Chapter 1

Introduction

A striking development in mathematics in recent decades has been use of ideas in mathematical physics to study the topology of low dimensional manifolds. In 1983, Donaldson used self-dual Yang-Mills theory to study 4-manifolds, leading to the development of new topological invariants of 4-manifolds [Do83, Do87, Do90]. In 1988, Witten introduced the concept of a topological quantum field theory (TQFT) [Wi88], which provided a unifying principle for Donaldson's 4-manifold invariants and Floer's invariants of oriented integral homology 3-spheres [Fl88]. In 1989, Witten showed a connection between the Jones polynomial of links and quantum Chern-Simons theory that *"gave a (formal) 3-dimensional interpretation of the Jones polynomial"* [Wi89, O02]. The quantum Chern-Simons theory has also been a rich source of topological invariants of 3-manifolds. Its study has led to much progress in the area of knot theory and 3-manifold theory. In particular, the influential work of Reshetikhin and Turaev [RT91] gave a mathematically rigorous construction of the 3-manifold invariants using quantum algebras.

This provides the first of the two general subject areas encompassing the work presented in this thesis: the study of topological invariants of 3-manifolds. The other general subject area is quite distinct from the first and can be easily studied without reference to the first. The genesis of this thesis is in some very interesting work by Reshetikhin amongst others who showed unexpected connections between this second area and the area of topological invariants of links and 3-manifolds, and it is this connection that has essentially led to the theme of our research. This second general subject area is the study of quantum algebras and quantum superalgebras and their representations.

Each of these two subject areas is very interesting and offers many unanswered questions, some of which we discuss below. In this thesis we obtain some new results in both of these areas and then weave them together yielding further results.

In this thesis, we firstly study the quantum superalgebra $U_q(osp(1|2n))$ and its representations at roots of unity in Chapters 3 and 4 and then use the results obtained to study topological invariants of 3-manifolds in Chapter 5.

Quantum algebras have been the subject of much research since Drinfel'd [Dr86] and Jimbo [Ji85, Ji86] introduced quantum groups to the mathematical world in approximately 1985. Quantum superalgebras were introduced in [BGZ90, Y94] and other papers. Quantum algebras and quantum superalgebras related to classical Lie algebras and simple basic Lie superalgebras, respectively, are q-deformations of the universal enveloping algebras of the relevant Lie algebra or Lie superalgebra. The representation theory of quantum algebras and quantum superalgebras has attracted a lot of attention, not only for the intrinsic interest in the representations of these new algebras, but also for the possibility that it could solve already existing problems, and in addition, potential applications. The potential for applications was quite strong as the quantum groups introduced by Drinfel'd admitted an element satisfying the Yang-Baxter equation, an important equation in statistical mechanics. The existence and explication of these elements, called universal R-matrices, has been one of the goals in the study of quantum algebras and quantum superalgebras.

The representation theories of quantum algebras and quantum superalgebras are at present not completely known, but aspects of them are somewhat known. In some ways the known representation theory depends dramatically on the nature of q: if q is non-zero and not a root of unity, the representation theory of quantum algebras and quantum superalgebras is much better known than if q is a root of unity. In one sense, this may arise from the fact that the centres of the quantum algebras and quantum superalgebras are much larger when q is a root of unity.

The representation theory of quantum algebras at roots of unity has attracted much research (eg [An92, AJS94, AP95]). However, the representation theory of quantum superalgebras at roots of unity is not nearly as well known. For example, the representation theory of $U_q(osp(1|2n))$ at roots of unity was barely studied before. The first part of this thesis studies the structure and representations of $U_q(osp(1|2n))$ at roots of unity.

In particular, we fix $q = \exp(2\pi i/N)$ where $N \geq 3$ is an integer. In this case, the quantum superalgebra $U_q(osp(1|2n))$ has a much larger centre than that at generic q. A two-sided ideal $\mathcal{I}$ is generated by certain central elements, which is also a Hopf ideal. The quotient quantum superalgebra $U_q^{(N)}(osp(1|2n)) = U_q(osp(1|2n))/\mathcal{I}$ is again a $\mathbb{Z}_2$-graded Hopf algebra, and, importantly, admits a universal R-matrix. It transpires that $U_q^{(N)}(osp(1|2n))$ is a $\mathbb{Z}_2$-graded ribbon Hopf algebra.

We construct a set of $U_q^{(N)}(osp(1|2n))$-modules, each of which is characterised by an element from a truncated Weyl chamber Λ_N^+. We prove that each of these modules is self-dual, and, more importantly, their tensor products decompose in a very nice way. It is the tensor product decomposition theorems which we need for the cosntruction of 3-manifold invariants.

The second major theme in this thesis is the construction of topological invariants of closed, connected, orientable 3-manifolds. We review the construction of the topological invariants from modular Hopf algebras introduced by Reshetikhin and Turaev [RT91], and show that topological invariants, similar to those introduced in [RT91], can be constructed from a more general class of algebras which we call pseudo-modular Hopf algebras. The conditions for an algebra to be modular are quite prescriptive, and we define pseudo-modular Hopf algebras to be much more general, so that topological invariants can be constructed following our work for as many ribbon Hopf algebras as possible.

After these two independent strands of study, we tie them together by showing that the quotient quantum superalgebra $U_q^{(N)}(osp(1|2n))$, when $q = \exp(2\pi i/N)$ and $N \geq 6$ satisfies $N \equiv 2 \pmod 4$, together with a set of finite dimensional representations, is a pseudo-modular Hopf algebra, and thus yields topological invariants of closed, connected, orientable 3-manifolds.

We now turn to a discussion of 3-manifold invariants, proper.

Despite the relative age of the programme of studying 3-manifolds (it being initiated in the 1880s [Gor99]), there are still many avenues of research, as evidenced by the recent claim of the proof of the Poincaré Conjecture [M04]. In addition, it is still unknown whether the problem of classifying 3-manifolds into homeomorphism equivalence classes, one of the major goals of the research programme, is solvable [O02]. This contrasts with what is known for simply connected n-manifolds for all integers $n \geq 2$ except $n = 3$. Here the classification problem has been solved one way or the other: it has been solved for $n = 2$ and $n \geq 5$, and is known to be algorithmically unsolvable for $n = 4$, as *"the set of fundamental groups of 4-manifolds is sufficiently large to be algorithmically unsolvable as a word problem in group theory"* [O02]. The discovery of new topological invariants of 3-manifolds should shed light on the classification problem; this is the general goal towards which part of the work presented in this thesis is directed.

While it is not known whether the classification problem for 3-manifolds is solvable, Lins has discussed a potential conjecture for completely classifying connected 3-manifolds [Lin95]. This potential conjecture uses a combinatorial algorithm for classifying 3-manifolds using the so-called 3-gems, or *3-dimensional graph-encoded manifolds* [KL94, Lin95]. Using 3-gems, Kauffman and Lins calculated topological invariants for a large number of 3-manifolds [KL94] and while the algorithm works on the elements of a proper subset of all connected 3-manifolds where the calculations are computationally tractable, it is not known whether the algorithm works in general.

Even if one does not have a complete topological invariant of connected 3-manifolds, a collection of topological invariants may altogether be a complete topological invariant. In any event, different topological invariants may have different properties, so it is useful to have as many topological invariants as possible when studying individual 3-manifolds and the classification problem.

This provides the raison d'etre for part of this thesis: we will construct a topological invariant of closed, connected, orientable 3-manifolds. Our invariants are one of the class of *quantum invariants* [O02], which are a large collection of topological invariants of knots and 3-manifolds that have been discovered since Jones discovered a new polynomial of links in 1985 [Jo85] (for examples of quantum invariants of tangles and knots see [FYHLMO85, BLM86, Ho86, LM87, T88, PT87, DA90, Re90, RT90, Kau91, Zh91, ZGB91, LG92, Zh92a, Gou93, GTB93, T94, GLZ96, ZL96, DeW99]). The Jones polynomial was the first link invariant discovered since Alexander discovered, in 1928, the Alexander-Conway polynomial [Al28] (see also [C70]). The topological invariants of 3-manifolds generated by TQFTs are also quantum invariants.

The topological invariants of 3-manifolds we construct in this thesis were inspired by the construction of Reshetikhin and Turaev in 1991 [RT91]. Their construction was formulated using a *modular Hopf algebra*, which is a ribbon Hopf algebra together with a finite collection of finite dimensional irreducible representations with non-zero quantum dimension, satisfying certain criteria including a subtle tensor product rule. These criteria encapsulate the necessary conditions for the construction of the 3-manifold invariants following [RT91], although, as we shall discuss below, it is possible to construct 3-manifold invariants using ribbon Hopf algebras satisfying less restrictive criteria than those satisfied by modular Hopf algebras.

The essential method for constructing the topological invariants presented in [RT91] is as follows. One uses a result of Lickorish [Lic62] to present each closed, connected, orientable 3-manifold M_L as the result of performing surgery on the 3-sphere S^3 along a framed link $L \subset S^3$. From the work of Kirby [Kirb78] and Fenn and Rourke [FR79], two such 3-manifolds, M_{L_1} and M_{L_2}, are homeomorphic if and only if the links L_1 and L_2 are equivalent with respect to the so-called Kirby moves, which are a set of equivalence relations on planar projections of a link. This enables one to turn the study of homeomorphism classes of closed, connected, orientable 3-manifolds into the study of equivalence classes of framed links in S^3 where the equivalence is generated by the Kirby moves. Then, upon taking any isotopy invariant of framed links that is invariant under the Kirby moves, one obtains a topological invariant of closed, connected, orientable 3-manifolds. Reshetikhin and Turaev put this scheme into effect in [RT91] by using an isotopy invariant of framed links they had constructed in 1990 from quantum algebras [RT90]. Interestingly, the isotopy invariant that they used from [RT90] can be seen as a generalisation of the Jones polynomial. By taking a linear sum of isotopy invariants of framed links that remained unchanged if the Kirby moves were applied to the framed link, Reshetikhin and Turaev obtained a topological invariant of closed, connected, orientable 3-manifolds.

Since Reshetikhin and Turaev's work in 1991, many topological invariants of 3-manifolds have been constructed from a variety of different viewpoints, for example, they have been constructed following the general idea of [RT91] by taking linear sums of invariants of framed links that are unchanged under the Kirby moves (eg see [Lic91a, Lic91b, Wa91, BHMV92, Lic92, TW93, Zh95]), by taking averages over all possible cablings of invariants of framed links [We93], by taking a state-sum over an arbitrary triangulation of the 3-manifold using quantum $6j$-symbols [TV92] and by using a Heegard decomposition of a 3-manifold [Ko92].

The topological invariants of 3-manifolds constructed since 1991 have been much studied: in particular the invariants of Reshetikhin and Turaev have been greatly studied due to the comparative ease of their calculation (eg see [KM90, Lic93, Ke97]). In addition to their status of new topological invariants, this study yielded applications to classical 3-manifold topology – for example the Reshetikhin-Turaev invariants were used to develop obstructions to embedding one 3-manifold in another [FKB01].

The construction of 3-manifold invariants in [RT91] is general, in that it provides a way to construct the invariants from any modular Hopf algebra. It was shown in [RT91] that a quotient of $U_q(sl_2)$ at $4k^{th}$ primitive roots of unity was modular. This naturally led researchers to ask whether other quantum algebras were also modular. But the modularity criteria involves representation-theoretic conditions, and the relevant representation theory of quotients of other quantum algebras at primitive roots of unity was not known.

However, Turaev and Wenzl managed to somewhat circumvent this problem in 1993 by constructing topological invariants of closed, connected, orientable 3-manifolds from *quasimodular Hopf algebras*, which are more general than modular Hopf algebras [TW93]. The essential difference between modular and quasimodular Hopf algebras is that the condition that the modules be irreducible is relaxed for quasimodular Hopf algebras. The quantum algebras related to the families of complex Lie algebras A_n, B_n, C_n and D_n are quasimodular at even primitive roots of unity, and the authors constructed 3-manifold invariants using these algebras [TW93].

In 1994 Turaev reformulated modular Hopf algebras using the language of categories, and showed that modular categories, which are tensor categories with additional structures, could be used to construct 3-manifold invariants [T94]. Informally, the key features of a modular category are a finite set of objects satisfying a certain tensor product property and the invertibility of a certain matrix commonly called the S-matrix. This was followed by the study of pre-modular categories by Bruguières [Br00a, Br00b], which are a class of categories more general than modular categories, in which the S-matrix is not necessarily invertible. Invariants of links and tangles and sometimes 3-manifolds can be constructed using a pre-modular category [BB01]. Although we have not explored the possibility, it appears plausible that the pseudo-modular Hopf algebras we define in this thesis may yield examples of pre-modular categories.

In 1995, the construction of Reshetikhin-Turaev 3-manifold invariants was extended to modular Hopf algebras derived from $\mathbb{Z}_2$-graded ribbon Hopf algebras by Zhang [Zh95], and the first such topological invariants were constructed from quotients of the quantum superalgebras $U_q(osp(1|2))$ [Zh94], and $U_q(gl(2|1))$ [Zh95] at odd roots of unity. This construction slightly differed from that of [RT91], in that it was not shown that the algebras were modular, but the author found a way to avoid this requirement by using known features of the representation theory of these algebras.

In contrast to the construction of topological invariants of 3-manifolds using quotients of quantum algebras at roots of unity and their representations, the construction of topological invariants of 3-manifolds using quotients of quantum superalgebras at roots of unity and their representations, has, unfortunately, been quite limited, this limitation having arisen from the limited knowledge of the representation theory of quantum superalgebras and their quotients at roots of unity. This was the motivation for our earlier-mentioned study of representations of a quotient of $U_q(osp(1|2n))$ at roots of unity in this thesis.

As mentioned previously, we then show that topological invariants of closed, connected, orientable 3-manifolds can be constructed from a class of $\mathbb{Z}_2$-graded ribbon Hopf algebras (together with a finite set of their representations) satisfying more general conditions than those satisfied by a modular or quasimodular Hopf algebras. We call these algebras *pseudo-modular Hopf algebras*. The key difference between a pseudo-modular and a modular Hopf algebra is that the S-matrix of a pseudo-modular Hopf algebra is not necessarily invertible, while it is invertible for modular Hopf algebras.

After defining pseudo-modular Hopf algebras and showing that 3-manifold invariants can be constructed from pseudo-modular Hopf algebras, we show that the quotient algebra

$U_q^{(N)}(osp(1|2n))$, where $q = \exp(2\pi i/N)$ and $N \geq 6$ satisfies $N \equiv 2 \pmod 4$, together with a set of its representations, is pseudo-modular and thus yields 3-manifold invariants.

A preliminary announcement of the work presented in this thesis was given in [Bl03].

1.1 Summary of new results

We now briefly summarise the new results of each chapter, and refer to the body of the thesis for the precise statement of each result mentioned here.

Chapter 3

Let $U_{\mathbb{C}[[h]]}(osp(1|2n))$ denote the Drinfel'd version of the quantum superalgebra over the ring $\mathbb{C}[[h]]$, for an indeterminate h. Write the universal R-matrix of $U_{\mathbb{C}[[h]]}(osp(1|2n))$ as $R = \widetilde{K}\widetilde{R}$ (eg see [KT91]), where $\widetilde{R}$ is an infinite sum of root vectors in $U_{\mathbb{C}[[h]]}(osp(1|2n))$. One of the problems in dealing with Jimbo's quantum superalgebra $U_q(osp(1|2n))$ over $\mathbb{C}$ is that $U_q(osp(1|2n))$ is not complete and thus does not admit a universal R-matrix.

In Section 3.3 we define R-matrices for finite dimensional representations of $U_q(osp(1|2n))$ at generic q, as is done in quantum algebras over $\mathbb{C}$ [CP94, KS97]. We do this by defining a completion $\overline{U}_q^+(osp(1|2n))$ of $U_q(osp(1|2n))$ which contains the factor $\widetilde{R}$ of the universal R-matrix. There is the problem of dealing with the factor $\widetilde{K}$ of the universal R-matrix of $U_{\mathbb{C}[[h]]}(osp(1|2n))$, as it is not clear which element of $\overline{U}_q^+(osp(1|2n))$, if any, corresponds to $\widetilde{K}$. We define an element, $\mathcal{E}_\mu \in U_q(osp(1|2n))$ for each integral dominant weight μ such that $\mathcal{E}_\mu$ has the same action on $V_\mu \otimes V_\lambda$, for any integral dominant λ, as would $\widetilde{K}$ if it existed. Then $\mathcal{E}_\mu\widetilde{R}$ is an 'R-matrix' for $V_\mu \otimes V_\lambda$ in that it is an R-matrix for representations.

In Section 3.4 we define some useful elements of $\overline{U}_q^+(osp(1|2n))$. These elements act in finite dimensional irreducible representations of $U_q(osp(1|2n))$ in a similar way as the elements u and v of a $\mathbb{Z}_2$-graded ribbon Hopf algebra. Let V_λ be the finite dimensional irreducible $U_q(osp(1|2n))$-module with integral dominant highest weight λ. For each integral dominant λ, define the elements $u_\lambda, v_\lambda \in \overline{U}_q^+(osp(1|2n))$. Even though v_λ does not necessarily commute with each element of $U_q(osp(1|2n))$, it acts on V_λ as multiplication by the scalar $q^{-(\lambda+2\rho,\lambda)}$.

In Section 3.5 we state and prove the spectral decomposition of the element

$$\check{\mathcal{R}}_{V,V} \in End_{U_q(osp(1|2n))}(V \otimes V)$$

defined by $\check{\mathcal{R}}_{V,V}(x \otimes y) = P \circ R_{V,V}(x \otimes y)$, where P is the graded permutation operator and $R_{V,V}$ is the R-matrix for the tensor product of $U_q(osp(1|2n))$-representations $V \otimes V$.

In Section 3.6 we show that there exists a representation of the Birman-Wenzl-Murakami algebra $\mathscr{BW}_f(-q^{2n}, q)$ in $End_{U_q(osp(1|2n))}(V^{\otimes f})$.

In Section 3.8 we use the elements v_λ of $\overline{U}_q^+(osp(1|2n))$ to define a set of mutually orthogonal idempotents in $End_{U_q(osp(1|2n))}(V^{\otimes f})$ that project down from $V^{\otimes f}$ (which is completely reducible) onto its irreducible $U_q(osp(1|2n))$-submodules.

In Section 3.9 we define matrix units in $End_{U_q(osp(1|2n))}(V^{\otimes f})$ which we obtain from the projections in Section 3.8 and the matrix units in a semisimple quotient of $\mathscr{BW}_f(-q^{2n}, q)$. The intertwiner matrix units in $End_{U_q(osp(1|2n))}(V^{\otimes f})$ are $U_q(osp(1|2n))$-linear maps between isomorphic irreducible $U_q(osp(1|2n))$-submodules of $V^{\otimes f}$.

Chapter 4

In this chapter we define the quotient algebra $U_q^{(N)}(osp(1|2n))$ and its relevant representations. The new results in this chapter are as follows. Fix $q = \exp(2\pi i/N)$ where $N \geq 3$ is an integer. In Theorem 4.1.1 we prove that a certain left ideal $\mathcal{I} \subset U_q(osp(1|2n))$ is a two-sided Hopf ideal, thus

$$U_q^{(N)}(osp(1|2n)) = U_q(osp(1|2n))/\mathcal{I},$$

is a $\mathbb{Z}_2$-graded Hopf algebra. This quotient algebra has apeared in the literature but only for the case $n = 1$ where $N \geq 3$ is odd [Zh94]. The proof of Theorem 4.1.1 involves many intricate and involved calculations and we leave these calculations to Appendix D.

In Proposition 4.1.1 we state the universal R-matrix of $U_q^{(N)}(osp(1|2n))$, which was originally given in [Zh92a]. In Theorem 4.1.3 we prove that $U_q^{(N)}(osp(1|2n))$ is a $\mathbb{Z}_2$-graded ribbon Hopf algebra.

We then construct representations of $U_q^{(N)}(osp(1|2n))$ that we will use in constructing the topological invariant. Lemma 4.1.1 discusses the fundamental irreducible $U_q^{(N)}(osp(1|2n))$-module V; all the $U_q^{(N)}(osp(1|2n))$-modules that we later define are submodules of tensor powers of V. In Subsection 4.2.1 we introduce the set $\overline{\Lambda_N^+}$ of integral weights in the truncated fundamental Weyl alcove. In Subsection 4.2.2 we define a $U_q^{(N)}(osp(1|2n))$-module V_λ associated with each integral dominant $\lambda \in \overline{\Lambda_N^+}$. In Lemma 4.2.6 we give the quantum superdimension of each of these $U_q^{(N)}(osp(1|2n))$-modules. The quantum superdimension of V_λ has the same expression as does the quantum superdimension of the irreducible $U_q(osp(1|2n))$-module with highest weight λ but we specialise q to the appropriate root of unity. We do not know whether the $U_q^{(N)}(osp(1|2n))$-modules defined in Subsection 4.2.2 are irreducible or even indecomposable, but we conjecture that they are all irreducible.

In Proposition 4.2.1 we prove that each $U_q^{(N)}(osp(1|2n))$-module defined in Subsection 4.2.2 is self-dual.

In Section 4.3 we prove tensor product theorems for the $U_q^{(N)}(osp(1|2n))$-modules defined in Subsection 4.2.2 at even N. The proofs are based on the proofs of similar tensor product theorems for representations of quantum algebras at even roots of unity [TW93, Thm. 5.2.2]. The new tensor product theorems are reminiscent of similar tensor product theorems for modular and quasimodular Hopf algebras, and are perhaps analogues of them.

Chapter 5

In this chapter we construct a topological invariant of closed, connected, orientable 3-manifolds. In Subsection 5.4.2 we define *pseudo-modular Hopf algebras* which generalise modular Hopf algebras. In Section 5.5 we prove that a topological invariant of closed, connected, orientable 3-manifolds can be constructed from a pseudo-modular Hopf algebra and its representations.

The essential differences between a pseudo-modular Hopf algebra and a modular Hopf algebra are three-fold: (i) the irreducibility conditions on the modules are relaxed for a pseudo-modular Hopf algebra, (ii) the condition that a certain matrix equation has a unique set of solutions is relaxed to a condition that the matrix equation has *at least one* set of solutions, and (iii) a certain sum must be non-zero. In modular Hopf algebras the result corresponding to (iii) is automatically true, which is a consequence of the fact that the matrix equation has a unique set of solutions (see [TW93]). The condition in (ii) for modular Hopf algebras that the matrix equation has a unique set of solutions boils down to a condition that the S-matrix be invertible, and the importance of this is that the solutions to the matrix equation are exactly the weights in the linear sum of isotopy invariants of framed links giving rise to the 3-manifold invariant. For pseudo-modular Hopf algebras we merely require that there exists at least one set of (non-zero) solutions to the matrix equation.

Theorem 5.6.2 contains the core result of this thesis: we prove that $U_q^{(N)}(osp(1|2n))$, together with a set of its representations $\{V_\lambda |\ \lambda \in \Lambda_N^+\}$ (here $\Lambda_N^+ \subset \overline{\Lambda_N^+}$), and a collection of other data, is a pseudo-modular Hopf algebra when $N \geq 6$ satisfies $N \equiv 2 \pmod 4$. Furthermore, in Theorem 5.8.1 we prove that $U_q^{(N)}(osp(1|2n))$ is *not* a pseudo-modular Hopf algebra when $N \geq 4$ satisfies $N \equiv 0 \pmod 4$, and that 3-manifold invariants cannot be constructed from $U_q^{(N)}(osp(1|2n))$ when $N \equiv 0 \pmod 4$.

A relevant question to ask is whether our topological invariants are complete or even new invariants. These are very difficult questions to answer, and it is more tractable to ask whether our topological invariants are the same as existing invariants. The set Λ_N^+ for $U_q^{(N)}(osp(1|2n))$ when $N \geq 6$ satisfies $N \equiv 2 \pmod 4$ is identical to the corresponding set for $U_q^{(N/2)}(so(2n+1))$, and in Section 5.7 we compare the topological invariants constructed using $U_q^{(N)}(osp(1|2n))$ to the invariants constructed using $U_q^{(N/2)}(so(2n+1))$. We show that the invariants from these algebras are *not* the same.

Appendices A–D

In Appendix B we prove two generalisations of the q-binomial theorem that we use in Appendix D. In Appendix D we prove that the left ideal $\mathcal{I} \subset U_q(osp(1|2n))$ at roots of unity is a two-sided Hopf ideal.

Chapter 2

Notation and background

In this chapter we introduce some notation that we use in this thesis and we also give relevent background material.

The plan of this chapter is as follows. In Section 2.1 we introduce some of the notations and conventions that we use in this thesis. In Section 2.2 we introduce the elementary algebraic structures that we will refer to and often use in this thesis. In Section 2.3 we define $\mathbb{Z}_2$-graded quasitriangular Hopf algebras, which are a class of $\mathbb{Z}_2$-graded Hopf algebras admitting a certain element called the universal R-matrix. These algebras are intrinsically mathematically interesting and are also important in physics (eg in statistical mechanics [KS82]), as the universal R-matrix furnishes a solution to the Yang-Baxter equation in each representation of the algebra.

We then define $\mathbb{Z}_2$-graded ribbon Hopf algebras. This last class of algebras is useful in constructing knot invariants (eg see [O02]) and is very important in our construction of topological invariants of 3-manifolds in Chapter 5.

2.1 Notation

In this section we introduce the notation and conventions that we will use in this thesis:

$\mathbb{Z}$ the set of integers,

$\mathbb{Z}_+$ the set of non-negative integers,

$\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$,

$\mathbb{N}$ the set of positive integers,

$\mathbb{C}$ the field of complex numbers,

$\mathbb{R}$ the field of real numbers.

For each $N \in \mathbb{Z}$, set

$$N' = \begin{cases} N, & \text{if } N \text{ is odd}, \\ N/2, & \text{if } N \text{ is even}, \end{cases} \qquad \overline{N} = \begin{cases} 2N, & \text{if } N \text{ is odd}, \\ N, & \text{if } N \equiv 0 \pmod 4, \\ N/2, & \text{if } N \equiv 2 \pmod 4. \end{cases}$$

For each $q = \rho e^{i\theta} \in \mathbb{C}$ where $0 \leq \theta < 2\pi$ and $\rho > 0$, set $q^{1/2} = +\sqrt{\rho} e^{i\theta/2}$.
For each $q \in \mathbb{C}$ not equal to 0 or 1, define

$$[n]^q = \frac{1-q^n}{1-q}, \qquad [n]^q! = [n]^q [n-1]^q \cdots [1]^q, \quad n \in \mathbb{N}, \qquad [0]^q! = 1,$$

and the Gaussian binomial

$$\left[\begin{array}{c} n \\ i \end{array} \right]^q = \frac{[n]^q!}{[i]^q!\,[n-i]^q!}, \qquad i \leq n. \tag{2.1}$$

We also define

$$(n)_q = \frac{1-(-q)^n}{1+q}, \qquad (n)_q! = (n)_q (n-1)_q \cdots (1)_q, \quad n \in \mathbb{N}, \qquad (0)_q! = 1,$$

and the pseudo-Gaussian binomial

$$\left(\begin{array}{c} n \\ i \end{array} \right)_q = \frac{(n)_q!}{(i)_q!(n-i)_q!}, \qquad i \leq n. \tag{2.2}$$

Note that $(n)_q = [n]^{-q}$ but we introduce the two notations as this is convenient when q is a root of unity. We also define

$$[n]_q = q^{(1-n)/2}(n)_q = \frac{q^{-n/2} - (-1)^n q^{n/2}}{q^{-1/2} + q^{1/2}}, \qquad n \in \mathbb{N},$$

and $[n]_q! = [n]_q [n-1]_q \cdots [1]_q$, $[0]_q! = 1$ and $\left[\begin{array}{c} n \\ i \end{array} \right]_q = \frac{[n]_q!}{[i]_q![n-i]_q!}$, $i \leq n$.
For each non-empty set A, we define the delta function $\delta : A \times A \to \mathbb{C}$ by

$$\delta_{x,y} = \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{if } x \neq y, \end{cases} \qquad \text{for all } x, y \in A.$$

2.2 Background algebraic structures

In this section we introduce the background algebraic structures that we refer to and often use in this thesis [Sc79, GZB93]. Throughout this section we denote the two elements of $\mathbb{Z}_2$ by 0 and 1.

Definition 2.2.1. *Let V be a vector space over $\mathbb{C}$. A $\mathbb{Z}_2$-gradation of the vector space V is a family $(V_\gamma)_{\gamma\in\mathbb{Z}_2}$ of subspaces of V such that*

$$V = \bigoplus_{\gamma\in\mathbb{Z}_2} V_\gamma.$$

The vector space V is said to be $\mathbb{Z}_2$-graded if it is equipped with a $\mathbb{Z}_2$-gradation.

An element of V is called *homogeneous of degree* γ, $\gamma \in \mathbb{Z}_2$, if it is an element of V_γ. The elements of V_0 (resp. V_1) are called even (resp. odd). We define a mapping $[\cdot] : V_\gamma \to \{0, 1\}$ for each $\gamma \in \mathbb{Z}_2$ by setting $[y] = \gamma$ for each $y \in V_\gamma$.

Each element $y \in V$ has a unique decomposition of the form

$$y = \sum_{\gamma\in\mathbb{Z}_2} y_\gamma, \qquad y_\gamma \in V_\gamma.$$

The element y_γ is called the *homogeneous component* of y of degree γ.

A subspace U of V is called $\mathbb{Z}_2$-graded if it contains the homogeneous components of all of its elements, that is, if

$$U = \bigoplus_{\gamma\in\mathbb{Z}_2} U_\gamma, \qquad U_\gamma = (U \cap V_\gamma).$$

Definition 2.2.2. *Let $V = \bigoplus_{\gamma\in\mathbb{Z}_2} V_\gamma$ and $W = \bigoplus_{\delta\in\mathbb{Z}_2} W_\delta$ be two $\mathbb{Z}_2$-graded vector spaces over $\mathbb{C}$. A $\mathbb{C}$-linear mapping*

$$\psi : V \to W,$$

is said to be homogeneous of degree $\gamma \in \mathbb{Z}_2$, if

$$\psi(V_\alpha) \subseteq W_{\alpha+\gamma}, \qquad \textit{for all } \alpha \in \mathbb{Z}_2.$$

The mapping ψ is called a homomorphism of the $\mathbb{Z}_2$-graded vector space V into the $\mathbb{Z}_2$-graded vector space W if ψ is homogeneous of degree 0.

Let $V = \bigoplus_{\gamma\in\mathbb{Z}_2} V_\gamma$ and $W = \bigoplus_{\delta\in\mathbb{Z}_2} W_\delta$ be two $\mathbb{Z}_2$-graded vector spaces over $\mathbb{C}$. The vector space $\mathrm{Hom}_{\mathbb{C}}(V, W)$ of $\mathbb{C}$-linear maps from V to W admits a $\mathbb{Z}_2$-gradation

$$\mathrm{Hom}_{\mathbb{C}}(V, W) = \bigoplus_{\alpha\in\mathbb{Z}_2} \mathrm{Hom}_{\mathbb{C}}(V, W)_\alpha,$$

where $\mathrm{Hom}_{\mathbb{C}}(V, W)_\alpha = \{\psi \in \mathrm{Hom}_{\mathbb{C}}(V, W) |\ \psi \text{ is homogeneous of degree } \alpha \in \mathbb{Z}_2\}$. We denote $End_{\mathbb{C}}(V) = \mathrm{Hom}_{\mathbb{C}}(V, V)$.

If $W = \mathbb{C}$ is regarded as a $\mathbb{Z}_2$-graded vector space with zero odd subspace, $\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ is called the dual space of V and is denoted by V^*, which is clearly $\mathbb{Z}_2$-graded.

Definition 2.2.3. *Let V be a $\mathbb{Z}_2$-graded vector space over $\mathbb{C}$. Define the map $\gamma \in End_{\mathbb{C}}(V)$ by*

$$\gamma(v) = (-1)^{[v]}(v),$$

for all homogeneous $v \in V$. We define the supertrace by

$$\mathrm{str} : End_{\mathbb{C}}(V) \to \mathbb{C}, \qquad \mathrm{str}(x) = \mathrm{tr}(\gamma x),$$

where $\mathrm{tr} : End_{\mathbb{C}}(V) \to \mathbb{C}$ is the standard trace functional.

We write $V \otimes W$ to denote $V \otimes_{\mathbb{C}} W$ where each of V and W are vector spaces over $\mathbb{C}$, unless otherwise specified.

Remark 2.2.1. *Let V and W be two $\mathbb{Z}_2$-graded vector spaces over $\mathbb{C}$, then $V \otimes W$ has a natural $\mathbb{Z}_2$-gradation defined by*

$$(V \otimes W)_\gamma = \bigoplus_{(\alpha+\beta)\equiv\gamma \pmod 2} V_\alpha \otimes W_\beta, \qquad \gamma \in \mathbb{Z}_2.$$

Let V and W be two $\mathbb{Z}_2$-graded vector spaces over $\mathbb{C}$. The graded permutation operator $P : V \otimes W \to W \otimes V$ is defined for all homogeneous $v \in V$ and $w \in W$ by

$$P(v \otimes w) = (-1)^{[v][w]} w \otimes v,$$

and extended to all inhomogeneous elements by linearity.

Definition 2.2.4. *An associative $\mathbb{Z}_2$-graded algebra A over $\mathbb{C}$ is a $\mathbb{Z}_2$-graded vector space equipped with graded vector space homomorphisms $m : A \otimes A \to A$ and $u : \mathbb{C} \to A$ satisfying the following properties:*

$$m \circ (m \otimes \mathrm{id}) = m \circ (\mathrm{id} \otimes m), \tag{2.3}$$

where $\mathrm{id} : A \to A$ is the identity homomorphism and we regard both sides of (2.3) as maps $A \otimes A \otimes A \to A$, and

$$m \circ (u \otimes \mathrm{id}) = m \circ (\mathrm{id} \otimes u), \tag{2.4}$$

where $A \to \mathbb{C} \otimes A$ and $A \to A \otimes \mathbb{C}$ are the natural maps and we regard both sides of (2.4) as maps $A \to A$. The maps m and u are called the multiplication and unit of A, respectively.

A $\mathbb{Z}_2$-graded algebra A is called commutative if

$$m \circ (v \otimes w) = m \circ P(v \otimes w), \qquad \forall v, w \in A.$$

In this thesis we assume that all $\mathbb{Z}_2$-graded algebras are associative algebras.

Definition 2.2.5. *Let A be a $\mathbb{Z}_2$-graded algebra with unit and let V be a left A-module. The A-module V is said to be $\mathbb{Z}_2$-graded if the underlying vector space is $\mathbb{Z}_2$-graded and if*

$$A_\alpha V_\beta \subseteq V_{\alpha+\beta}, \qquad A_\alpha \in A, \qquad \text{for all } \alpha, \beta \in \mathbb{Z}_2.$$

Right $\mathbb{Z}_2$-graded A-modules are similarly defined. In this thesis we assume that all modules are finite dimensional left modules unless otherwise stated.

A homomorphism of $\mathbb{Z}_2$-graded A-modules is by definition a homomorphism of the A-modules as well as of the underlying $\mathbb{Z}_2$-graded vector spaces. Each such homomorphism is A-linear and homogeneous of degree 0. Let $V = V_0 \oplus V_1$ be a $\mathbb{Z}_2$-graded A-module. If $V' = V'_0 \oplus V'_1$ is a $\mathbb{Z}_2$-graded A-module with $V'_0 = V_1$ and $V'_1 = V_0$, then we regard V and V' as *not* being isomorphic as A-modules.

Note that if A and B are two $\mathbb{Z}_2$-graded algebras, then $A \otimes B$ is a $\mathbb{Z}_2$-graded algebra with the multiplication

$$m_{A\otimes B} = (m_A \otimes m_B) \circ (\mathrm{id} \otimes P \otimes \mathrm{id}),$$

and the unit

$$u_{A\otimes B} = u_A \otimes u_B.$$

Let V (resp. W) be a $\mathbb{Z}_2$-graded A-module (resp. B-module), then $V \otimes W$ has a unique structure as a $\mathbb{Z}_2$-graded $(A \otimes B)$-module given by

$$(a \otimes b)(v \otimes w) = (-1)^{[b][v]} av \otimes bw, \qquad a \in A, \quad b \in B, \quad v \in V, \quad w \in W.$$

Definition 2.2.6. *A $\mathbb{Z}_2$-graded co-algebra C is a $\mathbb{Z}_2$-graded vector space V together with $\mathbb{Z}_2$-graded vector space homomorphisms $\Delta : C \to C \otimes C$ and $\epsilon : C \to \mathbb{C}$, where Δ satisfies*

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta, \tag{2.5}$$

with $\mathrm{id} : C \to C$ *being the identity homomorphism (we regard both sides of (2.5) as maps $C \to C \otimes C \otimes C$) and ϵ and Δ satisfy*

$$(\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \epsilon) \circ \Delta, \tag{2.6}$$

where we regard each side in (2.6) as a map $C \to C$. The maps Δ and ϵ are called the co-multiplication and co-unit of C, respectively. We refer to Eq. (2.5) by saying that the co-multiplication $\Delta : C \to C \otimes C$ is co-associative.

For a co-multiplication $\Delta : C \to C \otimes C$ we define the opposite co-multiplication $\Delta' : C \to C \otimes C$ by $\Delta' = P \circ \Delta$. A $\mathbb{Z}_2$-graded co-algebra C is called co-commutative if $\Delta(x) = \Delta'(x)$ for all $x \in C$. A $\mathbb{C}$-linear subspace I of a $\mathbb{Z}_2$-graded co-algebra C is called a two-sided co-ideal if

$$\Delta(x) \subseteq I \otimes A + A \otimes I, \quad \text{and} \quad \epsilon(x) = 0, \qquad \forall x \in I. \tag{2.7}$$

We inductively define the n^{th} co-multiplication $\Delta^{(n)} : C^{\otimes n} \to C^{\otimes(n+1)}$ by

$$\Delta^{(n)} = (\Delta \otimes \mathrm{id}^{\otimes(n-1)}) \circ \Delta^{(n-1)},$$

where we fix $\Delta^{(1)} = \Delta$. Similarly, we inductively define the n^{th} opposite co-multiplication $\Delta'^{(n)} : C^{\otimes n} \to C^{\otimes(n+1)}$ by

$$\Delta'^{(n)} = (\Delta' \otimes \mathrm{id}^{\otimes(n-1)}) \circ \Delta'^{(n-1)},$$

where we fix $\Delta'^{(1)} = \Delta'$.

We may use *Sweedler's Sigma notation* in writing down the co-multiplication of an element of a $\mathbb{Z}_2$-graded co-algebra C [Sw69]. In this notation we write

$$\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}, \qquad \Delta^{(2)}(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)} \otimes x_{(3)},$$

$$\Delta^{(n)}(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)} \otimes \cdots \otimes x_{(n+1)}, \quad n \geq 2, \qquad \forall x \in C.$$

A consequence of the co-associativity of the co-multiplication is that

$$\Delta^{(n)}(x) = (\mathrm{id}^{\otimes m} \otimes \Delta \otimes \mathrm{id}^{\otimes(n-m-1)}) \circ \Delta^{(n-1)}(x), \quad \text{for each } m = 0, 1, \ldots, n-1,$$

for each $x \in C$.

Observe that if A and B are $\mathbb{Z}_2$-graded co-algebras with co-multiplications Δ_A, Δ_B, and co-units ϵ_A, ϵ_B, respectively, then $A \otimes B$ is a $\mathbb{Z}_2$-graded co-algebra with the co-multiplication

$$\Delta_{A\otimes B} = (\mathrm{id} \otimes P \otimes \mathrm{id}) \circ (\Delta_A \otimes \Delta_B),$$

and the co-unit

$$\epsilon_{A\otimes B} = \epsilon_A \otimes \epsilon_B.$$

Definition 2.2.7. *Let A and B be $\mathbb{Z}_2$-graded co-algebras over $\mathbb{C}$. A homomorphism $f : A \to B$ of the co-algebras is defined to be a homomorphism of the underlying $\mathbb{Z}_2$-graded vector spaces satisfying*

$$(f \otimes f) \circ \Delta_A = \Delta_B \circ f, \tag{2.8}$$

and

$$\epsilon_B \circ f = \epsilon_A, \tag{2.9}$$

where both sides of (2.8) are maps $A \to B \otimes B$, and both sides of (2.9) are maps $A \to \mathbb{C}$.

Definition 2.2.8. *Let A be a $\mathbb{Z}_2$-graded algebra with multiplication m and unit u, and also a $\mathbb{Z}_2$-graded co-algebra with co-multiplication Δ and co-unit ϵ. Then A is called a $\mathbb{Z}_2$-graded bi-algebra if one of the following equivalent conditions is satisfied:*

(i) Δ and ϵ are $\mathbb{Z}_2$-graded algebra homomorphisms,

(ii) m and u are $\mathbb{Z}_2$-graded co-algebra homomorphisms.

If, furthermore, there exists a $\mathbb{Z}_2$-graded vector space homomorphism $S : A \to A$ satisfying

$$m \circ (\mathrm{id} \otimes S) \circ \Delta = u \circ \epsilon = m \circ (S \otimes \mathrm{id}) \circ \Delta, \tag{2.10}$$

then A is called a $\mathbb{Z}_2$-graded Hopf algebra. Such a $\mathbb{Z}_2$-graded vector space homomorphism is called the antipode of A.

The antipode S of A has the following properties:

(i) $S \circ m = m \circ P \circ (S \otimes S)$,

(ii) $S \circ u = u$,

(iii) $\epsilon \circ S = \epsilon$,

(iv) $P \circ (S \otimes S) \circ \Delta = \Delta \circ S$,

(v) if H is commutative or co-commutative, $S^2 = \mathrm{id}$.

Definition 2.2.9. *Let A be a $\mathbb{Z}_2$-graded Hopf algebra over $\mathbb{C}$ with antipode S. Given the A-module V, we define the dual A-module V^* as follows. Let $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ and let $\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{C}$ denote the dual space pairing. We define the action of A on V^* by*

$$\langle av^*, w\rangle = (-1)^{[a][v^*]}\langle v^*, S(a)w\rangle, \quad \text{for all} \;\; a \in A,\; v^* \in V^*,\; w \in V.$$

2.3 $\mathbb{Z}_2$-graded quasitriangular Hopf algebras

Definition 2.3.1. *We call a $\mathbb{Z}_2$-graded Hopf algebra A a $\mathbb{Z}_2$-graded quasitriangular Hopf algebra if there exists an invertible even element $R \in A \otimes A$ satisfying the equations*

$$\begin{aligned} R\Delta(x) &= \Delta'(x)R, \qquad \forall x \in A, && (2.11)\\ (\Delta \otimes \mathrm{id})R &= R_{13}R_{23}, && (2.12)\\ (\mathrm{id} \otimes \Delta)R &= R_{13}R_{12}, && (2.13) \end{aligned}$$

where id *is the identity map on A. Write $R = \sum_t \alpha_t \otimes \beta_t$, then $R_{12} = \sum_t \alpha_t \otimes \beta_t \otimes 1$, $R_{13} = \sum_t \alpha_t \otimes 1 \otimes \beta_t$ and $R_{23} = \sum_t 1 \otimes \alpha_t \otimes \beta_t$. The element R is called the universal R-matrix of A.*

Lemma 2.3.1. *Let A be a $\mathbb{Z}_2$-graded quasitriangular Hopf algebra with universal R-matrix $R \in A \otimes A$. The universal R-matrix satisfies the following equations*

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \tag{2.14}$$

$$(S \otimes S)R = R, \qquad (S \otimes 1)R = R^{-1}, \qquad (1 \otimes S)R^{-1} = R, \tag{2.15}$$

$$(\epsilon \otimes \mathrm{id})R = (\mathrm{id} \otimes \epsilon)R = 1. \tag{2.16}$$

Equation (2.14) is called the graded Quantum Yang-Baxter equation.

Proof. The proofs are standard (eg see [KS97]), for instance the proof of Eq. (2.14) is

$$R_{12}R_{13}R_{23} = R_{12}\big((\Delta \otimes \mathrm{id})R\big) = \big((\Delta' \otimes \mathrm{id})R\big)R_{12} = R_{23}R_{13}R_{12}.$$

The proofs of the other equations similarly follow those of the corresponding equations for ungraded quasitriangular Hopf algebras [KS97]. ☐

Let A be a $\mathbb{Z}_2$-graded quasitriangular Hopf algebra over $\mathbb{C}$ with universal R-matrix R. Let V and W be finite dimensional A-modules and let π_V, π_W be the representations of A afforded by V and W, respectively. Fix $R_{VW} = (\pi_V \otimes \pi_W)R$ and let $\check{R}_{VW} \in \mathrm{Hom}_{\mathbb{C}}(V \otimes W, W \otimes V)$ be the map defined by

$$\check{R}_{VW}(v \otimes w) = P \circ \big(R_{VW}(v \otimes w)\big), \quad \text{for all } v \in V,\ w \in W.$$

The actions of R_{VW} and $\check{R}_{VW}$ on $v \otimes w \in V \otimes W$ are respectively

$$R_{VW}(v \otimes w) = \sum_t \pi_V(\alpha_t)v \otimes \pi_W(\beta_t)w(-1)^{[\beta_t][v]},$$

$$\check{R}_{VW}(v \otimes w) = \sum_t \pi_W(\beta_t)w \otimes \pi_V(\alpha_t)v(-1)^{[\alpha_t]+[w]([v]+[\alpha_t])}.$$

Lemma 2.3.2. *Let A be a $\mathbb{Z}_2$-graded quasitriangular Hopf algebra with universal R-matrix R. Let V be a finite dimensional A-module and let π be the representation of A afforded by V. For each integer $t \geq 2$, define*

$$\check{R}_i = \mathrm{id}^{\otimes(i-1)} \otimes \check{R}_{VV} \otimes \mathrm{id}^{\otimes(t-(i+1))} \in End_{\mathbb{C}}(V^{\otimes t}),$$

$$P_{i,i+1} = \mathrm{id}^{\otimes(i-1)} \otimes P \otimes \mathrm{id}^{\otimes(t-(i+1))} \in End_{\mathbb{C}}(V^{\otimes t}),$$

for each $1 \leq i \leq t-1$, where $\mathrm{id} = \mathrm{id}_V$, then

(i) $\check{R}_i$ is an element of the centraliser algebra $\mathcal{L}_t = End_A(V^{\otimes t})$, for each $1 \leq i \leq t-1$,

(ii) $\check{R}_i\check{R}_j = \check{R}_j\check{R}_i$, for all $|i-j| > 1$,

(iii) $\check{R}_i\check{R}_{i+1}\check{R}_i = \check{R}_{i+1}\check{R}_i\check{R}_{i+1}$, for all $1 \leq i \leq t-2$.

Proof. (i) The proof is standard. For an arbitrary $x \in A$, we have

$$\begin{aligned}
\check{R}_{VV}\big((\pi \otimes \pi)\Delta(x)\big) &= P\big((\pi \otimes \pi)R\Delta(x)\big) \\
&= P\big((\pi \otimes \pi)\Delta'(x)R\big) \\
&= \big((\pi \otimes \pi)\Delta(x)\big)\big(P(\pi \otimes \pi)R\big) \\
&= \big((\pi \otimes \pi)\Delta(x)\big)\check{R}_{VV}.
\end{aligned}$$

(ii) This is obvious.

(iii) We consider the case $i = 1$, the other cases are similar. We write R_{ij} to mean $(\pi_V \otimes \pi_V \otimes \pi_V)R_{ij}$. Now

$$\begin{aligned}
\check{R}_1\check{R}_2\check{R}_1 &= P_{12}R_{12}P_{23}R_{23}P_{12}R_{12} \\
&= P_{12}P_{23}R_{13}R_{23}P_{12}R_{12} \\
&= P_{12}P_{23}P_{12}R_{23}R_{13}R_{12},
\end{aligned}$$

where we have used the easily verified results: $R_{12}P_{23} = P_{23}R_{13}$ and $R_{13}R_{23}P_{12} = P_{12}R_{23}R_{13}$. A similar calculation shows that

$$\check{R}_2\check{R}_1\check{R}_2 = P_{23}P_{12}P_{23}R_{12}R_{13}R_{23},$$

The equality $P_{12}P_{23}P_{12} = P_{23}P_{12}P_{23}$ then yields what we seek to prove.

□

From [LR97] we obtain the following lemma.

Lemma 2.3.3. *Let A be a $\mathbb{Z}_2$-graded quasitriangular Hopf algebra with universal R-matrix $R = \sum_t a_t \otimes b_t$, and let $R^T = \sum_t b_t \otimes a_t(-1)^{[a_t][b_t]}$. Let V and W be finite dimensional irreducible A-modules and let π_V, π_W be the representations of A afforded by V and W, respectively. Then*

(i) $(\pi_V \otimes \pi_W)\big(R^T R\big) \in End_A(V \otimes W)$, *and*

(ii) for each integer $t \geq 3$, the element

$$\big(\pi_V^{\otimes t} \otimes \pi_V\big)(\Delta^{(t-1)} \otimes \mathrm{id})\big(R^T R\big) = \check{R}_t\check{R}_{t-1}\cdots\check{R}_1\check{R}_1\cdots\check{R}_{t-1}\check{R}_t \in End_{\mathbb{C}}(V^{\otimes(t+1)}),$$

is an element of $End_A(V^{\otimes(t+1)})$.

Proof. (i) Let $x \in A$ be arbitrary. Applying P to the equation $R\Delta(x) = \Delta'(x)R$ gives $R^T\Delta'(x) = \Delta(x)R^T$. Hence $\Delta(x)R^TR = R^T\Delta'(x)R = R^TR\Delta(x)$.

(ii) We introduce some notation we will use later on. For each integer $t \geq 3$, let $i, j \in \mathbb{N}$ satisfy $1 \leq i < j \leq t$, and fix

$$R_{ij} = \sum_t \mathrm{id}^{\otimes(i-1)} \otimes a_t \otimes \mathrm{id}^{(j-1-i)} \otimes b_t \otimes \mathrm{id}^{(t-j)} \in A^{\otimes t},$$

$$R_{ji} = \sum_t \mathrm{id}^{\otimes(i-1)} \otimes b_t \otimes \mathrm{id}^{(j-1-i)} \otimes a_t \otimes \mathrm{id}^{(t-j)}(-1)^{[a_t][b_t]} \in A^{\otimes t}.$$

We inductively prove that $(\Delta^{(t-2)}\otimes\mathrm{id})R_{12} = R_{1t}R_{2t}\cdots R_{(t-1)t}$ for each $t \geq 3$. The induction hypothesis is true for $t = 3$ from (2.12) and assume that it is true for some $t \geq 3$, then

$$\begin{aligned}(\Delta^{(t-1)} \otimes \mathrm{id})R &= (\Delta \otimes \mathrm{id}^{\otimes(t-1)})(\Delta^{(t-2)} \otimes \mathrm{id})R \\ &= (\Delta \otimes \mathrm{id}^{\otimes(t-1)})R_{1t}R_{2t}\cdots R_{(t-1)t} \\ &= R_{1(t+1)}R_{2(t+1)}\cdots R_{t(t+1)}.\end{aligned}$$

In a similar way, we can prove that

$$(\Delta^{(t-1)} \otimes \mathrm{id})R^T = R_{(t+1)t}R_{(t+1)(t-1)}\cdots R_{(t+1)1}.$$

Let $\widetilde{P} = P_{12}P_{23}\cdots P_{t(t+1)}$. Then the above formulae lead to

$$\widetilde{P}\big(\pi^{\otimes(t+1)}(\Delta^{(t-1)} \otimes \mathrm{id})R\big) = \check{R}_1\check{R}_2\cdots\check{R}_t,$$

$$\pi^{\otimes(t+1)}\big((\Delta^{(t-1)} \otimes \mathrm{id})R^T\big)\widetilde{P}^{-1} = \check{R}_t\check{R}_{t-1}\cdots\check{R}_1.$$

Combining these equations together completes the proof.

□

Lemma 2.3.4. *Let A be a $\mathbb{Z}_2$-graded quasitriangular Hopf algebra with universal R-matrix $R = \sum_t a_t \otimes b_t$, and let $u \in A$ be defined by $u = m \circ P \circ (\mathrm{id} \otimes S)R$, that is*

$$u = \sum_t (-1)^{[a_t]} S(b_t) a_t.$$

Then u is invertible, and

$$\epsilon(u) = 1, \qquad \Delta(u) = (u \otimes u) \left(R^T R\right)^{-1}.$$

Furthermore,

$$S^2(x) = uxu^{-1}, \qquad \text{for all } x \in A.$$

Proof. The proofs are standard: we follow the proofs of the corresponding equations for ungraded quasitriangular Hopf algebras [Dr90, KS97]. From Lemma 2.3.5, $S^2(x)u = ux$ for all $x \in A$. Let $\widetilde{u} = \sum_s S^{-1}(d_s) c_s (-1)^{[d_s]}$ where $R^{-1} = \sum_s c_s \otimes d_s$; we will show that $\widetilde{u}$ is a two-sided inverse of u. Firstly

$$\begin{aligned} u\widetilde{u} &= \sum_s u S^{-1}(d_s) c_s (-1)^{[d_s]} \\ &= \sum_s S(d_s) u c_s (-1)^{[d_s]} \\ &= \sum_{s,t} S(b_t d_s) a_t c_s (-1)^{[d_s]+[b_t]+[d_s][b_t]}. \end{aligned}$$

Applying the map $m \circ P \circ (1 \otimes S)$ to $RR^{-1} = \sum_{s,t} a_t c_s \otimes b_t d_s (-1)^{[b_t][c_s]}$ gives

$$\sum_{s,t} S(b_t d_s) a_t c_s (-1)^{[b_t]+[d_s]+[a_t][d_s]} = 1.$$

Hence $u\widetilde{u} = 1$.

In exactly the same way we can also show that $\widetilde{u}$ is a left inverse of u, thus $\widetilde{u}$ is a two-sided inverse of u and we write $\widetilde{u} = u^{-1}$. We now show that $\epsilon(u) = 1$: firstly

$$\epsilon(u) = \sum_t \epsilon\big(S(b_t)\big)\epsilon(a_t)(-1)^{[a_t][b_t]} = \sum_t \epsilon(b_t)\epsilon(a_t) = \sum_t \epsilon(a_t)\epsilon(b_t),$$

as $\epsilon \circ S = \epsilon$ and $\epsilon(x) = 0$ if $[x] = 1$. Applying ϵ to the equation $(\epsilon \otimes \mathrm{id})R = 1$ gives

$$\epsilon(u) = \sum_t \epsilon(a_t)\epsilon(b_t) = 1.$$

The proof of the equation $\Delta(u) = (u \otimes u) \left(R^T R\right)^{-1}$ is given explicitly in [ZG91] and will not be repeated.

□

Lemma 2.3.5. *Let A be a $\mathbb{Z}_2$-graded quasitriangular Hopf algebra with universal R-matrix $R = \sum_t a_t \otimes b_t$, then $S^2(x)u = ux$ for all $x \in A$.*

Proof. The proof is well known. But as this is important to us later, we give a proof here. For each $x \in A$,

$$\sum_{(x)} R\Delta(x_{(1)}) \otimes x_{(2)} = \sum_{(x)} \Delta'(x_{(1)})R \otimes x_{(2)},$$

which we rewrite explicitly as

$$\sum_{t,(x)} a_t x_{(1)} \otimes b_t x_{(2)} \otimes x_{(3)} (-1)^{[b_t][x_{(1)}]} = \sum_{t,(x)} x_{(2)} a_t \otimes x_{(1)} b_t \otimes x_{(3)} (-1)^{[x_{(1)}][x_{(2)}]+[x_{(1)}][a_t]}.$$

Using this we obtain

$$\begin{aligned}\sum_{t,(x)} x_{(3)} \otimes b_t x_{(2)} \otimes a_t x_{(1)} (-1)^{[x_{(3)}]([x_{(1)}]+[x_{(2)}])+([a_t]+[x_{(1)}])([b_t]+[x_{(2)}])+[b_t][x_{(1)}]} \\ = \sum_{t,(x)} x_{(3)} \otimes x_{(1)} b_t \otimes x_{(2)} a_t (-1)^{[x_{(3)}]([x_{(1)}]+[x_{(2)}])+([a_t]+[x_{(2)}])([b_t]+[x_{(1)}])} \\ \times (-1)^{[x_{(1)}][x_{(2)}]+[a_t][x_{(1)}]}. \qquad (2.17)\end{aligned}$$

Note that R is even, thus $([a_t]+[b_t]) \equiv 0 \pmod 2$ for all t. In addition, $\big([x_{(1)}]+[x_{(2)}]+[x_{(3)}]\big) \equiv [x] \pmod 2$. Using these facts we can simplify the sign factors in the equation considerably. Applying the map $(\mathrm{id} \otimes m) \circ (\mathrm{id} \otimes S \otimes \mathrm{id})$ to both sides of (2.17) gives

$$\begin{aligned}\sum_{(x)} x_{(3)} \otimes S(x_{(2)}) u x_{(1)} (-1)^{[x_{(3)}]([x]+[x_{(3)}])+[x_{(1)}][x_{(2)}]} \\ = \sum_{t,(x)} x_{(3)} \otimes S(b_t) S(x_{(1)}) x_{(2)} a_t (-1)^{([a_t]+[x_{(3)}])([x]+[x_{(3)}])+[a_t]} \\ = \sum_{t,(x)} x_{(2)} \otimes S(b_t) a_t \epsilon(x_{(1)}) (-1)^{([a_t]+[x_{(2)}])([x]+[x_{(2)}])+[a_t]} \\ = x \otimes u. \qquad (2.18)\end{aligned}$$

Applying the map $m \circ (S^2 \otimes \mathrm{id})$ to (2.18) gives

$$\begin{aligned} S^2(x)u &= \sum_{(x)} S(x_{(2)} S(x_{(3)})) u x_{(1)} (-1)^{[x_{(1)}]([x_{(3)}]+[x_{(2)}])} \\ &= \sum_{(x)} S(x_{(2)} S(x_{(3)})) u x_{(1)} (-1)^{[x_{(1)}]([x]+[x_{(1)}])} \\ &= ux. \end{aligned}$$

□

We can readily prove the following lemma using induction.

Lemma 2.3.6. *Let A be a $\mathbb{Z}_2$-graded quasitriangular Hopf algebra with universal R-matrix R and the element $u = m \circ P \circ (\mathrm{id} \otimes S)R$, then*

$$\Delta^{(i)}(u) = u^{\otimes(i+1)}\big((R^T R)^{-1} \otimes \mathrm{id}^{\otimes(i-1)}\big) \prod_{j=1}^{i-1} \big((\Delta^{(j)} \otimes \mathrm{id}^{\otimes(i-j)}) \left(R^T R\right)^{-1}\big), \quad \textit{for all } i \geq 1,$$

where we fix the product to be equal to $1 \otimes 1$ if $i = 1$.

Definition 2.3.2. *A $\mathbb{Z}_2$-graded quasitriangular Hopf algebra A is called a $\mathbb{Z}_2$-graded ribbon Hopf algebra if it is equipped with an invertible central even element $v \in A$ satisfying*

$$v^2 = uS(u), \qquad S(v) = v, \qquad \epsilon(v) = 1, \tag{2.19}$$

$$\Delta(v) = (v \otimes v) \left(R^T R\right)^{-1}. \tag{2.20}$$

Definition 2.3.3. *Let A be a $\mathbb{Z}_2$-graded ribbon Hopf algebra over $\mathbb{C}$. Let V be an A-module, π the representation of A afforded by V, and f an element of $End_{\mathbb{C}}(V)$. The quantum supertrace of f is defined to be*

$$str_q(f) = \mathrm{str}\left(\pi(v^{-1}u) \circ f\right).$$

The quantum superdimension of V is defined to be the quantum supertrace of the identity endomorphism on V:

$$sdim_q(V) = \mathrm{str}\left(\pi(v^{-1}u)\right).$$

Chapter 3

Quantum $osp(1|2n)$ at generic q

In this chapter we introduce the quantum superalgebra $U_q(osp(1|2n))$ over $\mathbb{C}$ and discuss its finite dimensional irreducible representations. We define R-matrices for representations of $U_q(osp(1|2n))$, and construct projections from tensor powers of the fundamental irreducible $U_q(osp(1|2n))$-module V onto irreducible $U_q(osp(1|2n))$-submodules of $V^{\otimes t}$. By using matrix units in the Birman-Wenzl-Murakami algebra $\mathscr{BW}_t(-q^{2n}, q)$, we construct $U_q(osp(1|2n))$-linear maps between isomorphic irreducible $U_q(osp(1|2n))$-submodules of $V^{\otimes t}$.

The structure of this chapter is as follows. In Section 3.1 we introduce the quantum superalgebra $U_q(osp(1|2n))$. In Section 3.2 we discuss finite dimensional irreducible $U_q(osp(1|2n))$-modules, including the fundamental irreducible module V, and show that $V^{\otimes t}$ is completely reducible. In Section 3.3 we introduce R-matrices for representations of $U_q(osp(1|2n))$. In Section 3.4 we investigate the properties of two useful elements of a completion $\overline{U}_q^+(osp(1|2n))$ of $U_q(osp(1|2n))$. In Section 3.5 we determine the spectral decomposition of $\check{\mathcal{R}}_{V,V}$. In Section 3.6 we show that there is a representation of $\mathscr{BW}_t(-q^{2n}, q)$ in an algebra generated by the $\check{\mathcal{R}}_{V,V}$-matrices acting on the i^{th} and $(i+1)^{st}$ tensor powers of $V^{\otimes t}$ for $i = 1, \ldots, t-1$. In Section 3.7 we recall Bratteli diagrams and path algebras. In Section 3.8 we construct projections from $V^{\otimes t}$ onto irreducible $U_q(osp(1|2n))$-submodules of $V^{\otimes t}$. In Section 3.9 we construct matrix units in $End_{U_q(osp(1|2n))}(V^{\otimes t})$ from matrix units in $\mathscr{BW}_t(-q^{2n}, q)$ and prove that the algebra $End_{U_q(osp(1|2n))}(V^{\otimes t})$ is generated by the $\check{\mathcal{R}}_{V,V}$-matrices.

3.1 The quantum superalgebra $U_q(osp(1|2n))$

In this section we introduce the algebra that is at the core of this thesis: the quantum superalgebra $U_q(osp(1|2n))$ [Zh92a, KT91, Y94].

Let us begin by describing the root system of $osp(1|2n)$. Let H^* be a vector space over $\mathbb{C}$ with a basis $\{\epsilon_i |\ 1 \leq i \leq n\}$ and let

$$(\cdot, \cdot) : H^* \times H^* \to \mathbb{C}, \tag{3.1}$$

be a non-degenerate bilinear form defined by $(\epsilon_i, \epsilon_j) = \delta_{i,j}$.

The set of simple roots of $osp(1|2n)$ is $\{\alpha_i |\ 1 \le i \le n\}$ where

$$\alpha_i \quad = \quad \begin{cases} \epsilon_i - \epsilon_{i+1}, & i = 1, \dots, n-1, \\ \epsilon_n, & i = n, \end{cases}$$

which forms another basis of H^*.

Let Φ^+ denote the set of the positive roots of $osp(1|2n)$, we have

$$\Phi^+ = \{\epsilon_i \pm \epsilon_j, \epsilon_k, 2\epsilon_k |\ \ 1 \le i < j \le n, \ \ 1 \le k \le n\}\,.$$

We further define the subsets $\Phi_0^+, \Phi_1^+ \subset \Phi^+$ and $\overline{\Phi}_0^+ \subset \Phi_0^+$ by

$$\Phi_0^+ = \{\epsilon_i \pm \epsilon_j, 2\epsilon_k |\ \ 1 \le i < j \le n, \ \ 1 \le k \le n\}, \qquad \Phi_1^+ = \{\epsilon_k |\ \ 1 \le k \le n\},$$
$$\overline{\Phi}_0^+ = \{\alpha \in \Phi_0^+ |\ \ \alpha/2 \notin \Phi_1^+\}.$$

We call Φ_0^+ (resp. Φ_1^+) the set of *positive even roots* (resp. *positive odd roots*). The set of negative roots of $osp(1|2n)$ is $\Phi^- = -\Phi^+$, and $\Phi = \Phi^+ \cup \Phi^-$ is the set of all the roots.

We denote by $2\rho \in H^*$ the graded sum of the positive roots of $osp(1|2n)$:

$$2\rho = \sum_{\alpha \in \Phi_0^+} \alpha - \sum_{\beta \in \Phi_1^+} \beta.$$

Explicitly, $2\rho = \sum_{i=1}^n (2n - 2i + 1)\epsilon_i$. The element 2ρ satisfies $(2\rho, \alpha_i) = (\alpha_i, \alpha_i)$ for each $1 \le i \le n$, and will play an important role in this thesis.

Let $A = (a_{ij})_{i,j=1}^n$ be the Cartan matrix of $osp(1|2n)$. The components of A are defined by $a_{ij} = 2\,(\alpha_i, \alpha_j)\,/\,(\alpha_i, \alpha_i)$, and we have explicitly

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -2 & 2 \end{pmatrix}.$$

The Lie superalgebra $\mathfrak{g} = osp(1|2n)$ over $\mathbb{C}$ can be defined in terms of a Serre presentation with generators $\{E_i, F_i, H_i |\ 1 \le i \le n\}$ subject to the relations

$$[E_i, F_j] = \delta_{ij} H_i, \qquad [H_i, H_j] = 0, \qquad \forall i, j,$$
$$[H_i, E_j] = (\alpha_i, \alpha_j) E_j, \qquad [H_i, F_j] = -(\alpha_i, \alpha_j) F_j, \qquad \forall i, j,$$
$$(\text{ad } E_i)^{1-a_{ij}} E_j = 0, \qquad (\text{ad } F_i)^{1-a_{ij}} F_j = 0, \qquad i \neq j, \tag{3.2}$$

where $[\cdot, \cdot]$ represents the $\mathbb{Z}_2$-graded Lie bracket and $(\text{ad } a)b = [a, b]$. The $\mathbb{Z}_2$-grading of the generators is

$$[E_i] = [F_i] = [H_j] = 0, \qquad [E_n] = [F_n] = 1, \qquad 1 \le i \le n-1, \qquad 1 \le j \le n.$$

The universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ is a unital associative $\mathbb{Z}_2$-graded algebra, which may be considered as being generated by $\{E_i, F_i, H_i|\ 1 \le i \le n\}$ subject to relations that are formally the same as (3.2) but with the bracket $[\cdot,\cdot]$ interpreted as a $\mathbb{Z}_2$-graded commutator $[\cdot,\cdot] : U(\mathfrak{g}) \times U(\mathfrak{g}) \to U(\mathfrak{g})$ defined by

$$[X, Y] = XY - (-1)^{[X][Y]}YX. \tag{3.3}$$

If each of two elements $X, Y \in U(\mathfrak{g})$ has a grading, then the grading of $XY \in U(\mathfrak{g})$ is defined by

$$[XY] = \big([X] + [Y]\big) \bmod 2.$$

The graded commutator $[X, Y]$ of any two homogeneous elements of $U(\mathfrak{g})$ is defined by (3.3) and is extended to inhomogeneous elements of $U(\mathfrak{g})$ by linearity. Note that $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ has a natural $\mathbb{Z}_2$-graded associative algebra structure, with the grading defined for homogeneous $X, Y \in U(\mathfrak{g})$ by

$$[X \otimes Y] = \big([X] + [Y]\big) \bmod 2.$$

The universal enveloping algebra $U(\mathfrak{g})$ has the structures of a $\mathbb{Z}_2$-graded Hopf algebra. The co-multiplication Δ, co-unit ϵ and antipode S are defined on each generator $X \in \mathfrak{g} \hookrightarrow U(\mathfrak{g})$ by

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \qquad \epsilon(X) = 0, \qquad \epsilon(1) = 1, \qquad S(X) = -X.$$

The quantum superalgebra $U_q(\mathfrak{g})$ is some "q-deformation" of $U(\mathfrak{g})$. We describe its Jimbo version here.

Definition 3.1.1. *The quantum superalgebra $U_q(\mathfrak{g})$ over $\mathbb{C}$, in the sense of Jimbo, is an associative $\mathbb{Z}_2$-graded unital algebra generated by the elements $\{e_i, f_i, K_i, K_i^{-1}|\ 1 \le i \le n\}$ subject to the relations*

$$\begin{gathered}
[e_i, f_j] = \delta_{ij}\frac{K_i - K_i^{-1}}{q - q^{-1}}, \\
K_i e_j K_i^{-1} = q^{(\alpha_i,\alpha_j)} e_j, \qquad K_i f_j K_i^{-1} = q^{-(\alpha_i,\alpha_j)} f_j, \\
\big[K_i^{\pm 1}, K_j^{\pm 1}\big] = \big[K_i^{\pm 1}, K_j^{\mp 1}\big] = 0, \qquad K_i^{\pm 1}K_i^{\mp 1} = 1, \\
(ad_q e_i)^{1-a_{ij}} e_j = 0, \qquad (ad_q f_i)^{1-a_{ij}} f_j = 0, \qquad \forall i, j,
\end{gathered} \tag{3.4}$$

where $0 \ne q \in \mathbb{C}$ and $q^2 \ne 1$, and the adjoint actions are defined by

$$(ad_q e_i)X = e_i X - (-1)^{[e_i][X]} K_i X K_i^{-1} e_i,$$

$$(ad_q f_i)X = f_i X - (-1)^{[f_i][X]} K_i^{-1} X K_i f_i,$$

for all $X \in U_q(\mathfrak{g})$. In (3.4), the bracket $[\cdot,\cdot]$ is as defined in Eq. (3.3).

