WORKING PAPERS IN ECONOMICS

RUBINSTEIN'S SOLUTION OF THE BARGAINING PROBLEM: SOME GENERALISATIONS AND EXTENSIONS

by Michael C. Blad and Nicholas Oulton

No.121 December 1988

DEPARTMENT OF ECONOMICS

The University of Sydney
Australia 2006

WORKING PAPERS IN ECONOMICS

RUBINSTEIN'S SOLUTION OF THE BARGAINING PROBLEM: SOME GENERALISATIONS AND EXTENSIONS

by Michael C. Blad and Nicholas Oulton

No.121 December 1988

DEPARTMENT OF ECONOMICS

The University of Sydney
Australia 2006

RUBINSTEIN'S SOLUTION OF THE BARGAINING PROBLEM: SOME GENERALISATIONS AND EXTENSIONS

by Michael C. Blad and Nicholas Oulton

No.121 December 1988

Comments on an earlier draft by Robert Anderson and Eddie Dekel-Tabak are greatly appreciated.

Nat.Lib. of Aust. Card No. and ISBN 0 949269 59

1. INTRODUCTION

The outcome of a bargaining process has in traditional economics frequently been claimed to be indeterminate. Nash (1950) proposed a unique <u>axiomatic</u> solution, but it was not until Rubinstein (1982) that a satisfactory solution based on individually rational behaviour emerged. By imposing a particular structure on the bargaining game, Rubinstein showed that a unique perfect equilibrium solution exists. The basic idea behind Rubinstein's approach is that bargaining takes time and time is valuable, either because the players discount the future or because there is a cost incurred in each bargaining round. Hence the players in the bargaining game will rationally be impatient and it turns out that, at least in the special cases Rubinstein considered, this impatience is enough to induce a unique, perfect equilibrium (PE) outcome.

The structure assumed by Rubinstein is as follows. Two players bargain for shares in a cake of size 1. If they cannot agree, they get nothing. Each player in turn makes an offer to (demands a share of the cake from) the other. If an offer is refused, the bargaining round ends and in the next (discrete) time period the player who refused the offer must make an offer to his opponent. If an offer is accepted, the game ends with the cake being divided on the agreed basis.

The cases considered by Rubinstein were:

(a) the discounting case, where each player discounts the future at a constant rate and where the players' utility functions are linear in the cake to be divided between them; and

(b) the fixed cost case, where after each bargaining round each player must pay a fixed cost.

In both these cases, Rubinstein presented solutions only for the two-person game. In this paper we generalise his results in a number of directions. First, we extend his result by allowing the players' utility functions to be non-linear, so that the utility frontier is also non-linear. A similar result is contained in Hoel (1986), though the proof is different. Secondly, we show how our method of proof provides a solution of one natural extension of the setting to an N-person game ($N \geq 2$). Finally, we discuss the implication of allowing coalition formation in the general case. Throughout the paper we make use of simple economic principles and simple mathematics. Our method, which considers outcomes rather than strategies, is related to, though not identical with, that of Shaked and Sutton (1984), who give a simple proof of case (a) when the number of players is two.

The plan of the paper is as follows. In section 2 we introduce our method of proof by providing simple proofs of Rubinstein's results in his original cases (a) and (b). In section 3, we consider non-linear utility functions in the two-person discounting case. We show in particular that if the utility frontier is concave, then there is a unique perfect equilibrium outcome. In section 4, we show that in an N-person ($N \geq 2$) game with discounting, there are again perfect equilibrium solutions which parallel those for the $N = 2$ case. In particular, in the discounting case the PE outcome is, under certain conditions, unique and each player's share approaches $1/N$ as the players' discount factors approach one. These solutions for the N-person game assume no coalitions. So in section 5 we consider whether the possibility of coalitions formation affects our results. Somewhat surprisingly, we find that, at least in the case where all players have identical discount factors, coalitions are unprofitable. This result may be viewed as strengthening the case for considering the unique PE outcome of the N-person version of Rubinstein's game to be a satisfactory solution.

2. A NEW METHOD OF DERIVING THE SOLUTION OF RUBINSTEIN'S BARGAINING GAME

In the present section we shall introduce a simple method of proving existence of a (unique) pefect equilibrium in the original bargaining game considered by Rubinstein (1982). This proof will form the basis for the various extensions to be discussed in the following sections.

For an exact definition of a perfect equilibrium the reader is referred to Rubinstein (1982). Loosely stated a set of strategies for the 2 players form a perfect equilibrium if in every subgame of the original game the corresponding strategies form a Nash equilibrium.

We shall in the following only deal with the question of existence of stationary perfect equilibria, i.e. perfect equilibria, where the supporting strategies are history independent.

An outcome achieved in a PE of the game will be referred to as a perfect equilibrium partition (PEP) of the cake. As Selten (1975) has shown, at least one PE exists. Let y_i be the outcome achieveable by player i ($i = 1, 2$) in some (stationary) PE. Obviously, $0 \leq y_i \leq 1$. By the basic structure of the game, y_i is invariant over time - for example, what player 1, who moves first, can achieve in a PE if the game ends in the first time period (when $t = 0$, where t is time) must be the same as what he could achieve were the game to continue until it is his turn to make an offer again (when $t = 2$).

2.1 The discounting case when N = 2

Following Rubinstein we now assume that the utility functions of the players are linear in cake and the players apply a constant discount factor δ_i ($0 \leq \delta_i < 1$, $i = 1, 2$) to future cake.

