

Semigroups

In this chapter we introduce semigroups and Green's relations. We then give some well known semigroup results that will be used later. Finally we conclude with three examples of semigroups, which are important for the topic of the thesis.

Definition 1. A semigroup is a nonempty set S together with a binary operation, usually denoted by juxtaposition, which is associative. That is,

$$x(yz) = (xy)z$$

for all $x, y, z \in S$.

Throughout S will denote a semigroup. We say y is an *inverse* of x if $xyx = x$ and $yxy = y$. A semigroup S is *regular* if, for each $x \in S$, there exists $y \in S$ with $xyx = x$; in this case yxy is an inverse of x . An element $x \in S$ is called an *identity* if $xy = yx = y$ for all $y \in S$. If an identity exists it is clearly unique; we denote it by 1_S or just 1 , and say that S is a *monoid*. Given a semigroup S , we can construct a monoid S^1 which contains it as follows. If S is a monoid we put $S^1 = S$. Otherwise we form $S^1 = S \cup \{1\}$ by adjoining a single element 1 , and define the operation \cdot on S^1 by

$$\begin{aligned} 1 \cdot x &= x && \text{for } x \in S^1, \\ x \cdot 1 &= x && \text{for } x \in S, \\ x \cdot y &= xy && \text{for } x, y \in S. \end{aligned}$$

An element $x \in S$ is called a *zero* if $xy = yx = x$ for all $x \in S$. Again a zero is unique if it exists, and is denoted 0_S or 0 .

As usual we denote

$$T_1T_2 = \{xy \mid x \in T_1 \text{ and } y \in T_2\}$$

for subsets T_1 and T_2 of S , and so on. We say a nonempty subset $T \subseteq S$ is a *subsemigroup* if $T^2 \subseteq T$; then T becomes a semigroup by restricting the operation. A subsemigroup is called a *subgroup* if it happens to be a group with respect to the restricted operation. Also T is a *left (right) ideal* if $ST \subseteq T$ ($TS \subseteq T$). Finally T is a *two sided ideal* or simply *ideal* if it is both a left and right ideal. Given an ideal T , we can form a *quotient semigroup* $S/T = S \setminus T \cup \{0\}$ with operation \cdot defined by

$$\begin{aligned} x \cdot y &= xy && \text{if } xy \notin T, \\ x \cdot y &= 0 && \text{if } xy \in T, \\ 0 \cdot x &= x \cdot 0 = 0 \cdot 0 = 0, \end{aligned}$$

for $x, y \in S \setminus T$. Note that 0 is a zero of S/T , as the notation suggests.

Fundamental to the study of any semigroup are certain equivalence relations introduced by Green [7]. These *Green's relations* are defined as follows.

Definition 2. Define relations $\leq_{\mathcal{R}}, \leq_{\mathcal{L}}, \leq_{\mathcal{J}}, \mathcal{R}, \mathcal{L}, \mathcal{J}, \mathcal{H}$ and \mathcal{D} on S , as follows.

- (i) For $x, y \in S$, write $x \leq_{\mathcal{R}} y$ if $x \in yS^1$.
- (ii) For $x, y \in S$, write $x \leq_{\mathcal{L}} y$ if $x \in S^1y$.

- (iii) For $x, y \in S$, write $x \leq_{\mathcal{J}} y$ if $x \in S^1 y S^1$.
- (iv) Let $\mathcal{R} = \leq_{\mathcal{R}} \cap \geq_{\mathcal{R}}$, and similarly for \mathcal{L} and \mathcal{J} .
- (v) Let $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$.
- (vi) Let \mathcal{D} be the equivalence relation generated by \mathcal{R} and \mathcal{L} .

Because $1 \in S^1$, certainly $\leq_{\mathcal{R}}$, $\leq_{\mathcal{L}}$ and $\leq_{\mathcal{J}}$ are transitive and reflexive. Thus \mathcal{R} , \mathcal{L} , \mathcal{J} and \mathcal{H} are equivalence relations. In particular, $\mathcal{D} \subseteq \mathcal{J}$. An equivalence class under \mathcal{R} is called an \mathcal{R} class, and so on. We use \mathbb{D} to denote the set of \mathcal{D} classes in S . Also for $D \in \mathbb{D}$, let \mathbb{L}_D and \mathbb{R}_D denote the sets of \mathcal{L} and \mathcal{R} classes in D respectively. We say an equivalence relation σ on S is a *left congruence* if $x \sigma y$ implies $ax \sigma ay$ for $a \in S^1$. Define a *right congruence* similarly. The following properties of Green's relations can be found in any standard semigroup text (see for example [10] or [11]). Properties (ii) and (iii) constitute a fundamental result in semigroup theory known as Green's Lemma. Property (iv) says that \mathcal{R} and \mathcal{L} commute under composition, and that their composite is exactly \mathcal{D} .

Proposition 3. *Let S be any semigroup.*

- (i) *The relation \mathcal{R} is a left congruence, while \mathcal{L} is a right congruence.*
- (ii) *Suppose that $x \in S$ and $a \in S^1$ are such that $xa \mathcal{R} x$. Then right multiplication by a gives an \mathcal{R} class preserving bijection from the \mathcal{L} class of x to the \mathcal{L} class of xa .*
- (iii) *Suppose that $x \in S$ and $a \in S^1$ are such that $ax \mathcal{L} x$. Then left multiplication by a gives an \mathcal{L} class preserving bijection from the \mathcal{R} class of x to the \mathcal{R} class of ax .*
- (iv) *For $x, y \in S$, we have*

$$\begin{aligned}
 & x \mathcal{D} y \\
 & \text{if and only if } x \mathcal{R} z \mathcal{L} y \text{ for some } z \in S, \\
 & \text{if and only if } x \mathcal{L} w \mathcal{R} y \text{ for some } w \in S.
 \end{aligned}$$

- (v) *If one element of a \mathcal{D} class D has an inverse then each element of D has an inverse, in which case we say that D is a regular \mathcal{D} -class. A \mathcal{D} class is regular if and only if it contains an idempotent. Thus a semigroup is regular if and only if each \mathcal{D} class contains an idempotent.*
- (vi) *The \mathcal{H} class of any idempotent is a subgroup.*
- (vii) *Suppose $x \mathcal{D} y$. Then $x \mathcal{R} xy \mathcal{L} y$ exactly when there exists an idempotent e with $x \mathcal{L} e \mathcal{R} y$.*

We are mostly concerned with finite semigroups in this thesis. However, many of the results hold in a more general class of semigroups, which we now define.

Definition 4. *A semigroup S is group bound if, for every element $x \in S$, there is a positive integer n such that x^n is contained in a subgroup.*

As was indicated above, finite semigroups are a special case of group bound semigroups. Indeed suppose S is finite, and consider an element $x \in S$. Then the infinite sequence x^1, x^2, x^3, \dots must contain a repetition. Suppose $x^a = x^{a+b}$ where a and b are positive integers. It follows easily by induction that $x^a = x^{a+nb}$ for any positive integer n . Because $b > 0$, we can choose n large enough that $nb > a$. Then multiplying $x^a = x^{a+nb}$ by x^{nb-a} , we obtain $x^{nb} = x^{2nb}$, so that x^{nb} is idempotent. Thus $\{x^{nb}\}$ is a (trivial) subgroup. Since $x \in S$ was arbitrary, S is indeed group bound. It is clear that S^1 is group bound whenever S is. The following results are well known, and we provide proofs for completeness.

Theorem 5. *Suppose S is a group bound semigroup.*

- (i) *If $x \in S$ and $a, b \in S^1$ are such that $axb = x$ then $xb \mathcal{R} x \mathcal{L} ax$.*
- (ii) *If $x \mathcal{J} ax$ then $x \mathcal{L} ax$, for $x \in S$ and $a \in S^1$.*
- (iii) *The relations \mathcal{J} and \mathcal{D} coincide.*
- (iv) *If $x, y \in S$ are such that $x \mathcal{D} xy \mathcal{D} y$, then $x \mathcal{R} xy \mathcal{L} y$.*

Proof.

- (i) It is clear by induction on n that $a^n x b^n = x$ for all $n \geq 1$. Because S is group bound, there exists $n \geq 1$ such that $b^n = c$ is contained in a subgroup, in which it has an inverse c^{-1} . Then $b^{2n} c^{-1} = c^2 c^{-1} = c = b^n$, so that

$$x = a^n x b^n = a^n x b^{2n} c^{-1} = a^n x b^n b^n c^{-1} = x b^n c^{-1} = (xb) b^{n-1} c^{-1}.$$

Thus $x \leq_{\mathcal{R}} xb$. Clearly $xb \leq_{\mathcal{R}} x$, so $xb \mathcal{R} x$. The other relation follows similarly.

- (ii) By assumption, there exists $b, c \in S^1$ with $x = baxc$. By (i), $x \mathcal{L} bax$. Thus

$$x \mathcal{L} bax \leq_{\mathcal{L}} ax \leq_{\mathcal{L}} x,$$

whence $x \mathcal{L} ax$.

- (iii) In any semigroup $x \mathcal{D} y$ implies $x \mathcal{J} y$. Suppose conversely that $x \mathcal{J} y$, so that there exist $a, b \in S^1$ such that $axb = y$. Then

$$x \mathcal{J} y = axb \leq_{\mathcal{J}} ax \leq_{\mathcal{J}} x.$$

Thus $x \mathcal{J} ax$, so that $x \mathcal{L} ax$ by (ii). Similarly $x \mathcal{R} xb$, and multiplying this on the left by a gives

$$ax \mathcal{R} axb = y.$$

Hence $x \mathcal{L} ax \mathcal{R} y$, so that $x \mathcal{D} y$ as required.

- (iv) This follows immediately from (ii) and its dual.

□

We now define the partition, Brauer and Temperley-Lieb semigroups. First recall some simple properties of equivalence relations. If I is a set and x is any binary relation on I , then there is a unique minimal equivalence relation on I which contains x , which we denote $\langle x \rangle$ and call the *equivalence relation generated by x* . Also if x is an equivalence relation on I and $J \subseteq I$, then the restriction

$$x_J = \{(i, j) \in x \mid i, j \in J\}$$

of x to J is an equivalence relation on J . Finally for any equivalence relation x , let \tilde{x} denote the set of equivalence classes of x . Now fix an integer $n \geq 1$. For convenience, we denote

$$\begin{aligned} I &= \{1, 2, 3, \dots, n\}, \\ I' &= \{1', 2', 3', \dots, n'\}, \\ I'' &= \{1'', 2'', 3'', \dots, n''\}. \end{aligned}$$

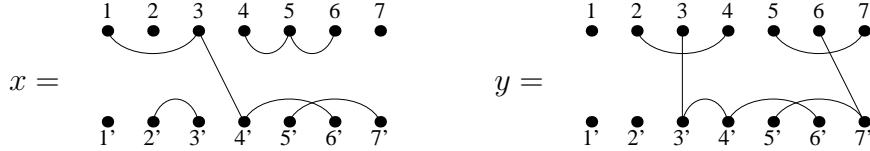
Let A_n denote the set of equivalence relations on the set $I \cup I'$. We represent elements of A_n by diagrams as follows. First arrange $2n$ dots in the plane and label them as shown.

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & \dots & n-1 & n & \\ \bullet & \bullet & \bullet & \bullet & \dots & \bullet & \bullet & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ \bullet & \bullet & \bullet & \bullet & \dots & \bullet & \bullet & \\ 1' & 2' & 3' & 4' & \dots & (n-1)' & n' & \end{array}$$

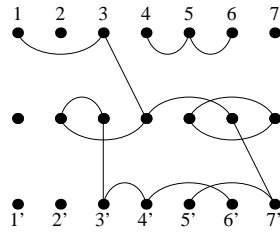
Now to represent $x \in A_n$, join up pairs of dots in such a way that $(i, j) \in x$ exactly when i and j are joined by a path. For instance, the elements $x, y \in A_7$ whose equivalence classes are

$$\begin{aligned} \tilde{x} &= \{\{1, 3, 4', 6'\}, \{2\}, \{4, 5, 6\}, \{7\}, \{1'\}, \{2', 3'\}, \{5', 7'\}\}, \\ \tilde{y} &= \{\{1\}, \{2, 4\}, \{3, 3', 4', 6'\}, \{5, 7\}, \{6, 5', 7'\}, \{1'\}, \{2'\}\} \end{aligned}$$

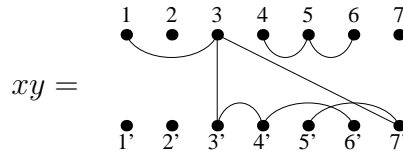
can be represented by the diagrams



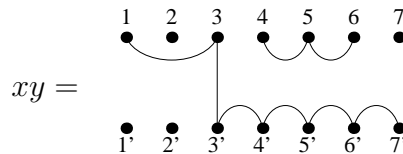
Of course the diagram representing a particular element is not unique. We define a binary operation on A_n as follows. Consider two elements $x, y \in A_n$. Let y' denote the equivalence relation on the set $I' \cup I''$ which is obtained from y by appending a $'$ to each number. Consider the equivalence relation $\langle x \cup y' \rangle$ generated by x and y' , and restrict to $I \cup I''$ to obtain $\langle x \cup y' \rangle_{I \cup I''}$. Let xy be the equivalence relation on $I \cup I'$ obtained from $\langle x \cup y' \rangle_{I \cup I''}$ by replacing each double dashed element i'' with the single dashed element i' , for $i \in I$. This operation $(x, y) \mapsto xy$ can also be described in terms of the diagrams. We first concatenate x and y , identifying the dot i' in x with the dot i in y . This is shown below for the above elements of A_7 .



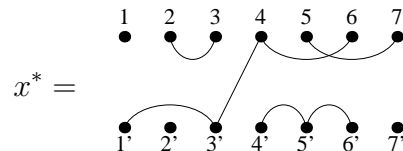
Equivalence classes now correspond to connected components. We remove those connected components which only contain dots from the central row. We then remove the remaining central dots, retaining the “edges” that pass through them. In this way we obtain a diagram representing the product xy , shown below for the example.



Of course this element is more naturally represented by the diagram



It is clear from this geometric construction that A_n is a semigroup under this operation. We call A_n the *partition semigroup* or *partition monoid*. There is a natural anti-involution $*$ on A_n which swaps i and i' , for each $i \in I$. Geometrically this simply reflects the diagrams vertically. Thus in the above example,



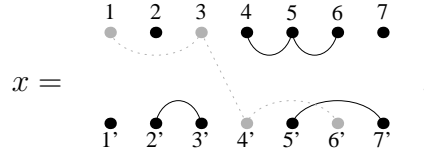
To describe Green's relations in A_n , define the following functions, for $x \in A_n$:

$$\begin{aligned} d(x) &= \#\{J \in \tilde{x} \mid J \cap I \neq \emptyset \neq J \cap I'\}. \\ r(x) &= (\{J \in \tilde{x} \mid J \subseteq I\}, \{J \cap I \mid J \in \tilde{x} \text{ and } J \cap I \neq \emptyset \neq J \cap I'\}). \\ l(x) &= (\{J \in \tilde{x} \mid J \subseteq I'\}, \{J \cap I' \mid J \in \tilde{x} \text{ and } J \cap I \neq \emptyset \neq J \cap I'\}). \end{aligned}$$

We use $r(x)_1$ and $r(x)_2$ to denote the two components of $r(x)$, and similarly for $l(x)$. It is clear that

$$\tilde{x}_I = r(x)_1 \cup r(x)_2,$$

so that in particular, I is a disjoint union of the elements of $r(x)_1 \cup r(x)_2$. Similarly I' is a disjoint union of the elements of $l(x)_1 \cup l(x)_2$. To illuminate these definitions, consider a diagram representing an element x . We draw in black those equivalence classes which are contained entirely within I or I' , and in grey those that contain elements from both I and I' . In the above example, we obtain the following diagram (the grey lines are also dotted for clarity).



Now $d(x)$ is simply the number of grey equivalence classes. Also $r(x)$ is represented by the diagram obtained by removing the bottom dots, keeping only those arcs with both dots on the top row, but retaining the colours. The black components represent $r(x)_1$, and the grey $r(x)_2$. We represent $l(x)$ diagrammatically in a similar fashion. In the example, $d(x) = 1$ and $r(x)$ and $l(x)$ are represented by

$$\begin{aligned} r(x) &= \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \quad \text{and} \\ l(x) &= \begin{array}{ccccccc} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1' & 2' & 3' & 4' & 5' & 6' & 7' \end{array}. \end{aligned}$$

Note that $r(x)$ and $l(x)$ can be naturally identified with elements in the set $S_n(k)$ of [15], where $k = d(x)$. Green's relations in A_n are now described by the following theorem.

Theorem 6. For each $x, y \in A_n$,

- (i) $x \mathcal{D} y$ exactly when $d(x) = d(y)$.
- (ii) $x \mathcal{R} y$ exactly when $r(x) = r(y)$.
- (iii) $x \mathcal{L} y$ exactly when $l(x) = l(y)$.
- (iv) $x \leq_{\mathcal{R}} y$ exactly when $r(x)_1 \supseteq r(y)_1$ and $x_I \supseteq y_I$. In this case $x = yy^*x$.
- (v) $x \leq_{\mathcal{L}} y$ exactly when $l(x)_1 \supseteq l(y)_1$ and $x_{I'} \supseteq y_{I'}$. In this case $x = xy^*y$.

Proof. The reader is advised to keep the above diagrammatic interpretation in mind throughout the proof. Note that $A_n^1 = A_n$ because A_n is a monoid. Suppose that $x \leq_{\mathcal{R}} y$, so that $x = yz$ for some $z \in A_n$. Consider any $J \in r(y)_1$, so that $J \in \tilde{y}$. Because $J \subseteq I$ and z' is an equivalence relation on the elements $I' \cup I''$ only, J is also an equivalence class of $\langle y \cup z' \rangle$. Now x is obtained from the latter by restricting to $I \cup I''$ and replacing i'' by i' for $i \in I$, none of which affects the elements of J which are undashed. Therefore J is an equivalence class in x , so $J \in r(x)_1$. This shows that $r(y)_1 \subseteq r(x)_1$. Also note that

$$y_I \subseteq y \subseteq \langle y \cup z' \rangle.$$

Hence $y_I \subseteq \langle y \cup z' \rangle_{I \cup I''}$. Again replacing i'' by i' does not affect y_I , which is an equivalence relation on I , so we have $y_I \subseteq yz = x$. Therefore $y_I \subseteq x_I$, thus verifying one direction of (iv).