As is well known, there exists a $\mathbb{Z}_2$-graded Hopf algebra structure on $U_q(\mathfrak{g})$ with the co-multiplication Δ, the co-unit ϵ, and the antipode S defined on each generator by

$$\Delta(e_i) = e_i \otimes K_i + 1 \otimes e_i, \quad \Delta(f_i) = f_i \otimes 1 + K_i^{-1} \otimes f_i, \quad \Delta\left(K_i^{\pm 1}\right) = K_i^{\pm 1} \otimes K_i^{\pm 1}.$$

$$\epsilon(e_i) = \epsilon(f_i) = 0, \quad \epsilon\left(K_i^{\pm 1}\right) = \epsilon(1) = 1.$$

$$S(e_i) = -e_i K_i^{-1}, \quad S(f_i) = -K_i f_i, \quad S(K_i^{\pm 1}) = K_i^{\mp 1}.$$

There are a number of subalgebras of $U_q(\mathfrak{g})$ which will be quite useful. We define $U_q(\mathfrak{b}_+)$ to be the subalgebra of $U_q(\mathfrak{g})$ generated by $\left\{e_i, K_i^{\pm 1} |\ 1 \leq i \leq n\right\}$, and $U_q(\mathfrak{b}_-)$ to be the subalgebra of $U_q(\mathfrak{g})$ generated by $\left\{f_i, K_i^{\pm 1} |\ 1 \leq i \leq n\right\}$.

We note that the action of S^{-1} on each generator of $U_q(\mathfrak{g})$ is

$$S^{-1}(e_i) = -K_i^{-1} e_i, \qquad S^{-1}(f_i) = -f_i K_i, \qquad S^{-1}(K_i^{\pm 1}) = K_i^{\mp 1}.$$

Finally, for each $\beta = \sum_{i=1}^{n} m_i \alpha_i$ where $m_i \in \mathbb{Z}$, we define $K_\beta = \prod_{i=1}^{n} (K_i)^{m_i}$.

The quantum superalgebra $U_h(\mathfrak{g})$, in the sense of Drinfel'd, is a $\mathbb{Z}_2$-graded algebra over the ring $\mathbb{C}[[h]]$ for an indeterminate h, completed with respect to the h-adic topology [KT91]. The $\mathbb{Z}_2$-graded algebra $U_h(\mathfrak{g})$ is generated by $\{E_i, F_i, H_i |\ 1 \leq i \leq n\}$ subject to the relations

$$[E_i, F_j] = \delta_{ij} \frac{e^{hH_i} - e^{-hH_i}}{e^h - e^{-h}}, \qquad [H_i, H_j] = 0,$$

$$[H_i, E_j] = (\alpha_i, \alpha_j) E_j, \qquad [H_i, F_j] = -(\alpha_i, \alpha_j) F_j,$$

$$(ad_q E_i)^{1 - a_{ij}} E_j = 0, \qquad (ad_q F_i)^{1 - a_{ij}} F_j = 0, \qquad \forall i, j,$$

where the adjoint functions are defined by

$$(ad_q E_i) X = E_i X - (-1)^{[E_i][X]} e^{hH_i} X e^{-hH_i} E_i,$$

$$(ad_q F_i) X = F_i X - (-1)^{[F_i][X]} e^{-hH_i} X e^{hH_i} F_i,$$

where we fix $q = e^h$.

There is a $\mathbb{Z}_2$-graded Hopf algebra structure on $U_h(\mathfrak{g})$ with the co-multiplication $\Delta : U_h(\mathfrak{g}) \to U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g})$, the co-unit ϵ, and the antipode S defined on each generator by

$$\Delta(E_i) = E_i \otimes e^{hH_i} + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + e^{-hH_i} \otimes F_i, \quad \Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i.$$

$$\epsilon(E_i) = \epsilon(F_i) = \epsilon(H_i) = 0, \qquad \epsilon(1) = 1,$$

$$S(E_i) = -E_i e^{-hH_i}, \qquad S(F_i) = -e^{hH_i} F_i, \qquad S(H_i) = -H_i.$$

The quantum superalgebra $U_h(\mathfrak{g})$ admits a universal R-matrix [KT91] and is a $\mathbb{Z}_2$-graded ribbon Hopf algebra [ZG91].

3.2 Finite dimensional irreducible $U_q(osp(1|2n))$-modules

Throughout this section we assume that q is non-zero and not a root of unity. In this case the representation theory of $U_q(\mathfrak{g})$ is completely understood [Zh92b, Zo98].

For a $\mathbb{Z}_2$-graded algebra A, we will write V_λ in this thesis to denote an A-module labelled by $\lambda \in I$, for some index set I, and we write π_λ to denote the representation of A afforded by V_λ.

We say that an element $\lambda \in H^*$ is *integral* if

$$l_i = \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)} \in \mathbb{Z}, \quad \forall i < n, \qquad l_n = \frac{(\lambda, \alpha_n)}{(\alpha_n, \alpha_n)} \in \mathbb{Z},$$

where $(\cdot,\cdot) : H^* \times H^* \to \mathbb{C}$ is the bilinear form in (3.1). Let $\mathcal{P}$ denote the set of all integral elements of H^*. We say that an element $\lambda \in \mathcal{P}$ is *integral dominant* if $l_i \in \mathbb{Z}_+$ for all i. We denote the set of all integral dominant elements of H^* by $\mathcal{P}^+$.

We call a $U_q(\mathfrak{g})$-module V a *highest weight module* if there is a non-zero vector $v \in V$ satisfying

(i) $e_i v = 0, \quad 1 \le i \le n,$

(ii) $K_i v = \omega_i v, \quad \omega_i \in \mathbb{C}, \quad 1 \le i \le n,$

(iii) $V \subseteq U_q(\mathfrak{b}_-)v.$

The vector v is called a *highest weight vector* of V. Similarly, we call a $U_q(\mathfrak{g})$-module V a *lowest weight module* if there is a non-zero vector $w \in V$ satisfying

(i) $f_i w = 0, \quad 1 \le i \le n,$

(ii) $K_i w = \omega_i' w, \quad \omega_i' \in \mathbb{C}, \quad 1 \le i \le n,$

(iii) $V \subseteq U_q(\mathfrak{b}_+)w.$

In this case w is called a *lowest weight vector*.

Let V be a $U_q(\mathfrak{g})$-module on which all the K_i act semisimply. For each sequence $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ where $\beta_j \in \mathbb{C}$ for each j, define

$$V_\beta = \{x \in V |\ K_i x = \beta_i x,\ 1 \le i \le n\}.$$

If $V_\beta \ne 0$, we say that β is a weight of V, and that V_β is a weight space of V. The nonzero elements of V_β are called weight vectors.

Zou investigated the representation theory of the quantum superalgebra $U_q(\mathfrak{g})$ over the quotient field $\mathbb{C}(v)$ for an indeterminate v [Zo98], which is related to q via $q = v^2$. Zou's results can be adapted to our setting where we take q to be a transcendental number. Let $\sqrt{q}$ be any square root of the complex number q. Call a $U_q(\mathfrak{g})$-module V *integrable* if V is a direct sum of its weight spaces and if e_i and f_i act as locally nilpotent endomorphisms of V for each $i = 1, \ldots, n$.

Let $\overline{V}(\omega)$ be a highest weight $U_q(\mathfrak{g})$-module with highest weight vector v where $K_i v = \omega_i v$, $\omega_i \in \mathbb{C}$, for each $i = 1, \ldots, n$. The $U_q(\mathfrak{g})$-module $\overline{V}(\omega)$ has a unique maximal proper $U_q(\mathfrak{g})$-submodule $\overline{M}(\omega)$, and the quotient

$$V(\omega) = \overline{V}(\omega)/\overline{M}(\omega)$$

is an irreducible $U_q(\mathfrak{g})$-module with highest weight $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$. Theorem 3.1 of [Zo98] can be stated in our setting as follows.

Theorem 3.2.1. *The irreducible highest weight $U_q(\mathfrak{g})$-module $V(\omega)$, with $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$, is integrable if and only if*

$$\omega_i = \zeta_i q^{m_i}, \quad 1 \le i \le n-1,$$

where $m_i \in \mathbb{Z}_+$, $\zeta_i^2 = 1$, and

$$\omega_n = \begin{cases} \pm q^m, & \text{if } m \in \mathbb{Z}_+, \\ \pm\sqrt{-1}\ q^m, & \text{if } m \in \mathbb{Z}_+ + \frac{1}{2}. \end{cases}$$

Note that every finite dimensional integrable $U_q(\mathfrak{g})$-module is semisimple [Zo98, Sec. 5].

If $\omega = (q^{m_1}, q^{m_2}, \ldots, q^{m_n})$ with $m_i \in \mathbb{Z}_+$ for each i, there exists an irreducible $U(\mathfrak{g})$-module $V(\omega)_{\mathfrak{g}}$ with highest weight $\lambda \in \mathcal{P}^+$ satisfying $(\lambda, \alpha_i) = m_i$ for each i, and $V(\omega)$ and $V(\omega)_{\mathfrak{g}}$ have the same weight space decomposition [Zh92b]. In this thesis, we are mostly interested in these irreducible $U_q(\mathfrak{g})$-modules, and for $\lambda \in \mathcal{P}^+$ we let V_λ denote $V(\omega)$ and call λ the highest weight of V_λ.

The following lemma is from [Zh92b, Thms. 2.2, 3.3–3.5].

Lemma 3.2.1. *Let V_μ and V_ν be finite dimensional irreducible $U_q(\mathfrak{g})$-modules with highest weights μ and ν respectively, where $\mu, \nu \in \mathcal{P}^+$. Then $V_\mu \otimes V_\nu$ can be decomposed into a direct sum of irreducible $U_q(\mathfrak{g})$-modules V_λ with highest weights $\lambda \in \mathcal{P}^+$.*

Throughout this thesis we always take the grading of the highest weight vector of the finite dimensional irreducible $U_q(\mathfrak{g})$-module V_λ with integral dominant highest weight $\lambda = \sum_{i=1}^n \lambda_i \epsilon_i \in \mathcal{P}^+$ to be $\begin{cases} \text{even}, & \text{if } (\sum_{i=1}^n \lambda_i) \bmod 2 = 0, \\ \text{odd}, & \text{if } (\sum_{i=1}^n \lambda_i) \bmod 2 = 1. \end{cases}$

In the next lemma adapted from [LZ99] we consider the *fundamental* $U_q(\mathfrak{g})$-module V which will play a very important role in this thesis. This lemma is proved by elementary calculations.

Lemma 3.2.2. *There exists a $(2n+1)$-dimensional irreducible $U_q(osp(1|2n))$-module $V = V_{\epsilon_1}$ with highest weight ϵ_1. This module admits a basis $\{v_i | -n \le i \le n\}$ with v_1 being the highest weight vector. The actions of the generators of $U_q(osp(1|2n))$ on the basis elements are*

$$f_i v_i = v_{i+1}, \quad f_n v_n = v_0, \quad f_n v_0 = v_{-n}, \quad f_i v_{-i-1} = v_{-i},$$
$$e_i v_{i+1} = v_i, \quad e_n v_0 = v_n, \quad e_n v_{-n} = -v_0, \quad e_i v_{-i} = v_{-i-1},$$
$$K_j^{\pm 1} v_k = q^{\pm(\alpha_j, \epsilon_k)} v_k,$$

where $1 \le i < n$, $1 \le j \le n$, $-n \le k \le n$, $\epsilon_0 = 0$, and $\epsilon_{-i} = -\epsilon_i$. All remaining actions are zero.

Note that the highest weight vector v_1 of the fundamental irreducible $U_q(osp(1|2n))$-module V always has an odd grading in this thesis.

Proposition 3.2.1. *There exists a $U_q(\mathfrak{g})$-invariant, non-degenerate bilinear form $\langle\langle\ ,\ \rangle\rangle : V \times V \to \mathbb{C}$. Thus the dual $U_q(\mathfrak{g})$-module of V is isomorphic to V.*

Proof. Let $\{v_i| \ -n \leq i \leq n\}$ be the basis of V given in Lemma 3.2.2. Now define a non-degenerate $\mathbb{C}$-bilinear form

$$\langle\langle\ ,\ \rangle\rangle : V \times V \to \mathbb{C},$$

by

$$\begin{array}{rcll} \langle\langle v_1, v_{-1}\rangle\rangle &=& 1, & \\ \langle\langle v_i, v_{-i}\rangle\rangle &=& -q^{-1}\langle\langle v_{i-1}, v_{-(i-1)}\rangle\rangle, & 2 \leq i \leq n, \\ \langle\langle v_0, v_0\rangle\rangle &=& q^{-1}\langle\langle v_n, v_{-n}\rangle\rangle, & \\ \langle\langle v_{-n}, v_n\rangle\rangle &=& -\langle\langle v_0, v_0\rangle\rangle, & \\ \langle\langle v_{-j}, v_j\rangle\rangle &=& -q^{-1}\langle\langle v_{-(j+1)}, v_{j+1}\rangle\rangle, & 1 \leq j \leq n-1, \\ \langle\langle v_k, v_l\rangle\rangle &=& 0, & \text{if } k+l \neq 0. \end{array}$$

A direct calculation shows that

$$\langle\langle xv_i, v_j\rangle\rangle = (-1)^{[x][v_i]}\langle\langle v_i, S(x)v_j\rangle\rangle, \quad \forall x \in U_q(\mathfrak{g}), \quad v_i, v_j \in V,$$

thus proving the $U_q(\mathfrak{g})$-invariance of the bilinear form. This form identifies V with its dual module. □

Let us discuss in more detail the dual module of V. Recall the definition of the dual $U_q(\mathfrak{g})$-module V^* to V. Let $\{v_i^*| -n \leq i \leq n\}$ be a basis of V^* such that $\langle v_i^*, v_j\rangle = \delta_{ij}$ and $[v_i^*] = [v_i]$ where $\langle\ ,\ \rangle : V^* \times V \to \mathbb{C}$ is the dual space pairing. Now define a homogeneous bijection $T \in \mathrm{Hom}_{\mathbb{C}}(V, V^*)$ of degree 0 by

$$T : v_i \mapsto (-1)^{i-1}q^{-(i-1)}v_{-i}^*, \quad v_0 \mapsto (-1)^{n-1}q^{-n}v_0^*, \quad v_{-i} \mapsto (-1)^i q^{-(2n-i)}v_i^*, \quad 1 \leq i \leq n. \tag{3.5}$$

A direct calculation shows that this map is an element of $\mathrm{Hom}_{U_q(\mathfrak{g})}(V, V^*)$ and that it satisfies

$$\langle T(v_i), v_j\rangle = \langle\langle v_i, v_j\rangle\rangle, \qquad \text{for all } v_i, v_j \in V.$$

3.3 R-matrices for representations of $U_q(osp(1|2n))$

Drinfel'd's quantum superalgebra $U_h(\mathfrak{g})$ has a universal R-matrix [KT91]. We will show that there does not exist an element of $U_q(\mathfrak{g})$ over $\mathbb{C}$ that corresponds to the universal R-matrix of $U_h(\mathfrak{g})$ in an obvious way. However, there is a completion $\overline{U_q^+}(\mathfrak{g})$ of $U_q(\mathfrak{g})$ such that one of the multiplicative factors of the universal R-matrix of $U_h(\mathfrak{g})$ maps to an element $\widetilde{R}$ of $\overline{U_q^+}(\mathfrak{g})$. Although $\widetilde{R}$ is not an element of $U_q(\mathfrak{g})$, only a finite number of terms of $\widetilde{R}$ act as non-zero endomorphisms on each tensor product of finite dimensional irreducible $U_q(\mathfrak{g})$-modules, and thus the action of $\widetilde{R}$ on these tensor products is well-defined.

3.3.1 The universal R-matrix of $U_h(osp(1|2n))$

The Drinfel'd quantum algebras admit universal R-matrices [LS90, KR90], and the explicit expressions of these universal R-matrices employ infinite sums of root vectors in the quantum algebra corresponding to the positive roots of the associated Lie algebra [CP94].

The simple root vectors in the quantum algebra are just the generators e_i and f_i, and the non-simple root vectors are obtained by applying Lusztig's automorphisms to the generators [Lu90, CP94]. The non-simple root vectors are not uniquely defined in general [DeCK90, CP94]. Different choices for the decomposition of the longest element of the Weyl group of the associated Lie algebra into a product of reflections generated by the simple roots may lead to non-simple root vectors that may not even be proportional to each other, and *a priori* there is no canonical choice for the decomposition of the longest element of the Weyl group into a product of such reflections [CP94].

Khoroshkin and Tolstoy wrote down the universal R-matrix of $U_h(osp(1|2n))$ [KT91] using a different method. They employed infinite sums of root vectors in $U_h(\mathfrak{g})$, but these root vectors were defined in a different way to how the root vectors in quantum algebras were defined. Khoroshkin and Tolstoy's procedure is general for quantum superalgebras and we write it down for $U_h(\mathfrak{g})$ here.

In $U_h(\mathfrak{g})$, root vectors are only defined for the elements of the *reduced root system* ϕ of $\mathfrak{g}$. The reduced root system ϕ of $\mathfrak{g}$ is the set of all positive roots of $\mathfrak{g}$ except those roots α for which $\alpha/2$ is also a positive root, and the reduced root system of $\mathfrak{g} = osp(1|2n)$ is just $\phi = \overline{\Phi}_0^+ \cup \Phi_1^+$.

A total ordering of ϕ called a *normal ordering* is then introduced, and the root vectors of $U_h(\mathfrak{g})$ are recursively defined using the normal ordering of ϕ and a map involving the q-bracket that we introduce below. We denote a normal ordering of ϕ by $\mathcal{N}(\phi)$. A difference between the root vectors in quantum algebras and the root vectors in $U_q(\mathfrak{g})$ is that the latter are defined by a map that is not necessarily an algebra automorphism. The way the universal R-matrix of $U_h(\mathfrak{g})$ is formally written down depends on $\mathcal{N}(\phi)$.

A normal ordering of a reduced root system of $\mathfrak{g}$ is defined as follows [KT91, Def. 3.1].

Definition 3.3.1. *A normal ordering $\mathcal{N}(\phi)$ of $\phi = \overline{\Phi}_0^+ \cup \Phi_1^+$ is a total order $\prec$ of the set ϕ such that if $\alpha \prec \beta$ and $\alpha + \beta \in \phi$, then $\alpha \prec \alpha + \beta \prec \beta$.*

In general, there is more than one normal ordering of ϕ [KT91]. For example, the reduced root system of $osp(1|4)$ is $\phi = \{\epsilon_1, \epsilon_2, \epsilon_1 \pm \epsilon_2\}$ and there are two different normal orderings of ϕ:

$$\alpha_1 \prec \alpha_1 + \alpha_2 \prec \alpha_1 + 2\alpha_2 \prec \alpha_2,$$

$$\alpha_2 \prec \alpha_1 + 2\alpha_2 \prec \alpha_1 + \alpha_2 \prec \alpha_1,$$

where we write the elements of ϕ as sums of the simple roots.

We now write down the universal R-matrix of $U_h(osp(1|2n))$ [KT91], which we adapt slightly to take account of the different co-multiplication used in this thesis. Writing $q = e^h \in \mathbb{C}[[h]]$, let us firstly define $\exp_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]^q!}$.

We now construct the root vectors in $U_h(osp(1|2n))$. The easiest root vectors are the simple root vectors: we fix $E_{\alpha_i} = E_i$, $F_{\alpha_i} = F_i$ and $H_{\alpha_i} = H_i$ for each simple root α_i. Now we recursively construct the non-simple root vectors. Let $\alpha, \beta, \gamma \in \phi$ be roots such that $\gamma = \alpha + \beta$ and $\alpha \prec \beta$, and in addition let no other roots $\alpha', \beta' \in \phi$ exist which satisfy (i) $\alpha' + \beta' = \gamma$, (ii) $\alpha \prec \alpha' \prec \beta$ and (iii) $\alpha \prec \beta' \prec \beta$. Then, if all of the root vectors $E_\alpha, E_\beta, F_\alpha, F_\beta \in U_h(\mathfrak{g})$ have already been defined, we define

$$E_\gamma = [E_\alpha, E_\beta]_q\,, \qquad F_\gamma = [F_\beta, F_\alpha]_{q^{-1}}\,,$$

where the q-bracket $[\cdot,\cdot]_q$ is defined by

$$[X_\alpha, X_\beta]_q = X_\alpha X_\beta - (-1)^{[X_\alpha][X_\beta]} q^{(\alpha,\beta)} X_\beta X_\alpha,$$

where X stands for E or F.

For each $\gamma \in \phi$, set

$$R_\gamma = \exp_{q_\gamma}\left((-1)^{[\gamma]}(a_\gamma)^{-1}(q - q^{-1}) E_\gamma \otimes F_\gamma\right) \in U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g}), \tag{3.6}$$

where $q_\gamma = (-1)^{[\gamma]} q^{-(\gamma,\gamma)}$ and $a_\gamma \in \mathbb{C}[[h]]$ is defined by

$$E_\gamma F_\gamma - (-1)^{[E_\gamma]} F_\gamma E_\gamma = \frac{a_\gamma\left(q^{H_\gamma} - q^{-H_\gamma}\right)}{q - q^{-1}}.$$

It is important to observe that a_γ is a rational function of q. Now we can write down the universal R-matrix of $U_h(\mathfrak{g})$ [KT91, Thm. 8.1].

Theorem 3.3.1. *Define* $\mathbf{H}_i = \sum_{j=i}^n H_j \in U_h(\mathfrak{g})$ *for each* $i = 1, \ldots, n$*, then*

$$R = \exp\left(h \sum_{i=1}^n \mathbf{H}_i \otimes \mathbf{H}_i\right) \prod_{\gamma \in \phi} R_\gamma, \tag{3.7}$$

is the universal R-matrix of $U_h(\mathfrak{g})$*, where the product is ordered with respect to the same normal ordering* $\mathcal{N}(\phi)$ *used to define the root vectors in* $U_h(\mathfrak{g})$ *so that* $\prod_{\gamma\in\phi} R_\gamma = R_{\gamma_1} R_{\gamma_2} \cdots R_{\gamma_k}$ *where* $\gamma_1 \prec \gamma_2 \prec \cdots \prec \gamma_k$ *in* $\mathcal{N}(\phi)$*.*

3.3.2 R-matrices for representations of $U_q(osp(1|2n))$

It is unknown whether Jimbo's quantum algebras over $\mathbb{C}$ have universal R-matrices. However, there exist *R-matrices for representations* of these quantum algebras. Let π_λ and π_μ be finite dimensional irreducible representations of the quantum algebra A, then there is an invertible element $\mathcal{R}_{\lambda,\mu} \in End_{\mathbb{C}}(V_\lambda \otimes V_\mu)$ satisfying

$$\mathcal{R}_{\lambda,\mu} \cdot (\pi_\lambda \otimes \pi_\mu)\big(\Delta(x)\big) = (\pi_\lambda \otimes \pi_\mu)\big(\Delta'(x)\big) \cdot \mathcal{R}_{\lambda,\mu} \qquad \forall x \in A, \tag{3.8}$$

which we note is just Eq. (2.11) in $\pi_\lambda \otimes \pi_\mu$. We follow a similar scheme for $U_q(osp(1|2n))$.

For each pair of finite dimensional irreducible $U_q(\mathfrak{g})$-modules, we will construct an element $\mathcal{R}_{\lambda,\mu} \in End_{\mathbb{C}}(V_\lambda \otimes V_\mu)$ satisfying (3.8) for all $x \in U_q(\mathfrak{g})$. We do this following the method laid out in [CP94, KS97] for quantum algebras.

We firstly define a completion $\overline{U}_q^+(\mathfrak{g})$ of $U_q(\mathfrak{g})$ following [KS97, Subsec. 6.3.3]. Let $U_q(\mathfrak{n}_+)$ (resp. $U_q(\mathfrak{n}_-)$) be the subalgebra of $U_q(\mathfrak{g})$ generated by the elements $\{e_i|\ 1 \leq i \leq n\}$ (resp. $\{f_i|\ 1 \leq i \leq n\}$). We say that a non-zero element $x \in U_q(\mathfrak{g})$ has degree $\lambda = \sum_{i=1}^n m_i\alpha_i,\ m_i \in \mathbb{Z}$, if $K_i x K_i^{-1} = q^{(\lambda,\alpha_i)}x$ for all $i = 1, 2, \ldots, n$. We define $\overline{U}_q^\pm(\mathfrak{g})$ by

$$\overline{U}_q^\pm(\mathfrak{g}) = \prod_{\beta \in Q_+} U_q(\mathfrak{b}_\pm) U_q^{\mp\beta}(\mathfrak{n}_\mp),$$

where $U_q^{\mp\beta}(\mathfrak{n}_\mp)$ is defined by

$$U_q^{\pm\beta}(\mathfrak{n}_\pm) = \{x \in U_q(\mathfrak{n}_\pm)|\ K_i x K_i^{-1} = q^{\pm(\alpha_i,\beta)}x\}, \qquad i = 1, 2, \ldots, n,$$

and Q_+ is defined by $Q_+ = \{\sum_{i=1}^n n_i\alpha_i|\ n_i \in \mathbb{Z}_+\}$.

The elements of $\overline{U}_q^\pm(\mathfrak{g})$ are sequences $x = (x_\beta)_{\beta\in Q_+}$ where $x_\beta \in U_q(\mathfrak{b}_\pm)U_q^{\mp\beta}(\mathfrak{n}_\mp)$. Let us write this sequence formally as an infinite sum $x = \sum_\beta x_\beta$. The results of [KS97, Subsec. 6.1.5]) imply that $U_q(\mathfrak{g})$ can be expressed as a direct sum

$$U_q(\mathfrak{g}) = \bigoplus_{\beta\in Q_+} U_q(\mathfrak{b}_+)U_q^{-\beta}(\mathfrak{n}_-),$$

and thus $U_q(\mathfrak{g})$ can be considered as the subspace of $\overline{U}_q^+(\mathfrak{g})$ formed by the sums $x = \sum_\beta x_\beta$ for which all but finitely many terms vanish.

The multiplication in $U_q(\mathfrak{g})$ extends to a multiplication in $\overline{U}_q^\pm(\mathfrak{g})$. Let $\beta, \gamma \in Q_+$, let $x_\beta \in U_q^{-\beta}(\mathfrak{n}_-)$, and let $y_\gamma \in U_q(\mathfrak{b}_+)$ have degree γ. From the commutation relations of $U_q(\mathfrak{g})$, we have

$$x_\beta y_\gamma \in \bigoplus_\delta U_q(\mathfrak{b}_+)U_q^{-\delta}(\mathfrak{n}_-),$$

where the direct sum ranges over all $\delta \in Q_+$ satisfying $|\beta| - |\gamma| \leq |\delta| \leq |\beta|$, where $|\beta| = \sum_{i=1}^n m_i$ for $\beta = \sum_{i=1}^n m_i\alpha_i$. Thus $\overline{U}_q^+(\mathfrak{g})$ is an algebra, and similarly $\overline{U}_q^-(\mathfrak{g})$ and $\overline{U}_q^\pm(\mathfrak{g})\overline{\otimes}\cdots\overline{\otimes}\overline{U}_q^\pm(\mathfrak{g})$ are algebras. The algebras $\overline{U}_q^\pm(\mathfrak{g})$ and $\overline{U}_q^\pm(\mathfrak{g})\overline{\otimes}\cdots\overline{\otimes}\overline{U}_q^\pm(\mathfrak{g})$ (with m factors) contain $U_q(\mathfrak{g})$ and $U_q(\mathfrak{g})\otimes\cdots\otimes U_q(\mathfrak{g})$ (with m factors), respectively, as subalgebras.

We now construct an element in $\overline{U}_q^+(\mathfrak{g})$ corresponding to the element $\prod_{\gamma\in\phi} R_\gamma$ in (3.7). Given a normal ordering $\mathcal{N}(\phi)$ for a reduced root system ϕ, we construct root vectors $E_\gamma, F_\gamma \in U_q(\mathfrak{g})$ following the same procedure as in $U_h(\mathfrak{g})$ by setting $E_{\alpha_i} = e_i$ and $F_{\alpha_i} = f_i$ and thinking of q a complex number. Then R_γ is well-defined as an element of $\overline{U}_q^+(\mathfrak{g})\overline{\otimes}\overline{U}_q^+(\mathfrak{g})$, and to simplify R_γ we normalise the root vectors:

$$e_\gamma = E_\gamma, \qquad f_\gamma = F_\gamma/a_\gamma.$$

This is well-defined as $a_\gamma \neq 0$ [KT91, Eqs. (8.3)–(8.4)]. Simplifying the expression for R_γ, we have

$$R_\gamma = \begin{cases} \displaystyle\sum_{k=0}^{\infty} \frac{(q-q^{-1})^k (e_\gamma \otimes f_\gamma)^k}{[k]^{q^{-2}}!}, & \text{if } [e_\gamma]=0, \\ \displaystyle\sum_{k=0}^{\infty} \frac{(q^{-1}-q)^k (e_\gamma \otimes f_\gamma)^k}{[k]^{-q^{-1}}!}, & \text{if } [e_\gamma]=1. \end{cases}$$

Define $\widetilde{R} \in \overline{U}_q^+(\mathfrak{g}) \overline{\otimes} \overline{U}_q^+(\mathfrak{g})$ by $\widetilde{R} = \prod_{\gamma\in\phi} R_\gamma$ where the product is ordered using the same normal order $\mathcal{N}(\phi)$ that we used to define the root vectors in $U_q(\mathfrak{g})$ and such that $\prod_{\gamma\in\phi} R_\gamma = R_{\gamma_1} R_{\gamma_2} \cdots R_{\gamma_k}$ where $\mathcal{N}(\phi) = \gamma_1 \prec \gamma_2 \prec \cdots \prec \gamma_k$. Clearly $\widetilde{R}$ is invertible as $\prod_{\gamma\in\phi} R_\gamma \in U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g})$ is invertible and q is not a root of unity.

Lemma 3.3.1. *Define an automorphism Ψ of $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ by*

$$\begin{aligned} &\Psi(K_i^{\pm 1} \otimes 1) = K_i^{\pm 1} \otimes 1, && \Psi(1 \otimes K_i^{\pm 1}) = 1 \otimes K_i^{\pm 1}, \\ &\Psi(e_i \otimes 1) = e_i \otimes K_i^{-1}, && \Psi(1 \otimes e_i) = K_i^{-1} \otimes e_i, \\ &\Psi(f_i \otimes 1) = f_i \otimes K_i, && \Psi(1 \otimes f_i) = K_i \otimes f_i. \end{aligned}$$

The automorphism Ψ satisfies the following relations:

(i) $\widetilde{R}\Delta(x) = \Psi\big(\Delta'(x)\big) \cdot \widetilde{R}, \qquad$ *for all* $x \in U_q(\mathfrak{g})$,

(ii) $(\Delta \otimes \mathrm{id})\widetilde{R} = \Psi_{23}(\widetilde{R}_{13}) \cdot \widetilde{R}_{23}$,

(iii) $(\mathrm{id} \otimes \Delta)\widetilde{R} = \Psi_{12}(\widetilde{R}_{13}) \cdot \widetilde{R}_{12}$,

where $\Psi_{12} = \Psi \otimes \mathrm{id}$ *and* $\Psi_{23} = \mathrm{id} \otimes \Psi$.

Proof. We prove (i) for each generator of $U_q(\mathfrak{g})$. We firstly wish to prove the following equations:

$$\widetilde{R}(e_i \otimes K_i + 1 \otimes e_i) = (e_i \otimes K_i^{-1} + 1 \otimes e_i)\widetilde{R}, \tag{3.9}$$

$$\widetilde{R}(f_i \otimes 1 + K_i^{-1} \otimes f_i) = (f_i \otimes 1 + K_i \otimes f_i)\widetilde{R}, \tag{3.10}$$

$$\widetilde{R}(K_i^{\pm 1} \otimes K_i^{\pm 1}) = (K_i^{\pm 1} \otimes K_i^{\pm 1})\widetilde{R}. \tag{3.11}$$

Now Eq. (3.11) is true by inspection and Eqs. (3.9)–(3.10) follow from the corresponding results in $U_h(\mathfrak{g})$ [KT91, Prop. 6.2]. The proof of (i) then follows from the definition of Ψ and the proofs of (ii) and (iii) follow similarly from [KT91]. □

We now examine the usual approach used to create R-matrices for representations of a quantum algebra A. For each tensor product $W_1 \otimes W_2$ of finite dimensional integrable A-modules, an invertible element $\mathcal{E}_{W_1,W_2} \in End_{\mathbb{C}}(W_1 \otimes W_2)$ is constructed implementing the automorphism Ψ, in the sense that $\mathcal{E}_{W_1,W_2}$ satisfies

$$\mathcal{E}_{W_1,W_2}^{-1} \cdot (\pi_{W_1} \otimes \pi_{W_2})(x) \cdot \mathcal{E}_{W_1,W_2} = (\pi_{W_1} \otimes \pi_{W_2})\Psi(x), \quad \text{for all } x \in A \otimes A.$$

This $\mathcal{E}_{W_1,W_2}$ is defined by fixing its action to be

$$\mathcal{E}_{W_1,W_2}(w_\lambda \otimes w_\mu) = q^{(\lambda,\mu)}(w_\lambda \otimes w_\mu),$$

on all weight vectors $w_\lambda \in W_1$, $w_\mu \in W_2$ with weights λ and μ, respectively [CP94, Prop. 10.1.19].

We could have used the same method to construct R-matrices for representations of $U_q(osp(1|2n))$ but we have found a more useful approach. Note that in the above we need to know the weight space decompositions of both W_1 and W_2 before defining $\mathcal{E}_{W_1,W_2}$, but if we have this information we can use it in a more interesting way: instead of defining an element of $End_{\mathbb{C}}(W_1 \otimes W_2)$ we can define an element $E_{W_1,W_2} \in U_q(\mathfrak{g})$ with the property that $(\pi_{W_1} \otimes \pi_{W_2})E_{W_1,W_2} = \mathcal{E}_{W_1,W_2}$. We define this E_{W_1,W_2} for each tensor product of finite dimensional irreducible $U_q(\mathfrak{g})$-modules following a related idea in [Zh92a].

For each $i = 1, \ldots, n$, set

$$J_i = K_i K_{i+1} \cdots K_n.$$

The action of J_i on a weight vector w_ξ with weight $\xi = \sum_{j=1}^n \xi_j \epsilon_j \in H^*$ of a $U_q(\mathfrak{g})$-module is

$$J_i w_\xi = q^{\xi_i} w_\xi.$$

Consider the weight space decomposition of a finite dimensional irreducible $U_q(\mathfrak{g})$-module V_μ with integral dominant highest weight μ. The weight of the weight vector $w_\xi \in V_\mu$ is $\xi = \sum_{i=1}^n \xi_i \epsilon_i \in \bigoplus_{i=1}^n \mathbb{Z}\epsilon_i$. Now define

$$E_\mu = \prod_{a=1}^{n} \sum_{b=p}^{s} (J_a)^b \otimes P_a[b], \qquad P_a[b] = \prod_{\substack{c=p \\ c\neq b}}^{s} \frac{J_a - q^c}{q^b - q^c}, \quad c \in \mathbb{Z}, \tag{3.12}$$

where p and s are any integers satisfying $p \leq s$ and the following condition:

- $J_i w_\xi = q^{\xi_i} w_\xi$ for some ξ_i satisfying $p \leq \xi_i \leq s$, for each weight vector $w_\xi \in V_\mu$.

Once we have any such p and s, we can use any p' and s' satisfying $p' \leq p$ and $s' \geq s$ in (3.12) instead of p and s, respectively.

The element E_μ is well-defined and invertible in $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$, and for all weight vectors $v_{\lambda'} \in V_\lambda$ and $v_{\mu'} \in V_\mu$ we have

$$E_\mu(v_{\lambda'} \otimes v_{\mu'}) = q^{(\lambda',\mu')}\ (v_{\lambda'} \otimes v_{\mu'}), \tag{3.13}$$

where the weights of $v_{\lambda'}$ and $v_{\mu'}$ are λ' and μ', respectively. The element E_μ is not a *universal* element in that it does not satisfy (3.13) for all representations of $U_q(\mathfrak{g})$. It would be very useful if one could construct such a universal element in $U_q(\mathfrak{g})$.

From this we obtain R-matrices for tensor products of finite dimensional irreducible $U_q(\mathfrak{g})$-modules in the following sense. Let V_λ and V_μ be irreducible $U_q(\mathfrak{g})$-modules with integral dominant highest weights λ and μ, respectively. Then we have the following important theorem.

Theorem 3.3.2. *Define $R_{\lambda,\mu} = E_\mu \widetilde{R} \in \overline{U}_q^+(\mathfrak{g}) \overline{\otimes} \overline{U}_q^+(\mathfrak{g})$ and $\mathcal{R}_{\lambda,\mu} = (\pi_\lambda \otimes \pi_\mu) R_{\lambda,\mu}$, then*

$$\mathcal{R}_{\lambda,\mu} \cdot (\pi_\lambda \otimes \pi_\mu)\big(\Delta(x)\big) = (\pi_\lambda \otimes \pi_\mu)\big(\Delta'(x)\big) \cdot \mathcal{R}_{\lambda,\mu}, \qquad \forall x \in U_q(\mathfrak{g}). \tag{3.14}$$

Proof. This is similar to the proof of the corresponding theorem in quantum algebras [CP94, Prop. 10.1.19], and we follow that proof. By direct calculation we can readily show that

$$(\pi_\lambda \otimes \pi_\mu)\Psi\big(\Delta(x)\big) = (\pi_\lambda \otimes \pi_\mu)\big(E_\mu^{-1} \cdot \Delta(x) \cdot E_\mu\big), \qquad \forall x \in U_q(\mathfrak{g}),$$

then by using $\widetilde{R}\Delta(x) = \Psi\big(\Delta'(x)\big) \cdot \widetilde{R}$ from Lemma 3.3.1 we have

$$(\pi_\lambda \otimes \pi_\mu)\big(R_{\lambda,\mu}\Delta(x)\big) = (\pi_\lambda \otimes \pi_\mu)\big(\Delta'(x) R_{\lambda,\mu}\big), \tag{3.15}$$

which is precisely Eq. (3.14). □

A similar calculation shows that

$$(\pi_\lambda \otimes \pi_\mu)\big(R^T_{\mu,\lambda}\Delta'(x)\big) = (\pi_\lambda \otimes \pi_\mu)\big(\Delta(x) R^T_{\mu,\lambda}\big), \qquad \forall x \in U_q(\mathfrak{g}).$$

We now determine some useful results involving $R_{\lambda,\mu}$.

Proposition 3.3.1. *The element $R_{\lambda,\mu}$ has the following properties:*

$$(\epsilon \otimes \mathrm{id}) R_{\lambda,\mu} = (\mathrm{id} \otimes \epsilon) R_{\lambda,\mu} = 1, \tag{3.16}$$

$$(\pi_\lambda \otimes \pi_\mu)\big((S \otimes \mathrm{id}) R_{\lambda,\mu}\big) = (\pi_\lambda \otimes \pi_\mu) R^{-1}_{\lambda,\mu}, \quad (\pi_\lambda \otimes \pi_\mu)\big((\mathrm{id} \otimes S) R^{-1}_{\lambda,\mu}\big) = R_{\lambda,\mu}, \tag{3.17}$$

$$(\pi_\lambda \otimes \pi_\mu)\big((S \otimes S) R_{\lambda,\mu}\big) = (\pi_\lambda \otimes \pi_\mu) R_{\lambda,\mu}. \tag{3.18}$$

Proof. Eq. (3.16) is proved by inspection. The proofs of (3.17)–(3.18) are straightforward and almost identical to the proofs of the corresponding equations in $\mathbb{Z}_2$-graded quasitriangular Hopf algebras. □

Let v_λ and v_ν be weight vectors of $U_q(\mathfrak{g})$-modules with weights λ, $\nu \in \bigoplus_{i=1}^n \mathbb{Z}\epsilon_i$, respectively and let $v_{\mu'} \in V_\mu$ be a weight vector with weight μ', then it can be easily shown that

$$\big[(\Delta \otimes 1)E_\mu\big](v_\lambda \otimes v_\nu \otimes v_{\mu'}) = q^{(\mu',\lambda+\nu)}(v_\lambda \otimes v_\nu \otimes v_{\mu'}),$$
$$\big[(1 \otimes \Delta)E_\mu\big](v_\lambda \otimes v_\nu \otimes v_{\mu'}) = q^{(\lambda,\nu+\mu')}(v_\lambda \otimes v_\nu \otimes v_{\mu'}),$$

where in $(1 \otimes \Delta)E_\mu$ we change the limits p and s if necessary.

We now consider analogues in $U_q(\mathfrak{g})$ of the equations $(\Delta\otimes 1)R = R_{13}R_{23}$ and $(1\otimes\Delta)R = R_{13}R_{12}$ of a $\mathbb{Z}_2$-graded quasitriangular Hopf algebra. By definition, we have $(\Delta \otimes 1)R_{\lambda,\mu} = \big[(\Delta\otimes 1)E_\mu\big] \cdot \Psi_{23}(\widetilde{R}_{13}) \cdot \widetilde{R}_{23}$. Let V_λ and V_ν be finite dimensional irreducible $U_q(\mathfrak{g})$-modules, then from the properties of E_μ we have

$$\begin{aligned}(\pi_\lambda \otimes \pi_\nu \otimes \pi_\mu)\big[(\Delta \otimes 1)R_{\lambda,\mu}\big] &= (\pi_\lambda \otimes \pi_\nu \otimes \pi_\mu)\left[(\Delta \otimes 1)E_\mu \cdot (E_\mu^{23})^{-1}\widetilde{R}_{13}E_\mu^{23}\widetilde{R}_{23}\right] \\ &= (\pi_\lambda \otimes \pi_\nu \otimes \pi_\mu)\left[E_\mu^{13}\widetilde{R}_{13}E_\mu^{23}\widetilde{R}_{23}\right],\end{aligned} \tag{3.19}$$

where writing $E_\mu = \sum_t \alpha_t \otimes \beta_t$ we have $E_\mu^{13} = \sum_t \alpha_t \otimes \mathrm{id} \otimes \beta_t$ and $E_\mu^{23} = \sum_t \mathrm{id} \otimes \alpha_t \otimes \beta_t$. Note that (3.19) uses the result

$$(\pi_\lambda \otimes \pi_\nu \otimes \pi_\mu)\left[(\Delta \otimes 1)E_\mu \cdot (E_\mu^{23})^{-1}\right] = (\pi_\lambda \otimes \pi_\nu \otimes \pi_\mu)E_\mu^{13},$$

rather than an equality in $U_q(\mathfrak{g})^{\otimes 3}$. Similarly, we have

$$(1 \otimes \Delta)R_{\lambda,\mu} = [(1 \otimes \Delta)E_\mu] \cdot \Psi_{12}(\widetilde{R}_{13}) \cdot \widetilde{R}_{12},$$

and

$$(\pi_\lambda \otimes \pi_\nu \otimes \pi_\mu)\left[(1 \otimes \Delta)R_{\lambda,\mu}\right] = (\pi_\lambda \otimes \pi_\nu \otimes \pi_\mu)\left[E_\mu^{13}\widetilde{R}_{13}E_\nu^{12}\widetilde{R}_{12}\right]. \tag{3.20}$$

Together with Theorem 3.3.2, this shows that $R_{\lambda,\mu}$ satisfies the defining relations (2.11)–(2.13) of the universal R-matrix of a $\mathbb{Z}_2$-graded quasitriangular Hopf algebra in *each triple tensor product* of finite dimensional irreducible $U_q(\mathfrak{g})$ representations, if one carefully chooses the limits in the definition of E_μ (which can always be done). Furthermore, Eqs. (3.15) and (3.19)–(3.20) imply that

$$(\pi_\lambda \otimes \pi_\lambda \otimes \pi_\lambda)R_{12}R_{13}R_{23} = (\pi_\lambda \otimes \pi_\lambda \otimes \pi_\lambda)R_{23}R_{13}R_{12}, \tag{3.21}$$

where we have fixed $R = R_{\lambda,\lambda}$.

For later use we note the following easily proved results. Define an automorphism $\Psi_m : U_q(\mathfrak{g})^{\otimes m} \to U_q(\mathfrak{g})^{\otimes m}$ generalising the automorphism $\Psi : U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ in Lemma 3.3.1 by

$$\begin{aligned}
\Psi_m(1^{\otimes j} \otimes K_i^{\pm 1} \otimes 1^{\otimes(m-j-1)}) &= 1^{\otimes j} \otimes K_i^{\pm 1} \otimes 1^{\otimes(m-j-1)}, \\
\Psi_m(1^{\otimes j} \otimes e_i \otimes 1^{\otimes(m-j-1)}) &= (K_i^{-1})^{\otimes j} \otimes e_i \otimes (K_i^{-1})^{\otimes(m-j-1)}, \\
\Psi_m(1^{\otimes j} \otimes f_i \otimes 1^{\otimes(m-j-1)}) &= (K_i)^{\otimes j} \otimes f_i \otimes (K_i)^{\otimes(m-j-1)},
\end{aligned}$$

for each $1 \leq i \leq n$ and all $0 \leq j \leq m-1$. Then

$$\big(\Delta \otimes \mathrm{id}^{\otimes t}\big)\Psi_{2,3,\ldots,t+1}(\widetilde{R}_{1(t+1)}) = \Psi_{2,3,\ldots,t+2}(\widetilde{R}_{1(t+2)}) \cdot \Psi_{3,4,\ldots,t+2}(\widetilde{R}_{2(t+2)}), \tag{3.22}$$

$$\big(\mathrm{id}^{\otimes t} \otimes \Delta\big)\Psi_{1,2,\ldots,t}(\widetilde{R}_{1(t+1)}) = \Psi_{1,2,\ldots,t+1}(\widetilde{R}_{1(t+2)}) \cdot \Psi_{1,2,\ldots,t}(\widetilde{R}_{1(t+1)}), \tag{3.23}$$

where $\Psi_{k,\ldots,m} = \mathrm{id}^{\otimes(k-1)} \otimes \Psi_{m-k+1}$, $k \geq 2$, in (3.22) and $\Psi_{1,\ldots,m} = \Psi_m \otimes \mathrm{id}$ in (3.23). Then it may be easily shown that

$$\begin{aligned}
\left(\pi^{\otimes t} \otimes \pi\right)\left(\Delta^{(t-1)} \otimes \mathrm{id}\right)R &= \left(\pi^{\otimes t} \otimes \pi\right) R_{1(t+1)}R_{2(t+1)} \cdots R_{t(t+1)}, \\
\left(\pi \otimes \pi^{\otimes t}\right)\left(\mathrm{id} \otimes \Delta^{(t-1)}\right)R &= \left(\pi \otimes \pi^{\otimes t}\right) R_{1(t+1)}R_{1t} \cdots R_{12},
\end{aligned}$$

where we fix $R = R_{V,V}$.

3.4 Useful elements of $\overline{U}_q^+(\mathfrak{g})$

Define a set of elements $\{u_\lambda \in \overline{U}_q^+(\mathfrak{g}) |\ \lambda \in \mathcal{P}^+\}$ by $u_\lambda = \sum_t S(b_{\lambda_t}) a_{\lambda_t} (-1)^{[a_{\lambda_t}]}$ upon writing $R_{\lambda,\lambda} = \sum_t a_{\lambda_t} \otimes b_{\lambda_t}$.

Lemma 3.4.1. *The element u_λ has the following properties:*

(i) $\epsilon(u_\lambda) = 1$,

(ii) $\pi_\lambda\big(S^2(x) u_\lambda\big) = \pi_\lambda(u_\lambda x)$, $\forall x \in U_q(\mathfrak{g})$.

(iii) $\pi_\lambda\big(u_\lambda \widetilde{u}_\lambda\big) = \pi_\lambda(1) = \pi_\lambda\big(\widetilde{u}_\lambda u_\lambda\big)$, *where $\widetilde{u}_\lambda$ is defined by* $\widetilde{u}_\lambda = \sum_s S^{-1}(d_{\lambda_s}) c_{\lambda_s} (-1)^{[c_{\lambda_s}]}$, *where* $R_{\lambda,\lambda}^{-1} = \sum_s c_{\lambda_s} \otimes d_{\lambda_s}$,

(iv) $(\pi_\lambda \otimes \pi_\lambda)\big(\Delta(u_\lambda)\big) = (\pi_\lambda \otimes \pi_\lambda)\Big[(u_\lambda \otimes u_\lambda)\big(R_{\lambda,\lambda}^T R_{\lambda,\lambda}\big)^{-1}\Big]$,

where π_λ is the representation of $U_q(\mathfrak{g})$ afforded by the finite dimensional irreducible $U_q(\mathfrak{g})$-module V_λ.

Proof. Part (i) is proved is by inspection. To prove part (ii), $u \in U_h(\mathfrak{g})$ can be written as $u = \sum_t c_t S(F_t) q^{-(\sum_{i=1}^n \mathbf{H}_i^2)} E_t (-1)^{[E_t]}$, where the universal R-matrix is written as $R = q^{\sum_{i=1}^n \mathbf{H}_i \otimes \mathbf{H}_i} \sum_t c_t E_t \otimes F_t$. In $\overline{U}_q^+(\mathfrak{g})$,

$$u_\lambda = \sum_{t=0}^{\infty} c_t S(F_t) \left(\prod_{a=1}^{n} \sum_{b=p}^{s} P_a[b] (J_a)^{-b} \right) E_t (-1)^{[F_t]}.$$

The proof follows by noting that $\prod_{a=1}^n \sum_{b=p}^s P_a[b](J_a)^{-b}$ 'implements' the action of $q^{-(\sum_{i=1}^t \mathbf{H}_i^2)}$ in the $U_q(\mathfrak{g})$ representation π_λ. Part (iii) is proved by formally undertaking the proof of $u\widetilde{u} = 1$ in Lemma 2.3.4 for $\mathbb{Z}_2$-graded quasitriangular Hopf algebras but using the expression for $R_{\lambda,\lambda}$ instead of R and also using (ii) above.

The proof of part (iv) is similar to the proof of $\Delta(u) = (u \otimes u)\left(R^T R\right)^{-1}$ in $\mathbb{Z}_2$-graded quasitriangular Hopf algebras, but much of the calculation is done in the representations of $U_q(\mathfrak{g})$ afforded by tensor products of finite dimensional irreducible $U_q(\mathfrak{g})$-modules. Although the proof can be understood by following [ZG91, Lem. 1], we give it here for completeness. Firstly $(\pi_\lambda \otimes \pi_\lambda) R_{\lambda,\lambda}^T R_{\lambda,\lambda} \in End_{U_q(\mathfrak{g})}(V_\lambda \otimes V_\lambda)$, thus

$$(\pi_\lambda \otimes \pi_\lambda)\Big[R_{\lambda,\lambda}^T R_{\lambda,\lambda} \Delta(u_\lambda)\Big] = (\pi_\lambda \otimes \pi_\lambda) \sum_t (S \otimes S)\Delta'(b_t) \cdot R_{\lambda,\lambda}^T R_{\lambda,\lambda} \cdot \Delta(a_t)(-1)^{[a_t]}.$$

Introduce the operation

$$\begin{aligned}
&(x_1 \otimes x_2) \circ (a_1 \otimes a_2 \otimes a_3 \otimes a_4) \\
&= S(a_3) x_1 a_1 \otimes S(a_4) x_2 a_2 (-1)^{[x_2][a_1] + ([a_3]+[a_4])([x_1]+[x_2]+[a_1]+[a_2]) + [a_4]([x_1]+[a_1])},
\end{aligned}$$

where x_1, x_2, a_i, $i = 1, 2, 3, 4$, are homogeneous elements of $U_q(\mathfrak{g})$. Straightforward calculations show that

$$(x_1 \otimes x_2) \circ (bc) = [(x_1 \otimes x_2) \circ b] \circ c, \qquad \forall b, c \in U_q(\mathfrak{g})^{\otimes 4},$$

and that

$$\Delta(u_\lambda) R^T_{\lambda,\lambda} R_{\lambda,\lambda} = R^{21}_{\lambda,\lambda} \circ \left[R^{12}_{\lambda,\lambda} (\Delta \otimes \Delta') R_{\lambda,\lambda} \right].$$

By using Eqs. (3.19)–(3.21) we obtain

$$\begin{aligned}(\pi_\lambda \otimes \pi_\lambda)\Big[\Delta(u_\lambda) R^T_{\lambda,\lambda} R_{\lambda,\lambda}\Big] &= (\pi_\lambda \otimes \pi_\lambda)\left(R^{21}_{\lambda,\lambda} \circ \left[R^{12}_{\lambda,\lambda} R^{13}_{\lambda,\lambda} R^{23}_{\lambda,\lambda} R^{14}_{\lambda,\lambda} R^{24}_{\lambda,\lambda}\right]\right), \\ &= (\pi_\lambda \otimes \pi_\lambda)\left(R^{21}_{\lambda,\lambda} \circ \left[R^{23}_{\lambda,\lambda} R^{13}_{\lambda,\lambda} R^{12}_{\lambda,\lambda} R^{14}_{\lambda,\lambda} R^{24}_{\lambda,\lambda}\right]\right).\end{aligned}$$

Straightforward calculations then show that

$$(\pi_\lambda \otimes \pi_\lambda)\left(R^{21}_{\lambda,\lambda} \circ R^{23}_{\lambda,\lambda}\right) = (\pi_\lambda \otimes \pi_\lambda)(1 \otimes 1), \qquad (1 \otimes 1) \circ R^{13}_{\lambda,\lambda} = u_\lambda \otimes 1, \quad \text{and}$$

$$(\pi_\lambda \otimes \pi_\lambda)\left[(u_\lambda \otimes 1) \circ (R^{12}_{\lambda,\lambda} R^{14}_{\lambda,\lambda})\right] = (\pi_\lambda \otimes \pi_\lambda)(u_\lambda \otimes 1),$$

and thus it follows that

$$(\pi_\lambda \otimes \pi_\lambda)\Big[\Delta(u_\lambda) R^T_{\lambda,\lambda} R_{\lambda,\lambda}\Big] = (\pi_\lambda \otimes \pi_\lambda)\left[(u \otimes 1) \circ R^{24}_{\lambda,\lambda}\right] = (\pi_\lambda \otimes \pi_\lambda)(u_\lambda \otimes u_\lambda),$$

completing the proof.

□

Define a set of elements $\{v_\lambda \in \overline{U}_q^+(\mathfrak{g}) |\ \lambda \in \mathcal{P}^+\}$ by

$$v_\lambda = u_\lambda K^{-1}_{2\rho}. \tag{3.24}$$

Lemma 3.4.2. *The element v_λ has the following properties:*

$$\epsilon(v_\lambda) = 1, \qquad \pi_\lambda(v_\lambda x) = \pi_\lambda(x v_\lambda), \quad \forall x \in U_q(\mathfrak{g}),$$

$$(\pi_\lambda \otimes \pi_\lambda)\Delta(v_\lambda) = (\pi_\lambda \otimes \pi_\lambda)\left[(v_\lambda \otimes v_\lambda)\left(R^T_{\lambda,\lambda} R_{\lambda,\lambda}\right)^{-1}\right]. \tag{3.25}$$

Proof. The proofs of $\epsilon(v_\lambda) = 1$ and (3.25) follow from the definition of v_λ. To prove the remaining relation, note that $S^2(e_i) = K_i e_i K_i^{-1} = K_{2\rho} e_i K_{2\rho}^{-1}$, $S^2(f_i) = K_i f_i K_i^{-1} = K_{2\rho} f_i K_{2\rho}^{-1}$, and $S^2(K_i^{\pm 1}) = K_{2\rho} K_i^{\pm 1} K_{2\rho}^{-1}$. As S^2 is a homomorphism we have $S^2(x) = K_{2\rho} x K_{2\rho}^{-1}$ for all $x \in U_q(\mathfrak{g})$ and then

$$\pi_\lambda\big(v_\lambda x v_\lambda^{-1}\big) = \pi_\lambda\big(u_\lambda K_{2\rho}^{-1} x K_{2\rho} u_\lambda^{-1}\big) = \pi_\lambda\big(u_\lambda S^{-2}(x) u_\lambda^{-1}\big) = \pi_\lambda\big(S^2(S^{-2}(x))\big) = \pi_\lambda(x),$$

completing the proof. □

Let V_λ be an irreducible $U_q(\mathfrak{g})$-module with integral dominant highest weight λ.

Lemma 3.4.3. *The element* v_λ *acts on each vector in* V_λ *as the multiplication by the scalar* $q^{-(\lambda+2\rho,\lambda)}$.

Proof. Note that v_λ is even and that $\pi_\lambda(v_\lambda) \in End_{U_q(\mathfrak{g})}(V_\lambda)$. Write $R_{\lambda,\lambda} = E_\lambda \widetilde{R}$ and $\widetilde{R} = \sum_{t=0}^{\infty} a_t \otimes b_t$ where $a_t \in U_q(\mathfrak{b}_+)$, $b_t \in U_q(\mathfrak{b}_-)$ and $a_0 = b_0 = 1$. Then

$$\pi_\lambda(v_\lambda) = \pi_\lambda \left(\sum_{t=0}^{\infty} S(b_t) E a_t K_{2\rho}^{-1} (-1)^{[a_t]} \right),$$

where E is an even element of $U_q(\mathfrak{g})$ satisfying $Ew_\xi = q^{-(\xi,\xi)} w_\xi$ for each weight vector $w_\xi \in V_\lambda$ of weight $\xi \in \bigoplus_{i=1}^n \mathbb{Z}\epsilon_i$. Let w_λ be a non-zero highest weight vector of V_λ, then $a_t w_\lambda = 0$ for all $t > 0$, which yields

$$v_\lambda \cdot w_\lambda = E K_{2\rho}^{-1} w_\lambda = q^{-(\lambda+2\rho,\lambda)} w_\lambda.$$

As V_λ is irreducible and $\pi_\lambda(v_\lambda) \in End_{U_q(\mathfrak{g})}(V_\lambda)$ (and $\pi_\lambda(v_\lambda)$ is a homogeneous map of degree zero), v_λ acts on each weight vector w in V_λ as the multiplication by the claimed scalar from Schur's lemma. □

We may denote $q^{-(\lambda+2\rho,\lambda)}$ by $\chi_\lambda(v_\lambda)$. Note that it may be true that v_μ acts on each weight vector w in V_λ as the multiplication by the salar $q^{-(\lambda+2\rho,\lambda)}$ even if $\mu \neq \lambda$, and in this case we also write $\chi_\lambda(v_\mu)$ to denote $q^{-(\lambda+2\rho,\lambda)}$.

Following [Zh95], we define the quantum superdimension of the finite dimensional $U_q(\mathfrak{g})$-module W to be

$$sdim_q(W) = \mathrm{str}\big(\pi_W(K_{2\rho})\big).$$

We now prove the following lemma originally given in [Zh92b, p. 323].

Lemma 3.4.4. *Let* V_λ *be a finite dimensional irreducible* $U_q(\mathfrak{g})$*-module with integral dominant highest weight* λ*. The quantum superdimension of* V_λ *is*

$$sdim_q(V_\lambda) = (-1)^{[\lambda]} q^{-(\lambda,2\rho)} \prod_{\alpha \in \overline{\Phi}_0^+} \left(\frac{q^{2(\lambda+\rho,\alpha)} - 1}{q^{2(\rho,\alpha)} - 1} \right) \prod_{\beta \in \Phi_1^+} \left(\frac{q^{2(\lambda+\rho,\beta)} + 1}{q^{2(\rho,\beta)} + 1} \right), \tag{3.26}$$

where $[\lambda]$ *is the grading of the highest weight vector of* V_λ.

Proof. We follow the usual proof for the formula for the quantum dimension of a finite dimensional irreducible module over a quantum algebra. We have not seen the proof of Eq. (3.26) in the literature and so write it down explicitly here.

The weight space decomposition of the $U_q(\mathfrak{g})$-module V_λ is the same as for a $U(\mathfrak{g})$-module with highest weight λ, so we can use Kac's supercharacter formula (see Appendix C) to calculate $\mathrm{str}\big(\pi_\lambda(K_{2\rho})\big)$. Fix $e^h \in \mathbb{C}$ by $e^h = q$ and define in the notations of Appendix C a homomorphism $f : H^* \times E' \to \mathbb{C}[[h]]$ by $f_\eta(e^\nu) = e^{h(\eta,\nu)}$, $\eta, \nu \in H^*$.

By definition, $sch_\lambda = \sum_\mu (-1)^{[\mu]} m(\mu) e^\mu$, where the sum is over all weight spaces of V_λ, where $[\mu]$ is the grading of the vectors in the weight space μ, and $m(\mu)$ is the multiplicity of the weight space μ. Applying $f_{2\rho}$ to sch_λ gives

$$f_{2\rho}(sch_\lambda) = \sum_\mu (-1)^{[\mu]} m(\mu) e^{h(2\rho,\mu)},$$

which is just $sdim_q(V_\lambda)$.

Applying $f_{2\rho}$ to sch_λ, and using the variant of Weyl's denominator formula in Appendix C, gives

$$\begin{aligned}\prod_{\alpha\in\overline{\Phi}_0^+}(e^{h(\alpha,\rho)} - e^{-h(\alpha,\rho)}) \prod_{\beta\in\Phi_1^+}(e^{h(\beta,\rho)} + e^{-h(\beta,\rho)}) \sum_\mu (-1)^{[\mu]} m(\mu) e^{h(2\rho,\mu)} \\ = (-1)^{[\lambda]} \sum_{\sigma\in\mathcal{W}} \epsilon'(\sigma) e^{h(2\rho,\sigma(\lambda+\rho))}, \\ = (-1)^{[\lambda]} f_{2\lambda+2\rho} \sum_{\sigma\in\mathcal{W}} \epsilon'(\sigma) e^{\sigma(\rho)}.\end{aligned} \tag{3.27}$$

Rewriting the right hand side of (3.27) as

$$(-1)^{[\lambda]} \prod_{\alpha\in\overline{\Phi}_0^+}(e^{h(\alpha,\lambda+\rho)} - e^{-h(\alpha,\lambda+\rho)}) \prod_{\beta\in\Phi_1^+}(e^{h(\beta,\lambda+\rho)} + e^{-h(\beta,\lambda+\rho)}),$$

yields the desired result. □

The quantum superdimension of the fundamental $U_q(osp(1|2n))$-module V is

$$sdim_q(V) = 1 - \frac{q^{2n} - q^{-2n}}{q - q^{-1}}, \tag{3.28}$$

where the grading of the highest weight vector of V is odd.

3.5 Spectral decomposition of $\check{\mathcal{R}}_{V,V}$

Let V_λ and V_μ be finite dimensional irreducible $U_q(\mathfrak{g})$-modules with integral dominant highest weights λ and μ, respectively. Let $R_{\lambda,\mu}$ be as in Theorem 3.3.2 and define $\check{\mathcal{R}}_{V_\lambda,V_\mu} \in \mathrm{Hom}_\mathbb{C}(V_\lambda \otimes V_\mu, V_\mu \otimes V_\lambda)$ by

$$\check{\mathcal{R}}_{V_\lambda,V_\mu}(v_\lambda \otimes v_\mu) = P \circ \big(R_{\lambda,\mu}(v_\lambda \otimes v_\mu)\big), \tag{3.29}$$

where $v_\lambda \in V_\lambda$ and $v_\mu \in V_\mu$ are weight vectors and P is the graded permutation operator.

Lemma 3.5.1. *Let V_λ be an irreducible $U_q(\mathfrak{g})$-module with integral dominant highest weight λ, then*

$$\check{\mathcal{R}}_{V_\lambda,V_\lambda} \in End_{U_q(\mathfrak{g})}(V_\lambda \otimes V_\lambda).$$

Proof. We follow Lemma 2.3.2 (i) to prove that

$$\check{\mathcal{R}}_{V_\lambda,V_\lambda} \cdot (\pi_\lambda \otimes \pi_\lambda)\big(\Delta(x)\big) = (\pi_\lambda \otimes \pi_\lambda)\big(\Delta(x)\big) \cdot \check{\mathcal{R}}_{V_\lambda,V_\lambda},$$

then the result follows. □

For $n = 1$, $V \otimes V$ decomposes into a direct sum of irreducible $U_q(\mathfrak{g})$-modules (see [Zh92b]):

$$V \otimes V = V_{2\epsilon_1} \oplus V_{\epsilon_1} \oplus V_0, \tag{3.30}$$

and for $n \geq 2$, we have

$$V \otimes V = V_{2\epsilon_1} \oplus V_{\epsilon_1+\epsilon_2} \oplus V_0. \tag{3.31}$$

Lemma 3.5.2. *Let $n \geq 2$ and let $\{P[\mu] \in End_{U_q(\mathfrak{g})}(V \otimes V) |\ \mu = 2\epsilon_1, \epsilon_1 + \epsilon_2, 0\}$ be a set of even $U_q(\mathfrak{g})$-linear maps: $P[\mu] : V \otimes V \to V \otimes V$, where the image of $P[\mu]$ is V_μ and the maps satisfy $\big(P[\mu]\big)^2 = P[\mu]$ and $P[\mu]P[\nu] = \delta_{\mu\nu}P[\mu]$. Then there is a spectral decomposition of $\check{\mathcal{R}}_{V,V}$:*

$$\check{\mathcal{R}}_{V,V} = -qP[2\epsilon_1] + q^{-1}P[\epsilon_1 + \epsilon_2] + q^{-2n}P[0].$$

Proof. As $\check{\mathcal{R}}_{V,V} \in End_{U_q(\mathfrak{g})}(V \otimes V)$, we can write

$$\check{\mathcal{R}}_{V,V} = \beta_{2\epsilon_1}P[2\epsilon_1] + \beta_{\epsilon_1+\epsilon_2}P[\epsilon_1 + \epsilon_2] + \beta_0 P[0],$$

for some set of constants $\{\beta_\mu \in \mathbb{C} |\ \mu = 2\epsilon_1, \epsilon_1 + \epsilon_2, 0\}$, where β_μ is the scalar action of $\check{\mathcal{R}}_{V,V}$ on the irreducible $U_q(\mathfrak{g})$-submodule $V_\mu \subset V \otimes V$. We explicitly calculate each β_μ using $R_{V,V}$.

Let $\{v_i |\ -n \leq i \leq n\}$ be the basis of weight vectors of V given in Lemma 3.2.2. The highest weight vector of $V_{2\epsilon_1}$ is $w_{2\epsilon_1} = v_1 \otimes v_1$, the highest weight vector of $V_{\epsilon_1+\epsilon_2}$ is $w_{\epsilon_1+\epsilon_2} = v_1 \otimes v_2 - q^{-1} v_2 \otimes v_1$ and the highest weight vector of the trivial module $V_0 \subset V \otimes V$ is

$$w_0 = \sum_{i=-n}^{n} c_i v_i \otimes v_{-i},$$

where $\{c_i \in \mathbb{C} |\ -n \leq i \leq n\}$ is a set of non-zero constants inductively defined by

$$\begin{aligned} c_n &= -c_0, & c_{-n} &= q^{-1}c_0, \\ c_{n-1} &= -qc_n, & c_{-(n-1)} &= -q^{-1}c_{-n}, \\ c_i &= -qc_{i+1}, & c_{-i} &= -q^{-1}c_{-(i+1)}, \end{aligned}$$

where $i = 1, 2, \ldots, n-2$ and we fix $c_0 \neq 0$.

