GORMAN POLAR FORMS AND THE S-BRANCH UTILITY TREK

by

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Abstract

With the advent of flexible functional forms and their superior econometric performance, the use of closed form utility functions is now limited to general equilibrium or simulation models. This is because they automatically satisfy the axioms of consumer behaviour and thus guarantee that a correctly characterised numerical solution will represent a utility maximum.

In this context the Gorman Polar Form proposed by Brown and Heien in their S-Branch paper (1972) is re-examined and it is argued that a reformulation enhances its use considerably.
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Addendum
I. Introduction

In a celebrated paper on demand systems, Brown and Helen (1972) developed the "S-branch" utility function. Its advantages were twofold: (i) it permitted the resulting demand equations to have greater flexibility than earlier generation functions in respect of their elasticity values. (ii) Estimation remained relatively simple as a result of the two stage budgeting procedure which the separable utility function permitted and the limited number of additional parameters needed to achieve the flexibility in the elasticities.

While the last decade has seen the use of flexible functional forms, and consequent improvement in elasticity estimates, the use of closed form utility functions which permit a reasonable range of elasticity values continues to be of importance. In particular, in general equilibrium or simulation models it is desirable that the functions have the appropriate concavity and homogeneity properties regardless of the values taken by the data – a property not generally ascribable to forms such as the translog or generalized Leontief.

The purpose of this paper is to present an alternative formulation of the S-branch which, in addition to having most of the properties of the Brown-Helen formulation, has two additional advantages. In Section II we briefly state the Brown-Helen results. In Section III we reformulate the system and describe its properties.

II. The S-branch Utility Tree

The S-branch utility function is a member of the class of Gorman polar forms (GPF) (Gorman, 1959) and is defined as a two-level CES function with "committed" expenditures at the lower level. In Brown and Helen's terminology
it is given by

\[ U = \sum_{r=1}^{s} \sum_{j=1}^{n} \left[ \frac{p_{s}}{p_{s,j}} \left( q_{s} - y_{s} \right) \right]^{\frac{\beta_{s}}{\beta_{s,j}}} \]  

(1)

where \( \beta_{s} > 0 \), \( (q_{s} - y_{s}) > 0 \), \( y_{s} \leq 0 \), \( p_{s} = 1 \), \( \rho - \left( \frac{1}{\rho} \right) < 1 \), \( \rho_{s} = 1 \), \( \rho = 1 \).

The weak separability of \( U \) with respect to its partitions permits two-stage budgeting and the resulting demand equations are given by equation (9) in Brown and Helen,

\[ q_{s} = y_{s} \left( \frac{p_{s}}{p_{s,j}} \right)^{\frac{\rho}{\rho_{s}}} \left[ \sum_{r=1}^{n} \left( \frac{x_{s}}{x_{s,j}} \right)^{\frac{\rho}{\rho_{s}}} \right] \left[ \sum_{r=1}^{n} \left( \frac{x_{s}}{x_{s,j}} \right)^{\frac{\rho}{\rho_{s}}} \right]^{-1} \]

(2)

\[ m - \sum_{r=1}^{n} \sum_{j=1}^{n} p_{r,j} y_{r} \]

where \( x_{s} \) is defined as

\[ x_{s} = \sum_{j=1}^{n} p_{s,j} \left( \frac{p_{s}}{p_{s,j}} \right)^{\frac{\rho}{\rho_{s}}} \]

(3)

Brown and Helen derive the own price elasticities and the Slutsky terms for these demands and illustrate that their system, in contrast to the linear expenditure system of Stone (1954), (1) permits net complements as well as substitues and (11) permits the own price elasticities to lie in the range \((0, \infty)\). These superior properties are attained at the estimation cost of \((s + 1)\) additional parameters, i.e. the \( s \) substitution elasticities for each group and the overall elasticity of substitution \( \rho \), none of which is assumed to be unity. In the final section of their paper Brown and Helen present estimates of a demand system for twenty-eight elementary food commodities in five groups.

Blackorby, Boyce and Russell (BBR) subsequently re-examined the empirical

properties of this system. To test the restrictive ness of the Brown and Helen model they proposed that the component \( \sum_{r} \sum_{j} p_{r,j} y_{r} \) be replaced with a generalized Leontief form:

\[ \sum_{r} \sum_{j} \sum_{r} \left( \frac{y_{r}}{y_{r,j}} \right)^{\frac{1}{\rho_{r,j}}} \left( \frac{p_{r,j}}{p_{s,j}} \right)^{\frac{1}{\rho_{s,j}}} \]

(4)

Upon estimating the more general system, BBR concluded that the data reject the Brown and Helen formulation in favour of their more general specification.

In the long discussion of the results they note that the estimated parameter values violated the global concavity conditions - a frequent result when flexible functional forms are used.

III. An Inverted GPE

Given that flexible functional forms almost invariably out perform the type of closed form utility function developed by Brown and Helen, the real interest in such functions must lie in their use in GE type models. For example, if we wish to ensure that consumer responses to simulated price changes will always reflect a concave cost function, then it is necessary to use the most general closed-form utility function which will permit this. More specifically, it seems to us that the only important use of two level closed form functions is in disaggregated simulation analysis.