Conversely suppose the latter two conditions of (iv) hold. We first claim that

$$\begin{aligned} \langle y \cup y^{*'} \rangle &= y \cup y^{*'} \\ &\cup \{(i, j'') \mid i, j \in K \text{ for some } K \in r(y)_2\} \\ &\cup \{(j'', i) \mid i, j \in K \text{ for some } K \in r(y)_2\}. \end{aligned}$$

The right hand side is certainly reflexive and symmetric because y and $y^{*'}$ are. It is straightforward to check transitivity, so it is an equivalence relation on $I \cup I' \cup I''$ containing $y \cup y^{*'}$. Hence the right hand side contains the left hand side. Suppose $K \in r(y)_2$ and $i, j \in K$. Then $K = J \cap I$ for some $J \in \tilde{y}$ with $J \cap I' \neq \emptyset$. Suppose $k' \in J \cap I'$. Since $i, j, k' \in J$ we have $(i, k'), (j, k') \in y$. The latter gives $(k, j') \in y^*$, so that $(k', j'') \in y^{*'}$. Combining with the former,

$$(i, j'') \in \langle y \cup y^{*' \rangle}.$$

This also shows $(j'', i) \in \langle y \cup y^{*' \rangle}$. Hence the right hand side is contained in the left hand side, and the above equality is verified. It now follows that

$$\begin{aligned} yy^* &= y_I \cup (y_I)' \\ &\cup \{(i, j') \mid i, j \in K \text{ for some } K \in r(y)_2\} \\ &\cup \{(j', i) \mid i, j \in K \text{ for some } K \in r(y)_2\} \end{aligned}$$

where, as usual, $(y_I)'$ is obtained from y_I by replacing i with i' for $i \in I$. We next claim that

$$\begin{aligned} \langle yy^* \cup x' \rangle &= x_I \cup x' \\ &\cup \{(i, j') \mid (i, j) \in x \text{ and } i \in K \text{ for some } K \in r(y)_2\} \\ &\cup \{(j', i) \mid (i, j) \in x \text{ and } i \in K \text{ for some } K \in r(y)_2\}. \end{aligned}$$

Note that j is allowed to be in I' in the last two sets. Certainly the right hand side is reflexive and symmetric because x_I and x' are. Again transitivity is straightforward but tedious to verify using the inclusion $r(y)_1 \subseteq r(x)_1$. Now we have assumed $y_I \subseteq x_I$, so the right hand side contains y_I . This also implies $y_I \subseteq x$, so that $(y_I)' \subseteq x'$ and the right hand side contains $(y_I)'$. To show it contains the third set in the above decomposition of yy^* , suppose $i, j \in K$ for some $K \in r(y)_2$. Then $(i, j) \in y_I \subseteq x$, so that

$$(i, j') \in \{(i, j') \mid (i, j) \in x \text{ and } i \in K \text{ for some } K \in r(y)_2\}.$$

Similarly (j', i) is an element of the right hand side. Therefore the right hand side is an equivalence relation containing yy^* and x' , and so contains the left hand side. It remains to show the reverse inclusion. Suppose $(i, j) \in x_I$. If $i \in J$ for some $J \in r(y)_1$, then $J \in r(y)_1 \subseteq r(x)_1 \subseteq \tilde{x}_I$, so that $i, j \in J$, giving $(i, j) \in y_I \subseteq yy^* \subseteq \langle yy^* \cup x' \rangle$. Otherwise $i \in K$ for some $K \in r(y)_2$, so that

$$(i, i') \in \{(i, k') \mid i, k \in K \text{ for some } K \in r(y)_2\} \subseteq yy^*.$$

Similarly we may assume $(j, j') \in yy^*$. But $(i', j') \in (x_I)' \subseteq x'$, so combining the three ordered pairs we have $(i, j) \in \langle yy^* \cup x' \rangle$ in this case also. Therefore the left hand side contains x_I . It clearly contains x' . Finally suppose $i \in I$ and $j \in I \cup I'$ are such that $(i, j) \in x$ and $i \in K$ for some $K \in r(y)_2$. As above we obtain $(i, i') \in yy^*$ and $(i', j') \in x'$, so that $(i, j') \in \langle yy^* \cup x' \rangle$. Hence

$$\{(i, j') \mid (i, j) \in x \text{ and } i \in K \text{ for some } K \in r(y)_2\} \subseteq \langle yy^* \cup x' \rangle$$

and similarly for the final set. This gives the reverse inclusion, verifying the claim. It now follows that $yy^*x = x$. In particular $x \leq_{\mathcal{R}} y$, thus proving (iv). Statement (v) can be proven similarly, or deduced from (iv) using $*$.

Statement (ii) now follows from (iv). Indeed suppose $r(x) = r(y)$. Then

$$\tilde{x}_I = r(x)_1 \cup r(x)_2 = r(y)_1 \cup r(y)_2 = \tilde{y}_I.$$

Thus $x_I = y_I$. Also $r(x)_1 = r(y)_1$, so (iv) implies $x \leq_{\mathcal{R}} y \leq_{\mathcal{R}} x$, so that $x \mathcal{R} y$. Conversely suppose $x \mathcal{R} y$ and apply (iv) twice. Then the first condition gives $r(x)_1 \subseteq r(y)_1 \subseteq r(x)_1$, so the two are equal. Similarly the second condition gives $x_I \subseteq y_I \subseteq x_I$, so $x_I = y_I$. Hence

$$r(x)_1 \cup r(x)_2 = \tilde{x}_I = \tilde{y}_I = r(y)_1 \cup r(y)_2,$$

so that $r(x)_2 = r(y)_2$, whence $r(x) = r(y)$, as required. Statement (iii) follows similarly.

Finally we verify (i). Note that $d(x) = |r(x)_2| = |l(x)_2|$. Thus by (ii) and (iii), we have $d(x) = d(y)$ whenever $x \mathcal{R} y$ or $x \mathcal{L} y$. Hence $d(x) = d(y)$ whenever $x \mathcal{D} y$ by (iv) of Proposition 3. Conversely suppose $d(x) = d(y) = k$, so that $|r(x)_2| = k = |l(y)_2|$. Write

$$\begin{aligned} r(x)_2 &= \{K_1, K_2, \dots, K_k\} \text{ and} \\ l(y)_2 &= \{L_1, L_2, \dots, L_k\}. \end{aligned}$$

Let $z \in A_n$ denote the equivalence relation whose equivalence classes are

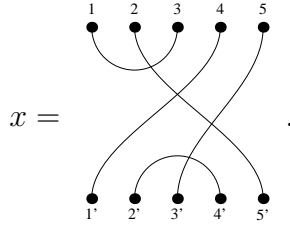
$$\tilde{z} = r(x)_1 \cup l(y)_1 \cup \{K_1 \cup L_1, K_2 \cup L_2, \dots, K_k \cup L_k\}.$$

It is easy to verify that $r(z) = r(x)$ and $l(z) = l(y)$, so that $x \mathcal{R} z \mathcal{L} y$. Hence $x \mathcal{D} y$, proving (i). \square

Let BR_n denote the set of elements of A_n whose equivalence classes each contain 2 elements. Thus BR_n essentially consists of all partitions of the set $I \cup I'$ into pairs. For example, BR_5 contains the element

$$x = \{\{1, 3\}, \{2, 5'\}, \{4, 1'\}, \{5, 3'\}, \{2', 4'\}\}$$

which is represented by the diagram



Note that the diagram representing an element of BR_n is unique (up to a continuous distortion of the arcs). In fact BR_n forms a subsemigroup of A_n , called the *Brauer semigroup* [14, 16]. Moreover the proof of Theorem 6 shows that the \mathcal{R} , \mathcal{L} and \mathcal{D} relations in BR_n are simply the restrictions of those in A_n . We therefore have the following description of these relations, which is given in Theorem 7 of [16].

Theorem 7. For $x \in BR_n$, define the functions

$$\begin{aligned} r(x) &= \{\{i, j\} \in x \mid 1 \leq i, j \leq n\}, \\ l(x) &= \{\{i', j'\} \in x \mid 1 \leq i, j \leq n\}, \\ d(x) &= \#\{\{i, j'\} \in x \mid 1 \leq i, j \leq n\}. \end{aligned}$$

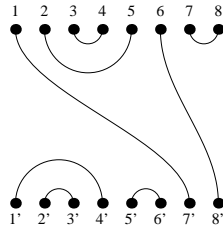
Note that

$$d(x) = n - 2|r(x)| = n - 2|l(x)| \in \{n, n - 2, n - 4, \dots\}.$$

Suppose $x, y \in BR_n$. Then

- (i) $x \mathcal{D} y$ exactly when $d(x) = d(y)$.
- (ii) $x \mathcal{R} y$ exactly when $r(x) = r(y)$.
- (iii) $x \mathcal{L} y$ exactly when $l(x) = l(y)$.

Finally the *Temperley-Lieb semigroup* TL_n is the subsemigroup of BR_n consisting of the diagrams that can be drawn without intersecting curves. For example, an element of TL_8 is shown below.



Again the \mathcal{R} , \mathcal{L} and \mathcal{D} relations in this semigroup are just the restriction of those in A_n (or equivalently those in BR_n). This also follows from the proof of Theorem 6, provided the elements of $r(x)_2$ and $l(y)_1$ are labelled appropriately in the proof of (i).

Twisted Semigroup Algebras

In this chapter we first define the concept of a semigroup algebra, an object which occurs naturally when studying the representations of a semigroup. By altering the multiplication in this algebra, we obtain a twisted semigroup algebra. We prove some general results about twisted semigroup algebras of regular semigroups. Finally we continue the three examples of the previous chapter.

Definition 8. *Suppose S is a semigroup and R is a commutative ring with 1. The semigroup algebra of S over R , denoted by $R[S]$, is the R -algebra with R -basis S and multiplication \cdot defined by*

$$x \cdot y = xy$$

for $x, y \in S$, and extended by linearity.

Because the operation in S is associative, it follows immediately that $R[S]$ is an associative algebra. These objects are of interest because semigroup representations of S over R correspond naturally to algebra representations of $R[S]$.

By analogy with twisted group algebras [19], we now “twist” the multiplication in $R[S]$, so that the product of basis elements x and y is not simply the basis element xy , but a scalar multiple of it, $\alpha(x, y)(xy)$. In order for the resulting multiplication to be associative, we require

$$(1) \quad \alpha(x, y)\alpha(xy, z) = \alpha(x, yz)\alpha(y, z)$$

for all $x, y, z \in S$. The following definition is essentially that in [3], except that we give no special treatment to the zero of the semigroup (if it exists).

Definition 9. *Suppose S is a semigroup and R is a commutative ring with 1. A twisting from S into R is a map*

$$\alpha : S \times S \rightarrow R$$

which satisfies (1). The twisted semigroup algebra of S over R , with twisting α , denoted by $R^\alpha[S]$, is the R -algebra with R -basis S and multiplication $\cdot_\alpha = \cdot$ defined by

$$x \cdot_\alpha y = \alpha(x, y)(xy)$$

for $x, y \in S$, and extended by linearity. It follows easily from (1) that $R^\alpha[S]$ is associative.

For any $T \subseteq S$, let $R^\alpha[T]$ denote the R -span of T in $R^\alpha[S]$, so that T forms an R -basis for $R^\alpha[T]$. It is clear that if T is a subsemigroup of S , then $R^\alpha[T]$ is a subalgebra, and moreover is isomorphic to the twisted semigroup algebra of T whose twisting is the restriction of α to T , thus justifying the notation. For example, suppose $R = \mathbb{R}$ is the field of real numbers and $S = \{1, i, j, k\}$ is the Klein four-group (where $i^2 = j^2 = k^2 = 1$, $ij = ji = k$ and so on). Also consider the subsemigroup $T = \{1, i\}$ of S , which is isomorphic to a cyclic group of order 2. Define $\alpha : S \times S \rightarrow R$ by setting

$$\alpha(i, i) = \alpha(j, j) = \alpha(k, k) = \alpha(i, k) = \alpha(k, j) = \alpha(j, i) = -1,$$

and putting the remaining values $\alpha(x, y)$ equal to 1. It is routine to check that α is a twisting from S into R , that $R^\alpha[S] \cong \mathbb{H}$ is the division algebra of quaternions, and that the subalgebra

$R^\alpha[T] \cong \mathbb{C}$ is the field of complex numbers. In fact both of these examples are twisted group algebras; we will see examples in which the underlying semigroup is not a group at the end of the chapter.

Returning to a general twisted semigroup algebra, if T is an ideal of S , then $R^\alpha[T]$ is an ideal of $R^\alpha[S]$. If S has a zero, define the contracted twisted semigroup algebra by

$$R_0^\alpha[S] = R^\alpha[S]/R^\alpha[\{0\}].$$

Recalling the definition of the quotient S/T from the previous chapter, we have the following easy result.

Lemma 10. *If T is an ideal of S then*

$$R^\alpha[S]/R^\alpha[T] \cong R_0^{\alpha_0}[S/T],$$

where α_0 is the twisting from S/T into R defined by

$$\alpha_0(x, y) = \begin{cases} 0 & \text{if } xy = 0, \\ \alpha(x, y) & \text{otherwise.} \end{cases}$$

Proof. We first check that α_0 is a twisting. That is,

$$\alpha_0(x, y)\alpha_0(xy, z) = \alpha_0(x, yz)\alpha_0(y, z)$$

for all $x, y, z \in S/T$. If $xyz = 0$ then $\alpha_0(xy, z) = 0 = \alpha_0(x, yz)$, so that both sides are 0. Otherwise the elements xy and yz must both be nonzero, so the required equality becomes

$$\alpha(x, y)\alpha(xy, z) = \alpha(x, yz)\alpha(y, z),$$

which holds because α is a twisting. Therefore the algebra $R_0^{\alpha_0}[S/T]$ is indeed defined.

Since S and T form R -bases for $R^\alpha[S]$ and $R^\alpha[T]$ respectively, the elements

$$\{x + R^\alpha[T] \mid x \in S \setminus T\}$$

form an R -basis for $R^\alpha[S]/R^\alpha[T]$. Similarly the elements

$$\{x + R^{\alpha_0}[\{0\}] \mid x \in (S/T) \setminus \{0\}\} = \{x + R^{\alpha_0}[\{0\}] \mid x \in S \setminus T\}$$

form an R -basis for $R_0^{\alpha_0}[S/T]$. We therefore have an R -module isomorphism

$$\theta : R^\alpha[S]/R^\alpha[T] \rightarrow R_0^{\alpha_0}[S/T]$$

defined by

$$\theta : x + R^\alpha[T] \mapsto x + R^{\alpha_0}[\{0\}]$$

for $x \in S \setminus T$. Suppose $x, y \in S \setminus T$. If $xy \notin T$ then

$$\theta((x + R^\alpha[T])(y + R^\alpha[T])) = \theta(xy + R^\alpha[T]) = xy + R^{\alpha_0}[\{0\}]$$

and

$$\theta(x + R^\alpha[T])\theta(y + R^\alpha[T]) = (x + R^{\alpha_0}[\{0\}])(y + R^{\alpha_0}[\{0\}]) = xy + R^{\alpha_0}[\{0\}].$$

Otherwise $xy \in T$, so that $xy = 0$ in S/T , giving

$$\theta((x + R^\alpha[T])(y + R^\alpha[T])) = \theta(xy + R^\alpha[T]) = \theta(R^\alpha[T]) = 0$$

and

$$\theta(x + R^\alpha[T])\theta(y + R^\alpha[T]) = (x + R^{\alpha_0}[\{0\}])(y + R^{\alpha_0}[\{0\}]) = 0.$$

Therefore θ is an isomorphism of R -algebras, as required. \square

In Definition 9, we have not assumed that the elements $\alpha(x, y)$ are invertible. Although such an assumption would greatly simplify the proofs, it would turn out to be too restrictive for our purposes. We will instead consider the much weaker condition given in Assumption 11 below. It essentially says that although α may not be invertible, when restricted appropriately it can be decomposed into a constant part and an invertible part. This assumption depends on an idempotent 1_D , which is to be specified when the assumption is actually made.

Assumption 11. *Let L_0 and K_0 denote the \mathcal{L} and \mathcal{R} classes of 1_D respectively. Assume there exists a map*

$$\beta_D : L_0 \times K_0 \rightarrow G(R),$$

where $G(R)$ is the group of units in R . Suppose moreover that β_D satisfies the following analogues of (1):

- (2) $\alpha(x, y)\beta_D(xy, z) = \alpha(x, yz)\beta_D(y, z)$ if $yz \mathcal{R} y$,
- (3) $\beta_D(x, y)\alpha(xy, z) = \beta_D(x, yz)\alpha(y, z)$ if $xy \mathcal{L} y$, and
- (4) $\beta_D(x, y)\beta_D(xy, z) = \beta_D(x, yz)\beta_D(y, z)$

whenever the relevant values of β_D are defined.

Note that if the elements $\alpha(x, y)$ are invertible, even just for $(x, y) \in L_0 \times K_0$, then the assumption holds by setting β_D to be the restriction of α . Before proceeding we discuss some implications of the assumption. Let G_D denote the \mathcal{H} class of 1_D , which is a subgroup. By (4), the restriction of β_D defines a twisting from G_D into R , so we can consider the algebra $R^{\beta_D}[G_D]$. The assumption also implies that the restriction of α to $L_0 \times K_0$ can be obtained from β_D by multiplying by a constant, as we now demonstrate. Indeed, putting $x = y = 1_D$ in (2), we obtain

$$\alpha(1_D, 1_D)\beta_D(1_D, z) = \alpha(1_D, z)\beta_D(1_D, z).$$

Since $\beta_D(1_D, z)$ is invertible, this gives $\alpha(1_D, z) = \alpha(1_D, 1_D)$. Similarly putting $y = z = 1_D$ in (4) gives $\beta_D(x, 1_D) = \beta_D(1_D, 1_D)$. Finally putting $y = 1_D$ in (3) gives

$$\beta_D(x, 1_D)\alpha(x, z) = \beta_D(x, z)\alpha(1_D, z).$$

Thus $\beta_D(1_D, 1_D)\alpha(x, z) = \beta_D(x, z)\alpha(1_D, 1_D)$, so that

$$\alpha(x, z) = \alpha(D)\beta_D(x, z)$$

for all $x \in L_0$ and $z \in K_0$, where $\alpha(D) = \alpha(1_D, 1_D)\beta_D(1_D, 1_D)^{-1}$. In particular, multiplication by $\alpha(D)$ gives a homomorphism $R^\alpha[G_D] \rightarrow R^{\beta_D}[G_D]$.

Now suppose that S is group bound, so that $\mathcal{D} = \mathcal{J}$, by (iii) of Theorem 5, and we have a pre-order $\leq_{\mathcal{D}}$ on S . If $T_1 \subseteq T_2$ are ideals in S such that $T_2 \setminus T_1$ is a single \mathcal{D} class D , then Lemma 10 shows that $R^\alpha[T_2]/R^\alpha[T_1] \cong R_0^{\alpha_0}[T_2/T_1]$. By definition, T_1/T_2 is the semigroup with $D \cup \{0\}$ as its elements and with multiplication

$$x \cdot y = \begin{cases} xy & \text{if } xy \in D \\ 0 & \text{otherwise} \end{cases} = \begin{cases} xy & \text{if } xy \in D \\ 0 & \text{if } xy <_{\mathcal{D}} D \end{cases}$$

for $x, y \in D$. Denote this semigroup by D^0 . Note that if x, y and xy are in D , then $x \mathcal{R} xy \mathcal{L} y$ by (iv) of Theorem 5, so that by Proposition 3, D contains an idempotent and is therefore regular. Thus if D is not regular, all products in D^0 are 0, so all products in $R_0^{\alpha_0}[D^0]$ are 0. We will see that if D is regular, then under Assumption 11, $R_0^{\alpha_0}[D^0]$ can be described by the following construction.

Definition 12. Let A be an R -algebra, I and Λ be any indexing sets, and P be a $\Lambda \times I$ matrix over A . The Munn algebra $\mathcal{M}(A, I, \Lambda, P)$ over A with sandwich matrix P is the R -algebra consisting of all $I \times \Lambda$ matrices over A with finitely many nonzero entries, with multiplication

$$X \circ Y = XPY$$

for $X, Y \in \mathcal{M}(A, I, \Lambda, P)$. The juxtaposition on the right hand side denotes usual matrix multiplication.

Assume D is a regular \mathcal{D} class. By Proposition 3, there exists an idempotent $1_D \in D$, and its \mathcal{H} class G_D is a group. Suppose that 1_D satisfies Assumption 11. For each $L \in \mathbb{L}_D$, pick an element $u_L \in L$ with $u_L \mathcal{R} 1_D$. Similarly for $K \in \mathbb{R}_D$, pick $v_K \in K$ with $v_K \mathcal{L} 1_D$. Note that for each $L \in \mathbb{L}_D$ and $K \in \mathbb{R}_D$, we either have $u_L v_K <_{\mathcal{D}} D$ or $u_L v_K \in G_D$, by (iv) of Theorem 5. Define the twisted sandwich matrix P_D^α to be the $\mathbb{L}_D \times \mathbb{R}_D$ matrix with entries in $R^{\beta_D}[G_D]$ given by

$$(P_D^\alpha)_{LK} = \begin{cases} 0 & \text{if } u_L v_K <_{\mathcal{D}} D \\ \alpha(u_L, v_K) u_L v_K & \text{if } u_L v_K \in G_D. \end{cases}$$

Of course when α is trivial, this reduces to the usual sandwich matrix, on identifying $G_D \cup \{0\}$ with a subset of $R^{\beta_D}[G_D]$.