To study the action of $\check{\mathcal{R}}_{V,V}$ on the highest weight vectors $w_{2\epsilon_1}, w_{\epsilon_1+\epsilon_2}$ and w_0, we note that the weight space decomposition of V gives rise to the following results:

$$\pi(f_{\epsilon_i})^3 = \pi(e_{\epsilon_i})^3 = 0, \text{ for all } i = 1, \ldots, n, \tag{3.32}$$

$$\pi(f_\gamma)^2 = \pi(e_\gamma)^2 = 0, \text{ for all } \gamma \in \phi \text{ where } \gamma \neq \epsilon_i. \tag{3.33}$$

Using (3.32) and (3.33), we have

$$(\pi \otimes \pi)\widetilde{R} = (\pi \otimes \pi) \prod_{\gamma \in \phi} \mathfrak{R}_\gamma \tag{3.34}$$

where the product is ordered according to the same normal ordering $\mathcal{N}(\phi)$ used to construct the root vectors so that given $\mathcal{N}(\phi) = \gamma_1 \prec \gamma_2 \prec \cdots \prec \gamma_k$, we fix $\prod_{\gamma \in \phi} \mathfrak{R}_\gamma = \mathfrak{R}_{\gamma_1}\mathfrak{R}_{\gamma_2} \cdots \mathfrak{R}_{\gamma_k}$, where

$$\begin{aligned}
\mathfrak{R}_\gamma &= \begin{cases} \displaystyle\sum_{k=0}^{2} \frac{(q^{-1}-q)^k (e_\gamma \otimes f_\gamma)^k}{[k]^{-q^{-1}}!}, & \text{if } \gamma = \epsilon_i, \\ \displaystyle\sum_{k=0}^{1} \frac{(q-q^{-1})^k (e_\gamma \otimes f_\gamma)^k}{[k]^{q^{-2}}!}, & \text{if } \gamma \neq \epsilon_i. \end{cases} \\
&= \begin{cases} 1 \otimes 1 + (q^{-1}-q)(e_\gamma \otimes f_\gamma) + \dfrac{(q^{-1}-q)^2 (e_\gamma \otimes f_\gamma)^2}{(1-q^{-1})}, & \text{if } \gamma = \epsilon_i, \\ 1 \otimes 1 + (q-q^{-1})(e_\gamma \otimes f_\gamma), & \text{if } \gamma \neq \epsilon_i. \end{cases}
\end{aligned}$$

Using this, we have $\check{\mathcal{R}}_{V,V}(w_{2\epsilon_1}) = -q w_{2\epsilon_1}$ and $\check{\mathcal{R}}_{V,V}(w_{\epsilon_1+\epsilon_2}) = q^{-1} w_{\epsilon_1+\epsilon_2}$. Calculating β_0 is more difficult: note that

$$\check{\mathcal{R}}_{V,V}\left(c_{-1} v_{-1} \otimes v_1 + \sum_{\substack{i=-n \\ i \neq -1}}^{n} c_i v_i \otimes v_{-i} \right) = -q^{-1} c_{-1} v_1 \otimes v_{-1} + \sum_{\substack{j=-n \\ j \neq -1}}^{n} c'_j v_{-j} \otimes v_j,$$

for some set of non-zero constants $\left\{c'_j \in \mathbb{C} \mid -n \leq j \leq n,\ j \neq -1\right\}$. Recall that $\check{\mathcal{R}}_{V,V}(w_0) = \beta_0 w_0$, so we obtain β_0 by comparing $-q^{-1}c_{-1}$ and c_1. Now $c_{-1} = (-1)^{n-1} q^{-n} c_0$ and $c_1 = (-1)^n q^{n-1} c_0$, thus $\beta_0 = q^{-2n}$. □

Lemma 3.5.3. *Let $n = 1$ and let $\{P[\mu] \in End_{U_q(\mathfrak{g})}(V \otimes V) \mid \mu = 2\epsilon_1, \epsilon_1, 0\}$ be a set of even $U_q(\mathfrak{g})$-linear maps: $P[\mu] : V \otimes V \to V \otimes V$, where the image of $P[\mu]$ is V_μ and the maps satisfy $\left(P[\mu]\right)^2 = P[\mu]$ and $P[\mu]P[\nu] = \delta_{\mu\nu} P[\mu]$. Then there is a spectral decomposition of $\check{\mathcal{R}}_{V,V}$:*

$$\check{\mathcal{R}}_{V,V} = -qP[2\epsilon_1] + q^{-1}P[\epsilon_1] + q^{-2}P[0].$$

Proof. The proof is almost identical to the proof of Lemma 3.5.2 with just the following minor difference. The decomposition of $V \otimes V$ is given in (3.30) and the highest weight vector of V_{ϵ_1} is $w_{\epsilon_1} = v_1 \otimes v_0 + q^{-1} v_0 \otimes v_1$. To complete the proof we note that $\check{\mathcal{R}}_{V,V}(w_{\epsilon_1}) = q^{-1} w_{\epsilon_1}$. □

From this we can write down the following important result:

Corollary 3.5.1. *For each $n \geq 1$, $\check{\mathcal{R}}_{V,V}$ satisfies*

$$(\check{\mathcal{R}}_{V,V} + q)(\check{\mathcal{R}}_{V,V} - q^{-1})(\check{\mathcal{R}}_{V,V} - q^{-2n}) = 0. \tag{3.35}$$

Note that the expression for $(\pi \otimes \pi)\widetilde{R}$ in (3.34) readily allows the use of $(\pi \otimes \pi)R_{V,V}$ and $\check{\mathcal{R}}_{V,V}$ in calculations.

3.6 A representation of the Birman-Wenzl-Murakami algebra

In this section we will define a map from the Birman-Wenzl-Murakami algebra $\mathscr{BW}_f(r,q)$ [BW89, We90] to a subalgebra $\mathcal{C}_f$ of the centraliser algebra of $V^{\otimes f}$. This will allow us to map matrix units in $\mathscr{BW}_f(r,q)$ (see [RW92]) to matrix units of $\mathcal{C}_f$.

Before defining the map from $\mathscr{BW}_f(r,q)$ to $\mathcal{C}_f$, we need to obtain some preliminary results.

Definition 3.6.1. *Define $\mathcal{C}_t$ to be the algebra over $\mathbb{C}$ generated by the elements*

$$\left\{\check{R}_i^{\pm 1} \in End_{U_q(\mathfrak{g})}(V^{\otimes t}) \middle| \ 1 \le i \le t-1\right\}, \quad \textit{where}$$

$$\check{R}_i = \mathrm{id}^{\otimes(i-1)} \otimes \check{\mathcal{R}}_{V,V} \otimes \mathrm{id}^{\otimes(t-(i+1))} \in End_{U_q(\mathfrak{g})}(V^{\otimes t}). \tag{3.36}$$

Let us investigate $\mathcal{C}_t$. Let $\{v_i| \ -n \le i \le n\}$ be the basis of weight vectors of V given in Lemma 3.2.2 and let $\{v_i^*| \ -n \le i \le n\}$ be a basis of V^* such that $\langle v_i^*, v_j\rangle = \delta_{ij}$ and $[v_i^*] = [v_i]$; we have

$$av_i = \sum_j \langle v_j^*, av_i\rangle v_j, \qquad av_i^* = \sum_j \langle av_i^*, v_j\rangle v_j^*, \qquad \forall a \in U_q(\mathfrak{g}).$$

Define $\check{e} \in End_{\mathbb{C}}(V \otimes V^*)$ by

$$\check{e}(x \otimes y^*) = (-1)^{[y^*][x]}\langle y^*, v^{-1}u\ x\rangle \sum_{i=-n}^{n} v_i \otimes v_i^*,$$

where v and u are the elements $v_{\epsilon_1}, u_{\epsilon_1} \in \overline{U}_q^+(\mathfrak{g})$ respectively.

Lemma 3.6.1. *The map $\check{e}$ satisfies*

(i) $(\check{e})^2 = sdim_q(V)\check{e}$,

(ii) $a\check{e} = \epsilon(a)\check{e}, \quad \forall a \in U_q(\mathfrak{g})$,

(iii) $\check{e}a = \epsilon(a)\check{e}, \quad \forall a \in U_q(\mathfrak{g})$,

(iv) $\check{e}_2\check{R}_1\check{e}_2 = q^{2n}\check{e}_2$, *where*

$$\check{e}_2 = \mathrm{id}_V \otimes \check{e} : V \otimes V \otimes V^* \to V \otimes V \otimes V^*,$$
$$\check{R}_1 = \check{\mathcal{R}}_{V,V} \otimes \mathrm{id}_{V^*} : V \otimes V \otimes V^* \to V \otimes V \otimes V^*.$$

Proof. (i)

$$\begin{aligned}(\check{e})^2(x \otimes y^*) &= (-1)^{[y^*][x]}\langle y^*, v^{-1}ux\rangle \sum_i (-1)^{[v_i]}\langle v_i^*, v^{-1}u\ v_i\rangle \sum_j v_j \otimes v_j^* \\ &= sdim_q(V)(-1)^{[y^*][x]}\langle y^*, v^{-1}u\ x\rangle \sum_j v_j \otimes v_j^* = sdim_q(V)\check{e}(x \otimes y^*).\end{aligned}$$

(ii) By definition, $a\check{e} = a_{V\otimes V^*} \circ \check{e}$; we calculate that

$$\begin{aligned} a\check{e}(x\otimes y^*) &= (-1)^{[y^*][x]}\langle y^*, v^{-1}ux\rangle \sum_{(a),i,j,k} \langle v_j^*, a_{(1)}v_i\rangle\langle v_i^*, S(a_{(2)})v_k\rangle v_j \otimes v_k^* \\ &= (-1)^{[y^*][x]}\langle y^*, v^{-1}ux\rangle \sum_{(a),j,k} \langle v_j^*, a_{(1)}S(a_{(2)})v_k\rangle v_j \otimes v_k^* \\ &= \epsilon(a)(-1)^{[y^*][x]}\langle y^*, v^{-1}ux\rangle \sum_k v_k \otimes v_k^* = \epsilon(a)\check{e}(x\otimes y^*). \end{aligned}$$

Similar calculations prove (iii) and (iv) (see [LR97] for the corresponding calculations in ungraded quasitriangular Hopf algebras). □

Define $\hat{e} \in End_{\mathbb{C}}(V^* \otimes V)$ by

$$\hat{e}(x^* \otimes y) = \langle x^*, y\rangle \sum_{i=-n}^{n} (-1)^{[v_i]} v_i^* \otimes vu^{-1}\, v_i,$$

where v and u are set to be v_{ϵ_1} and u_{ϵ_1}, respectively.

Lemma 3.6.2. *The map $\hat{e}$ satisfies*

(i) $(\hat{e})^2 = sdim_q(V)\hat{e}$,

(ii) $a\hat{e} = \epsilon(a)\hat{e}, \quad \forall a \in U_q(\mathfrak{g})$,

(iii) $\hat{e}a = \epsilon(a)\hat{e}, \quad \forall a \in U_q(\mathfrak{g})$,

(iv) $\hat{e}_2\check{R}_1^{-1}\hat{e}_2 = q^{-2n}\hat{e}_2$ *where*

$$\hat{e}_2 = \mathrm{id}_{V^*} \otimes \hat{e} : V^* \otimes V^* \otimes V \to V^* \otimes V^* \otimes V.$$

Proof. The proofs of (i)–(iv) are similar to the proofs of parts (i)–(iv) of Lemma 3.6.1. □

Remark 3.6.1. *The maps $\check{e}$ and $\hat{e}$ are $U_q(\mathfrak{g})$-invariant maps onto one-dimensional $U_q(\mathfrak{g})$-submodules in $V \otimes V^*$ and $V^* \otimes V$, respectively.*

Recall that $V\otimes V$ has the decomposition into irreducible $U_q(\mathfrak{g})$-modules given in (3.30)–(3.31) and that there exists an even $U_q(\mathfrak{g})$-invariant map $P[0] : V\otimes V \to V\otimes V$, the image of which is $V_0 \subset V \otimes V$, defined in Lemmas 3.5.2–3.5.3. Recall that $V^* \cong V$ and define

$$E = (\mathrm{id} \otimes T^{-1}) \circ \check{e} \circ (\mathrm{id} \otimes T) = (T^{-1} \otimes \mathrm{id}) \circ \hat{e} \circ (T \otimes \mathrm{id}) = sdim_q(V)P[0],$$

where T is the isomorphism $T : V \to V^*$ given in Eq. (3.5). Furthermore, define the elements

$$E_i = \mathrm{id}^{\otimes(i-1)} \otimes E \otimes \mathrm{id}^{\otimes(t-(i+1))} \in End_{U_q(\mathfrak{g})}(V^{\otimes t}), \qquad i = 1, \ldots, t-1.$$

Lemma 3.6.3. *The elements $\check{R}_i$, $E_i \in End_{U_q(\mathfrak{g})}(V^{\otimes t})$ satisfy the relations*

(i) $\check{R}_i\check{R}_{i+1}\check{R}_i = \check{R}_{i+1}\check{R}_i\check{R}_{i+1},\ \ 1 \le i \le t-2,$

(ii) $\check{R}_i\check{R}_j = \check{R}_j\check{R}_i,\ \ |i-j| > 1,$

(iii) $(\check{R}_i + q)(\check{R}_i - q^{-1})(\check{R}_i - q^{-2n}) = 0,\ \ 1 \le i \le t-1,$

(iv) $-\check{R}_i + \check{R}_i^{-1} = (q - q^{-1})(1 - E_i),$

(v) $E_i\check{R}_{i-1}^{\pm 1}E_i = q^{\pm 2n}E_i,$

(vi) $E_i\check{R}_i^{\pm 1} = \check{R}_i^{\pm 1}E_i = q^{\mp 2n}E_i,\ \ 1 \le i \le t-1.$

Proof. The proofs of (i) and (ii) follow from Lemma 2.3.2 and the proof of (iii) follows from Corollary 3.5.1. The proof of (v) follows from Lemmas 3.6.1–3.6.2. The proofs of (iv) and (vi) follow from the definition of E_i, Eq. (3.28) and the fact that $\check{R}_1$ acts on $V_0 \subset V \otimes V$ as a scalar multiple of the identity: $\check{R}_1 w = q^{-2n}w$ for all $w \in V_0$. □

We recall the definition of the Birman-Wenzl-Murakami algebra $\mathscr{BW}_f(r,q)$ from [RW92]. Let r and q be non-zero complex parameters and let $f \ge 2$ be an integer. The Birman-Wenzl-Murakami algebra $\mathscr{BW}_f(r,q)$ is the algebra over $\mathbb{C}$ generated by the invertible elements $\{g_i |\ 1 \le i \le f-1\}$ subject to the relations

$$\begin{array}{ll} g_ig_j = g_jg_i, & |i-j| > 1, \\ g_ig_{i+1}g_i = g_{i+1}g_ig_{i+1}, & 1 \le i \le f-2, \\ e_ig_i = r^{-1}e_i, & 1 \le i \le f-1, \\ e_ig_{i-1}^{\pm 1}e_i = r^{\pm 1}e_i, & 1 \le i \le f-1, \end{array}$$

where e_i is defined by

$$(q - q^{-1})(1 - e_i) = g_i - g_i^{-1}, \qquad 1 \le i \le f-1.$$

It can be shown that each g_i also satisfies

$$(g_i - r^{-1})(g_i + q^{-1})(g_i - q) = 0.$$

From Lemma 3.6.3 we have the following result:

Lemma 3.6.4. *Let $q \in \mathbb{C}$ be non-zero and not a root of unity. Then there is an algebra homomorphism $\Upsilon : \mathscr{BW}_f(-q^{2n}, q) \to \mathcal{C}_f \subseteq End_{U_q(\mathfrak{g})}(V^{\otimes f})$ defined by*

$$\Upsilon : g_i \mapsto -\check{R}_i.$$

3.7 Bratteli diagrams and path algebras

3.7.1 Bratteli diagrams

To proceed further with the study of $\mathscr{BW}_f(r,q)$ we now need the notions of *Bratelli diagrams* and *Path algebras related to Bratelli diagrams*, both of which we take from [LR97]. (The reader is also referred to [GHJ89, Chap. 2]).

A Bratteli diagram is an undirected graph that encodes information about a sequence $\mathbb{C} = A_0 \subset A_1 \subset A_2 \subset \cdots$ of inclusions of finite dimensional semisimple algebras [RW92]. In a graph-theoretic sense, the properties of a Bratteli diagram are that

(i) the vertices are elements of certain sets $\widetilde{A}_t$, $t \in \mathbb{Z}_+$, and

(ii) if we let $n(a,b) \in \mathbb{Z}_+$ denote the number of edges between the vertices a and b, then $n(a,b) = 0$ for any vertices $a \in \widetilde{A}_i$ and $b \in \widetilde{A}_j$ where $|i-j| \neq 1$.

Assume that $\widetilde{A}_0$ consists of a unique vertex we denote by $\emptyset$. We call the elements of $\widetilde{A}_t$ *shapes* and say that $\widetilde{A}_t$ is the set of shapes on the t^{th} level of the Bratteli diagram. If $\lambda \in \widetilde{A}_t$ is connected to $\mu \in \widetilde{A}_{t+1}$ in the Bratteli diagram we write $\lambda \leq \mu$.

A *multiplicity free Bratteli diagram* is a Bratteli diagram in which any two vertices are connected by no more than one edge. All Bratteli diagrams considered in this thesis are multiplicity free.

Let A be a Bratteli diagram and let $\lambda \in \widetilde{A}_r$ and $\mu \in \widetilde{A}_t$ for some $0 \leq r < t$. We define a *path from λ to μ* to be a sequence of shapes

$$P = (s_r, s_{r+1}, \ldots, s_t),$$

where $\lambda = s_r \leq s_{r+1} \leq \cdots \leq s_{t-1} \leq s_t = \mu$ and each s_i is a shape on the i^{th} level of A.

Given a path $T = (\lambda, \ldots, \mu)$ from λ to μ and a path $S = (\mu, \ldots, \nu)$ from μ to ν we define the concatenation of T and S to be the path from λ to ν defined by

$$T \circ S = (\lambda, \ldots, \mu, \ldots, \nu).$$

We define a tableau T of shape λ to be a path from $\emptyset \in \widetilde{A}_0$ to λ and we write $shp(T) = \lambda$. We say that the length of T is t if there are $t+1$ shapes in the tableau.

3.7.2 Path algebras related to Bratteli diagrams

We now define the concept of a *Path algebra* for a Bratteli diagram A. For each $t \in \mathbb{Z}_+$, let $\mathcal{T}^t$ be the set of tableaux of length t in A and let $\Omega^t \subset \mathcal{T}^t \times \mathcal{T}^t$ be the set of pairs (S,T) of tableaux where $shp(S) = shp(T)$, that is S and T both end in the same shape. Let us also define an algebra A_t over $\mathbb{C}$ generated by

$$\{E_{ST}|\ (S,T) \in \Omega^t\},$$

where the algebra multiplication is defined by

$$E_{ST}E_{PQ} = \delta_{TP}E_{SQ}.$$

Note that $A_0 \cong \mathbb{C}$. Each element $a \in A_t$ can be written in the form

$$a = \sum_{(S,T)\in\Omega^t} a_{ST}E_{ST}, \qquad a_{ST} \in \mathbb{C}.$$

We refer to the collection of algebras A_t, $t \in \mathbb{Z}_+$, as the *tower of path algebras corresponding to the Bratteli diagram* A.

Each of the algebras A_t is isomorphic to a direct sum of matrix algebras. The irreducible representations of A_t are indexed by the elements of $\widetilde{A}_t$, that is, the set of shapes on the t^{th} level of A. Let $\mathcal{T}^\lambda$ denote the set of tableaux of shape λ, then the cardinality d_λ of $\mathcal{T}^\lambda \cap \mathcal{T}^t$ is equal to the dimension of the irreducible A_t-module indexed by $\lambda \in \widetilde{A}_t$. We record this in the formula

$$A_t \cong \bigoplus_{\lambda\in\widetilde{A}_t} M_{d_\lambda}(\mathbb{C}),$$

where $M_d(\mathbb{C})$ denotes the algebra of $d \times d$ matrices with complex entries.

We now define some useful sets. Let $\mathcal{T}_\lambda^\mu$ be the set of paths in A from the shape λ to the shape μ and let $\mathcal{T}_r^t$ be the set of paths starting on the r^{th} level of A and going down to the t^{th} level. Furthermore, let $\mathcal{T}_\lambda^t$ be the set of paths in A from the shape λ to any shape on the t^{th} level of A.

We also define $\Omega_\lambda^\mu \subset \mathcal{T}_\lambda^\mu \times \mathcal{T}_\lambda^\mu$ to be the set of pairs (S,T) of paths $S,T \in \mathcal{T}_\lambda^\mu$ and $\Omega_r^t \subset \mathcal{T}_r^t \times \mathcal{T}_r^t$ to be the set of pairs (S,T) of paths where in both situations we have $shp(S) = shp(T)$.

We define the inclusion of path algebras $A_r \subseteq A_t$ for $0 \le r < t$ as follows: for each pair $(P,Q) \in \Omega^r$ we fix $E_{PQ} \in A_t$ by

$$E_{PQ} = \sum_{T\in\mathcal{T}_\lambda^t\cap\mathcal{T}_r^t} E_{P\circ T,Q\circ T}, \quad \text{where } \lambda = shp(P) = shp(Q).$$

In particular, we have $A_s \subseteq A_{s+1}$ for each $s \in \mathbb{Z}_+$.

Let $\lambda \in \widetilde{A}_t$ and let $\mathcal{V}_\lambda$ be an irreducible representation of A_t indexed by λ. The restriction of $\mathcal{V}_\lambda$ to the subalgebra $A_{t-1} \subseteq A_t$ decomposes into irreducible representations of A_{t-1} according to

$$\mathcal{V}_\lambda \downarrow_{A_{t-1}}^{A_t} \cong \bigoplus_{\mu\in\lambda^-} \mathcal{V}_\mu, \qquad \text{where } \lambda^- = \{\nu \in \widetilde{A}_{t-1} |\ \nu \le \lambda\}.$$

This decomposition is multiplicity free as the Bratteli diagram A is multiplicity free.

For each $r \in \mathbb{Z}_+$ satisfying $r < t$, the *centraliser of* A_r *contained in* A_t is defined by

$$\mathcal{L}(A_r \subseteq A_t) = \{a \in A_t |\ ab = ba,\ \forall b \in A_r\}.$$

Let now (S,T) be a pair of paths each starting on the r^{th} level of A at the shape λ and ending on the t^{th} level of A at the shape μ. For each such pair we define $E_{ST} \in A_t$ by

$$E_{ST} = \sum_{P \in \mathcal{T}^\lambda \cap \mathcal{T}^r} E_{P\circ S, P\circ T},$$

which we can think of as the sum of all 'paths' ending in (S,T). We then have the following lemma, stated in [LR97, Prop. (1.4)] and proved in [GHJ89, Sect. 2.3].

Lemma 3.7.1. *A basis of $\mathcal{L}(A_r \subseteq A_t)$ is given by the elements*

$$\{E_{ST}|\ (S,T) \in \Omega^\mu_\lambda \cap \Omega^t_r,\ \lambda \in \widetilde{A}_r,\ \mu \in \widetilde{A}_t\}.$$

From Lemma 3.7.1 we have the following corollary, which is proved in [LR97, Cor. (1.5) – (1.6)].

Corollary 3.7.1. *Let the collection of algebras A_t, $t \in \mathbb{Z}_+$, be a tower of path algebras corresponding to a Bratteli diagram A and suppose that $g_i \in A_{i+1}$, $i \in \mathbb{N}$, are elements such that*

(i) for each t, the elements $\{g_i|\ 1 \le i \le t-1\}$ generate A_t, and

(ii) $g_i g_j = g_j g_i$ for all i,j satisfying $|i-j| > 1$,

then

$$g_{t-1} = \sum_{(P,Q)\in\Omega^t_{t-2}} (g_{t-1})_{PQ} E_{PQ}, \qquad (g_{t-1})_{PQ} \in \mathbb{C}.$$

Furthermore, if the elements $g_1, g_2, \ldots, g_{t-1}$ satisfy the relation $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$ for all $1 \le i \le t-2$, then the element

$$D_t = g_{t-1} g_{t-2} \cdots g_1 g_1 \cdots g_{t-2} g_{t-1} \in A_t$$

can be expressed as

$$D_t = \sum_{S \in \mathcal{T}^t_{t-1}} D_{SS} E_{SS}, \qquad D_{SS} \in \mathbb{C}.$$

3.7.3 Centraliser algebras

Let U be a $\mathbb{Z}_2$-graded Hopf algebra over $\mathbb{C}$. Let V be a finite dimensional U-module with the property that $V^{\otimes t}$ is completely reducible for each $t \in \mathbb{Z}_+$. We will now define the concepts of a *Bratteli diagram for tensor powers of* V and the *Bratteli diagram for* $V^{\otimes t}$. Furthermore, we aim to show that the centraliser $\mathcal{L}_t$ of U in $End_{\mathbb{C}}(V^{\otimes t})$ defined by $\mathcal{L}_t = End_U(V^{\otimes t})$ is isomorphic to the path algebra A_t of the Bratteli diagram for $V^{\otimes t}$.

In this subsection we regard all modules as being graded. By convention $V^{\otimes 0} \cong \mathbb{C}$ and thus $\mathcal{L}_0 = \mathbb{C}$. If V is an irreducible U-module then $\mathcal{L}_1 \cong \mathbb{C}$ by Schur's lemma. For all $0 \le r < t$ we define the inclusion $\mathcal{L}_r \subseteq \mathcal{L}_t$ by $a \mapsto a \otimes \mathrm{id}^{\otimes(t-r)}$ for all $a \in \mathcal{L}_r$. Now $\mathcal{L}_t$ acts on $V^{\otimes t}$ in the obvious way. Since U and $\mathcal{L}_t$ commute, $V^{\otimes t}$ has a natural $\mathcal{L}_t \otimes U$-module structure.

Let $\{\Lambda_\lambda |\ \lambda \in I\}$ be the set of non-isomorphic finite dimensional irreducible U-modules. Then by the double centraliser theorem there exists a finite subset $\widetilde{\mathcal{L}}_t$ of I such that

$$V^{\otimes t} \cong \bigoplus_{\lambda \in \widetilde{\mathcal{L}}_t} \mathcal{L}^\lambda \otimes \Lambda_\lambda,$$

where each $\mathcal{L}^\lambda$ is an irreducible $\mathcal{L}_t$-module such that $\mathcal{L}^\lambda \ncong \mathcal{L}^\mu$ if $\lambda \neq \mu$.

We now assume that V is an irreducible U-module and we consider the Bratteli diagram for tensor powers of V. Let $\lambda \in \widetilde{\mathcal{L}}_t$ for some t. Then we have the decomposition

$$\Lambda_\lambda \otimes V = \bigoplus_{\mu \in \widetilde{\mathcal{L}}_{t+1}} (\Lambda_\mu)^{\oplus n_\lambda(\mu)}, \qquad n_\lambda(\mu) \in \mathbb{Z}_+, \tag{3.37}$$

of $\Lambda_\lambda \otimes V$ into a direct sum of irreducible U-modules. The non-negative integer $n_\lambda(\mu)$ is the multiplicity of Λ_μ in the decomposition. We say that the decomposition of $\Lambda_\lambda \otimes V$ is *multiplicity free* if $n_\lambda(\mu) \leq 1$ for all $\mu \in \widetilde{\mathcal{L}}_{t+1}$.

The *branching rule for inclusion* $\mathcal{L}_t \subseteq \mathcal{L}_{t+1}$ describes the decomposition of the $\mathcal{L}_{t+1}$-module $\mathcal{L}^\nu$ into $\mathcal{L}_t$-modules

$$\mathcal{L}^\nu = \bigoplus_{\lambda \in \widetilde{\mathcal{L}}_t} \left(\mathcal{L}^\lambda\right)^{\oplus n_\lambda(\nu)}, \qquad n_\lambda(\nu) \in \mathbb{Z}_+. \tag{3.38}$$

Note that the positive integers $n_\lambda(\nu)$ appearing in (3.37) and (3.38) are the same [LR97].

The *Bratteli diagram for tensor powers of* V is defined as follows: for each $t \in \mathbb{Z}_+$ fix the vertices on the t^{th} level of the Bratteli diagram to be the elements of $\widetilde{\mathcal{L}}_t$. Then a vertex $\lambda \in \widetilde{\mathcal{L}}_t$ is connected to a vertex $\mu \in \widetilde{\mathcal{L}}_{t+1}$ by $n_\lambda(\mu)$ edges.

For a fixed t, the *Bratteli diagram for* $V^{\otimes t}$ is an undirected graph with vertices given by the elements of $\bigcup_{i=0}^t \widetilde{\mathcal{L}}_i$, and the edges are such that a vertex $\lambda \in \widetilde{\mathcal{L}}_i$ is connected to a vertex $\mu \in \widetilde{\mathcal{L}}_{i+1}$ by $n_\lambda(\mu)$ edges for each $0 \leq i \leq t-1$.

Let V be a finite dimensional irreducible U-module with the property that for every irreducible U-module W, the decomposition of the tensor product $W \otimes V$ is multiplicity free. In this case, we say that tensoring by V is multiplicity free. We will show that the centraliser algebra $\mathcal{L}_t = End_U(V^{\otimes t})$ is isomorphic to the path algebra A_t associated with the Bratteli diagram for $V^{\otimes t}$.

We construct an algebra isomorphism $A_t \to \mathcal{L}_t$ inductively. Assume that there is an identification of $\mathcal{L}_t$ with the path algebra A_t for some $t \geq 0$, so that

$$V^{\otimes t} = \bigoplus_{\lambda \in \widetilde{\mathcal{L}}_t} \left(\bigoplus_{T \in \mathcal{T}^\lambda \cap \mathcal{T}^t} E_{TT} V^{\otimes t} \right)$$

is a decomposition of $V^{\otimes t}$ into irreducible U-modules Λ_λ where the U-submodule $E_{TT}V^{\otimes t}$ is isomorphic to Λ_λ when $shp(T) = \lambda$. The map E_{TT} is a U-invariant map from $V^{\otimes t}$ onto a U-submodule isomorphic to Λ_λ.

Let $T = (\emptyset, s_1, s_2, \ldots, \lambda) \in \mathcal{T}^\lambda$ be a tableau of length t and let $E_{TT}V^{\otimes t} \cong \Lambda_\lambda$ for some $\lambda \in \widetilde{\mathcal{L}}_t$. As tensoring by V is multiplicity free, the decomposition

$$(E_{TT}V^{\otimes t}) \otimes V = \bigoplus_{\substack{\nu \in \widetilde{\mathcal{L}}_{t+1} \\ \lambda \leq \nu}} V_{T\circ\nu}, \tag{3.39}$$

is multiplicity free and thus unique, where $T \circ \nu$ is the tableau

$$T \circ \nu = (\emptyset, s_1, s_2, \ldots, \lambda, \nu), \quad \lambda \leq \nu,$$

and $V_{T\circ\nu} \cong \Lambda_\nu$.

The next step is to identify $E_{T\circ\nu,T\circ\nu}$ with the unique U-invariant projection operator mapping $(E_{TT}V^{\otimes t}) \otimes V$ onto $V_{T\circ\nu}$. This way we identify each element E_{SS} of the path algebra A_{t+1}, where $S \in \mathcal{T}^{t+1}$, with an element of $\mathcal{L}_{t+1}$. Thus we have the decomposition

$$V^{\otimes(t+1)} = \bigoplus_{\nu \in \widetilde{\mathcal{L}}_{t+1}} \left(\bigoplus_{S \in \mathcal{T}^\nu \cap \mathcal{T}^{t+1}} E_{SS}V^{\otimes(t+1)} \right),$$

of $V^{\otimes(t+1)}$ into irreducible U-modules $E_{SS}V^{\otimes(t+1)} = V_S \cong \Lambda_\nu$, where $\nu \in \widetilde{\mathcal{L}}_{t+1}$ and $S \in \mathcal{T}^\nu \cap \mathcal{T}^{t+1}$.

We now identify the other elements in the basis $\{E_{PQ} \in A_{t+1} |\ (P,Q) \in \Omega^{t+1}\}$ with elements of $\mathcal{L}_{t+1}$. For each pair of paths $(P,Q) \in \Omega^{t+1}$ we choose non-zero elements

$$E_{PQ} \in E_{PP}\mathcal{L}_{t+1}E_{QQ}, \qquad E_{QP} \in E_{QQ}\mathcal{L}_{t+1}E_{PP},$$

normalised in such a way that $E_{PQ}E_{QP} = E_{PP}$ and $E_{QP}E_{PQ} = E_{QQ}$. Thus there is an algebra isomorphism $A_{t+1} \to \mathcal{L}_{t+1}$.

We then have the following theorem.

Theorem 3.7.1. *Let V be a finite dimensional irreducible U-module such that $V^{\otimes t}$ is completely reducible for each $t \in \mathbb{Z}_+$ and such that tensoring by V is multiplicity free. Then for any $t \in \mathbb{Z}_+$, the centraliser algebra $\mathcal{L}_t = End_U(V^{\otimes t})$ is isomorphic to the path algebra A_t corresponding to the Bratteli diagram for $V^{\otimes t}$.*

3.8 Projections from $V^{\otimes t}$ onto irreducible $U_q(\mathfrak{g})$-modules

In this section we define projections from $V^{\otimes t}$ onto irreducible $U_q(\mathfrak{g})$-modules $V_\lambda \subset V^{\otimes t}$, $\lambda \in \mathcal{P}^+$, using elements of $\mathcal{C}_t$. No substantially new results appear in this section, however, we are not aware of this specific formulation of the projections in the literature. In addition, this work provides a model for the definition of the $U_q^{(N)}(\mathfrak{g})$-modules in Chapter 4.

Let V_μ be a finite dimensional irreducible $U_q(\mathfrak{g})$-module with highest weight $\mu \in \mathcal{P}^+$. Since each weight space of V is one-dimensional, $V_\mu \otimes V$ is multiplicity free.

Definition 3.8.1. *We define $\mathcal{P}^+_\mu \subset \mathcal{P}^+$ to be the set such that for each $\lambda \in \mathcal{P}^+_\mu$, V_λ appears in $V_\mu \otimes V$ as an irreducible submodule.*

Now each $\lambda \in \mathcal{P}^+_\mu$ can only have one of the following three forms: $\mu, \mu + \epsilon_i, \mu - \epsilon_i$ for some i. Thus

$$\mathcal{P}^+_\mu \subseteq \mathcal{P}^0_\mu = \{\mu, \mu \pm \epsilon_i \in \mathcal{P}^+ |\ 1 \leq i \leq n\}.$$

Definition 3.8.2. *Let V_μ be an irreducible $U_q(\mathfrak{g})$-module with highest weight $\mu \in \mathcal{P}^+$, then $V_\mu \otimes V = \bigoplus_{\lambda \in \mathcal{P}^+_\mu} V_\lambda$. Let $\{p_\mu[\lambda] \in End_{U_q(\mathfrak{g})}(V_\mu \otimes V)|\ \lambda \in \mathcal{P}^+_\mu\}$ be a set of even maps*

$$p_\mu[\lambda] : V_\mu \otimes V \to V_\mu \otimes V$$

such that

(i) the image of $p_\mu[\lambda]$ is V_λ,

(ii) $\left(p_\mu[\lambda]\right)^2 = p_\mu[\lambda]$,

(iii) $p_\mu[\lambda] \cdot p_\mu[\nu] = \delta_{\lambda\nu} p_\mu[\lambda]$.

We call each such $p_\mu[\lambda]$ a projection.

Recall that for each integral dominant λ, there exists an element $v_\lambda \in \overline{U}^+_q(\mathfrak{g})$ defined in Eq. (3.24) that acts on each vector in the finite dimensional irreducible $U_q(\mathfrak{g})$-module V_λ as the multiplication by the scalar $q^{-(\lambda+2\rho,\lambda)}$.

For each $\mu \in \mathcal{P}^+$ and each $\lambda \in \mathcal{P}^+_\mu$, define $p_\mu[\lambda] \in End_{U_q(\mathfrak{g})}(V_\mu \otimes V)$ by

$$p_\mu[\lambda] = (\pi_\mu \otimes \pi) \left(\prod_{\substack{\nu \in \mathcal{P}^+_\mu \\ \nu \neq \lambda}} \frac{\Delta(v_\xi) - q^{-(\nu+2\rho,\nu)}}{q^{-(\lambda+2\rho,\lambda)} - q^{-(\nu+2\rho,\nu)}} \right), \tag{3.40}$$

where v_ξ is the element $v_\lambda \in \mathcal{P}^+$ with $\lambda = \xi$, for some integral dominant ξ which is chosen so that v_ξ acts as the multiplication by the scalar $q^{-(\nu+2\rho,\nu)}$ on each vector in the irreducible $U_q(\mathfrak{g})$-module V_ν, for each $\nu \in \mathcal{P}^+_\mu$. For each integral dominant μ there always exists at least one such ξ. To see this, all we need is some E_ξ given by Eq. (3.12): $E_\xi = \prod_{a=1}^{n} \sum_{b=p}^{s} (J_a)^b \otimes P_a[b]$, such that the element $E = \prod_{a=1}^{n} \sum_{b=p}^{s} P_a[b](J_a)^{-b}$ acts as the multiplication by the scalar $q^{-(\zeta,\zeta)}$ on each weight vector $w_\zeta \in V_\nu \subseteq V_\mu \otimes V$, where w_ζ has the weight ζ, and this is true for each $\nu \in \mathcal{P}^+_\mu$.

The element E has this action whenever s is sufficiently large enough and the absolute value of the negative integer p is sufficiently large enough, and so all we need do is to choose some ξ for which this is true.

To do this, for each $\nu \in \mathcal{P}_\mu^+$ let I_ν be the set of distinct weights of the weight vectors of V_ν, then $(\zeta_\nu, \epsilon_i) \in \mathbb{Z}$ for each weight $\zeta_\nu \in I_\nu$ and each $i = 1, \ldots, n$. Let

$$m = \max\left\{|(\zeta_\nu, \epsilon_i)| \in \mathbb{Z}_+ \middle| \zeta_\nu \in I_\nu, \nu \in \mathcal{P}_\mu^+, i = 1, \ldots, n\right\},$$

then fixing $\xi = \sum_{i=1}^n m\epsilon_i$ yields elements E_ξ and E with the desired properties.

Note that $(\pi_\mu \otimes \pi)\Delta(v_\xi)$ in (3.40) is diagonalisable as $V_\mu \otimes V$ is completely reducible and $\Delta(v_\xi)$ acts on each irreducible $U_q(\mathfrak{g})$-submodule $V_\nu \subset V_\mu \otimes V$ as the multiplication by the scalar $q^{-(\nu+2\rho,\nu)}$.

Lemma 3.8.1. *The maps $p_\mu[\lambda]$ are well-defined and satisfy*

(i) $\left(p_\mu[\lambda]\right)^2 = p_\mu[\lambda]$,

(ii) $p_\mu[\lambda] \cdot p_\mu[\nu] = \delta_{\lambda\nu} p_\mu[\lambda]$,

(iii) $\displaystyle\sum_{\lambda \in \mathcal{P}_\mu^+} p_\mu[\lambda] = \mathrm{id}_{V_\mu \otimes V}$.

Proof. If α and β are the highest weights of irreducible $U_q(\mathfrak{g})$-submodules in $V_\mu \otimes V$, then $(\alpha + 2\rho, \alpha) = (\beta + 2\rho, \beta)$ implies that $\alpha = \beta$. Then $p_\mu[\lambda]$ is well defined as tensoring by V is multiplicity free and q is not a root of unity. The proof of (i) follows from the result $\left(p_\mu[\lambda]\right)^2 (V_\mu \otimes V) = p_\mu[\lambda](V_\lambda) = V_\lambda$. For (ii) the case $\lambda = \nu$ reduces to (i), and for $\lambda \neq \nu$ we have

$$p_\mu[\lambda] \cdot p_\mu[\nu] = (\pi_\mu \otimes \pi) \left(\prod_{\substack{\lambda' \in \mathcal{P}_\mu^+ \\ \lambda' \neq \lambda}} \frac{\Delta(v_\xi) - q^{-(\lambda'+2\rho,\lambda')}}{q^{-(\lambda+2\rho,\lambda)} - q^{-(\lambda'+2\rho,\lambda')}} \prod_{\substack{\nu' \in \mathcal{P}_\mu^+ \\ \nu' \neq \nu}} \frac{\Delta(v_\xi) - q^{-(\nu'+2\rho,\nu')}}{q^{-(\nu+2\rho,\nu)} - q^{-(\nu'+2\rho,\nu')}} \right) = 0.$$

(iii) $\displaystyle\sum_{\lambda \in \mathcal{P}_\mu^+} p_\mu[\lambda]\,(V_\mu \otimes V) = \bigoplus_{\lambda \in \mathcal{P}_\mu^+} V_\lambda = V_\mu \otimes V.$

□

Note that $(\pi_\mu \otimes \pi) \displaystyle\prod_{\lambda \in \mathcal{P}_\mu^+} \left(\Delta(v_\xi) - q^{-(\lambda+2\rho,\lambda)}\right) = 0$.

We introduce some notation. Let $\mathcal{T}^t$ be the set of tableaux of length t derived from the Bratteli diagram for $V^{\otimes t}$. Let

$$i^t = (0, s_1, s_2, \ldots, s_t) \in \mathcal{T}^t.$$

We write $\lambda_i^t = i^t$ where $\lambda = s_t$.

Let $i^t \in \mathcal{T}^t$ and $s_j, s_{j+1} \in i^t$. Define a map

$$p_{s_j}^{t-(j+1)}[s_{j+1}] : \left(V_{s_j} \otimes V\right) \otimes V^{\otimes(t-(j+1))} \to V_{s_{j+1}} \otimes V^{\otimes(t-(j+1))}$$

by

$$p_{s_j}^{t-(j+1)}[s_{j+1}] = p_{s_j}[s_{j+1}] \otimes \mathrm{id}^{\otimes(t-(j+1))}.$$

Lemma 3.8.2. *The map $p_{s_j}^{t-(j+1)}[s_{j+1}]$ satisfies*

(i) $\left(p_{s_j}^{t-(j+1)}[s_{j+1}]\right)^2 = p_{s_j}^{t-(j+1)}[s_{j+1}]$,

(ii) $p_{s_j}^{t-(j+1)}[s_{j+1}] \cdot p_{s_j}^{t-(j+1)}[r_{j+1}] = \delta_{s_{j+1},r_{j+1}} p_{s_j}^{t-(j+1)}[s_{j+1}]$,

(iii) $\sum_{s_{j+1}\in\mathcal{P}_{s_j}^+} p_{s_j}^{t-(j+1)}[s_{j+1}] = \mathrm{id}_{V_{s_j}\otimes V^{\otimes(t-j)}}$.

Proof. The proofs of parts (i) and (ii) follow from Lemma 3.8.1 (i) and (ii), respectively. The proof of (iii) follows from Lemma 3.8.1 (iii): explicitly, we have

$$\sum_{s_{j+1}\in\mathcal{P}_{s_j}^+} p_{s_j}^{t-(j+1)}[s_{j+1}] \cdot \left(V_{s_j}\otimes V\right)\otimes V^{\otimes(t-(j+1))} = V_{s_j}\otimes V\otimes V^{\otimes t-(j+1)}.$$

□

Definition 3.8.3. *Let $\tilde{p}_i^t[\lambda] \in End_{U_q(\mathfrak{g})}(V^{\otimes t})$ be a map $\tilde{p}_i^t[\lambda] : V^{\otimes t} \to V_\lambda \subset V^{\otimes t}$ defined by*

$$\tilde{p}_i^t[\lambda] = p_{s_{t-1}}^0[\lambda] p_{s_{t-2}}^1[s_{t-1}] \cdots p_{\epsilon_1}^{t-2}[s_2],$$

where $\lambda_i^t \in \mathcal{T}^t$. We say that $\tilde{p}_i^t[\lambda]$ projects from $V^{\otimes t}$ onto V_λ by the path $\lambda_i^t \in \mathcal{T}^t$ *and we call $\tilde{p}_i^t[\lambda]$ a* path projection of length t.

Lemma 3.8.3. *The map $\tilde{p}_i^t[\lambda]$ satisfies*

(i) $\left(\tilde{p}_i^t[\lambda]\right)^2 = \tilde{p}_i^t[\lambda]$,

(ii) $\tilde{p}_i^t[\lambda] \cdot \tilde{p}_j^t[\lambda] = \begin{cases} 0, & \text{if } i^t \neq j^t, \\ \tilde{p}_i^t[\lambda], & \text{if } i^t = j^t, \end{cases}$

(iii) $\tilde{p}_i^t[\lambda] \cdot \tilde{p}_j^t[\mu] = 0$ *if* $\lambda \neq \mu$.

Furthermore, the map $P_t = \sum_{i^t\in\mathcal{T}^t} \tilde{p}_i^t[\lambda]$ is the identity on $V^{\otimes t}$.

Proof.

(i) This follows from Lemma 3.8.2 (i).

(ii) For $i^t = j^t$ the case reduces to (i), let $i^t \neq j^t$ where

$$\begin{aligned} i^t &= (0, s_1, s_2, \ldots, s_{k-1}, s_k, r_{k+1}, \ldots, r_{t-2}, r_{t-1}, \lambda), \\ j^t &= (0, s_1, s_2, \ldots, s_{k-1}, s_k, s_{k+1}, \ldots, s_{t-2}, s_{t-1}, \lambda). \end{aligned}$$

Now i^t, $j^t \in \mathcal{T}^t$ and $r_{k+1} \neq s_{k+1}$ for some $2 \leq k+1 \leq t$, then

$$\begin{aligned}\tilde{p}_i^t[\lambda]\cdot\tilde{p}_j^t[\lambda] &= p^0_{r_{t-1}}[\lambda]p^1_{r_{t-2}}[r_{t-1}]\cdots p^{t-k-1}_{s_k}[r_{k+1}]p^{t-k}_{s_{k-1}}[s_k]p^{t-k+1}_{s_{k-2}}[s_{k-1}]\cdots p^{t-2}_{s_1}[s_2]\\ &\quad\times p^0_{s_{t-1}}[\lambda]p^1_{s_{t-2}}[s_{t-1}]\cdots p^{t-k-1}_{s_k}[s_{k+1}]p^{t-k}_{s_{k-1}}[s_k]p^{t-k+1}_{s_{k-2}}[s_{k-1}]\cdots p^{t-2}_{s_1}[s_2]\\ &= p^0_{r_{t-1}}[\lambda]p^0_{s_{t-1}}[\lambda]p^1_{r_{t-2}}[r_{t-1}]p^1_{s_{t-2}}[s_{t-1}]\cdots p^{t-k-1}_{s_k}[r_{k+1}]p^{t-k-1}_{s_k}[s_{k+1}]\\ &\quad\times p^{t-k}_{s_{k-1}}[s_k]p^{t-k}_{s_{k-1}}[s_k]p^{t-k+1}_{s_{k-2}}[s_{k-1}]p^{t-k+1}_{s_{k-2}}[s_{k-1}]\cdots p^{t-2}_{s_1}[s_2]p^{t-2}_{s_1}[s_2]\\ &= 0,\end{aligned}$$

as $p^{t-k-1}_{s_k}[r_{k+1}]\cdot p^{t-k-1}_{s_k}[s_{k+1}] = 0$.

(iii) Assume that

$$\begin{aligned} i^t &= (0, s_1, s_2, \ldots, s_{k-1}, s_k, r_{k+1}, \ldots, r_{t-2}, r_{t-1}, \lambda),\\ j^t &= (0, s_1, s_2, \ldots, s_{k-1}, s_k, s_{k+1}, \ldots, s_{t-2}, s_{t-1}, \mu),\end{aligned}$$

where $r_{k+1} \neq s_{k+1}$ for some $2 \leq k+1 \leq t$. The calculations are similar to those of (ii) and we have $\tilde{p}_i^t[\lambda]\cdot\tilde{p}_j^t[\mu] = 0$.

The last claim follows inductively from the result in Lemma 3.8.1 that $\sum_{\lambda\in\mathcal{P}^+_\mu} p_\mu[\lambda] = \mathrm{id}_{V_\mu\otimes V}$.

□

Recall that $\mathcal{C}_t$ is the algebra over $\mathbb{C}$ generated by the elements

$$\left\{\check{R}_i^{\pm1} \in End_{U_q(\mathfrak{g})}(V^{\otimes t}) \mid 1 \leq i \leq t-1\right\}.$$

Proposition 3.8.1. *For each path $\lambda_i^t \in \mathcal{T}^t$, $\tilde{p}_i^t[\lambda] \in \mathcal{C}_t$.*

Proof. We prove the proposition inductively. Firstly, for some appropriately chosen integral dominant weight ξ,

$$(\pi\otimes\pi)\Delta(v_\xi) = (\pi\otimes\pi)\left[(v_\xi\otimes v_\xi)\left(R^T_{\xi,\xi}R_{\xi,\xi}\right)^{-1}\right] = q^{-2(\epsilon_1+2\rho,\epsilon_1)}\check{\mathcal{R}}^{-2} \in \mathcal{C}_2.$$

Now assume that $\tilde{p}_i^{(t-1)}[\mu] \in \mathcal{C}_{(t-1)}$ where $\tilde{p}_i^{(t-1)}[\mu]$ is a path projection $\tilde{p}_i^{(t-1)}[\mu] : V^{\otimes(t-1)} \to V_\mu$ and V_μ is an irreducible $U_q(\mathfrak{g})$-submodule of $V^{\otimes(t-1)}$. We will show that $(\pi_\mu\otimes\pi)\Delta(v_\zeta)$ is an element of $\mathcal{C}_t$ for some appropriately chosen ζ. Let ζ be an integral dominant weight such that the element $v_\zeta \in \overline{U}_q^+(\mathfrak{g})$ acts as the multiplication by the scalar $q^{-(\lambda+2\rho,\lambda)}$ on each vector in the irreducible $U_q(\mathfrak{g})$-submodule $V_\lambda \subset V_\mu\otimes V$ for each $\lambda\in\mathcal{P}^+_\mu$. Now

$$\begin{aligned}(\pi_\mu\otimes\pi)\Delta(v_\zeta) &= (\pi_\mu\otimes\pi)\left[(v_\zeta\otimes v_\zeta)\left(R^T_{\zeta,\zeta}R_{\zeta,\zeta}\right)^{-1}\right]\\ &= q^{-(\mu+2\rho,\mu)-(\epsilon_1+2\rho,\epsilon_1)}(\tilde{p}_i^{(t-1)}[\mu]\otimes\mathrm{id})\left(\pi^{\otimes(t-1)}\otimes\pi\right)\left(\Delta^{(t-2)}\otimes\mathrm{id}\right)\left(R^T_{\zeta,\zeta}R_{\zeta,\zeta}\right)^{-1}\\ &= q^{-(\mu+2\rho,\mu)-(\epsilon_1+2\rho,\epsilon_1)}(\tilde{p}_i^{(t-1)}[\mu]\otimes\mathrm{id})\check{R}^{-1}_{t-1}\check{R}^{-1}_{t-2}\cdots\check{R}^{-1}_1\check{R}^{-1}_1\cdots\check{R}^{-1}_{t-2}\check{R}^{-1}_{t-1},\end{aligned}$$

where we have used the identity (fixing $R = R_{\zeta,\zeta}$):

$$\left(\pi^{\otimes(t-1)}\otimes\pi\right)\left(\Delta^{(t-2)}\otimes\mathrm{id}\right)R = \left(\pi^{\otimes(t-1)}\otimes\pi\right)R_{1t}R_{2t}\cdots R_{(t-1)t},$$

which arises from (3.22).

□

3.9 Matrix units for $\mathcal{C}_t$

It is clear that the Bratteli diagram for $V^{\otimes t}$ is multiplicity free, as tensoring by the fundamental $U_q(\mathfrak{g})$-module V is multiplicity free. It follows then from Theorem 3.7.1 that the centraliser algebra $\mathcal{L}_t = End_{U_q(\mathfrak{g})}(V^{\otimes t})$ is isomorphic to the path algebra A_t obtained from the Bratteli diagram for $V^{\otimes t}$. Clearly, we have the inclusion $\mathcal{C}_t \subseteq \mathcal{L}_t$. The aim of this section is to show that $\mathcal{C}_t$ and $\mathcal{L}_t$ are in fact equal:

Theorem 3.9.1. *The centraliser algebra $\mathcal{L}_t = End_{U_q(\mathfrak{g})}\big(V^{\otimes t}\big)$ is generated by the elements*

$$\big\{\check{R}_i^{\pm 1} \in End_{U_q(\mathfrak{g})}\big(V^{\otimes t}\big)\big|\ i = 1, 2, \ldots, t-1\big\}.$$

To prove this theorem we firstly partition the matrix units in A_t into two groups: the *projectors* $\{E_{SS} \in A_t|\ (S,S) \in \Omega^t\}$ and the *intertwiners* $\{E_{ST} \in A_t|\ (S,T) \in \Omega^t, S \neq T\}$ and we use an invertible homomorphism to map matrix units in A_t to matrix units in $\mathcal{C}_t$.

Recall that $V^{\otimes t}$ is completely reducible. Each matrix unit in $\mathcal{C}_t$ corresponding to a projector in A_t projects down from $V^{\otimes t}$ onto an irreducible $U_q(\mathfrak{g})$-submodule $V_\lambda \subset V^{\otimes t}$. Each matrix unit in $\mathcal{C}_t$ corresponding to an intertwiner in A_t maps between isomorphic irreducible $U_q(\mathfrak{g})$-submodules of $V^{\otimes t}$.

Recall that the homomorphism $\Upsilon : g_i \mapsto -\check{R}_i$, given in Lemma 3.6.4, yields a representation of $\mathscr{BW}_t(-q^{2n}, q)$ in $\mathcal{C}_t$. In Subsection 3.9.1 we will write down the matrix units in a semisimple quotient of $\mathscr{BW}_t(-q^{2n}, q)$ that map via Υ onto the projectors and intertwiners in $\mathcal{C}_t$. We will do this for the intertwiners, but we choose to define the projectors more straightforwardly using our previous work.

The projections E_{SS} that project down from $V^{\otimes t}$ onto irreducible $U_q(\mathfrak{g})$-submodules $V_{shp(S)} \subset V^{\otimes t}$ that we defined in Section 3.8, are elements of $\mathcal{C}_t$, and they satisfy the equations satisfied by the projector matrix units: $(E_{SS})^2 = E_{SS}$ and $\sum_{S \in \mathcal{T}^t} E_{SS} = \mathrm{id}_{V^{\otimes t}}$. We fix the projectors in $\mathcal{C}_t$ as follows: we map the projector $E_{SS} \in A_t$ to $\tilde{p}_i^t[\lambda] \in \mathcal{C}_t$, where $\lambda_i^t = S \in \mathcal{T}^t$ is a path of length t: $E_{SS} \leftrightarrow \tilde{p}_i^t[\lambda]$.

All we need to do now is to construct the matrix units in $\mathcal{C}_t$ corresponding to the intertwiners in A_t. We denote the matrix unit in $\mathcal{C}_t$ corresponding to $E_{MP} \in A_t$ by the same label E_{MP}. This should not cause confusion: the precise meaning of E_{MP} in any given situation will be clear.

3.9.1 Matrix units in $\mathscr{BW}_t(-q^{2n}, q)$

In this subsection, we say that an algebra B is semisimple if it is isomorphic to a direct sum of matrix algebras, ie $B \cong \bigoplus_{i \in I} M_{b_i}(\mathbb{C})$, where $M_{b_i}(\mathbb{C})$ is the algebra of $b_i \times b_i$ matrices with complex entries. The algebra $\mathscr{BW}_t(-q^{2n}, q)$ is not semisimple at generic q [We90, Cor. 5.6] but Ram and Wenzl have constructed matrix units for the semisimple Birman-Wenzl-Murakami algebra $\mathscr{BW}_t$ defined over $\mathbb{C}(r, q)$ (the field of rational functions in r and q) for indeterminates r and q [RW92]. By replacing the indeterminates r and q with the complex numbers $-q^{2n}$ and q, respectively, we obtain matrix units in a semisimple quotient of $\mathscr{BW}_t(-q^{2n}, q)$. By then applying the map Υ to these matrix units, we obtain matrix units in $\mathcal{C}_t$.

Before doing this, let us introduce Young diagrams and discuss a relation between certain Young diagrams and the integral dominant highest weights of irreducible $U_q(osp(1|2n))$-modules. For each non-negative integer m, there exists a Young diagram for each partition of m. Let $m = m_1 + m_2 + \cdots + m_l$ be a partition of m, where $m_i - m_{i+1} \in \mathbb{Z}_+$ for each $i = 1, 2, \ldots, l-1$ and $m_l \in \mathbb{Z}_+$. The Young diagram representing this partition is a collection of m boxes arranged in l left-aligned rows where the i^{th} row from the top contains exactly m_i boxes. If $m \geq 1$, let c_i, $i = 1, 2, \ldots, m_1$, be the number of boxes in the i^{th} column from the left in the Young diagram, then $c_i - c_{i+1} \in \mathbb{Z}_+$ for each $i = 1, 2, \ldots, m_1 - 1$ and $c_{m_1} \in \{1, 2, \ldots, l\}$.

Recall that an integral dominant highest weight λ of an irreducible $U_q(osp(1|2n))$-module V_λ is of the form $\lambda = \sum_{i=1}^n \lambda_i \epsilon_i \in \mathcal{P}^+$ where $\lambda_i - \lambda_{i+1} \in \mathbb{Z}_+$ for each $i = 1, 2, \ldots, n-1$ and $\lambda_n \in \mathbb{Z}_+$. We can use a Young diagram to label the highest weight of V_λ: this Young diagram consists of $\sum_{i=1}^n \lambda_i$ boxes arranged in n left-aligned rows, where the i^{th} row contains exactly λ_i boxes. Let μ be a Young diagram containing no more than n rows of boxes and let μ_i be the number of boxes in the i^{th} row from the top for each $i = 1, 2, \ldots, n$. We can use μ to label an integral dominant highest weight of an irreducible representation of $U_q(osp(1|2n))$: the integral dominant highest weight that μ represents is $\sum_{i=1}^n \mu_i \epsilon_i \in \mathcal{P}^+$.

The algebra $\mathscr{BW}_t$

The Birman-Wenzl-Murakami algebra $\mathscr{BW}_t$, with r and q indeterminates, is equipped with a functional $\mathrm{tr} : \mathscr{BW}_t \to \mathbb{C}(r, q)$ which satisfies, amongst other relations [We90, Lem. 3.4 (d)],

$$\mathrm{tr}(a\chi b) = \mathrm{tr}(\chi)\mathrm{tr}(ab), \qquad \forall a, b \in \mathscr{BW}_{t-1}, \quad \chi \in \{g_{t-1}, e_{t-1}\}, \tag{3.41}$$

where we regard each element of $\mathscr{BW}_{t-1}$ as an element of $\mathscr{BW}_t$ under the obvious inclusion. The algebra $\mathscr{BW}_t$ is semisimple [We90, Thm. 3.5]. To discuss its structure, we introduce the Young lattice.

The Young lattice is the following infinite graph [We90, Sec. 1]. The vertices of the Young lattice are the Young diagrams; the vertices are grouped into levels so that each Young diagram with exactly t boxes labels a vertex on the t^{th} level of the Young lattice. The edges of the Young lattice are completely determined as follows: a vertex λ on the t^{th} level is connected to a vertex μ on the $(t+1)^{st}$ level by one edge if and only if λ and μ differ by exactly one box. We show the Young lattice up to the 4^{th} level in Figure 3.1, where the circle represents the Young diagram with no boxes. We say that the level containing the Young diagram with no boxes is the 0^{th} level. Note that the Young lattice is (apart from the 0^{th} level) identical to the Bratteli diagram for the sequence of inclusions of group algebras of the symmetric group: $\mathbb{C}S_1 \subset \mathbb{C}S_2 \subset \mathbb{C}S_3 \subset \cdots$.

Let Γ_t be the set of vertices on the t^{th} level of the Young lattice, that is, Γ_t is the set of Young diagrams containing $t - 2k \geq 0$ boxes, where k ranges over all of $\mathbb{Z}_+$. Then $\mathscr{BW}_t$

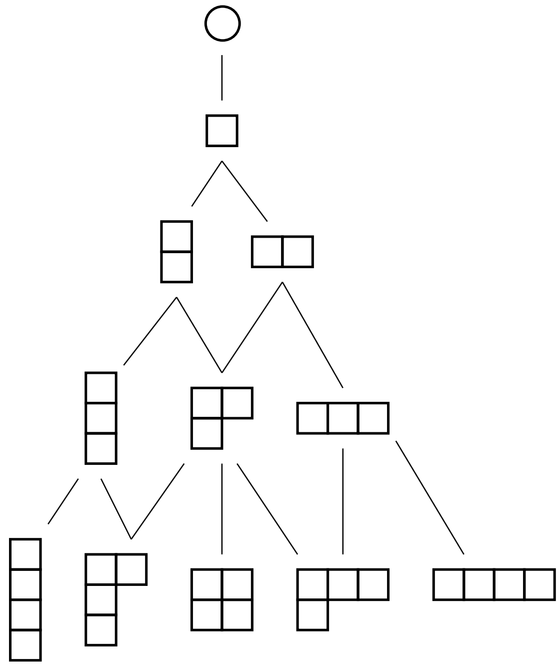


Figure 3.1: The Young lattice up to the 4^{th} level

is isomorphic to a direct sum of matrix algebras [We90, Thm. 3.5]:

$$\mathscr{BW}_t \cong \bigoplus_{\mu \in \Gamma_t} M_{b_\mu}(\mathbb{C}).$$

Ram and Wenzl defined matrix units for $\mathscr{BW}_t$ [RW92]. We will write down these matrix units below.

To label the matrix units of $\mathscr{BW}_t$ we need to discuss the Bratteli diagram of $\mathscr{BW}_t$, which is the following graph. The vertices of the Bratteli diagram of $\mathscr{BW}_t$ are divided into levels; for each $s = 0, 1, \ldots, t$, the elements of Γ_s are the vertices on the s^{th} level of the Bratteli diagram of $\mathscr{BW}_t$. The edges are as follows: a vertex μ on the s^{th} level is connected to a vertex λ on the $(s+1)^{st}$ level if and only if μ and λ differ by exactly one box.

We say that R is a path of length t in the Bratteli diagram of $\mathscr{BW}_t$ if R is a sequence of $t+1$ Young diagrams: $R = ([0], [1], r_2, \ldots, r_t)$ where $r_s \in \Gamma_s$ for each $s = 0, 1, \ldots, t$ and where r_i is connected to r_{i+1} for each $0 \leq i \leq t-1$. We say that $shp(R) = r_t$. Let ω_t be the set of pairs (R, S) of paths of length t in the Bratteli diagram of $\mathscr{BW}_t$ satisfying $r_t = s_t$.

Ram and Wenzl defined a set of matrix units $\{e_{ST} \in \mathscr{BW}_t | \ (S, T) \in \omega_t\}$ in [RW92]. This set is a basis of $\mathscr{BW}_t$ and the matrix units satisfy

$$e_{QR} e_{ST} = \delta_{RS} e_{QT}.$$

We obtain the matrix units in $\mathscr{BW}_t(-q^{2n}, q)/\mathscr{J}_t(-q^{2n}, q)$ by taking a certain proper subset of the matrix units in $\mathscr{BW}_t$ and replacing the indeterminates r and q with the complex numbers $-q^{2n}$ and q, respectively.

Let us fix some notation that we will use in the rest of this chapter and in Chapter 4. Given a sequence T of $t+1$ elements

$$T = (0, s_1, s_2, \ldots, s_{t-1}, s_t),$$

we fix T' to be the following sequence of t elements:

$$T' = (0, s_1, s_2, \ldots, s_{t-1}).$$

If T is a path of length t, then T' is the path of length $t-1$ obtained by removing the last vertex and edge of T.

Before defining the matrix units of $\mathscr{BW}_t$ we define some 'pre-matrix units'. Let T be a path of length t in the Bratteli diagram for $\mathscr{BW}_t$ such that $shp(T)$ has t boxes. We can identify T with a standard tableau containing the numbers $1, 2, \ldots, t$ in a canonical way. We do this by placing the number 1 in the top left hand box of $shp(T)$ and we then fill each box of $shp(T)$ with increasing numbers according to the path T [TW93, Sec. 4.2].

For each path T of length t in the Bratteli diagram for $\mathscr{BW}_t$, we define the number $d(T,i)$, for each $i = 1, 2, \ldots, t-1$, by

$$d(T,i) = c(i+1) - c(i) - r(i+1) + r(i), \tag{3.42}$$

where $c(j)$ and $r(j)$ denote the column and row, respectively, of the box containing the number j in the standard tableau corresponding to T. For each $d \in \mathbb{Z}\backslash\{0\}$, we define

$$b_d(q) = \frac{q^d(1-q)}{1-q^d}.$$

Let T be a path of length t in the Bratteli diagram for $\mathscr{BW}_t$. Firstly fix $o_{[1]} = 1 \in \mathscr{BW}_t$. Let R be a path of length $t-1$ defined by $R = T'$ and inductively define

$$o_T = \prod_S \frac{o_R g_{t-1} o_R - b_{d(S,t-1)}(q^2) o_R}{b_{d(T,t-1)}(q^2) - b_{d(S,t-1)}(q^2)} \in \mathscr{BW}_t,$$

where the product is over all paths S of length t where $shp(S)$ contains t boxes such that $S \neq T$ and $S' = R$. We write $o_{TT} = o_T$.

Let M and P be paths of length t in the Bratteli diagram for $\mathscr{BW}_t$ where $(M,P) \in \omega_t$ and $shp(M) = shp(P)$ has exactly t boxes and $shp(M') = shp(P')$, then we define

$$o_{MP} = o_{M'P'} o_{PP}.$$

Let M and P be paths of length t where $(M,P) \in \omega_t$ and $shp(M) = shp(P)$ has exactly t boxes and $shp(M') \neq shp(P')$, then the pre-matrix unit o_{MP} is defined more intricately. Choose paths $\overline{M}$ and $\overline{P}$ of length t that satisfy $shp(\overline{M}) = shp(M)$, $shp(\overline{P}) = shp(P)$ and the following three conditions:

(i) $\overline{M}'' = \overline{P}''$,

(ii) $shp(\overline{M}') = shp(M')$,

(iii) $shp(\overline{P}') = shp(P')$.

It may appear that these conditions cannot always be satisfied. However, paths $\overline{M}$ and $\overline{P}$ can always be obtained satisfying these conditions from the following construction [RW92].

By considering $\overline{M}$ and $\overline{P}$ as standard tableaux, we obtain the desired paths $\overline{M}$ and $\overline{P}$ by ensuring the following is true. Firstly, fix t to be in the same box in $\overline{M}$ (resp. $\overline{P}$) that t is in M (resp. P). Then, fix $(t-1)$ to be in the same box in $\overline{M}$ (resp. $\overline{P}$) that t is in P (resp. M). Lastly, for each $i = 1, 2, \ldots, t-2$, fix i to be in the same box in $\overline{M}$ that it is in $\overline{P}$.

We then define

$$o_{MP} = \frac{1 - q^{2d}}{\sqrt{(1 - q^{2(d+1)})(1 - q^{2(d-1)})}} o_{M'\overline{M}'} g_{t-1} o_{\overline{P}'P'} o_{PP},$$

where $d = d(\overline{M}, t-1)$ is as given in (3.42).

This completes the definition of the 'pre-matrix units'; now we define the matrix units for $\mathscr{BW}_t$.

Assume that the matrix units are known for $\mathscr{BW}_{t-1}$. Let M and P be paths of length t in the Bratteli diagram for $\mathscr{BW}_t$ where $shp(M) = \lambda = shp(P)$ and λ contains strictly fewer than t boxes, then we define

$$e_{MP} = \frac{Q_\lambda(r,q)}{\sqrt{Q_\mu(r,q) Q_{\widetilde{\mu}}(r,q)}} e_{M'S} e_{t-1} e_{TP'},$$

where S and T are paths of length $t-1$ satisfying

(i) $shp(S) = shp(M') = \mu$, and

(ii) $shp(T) = shp(P') = \widetilde{\mu}$, and

(iii) $S' = T'$, and

(iv) $shp(S') = \lambda = shp(T')$.

It may appear that these conditions cannot always be satisfied. However, there always exists a pair of paths S and T of length $t-1$ satisfying these conditions for the following reasons. Firstly, by examining the relevant Bratteli diagrams, it is clear that there are no intertwiner matrix units in $\mathscr{BW}_1$ and $\mathscr{BW}_2$. Now for each $t \geq 3$, a shape λ that has at most $t-2$ boxes and which labels a vertex on the t^{th} level of the Bratteli diagram for $\mathscr{BW}_t$ also labels a vertex on the $(t-2)^{nd}$ level of the Bratteli diagram. Hence there always exists at least one path of length $t-2$ in the Bratteli diagram for $\mathscr{BW}_t$ ending at the vertex

$shp(M)$ on the $(t-2)^{nd}$ level, as $shp(M)$ contains no more than $t-2$ boxes (this shows that (iii) and (iv) might be satisfied).

In the Bratteli diagram for $\mathscr{BW}_t$, two vertices λ and μ are connected by an edge only if their shapes differ by exactly one box. Now the vertices $shp(M')$ and $shp(P')$ on the $(t-1)^{st}$ level are connected to the vertex $shp(M)$ on the t^{th} level by one edge each, and they are also connected to the vertex $shp(M)$ on the $(t-2)^{nd}$ level by one edge each. It follows, then, that by fixing S and T to be paths of length $t-1$ that coincide on the first $t-2$ levels of the Bratteli diagram and that pass through the vertex $shp(M)$ on the $(t-2)^{nd}$ level, and also fixing $shp(S) = shp(M')$ and $shp(T) = shp(P')$ (which is always possible), we obtain the desired paths S and T.

Let M and P be paths of length t in the Bratteli diagram for $\mathscr{BW}_t$, where $(M, P) \in \omega_t$, and where $shp(M)$ contains t boxes. Then we define

$$e_{MP} = (1 - z_t) o_{MP},$$

where $z_t = \sum_P e_{PP}$ with the summation going over all paths P of length t such that $shp(P)$ contains fewer than t boxes.

The following fact is important [We90, Lem. 4.2]: let M be a path of length t in the Bratteli diagram for $\mathscr{BW}_t$ where $shp(M) = \lambda$, then

$$\mathrm{tr}(e_{MM}) = Q_\lambda(r, q)/x^t, \tag{3.43}$$

where $x = \dfrac{r - r^{-1}}{q - q^{-1}} + 1$ and $Q_\lambda(r, q)$ is the polynomial given in (3.45).