In the first instance, this is because available priors on consumer behaviour generally come in the form of elasticity estimates for aggregate products rather than the elementary commodities e.g. for "meat", rather than "lamb". In order to model disaggregated systems, a valid strategy would be to
impose the available priors on the aggregate quantities and simulate the model results over different possible elasticity values for the elementary commodities. The key issue is how this can be done in a two-stage budgeting process.

Frequently, the analyst is faced with a problem related to the demand consequences of a policy change (e.g., commodity taxation) which relates to certain elementary commodities, but finds that relevant information on price elasticities is available only at a much broader level of aggregation. The problem which motivated the present paper is that of determining the effects of differential tax treatment of various alcoholic beverages within a broad beverage aggregate. (For example, how could we determine the revenue consequences of taxing Californian and French wines differentially, where the only prior information relates to the demand elasticity for all wines?) The two-stage budgeting approach discussed in Gorman (1959) is well suited to such a problem.

The second important issue is the way in which relatively unrestrictive price elasticities can be built into such models. Of course, this is the role played by the "r" parameters in the work of Brown and Helen and BBY. The presence of the r = r values permits the systems to have complementary goods and price elasticities of demand in excess of unity. However, from the standpoint of economic decision making, one must question the practice of specifying tastes in a manner that depends upon necessary or committed quantities of the elementary goods. A considerably more appealing approach, particularly in nutritional examples where a balance between different food types is key, is to define the committed expenditures in terms of broad aggregates (e.g. "meat" or "vegetables"). This means that households would commit a specific component of their budget to each aggregate and this commitment would then be translated into demands at the lower level.

To meet both of the above requirements it is clearly necessary that the utility function permits linearly homogeneous price and quantity aggregates. While a GMM at the lower level yields a linearly homogeneous price aggregator, a similar quantity aggregator with the property that price times quantity for a particular aggregate equal expenditure (at any set of prices) will not necessarily exist unless the lower level utility function is homothetic. This is referred to by Blackorby, Primont and Russell (1978) as additive (as opposed to strong) price aggregation. The S-Branch utility function does not satisfy these requirements because of the non-homotheticity of the lower level utility functions. However, a reformulation of it would. Consider the function

$$U = \left[ \sum r_i (q_i - \gamma_i)^2 \right]^{\frac{1}{2}}$$

(5)

where $q_i$ is a linear homogeneous aggregator function

$$q_i = \left[ \sum \frac{P_i}{P} \right]^{\frac{1}{2}}$$

(6)

and $U$ is defined for $(q_i - \gamma_i) > 0 \forall i$. We will term the function defined by (5) and (6) the 'Inverted S-Branch'.

The lower level demands, conditional upon expenditure on group $i$ ($q_i$), are given by

$$q_i = (P_i/P_1)^{\sigma_i} \left[ \sum \frac{P_i}{P_1} (R_i/P_1) \right]^{\sigma_i - 1} q_i$$

(7)

The upper level demand functions for the aggregate $q_i$ are easily shown to be of the form

$$q_i = \gamma_i + (P_i/P_1)^{\sigma_i} \left[ \sum \frac{P_i}{P_1} (P_i/P_1) \right]^{\sigma_i - 1} (\gamma_i - \sum \frac{P_i}{P_1} q_i)$$

(8)
where $m$ is total expenditure. The unconditional demands are simply $q_i$ in
(7) where $q_i$ is substituted by means of $p_i$'s in (8)

$$q_i = \left( \frac{\sigma}{\alpha_i} / \sum_{j \neq i} p_j \left( \frac{\sigma_j}{\alpha_j} \right)^{1-\sigma_j} \right)^{-1} \left[ p_i \left( 1 - \sigma_i \right) \right]^{\frac{1}{1-\sigma_i}}$$

(9)

where $\alpha_i$ is the marginal propensity to spend on aggregate $i$, i.e.

$$\alpha_i = p_i (\alpha_i / p_i)^{1-\sigma_i}$$

(10)

We note that the lower level utility function is a monotonic transformation of the quantity aggregator function, i.e. $U_s = (q_s - \gamma_s)^{\theta_s}$

it is straightforward to establish that the form of the price index for each aggregate is

$$p_s = \left[ \left( \frac{\sigma}{\alpha_s} / \sum_{j \neq i} p_j \left( \frac{\sigma_j}{\alpha_j} \right)^{1-\sigma_j} \right)^{-1} \right]^{\frac{1}{1-\sigma_s}}$$

(11)

Limiting cases for these demand functions can be obtained by setting the number of partitions to $s = 1$ and by setting the elasticities of substitution equal to unity or zero.

$\Delta_1$. Properties of Individual Commodity Demands.