The following is a table of the outcomes that each player could achieve in some PE if the game were to end in time period t:

TABLE 1

Time (t)	Player making first demand	Outcome attainable in some PE by player: 1	2
0	1	y_1	$\delta_2 y_2$
1	2	$\delta_1 y_1$	y_2
2	1	y_1	$\delta_2 y_2$
.	.	.	.
.	.	.	.
.	.	.	.

The explanation of the entries in the table is as follows. Consider first player 1. At $t = 2$, by definition y_1 is the outcome achieveable by player 1 in some PE. In the previous time period, player 1 would therefore accept any offer from player 2 which is greater than or equal to the present value of what 1 could obtain by waiting one time period, namely $\delta_1 y_1$. On the other hand in a PE player 1 would not be offered more than this amount by player 2, were the game to continue until $t = 1$. Therefore $\delta_1 y_1$ is player 1's outcome at $t = 1$. An identical argument establishes that $\delta_2 y_2$ is player 2's outcome at $t = 0$. The other entries in the table can be filled in by symmetry.

Suppose now that player 2 accepted 1's offer at $t = 0$ and that 2 actually received his achieveable outcome $\delta_2 y_2$. Then 1 would receive 1 - $\delta_2 y_2$, so we have

$$y_1 = 1 - \delta_2 y_2 \tag{1}$$

Suppose that the game did not end at $t = 0$ but continued until $t = 1$. Then by identical reasoning

$$y_2 = 1 - \delta_1 y_1 \tag{2}$$

These two equations have the unique solution

$$y_1 = (1 - \delta_2)/(1 - \delta_1\delta_2)$$

and

$$y_2 = (1 - \delta_1)/(1 - \delta_1\delta_2) \tag{3}$$

Therefore, there is a unique perfect equilibrium, that is, a unique division of the cake, which can be supported as a P.E. Furthermore the game will end at t=0 with player 1 receiving $y_1 = (1 - \delta_2)/(1 - \delta_1\delta_2)$ and player 2 receiving $\delta_2 y_2 = \delta_2(1 - \delta_1)/(1 - \delta_1\delta_2)$. These formulae are of course the ones given in Rubinstein (1982).

We stress that our very simple proof only applies to stationary PE. Rubinstein's original proof (and the simplified version by Shaked and Sutton (1984)) does not <u>assume</u> stationarity. There it is part of the result on the existence of a unique PE.

2.2 The fixed costs case when N = 2

As before, let y_i be the outcome attainable in some PE by player i if it is his turn to move. Let $c_i > 0$ be the additional cost incurred by player i if the bargaining continues an extra time period. Therefore, if agreement is reached in period t, then player i will have incurred total bargaining costs of $t\,c_i$ ($t = 0, 1, \ldots$). However, if the game does continue into time period t+1, then the costs incurred up to this point by player i (tc_i) are bygones and so have no effect on his actions at t+1. Players need only take into account <u>future</u> bargaining costs.

The players' utility functions are no longer necessarily linear in cake but the bargaining costs are assumed to be paid in effect in cake. Players do not discount the future.

With these assumptions a table analogous to table 1 can be drawn up.

TABLE 2

Time (t)	Player making first demand	Outcome attainable in some PE by player: 1	2
0	1	y_1	$\max(0,\ y_2-c_2)$
1	2	$\max(0,\ y_1-c_1)$	y_2
2	1	y_1	$\max(0,\ y_2-c_2)$
.	.	.	.
.	.	.	.
.	.	.	.

Consider player 1 and suppose first that $y_1 - c_1 \geq 0$. At $t = 2$, by definition his achieveable outcome in some PE is y_1. At $t = 1$, he would accept any offer from player 2 greater than or equal to $y_1 - c_1$, which is the share of the cake he could obtain by waiting one extra period (by waiting, he would obtain y_1, at a cost of c_1, for a net gain of $y_1 - c_1$). On the other hand, since he would accept $y_1 - c_1$, in a PE player 2 would not offer him more than this. Hence $y_1 - c_1$ is 1's maximum potential payout at $t = 1$. Suppose now that $y_1 - c_1 < 0$. Since 2 cannot make 1 accept a negative amount of cake, 1 would accept an offer from 2 greater than or equal to zero. The (clearly inferior) alternative would be for 1 to wait an extra period and gain y_1 at a cost of c_1, a net gain of only $y_1 - c_1 < 0$. Since player 2 will not offer 1 more than zero in a PE, zero is 1's outcome, if $y_1 - c_1 < 0$. Therefore in general 1's maximum potential payout is max $(0, y_1 - c_1)$.

A parallel argument shows that max $(0, y_2 - c_2)$ is 2's maximum potential payout at $t = 0$. The other entries in the table follow from symmetry considerations.

Suppose now that player 2 accepted 1's offer at $t = 0$ and that 2 actually received his maximum potential payout, max $(0, y_2 - c_2)$. Then 1 would receive $1 - \max(0, y_2 - c_2)$, so

$$y_1 = 1 - \max(0, y_2 - c_2) \tag{4}$$

Suppose instead that the game continued into the next time period $(t = 1)$. Then a parallel argument shows that

$$y_2 = 1 - \max(0, y_1 - c_1) \tag{5}$$

By substitution, these two equations can be re-written as follows:

$$y_1 = 1 - \max(0, 1 - \max(0, y_1 - c_1) - c_2)$$
$$y_2 = 1 - \max(0, 1 - \max(0, y_2 - c_2) - c_1) \tag{6}$$

These equations can be solved by an exhaustive consideration of all possible cases. For example, for the first equation the cases to consider are:

Case A: $y_1 - c_1 \geq 0$	Case B: $y_1 - c_1 < 0$
(i) $c_1 > c_2$	(i) $c_1 > c_2$
(ii) $c_1 = c_2$	(ii) $c_1 = c_2$
(iii) $c_1 < c_2$	(iii) $c_1 < c_2$

The solutions for y_1 are:

$$
\begin{array}{lll}
(1) & c_1 > c_2 : & y_1 = \min(c_2, 1) \\
(2) & c_1 = c_2 = c : & \begin{cases} c \leq y_1 \leq 1 \text{ if } c \leq 1 \\ y_1 = 1 \text{ if } c > 1 \end{cases} \\
(3) & c_1 < c_2 : & y_1 = 1
\end{array}
\qquad (7)
$$

Interpreting these solutions we obtain that if <u>either</u> $c_1 > c_2$ <u>or</u> $c_1 < c_2$ <u>or</u> $c_1 = c_2 = c \geq 1$, then there is a unique PEP.

If $c_1 = c_2 = c < 1$, then any solution for player 1 in the closed interval $[c, 1]$ is a PEP. These results differ slightly from those given by Rubinstein, who implicitly assumes that $c_i < 1$, $i = 1, 2$.

We might add that there seems to be a hidden (rather strange) assumption concerning the structure of costs in this type of bargaining situation, a point not mentioned in Rubinstein's original paper. In connection with each bargaining round the players incur costs c_i; in general these costs cannot be paid out of the cake, hence the players must have other resources. However, as the game in principle might involve an infinity of bargaining rounds, it is an implicit assumption that both players have infinitely large resources.

3. MORE GENERAL UTILITY FUNCTIONS

In the section above we discussed the bargaining problem in the original setting, given by Rubinstein. There utility functions were assumed to be linear in 'cake' in the discounting case, whereas in the fixed costs case, utility could be any monotonically increasing function of 'cake', but only of 'cake' i.e. a single argument utility function. In this section, we consider more general utility functions. Our results in this section are partly contained in Hoel (1986), though our method of proof is different. Also, while Hoel only considers situations which guarantee existence of a unique PE, we also illustrate the situation when multiple equilibria may be present.

3.1 The discounting case

Two players are assumed to bargain with each other in exactly the same way as in section 2. But now there is not necessarily only one source of utility (cake), nor need the utility function be linear in its arguments, whatever these are. The bargaining situation facing the two players can be summed up in the utility frontier, f:

$$u_2 = f(u_1), \quad f' < 0 \tag{8}$$

where u_i is player i's utility level, assumed to be time invariant. There is some maximum level of utility which each player could in principle attain, if his opponent gained nothing from the bargaining outcome. By choice of utility units, each player's maximum utility level is 1 and his minimum level is 0, that is, both u_1 and u_2 lie in the closed interval [0, 1]. Hence we have

$$f(1) = 0,\ f(0) = 1,\ f^{-1}(0) = 1 \text{ and } f^{-1}(1) = 0 \tag{9}$$

If the players fail to agree at some time t, then both receive zero utility at that time. The utility today of receiving utility $\bar{u}_i$ in the next time period is $\delta_i \bar{u}_i$, where δ_i is player i's discount factor ($0 \leq \delta_i < 1$).

Let y_i be the level of <u>utility</u>, achievable by player 1 in some PE when it is his turn to make an offer to the other. Applying the method of section 2, we start by drawing up a corresponding table. It will be identical to table 1, except that we must interpret y_1 as a quantity of utility, not cake.

Take player 1. At $t = 2$, by definition the achievable level of his utility is y_1. In the previous time period, if the game reaches this point, he would accept an offer of $\delta_1 y_1$ or greater, since $\delta_1 y_1$ is the present value of the level of utility achievable by waiting one period. Since he would accept $\delta_1 y_1$, player 2 would not offer him more than this in a PE, so $\delta_1 y_1$ is his outcome. Similarly, $\delta_2 y_2$ is player 2's achieveable level at $t = 0$.

Now if at time $t = 0$, player 2 were to receive his achievable level of utility, $\delta_2 y_2$, then player 1 would get $f^{-1}(\delta_2 y_2)$. Hence

$$y_1 = f^{-1}(\delta_2 y_2) \tag{10}$$

If the game lasted until $t = 1$, then the same argument shows that

$$y_2 = f(\delta_1 y_1) \tag{11}$$

By substituting (11) into (10) and vice versa, we obtain

$$y_1 = f^{-1}(\delta_2 f(\delta_1 y_1))$$

and

$$y_2 = f(\delta_1 f^{-1}(\delta_2 y_2)) \tag{12}$$

So y_1 is a fixed point of the function g(.), where

$$g(z) = f^{-1}(\delta_2 f(\delta_1 z)), \quad 0 \leq z \leq 1 \tag{13}$$

and y_2 is a fixed point of the function h(.), where

$$h(z) = f(\delta_1 f^{-1}(\delta_2 z)),\ 0 \leq z \leq 1 \tag{14}$$

We may remark that the functions g(.) and h(.) both have at least one fixed point, since they are continuous maps of a closed interval into itself. If, however, each function possesses only one fixed point, then the game has a unique PEP. This is the content of:

<u>Proposition 1</u> In the generalised, two-person Rubinstein bargaining game, where the players' utility frontier is given by $u_2 = f(u_1)$, $f' < 0$, there is a unique PEP if <u>either</u> (a) $\delta_2 = 0$ <u>or</u> (b) $f'' \leq 0$ (the utility frontier is concave). In both these cases the game ends at $t = 0$, with player 1 obtaining z_1, where z_1 is the (unique) solution to

$$z_1 = g(z_1),$$

and with player 2 obtaining $f(z_1)$. If (a) holds and $\delta_2 = 0$, then $z_1 = 1$ and player 2 obtains $f(1) = 0$.