It is interesting to note that the quantum superdimension of the fundamental irreducible $U_q(osp(1|2n))$-module V is $(-q^{2n} + q^{-2n})/(q - q^{-1}) + 1$, which is just the polynomial x introduced in the preceding paragraph with the indeterminates q and r replaced with the complex numbers q and $-q^{2n}$, respectively.

The algebra $\mathscr{BW}_t(r, q)$

The algebra $\mathscr{BW}_t(r, q)$, with $r, q \in \mathbb{C}$, is equipped with a functional $\mathrm{tr} : \mathscr{BW}_t(r, q) \to \mathbb{C}$ which satisfies, amongst other relations [We90, Lem. 3.4 (d)],

$$\mathrm{tr}(a\chi b) = \mathrm{tr}(\chi)\mathrm{tr}(ab), \qquad \forall a, b \in \mathscr{BW}_{t-1}(r, q), \quad \chi \in \{g_{t-1}, e_{t-1}\}, \tag{3.44}$$

where we regard each element of $\mathscr{BW}_{t-1}(r, q)$ as an element of $\mathscr{BW}_t(r, q)$ under the obvious inclusion.

Define the annihilator ideal $\mathscr{J}_t(r, q) \subset \mathscr{BW}_t(r, q)$ with respect to tr by

$$\mathscr{J}_t(r, q) = \{b \in \mathscr{BW}_t(r, q) |\ \mathrm{tr}(ab) = 0,\ \forall a \in \mathscr{BW}_t(r, q)\}.$$

If q is not a root of unity and $r \neq \pm q^k$ for any $k \in \mathbb{Z}$, $\mathscr{J}_t(r, q) = 0$ and $\mathscr{BW}_t(r, q)$ is semisimple [We90, Cor. 5.6]. If $r = \pm q^k$ for some $k \in \mathbb{Z}$, then $\mathscr{J}_t(\pm q^k, q) \neq 0$ and the quotient $\mathscr{BW}_t(\pm q^k, q)/\mathscr{J}_t(\pm q^k, q)$ is semisimple [We90, Cor. 5.6]. Let us now fix

$k = 2n$ and $r = -q^{2n}$; recall that the homomorphism $\Upsilon : g_i \mapsto -\check{R}_i$ yields a representation of $\mathscr{BW}_t(-q^{2n}, q)$ in $\mathcal{C}_t$. The next task is to determine the structure of the quotient $\mathscr{BW}_t(-q^{2n}, q)/\mathscr{J}_t(-q^{2n}, q)$. We do this in the following work.

We now introduce a subgraph $\Gamma(-q^{2n}, q)$ of the Young lattice that we will use to describe the structure of $\mathscr{BW}_t(-q^{2n}, q)/\mathscr{J}_t(-q^{2n}, q)$. We inductively obtain the vertices of $\Gamma(-q^{2n}, q)$ as follows. Firstly fix the Young diagram with no boxes to belong to $\Gamma(-q^{2n}, q)$. The inductive step is that if the Young diagram μ belongs to $\Gamma(-q^{2n}, q)$, the Young diagram λ then also belongs to $\Gamma(-q^{2n}, q)$ if λ differs from μ by exactly one box and if $Q_\lambda(-q^{2n}, q) \neq 0$, where $Q_\lambda(r, q)$ is given in (3.45).

We now give the polynomial $Q_\lambda(r, q)$. Given a Young diagram λ, let (i, j) denote the box in the i^{th} row and the j^{th} column of λ, and let λ_i (resp. λ'_j) denote the number of boxes in the i^{th} row (resp. j^{th} column) of λ. We introduce some notation: we may denote the Young diagram λ by $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_k]$ where the i^{th} row contains λ_i boxes for each $i = 1, 2, \ldots, k$, and the l^{th} row contains no boxes for each $l > k$. The polynomial $Q_\lambda(r, q)$ is

$$\begin{aligned} Q_\lambda(r,q) &= \prod_{(j,j)\in\lambda} \frac{rq^{\lambda_j-\lambda'_j} - r^{-1}q^{-\lambda_j+\lambda'_j} + q^{\lambda_j+\lambda'_j-2j+1} - q^{-\lambda_j-\lambda'_j+2j-1}}{q^{h(j,j)} - q^{-h(j,j)}} \\ &\quad \times \prod_{(i,j)\in\lambda, i\neq j} \frac{rq^{d(i,j)} - r^{-1}q^{-d(i,j)}}{q^{h(i,j)} - q^{-h(i,j)}}, \end{aligned} \tag{3.45}$$

where the hooklength $h(i, j)$ is defined by $h(i, j) = \lambda_i - i + \lambda'_j - j + 1$, and where

$$d(i,j) = \begin{cases} \lambda_i + \lambda_j - i - j + 1, & \text{if } i \leq j, \\ -\lambda'_i - \lambda'_j + i + j - 1, & \text{if } i > j. \end{cases}$$

More intuitively, the hooklength $h(i, j)$ is the number of boxes below the (i, j) box in the j^{th} column plus the number of boxes to the right of the (i, j) box in the i^{th} row, plus one.

Now $h(i, j) \geq 1$ for all $(i, j) \in \lambda$, so $Q_\lambda(-q^{2n}, q)$ is well-defined for all λ. Also, for each $(j, j) \in \lambda$ we have

$$\begin{aligned} -q^{2n+\lambda_j-\lambda'_j} &+ q^{-2n-\lambda_j+\lambda'_j} + q^{\lambda_j+\lambda'_j-2j+1} - q^{-\lambda_j-\lambda'_j+2j-1} \\ &= (q^{-n+\lambda'_j-j+1/2} - q^{n-\lambda'_j+j-1/2})(q^{n+\lambda_j-j+1/2} + q^{-n-\lambda_j+j-1/2}), \end{aligned}$$

so $Q_\lambda(-q^{2n}, q) = 0$ if and only if one (or both) of the following conditions is satisfied:

(a) $q^{4n+2d(i,j)} = 1$ for some $(i, j) \in \lambda$ where $i \neq j$,

(b) $q^{2n-2\lambda'_j+2j-1} = 1$ or $q^{2n+2\lambda_j-2j+1} = -1$ for some j.

Now q is non-zero and not a root of unity, so (b) is never satisfied for any λ, and (a) is only satisfied if $d(i, j) = -2n$. We now determine the circumstances for which $d(i, j) = -2n$. If $i > j$, we can see that $\min(d(i, j)) = d(2, 1) = -\lambda'_1 - \lambda'_2 + 2$ from the constraints on the lengths of the columns of a Young diagram and it follows that $Q_\lambda(-q^{2n}, q) = 0$ if $\lambda'_1 + \lambda'_2 = 2n + 2$. Let us call a Young diagram λ *allowable* if $\lambda'_1 + \lambda'_2 \leq 2n + 1$.

Across all the allowable Young diagrams, let us calculate $\min(d(i,j))$ where $i < j$. If the first column of the allowable diagram λ contains $2n+1$ boxes, ie $\lambda'_1 = 2n+1$, then all the other columns must contain no boxes from the definition of an allowable diagram. For such a λ, there does not exist any box (i,j) in the i^{th} row and the j^{th} column with $i < j$ and so there is nothing more to consider in this case. Now if the first column of λ contains strictly fewer than $2n+1$ boxes, ie $\lambda'_1 \leq 2n$, then the following relations hold: $i \leq 2n$, $\lambda_i - j \geq 0$ and $\lambda_j \geq 0$. Then $d(i,j) = \lambda_i + \lambda_j - i - j + 1 \geq -2n + 1$, which means that $d(i,j) \neq -2n$ for all $i < j$.

It follows that $Q_\lambda(-q^{2n}, q) = 0$ if $\lambda'_1 + \lambda'_2 = 2n + 2$ and that $Q_\lambda(-q^{2n}, q) \neq 0$ for all allowable Young diagrams λ. Consequently, the vertices of $\Gamma(-q^{2n}, q)$ are all the allowable Young diagrams, that is, all the Young diagrams λ satisfying $\lambda'_1 + \lambda'_2 \leq 2n + 1$.

Now $\mathscr{I}_t(-q^{2n}, q) \neq 0$ and $\mathscr{BW}_t(-q^{2n}, q)$ is not semisimple. However, the quotient $\mathscr{BW}_t(-q^{2n}, q)/\mathscr{I}_t(-q^{2n}, q)$ is semisimple:

$$\mathscr{BW}_t(-q^{2n}, q)/\mathscr{I}_t(-q^{2n}, q) \cong \bigoplus_{\lambda \in \Gamma(-q^{2n}, q)_t} M_{b_\lambda}(\mathbb{C}),$$

where $\Gamma(-q^{2n}, q)_t$ is the set of Young diagrams belonging to $\Gamma(-q^{2n}, q)$ with $t - 2k \geq 0$ boxes, where k ranges over all of $\mathbb{Z}_+$ [We90, Cor. 5.6].

We can obtain matrix units for $\mathscr{BW}_t(-q^{2n}, q)/\mathscr{I}_t(-q^{2n}, q)$ from the matrix units of $\mathscr{BW}_t$. We replace the indeterminates r and q in some of the latter matrix units with the complex numbers $-q^{2n}$ and q, respectively, to obtain matrix units for $\mathscr{BW}_t(-q^{2n}, q)/\mathscr{I}_t(-q^{2n}, q)$.

To label the matrix units for $\mathscr{BW}_t(-q^{2n}, q)/\mathscr{I}_t(-q^{2n}, q)$, we use the Bratteli diagram for $\mathscr{BW}_t(-q^{2n}, q)/\mathscr{I}_t(-q^{2n}, q)$, which we define in the same way as we defined the Bratteli diagram for $\mathscr{BW}_t$ but we replace Γ_s with the $\Gamma(-q^{2n}, q)_s$ detailed in the next paragraph, for each $s = 0, 1, \ldots, t$.

Recall that the sets $\Gamma(-q^{2n}, q)_s$ are given as follows. The graph $\Gamma(-q^{2n}, q)$ is a subgraph of the Young lattice and the vertices of $\Gamma(-q^{2n}, q)$ are all the allowable Young diagrams, that is, all the Young diagrams λ satisfying $\lambda'_1 + \lambda'_2 \leq 2n+1$. Then, for each $s = 0, 1, \ldots, t$, $\Gamma(-q^{2n}, q)_s$ is the set of Young diagrams belonging to $\Gamma(-q^{2n}, q)$ that contain exactly $s - 2k \geq 0$ boxes, where k ranges over all of $\mathbb{Z}_+$.

We say that $T = (0, s_1, s_2, \ldots, s_t)$ is a path of length t in the Bratteli diagram for $\mathscr{BW}_t(-q^{2n}, q)/\mathscr{I}_t(-q^{2n}, q)$ if $s_i \in \Gamma(-q^{2n}, q)_i$ for each i and if s_j is joined to s_{j+1} for each $j = 0, \ldots, t-1$.

Let $\omega(-q^{2n}, q)_t$ be the set of pairs (R, S) of paths of length t in the Bratteli diagram for $\mathscr{BW}_t(-q^{2n}, q)/\mathscr{I}_t(-q^{2n}, q)$ where $r_t = s_t$, that is $shp(R) = shp(S)$. The matrix units

$$\{e_{RS} \in \mathscr{BW}_t |\ (R, S) \in \omega(-q^{2n}, q)_t\}$$

are all well-defined and non-zero if the indeterminates r and q are replaced with the complex numbers $-q^{2n}$ and q, respectively. Henceforth we write e_{RS} to mean the matrix unit $e_{RS}(-q^{2n}, q) \in \mathscr{BW}_t(-q^{2n}, q)/\mathscr{I}_t(-q^{2n}, q)$. It is very important to note that $\mathrm{tr}(e_{SS}) \neq 0$ for all $(S, S) \in \omega(-q^{2n}, q)_t$ and that $e_{RS} \notin \mathscr{I}_t(-q^{2n}, q)$ for all $(R, S) \in \omega(-q^{2n}, q)_t$.

Matrix units in $\mathscr{BW}_t(-q^{2n},q)/\mathscr{J}_t(-q^{2n},q)$ and $\mathcal{C}_t$

We now relate the idempotent matrix units in $\mathscr{BW}_t(-q^{2n},q)/\mathscr{J}_t(-q^{2n},q)$ to the projectors in $\mathcal{C}_t$ we defined at the start of this section. Let $\widetilde{\mathscr{BW}}_t(-q^{2n},q)$ be the semisimple subalgebra of $\mathscr{BW}_t(-q^{2n},q)$ spanned by the matrix units in $\mathscr{BW}_t(-q^{2n},q)/\mathscr{J}_t(-q^{2n},q)$, ie $\{e_{RS}|\ (R,S)\in\omega(-q^{2n},q)_t\}$.

Firstly, we will show that $\mathscr{BW}_t(-q^{2n},q) = \widetilde{\mathscr{BW}}_t(-q^{2n},q)\oplus\mathscr{J}_t(-q^{2n},q)$. Any $f\in\widetilde{\mathscr{BW}}_t(-q^{2n},q)$ can be written as

$$f = \sum_{(S,T)\in\omega(-q^{2n},q)_t} f_{ST}e_{ST}, \qquad f_{ST}\in\mathbb{C},$$

where $f_{ST}\neq 0$ for at least one pair (S,T) of paths. Fix (A,B) to be such a pair, then

$$\mathrm{tr}(e_{BA}f) = \mathrm{tr}(f_{AB}e_{BA}e_{AB}) = f_{AB}\mathrm{tr}(e_{BB})\neq 0,$$

as $\mathrm{tr}(e_{BB})\neq 0$. Thus any non-zero f belonging to $\widetilde{\mathscr{BW}}_t(-q^{2n},q)$ does not belong to $\mathscr{J}_t(-q^{2n},q)$, yielding

$$\mathscr{BW}_t(-q^{2n},q) = \widetilde{\mathscr{BW}}_t(-q^{2n},q)\oplus\mathscr{J}_t(-q^{2n},q).$$

Then we can write $a=\widetilde{a}+a_j$ for each $a\in\mathscr{BW}_t(-q^{2n},q)$, where $\widetilde{a}\in\widetilde{\mathscr{BW}}_t(-q^{2n},q)$ and $a_j\in\mathscr{J}_t(-q^{2n},q)$.

Now define

$$\mathscr{P}_t = \sum_{(S,S)\in\omega(-q^{2n},q)_t} e_{SS}\in\widetilde{\mathscr{BW}}_t(-q^{2n},q),$$

then $\mathscr{P}_t a\mathscr{P}_t=\widetilde{a}$, which can be seen by regarding $\mathscr{BW}_t(-q^{2n},q)$ as a matrix algebra.

Let us now turn our attention to $\mathcal{C}_t$. Define $J_t\subset\mathcal{C}_t$ to be the annihilator ideal of $\mathcal{C}_t$ with respect to the quantum supertrace:

$$J_t = \{b\in\mathcal{C}_t|\ str_q(ab)=0,\ \forall a\in\mathcal{C}_t\}.$$

Now define a map $\psi:\mathcal{C}_t\to\mathbb{C}$ by

$$\psi(X) = str_q(X)/\big(sdim_q(V)\big)^t,$$

then $\psi(X)=0$ if and only if $str_q(X)=0$, and furthermore,

$$\psi\big(\Upsilon(a)\big) = \mathrm{tr}(a), \qquad \forall a\in\mathscr{BW}_t(-q^{2n},q), \tag{3.46}$$

from Lemma 3.9.1. Thus we can regard J_t as the annihilator ideal of $\mathcal{C}_t$ with respect to ψ.

Now we will use Eq. (3.46) to show that

$$\Upsilon\big(\mathscr{J}_t(-q^{2n},q)\big) = J_t. \tag{3.47}$$

We firstly show that $\Upsilon\big(\mathscr{J}_t(-q^{2n},q)\big) \subseteq J_t$. Let b be an arbitrary element of $\mathscr{J}_t(-q^{2n},q)$, then $\mathrm{tr}(ab)=0$ for all $a \in \mathscr{BW}_t(-q^{2n},q)$, and the surjectivity of Υ, in addition to the fact that $\psi\big(\Upsilon(ab)\big)=\mathrm{tr}(ab)$, means that $\Upsilon(a) \in J_t$.

Now let B be an arbitrary element of J_t, then there is some b belonging to $\mathscr{BW}_t(-q^{2n},q)$ satisfying $B=\Upsilon(b)$, and furthermore, $b \in \mathscr{J}_t(-q^{2n},q)$ as $\mathrm{tr}(ab)=\psi\big(\Upsilon(a)\Upsilon(b)\big)=0$ for all $a \in \mathscr{BW}_t(-q^{2n},q)$. Then $\Upsilon\big(\mathscr{J}_t(-q^{2n},q)\big) \supseteq J_t$, proving Eq. (3.47).

The surjectivity of Υ implies that

$$\mathcal{C}_t = \Upsilon\big(\widetilde{\mathscr{BW}}_t(-q^{2n},q)\big) + J_t,$$

and we will show that this sum is direct. To see this, assume that there exists some non-zero element F of $\mathcal{C}_t$ belonging to $\Upsilon\big(\widetilde{\mathscr{BW}}_t(-q^{2n},q)\big)$ and also to J_t, then $str_q(XF)=0$ for all $X \in \mathcal{C}_t$. However, F is the image of a linear combination of matrix units:

$$F = \sum_{(S,T)\in\omega(-q^{2n},q)_t} f_{ST}\Upsilon(e_{ST}), \qquad f_{ST} \in \mathbb{C},$$

where $f_{ST} \neq 0$ for at least one pair (S,T). Assume that (A,B) is such a pair, then by similar reasoning as previously, $str_q(\Upsilon(e_{BA})F) \neq 0$ which contradicts the assumption that $F \in J_t$. Thus $\Upsilon\big(\widetilde{\mathscr{BW}}_t(-q^{2n},q)\big) \cap J_t = 0$, and

$$\mathcal{C}_t = \Upsilon\big(\widetilde{\mathscr{BW}}_t(-q^{2n},q)\big) \oplus J_t. \tag{3.48}$$

It is clear that the image of each matrix unit $e_{ST} \in \widetilde{\mathscr{BW}}_t(-q^{2n},q)$ in $\mathcal{C}_t$ under the map Υ is again a matrix unit. Each matrix unit $e_{SS} \in \widetilde{\mathscr{BW}}_t(-q^{2n},q)$ is an idempotent, thus each $\Upsilon(e_{SS})$ is an idempotent that is also $U_q(\mathfrak{g})$-linear. Now $\big(\Upsilon(e_{SS})\big)V^{\otimes t} \neq 0$ as $str_q\big(\Upsilon(e_{SS})\big) = (sdim_q(V))^t\mathrm{tr}(e_{SS}) \neq 0$, and as $V^{\otimes t}$ is completely reducible, $\Upsilon(e_{SS})$ projects down from $V^{\otimes t}$ onto a direct sum of irreducible $U_q(\mathfrak{g})$-submodules of $V^{\otimes t}$. The matrix units $\{e_{SS} | \ (S,S) \in \omega(-q^{2n},q)_t\}$ are all orthogonal, thus all the $\Upsilon(e_{SS})$ are orthogonal.

Let S be a path of length t in the Bratteli diagram for $\widetilde{\mathscr{BW}}_t(-q^{2n},q)$; let $\lambda = shp(S)$. If λ, as a Young diagram, contains no more than n rows of boxes, then we can interpret $\Upsilon(e_{SS})$ as the projection from $V^{\otimes t}$ onto an irreducible $U_q(\mathfrak{g})$-submodule $V_\lambda \subseteq V^{\otimes t}$, where we use the Young diagram λ to label the integral dominant highest weight of V_λ as discussed in the third paragraph of Subsection 3.9.1. If λ, as a Young diagram, has more than n rows of boxes (that is, $\lambda'_1 > n$), then we must consider $\Upsilon(e_{SS})$ more carefully. In this case, λ does not have an immediate interpretation as the highest weight of a finite dimensional irreducible representation of $U_q(\mathfrak{g})$. However, there is a completely standard way of dealing with this problem. Each such λ satisfies $\lambda'_1+\lambda'_2 \leq 2n+1$ and we can regard λ as labelling an irreducible representation of the Lie group $SO(2n+1)$ as follows. Let $\widetilde{\lambda}$ be the following diagram: fix $\widetilde{\lambda}'_1 = 2n+1-\lambda'_1$ and $\widetilde{\lambda}'_j = \lambda'_j$ for all $j \geq 2$, then $\widetilde{\lambda}$ is a Young diagram (see the next paragraph), and the characters associated with the $SO(2n+1)$ representations labelled by λ and $\widetilde{\lambda}$ are the same [CK87, Sec. 2]. Furthermore, the $OSp(1|2n)$ supercharacters of the representations labelled by λ and $\widetilde{\lambda}$ are the same up to a factor of ± 1 [Fa86, CK87].

If R and S are paths of length t in the Bratteli diagram for $\widetilde{\mathscr{BW}}_t(-q^{2n},q)$ satisfying $shp(R) = \lambda$ and $shp(S) = \widetilde{\lambda}$, then $\Upsilon(e_{RR})$ and $\Upsilon(e_{SS})$ project down from $V^{\otimes t}$ onto isomorphic irreducible $U_q(\mathfrak{g})$-submodules of $V^{\otimes t}$.

We now show that the $\widetilde{\lambda}$ mentioned above is in fact a Young diagram. Write $\lambda'_1 = \lambda'_2 + k$ where $k \geq 1$, then $\widetilde{\lambda}'_1 = 2n+1-(\lambda'_2+k)$. Now $\widetilde{\lambda}$ is a Young diagram if $\widetilde{\lambda}'_1 \geq \widetilde{\lambda}'_2$ (which is just $2n+1-k \geq 2\lambda'_2$) and this is true as $\lambda'_1+\lambda'_2 = 2\lambda'_2+k \leq 2n+1$.

Let λ be a Young diagram with more than n rows of boxes satisfying $\lambda'_1+\lambda'_2 \leq 2n+1$ and let $\widetilde{\lambda}$ be the Young diagram given by $\widetilde{\lambda}'_1 = 2n+1-\lambda'_1$ and $\widetilde{\lambda}'_j = \lambda'_j$ for all $j \geq 2$. We now show that there do not exist idempotent matrix units e_{RR} and e_{SS} in $\widetilde{\mathscr{BW}}_t(-q^{2n},q)$ where $shp(R) = \lambda$ and $shp(S) = \widetilde{\lambda}$. We show this important result using an easy even/odd number argument. If the number of boxes in λ is even (resp. odd), then the number of boxes in $\widetilde{\lambda}$ is odd (resp. even), as

$$\widetilde{\lambda}'_1 \bmod 2 = (2n+1-\lambda'_1) \bmod 2 = (\lambda'_1+1) \bmod 2 \quad \text{and} \quad \widetilde{\lambda}'_j = \lambda'_j, \quad j \geq 2.$$

Now let r be an even (resp. odd) number satisfying $0 \leq r \leq t$, then the vertices on the r^{th} level of the Bratteli diagram for $\widetilde{\mathscr{BW}}_t(-q^{2n},q)$ are all the Young diagrams in $\Gamma(-q^{2n},q)$ with k boxes where $k \leq r$ is an even (resp. odd) number. Let $|\lambda|$ denote the number of boxes in the Young diagram λ, then $|\lambda| \bmod 2 = (|\widetilde{\lambda}|+1) \bmod 2$, and consequently it is not possible that λ and $\widetilde{\lambda}$ are vertices on the same level of the Bratteli diagram for $\widetilde{\mathscr{BW}}_t(-q^{2n},q)$.

As it is not possible that both λ and $\widetilde{\lambda}$ are vertices on the t^{th} level of Bratteli diagram for $\widetilde{\mathscr{BW}}_t(-q^{2n},q)$, at most only one of e_{RR} and e_{SS} exists in $\widetilde{\mathscr{BW}}_t(-q^{2n},q)$ where $shp(R) = \lambda$ and $shp(S) = \widetilde{\lambda}$, and thus no more than one of $\Upsilon(e_{RR})$ and $\Upsilon(e_{SS})$ exists for any t.

Let S be a path of length t in the Bratteli diagram for $\widetilde{\mathscr{BW}}_t(-q^{2n},q)$ and let R be a path of length t in the Bratteli diagram for $V^{\otimes t}$. We can directly compare the orthogonal idempotents $\Upsilon(e_{SS})$ with the orthogonal projectors $E_{RR} \in \mathcal{C}_t$ by examining the paths S and R. It can be seen from the definitions of the idempotents e_{SS} and the map Υ that the idempotents $\Upsilon(e_{SS})$ act on $V^{\otimes t}$ from the left-most tensor powers as follows. If R is a path of length 2, then $\Upsilon(e_{RR})$ as an element of $\mathcal{C}_t$ is $\Upsilon(e_{RR}) \otimes \mathrm{id}^{\otimes(t-2)}$. If S is a path of length i where $2 \leq i \leq t$, then $\Upsilon(e_{SS})$ as an element of $\mathcal{C}_t$ is $\Upsilon(e_{SS}) \otimes \mathrm{id}^{\otimes(t-i)}$.

Now let R be a path of length t in the Bratteli diagram for $\widetilde{\mathscr{BW}}_t(-q^{2n},q)$. Let S be the same path of length t as R except that if a Young diagram λ on the path R has more than n rows of boxes, we take the transformed Young diagram $\widetilde{\lambda}$ to be in S instead of λ. Then S is a path of length t in the Bratteli diagram for $V^{\otimes t}$. Recall that the integral dominant highest weights of the irreducible $U_q(\mathfrak{g})$-submodules of $V^{\otimes k}$, where $k \leq t$, are the vertices on the k^{th} level of the Bratteli diagram for $V^{\otimes t}$. Then $\Upsilon(e_{RR}) \in \mathcal{C}_t$ and $E_{SS} \in \mathcal{C}_t$ project onto the same irreducible $U_q(\mathfrak{g})$-submodule of $V^{\otimes t}$, and we also have $Q_{shp(R)}(-q^{2n},q) = sdim_q(V_{shp(S)})$ from (3.43) and (3.46).

Let $E_{SS} \in \mathcal{C}_t$ be a projector with the property that $(E_{SS}V^{\otimes t}) \cap (\Upsilon(e_{RR})V^{\otimes t}) = 0$ for all idempotent matrix units $e_{RR} \in \widetilde{\mathscr{BW}}_t(-q^{2n},q)$. We will show that no such projector exists.

Suppose that such a projector does exist, that E_{SS} is a projector that is orthogonal to $\Upsilon(e_{RR})$ for each idempotent matrix unit $e_{RR} \in \widetilde{\mathscr{BW}}_t(-q^{2n}, q)$. Then E_{SS} is orthogonal to $\Upsilon(e_{RT})$ for each matrix unit $e_{RT} \in \widetilde{\mathscr{BW}}_t(-q^{2n}, q)$ where $R \neq T$ and $(R,T) \in \omega(-q^{2n}, q)_t$, as

$$E_{SS}\Upsilon(e_{RT}) = E_{SS}\Upsilon(e_{RR}e_{RT}e_{TT}) = 0 = \Upsilon(e_{RR}e_{RT}e_{TT})E_{SS} = \Upsilon(e_{RT})E_{SS},$$

as Υ is a homomorphism.

As E_{SS} is orthogonal to $\Upsilon(e_{RT})$ for each matrix unit $e_{RT} \in \widetilde{\mathscr{BW}}_t(-q^{2n}, q)$, $(R,T) \in \omega(-q^{2n}, q)_t$, it is true that $E_{SS} \in J_t$. To see this, assume the contrary, then

$$E_{SS} = \widetilde{E}_{SS} + E_j, \tag{3.49}$$

where $\widetilde{E}_{SS} \in \Upsilon\big(\widetilde{\mathscr{BW}}_t(-q^{2n}, q)\big)$ and $E_j \in J_t$ (we can write any element of $\mathcal{C}_t$ as a sum of elements of $\Upsilon\big(\widetilde{\mathscr{BW}}_t(-q^{2n}, q)$ and J_t from (3.48)). Then

$$\widetilde{E}_{SS} = \sum_{(R,T)\in\omega(-q^{2n},q)_t} c_{RT}\Upsilon(e_{RT}), \quad c_{RT} \in \mathbb{C}, \tag{3.50}$$

where $c_{RT} \neq 0$ for at least one pair $(R,T) \in \omega(-q^{2n}, q)_t$. Assume that $(A,B) \in \omega(-q^{2n}, q)_t$ is such a pair so that $c_{AB} \neq 0$, then

$$str_q\left(\Upsilon(e_{BA})\widetilde{E}_{SS}\right) = str_q\left(\sum_T c_{AT}\Upsilon(e_{BT})\right) = c_{AB}str_q\left(\Upsilon(e_{BB})\right) \neq 0, \tag{3.51}$$

as $c_{AB} \neq 0$ and $str_q(\Upsilon(e_{BB})) \neq 0$.

We will show that (3.51) is not true. Recall that E_{SS} satisfies

$$\Upsilon(e_{BA})E_{SS} = \Upsilon(e_{BA})(\widetilde{E}_{SS} + E_j) = 0, \tag{3.52}$$

and note that $\big(\Upsilon(e_{BA})\widetilde{E}_{SS}\big) \in \Upsilon\big(\widetilde{\mathscr{BW}}_t(-q^{2n}, q)$ and $\big(\Upsilon(e_{BA})E_j\big) \in J_t$. This last fact means that $\Upsilon(e_{BA})\widetilde{E}_{SS} \neq -\Upsilon(e_{BA})E_j$ if $\Upsilon(e_{BA})\widetilde{E}_{SS} \neq 0$ and $\Upsilon(e_{BA})E_j \neq 0$, and by re-examining (3.52) it is then clear that $\Upsilon(e_{BA})\widetilde{E}_{SS} = \Upsilon(e_{BA})E_j = 0$.

An implication of the result $\Upsilon(e_{BA})\widetilde{E}_{SS} = 0$ is that $str_q\left(\Upsilon(e_{BA})\widetilde{E}_{SS}\right) = 0$, however, this contradicts Eq. (3.51). This then implies that the assumption in (3.50) that $c_{RT} \neq 0$ for at least one pair $(R,T) \in \omega(-q^{2n}, q)_t$ is false, thus we have $\widetilde{E}_{SS} = 0$.

It then follows that it must be true that $E_{SS} = E_j \in J_t$ from (3.49). However, this is not true as $str_q(E_{SS}) = sdim_q(E_{SS}V^{\otimes t}) \neq 0$. Thus, our original assumption that there exists a projector $E_{SS} \in \mathcal{C}_t$ with the property that $(E_{SS}V^{\otimes t}) \cap (\Upsilon(e_{RR})V^{\otimes t}) = 0$ for all idempotent matrix units $e_{RR} \in \widetilde{\mathscr{BW}}_t(-q^{2n}, q)$ is not true.

3.9.2 Matrix units in $\mathcal{C}_t$

We have not yet proved that $\mathcal{L}_t = \mathcal{C}_t$. We will complete the proof in this subsection by defining a complete set of intertwiners in $\mathcal{C}_t$, which we will obtain by applying the map Υ to the intertwiner matrix units in $\widetilde{\mathscr{BW}}_t(-q^{2n}, q)$.

There is a one-to-one map between paths in the Bratteli diagram for $\widetilde{\mathscr{BW}}_t(-q^{2n}, q)$ and paths in the Bratteli diagram for $V^{\otimes t}$. Recall that each Young diagram on the k^{th} level of the Bratteli diagram for $\widetilde{\mathscr{BW}}_t(-q^{2n}, q)$ contains an even (resp. odd) number of boxes if k is an even (resp. odd) number. Each vertex λ on the k^{th} level of the Bratteli diagram for $\widetilde{\mathscr{BW}}_t(-q^{2n}, q)$ appears on the k^{th} level of the Bratteli diagram for $V^{\otimes t}$ unless λ has more than n rows of boxes, in which case the Young diagram $\widetilde{\lambda}$ appears instead, where $\widetilde{\lambda}$ is the Young diagram defined in the second paragraph after Eq. (3.48).

Given a path $\widetilde{T}$ of length t in the Bratteli diagram for $V^{\otimes t}$, we can write down the corresponding path T of length t in the Bratteli diagram for $\widetilde{\mathscr{BW}}_t(-q^{2n}, q)$ as follows. Write $\widetilde{T} = (0, s_1, s_2, \ldots, s_t)$ where s_i is a Young diagram on the i^{th} level of the Bratteli diagram for $V^{\otimes t}$. If i is an even (resp. odd) number and s_i contains an even (resp. odd) number of boxes, then s_i is also a vertex on the i^{th} level of the Bratteli diagram for $\widetilde{\mathscr{BW}}_t(-q^{2n}, q)$. If, however, i is an even (resp. odd) number and s_i contains an odd (resp. even) number of boxes, then $s_i = \widetilde{\lambda}$ is the vertex that is obtained by taking a vertex λ on the i^{th} level of the Bratteli diagram for $\widetilde{\mathscr{BW}}_t(-q^{2n}, q)$ and defining $\widetilde{\lambda}$ by $\widetilde{\lambda}'_1 = 2n + 1 - \lambda'_1$ and $\widetilde{\lambda}'_j = \lambda'_j$ for $j \geq 2$. Using this, we can define a path T of length t in the Bratteli diagram for $\widetilde{\mathscr{BW}}_t(-q^{2n}, q)$ corresponding to the path $\widetilde{T}$ of length t in the Bratteli diagram for $V^{\otimes t}$.

It is easy to define the intertwiners in $\mathcal{C}_t$ between the isomorphic irreducible $U_q(\mathfrak{g})$-submodules of $V^{\otimes t}$ obtained by using the projectors $E_{RR} \in \mathcal{C}_t$, where R is a path of length t in the Bratteli diagram for $V^{\otimes t}$. All we need do is to check that the images in $\mathcal{C}_t$ of the intertwiner matrix units are well-defined and non-zero.

We construct the intertwiners in $\mathcal{C}_t$ recursively. To do this, assume that all the matrix units in $\mathcal{C}_{t-1}$ have already been defined, and that they are non-zero. Note that the decomposition of $V \otimes V$ into irreducible $U_q(\mathfrak{g})$-submodules is multiplicity free, so no intertwiners exist in $\mathcal{C}_2$.

In the remainder of the subsection, let $\widetilde{M}$ and $\widetilde{P}$ be a pair of paths of length t in the Bratteli diagram for $V^{\otimes t}$ where $shp(\widetilde{M}) = shp(\widetilde{P})$ and $\widetilde{M} \neq \widetilde{P}$. Let M and P be the corresponding paths in the Bratteli diagram for $\widetilde{\mathscr{BW}}_t(-q^{2n}, q)$. The intertwiner $E_{\widetilde{M}\widetilde{P}} \in \mathcal{C}_t$ is precisely $E_{\widetilde{M}\widetilde{P}} = \Upsilon(e_{MP})$.

Let us firstly deal with the situation that $shp(M) = shp(P) = \lambda$ where λ contains strictly fewer than t boxes. Referring back to Subsection 3.9.1 we see that

$$E_{\widetilde{M}\widetilde{P}} = \Upsilon(e_{MP}) = \frac{Q_\lambda(-q^{2n}, q)}{\sqrt{Q_\mu(-q^{2n}, q) Q_{\widetilde{\mu}}(-q^{2n}, q)}} E_{\widetilde{M'}\widetilde{S}} \Upsilon(e_{t-1}) E_{\widetilde{T}\widetilde{P'}}, \quad e_{t-1} \in \mathscr{BW}_t(-q^{2n}, q),$$

where S and T are paths of length $t-1$ in the Bratteli diagram for $\widetilde{\mathscr{BW}}_t(-q^{2n}, q)$ such that

(i) $shp(S) = shp(M') = \mu$, and

(ii) $shp(T) = shp(P') = \widetilde{\mu}$, and

(iii) $S' = T'$, and

(iv) $shp(S') = \lambda = shp(T')$.

(Recall that such paths always exist.) Note that $Q_\mu(-q^{2n}, q) \neq 0$ and $Q_{\widetilde{\mu}}(-q^{2n}, q) \neq 0$; if it were true that $Q_\mu(-q^{2n}, q) = 0$ then μ would not be a vertex in the Bratteli diagram for $\widetilde{\mathscr{BW}}_t(-q^{2n}, q)$. Similar remarks hold for $\widetilde{\mu}$ and λ. Thus $E_{\widetilde{M}\widetilde{P}}$ is well-defined. (We will later show that $E_{\widetilde{M}\widetilde{P}}$ is non-zero.)

Now let us deal with the situation that $shp(M) = shp(P) = \lambda$ where λ contains exactly t boxes and $shp(M') = shp(P')$. Referring back to Subsection 3.9.1, we see that

$$E_{\widetilde{M}\widetilde{P}} = \Upsilon(e_{MP}) = \Upsilon((1 - z_t)o_{MP}),$$

where $o_{MP} = o_{M'P'}o_{PP}$ and $z_t = \sum_S e_{SS}$ with the summation going over all paths S of length t such that $shp(S)$ contains fewer than t boxes. It is not difficult to see that each such $\Upsilon(e_{SS})$ is some projection $E_{\widetilde{S}\widetilde{S}} \in \mathcal{C}_t$.

Now let us deal with the situation that $shp(M) = shp(P) = \lambda$ where λ contains exactly t boxes and $shp(M') \neq shp(P')$. Choose paths $\overline{M}$ and $\overline{P}$ of length t such that $shp(\overline{M}) = shp(M)$ and $shp(\overline{P}) = shp(P)$ and

(i) $\overline{M}'' = \overline{P}''$, and

(ii) $shp(\overline{M}') = shp(M')$, and

(iii) $shp(\overline{P}') = shp(P')$.

Such paths can always be chosen. Then

$$E_{\widetilde{M}\widetilde{P}} = \Upsilon(e_{MP}) = \Upsilon((1 - z_t)o_{MP}),$$

where

$$o_{MP} = \frac{1 - q^{2d}}{\sqrt{(1 - q^{2(d+1)})(1 - q^{2(d-1)})}} o_{M'\overline{M}'} g_{t-1} o_{\overline{P}'P'} o_{PP}, \qquad g_{t-1} \in \mathscr{BW}_t(-q^{2n}, q), \tag{3.53}$$

where $d = d(\overline{M}, t-1)$ is the integer defined by (3.42). The integer $|d(\overline{M}, i)| + 1$ is the number of boxes in the hook going through the boxes containing the numbers i and $(i+1)$ [TW93].

We now prove that the coefficient on the right hand side of (3.53) is well-defined and non-zero. It is not difficult to see that the coefficient is well-defined if $|d| \neq 1$, and we now show that this is always true. As $|d| + 1$ is the length of the hook going through the boxes containing the numbers $(t-1)$ and t, it is always true that $|d|+1 \geq 2$ as each such hook contains at least two boxes. Now the only situation in which it could possibly be true that $|d| = 1$ is when the boxes containing the numbers $(t-1)$ and t are immediately horizontally or vertically adjacent. However this cannot occur for the following reason: from the above construction, the number t is in the same box in $\overline{M}$ as the number $(t-1)$ is in $\overline{P}$, and the number t is in the same box in $\overline{P}$ that the number $(t-1)$ is in $\overline{M}$. It follows that if the numbers $(t-1)$ and t are immediately horizontally or vertically adjacent in $\overline{M}$, each must be in the corresponding 'swapped' box in $\overline{P}$, and then at least one of $\overline{M}$ or $\overline{P}$ *cannot* be a standard tableau. This contradicts the assumption that both $\overline{M}$ and $\overline{P}$ are standard tableaux, thus $|d| \neq 1$ and the coefficient in (3.53) is well-defined.

It remains for us to show that the coefficient in (3.53) is non-zero. This follows immediately from the fact that $|d| \neq 0$. Note that we have not yet proved that the matrix units are all non-zero.

Let us write E_{MP} to denote $E_{\widetilde{M}\widetilde{P}}$. We note that the matrix unit $E_{MP} \in \mathcal{C}_t$, where $M \neq P$, is an intertwiner between the isomorphic irreducible $U_q(\mathfrak{g})$-modules $E_{PP}(V^{\otimes t})$ and $E_{MM}(V^{\otimes t})$:

$$E_{MP} : E_{PP}(V^{\otimes t}) \to E_{MM}(V^{\otimes t}),$$

and that the whole collection of matrix units satisfy

$$E_{QR}E_{ST} = \delta_{RS}E_{QT}.$$

To show that each intertwiner E_{MP} is non-zero, it suffices to note that each projector E_{PP} is non-zero and that $E_{PP} = E_{PM}E_{MP}$.

We then have the complete sets of projectors and intertwiners in $\mathcal{C}_t$. This means that $\mathcal{L}_t = \mathcal{C}_t$, and also that $J_t = 0$. To see this last claim, note that the matrix units $\{E_{ST}|\ (S,T) \in \Omega^t\}$ are a basis for $\mathcal{C}_t$. Let X be an arbitrary element of $\mathcal{C}_t$, then

$$X = \sum_{(S,T)\in\Omega^t} x_{ST}E_{ST}, \qquad x_{ST} \in \mathbb{C},$$

where $x_{ST} \neq 0$ for at least one pair (S,T). Let (A,B) be such a pair, then

$$str_q(E_{BA}X) = str_q(x_{AB}E_{BA}E_{AB}) = x_{AB}str_q(E_{BB}) \neq 0,$$

thus $X \notin J_t$. As X is arbitrary, $J_t = 0$.

Note that we obtained $\mathcal{C}_t = \mathcal{L}_t$ by using the fact that λ and $\widetilde{\lambda}$ do not appear on the same level of the Bratteli diagram for $\widetilde{\mathscr{BW}}_t(-q^{2n}, q)$. If λ and $\widetilde{\lambda}$ did appear on the same level, we could only conclude from our work that there is a *proper* inclusion of $\mathcal{C}_t$ in $\mathcal{L}_t$ rather than an equality. Of course, in that event, there may actually be an equality, but a different method would have to be used to obtain all the intertwiners.

We now present the two lemmas used in this section.

Lemma 3.9.1. *Let* $\psi : \mathcal{C}_t \to \mathbb{C}$ *be a map defined by*

$$\psi(X) = str_q(X)/\big(sdim(V)\big)^t,$$

and let tr *be the trace functional on* $\mathscr{BW}_t(-q^{2n}, q)$ *mentioned in (3.44). Then*

$$\psi\big(\Upsilon(a)\big) = \mathrm{tr}(a), \qquad \forall a \in \mathscr{BW}_t(-q^{2n}, q).$$

Proof. Any functional ϕ on $\mathscr{BW}_\infty(-q^{2n}, q)$ satisfying Eq. (3.44) for all $t \in \mathbb{N}$ is identical to tr [We90, Lem. 3.4 (d)], and we will show that $\psi \circ \Upsilon$ has this property.

To show that $\psi \circ \Upsilon$ satisfies Eq. (3.44), it suffices to show that for each $t \in \mathbb{N}$, we have

$$-\psi\big(\Upsilon(a)\check{R}_{t-1}\Upsilon(b)\big) = -\psi\big(\check{R}_{t-1}\big)\psi\big(\Upsilon(ab)\big), \qquad \forall a, b \in \mathscr{BW}_{t-1}(-q^{2n}, q), \tag{3.54}$$

as the element $e_{t-1} \in \mathscr{BW}_t(-q^{2n}, q)$ can be written as a function of the g_{t-1}'s. We will show that Eq. (3.54) is true using Lemma 3.9.2, which we give after this proof.

The left hand side of Eq. (3.54) is

$$-str_q^{\otimes t}\big(A\check{R}_{t-1}B\big)/\big(sdim_q(V)\big)^t, \tag{3.55}$$

where we write $str_q^{\otimes t}$ to mean that we take the quantum supertrace over all t tensor factors, and we also write $A = \Upsilon(a)$ and $B = \Upsilon(b)$. Now we can regard each $X \in \mathcal{C}_{t-1}$ as an element of $\mathcal{C}_t$ under the mapping $X \mapsto X \otimes \mathrm{id}$, then by applying the identity to the first $t-1$ tensor powers of (3.55) and taking the quantum supertrace over the t^{th} tensor power of (3.55), we obtain, using Lemma 3.9.2 and applying some simple but tedious calculations,

$$-str_q^{\otimes t}\big(A\check{R}_{t-1}B\big)/\big(sdim_q(V)\big)^t = \frac{-\chi_V(v^{\mp 1})}{sdim_q(V)} \, \frac{str_q^{\otimes(t-1)}\big(AB\big)}{\big(sdim_q(V)\big)^{t-1}}. \tag{3.56}$$

Now

$$\psi\big(\check{R}_{t-1}\big) = \chi_V(v^{\mp 1})/sdim_q(V),$$

and the right hand side of Eq. (3.56) equals the right hand side of Eq. (3.54). Now Eq. (3.54) is true for all a and b belonging to $\mathscr{BW}_{t-1}(-q^{2n}, q)$, and it remains to show that $\psi \circ \Upsilon$ is a functional on $\mathscr{BW}_\infty(-q^{2n}, q)$ satisfying Eq. (3.44) for all natural numbers t. This follows from the fact that $\psi(A \otimes \mathrm{id}) = \psi(A)$ for all $A \in \mathcal{C}_t$, thus we can regard ψ as well-defined in the inductive limit $\mathcal{C}_2 \subset \mathcal{C}_3 \subset \mathcal{C}_4 \subset \cdots$. This completes the proof.

□

The following lemma, which we used in the proof of Lemma 3.9.1, appears in [LG92, Lem. 2] and is proved in [Zh92a, Lem. 3.1].

Lemma 3.9.2. *Let* V *be the fundamental irreducible* $U_q(osp(1|2n))$*-module with highest weight* ϵ_1 *and let* π *be the representation of* $U_q(osp(1|2n))$ *afforded by* V*. Let* $\check{\mathcal{R}}_{V,V} \in End_{U_q(osp(1|2n))}(V \otimes V)$ *be as given in Eq. (3.29). Then*

$$(\mathrm{id} \otimes \mathrm{str})\big[(\mathrm{id} \otimes \pi)(\mathrm{id} \otimes K_{2\rho})\big]\check{\mathcal{R}}^{\pm 1}_{V,V} = q^{\pm(\epsilon_1, \epsilon_1 + 2\rho)}\mathrm{id} = \chi_V(v^{\mp 1})\mathrm{id}.$$

Chapter 4

Quantum $osp(1|2n)$ at roots of unity

In this chapter we define a $\mathbb{Z}_2$-graded ribbon Hopf algebra $U_q^{(N)}(osp(1|2n))$ that is a certain quotient of $U_q(osp(1|2n))$ where $q = \exp(2\pi i/N)$ for some integer $N \geq 3$, and we also study aspects of its representation theory. We define certain representations of $U_q^{(N)}(osp(1|2n))$, show that each of these representations is self-dual, and most importantly, prove tensor product decomposition theorems for these representations at even N. The results in this chapter are almost entirely new.

The structure of this chapter is as follows. In Section 4.1 we define the quotient algebra $U_q^{(N)}(osp(1|2n))$ and prove that it is a $\mathbb{Z}_2$-graded ribbon Hopf algebra. In Section 4.2 we define a finite number of finite dimensional $U_q^{(N)}(osp(1|2n))$-modules and prove that the dual $U_q^{(N)}(osp(1|2n))$-module to each of these modules is isomorphic to the original module. In Section 4.3 we prove tensor product theorems for these modules at even N. In Section 4.4 we present the technical proof that the projections defining the $U_q^{(N)}(osp(1|2n))$-modules are all well-defined.

In this chapter we use $\mathfrak{g}$ to denote $osp(1|2n)$, and fix $q = \exp(2\pi i/N)$ where $N \geq 3$ is an integer.

4.1 The $\mathbb{Z}_2$-graded ribbon Hopf algebra $U_q^{(N)}(osp(1|2n))$

In this section we introduce a quotient algebra of $U_q(\mathfrak{g})$ and prove that it is a $\mathbb{Z}_2$-graded ribbon Hopf algebra.

We firstly generalise the q-bracket from Chapter 3 to arbitrary elements of $U_q(\mathfrak{g})$. Each element in $U_q(\mathfrak{g})$ is a linear combination of products of $K_i^{\pm 1}, e_i, f_i,\ i = 1, 2, \ldots, n$, and every product X of the generators satisfies

$$K_i X K_i^{-1} = q^{(wt(X),\alpha_i)} X, \qquad i = 1, 2, \ldots, n, \tag{4.1}$$

for some integral element $wt(X) \in H^*$. Needless to say, $(wt(X), \alpha_i) \in \mathbb{Z}$ for all i. Then the q-bracket is a bilinear map $[\cdot,\cdot]_q : U_q(\mathfrak{g}) \times U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ defined by

$$[X, Y]_q = XY - (-1)^{[X][Y]} q^{(wt(X),wt(Y))} YX.$$

If both X and Y satisfy (4.1) for some $wt(X)$ and $wt(Y)$, the meaning of $[X,Y]_q$ is clear. The definition is generalised to arbitrary elements by linearity.

4.1.1 Definition of $U_q^{(N)}(osp(1|2n))$

We now define root vectors in $U_q(\mathfrak{g})$ using a particular normal ordering of the elements of ϕ, the set of positive roots of the reduced root system of $\mathfrak{g}$. Recall that $\phi = \{\epsilon_i, \epsilon_j \pm \epsilon_k | \ 1 \leq i \leq n,\ 1 \leq j < k \leq n\}$. We use the following notation to help in writing elements of ϕ in terms of the simple roots:

$$\begin{aligned}
\alpha_i + \cdots + \alpha_j &= \sum_{k=i}^{j} \alpha_k, \\
\alpha_i + \cdots + 2\alpha_j &= \sum_{k=i}^{j-1} \alpha_k + 2\alpha_j, \\
\alpha_i + \cdots + 2\alpha_j + \cdots + 2\alpha_n &= \sum_{k=i}^{j-1} \alpha_k + \sum_{m=j}^{n} 2\alpha_m.
\end{aligned}$$

Then

$$\begin{aligned}
\epsilon_i &= \alpha_i + \cdots + \alpha_n, && i = 1, \ldots, n, \\
\epsilon_i - \epsilon_j &= \alpha_i + \cdots + \alpha_{j-1}, && 1 \leq i < j \leq n, \\
\epsilon_i + \epsilon_j &= \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_n, && 1 \leq i < j \leq n.
\end{aligned}$$

We fix the normal order $\mathcal{N}(\phi)$ that we use in this chapter to be:

$$\begin{aligned}
&\alpha_1 \prec \alpha_1 + \alpha_2 \prec \alpha_1 + \alpha_2 + \alpha_3 \prec \ldots \prec \alpha_1 + \alpha_2 + \cdots + \alpha_k \prec \ldots \prec \alpha_1 + \cdots + \alpha_n \prec \\
&\quad \alpha_1 + \cdots + 2\alpha_n \prec \alpha_1 + \cdots + 2\alpha_{n-1} + 2\alpha_n \prec \ldots \prec \alpha_1 + \cdots + 2\alpha_k + \cdots + 2\alpha_n \prec \\
&\quad \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n \prec \\
&\alpha_2 \prec \alpha_2 + \alpha_3 \prec \alpha_2 + \alpha_3 + \alpha_4 \prec \ldots \prec \alpha_2 + \alpha_3 + \cdots + \alpha_k \prec \ldots \prec \alpha_2 + \cdots + \alpha_n \prec \\
&\quad \alpha_2 + \cdots + 2\alpha_n \prec \alpha_2 + \cdots + 2\alpha_{n-1} + 2\alpha_n \prec \ldots \prec \alpha_2 + \cdots + 2\alpha_k + \cdots + 2\alpha_n \prec \\
&\quad \alpha_2 + 2\alpha_3 + \cdots + 2\alpha_n \prec \\
&\qquad \vdots \\
&\alpha_j \prec \alpha_j + \alpha_{j+1} \prec \ldots \prec \alpha_j + \alpha_{j+1} + \cdots + \alpha_k \prec \ldots \prec \alpha_j + \cdots + \alpha_n \prec \\
&\quad \alpha_j + \cdots + 2\alpha_n \prec \alpha_j + \cdots + 2\alpha_{n-1} + 2\alpha_n \prec \ldots \prec \alpha_j + \cdots + 2\alpha_k + \cdots + 2\alpha_n \prec \\
&\quad \alpha_j + 2\alpha_{j+1} + \cdots + 2\alpha_n \prec \\
&\qquad \vdots \\
&\alpha_{n-1} \prec \alpha_{n-1} + \alpha_n \prec \alpha_{n-1} + 2\alpha_n \prec \alpha_n.
\end{aligned}$$

We also define a second normal order $\overline{\mathcal{N}}(\phi)$, which we call the *opposite normal order* to $\mathcal{N}(\phi)$, by $y \prec x$ whenever $x \prec y$ in $\mathcal{N}(\phi)$.

Using $\mathcal{N}(\phi)$, we recursively define the root vectors $e_\mu, f_\mu \in U_q(\mathfrak{g})$ for each $\mu \in \phi$ as follows: firstly fix $e_{\alpha_i}=e_i$ and $f_{\alpha_i}=f_i$ for each $i=1,\dots,n$, then recursively define

$$\begin{aligned}
e_{\alpha_i+\alpha_{i+1}} &= [e_i,e_{i+1}]_q, && i=1,\dots,n-1,\\
e_{\alpha_i+\cdots+\alpha_j} &= [e_{\alpha_i+\cdots+\alpha_{j-1}},e_j]_q, && j=i+1,\dots,n,\\
e_{\alpha_i+\cdots+2\alpha_n} &= [e_{\alpha_i+\cdots+\alpha_n},e_n]_q, &&\\
e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n} &= [e_{\alpha_i+\cdots+2\alpha_{j+1}+\cdots+2\alpha_n},e_j]_q, && j=i+1,\dots,n-1,
\end{aligned}$$

$$\begin{aligned}
f_{\alpha_i+\alpha_{i+1}} &= [f_{i+1},f_i]_{q^{-1}}, && i=1,\dots,n-1,\\
f_{\alpha_i+\cdots+\alpha_j} &= [f_j,f_{\alpha_i+\cdots+\alpha_{j-1}}]_{q^{-1}}, && j=i+1,\dots,n,\\
f_{\alpha_i+\cdots+2\alpha_n} &= [f_n,f_{\alpha_i+\cdots+\alpha_n}]_{q^{-1}}, &&\\
f_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n} &= [f_j,f_{\alpha_i+\cdots+2\alpha_{j+1}+\cdots+2\alpha_n}]_{q^{-1}}, && j=i+1,\dots,n-1.
\end{aligned}$$

Using $\overline{\mathcal{N}}(\phi)$, we define a further set of elements $\overline{e}_\mu,\overline{f}_\mu\in U_q(\mathfrak{g})$ for each $\mu\in\phi$. Firstly fix $\overline{e}_i=e_i$ and $\overline{f}_i=f_i$ for each $i=1,\dots,n$, then $\overline{e}_\mu$ and $\overline{f}_\mu$ are recursively defined by

$$\begin{aligned}
\overline{e}_{\alpha_i+\alpha_{i+1}} &= [e_{i+1},e_i]_q, && i=1,\dots,n-1,\\
\overline{e}_{\alpha_i+\cdots+\alpha_j} &= [e_j,\overline{e}_{\alpha_i+\cdots+\alpha_{j-1}}]_q, && j=i+1,\dots,n,\\
\overline{e}_{\alpha_i+\cdots+2\alpha_n} &= [e_n,\overline{e}_{\alpha_i+\cdots+\alpha_n}]_q, &&\\
\overline{e}_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n} &= [e_j,\overline{e}_{\alpha_i+\cdots+2\alpha_{j+1}+\cdots+2\alpha_n}]_q, && j=i+1,\dots,n-1,
\end{aligned}$$

$$\begin{aligned}
\overline{f}_{\alpha_i+\alpha_{i+1}} &= [f_i,f_{i+1}]_{q^{-1}}, && i=1,\dots,n-1,\\
\overline{f}_{\alpha_i+\cdots+\alpha_j} &= \left[\overline{f}_{\alpha_i+\cdots+\alpha_{j-1}},f_j\right]_{q^{-1}}, && j=i+1,\dots,n,\\
\overline{f}_{\alpha_i+\cdots+2\alpha_n} &= \left[\overline{f}_{\alpha_i+\cdots+\alpha_n},f_n\right]_{q^{-1}}, &&\\
\overline{f}_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n} &= \left[\overline{f}_{\alpha_i+\cdots+2\alpha_{j+1}+\cdots+2\alpha_n},f_j\right]_{q^{-1}}, && j=i+1,\dots,n-1.
\end{aligned}$$

With these elements of $U_q(\mathfrak{g})$ we define the quotient algebra $U_q^{(N)}(\mathfrak{g})$ in Theorem 4.1.2 below. Recall that we fix

$$N'=\begin{cases} N, & \text{if } N \text{ is odd,}\\ N/2, & \text{if } N \text{ is even,}\end{cases} \quad\text{and } \overline{N}=\begin{cases} 2N, & \text{if } N \text{ is odd,}\\ N, & \text{if } N\equiv 0 \pmod 4,\\ N/2, & \text{if } N\equiv 2 \pmod 4.\end{cases}$$

Theorem 4.1.1. *The left ideal $\mathcal{I}\subset U_q(\mathfrak{g})$ generated by the elements of the set I below is a two-sided Hopf ideal of $U_q(\mathfrak{g})$:*

$$I=\left\{(e_\gamma)^{N'},(e_\beta)^{\overline{N}},(\overline{e}_\gamma)^{N'},(\overline{e}_\beta)^{\overline{N}},(f_\gamma)^{N'},(f_\beta)^{\overline{N}},(\overline{f}_\gamma)^{N'},(\overline{f}_\beta)^{\overline{N}},(J_i)^{\pm N}-1 \mid 1\le i\le n\right\}, \tag{4.2}$$

where $J_i=K_iK_{i+1}\cdots K_n$ for each $i=1,2,\dots,n$, and γ (resp. β) ranges over all the even (resp. odd) elements of $\phi=\{\epsilon_i,\epsilon_j\pm\epsilon_k \mid 1\le i\le n,\ 1\le j<k\le n\}$.

Proof. The proof of this result is technical and very lengthy. Thus we relegate it to Appendix D. □

It immediately follows from this theorem that

Theorem 4.1.2. *The quotient algebra $U_q^{(N)}(\mathfrak{g})$ defined by*

$$U_q^{(N)}(\mathfrak{g}) = U_q(\mathfrak{g})/\mathcal{I},$$

is a $\mathbb{Z}_2$-graded Hopf algebra.

We denote the image of $x \in U_q(\mathfrak{g})$ in $U_q^{(N)}(\mathfrak{g})$ under the canonical homomorphism by x.

The algebra $U_q^{(N)}(\mathfrak{g})$ has appeared in the literature, but only for the case $n = 1$ and then only for $N \geq 3$ an odd integer [Zh94]. The representation theory of $U_q^{(N)}(\mathfrak{g})$ is unknown except in this case, and in this case it is only partially known [Zh94, AB97].

We now introduce a very important representation of $U_q^{(N)}(\mathfrak{g})$ that we will extensively use in this thesis: the representation afforded by the *fundamental* irreducible $U_q^{(N)}(\mathfrak{g})$-module V. As a matter of notation, we will henceforth write V^{gen} to denote the fundamental irreducible module over $U_q(\mathfrak{g})$ where $q \neq 0$ is not a root of unity.

Lemma 4.1.1. *There exists a $(2n+1)$-dimensional irreducible $U_q^{(N)}(osp(1|2n))$-module V with highest weight $\epsilon_1 \in \mathcal{P}^+$. Let a basis of V be $\{v_i| \ -n \leq i \leq n\}$ where each v_i is a weight vector of weight ϵ_i, where we fix $\epsilon_{-i} = -\epsilon_i$ and $\epsilon_0 = 0$. Let v_1 be the highest weight vector of V. The action of an element $x \in U_q^{(N)}(osp(1|2n))$ on the weight vector v_i is identical to the action of the pre-image of x in $U_q(osp(1|2n))$ on the weight vector v_i of the fundamental $U_q(osp(1|2n))$-module V in Lemma 3.2.2 if we fix $q = \exp(2\pi i/N)$.*

Proof. Besides what is contained in the proof of Lemma 3.2.2, the only additional matter that we need to prove is that elements of I all act by zero and this is easily seen to be true. □

We always take the grading of the highest weight vector $v_1 \in V$ to be odd.

4.1.2 The universal R-matrix of $U_q^{(N)}(osp(1|2n))$

As the ideal $\mathcal{I} \subset U_q(\mathfrak{g})$ is a two-sided Hopf ideal, we can immediately write down the universal R-matrix of $U_q^{(N)}(\mathfrak{g})$ following [Zh92a]:

Proposition 4.1.1. *The universal R-matrix of $U_q^{(N)}(\mathfrak{g})$ is*

$$R = \left(\prod_{a=1}^{n} \sum_{b=0}^{N-1} (J_a)^b \otimes P_a[b] \right) \cdot \prod_{\gamma \in \phi} R_\gamma,$$

where

$$P_a[b] = \prod_{\substack{c=0 \\ c \neq b}}^{N-1} \frac{J_a - q^c}{q^b - q^c},$$

and where the product $\prod_{\gamma\in\phi} R_\gamma$ *is ordered in accordance with the normal order* $\mathcal{N}(\phi)$ *used to define the root vectors, so that* $\prod_{\gamma\in\phi} R_\gamma = R_{\gamma_1}R_{\gamma_2}\cdots R_{\gamma_k}$ *where* $\mathcal{N}(\phi) = \gamma_1 \prec \gamma_2 \prec \cdots \prec \gamma_k$. *Here,* R_γ *is*

$$R_\gamma = \begin{cases} \displaystyle\sum_{k=0}^{N'-1} \frac{(q-q^{-1})^k (e_\gamma \otimes f_\gamma)^k}{[k]^{q^{-2}}!}, & \text{if } [e_\gamma]=0, \\ \displaystyle\sum_{k=0}^{N-1} \frac{(q^{-1}-q)^k (e_\gamma \otimes f_\gamma)^k}{[k]^{-q^{-1}}!}, & \text{if } [e_\gamma]=1. \end{cases}$$

Note that $\prod_{\gamma\in\phi} R_\gamma$ is well-defined and each R_γ can be thought of as a truncation of the infinite sum of the corresponding factor of $\widetilde{R}$ in the universal R-matrix of $U_h(\mathfrak{g})$ so that it is well-defined for q at a root of unity.

Universal R-matrices have been written down for quotients of other quantum algebras and quantum superalgebras at roots of unity. This was first done for $U_q(sl_2)$ at $4k^{th}$ roots of unity where $k \in \mathbb{N}$ [RT91], then for the quantum algebras connected with the classical series of Lie algebras at odd roots of unity [ZC96] (also implied in [Kiri96]) and then for the quantum algebras connected with the exceptional Lie algebras G_2, F_4, E_8 at odd roots of unity [Zh97]. Amongst quantum superalgebras, universal R-matrices have been written down for quotients of $U_q(osp(1|2))$ [Zh94] and $U_q(gl(2|1))$ both at odd roots of unity [Zh95].

An immediate consequence of Proposition 4.1.1 is the

Corollary 4.1.1. *The quotient algebra* $U_q^{(N)}(\mathfrak{g})$ *is a* $\mathbb{Z}_2$*-graded quasitriangular Hopf algebra.*

Write the universal R-matrix of $U_q^{(N)}(\mathfrak{g})$ as $R = \sum_t a_t \otimes b_t$, then the element $u = \sum_t S(b_t)a_t(-1)^{[a_t]}$ satisfies

$$\epsilon(u) = 1, \qquad \Delta(u) = (u\otimes u)\left(R^T R\right)^{-1}, \qquad S^2(x) = uxu^{-1}, \quad \forall x \in U_q^{(N)}(\mathfrak{g}).$$

Furthermore, we have the following important theorem.

Theorem 4.1.3. *The quotient algebra* $U_q^{(N)}(\mathfrak{g})$ *is a* $\mathbb{Z}_2$*-graded ribbon Hopf algebra.*

Proof. Define the even element

$$v = uK_{2\rho}^{-1} \in U_q^{(N)}(\mathfrak{g}).$$

It suffices to prove that v is central in $U_q^{(N)}(\mathfrak{g})$ and that it satisfies the following relations:

$$\epsilon(v) = 1, \qquad v^2 = S(u)u, \qquad S(v) = v, \tag{4.3}$$

$$\Delta(v) = (v\otimes v)\left(R^T R\right)^{-1}. \tag{4.4}$$

We firstly prove that v is central in $U_q^{(N)}(\mathfrak{g})$. The proof is standard but we repeat it here as the elements u and v are crucial for later applications. Firstly, the homomorphism S^2 satisfies $S^2(x) = K_{2\rho}xK_{2\rho}^{-1}$ for all $x \in U_q^{(N)}(\mathfrak{g})$ (see Lemma 3.4.2). As v is invertible,

$$vxv^{-1} = uK_{2\rho}^{-1}xK_{2\rho}u^{-1} = uS^{-2}(x)u^{-1} = S^2(S^{-2}(x)) = x, \qquad \forall x \in U_q^{(N)}(\mathfrak{g}),$$

proving that v is central in $U_q^{(N)}(\mathfrak{g})$. The proofs of Eq. (4.4) and the first equation in (4.3) follow from the properties of u, and the proof of the third equation in (4.3) is similar to the proof of the corresponding equation in quantum algebras [Dr90, Prop. 5.1]. The second equation in (4.3) follows from the third. □

Definition 4.1.1. *Define $\check{\mathcal{R}}_{V,V} \in End_{\mathbb{C}}(V \otimes V)$ by $\check{\mathcal{R}}_{V,V}(v_i \otimes v_j) = P \circ (\pi \otimes \pi)R(v_i \otimes v_j)$ for all $v_i, v_j \in V$; $\check{\mathcal{R}}_{V,V}$ is an element of $End_{U_q^{(N)}(\mathfrak{g})}(V \otimes V)$. Define $\mathcal{C}_t$ to be the subalgebra of $End_{U_q^{(N)}(\mathfrak{g})}(V^{\otimes t})$ generated by the elements*

$$\left\{ \check{R}_i^{\pm 1} \in End_{U_q^{(N)}(\mathfrak{g})}(V^{\otimes t}) \middle| \ 1 \leq i \leq t-1 \right\},$$

where

$$\check{R}_i = \mathrm{id}^{\otimes(i-1)} \otimes \check{\mathcal{R}}_{V,V} \otimes \mathrm{id}^{\otimes(t-(i+1))}. \tag{4.5}$$

Remark 4.1.1. *Note that if we take the explicit expression of $\check{R}_i$ in (3.36) and set q to the appropriate root of unity, we obtain the $\check{R}_i$ defined here.*

4.2 $U_q^{(N)}(osp(1|2n))$-modules

In this section we define certain $U_q^{(N)}(\mathfrak{g})$-modules for each $N \geq 3$, we calculate their quantum superdimensions, and we also show that each of these modules is self-dual, that is, each of these modules is isomorphic to its dual $U_q^{(N)}(\mathfrak{g})$-module.

4.2.1 The truncated Weyl alcoves

In this subsection we define the *truncated Weyl alcoves* $\Lambda_N^+ \subseteq \overline{\Lambda_N^+}$, which are proper subsets of $X = \bigoplus_{i=1}^n \mathbb{Z}\epsilon_i \subset H^*$ that we will extensively use in the definition of the $U_q^{(N)}(\mathfrak{g})$-modules in Subsection 4.2.2. In the usual definition, many non-integral elements of H^* belong to the truncated Weyl alcoves. However, we only ever use the term in this thesis to refer to the relevant integral elements given below. The truncated Weyl alcoves are defined in terms of inequalities connected with the formula for the quantum superdimension of finite dimensional irreducible $U_q(\mathfrak{g})$-modules for $q \neq 0$ not a root of unity (see Eq. (3.26)). In this subsection, $(\cdot,\cdot) : H^* \times H^* \to \mathbb{C}$ is the non-degenerate, bilinear form given by (3.1) and n is the rank of $U_q^{(N)}(\mathfrak{g})$.

In this work it proves convenient to introduce the following notation: we write $\overline{\mathcal{P}}_N^+$ to denote $\overline{\Lambda_N^+} \cap \mathcal{P}^+$.