Lemma 13. Under the above assumptions and notations,

$$R_0^{\alpha_0}[D^0] \cong \mathcal{M}(R^{\beta_D}[G_D], \mathbb{R}_D, \mathbb{L}_D, P_D^\alpha).$$

Proof. For each $L \in \mathbb{L}_D$, we have $u_L \mathcal{R} 1_D$ by choice of u_L . Since 1_D is idempotent, this implies that $1_D u_L = u_L$. By Green's Lemma, right multiplication by u_L then gives an \mathcal{R} class preserving bijection from the \mathcal{L} class of 1_D to the \mathcal{L} class of u_L , which is L . Similarly left multiplication by v_K gives an \mathcal{L} class preserving bijection from the \mathcal{R} class of 1_D to K . Thus we have a bijection $G_D \rightarrow K \cap L$ given by $g \mapsto v_K g u_L$. Since D is a disjoint union of its \mathcal{H} classes $K \cap L$ for $K \in \mathbb{R}_D$ and $L \in \mathbb{L}_D$, this gives a bijection $\mathbb{R}_D \times \mathbb{L}_D \times G_D \rightarrow D$ defined by $(K, L, g) \mapsto v_K g u_L$. Therefore $R_0^{\alpha_0}[D^0]$ has basis

$$D = \{v_K g u_L \mid K \in \mathbb{R}_D, L \in \mathbb{L}_D \text{ and } g \in G_D\}.$$

Now because G_D is a basis for $R^{\beta_D}[G_D]$, the algebra $\mathcal{M}(R^{\beta_D}[G_D], \mathbb{R}_D, \mathbb{L}_D, P_D^\alpha)$ has a basis consisting of the elements

$$\{(g)_{KL} \mid K \in \mathbb{R}_D, L \in \mathbb{L}_D \text{ and } g \in G_D\},$$

where $(a)_{KL}$ denotes the matrix with a in position (K, L) and zeros elsewhere, for any element $a \in R^{\beta_D}[G_D]$. Thus there is an R -module homomorphism

$$\theta : \mathcal{M}(R^{\beta_D}[G_D], \mathbb{R}_D, \mathbb{L}_D, P_D^\alpha) \rightarrow R_0^{\alpha_0}[D^0]$$

defined on the basis by

$$\theta((g)_{KL}) = \beta_D(v_K, g) \beta_D(v_K g, u_L) (v_K g u_L).$$

With respect to these bases, the matrix representing θ is diagonal. Moreover each diagonal entry, namely the scalar factor on the right, is invertible because β_D takes values in $G(R)$. Thus θ is an R -module isomorphism. Consider any two basis elements $(g_1)_{K_1 L_1}$ and $(g_2)_{K_2 L_2}$. To show that θ preserves multiplication, we need only check that

$$\theta((g_1)_{K_1 L_1} \circ (g_2)_{K_2 L_2}) = \theta((g_1)_{K_1 L_1}) \cdot_{\alpha_0} \theta((g_2)_{K_2 L_2}).$$

First suppose that $u_{L_1} v_{K_2} <_{\mathcal{D}} D$. Then $(P_D^\alpha)_{L_1 K_2} = 0$, so that $(g_1)_{K_1 L_1} \circ (g_2)_{K_2 L_2} = 0$. Also

$$(v_{K_1} g_1 u_{L_1})(v_{K_2} g_2 u_{L_2}) \leq_{\mathcal{D}} u_{L_1} v_{K_2} <_{\mathcal{D}} D,$$

so that $(v_{K_1}g_1u_{L_1})(v_{K_2}g_2u_{L_2}) = 0$ in D^0 . Hence

$$(v_{K_1}g_1u_{L_1}) \cdot_{\alpha_0} (v_{K_2}g_2u_{L_2}) = 0$$

in $R_0^{\alpha_0}[D^0]$. Both sides of the equality are then 0. The other case is when $u_{L_1}v_{K_2} \in G_D$, so that $g_1u_{L_1}v_{K_2}g_2 \in G_D$. Hence

$$v_{K_1}g_1u_{L_1}v_{K_2}g_2u_{L_2} \in D.$$

Using Assumption 11 repeatedly, we calculate

$$\begin{aligned} & \beta_D(v_{K_1}, g_1u_{L_1}v_{K_2}g_2)\beta_D(g_1, u_{L_1}v_{K_2})\alpha(u_{L_1}, v_{K_2})\beta_D(g_1u_{L_1}v_{K_2}, g_2)\beta_D(v_{K_1}g_1u_{L_1}v_{K_2}g_2, u_{L_2}) \\ &= \beta_D(v_{K_1}, g_1u_{L_1}v_{K_2}g_2)\beta_D(g_1, u_{L_1})\alpha(g_1u_{L_1}, v_{K_2}) \\ & \quad \times \beta_D(g_1u_{L_1}v_{K_2}, g_2)\beta_D(v_{K_1}g_1u_{L_1}v_{K_2}g_2, u_{L_2}) \text{ by (3)} \\ &= \beta_D(v_{K_1}, g_1u_{L_1}v_{K_2}g_2)\beta_D(g_1, u_{L_1})\alpha(g_1u_{L_1}, v_{K_2}g_2) \\ & \quad \times \beta_D(v_{K_2}, g_2)\beta_D(v_{K_1}g_1u_{L_1}v_{K_2}g_2, u_{L_2}) \text{ by (2)} \\ &= \beta_D(v_{K_1}, g_1u_{L_1})\beta_D(g_1, u_{L_1})\alpha(v_{K_1}g_1u_{L_1}, v_{K_2}g_2) \\ & \quad \times \beta_D(v_{K_2}, g_2)\beta_D(v_{K_1}g_1u_{L_1}v_{K_2}g_2, u_{L_2}) \text{ by (3)} \\ &= \beta_D(v_{K_1}, g_1u_{L_1})\beta_D(g_1, u_{L_1})\alpha(v_{K_1}g_1u_{L_1}, v_{K_2}g_2u_{L_2})\beta_D(v_{K_2}, g_2)\beta_D(v_{K_2}g_2, u_{L_2}) \text{ by (2)} \\ &= \beta_D(v_{K_1}, g_1)\beta_D(v_{K_1}g_1, u_{L_1})\alpha(v_{K_1}g_1u_{L_1}, v_{K_2}g_2u_{L_2})\beta_D(v_{K_2}, g_2)\beta_D(v_{K_2}g_2, u_{L_2}) \text{ by (4)}. \end{aligned}$$

Multiplying both sides by $(v_{K_1}g_1u_{L_1}v_{K_2}g_2u_{L_2})$, these become, by definition of θ and \cdot_{α_0} ,

$$\begin{aligned} & \beta_D(g_1, u_{L_1}v_{K_2})\alpha(u_{L_1}, v_{K_2})\beta_D(g_1u_{L_1}v_{K_2}, g_2)\theta\left((g_1u_{L_1}v_{K_2}g_2)_{K_1L_2}\right) \\ &= \beta_D(v_{K_1}, g_1)\beta_D(v_{K_1}g_1, u_{L_1})\beta_D(v_{K_2}, g_2)\beta_D(v_{K_2}g_2, u_{L_2})\left[(v_{K_1}g_1u_{L_1}) \cdot_{\alpha_0} (v_{K_2}g_2u_{L_2})\right]. \end{aligned}$$

That is,

$$\begin{aligned} & \alpha(u_{L_1}, v_{K_2})\theta\left((g_1 \cdot_{\beta_D} (u_{L_1}v_{K_2}) \cdot_{\beta_D} g_2)_{K_1L_2}\right) \\ &= \left[\beta_D(v_{K_1}, g_1)\beta_D(v_{K_1}g_1, u_{L_1})(v_{K_1}g_1u_{L_1})\right] \cdot_{\alpha_0} \left[\beta_D(v_{K_2}, g_2)\beta_D(v_{K_2}g_2, u_{L_2})(v_{K_2}g_2u_{L_2})\right]. \end{aligned}$$

Substituting $(P_D^\alpha)_{L_1K_2} = \alpha(u_{L_1}, v_{K_2})(u_{L_1}v_{K_2})$, this becomes

$$\theta\left((g_1 \cdot_{\beta_D} (P_D^\alpha)_{L_1K_2} \cdot_{\beta_D} g_2)_{K_1L_2}\right) = \theta\left((g_1)_{K_1L_1}\right) \cdot_{\alpha_0} \theta\left((g_2)_{K_2L_2}\right).$$

This gives the required equality, so that θ preserves multiplication. It is therefore an isomorphism of R -algebras. \square

Remark. In the special case where the elements $\alpha(x, y)$ are invertible, β_D is just the restriction of α . Then $R^{\beta_D}[G_D] = R^\alpha[G_D]$ embeds in $R_0^{\alpha_0}[D^0]$, and the above map θ becomes the map defined by

$$\theta((a)_{KL}) = v_K \cdot a \cdot u_L$$

for any $a \in R^\alpha[G_D]$. The proof then becomes much more elegant and transparent, and still demonstrates all the essential ideas. For completeness we have offered a proof of the general case, although it is less illustrative.

The semisimplicity of a finite dimensional Munn algebra is determined by the following result of Munn [18] (see also [20]).

Theorem 14 (Theorem 4.7 of [18]). *Consider the Munn algebra $\mathcal{M}(A, I, \Lambda, P)$, where I and Λ are finite. This algebra is semisimple exactly when the following two conditions hold.*

- (i) *The algebra A is semisimple, and*
- (ii) *The matrix P is invertible over A (so in particular it is square).*

Under Assumption 11, we can now characterise the semisimplicity of twisted semigroup algebras of finite semigroups.

Theorem 15. *Suppose $R^\alpha[S]$ is a twisted semigroup algebra over a finite semigroup S . Suppose moreover that each regular \mathcal{D} class D in S contains an idempotent 1_D which satisfies Assumption 11. Then $R^\alpha[S]$ is semisimple exactly when the following conditions hold for each \mathcal{D} class D .*

- (i) D is regular.
- (ii) The algebra $R^{\beta_D}[G_D]$ is semisimple (where β_D is defined by Assumption 11).
- (iii) The twisted sandwich matrix P_D^α is invertible (so in particular it is square).

Proof. Because S is finite, it is group bound and has finitely many \mathcal{D} classes. By refining the order $\leq_{\mathcal{D}}$ on the \mathcal{D} classes to a total order, we can label the \mathcal{D} classes D_1, D_2, \dots, D_k such that $D_i \leq_{\mathcal{D}} D_j$ implies $i \leq j$. Now the set

$$S_i = \bigcup_{j=1}^i D_j$$

is an ideal of S . We have therefore constructed an ideal chain

$$\emptyset = S_0 \subset S_1 \subset S_2 \subset \dots \subset S_k = S$$

in S , such that $S_i \setminus S_{i-1}$ is the single \mathcal{D} class D_i . This gives an ideal chain

$$0 = R^\alpha[S_0] \subset R^\alpha[S_1] \subset R^\alpha[S_2] \subset \dots \subset R^\alpha[S_k] = R^\alpha[S]$$

in $R^\alpha[S]$. It is well known that $R^\alpha[S]$ is semisimple exactly when each of the quotients

$$R^\alpha[S_i]/R^\alpha[S_{i-1}] \cong R_0^{\alpha_0}[D_i^0]$$

is semisimple, for $1 \leq i \leq k$. That is, $R^\alpha[S]$ is semisimple exactly when $R_0^{\alpha_0}[D^0]$ is semisimple for each \mathcal{D} class D . It therefore suffices to show that the semisimplicity of $R_0^{\alpha_0}[D^0]$ is equivalent to the three conditions holding for D . If D is not regular, then all products in $R_0^{\alpha_0}[D^0]$ are 0 as discussed previously, so that $R_0^{\alpha_0}[D^0]$ is not semisimple and the equivalence holds. Suppose that D is regular. Then

$$R_0^{\alpha_0}[D^0] \cong \mathcal{M}(R^{\beta_D}[G_D], \mathbb{R}_D, \mathbb{L}_D, P_D^\alpha)$$

by Lemma 13. Now by Theorem 14, this algebra is semisimple if and only if $R^{\beta_D}[G_D]$ is semisimple and P_D^α is invertible. Thus the equivalence holds in this case also, completing the proof. \square

The above theorem relates a property of the twisted semigroup algebra $R^\alpha[S]$ to the corresponding property of the twisted group algebras $R^{\beta_D}[G_D]$ of the maximal subgroups. This theme will recur throughout the thesis; the following proposition gives another example (see Theorem 2.3 of [17]).

Proposition 16. *Suppose that S is any group bound semigroup and that D is a regular \mathcal{D} class in S . Suppose that 1_D is an idempotent in D satisfying Assumption 11. Let G_D denote the \mathcal{H} class of 1_D , and suppose M is a left $R^{\beta_D}[G_D]$ -module. For each $K \in \mathbb{R}_D$, pick an element $v_K \in K$ in the same \mathcal{L} class as 1_D , and let*

$$M_K = \{m_K \mid m \in M\}$$

be a set in bijection with M . Then

$$W = \bigoplus_{K \in \mathbb{R}_D} M_K$$

is a left $R^\alpha[S]$ -module under the action which is defined on S by

$$x \cdot m_K = \begin{cases} 0 & \text{if } xv_K <_{\mathcal{D}} D \\ \alpha(x, v_K)\beta_D(v_{K'}, g)^{-1}(gm)_{K'} & \text{if } xv_K = v_{K'}g \text{ where } g \in G_D \end{cases}$$

for $m \in M$, $K \in \mathbb{R}_D$ and $x \in S$, and which is extended to $R^\alpha[S]$ by R -linearity.

Proof. Note that for any $x \in S$ we either have $xv_K <_{\mathcal{D}} D$ or $xv_K \in D$. In the latter case $xv_K \mathcal{L} 1_D$ by (ii) of Theorem 5, so that, as in the proof of Lemma 13, there is by Green's Lemma a unique expression $xv_K = v_{K'}g$ for some $K' \in \mathbb{R}_D$ and $g \in G_D$. Therefore the action defined above does make sense. We need to check that

$$(x \cdot y) \cdot m_K = x \cdot (y \cdot m_K)$$

for any $x, y \in S$, $m \in M$ and $K \in \mathbb{R}_D$. Suppose first that $yv_K <_{\mathcal{D}} D$, so that $y \cdot m_K = 0$. Then

$$(xy)v_K \leq_{\mathcal{D}} yv_K <_{\mathcal{D}} D,$$

so that $(xy) \cdot m_K = 0$. Since $x \cdot y = \alpha(x, y)(xy)$, both sides are then 0, as required. Next assume $yv_K = v_{K'}g$ for some $K' \in \mathbb{R}_D$ and $g \in G_D$. The required equality now becomes

$$\alpha(x, y)(xy) \cdot m_K = \alpha(y, v_K)\beta_D(v_{K'}, g)^{-1}x \cdot (gm)_{K'}.$$

Now suppose $xv_{K'} <_{\mathcal{D}} D$, so that $x \cdot (gm)_{K'} = 0$. Then

$$xyv_K = xv_{K'}g \leq_{\mathcal{D}} xv_{K'} <_{\mathcal{D}} D,$$

so that $(xy) \cdot m_K = 0$ and again both sides are 0. Finally suppose $xv_{K'} = v_{K''}h$ for some $K'' \in \mathbb{R}_D$ and $h \in G_D$. Then

$$(xy)v_K = xv_{K'}g = v_{K''}(hg)$$

and

$$h(gm) = (h \cdot_{\beta_D} g)m = \beta_D(h, g)(hg)m,$$

so the required equality becomes

$$\begin{aligned} \alpha(x, y)\alpha(xy, v_K)\beta_D(v_{K''}, hg)^{-1}((hg)m)_{K''} \\ = \alpha(y, v_K)\beta_D(v_{K'}, g)^{-1}\alpha(x, v_{K'})\beta_D(v_{K''}, h)^{-1}\beta_D(h, g)((hg)m)_{K''}. \end{aligned}$$

Clearly it suffices to prove

$$\alpha(x, y)\alpha(xy, v_K)\beta_D(v_{K'}, g)\beta_D(v_{K''}, h) = \alpha(x, v_{K'})\alpha(y, v_K)\beta_D(h, g)\beta_D(v_{K''}, hg).$$

This follows by repeated application of Assumption 11, as follows:

$$\begin{aligned} \alpha(x, y)\alpha(xy, v_K)\beta_D(v_{K'}, g)\beta_D(v_{K''}, h) \\ &= \alpha(x, yv_K)\alpha(y, v_K)\beta_D(v_{K'}, g)\beta_D(v_{K''}, h) \text{ by (1)} \\ &= \alpha(x, v_{K'}g)\alpha(y, v_K)\beta_D(v_{K'}, g)\beta_D(v_{K''}, h) \\ &= \alpha(x, v_{K'})\alpha(y, v_K)\beta_D(xv_{K'}, g)\beta_D(v_{K''}, h) \text{ by (2)} \\ &= \alpha(x, v_{K'})\alpha(y, v_K)\beta_D(v_{K''}h, g)\beta_D(v_{K''}, h) \\ &= \alpha(x, v_{K'})\alpha(y, v_K)\beta_D(h, g)\beta_D(v_{K''}, hg) \text{ by (4)}, \end{aligned}$$

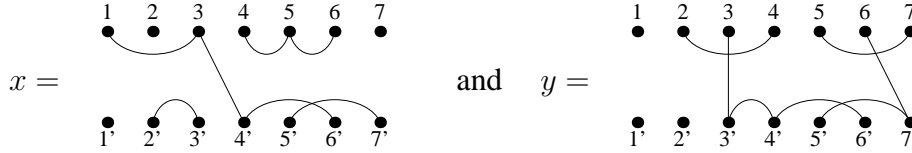
as required. This completes the final case. \square

Remark. In fact the above proof did not use (3) of Assumption 11; the proposition would hold without this condition. This reflects the fact that the proposition constructs a left module using \mathcal{R} classes. Clearly we could analogously construct right modules using \mathcal{L} classes.

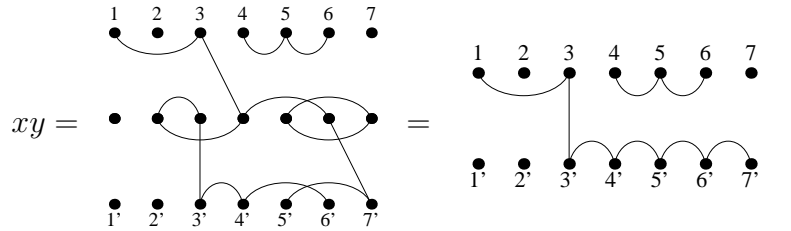
We conclude this chapter by constructing twisted semigroup algebras over the three example semigroups of the previous chapter. Recall that the product xy of two equivalence relations $x, y \in A_n$ is obtained from $\langle x \cup y' \rangle_{I \cup I''}$ by replacing i'' with i' for $i \in I$. When restricting to $I \cup I''$, any equivalence class of $\langle x \cup y' \rangle$ that is contained entirely within I' is removed. Denote the number of such equivalence classes by

$$m(x, y) = \# \left\{ J \in \langle x \cup y' \rangle \mid J \subseteq I' \right\}.$$

For example, recall the product of the elements



is



From a diagrammatic perspective, $m(x, y)$ is the number of connected components in the “concatenated” diagram involving dots from the central row only. In this case there are two such components, namely those corresponding to the dots $\{1\}$ and $\{5, 7\}$, so that $m(x, y) = 2$. It is clear from this geometric construction that

$$m(x, y) + m(xy, z) = m(x, yz) + m(y, z)$$

for any $x, y, z \in A_n$. Thus for any δ in a commutative ring R , we can define a twisting from A_n into R by

$$\alpha(x, y) = \delta^{m(x, y)}.$$

The resulting twisted semigroup algebra $R^\alpha[A_n]$ is called the *partition algebra* [15]. Recall that BR_n and TL_n are subsemigroups of A_n . The subalgebra $R^\alpha[BR_n]$ is called the *Brauer algebra*. This has been studied extensively in the literature [2, 8, 23]. It was realised as a twisted semigroup algebra as above in [14]. Similarly $R^\alpha[TL_n]$ is called the *Temperley-Lieb algebra* [5, 12].

Conjugate Cellular Algebras

Cellular algebras were introduced in the famous paper of Graham and Lehrer [6]. In this chapter we give an extended definition, which accommodates rings which are equipped with an involution, and which does not assume that the indexing sets $M(\lambda)$ are finite. Over the next two chapters we develop the theory of such algebras by exact analogy with [6]. Finally in Chapter 7 we give some examples of algebras which are conjugate cellular (and one which is not).