<u>Proof</u> Consider the function $g(z) = f^{-1}(\delta_2 f(\delta_1 z))$:

$$g'(z) = \delta_1\delta_2 f'(\delta_1 z)/f'(g(z)) \tag{15}$$
$$\geq 0,\ \text{since } f'(.) < 0$$

(a) First we note that if $\delta_2 = 0$, then $g(z) = f^{-1}(0) = 1$, all z, and $g'(z) = 0$. So the only fixed point is $z_1 = 1$ (see Figure 1(a) below).

(b) Now suppose δ_2 is different from zero and consider g(z) at its endpoints, g(0) and g(1):

$$g(0) = f^{-1}(\delta_2 f(0)) = f^{-1}(\delta_2) > 0$$

and $$g(1) = f^{-1}(\delta_2 f(\delta_1)) < 1.$$

So the graph of $g(z)$ would start above the 45° line and finish below it as shown in Figure 1(b). Now consider the slope of $g(z)$ at a fixed point (when $g(z) = z$).

From (15),

$$g'(z) = \delta_1 \delta_2 f'(\delta_1 z)/f'(z). \qquad (16)$$

If $f'' \leq 0$, then $- f'(\delta_1 z) < - f'(z)$, and so

$$g'(z) < 1.$$

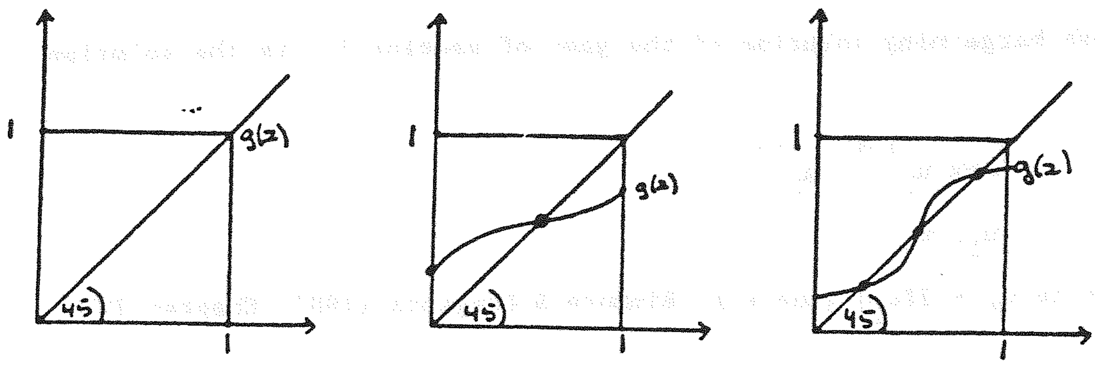
Now consider Figure 1(b) again. Since $g'(z) < 1$ at a fixed point, it follows that when $g(z)$ intersects the 45° line it must do so from above. But we have already shown that $g(z)$ starts above the 45° line and finishes below it. Therefore $g(z)$ intersects the 45° line only once, i.e. there is a unique fixed point.

Thus $g(z)$ has a unique fixed point if <u>either</u> $\delta_2 = 0$ <u>or</u> $f'' \leq 0$. So there is a unique PEP, which completes the proof of Proposition 1. □

<u>Remark 1</u> This result was originally established by Binmore (See Binmore and Dasgupta (1987, Chapter IV)). A proof similar to ours has independently been presented in van Damme (1987).

<u>Remark 2</u> If no sign restrictions are put on f'', then more than one fixed point may exist. However it is still the case that $g(0) > 0$, $g(1) < 1$ and $g' > 0$. So generically the g-curve must intersect the 45° line an odd number of times, an example of three intersections being depicted in figure 1(c). However, even if f is convex ($f'' > 0$), consideration of (16) shows that assuming a sufficiently small curvature of f (i.e. allowing f to be only "slightly convex") would still imply that $g'(z) < 1$ for all $z \in [0,1]$, and hence there would still be only one fixed point.

Figure 1



(a) $\delta_2 = 0$ (b) $0<\delta_2, f'' \leq 0$ (c) f not concave

Remark 3 Rubinstein proved that there is a unique PEP in the special case of a linear utility frontier ($f'' = 0$). Proposition 1 thus generalises his result.

Remark 4 The solution to the generalised game of Proposition 1 has the same qualitative properties as Rubinstein's original game. In particular, the more patient is a player and the less patient his opponent, then the higher is the former's share:

$$\partial z_1/\partial \delta_1 = [1/f'(g(z_1))]\delta_2 f'(z_1)z_1 > 0$$
$$= \delta_2 z_1 > 0$$

and

$$\partial z_1/\partial \delta_2 = [1/f'(g(z_1))]\, f(\delta_1 z_1)$$
$$= f(\delta_1 z_1)/f'(z_1) < 0,$$

both derivatives being evaluated at the fixed point.

3.2 A Comparison with the Nash bargaining solution

In the discounting case analysed by Rubinstein, which corresponds to the special case $f'' = 0$ in our analysis, the solution approached the Nash bargaining solution as the discount factor of both players approached 1.

We shall now show that the generalisation given in section 3.1 possesses the same property.

The Nash bargaining solution of the game of section 3.1 is the solution to

$$\max_{u_1,\, u_2} u_1^{1/2} \; u_2^{1/2}$$

subject to $u_2 = f(u_1)$ (see e.g. Binmore & Dasgupta (1987, Chapter IV)).