Definition 4.2.1. *We define $\overline{\Lambda_N^+} \subset X$ as follows:*

(i) for $N \equiv 0, 1, 3 \pmod 4$,

$$\overline{\Lambda_N^+} = \left\{ \lambda \in X \middle| \ 0 \leq \frac{2(\lambda+\rho,\alpha)}{(\alpha,\alpha)} \leq N', \ \forall \alpha \in \Phi_0^+ \right\},$$

(ii) for $N \equiv 2 \pmod 4$,

$$\overline{\Lambda_N^+} = \left\{ \lambda \in X \,\middle|\, 0 \leq \frac{2(\lambda+\rho,\alpha)}{(\alpha,\alpha)} \leq N',\ \forall \alpha \in \overline{\Phi}_0^+ \cup \Phi_1^+ \right\}.$$

Definition 4.2.2. *We define* $\Lambda_N^+ \subseteq \overline{\Lambda_N^+}$ *as follows:*

(i) for $N \equiv 0,1,3 \pmod 4$,

$$\Lambda_N^+ = \left\{ \lambda \in X \,\middle|\, 0 < \frac{2(\lambda+\rho,\alpha)}{(\alpha,\alpha)} < N',\ \forall \alpha \in \Phi_0^+ \right\}, \tag{4.6}$$

(ii) for $N \equiv 2 \pmod 4$,

$$\Lambda_N^+ = \left\{ \lambda \in X \,\middle|\, 0 < \frac{2(\lambda+\rho,\alpha)}{(\alpha,\alpha)} < N',\ \forall \alpha \in \overline{\Phi}_0^+ \cup \Phi_1^+ \right\}. \tag{4.7}$$

Lemma 4.2.1. *For each* $\lambda \in X$ *let* $\lambda_i = (\lambda, \epsilon_i)$ *for each* $i = 1, \ldots, n$. *There is an alternative description of* Λ_N^+ *and* $\overline{\mathcal{P}}_N^+ = \overline{\Lambda_N^+} \cap \mathcal{P}^+$*:*

$$\text{(i) } \Lambda_N^+ = \begin{cases} \{\lambda \in \mathcal{P}^+ |\ \lambda_1 + \lambda_2 < N' - 2n + 2\}, & \text{when } N \equiv 0,1,3 \pmod 4, \quad \text{for } n \geq 2, \\ \{\lambda \in \mathcal{P}^+ |\ \lambda_1 < N'\}, & \text{when } N \equiv 0,1,3 \pmod 4, \quad \text{for } n = 1, \\ \{\lambda \in \mathcal{P}^+ |\ \lambda_1 < N/4 - n + 1/2\}, & \text{when } N \equiv 2 \pmod 4. \end{cases}$$

$$\text{(ii) } \overline{\mathcal{P}}_N^+ = \begin{cases} \{\lambda \in \mathcal{P}^+ |\ \lambda_1 + \lambda_2 \leq N' - 2n + 2\}, & \text{when } N \equiv 0,1,3 \pmod 4, \quad \text{for } n \geq 2, \\ \{\lambda \in \mathcal{P}^+ |\ \lambda_1 \leq N'\}, & \text{when } N \equiv 0,1,3 \pmod 4, \quad \text{for } n = 1, \\ \{\lambda \in \mathcal{P}^+ |\ \lambda_1 \leq N/4 - n + 1/2\}, & \text{when } N \equiv 2 \pmod 4. \end{cases}$$

Proof. We will only prove (i). The proof of (ii) is similar and will be omitted. We will assume throughout this proof that $\lambda \in \Lambda_N^+$. Set $N \equiv 0,1,3 \pmod 4$. We will firstly show that λ must be an element of $\mathcal{P}^+$ if it is an element of Λ_N^+. Fixing $\alpha = 2\epsilon_n$ in (4.6) shows that λ satisfies

$$0 < \frac{2(\lambda+\rho, 2\epsilon_n)}{(2\epsilon_n, 2\epsilon_n)} < N',$$

which is just $0 < \lambda_n + 1/2 < N'$; note that $\lambda_n \geq 0$ as $\lambda \in X$. This completes the proof for $n = 1$ for $N \equiv 0,1,3 \pmod 4$. For $n \geq 2$ we use the following arguments. Fixing $\alpha = \epsilon_i - \epsilon_{i+1}$ for $i = 1, \ldots, n-1$ in (4.6) shows that λ satisfies

$$0 < \frac{2(\lambda+\rho, \epsilon_i - \epsilon_{i+1})}{(\epsilon_i - \epsilon_{i+1}, \epsilon_i - \epsilon_{i+1})} < N',$$

which tells us that $\lambda_i - \lambda_{i+1} \geq 0$ for each i. Then λ must be an element of $\mathcal{P}^+$ as $\lambda \in X$ and $\lambda_n \geq 0$. Fixing $\alpha = \epsilon_1 + \epsilon_2$ in (4.6) shows that

$$0 < \frac{2(\lambda+\rho, \epsilon_1 + \epsilon_2)}{(\epsilon_1 + \epsilon_2, \epsilon_1 + \epsilon_2)} < N', \tag{4.8}$$

and so $0 \leq \lambda_1 + \lambda_2 < N' - 2n + 2$ as $\lambda \in \mathcal{P}^+$. Fix $\alpha = \epsilon_i \pm \epsilon_j$ for $i < j$ where $\alpha \neq \epsilon_1 + \epsilon_2$, then

$$\frac{2(\lambda+\rho,\alpha)}{(\alpha,\alpha)} < \frac{2(\lambda+\rho,\epsilon_1+\epsilon_2)}{(\epsilon_1+\epsilon_2,\epsilon_1+\epsilon_2)}, \qquad \text{for each } \alpha \neq \epsilon_1+\epsilon_2.$$

Finally, note that

$$\frac{2(\lambda+\rho,2\epsilon_i)}{(2\epsilon_i,2\epsilon_i)} = (\lambda+\rho,\epsilon_i) < \frac{2(\lambda+\rho,\epsilon_1+\epsilon_2)}{(\epsilon_1+\epsilon_2,\epsilon_1+\epsilon_2)}.$$

From this, any $\lambda \in \mathcal{P}^+$ satisfying (4.8) also belongs to Λ_N^+, and so

$$\Lambda_N^+ = \{\lambda \in \mathcal{P}^+ |\ \lambda_1 + \lambda_2 < N' - 2n + 2\}, \qquad \text{when } N \equiv 0,1,3 \pmod 4.$$

Let us consider the case that $N \equiv 2 \pmod 4$; again, fix $\lambda \in \Lambda_N^+$. As above, we can show that $\lambda \in \mathcal{P}^+$. Fixing $\alpha = \epsilon_1$ in (4.7) shows that λ satisfies

$$0 < \frac{2(\lambda+\rho,\epsilon_1)}{(\epsilon_1,\epsilon_1)} < N', \tag{4.9}$$

which means that $0 \leq \lambda_1 < N/4 - n + 1/2$ as $\lambda \in \mathcal{P}^+$. It is also true that

$$\frac{2(\lambda+\rho,\alpha)}{(\alpha,\alpha)} < \frac{2(\lambda+\rho,\epsilon_1)}{(\epsilon_1,\epsilon_1)}, \qquad \forall \alpha = \epsilon_i \pm \epsilon_j, \epsilon_k \quad i<j,\ k \geq 2.$$

From this, any $\lambda \in \mathcal{P}^+$ satisfying (4.9) also belongs to Λ_N^+, and so

$$\Lambda_N^+ = \{\lambda \in \mathcal{P}^+ |\ \lambda_1 < N/4 - n + 1/2\}, \quad \text{for } N \equiv 2 \pmod 4,$$

which completes the proof of part (i). □

4.2.2 $U_q^{(N)}(osp(1|2n))$-modules

We now define the $U_q^{(N)}(\mathfrak{g})$-modules of interest in our work, and in doing so we use the truncated Weyl alcoves.

We have already defined the fundamental irreducible $U_q^{(N)}(\mathfrak{g})$-module V for all $n \geq 1$ and all $N \geq 3$. In this subsection we define a further set of $U_q^{(N)}(\mathfrak{g})$-modules V_λ for all $\lambda \in \overline{\mathcal{P}}_N^+$ when $N \geq 4$ is even. For technical reasons, we only define the $U_q^{(N)}(\mathfrak{g})$-modules for λ an element of a proper subset of $\overline{\mathcal{P}}_N^+$ when $N \geq 3$ is odd.

We define each of these new $U_q^{(N)}(\mathfrak{g})$-modules V_λ by using a projection operator $V^{\otimes t} \to V_\lambda$. This projection operator is an element of $End_{U_q^{(N)}(\mathfrak{g})}(V^{\otimes t})$ that we obtain by setting q to the appropriate root of unity in the projection $(V^{gen})^{\otimes t} \to V_\lambda^{gen}$ we defined in Section 3.8. Here, V^{gen} and V_λ^{gen} are finite dimensional irreducible $U_q(\mathfrak{g})$-modules where $q \neq 0$ is not a root of unity, and the highest weight of V_λ^{gen} is $\lambda \in \mathcal{P}^+$. It is crucially important that these projections are well defined and we deal with this problem in Section 4.4.

The projections are all well-defined for even N, but for odd N they are only well-defined if λ belongs to a proper subset of $\overline{\mathcal{P}}_N^+$; this is the reason we only define these modules for

λ an element of a proper subset of $\overline{\mathcal{P}}_N^+$ when N is odd. We have not been able to resolve this ill-definedness problem, but we conjecture that well-defined projections do exist for all $\lambda \in \overline{\mathcal{P}}_N^+$ at odd N.

Recall that the Bratteli diagram for $(V^{gen})^{\otimes t}$ defined in Chapter 3 encodes the decomposition of $(V^{gen})^{\otimes t}$ into irreducible $U_q(\mathfrak{g})$-submodules for $q \neq 0$ not a root of unity. Recall further that the elements on the j^{th} level of the Bratteli diagram for $(V^{gen})^{\otimes t}$, for each $j \leq t$, are called shapes, and that a sequence of $(t+1)$ elements:

$$\lambda_i^t = (0, \epsilon_1, s_2, \dots, s_{t-1}, \lambda),$$

is called a tableau of length t if s_j is a shape on the j^{th} level of the Bratteli diagram for each j. We let $\mathcal{T}^t$ denote the set of all tableaux of length t derived from the Bratteli diagram for $(V^{gen})^{\otimes t}$.

We now define two proper subsets of $\mathcal{T}^t$ that we will use in creating the projections from $V^{\otimes t}$ onto $U_q^{(N)}(\mathfrak{g})$-submodules of $V^{\otimes t}$.

Definition 4.2.3. *Define the two proper subsets $\widetilde{\mathcal{T}}^t$ and $\widehat{\mathcal{T}}^t$ of $\mathcal{T}^t$ by*

$$\widetilde{\mathcal{T}}^t = \left\{ i^t = (s_0, s_1, \dots, s_t) \in \mathcal{T}^t \middle|\ s_j \in \Lambda_N^+ \text{ for all } 0 \leq j \leq t \right\},$$

$$\widehat{\mathcal{T}}^t = \left\{ i^t = (s_0, s_1, \dots, s_t) \in \mathcal{T}^t \middle|\ s_j \in \Lambda_N^+ \text{ for each } 0 \leq j \leq t-1, \text{ and } s_t \in \overline{\mathcal{P}}_N^+ \right\}.$$

- The set $\widetilde{\mathcal{T}}^t$ is the set of all tableaux of length t where each shape in each tableau is an element of Λ_N^+.
- The set $\widehat{\mathcal{T}}^t$ is the set of all tableaux of length t where the first t shapes in each tableau are elements of Λ_N^+ and the last shape in each tableau is an element of $\overline{\mathcal{P}}_N^+$.
- Note that $\widetilde{\mathcal{T}}^t$ is a proper subset of $\widehat{\mathcal{T}}^t$. Also, it is convenient to write λ_i^t to mean i^t if $shp(i^t) = \lambda$.

Let $\lambda_i^t = (0, \epsilon_1, s_2, \dots, s_{t-1}, \lambda) \in \widehat{\mathcal{T}}^t$ be a tableau and let $\tilde{p}_i^t[\lambda]^{gen} \in End_{U_q(\mathfrak{g})}(V^{gen})^{\otimes t}$ be a path projection at generic q:

$$\tilde{p}_i^t[\lambda]^{gen} : (V^{gen})^{\otimes t} \to V_\lambda^{gen} \subset (V^{gen})^{\otimes t}, \tag{4.10}$$

then $\tilde{p}_i^t[\lambda]^{gen}$ can be written as the ratio z_1/z_2 where z_1 is an ordered polynomial in the elements $\left\{\check{R}_i^{\pm 1} \in End_{U_q(\mathfrak{g})}(V^{gen})^{\otimes t} \mid i = 1, \dots, t-1\right\}$ with coefficients in $\left\{\pm q^{-(\mu+2\rho,\mu)} \middle|\ \mu \in \mathcal{P}^+\right\}$, and z_2 is a non-zero product of elements of the form $\left\{q^{-(\mu+2\rho,\mu)} - q^{-(\nu+2\rho,\nu)} \middle|\ \mu, \nu \in \mathcal{P}^+\right\}$.

Definition 4.2.4. *For each tableau $\lambda_i^t = (0, \epsilon_1, s_2, \dots, s_{t-1}, \lambda) \in \widehat{\mathcal{T}}^t$, we define the path projection*

$$\tilde{p}_i^t[\lambda] \in End_{U_q^{(N)}(\mathfrak{g})}\left(V^{\otimes t}\right), \tag{4.11}$$

to be an identical expression in the $\check{R}_i^{\pm 1}$ and the $\pm q^{-(\mu+2\rho,\mu)}$ as in the definition of $\tilde{p}_i^t[\lambda]^{gen}$ in (4.10) except that we fix $q = \exp(2\pi i/N)$ and we also fix $\check{R}_i^{\pm 1} \in End_{U_q^{(N)}(\mathfrak{g})}(V^{\otimes t})$.

Explicitly, the projection $\tilde{p}_i^t[\lambda]$ from (4.11) is $\tilde{p}_i^t[\lambda] = p^0_{s_{t-1}}[\lambda]p^1_{s_{t-2}}[s_{t-1}]\cdots p^{t-2}_{\epsilon_1}[s_2]$, where

$$p^j_{s_{t-(j+1)}}[s_{t-j}] = \pi^{\otimes t}\left(\prod_{\substack{\nu\in\mathcal{P}^+_{s_{t-(j+1)}}\\ \nu\neq s_{t-j}}} \frac{\Delta^{(t-1-j)}(v) - q^{-(\nu+2\rho,\nu)}}{q^{-(s_{t-j}+2\rho,s_{t-j})} - q^{-(\nu+2\rho,\nu)}} \otimes \mathrm{id}^{\otimes j}\right), \qquad j = 0,1,\dots,t-2,$$

where π is the representation of $U_q^{(N)}(\mathfrak{g})$ afforded by the fundamental irreducible module V.

Lemma 4.2.2. *For each tableau $\lambda_i^t = (0,\epsilon_1,s_2,\dots,s_{t-1},\lambda) \in \widehat{\mathcal{T}}^t$, the path projection $\tilde{p}_i^t[\lambda] \in End_{U_q^{(N)}(\mathfrak{g})}(V^{\otimes t})$ is well-defined if either one of the two following conditions is met:*

(a) $N \geq 4$ is even, or

(b) $N \geq 3$ is odd and

  (i) $\lambda_1 \leq (N-1)/2 - n + 1$, or

  (ii) the components of $s_{t-1} = \overline{\lambda} \in \Lambda_N^+$ satisfy $\overline{\lambda}_1 = (N-1)/2 - n + 1$ and $\overline{\lambda}_2 = \overline{\lambda}_1$, and λ is given by $\lambda = \overline{\lambda} + \epsilon_1$.

The proof of this lemma is easy but very lengthy. To avoid interrupting the flow of thought we relegate the proof to Section 4.4 at the very end of this chapter. Here we wish to make the following remarks. Part (a) of the lemma means that at even $N \geq 4$, all projections that project down by a path in $\widehat{\mathcal{T}}^t$ onto a $U_q^{(N)}(\mathfrak{g})$-module labelled by an element of $\overline{\mathcal{P}}_N^+$ are well-defined. Part (b) means that at odd $N \geq 3$, only certain of the path projections are well defined. The significance of (b) is that we cannot prove tensor product theorems of the form of Theorem 4.3.3 for $U_q^{(N)}(\mathfrak{g})$-modules at odd N. As mentioned previously, we conjecture that at odd N there exist well-defined projections onto each module labelled by an element of $\overline{\mathcal{P}}_N^+$.

In the proof of Lemma 4.2.3 we use an argument that we will repeatedly use throughout this thesis, and here we wish to make some relevant comments. The idea in this argument is to take an element p^{gen} belonging to $End_{\mathbb{C}}(V^{gen})^{\otimes t}$ at generic $q \neq 0$, and to then specialise q to a root of unity yielding the element $p \in End_{\mathbb{C}}(V^{\otimes t})$. It is essential that p^{gen} is well-defined upon specialising q to the desired root of unity.

Let us write $\mathbb{C}[[q,q^{-1}]]$ to mean the ring of power series in q and q^{-1} with coefficients in $\mathbb{C}$. Let $\mathscr{V}$ be a vector subspace of $V^{\otimes t}$ over $\mathbb{C}[[q,q^{-1}]]$ with a basis

$$\{v_{i_1}\otimes v_{i_2}\otimes\cdots\otimes v_{i_t} \mid i_1,i_2,\dots,i_t = -n,\dots,n\},$$

where each v_i is a weight vector of the fundamental $U_q(\mathfrak{g})$-module V^{gen}. In order that p^{gen} is well-defined when q is specialised to the root of unity, we only ever apply the specialisation argument to

(i) those p^{gen} that we know are well-defined if q is specialised, or

(ii) those p^{gen} belonging to $End_{\mathbb{C}[[q,q^{-1}]]}(\mathscr{V})$.

In the proof of Lemma 4.2.3 (amongst other lemmas and propositions in this thesis), we claim that certain elements in $End_{U_q^{(N)}(\mathfrak{g})}(V^{\otimes t})$ can be obtained by taking corresponding elements in $End_{U_q(\mathfrak{g})}(V^{gen})^{\otimes t}$ and specialising q to the appropriate root of unity. In particular, we claim this for $\pi^{\otimes t}(\Delta^{(t-1)}(v))$ (where $v \in U_q^{(N)}(\mathfrak{g})$) and for the well-defined projections $\tilde{p}_i^t[\lambda]$ of $\mathcal{C}_t$ from Lemma 4.2.2.

To see this, firstly note that the matrices $\check{R}_i^{\pm 1}$ of $End_{U_q^{(N)}(\mathfrak{g})}(V^{\otimes t})$ and $(\check{R}_i^{gen})^{\pm 1}$ of $End_{U_q(\mathfrak{g})}(V^{gen})^{\otimes t}$ are exactly the same relative to the basis

$$\left\{v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_t} \middle|\, i_1, i_2, \ldots, i_t = -n, \ldots, n\right\}$$

of both $V^{\otimes t}$ and $(V^{gen})^{\otimes t}$ if we consider q to be an indeterminate (see Remark 4.1.1). The same is true for $\pi^{\otimes t}\big(\Delta^{(k)}(v) \otimes \mathrm{id}^{\otimes(t-k-1)}\big)$ for each $k = 0, 1, \ldots, t-1$ as it is a product in the $\check{R}_i^{\pm 1}$ with coefficients in $\mathbb{C}[[q, q^{-1}]]$ in both cases.

Now the projections $\tilde{p}_i^t[\lambda]$ of $\mathcal{C}_t$ in Lemma 4.2.2 are polynomials in products of the $\pi^{\otimes t}\big(\Delta^{(k)}(v) \otimes \mathrm{id}^{\otimes(t-k-1)}\big)$ with coefficients in $\mathbb{C}(q)$, but we know from Lemma 4.2.2 that they are all well-defined if q is specialised to the appropriate root of unity.

We will use specialisation arguments to prove various claims throughout this thesis. In each of these proofs we will rely on the comments here and we will show that any coefficients in $\mathbb{C}(q)$ appearing in the calculations are well-defined if q is specialised to the appropriate root of unity.

Lemma 4.2.3. *Let $\lambda_i^t = (0, \epsilon_1, s_2, \ldots, s_{t-1}, \lambda) \in \widehat{\mathcal{T}}^t$ be a tableau of length t and $\tilde{p}_i^t[\lambda] \in \mathcal{C}_t$ be a well-defined projection referred to in Lemma 4.2.2, then*

(i) $\big(\tilde{p}_i^t[\lambda]\big)^2 = \tilde{p}_i^t[\lambda]$,

(ii) $\tilde{p}_i^t[\lambda] \cdot \tilde{p}_j^t[\lambda] = \begin{cases} 0, & \text{if } i^t \neq j^t, \\ \tilde{p}_i^t[\lambda], & \text{if } i^t = j^t, \end{cases}$

(iii) $\tilde{p}_i^t[\lambda] \cdot \tilde{p}_j^t[\mu] = 0$ *if* $\lambda \neq \mu$.

Proof. To prove this lemma we consider similar equations in $End_{U_q(\mathfrak{g})}(V^{gen})^{\otimes t}$ where $q \neq 0$ is not a root of unity, and we apply a specialisation argument.

Consider the elements $\tilde{p}_i^t[\lambda]^{gen} \in End_{U_q(\mathfrak{g})}(V^{gen})^{\otimes t}$ that project down from $(V^{gen})^{\otimes t}$ onto irreducible $U_q(\mathfrak{g})$-submodules with highest weights in $\mathcal{P}^+$. For each such $\tilde{p}_i^t[\lambda]^{gen}$, we can define a matrix valued function $M_{\overline{q}}(\tilde{p}_i^t[\lambda]^{gen})$ of a complex parameter $\overline{q}$, for all $\overline{q} \neq 0$ that are not roots of unity, such that

$$M_{\overline{q}}\left(\tilde{p}_i^t[\lambda]^{gen}\right)\Big|_{\overline{q}=q} = \tilde{p}_i^t[\lambda]^{gen} \in End_{U_q(\mathfrak{g})}\left(V^{gen}\right)^{\otimes t},$$

for each $q \neq 0$ that is not a root of unity. Each component of the matrix $M_{\overline{q}}\left(\tilde{p}_i^t[\lambda]^{gen}\right)$ is a continuous function in $\overline{q}$ and has no poles for all non-zero $\overline{q}$ that are not roots of unity. Furthermore,

$$\lim_{\overline{q}\to q}\left[M_{\overline{q}}\left(\tilde{p}_i^t[\lambda]^{gen}\right)\right] = M_{\overline{q}}\left(\tilde{p}_i^t[\lambda]^{gen}\right)\Big|_{\overline{q}=q},$$

where in taking the limit we take the limit of each component of $M_{\overline{q}}\left(\tilde{p}_i^t[\lambda]^{gen}\right)$.

Let $\lambda_i^t = (0, \epsilon_1, s_2, \ldots, s_{t-1}, \lambda) \in \widehat{\mathcal{T}}^t$ be a tableau of length t where the path projection $\tilde{p}_i^t[\lambda] \in End_{U_q^{(N)}(\mathfrak{g})}\left(V^{\otimes t}\right)$ is well-defined from Lemma 4.2.2. Then we can extend the domain in $\overline{q}$ of $M_{\overline{q}}\left(\tilde{p}_i^t[\lambda]^{gen}\right)$ to include $\overline{q} = \exp\left(2\pi i/N\right)$ by defining

$$M_{e^{2\pi i/N}}\left(\tilde{p}_i^t[\lambda]^{gen}\right) = \tilde{p}_i^t[\lambda] \in End_{U_q^{(N)}(\mathfrak{g})}\left(V\right)^{\otimes t}.$$

No component of $M_{e^{2\pi i/N}}\left(\tilde{p}_i^t[\lambda]^{gen}\right)$ has a pole, and furthermore, $\tilde{p}_i^t[\lambda] = M_{e^{2\pi i/N}}\left(\tilde{p}_i^t[\lambda]^{gen}\right) = \lim_{\overline{q}\to e^{2\pi i/N}}\left[M_{\overline{q}}\left(\tilde{p}_i^t[\lambda]^{gen}\right)\right]$.

Now consider the equations in $End_{U_q(\mathfrak{g})}\left(V^{gen}\right)^{\otimes t}$ given in Lemma 3.8.3 (i)–(iii) corresponding to parts (i)–(iii) of this lemma. For each non-zero q that is not a root of unity, the equations in Lemma 3.8.3 (i)–(iii) are true. As each component of $M_{\overline{q}}\left(\tilde{p}_i^t[\lambda]^{gen}\right)$ is a continuous function of $\overline{q}$ and has no poles for all $\overline{q}$, the product of two such matrix valued functions is again a matrix valued function where each component of the product is a continuous function in $\overline{q}$ with no poles for all $\overline{q}$. It follows then that each equation in Lemma 3.8.3 (i)–(iii) can be obtained by taking the limit $\overline{q} \to q \in \mathbb{C}$ of the products of the matrix valued functions $M_{\overline{q}}\left(\tilde{p}_i^t[\lambda]^{gen}\right)$.

The proof of Eqs. (i)–(iii) of this lemma is then given by taking the limit $\overline{q} \to e^{2\pi i/N}$ of the corresponding equations in $End_{U_q(\mathfrak{g})}\left(V^{gen}\right)^{\otimes t}$. □

We now define the specific collection of new $U_q^{(N)}(\mathfrak{g})$-modules of interest.

Definition 4.2.5. *Let $\lambda_i^t = (0, \epsilon_1, s_2, \ldots, s_{t-1}, \lambda) \in \widehat{\mathcal{T}}^t$ be a tableau and let $\tilde{p}_i^t[\lambda] \in \mathcal{C}_t$ be a well-defined projection referred to in Lemma 4.2.2. We define the finite dimensional $U_q^{(N)}(\mathfrak{g})$-module V_λ depending on the tableau λ_i^t by*

$$V_\lambda = \tilde{p}_i^t[\lambda]\left(V^{\otimes t}\right). \tag{4.12}$$

We conjecture that each of the $U_q^{(N)}(\mathfrak{g})$-modules defined in (4.12) is an irreducible $U_q^{(N)}(\mathfrak{g})$-module. We now investigate properties of these $U_q^{(N)}(\mathfrak{g})$-modules. Note that $\mathcal{P}_\mu^+ \subseteq \overline{\Lambda_N^+}$ for each $\mu \in \Lambda_N^+$, and we write $\mathcal{P}_\mu^+ \cap \overline{\Lambda_N^+}$ below to make clear that each $\lambda \in \mathcal{P}_\mu^+$ is also an element of $\overline{\Lambda_N^+}$.

Lemma 4.2.4. *Let $i^t = (0, \epsilon_1, s_2, \ldots, s_{t-1}, \mu) \in \widetilde{\mathcal{T}}^t$ be a tableau of length t and let V_μ be a $U_q^{(N)}(\mathfrak{g})$-module defined by $V_\mu = \tilde{p}_i^t[\mu]\left(V^{\otimes t}\right)$ as given in Definition 4.2.5. Then there is a decomposition of $V_\mu \otimes V$ into a direct sum of $U_q^{(N)}(\mathfrak{g})$-submodules:*

$$V_\mu \otimes V = \bigoplus_{\lambda \in \mathcal{P}_\mu^+ \cap \overline{\Lambda_N^+}} V_\lambda, \tag{4.13}$$

where each V_λ is a $U_q^{(N)}(\mathfrak{g})$-submodule defined by $V_\lambda = \tilde{p}_j^{(t+1)}[\lambda]\left(V^{\otimes(t+1)}\right)$, where $\tilde{p}_j^{(t+1)}[\lambda] \in End_{U_q^{(N)}(\mathfrak{g})}(V^{\otimes(t+1)})$ is a path projection of length $t+1$, and $j^{(t+1)} = (0, \epsilon_1, s_2, \ldots, s_{t-1}, \mu, \lambda) \in \widehat{\mathcal{T}}^{(t+1)}$ is a tableau of length $t+1$.

Proof. Firstly, note that $\mathcal{P}_\mu^+ \subseteq \overline{\Lambda_N^+}$ for each $\mu \in \Lambda_N^+$. Lemma 4.2.3 (ii) implies the only vector belonging to any pair of distinct summands on the right hand side of Eq. (4.13) is the zero vector. As $j^{(t+1)} \in \widehat{\mathcal{T}}^{(t+1)}$, the path projection $\tilde{p}_j^{(t+1)}[\lambda] \in End_{U_q^{(N)}(\mathfrak{g})}(V^{\otimes(t+1)})$ is well-defined for each $\lambda \in \mathcal{P}_\mu^+ \cap \overline{\Lambda_N^+}$ from Lemma 4.2.2. To complete the proof we need only show that the inclusion $\bigcup_{\lambda \in \mathcal{P}_\mu^+ \cap \overline{\Lambda_N^+}} V_\lambda \subseteq V_\mu \otimes V$ is actually an equality, and this follows from the equation

$$\sum_{\lambda \in \mathcal{P}_\mu^+ \cap \overline{\Lambda_N^+}} \tilde{p}_j^{(t+1)}[\lambda] = \mathrm{id}_{V_\mu \otimes V},$$

which can be shown to be true by applying an argument similar to the one we used in the proof of Lemma 4.2.3 to the corresponding matrix equations at generic q. □

We can prove the following lemma by also applying a similar idea to the one we used in the proof of Lemma 4.2.3 to the corresponding matrix equations at generic q.

Lemma 4.2.5. *Let $\lambda_i^t \in \widehat{\mathcal{T}}^t$ be a tableau of length t and let V_λ be a $U_q^{(N)}(\mathfrak{g})$-module defined by $V_\lambda = \tilde{p}_i^t[\lambda]\left(V^{\otimes t}\right)$. Then*

$$v \cdot w = q^{-(\lambda+2\rho,\lambda)} w, \qquad \forall w \in V_\lambda,$$

where $v = uK_{2\rho}^{-1} \in U_q^{(N)}(\mathfrak{g})$.

Lemma 4.2.6. *Let $\lambda \in \overline{\mathcal{P}}_N^+$ and let V_λ be the $U_q^{(N)}(\mathfrak{g})$-module given in Definition 4.2.5, then*

(i) the quantum superdimension of V_λ is

$$sdim_q(V_\lambda) = (-1)^{[\lambda]} q^{-(\lambda,2\rho)} \prod_{\alpha \in \overline{\Phi}_0^+} \left(\frac{q^{2(\lambda+\rho,\alpha)} - 1}{q^{2(\rho,\alpha)} - 1} \right) \prod_{\beta \in \Phi_1^+} \left(\frac{q^{2(\lambda+\rho,\beta)} + 1}{q^{2(\rho,\beta)} + 1} \right), \qquad (4.14)$$

where $[\lambda]$ is the grading of the highest weight vector of the irreducible $U_q(\mathfrak{g})$-module V_λ^{gen} with highest weight $\lambda \in \mathcal{P}^+$ where $q \neq 0$ is not a root of unity,

(ii) $sdim_q(V_\lambda) \neq 0$ if $\lambda \in \Lambda_N^+$,

(iii) $sdim_q(V_\lambda) = 0$ if $\lambda \in \overline{\mathcal{P}}_N^+ \backslash \Lambda_N^+$.

Proof. (i) The quantum superdimension of $V_\lambda \subseteq V^{\otimes t}$ is

$$sdim_q(V_\lambda) = \mathrm{str}\big(\tilde{p}_i^t[\lambda] \cdot \pi^{\otimes t}\big(\Delta^{(t-1)}(K_{2\rho})\big)\big),$$

and by using the idea in the proof of Lemma 4.2.3, one can show that this gives the right hand side of (4.14) provided that it is well-defined. This is indeed the case for all $\lambda \in \overline{\mathcal{P}}_N^+$ as the denominator of the right hand side of (4.14) is non-zero. This is very easy to see, but nevertheless we present the detailed proof.

Set $N \equiv 0, 1, 3 \pmod 4$. For each $\alpha \in \overline{\Phi}_0^+$ we have $2(\rho, \alpha) \in 2\mathbb{Z}$ and $0 < 2(\rho, \alpha) < 2N'$, which implies that $q^{2(\rho,\alpha)} \neq 1$. Now if N is odd, $q^{2(\rho,\beta)} \neq -1$ for all $\beta \in \Phi_1^+$. If $N \equiv 0 \pmod 4$ then $q^{2(\rho,\beta)} = -1$ for some $\beta \in \Phi_1^+$ if and only if $2(\rho, \beta) \equiv N/2 \pmod N$. However, it is not possible that $2(\rho, \beta) \equiv N/2 \pmod N$, as $2(\rho, \beta)$ is odd and both of N and $N/2$ are even, thus $q^{2(\rho,\beta)} \neq -1$ for all $\beta \in \Phi_1^+$. Now if $N \equiv 2 \pmod 4$ then $0 < 2(\rho, \alpha) < N$ for all $\alpha \in \overline{\Phi}_0^+$, and $0 < 2(\rho, \beta) < N/2$ for all $\beta \in \Phi_1^+$.

It follows that the right hand side of (4.14) is well defined.

(ii) If $\lambda \in \Lambda_N^+$, none of the factors in the numerator of the right hand side of Eq. (4.14) is zero. The proof of this fact is easy thus omitted.

(iii) Consider $\lambda \in \overline{\mathcal{P}}_N^+ \backslash \Lambda_N^+$. From the definitions of $\overline{\Lambda_N^+}$ and Λ_N^+, we obtain

$$\begin{array}{lll} 2(\lambda + \rho, \alpha) = 2N', & \text{for some } \alpha \in \overline{\Phi}_0^+, & \text{if } N \equiv 0, 1, 3 \pmod 4, \\ 2(\lambda + \rho, \epsilon_1) = N/2, & & \text{if } N \equiv 2 \pmod 4. \end{array}$$

Thus the right hand side of Eq. (4.14) is zero.

□

Proposition 4.2.1. *Let $\lambda \in \overline{\mathcal{P}}_N^+$ and let V_λ be the $U_q^{(N)}(\mathfrak{g})$-module given in Definition 4.2.5, then V_λ is self-dual.*

Proof. It suffices to show that there is a non-degenerate, $U_q^{(N)}(\mathfrak{g})$-invariant, bilinear form $(\cdot, \cdot) : V_\lambda \times V_\lambda \to \mathbb{C}$; and we do this inductively.

Firstly, we will show that there exists such a form on $V_\nu \times V_\nu$ for any $U_q^{(N)}(\mathfrak{g})$-summand V_ν on the right hand side of the decomposition

$$V \otimes V = V_{2\epsilon_1} \oplus V_{\epsilon_1+\epsilon_2} \oplus V_0, \tag{4.15}$$

where we write $\epsilon_1 + \epsilon_2$ to mean ϵ_1 if $n = 1$. Let $\langle\langle\ ,\ \rangle\rangle : V \times V \to \mathbb{C}$ be the non-degenerate bilinear form from Proposition 3.2.1, the $U_q^{(N)}(\mathfrak{g})$-invariance of which is given by

$$\langle\langle a \cdot x, y\rangle\rangle = (-1)^{[a][x]} \langle\langle x, S(a)y\rangle\rangle, \qquad \forall a \in U_q^{(N)}(\mathfrak{g}), \quad x, y \in V.$$

Now define a new bilinear form $\langle\langle\ ,\ \rangle\rangle : (V \otimes V) \times (V \otimes V) \to \mathbb{C}$ by

$$\langle\langle x_1 \otimes y_1, x_2 \otimes y_2\rangle\rangle = (-1)^{[y_1][x_2]} \langle\langle x_1, x_2\rangle\rangle \langle\langle y_1, y_2\rangle\rangle, \qquad x_1, x_2, y_1, y_2 \in V,$$

which is evidently non-degenerate. Elementary calculations show that this form satisfies

$$\big\langle\big\langle a\cdot(x_1\otimes y_1), x_2\otimes y_2\big\rangle\big\rangle = (-1)^{[a]([x_1]+[y_1])}\big\langle\big\langle x_1\otimes y_1, \Delta'(S(a))(x_2\otimes y_2)\big\rangle\big\rangle, \quad \forall a\in U_q^{(N)}(\mathfrak{g}),$$

but this is not our desired $U_q^{(N)}(\mathfrak{g})$-invariance. To deal with this, we introduce a new bilinear form that is non-degenerate and has the desired $U_q^{(N)}(\mathfrak{g})$-invariance. Define the new bilinear form $\langle\langle\ ,\ \rangle\rangle^{new} : (V\otimes V)\times(V\otimes V)\to\mathbb{C}$ by

$$\big\langle\big\langle x_1\otimes y_1, x_2\otimes y_2\big\rangle\big\rangle^{new} = \big\langle\big\langle x_1\otimes y_1, R\cdot(x_2\otimes y_2)\big\rangle\big\rangle, \qquad x_1,x_2,y_1,y_2\in V,$$

where R is the universal R-matrix of $U_q^{(N)}(\mathfrak{g})$. This new form is non-degenerate as R is invertible and $\langle\langle\ ,\ \rangle\rangle$ is non-degenerate. We now claim that $\langle\langle\ ,\ \rangle\rangle^{new}$ is $U_q^{(N)}(\mathfrak{g})$-invariant. To see this, note that for each $a\in U_q^{(N)}(\mathfrak{g})$ we have

$$\begin{aligned}
&\langle\langle a\cdot(x_1\otimes y_1), x_2\otimes y_2\rangle\rangle^{new}\\
&\quad= \langle\langle a\cdot(x_1\otimes y_1), R\cdot(x_2\otimes y_2)\rangle\rangle\\
&\quad= \sum_{(a),t}\langle\langle a_{(1)}x_1\otimes a_{(2)}y_1, \alpha_t x_2\otimes\beta_t y_2\rangle\rangle(-1)^{[a_2][x_1]+[\beta_t][x_2]}\\
&\quad= \sum_{(a),t}\langle\langle a_{(1)}x_1, \alpha_t x_2\rangle\rangle\langle\langle a_{(2)}y_1, \beta_t y_2\rangle\rangle(-1)^{([a_{(2)}]+[y_1])([\alpha_t]+[x_2])+[a_{(2)}][x_1]+[\beta_t][x_2]}, \quad (4.16)
\end{aligned}$$

where we write the universal R-matrix as $R=\sum_t \alpha_t\otimes\beta_t$. Using the $U_q^{(N)}(\mathfrak{g})$-invariance of $\langle\langle\ ,\ \rangle\rangle : V\times V\to\mathbb{C}$, we can rewrite (4.16) as

$$\begin{aligned}
&\langle\langle x_1\otimes y_1, ((S\otimes S)\Delta(a))R\cdot(x_2\otimes y_2)\rangle\rangle(-1)^{[a]([y_1]+[x_1])}\\
&\quad= \langle\langle x_1\otimes y_1, \Delta'(S(a))R\cdot(x_2\otimes y_2)\rangle\rangle(-1)^{[a]([y_1]+[x_1])}\\
&\quad= \langle\langle x_1\otimes y_1, R\cdot\Delta(S(a))(x_2\otimes y_2)\rangle\rangle(-1)^{[a]([y_1]+[x_1])}\\
&\quad= \langle\langle x_1\otimes y_1, \Delta(S(a))(x_2\otimes y_2)\rangle\rangle^{new}(-1)^{[a]([y_1]+[x_1])}\\
&\quad= \langle\langle x_1\otimes y_1, S(a)\cdot(x_2\otimes y_2)\rangle\rangle^{new}(-1)^{[a]([y_1]+[x_1])},
\end{aligned}$$

thus $\langle\langle\ ,\ \rangle\rangle^{new} : V^{\otimes 2}\times V^{\otimes 2}\to\mathbb{C}$ is $U_q^{(N)}(\mathfrak{g})$-invariant. We now show that $\langle\langle\ ,\ \rangle\rangle^{new}$ is non-degenerate on each of the $U_q^{(N)}(\mathfrak{g})$-summands of $V\otimes V$ on the right hand side of (4.15). Let $\lambda,\nu\in\mathcal{P}^+_{\epsilon_1}\cap\overline{\Lambda^+_N}$ be non-equal and let $x\in V_\lambda$ and $y\in V_\nu$ be arbitrary non-zero vectors. The fact that the projections from $V\otimes V$ onto its summands are well-defined comes about from the fact that $\chi_\lambda(v)\neq\chi_\nu(v)$ for all these λ and ν satisfying $\lambda\neq\nu$. Furthermore, we have $\chi_\lambda(v)\neq 0\neq\chi_\nu(v)$, and it follows that

$$\chi_\lambda(v)\langle\langle x,y\rangle\rangle = \langle\langle v\cdot x,y\rangle\rangle = \langle\langle x, v\cdot y\rangle\rangle = \chi_\nu(v)\langle\langle x,y\rangle\rangle, \quad \text{if and only if} \quad \langle\langle x,y\rangle\rangle = 0,$$

where we have used the fact that $S(v)=v$. The non-degeneracy of $\langle\langle\ ,\ \rangle\rangle^{new} : V^{\otimes 2}\times V^{\otimes 2}\to\mathbb{C}$ then implies that $\langle\langle\ ,\ \rangle\rangle^{new}$ is non-degenerate on $V_\nu\times V_\nu$ for each summand V_ν on the right hand side of (4.15).

We have shown that $\langle\langle\ ,\ \rangle\rangle^{new} : V^{\otimes 2} \times V^{\otimes 2} \to \mathbb{C}$ is $U_q^{(N)}(\mathfrak{g})$-invariant, thus the form $\langle\langle\ ,\ \rangle\rangle^{new} : V_\nu \otimes V_\nu \to \mathbb{C}$ is non-degenerate, $U_q^{(N)}(\mathfrak{g})$-invariant, and bilinear as desired.

Now we do the inductive step, using an almost identical argument. Let V_μ, $\mu \in \Lambda_N^+$, be a $U_q^{(N)}(\mathfrak{g})$-module from Definition 4.2.5 and let V_μ be equipped with a non-degenerate, $U_q^{(N)}(\mathfrak{g})$-invariant, bilinear form $\langle\langle\ ,\ \rangle\rangle_\mu : V_\mu \times V_\mu \to \mathbb{C}$, where the $U_q^{(N)}(\mathfrak{g})$-invariance is

$$\langle\langle a \cdot x, y\rangle\rangle_\mu = (-1)^{[a][x]}\langle\langle x, S(a)y\rangle\rangle_\mu, \qquad \forall a \in U_q^{(N)}(\mathfrak{g}),\ x, y \in V_\mu.$$

Now let V_λ, $\lambda \in \mathcal{P}_\mu^+ \cap \overline{\Lambda_N^+}$, be the $U_q^{(N)}(\mathfrak{g})$-module defined in Definition 4.2.5 by $V_\lambda = \tilde{p}_j^{(t+1)}[\lambda](V_\mu \otimes V)$. Define a bilinear form

$$\langle\langle\ ,\ \rangle\rangle : (V_\mu \otimes V) \times (V_\mu \otimes V) \to \mathbb{C}, \quad \text{by} \tag{4.17}$$

$$\langle\langle x_1 \otimes y_1, x_2 \otimes y_2\rangle\rangle = (-1)^{[y_1][x_2]}\langle\langle x_1, x_2\rangle\rangle_\mu\langle\langle y_1, y_2\rangle\rangle, \quad x_1, x_2 \in V_\mu, \quad y_1, y_2 \in V.$$

As $\langle\langle\ ,\ \rangle\rangle_\mu : V_\mu \times V_\mu \to \mathbb{C}$ is non-degenerate, so is the form in (4.17). Now define a new bilinear form $\langle\langle\ ,\ \rangle\rangle^{new} : (V_\mu \otimes V) \times (V_\mu \otimes V) \to \mathbb{C}$ by

$$\langle\langle x_1 \otimes y_1, x_2 \otimes y_2\rangle\rangle^{new} = \langle\langle x_1 \otimes y_1, R \cdot (x_2 \otimes y_2)\rangle\rangle,$$

where R is the universal R-matrix, then by an almost identical argument as before, $\langle\langle\ ,\ \rangle\rangle^{new}$ is also non-degenerate, and furthermore, is $U_q^{(N)}(\mathfrak{g})$-invariant:

$$\big\langle\big\langle a\cdot(x_1\otimes y_1), x_2\otimes y_2\big\rangle\big\rangle^{new} = (-1)^{[a]([x_1]+[y_1])}\big\langle\big\langle x_1\otimes y_1, \Delta(S(a))(x_2\otimes y_2)\big\rangle\big\rangle^{new}, \quad \forall a \in U_q^{(N)}(\mathfrak{g}).$$

Recall that $V_\mu \otimes V$ decomposes into the following direct sum of $U_q^{(N)}(\mathfrak{g})$-modules:

$$V_\mu \otimes V = \bigoplus_{\lambda \in \mathcal{P}_\mu^+ \cap \overline{\Lambda_N^+}} V_\lambda. \tag{4.18}$$

Using almost exactly the same argument as previously, we can show that $\langle\langle\ ,\ \rangle\rangle^{new}$ is non-degenerate on each of the summands on the right hand side of (4.18). Now the fact that $\langle\langle\ ,\ \rangle\rangle^{new} : (V_\mu\otimes V)\times(V_\mu\otimes V) \to \mathbb{C}$ is $U_q^{(N)}(\mathfrak{g})$-invariant implies that $\langle\langle\ ,\ \rangle\rangle^{new} : V_\lambda\times V_\lambda \to \mathbb{C}$ is also $U_q^{(N)}(\mathfrak{g})$-invariant, thus we have our non-degenerate, $U_q^{(N)}(\mathfrak{g})$-invariant, bilinear form $\langle\langle\ ,\ \rangle\rangle^{new} : V_\lambda \times V_\lambda \to \mathbb{C}$ as desired.

One easily completes the proof of the proposition using induction.

□

4.3 Tensor products of $U_q^{(N)}(osp(1|2n))$-modules

In this section we prove some of the most important results of this chapter. We present certain tensor product theorems for the $U_q^{(N)}(\mathfrak{g})$-modules V_λ, where $\lambda \in \Lambda_N^+$, in the following cases:

(i) $n = 1$ and $N \geq 6$ satisfies $N \equiv 2 \pmod 4$, and

(ii) $n \geq 2$ and $N \geq 4$ is even.

We then give the detailed proofs of these tensor product theorems. Note that we do not consider the case that $n = 1$ and $N \geq 4$ satisfies $N \equiv 0 \pmod 4$. We leave this case for future study, but conjecture that the results in this case are similar to the results for the cases that we do consider.

In obtaining this result we use many of the techniques of Sections 4 and 5 of [TW93], in which a corresponding result was obtained for modules of quantum algebras associated with the A, B, C and D families of Lie algebras at even roots of unity.

4.3.1 Technical Lemmas

Recall that $\mathcal{C}_t$ is the subalgebra over $\mathbb{C}$ of $End_{U_q^{(N)}(\mathfrak{g})}(V^{\otimes t})$ generated by

$$\check{R}_i = \mathrm{id}^{\otimes(i-1)} \otimes \check{\mathcal{R}}_{V,V} \otimes \mathrm{id}^{\otimes(t-(i+1))}, \quad 1 \leq i \leq t-1.$$

Also recall that

- $\mathcal{T}^t$ is the set of all tableaux of length t derived from the Bratteli diagram for $(V^{gen})^{\otimes t}$,
- $\widetilde{\mathcal{T}}^t$ is the proper subset of $\mathcal{T}^t$ consisting of all those sequences $\lambda_i^t = (0, \epsilon_1, s_2, \ldots, s_t) \in \mathcal{T}^t$ where $s_i \in \Lambda_N^+$ for each $1 \leq i \leq t$,
- $\widehat{\mathcal{T}}^t$ is a further proper subset of $\mathcal{T}^t$ consisting of all those sequences $\lambda_i^t = (0, \epsilon_1, s_2, \ldots, s_t) \in \mathcal{T}^t$ where $s_i \in \Lambda_N^+$ for each $1 \leq i \leq t-1$ and $s_t \in \overline{\mathcal{P}}_N^+$.

We now define some matrix units in $\mathcal{C}_t$ that will play a key role in the proof of the tensor product theorems for certain $U_q^{(N)}(\mathfrak{g})$-modules later in this section. In constructing these matrix units in $\mathcal{C}_t$, it is convenient to define the set

$$\widetilde{\Omega}^t = \left\{(S,T) \in \Omega^t |\ S, T \in \widetilde{\mathcal{T}}^t, shp(S) = shp(T)\right\}$$

of pairs of paths in $\widetilde{\mathcal{T}}^t$ that end at the same shape.

Now for each $(S,T) \in \widetilde{\Omega}^t$ we define a matrix unit $E_{ST} \in \mathcal{C}_t$ using precisely the same method we used to define the matrix unit $E_{ST} \in End_{U_q(\mathfrak{g})}(V^{gen})^{\otimes t}$ in Subsection 3.9.2, except that here we fix $q = \exp(2\pi i/N)$ and also fix $\check{R}_i$ to be the appropriate element of $End_{U_q^{(N)}(\mathfrak{g})}(V^{\otimes t})$. Note that we only define a matrix unit E_{ST} in $\mathcal{C}_t$ here for each pair $(S,T) \in \widetilde{\Omega}^t$.

We need to show that these matrix units in $\mathcal{C}_t$ are all well-defined and non-zero. Firstly, each of the projectors $\left\{E_{SS} \in \mathcal{C}_t |\ (S,S) \in \widetilde{\Omega}^t\right\}$ is defined by $E_{SS} = \tilde{p}_i^t[\lambda] \in \mathcal{C}_t$ where $S = \lambda_i^t \in \widetilde{\mathcal{T}}^t$, and these are well-defined and non-zero by construction. It takes more work to prove the corresponding result for the intertwiners $\left\{E_{ST} \in \mathcal{C}_t |\ (S,T) \in \widetilde{\Omega}^t, S \neq T\right\}$, and we will do this inductively.

Assume firstly that the matrix units $\left\{E_{ST} \in \mathcal{C}_r |\ (S,T) \in \widetilde{\Omega}^r\right\}$, are all well-defined and non-zero for some positive integer $r \geq 1$. We will show that the intertwiners

$$\left\{E_{MP} \in \mathcal{C}_{r+1} |\ (M,P) \in \widetilde{\Omega}^{r+1}, M \neq P\right\}$$

are also all well-defined and non-zero. To do this, we firstly recall from Subsection 3.9.2 that each path $\widetilde{S}$ of length t in the Bratteli diagram for $(V^{gen})^{\otimes t}$ corresponds to a path S of length t in the Bratteli diagram for $\widetilde{\mathscr{BW}}_t(-q^{2n}, q)$ and that this is a one-to-one relationship. In the following discussion concerning intertwiner matrix units, in discussing a path $\widetilde{S}$ of length t in $\widetilde{\mathcal{T}}^t$, we always use this instead to refer to the corresponding path S of length t in the Bratteli diagram for $\widetilde{\mathscr{BW}}_t(-q^{2n}, q)$. The reason we do this is that although the intertwiner matrix units are defined using paths in the Bratteli diagram for $\widetilde{\mathscr{BW}}_t(-q^{2n}, q)$, it is easier to discuss paths in $\widetilde{\mathcal{T}}^t$.

We now define the intertwiners $E_{MP} \in \mathcal{C}_{r+1}$; let us partition them into two sets:

(a) the intertwiners E_{MP} for which $|shp(M)| = |shp(P)| = r + 1$,

(b) the intertwiners for which $|shp(M)| = |shp(P)| < r + 1$.

We will show that the intertwiners in each set are well-defined and non-zero. Firstly consider (a): let the paths M and P satisfy $|shp(M)| = |shp(P)| = r + 1$ and $shp(M') \neq shp(P')$, then the intertwiner $E_{MP} \in \mathcal{C}_{r+1}$ is well-defined if the coefficient

$$\frac{1 - q^{2d}}{\sqrt{(1 - q^{2d+2})(1 - q^{2d-2})}}, \tag{4.19}$$

is well-defined, where $d = d(\overline{M}, r)$ is an integer defined in Subsection 3.9 and $\overline{M}$ is a particular path in $\widetilde{\mathcal{T}}^{r+1}$ also defined in Subsection 3.9 such that $shp(\overline{M}) = shp(M)$. The number $|d(\overline{M}, r)|+1$ is the length of a hook going through the boxes containing the numbers r and $r + 1$ in the standard tableau obtained from $\overline{M}$ in the canonical way.

The question of whether the coefficient (4.19) is well-defined and non-zero evidently depends on the values that d can take, and we will show that (4.19) is indeed well-defined and non-zero. As $|d| + 1$ is the length of a hook in the standard tableau obtained from $\overline{M}$, we can calculate the values that d can take by considering all possible hooks of $shp(\overline{M})$.

Clearly, the minimum length of a hook is 2, ie $|d| + 1 \geq 2$. We now break the problem down into a number of sub-cases:

(i) $n \geq 2$ and $N \equiv 0 \pmod 4$.

Here $\lambda \in \Lambda_N^+$ if and only if λ is an element of $\mathcal{P}^+$ satisfying $0 \leq \lambda_1 + \lambda_2 \leq N/2 - 2n + 1$. We want to find the greatest possible length of a hook over all allowable Young diagrams that also satisfy $0 \leq \lambda_1 + \lambda_2 \leq N/2 - 2n + 1$. Recall from Chapter 3 that a Young diagram μ is said to be allowable if $\mu'_1 + \mu'_2 \leq 2n+1$. By considering the geometry of the relevant Young diagrams we can see that the greatest such length appears in a hook in any allowable Young diagram λ satisfying $\lambda_1 = N/2 - 2n$ and $\lambda'_1 = 2n$. Now the greatest possible length of any hook in any such Young diagram is $2n + N/2 - 2n - 1$, thus d satisfies $2 \leq |d| + 1 \leq N/2 - 1$.

(ii) $n \geq 2$ and $N \equiv 2 \pmod 4$.

Here $\lambda \in \Lambda_N^+$ if and only if λ is an element of $\mathcal{P}^+$ satisfying $0 \leq \lambda_1 \leq N/4 - n - 1/2$. We want to find the greatest possible length of a hook over all allowable Young diagrams that also satisfy $0 \leq \lambda_1 \leq N/4 - n - 1/2$. By considering the geometry of the relevant Young diagrams we can easily see that the greatest such length appears in a hook in any allowable Young diagram λ satisfying $\lambda_1 = N/4 - n - 1/2$ and $\lambda'_1 = 2n$. Now the greatest possible length of a hook in any such Young diagram is $2n + N/4 - n - 3/2$, thus d satisfies $2 \leq |d| + 1 \leq N/4 + n - 3/2$.

(iii) $n = 1$ and N is even.

Intertwiner matrix units $E_{MP} \in \mathcal{C}_{r+1}$ do not exist in this sub-case. Each element of Λ_N^+ belongs to $\mathbb{Z}_+\epsilon_1$ and after examining the geometry of the relevant Young diagrams, the Bratteli diagram shows us that the path M is $M = (0, \epsilon_1, 2\epsilon_1, \ldots, (r+1)\epsilon_1) \in \widetilde{\mathcal{T}}^{r+1}$ as this is the only path in $\widetilde{\mathcal{T}}^{r+1}$ that ends on a shape with $r+1$ boxes. As this is the only such path, there is no P and no intertwiner E_{MP}.

This allows us to analyse the cofficient (4.19) of E_{MP}. Firstly, (4.19) is well-defined if $q^{2d\pm2} \neq 1$. By examining the values of d from (i)–(ii) above, it is not difficult to see that $q^{2d+2} \neq 1$ and $q^{2d-2} \neq 1$ unless $|d| = 1$. However, in Subsection 3.9 we showed that $|d| \geq 2$, thus $q^{2d\pm2} \neq 1$.

We now show that (4.19) is non-zero: to do this it suffices to show that $q^{2d} \neq 1$. In sub-case (i) above we have

$$2 \leq |2d| \leq N - 4,$$

and in (ii) we have

$$2 \leq |2d| \leq N - 8,$$

thus it is indeed true that $q^{2d} \neq 1$ and therefore (4.19) is non-zero. Note that we have not proved that the intertwiner E_{MP} itself is non-zero, but we will show this momentarily.

Now we consider the remaining intertwiners in case (b): the intertwiners E_{MP} where $|shp(M)| = |shp(P)| < r+1$. The coefficient of E_{MP} is

$$\frac{sdim_q(V_{shp(M)})}{\sqrt{sdim_q(V_{shp(M')})sdim_q(V_{shp(P')})}}, \tag{4.20}$$

which is well-defined and non-zero as $M, P \in \widetilde{\mathcal{T}}^{r+1}$, each vertex on the paths M and P is an element of Λ_N^+ and $sdim_q(V_\lambda) \neq 0$ for all $\lambda \in \Lambda_N^+$.

Now we need to prove that $E_{MP} \in \mathcal{C}_{r+1}$ is non-zero at the appropriate root of unity. We showed that the coefficient of E_{MP} is non-zero at roots of unity in our work above.

Now recall that $E_{MP}E_{PM} = E_{MM}$ at all generic q, then by using arguments similar to those we used in the proof of Lemma 4.2.3, we can show that E_{MP} is non-zero when q is specialised to the appropriate root of unity, as E_{PM} has no poles at these roots of unity. Furthermore, $str_q(E_{PM}E_{MP}) = str_q(E_{PP}) \neq 0$, so $E_{MP} \neq 0$. Thus the intertwiner E_{MP} is well-defined and non-zero at roots of unity. This completes the proof that the intertwiners in $\left\{E_{MP} \in \mathcal{C}_{r+1} \mid \ (M,P) \in \widetilde{\Omega}^{r+1}\right\}$ are all well-defined and non-zero.

We have shown that all the $\left\{E_{ST} \in \mathcal{C}_t \mid \ (S,T) \in \widetilde{\Omega}^{t}\right\}$ are well-defined and non-zero when $q = \exp(2\pi i/N)$. Now we wish to show that

Lemma 4.3.1. *Each projector $E_{SS} \in \mathcal{C}_t$, $S \in \widetilde{\mathcal{T}}^t$, is a minimal idempotent in $\mathcal{C}_t$, ie, E_{SS} cannot be written as a sum of orthogonal idempotents $E_{SS}(1)$ and $E_{SS}(2)$ in $\mathcal{C}_t$ where both $E_{SS}(1)$ and $E_{SS}(2)$ are non-zero.*

We will show this by drawing on work of Wenzl [We90]. Before proving Lemma 4.3.1, we define the annihilator ideal $J_t \subset \mathcal{C}_t$ with respect to the quantum supertrace, which we will use in the proof of Lemma 4.3.1 and in the proof of the tensor product theorems later in this chapter.

Definition 4.3.1. *Define the ideal $J_t \subset \mathcal{C}_t$ by $J_t = \{y \in \mathcal{C}_t \mid \ str_q(xy) = 0, \ \forall x \in \mathcal{C}_t\}$.*

Note that for any $y \in J_t$, $str_q(xy) = (-1)^{[x][y]} str_q(yx) = 0$ for all $x \in \mathcal{C}_t$, thus J_t is a two-sided ideal.

Recall from Lemma 3.6.4 that for $\hat{q} \neq 0$ not a root of unity, the algebra homomorphism

$$\Upsilon : g_i \mapsto -\check{R}_i^{gen}, \qquad i = 1, \ldots, t-1,$$

furnishes a representation of the Birman-Wenzl-Murakami algebra $\mathscr{BW}_t(-\hat{q}^{2n}, \hat{q})$ in $\mathcal{C}_t^{gen}$. In a similar way, we can show that for $q = \exp(2\pi i/N)$ where $N \geq 3$ is an integer, the algebra homomorphism

$$\Upsilon : g_i \mapsto -\check{R}_i, \qquad i = 1, \ldots, t-1,$$

furnishes a representation of $\mathscr{BW}_t(-q^{2n}, q)$ in $\mathcal{C}_t$.

We now prove Lemma 4.3.1.

Proof. In this proof we shall say that an algebra B is semisimple if it is isomorphic to a direct sum of matrix algebras: $B \cong \bigoplus_{i \in I} M_{b_i}(\mathbb{C})$, where $M_{b_i}(\mathbb{C})$ is the algebra of $b_i \times b_i$ matrices with complex entries [We90, Sec. 1].

As $q = \exp(2\pi i/N)$, $-q$ is a root of unity and [We90, Thm. 4.4] states that there is an isomorphism

$$\mathscr{BW}_t\big(-q^{2n}, q\big)/\mathscr{J}_t\big(-q^{2n}, q\big) \cong \mathscr{BW}_t\big((-q)^{2n}, -q\big)/\mathscr{J}_t\big((-q)^{2n}, -q\big), \tag{4.21}$$

where right hand side of Eq. (4.21) is semisimple [We90, Thm. 4.4 (d), Thm. 6.4], thus

$$\mathscr{BW}_t\big(-q^{2n}, q\big)/\mathscr{J}_t\big(-q^{2n}, q\big) \cong \bigoplus_{i \in \Gamma(-q^{2n}, q)_t} M_{b_i}(\mathbb{C}),$$

where $\Gamma(-q^{2n},q)_t$ is the set of Young diagrams containing $t-2k \geq 0$ boxes where $k \in \mathbb{Z}_+$ appearing in a certain graph $\Gamma(-q^{2n},q)$ [We90, Thm. 4.4 (d)]. We obtain the graph $\Gamma(-q^{2n},q)$ by following [We90, Thm. 4.4]. To obtain $\Gamma(-q^{2n},q)$, we firstly define a subgraph $\widetilde{\Gamma}(-q^{2n},q)$ of the Young lattice [We90, p. 407]. Firstly, let the Young diagram with no boxes belong to $\widetilde{\Gamma}(-q^{2n},q)$, then a Young diagram λ belongs to $\widetilde{\Gamma}(-q^{2n},q)$ if

(a) $Q_\lambda(-q^{2n},q) \neq 0$, and

(b) there is at least one subdiagram of λ with $|\lambda|-1$ boxes that belongs to $\widetilde{\Gamma}(-q^{2n},q)$.

From $\widetilde{\Gamma}(-q^{2n},q)$, we obtain the graph $\Gamma(-q^{2n},q)$ using [We90, Thm. 4.4], the relevant parts of which we now quote. In this theorem, the concept of an $N/2$ regular diagram is used. We say that a Young diagram λ is $N/2$ regular if its largest hook has fewer than $N/2$ boxes, ie $\lambda_1+\lambda'_1-1<N/2$ [We90, Eq. (2.5)].

Before stating the relevant parts of [We90, Thm. 4.4], we again stress the important fact that $\mathscr{BW}_t(-q^{2n},q)/\mathscr{J}_t(-q^{2n},q)$ is semisimple for all $t \in \mathbb{N}$.

Theorem 4.3.1. *Theorem 4.4 of [We90]. Let q^2 be a primitive $(N/2)^{th}$ root of unity, then*

(b) Assume that $\widetilde{\Gamma}(-q^{2n},q)$, as defined above, does not contain a hook diagram with $N/2-1$ boxes. Then $\Gamma(-q^{2n},q)=\widetilde{\Gamma}(-q^{2n},q)$ with edges inherited from the Young lattice and it only contains $N/2$ regular diagrams.

(c) If $\widetilde{\Gamma}(-q^{2n},q)$ contains only one hook diagram μ with $N/2-1$ boxes with, say, $\mu=[N/2-2n,1^{2n-1}]$ and it does not contain its successor $[N/2-2n,2,1^{2n-2}]$, then $\Gamma(-q^{2n},q)$ consists of all $N/2$ regular diagrams in $\widetilde{\Gamma}(-q^{2n},q)$ and, if $Q_\lambda(-q^{2n},q)\neq 0$, also the diagram $\lambda=[N/2-2n+1,1^{2n-1}]$. The edges of $\Gamma(-q^{2n},q)$ are exactly those inherited from the Young lattice.

(d) These graphs completely determine the direct sum of matrix rings that is isomorphic to $\mathscr{BW}_t(-q^{2n},q)/\mathscr{J}_t(-q^{2n},q)$.

The first step is to construct $\widetilde{\Gamma}(-q^{2n},q)$. Fix the Young diagram with no boxes to belong to $\widetilde{\Gamma}(-q^{2n},q)$. For convenience, we again state $Q_\lambda(-q^{2n},q)$ from (3.45):

$$Q_\lambda(-q^{2n},q) = \prod_{(j,j)\in\lambda} \frac{-q^{2n+\lambda_j-\lambda'_j}+q^{-2n-\lambda_j+\lambda'_j}+q^{\lambda_j+\lambda'_j-2j+1}-q^{-\lambda_j-\lambda'_j+2j-1}}{q^{h(j,j)}-q^{-h(j,j)}} \times \prod_{(i,j)\in\lambda, i\neq j} \frac{-q^{2n+d(i,j)}+q^{-2n-d(i,j)}}{q^{h(i,j)}-q^{-h(i,j)}}, \tag{4.22}$$

where we recall the meanings of $d(i,j)$ and $h(i,j)$ from Eq. (3.45). As discussed after Eq. (3.45), $Q_\lambda(-q^{2n},q)=0$ if and only if one (or both) of the following conditions is satisfied:

(a) $q^{4n+2d(i,j)}=1$ for some $(i,j)\in\lambda$ where $i\neq j$,

(b) $q^{2n-2\lambda'_j+2j-1} = 1$ or $q^{2n+2\lambda_j-2j+1} = -1$ for some j.

Fix $q = \exp(2\pi i/N)$ where $N \geq 4$ is an even number. Note the overarching result that $Q_\lambda(-q^{2n}, q) = 0$ if $\lambda'_1 + \lambda'_2 \geq 2n + 2$ [We90, p. 422]. We firstly determine when the numerator of $Q_\lambda(-q^{2n}, q)$ vanishes. Let us determine for which λ the first equation in condition (b) above is true. Here $q^{2n-2\lambda'_j+2j-1} = 1$ if and only if $\lambda'_j = rN/2+n+j-1/2$ for some $r \in \mathbb{Z}$, but this is not possible as λ'_j must be an integer. Similarly in relation to the second equation in condition (b) we have $q^{2n+2\lambda_j-2j+1} \neq 1$ for $N \equiv 0 \pmod 4$. However, for $N \equiv 2 \pmod 4$ we have $q^{2n+2\lambda_j-2j+1} = -1$ if

$$\lambda_j = N/4 + rN/2 - n + j - 1/2, \qquad \text{for any } r \in \mathbb{Z}, \tag{4.23}$$

where we note that the right hand side of (4.23) must be an integer. Recall that we have fixed N to satisfy the inequality $N/4 \geq n+1/2$ so that $\Lambda^+_N \neq \emptyset$, then $N/4-n+j-1/2 \geq j$. Also, we have $-N/4 - n + j - 1/2 \leq -2n + j - 1 \leq 0$, so the least non-negative value on the right hand side of (4.23) is obtained by fixing $r = 0$. Now $\max\{\lambda_j\} = \lambda_1$, and so the numerator of $Q_\lambda(-q^{2n}, q)$ is zero if

$$\lambda_1 = N/4 - n + 1/2. \tag{4.24}$$

We claim that $Q_\lambda(-q^{2n}, q) \neq 0$ for $N \equiv 2 \pmod 4$ if λ satisfies

$$\lambda_1 \leq N/4 - n - 1/2. \tag{4.25}$$

It may be observed that (4.23) is never satisfied by any λ satisfying (4.25). Furthermore, any λ satisfying (4.25) and $\lambda'_1 + \lambda'_2 \leq 2n + 1$ also satisfies the hook length condition that $h(1,1) \leq N/2 - 1$ as

$$\max(h(1,1)) = \begin{cases} 2n+1 \leq N/2 - 2, & \text{if } \lambda_1 = 1, \\ N/4 + n - 3/2 \leq N/2 - 4, & \text{if } \lambda_1 > 1. \end{cases}$$

The statement for $\lambda_1 = 1$ comes about from the condition in (4.24) that $1 \leq N/4-n-1/2$. The statement for $\lambda_1 > 1$ comes about from the condition in (4.24) that $n - 3/2 \leq N/4 - \lambda_1 - 2 \leq N/4 - 4$.

Now we will show that for $N \equiv 2 \pmod 4$, condition (a) mentioned just after Eq. (4.22) is never satisfied by any Young diagram λ satisfying (4.25) and $\lambda'_1 + \lambda'_2 \leq 2n + 1$. Fix $N \equiv 2 \pmod 4$. Condition (a) is true if and only if $d(i,j) = rN/2 - 2n$ for some $r \in \mathbb{Z}$ where $i \neq j$. Recall that

$$d(i,j) = \begin{cases} \lambda_i + \lambda_j - i - j + 1, & \text{if } i \leq j, \\ -\lambda'_i - \lambda'_j + i + j - 1, & \text{if } i > j, \end{cases}$$

then for all $i \neq j$,

- $\max(d(i,j)) = d(1,2) = \lambda_1 + \lambda_2 - 2$, and

- $\min(d(i,j)) = d(2,1) = -\lambda'_1 - \lambda'_2 + 2.$

It follows that $q^{4n+2d(i,j)} = 1$ for some $i \neq j$ if and only if

$$\lambda_i + \lambda_j = rN/2 - 2n + i + j - 1, \qquad \text{if } i < j, \tag{4.26}$$
$$\lambda'_i + \lambda'_j = rN/2 + 2n + i + j - 1, \qquad \text{if } i > j, \tag{4.27}$$

for some $r \in \mathbb{Z}$.

Let us consider the case where $i < j$. From the previous condition in (4.25) that $\lambda_1 \leq N/4 - n - 1/2$, we have

$$0 \leq \lambda_i + \lambda_j \leq N/2 - 2n - 1, \quad \forall i < j. \tag{4.28}$$

Note that for these i and j, we have $i \leq n$ and $j \leq N/4 - n - 1/2$. The right hand side of (4.26) with $r = -1$ is $-N/2 - 2n + i + j - 1 \leq -N/4 - 2n - 3/2 < 0$, and thus $\lambda_i + \lambda_j > -N/2 - 2n + i + j - 1$. The right hand side of (4.26) with $r = 1$ is $N/2 - 2n + i + j - 1 \geq N/2 - 2n + 2$, but we always have $\lambda_i + \lambda_j < N/2 - 2n + 2$ from (4.28) and so $\lambda_i + \lambda_j < N/2 - 2n + i + j - 1$. Eq. (4.26) with $r = 0$ is $\lambda_i + \lambda_j - i - j + 1 = -2n$, and as at generic q, this is not true. (Recall the argument from generic q: we have $i \leq 2n$, $\lambda_i - j \geq 0$ and $\lambda_j \geq 0$, thus $\lambda_i + \lambda_j - i - j + 1 \geq -2n + 1$.) It follows that (4.26) is not true at these roots of unity.