Definition 17. Suppose that R is a commutative ring with identity equipped with an involution $\bar{}$. An associative R -algebra A is conjugate cellular, with cell datum $(\Lambda, M, C, *)$, if

(C1) Λ is a finite poset, and for each $\lambda \in \Lambda$ we have a nonempty indexing set $M(\lambda)$ and elements $C_{st}^\lambda \in A$ for $s, t \in M(\lambda)$. The elements

$$\{C_{st}^\lambda \mid \lambda \in \Lambda \text{ and } s, t \in M(\lambda)\}$$

form an R -basis of A .

(C2) The map $* : A \rightarrow A$ is an R -conjugate linear anti-involution; that is,

$$\begin{aligned} (a_1^*)^* &= a_1, \\ (r_1 a_1 + r_2 a_2)^* &= \bar{r}_1 a_1^* + \bar{r}_2 a_2^*, \text{ and} \\ (a_1 a_2)^* &= a_2^* a_1^* \end{aligned}$$

for $r_1, r_2 \in R$ and $a_1, a_2 \in A$. The action of $*$ on the above basis is given by

$$(C_{st}^\lambda)^* = C_{ts}^\lambda.$$

(C3) For any $\lambda \in \Lambda$, $s \in M(\lambda)$ and $a \in A$, there exist elements $r_a(u, s) \in R$ for $u \in M(\lambda)$ such that, for each $t \in M(\lambda)$,

$$(5) \quad a C_{st}^\lambda \in \sum_{u \in M(\lambda)} r_a(u, s) C_{ut}^\lambda + A(< \lambda)$$

where

$$A(< \lambda) = \text{span}_R \{C_{s't'}^\mu \mid \mu < \lambda \text{ and } s', t' \in M(\mu)\}.$$

Of course when $\bar{}$ is the identity on R and $M(\lambda)$ is finite, this becomes Definition (1.1) of [6]. Note also we have not assumed that A possesses a multiplicative identity. However, it can be immediately seen that this will have little effect. Indeed, we define the algebra A^1 obtained from A by “appending” a 1 to A if necessary. If A has a 1 we simply set $A^1 = A$. Note that in this case

$$1 = (1^*)^* = (1^* 1)^* = 1^* (1^*)^* = 1^* 1 = 1^*$$

because $*$ is an anti-involution. Suppose A does not have a 1. As an R -module, we define

$$A^1 = A \oplus R.$$

This becomes an R -algebra under the multiplication

$$(a_1, r_1)(a_2, r_2) = (a_1 a_2 + r_1 a_2 + r_2 a_1, r_1 r_2).$$

We identify $a \in A$ with $(a, 0)$ and denote $(0, 1)$ by 1. Clearly $*$ can be extended to A^1 by defining $(a, r)^* = (a^*, \bar{r})$ for $a \in A$ and $r \in R$. In fact A^1 then becomes conjugate cellular with cell datum $(\Lambda^1, M^1, C, *)$ by appending a greatest element 1 to Λ and setting

$$M(1) = \{1\} \quad \text{and} \quad C_{11}^1 = 1.$$

Thus one can deduce many of the results for algebras without identities directly from [6]. Nevertheless we offer complete proofs here.

Suppose from this point that A is conjugate cellular with cell datum $(\Lambda, M, C, *)$. If we apply $*$ to (5), replace a with a^* and swap s and t , we obtain

$$(6) \quad C_{st}^\lambda a \in \sum_{u \in M(\lambda)} \overline{r_{a^*}(u, t)} C_{su}^\lambda + A(< \lambda).$$

For any subset $\Lambda' \subseteq \Lambda$, let

$$A(\Lambda') = \text{span}_R \{ C_{st}^\lambda \mid \lambda \in \Lambda' \text{ and } s, t \in M(\lambda) \}.$$

The following is then immediate from (5) and (6).

Lemma 18. *Suppose that Λ' is an ideal of Λ ; that is, if $\lambda \in \Lambda'$ and $\mu \leq \lambda$ then $\mu \in \Lambda'$. Then $A(\Lambda')$ is an ideal of A .*

For instance,

$$A(\leq \lambda) = A(\{\mu \in \Lambda \mid \mu \leq \lambda\}) \quad \text{and} \quad A(< \lambda) = A(\{\mu \in \Lambda \mid \mu < \lambda\})$$

are ideals. We can therefore define the (A, A) -bimodule

$$Q(\lambda) = A(\leq \lambda)/A(< \lambda).$$

Now $A(\leq \lambda) = A(\lambda) \oplus A(< \lambda)$ as R -modules, where we have denoted $A(\lambda) = A(\{\lambda\})$ for convenience. We therefore have a natural R -module isomorphism

$$(7) \quad Q(\lambda) \cong A(\lambda).$$

We conclude this chapter with the obvious analogue of [6] Theorem 1.8.

Theorem 19 (Specialisation). *Suppose $\sigma : R \rightarrow R'$ is a homomorphism of commutative rings, so that R' becomes an R -module in the obvious way. Suppose moreover that R and R' are equipped with involutions, both of which will be denoted $\bar{}$, and that σ is compatible with these in the sense that*

$$\sigma(\bar{r}) = \overline{\sigma(r)}$$

for all $r \in R$. The R' -algebra $A^\sigma = R' \otimes_R A$ is the specialisation of A at σ . If A is conjugate cellular with cell datum $(\Lambda, M, C, *)$, then A^σ is conjugate cellular with datum $(\Lambda, M, C^\sigma, *^\sigma)$, where

$$(C^\sigma)_{st}^\lambda = 1 \otimes_R C_{st}^\lambda$$

and

$$(r' \otimes_R a)^{\sigma} = \bar{r}' \otimes_R a^*.$$

Cell Representations

Because the C_{st}^λ form an R -basis of A , the elements $r_a(s, t)$ are clearly unique. It follows that they are R -linear in a , and that

$$r_{ab}(s, t) = \sum_{u \in M(\lambda)} r_a(s, u)r_b(u, t).$$

We can therefore define an A -module as follows.

Definition 20. For each $\lambda \in \Lambda$, let $W(\lambda)$ be the left A -module with R -basis

$$\{C_s \mid s \in M(\lambda)\}$$

and A action

$$aC_s = \sum_{t \in M(\lambda)} r_a(t, s)C_t.$$

We use

$$\rho^\lambda : A \rightarrow \text{Mat}_{M(\lambda)}(R)$$

to denote the corresponding representation relative to the natural basis. That is,

$$\rho^\lambda(a)_{st} = r_a(s, t)$$

for $a \in A$ and $s, t \in M(\lambda)$. Similarly let $W(\lambda)^*$ denote the right A -module with R -basis

$$\{C_s^* \mid s \in M(\lambda)\}$$

and A action

$$C_s^*a = \sum_{t \in M(\lambda)} \overline{r_{a^*}(t, s)}C_t^*.$$

In fact because $r_a(s, t)$ can be defined for $a \in A^1$, we often think of $W(\lambda)$ as an A^1 -module. We make use of the following simple results later.

Lemma 21.

(i) There is a natural isomorphism of R -modules $C^\lambda : W(\lambda) \otimes_R W(\lambda)^* \rightarrow A(\lambda)$ given by

$$C^\lambda(C_s \otimes_R C_t^*) = C_{st}^\lambda$$

for $s, t \in M(\lambda)$. On identifying $A(\lambda)$ with $Q(\lambda)$ via (7), this becomes an (A, A) -bimodule isomorphism $C^\lambda : W(\lambda) \otimes_R W(\lambda)^* \rightarrow Q(\lambda)$.

(ii) We have an R -module decomposition

$$A = \bigoplus_{\lambda \in \Lambda} A(\lambda).$$

(iii) If $a \in A(\lambda)$ and $s, t \in M(\mu)$ then $r_a(s, t) = 0$ unless $\mu \leq \lambda$.

Proof.

- (i) Because $\{C_s \mid s \in M(\lambda)\}$ is an R -basis for $W(\lambda)$ and $\{C_s^* \mid s \in M(\lambda)\}$ is an R -basis for $W(\lambda)^*$, certainly

$$\{C_s \otimes_R C_t^* \mid s, t \in M(\lambda)\}$$

is an R -basis for $W(\lambda) \otimes_R W(\lambda)^*$. Also by definition

$$\{C_{st}^\lambda \mid s, t \in M(\lambda)\}$$

is an R -basis for $A(\lambda)$, so certainly C^λ is an R -module isomorphism. To check that it preserves the (A, A) -bimodule action, we need only consider a basis element of $W(\lambda) \otimes_R W(\lambda)^*$. Note that we are now treating C^λ as a map to $Q(\lambda)$, under the identification (7). For each $a \in A$ and $s, t \in M(\lambda)$, we have

$$\begin{aligned} aC^\lambda(C_s \otimes_R C_t^*) &= aC_{st}^\lambda + A(< \lambda) \\ &= \sum_{u \in M(\lambda)} r_a(u, s)C_{ut}^\lambda + A(< \lambda) \\ &= \sum_{u \in M(\lambda)} r_a(u, s)C^\lambda(C_u \otimes_R C_t^*) \\ &= C^\lambda \left(\sum_{u \in M(\lambda)} r_a(u, s)C_u \otimes_R C_t^* \right) \\ &= C^\lambda((aC_s) \otimes_R C_t^*) \\ &= C^\lambda(a(C_s \otimes_R C_t^*)) \end{aligned}$$

as required. Preservation of the right action follows similarly.

- (ii) By assumption, A has an R -basis

$$\{C_{st}^\lambda \mid \lambda \in \Lambda \text{ and } s, t \in M(\lambda)\} = \coprod_{\lambda \in \Lambda} \{C_{st}^\lambda \mid s, t \in M(\lambda)\}.$$

Thus

$$A = \bigoplus_{\lambda \in \Lambda} \text{span}_R \{C_{st}^\lambda \mid s, t \in M(\lambda)\} = \bigoplus_{\lambda \in \Lambda} A(\lambda).$$

- (iii) Since $a \in A(\lambda) \subseteq A(\leq \lambda)$, and the latter is an ideal, we have

$$aC_{tt}^\mu \in A(\leq \lambda).$$

By (5), the coefficient of C_{st}^μ in the left hand product is $r_a(s, t)$. Thus if this is nonzero, we must have $\mu \leq \lambda$.

□

We now aim to define an important bilinear form on $W(\lambda)$.

Lemma 22. *For $\lambda \in \Lambda$, $s, t, u, v \in M(\lambda)$ and $a \in A^1$, there is a unique element $\phi_a^\lambda(C_s, C_t)$ in R such that*

$$C_{us}^\lambda a C_{tv}^\lambda \in \phi_a^\lambda(C_s, C_t) C_{uv}^\lambda + A(< \lambda).$$

This element is independent of u and v .

Proof. By (5) and (6), we have

$$C_{us}^\lambda a C_{tv}^\lambda \in \sum_{w \in M(\lambda)} r_{C_{us}^\lambda a}(w, t) C_{wv}^\lambda + A(< \lambda)$$

and

$$C_{us}^\lambda a C_{tv}^\lambda \in \sum_{w \in M(\lambda)} \overline{r_{C_{vt}^\lambda a^*}(w, s)} C_{uw}^\lambda + A(< \lambda).$$

Hence

$$\sum_{w \in M(\lambda)} r_{C_{us}^\lambda a}(w, t) C_{vw}^\lambda + A(< \lambda) = \sum_{w \in M(\lambda)} \overline{r_{C_{vt}^\lambda a^*}(w, s)} C_{uw}^\lambda + A(< \lambda).$$

Now the C_{st}^μ form an R -basis for A . Thus the elements $C_{st}^\lambda + A(< \lambda)$ are linearly independent in $A/A(< \lambda)$. Comparing their coefficients in the above, we have

$$\begin{aligned} r_{C_{us}^\lambda a}(w, t) &= 0 && \text{for } w \neq u, \\ r_{C_{vt}^\lambda a^*}(w, s) &= 0 && \text{for } w \neq v, \\ r_{C_{us}^\lambda a}(u, t) &= \overline{r_{C_{vt}^\lambda a^*}(v, s)}. \end{aligned}$$

In the third equality, the left hand side is independent of v , while the right hand side is independent of u . Therefore both sides are equal to an element

$$\phi_a^\lambda(C_s, C_t) \in R$$

which is independent of both u and v . Substituting these coefficients into either of the above expressions for $C_{us}^\lambda a C_{tv}^\lambda$, we obtain the required equation

$$C_{us}^\lambda a C_{tv}^\lambda \in \phi_a^\lambda(C_s, C_t) C_{uv}^\lambda + A(< \lambda).$$

The uniqueness of $\phi_a^\lambda(C_s, C_t)$ again follows from the linear independence of the $C_{st}^\lambda + A(< \lambda)$ in $A/A(< \lambda)$. \square

The above Lemma defines $\phi_a^\lambda(C_s, C_t)$ for any two basis elements $C_s, C_t \in W(\lambda)$. We extend ϕ_a^λ to an R -sesquilinear form on $W(\lambda)$; that is, we extend it to be R -conjugate linear in the first variable and R -linear in the second. Also we are primarily interested in the form ϕ_1^λ , which we denote ϕ^λ for convenience. We use Φ^λ to denote the matrix representation of ϕ^λ relative to the natural basis. That is, $\Phi^\lambda \in \text{Mat}_{M(\lambda)}(R)$ is defined by

$$\Phi_{st}^\lambda = \phi^\lambda(C_s, C_t)$$

for $s, t \in M(\lambda)$.

Proposition 23. *The form ϕ_a^λ has the following properties.*

(i) For $a \in A^1$ and $x, y \in W(\lambda)$,

$$\phi_a^\lambda(x, y) = \overline{\phi_{a^*}^\lambda(y, x)}.$$

(ii) The form ϕ^λ is conjugate symmetric; that is, for $x, y \in W(\lambda)$,

$$\phi^\lambda(x, y) = \overline{\phi^\lambda(y, x)}.$$

(iii) For $x, y \in W(\lambda)$ and $a, b \in A^1$,

$$\phi_a^\lambda(x, by) = \phi_{ab}^\lambda(x, y).$$

(iv) For $x, y \in W(\lambda)$ and $a \in A^1$,

$$\phi^\lambda(x, ay) = \phi^\lambda(a^*x, y).$$

(v) For $a \in A^1$ and $s, t \in M(\lambda)$,

$$\phi_a^\lambda(C_s, C_t) = \sum_{u \in M(\lambda)} \Phi_{su}^\lambda \rho^\lambda(a)_{ut}.$$

(vi) For $x, y, z \in W(\lambda)$, we have

$$C^\lambda(x \otimes_R y^*)z = \phi^\lambda(y, z)x$$

where $*$: $W(\lambda) \rightarrow W(\lambda)^*$ denotes the R -conjugate linear map

$$\left(\sum_{s \in M(\lambda)} r_s C_s \right)^* = \sum_{s \in M(\lambda)} \overline{r_s} C_s^*.$$

Proof.

(i) Since both sides are linear in y and conjugate linear in x , it suffices to prove the equality for $x = C_s$ and $y = C_t$. By definition,

$$C_{st}^\lambda a^* C_{st}^\lambda \in \phi_{a^*}^\lambda(C_t, C_s) C_{st}^\lambda + A(< \lambda).$$

Applying $*$,

$$C_{ts}^\lambda a C_{ts}^\lambda \in \overline{\phi_{a^*}^\lambda(C_t, C_s)} C_{ts}^\lambda + A(< \lambda).$$

By uniqueness, we then have

$$\phi_a(C_s, C_t) = \overline{\phi_{a^*}^\lambda(C_t, C_s)}$$

as required.

(ii) This follows immediately from (i), since $1^* = 1$.

(iii) Again both sides are linear in y and conjugate linear in x , so we can take $x = C_s$ and $y = C_t$. We calculate

$$\begin{aligned} C_{ts}^\lambda a b C_{ts}^\lambda &\in C_{ts}^\lambda a \sum_{u \in M(\lambda)} r_b(u, t) C_{us}^\lambda + A(< \lambda) \\ &= \sum_{u \in M(\lambda)} r_b(u, t) C_{ts}^\lambda a C_{us}^\lambda + A(< \lambda) \\ &= \sum_{u \in M(\lambda)} r_b(u, t) \phi_a^\lambda(C_s, C_u) C_{ts}^\lambda + A(< \lambda). \end{aligned}$$

Thus by uniqueness,

$$\begin{aligned} \phi_{ab}^\lambda(C_s, C_t) &= \sum_{u \in M(\lambda)} r_b(u, t) \phi_a^\lambda(C_s, C_u) \\ &= \phi_a^\lambda \left(C_s, \sum_{u \in M(\lambda)} r_b(u, t) C_u \right) \\ &= \phi_a^\lambda(C_s, bC_t), \end{aligned}$$

as required.

(iv) Combining (i) and (iii),

$$\phi^\lambda(x, ay) = \phi_a^\lambda(x, y) = \overline{\phi_{a^*}^\lambda(y, x)} = \overline{\phi^\lambda(y, a^*x)} = \phi^\lambda(a^*x, y).$$

(v) This follows from (iii), as

$$\begin{aligned}
\phi_a^\lambda(C_s, C_t) &= \phi^\lambda(C_s, aC_t) \\
&= \phi^\lambda\left(C_s, \sum_{u \in M(\lambda)} r_a(u, t)C_u\right) \\
&= \sum_{u \in M(\lambda)} r_a(u, t)\phi^\lambda(C_s, C_u) \\
&= \sum_{u \in M(\lambda)} \Phi_{su}^\lambda \rho^\lambda(a)_{ut}.
\end{aligned}$$

(vi) As above we can assume $x = C_s$, $y = C_t$ and $z = C_u$. Now by definition,

$$C_{st}^\lambda C_{uu}^\lambda \in \phi^\lambda(C_t, C_u)C_{su}^\lambda + A(< \lambda).$$

By the definition of the A -action on $W(\lambda)$, this implies

$$C_{st}^\lambda C_u = \phi^\lambda(C_t, C_u)C_s.$$

Employing the definition of C^λ , this becomes

$$C^\lambda(C_s \otimes_R C_t^*)C_u = \phi^\lambda(C_t, C_u)C_s$$

as required. □

Corollary 24. For $z \in W(\lambda)$, let

$$R_z = \{\phi^\lambda(y, z) \mid y \in W(\lambda)\}.$$

- (i) R_z is an ideal of R .
- (ii) If $z_1, z_2 \in W(\lambda)$ then $R_{z_1+z_2} \subseteq R_{z_1} + R_{z_2}$.
- (iii) If $a \in A^1$ then $R_{az} \subseteq R_z$.
- (iv) We have $R_z W(\lambda) = A(\lambda)z$. In particular, if $R_z = R$ then $W(\lambda) = Az$.

Proof.

- (i) For $\phi^\lambda(y_1, z)$ and $\phi^\lambda(y_2, z) \in R_z$, and $r_1, r_2 \in R$, we have

$$r_1\phi^\lambda(y_1, z) + r_2\phi^\lambda(y_2, z) = \phi^\lambda(\bar{r}_1 y_1 + \bar{r}_2 y_2, z) \in R_z.$$

Therefore R_z is an ideal of R .

- (ii) Any element of $R_{z_1+z_2}$ is of the form $\phi^\lambda(y, z_1 + z_2)$. But

$$\phi^\lambda(y, z_1 + z_2) = \phi^\lambda(y, z_1) + \phi^\lambda(y, z_2) \in R_{z_1} + R_{z_2}.$$

Thus $R_{z_1+z_2} \subseteq R_{z_1} + R_{z_2}$.

- (iii) For any element $\phi^\lambda(y, az) \in R_{az}$, we have

$$\phi^\lambda(y, az) = \phi^\lambda(a^*y, z) \in R_z$$

by (iv) of Proposition 23. Hence $R_{az} \subseteq R_z$.

- (iv) Recall from (i) of Lemma 21 that C^λ is an R -module isomorphism. Also the map $*$: $W(\lambda) \rightarrow W(\lambda)^*$ is clearly bijective, so

$$\begin{aligned}
A(\lambda)z &= C^\lambda(W(\lambda) \otimes_R W(\lambda)^*)z \\
&= \text{span}_R \{C^\lambda(x \otimes_R y)z \mid x \in W(\lambda) \text{ and } y \in W(\lambda)^*\} \\
&= \text{span}_R \{C^\lambda(x \otimes_R y^*)z \mid x, y \in W(\lambda)\} \\
&= \text{span}_R \{\phi^\lambda(y, z)x \mid x, y \in W(\lambda)\} \text{ by (vi) of Proposition 23} \\
&= \{\phi^\lambda(y, z) \mid y \in W(\lambda)\} \cdot \{x \mid x \in W(\lambda)\} \\
&= R_z W(\lambda)
\end{aligned}$$

as required. □

We use the above to investigate the representations $W(\lambda)$.