If f(.) is concave, then a necessary and sufficient condition for the Nash bargaining solution to obtain is

$$f'(u_1) = -u_2/u_1 \qquad (17)$$

<u>Proposition 2</u> If there is a unique PE solution to the generalised Rubinstein game of section 3.1, then this solution approaches the Nash bargaining solution as $\delta_1, \delta_2 \to 1$.

<u>Proof</u> We first note that by definition, $z_1 = f^{-1}(\delta_2 f(\delta_1 z_1))$, so applying f to both sides,

$$f(z_1) = \delta_2 f(\delta_1 z_1)$$

Now put $\delta_1 = \delta_2 = \delta$. Using the definition of a derivative,

$$f'(z_1) = \lim_{\delta \to 1} [f(\delta z_1) - f(z_1)]/[\delta z_1 - z_1]$$

$$= \lim_{\delta \to 1} [f(\delta z_1) - \delta f(\delta z_1)]/[-z_1(1-\delta)],$$

and using the remark about z_1 from above,

$$= \lim_{\delta \to 1} [-f(\delta z_1)/z_1]$$

$$= -f(z_1)/z_1$$

Next consider (17) again. At the PE solution, $u_1 = z_1$ and $u_2 = f(z_1)$, so the Proposition is proved. □

Remark. This result originally appeared in Binmore (1980).

4. THE N-PERSON GAME, WITH DISCOUNTING

In the present section we shall investigate an extension of Rubinstein's bargaining game to the case where there are $N (N \geq 2)$ players to share a cake of size 1. Extensions of Rubinstein's original game to situations with more than 2 players have already appeared in the literature. Among these suffice it here to mention two representative contributions by Herrero (1985) and Haller (1986). They both apply the same basic structure of the extended game: in the first round player 1 suggests a complete partitioning of the cake, and player 2, ..., N then vote on this partitioning. If all players accept the suggested division the game ends (and everybody receives a share of the cake in accordance with the agreed division). If, however, one (or more) player(s) reject(s), the partitioning is rejected and the game moves to the second round, where player 2 suggests a division (the size of the cake is now reduced to $\delta .1$, as all players are assumed to have the same discount factor δ, $0 \leq \delta < 1$). The procedure now repeats itself as above etc. The only substantial difference between Herrero's and Haller's description of the game lies in the voting procedure. In Herrero players vote successively (with perfect information) whereas Haller assumes that all players vote simultaneously (i.e. not knowing how the other players vote). This difference does seem to have implications for which of the suggested divisions of cake can be realized as a PEP, as only Haller gets the (negative) result that any partitioning is a PEP, independent of the size of the discount factor. Herrero's approach only leads to non-uniqueness of PEP, if the discount factor $\delta \geq 1/(N-1)$. However, if the equilibrium concept is changed to that of strong perfect equilibrium, Herrero shows that Rubinstein's uniqueness result still applies to the case with N ($N \geq 2$) players, where the players now are allowed to split into two coalitions. This result should therefore be compared to Proposition 3 of this section and Proposition 5 of section 5, where we for our different version of the N-person bargaining game show that there is a unique stationary PEP, and that any type of coalition formation is without benefits to the players.

In the present paper we consider a slightly different (though natural) extension of Rubinstein's original game. Our extension has the advantage that the method of proof introduced in section 2 can be applied straightforwardly to the N-player case ($N \geq 2$). The structure of the generalized game is as follows.

At $t = 0$, player 1 demands a share of the cake for himself. Player 2 can either accept or reject this demand. If he accepts, player 2 then demands a share of what is left of the cake for himself. Player 3 can either accept or reject this, and so on. If any player rejects a demand, this round of bargaining comes to an end. If no player has rejected a demand, the game ends and the cake is shared out on the agreed basis. If agreement has not been reached at $t = 0$, then at $t = 1$ a new round of bargaining commences in which player 2 is the first to make an offer to player 3. If it ran its full course, this round of bargaining would finish with player N making an offer to player 1. If the game continues sufficiently long, each player will get the chance to make the first demand in a bargaining round - in time period i

($i \geq 0$) the j^{th} player is the first to make a demand, where $j - 1 = i$ mod N.

In this section we consider non-cooperative solutions, deferring to section 5 the possibility of allowing coalition formation. As before let y_i be the outcome achieveable by player i ($i = 1, 2, \ldots, N$) in some PE, if the game ends after a bargaining round in which it was i's turn to make the first demand. Let δ_i ($0 \leq \delta_i < 1$) be player i's discount factor. As above we get a table of achieveable outcomes of the following form:

TABLE 3

Time (t)	Player making first demand	Achieveable outcome in some PE by player: 1	2		N
0	1	y_1	$\delta_2 y_2$	. . .	$\delta_N^{N-1} y_N$
1	2	$\delta_1^{N-1} y_1$	y_2	. . .	$\delta_N^{N-2} y_N$
2	3	$\delta_1^{N-2} y_1$	$\delta_2^{N-1} y_2$	. . .	$\delta_N^{N-3} y_N$
.	.	.	.		.
.	.	.	.		.
.	.	.	.		.
N-1	N	$\delta_1 y_1$	$\delta_2^2 y_2$	. . .	y_N

The argument is exactly the same as in the two-person case. Consider as usual player 1. At $t = N-1$, he is one period away from being the first to make a demand, when by definition his achieveable outcome will be y_1. So at $t = N-1$, his outcome is $\delta_1 y_1$ and in general his achieveable outcome at time t is δ_1 times his outcome at time $t + 1$ $(1 \leq t \leq N-1)$.

