} (1 - f(p)) \geq F(p - \delta_n) &\geq \frac{(1-k)\,[w(p) - w(p - \delta_n)]}{(1-k)w(p) + kw(p - \delta_n)} \\ &= r_n^s(p). \end{aligned}$$

■

Lemma 5. *Let F be a distribution of ask prices such that $V_b(\tilde{p}, F) \geq V_b(p, F)$ for some $\tilde{p} \neq p$ Then,* $(1 - f(p)) \geq \min\left\{r_n^b(p), \frac{v(1-\delta_n)}{v(0)}\right\}$.

Proof. This is similar to the proof of the previous Lemma and is hence omitted. ■

Proof. (**Proposition** 1) Let F be a strategy such that $V_i(\tilde{p}, F) \geq V_i(p, F)$ for some $\tilde{p} \neq p$, for some $i = s, b$. Then, by Lemma 4-5, it must be the case that $1-f(p)\geq\min\left\{r_n^b(p),r_n^s(p)),\frac{v(1-\delta_n)}{v(0)},\frac{w(\delta_n)}{w(1)}\right\}$. If we show that

$$\min\left\{r_n^b(p),r_n^s(p)),\frac{v(1-\delta_n)}{v(0)},\frac{w(\delta_n)}{w(1)}\right\}=r_n(p),\tag{23}$$

then the contrapositive of the previous sentence is

$$(f(p)>(1-r_n(p)))\quad\Rightarrow\quad(V_i(p,F)>V_i(\tilde{p},F)\ \ \forall\tilde{p}\neq p,\forall i)\,.$$

We will argue that Eq. (23) holds for all $p\in\Sigma_n$ provided n is sufficiently large.

By Lemma 2, $\lim_{n\to\infty}\frac{r_n(q_{n,k})}{\delta_n}<\infty$ whereas because of the assumption that $-v'(1)=w'(0)=\infty$, we have $\lim_{n\to\infty}\frac{v(1-\delta_n)}{\delta_n v(0)}=\infty$ and $\lim_{n\to\infty}\frac{w(\delta_n)}{\delta_n w(1)}=\infty$. Therefore,

$$\frac{r_n(q_{n,k})}{\delta_n}<\min\left\{\frac{v(1-\delta_n)}{\delta_n v(0)},\frac{w(\delta_n)}{\delta_n w(1)}\right\}\quad\text{for all } n \text{ sufficiently large.}\tag{24}$$

That Eq. (23) holds for all n sufficiently large is immediate on noting that by definition of $q_{n,k}$ we have $r_n(p)\leq r_n(q_{n,k})$ for all $p\in\Sigma_n$. ■

Proofs of Results in Section 3.2

Proof. (Theorem 2) Proof of Eq. (18a- 18b).

Let $p>q_{n,k}$ where $q_{n,k}$ is as defined in Eq. (7). Then $r_n(p)=r_n^s(p)$ is the risk-potential of the equilibrium (p,p). Now let $F^{\hat{m}}$ denote a distribution of prices where the price p is observed with a frequency $1-\hat{m}/m$ and the price $p-\delta_n$ is observed with the remaining frequency of $\hat{m}/m$. We shall argue that the transition $(F^{\hat{m}},\mathbf{p})\to(F^{\hat{m}+1},\mathbf{p})$ involves exactly one mistake, provided $\hat{m}<mr_n^s(p)$.

When the current state is $(F^{\hat{m}},\mathbf{p})$, there is a positive probability that all the sellers choose a best response to $F^{\hat{m}}$. Since $\hat{m}/m<r_n^s(p)$, the definition of risk-potential tells us that p is the unique best response. Therefore, the ask distribution in the next period continues to be $\mathbf{p}$.

The behavior of the buyers in the next period is a little different. There are $\hat{m}$ buyers in the preceding period who had bid $p-\delta_n$. Each of these buyers would have been matched with sellers who asked for p. Consequently, each of these buyers would not have traded and returned to the pool of unsuccessful buyers. With a positive probability they are rematched in the current period and, due to inertia, choose $p-\delta_n$ again. For the remaining $m-\hat{m}$ buyers, the unique best response is to choose p. However, if exactly one of these buyers err and choose $p-\delta_n$ instead, then the new bid distribution would contain $\hat{m}+1$ observations of $p-\delta_n$ with the remaining being p. In other words, the new state would be $(F^{\hat{m}+1},\mathbf{p})$ and the transition from $(F^{\hat{m}},\mathbf{p})$ involved exactly one mistake.

Now on the other hand if $\hat{m}\geq[mr_n^s(p)]$, then clearly $p-\delta_n$ is also a best response to a typical seller. In this case there would be positive probability of transition from $(F^{\hat{m}},\mathbf{p})$ to $(\mathbf{p},\mathbf{p}-\delta_\mathbf{n})$. By Lemma 3, the PPFP can transit to $(\mathbf{p}-\delta_n)\mathbf{p}-\delta_\mathbf{n})$ with no further mistakes.

Thus, through the following sequence of transitions,

$$(\mathbf{p},\mathbf{p})\to(F^1,\mathbf{p})\to\cdots\to(F^{[mr_n^s(p)]-1},\mathbf{p})\to(F^{[mr_n^s(p)]},\mathbf{p})\to(\mathbf{p},\mathbf{p}-\delta_\mathbf{n})\to($$

which involves exactly $[mr_n^s(p)]$ mistakes and therefore $c(p,p-\delta_n)=[mr_n^s(p)]$.

A symmetric argument for the case when $p<q_{n,k}$ establishes that $c(p,p+\delta_n)=[mr_n^b(p)]$. ■

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