Let us consider the case where $i > j$ with N again fixed as $N \equiv 2 \pmod 4$. We have already imposed the condition $\lambda'_1 + \lambda'_2 \leq 2n + 1$, so $\lambda'_i + \lambda'_j \leq 2n$ for all $(i,j) \neq (2,1)$. As $\lambda'_1 + \lambda'_2 \leq 2n+1$, a necessary condition on the integer r for $\lambda'_1 + \lambda'_2 = rN/2 + 2n + 2$ to be true is that $r \leq -1$. Setting $r = -2$ into this equation gives $\lambda'_1 + \lambda'_2 = -N + 2n + 2$, however, it is true that $-N + 2n + 2 \leq -2n$ as N satisfies $N \geq 4n + 2$, and thus $\lambda'_1 + \lambda'_2 > -N + 2n + 2$. Fixing then $r = -1$, the equation is $\lambda'_1 + \lambda'_2 = -N/2 + 2n + 2$, and from the conditions on N we have $\lambda'_1 + \lambda'_2 = -N/2 + 2n + 2 \leq 1$. Now $-N/2 + 2n + 2$ is odd so this condition is $\lambda'_1 + \lambda'_2 = -N/2 + 2n + 2 = 1$ which is precisely the Young diagram [1]. But this Young diagram has already been ruled out for $N/2 = 2n + 1$ as we have the condition $\lambda_1 \leq N/4 - n - 1/2 = 0$. Thus (4.27) is not true for $(i,j) = (2,1)$ at these roots of unity.

Now consider (4.27) for $i > j$ where $(i,j) \neq (2,1)$: recall that $0 \leq \lambda'_i + \lambda'_j \leq 2n$. Here $2n + 2 \geq i + j \geq 4$, so $rN/2 + 2n + i + j - 1 \geq rN/2 + 2n + 3$, and then a necessary condition on r for Eq. (4.27) to be true is that $r \leq -1$. For $r \leq -2$ we have

$$rN/2 + 2n + i + j - 1 \leq -N + 2n + i + j - 1 \leq -N + 4n + 1 \leq -1,$$

and so a necessary condition on r for Eq. (4.27) to be true is that $r = -1$.

We now show that (4.27) is not true with $r = -1$ for all $i > j$ where $(i,j) \neq (2,1)$ by considering all the possible relevant Young diagrams. Consider the Young diagram $[1^k]$ where $k \in \{3, 4, \ldots, 2n+1\}$. Here $(i,j) = (i,1)$ where $i \in \{3, 4, \ldots, k\}$ and $\lambda'_i = 0$ and $\lambda'_1 = k$. Eq. (4.27) is true here only if

$$\lambda'_1 - i = -N/2 + 2n. \tag{4.29}$$

However, $\lambda'_1 - i \geq 0$ and $N/2 \geq 2n+3$, thus the left hand of (4.29) is non-negative and the right hand side is strictly negative, thus (4.29) is not true. Consider now a Young diagram λ with more than one column of boxes. Fix $i \geq n+1$, then $j = 1$ (note that $i \leq 2n$). Then (4.27) is true here only if

$$\lambda'_i + \lambda'_1 - i = -N/2 + 2n. \tag{4.30}$$

Now the left hand side of (4.30) is non-negative as $\lambda'_i \geq 0$ and $\lambda'_1 - i \geq 0$, but $N/2 \geq 2n+3$, so the right hand side is strictly negative, thus (4.30) is not true. Consider again a Young diagram λ with more than one column of boxes. Fix $i \leq n$, then $j \leq i-1$, and (4.27) is true only if

$$\lambda'_i + \lambda'_j - i = -N/2 + 2n + j - 1. \tag{4.31}$$

Now $\lambda'_i \geq 0$ and $\lambda'_j - i \geq 0$ so the left hand side of (4.31) is non-negative. Note that

$$-N/2 + 2n + j - 1 \leq -N/4 + n - 3/2 \leq -3,$$

as $j \leq N/4 - n - 1/2$ and $N/2 \geq 2n+3$, so the right hand side of (4.31) is strictly negative, thus (4.31) is not true.

It follows from these calculations that for $N \equiv 2 \pmod 4$, $Q_\lambda(-q^{2n}, q) \neq 0$ for all the Young diagrams λ satisfying

$$\lambda_1 \leq N/4 - n - 1/2, \quad \text{and} \quad \lambda'_1 + \lambda'_2 \leq 2n + 1. \tag{4.32}$$

We can then write down the Young diagrams that comprise the vertices of $\widetilde{\Gamma}(-q^{2n}, q)$ at these roots of unity; these are all the Young diagrams λ satisfying (4.32). Note that $\widetilde{\Gamma}(-q^{2n}, q)$ contains no hook diagrams with exactly $N/2 - 1$ boxes.

Now we determine the vertices of $\widetilde{\Gamma}(-q^{2n}, q)$ when N satisfies $N \equiv 0 \pmod 4$; fix such an N. As previously mentioned, neither of the equations in condition (b) is true at these roots of unity; we now determine whether the equation in condition (a) is true at these roots of unity. This condition can be expressed as the two equations (4.26)–(4.27). Recall the overarching result that $Q_\lambda(-q^{2n}, q) = 0$ if $\lambda'_1 + \lambda'_2 \geq 2n + 2$. Let us consider Eqs. (4.26)–(4.27) for $i < j$.

Firstly, fix $(i, j) = (1, 2)$, then Eq. (4.26) is

$$\lambda_1 + \lambda_2 = rN/2 - 2n + 2, \tag{4.33}$$

for some $r \in \mathbb{Z}$. Now (4.33) can only be true for Young diagrams containing at least one box if $r \geq 1$, and we have $Q_\lambda(-q^{2n}, q) = 0$ if $\lambda_1 + \lambda_2 = N/2 - 2n + 2$. We claim that Eqs. (4.26)–(4.27) are not true for all Young diagrams λ satisfying

$$\lambda_1 + \lambda_2 \leq N/2 - 2n + 1, \quad \text{and} \quad \lambda'_1 + \lambda'_2 \leq 2n + 1. \tag{4.34}$$

The proof of this claim is very similar to the proof of the corresponding result for $N \equiv 2 \pmod 4$ for all Young diagrams satisfying (4.32), thus we omit it.

Then for $N \equiv 0 \pmod 4$, the vertices of the graph $\widetilde{\Gamma}(-q^{2n}, q)$ are all the Young diagrams λ satisfying (4.34). We need to check that all such λ also satisfy the hook length condition. However this is easy to do and we omit the proof.

Now we wish to determine the number of vertices of $\widetilde{\Gamma}(-q^{2n}, q)$ for $N \equiv 0 \pmod 4$ that are hook diagrams with exactly $N/2-1$ boxes. As $N \geq 4$, we are examining the Young diagrams with at least one box. For all of these Young diagrams, $\lambda_1 \geq 1$ and thus $N/2 \geq 2n$.

Consider the case that $N/2 = 2n$. Here $\lambda_1 + \lambda_2 \leq 1$ which means that the only non-empty Young diagram in $\widetilde{\Gamma}(-q^{2n}, q)$ is $[1]$. This is a hook diagram, but it contains exactly $N/2-1$ boxes only if $N = 4$ and $n = 1$.

Consider the case that $N/2 = 2n+2$. Here $\lambda_1 + \lambda_2 \leq 3$, and the hook diagrams in $\widetilde{\Gamma}(-q^{2n}, q)$ with exactly $N/2-1$ boxes are

- $[3]$, $[2,1]$ and $[1^3]$, if $n = 1$,
- $[1^{2n+1}]$ and $[2, 1^{2n-1}]$, if $n \geq 2$.

Consider the case that $N/2 \geq 2n+4$. The hook diagrams in $\widetilde{\Gamma}(-q^{2n}, q)$ with exactly $N/2-1$ boxes are

- $[N/2-1]$ and $[N/2-2, 1]$, if $n = 1$,
- $[N/2-2n, 1^{2n-1}]$, if $n \geq 2$.

We can now work out the vertices belonging to the graph $\Gamma(-q^{2n}, q)$ using Theorem 4.3.1. For $N \equiv 2 \pmod 4$, $\widetilde{\Gamma}(-q^{2n}, q)$ contains no hook diagrams with exactly $N/2-1$ boxes, and so the vertices of $\Gamma(-q^{2n}, q)$ are all the Young diagrams satisfying

$$\lambda_1 \leq N/4 - n - 1/2, \quad \text{and} \quad \lambda_1' + \lambda_2' \leq 2n+1.$$

Now for $N \equiv 0 \pmod 4$, we have the following cases. For $N/2 = 2n$, the Young diagram $[1]$ is a hook diagram with $N/2-1$ boxes if $N = 4$ and $n = 1$. However, in this case $Q_{[2]}(-q^{2n}, q) = 0$ and so the hook diagram $[2]$ is not a vertex in the graph $\Gamma(-q^{2n}, q)$. Then, for $N/2 = 2n$ and $N/2 = 2n+2$, and also $N/2 \geq 2n+4$ where $n = 1$, the vertices of $\Gamma(-q^{2n}, q)$ are all the Young diagrams satisfying

$$\lambda_1 + \lambda_2 \leq N/2 - 2n + 1, \quad \text{and} \quad \lambda_1' + \lambda_2' \leq 2n+1.$$

For $N/2 \geq 2n+4$ where $n \geq 2$, $\widetilde{\Gamma}(-q^{2n}, q)$ contains one hook diagram with $N/2-1$ boxes: $[N/2-2n, 1^{2n-1}]$ and it does not contain its successor $[N/2-2n, 2, 1^{2n-2}]$. In addition, the $OSp(1|2n)$ supercharacters of $[N/2-2n+1, 1^{2n-1}]$ and $[N/2-2n+1]$ are the same up to a sign, so $Q_{[N/2-2n+1, 1^{2n-1}]}(-q^{2n}, q) \neq 0$. The vertices of $\Gamma(-q^{2n}, q)$ are then all the Young diagrams λ in the set

$$\{\lambda |\ \lambda_1 + \lambda_2 \leq N/2 - 2n + 1,\ \lambda_1' + \lambda_2' \leq 2n+1\} \cup \{[N/2-2n+1, 1^{2n-1}]\}.$$

Now we can write down the Bratteli diagram for $\mathscr{BW}_t(-q^{2n}, q)$ as we did at generic q, and then write down the matrix units in $\mathscr{BW}_t(-q^{2n}, q)/\mathscr{J}_t(-q^{2n}, q)$. For each $s = 0, 1, \ldots, t$, let $\Gamma(-q^{2n}, q)_s$ be the set of Young diagrams belonging to $\Gamma(-q^{2n}, q)$ with

$s-2k \geq 0$ boxes, where k ranges over all of $\mathbb{Z}_+$. We say that R is a path of length t if R is a sequence of $t+1$ Young diagrams: $R = ([0],[1],r_2,\ldots,r_t)$ where $r_s \in \Gamma(-q^{2n},q)_s$ for each s and r_j is connected to r_{j+1} for each $j=0,1,\ldots,t-1$. Let $\omega(-q^{2n},q)_t$ be the set of pairs (R,S) of paths of length t such that $r_t = s_t$.

We obtain a complete set of matrix units for $\mathscr{BW}_t(-q^{2n},q)/\mathscr{J}_t(-q^{2n},q)$ by taking all the matrix units $e_{RS} \in \mathscr{BW}_t$ where $(R,S) \in \omega(-q^{2n},q)_t$ and fixing $q = \exp(2\pi i/N)$ and $r = -q^{2n}$. These are all well-defined and non-zero.

Let us define $\widetilde{\mathscr{BW}}_t(-q^{2n},q)$ to be the semisimple subalgebra of $\mathscr{BW}_t(-q^{2n},q)$ spanned by the matrix units $\{e_{ST} \in \mathscr{BW}_t(-q^{2n},q) |\ (S,T) \in \omega(-q^{2n},q)_t\}$. Note that $\Upsilon(e_{ST}) = E_{ST} \in \mathcal{C}_t$ for each $(S,T) \in \omega(-q^{2n},q)_t$, where we recall that Υ is the algebra homomorphism $\Upsilon : g_i \mapsto -\check{R}_i$.

Now define a map $\psi : \mathcal{C}_t \to \mathbb{C}$ by

$$\psi(X) = str_q(X)/\big(sdim_q(V)\big)^t,$$

then

$$\psi\big(\Upsilon(a)\big) = \mathrm{tr}(a), \qquad \text{for all } a \in \mathscr{BW}_t(-q^{2n},q), \tag{4.35}$$

which we can prove in the same way that we proved Lemma 3.9.1. Note here that Lemma 3.9.2 holds true if we write $U_q^{(N)}(\mathfrak{g})$ instead of $U_q(\mathfrak{g})$ and fix $\check{\mathcal{R}}_{V,V} \in End_{U_q^{(N)}(\mathfrak{g})}(V \otimes V)$ to be as given in Definition 4.1.1. Furthermore, note that $\psi(X) = 0$ if and only if $str_q(X) = 0$, thus we can regard J_t as the annihilator ideal of $\mathcal{C}_t$ with respect to ψ.

Note that $\widetilde{\mathscr{BW}}_t(-q^{2n},q) \cap \mathscr{J}_t(-q^{2n},q) = 0$ for the following reasons. Any element $f \in \widetilde{\mathscr{BW}}_t(-q^{2n},q)$ is a linear combination of the matrix units:

$$f = \sum_{(S,T)\in\omega(-q^{2n},q)_t} f_{ST} e_{ST}, \qquad f_{ST} \in \mathbb{C},$$

where $f_{ST} \neq 0$ for at least one pair (S,T). Fix (A,B) to be such a pair, then Eq. (4.35) implies that

$$\mathrm{tr}(e_{BA}f) = str_q\big(f_{AB}E_{BA}E_{AB}\big)/\big(sdim_q(V)\big)^t = str_q\big(f_{AB}E_{BB}\big)/\big(sdim_q(V)\big)^t \neq 0,$$

as $str_q(E_{BB}) \neq 0$ for all $(B,B) \in \omega(-q^{2n},q)_t$. Thus any non-zero f belonging to $\widetilde{\mathscr{BW}}_t(-q^{2n},q)$ does not belong to $\mathscr{J}_t(-q^{2n},q)$, giving the direct sum decomposition

$$\mathscr{BW}_t(-q^{2n},q) = \widetilde{\mathscr{BW}}_t(-q^{2n},q) \oplus \mathscr{J}_t(-q^{2n},q). \tag{4.36}$$

Then for all $a \in \mathscr{BW}_t(-q^{2n},q)$, we have $a = \widetilde{a} + a_j$, where $\widetilde{a} \in \widetilde{\mathscr{BW}}_t(-q^{2n},q)$ and $a_j \in \mathscr{J}_t(-q^{2n},q)$.

Let us define

$$\mathscr{P}_t = \sum_{(T,T)\in\omega(-q^{2n},q)_t} e_{TT} \in \mathscr{BW}_t(-q^{2n},q),$$

then we have $\mathscr{P}_t a \mathscr{P}_t = \widetilde{a}$ for all $a \in \mathscr{BW}_t(-q^{2n}, q)$, which can be seen by regarding $\mathscr{BW}_t(-q^{2n}, q)$ as a matrix algebra.

Using Eq. (4.35), we can prove that

$$\Upsilon\big(\mathscr{J}_t(-q^{2n}, q)\big) = J_t, \tag{4.37}$$

in the same way that we proved Eq. (3.47). The surjectivity of Υ implies that

$$\mathcal{C}_t = \Upsilon\big(\widetilde{\mathscr{BW}}_t(-q^{2n}, q)\big) + J_t,$$

and we will show that this sum is direct. To see this, assume that there exists some non-zero element F of $\mathcal{C}_t$ belonging to $\Upsilon\big(\widetilde{\mathscr{BW}}_t(-q^{2n}, q)\big)$ and also to J_t, then $str_q(XF) = 0$ for all $X \in \mathcal{C}_t$. However, F is the image of a linear combination of matrix units: $F = \sum_{(S,T)\in\omega(-q^{2n},q)_t} f_{ST}\Upsilon(e_{ST})$, where $f_{ST} \in \mathbb{C}$ and f_{ST} is non-zero for at least one pair (S, T). Assume that (A, B) is such a pair, then by similar reasoning as previously, $str_q(\Upsilon(e_{BA})F) \neq 0$ contradicting the assumption that $F \in J_t$. Thus $\Upsilon\big(\widetilde{\mathscr{BW}}_t(-q^{2n}, q)\big) \cap J_t = 0$, and we have

$$\mathcal{C}_t = \Upsilon\big(\widetilde{\mathscr{BW}}_t(-q^{2n}, q)\big) \oplus J_t.$$

We note the important fact that each element of $\Upsilon\big(\widetilde{\mathscr{BW}}_t(-q^{2n}, q)\big)$ is a linear combination of the matrix units $\{E_{ST} \in \mathcal{C}_t |\ (S, T) \in \widetilde{\Omega}^t\}$ we defined previously. Note that the two sets $\omega(-q^{2n}, q)_t$ and $\widetilde{\Omega}^t$ are identical if we apply the map $\lambda \mapsto \widetilde{\lambda}$ to the Young diagrams in the relevant paths if appropriate, where $\widetilde{\lambda}'_1 = 2n + 1 - \lambda_1$ and $\widetilde{\lambda}'_j = \lambda'_j$ for $j \geq 2$. As $E_{BB} = \Upsilon(e_{BB})$ for each $(B, B) \in \widetilde{\Omega}^t$ (where we think of the vertices on each path appropriately), this completes the proof.

□

After showing that the projectors in $\mathcal{C}_t$ are minimal idempotents, we move back to the main track of our argument and define P_t, which will be an extremely useful element of $\mathcal{C}_t$.

Lemma 4.3.2. *For each $t \in \mathbb{N}$, define*

$$P_t = \sum_{T\in\widetilde{\mathcal{T}}^t} E_{TT} \in \mathcal{C}_t,$$

then P_t satisfies

(i) $(P_t)^2 = P_t$,

(ii) $str_q(P_t) = sdim_q(V^{\otimes t})$,

(iii) $str_q(1 - P_t) = 0$.

Note that $P_t(V^{\otimes t}) = \sum_{T\in\widetilde{\mathcal{T}}^t} E_{TT}(V^{\otimes t}) \cong \bigoplus_{T\in\widetilde{\mathcal{T}}^t} V_{shp(T)}$.

Proof. The proof of (i) follows from the fact that the set of projections $\{E_{TT} \in \mathcal{C}_t |\, T \in \widetilde{\mathcal{T}}^t\}$ is a set of mutually orthogonal idempotents. We now inductively show that $str_q(P_t) = sdim_q(V^{\otimes t})$.

Firstly $P_1 = \mathrm{id}_V$ and $str_q(P_1) = sdim_q(V)$. Secondly $V \otimes V = V_{2\epsilon_1} \oplus V_{\epsilon_1+\epsilon_2} \oplus V_0$, where each of $2\epsilon_1, \epsilon_1 + \epsilon_2, 0$ are elements of $\overline{\mathcal{P}}_N^+$; obviously $sdim_q(V \otimes V) = \sum_{\mu \in \mathcal{P}_{\epsilon_1}^+} sdim_q(V_\mu)$ where $\mathcal{P}_{\epsilon_1}^+ = \{2\epsilon_1, \epsilon_1+\epsilon_2, 0\}$. (Note that for $n = 1$ we write $\epsilon_1+\epsilon_2$ to mean ϵ_1.) Now if each element of $\mathcal{P}_{\epsilon_1}^+$ is an element of Λ_N^+, then $P_2 = \mathrm{id}_{V\otimes V}$ and $str_q(P_2) = sdim_q(V^{\otimes 2})$. Now any element of $\mathcal{P}_{\epsilon_1}^+$ that is not in Λ_N^+ is an element of $\overline{\mathcal{P}}_N^+ \backslash \Lambda_N^+$; denote any such element by μ, then $sdim_q(V_\mu) = 0$. Now

$$P_2 = \sum_{\lambda \in \mathcal{P}_{\epsilon_1}^+ \cap \Lambda_N^+} \tilde{p}_i^2[\lambda],$$

and if there is at least one element of $\mathcal{P}_{\epsilon_1}^+$ that is not in Λ_N^+, then $P_2 \neq \mathrm{id}_{V\otimes V}$. Let $\mathcal{S}_{\epsilon_1}$ be the subset of $\{2\epsilon_1, \epsilon_1 + \epsilon_2, 0\}$ consisting of those elements not in Λ_N^+, then $str_q\big(\tilde{p}_i^2[\lambda]\big) = 0$ for each $\lambda \in \mathcal{S}_{\epsilon_1}$. As

$$V \otimes V = V_{2\epsilon_1} \oplus V_{\epsilon_1+\epsilon_2} \oplus V_0 = \left(P_2 + \sum_{\lambda \in \mathcal{S}_{\epsilon_1}} \tilde{p}_i^2[\lambda] \right) V \otimes V,$$

we have

$$str_q(P_2) = str_q \left(P_2 + \sum_{\lambda \in \mathcal{S}_{\epsilon_1}} \tilde{p}_i^2[\lambda] \right) = sdim_q\big(V \otimes V\big).$$

Now we do the inductive step. Assume that $str_q(P_j) = sdim_q(V^{\otimes j})$ for some $j \geq 2$, and let $V_\mu \subseteq V^{\otimes j}$, $\mu \in \Lambda_N^+$, be a $U_q^{(N)}(\mathfrak{g})$-module defined by $V_\mu = E_{SS}\big(V^{\otimes j}\big)$ where $S \in \widetilde{\mathcal{T}}^j$ and E_{SS} is a path projection of length j. As $S \in \widetilde{\mathcal{T}}^j$, we have $sdim_q(V_\mu) \neq 0$.

Let us define a useful set:

$$\mathcal{S}_\mu = \left\{ \lambda \in \mathcal{P}_\mu^+ \cap \overline{\Lambda_N^+} \middle|\ \lambda \notin \Lambda_N^+ \right\}. \tag{4.38}$$

Recall from Lemma 4.2.4 the following result: $V_\mu \otimes V = \bigoplus_{\lambda \in \mathcal{P}_\mu^+ \cap \overline{\Lambda_N^+}} V_\lambda$ where we have the important facts that $sdim_q(V_\lambda) \neq 0$ if $\lambda \in \Lambda_N^+$ and $sdim_q(V_\lambda) = 0$ if $\lambda \in \overline{\mathcal{P}}_N^+ \backslash \Lambda_N^+$. Then

$$\begin{aligned} sdim_q(V_\mu \otimes V) &= \sum_{\lambda \in \mathcal{P}_\mu^+ \cap \overline{\Lambda_N^+}} str_q\big(E_{S\circ\lambda, S\circ\lambda}\big) \\ &= \sum_{\xi \in \mathcal{P}_\mu^+ \cap \Lambda_N^+} str_q\big(E_{S\circ\xi, S\circ\xi}\big) + \sum_{\zeta \in \mathcal{S}_\mu} str_q\big(E_{S\circ\zeta, S\circ\zeta}\big) \\ &= \sum_{\xi \in \mathcal{P}_\mu^+ \cap \Lambda_N^+} str_q\big(E_{S\circ\xi, S\circ\xi}\big), \end{aligned}$$

as $sdim_q(V_\zeta) = 0$ for all $\zeta \in \mathcal{S}_\mu$.

Now this result is true for all $U_q^{(N)}(\mathfrak{g})$-modules $V_{\mu'} \subseteq V^{\otimes j}$ defined by $V_{\mu'} = E_{RR}\big(V^{\otimes j}\big)$ where $R \in \widetilde{\mathcal{T}}^j$ and E_{RR} is a path projection of length j, that is

$$sdim_q\big(E_{RR}V^{\otimes j} \otimes V\big) = \sum_{\xi \in \mathcal{P}^+_{shp(R)} \cap \Lambda^+_N} str_q\big(E_{R\circ\xi,R\circ\xi}\big). \tag{4.39}$$

Summing over all distinct $R \in \widetilde{\mathcal{T}}^j$ on both sides of Eq. (4.39) gives

$$sdim_q\big(P_j V^{\otimes j} \otimes V\big) = str_q\big(P_{j+1}\big), \tag{4.40}$$

as

$$P_{j+1} = \sum_{Q \in \widetilde{\mathcal{T}}^{(j+1)}} E_{QQ} = \sum_{R \in \widetilde{\mathcal{T}}^j} \left(\sum_{\xi \in \mathcal{P}^+_{shp(R)} \cap \Lambda^+_N} E_{R\circ\xi,R\circ\xi} \right),$$

and the left hand side of Eq. (4.40) is $sdim_q\big(V^{\otimes(j+1)}\big)$ by assumption. This completes the induction and the proof of (ii), and the proof of (iii) then follows.

□

We now consider a proposition that will be extremely useful in proving the tensor product theorems in this chapter:

Proposition 4.3.1.

(i) $(1 - P_t)$ *generates* J_t *as a two-sided ideal in* $\mathcal{C}_t$*,*

(ii) the mapping $\mathcal{C}_t \to P_t\mathcal{C}_tP_t$ *defined by* $a \mapsto P_taP_t$ *is an algebra homomorphism.*

We now present two technical lemmas that we will use, in addition to Lemma 3.9.2, in the proof of Proposition 4.3.1.

Lemma 4.3.3. *Each element in* $\mathcal{C}_t$ *can be written as a linear combination of elements* $(a \otimes \mathrm{id})$ *and* $(a \otimes \mathrm{id})\check{R}^{\pm 1}_{t-1}(b \otimes \mathrm{id})$ *where* $a, b \in \mathcal{C}_{t-1}$*.*

Proof. From [BW89, Lem. 3.1], each element of the Birman-Wenzl-Murakami algebra $\mathscr{BW}_t(z,q)$ can be written as a linear combination of elements $a\gamma b$ where $\gamma \in \{g_{t-1}, e_{t-1}, 1\}$ and a, b are elements of $\mathscr{BW}_{t-1}(z,q)$. We complete the proof by applying the algebra homomorphism $\Upsilon : \mathscr{BW}_t(-q^{2n},q) \to \mathcal{C}_t$ to the appropriate equations in $\mathscr{BW}_t(-q^{2n},q)$. □

Lemma 4.3.4. *Let* $B \in \widehat{\mathcal{T}}^t \backslash \widetilde{\mathcal{T}}^t$ *and* $E_{BB} = \tilde{p}^t_i[\lambda]$*, where* $\tilde{p}^t_i[\lambda] : V^{\otimes t} \to V_\lambda$ *is a well-defined projection. Then* $E_{BB} \in J_t$*.*

Proof. Let $B \in \widehat{\mathcal{T}}^t \backslash \widetilde{\mathcal{T}}^t$, then $shp(B) = \lambda \in \overline{\mathcal{P}}^+_N \backslash \Lambda^+_N$ and $sdim_q(V_\lambda) = 0$, which implies that $str_q(E_{BB}) = 0$. Now let $f \in \mathcal{C}_t$ be arbitrary, then

$$str_q(fE_{BB}) = str_q(E_{BB}fE_{BB}) = \beta str_q(E_{BB}) = 0,$$

for some complex constant β. □

We now prove Proposition 4.3.1.

Proof. We prove (i). We will firstly show that $P_t\mathcal{C}_tP_t \cap J_t = 0$. Assume that this is not true, that is there exists some $f \in \mathcal{C}_t$ such that $P_tfP_t \neq 0$ and $P_tfP_t \in J_t$, then

$$P_tfP_t = \sum_{S,T\in\widetilde{\mathcal{T}}^t} E_{SS}fE_{TT} = \sum_{(S,T)\in\widetilde{\Omega}^t} f_{ST}E_{ST}, \qquad f_{ST} \in \mathbb{C}.$$

The fact that $P_tfP_t \neq 0$ implies that $f_{ST} \neq 0$ for at least one pair $(S,T) \in \widetilde{\Omega}^t$. Fix (S,T) to be such a pair, then

$$\begin{aligned} str_q\big(E_{TS}P_tfP_tE_{TT}\big) &= str_q\left(E_{TS}\sum_{(A,B)\in\widetilde{\Omega}^t} f_{AB}E_{AB}E_{TT}\right) \\ &= str_q\left(\sum_{B\in\widetilde{\mathcal{T}}^t} f_{SB}E_{TB}E_{TT}\right) \\ &= str_q(E_{TT})f_{ST} \\ &\neq 0, \end{aligned}$$

as $str_q(E_{TT}) \neq 0$ for all $T \in \widetilde{\mathcal{T}}^t$. However, the fact that $P_tfP_t \in J_t$ implies that

$$str_q\big(E_{TS}P_tfP_tE_{TT}\big) = str_q\big(E_{TT}E_{TS}P_tfP_t\big) = 0,$$

which is a contradiction, thus there does not exist any such $f \in \mathcal{C}_t$ and thus $P_t\mathcal{C}_tP_t \cap J_t = 0$.

We define the inclusion $a \in \mathcal{C}_t \hookrightarrow \mathcal{C}_{t+1}$ by $a \mapsto a\otimes\mathrm{id}$, and we can regard each element of J_t as an element of J_{t+1} under this inclusion. From Lemma 4.3.3 each element in $\mathcal{C}_{t+1}$ can be written as a linear combination of elements $(a\otimes\mathrm{id})$ and $(a\otimes\mathrm{id})\check{R}_t^{\pm1}(b\otimes\mathrm{id})$ where $a, b \in \mathcal{C}_t$ and $\check{R}_t^{\pm1} = \mathrm{id}^{\otimes(t-1)} \otimes \check{\mathcal{R}}_{V,V} \in \mathcal{C}_{t+1}$. Let $x \in J_t$, then we claim that $str_q\big(a\check{R}_t^{\pm1}bx\big) = 0$ for all $a, b \in \mathcal{C}_t$, which we now prove using Lemma 3.9.2, which we recall is still valid for $U_q^{(N)}(\mathfrak{g})$ as discussed after Eq. (4.35):

$$\begin{aligned} str_q^{\otimes(t+1)}\big(a\check{R}_t^{\pm1}bx\big) &= str_q^{\otimes t}(\mathrm{id}^{\otimes t}\otimes str_q)\big(bxa\check{R}_t^{\pm1}\big) \\ &= \chi_V(v^{\mp1})str_q^{\otimes t}(bxa) = \chi_V(v^{\mp1})str_q^{\otimes t}(abx) = 0, \end{aligned}$$

where $str_q^{\otimes t}$ means that we take the quantum supertrace over $V^{\otimes t}$.

Let $T \in \widetilde{\mathcal{T}}^t$. Under the inclusion $\mathcal{C}_t \hookrightarrow \mathcal{C}_{t+1}$ discussed above we have

$$E_{TT} \mapsto \sum_{P\in\mathcal{P}^+_{shp(T)}} E_{T\circ P,T\circ P},$$

which corresponds to the decomposition of $V_\mu\otimes V$ into a direct sum of $U_q^{(N)}(\mathfrak{g})$-submodules in Eq. (4.13) where $\mu = shp(T)$ and $V_\mu = E_{TT}\big(V^{\otimes t}\big)$:

$$\big(E_{TT}V^{\otimes t}\big)\otimes V = V_\mu\otimes V = \bigoplus_{P\in\mathcal{P}^+_\mu\cap\overline{\Lambda^+_N}} V_P = \sum_{P\in\mathcal{P}^+_\mu\cap\overline{\Lambda^+_N}} E_{T\circ P,T\circ P}\left(V^{\otimes(t+1)}\right). \tag{4.41}$$

Using the set $\mathcal{S}_\mu$ defined by (4.38), we can rewrite (4.41) as

$$\begin{aligned} V_\mu \otimes V &= \left(\bigoplus_{P \in \mathcal{P}_\mu^+ \cap \Lambda_N^+} V_P \right) \oplus \left(\bigoplus_{Q \in \mathcal{S}_\mu} V_Q \right) \\ &= \sum_{P \in \mathcal{P}_\mu^+ \cap \Lambda_N^+} E_{T\circ P, T\circ P} V^{\otimes(t+1)} + \sum_{Q \in \mathcal{S}_\mu} E_{T\circ Q, T\circ Q} V^{\otimes(t+1)}, \end{aligned}$$

where the quantum superdimension of the module $E_{T\circ Q,T\circ Q}\left(V^{\otimes(t+1)}\right)$ is zero for $Q \in \mathcal{S}_\mu$ as $Q \in \overline{\mathcal{P}}_N^+ \backslash \Lambda_N^+$ and $T \circ Q$ is a path belonging to $\widehat{\mathcal{T}}^{t+1} \backslash \widetilde{\mathcal{T}}^{t+1}$. It is very important to note that $E_{T\circ Q,T\circ Q}$ belongs to J_{t+1} from Lemma 4.3.4, and that $\sum_{Q\in\mathcal{S}_\mu} E_{T\circ Q,T\circ Q}$ also belongs to J_{t+1}.

We will now show that there is some integer $r \geq 2$ such that $(1 - P_s)$ belongs to J_s for each integer $s \geq r$. To prove this we firstly note that $\epsilon_1 \in \Lambda_N^+$ and that $P_1 = \mathrm{id}_V \notin J_1$. Now Λ_N^+ is a proper subset of $\overline{\Lambda_N^+}$ and of $\mathcal{P}_N^+$, thus there is some integer $m \geq 2$ such that $\widehat{\mathcal{T}}^m$ contains at least one path that is not in $\widetilde{\mathcal{T}}^m$. Let us fix $r \geq 2$ to be the smallest integer such that $\widehat{\mathcal{T}}^r$ contains at least one path that is not in $\widetilde{\mathcal{T}}^r$, then

$$P_{(r-1)} = \mathrm{id}_{V^{\otimes(r-1)}}, \quad \text{and} \quad P_r \neq \mathrm{id}_{V^{\otimes r}}.$$

Note that $P_i = \mathrm{id}_{V^{\otimes i}}$ for all integers $i = 1, 2, \ldots, r-1$.

Recall the definition (4.38) of $\mathcal{S}_{shp(T)}$ for a path T of length $r-1$. Under the inclusion $\mathcal{C}_{r-1} \hookrightarrow \mathcal{C}_r$ we have

$$\begin{aligned} P_{(r-1)} &\mapsto \sum_{T \in \widetilde{\mathcal{T}}^{(r-1)}} \left(\sum_{P \in \mathcal{P}_{shp(T)}^+ \cap \Lambda_N^+} E_{T\circ P, T\circ P} + \sum_{Q \in \mathcal{S}_{shp(T)}} E_{T\circ Q, T\circ Q} \right), \\ &= \sum_{T \in \widetilde{\mathcal{T}}^r} E_{TT} + \sum_{T \in \widetilde{\mathcal{T}}^{(r-1)}} \left(\sum_{Q \in \mathcal{S}_{shp(T)}} E_{T\circ Q, T\circ Q} \right) \\ &= P_r + \sum_{T \in \widetilde{\mathcal{T}}^{(r-1)}} \left(\sum_{Q \in \mathcal{S}_{shp(T)}} E_{T\circ Q, T\circ Q} \right). \end{aligned} \tag{4.42}$$

As $P_{(r-1)} = \mathrm{id}_{V^{\otimes(r-1)}}$ and we have $P_{(r-1)} \mapsto P_{(r-1)} \otimes \mathrm{id}$ under the inclusion $\mathcal{C}_{r-1} \hookrightarrow \mathcal{C}_r$, (4.42) equals $\mathrm{id}_{V^{\otimes r}}$. We reiterate once more that $P_r \neq \mathrm{id}_{V^{\otimes r}}$.

As $P_{(r-1)} \otimes \mathrm{id} = \mathrm{id}_{V^{\otimes r}}$, rewriting (4.42) gives us

$$(1 - P_r) = \sum_{T \in \widetilde{\mathcal{T}}^{(r-1)}} \left(\sum_{Q \in \mathcal{S}_{shp(T)}} E_{T\circ Q, T\circ Q} \right),$$

and the two summations on the right hand side of this expression can be rewritten as a single sum:

$$\sum_{T\in\widetilde{\mathcal{T}}^{(r-1)}}\left(\sum_{Q\in\mathcal{S}_{shp(T)}} E_{T\circ Q,T\circ Q}\right)=\sum_{D\in\widehat{\mathcal{T}}^{r}\setminus\widetilde{\mathcal{T}}^{r}} E_{DD},$$

and therefore

$$(1-P_r)=\sum_{D\in\widehat{\mathcal{T}}^{r}\setminus\widetilde{\mathcal{T}}^{r}} E_{DD},$$

where each E_{DD} is an element of J_r from Lemma 4.3.4.

Now for each integer $s\geq r$ we can similarly show that

$$P_s-P_{(s+1)}=\sum_{Q\in\widehat{\mathcal{T}}^{s+1}\setminus\widetilde{\mathcal{T}}^{s+1}} E_{QQ}\in J_{(s+1)},$$

and by expressing $(1-P_{(s+1)})$ as the sum:

$$(1-P_{(s+1)})=(1-P_r)+(P_r-P_{(r+1)})+\cdots+(P_s-P_{(s+1)}),$$

we have $(1-P_{(s+1)})\in J_{(s+1)}$ as $(1-P_r)$ and $(P_i-P_{(i+1)})$ belong to $J_{(i+1)}$ for all $i=r,\ldots,s$ under the inclusion $\mathcal{C}_i\hookrightarrow\mathcal{C}_{(i+1)}$. This proves that $(1-P_j)\in J_j$ for all $j\in\mathbb{N}$.

Note that

$$\begin{aligned}\mathcal{C}_t &= \big(P_t+(1-P_t)\big)\mathcal{C}_t\big(P_t+(1-P_t)\big)\\ &= P_t\mathcal{C}_tP_t+(1-P_t)\mathcal{C}_tP_t+P_t\mathcal{C}_t(1-P_t)+(1-P_t)\mathcal{C}_t(1-P_t).\end{aligned}\tag{4.43}$$

Now $(1-P_t)$ is in J_t, and each of $(1-P_t)xP_t$, $P_tx(1-P_t)$ and $(1-P_t)x(1-P_t)$ are in J_t for each $x\in\mathcal{C}_t$, thus

$$\big((1-P_t)\mathcal{C}_tP_t+P_t\mathcal{C}_t(1-P_t)+(1-P_t)\mathcal{C}_t(1-P_t)\big)\subseteq J_t.$$

We previously proved that $P_t\mathcal{C}_tP_t\cap J_t=0$, thus any element $x\in\mathcal{C}_t$ belonging to J_t must also belong to $\big((1-P_t)\mathcal{C}_tP_t+P_t\mathcal{C}_t(1-P_t)+(1-P_t)\mathcal{C}_t(1-P_t)\big)$ from (4.43). We thus obtain

$$\mathcal{C}_t=P_t\mathcal{C}_tP_t\oplus J_t.$$

We wish to show that $(1-P_t)$ generates J_t as a two-sided ideal. To do this we will prove two assertions:

(a) $x(1-P_t)y$ belongs to J_t for all $x,y\in\mathcal{C}_t$, and

(b) each element of J_t belongs to $\mathcal{C}_t(1-P_t)\mathcal{C}_t$.

The proof of (a) follows easily from the fact that $(1-P_t)\in J_t$ and the properties of the quantum supertrace, but the proof of (b) is more involved. Let z be an arbitrary element of J_t, then

$$z=P_tzP_t+(1-P_t)zP_t+P_tz(1-P_t)+(1-P_t)z(1-P_t).$$

As $P_t z P_t \in J_t$, but $P_t \mathcal{C}_t P_t \cap J_t = 0$, we have $P_t z P_t = 0$. Then

$$z = (1 - P_t) z P_t + P_t z (1 - P_t) + (1 - P_t) z (1 - P_t) = z(1 - P_t) + (1 - P_t) z P_t,$$

which belongs to $\mathcal{C}_t(1-P_t)\mathcal{C}_t$, proving (b). This completes the proof that $(1-P_t)$ generates J_t as a two-sided ideal.

We now prove part (ii) of Proposition 4.3.1. Let a, $b \in \mathcal{C}_t$ be arbitrary, then

$$P_t ab P_t = P_t a\big(P_t + (1 - P_t)\big) b P_t = P_t a P_t b P_t = (P_t a P_t)(P_t b P_t),$$

as $P_t a(1 - P_t) b P_t \in J_t$ and $P_t \mathcal{C}_t P_t \cap J_t = 0$. □

4.3.2 The Tensor Product Theorems

We now prove the tensor product theorems.

Theorem 4.3.2. *Let $n \geq 2$ and $N \geq 4$ be even, or let $n = 1$ and $N \geq 6$ satisfy $N \equiv 2 \pmod 4$. Furthermore, let N be sufficiently large enough so that $\epsilon_1 \in \Lambda_N^+$. Then for each $t \in \mathbb{Z}_+$, there is a decomposition of $V^{\otimes t}$ into a direct sum of $U_q^{(N)}(\mathfrak{g})$-submodules*

$$V^{\otimes t} = \mathcal{V} \oplus \mathcal{Z}, \tag{4.44}$$

where $\mathcal{V}$ is of the form

$$\mathcal{V} = \bigoplus_{\lambda \in \Lambda_N^+} (V_\lambda)^{\oplus n_t(\lambda)},$$

with $n_t(\lambda) \in \mathbb{Z}_+$ being the (possibly zero) number of copies of the $U_q^{(N)}(\mathfrak{g})$-submodule V_λ in $\mathcal{V}$. Here $\mathcal{Z}$ is a possibly vanishing $U_q^{(N)}(\mathfrak{g})$-submodule with the property that $str_q(f) = 0$ for all $f \in \mathcal{C}_t$ satisfying $f(V^{\otimes t}) \subseteq \mathcal{Z}$.

Proof. Let P_t be as given in Lemma 4.3.2 and fix $\mathcal{V} = P_t V^{\otimes t}$ and $\mathcal{Z} = (1 - P_t)V^{\otimes t}$. As P_t and $(1 - P_t)$ are orthogonal idempotents, the sum on the right hand side of (4.44) is direct. To prove that $\mathcal{Z}$ has the claimed property, let $f \in \mathcal{C}_t$ satisfy $f(V^{\otimes t}) \subseteq \mathcal{Z}$. Then $f = \big(P_t + (1 - P_t)\big)f$. Now $P_t f(V^{\otimes t}) \subseteq P_t\mathcal{Z} = P_t(1 - P_t)V^{\otimes t} = 0$, thus $P_t f = 0$ and $f = (1 - P_t)f$. Then $str_q(f) = str_q\big((1 - P_t)f\big) = 0$ as $(1 - P_t) \in J_t$, completing the proof. □

Lemma 4.3.5. *Assume that the given conditions on n and N are the same as in Theorem 4.3.2. Let $V^{\otimes t} = \mathcal{V}_t \oplus \mathcal{Z}_t$ be the decomposition of $V^{\otimes t}$ into $U_q^{(N)}(\mathfrak{g})$-submodules for each $t \in \mathbb{N}$ given in Eq. (4.44). Then $\mathcal{Z}_t \otimes V \subseteq \mathcal{Z}_{t+1}$.*

Proof. This is similar to the proof of [TW93, Thm. 5.5.2]. From the definition of P_t and the inclusion $P_t \hookrightarrow P_t \otimes \mathrm{id} \in \mathcal{C}_{t+1}$ we have

$$P_t \otimes \mathrm{id} = \sum_{S\in\widetilde{\mathcal{T}}^t} E_{SS} \otimes \mathrm{id} = \sum_{S\in\widetilde{\mathcal{T}}^t} \left(\sum_{\substack{\mu\in\overline{\Lambda_N^+} \\ shp(S)\le\mu}} E_{S\circ\mu,S\circ\mu} \right) \tag{4.45}$$

$$= P_{(t+1)} + \sum_{T\in\widehat{\mathcal{T}}^{t+1}\setminus\widetilde{\mathcal{T}}^{t+1}} E_{TT}. \tag{4.46}$$

To be certain that (4.45) is true, note that E_{SS} is a path projection of length t projecting down from $V^{\otimes t}$ onto a $U_q^{(N)}(\mathfrak{g})$-submodule V_ν where $\nu = shp(S) \in \Lambda_N^+$, and recall from (4.13) that

$$V_\nu \otimes V = \bigoplus_{\mu\in\mathcal{P}_\nu^+\cap\overline{\Lambda_N^+}} V_\mu.$$

Then (4.45) follows from the fact that $\mathcal{P}_\nu^+ \cap \overline{\Lambda_N^+}$ is the set of all elements ξ of $\overline{\Lambda_N^+}$ where $\nu \le \xi$, that is, $\mathcal{P}_\nu^+ \cap \overline{\Lambda_N^+}$ is the set of all the ξ connected to ν in the relevant Bratelli diagram where ξ is on the level of the Bratteli diagram immediately below the level containing ν.

To see that (4.46) is correct, note that each vertex $\nu \in \Lambda_N^+$ on the t^{th} level of the Bratteli diagram is connected to vertices on the $(t+1)^{st}$ level of the Bratteli diagram where all of these latter vertices are elements of $\overline{\mathcal{P}}_N^+$.

Note that the image of $\sum_{T\in\widehat{\mathcal{T}}^{t+1}\setminus\widetilde{\mathcal{T}}^{t+1}} E_{TT}$ is contained in $\mathcal{Z}_{t+1}$. Hence

$$\mathcal{Z}_t \otimes V = \left((1 - P_{t+1}) - \sum_{T\in\widehat{\mathcal{T}}^{t+1}\setminus\widetilde{\mathcal{T}}^{t+1}} E_{TT} \right) V^{\otimes(t+1)} \subseteq \mathcal{Z}_{t+1}.$$

□

Theorem 4.3.3. *Assume that the given conditions on n and N are the same as in Theorem 4.3.2. Let $s \in \mathbb{N}$ and $\lambda_i \in \Lambda_N^+$ for each $1 \le i \le s$. Let V_{λ_i} be a $U_q^{(N)}(\mathfrak{g})$-module defined in Definition 4.2.5:*

$$\tilde{p}_{j_i}^{t_i}[\lambda_i] : V^{\otimes t_i} \to V_{\lambda_i}, \qquad i = 1, \ldots, s.$$

Then there is a decomposition of $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_s}$ into a direct sum of $U_q^{(N)}(\mathfrak{g})$-submodules:

$$V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_s} = \mathcal{V} \oplus \mathcal{Z}, \tag{4.47}$$

where $\mathcal{V}$ is a direct sum of q-admissible submodules and $\mathcal{Z}$ is a possibly vanishing submodule with the property that $str_q(f) = 0$ for all $f \in \mathcal{C}_t$ satisfying $f\big(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_s}\big) \subseteq \mathcal{Z}$.

Proof. We can write the left hand side of (4.47) as

$$V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_s} = e\big(V^{\otimes t_1} \otimes \cdots \otimes V^{\otimes t_s}\big) \subseteq V^{\otimes t},$$

where $t = \sum_{i=1}^{s} t_i$, and

$$e = \tilde{p}_{j_1}^{t_1}[\lambda_1] \otimes \cdots \otimes \tilde{p}_{j_s}^{t_s}[\lambda_s] \in \mathcal{C}_t.$$

Then $e^2 = e$, and we can use Proposition 4.3.1 (ii) to prove that

$$\big(P_t e P_t\big)^2 = P_t e P_t e P_t = P_t e P_t. \tag{4.48}$$

In the same way, we can show that eP_teP_te is an idempotent, and thus so is $e - eP_teP_te$. Clearly these two idempotents are orthogonal to each other. The method we use to prove the theorem is to show that $eP_teP_te\big(V^{\otimes t}\big)$ is isomorphic to $P_teP_t(V^{\otimes t})$, which is a direct sum of q-admissible modules, and that $(e - eP_teP_te)V^{\otimes t}$ satisfies the given properties of $\mathcal{Z}$ in (4.47).

Firstly, we will show that $(e - eP_teP_te) \in J_t$. Clearly, $(1-P_t)(e-eP_teP_te)(1-P_t) \in J_t$, and

$$\begin{aligned}(1 - P_t)&(e - eP_teP_te)(1 - P_t)\\ &= (e - eP_teP_te)(1 - P_t) - (P_te - P_teP_te)(1 - P_t)\\ &= (e - eP_teP_te) - (eP_t - eP_teP_t) + (P_teP_te - P_te)(1 - P_t).\end{aligned}$$

Now $P_t(eP_t - eP_teP_t) = 0$ as P_t and P_teP_t are idempotents, and so we can write

$$\begin{aligned}(1 - P_t)&(e - eP_teP_te)(1 - P_t)\\ &= (e - eP_teP_te) + (P_teP_te - P_te)(1 - P_t) - (1 - P_t)(eP_t - eP_teP_t),\end{aligned}$$

which implies that

$$(e - eP_teP_te) \in J_t,$$

as both $(P_teP_te - P_te)(1 - P_t)$ and $(1 - P_t)(eP_t - eP_teP_t)$ are elements of J_t. Clearly $str_q(f) = 0$ for all $f \in \mathcal{C}_t$ satisfying $f\big(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_s}\big) \subseteq (e - eP_teP_te)V^{\otimes t}$.

Fix $A = eP_teP_te$ and $B = P_teP_t$, then A and B are $U_q^{(N)}(\mathfrak{g})$-linear idempotents satisfying

$$ABA = A, \quad \text{and} \quad BAB = B. \tag{4.49}$$

Write

$$V_A = A\big(V^{\otimes t}\big), \quad \text{and} \quad V_B = B\big(V^{\otimes t}\big),$$

then $V_A \cong V_B$. While this is easy to show, we prove it here for clarity. Clearly $A\big(V_B\big) \subseteq V_A$, so $BA\big(V_B\big) \subseteq B\big(V_A\big)$, and we can rewrite this using (4.49) as $V_B \subseteq B\big(V_A\big)$. In addition, $B\big(V_A\big) \subseteq V_B$, so $AB\big(V_A\big) \subseteq A\big(V_B\big)$, and we can rewrite this using (4.49) as $V_A \subseteq A\big(V_B\big)$. Then

$$A\big(V_B\big) = V_A \quad \text{and} \quad B\big(V_A\big) = V_B.$$

The idempotents A and B are $U_q^{(N)}(\mathfrak{g})$-linear intertwiners between V_A and V_B, so $V_A \cong V_B$. Now $V_B = P_t e P_t\big(V^{\otimes t}\big)$ is a direct sum of q-admissible modules, so $V_A = eP_teP_te\big(V^{\otimes t}\big)$ is also a direct sum of q-admissible modules. Fixing $\mathcal{V} = eP_teP_te\big(V^{\otimes t}\big)$ and $\mathcal{Z} = (e - eP_teP_te)V^{\otimes t}$ completes the proof.

□

We say that a $U_q^{(N)}(\mathfrak{g})$-module W_μ is *q-admissible* if $W_\mu = p_\mu(V^{\otimes t})$ for some idempotent $p_\mu \in \mathcal{C}_t$ and if W_μ is isomorphic to a $U_q^{(N)}(\mathfrak{g})$-module $V_\lambda = E_{TT}\big(V^{\otimes t}\big)$ for some $\lambda \in \Lambda_N^+$. Here $E_{TT} \in End_{U_q^{(N)}(\mathfrak{g})}(V^{\otimes t})$ is the path projection associated with the path $T \in \widetilde{\mathcal{T}}^t$ of length t with $shp(T) = \lambda$. Each q-admissible $U_q^{(N)}(\mathfrak{g})$-module has non-vanishing quantum superdimension.

We finish off this section by making the following conjecture.

Conjecture 4.3.1. *Assume that the given conditions on n and N are the same as in Theorem 4.3.2. Let $s \in \mathbb{N}$ and $\lambda_i \in \Lambda_N^+$ for each $1 \le i \le s$. Let V_{λ_i} be a $U_q^{(N)}(\mathfrak{g})$-module defined in Definition 4.2.5:*

$$\tilde{p}_{j_i}^{t_i}[\lambda_i] : V^{\otimes t_i} \to V_{\lambda_i}, \qquad i = 1, \dots, s.$$

Let $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_s} \subseteq V^{\otimes t}$ where $t = \sum_{i=1}^s t_i$, then given any decomposition of $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_s}$ into a direct sum of $U_q^{(N)}(\mathfrak{g})$-submodules:

$$V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_s} = \widetilde{\mathcal{V}} \oplus \widetilde{\mathcal{Z}}, \tag{4.50}$$

where $\widetilde{\mathcal{V}}$ is a direct sum of q-admissible submodules and $\widetilde{\mathcal{Z}}$ is a possibly vanishing submodule with the property that $str_q(f) = 0$ for all $f \in \mathcal{C}_t$ satisfying $f(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_s}) \subseteq \widetilde{\mathcal{Z}}$, then $\widetilde{\mathcal{V}} \cong \mathcal{V}$ and $\widetilde{\mathcal{Z}} \cong \mathcal{Z}$ where $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_s} = \mathcal{V} \oplus \mathcal{Z}$ is the decomposition of $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_s}$ into $U_q^{(N)}(\mathfrak{g})$-submodules given in Eq. (4.47).

4.4 The well-definedness of the projection operators

In this section we present the detailed proof of Lemma 4.2.2. Let $\lambda_i^t = (0, \epsilon_1, s_2, s_3, \dots, s_{t-1}, \lambda) \in \widehat{\mathcal{T}}^t$. The projection $\tilde{p}_i^t[\lambda] : V^{\otimes t} \to V_\lambda$ is well defined if for each $s_j \in \lambda_i^t$,

$$\prod_{\substack{\mu \in \mathcal{P}_{s_j}^+ \\ \mu \ne s_{j+1}}} \big(\chi_{s_{j+1}}(v) - \chi_\mu(v)\big) \ne 0. \tag{4.51}$$

We now write λ instead of s_j. From Def. 3.8.1, $\mathcal{P}_\lambda^+ \subseteq \mathcal{P}_\lambda^0 = \big\{\lambda, \lambda \pm \epsilon_j \in \mathcal{P}^+ | \; 1 \le j \le n\big\}$. Clearly the following equation

$$\prod_{\substack{\mu \in \mathcal{P}_\lambda^0 \\ \mu \ne s_{j+1}}} \big(\chi_{s_{j+1}}(v) - \chi_\mu(v)\big) \ne 0, \tag{4.52}$$

implies (4.51). Note that (4.52) is true if and only if

$$\big((s_{j+1}+2\rho, s_{j+1}) - (\mu+2\rho,\mu)\big) \not\equiv 0 \pmod N, \tag{4.53}$$

for all $s_{j+1}, \mu \in \mathcal{P}^0_\lambda$ where $\mu \neq s_{j+1}$. Let us write ξ instead of s_{j+1} for convenience.

In order to show that (4.53) is true, we will find the largest and smallest elements of

$$\mathcal{S} = \Big\{ \big|(\mu+2\rho,\mu) - (\xi+2\rho,\xi)\big| \in \mathbb{R} \;\Big|\; \mu,\xi \in \mathcal{P}^0_\lambda,\ \mu\neq\xi \Big\}.$$

To help with this we note the following inequalities for each $\lambda \in \mathcal{P}^+$ and each $i = 1,2,\ldots,n-1$:

$$(\lambda+2\rho,\lambda) \le (\lambda+\epsilon_{i+1}+2\rho, \lambda+\epsilon_{i+1}) \le (\lambda+\epsilon_i+2\rho,\lambda+\epsilon_i),$$
$$(\lambda-\epsilon_i+2\rho,\lambda-\epsilon_i) \le (\lambda-\epsilon_{i+1}+2\rho,\lambda-\epsilon_{i+1}) \le (\lambda+2\rho,\lambda).$$

From these inequalities, the largest element of $\mathcal{S}$ is

$$(\lambda+\epsilon_1+2\rho,\lambda+\epsilon_1) - (\lambda-\epsilon_1+2\rho,\lambda-\epsilon_1) = 4\lambda_1+4n-2,$$

and the second largest element of $\mathcal{S}$ is

$$(\lambda+\epsilon_1+2\rho,\lambda+\epsilon_1) - (\lambda-\epsilon_2+2\rho,\lambda-\epsilon_2) = (2\lambda+2\rho,\epsilon_1+\epsilon_2) = 2\lambda_1+2\lambda_2+4n-4.$$

We now determine the smallest element of $\mathcal{S}$. For each $\lambda\in\mathcal{P}^+$ and $i=1,2,\ldots,n-1$, we have

$$2 \le (\lambda+\epsilon_i+2\rho,\lambda+\epsilon_i) - (\lambda+\epsilon_{i+1}+2\rho,\lambda+\epsilon_{i+1}),$$
$$2 \le (\lambda-\epsilon_{i+1}+2\rho,\lambda-\epsilon_{i+1}) - (\lambda-\epsilon_i+2\rho,\lambda-\epsilon_i),$$

and

$$2 \le (\lambda+\epsilon_n+2\rho,\lambda+\epsilon_n) - (\lambda+2\rho,\lambda),$$
$$2 \le 2\lambda_n \le (\lambda+2\rho,\lambda) - (\lambda-\epsilon_n+2\rho,\lambda-\epsilon_n),$$

as $\lambda-\epsilon_n$ is an integral dominant weight.

Lemma 4.4.1. *The projections from Lemma 4.2.2 (a) are well defined when* $N \equiv 0 \pmod 4$.

Proof. The projections are well-defined if for each $\lambda \in \Lambda^+_N$ we have

$$\big|(\mu+2\rho,\mu) - (\xi+2\rho,\xi)\big| \not\equiv 0 \pmod N, \tag{4.54}$$

for all $\mu,\xi\in\mathcal{P}^0_\lambda$ where $\mu\neq\xi$. As before, the smallest value of the left hand side of (4.54) is 2 and the largest is $4\lambda_1+4n-2$. From the definition of Λ^+_N, the components of λ satisfy $0 \le \lambda_1+\lambda_2 \le N/2-2n+1$, thus

$$0 \le 4\lambda_1+4n-2 \le 2N-4n+2,$$

and $2N-4n+2 < 2N$. As $\lambda_1\in\mathbb{Z}_+$, we have $4\lambda_1+4n-2 \in 4\mathbb{Z}_+ + 2$, then it follows that $4\lambda_1+4n-2 \neq N$ as $N\equiv 0 \pmod 4$.

To complete the proof, we note that the second largest value of the left hand side of (4.54) is $2\lambda_1+2\lambda_2+4n-4$. Now from the inequality $\lambda_1+\lambda_2 \le N/2-2n+1$ we obtain $2\lambda_1+2\lambda_2+4n-4 \le N-2$, completing the proof. ☐

Lemma 4.4.2. *The projections from Lemma 4.2.2 (a) are well defined for* $N \equiv 2 \pmod 4$.

Proof. For $\lambda \in \Lambda_N^+$, we have $0 \le \lambda_1 \le N/4 - n - 1/2$. As previously, the projections are well-defined if (4.54) is true. The smallest value of the left hand side of (4.54) is 2 and the largest value is $4\lambda_1 + 4n - 2$. Thus

$$6 \le 4\lambda_1 + 4n - 2 \le N - 4,$$

and the projections are well-defined.

□

Lemma 4.4.3. *For each odd* $N \ge 3$, *the projection* $\tilde{p}_i^t[\lambda]$ *is well-defined for each* $\lambda_i^t \in \widehat{\mathcal{T}}^t$ *if*

(i) $\lambda_1 \le (N-1)/2 - n + 1$, *or if*

(ii) the components of $s_{t-1} = \overline{\lambda} \in \Lambda_N^+$ *satisfy* $\overline{\lambda}_1 = (N-1)/2 - n + 1$ *and* $\overline{\lambda}_2 = \overline{\lambda}_1$, *and* λ *is such that* $\lambda = \overline{\lambda} + \epsilon_1$.

Proof. As previously, the projection is well-defined if (4.54) is true for all pairs $(\mu, \xi) \in \mathcal{P}_\lambda^0 \times \mathcal{P}_\lambda^0$ where $\mu \ne \xi$, for each $\lambda \in \Lambda_N^+$. We will show that this is true only when parts (i) or (ii) of the lemma are satisfied. As before, the smallest value of the left hand side of (4.54) is 2, the largest value is $4\lambda_1 + 4n - 2$, and the second largest value is $2\lambda_1 + 2\lambda_2 + 4n - 4$.

Consider the largest value. Let $\lambda \in \Lambda_N^+$, then $0 \le \lambda_1 + \lambda_2 \le N - 2n + 1$, thus

$$0 \le 4\lambda_1 + 4n - 2 < 4N.$$

Note that $4\lambda_1 + 4n - 2 \ne N$ and that $4\lambda_1 + 4n - 2 \ne 3N$ as the left hand sides of each of these is even. However, we may have $4\lambda_1 + 4n - 2 = 2N$ and it transpires that this results in part (i) of the lemma.

Let us consider the second largest value of the left hand side of (4.54). From the relations $0 \le \lambda_1 + \lambda_2 \le N - 2n - 1$ we have

$$4 \le 2\lambda_1 + 2\lambda_2 + 4n - 4 \le 2N - 2.$$

Thus if we can show that the left hand side of (4.54) is always even, the only possible case in which (4.54) is not satisfied is when $4\lambda_1 + 4n - 2 = 2N$. We will now show that the left hand side of (4.54) is always even. To do this, we consider all the possible cases below, in which we let $i, j \in \{1, 2, \ldots, n\}$.

(i) Set $\mu = \lambda + \epsilon_i$ and $\xi = \lambda$. Then $(\lambda + \epsilon_i + 2\rho, \lambda + \epsilon_i) - (\lambda + 2\rho, \lambda) = (2\lambda + 2\rho, \epsilon_i) + 1 \in 2\mathbb{Z}_+$.

(ii) Set $\mu = \lambda$ and $\xi = \lambda - \epsilon_i$. Then $(\lambda + 2\rho, \lambda) - (\lambda - \epsilon_i + 2\rho, \lambda - \epsilon_i) = (2\lambda + 2\rho, \epsilon_i) - 1 \in 2\mathbb{Z}_+$.

(iii) Set $\mu = \lambda + \epsilon_i$ and $\xi = \lambda - \epsilon_j$ where $i \ne j$. Then $(\lambda + \epsilon_i + 2\rho, \lambda + \epsilon_i) - (\lambda - \epsilon_j + 2\rho, \lambda - \epsilon_j) = (2\lambda + 2\rho, \epsilon_i + \epsilon_j) \in 2\mathbb{Z}_+$.

(iv) Set $\mu = \lambda - \epsilon_i$ and $\xi = \lambda - \epsilon_j$ where $i \ne j$. Then $(\lambda - \epsilon_i + 2\rho, \lambda - \epsilon_i) - (\lambda - \epsilon_j + 2\rho, \lambda - \epsilon_j) = (2\lambda + 2\rho, \epsilon_j - \epsilon_i) \in 2\mathbb{Z}$.

(v) Set $\mu = \lambda+\epsilon_i$ and $\xi = \lambda+\epsilon_j$ where $i \neq j$. Then $(\lambda+\epsilon_i+2\rho, \lambda+\epsilon_i)-(\lambda+\epsilon_j+2\rho, \lambda+\epsilon_j) = (2\lambda + 2\rho, \epsilon_i - \epsilon_j) \in 2\mathbb{Z}$.

(vi) Set $\mu = \lambda + \epsilon_i$ and $\xi = \lambda - \epsilon_i$. Then $(\lambda + \epsilon_i + 2\rho, \lambda + \epsilon_i) - (\lambda - \epsilon_i + 2\rho, \lambda - \epsilon_i) = (2\lambda + 2\rho, 2\epsilon_i) \in 2\mathbb{Z}_+$ and

$$(2\lambda + 2\rho, 2\epsilon_i) = 2(2\lambda_i + 2n - 2i + 1) < \begin{cases} 4N, & \text{when } i = 1, \\ 2N, & \text{when } i \geq 2. \end{cases}$$

The only possible case in which (4.54) is *not* satisfied is when $\mu = \lambda + \epsilon_1$ and $\xi = \lambda - \epsilon_1$, and here it may be possible that $(\xi + 2\rho, \xi) - (\mu + 2\rho, \mu) = 2N$. We will show that for each odd $N \geq 3$ there always exists at least one element λ of Λ_N^+ with the property that

$$(\lambda + \epsilon_i + 2\rho, \lambda + \epsilon_i) - (\lambda - \epsilon_i + 2\rho, \lambda - \epsilon_i) = 2N. \tag{4.55}$$

If (4.55) is satisfied, the notionally existing projection $P[\lambda + \epsilon_1] \in End_{U_q^{(N)}(\mathfrak{g})}(V_\lambda \otimes V)$ that would act if it was well-defined as

$$P[\lambda + \epsilon_1] : V_\lambda \otimes V \to V_{\lambda+\epsilon_1} \subset V_\lambda \otimes V,$$

is only well-defined if $\lambda - \epsilon_1 \notin \mathcal{P}_\lambda^+ \cap \overline{\Lambda_N^+}$. Additionally, if (4.55) is satisfied, the notionally existing projection $P[\lambda - \epsilon_1] \in End_{U_q^{(N)}(\mathfrak{g})}(V_\lambda \otimes V)$ that would act if it was well-defined as

$$P[\lambda - \epsilon_1] : V_\lambda \otimes V \to V_{\lambda-\epsilon_1} \subset V_\lambda \otimes V,$$

is only well-defined if $\lambda + \epsilon_1 \notin \mathcal{P}_\lambda^+ \cap \overline{\Lambda_N^+}$.

If $\lambda - \epsilon_1$ and $\lambda + \epsilon_1$ are elements of $\mathcal{P}_\lambda^0$ for any given $\lambda \in \Lambda_N^+$, then they are elements of $\mathcal{P}_\lambda^+ \cap \overline{\Lambda_N^+}$. By inspection, both $\lambda - \epsilon_1$ and $\lambda + \epsilon_1$ are elements of $\mathcal{P}_\lambda^0$ if and only if the components of λ satisfy

$$\lambda_2 \leq \lambda_1 - 1, \quad \text{and} \quad (\lambda_1 + 1) + \lambda_2 \leq N - 2n + 2.$$

The first condition implies that both $\lambda - \epsilon_1$ and $\lambda + \epsilon_1$ are elements of $\mathcal{P}_\lambda^+$ and the second condition is necessary so that $\lambda + \epsilon_1$ is an element of $\overline{\Lambda_N^+}$.

Let us determine the conditions on the components of λ so that (4.55) is true. This equation states that $4\lambda_1 + 4n - 2 = 2N$ which we rewrite as $\lambda_1 = N/2 - n + 1/2$. Consider an integral dominant weight λ where $\lambda_1 = N/2 - n + 1/2$ and $\lambda_2 \leq \lambda_1 - 1$. Then $\lambda_1 + \lambda_2 \leq N - 2n$ and λ is in Λ_N^+. This means that for each Λ_N^+ there is *at least one* $\lambda \in \Lambda_N^+$ where the notional projections $P[\lambda + \epsilon_1]$ and $P[\lambda - \epsilon_1]$ are *not well-defined*. In general there are many such elements of Λ_N^+ for which this is true. This proves (i).

We now prove (ii). Consider an element $\overline{\lambda} \in \Lambda_N^+$ where $\overline{\lambda}_1 = N/2 - n + 1/2$ and $\overline{\lambda}_2 = \overline{\lambda}_1$. Then $\overline{\lambda} - \epsilon_1 \notin \mathcal{P}_{\overline{\lambda}}^0$ and $(\overline{\lambda}_1 + 1) + \overline{\lambda}_2 = N - 2n + 2$, thus we have $\overline{\lambda} + \epsilon_1 \in \overline{\Lambda_N^+} \backslash \Lambda_N^+$. Then the projection $P\left[\overline{\lambda} + \epsilon_1\right] : V_{\overline{\lambda}} \otimes V \to V_{\overline{\lambda}+\epsilon_1}$ is well-defined as $\overline{\lambda} - \epsilon_1 \notin \mathcal{P}_{\overline{\lambda}}^0$, proving (ii). Note that $sdim_q(V_{\overline{\lambda}+\epsilon_1}) = 0$.

□

Chapter 5

Topological invariants of 3-manifolds from $U_q^{(N)}(osp(1|2n))$

The structure of this chapter is as follows. In Section 5.1 we introduce knots and links and the equivalence relations of ambient and regular isotopy generated by the Reidemeister moves on planar projections of links. We describe how each closed, connected, orientable 3-manifold M_L can be obtained by *performing surgery on the* 3*-sphere* S^3 *along a link* $L \subset S^3$, and we describe the equivalence relations, called the *Kirby moves*, on two links L and L' embedded in S^3 such that the 3-manifolds they give rise to upon performing surgery are homeomorphic.

In Section 5.2 we introduce directed ribbon tangles and the category of coloured directed ribbon tangles following Reshetikhin [Re90] and Reshetikhin and Turaev [RT90], as a precursor to constructing isotopy invariants of coloured directed ribbon tangles, and thereby regular isotopy invariants of links later in this chapter.

In Section 5.3 we state the Reshetikhin-Turaev functor F, a covariant functor from the category of coloured directed ribbon tangles to the category of finite dimensional representations of a $\mathbb{Z}_2$-graded ribbon Hopf algebra. This definition follows directly from [Zh95] and is a generalisation of the covariant functor from the category of coloured directed ribbon tangles to the category of finite dimensional representations of an ungraded ribbon Hopf algebra [Re90, RT90, RT91]. The functor yields isotopy invariants of coloured directed ribbon tangles and thereby regular isotopy invariants of links.

Sections 5.4–5.8 contain the new results in this chapter.