Now suppose that at $t = 0$ all players except player 1 receive their achieveable outcome. Then player 1, who gets the remainder, must receive

$$y_1 = 1 - \sum_{j=2}^{N} \delta_j^{j-1} y_j \qquad (18)$$

By applying the same argument to each player in turn at respectively $t = 0, 1, \ldots, N-1$, we obtain in general

$$y_i = 1 - \sum_{j=1}^{i-1} \delta_j^{N-i+j} y_j - \sum_{j=i+1}^{N} \delta_j^{j-i} y_j \qquad i=1, 2, \ldots, N \quad (19)$$

4.1 The special case of equal discount factors

Before seeking a general solution of these N equations, consider first the special case where $\delta_j = \delta$, all j. Then the players are identical except for the order in which they make their demands. Hence it must be the case that all the y_i are the same, so we can assume $y_i = y$, all i. (For example, if all players have the same discount factor, then if player 2 gets the opportunity to make the first demand, his chances must be exactly the same as those of player 1 when it was the latter's turn to make the first demand.) Therefore in this special case all but one of the equations (19) become redundant and we need to consider only (19) for $i = 1$, which can be rewritten as

$$y = 1 - Ky \qquad (20)$$

where
$$K = (\delta + \delta^2 + \ldots + \delta^{N-1})$$
$$= \delta(1 - \delta^{N-1})/(1 - \delta) > 0 \qquad (21)$$

So we have the following result

Proposition 3 In the generalised, N-person, Rubinstein bargaining game, if all players have the same discount factor ($0 \leq \delta < 1$), then there is a unique (non-cooperative) PEP in which player 1 receives

$$y = (1 - \delta)/(1 - \delta^N) \tag{22}$$

and in general player i's share is given by

$$\delta^{i-1}y, \quad i = 1, 2, \ldots, N \tag{23}$$

The game ends at $t = 0$ with agreement on these shares.

Remark 1. This proposition appears in Herrero (1985) with a different proof.

Remark 2. We stress again that we here only prove uniqueness of a stationary PE. However, it may be worth noting that the examples, which have been constructed to demonstrate non-uniqueness of PE in the $N>2$ case (see e.g. Sutton (1986)) all require rather peculiar strategies like demanding a large share of the cake for a third party rather than for oneself. Under our structure, this is not allowed, so maybe the stationary PE is the only PE, if our kind of restriction is put on the allowable strategies.

Remark 3. Herrero (1985) shows that if $\delta<1/(N-1)$ then the unique PE is the stationary PE.

Corollary If the conditions of Proposition 3 are satisfied then

(1) a rise in δ lowers the share of player 1;

(2) as $\delta \to 1$, the share of each player approaches $1/N$.

Proof of corollary Obvious, from consideration of (22) and from the fact that $(1 - \delta)/(1 - \delta^N) = 1/(1 + \delta + \delta^2 + \ldots + \delta^{N-1})$. □

4.2 The general case

We now revert to the general case where the discount factors δ_j may differ and the set of equations to solve is given by (19).

Let $\underline{y} = [y_1, y_2, \ldots, y_N]'$ and $\underline{i} = [1, 1, \ldots, 1]'$ be Nx1 column vectors, let I be the NxN identity matrix and define the NxN matrix D by

$$D = \begin{bmatrix} 0 & \delta_2 & \delta_3^2 & \cdots & \delta_N^{N-1} \\ \delta_1^{N-1} & 0 & \delta_3 & \cdots & \delta_N^{N-2} \\ \delta_1^{N-2} & \delta_2^{N-1} & 0 & \cdots & \delta_N^{N-3} \\ \vdots & \vdots & \vdots & & \vdots \\ \delta_1^2 & \delta_2^3 & \delta_3^4 & \cdots & \delta_N \\ \delta_1 & \delta_2^2 & \delta_3^3 & \cdots & 0 \end{bmatrix} \qquad (24)$$

The equations (19) can thus be written in matrix form as

$$\underline{y} = \underline{i} - D\underline{y} \qquad (25)$$

and we get immediately the following result.

<u>Proposition 4</u> In the generalised, N-person, Rubinstein bargaining game, where the discount factors δ_j may differ, if the matrix (I + D) is non-singular then there is a unique (non-cooperative) PE in which

$$\underline{y} = (I + D)^{-1}\underline{i} \qquad (26)$$

Denoting the typical element of $(I + D)^{-1}$ by h_{ij}, the solution for player i is

$$y_i = \sum_{j=1}^{N} h_{ij}, \qquad i = 1, 2, \ldots N \qquad (27)$$

Consequently, the actual shares which will be agreed at t = 0 are:

$$\delta_i^{i-1} \sum_{j=1}^{N} h_{ij}, \quad i = 1, \ldots, N \qquad (28)$$

Working Papers in Economics

65.	A.J. Phipps	Australian Unemployment: Some Evidence from Industry Labour Demand Functions; November 1982
66.	E.M.A. Gross & W.P. Hogan	Short Term Management of the Australian Exchange Rate, 1977-82; December 1982
67.	V.B. Hall	Industrial Sector Interfuel Substitution Following the First Major Oil Shock; January 1983
68.	J. Yates	Access to Housing Finance and Alternative Forms of Housing Loans in the 1980s; July 1983
69.	V.B. Hall	Major OECD Country Industrial Sector-Interfuel Substitution Estimates: 1960-79; August 1983
70.	F. Gill	Inequality and Arbitration of Wages in Australia: An Historical Perspective; December 1983
71.	W.J.Merrilees	Do Wage Subsidies Stimulate Training? An Evaluation of the Craft Rebate Scheme; November 1983
72.	M.C. Blad	Economic Policy and Catastrophe Theory; November 1983
73.	C.G.F. Simkin	Does Money Matter in Singapore?; April 1984
74.	J. Yates	Home Purchase Assistance for Low Income Earners; March 1984
75.	C.G.F. Simkin	Long-term Aspects of New Zealand's External Deficits; April 1984
76.	C.G.F. Simkin	Methodological Scepticism; July 1984
77.	V.B. Hall	Industrial Sector Fuel Price Elasticities of Demand Following the First and Second Major Oil Shocks; August 1984
78.	S.S. Joson	Substitutability of 'Buy Local' Policy for Tariff Protection in Small Economies; January 1985
79.	R.T. Ross	Analysis of the 1980 Sydney Survey of Work Patterns of Married Women: Further Results; January 1985
80.	J. Yates	Discrimination in Lending; May 1985
81.	R.T. Ross	Measuring Underutilisation of Labour: Beyond Unemployment Statistics; May 1985
82.	P.D. Groenewegen	Alfred Marshall as Professor of Political Economy at Cambridge 1885-1908; June 1985
83.	C.G.F. Simkin	Popper's Methodology and Economic Theory; July 1985
84.	E.M.A. Gross, W.P. Hogan & I.G. Sharpe	Market Information and Potential Insolvency of Australian Financial Institutions; July 1985
85.	F. Gill	Over-Award Payments; Result of a Survey conducted in 1982; December 1985
86.	S.K. Kim	Short Run Policy Analysis of Employment, Food Price and Rural-Urban Migration for a Labour-Abundant Developing Economy; January 1986
87.	E. Kiernan & D.B. Madan	Stochastic Stability in a Rational Expectations Model of a Small Open Economy; March 1986
88.	E. Gross	A Note on the Testability of Fama's Efficient Capital Market Hypothesis; February 1986
89.	M.C. Blad & E. Gross	Multinational Producers in an Arrow-Debreu type General Equilibrium Model; March 1987
90.	P. Saunders	Explaining International Differences in Public Expenditure: An Empirical Study; August 1986
91.	W.P. Hogan	International Debt and Foreign Exchange Markets; January 1987
92.	Michael C. Blad & Nicholas Oulton	Union-Firm Bargaining as a Repeated Prisoner's Dilemma; January 1987
93.	R.T.Ross	The Heckman Procedure for Estimating Static Disaggregate Labour Supply Functions: An International Comparison of Estimates for Married Women; March 1987
94.	W.P. Hogan	Assessing Insider Trading; June 1987
95.	J. Yates	Housing Policy Reform: A Constructive Critique; June 1987
96.	B.W. Ross	The Leisure Factor in Entrepreneurial Success during the 'Robber Baron' Era; July 1987

97.	F. Gill	Determination of Wage Relativities under the Federal Tribunal: 1953-1974; August 1987
98.	A.J. Phipps	Union Objectives, Wage Bargaining and the Phillips Curve; October 1987
99.	R.T. Ross	The Labour Market Position of Aboriginal People in New South Wales; November 1987
100.	L. Haddad	List Revisited: Dynamic Consideration of Trade and Protection; November 1987
101.	John Piggott	General Equilibrium Computation Applied to Public Sector Issues; December 1987
102.	J.A. Carlson & D.W. Findlay	Relative Prices, Wage Indexation and Unemployment; December 1987
103.	M. Waterson	A Model of Product Differentiation and Profitability; December 1987
104.	P.D. Groenewegen	Taxation and Decentralisation: A Reconsideration of the Costs and Benefits of a Decentralised Tax System; March 1988
105.	L. Ermini	Some New Evidence on the Timing of Consumption Decisions and on Their Generating Process; March 1988
106.	G. Mills	Spatially-Differentiated Trucking Markets: Equilibria under Price Regulation without Entry Restrictions; April 1988
107.	B.W. Ross	Strategic Commitment, Unknowledge and the Nature of Entreprenuerial Activity; April 1988
108.	S.S. Joson	Offsets and Development of Defence Support Industries in Small Economies; April 1988
109.	B.W. Ross	The Conglomerate and the Focussed Agglomerate: Modern Forms of the Leader-Commanded Firm; July 1988
110.	L. Ermini	Inertial Behavior on Schedule and Hierarchical Decomposition; July 1988
111.	L. Ermini	The Limits of Systems Control Theory in Economic Policy-Making; July-October 1988
112.	P.D. Groenewegen	Neo-Classical Value and Distribution Theory: The English Speaking Pioneers; September 1988
113.	V.B. Hall, T.P. Truong & V.A. Nguyen	An Australian Fuel Substitution Tax Model: ORANI-LFT; October 1988
114.	V.B. Hall, T.P. Truong & V.A. Nguyen	Responses to World Oil and Coal Shocks, in an Australian Short-Run Fuel Substitution Tax Model; October 1988
115.	F. Gill	Social Justice and the Low-Paid Worker; October 1988
116.	G. Kingston	Theoretical Foundations of Constant-Proportion Portfolio Insurance; October 1988
117.	V.B. Hall & D.R. Mills	Is Medium Temperature Solar Thermal Process Steam Viable for Australia? Some Preliminary Results; November 1988
118.	W.P. Hogan	Insider Information and Market Adjustment; November 1988
119.	L. Ermini	Reinterpreting a Recent Temporally Aggregated Consumption-Cap Model; December 1988
120.	P.D. Groenewegen	Progressive Personal Income Tax - A Historical Perspective; December 1988
121.	M.C. Blad & N. Oulton	Rubinstein's Solution of the Bargaining Problem: Some Generalisations and Extensions; December 1988

Copies are available upon request from:

Department of Economics,
The University of Sydney,
N.S.W. 2006, Australia.