Proposition 25. *Let $\lambda, \mu \in \Lambda$ and suppose $\theta : W(\lambda) \rightarrow W(\mu)/W$ is an A -module homomorphism, where W is an A -submodule of $W(\mu)$. Consider any $r \in R$ of the form $r = \phi^\lambda(y, z)$ for some $y, z \in W(\lambda)$.*

- (i) *Suppose the only element of $W(\mu)/W$ annihilated by r is 0. Then either $\mu \leq \lambda$ or $\theta = 0$. In particular this applies if $r \neq 0$ and $W(\mu)/W$ is torsion free as an R -module.*
- (ii) *If $\mu = \lambda$, there exists $r' \in R$ such that $r\theta(x) = r'x + W$ for all $x \in W(\lambda)$. Indeed we can take $r' = \phi^\lambda(y, z')$ where $\theta(z) = z' + W$.*

Proof.

- (i) Suppose 0 is the only element of $W(\mu)/W$ annihilated by r , and suppose $\mu \not\leq \lambda$. Consider any $x \in W(\lambda)$, and let $a = C^\lambda(x \otimes_R y^*) \in A(\lambda)$. By (iii) of Lemma 21, we have $r_a(s, t) = 0$ for all $s, t \in M(\mu)$. By definition of the A -action on $W(\mu)$, this implies that a annihilates $W(\mu)$. Hence a annihilates $W(\mu)/W$, so

$$\begin{aligned}
r\theta(x) &= \theta(\phi^\lambda(y, z)x) \\
&= \theta(C^\lambda(x \otimes_R y^*)z) \text{ by (vi) of Proposition 23} \\
&= \theta(az) \\
&= a\theta(z) \\
&= 0.
\end{aligned}$$

Hence $\theta(x) = 0$. But x was arbitrary, so $\theta = 0$ as required. Note that if $r \neq 0$ and $W(\mu)/W$ is torsion free as an R -module, then by definition the only element annihilated by r is 0.

- (ii) Suppose $\mu = \lambda$. Let $\theta(z) = z' + W$, and $r' = \phi^\lambda(y, z')$. Then for $x \in W(\lambda)$,

$$\begin{aligned}
r\theta(x) &= \theta(\phi^\lambda(y, z)x) \\
&= \theta(C^\lambda(x \otimes_R y^*)z) \text{ by (vi) of Proposition 23} \\
&= C^\lambda(x \otimes_R y^*)\theta(z) \\
&= C^\lambda(x \otimes_R y^*)z' + W \\
&= \phi^\lambda(y, z')x + W \\
&= r'x + W,
\end{aligned}$$

as required.

□

Corollary 26. *Suppose $y, z \in W(\lambda)$ are such that $r = \phi^\lambda(y, z) \in R$ is not a zero divisor. Then the obvious inclusion*

$$R \hookrightarrow \text{hom}_A(W(\lambda), W(\lambda))$$

is an isomorphism.

Proof. Note first that the above is indeed an inclusion, because $W(\lambda)$ is free as an R -module. Now suppose $\theta \in \text{hom}_A(W(\lambda), W(\lambda))$. Let $\theta(z) = z'$ and $r' = \phi^\lambda(y, z')$. By (ii) of Proposition 25, we have

$$(8) \quad r\theta(x) = r'x$$

for all $x \in W(\lambda)$. Pick any $s \in M(\lambda)$. Then (8) gives

$$r\theta(C_s) = r'C_s.$$

But the C_t form an R -basis for $W(\lambda)$, so this implies that $r' = rr''$, where $r'' \in R$ is the coefficient of C_s in $\theta(C_s)$. Thus (8) becomes

$$r\theta(x) = rr''x.$$

Because r is not a zero divisor and $W(\lambda)$ is free, this gives

$$\theta(x) = r''x$$

for $x \in W(\lambda)$. Therefore the above inclusion is surjective, whence the statement. □

We conclude this chapter with a result about irreducible modules. We first clarify our terminology. An A -module V is *simple* if it has no A -submodules other than 0 and V . We say V is *irreducible* if it is simple and satisfies $AV \neq 0$. Because A is not necessarily unital, we need the latter condition to exclude modules with a trivial A action, as well as the zero module.

Proposition 27. *Any irreducible A -module V is a quotient of $W(\lambda)$ for some $\lambda \in \Lambda$ such that $A(\lambda)V \neq 0$.*

Proof. From (ii) of Lemma 21, we have

$$0 \neq AV = \sum_{\lambda \in \Lambda} A(\lambda)V.$$

Thus

$$(9) \quad A(\lambda)V \neq 0$$

for some $\lambda \in \Lambda$. Because Λ is finite, we can pick λ to be minimal subject to (9), with respect to the partial order on Λ . Because the elements C_{st}^λ form an R -basis for $A(\lambda)$, there exist $s, t \in M(\lambda)$ and $x \in V$ with

$$C_{st}^\lambda x \neq 0.$$

Let θ be the R -linear map $\theta : W(\lambda) \rightarrow V$ defined on the basis by

$$\theta(C_u) = C_{ut}^\lambda x \in V.$$

We will show that θ is an A -module homomorphism. Consider any $a \in A$. Then

$$aC_{ut}^\lambda \in \sum_{v \in M(\lambda)} r_a(v, u)C_{vt}^\lambda + A(< \lambda).$$

Now because λ was chosen to be minimal subject to (9), we have

$$A(\mu)V = 0$$

for any $\mu < \lambda$. Thus

$$A(< \lambda)V = \sum_{\mu < \lambda} A(\mu)V = 0.$$

Hence applying the above expression for aC_{ut}^λ to x , we have

$$aC_{ut}^\lambda x = \sum_{v \in M(\lambda)} r_a(v, u)C_{vt}^\lambda x.$$

Thus

$$\begin{aligned} a\theta(C_u) &= aC_{ut}^\lambda x \\ &= \sum_{v \in M(\lambda)} r_a(v, u)C_{vt}^\lambda x \\ &= \sum_{v \in M(\lambda)} r_a(v, u)\theta(C_v) \\ &= \theta\left(\sum_{v \in M(\lambda)} r_a(v, u)C_v\right) \\ &= \theta(aC_u). \end{aligned}$$

Thus θ preserves the A -action, as required. Now

$$0 \neq C_{st}^\lambda x = \theta(C_s) \in \theta(W(\lambda)),$$

so that $\theta(W(\lambda))$ is nonzero. But this is an A -submodule of V , so $\theta(W(\lambda)) = V$ as V is simple. Hence θ is a surjective A -module homomorphism, so V is a quotient of $W(\lambda)$. \square

Representation Theory over a Field

We suppose in this chapter that R is a field. In this case the above theory allows us to completely determine the irreducible representations of A . First we introduce some notation.

Definition 28. For each $\lambda \in \Lambda$, define

$$\begin{aligned} \text{rad}(\lambda) &= \{z \in W(\lambda) \mid R_z = 0\} \\ &= \{z \in W(\lambda) \mid \phi^\lambda(y, z) = 0 \text{ for all } y \in W(\lambda)\}. \end{aligned}$$

Also let

$$\begin{aligned} \Lambda_0 &= \{\lambda \in \Lambda \mid \text{rad}(\lambda) \neq W(\lambda)\} \\ &= \{\lambda \in \Lambda \mid \phi^\lambda \neq 0\}. \end{aligned}$$

The notation $\text{rad}(\lambda)$ is justified by part (iii) of the following Theorem.

Theorem 29.

- (i) For each $\lambda \in \Lambda$, the set $\text{rad}(\lambda)$ is an A -submodule of $W(\lambda)$.
- (ii) For $\lambda \in \Lambda_0$, the quotient $L_\lambda = W(\lambda)/\text{rad}(\lambda)$ is irreducible.
- (iii) For $\lambda \in \Lambda_0$, the submodule $\text{rad}(\lambda)$ is the minimal submodule of $W(\lambda)$ with semisimple quotient, that is, the radical of $W(\lambda)$.
- (iv) The modules

$$\{L_\lambda \mid \lambda \in \Lambda_0\}$$

form a complete set of distinct irreducible modules over A .

Proof.

- (i) It follows from (ii) of Corollary 24 that $\text{rad}(\lambda)$ is closed under addition, and from (iii) that it is invariant under the action of A^1 .
- (ii) Suppose $\lambda \in \Lambda_0$, so that $L_\lambda = W(\lambda)/\text{rad}(\lambda) \neq 0$. Consider any nonzero element

$$z + \text{rad}(\lambda) \in L_\lambda.$$

Then $z \notin \text{rad}(\lambda)$, so that $R_z \neq 0$. Since R is a field and R_z is an ideal, this gives $R_z = R$. But then $Az = W(\lambda)$ by (iv) of Corollary 24. Hence

$$A(z + \text{rad}(\lambda)) = W(\lambda)/\text{rad}(\lambda) = L_\lambda.$$

This shows that L_λ is generated by any nonzero element. It is therefore irreducible.

- (iii) Certainly the quotient $W(\lambda)/\text{rad}(\lambda)$ is semisimple by (ii). Suppose that U is any submodule of $W(\lambda)$ with semisimple quotient. Let $V = U \cap \text{rad}(\lambda)$. We have an inclusion

$$W(\lambda)/V \hookrightarrow W(\lambda)/U \oplus W(\lambda)/\text{rad}(\lambda),$$

so that $W(\lambda)/V$ is semisimple. We can therefore write

$$W(\lambda)/V = \text{rad}(\lambda)/V \oplus W$$

for some A -submodule $W \subseteq W(\lambda)/V$. Since $\lambda \in \Lambda_0$, certainly $\text{rad}(\lambda)$ is a proper submodule of $W(\lambda)$, so that $\text{rad}(\lambda)/V$ is a proper submodule of $W(\lambda)/V$. Thus $W \neq 0$. Pick any nonzero element

$$z + V \in W.$$

Because the sum is direct, we must have

$$z + V \notin \text{rad}(\lambda)/V.$$

Thus $z \notin \text{rad}(\lambda)$. As above, this implies $Az = W(\lambda)$. But W is an A -submodule, so

$$W \supseteq A(z + V) = W(\lambda)/V.$$

Hence $W = W(\lambda)/V$, so that $\text{rad}(\lambda)/V = 0$. This implies $\text{rad}(\lambda) = V$, so that $\text{rad}(\lambda) \subseteq U$, as required.

(iv) Certainly each L_λ is irreducible by (ii). Consider any irreducible A -module V . By Proposition 27, we have

$$V \cong W(\lambda)/W$$

for some $\lambda \in \Lambda$ and $W \subseteq W(\lambda)$ with $A(\lambda)V \neq 0$. Suppose $\text{rad}(\lambda) = W(\lambda)$. Then for each $z \in W(\lambda)$, we have $R_z = 0$, so that $A(\lambda)z = 0$ by (iv) of Corollary 24. As z was arbitrary, this gives $A(\lambda)W(\lambda) = 0$, so that $A(\lambda)V = A(\lambda)(W(\lambda)/W) = 0$, a contradiction. Hence $\text{rad}(\lambda) \neq W(\lambda)$, so that $\lambda \in \Lambda_0$. Now by (iii), $\text{rad}(\lambda)$ is the radical of $W(\lambda)$. But $W(\lambda)/W \cong V$ is irreducible, and therefore semisimple, so

$$\text{rad}(\lambda) \subseteq W.$$

Thus V is a quotient of $W(\lambda)/\text{rad}(\lambda) = L_\lambda$. But the latter is irreducible by (ii), so $V \cong L_\lambda$. Hence the L_λ form a complete set of irreducibles. It remains to show that they are inequivalent. Suppose that

$$L_\lambda \cong L_\mu$$

for some $\lambda, \mu \in \Lambda_0$. Composing this isomorphism with the natural surjection

$$W(\lambda) \twoheadrightarrow W(\lambda)/\text{rad}(\lambda) = L_\lambda,$$

we obtain a surjective A -module homomorphism

$$\theta : W(\lambda) \twoheadrightarrow L_\mu = W(\mu)/\text{rad}(\mu).$$

Because R is a field, certainly $W(\mu)/\text{rad}(\mu)$ is torsion free as an R -module. Also $\lambda \in \Lambda_0$ implies $\phi^\lambda \neq 0$, so for some $y, z \in W(\lambda)$ we have $r = \phi^\lambda(y, z) \neq 0$. We can then apply (i) of Proposition 25. But $L_\mu \neq 0$ and θ is surjective, so certainly $\theta \neq 0$. We can therefore conclude that $\mu \leq \lambda$. Similarly we obtain $\lambda \leq \mu$, so $\lambda = \mu$ as required.

□

Before proceeding, we prove a well known lemma about algebra ideals.

Lemma 30. *Suppose A is an algebra and B is an ideal of A . Suppose moreover that B has an identity. Then*

$$A = B \oplus C$$

for some ideal C of A . In particular, A/B is isomorphic to an ideal of A .

Proof. Denote the identity of B by e . Consider any $a \in A$. Certainly $ea \in B$, because B is an ideal. Because e is an identity for B , this gives $eae = ea$. Similarly $eae = ae$, so $ea = ae$. Thus e is a central idempotent in A . Let

$$C = \{a \in A \mid ae = 0\}.$$

Then C is an ideal because e is central. Moreover if $a \in B \cap C$, then $a = ae = 0$. Hence $B \cap C = 0$. Finally for any $a \in A$, we have $ae \in B$ because B is an ideal, and

$$(a - ae)e = ae - ae^2 = ae - ae = 0,$$

so that $a - ae \in C$. Thus $a = ae + (a - ae) \in B + C$, and so $A = B \oplus C$ as claimed. It is now clear that $A/B \cong C$ is an ideal of A . \square

Finally we prove an analogue of Theorem (3.8) of [6]. Unfortunately, it only partly generalises to the case where the $M(\lambda)$ may be infinite.

Theorem 31. *Consider the following four statements about the conjugate cellular algebra A .*

- (A) *The algebra A is semiprimitive.*
- (B) *The algebra A is semiprime.*
- (C) *For each $\lambda \in \Lambda$, the form ϕ^λ is nondegenerate, that is $\text{rad}(\lambda) = 0$.*
- (D) *The cell representations $W(\lambda)$ are irreducible and pairwise inequivalent.*

Then (A) is equivalent to (B), (C) is equivalent to (D), and (C) implies (A). If the $M(\lambda)$ are finite whenever λ is not maximal in Λ (so in particular if all the $M(\lambda)$ are finite), then all four statements are equivalent.

Proof. We first note that by (iv) of Corollary 24, the definition of $\text{rad}(\lambda)$ can be rewritten as

$$(10) \quad \text{rad}(\lambda) = \{z \in W(\lambda) \mid A(\lambda)z = 0\}.$$

Certainly (A) implies (B) for any algebra. Suppose (B) holds, so that A is semiprime. Recall that an algebra is semiprimitive if the intersection of the annihilator ideals of its irreducible representations is trivial. Suppose an element $a \in A$ annihilates every irreducible representation of A . Suppose $\text{rad}(\lambda) \neq W(\lambda)$, so the module $W(\lambda)/\text{rad}(\lambda)$ is irreducible by (ii) of Theorem 29. Then

$$a(W(\lambda)/\text{rad}(\lambda)) = 0,$$

so that

$$aW(\lambda) \subseteq \text{rad}(\lambda).$$

Clearly this also holds when $\text{rad}(\lambda) = W(\lambda)$. Thus for any λ , we have

$$A(\lambda)aW(\lambda) \subseteq A(\lambda)\text{rad}(\lambda) = 0$$

by (10). Therefore

$$A(\lambda)a(W(\lambda) \otimes_R W(\lambda)^*) = 0.$$

Recall from (i) of Lemma 21 that C^λ is an (A, A) -bimodule isomorphism from $W(\lambda) \otimes_R W(\lambda)^*$ to $Q(\lambda) = A(\leq \lambda)/A(< \lambda)$. The above then implies

$$A(\lambda)aA(\leq \lambda) \subseteq A(< \lambda).$$

We now write the elements of Λ as $\lambda^1, \lambda^2, \dots, \lambda^d$ by refining the partial order on Λ to a total order; that is, the labels are chosen so that $\lambda^i \leq \lambda^j$ implies $i \leq j$. Then

$$\Lambda^i = \{\lambda^1, \lambda^2, \dots, \lambda^i\}$$

is an ideal of Λ , where we take Λ^0 to be the empty set. The above gives

$$A(\lambda^i)aA(\lambda^i) \subseteq A(\lambda^i)aA(\leq \lambda^i) \subseteq A(< \lambda^i) \subseteq A(\Lambda^{i-1})$$

for $1 \leq i \leq d$. Now $A(\Lambda^i) = A(\lambda^i) \oplus A(\Lambda^{i-1})$ and $A(\Lambda^{i-1})$ is an ideal, so this gives

$$A(\Lambda^i)aA(\Lambda^i) \subseteq A(\Lambda^{i-1}).$$

Since $A(\Lambda^d) = A$, it now follows easily by induction that

$$(A^1 a A^1)^{2^{i+1}-1} \subseteq A(\Lambda^{d-i})$$

for $0 \leq i \leq d$. In particular

$$(A^1 a A^1)^{2^{d+1}-1} \subseteq A(\Lambda^0) = 0.$$

The ideal $A^1 a A^1$ is therefore nilpotent, so $A^1 a A^1 = 0$ since A is semiprime. Hence $a = 0$, so A is indeed semiprimitive.

Recall that each $M(\lambda)$ was assumed to be nonempty in the definition of conjugate cellular. Since the dimension of $W(\lambda)$ is just $|M(\lambda)|$, this implies that each $W(\lambda)$ is nonzero. Having noted this convention, the equivalence of (C) and (D) follows easily from Theorem 29. Indeed suppose (D) holds. By (ii) of Theorem 29, if $\lambda \in \Lambda_0$ then $W(\lambda)$ and $W(\lambda)/\text{rad}(\lambda)$ are both irreducible, so certainly $\text{rad}(\lambda) = 0$. However, if $\lambda \notin \Lambda_0$ then by (iv) of the same theorem, $W(\lambda) \cong L_\mu = W(\mu)$ for some $\mu \in \Lambda_0$, thus contradicting the assumption that the $W(\lambda)$ are pairwise inequivalent. Hence $\Lambda_0 = \Lambda$, so (C) does indeed hold. Conversely suppose (C) holds, so that $\text{rad}(\lambda) = 0 \neq W(\lambda)$ for each $\lambda \in \Lambda$. Thus $\Lambda_0 = \Lambda$ and $L_\lambda = W(\lambda)/\text{rad}(\lambda) = W(\lambda)$ for each λ , so (D) follows immediately from (iv) of Theorem 29.

Next suppose conditions (C) and (D) hold, and consider an element $a \in A$ which annihilates every irreducible representation of A . By (ii) of Lemma 21, we can express a uniquely as a sum

$$a = \sum_{\lambda \in \Lambda} a_\lambda$$

where $a_\lambda \in A(\lambda)$. Fix any $\lambda \in \Lambda$, and suppose $a_\mu = 0$ whenever $\mu > \lambda$. Certainly then $a_\mu W(\lambda) = 0$ for $\mu > \lambda$. Also if $\mu \not\geq \lambda$ then $A(\mu)W(\lambda) = 0$ by (iii) of Lemma 21, and we again have $a_\mu W(\lambda) = 0$. Finally $W(\lambda)$ is irreducible by condition (D), so that $aW(\lambda) = 0$ by assumption. Substituting the above sum for a , we therefore have

$$(11) \quad a_\lambda W(\lambda) = 0.$$

By (i) of Lemma 21, we can write $a_\lambda = C^\lambda(a')$ for some $a' \in W(\lambda) \otimes_R W(\lambda)^*$. Now recall that the $\{C_s \mid s \in M(\lambda)\}$ form a basis for $W(\lambda)$, and that each element of $W(\lambda)^*$ is uniquely expressible as y^* for some $y \in W(\lambda)$. Thus a' can be written uniquely in the form

$$a' = \sum_{s \in M(\lambda)} C_s \otimes_R y_s^*$$

for some $y_s \in W(\lambda)$. Given $z \in W(\lambda)$, we use (vi) of Proposition 23 to calculate

$$\begin{aligned} a_\lambda z &= C^\lambda(a')z \\ &= C^\lambda \left(\sum_{s \in M(\lambda)} C_s \otimes_R y_s^* \right) z \\ &= \sum_{s \in M(\lambda)} C^\lambda(C_s \otimes_R y_s^*)z \\ &= \sum_{s \in M(\lambda)} \phi^\lambda(y_s, z) C_s. \end{aligned}$$

However, $a_\lambda z = 0$ by (11), so the final expression above is 0. Since the C_s form a basis for $W(\lambda)$, this implies that $\phi^\lambda(y_s, z) = 0$ for each $s \in M(\lambda)$. This holds for all $z \in W(\lambda)$, so $y_s \in \text{rad}(\lambda)$ for each $s \in M(\lambda)$. This gives $y_s = 0$ by condition (C), so $a' = 0$. Hence $a_\lambda = C^\lambda(a') = 0$. Because Λ is finite, it follows by induction on λ that $a_\lambda = 0$ for each $\lambda \in \Lambda$, so that $a = 0$ and condition (A) holds.