We now examine the properties of the demand equations in (9) and compare these with the Brown and Helen results. The own price elasticity for $q_i$ is given by

$$e_{q_i} = \left( \frac{1}{\theta_s} \left( 1 - \sigma_i \right) \right) \left[ p_i \left( 1 - \sigma_i \right) \theta_s \left( m - \sum_j p_j \right) \right]^{\frac{1}{1-\sigma_i}}$$

(12)

where $\phi_i$ is defined as good i's share of expenditure in group $s$, i.e.

$$\phi_i = q_i / p_i \theta_s \quad \Delta_{s_i}$$

and $e_{q_i}$ is necessarily negative.

The cross-price elasticities are given by

$$e_{q_i, q_j} = \left( 1 - \sigma_i \right) \phi_i \left( 1 - \sigma_j \right) \phi_j \left[ p_i \left( 1 - \sigma_i \right) \theta_s \left( m - \sum_j p_j \right) \right]^{\frac{1}{1-\sigma_i}}$$

(13)

$$e_{q_i, r} = \left( 1 - \sigma_i \right) \phi_i \left[ p_r \left( 1 - \sigma_i \right) \theta_s \left( m - \sum_j p_j \right) \right]^{\frac{1}{1-\sigma_i}}$$

(14)

and these expressions permit both gross complements and substitutes. The Slutsky elasticities are obtained by noting that

$$\frac{\delta \ln q_i}{\delta \ln p_i} = \frac{\delta \ln q_i}{\delta \ln p_i} + \frac{\delta \ln q_i}{\delta \ln m}$$

(15)

Since the income elasticities are given by

$$\frac{\delta \ln q_i}{\delta \ln m} = \theta_s \quad \forall 1 \leq s$$

(16)

the set of Slutsky elasticities can be derived using this result and (12).

$^2$ If we take logarithms of both sides of (8) and partially differentiate with respect to $p_i$, we obtain an expression involving $\delta \theta / \delta p_i$ and $\delta p_i / \delta p_i$. But the elasticity $(\delta p_i / \delta p_i)(p_i / p_s)$ can easily be shown to equal $\phi_i$ by differentiating the price index $p_i$ defined in (10) and this is then substituted into $\delta \theta / \delta p_i$ and the result is obtained that

$$\frac{\delta \theta}{\delta p_i} = (1 - \sigma) \phi_i \left( 1 - \phi_i \right).$$
These are given by

\[ \tilde{c}_{s,s} = \phi_s \left( \sigma_s - 1 + (1-\theta_s) \Sigma p_j y_j \right) \]  \hspace{1cm} (17)

\[ \tilde{c}_{s,r} = -\theta_s \phi_r \left( m_s - \Sigma p_j y_j \right) \frac{1}{m_s} \frac{\Sigma p_j y_j}{n_s} \]  \hspace{1cm} (18)

In contrast to the Brown and Helen results these permit both net substitutes and complements.

**B. Properties of aggregate commodity demands.**

The demand elasticities for the aggregate quantities in this system are

\[ e_{rr} = \left( \frac{q_r - y_r}{q_r} \right) \left( (\sigma_r - 1) - \theta_r - \frac{\Sigma p_j y_j}{\Sigma p_j y_j} \right) \]  \hspace{1cm} (19)

\[ e_{rs} = \left( \frac{q_s - y_s}{q_s} \right) \left( (\sigma_r - 1) - \frac{\Sigma p_j y_j}{\Sigma p_j y_j} \right) \]  \hspace{1cm} (20)

Clearly \( e_{rr} \) is necessarily negative, while \( e_{rs} \) can be positive or negative and hence \( r \) and \( s \) may be gross substitutes or complements. It is straightforward to show that the compensated cross price elasticity reduces to

\[ \tilde{e}_{rs} = \left( \frac{q_r - y_r}{q_r} \right) \frac{\Sigma p_j y_j}{\Sigma p_j y_j} \]  \hspace{1cm} (21)

What are the drawbacks to reformulating the S-branch in this manner? The most obvious one is the constraint placed upon the income elasticities of demand. The cost of introducing a homogeneous of degree one function at the lower level is that the income elasticities of all elementary commodities in a given group are necessarily equal - though they may be greater or less than unity. This can be seen from equation (16) above. How serious a restriction this is will clearly depend upon the number of goods in each partition.

**Conclusion**

We have argued that, with the advent of flexible functional forms for purposes of estimation, the most important use of two stage budgeting systems, such as the S-branch, lies in simulation analysis at a disaggregated level. Given this, we have proposed a form of utility function which (a) permits generally available prior to be introduced, (b) satisfies the requirements of additive price aggregation in order that linear homogeneous price and quantity aggregates exist at the intermediate level of the system and (c) corresponds to a more reasonable notion of how consumers distribute their income to "committed" expenditures. This involves a lower level utility function which is homothetic and an upper level which is a Gorman polar form. In contrast to the properties of the Brown and Helen system, the inverted S-Branch has a broader range of possible values for cross price effects.

While we see no value in reestimating the demand equations, given our discussion of flexible functional forms, a comment regarding the number of parameters to be estimated is in order. Brown and Helen's system involves \((2n + s)\) parameters whereas the inverted S-Branch would require \((n + 2s)\). Whichever system has the greater number of parameters, depends upon the number of partitions into which the \( n \) goods are classified.
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