Working Papers in Economics Published or Accepted for Publication Elsewhere

2.	I.G. Sharpe & R.G.Walker	Journal of Accounting Research, Vol. 13, No.2, Autumn 1975
3.	N. V. Lam	Journal of the Developing Economies, Vol. 17, No.1, March 1979
4.	V.B. Hall & M.L. King	New Zealand Economic Papers, Vol. 10, 1976
5.	A.J. Phipps	Economic Record, Vol. 53, No. 143, September 1977
6.	N.V. Lam	Journal of Development Studies, Vol. 14, No. 1, October 1977
7.	I.G. Sharpe	Australian Journal of Management, April 1976
9.	W.P. Hogan	Economic Papers, No. 55, The Economic Society of Australia and New Zealand
12.	I.G. Sharpe & P.A. Volker	Economics Letters, 2, (1979)
13.	I.G. Sharpe & P.A. Volker	Kredit and Kapital, Vol. 12, No.1, 1979
14.	W.P. Hogan	Some Calculations in Stability and Inflation, A.R.Bergstrom et.al. (eds), John Wiley and Sons, 1978
15.	F. Gill	Australian Economic Papers, Vol. 19, No. 35, December 1980
18.	I.G. Sharpe	Journal of Banking and Finance, 4, 1980
21.	R.L. Brown	Australian Journal of Management, Vol. 3, No. 1, April 1978
23.	I.G. Sharpe & P.A. Volker	The Australian Monetary System in the 1970s, M. Porter (ed.), Supplement to the Economic Board 1978
24.	V.B. Hall	Economic Record, Vol. 56, No. 152, March 1980
25.	I.G. Sharpe & P.A. Volker	Australian Journal of Management, October 1979
27.	W.P. Hogan	Malayan Economic Review, Vol. 24, No. 1, April 1979
28.	P. Saunders	Australian Economic Papers, Vol. 19, No. 34, June 1980
29.	W.P. Hogan I.G. Sharpe & P.A. Volker	Economics Letters, 6 (1980)
29.	W.P. Hogan I.G. Sharpe & P.A. Volker	Economics Letters, 7 (1981)
30.	W.P. Hogan	Australian Economic Papers, Vol. 18, NO. 33, December 1979
32.	R.W. Bailey, V.B. Hall & P.C.B. Phillips	Keynesian Theory, Planning Models, and Quantitative Economics, G. Gandolfo and F. Marzano (eds.), Vol. II, 703-767, 1987
38.	U.R. Kohli	Australian Economic Papers, Vol. 21, No. 39, December 1982
39.	G. Mills	Journal of the Operational Research Society 33, (1982)
41.	U.R. Kohli	Canadian Journal of Economics, Vol. XV, No. 2, May 1982
42.	W.J. Merrilees	Applied Economics, Vol. 15, February 1983
43.	P. Saunders	Australian Economic Papers, Vol 20, No. 37, December 1981
45.	W.J. Merrilees	Canadian Journal of Economics, Vol. XV, No. 3, August 1982
46.	W.J. Merrilees	Journal of Industrial Economics, Vol. XXXI, March 1983
49.	U.R. Kohli	Review of Economic Studies, Vol. L(1) No. 160, January 1983
50.	P. Saunders	Economic Record, Vol. 57, No. 159, December 1981
53.	J. Yates	AFSI, Commissioned Studies and Selected Papers, AGPS, IV 1982
54.	J. Yates	Economic Record, Vol. 58, No. 161, June 1982
55.	G. Mills	Seventh Australian Transport Research Forum-Papers, Hobart, 1982
56.	V. B.Hall & P. Saunders	Economic Record, Vol. 60, No. 168, March 1984
57.	P. Saunders	Economic Record, Vol. 59, No. 166, September 1983
58.	F. Gill	Economie Appliquee, Vol XXXVII, 1984, No's 3-4, pp. 523-541
59.	G. Mills & W. Coleman	Journal of Transport Economics and Policy, Vol. XVI, No. 3, September 1982
60.	J. Yates	Economic Papers, Special Edition, April 1983

61.	S.S. Joson	Australian Economic Papers, Vol. 24, No. 44, June 1985
62.	R.T. Ross	Australian Quarterly, Vol. 56(3), Spring 1984
63.	W.J. Merrilees	Economic Record, Vol. 59, No. 166, September 1983
65.	A.J. Phipps	Australian Economic Papers, Vol. 22, No.41, December 1983
67.	V.B. Hall	Economics Letters, 12, (1983)
69.	V.B. Hall	Energy Economics, Vol. 8, No. 2, April 1986
70.	F. Gill	Australian Quarterly, Vol. 59, No.2, Winter 1987
71.	W.J. Merrilees	Australian Economic Papers, Vol. 23, No. 43, December 1984
74.	J. Yates	Australian Quarterly, Vol. 56 (2), Winter 1984
77.	V.B. Hall	Economics Letters, 20, (1986)
78.	S.S. Joson	Journal of Policy Modeling, Vol. 8., No. 2, Summer 1986
79.	R.T. Ross	Economic Record, Vol. 62, No. 178, September 1986
81.	R.T. Ross	Australian Bulletin of Labour, Vol. 11(4), September 1985
82.	P.D. Groenewegen	History of Political Economy, Vol. 20(4), Winter 1988
84.	E.M.A. Gross, W.P. Hogan & I.G. Sharpe	Australian Economic Papers, Vol. 27, No. 50, June 1988
94.	W.P. Hogan	Company and Securities Law Journal, Vol. 6, No. 1, February 1988