Finally, under the assumption that $M(\lambda)$ is finite whenever λ is not maximal, we must show that (B) implies (C). So suppose that A is semiprime. Recall that an ideal of a semiprime algebra is semiprime. Fix some $\lambda \in \Lambda$. Then $A(\leq \lambda)$ and $A(< \lambda)$ are semiprime. Now if $\mu < \lambda$ then μ is not maximal, so that $M(\mu)$ is finite, and so $A(\mu)$ has finite dimension. Hence

$$A(< \lambda) = \bigoplus_{\mu < \lambda} A(\mu)$$

is a finite dimensional semiprime algebra. It therefore has an identity, so by Lemma 30, the quotient $Q(\lambda) = A(\leq \lambda)/A(< \lambda)$ is isomorphic as an algebra to an ideal of $A(\leq \lambda)$. In particular, $Q(\lambda)$ is semiprime. Consider an element $x \in \text{rad}(\lambda)$, so that $A(\lambda)x = 0$ by (10). It is easy to see from the definitions that, for any $a \in A$,

$$x^*a^* = (ax)^*$$

in $W(\lambda)^*$. Thus we also have $x^*A(\lambda) = 0$. Now $C^\lambda : W(\lambda) \otimes_R W(\lambda)^* \rightarrow Q(\lambda)$ is an (A, A) -bimodule isomorphism by (i) of Lemma 21, so we obtain

$$A(\lambda)C^\lambda(x \otimes_R x^*) = C^\lambda(x \otimes_R x^*)A(\lambda) = 0.$$

It is now clear from the multiplication in $Q(\lambda)$ and from (7) that

$$Q(\lambda)C^\lambda(x \otimes_R x^*) = C^\lambda(x \otimes_R x^*)Q(\lambda) = 0.$$

Hence the R -span of this element, that is $RC^\lambda(x \otimes_R x^*)$, is a nilpotent ideal of $Q(\lambda)$. Since $Q(\lambda)$ is semiprime, it must be the zero ideal. That is, we must have $C^\lambda(x \otimes_R x^*) = 0$. But C^λ is an isomorphism, so that $x \otimes_R x^* = 0$. Finally because the tensor product is over a field, and because $x^* = 0$ implies $x = 0$, we must have $x = 0$. But x was an arbitrary element of $\text{rad}(\lambda)$, so we have shown $\text{rad}(\lambda) = 0$ for each $\lambda \in \Lambda$. This is exactly condition (C). \square

Examples of Conjugate Cellular Algebras

We now give some examples to illustrate the last three chapters. Consider the commutative ring $R = \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$, which is equipped with an involution $\bar{}$ defined by $\overline{q^{\frac{1}{2}}} = q^{-\frac{1}{2}}$. The *Hecke algebra of type A_n* , denoted $H_n(q)$ (see [1], §2, Ex. 34), is the associative unital R -algebra with generators T_1, T_2, \dots, T_{n-1} and relations

$$\begin{aligned} T_i^2 &= (q-1)T_i + q, \\ T_i T_j &= T_j T_i && \text{if } |i-j| > 1, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}. \end{aligned}$$

This algebra is equipped with an R -linear anti-involution $*$: $H_n(q) \rightarrow H_n(q)$ defined on the generators by $T_i^* = T_i$, and an R -conjugate linear involution $\bar{}$ defined on the generators by

$$\overline{T_i} = T_i^{-1} = q^{-1}T_i + (q^{-1} - 1).$$

The Hecke algebra has several natural bases indexed by the symmetric group S_n . One such basis is the Kazhdan-Lusztig basis [13]

$$\{C_w \mid w \in S_n\}$$

which has the property that $\overline{C_w} = C_w$ for each $w \in S_n$. It is shown in [6] that in fact this is a cellular basis for $H_n(q)$. That is, for some poset Λ and indexing sets $M(\lambda)$, there is a bijection

$$RS : \bigcup_{\lambda \in \Lambda} M(\lambda) \times M(\lambda) \rightarrow S_n$$

such that, defining $C_{st}^\lambda = C_{RS(s,t)}$, the Hecke algebra is cellular with cell datum $(\Lambda, M, C, *)$. In fact Λ is the set of partitions of n , $M(\lambda)$ is the set of standard tableaux of shape λ , and RS is the Robinson-Schensted correspondence. Now consider the R -conjugate linear anti-involution $\bar{*}$ given by composing $\bar{}$ and $*$. That is,

$$a^{\bar{*}} = \overline{a^*}$$

for each $a \in H_n(q)$. Then

$$(C_{st}^\lambda)^{\bar{*}} = (\overline{C_{RS(s,t)}})^* = C_{RS(s,t)}^* = (C_{st}^\lambda)^* = C_{ts}^\lambda.$$

It now follows that $H_n(q)$ is conjugate cellular with cell datum $(\Lambda, M, C, \bar{*})$, where the involution on R is $\bar{}$.

The final claim of Theorem 31 is that, when the $M(\lambda)$ are finite, the equivalent conditions (A) and (B) imply the equivalent conditions (C) and (D). Our second example investigates the necessity of this finiteness condition. We first define an indexing set

$$I = \mathbb{Z} \times \{1, 2\}$$

which essentially consists of two copies of \mathbb{Z} . Let R be any field, and consider the algebra $\text{Mat}_I(R)$ consisting of $I \times I$ matrices over R , which have only finitely many nonzero entries in any fixed row or column. As usual, let E_{ij} denote the matrix which has 1 in position (i, j) and

zeroes elsewhere. Thus the R -span of the E_{ij} is the ideal consisting of matrices with finitely many nonzero entries. Also let X and P denote the matrices with entries

$$X_{(n,a)(m,b)} = \delta_{nm} \quad \text{and} \quad P_{(n,a)(m,b)} = \delta_{nm}\delta_{ab}(-1)^a.$$

Thus X and P are essentially 2×2 block matrices obtained from the matrices

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

by replacing each entry with the corresponding scalar matrix indexed by \mathbb{Z} (in these expressions we order the indexing set I as $\mathbb{Z} \times \{1\}$ followed by $\mathbb{Z} \times \{2\}$). Define the operation \circ on $\text{Mat}_I(R)$ using the sandwich matrix P so that

$$Y \circ Z = YPZ$$

for any $Y, Z \in \text{Mat}_I(R)$, where the juxtaposition on the right is usual matrix multiplication. Let A denote the R -span of the E_{ij} and X . Then A is closed under \circ because the span of the E_{ij} is an ideal and

$$X \circ X = XPX = 0.$$

Thus A is an R -algebra with multiplication \circ . Moreover A is conjugate cellular (with $\bar{}$ the identity) with the following cell datum. Let $\Lambda = \{0, 1\}$ under the usual ordering. Let $M(0) = I$ and $M(1) = \{1\}$. Let $C_{ij}^0 = E_{ij}$ for $i, j \in I$, and let $C_{11}^1 = X$. Finally let $*$ denote the transposition map, noting that X and P are symmetric matrices. Now the ideal generated by any nonzero element in A contains a nonzero idempotent. Indeed consider any nonzero element $Y \in \text{Mat}_I(R)$, so that $Y_{ij} \neq 0$ for some $i, j \in I$. Then

$$E_{jj} = \pm \frac{1}{Y_{ij}} E_{ji} \circ Y \circ E_{jj}$$

is idempotent and nonzero (in fact this would be true without the $\circ E_{jj}$ on the right). Therefore A is certainly semiprime and semiprimitive. However, because $X \circ X = 0$, the form ϕ^1 is identically zero, so conditions (C) and (D) clearly fail.

Our final example demonstrates the difficulty in removing the condition that Λ be finite. Note that this condition was only needed for Proposition 27, the word ‘‘complete’’ in (iv) of Theorem 29, and Theorem 31; the proofs of all the other results still hold when Λ is allowed to be infinite. However, these three are arguably the most important results. Let R be any algebraically closed field, and let $A = R[x]$ be the polynomial ring in one variable over R . Let $\Lambda = \mathbb{Z}_{\leq 0}$, the set of nonpositive integers, under its usual ordering. For $n \in \Lambda$, let $M(n) = \{n\}$, and let

$$C_{nn}^n = x^{-n} \in A.$$

Finally let $*$: $A \rightarrow A$ be the identity, which is an anti-involution because A is commutative. It is easy to verify that A satisfies all the criteria for a conjugate cellular algebra with cell datum $(\Lambda, M, C, *)$, except the finiteness of Λ . Now for each $r \in R$, there is an irreducible A -module V_r which is one dimensional, and on which x acts as r . In fact these form a complete set of distinct irreducible A -modules. However, every cell representation produced by this cell datum is isomorphic to V_0 . Thus Proposition 27 and (iv) of Theorem 29 both fail spectacularly.

The Main Theorem

In this chapter we prove a version of Theorem 15 of [4] for regular semigroups and for twisted semigroup algebras. As in [4], we will assume that the group algebras of the maximal subgroups of S are cellular, namely in Assumption 36. However, in [4] the anti-involutions on these algebras are woven together in the hope of creating an anti-involution on the semigroup algebra. In contrast, we start at the top by constructing an anti-involution $*$ on the semigroup algebra, and assuming that the anti-involution on each group algebra is a restriction of $*$. Therefore the assumptions we make will ensure that there is an anti-involution on the semigroup which induces an anti-involution on the twisted semigroup algebra, and which fixes certain maximal subgroups setwise.

We find it convenient to list the assumptions in the following discussion before stating the theorem. Firstly, we begin with the following objects.

Assumption 32. *Suppose that S is a finite regular semigroup, $*$: $S \rightarrow S$ an anti-involution, R a commutative ring with identity and with an involution $\bar{}$, and α a twisting from S into R .*

We suppose that $*$, $\bar{}$ and α are compatible in the following sense.

Assumption 33. *Assume that*

$$\alpha(x, y) = \overline{\alpha(y^*, x^*)}$$

for all $x, y \in S$.

This assumption implies that $*$ induces an R -conjugate linear anti-involution on $R^\alpha[S]$, which we also denote by $*$, given by

$$\left(\sum_{x \in S} r_x x \right)^* = \sum_{x \in S} \overline{r_x} x^*.$$

The next assumption ensures that $*$ fixes certain maximal subgroups.

Assumption 34. *Suppose that for each \mathcal{D} class $D \in \mathbb{D}$, we have an idempotent $1_D \in D$ which is fixed by $*$.*

The \mathcal{H} class of 1_D is then a group, which we denote G_D . Because $*$ is an anti-involution, it certainly preserves \mathcal{H} . Thus $*$ maps G_D into itself, and we use $*$ to denote its restriction to G_D . We will also need a twisted group algebra over G_D to be cellular. However, for this to give information about the rest of the \mathcal{D} class D , we need the scalar elements $\alpha(x, y)$ to be “sufficiently invertible”. Again assuming that $\alpha(x, y)$ is invertible is too restrictive, so we use a version of Assumption 11 which takes into account the involution $*$.

Assumption 35. *For each \mathcal{D} class D , we assume the existence of a map*

$$\beta_D : L_0 \times L_0^* \rightarrow G(R),$$

where L_0 is the \mathcal{L} class of 1_D . We suppose that β_D satisfies analogues of (1) and Assumption 33, namely

$$(12) \quad \alpha(x, y)\beta_D(xy, z) = \alpha(x, yz)\beta_D(y, z) \quad \text{if } yz \mathcal{R} y,$$

$$(13) \quad \beta_D(x, y)\beta_D(xy, z) = \beta_D(x, yz)\beta_D(y, z), \quad \text{and}$$

$$(14) \quad \beta_D(x, y) = \overline{\beta_D(y^*, x^*)}$$

whenever the relevant values of β_D are defined.

In fact this is a stronger version of Assumption 11; indeed (2) and (4) of Assumption 11 are exactly (12) and (13) of Assumption 35, while (3) of Assumption 11 can be obtained by applying $\bar{}$ to (12), replacing x, y and z with z^*, y^* and x^* respectively, and employing Assumption 33 and (14). Therefore the deductions discussed immediately after Assumption 11 apply here also; in particular,

$$\alpha(x, y) = \alpha(D)\beta_D(x, y)$$

for all $x \in L_0$ and $y \in L_0^*$, where $\alpha(D) = \alpha(1_D, 1_D)\beta_D(1_D, 1_D)^{-1}$, and the restriction of β_D defines a twisting on G_D . Note that by (14), $*$ induces an anti-involution on $R^{\beta_D}[G_D]$, which we again denote by $*$.

As foreshadowed, our final assumption is that certain twisted group algebras of the maximal subgroups are cellular.

Assumption 36. Suppose that the twisted group algebra $R^{\beta_D}[G_D]$ is conjugate cellular with cell datum

$$(\Lambda_D, M_D, C, *).$$

Remark. Note we have assumed that the anti-involution in this cell datum is exactly $*$.

Remark. It is interesting to note that, although we allow the twisting α for the semigroup to take values that are not units, we only consider twisted group algebras $R^{\beta_D}[G_D]$ in which the twisting is invertible.

Under the above assumptions, we will show that the twisted semigroup algebra $R^\alpha[S]$ is conjugate cellular. To be more precise, we describe the cell datum below. Because S is finite, $\mathcal{D} = \mathcal{J}$ by (i) of Theorem 5, so we have a relation $\leq_{\mathcal{D}}$ on S . Define the poset

$$\Lambda = \{(D, \lambda) \mid D \in \mathbb{D} \text{ and } \lambda \in \Lambda_D\}$$

with partial order

$$(D_1, \lambda_1) \leq (D_2, \lambda_2) \text{ iff } D_1 <_{\mathcal{D}} D_2 \text{ or } D_1 = D_2 \text{ and } \lambda_1 \leq \lambda_2 \text{ in } \Lambda_{D_1}.$$

Now for $(D, \lambda) \in \Lambda$, let

$$M(D, \lambda) = \mathbb{L}_D \times M_D(\lambda).$$

Finally for each $L \in \mathbb{L}_D$, choose any $u_L \in L$ with $u_L \mathcal{R} 1_D$. The basis elements that result from the cell datum of $R^{\beta_D}[G_D]$ can be written uniquely as

$$C_{st}^\lambda = \sum_{g \in G_D} c_{st}^\lambda(g)g$$

for some coefficients $c_{st}^\lambda(g) \in R$. Define

$$C_{(L,s)(K,t)}^{(D,\lambda)} = \sum_{g \in G_D} c_{st}^\lambda(g)\beta_D(g, u_K)\beta_D(u_L^*, gu_K)(u_L^*gu_K) \in R^\alpha[S]$$

for each $(D, \lambda) \in \Lambda_D$ and $(L, s), (K, t) \in M(D, \lambda)$.

Theorem 37. *Under Assumptions 32, 33, 34, 35 and 36, the algebra $R^\alpha[S]$ is conjugate cellular with the cell datum*

$$(\Lambda, M, C, *)$$

as given above.

Proof. Consider a \mathcal{D} class $D \in \mathbb{D}$. Now $*$ preserves \mathcal{D} and $1_D^* = 1_D$, so $*$ maps D onto D . Since $*$ is an anti-involution, it therefore maps the \mathcal{L} classes in D bijectively onto the \mathcal{R} classes in D . That is, each \mathcal{R} class in D is uniquely expressible as L^* for some $L \in \mathbb{L}_D$. Thus each \mathcal{H} -class in D is uniquely expressible as $L^* \cap K$ for some $L, K \in \mathbb{L}_D$.

For each $L \in \mathbb{L}_D$, we have $u_L \mathcal{R} 1_D$ by choice of u_L . Since 1_D is idempotent, this implies that $1_D u_L = u_L$. By Green's Lemma, right multiplication by u_L then gives an \mathcal{R} class preserving bijection from the \mathcal{L} class of 1_D to the \mathcal{L} class of u_L , which is L . Also applying $*$, we obtain $u_L^* 1_D = u_L^*$, so left multiplication by u_L^* gives an \mathcal{L} class preserving bijection from the \mathcal{R} class of 1_D to the \mathcal{R} class of u_L^* , namely L^* . Thus for each \mathcal{H} -class $L^* \cap K$ in D , we have a bijection

$$\begin{aligned} G_D &\rightarrow L^* \cap K && \text{given by} \\ g &\mapsto u_L^* g u_K. \end{aligned}$$

The above deductions from Green's Lemma also imply that, for each $g \in G_D$, the elements g and u_L^* are \mathcal{L} related to 1_D , while u_K and $g u_K$ are \mathcal{R} related to 1_D . Therefore $\beta_D(g, u_K)$ and $\beta_D(u_L^*, g u_K)$ are defined, so we can define an R -module homomorphism

$$\theta_{LK} : R^\alpha[G_D] \rightarrow R^\alpha[L^* \cap K]$$

on the natural basis by

$$\theta_{LK} : g \mapsto \beta_D(g, u_K) \beta_D(u_L^*, g u_K) (u_L^* g u_K).$$

Now these scalar factors are invertible by Assumption 35. Therefore by the above bijection $G_D \rightarrow L^* \cap K$, the linear map θ_{LK} has an invertible diagonal matrix with respect to the natural bases. Thus it is an isomorphism of R -modules. By assumption the elements

$$\left\{ C_{st}^\lambda \mid \lambda \in \Lambda_D \text{ and } s, t \in M_D(\lambda) \right\} = \left\{ \sum_{g \in G_D} c_{st}^\lambda(g) g \mid \lambda \in \Lambda_D \text{ and } s, t \in M_D(\lambda) \right\}$$

form an R -basis for $R^{\beta_D}[G_D]$, which is isomorphic to $R^\alpha[G_D]$ as an R -module. Thus applying the isomorphism θ_{LK} , the elements

$$\begin{aligned} &\left\{ \theta_{LK} (C_{st}^\lambda) \mid \lambda \in \Lambda_D \text{ and } s, t \in M_D(\lambda) \right\} \\ &= \left\{ \sum_{g \in G_D} c_{st}^\lambda(g) \beta_D(g, u_K) \beta_D(u_L^*, g u_K) (u_L^* g u_K) \mid \lambda \in \Lambda_D \text{ and } s, t \in M_D(\lambda) \right\} \\ &= \left\{ C_{(L,s)(K,t)}^{(D,\lambda)} \mid \lambda \in \Lambda_D \text{ and } s, t \in M_D(\lambda) \right\} \end{aligned}$$

form an R -basis for $R^\alpha[L^* \cap K]$. Now D is a disjoint union of its \mathcal{H} classes

$$D = \coprod_{L, K \in \mathbb{L}_D} L^* \cap K,$$

and S is in turn a disjoint union of its \mathcal{D} classes

$$S = \coprod_{D \in \mathbb{D}} D.$$

Thus

$$R^\alpha[D] = \bigoplus_{L, K \in \mathbb{L}_D} R^\alpha[L^* \cap K]$$

and

$$R^\alpha[S] = \bigoplus_{D \in \mathbb{D}} R^\alpha[D],$$

so that

$$(15) \quad \left\{ C_{(L,s)(K,t)}^{(D,\lambda)} \mid \lambda \in \Lambda_D \text{ and } (L,s), (K,t) \in M(D,\lambda) \right\}$$

form an R -basis for $R^\alpha[D]$, and

$$\left\{ C_{(L,s)(K,t)}^{(D,\lambda)} \mid (D,\lambda) \in \Lambda \text{ and } (L,s), (K,t) \in M(D,\lambda) \right\}$$

form an R -basis for $R^\alpha[S]$. This verifies property (C1) in Definition 17. We know that $*$ is an R -conjugate linear anti-involution of $R^\alpha[S]$, so to verify (C2) we need only check that it has the required action on the basis elements C . That is, we must check that

$$\left(C_{(L,s)(K,t)}^{(D,\lambda)} \right)^* = C_{(K,t)(L,s)}^{(D,\lambda)}.$$

First note that, because $(\Lambda_D, M_D, C, *)$ form a cell datum for $R^{\beta_D}[G_D]$, we have

$$(C_{st}^\lambda)^* = C_{ts}^\lambda.$$

That is,

$$\sum_{g \in G_D} \overline{c_{st}^\lambda(g)} g^* = \sum_{g \in G_D} c_{ts}^\lambda(g) g.$$