In Section 5.4 we generalise the definition of a modular Hopf algebra to the $\mathbb{Z}_2$-graded case. Drawing on this generalisation, we define a new algebra that we call a *pseudo-modular Hopf algebra.* Pseudo-modular Hopf algebras are $\mathbb{Z}_2$-graded ribbon Hopf algebras together with a finite set of finite dimensional representations satisfying slightly weaker conditions than those satisfied by modular Hopf algebras. We define these algebras as there exist quotients of quantum algebras and quantum superalgebras that are not modular, or are not known to be modular, from which topological invariants of 3-manifolds can be constructed. Later in this chapter we construct topological invariants of 3-manifolds from pseudo-modular Hopf algebras.

In Section 5.5 we prove that topological invariants of closed, connected, orientable 3-manifolds can be constructed using pseudo-modular Hopf algebras.

In Section 5.6 we prove that the $\mathbb{Z}_2$-graded ribbon Hopf algebra $U_q^{(N)}(osp(1|2n))$, where $N \geq 6$ satisfies $N \equiv 2 \pmod 4$, together with a set of non-isomorphic finite dimensional $U_q^{(N)}(osp(1|2n))$-modules $\{V_\lambda |\ \lambda \in \Lambda_N^+\}$ defined in Chapter 4, is a pseudo-modular Hopf algebra. We also prove that 3-manifold invariants cannot be constructed from $U_q^{(N)}(osp(1|2n))$ when $N \geq 4$ satisfies $N \equiv 0 \pmod 4$.

In Section 5.7 we compare the 3-manifold invariants arising from $U_q^{(N)}(osp(1|2n))$ when $N \geq 6$ satisfies $N \equiv 2 \pmod 4$, with the invariants arising from other ribbon Hopf algebras. This is difficult to do, and for tractability we only compare our invariants with those arising from $U_q^{(N/2)}(so(2n+1))$ at the same N. We do this as the quantised universal enveloping algebras of $osp(1|2n)$ and $so(2n+1)$ are known to be related at generic q. We show that the 3-manifold invariants arising from $U_q^{(N)}(osp(1|2n))$ and $U_q^{(N/2)}(so(2n+1))$ are *not* the same.

In Section 5.8 we give some side results.

5.1 Knots and links

We take the following definitions from [Ro90, CP94, Lic97].

Definition 5.1.1. *A link $L = \bigcup_{i=1}^m L_i$ of m components embedded in S^3 is a subset of S^3 consisting of m disjoint smooth 1-dimensional submanifolds of S^3. A knot is a link with one component.*

This definition of a link ensures that we do not deal with wild links in this thesis.

Definition 5.1.2. *An oriented link L is a link with an orientation assigned to each connected component of L.*

Definition 5.1.3. *Two oriented links L, $L' \subset S^3$ are said to be equivalent if there exists an orientation preserving diffeomorphism f of S^3 such that $f(L) = L'$ and such that f takes the orientation of L into that of L'.*

Let S^3 denote the 3-sphere, then $S^3 = \mathbb{R}^3 \cup \{\infty\}$. Let a link $L \subset S^3$ be embedded in S^3 such that

(i) $L \cap \{\infty\} = \emptyset$, and

(ii) given the standard projection $p : \mathbb{R}^3 \to \mathbb{R}^2$, $p(L)$ has only transversal crossings where in some sufficiently small neighbourhood of each crossing there are projections of at most two *branches* of L.

Here we define a branch of L to be a closed proper subset of a single connected component of L. We define the *planar projection of a link* $L \subset \mathbb{R}^3 \subset S^3$ to be $p(L)$ together with information at each crossing in $p(L)$ specifying which of the branches is the overcrossing branch. The overcrossing branch is represented by an unbroken line and the undercrossing branch is represented by a broken line, the line being broken at the crossing [CP94].

We detail the *Reidemeister moves* in Figure 5.1 [Kau91]. Let $L \subset S^3$ be a link such that $L \cap \{\infty\} = \emptyset$. Each of the Reidemeister moves replaces a configuration of arcs and crossings in the intersection of $p(L)$ with a 2-disc D^2, with another collection of arcs and crossings, such that the complement of $p(L)$ is left unchanged [Lic97]. Each Reidemeister move is an equivalence relation: two links $L, L' \subset S^3$ are said to be *ambient isotopic* if their planar projections are elements of the equivalence class of planar projections of links generated by the Reidemeister I, II and III moves. Two links L, $L' \subset S^3$ are said to be *regularly isotopic* if their planar projections are elements of the equivalence class generated by the Reidemeister II and III moves.

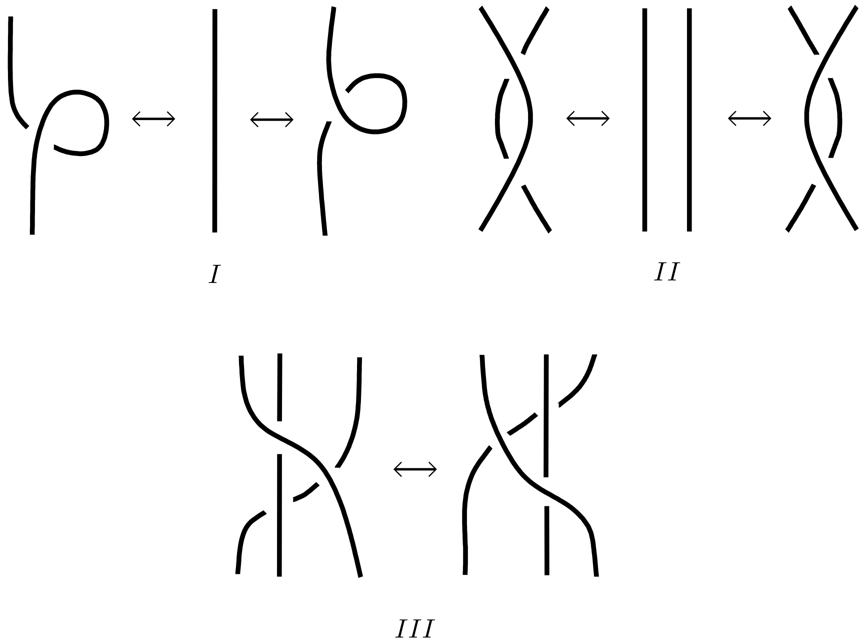


Figure 5.1: The Reidemeister I, II and III moves

We associate a linking number $+1$ or -1 with each crossing of any pair of components of an oriented link as given in Figure 5.2.

Definition 5.1.4. *Let L_i and L_j be connected components of an oriented link L. The linking number* $\mathrm{lk}(L_i, L_j)$ *of L_i and L_j is half of the sum of the linking numbers associated with each crossing of L_i and L_j in a planar projection of L.*

Definition 5.1.5. *Let $L \subset S^3$ be an unoriented link. The writhing number $w(L_i)$ of a connected component L_i of L is the sum of the linking numbers associated with each*

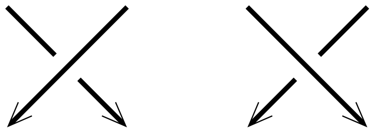


Figure 5.2: Linking numbers

crossing of L_i with itself in a planar projection of L, where L is assigned an arbitrary orientation.

The writhing number of a connected component of a link is independent of the orientation of the link, thus the writhing number of a connected component of an unoriented link is well-defined using the above definition [Kau91].

Definition 5.1.6. *A framed link $L \subset S^3$ is a link L together with an assignment of integers, each connected component of L being assigned an integer which we call the framing number of that component.*

Let $L = \bigcup_{i=1}^{m} L_i \subset S^3$ be a framed unoriented link with m connected components where the framing number of the connected component L_i is n_i. We can equip L_i with a normal vector field as follows: let $U_i \subset S^3$ be a small tubular neighbourhood of L_i such that $U_i \cap U_j = \emptyset$ if $i \neq j$. Let $K_i \subset U_i$ be a connected link such that we can equip L_i with a normal vector field where the tips of the vectors trace out K_i and such that if we assign an orientation to L_i and a parallel orientation to K_i, the linking number of L_i and K_i is n_i [Kau91]. For each such normal vector field there exists an integer n_i, and for each integer n_i there exists such a normal vector field.

A natural normal vector field to use gives the so-called *blackboard framing*. Set the normal vector field to L_i to lie in the planar projection of L. Then the tips of the vectors sweep out a link K_i parallel to L_i [Kau91] and the linking number of L_i and K_i is $w(L_i)$. The normal vector field of a framed link can always be presented in the blackboard framing.

The blackboard framing allows us to define a convenient notion of 'equivalence' of framed links. We say that two framed links are equivalent if the links are regularly isotopic when presented in the blackboard framing.

An important element in the study of 3-manifolds is the notion of surgery [Kau91, p. 252]:

Definition 5.1.7. *Performing surgery on a 3-manifold M^3 along a link $L \subset M^3$.*

Let M^3 be a 3-manifold and $L \subset M^3$ an unoriented link with one component and framing number n_L. Let $\alpha : S^1 \times D^2 \to M^3$ be an embedding and $\alpha(S^1 \times 0)$ an embedding of L. A longitude of $\alpha(S^1 \times D^2)$ is $\alpha(S^1 \times 1) \subset \alpha\big(\partial(S^1 \times D^2)\big)$ where $\partial(S^1 \times D^2)$ denotes the boundary of $S^1 \times D^2$. Let $L' \subset \alpha\big(\partial(S^1 \times D^2)\big)$ be a closed twisted longitude of $\alpha(S^1 \times D^2)$ such that $\mathrm{lk}(L, L') = n_L$ where we now fix L and L' to have the same orientation.

We perform surgery on M^3 along L by gluing $D^2 \times S^1$ to $M^3 - Int\big(\alpha(S^1 \times D^2)\big)$ along the boundary of $S^1 \times S^1 \subset M^3 - Int\big(\alpha(S^1 \times D^2)\big)$ so that the meridian $m = S^1 \times 1 \subset D^2 \times S^1$ matches the closed twisted longitude L'. We conduct surgery on links with more than one component by performing surgery on each component simultaneously.

The following theorem [Lic62] (see also [Lic97]) establishes a connection between closed, connected, orientable 3-manifolds and links embedded in S^3 that underpins our approach to developing topological invariants of such 3-manifolds.

Theorem 5.1.1. *Any closed, connected, orientable 3-manifold M_L can be obtained by performing surgery on S^3 along a framed link $L \subset S^3$.*

We refer to [Ro90, Chap. 9] and [Kau91, Part I, Chap. 16] for detailed discussion concerning, and examples of, surgery on S^3 along framed links.

Let L, $L' \subset S^3$ be two regularly isotopic links with the same framing. Let M_L (resp. $M_{L'}$) be the closed, connected, orientable 3-manifold obtained by performing surgery on S^3 along L (resp. L'), then there exists an orientation preserving homeomorphism $M_L \to M_{L'}$.

A question which immediately arises is whether there are any other relations between framed links such that the 3-manifolds they give rise to upon performing surgery are related by an orientation-preserving homeomorphism. Kirby answered that question by finding a complete collection of such relations between framed links and proving that closed, connected, orientable 3-manifolds M_L and $M_{L'}$ are related by an orientation-preserving homeomorphism if and only if the framed links L and L' are related by a certain set of transformations [Kirb78].

Fenn and Rourke subsequently proved a similar result but with the advantage that their transformations were *local* transformations, that is, their transformations occur on some subset of the planar projection of a link [FR79]. These transformations are called the *Kirby moves* and are detailed in Figures 5.3–5.5. Let L, $L' \subset S^3$ be two links that are regularly isotopic outside of their intersections with a 3-disc D^3. Each Kirby move in Figures 5.3–5.5 relates the intersection of a 2-disc D^2 with $p(L)$ and $p(L')$, respectively.

We say that L and L' are equivalent under the Kirby moves if the intersection of $p(L)$ with D^2 is regularly isotopic to one diagram of a move, and the intersection of $p(L')$ with D^2 is regularly isotopic to the other diagram of the move. The relation *equivalent under the Kirby moves* is an equivalence relation. The reader is referred to [Kau91, Part I, Chap. 16] and [KL94, Chap. 12] for examples of links equivalent under the Kirby moves.

The $\kappa_+^{(0)}$ and $\kappa_-^{(0)}$ moves are called the *special Kirby (+) move*, and the *special Kirby (-) move*, respectively. The κ_+ and κ_- moves are called the *Kirby (+) move*, and the *Kirby (-) move*, respectively.

We now explain the origin of the Kirby moves [Kau91, KL94, Lic97]. Let $L \subset S^3$ be a framed link and $L' \subset S^3$ an unknot with framing number ± 1 such that $L \cup L'$ is a split link, that is, that $L \cup L'$ is the disjoint union of links L, L' such that L and L' are mutually unlinked. Surgery on S^3 along L' gives rise to a 3-manifold homeomorphic to S^3 [KL94]. Let M_L be a closed, connected, orientable 3-manifold obtained by performing surgery on S^3 along the framed link $L \subset S^3$. Let $M_L \# S^3$ denote the connected sum of M_L and S^3

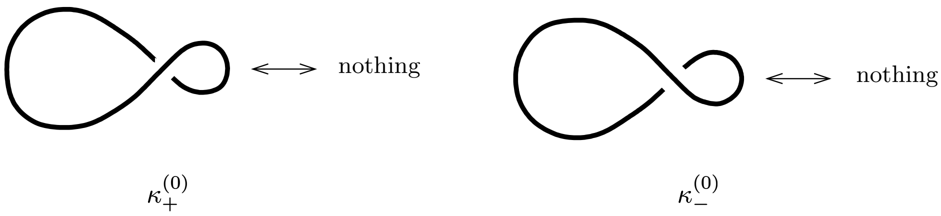


Figure 5.3: The Kirby $\kappa_+^{(0)}$ and $\kappa_-^{(0)}$ moves

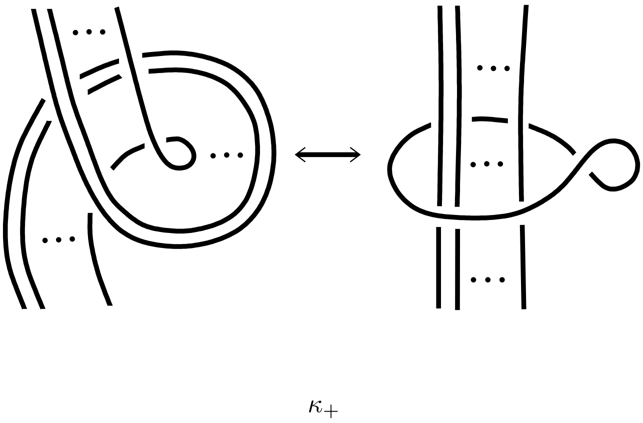


Figure 5.4: The Kirby κ_+ move

and let $\cong$ denote an orientation-preserving homeomorphism. Then $M_L \# S^3 \cong M_L$, and the relations $M_{L\cup L'} \cong M_L \# M_{L'} \cong M_L$ imply the existence of the $\kappa_+^{(0)}$ and $\kappa_-^{(0)}$ moves.

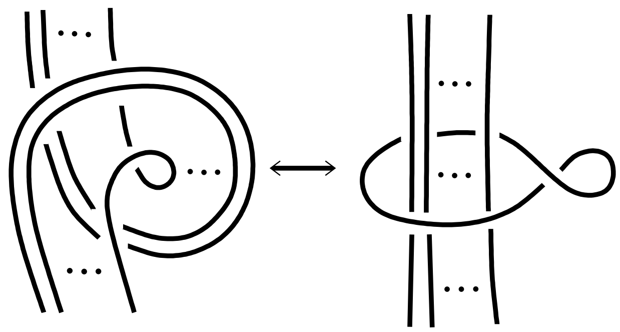
The κ_+ and κ_- moves arise from an application of the *band connected sum move* as detailed in the following theorem [Kau91].

Theorem 5.1.2. *Let $K, K' \subset S^3$ be two links related by a band connected sum move and let $M_K, M_{K'}$ be the closed, connected, orientable 3-manifolds obtained by performing surgery on S^3 along K and K', respectively. Then $M_K \cong M_{K'}$.*

We now explain the band connected sum move. Let L_i and L_j be two disjoint connected components of a link $L \subset S^3$. The action of the band connected sum move on L_i is to replace L_i with the band connected sum $L_i \#_b L_j$, which we define as follows. Let b be the image of a smooth embedding $e : [0,1] \times [0,1] \to S^3$ where $e\big([0,1] \times \{0\}\big)$ is glued smoothly to L_i and $e\big([0,1] \times \{1\}\big)$ is glued smoothly to L_j such that the linking number of $e\big(\{0,1\} \times [0,1]\big)$ is zero, and furthermore, we require that the only part of b that intersects L_i (resp. L_j) is $e\big([0,1] \times \{0\}\big)$ (resp. $e\big([0,1] \times \{1\}\big)$). We define

$$L_i \#_b L_j = \big(L_i \cup b \cup L_j\big) \backslash e\big((0,1) \times [0,1]\big).$$

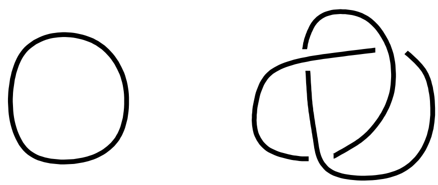
We now detail an example of the band connected sum move to be concrete. Let L_i and



κ_-

Figure 5.5: The Kirby κ_- move

L_j be components of a split link in Figure 5.6, then $L_i \#_b L_j$ is the left most component in Figure 5.7.



L_i L_j

Figure 5.6: Link components L_i and L_j

The band connected sum move gives rise to the κ_+ move. In Figure 5.8 we show this in the situation that the left hand side of the κ_+ move has one component. The multicomponent case is similar. In Figure 5.8, $\overset{b}{\longleftrightarrow}$ indicates the band connected sum move, $\overset{k}{\longleftrightarrow}$ indicates the $\kappa_+^{(0)}$ move and the remaining move is regular isotopy. The generation of the κ_- move is similar.

The existence of the band connected sum move suggests that there may be a large number of moves on links that give rise to homeomorphic 3-manifolds. However, a convenient result is that the Kirby moves are a sufficient generating set of such moves [FR79, Thm. p. 1]:

Theorem 5.1.3. *Orientation preserving homeomorphism classes of closed, connected, orientable 3-manifolds correspond bijectively to equivalence classes of framed links in* S^3 *where the equivalence is generated by the Kirby moves.*

It transpires that the four Kirby moves are not a minimal generating set of the Kirby

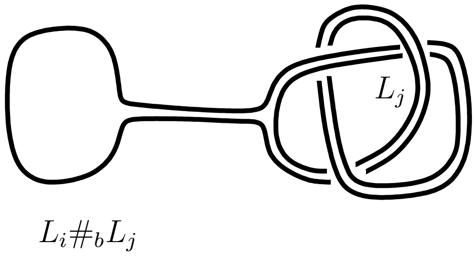


Figure 5.7: The links $L_i\#_bL_j$ and L_j

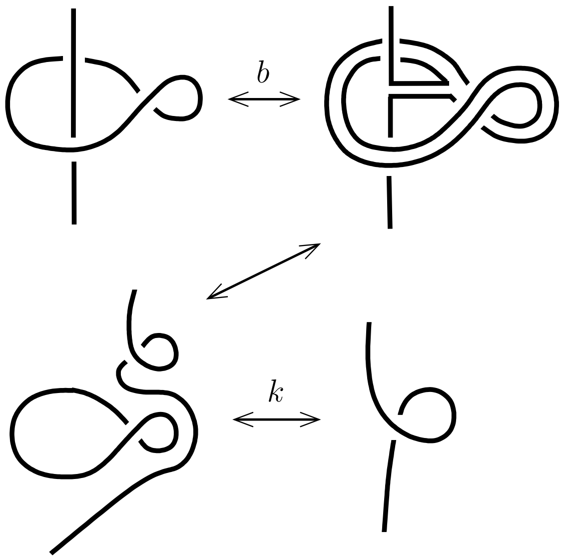


Figure 5.8: The derivation of the κ_+ move

calculus: the $\kappa_\pm^{(0)}$ moves in addition to either of the κ_+ or κ_- moves generates the entire Kirby calculus [Kau91].

We will use all of these facts to create topological invariants of closed, connected, orientable 3-manifolds later in this thesis. Let M_L be a closed, connected, orientable 3-manifold obtained by performing surgery on S^3 along a framed link $L \subset S^3$. We will create a topological invariant of M_L by taking such sums of regular isotopy invariants of L as are unchanged after applying each of the Kirby moves to L. A convenient way to study regular isotopy invariants of links is to consider isotopy invariants of *ribbon tangles*.

5.2 Tangles and ribbon tangles

We now examine tangles, directed ribbon tangles and coloured directed ribbon tangles before constructing isotopy invariants of directed ribbon tangles. The work in this section is well known: it first appeared in [Re90] and then in [RT90, RT91] and it often appears and is referenced in the literature. We state it here to be complete.

5.2.1 The category of directed ribbon tangles

The basic element in this work is the *ribbon*. A ribbon is defined to be a square $[0,1]\times[0,1]$ smoothly embedded in $\mathbb{R}^3$ [RT91], and the images of the segments $[0,1]\times\{0\}$ and $[0,1]\times\{1\}$ under the embeddings are the *bases* of the ribbon. We call the image of $\{1/2\}\times[0,1]$ the *core* of the ribbon.

We call the image of a cylinder $S^1\times[0,1]$ smoothly embedded in $\mathbb{R}^3$ an *annulus*, and the image of $S^1\times\{1/2\}$ the *core* of the annulus.

We define the *writhe* of a ribbon to be the linking number of the image of $\{0\}\times[0,1]$ and the image of $\{1\}\times[0,1]$ where we assign the edges of the ribbon parallel orientations.

Ribbons and annuli are orientable surfaces in $\mathbb{R}^3$ and an orientation of a ribbon or annulus is equivalent to a choice of one side of the ribbon or annulus.

We say that a ribbon or annulus is directed if its core is oriented, and we orient the core of a ribbon by labelling one of the bases of the ribbon the *initial base* and the other base the *final base*. Each ribbon and annulus can be directed in two ways and oriented in two ways.

For $k,l\in\mathbb{Z}_+$ we define a (k,l)-ribbon tangle Γ to be the union of a finite number of disjoint oriented ribbons and annuli embedded in $\mathbb{R}^2\times[0,1]$ such that Γ satisfies:

$$\Gamma\cap\big(\mathbb{R}^2\times\{1\}\big)=\big\{[i-1/4,i+1/4]\times\{0\}\times\{1\}\mid i=1,2,\ldots,k\big\},$$
$$\Gamma\cap\big(\mathbb{R}^2\times\{0\}\big)=\big\{[j-1/4,j+1/4]\times\{0\}\times\{0\}\mid j=1,2,\ldots,l\big\},$$

and such that the same side of each ribbon faces the reader at $\Gamma\cap(\mathbb{R}^2\times\{1\})$ and $\Gamma\cap(\mathbb{R}^2\times\{0\})$.

We define the (k,l)-tangle associated with a particular (k,l)-ribbon tangle Γ to be the union of the cores of all the components of Γ. This corresponds to our intuitive notion of a (k,l)-tangle.

So far we have not imposed any directions on the (ribbon) tangles. In Reshetikhin and Turaev's isotopy invariants of ribbons, one uses oriented and directed ribbon tangles, and so we now define a directed (k,l)-ribbon tangle.

We define a *directed (k,l)-ribbon tangle* Γ to be a (k,l)-ribbon tangle where all the ribbons and annuli of Γ are directed. We can intuitively think of this by drawing an arrow on each ribbon and annulus parallel to the core of the ribbon or annulus.

For each directed (k,l)-ribbon tangle Γ there are two sequences which specify the directions of the ribbons of Γ:

$$\begin{aligned}\epsilon^*(\Gamma) &= (\epsilon^1,\epsilon^2,\ldots,\epsilon^k),\\ \epsilon_*(\Gamma) &= (\epsilon_1,\epsilon_2,\ldots,\epsilon_l).\end{aligned}$$

Here each ϵ^i, ϵ_i is either -1 or $+1$. Intuitively, $\epsilon^i=+1$ (resp. $\epsilon^i=-1$) if the arrow on the i^{th} ribbon from the left at the top of the ribbon tangle is pointing down (resp. up), and

$\epsilon_i = +1$ (resp. $\epsilon_i = -1$) if the arrow on the i^{th} ribbon from the left at the bottom of the ribbon tangle is pointing down (resp. up). Technically, we set $\epsilon^i = +1$ if

$$[i - 1/4, i + 1/4] \times \{0\} \times \{1\}$$

is an initial base and $\epsilon^i = -1$ if it is a final base. Similarly, we set $\epsilon_j = +1$ if

$$[j - 1/4, j + 1/4] \times \{0\} \times \{0\}$$

is a final base and $\epsilon_j = -1$ if it is an initial base.

We depict directed ribbon tangles as the disjoint union of ribbons and annuli, the directions of the ribbons and annuli denoted by arrows drawn on them. On ribbons we point the arrows in the direction of the final base. An example of this is in Figure 5.9.

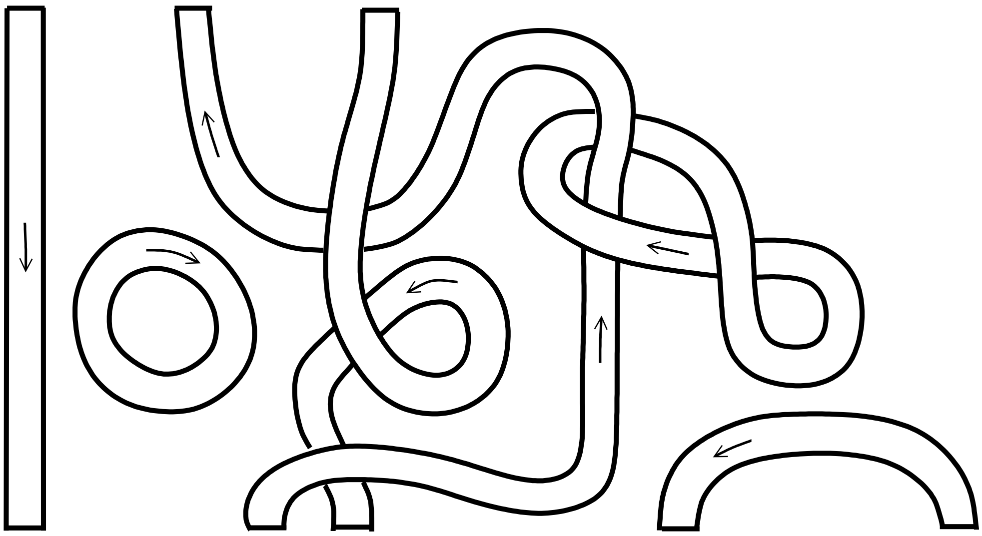


Figure 5.9: An example of directed ribbons and annuli

Two directed (k, l)-ribbon tangles Γ, Γ' are said to be isotopic if there exists a smooth isotopy

$$h_t : \mathbb{R}^2 \times [0, 1] \to \mathbb{R}^2 \times [0, 1], \qquad t \in [0, 1],$$

of the identity $h_0 = \text{id}$ such that each h_t is a diffeomorphism of the strip $\mathbb{R}^2 \times [0, 1]$ fixing its boundary $\mathbb{R}^2 \times \{0, 1\}$ and h_1 transforms Γ into Γ' preserving the decomposition into ribbons and annuli, the directions of cores and the orientations of the ribbons and annuli [RT91]. Isotopy is an equivalence relation.

In referring to a particular directed (k, l)-ribbon tangle Γ we are referring to any element of the equivalence class of directed (k, l)-ribbon tangles containing Γ where the equivalence is generated by isotopy.

If Γ is a directed (k, m)-ribbon tangle and Γ' is a directed (m, n)-ribbon tangle then we can vertically compose them to produce a new directed (k, n)-ribbon tangle $\Gamma \circ \Gamma'$ if the directions of Γ and Γ' are compatible. By 'compatible' we mean that $\epsilon_*(\Gamma) = \epsilon^*(\Gamma')$, or more intuitively, the 'up, down' sequence of arrows at the bottom of Γ is the same as the 'up, down' sequence of arrows at the top of Γ'.

We technically define $\Gamma \circ \Gamma'$ to be the directed (k, n)-ribbon tangle obtained as follows. Take the union $\Gamma \cup (\Gamma' + t)$ where t is the vector $t = (0, 0, -1) \in \mathbb{R}^3$ such that the bases of the ribbons are smoothly glued together. Then $\Gamma \circ \Gamma'$ is the image of $\Gamma \cup (\Gamma' + t)$ under the map $(x, y, z) \mapsto (x, y, (z + 1)/2)$, for all $(x, y, z) \in \mathbb{R}^2 \times [-1, 1]$, so that we have $\Gamma \circ \Gamma' \subset \mathbb{R}^2 \times [0, 1]$. Then $\Gamma \circ \Gamma'$ is a directed (k, n)-ribbon tangle with two sequences encoding the directions of the ribbons:

$$\epsilon^*(\Gamma \circ \Gamma') = \epsilon^*(\Gamma),$$

$$\epsilon_*(\Gamma \circ \Gamma') = \epsilon_*(\Gamma'),$$

as expected. We heuristically depict $\Gamma \circ \Gamma'$ in Figure 5.10.

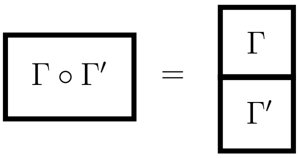


Figure 5.10: The vertical composition $\Gamma \circ \Gamma'$

Let Γ be a directed (k, l)-ribbon tangle and Γ' a directed (m, n)-ribbon tangle, then the horizontal composition $\Gamma \otimes \Gamma'$ is always defined. This $\Gamma \otimes \Gamma'$ is a directed $(k + m, l + n)$-ribbon tangle $\Gamma \otimes \Gamma'$ depicted by placing Γ' immediately to the right of Γ such that Γ and Γ' are mutually unlinked. We heuristically depict $\Gamma \otimes \Gamma'$ in Figure 5.11.

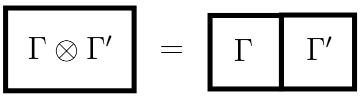


Figure 5.11: The horizontal composition $\Gamma \otimes \Gamma'$

Now let Γ_1, Γ_1', Γ_2 and Γ_2' be directed ribbon tangles where $\Gamma_1 \circ \Gamma_1'$ and $\Gamma_2 \circ \Gamma_2'$ exist, then

$$(\Gamma_1 \circ \Gamma_1') \otimes (\Gamma_2 \circ \Gamma_2') = (\Gamma_1 \otimes \Gamma_2) \circ (\Gamma_1' \otimes \Gamma_2').$$

Note that $(\Gamma_1 \otimes \Gamma_2) \circ (\Gamma_1' \otimes \Gamma_2')$ may exist even if $\Gamma_1 \circ \Gamma_1'$ and $\Gamma_2 \circ \Gamma_2'$ do not.

We now introduce the category of directed ribbon tangles **drib** [RT91, CP94]. The objects of **drib** are finite sequences:

$$\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_k),$$

where $\epsilon_i \in \{-1, +1\}$ for each $i = 1, 2, \ldots, k$. We can think of an object ϵ of **drib** as being $\epsilon^*(\Gamma)$ or $\epsilon_*(\Gamma)$ for some directed ribbon tangle Γ. Isotopy equivalence classes of directed ribbon tangles are the morphisms of **drib**.

If Γ and Γ' are two directed ribbon tangles where $\Gamma \circ \Gamma'$ exists, and $f : \epsilon^*(\Gamma) \to \epsilon_*(\Gamma)$ and $f' : \epsilon^*(\Gamma') \to \epsilon_*(\Gamma')$ are two morphisms of **drib**, then we define $f \circ f' : \epsilon^*(\Gamma \circ \Gamma') \to \epsilon_*(\Gamma \circ \Gamma')$ to be $\Gamma \circ \Gamma'$.

Let

$$\epsilon_a = (\epsilon_{a_1}, \epsilon_{a_2}, \ldots, \epsilon_{a_k}), \qquad \epsilon_b = (\epsilon_{b_1}, \epsilon_{b_2}, \ldots, \epsilon_{b_l}),$$

be two objects of **drib**. The tensor product of ϵ_a and ϵ_b is an object $\epsilon_a \otimes \epsilon_b$:

$$\epsilon_a \otimes \epsilon_b = (\epsilon_{a_1}, \epsilon_{a_2}, \ldots, \epsilon_{a_k}, \epsilon_{b_1}, \epsilon_{b_2}, \ldots, \epsilon_{b_l}).$$

Now let $f : \epsilon_a \to \epsilon'_a$ and $g : \epsilon_b \to \epsilon'_b$ be two morphisms of **drib** that are the directed ribbon tangles Γ and Γ', respectively. The tensor product of f and g is a morphism $f \otimes g$:

$$f \otimes g : \epsilon_a \otimes \epsilon_b \to \epsilon'_a \otimes \epsilon'_b,$$

which is just the directed ribbon tangle $\Gamma \otimes \Gamma'$.

Let f, f', g and g' be morphisms where $f \circ f'$ and $g \circ g'$ exist, then

$$(f \circ f') \otimes (g \circ g') = (f \otimes g) \circ (f' \otimes g').$$

Note that $(f \otimes g) \circ (f' \otimes g')$ may exist even if each of $f \circ f'$ and $g \circ g'$ do not exist.

A directed ribbon tangle can be expressed as some combination of vertical and horizontal compositions of the directed ribbon tangles in Figure 5.12 [RT91]. For convenience we call the directed ribbon tangles in Figure 5.12 the *directed ribbon tangle atoms*.

5.2.2 Coloured directed ribbon tangles

We now introduce *coloured directed ribbon tangles*, which we can intuitively think of as directed ribbon tangles where each component of the ribbon tangle has been 'coloured' with an element of some set.

Let $S = \{s_1, s_2, \ldots, s_t\}$ be a non-empty finite set and let Γ be a directed (k, l)-ribbon tangle with m disjoint directed ribbons:

$$\Gamma = \bigcup_{i=1}^{m} \Gamma_i.$$

We say that Γ_i is coloured with $s_{a_i} \in S$ if we associate s_{a_i} with Γ_i. Let each ribbon Γ_i be coloured with s_{a_i}, then we say that Γ is coloured with $(s_{a_1}, s_{a_2}, \ldots, s_{a_m}) \in S^{\times m}$. We denote any involution $*$ of S by $*(s_{a_i}) = (s_{a_i})^*$.

Two coloured directed (k, l)-ribbon tangles Γ, Γ' are said to be isotopic if they are isotopic as directed (k, l)-ribbon tangles and if the isotopy takes the colouring of Γ into the colouring of Γ' [RT91].

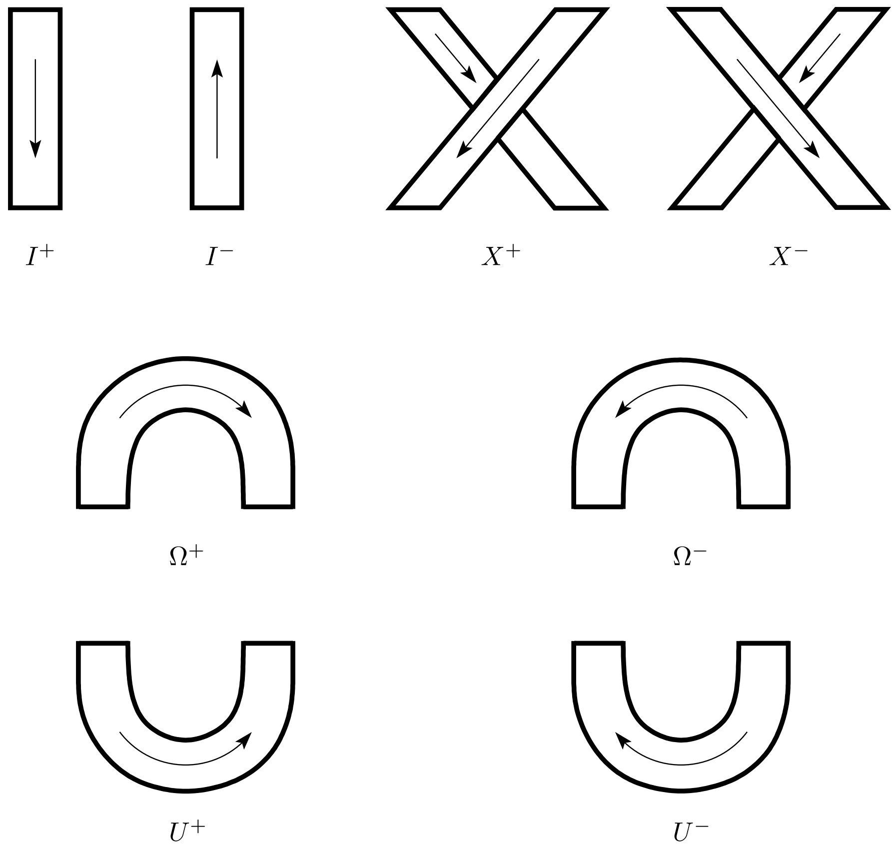


Figure 5.12: The directed ribbon tangle atoms

To each coloured directed (k, l)-ribbon tangle Γ we associate two sequences encoding the colourings and directions of its ribbons:

$$\begin{aligned}(X^*, \epsilon^*)_\Gamma &= \big((i^1, \epsilon^1), (i^2, \epsilon^2), \ldots, (i^k, \epsilon^k)\big),\\ (X_*, \epsilon_*)_\Gamma &= \big((i_1, \epsilon_1), (i_2, \epsilon_2), \ldots, (i_l, \epsilon_l)\big),\end{aligned}$$

where $i^j, i_j \in S$ and $\epsilon^j, \epsilon_j \in \{-1, +1\}$ for each j. Intuitively, the sequences $\epsilon^*(\Gamma) = (\epsilon^1, \epsilon^2, \ldots, \epsilon^k)$ and $\epsilon_*(\Gamma) = (\epsilon_1, \epsilon_2, \ldots, \epsilon_l)$ are just the corresponding sequences if we think of Γ as uncoloured. The sequences

$$X^*(\Gamma) = (i^1, i^2, \ldots, i^k), \qquad X_*(\Gamma) = (i_1, i_2, \ldots, i_l),$$

encode the colourings of the ribbons, such that i^j (resp. i_j) is the element of S that colours the component of Γ with non-empty intersection with $[j - 1/4, j + 1/4] \times \{0\} \times \{1\}$ (resp. $[j - 1/4, j + 1/4] \times \{0\} \times \{0\}$).

The vertical composition $\Gamma \circ \Gamma'$ of a coloured directed (k, m)-ribbon tangle Γ and a coloured directed (m, n)-ribbon tangle Γ' exists if the directions and colourings of Γ and Γ' are compatible, that is if $(X_*, \epsilon_*)_\Gamma = (X^*, \epsilon^*)_{\Gamma'}$. We technically define $\Gamma \circ \Gamma'$ to be the coloured directed (k, n)-ribbon tangle with the underlying directed ribbon tangle to be the vertical composition of Γ and Γ' and with two sequences encoding the colouring and directions of the ribbons:

$$(X^*, \epsilon^*)_{\Gamma \circ \Gamma'} = (X^*, \epsilon^*)_{\Gamma}, \qquad (X_*, \epsilon_*)_{\Gamma \circ \Gamma'} = (X_*, \epsilon_*)_{\Gamma'}.$$

The horizontal composition $\Gamma \otimes \Gamma'$ of a coloured directed (k, l)-ribbon tangle Γ and a coloured directed (m, n)-ribbon tangle Γ' is always defined to be the coloured directed $(k+m, l+n)$-ribbon tangle depicted by placing Γ' immediately to the right of Γ such that Γ and Γ' are mutually unlinked.

Let Γ_1, Γ_1', Γ_2 and Γ_2' be coloured directed ribbon tangles where $\Gamma_1 \circ \Gamma_1'$ and $\Gamma_2 \circ \Gamma_2'$ exist, then

$$(\Gamma_1 \circ \Gamma_1') \otimes (\Gamma_2 \circ \Gamma_2') = (\Gamma_1 \otimes \Gamma_2) \circ (\Gamma_1' \otimes \Gamma_2').$$

Note that $(\Gamma_1 \otimes \Gamma_2) \circ (\Gamma_1' \otimes \Gamma_2')$ may exist even if $\Gamma_1 \circ \Gamma_1'$ and $\Gamma_2 \circ \Gamma_2'$ do not.

Any coloured directed (k, l)-ribbon tangle, where the ribbon tangle is coloured with elements of a set S, can be written down as some combination of vertical and horizontal compositions of the directed ribbon tangle atoms, where each ribbon tangle atom is coloured with an element of S [RT91]. We denote a colouring of the directed ribbon tangle atoms as follows:

(i) $I_i^{\pm}$ means that $I^{\pm}$ is coloured with i,

(ii) $X_{i,j}^{+}$ means that X^{+} is coloured so that the ribbon passing from the top right hand corner to the bottom left corner is coloured with i and the other ribbon is coloured with j,

(iii) $X_{i,j}^{-}$ means that X^{-} is coloured in the same way that $X_{i,j}^{+}$ is,

(iv) W_i means that W is coloured with i, where W is any one of Ω^+, Ω^-, U^+ and U^-.

We now introduce the category of coloured directed ribbon tangles **cdrib**(S) for a finite non-empty set S [Re90, RT91, CP94]. The objects of **cdrib**(S) are sequences of pairs

$$(X, \epsilon) = \big((i_1, \epsilon_1), (i_2, \epsilon_2), \ldots, (i_k, \epsilon_k)\big),$$

where $i_j \in S$ and $\epsilon_i \in \{-1, +1\}$ for each $i = 1, 2, \ldots, k$. Each object (X, ϵ) of **cdrib**(S) can be thought of as $(X^*, \epsilon^*)_{\Gamma}$ or $(X_*, \epsilon_*)_{\Gamma}$ for some coloured directed ribbon tangle Γ. The morphisms of **cdrib**(S) are equivalence classes of coloured directed ribbon tangles coloured with elements of S.

Lemma 5.2.1. *All morphisms in* **cdrib***(S) can be expressed as vertical and horizontal compositions of the coloured directed ribbon tangles* I_i^+, I_i^-, $X_{i,j}^+$, $X_{i,j}^-$, Ω_i^+, Ω_i^-, U_i^+ *and* U_i^-, *where* $i, j \in S$.

Proof. This follows from [Re90, Prop. 1.4]. □

Let Γ and Γ' be two coloured directed ribbon tangles where $\Gamma \circ \Gamma'$ exists. Let $f : (X^*, \epsilon^*)_\Gamma \to (X_*, \epsilon_*)_\Gamma$ and $f' : (X^*, \epsilon^*)_{\Gamma'} \to (X_*, \epsilon_*)_{\Gamma'}$ be two morphisms. We define the morphism $f \circ f' : (X^*, \epsilon^*)_{\Gamma \circ \Gamma'} \to (X_*, \epsilon_*)_{\Gamma \circ \Gamma'}$ to be the coloured directed ribbon tangle $\Gamma \circ \Gamma'$.

Let (X_a, ϵ_a) and (X_b, ϵ_b) be two objects of **cdrib**(S):

$$\begin{aligned}(X_a, \epsilon_a) &= \big((i_{a_1}, \epsilon_{a_1}), (i_{a_2}, \epsilon_{a_2}), \ldots, (i_{a_k}, \epsilon_{a_k})\big),\\ (X_b, \epsilon_b) &= \big((i_{b_1}, \epsilon_{b_1}), (i_{b_2}, \epsilon_{b_2}), \ldots, (i_{b_m}, \epsilon_{b_m})\big).\end{aligned}$$

The tensor product of (X_a, ϵ_a) and (X_b, ϵ_b) is an object $(X_a, \epsilon_a) \otimes (X_b, \epsilon_b)$:

$$(X_a, \epsilon_a) \otimes (X_b, \epsilon_b) = \big((i_{a_1}, \epsilon_{a_1}), (i_{a_2}, \epsilon_{a_2}), \ldots, (i_{a_k}, \epsilon_{a_k}), (i_{b_1}, \epsilon_{b_1}), (i_{b_2}, \epsilon_{b_2}), \ldots, (i_{b_m}, \epsilon_{b_m})\big).$$

Let f and g be morphisms of **cdrib**(S):

$$f : (X_a, \epsilon_a)_\Gamma \to (X'_a, \epsilon'_a)_\Gamma, \qquad g : (X_b, \epsilon_b)_{\Gamma'} \to (X'_b, \epsilon'_b)_{\Gamma'}.$$

The tensor product of f and g is a morphism $f \otimes g$ which is the coloured directed ribbon tangle $\Gamma \otimes \Gamma'$:

$$f \otimes g : (X_a, \epsilon_a)_\Gamma \otimes (X_b, \epsilon_b)_{\Gamma'} \to (X'_a, \epsilon'_a)_\Gamma \otimes (X'_b, \epsilon'_b)_{\Gamma'}.$$

Note that if f, f', g and g' are morphisms of **cdrib**(S) where $f \circ f'$ and $g \circ g'$ exist, then

$$(f \circ f') \otimes (g \circ g') = (f \otimes g) \circ (f' \otimes g').$$

Closing a coloured directed ribbon tangle

Let Γ be a coloured directed (k,k)-ribbon tangle with $(X_*, \epsilon_*)_\Gamma = (X^*, \epsilon^*)_\Gamma$. For each $j \in \{1, 2, \ldots, k\}$ let r_j be a ribbon that is the image of a smooth embedding $e : [0,1] \times [0,1] \to \mathbb{R}^3$ given as follows. The ribbon r_j has disjoint intersection with Γ: call $e\big([0,1] \times \{0\}\big)$ the *beginning* of the ribbon and $e\big([0,1] \times \{1\}\big)$ the *end* of the ribbon. Smoothly glue the beginning of r_j to Γ at $[j - 1/4, j + 1/4] \times \{0\} \times \{1\}$ and the end of r_j to Γ at $[j - 1/4, j + 1/4] \times \{0\} \times \{0\}$ in such a way that r_j is unlinked with each component of Γ, and also so that the linking number of the two edges $e\big(\{0\} \times [0,1]\big)$ and $e\big(\{1\} \times [0,1]\big)$ of r_j is zero if they are given parallel orientations.

Now orient and direct r_j to be consistent with the orientation and direction of the components of Γ to which it is glued, and colour r_j with $i_j \in S$. After attaching, orienting, directing and colouring the k ribbons we obtain from Γ a coloured directed $(0,0)$-ribbon tangle $\hat{\Gamma}$ heuristically depicted in Figure 5.13 (here T is a coloured directed (k,k)-ribbon tangle). Sometimes we refer to $\hat{\Gamma}$ as the closure of Γ.

5.2.3 Representing framed links as $(0,0)$-ribbon tangles

Let $L = \bigcup_{i=1}^m L_i \subset S^3$ be a framed link with m connected components and let the framing number of the component L_i be n_i for each $i = 1, 2, \ldots, m$. We can associate L with a

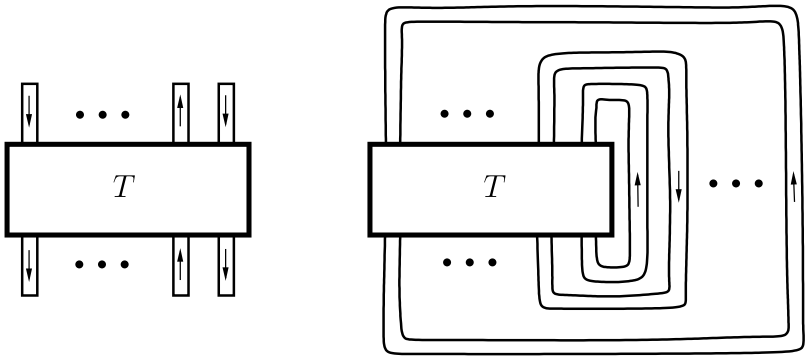


Figure 5.13: A coloured directed ribbon tangle Γ and its closure $\hat{\Gamma}$

$(0,0)$-ribbon tangle $\Gamma(L)$ in a natural way. Recall that each ribbon tangle is a subset of $\mathbb{R}^3$ and that $S^3 = \mathbb{R}^3 \cup \{\infty\}$. Any link $L \subset S^3$ can be deformed so that $L \subset \mathbb{R}^3$ and ambient isotopic links in S^3 give rise to ambient isotopic links in $\mathbb{R}^3$ [T94]. In the remainder of this subsection we consider L as a subset of $\mathbb{R}^3$.

We obtain the $(0,0)$-ribbon tangle $\Gamma(L)$ for each framed link $L \subset S^3$ as follows. For each connected component L_i of L fix $U_i \subset \mathbb{R}^3$ to be a small tubular neighbourhood of L_i such that $U_i \cap U_j = \emptyset$ if $i \neq j$. Let $K_i \subset \partial(U_i)$ be a parallel of L_i such that $\mathrm{lk}(L_i, K_i) = n_i$, then we can equip L_i with a normal vector field so that the tips of the vectors sweep out K_i.

The links L_i and K_i are the boundary of a $(0,0)$-ribbon tangle where each element of the normal vector field to L_i coincides with some proper subset of the $(0,0)$-ribbon tangle. Repeating this for each connected component of L defines a $(0,0)$-ribbon tangle for L.

5.3 The Reshetikhin-Turaev functor F

The Reshetikhin-Turaev functor F is a covariant functor from the category of coloured directed ribbon tangles to the category of finite dimensional representations of a $\mathbb{Z}_2$-graded ribbon Hopf algebra. The functor provides the machinery allowing us to construct isotopy invariants of ribbon tangles and thereby regular isotopy invariants of links. The functor was first defined for ungraded ribbon Hopf algebras and their representations [Re90, RT90, RT91] and extended to $\mathbb{Z}_2$-graded ribbon Hopf algebras and their representations [Zh95].

Let A be a $\mathbb{Z}_2$-graded ribbon Hopf algebra over $\mathbb{C}$. Recall from Section 2.3 that A admits an invertible even element, called the universal R-matrix,

$$R = \sum_t a_t \otimes b_t \in A \otimes A, \tag{5.1}$$

satisfying Eqs. (2.11)–(2.13). As usual we set

$$u = \sum_t S(b_t)a_t(-1)^{[a_t]}. \tag{5.2}$$

Recall that there exists an invertible even central element

$$v \in A, \tag{5.3}$$

with the following properties:

$$\epsilon(v) = 1, \quad v^2 = uS(u), \quad S(v) = v, \quad \Delta(v) = (v \otimes v)(R^T R)^{-1},$$

where $R^T = \sum_t b_t \otimes a_t(-1)^{[a_t]}$.

Let $Rep(A)$ be the category of finite dimensional $\mathbb{Z}_2$-graded left A-modules. The objects of $Rep(A)$ are $\mathbb{Z}_2$-graded left A-modules over $\mathbb{C}$. The morphisms of $Rep(A)$ are A-linear homomorphisms of degree 0. Let $\{V_i | \ i \in I\}$ be a set of objects of $Rep(A)$ for some index set I such that for each $i \in I$, the dual A-module of V_i, which we denote by $(V_i)^*$, is isomorphic to V_j for some $j \in I$. Now let

$$\eta = \big((i_1, \epsilon_1), (i_2, \epsilon_2), \ldots, (i_k, \epsilon_k)\big),$$

be an object of $\mathbf{cdrib}(I)$, then for each such object η we define the A-module

$$V_\eta = V_{i_1}^{\epsilon_1} \otimes V_{i_2}^{\epsilon_2} \otimes \cdots \otimes V_{i_k}^{\epsilon_k},$$

where we fix $V_i^{+1} = V_i$ and $V_i^{-1} = (V_i)^*$. If $\eta = \emptyset$ then we set $V_\emptyset = V_0 \cong \mathbb{C}$, which is the one-dimensional A-module.

We now present the theorem defining the covariant functor F [Re90, RT90, RT91, Zh95]. This theorem is a generalisation of a theorem defining F as a covariant functor from the category of coloured directed ribbon tangles to the category of finite dimensional representations of an ungraded ribbon Hopf algebra [Re90, RT90, RT91]. The proof of Theorem 5.3.1 is similar to the proof of the theorem it generalises.

Theorem 5.3.1. *Let $\mathcal{H} = \mathbf{cdrib}(I)$. Let A be a $\mathbb{Z}_2$-graded ribbon Hopf algebra over $\mathbb{C}$ with universal R-matrix $R \in A \otimes A$ stated in (5.1), the element $u \in A$ stated in (5.2), and the even central element $v \in A$ stated in (5.3).*

Let $\{V_i | \ i \in I\}$ be a set of objects of Rep(A) such that for each $i \in I$, $(V_i)^*$ is isomorphic to V_j for some $j \in I$. There exists a covariant functor $F : \mathcal{H} \to Rep(A)$ with the following properties:

(i) F transforms any object η of $\mathcal{H}$ into the object V_η of Rep(A),

(ii) For any two coloured directed ribbon tangles Γ, Γ',

$$F(\Gamma \otimes \Gamma') = F(\Gamma) \otimes F(\Gamma'),$$

(iii) F is defined by

$$
\begin{array}{ll}
F(I_i^+) = \mathrm{id_i} : V_i \to V_i; & F(I_i^+)(x) = x, \\
F(I_i^-) = \mathrm{id_{i^*}} : (V_i)^* \to (V_i)^*; & F(I_i^-)(x^*) = x^*,
\end{array}
$$

$$
\begin{array}{l}
F(X_{i,j}^+) = P \circ R : V_i \otimes V_j \to V_j \otimes V_i; \\
F(X_{i,j}^+)(x \otimes y) = \sum_t b_t y \otimes a_t x \ (-1)^{[x][y]+[a_t]([b_t]+[y])}, \\
F(X_{i,j}^-) = R^{-1} \circ P : V_i \otimes V_j \to V_j \otimes V_i; \\
F(X_{i,j}^-)(x \otimes y) = \sum_t S(a_t) y \otimes b_t x \ (-1)^{[y]([x]+[b_t])},
\end{array}
$$

$$
\begin{array}{ll}
F(\Omega_i^+) : (V_i)^* \otimes V_i \to \mathbb{C}; & F(\Omega_i^+)(x^* \otimes y) = \langle x^*, y \rangle, \\
F(\Omega_i^-) : V_i \otimes (V_i)^* \to \mathbb{C}; & F(\Omega_i^-)(x \otimes y^*) = (-1)^{[x][y^*]} \langle y^*, (v^{-1}u)\ x \rangle, \\
F(U_i^+) : \mathbb{C} \to V_i \otimes (V_i)^*; & F(U_i^+)(c) = c \sum_r v_r \otimes v_r^*, \\
F(U_i^-) : \mathbb{C} \to (V_i)^* \otimes V_i; & F(U_i^-)(c) = c \sum_r (-1)^{[v_r]} v_r^* \otimes (vu^{-1})\ v_r,
\end{array}
$$

where $\{v_r\}$ *and* $\{v_r^*\}$ *are dual bases of* V_i *and* $(V_i)^*$*, respectively, such that* $\langle v_r^*, v_s \rangle = \delta_{rs}$ *and* $[v_r] = [v_r^*]$.

Needless to say, if Γ and Γ' are coloured directed ribbon tangles where $\Gamma \circ \Gamma'$ exists, then we have

$$F(\Gamma \circ \Gamma') = F(\Gamma) \circ F(\Gamma').$$

Let L be an oriented (k,l)-tangle with m connected components (note that k, l and $m \geq 1$ can all be different non-negative integers). Given such an oriented (k,l)-tangle L, fix Γ to be the associated coloured directed (k,l)-ribbon tangle with m connected components, where there are two associated sequences of pairs that encode the directions and colourings of all the ribbons of Γ as given in Subsection 5.2.2:

$$
\begin{array}{rcl}
\big(X^*, \epsilon^*\big)_\Gamma & = & \big((\mu^1, \epsilon^1), (\mu^2, \epsilon^2), \ldots, (\mu^k, \epsilon^k)\big), \\
\big(X_*, \epsilon_*\big)_\Gamma & = & \big((\nu_1, \epsilon_1), (\nu_2, \epsilon_2), \ldots, (\nu_l, \epsilon_l)\big),
\end{array}
$$

where $\mu^i, \nu_j \in I$. The sequence $\big(X^*, \epsilon^*\big)_\Gamma$ uniquely specifies the colourings and directions of the ribbon tangles intersecting the top of the ribbon tangle diagram. In particular, the i^{th} ribbon tangle from the left at the top of the ribbon tangle diagram is coloured with $\mu^i \in I$ and is directed downwards (resp. upwards) if $\epsilon^i = +1$ (resp. $\epsilon^i = -1$).

Similarly, the sequence $\big(X_*, \epsilon_*\big)_\Gamma$ uniquely specifies the colourings and directions of the ribbon tangles intersecting the bottom of the ribbon tangle diagram: the j^{th} ribbon tangle from the left at the bottom of the ribbon tangle diagram is coloured with $\nu_j \in I$ and is directed downwards (resp. upwards) if $\epsilon_j = +1$ (resp. $\epsilon_j = -1$).

For such a Γ, $F(\Gamma)$ is a map

$$F(\Gamma) : V_{\mu^1}^{\epsilon^1} \otimes V_{\mu^2}^{\epsilon^2} \otimes \cdots \otimes V_{\mu^k}^{\epsilon^k} \longrightarrow V_{\nu_1}^{\epsilon_1} \otimes V_{\nu_2}^{\epsilon_2} \otimes \cdots \otimes V_{\nu_l}^{\epsilon_l}. \tag{5.4}$$

As the functor F maps morphisms of $\mathcal{H}$ to A-linear homomorphisms of degree 0, the map (5.4) must commute with the action of A, that is

$$F(\Gamma)\big(\pi_{\mu^1}^{\epsilon^1} \otimes \pi_{\mu^2}^{\epsilon^2} \otimes \cdots \otimes \pi_{\mu^k}^{\epsilon^k} \big(\Delta^{(k-1)}(a)\big)\big) = \big(\pi_{\nu_1}^{\epsilon_1} \otimes \pi_{\nu_2}^{\epsilon_2} \otimes \cdots \otimes \pi_{\nu_l}^{\epsilon_l}\big(\Delta^{(l-1)}(a)\big)\big)\, F(\Gamma),$$

for all $a \in A$, where π_i^{+1} (resp. π_i^{-1}) denotes the representation of A afforded by the A-module V_i (resp. the dual A-module $(V_i)^*$).

To illuminate this fact, we give a direct proof of it. Each coloured directed ribbon tangle Γ can be expressed as some combination of horizontal and vertical compositions of the coloured directed ribbon tangle atoms in Figure 5.12, thus the homomorphism $F(\Gamma)$ of A-modules can be expressed as some appropriate combination of tensor products and compositions of the homomorphisms of A-modules obtained by applying the functor F to the coloured directed ribbon tangle atoms in Figure 5.12. To prove the theorem it suffices to prove it for each case in which Γ is a coloured directed ribbon tangle atom.

The theorem is trivially true for $F(I_i^+)$ and $F(I_i^-)$, and it may be shown to be true for $F(X_{i,j}^+)$ and $F(X_{i,j}^-)$ by simple calculations using $R\Delta(x) = \Delta'(x)R$, $\forall x \in A$. Now

$$\begin{aligned} F(\Omega_i^+)a(x^* \otimes y) &= F(\Omega_i^+) \sum_{(a)} a_{(1)}x^* \otimes a_{(2)}y(-1)^{[x^*][a_{(2)}]} \\ &= \sum_{(a)} \langle x^*, S(a_{(1)})a_{(2)}y\rangle(-1)^{[x^*][a]} \\ &= \epsilon(a)F(\Omega_i^+)(x^* \otimes y), \end{aligned}$$

where we have used the fact that $\epsilon(a) = 0$ if $[a] = 1$. Also,

$$\begin{aligned} aF(U_i^+)(c) &= c\sum_{(a),r} a_{(1)}v_r \otimes a_{(2)}v_r^*(-1)^{[v_r][a_{(2)}]} \\ &= c\sum_{(a),r,i} a_{(1)}v_r \otimes \langle a_{(2)}v_r^*, v_i\rangle v_i^*(-1)^{[v_r][a_{(2)}]} \\ &= c\sum_{(a),r,i} a_{(1)}\langle v_r^*, S(a_{(2)})v_i\rangle v_r \otimes v_i^* \\ &= c\sum_{(a),i} a_{(1)}S(a_{(2)})v_i \otimes v_i^* \\ &= F(U_i^+)(c)\epsilon(a). \end{aligned}$$

The proofs for $F(\Omega_i^-)$ and $F(U_i^-)$ are similar.

Corollary 5.3.1. *Let* $\Gamma(L, \lambda)$ *be a coloured directed* $(1,1)$*-ribbon tangle with* m *components where the ribbon joining* $\mathbb{R}^2 \times \{0\}$ *and* $\mathbb{R}^2 \times \{1\}$ *is coloured with* $\lambda_i \in I$ *and directed downwards at its bases. Then the map* $F\big(\Gamma(L,\lambda)\big) : V_{\lambda_i} \to V_{\lambda_i}$ *is an element of* $End_A(V_{\lambda_i})$.

Remark 5.3.1. *Let* $\Gamma(L, \lambda)$ *be a coloured directed* $(0,0)$*-ribbon tangle associated with the framed oriented link* L*. Then from the definition of the functor* F*, the map* $F\big(\Gamma(L,\lambda)\big)$: $\mathbb{C} \to \mathbb{C}$ *is an invariant of isotopy of* $\Gamma(L)$ *and thus an invariant of regular isotopy of* L *[Re90].*

5.3.1 Calculations using the functor F

We now do some calculations using the functor F. These results will be needed in the later sections.

Let A be a $\mathbb{Z}_2$-graded ribbon Hopf algebra and $\{V_i|\ i \in I\}$ a set of non-isomorphic irreducible A-modules such that for each $i \in I$, $(V_i)^* \cong V_j$ for some $j \in I$. Consider the two framed oriented links L, $L' \subset S^3$ in Figure 5.14, the planar projections of which are equivalent with respect to the Kirby move κ_+. Here T is an arbitrary oriented (m, m)-tangle represented by the rectangle, the orientation of which is compatible with the orientations of $L\backslash T$ and $L'\backslash T$. Let the directed ribbon tangles associated with L and L' be coloured as

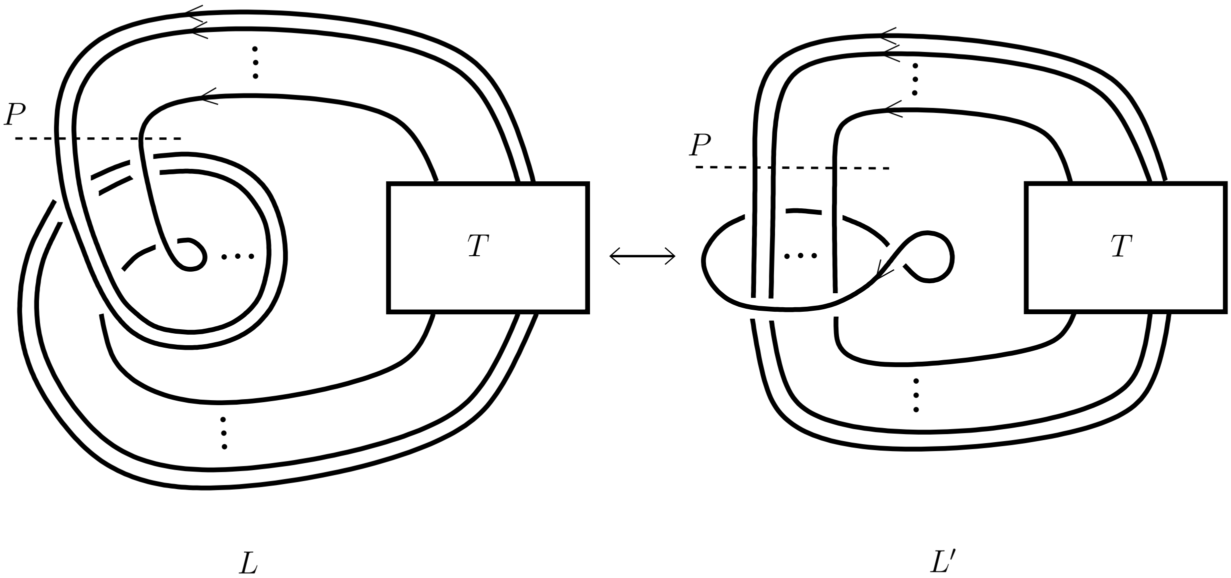


Figure 5.14: Two links L, L' related by the Kirby κ_+ move

follows: let $\Gamma(L_i)$ (resp. $\Gamma(L'_i)$) denote the directed $(1, 1)$-ribbon tangle derived from the i^{th} tangle of $L\backslash T$ (resp. $L'\backslash T$) intersecting the line P from the left. Colour both $\Gamma(L_i)$ and $\Gamma(L'_i)$ with $\lambda_i \in I$, and colour the unique component of $\Gamma(L'\backslash T)$ isotopic to an unknotted annulus with $\mu \in I$. We denote these coloured directed (m, m)-ribbon tangles, respectively, by

$$\Gamma\big(L\backslash T, (\lambda_1, \lambda_2, \ldots, \lambda_m)\big), \qquad \Gamma\big(L'\backslash T, (\lambda_1, \lambda_2, \ldots, \lambda_m, \mu)\big).$$

Lemma 5.3.1. *Assume that v acts on V_μ as the multiplication by a scalar. Fix $m = 1$, then*

$$F\big(\Gamma(L\backslash T, \lambda)\big) = v : V_\lambda \to V_\lambda, \tag{5.5}$$
$$F\big(\Gamma(L'\backslash T, (\lambda, \mu))\big) = \chi_\mu(v^{-1})C_\mu : V_\lambda \to V_\lambda, \tag{5.6}$$

where

$$C_\mu = (\mathrm{id} \otimes \mathrm{str}) \left[(\mathrm{id} \otimes \pi_\mu)(\mathrm{id} \otimes v^{-1}u) R^T R \right], \tag{5.7}$$

is a central element of A, and $\chi_\mu(v^{-1})$ is the eigenvalue of v^{-1} in the A-representation π_μ.

Lemma 5.3.2. *For each $m \geq 2$, the map*

$$F\big(\Gamma(L \backslash T, (\lambda_1, \ldots, \lambda_m))\big) : V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_m} \to V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_m},$$

acts as

$$\Delta^{(m-1)}(v) : V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_m} \to V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_m}.$$

Proof. Assume that Lemma 5.3.1 is true (we will prove it below), then Figure 5.15 proves the lemma for $m = 2$. Now assume that the inductive hypothesis is true for some $m \geq 2$, then the proof follows for $(m+1)$ by using Figure 5.16 and the representation of $\Delta^{(m)}(v)$ in tensor product representations of A. □

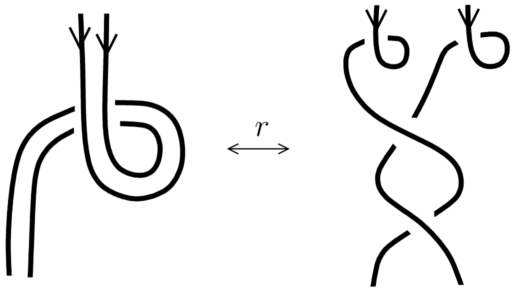


Figure 5.15: Regularly isotopic $(2,2)$-tangles

Remark 5.3.2. *The results of Eq. (5.5) in Lemma 5.3.1 and Lemma 5.3.2 still hold true if the modules are not irreducible.*

We now prove Lemma 5.3.1.

Proof. We consider the first claim. Theorem 5.3.1 implies that $F\big(\Gamma(L \backslash T, \lambda)\big) : V_\lambda \to V_\lambda$ is given by

$$F\big(\Gamma(L \backslash T, \lambda)\big) = \mathrm{id}_\lambda \circ (\mathrm{id}_\lambda \otimes \Omega_\lambda^-) \circ (X_{\lambda,\lambda}^- \otimes \mathrm{id}_\lambda) \circ (\mathrm{id}_\lambda \otimes U_\lambda^+) \circ \mathrm{id}_\lambda.$$

Let x be a basis vector of V_λ and let $\{v_r\}$, $\{v_r^*\}$ be dual bases of V_λ, $(V_\lambda)^*$, respectively, such that $\langle v_r^*, v_s \rangle = \delta_{rs}$ and $[v_r^*] = [v_r]$, where $\langle \cdot, \cdot \rangle : (V_\lambda)^* \times V_\lambda \to \mathbb{C}$ is the dual space

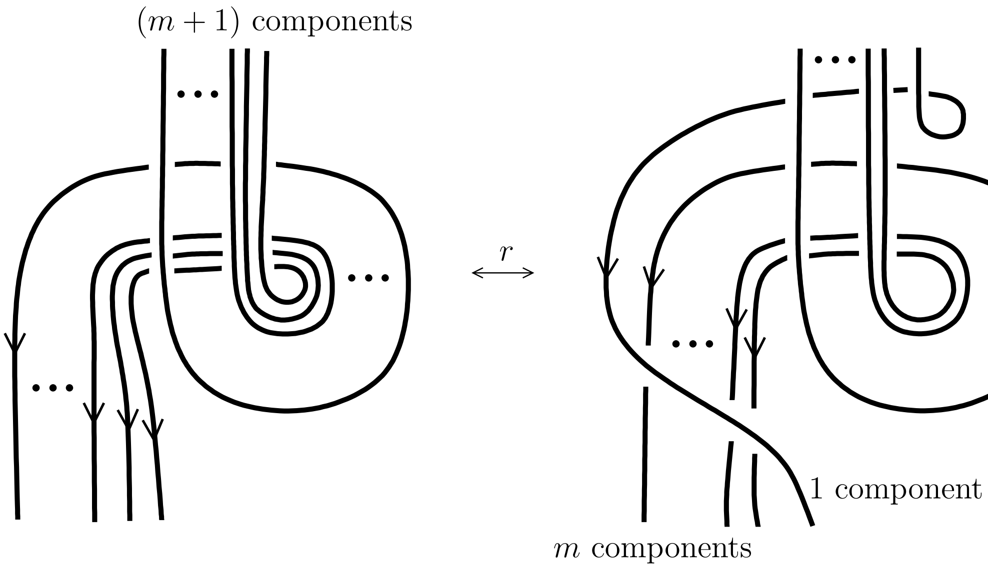


Figure 5.16: Regularly isotopic $(m+1,m+1)$-tangles

pairing. We calculate the action of $F\big(\Gamma(L\backslash T,\lambda)\big)$ on x to be

$$\begin{aligned}
x \quad &\overset{(\mathrm{id}_\lambda\otimes U_\lambda^+)}{\longmapsto} \quad \sum_r x\otimes v_r\otimes v_r^* \\
&\overset{(X^-_{\lambda,\lambda}\otimes\mathrm{id}_\lambda)}{\longmapsto} \quad \sum_{t,r} S(a_t)v_r\otimes b_t x\otimes v_r^*(-1)^{[x][v_r]+[b_t][v_r]} \\
&\overset{(\mathrm{id}_\lambda\otimes\Omega_\lambda^-)}{\longmapsto} \quad \sum_{t,r}\langle v_r^*, v^{-1}ub_t x\rangle S(a_t)v_r \\
&= \quad v^{-1}S(u)ux = vx,
\end{aligned}$$

as $v^2 = S(u)u$.