Because the elements of G_D form an R -basis for $R^{\beta_D}[G_D]$, this implies

$$\overline{c_{st}^\lambda(g)} = c_{ts}^\lambda(g^*).$$

Also applying (14) and (13) respectively, we have

$$\overline{\beta_D(g, u_K) \beta_D(u_L^*, gu_K)} = \beta_D(u_K^*, g^*) \beta_D(u_K^* g^*, u_L) = \beta_D(u_K^*, g^* u_L) \beta_D(g^*, u_L).$$

Hence

$$\begin{aligned} \left(C_{(L,s)(K,t)}^{(D,\lambda)} \right)^* &= \left(\sum_{g \in G_D} c_{st}^\lambda(g) \beta_D(g, u_K) \beta_D(u_L^*, gu_K) (u_L^* gu_K) \right)^* \\ &= \sum_{g \in G_D} \overline{c_{st}^\lambda(g) \beta_D(g, u_K) \beta_D(u_L^*, gu_K)} (u_L^* gu_K)^* \\ &= \sum_{g \in G_D} c_{ts}^\lambda(g^*) \beta_D(u_K^*, g^* u_L) \beta_D(g^*, u_L) (u_K^* g^* u_L) \\ &= \sum_{g \in G_D} c_{ts}^\lambda(g) \beta_D(u_K^*, gu_L) \beta_D(g, u_L) (u_K^* gu_L) \\ &= C_{(K,t)(L,s)}^{(D,\lambda)}, \end{aligned}$$

as required. Finally we must verify (C3). Denote

$$R^\alpha[S](\prec D) = \bigoplus_{D' \prec_{\mathcal{D}} D} R^\alpha[D']$$

for each $D \in \mathbb{D}$, and let $R^\alpha[S](\prec (D, \lambda))$ be as in Definition 17. It can be seen from (15) that

$$\begin{aligned} R^\alpha[S](\prec D) &= \text{span}_R \left\{ C_{(L,s)(K,t)}^{(D',\lambda')} \mid D' \prec_{\mathcal{D}} D, \lambda' \in \Lambda_{D'} \text{ and } (L,s), (K,t) \in M(D',\lambda') \right\} \\ &\subseteq \text{span}_R \left\{ C_{(L,s)(K,t)}^{(D',\lambda')} \mid (D',\lambda') \prec (D,\lambda) \text{ and } (L,s), (K,t) \in M(D',\lambda') \right\} \\ &= R^\alpha[S](\prec (D, \lambda)) \end{aligned}$$

for each $\lambda \in \Lambda_D$. Now for a given $(D, \lambda) \in \Lambda$ and $(L, s) \in M(D, \lambda)$, and for an element $a \in R^\alpha[S]$, we must find elements $r_a((L', s'), (L, s)) \in R$ for $(L', s') \in M(D, \lambda)$ such that

$$(16) \quad a \cdot C_{(L,s)(K,t)}^{(D,\lambda)} \in \sum_{(L',s') \in M(D,\lambda)} r_a((L', s'), (L, s)) C_{(L',s')(K,t)}^{(D,\lambda)} + R^\alpha[S](\langle (D, \lambda) \rangle)$$

for each $(K, t) \in M(D, \lambda)$. Because S spans $R^\alpha[S]$ as an R -module, it suffices to take $a \in S$. Clearly $au_L^* \leq_{\mathcal{D}} u_L^*$. First suppose that

$$au_L^* <_{\mathcal{D}} u_L^*.$$

In this case, put $r_a((L', s'), (L, s)) = 0$ for all $(L', s') \in M(D, \lambda)$. For any $g \in G_D$ and $K \in \mathbb{L}_D$, we have

$$au_L^* gu_K \leq_{\mathcal{D}} au_L^* <_{\mathcal{D}} D.$$

Hence

$$a \cdot (u_L^* gu_K) = \alpha(a, u_L^* gu_K)(au_L^* gu_K) \in R^\alpha[S](\langle D \rangle).$$

Thus for each $(K, t) \in M(D, \lambda)$ we have

$$\begin{aligned} a \cdot C_{(L,s)(K,t)}^{(D,\lambda)} &= a \cdot \left(\sum_{g \in G_D} c_{st}^\lambda(g) \beta_D(g, u_K) \beta_D(u_L^*, gu_K) (u_L^* gu_K) \right) \\ &= \sum_{g \in G_D} c_{st}^\lambda(g) \beta_D(g, u_K) \beta_D(u_L^*, gu_K) a \cdot (u_L^* gu_K) \\ &\in R^\alpha[S](\langle D \rangle) \\ &\subseteq R^\alpha[S](\langle (D, \lambda) \rangle) \\ &= \sum_{(L',s') \in M(D,\lambda)} r_a((L', s'), (L, s)) C_{(L',s')(K,t)}^{(D,\lambda)} + R^\alpha[S](\langle (D, \lambda) \rangle), \end{aligned}$$

as required for (16). The other case is when $au_L^* \mathcal{D} u_L^* \in D$. It follows from (ii) of Theorem 5 that $au_L^* \mathcal{L} u_L^* \mathcal{L} 1_D$. Thus if $L_1^* \in \mathbb{R}_D$ is the \mathcal{R} class of au_L^* , then $au_L^* \mathcal{H} u_{L_1}^*$. As above, it follows from Green's Lemma that

$$(17) \quad au_L^* = u_{L_1}^* h$$

for some $h \in G_D$. Now because $R^{\beta_D}[G_D]$ is cellular, there exist elements $r_h(s', s) \in R$ for $s' \in M_D(\lambda)$ such that

$$h \cdot_{\beta_D} C_{st}^\lambda \in \sum_{s' \in M_D(\lambda)} r_h(s', s) C_{s't}^\lambda + R^{\beta_D}[G_D](\langle \lambda \rangle)$$

for each $t \in M_D(\lambda)$. That is,

$$(18) \quad \sum_{g \in G_D} c_{st}^\lambda(g) \beta_D(h, g) (hg) \in \sum_{s' \in M_D(\lambda)} r_h(s', s) C_{s't}^\lambda + R^{\beta_D}[G_D](\langle \lambda \rangle)$$

for each $t \in M_D(\lambda)$. We will show that the elements

$$r_a((L', s'), (L, s)) = \begin{cases} \alpha(a, u_L^*) \beta_D(u_{L_1}^*, h)^{-1} r_h(s', s) & \text{if } L' = L_1 \\ 0 & \text{if } L' \neq L_1 \end{cases}$$

satisfy (16). Pick any $(K, t) \in M(D, \lambda)$. First note that

$$\begin{aligned} \theta_{L_1 K}(R^{\beta_D}[G_D](\langle \lambda \rangle)) &= \theta_{L_1 K} \left(\text{span}_R \left\{ C_{st}^{\lambda'} \mid \lambda' < \lambda \text{ and } s, t \in M_D(\lambda') \right\} \right) \\ &= \text{span}_R \left\{ \theta_{L_1 K} \left(C_{st}^{\lambda'} \right) \mid \lambda' < \lambda \text{ and } s, t \in M_D(\lambda') \right\} \\ &= \text{span}_R \left\{ C_{(L_1, s)(K, t)}^{(D, \lambda')} \mid \lambda' < \lambda \text{ and } s, t \in M_D(\lambda') \right\} \\ &\subseteq R^\alpha[S](\langle (D, \lambda) \rangle). \end{aligned}$$

Therefore applying θ_{L_1K} to (18), we obtain

$$\theta_{L_1K} \left(\sum_{g \in G_D} c_{st}^\lambda(g) \beta_D(h, g)(hg) \right) \in \sum_{s' \in M_D(\lambda)} r_h(s', s) \theta_{L_1K} (C_{s't}^\lambda) + R^\alpha[S](\langle (D, \lambda) \rangle).$$

That is,

$$\begin{aligned} \sum_{g \in G_D} c_{st}^\lambda(g) \beta_D(h, g) \beta_D(u_{L_1}^*, hgu_K) \beta_D(hg, u_K) (u_{L_1}^* hgu_K) \\ \in \sum_{s' \in M_D(\lambda)} r_h(s', s) C_{(L_1, s')(K, t)}^{(D, \lambda)} + R^\alpha[S](\langle (D, \lambda) \rangle). \end{aligned}$$

Now using Assumption 35, we obtain

$$\begin{aligned} \alpha(a, u_L^*) \beta_D(u_{L_1}^*, hgu_K) \beta_D(h, g) \beta_D(hg, u_K) \\ = \alpha(a, u_L^*) \beta_D(u_{L_1}^*, hgu_K) \beta_D(h, gu_K) \beta_D(g, u_K) \text{ by (13)} \\ = \alpha(a, u_L^*) \beta_D(u_{L_1}^*, h) \beta_D(u_{L_1}^* h, gu_K) \beta_D(g, u_K) \text{ by (13)} \\ = \alpha(a, u_L^*) \beta_D(au_L^*, gu_K) \beta_D(u_{L_1}^*, h) \beta_D(g, u_K) \text{ by (17)} \\ = \alpha(a, u_L^* gu_K) \beta_D(u_L^*, gu_K) \beta_D(u_{L_1}^*, h) \beta_D(g, u_K) \text{ by (12)} \end{aligned}$$

for each $g \in G_D$. Therefore multiplying the above by $\alpha(a, u_L^*)$ and using (17), we obtain

$$\begin{aligned} \sum_{g \in G_D} c_{st}^\lambda(g) \alpha(a, u_L^* gu_K) \beta_D(u_L^*, gu_K) \beta_D(u_{L_1}^*, h) \beta_D(g, u_K) (au_L^* gu_K) \\ \in \sum_{s' \in M_D(\lambda)} \alpha(a, u_L^*) r_h(s', s) C_{(L_1, s')(K, t)}^{(D, \lambda)} + R^\alpha[S](\langle (D, \lambda) \rangle). \end{aligned}$$

Further multiplying by $\beta_D(u_{L_1}^*, h)^{-1}$ gives

$$\begin{aligned} \sum_{g \in G_D} c_{st}^\lambda(g) \beta_D(u_L^*, gu_K) \beta_D(g, u_K) a \cdot (u_L^* gu_K) \\ \in \sum_{s' \in M_D(\lambda)} \alpha(a, u_L^*) \beta_D(u_{L_1}^*, h)^{-1} r_h(s', s) C_{(L_1, s')(K, t)}^{(D, \lambda)} + R^\alpha[S](\langle (D, \lambda) \rangle). \end{aligned}$$

Thus

$$\begin{aligned} a \cdot \left(C_{(L, s)(K, t)}^{(D, \lambda)} \right) &\in \sum_{s' \in M_D(\lambda)} r_a((L_1, s'), (L, s)) C_{(L_1, s')(K, t)}^{(D, \lambda)} + R^\alpha[S](\langle (D, \lambda) \rangle) \\ &= \sum_{(L', s') \in M(D, \lambda)} r_a((L', s'), (L, s)) C_{(L', s')(K, t)}^{(D, \lambda)} + R^\alpha[S](\langle (D, \lambda) \rangle), \end{aligned}$$

since $r_a((L', s'), (L, s)) = 0$ whenever $L' \neq L_1$. This is exactly (16), as required. \square

Remark. Again the proof under the somewhat unnatural Assumption 35 is notationally more complicated than the case where the elements $\alpha(x, y)$ are invertible, without requiring any extra ideas. It is much more illustrative to verify the latter case directly, as in Corollary 38 below.

We state two special cases of Theorem 37, which replace Assumption 35 with more palatable assumptions. Note that a semigroup algebra is just a twisted semigroup algebra with trivial twisting. It is therefore covered by this Corollary.

Corollary 38. *Suppose Assumptions 32, 33 and 34 hold. Suppose also that for each $D \in \mathbb{D}$ and for each $x \mathcal{L} 1_D$ and $y \mathcal{R} 1_D$, the element $\alpha(x, y) \in R$ is invertible. As in Assumption 36, suppose that $R^\alpha[G_D]$ is cellular with cell datum*

$$(\Lambda_D, M_D, C, *).$$

Then the algebra $R^\alpha[S]$ is conjugate cellular with the cell datum

$$(\Lambda, M, C, *),$$

where Λ , M and $*$ are as given above. The basis elements now take the more elegant form

$$C_{(L,s)(K,t)}^{(D,\lambda)} = u_L^* \cdot C_{st}^\lambda \cdot u_K.$$

Proof. Consider any $D \in \mathbb{D}$. Letting L_0 and L_0^* denote the \mathcal{L} and \mathcal{R} classes of 1_D respectively, we can define the map

$$\beta_D : L_0 \times L_0^* \rightarrow G(R).$$

to simply be the restriction of α to $L_0 \times L_0^*$; that is,

$$\beta_D(x, y) = \alpha(x, y)$$

for $x \in L_0$ and $y \in L_0^*$. Note $\beta_D(x, y)$ is indeed a unit in R by assumption. It is now clear by (1) and Assumption 33 that β_D satisfies (12), (13) and (14), so that Assumption 35 holds. Now $R^{\beta_D}[G_D]$ is exactly $R^\alpha[G_D]$, so Assumption 36 holds and Theorem 37 applies. Moreover the basis elements are given by

$$\begin{aligned} C_{(L,s)(K,t)}^{(D,\lambda)} &= \sum_{g \in G_D} c_{st}^\lambda(g) \beta_D(g, u_K) \beta_D(u_L^*, gu_K) (u_L^* gu_K) \\ &= \sum_{g \in G_D} c_{st}^\lambda(g) \alpha(g, u_K) \alpha(u_L^*, gu_K) (u_L^* gu_K) \\ &= \sum_{g \in G_D} c_{st}^\lambda(g) \alpha(g, u_K) u_L^* \cdot (gu_K) \\ &= \sum_{g \in G_D} c_{st}^\lambda(g) u_L^* \cdot g \cdot u_K \\ &= u_L^* \cdot \left(\sum_{g \in G_D} c_{st}^\lambda(g) g \right) \cdot u_K \\ &= u_L^* \cdot C_{st}^\lambda \cdot u_K \end{aligned}$$

for each $(D, \lambda) \in \Lambda_D$ and $(L, s), (K, t) \in M(D, \lambda)$. This completes the proof. \square

The second special case of Theorem 37 will aid our investigation of the Brauer, Temperley-Lieb and partition algebras.

Corollary 39. *Suppose Assumptions 32, 33 and 34 hold. Suppose also that $\alpha(x, y) = \alpha(x, z)$ whenever $y \mathcal{R} z$. Suppose that the group algebra $R[G_D]$ is cellular with cell datum*

$$(\Lambda_D, M_D, C, *).$$

Then the algebra $R^\alpha[S]$ is conjugate cellular with the cell datum

$$(\Lambda, M, C, *),$$

where Λ , M and $*$ are as given above. The basis elements now take the form

$$C_{(L,s)(K,t)}^{(D,\lambda)} = \sum_{g \in G_D} c_{st}^\lambda(g) (u_L^* gu_K).$$

Proof. Consider any $D \in \mathbb{D}$. Letting L_0 and L_0^* denote the \mathcal{L} and \mathcal{R} classes of 1_D respectively, we simply define the map

$$\beta_D : L_0 \times L_0^* \rightarrow G(R)$$

by $\beta_D(x, y) = 1$. This trivially satisfies (13) and (14). Moreover if $y \mathcal{R} yz$ then by assumption

$$\alpha(x, y)\beta_D(xy, z) = \alpha(x, y) = \alpha(x, yz) = \alpha(x, yz)\beta_D(y, z),$$

assuming the values of β_D are defined. Thus β_D satisfies (12), and Assumption 35 holds. Now the twisted group algebra $R^{\beta_D}[G_D]$ is simply the group algebra $R[G_D]$. Thus Assumption 36 holds and Theorem 37 applies. The final assertion follows immediately from the definition of $C_{(L,s)(K,t)}^{(D,\lambda)}$ in Theorem 37. \square

Cell Representations of Twisted Semigroup Algebras

Recall from Chapter 5 that a conjugate cellular datum for an algebra gives rise to *cell representations* for the algebra. Suppose that the assumptions of Theorem 37 hold. This theorem allows us to obtain cell representations of $R^\alpha[S]$ from the cell representations of the twisted group algebras $R^{\beta_D}[G_D]$ (in fact this is a special case of Proposition 16). In this chapter we determine the bilinear forms associated with the cell representations of $R^\alpha[S]$ in terms of the cell representations of $R^{\beta_D}[G_D]$, by analogy with Section 4 of [4]. Recall that, for each \mathcal{D} class D , the \mathcal{R} classes \mathbb{R}_D are exactly $\{L^* \mid L \in \mathbb{L}_D\}$. Thus, setting $v_{L^*} = u_L^*$, the twisted sandwich matrix of Chapter 3 becomes the matrix $P_D^\alpha \in \text{Mat}_{\mathbb{L}_D}(R^{\beta_D}[G_D])$ with entries

$$(P_D^\alpha)_{LK} = \begin{cases} 0 & \text{if } u_L u_K^* <_{\mathcal{D}} D \\ \alpha(u_L, u_K^*) u_L u_K^* & \text{if } u_L u_K^* \in G_D. \end{cases}$$

We can now state the analogue of Lemma 16 of [4].

Lemma 40. *Let $(D, \lambda) \in \Lambda$ and $(L, s), (K, t) \in M(D, \lambda)$. Then*

$$\phi^{(D, \lambda)}(C_{(L, s)}, C_{(K, t)}) = \phi_{(P_D^\alpha)_{LK}}^\lambda(C_s, C_t).$$

Proof. Suppose first that $u_L u_K^* <_{\mathcal{D}} D$. Then for each $g, h \in G_D$, we have

$$u_K^* g u_L u_K^* h u_L \leq_{\mathcal{D}} u_L u_K^* <_{\mathcal{D}} D.$$

Thus

$$(u_K^* g u_L) \cdot (u_K^* h u_L) = \alpha(u_K^* g u_L, u_K^* h u_L) (u_K^* g u_L u_K^* h u_L) \in R^\alpha[S](< D).$$

As in the proof of Theorem 37, it now follows that

$$\begin{aligned} C_{(L, s)(K, t)}^{(D, \lambda)} \cdot C_{(L, s)(K, t)}^{(D, \lambda)} &\in R^\alpha[S](< D) \\ &\subseteq R^\alpha[S](< (D, \lambda)). \end{aligned}$$

Therefore

$$\phi^{(D, \lambda)}(C_{(L, s)}, C_{(K, t)}) = 0$$

in this case. Also the definition of the twisted sandwich matrix gives $(P_D^\alpha)_{LK} = 0$, so that

$$\phi_{(P_D^\alpha)_{LK}}^\lambda(C_s, C_t) = 0.$$

Thus both sides of the equality are 0. The other case is when $u_L u_K^* \in G_D$. Then by definition of ϕ^λ ,

$$C_{ts}^\lambda \cdot_{\beta_D} (P_D^\alpha)_{LK} \cdot_{\beta_D} C_{ts}^\lambda \in \phi_{(P_D^\alpha)_{LK}}^\lambda(C_s, C_t) C_{ts}^\lambda + R^{\beta_D}[G_D](< \lambda).$$

The left hand side is

$$\begin{aligned}
& C_{ts}^\lambda \cdot_{\beta_D} (P_D^\alpha)_{LK} \cdot_{\beta_D} C_{ts}^\lambda \\
&= \alpha(u_L, u_K^*) \left(\sum_{g \in G_D} c_{st}^\lambda(g) g \right) \cdot_{\beta_D} (u_L u_K^*) \cdot_{\beta_D} \left(\sum_{h \in G_D} c_{st}^\lambda(h) h \right) \\
&= \alpha(u_L, u_K^*) \sum_{g, h \in G_D} c_{st}^\lambda(g) c_{st}^\lambda(h) g \cdot_{\beta_D} (u_L u_K^*) \cdot_{\beta_D} h \\
&= \sum_{g, h \in G_D} c_{st}^\lambda(g) c_{st}^\lambda(h) \alpha(u_L, u_K^*) \beta_D(g, u_L u_K^*) (g u_L u_K^*) \cdot_{\beta_D} h \\
&= \sum_{g, h \in G_D} c_{st}^\lambda(g) c_{st}^\lambda(h) \alpha(u_L, u_K^*) \beta_D(g, u_L u_K^*) \beta_D(g u_L u_K^*, h) (g u_L u_K^* h).
\end{aligned}$$

As in the proof of Theorem 37, we can now apply θ_{KL} to the above to obtain

$$\begin{aligned}
& \sum_{g, h \in G_D} c_{st}^\lambda(g) c_{st}^\lambda(h) \alpha(u_L, u_K^*) \beta_D(g, u_L u_K^*) \beta_D(g u_L u_K^*, h) \\
& \quad \times \beta_D(g u_L u_K^* h, u_L) \beta_D(u_K^*, g u_L u_K^* h u_L) (u_K^* g u_L u_K^* h u_L) \\
& \quad \in \phi_{(P_D^\alpha)_{LK}}^\lambda(C_s, C_t) \theta_{KL}(C_{ts}^\lambda) + R^\alpha[S](\langle (D, \lambda) \rangle).
\end{aligned}$$

As usual we rearrange the coefficient

$$\begin{aligned}
& \alpha(u_L, u_K^*) \beta_D(g, u_L u_K^*) \beta_D(g u_L u_K^*, h) \beta_D(g u_L u_K^* h, u_L) \beta_D(u_K^*, g u_L u_K^* h u_L) \\
& \stackrel{(3)}{=} \alpha(g u_L, u_K^*) \beta_D(g, u_L) \beta_D(g u_L u_K^*, h) \beta_D(g u_L u_K^* h, u_L) \beta_D(u_K^*, g u_L u_K^* h u_L) \\
& \stackrel{(2)}{=} \alpha(g u_L, u_K^* h) \beta_D(g, u_L) \beta_D(u_K^*, h) \beta_D(g u_L u_K^* h, u_L) \beta_D(u_K^*, g u_L u_K^* h u_L) \\
& \stackrel{(2)}{=} \alpha(g u_L, u_K^* h u_L) \beta_D(g, u_L) \beta_D(u_K^*, h) \beta_D(u_K^* h, u_L) \beta_D(u_K^*, g u_L u_K^* h u_L) \\
& \stackrel{(3)}{=} \alpha(u_K^* g u_L, u_K^* h u_L) \beta_D(g, u_L) \beta_D(u_K^*, h) \beta_D(u_K^* h, u_L) \beta_D(u_K^*, g u_L) \\
& \stackrel{(4)}{=} \alpha(u_K^* g u_L, u_K^* h u_L) \beta_D(g, u_L) \beta_D(u_K^*, h u_L) \beta_D(h, u_L) \beta_D(u_K^*, g u_L).
\end{aligned}$$

The above then becomes

$$\begin{aligned}
& \sum_{g, h \in G_D} c_{st}^\lambda(g) c_{st}^\lambda(h) \beta_D(g, u_L) \beta_D(u_K^*, g u_L) \beta_D(h, u_L) \beta_D(u_K^*, h u_L) \\
& \quad \times \alpha(u_K^* g u_L, u_K^* h u_L) (u_K^* g u_L u_K^* h u_L) \\
& \quad \in \phi_{(P_D^\alpha)_{LK}}^\lambda(C_s, C_t) C_{(K,t)(L,s)}^{(D,\lambda)} + R^\alpha[S](\langle (D, \lambda) \rangle).
\end{aligned}$$

Substituting $\alpha(u_K^* g u_L, u_K^* h u_L) (u_K^* g u_L u_K^* h u_L) = (u_K^* g u_L) \cdot (u_K^* h u_L)$, this is

$$\begin{aligned}
& \sum_{g, h \in G_D} [c_{st}^\lambda(g) \beta_D(g, u_L) \beta_D(u_K^*, g u_L) (u_K^* g u_L)] \cdot [\beta_D(h, u_L) \beta_D(u_K^*, h u_L) c_{st}^\lambda(h) (u_K^* h u_L)] \\
& \quad \in \phi_{(P_D^\alpha)_{LK}}^\lambda(C_s, C_t) C_{(K,t)(L,s)}^{(D,\lambda)} + R^\alpha[S](\langle (D, \lambda) \rangle).
\end{aligned}$$

That is,

$$\left(C_{(K,t)(L,s)}^{(D,\lambda)} \right) \cdot \left(C_{(K,t)(L,s)}^{(D,\lambda)} \right) \in \phi_{(P_D^\alpha)_{LK}}^\lambda(C_s, C_t) C_{(K,t)(L,s)}^{(D,\lambda)} + R^\alpha[S](\langle (D, \lambda) \rangle)$$

so that

$$\phi^{(D,\lambda)}(C_{(L,s)}, C_{(K,t)}) = \phi_{(P_D^\alpha)_{LK}}^\lambda(C_s, C_t),$$

as required. \square

Remark. Again the proof is much clearer when the elements $\alpha(x, y)$ are invertible.

For each $\lambda \in \Lambda_D$, the representation

$$\rho^\lambda : R^{\beta_D}[G_D] \rightarrow \text{Mat}_{M_D(\lambda)}(R)$$

naturally induces a homomorphism

$$\text{Mat}_{\mathbb{L}_D}(R^{\beta_D}[G_D]) \rightarrow \text{Mat}_{\mathbb{L}_D}(\text{Mat}_{M_D(\lambda)}(R)) \cong \text{Mat}_{\mathbb{L}_D \times M_D(\lambda)}(R) = \text{Mat}_{M(D,\lambda)}(R),$$

which we also denote by ρ^λ .

Corollary 41. *The matrix representation of $\phi^{(D,\lambda)}$ is given by*

$$\Phi^{(D,\lambda)} = \Phi'^\lambda \rho^\lambda(P_D^\alpha),$$

where Φ'^λ is the block diagonal matrix

$$\Phi'^\lambda = \begin{pmatrix} \Phi^\lambda & 0 & 0 & \cdots & 0 \\ 0 & \Phi^\lambda & 0 & \cdots & 0 \\ 0 & 0 & \Phi^\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Phi^\lambda \end{pmatrix} \in \text{Mat}_{\mathbb{L}_D}(\text{Mat}_{M_D(\lambda)}(R)) \cong \text{Mat}_{M(D,\lambda)}(R).$$

Thus

$$\det \Phi^{(D,\lambda)} = (\det \Phi^\lambda)^{|\mathbb{L}_D|} \det \rho^\lambda(P_D^\alpha).$$

Proof. Using (v) of Proposition 23, the previous Lemma gives

$$\begin{aligned} \Phi_{(L,s)(K,t)}^{(D,\lambda)} &= \phi^{(D,\lambda)}(C_{(L,s)}, C_{(K,t)}) \\ &= \phi_{(P_D^\alpha)_{LK}}^\lambda(C_s, C_t) \\ &= \sum_{u \in M_D(\lambda)} \Phi_{su}^\lambda \rho^\lambda((P_D^\alpha)_{LK})_{ut} \\ &= \sum_{u \in M_D(\lambda)} \Phi_{su}^\lambda \rho^\lambda(P_D^\alpha)_{(L,u),(K,t)} \\ &= \sum_{(L',u) \in M(D,\lambda)} \Phi_{su}^\lambda \delta_{LL'} \rho^\lambda(P_D^\alpha)_{(L',u),(K,t)} \\ &= \sum_{(L',u) \in M(D,\lambda)} \Phi_{(L,s),(L',u)}^\lambda \rho^\lambda(P_D^\alpha)_{(L',u),(K,t)} \\ &= (\Phi'^\lambda \rho^\lambda(P_D^\alpha))_{(L,s),(K,t)}. \end{aligned}$$

Hence

$$\Phi^{(D,\lambda)} = \Phi'^\lambda \rho^\lambda(P_D^\alpha),$$

as required. Taking the determinant, it is then clear that

$$\det \Phi^{(D,\lambda)} = \det \Phi'^\lambda \det \rho^\lambda(P_D^\alpha) = (\det \Phi^\lambda)^{|\mathbb{L}_D|} \det \rho^\lambda(P_D^\alpha).$$

□

By analogy with Theorem 17 of [4], the utility of cellular machinery will be illustrated by providing a short proof of the following special case of Theorem 15.

Theorem 42. *Suppose that the conditions of Corollary 38 hold, and that R is a field. Then $R^\alpha[S]$ is semisimple exactly when*

- (i) $R^\alpha[G_D]$ is semisimple and
- (ii) P_D^α is invertible,

for each $D \in \mathbb{D}$, where P_D^α is as defined immediately before Lemma 40.

Proof. Suppose that the two conditions hold, and consider any $(D, \lambda) \in \Lambda$. Since P_D^α is invertible, certainly $\rho^\lambda(P_D^\alpha)$ is invertible. Thus $\det \rho^\lambda(P_D^\alpha) \neq 0$. Also because $R^\alpha[G_D]$ is semisimple, by Theorem 31 we have $\det \Phi^\lambda \neq 0$. Hence Corollary 41 gives

$$\det \Phi^{(D, \lambda)} \neq 0.$$

As this holds for each $(D, \lambda) \in \Lambda$, the algebra $R^\alpha[S]$ is semisimple by Theorem 31.

Conversely suppose that $R^\alpha[S]$ is semisimple, so that $\det \Phi^{(D, \lambda)} \neq 0$ for each $(D, \lambda) \in \Lambda$ by Theorem 31. By Corollary 41, we then have

$$\det \Phi^\lambda \neq 0 \quad \text{and} \quad \det \rho^\lambda(P_D^\alpha) \neq 0.$$

Now the former holds for all $\lambda \in \Lambda_D$. Thus applying Theorem 31, statement (A) implies that $R^\alpha[G_D]$ is semisimple, while statement (D) implies that the $\{\rho^\lambda \mid \lambda \in \Lambda_D\}$ constitute all irreducible representations of $R^\alpha[G_D]$. Thus the map

$$\bigoplus_{\lambda \in \Lambda_D} \rho^\lambda : R^\alpha[G_D] \rightarrow \bigoplus_{\lambda \in \Lambda_D} \text{Mat}_{M_D(\lambda)}(R)$$

is injective. By counting dimensions, the map is also surjective, and is therefore an isomorphism. Because $\det \rho^\lambda(P_D^\alpha) \neq 0$, the matrix $\rho^\lambda(P_D^\alpha)$ is invertible for each $\lambda \in \Lambda_D$. Thus

$$\left(\bigoplus_{\lambda \in \Lambda_D} \rho^\lambda \right) (P_D^\alpha) \in \bigoplus_{\lambda \in \Lambda_D} \text{Mat}_{M(D, \lambda)}(R)$$

is invertible. The above isomorphism then implies that the matrix P_D^α is invertible. Thus both conditions hold, and we are done. \square

Applications of the Main Theorem

As foreshadowed, we can apply Theorem 37, or rather Corollary 39, to the three examples at the end of Chapter 3, as we detail in this chapter.

The partition algebra $R^\alpha[A_n]$ was shown to be cellular by Xi in [24]. In order to reproduce this result with the aid of Theorem 37, first recall that A_n has a natural anti-involution $*$ which swaps i and i' , for each $i \in I$. It is clear geometrically that α and $*$ satisfy Assumption 33, where $\bar{}$ is the identity. Recall also that Green's relations in A_n were described by Theorem 6. It is clear that $m(x, y)$ depends only on the first components of $l(x)$ and $r(y)$, for $x, y \in A_n$. Thus if $y \mathcal{R} z$ then $r(y) = r(z)$ by Theorem 6, so that $\alpha(x, y) = \alpha(x, z)$. Now consider a \mathcal{D} class D in A_n . Theorem 6 implies that

$$D = \{x \in A_n \mid d(x) = n - k\}$$

for some integer k with $0 \leq k \leq n$. Let 1_D denote the element of A_n whose equivalence classes are

$$\tilde{1}_D = \{\{1, 2, \dots, k\}, \{1', 2', \dots, k'\}\} \cup \{\{i, i'\} \mid k < i \leq n\}.$$

That is, 1_D is the element represented by the diagram

$$1_D = \begin{array}{c} \begin{array}{ccccccccccc} \bullet & \bullet & \bullet & \bullet & \dots & \bullet & \bullet & & \bullet & \bullet & \dots & \bullet & \bullet \\ 1 & 2 & 3 & 4 & \dots & k-1 & k & & k+2 & \dots & n-1 & n \\ \end{array} \\ \begin{array}{ccccccccccc} \bullet & \bullet & \bullet & \bullet & \dots & \bullet & \bullet & & \bullet & \bullet & \dots & \bullet & \bullet \\ 1' & 2' & 3' & 4' & \dots & (k-1)' & k' & & (k+2)' & \dots & (n-1)' & n' \\ \end{array} \end{array} .$$

It is clear that $1_D \in D$ is an idempotent invariant under $*$. Moreover $x \in G_D$ exactly when

$$r(x) = r(1_D) = (\{\{1, 2, \dots, k\}\}, \{\{i\} \mid k < i \leq n\})$$

and

$$l(x) = l(1_D) = (\{\{1', 2', \dots, k'\}\}, \{\{i'\} \mid k < i \leq n\}).$$

Thus x differs from 1_D only by how the dots labelled $k+1, k+2, \dots, n$ are joined to the dots labelled $(k+1)', (k+2)', \dots, n'$. It then follows quickly from the multiplication in A_n that there is a group isomorphism

$$\theta_D : S_{n-k} \rightarrow G_D$$

such that

$$\widetilde{\theta_D(\sigma)} = \{\{1, 2, \dots, k\}, \{1', 2', \dots, k'\}\} \cup \{\{k + \sigma(i), (k + i)'\} \mid 1 \leq i \leq n - k\}.$$

This induces an isomorphism

$$\theta_D : R[S_{n-k}] \rightarrow R[G_D].$$

Moreover, using \dagger to denote the anti-involution on S_{n-k} given by inversion, we have

$$\theta_D(\sigma^\dagger) = \theta_D(\sigma)^*$$

for each $\sigma \in S_{n-k}$, so that

$$\theta_D(a^\dagger) = \theta_D(a)^*$$

for $a \in R[S_{n-k}]$. From example (1.2) of [6], we know that $R[S_{n-k}]$ is cellular with anti-involution \dagger . Therefore $R[G_D]$ is cellular with anti-involution $*$. Observe that A_n is regular

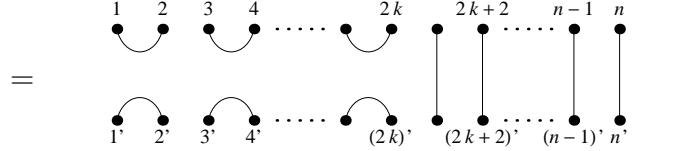
since every \mathcal{D} class contains an idempotent. The assumptions of Corollary 39 are then satisfied, so the partition algebra $R^\alpha[A_n]$ is cellular.

Similarly we can apply Corollary 39 to the Brauer algebra. By Theorem 7, a \mathcal{D} class D in BR_n is of the form

$$D = \{x \in BR_n \mid d(x) = n - 2k\}$$

for some integer k with $0 \leq k \leq \frac{n}{2}$. Let 1_D denote the following element of BR_n .

$$1_D = \{\{2i - 1, 2i\} \mid 1 \leq i \leq k\} \cup \{\{(2i - 1)', (2i)'\} \mid 1 \leq i \leq k\} \\ \cup \{\{i, i'\} \mid 2k + 1 \leq i \leq n\}$$



Again $1_D \in D$ is an idempotent invariant under $*$. Moreover $x \in G_D$ exactly when

$$r(x) = r(1_D) = \{\{2i - 1, 2i\} \mid 1 \leq i \leq k\}$$

and

$$l(x) = l(1_D) = \{\{(2i - 1)', (2i)'\} \mid 1 \leq i \leq k\}.$$

Thus x differs from 1_D only by how the dots $2k + 1, 2k + 2, \dots, n$ are joined to the dots $(2k + 1)', (2k + 2)', \dots, n'$. As for the partition algebra, this gives an isomorphism between S_{n-2k} and G_D under which inversion corresponds to $*$, so that $R[G_D]$ is cellular with anti-involution $*$. The condition that $\alpha(x, y) = \alpha(x, z)$ whenever $y \mathcal{R} z$ holds in BR_n because it holds in A_n . Therefore we can again apply Corollary 39 to show that the Brauer algebra is cellular.

To determine the actual cell datum, we must choose appropriate elements u_L for each $L \in \mathbb{L}_D$. Recall from Theorem 7 that L is determined uniquely by $l(L)$. Now $l(L)$ consists of k disjoint pairs of elements from the set $\{1', 2', \dots, n'\}$. Suppose that the remaining $n - 2k$ elements are $\{j'_1, j'_2, \dots, j'_{n-2k}\}$, where

$$j_1 < j_2 < \dots < j_{n-2k}.$$

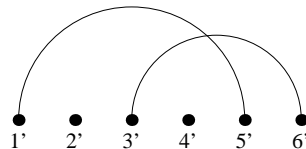
Let

$$u_L = \{\{2i - 1, 2i\} \mid 1 \leq i \leq k\} \cup l(L) \\ \cup \{\{2k + i, j'_i\} \mid 1 \leq i \leq n - 2k\}.$$

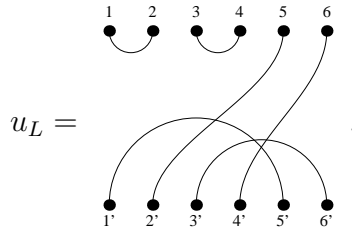
Diagrammatically, $l(L)$ determines the k edges which have both vertices on the bottom row, while $u_L \mathcal{R} 1_D$ implies that u_L must contain the k edges

$$r(1_D) = \{\{2i - 1, 2i\} \mid 1 \leq i \leq k\}$$

which have both vertices on the top row. The last $n - 2k$ dots on the top row are joined to the remaining $n - 2k$ dots on the bottom row in the natural way. For example, suppose that $n = 6$ and $k = 2$, and consider the \mathcal{L} class L such that $l(L)$ is represented by



Then



Having thus defined u_L , the cell datum produced by Corollary 39 is exactly that given in [6].

The application of Corollary 39 to the Temperley-Lieb algebra is almost identical to the Brauer algebra. Indeed for each \mathcal{D} class D in BR_n , the element 1_D defined above is an element of TL_n . Since Green's relations in TL_n are just the restrictions of those in BR_n , we can therefore choose 1_D as before. Similarly if $L \in \mathbb{L}_D$ in TL_n then the element u_L defined above is in TL_n , so we can again use these elements. In this case the group $G_D = \{1_D\}$ is trivial, so the group algebra $R[G_D]$ is trivially cellular. Thus the Temperley-Lieb algebra $R^\alpha[TL_n]$ is cellular, and again the cell datum produced by Corollary 39 is the same as in [6].

The cyclotomic Brauer [9] and Temperley-Lieb [21] algebras are variations on the Brauer and Temperley-Lieb algebras which depend on an additional positive integer parameter m . They were shown to be cellular in [22] and [21] respectively, provided the polynomial $x^m - 1$ can be decomposed into linear factors over the ground ring R . Again we can reproduce these results using Corollary 39. Indeed when realising these algebras as twisted semigroup algebras, the underlying semigroups of diagrams have the same \mathcal{D} class structures as BR_n and TL_n respectively, and we can use the same idempotents 1_D as above. In the case of the cyclotomic Brauer algebra, the maximal subgroups are wreath products $\mathbb{Z}_m \wr S_k$, the group algebra of which is cellular (with the appropriate anti-involution) by Theorem (5.5) of [6]. In the case of the cyclotomic Temperley-Lieb algebra, the maximal subgroups are direct sums of copies of \mathbb{Z}_m , the group algebra of which is easily shown to be cellular.

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