We now consider the second claim. We will calculate the action of $F\big(\Gamma(L'\backslash T,(\lambda,\mu))\big)$ on a basis vector $x\in V_\lambda$ where the annulus is coloured with μ. Consider the regularly isotopic oriented $(1,1)$-tangles in Figure 5.17, where $\overset{r}{\longleftrightarrow}$ indicates regular isotopy. As L' and M' are regularly isotopic, $F\big(\Gamma(L',\nu)\big) = F\big(\Gamma(M',\nu)\big)$, and we will now calculate $F\big(\Gamma(M',\nu)\big)$. Consider the coloured directed $(1,1)$-ribbon tangle $\Gamma(M'',(\lambda,\mu))$ in Figure 5.18: note that $\Gamma(M'',(\lambda,\mu))$ is identical to $\Gamma(M',(\lambda,\mu))$ 'modulo' a twist. The map $F\big(\Gamma(M'',(\lambda,\mu))\big): V_\lambda\to V_\lambda$ is

$$F\big(\Gamma(M'',(\lambda,\mu))\big) = \mathrm{id}_\lambda\circ(\mathrm{id}_\lambda\otimes\Omega_\mu^-)\circ(X^+_{\mu,\lambda}\otimes\mathrm{id}_{\mu^*})\circ(X^+_{\lambda,\mu}\otimes\mathrm{id}_{\mu^*})\circ(\mathrm{id}_\lambda\otimes U_\mu^+)\circ\mathrm{id}_\lambda.$$

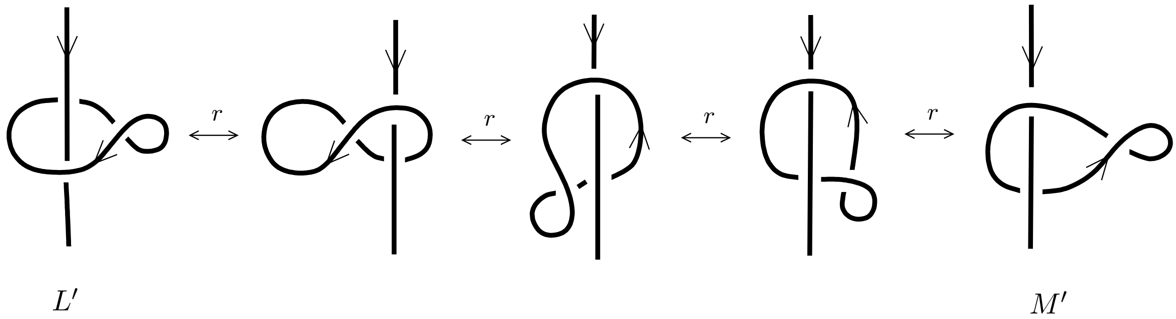


Figure 5.17: Regularly isotopic oriented $(1,1)$-tangles

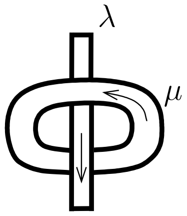


Figure 5.18: The coloured directed $(1,1)$-ribbon tangle $\Gamma(M'', (\lambda, \mu))$

Let x be a basis vector of V_λ, then we calculate $F\big(\Gamma(L, (\lambda, \mu))\big)(x)$ to be

$$
\begin{aligned}
x \quad &\overset{(\mathrm{id}_\lambda \otimes U_\mu^+)}{\longmapsto} \quad \sum_r x \otimes v_r \otimes v_r^* \\
&\overset{(X^+_{\lambda,\mu} \otimes \mathrm{id}_{\mu^*})}{\longmapsto} \quad \sum_{t,r} b_t v_r \otimes a_t x \otimes v_r^* (-1)^{[x][v_r]+[a_t](1+[v_r])} \\
&\overset{(X^+_{\mu,\lambda} \otimes \mathrm{id}_{\mu^*})}{\longmapsto} \quad \sum_{s,t,r} b_s a_t x \otimes a_s b_t v_r \otimes v_r^* (-1)^{[b_s]+[x]([a_s]+[b_t])+[a_t][a_s]} \\
&\overset{(\mathrm{id}_\lambda \otimes \Omega_\mu^-)}{\longmapsto} \quad \sum_{s,t,r} \langle v_r^*, (v^{-1}u) a_s b_t v_r \rangle b_s a_t x (-1)^{[b_s]+[x]([b_t]+[a_s])+[a_t][a_s]+[v_r^*]([a_s]+[b_t]+[v_r])} \\
&= \quad (\mathrm{id} \otimes \mathrm{str}) \left[(\mathrm{id} \otimes \pi_\mu)(\mathrm{id} \otimes v^{-1}u) R^T R \right] x = C_\mu(x).
\end{aligned}
$$

Note that C_μ is a central element of A [ZG91, Prop. 3] and $\chi_0(C_\mu) = sdim_q(V_\mu)$. Adding a twist with framing number $+1$ into the annulus coloured with μ gives precisely $\Gamma(L' \backslash T, (\lambda, \mu))$. Lemma 5.3.3 (ii) below implies that $F\big(\Gamma(L' \backslash T, (\lambda, \mu))\big) = \chi_{(V_\mu)^*}(v^{-1}) F\big(\Gamma(L, (\lambda, \mu))\big)$, where $\chi_{(V_\mu)^*}(v^{-1})$ is the scalar action of v^{-1} on $(V_\mu)^*$. As v is even and $S(v) = v$, we have $\chi_{(V_\mu)^*}(v^{-1}) = \chi_\mu(v^{-1})$. □

Lemma 5.3.3.

(i) Let $\Gamma(L, \lambda)$ *be the coloured directed* $(1,1)$*-ribbon tangle in Figure 5.19, then* $F\big(\Gamma(L, \lambda)\big) = v^{-1} : V_\lambda \to V_\lambda$.

(ii) Let $\Gamma(L',\lambda)$ *be the coloured directed* $(1,1)$*-ribbon tangle obtained by reversing the direction of* $\Gamma(L,\lambda)$*, then* $F\big(\Gamma(L',\lambda)\big) = v^{-1} : (V_\lambda)^* \to (V_\lambda)^*$.

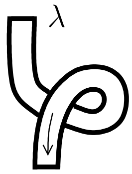


Figure 5.19: The coloured directed $(1,1)$-ribbon tangle $\Gamma(L,\lambda)$

Proof. We prove (i). The map $F\big(\Gamma(L,\lambda)\big) : V_\lambda \to V_\lambda$ is

$$F\big(\Gamma(L,\lambda)\big) = \mathrm{id}_\lambda \circ (\mathrm{id}_\lambda \otimes \Omega^-_\lambda) \circ (X^+_{\lambda,\lambda} \otimes \mathrm{id}_{\lambda^*}) \circ (\mathrm{id}_\lambda \otimes U^+_\lambda) \circ \mathrm{id}_\lambda.$$

Let x be a basis vector of V_λ, then we calculate $F\big(\Gamma(L,\lambda)\big)(x)$ as follows:

$$\begin{aligned}
x \quad &\overset{(\mathrm{id}_\lambda \otimes U^+_\lambda)}{\longmapsto} \quad \sum_r x \otimes v_r \otimes v_r^* \\
&\overset{(X^+_{\lambda,\lambda} \otimes \mathrm{id}_{\lambda^*})}{\longmapsto} \quad \sum_{t,r} b_t v_r \otimes a_t x \otimes v_r^* (-1)^{[a_t]+[v_r]([a_t]+[x])} \\
&\overset{(\mathrm{id}_\lambda \otimes \Omega^-_\lambda)}{\longmapsto} \quad \sum_{t,r} \langle v_r^*, v^{-1}u\ a_t x\rangle b_t v_r (-1)^{[a_t]} \\
&= \quad \sum_t v^{-1} b_t u a_t\ x(-1)^{[a_t]} \\
&= \quad \sum_t v^{-1} b_t S^2(a_t) u\ x(-1)^{[a_t]} = v^{-1}x.
\end{aligned}$$

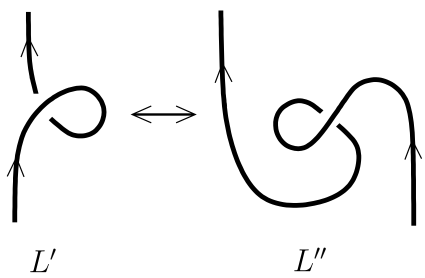


Figure 5.20: Two regularly isotopic oriented $(1,1)$-tangles

We now prove (ii). Consider the two regularly isotopic oriented $(1,1)$-tangles L' and L'' in Figure 5.20. As these tangles are regularly isotopic, we can rewrite the map $F\big(\Gamma(L',\lambda)\big) : (V_\lambda)^* \to (V_\lambda)^*$ as

$$\begin{aligned} F\big(\Gamma(L',\lambda)\big) &= F\big(\Gamma(L'',\lambda)\big) \\ &= \mathrm{id}_{\lambda^*} \circ (\mathrm{id}_{\lambda^*} \otimes \Omega^-_\lambda) \circ (\mathrm{id}_{\lambda^*} \otimes \Omega^+_\lambda \otimes \mathrm{id}_\lambda \otimes \mathrm{id}_{\lambda^*}) \circ (\mathrm{id}_{\lambda^*} \otimes \mathrm{id}_{\lambda^*} \otimes X^+_{\lambda,\lambda} \otimes \mathrm{id}_{\lambda^*}) \\ &\quad \circ (\mathrm{id}_{\lambda^*} \otimes U^-_\lambda \otimes \mathrm{id}_\lambda \otimes \mathrm{id}_{\lambda^*}) \circ (U^-_\lambda \otimes \mathrm{id}_{\lambda^*}) \circ \mathrm{id}_{\lambda^*}. \end{aligned} \tag{5.8}$$

Let x be a basis vector of $(V_\lambda)^*$, then the action of the right hand side of (5.8) on x is

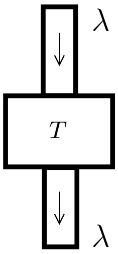
$$\begin{aligned} x \quad &\xmapsto{U^-_\lambda \otimes \mathrm{id}_{\lambda^*}} \quad \sum_r v_r^* \otimes (vu^{-1})v_r \otimes x(-1)^{[v_r]} \\ &\xmapsto{\mathrm{id}_{\lambda^*}\otimes U^-_\lambda \otimes \mathrm{id}_\lambda \otimes \mathrm{id}_{\lambda^*}} \quad \sum_{r,p} v_r^* \otimes v_p^* \otimes (vu^{-1})v_p \otimes (vu^{-1})v_r \otimes x(-1)^{[v_r]+[v_p]} \\ &\xmapsto{\mathrm{id}_{\lambda^*}\otimes \mathrm{id}_{\lambda^*} \otimes X^+_{\lambda,\lambda} \otimes \mathrm{id}_{\lambda^*}} \quad \sum_{r,p,t} v_r^* \otimes v_p^* \otimes b_t v u^{-1} v_r \otimes a_t v u^{-1} v_p \otimes x(-1)^{[v_r]+[v_p]+[a_t](1+[v_r])+[v_p][v_r]} \\ &\xmapsto{\mathrm{id}_{\lambda^*}\otimes \Omega^+_\lambda \otimes \mathrm{id}_\lambda \otimes \mathrm{id}_{\lambda^*}} \quad \sum_{r,p,t} \langle uv^{-1}S^{-1}(b_t)v_p^*, v_r\rangle v_r^* \otimes a_t v u^{-1} v_p \otimes x(-1)^{[v_p][b_t]+[v_p]+[a_t]+[v_r](1+[a_t]+[v_p])} \\ &= \quad \sum_{p,t} uv^{-1}S^{-1}(b_t)v_p^* \otimes a_t v u^{-1} v_p \otimes x(-1)^{[v_p]+[a_t]+[v_p][b_t]} \qquad (5.9) \\ &\xmapsto{\mathrm{id}_{\lambda^*}\otimes \Omega^-_\lambda} \quad \sum_{p,t} \langle x, v^{-1}u a_t v u^{-1} v_p\rangle uv^{-1}S^{-1}(b_t)v_p^*(-1)^{[x]([a_t]+[v_p])+[v_p]+[a_t]+[v_p][b_t]} \\ &= \quad \sum_{p,t} \langle S(a_t)x, v_p\rangle uv^{-1}S^{-1}(b_t)v_p^*(-1)^{[v_p](1+[x]+[b_t])+[a_t]} \\ &= \quad \sum_t uv^{-1}S^{-1}(b_t)S(a_t)x(-1)^{[a_t]} = v^{-1}x. \end{aligned}$$

Note that (5.9) is obtained by using the fact that $\langle uv^{-1}S^{-1}(b_t)v_p^*, v_r\rangle$ vanishes unless $[v_r] \equiv ([v_p]+[b_t]) \pmod 2$, allowing us to simplify the sign factors. □

Proposition 5.3.1. *Let $\Gamma(L,\lambda)$ be the coloured directed $(1,1)$-ribbon tangle in Figure 5.21 such that the map $F\big(\Gamma(L,\lambda)\big) : V_\lambda \to V_\lambda$ acts on V_λ as the multiplication by the scalar ζ. Now let $\hat{\Gamma}(L,\lambda)$ be the closure of $\Gamma(L,\lambda)$ in Figure 5.22, then $F\big(\hat{\Gamma}(L,\lambda)\big) : \mathbb{C} \to \mathbb{C}$ acts as the multiplication by $\zeta sdim_q(V_\lambda)$.*

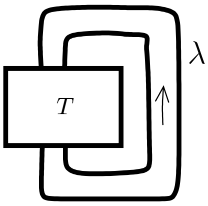
Proof. Explicitly, we have

$$F\big(\hat{\Gamma}(L,\lambda)\big) = \Omega^-_\lambda \circ (\zeta \mathrm{id}_\lambda \otimes \mathrm{id}_{\lambda^*}) \circ U^+_\lambda : \mathbb{C} \to \mathbb{C}, \tag{5.10}$$



$\Gamma(L, \mu)$

Figure 5.21: A coloured directed $(1, 1)$-ribbon tangle $\Gamma(L, \lambda)$



$\hat{\Gamma}(L, \mu)$

Figure 5.22: The closure $\hat{\Gamma}(L, \lambda)$ of the ribbon tangle $\Gamma(L, \lambda)$

and for any $c \in \mathbb{C}$,

$$
\begin{aligned}
c \quad & \overset{U_\lambda^+}{\longmapsto} \quad c \sum_r v_r \otimes v_r^* \overset{\zeta \mathrm{id}_\lambda \otimes \mathrm{id}_{\lambda^*}}{\longmapsto} c\,\zeta \sum_r v_r \otimes v_r^* \\
& \overset{\Omega_\lambda^-}{\longmapsto} \quad c\,\zeta \sum_r \langle v_r^*, (v^{-1}u)v_r \rangle (-1)^{[v_r]} = c\,\zeta sdim_q(V_\lambda).
\end{aligned}
$$

□

5.4 Pseudo-modular Hopf algebras

5.4.1 Modular Hopf algebras

We now introduce the notion of modular Hopf algebras that play a central role in the construction of topological invariants of 3-manifolds in [RT91]. We extend the definition to the $\mathbb{Z}_2$-graded case in the obvious way.

Definition 5.4.1. *Let A be a $\mathbb{Z}_2$-graded ribbon Hopf algebra over $\mathbb{C}$ with universal R-matrix $R \in A \otimes A$ stated in (5.1), the element $u \in A$ stated in (5.2), and the even central element $v \in A$ stated in (5.3).*

Let I be a finite index set with an involution $*: I \to I$ denoted by $*(i) = i^*$, and let there exist a distinguished element $0 \in I$ satisfying $*(0) = 0^* = 0$. Let $\{V_i |\ i \in I\}$ be a set of A-modules, where V_0 is the one-dimensional A-module, and let there exist a set of A-linear isomorphisms

$$\left\{\omega_i : (V_i)^* \to V_{i^*} |\ i \in I\right\},$$

where $\omega_0 = \mathrm{id}$ and $(V_i)^*$ is the dual A-module to V_i.

The $\mathbb{Z}_2$-graded ribbon Hopf algebra A is called a $\mathbb{Z}_2$-graded modular Hopf algebra if the following axioms are satisfied:

(i) The A-modules $\{V_i |\ i \in I\}$ are irreducible, finite-dimensional, mutually non-isomorphic and each V_i has non-vanishing quantum superdimension.

(ii) For each $i \in I$, let $\{x_j\}$, $\{x_j^*\}$ and $\{x_j^{**}\}$ be bases of V_i, $(V_i)^*$ and $((V_i)^*)^*$, respectively, such that $\langle\langle x_j^{**}, x_k^*\rangle\rangle = \langle x_k^*, x_j\rangle = \delta_{j,k}$ and $[x_j^{**}] = [x_j^*] = [x_j]$, where $\langle\langle \cdot, \cdot \rangle\rangle : ((V_i)^*)^* \times (V_i)^* \to \mathbb{C}$ is the dual space pairing. Then the map

$$(\omega_i)^* \circ (\omega_{i^*})^{-1} : V_i \to ((V_i)^*)^*,$$

is given by $x_j \mapsto (-1)^{[x_j]} u v^{-1} x_j^{**}$, $\forall j$.

(iii) For any $t \in \mathbb{N}$ and any sequence $\theta = (i_1, i_2, \ldots, i_t) \in I^{\times t}$,

$$V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_t} = \left(\bigoplus_{i \in I} (V_i)^{\oplus n_\theta(i)}\right) \oplus \mathcal{Z}_\theta,$$

where $n_\theta(i) \in \mathbb{Z}_+$ is the number of copies of the A-module V_i in the direct sum and $\mathcal{Z}_\theta$ is a possibly vanishing A-module that satisfies axiom (iv).

(iv) For each $\theta = (i_1, i_2, \ldots, i_t)$, $t \geq 2$, and all A-linear homomorphisms $\phi : \mathcal{Z}_\theta \to \mathcal{Z}_\theta$, $str_q(\phi) = 0$.

(v) Let $f = (f_{\lambda\mu})_{\lambda,\mu \in I}$ be the matrix with complex entries given by

$$f_{\lambda\mu} = (\mathrm{str} \otimes \mathrm{str}) \left[(\pi_\lambda \otimes \pi_\mu)(v^{-1}u \otimes v^{-1}u) R^T R\right].$$

The matrix f is invertible and there exists a unique set $\{d_i \in \mathbb{C} |\ i \in I\}$ of constants satisfying the relations

$$\chi_\lambda(v) sdim_q(V_\lambda) = \sum_{\mu \in I} d_\mu \chi_\mu(v^{-1}) f_{\lambda\mu}, \quad \text{for all } \lambda \in I. \tag{5.11}$$

Here π_μ is the irreducible A-representation furnished by V_μ, and $\chi_\mu(v)$ denotes the eigenvalue of v in π_μ.

Using results of Subsection 5.3.1, we have

$$f_{\lambda\mu} = sdim_q(V_\lambda)\chi_\lambda(C_\mu),$$

where C_μ is the central element of A defined by Eq. (5.7).

An additional axiom was included in the original definition of modular Hopf algebras [RT91]. This axiom stated that the scalar

$$z = \sum_{\lambda\in I} d_\lambda \chi_\lambda(v) sdim_q(V_\lambda) \tag{5.12}$$

should not vanish. However, this automatically follows from the other axioms [Wa91, Lem 8.20],[TW93, Sec. 1]. The proof for the $\mathbb{Z}_2$-graded case is exactly the same, thus we omit it here. However, note that the proof of Lemma 5.6.5 for pseudo-modular Hopf algebras bears much similarity to it.

Lemma 5.4.1. *For a $\mathbb{Z}_2$-graded modular Hopf algebra A, the complex constant z defined by*

$$z = \sum_{\lambda\in I} d_\lambda \chi_\lambda(v) sdim_q(V_\lambda), \tag{5.13}$$

is non-zero.

Another consequence of the axioms of a modular Hopf algebra is that $d_{\mu^*} = d_\mu$ for all $\mu \in I$. See [RT91, Sec. 5.2] for a detailed proof.

Finally, we note for interest that Turaev and Wenzl defined a slightly more general class of ribbon Hopf algebras that can be used in constructing 3-manifold invariants. They defined *quasimodular Hopf algebras* [TW93, Sec. 2.1] so as to construct 3-manifold invariants from the quantum algebras $U_q(\mathfrak{g})$ associated with the A, B, C and D families of Lie algebras at even roots of unity without knowing whether the relevant modules were irreducible [TW93]. The essential difference between quasimodular and modular Hopf algebras is that this irreducibility condition is relaxed for quasimodular Hopf algebras.

5.4.2 Pseudo-modular Hopf algebras

We now introduce the notion of pseudo-modular Hopf algebras that play a central role in this chapter. A *pseudo-modular Hopf algebra* is a $\mathbb{Z}_2$-graded ribbon Hopf algebra together with a collection of finite dimensional representations satisfying slightly weaker conditions than those satisfied by modular Hopf algebras. We will prove in Section 5.5 that one can construct topological invariants of closed, connected, orientable 3-manifolds from pseudo-modular Hopf algebras.

The essential difference between a pseudo-modular Hopf algebra and a modular Hopf algebra is that relations (5.11) do not necessarily have a unique set of solutions in pseudo-modular Hopf algebras, thus the set of constants $\{d_\nu \in \mathbb{C} | \ \nu \in I\}$ is not necessarily unique. Consequently, one must independently prove that the z in (5.12) is nonvanishing. Examples of algebras for which the constants $\{d_\nu | \ \nu \in I\}$ are not unique are $U_q^{(N)}(osp(1|2))$ at odd roots of unity [Zh94] and $U_q^{(N)}(gl_2)$ also at odd roots of unity [Zh95].

Definition 5.4.2. *Let A be a $\mathbb{Z}_2$-graded ribbon Hopf algebra over $\mathbb{C}$ with universal R-matrix $R \in A \otimes A$ stated in (5.1), the element $u \in A$ stated in (5.2), and the even central element $v \in A$ stated in (5.3).*

Let I be a finite index set with an involution $* : I \to I$ denoted by $*(i) = i^*$, and let there exist a distinguished element $0 \in I$ satisfying $*(0) = 0^* = 0$. Let $\{V_i |\ i \in I\}$ be a set of A-modules, where V_0 is the one-dimensional A-module, and let there exist a set of A-linear isomorphisms

$$\big\{\omega_i : (V_i)^* \to V_{i^*} |\ i \in I\big\},$$

where $\omega_0 = \mathrm{id}$ and $(V_i)^*$ is the dual A-module to V_i.

The $\mathbb{Z}_2$-graded ribbon Hopf algebra A is said to be a pseudo-modular Hopf algebra if the following axioms are satisfied:

(I) The A-modules $\{V_i |\ i \in I\}$ are finite dimensional, mutually non-isomorphic and $sdim_q(V_i) \neq 0$ for all $i \in I$.

(II) Let $\Gamma(L, \lambda)$ be a coloured directed $(1,1)$-ribbon tangle with m components, and let the ends of $\Gamma(L, \lambda)$ be directed downwards and lie in a component coloured with $i \in I$. Then the corresponding map $F\big(\Gamma(L, \lambda)\big) : V_i \to V_i$ acts by multiplication by a complex scalar. Furthermore, the even central element $v \in A$ acts on each A-module $V_i, i \in I$, as a complex scalar.

(III) For each $i \in I$, let $\{x_j\}$, $\{x_j^*\}$ and $\{x_j^{**}\}$ be bases of V_i, $(V_i)^*$ and $((V_i)^*)^*$, respectively, such that $\langle\langle x_j^{**}, x_k^*\rangle\rangle = \langle x_k^*, x_j\rangle = \delta_{j,k}$ and $[x_j^{**}] = [x_j^*] = [x_j]$, where $\langle\langle \cdot, \cdot\rangle\rangle : ((V_i)^*)^* \times (V_i)^* \to \mathbb{C}$ is the dual space pairing. Then the map

$$(\omega_i)^* \circ (\omega_{i^*})^{-1} : V_i \to ((V_i)^*)^*,$$

is given by $x_j \mapsto (-1)^{[x_j]} u v^{-1} x_j^{**},\ \forall j$.

(IV) For any $t \in \mathbb{N}$ and any sequence $\theta = (i_1, i_2, \ldots, i_t) \in I^{\times t}$,

$$V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_t} = \left(\bigoplus_{i \in I} (V_i)^{\oplus n_\theta(i)} \right) \oplus \mathcal{Z}_\theta,$$

where $n_\theta(i) \in \mathbb{Z}_+$ is the number of copies of the A-module V_i in the direct sum and $\mathcal{Z}_\theta$ is a possibly vanishing A-module such that $str_q(a) = 0$ for any A-linear map $a \in End_A(V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_t})$ obtained by applying the functor F to a directed oriented (m,m)-ribbon tangle, where the module map $a \in End_A(V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_t})$ also satisfies $a(V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_t}) \subseteq \mathcal{Z}_\theta$.

(V) Let $f = (f_{\lambda\nu})_{\lambda,\nu \in I}$ be a matrix with complex elements given by

$$f_{\lambda\nu} = (\mathrm{str} \otimes \mathrm{str}) \left[(\pi_\lambda \otimes \pi_\nu)(v^{-1}u \otimes v^{-1}u) R^T R \right].$$

Then there exists at least one set $\{d_\nu \in \mathbb{C} |\ \nu \in I\}$ *of constants satisfying the relations*

$$\chi_\lambda(v) sdim_q(V_\lambda) = \sum_{\nu\in I} d_\nu \chi_\nu(v^{-1}) f_{\lambda\nu}, \quad \textit{for all } \lambda \in I, \tag{5.14}$$

where $d_{\nu^*} = d_\nu$ *for all* $\nu \in I$*. Here* π_ν *is the* A*-representation furnished by* V_ν*, and* $\chi_\nu(v) = \pi_\nu(v)$*.*

(VI) The scalar $z = \sum_{\lambda\in I} d_\lambda \chi_\lambda(v) sdim_q(V_\lambda)$ *is non-zero.*

5.5 Reshetikhin-Turaev invariant arising from pseudo-modular Hopf algebras

We will construct a topological invariant of a closed, connected, orientable 3-manifold from each pseudo-modular Hopf algebra following the general approach of Reshetikhin and Turaev [RT91] and Turaev and Wenzl [TW93]. The main result of this section is Theorem 5.5.1.

Let us fix a pseudo-modular Hopf algebra as defined in Definition 5.4.2. Let $L \subset S^3$ be a framed oriented link with m connected components. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$, $\lambda_i \in I$, and fix $\mathcal{C}(L, I)$ to be the set of all different λ. Define

$$\sum(L) = \sum_{\lambda\in\mathcal{C}(L,I)} \prod_{i=1}^{m} d_{\lambda_i} F\big(\Gamma(L,\lambda)\big), \tag{5.15}$$

where $\{d_\nu |\ \nu \in I\}$ is a set of constants satisfying Eq. (5.14). Note that $\sum(L)$ is a regular isotopy invariant of L as each $F\big(\Gamma(L,\lambda)\big)$ is a regular isotopy invariant of L.

We now prove that $\sum(L) = \sum(L')$ if L and L' are two links that are equivalent with respect to the κ_+ move. This $\sum(L)$ is a core part of Reshetikhin and Turaev's 3-manifold invariants.

Consider the two framed oriented links $L, L' \subset S^3$ presented in the blackboard framing in Figure 5.23. The planar projections of L and L' are equivalent with respect to the Kirby move κ_+. In this figure, T is an arbitrary oriented (m, m)-tangle represented by the rectangle, where the orientation of T is compatible with the orientations of $L\backslash T$ and $L'\backslash T$. We colour the directed ribbon tangles $\Gamma(L)$ and $\Gamma(L')$ associated with L and L', respectively, as follows. Firstly, let $\Gamma(L_i)$ (resp. $\Gamma(L'_i)$) denote the directed $(1,1)$-ribbon tangle obtained from the i^{th} tangle of $L\backslash T$ (resp. $L'\backslash T$) intersecting the line P from the left in both diagrams. Each of $\Gamma(L_i)$ and $\Gamma(L'_i)$ is coloured with $\lambda_i \in I$. Lastly, colour the unique component of $\Gamma(L'\backslash T)$ isotopic to an unknotted annulus with $\xi \in I$. We denote the resulting coloured directed (m, m)-ribbon tangles as

$$\Gamma\big(L\backslash T, (\lambda_1, \lambda_2, \ldots, \lambda_m)\big), \qquad \Gamma\big(L'\backslash T, (\lambda_1, \lambda_2, \ldots, \lambda_m, \xi)\big),$$

respectively. We have the following result.

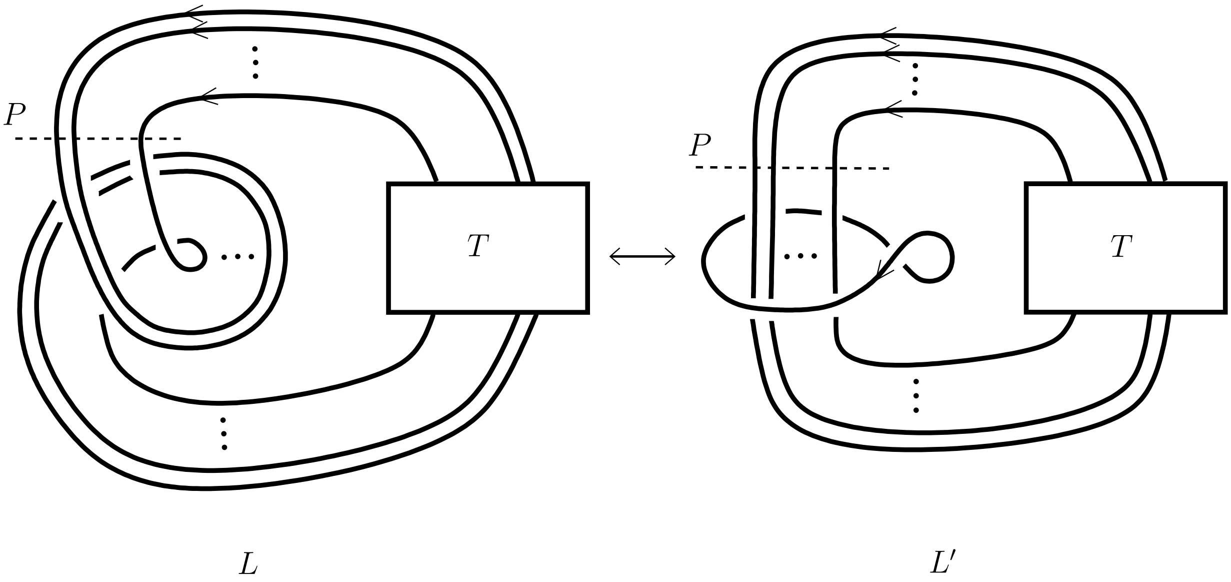


Figure 5.23: Two links L, L' that are equivalent with respect to the Kirby κ_+ move

Proposition 5.5.1. *For each $m \geq 1$,*

$$F\big(\Gamma(L\backslash T, (\lambda_1, \ldots, \lambda_m))\big) = \Delta^{(m-1)}(v) : V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_m} \to V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_m},$$

and

$$F\big(\Gamma(L'\backslash T, (\lambda_1, \ldots, \lambda_m, \xi))\big) = \chi_\xi(v^{-1})\Delta^{(m-1)}(C_\xi) : V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_m} \to V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_m},$$

where C_ξ is defined by (5.7).

Proof. The first claim follows from Remark 5.3.2. For $m = 1$, the second claim follows from Eq. (5.6), which is still true as v^{-1} acts as a scalar in the A-representation π_ξ, and for $m \geq 2$ by induction. □

Proposition 5.5.2. *Consider the two framed, oriented links L, L' in Figure 5.23 where $m \geq 1$ and T is an arbitrary oriented (m,m)-tangle the orientation of which is compatible with that of $L\backslash T$ and $L'\backslash T$, then*

$$\sum(L) = \sum(L'). \tag{5.16}$$

Proof. By definition,

$$\begin{aligned}\sum(L') \quad = \quad & \sum_{\lambda \in \mathcal{C}(L,I)} \sum_{\xi \in I} \prod_{i=1}^{m} d_{\lambda_i} \mathrm{str}\Big[\big(\pi_{\lambda_1}(v^{-1}u) \otimes \pi_{\lambda_2}(v^{-1}u) \cdots \otimes \pi_{\lambda_m}(v^{-1}u)\big) \\ & \times F\big(\Gamma(L'\backslash T, (\lambda_1, \lambda_2, \ldots, \lambda_m, \xi))\big) \circ F\big(\Gamma(T, (\lambda_1, \lambda_2, \ldots, \lambda_m))\big)\Big],\end{aligned} \tag{5.17}$$

where in writing str we mean that we take the supertrace over $V_{\lambda_1} \otimes V_{\lambda_2} \otimes \cdots \otimes V_{\lambda_m}$.

Let $\theta = (\lambda_1, \lambda_2, \ldots, \lambda_m) \in I^{\times m}$. From Axiom (IV) of a pseudo-modular Hopf algebra, $V_{\lambda_1} \otimes V_{\lambda_2} \otimes \cdots \otimes V_{\lambda_m} = \left(\bigoplus_{\xi \in I} (V_\xi)^{\oplus n_\theta(\xi)}\right) \oplus \mathcal{Z}_\theta$ for some non-negative constants $n_\theta(\xi)$, which allows us to rewrite (5.17) as

$$\sum_{\lambda \in \mathcal{C}(L,I)} \prod_{i=1}^{m} d_{\lambda_i} \left(\sum_{\xi \in I} n_\theta(\xi) \mathrm{str}_{V_\xi} + \mathrm{str}_{\mathcal{Z}_\theta} \right) \times \left[\left(v^{-1}u\right) \circ \left(\sum_{\zeta \in I} d_\zeta \chi_\zeta(v^{-1}) f_{\xi\zeta} / sdim_q(V_\xi) \right) \circ F\big(\Gamma(T, (\lambda_1, \lambda_2, \ldots, \lambda_m))\big) \right] \quad (5.18)$$

where in writing str_{V_ξ} we mean that we take the supertrace over the submodule $V_\xi \subseteq V_{\lambda_1} \otimes V_{\lambda_2} \otimes \cdots \otimes V_{\lambda_m}$, and $\mathrm{str}_{\mathcal{Z}_\theta}$ has a similar meaning. Axiom (V) allows us to rewrite the right hand side of (5.18) as

$$\sum_{\lambda \in \mathcal{C}(L,I)} \prod_{i=1}^{m} d_{\lambda_i} \left(\sum_{\xi \in I} n_\theta(\xi) \mathrm{str}_{V_\xi} + \mathrm{str}_{\mathcal{Z}_\theta} \right) \left[\left(v^{-1}u\right) \circ (\chi_\xi(v)) \, F\big(\Gamma(T, (\lambda_1, \lambda_2, \ldots, \lambda_m))\big) \right],$$

which is equal to $\sum(L)$. In the derivation of this equality, we used Eq. (5.14) and Axiom (II). Note that $\sum(L') = \sum(L)$ relies on the equality of the quantum supertraces of certain A-linear maps, not on an equality of the A-linear maps themselves. □

Remark 5.5.1. *The proof of Proposition 5.5.2 is similar to the proof of the corresponding theorem for modular Hopf algebras. See [RT91, Par. 7.2] for details.*

Proposition 5.5.3. *Let $\Gamma(L, (\lambda_1, \ldots, \lambda_m))$ be a coloured, directed $(0,0)$-ribbon tangle, with its i^{th} component coloured by λ_i. Let $\Gamma'(L, (\lambda_1, \ldots, \lambda_m))$ be the coloured, directed $(0,0)$-ribbon tangle obtained by changing the direction of the i^{th} component of $\Gamma(L(\lambda_1, \ldots, \lambda_m))$. Then*

$$F\big(\Gamma'(L, (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i, \lambda_{i+1}, \ldots, \lambda_m))\big) = F\big(\Gamma(L, (\lambda_1, \ldots, \lambda_{i-1}, (\lambda_i)^*, \lambda_{i+1}, \ldots, \lambda_m))\big).$$

Proof. The proof is almost identical to the proof of [RT91, Lem. 5.1], thus omitted. □

We can now define the topological invariant. Let $L = \bigcup_{i=1}^m L_i \subset S^3$ be an unoriented framed link with m connected components, and let M_L be the closed, connected, orientable 3-manifold obtained by performing surgery on S^3 along L. We introduce some notation by writing $A_L = (a_{ij})_{i,j=1}^m$ to mean the linking matrix of L, defined by

(i) $a_{ii} = w(L_i)$, the writhing number of L_i, for each i,

(ii) $a_{ij} = \mathrm{lk}(L_i, L_j)$, if $i \neq j$.

Note that A_L is real and symmetric. We define $\sigma(A_L)$ to be the number of non-positive eigenvalues of A_L. Following [RT91] we introduce the topological invariant $\mathcal{F}(M_L)$.

Theorem 5.5.1. *Let L be a framed link. Then*

$$\mathcal{F}(M_L) = z^{-\sigma(A_L)} \sum(L)$$

is a topological invariant of M_L, where in calculating $\sum(L)$ we assign an arbitrary orientation to L.

Proof. Observe that $\sum(L)$ does not depend on the orientation chosen for L. Let L have m components and assign an orientation to each component of L. Then

$$\sum(L) = \sum_{\lambda \in \mathcal{C}(L,I)} \prod_{j=1}^{m} d_{\lambda_j} F\big(\Gamma(L, (\lambda_1, \ldots, \lambda_m))\big).$$

Now let L' be the link obtained by reversing the orientation of the i^{th} component of L. By using Proposition 5.5.3 we have

$$\sum(L') = \sum \prod_{j=1}^{m} d_{\lambda_j} F\big(\Gamma(L, (\lambda_1, \ldots, \lambda_{i-1}, (\lambda_i)^*, \lambda_{i+1}, \ldots, \lambda_m))\big), \tag{5.19}$$

where the right hand side of (5.19) is a m-fold summation over all $\lambda_j \in I$, $j \neq i$, and $(\lambda_i)^* \in I$. Since $d_{\lambda_i} = d_{(\lambda_i)^*}$, we can rewrite $\sum(L')$ as

$$\sum(L') = \sum \prod_{\substack{j=1 \\ j \neq i}}^{m} d_{\lambda_j} d_{(\lambda_i)^*} F\big(\Gamma(L, (\lambda_1, \ldots, \lambda_{i-1}, (\lambda_i)^*, \lambda_{i+1}, \ldots, \lambda_m))\big). \tag{5.20}$$

On the right hand side, we sum all the λ_k's independently over I. Hence the right hand side of (5.20) is equal to $\sum(L)$, that is

$$\sum(L') = \sum(L').$$

We now need to show that $\mathcal{F}(M_L) = \mathcal{F}(M_{\mathcal{L}})$ if the links L and $\mathcal{L}$ are equivalent with respect to any of the Kirby moves. Each Kirby move can be expressed as some composition of the $\kappa_+^{(0)}$, $\kappa_-^{(0)}$ and κ_+ Kirby moves [RT91, Thm. 6.3]; we will show that $\mathcal{F}(M_L) = \mathcal{F}(M_{\mathcal{L}})$ if L and $\mathcal{L}$ are equivalent with respect to any of these moves.

We firstly show that $\mathcal{F}(M_L) = \mathcal{F}(M_{L'})$ if the two links L and L' are equivalent with respect to the $\kappa_+^{(0)}$ move. Let $\mathcal{O}_{+1}$ denote the unknot with framing $+1$ given in Figure 5.24 and let $L' = L \cup \mathcal{O}_{+1}$ be a split link.
It immediately follows from the definition (Eq. (5.15)) that $\sum(L') = \sum(L)\sum(\mathcal{O}_{+1})$, and

$$\sum(\mathcal{O}_{+1}) = \sum_{\lambda \in I} d_\lambda \chi_\lambda(v^{-1}) sdim_q(V_\lambda) = 1,$$

Figure 5.24: The unknots $\mathcal{O}_{+1}$ and $\mathcal{O}_{-1}$

where we have used the relation $f_{0\lambda} = sdim_q(V_\lambda)/sdim_q(V_0) = sdim_q(V_\lambda)$. As L' is a split link, $A_{L'}$ is the $(m+1)\times(m+1)$ matrix $A_{L'} = \big((1)\oplus A_L\big) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & A_L & \\ 0 & & & \end{pmatrix}$, thus $\sigma(A_{L'}) = \sigma(A_L)$ and $\mathcal{F}(M_L) = \mathcal{F}(M_{L'})$.

We now show that $\mathcal{F}(M_L) = \mathcal{F}(M_{\widetilde{L}})$ if the two links L and $\widetilde{L}$ are equivalent with respect to the $\kappa_-^{(0)}$ move. Let $\mathcal{O}_{-1}$ denote the unknot with framing -1 given in Figure 5.24 and let $\widetilde{L} = L \cup \mathcal{O}_{-1}$ be a split link, then

$$\sum(\widetilde{L}) = \sum(L)\sum(\mathcal{O}_{-1}) = \sum(L)\sum_{\lambda\in I} d_\lambda \chi_\lambda(v) sdim_q(V_\lambda) = z\sum(L).$$

Now $A_{\widetilde{L}}$ is an $(m+1)\times(m+1)$ matrix given by $A_{\widetilde{L}} = \big((-1)\oplus A_L\big) = \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & A_L & \\ 0 & & & \end{pmatrix}$, thus $\sigma(A_{\widetilde{L}}) = \sigma(A_L) + 1$ and $\mathcal{F}(M_L) = \mathcal{F}(M_{\widetilde{L}})$.

It remains to show that $\mathcal{F}(M_L) = \mathcal{F}(M_{L'})$ if the two links L and L' are equivalent with respect to the κ_+ move. Let L and L' be the two framed oriented links presented in the blackboard framing in Figure 5.23. The planar projections of L and L' are equivalent with respect to the κ_+ move. In Figure 5.23, T is an arbitrary oriented (m,m)-tangle represented by a rectangle, the orientation of which is compatible with the orientations of $L\backslash T$ and $L'\backslash T$. From Proposition 5.5.2, $\sum(L') = \sum(L)$. We now show that $\sigma(A_{L'}) = \sigma(A_L)$: A_L is an $m\times m$ matrix and $A_{L'}$ is an $(m+1)\times(m+1)$ matrix both of which we give below. In these matrices, A_T is the $m\times m$ linking matrix of T. We have

$$A_L = \begin{pmatrix} -1 & -1 & -1 & \cdots & -1 \\ -1 & -1 & -1 & \cdots & -1 \\ -1 & -1 & -1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & -1 \end{pmatrix} + A_T,$$

and

$$A_{L'} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} + \big((0) \oplus A_T\big),$$

where $\big((0) \oplus A_T\big) = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & A_T & \\ 0 & & & \end{pmatrix}$ is an $(m+1) \times (m+1)$-matrix. Let

$$X = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 0 \end{pmatrix} + \big((0) \oplus I_m\big),$$

then

$$XA_{L'}X^T = \big((1) \oplus A_L\big) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & A_L & \\ 0 & & & \end{pmatrix},$$

where X^T is the transpose of X and I_m is the $m \times m$ identity matrix. Now the matrices $A_{L'}$ and $\big((1) \oplus A_L\big)$ are real and symmetric. As X is invertible, $A_{L'}$ and $\big((1) \oplus A_L\big)$ are congruent. Congruent real symmetric matrices have the same numbers of positive, zero and negative eigenvalues [Fi66, Thms. 8.7, 8.9], thus $\sigma(A_{L'}) = \sigma(A_L)$ and $\mathcal{F}(M_L) = \mathcal{F}(M_{L'})$.

It follows that $\mathcal{F}(M_L) = \mathcal{F}(M_{\mathcal{L}})$ if the two links L and $\mathcal{L}$ are equivalent with respect to any of the Kirby moves, and thus $\mathcal{F}(M_L)$ is indeed a topological invariant of M_L. □

Note from our definition that $\mathcal{F}(M_L)$ is normalised to 1 on S^3.

Explicitly calculating $\mathcal{F}(M_L)$ for a particular closed, connected, orientable 3-manifold M_L is quite a difficult problem in general, and this is true for all the Reshetikhin-Turaev topological invariants. While topological invariants have been calculated for various classes of 3-manifolds (eg for the Lens spaces from quotients of the quantum algebras arising from the A, B, C, and D series of Lie algebras [ZC96] and the G_2, F_4, E_8 Lie algebras [Zh97] at odd roots of unity), we are only aware of one other collection of invariants that have been explicitly calculated for a substantial number of 3-manifolds [KL94, Chap. 14]. These invariants were constructed from $U_q^{(N)}(sl_2)$ where $N \equiv 0 \pmod 4$.

5.6 Invariants of 3-manifolds from $U_q^{(N)}(osp(1|2n))$

The following theorem is one of the main results of this thesis.

Theorem 5.6.1. *Set $q = \exp(2\pi i/N)$, where $N \geq 6$ satisfies $N \equiv 2 \pmod 4$. For the $\mathbb{Z}_2$-graded ribbon Hopf algebra $U_q^{(N)}(\mathfrak{g}) = U_q^{(N)}(osp(1|2n))$, fix the set of non-isomorphic $U_q^{(N)}(\mathfrak{g})$-modules $\{V_\lambda | \ \lambda \in \Lambda_N^+\}$ defined in Definition 4.2.5 by:*

$$V_\lambda = \tilde{p}_i^t[\lambda]\big(V^{\otimes t}\big), \tag{5.21}$$

where $t \in \mathbb{N}$ is given in Definition 4.2.5 and $\tilde{p}_i^t[\lambda] \in End_{U_q^{(N)}(\mathfrak{g})}(V^{\otimes t})$ is defined in Eq. (4.11). Fix an involution $* : \Lambda_N^+ \to \Lambda_N^+$ by $* = id_{\Lambda_N^+}$ and define a set of constants $\{d_\lambda \in \mathbb{C} | \ \lambda \in \Lambda_N^+\}$ by

$$d_\lambda = d_0 sdim_q(V_\lambda), \tag{5.22}$$

where $d_0 = \Omega Q(0)$ and

$$\Omega = \frac{2^n t^n q^{n^3 - n/2}}{\Big[(1+i)\sqrt{N}\Big]^n}, \qquad t = e^{\pi i/2N}, \tag{5.23}$$

$$Q(0) = \prod_{\alpha \in \overline{\Phi}_0^+} \left(q^{(\alpha,\rho)} - q^{-(\alpha,\rho)}\right) \prod_{\beta \in \Phi_1^+} \left(q^{(\beta,\rho)} + q^{-(\beta,\rho)}\right). \tag{5.24}$$

Let $z = (-i)^n q^{2n^3 - n} t^{2n}$. Let $L \subset S^3$ be a framed unoriented link with m connected components and let M_L be the closed, connected, orientable 3-manifold obtained by performing surgery on S^3 along L. Let A_L be the linking matrix of L and let $\sigma(A_L)$ denote the number of non-positive eigenvalues of A_L. Then

$$\mathcal{F}(M_L) = z^{-\sigma(A_L)} \sum_{\lambda \in \mathcal{C}(L, \Lambda_N^+)} \prod_{i=1}^m d_{\lambda_i} F\big(\Gamma(L, \lambda)\big),$$

is a topological invariant of M_L.

Proof. This is an immediate consequence of Theorems 5.5.1 and 5.6.2. □

Theorem 5.6.2. *Let $N \geq 6$ satisfy $N \equiv 2 \pmod 4$, then $U_q^{(N)}(osp(1|2n))$ and the following data give rise to a pseudo-modular Hopf algebra:*

(i) the set $\{V_\lambda | \ \lambda \in \Lambda_N^+\}$ of non-isomorphic $U_q^{(N)}(osp(1|2n))$-modules given in (5.21),

(ii) an involution $* : \Lambda_N^+ \to \Lambda_N^+$ defined by $* = id$,

(iii) an isomorphism $\omega : V^* \to V$ where we fix ω^{-1} to be the bijective homogeneous map of degree 0, $T \in End_{U_q^{(N)}(\mathfrak{g})}(V, V^*)$, defined in Eq. (3.5),

(iv) a set of constants $\{d_\lambda \in \mathbb{C} | \ \lambda \in \Lambda_N^+\}$ defined by Eq. (5.22).

Remark 5.6.1. *Unfortunately $U_q^{(N)}(\mathfrak{g})$ does not have a pseudo-modular Hopf algebra structure when $N \equiv 0 \pmod 4$. See Theorem 5.8.1 for details.*

We now prove that all the six axioms of pseudo-modular Hopf algebras are satisfied by $U_q^{(N)}(osp(1|2n))$ when $N \geq 6$ satisfies $N \equiv 2 \pmod 4$.

5.6.1 Proof of Axioms (I)–(IV)

In the proof of this theorem we let $N \geq 6$ satisfy $N \equiv 2 \pmod 4$.

(i) The proof that Axiom (I) is satisfied is contained in Chapter 4.

(ii) To prove that Axiom (II) is satisfied, we consider all of the $U_q^{(N)}(\mathfrak{g})$-modules V_λ, $\lambda \in \Lambda_N^+$, given in the data above, as being defined by $V_\lambda = \tilde{p}_i^t[\lambda](V^{\otimes t})$ for some t.

For each such module, there exists an irreducible $U_q(\mathfrak{g})$-module V_λ^{gen} with integral dominant highest weight λ, for all non-zero q that are not roots of unity, defined by $V_\lambda^{gen} = \tilde{p}_i^t[\lambda]^{gen}(V^{gen})^{\otimes t}$, where $\tilde{p}_i^t[\lambda]^{gen}$ is an element of the algebra $\mathcal{C}_t^{gen}$ over $\mathbb{C}$ generated by $\{\check{\mathcal{R}}_i \in End_{U_q(\mathfrak{g})}(V^{gen})^{\otimes t} | \ i = 1, \ldots, t-1\}$.

Recall from the proof of Lemma 4.2.3 that we obtain the matrix $\tilde{p}_i^t[\lambda]$ by fixing q to the appropriate root of unity in the matrix $\tilde{p}_i^t[\lambda]^{gen}$. We wish to show that the even central element $v \in U_q^{(N)}(\mathfrak{g})$ acts as a scalar on the $U_q^{(N)}(\mathfrak{g})$-module V_λ. To do this, recall from Lemma 3.4.3 that there exists an invertible even element $v_\lambda \in \overline{U}_q^+(\mathfrak{g})$ that acts as the scalar $q^{-(\lambda+2\rho,\lambda)}$ (here q is generic) on the finite dimensional irreducible $U_q(\mathfrak{g})$-module V_λ^{gen}. Furthermore, $\Delta^{(t-1)}(v_\lambda)$ acts as the same scalar on the finite dimensional irreducible $U_q(\mathfrak{g})$-module $V_\lambda^{gen} \subseteq V^{\otimes t}$ defined by $\tilde{p}_i^t[\lambda]^{gen} : (V^{gen})^{\otimes t} \to V_\lambda^{gen}$.

Consider the matrix $\pi^{\otimes t}\big(\Delta^{(t-1)}(v)\big)$ where $v \in U_q^{(N)}(\mathfrak{g})$. We can obtain this by specialising q to the appropriate root of unity in $(\pi^{gen})^{\otimes t}\big(\Delta^{(t-1)}(v_\lambda)\big)$, where $v_\lambda \in \overline{U}_q^+(\mathfrak{g})$. Thus $\pi^{\otimes t}\big(\Delta^{(t-1)}(v)\big)$ also acts as a scalar on V_λ, and this scalar is precisely $q^{-(\lambda+2\rho,\lambda)}$ where $q = \exp(2\pi i/N)$.

Now let $\Gamma(L,\lambda)$ be a coloured directed $(1,1)$-ribbon tangle with m components, and let the ends of $\Gamma(L,\lambda)$ be directed downwards and lie in a component coloured with $i \in I$. We want to show that the map $F\big(\Gamma(L,\lambda)\big) : V_i \to V_i$ acts as the multiplication by a scalar. Needless to say, $F\big(\Gamma(L,\lambda)\big)$ can be expressed in terms of the universal R-matrix of $U_q^{(N)}(\mathfrak{g})$.

We embed V_i in the $U_q^{(N)}(\mathfrak{g})$-module $V^{\otimes t}$. As $\tilde{p}_i^t[\lambda]$ belongs to $\mathcal{C}_t$, under the embedding $F\big(\Gamma(L,\lambda)\big)$ becomes an element of $\mathcal{C}_t$.

To determine the action of $F\big(\Gamma(L,\lambda)\big)$ on $V_i \subseteq V^{\otimes t}$, we take the element $f^{gen} \in \mathcal{C}_t^{gen}$ that corresponds to $F\big(\Gamma(L,\lambda)\big)$ at all generic q. Note that we obtain $F\big(\Gamma(L,\lambda)\big)$ by specialising q to $\exp(2\pi i/N)$ in f^{gen}. At all generic q, f^{gen} acts as the multiplication by a scalar on the irreducible $U_q(\mathfrak{g})$-module $V_i^{gen} \subseteq (V^{gen})^{\otimes t}$. We then just take the limit of f^{gen} as q goes to $\exp(2\pi i/N)$.

(iii) To prove that Axiom (III) is satisfied, firstly note that $V_\lambda \cong \tilde{p}_i^t[\lambda]\big(V^{\otimes t}\big)$ for some t for each $\lambda \in \Lambda_N^+$. Also note that $\lambda^* = \lambda$ for each $\lambda \in \Lambda_N^+$. Thus we need only show that Axiom (III) is satisfied for the map $\omega^* \circ \omega^{-1} : V \to (V^*)^*$, where $\omega : V^* \to V$ and $\omega^* : V^* \to (V^*)^*$ are isomorphisms. Note that we fix $\lambda^* = \lambda$ as $(V_\lambda)^* \cong V_\lambda$ for each $\lambda \in \Lambda_N^+$, which is proved in Chapter 4.

Let $\{v_i| -n \leq i \leq n\}$ be the basis of V given in Lemma 4.1.1 and let $\{v_i^*\}$ and $\{v_i^{**}\}$ be bases of V^* and $(V^*)^*$, respectively, such that $\langle\langle v_i^{**}, v_j^*\rangle\rangle = \langle v_j^*, v_i\rangle = \delta_{i,j}$ and $[v_i^{**}] = [v_i^*] = [v_i]$ for each i, where $\langle\langle\cdot,\cdot\rangle\rangle : (V^*)^* \times V^* \to \mathbb{C}$ is the dual space pairing. We want to show that the map $\omega^* \circ \omega^{-1} : V \to (V^*)^*$ has the following action:

$$(\omega^* \circ \omega^{-1})(v_i) = (-1)^{[v_i]} uv^{-1} v_i^{**}, \qquad -n \leq i \leq n. \tag{5.25}$$

For $U_q^{(N)}(\mathfrak{g})$, $uv^{-1} = K_{2\rho}$, with $K_{2\rho}$ a product of K_i's satisfying $K_{2\rho} e_i K_{2\rho}^{-1} = q^{(2\rho,\alpha_i)} e_i$, $\forall i$, where $2\rho = \sum_{i=1}^n (2n-2i+1)\epsilon_i$. The isomorphism $\omega^{-1} : V \to V^*$ is precisely the bijective map $T \in End_{U_q^{(N)}(\mathfrak{g})}(V, V^*)$ defined in Eq. (3.5). The map $\omega^* : V^* \to (V^*)^*$ is given by

$$\begin{array}{ll} v_i^* \mapsto (-1)^{i-1} q^{i-1} v_{-i}^{**}, & v_0^* \mapsto (-1)^{n-1} q^n v_0^{**}, \\ v_{-i}^* \mapsto (-1)^i q^{2n-i} v_i^{**}, & 1 \leq i \leq n. \end{array}$$

Thus

$$(\omega^* \circ \omega^{-1}) : v_i \mapsto -q^{2n-2i+1} v_i^{**}, \qquad v_0 \mapsto v_0^{**}, \qquad v_{-i} \mapsto -q^{-(2n-2i+1)} v_{-i}^{**}, \quad 1 \leq i \leq n,$$

which shows that (5.25) is true for all i.

(iv) Axiom (IV) directly follows from the tensor product theorems: Theorems 4.3.2 and 4.3.3.

5.6.2 Proof of Axiom (V)

We now find a set of constants $\{d_\lambda \in \mathbb{C}|\ \lambda \in \Lambda_N^+\}$ satisfying the relations (5.14). To do this, we need first to compute the $f_{\mu\lambda}$ for all $\mu, \lambda \in \Lambda_N^+$.

Recall Lemma 5.3.1. Because of part (ii) in Subsection 5.6.1, v^{-1} acts on V_λ as the multiplication by the scalar $\chi_\lambda(v^{-1}) = q^{(\lambda+2\rho,\lambda)}$ and C_λ also acts on V_μ as the multiplication by a scalar which we denote by $\chi_\mu(C_\lambda)$. It immediately follows that

$$f_{\mu\lambda} = \chi_\mu(C_\lambda) sdim_q(V_\mu), \qquad \forall \mu, \lambda \in \Lambda_N^+.$$

Using calculations similar to those in Subsection 5.3.1 and [ZG91, Lem. 2], we obtain

$$\chi_\mu(C_\lambda) = \mathrm{str}\big(\pi_\lambda(K_{2\mu+2\rho})\big),$$

where π_λ is the representation of $U_q^{(N)}(\mathfrak{g})$ furnished by the module V_λ. Here $K_{2\mu+2\rho}$ is a product of K_i's such that $K_{2\mu+2\rho} e_i K_{2\mu+2\rho}^{-1} = q^{2(\mu+\rho,\alpha_i)} e_i$, $\forall i$.

Now V_λ can always be embedded in $V^{\otimes t}$ as a direct summand. Let $\tilde{p}_i^t[\lambda]$ be the projection operator mapping $V^{\otimes t}$ onto V_λ. Then

$$\chi_\mu(C_\lambda) = str_{V^{\otimes t}}\big(K_{2\mu+2\rho} \cdot \tilde{p}_i^t[\lambda]\big).$$

As we have argued repeatedly, the right hand side can be obtained by first evaluating the corresponding quantity at generic q, then specialising q to a root of unity. This way we obtain, with the help of the supercharacter formula for irreducible $osp(1|2n)$-representations,

$$\chi_\mu(C_\lambda) = (-1)^{[\lambda]} \frac{\sum_{\sigma\in\mathcal{W}} \epsilon'(\sigma) q^{(2\mu+2\rho,\sigma(\lambda+\rho))}}{\sum_{\sigma\in\mathcal{W}} \epsilon'(\sigma) q^{(2\mu+2\rho,\sigma(\rho))}}, \qquad \lambda,\mu\in\Lambda_N^+, \tag{5.26}$$

where $\mathcal{W}$ is the Weyl group of $osp(1|2n)$. See Appendix D for further information. It is important to observe that for all $\mu\in\Lambda_N^+$, the denominator of the above formula is always non-zero.

For convenience, let us introduce some notation. Recall that $X = \bigoplus_{i=1}^n \mathbb{Z}\epsilon_i \subset H^*$. Let $S : X\times X \to \mathbb{C}$ and $Q : X \to \mathbb{C}$ be two mappings defined by

$$(\lambda,\mu) \mapsto S_{\lambda,\mu} = (-1)^{[\lambda]} \sum_{\sigma\in\mathcal{W}} \epsilon'(\sigma) q^{2(\lambda+\rho,\sigma(\mu+\rho))}, \tag{5.27}$$

$$\mu \mapsto Q(\mu) = \sum_{\sigma\in\mathcal{W}} \epsilon'(\sigma) q^{2(\mu+\rho,\sigma(\rho))}. \tag{5.28}$$

Then we have the following result.

Lemma 5.6.1. *$S_{\lambda,\mu}$ has the following properties:*

(i) $S_{\mu,\lambda} = (-1)^{[\lambda]+[\mu]} S_{\lambda,\mu}$,

(ii) $S_{\lambda,w(\mu+\rho)-\rho} = \epsilon'(w) S_{\lambda,\mu}$, $w\in\mathcal{W}$.

Proof. (i) $S_{\mu,\lambda} = (-1)^{[\mu]} \sum_{\sigma\in\mathcal{W}} \epsilon'(\sigma) q^{2(\mu+\rho,\sigma(\lambda+\rho))} = (-1)^{[\lambda]+[\mu]} S_{\lambda,\mu}$.

(ii) $S_{\lambda,w(\mu+\rho)-\rho} = (-1)^{[\lambda]} \sum_{\sigma\in\mathcal{W}} \epsilon'(\sigma) q^{2(\sigma(\lambda+\rho),w(\mu+\rho))} = (-1)^{[\lambda]}\epsilon'(w) \sum_{\sigma\in\mathcal{W}} \epsilon'(\sigma) q^{2(\sigma(\lambda+\rho),\mu+\rho)}$.

□

Restricting the domain of S to $\Lambda_N^+\times\Lambda_N^+$ and the domain of Q to Λ_N^+ gives mappings with non-empty image. We can rewrite (5.14) as

$$\chi_\mu(v) = \sum_{\lambda\in\Lambda_N^+} d_\lambda \chi_\lambda(v^{-1}) \left(S_{\lambda,\mu}/Q(\mu)\right), \qquad \forall\mu\in\Lambda_N^+. \tag{5.29}$$

The invertibility of the matrix $\left(S_{\lambda,\mu}/Q(\mu)\right)_{\lambda,\mu\in\Lambda_N^+}$ is unknown. However, $U_q^{(N)}(\mathfrak{g})$ may still satisfy Axiom (V) of a pseudo-modular Hopf algebra. This is the advantage of working with pseudo-modular Hopf algebras.

We shall now work towards proving Theorem 5.6.2. Fix $N\equiv 2 \pmod 4$ where $N\geq 6$, and also define $S'_{\lambda,\mu} = (-1)^{[\lambda]} S_{\lambda,\mu}$. We will solve the following set of linear equations for the constants $\{d'_\lambda\in\mathbb{C}|\ \lambda\in\Lambda_N^+\}$ where $d'_\lambda = (-1)^{[\lambda]} d_\lambda$:

$$Q(\mu)\chi_\mu(v) = \sum_{\lambda\in\Lambda_N^+} d'_\lambda \chi_\lambda(v^{-1}) S'_{\lambda,\mu}, \qquad \forall\mu\in\Lambda_N^+. \tag{5.30}$$

Note that $Q(\mu) \neq 0$ for all $\mu \in \Lambda_N^+$ and the set (5.30) of equations is identical to the set (5.29) of equations.

Define $X_N = X/NX$ and let $p : X \to X_N$ be the canonical projection defined by

$$p(\nu) = \sum_{i=1}^{n} \nu_i \epsilon_i + NX, \qquad \nu = \sum_{i=1}^{n} \nu_i \epsilon_i \in X.$$

We distinguish between elements of X and X_N by writing $\lambda \in X$ and $\overline{\lambda} = p(\lambda) \in X_N$. Note that $q^{(\lambda,\lambda+2\rho)}$ makes sense for all $\lambda \in X$. Furthermore, if $p(\lambda) = p(\mu)$, then $q^{(\lambda,\lambda+2\rho)} = q^{(\mu,\mu+2\rho)}$, thus we can regard $q^{(\lambda,\lambda+2\rho)}$ as a function $X_N \to \mathbb{C}$ defined by $p(\lambda) \mapsto q^{(\overline{\lambda},\overline{\lambda}+2\rho)}$, where to evaluate $q^{(\overline{\lambda},\overline{\lambda}+2\rho)}$ we take any representative $\lambda \in X$ of $\overline{\lambda}$ and calculate $q^{(\lambda,\lambda+2\rho)}$. Similarly we can regard $S'_{\lambda,\mu}$ as a function $X_N \times X_N \to \mathbb{C}$.

We will encounter expressions of the form $\sigma(\overline{\lambda})$, where $\sigma \in \mathcal{W}$, in our calculations. To be completely clear, we note that in writing $\sigma(\overline{\lambda})$ we actually mean $p\big(\sigma(\lambda)\big)$, where $\lambda \in X$ is any representative of $\overline{\lambda}$ in X_N.

Following [ZC96], we introduce the auxiliary equations:

$$Q(\overline{\mu})\chi_{\overline{\mu}}(v) = \sum_{\overline{\lambda} \in X_N} x_{\overline{\lambda}} \chi_{\overline{\lambda}}(v^{-1}) S'_{\overline{\lambda},\overline{\mu}}, \tag{5.31}$$

where $\overline{\mu}$ varies over all elements of $p(\Lambda_N^+)$ and we choose $\overline{\mu}$ to be the representative of $p(\mu)$ in Λ_N^+. Note that the sum in the right hand side of (5.31) is over all distinct elements of X_N.

We would like to solve (5.31) for the $x_{\overline{\lambda}}$. Let us try the following solution: $x_{\overline{\lambda}} \equiv cq^{-(\overline{\lambda},2\rho)}$, where $c \in \mathbb{C}$ is some nonzero constant yet to be determined. Substituting this into (5.31) gives

$$Q(\overline{\mu})q^{-(\overline{\mu},\overline{\mu}+2\rho)} = \sum_{\overline{\lambda} \in X_N} cq^{(\overline{\lambda},\overline{\lambda})} \sum_{\sigma \in \mathcal{W}} \epsilon'(\sigma) q^{2\left(\overline{\lambda}+\rho,\sigma(\overline{\mu}+\rho)\right)}. \tag{5.32}$$

We now aim to obtain an expression for c independent of $\overline{\mu}$. To help do this, we apply the following mapping to each term on the right hand side of (5.32) for each fixed pair $(\overline{\mu}, \sigma) \in p(\Lambda_N^+) \times \mathcal{W}$:

$$\overline{\lambda} \mapsto \sigma(\overline{\lambda}) - \sigma(\overline{\mu}) \in X_N.$$

Lemma 5.6.2. *Let $(\mu, \sigma) \in \Lambda_N^+ \times \mathcal{W}$ be a fixed pair. Then for any λ, $\lambda' \in X$,*

$$p\big(\sigma(\lambda) - \sigma(\mu)\big) = p\big(\sigma(\lambda') - \sigma(\mu)\big) \quad \text{if and only if} \quad p(\lambda) = p(\lambda'). \tag{5.33}$$

Proof. Write $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_n)$. Fix $\sigma \in \mathcal{W}$ and assume that $p(\lambda) = p(\lambda')$, then

$$\lambda_i \equiv \lambda'_i \pmod{N}, \quad \text{for each } i = 1, 2, \ldots, n,$$

and $\big(\sigma(\lambda)\big)_i \equiv \big(\sigma(\lambda')\big)_i \pmod{N}$ for all $i = 1, 2, \ldots, n$. It follows that $\big(\sigma(\lambda) - \sigma(\mu)\big)_i \equiv \big(\sigma(\lambda') - \sigma(\mu)\big)_i \pmod{N}$ for all $i = 1, 2, \ldots, n$, and that

$$p\big(\sigma(\lambda) - \sigma(\mu)\big) = p\big(\sigma(\lambda') - \sigma(\mu)\big).$$

Now let us assume that $p\big(\sigma(\lambda)-\sigma(\mu)\big)=p\big(\sigma(\lambda')-\sigma(\mu)\big)$, then clearly

$$\big(\sigma(\lambda)-\sigma(\mu)\big)_i \equiv \big(\sigma(\lambda')-\sigma(\mu)\big)_i \pmod N, \qquad i=1,2,\dots,n,$$

and

$$\big(\sigma(\lambda)\big)_i \equiv \big(\sigma(\lambda')\big)_i \pmod N, \qquad i=1,2,\dots,n. \tag{5.34}$$

From Eq. (5.34) we obtain

$$\lambda_i \equiv \lambda'_i \pmod N, \qquad i=1,2,\dots,n,$$

and thus $p(\lambda)=p(\lambda')$ as desired. □

It follows from Lemma 5.6.2 that for each fixed pair $(\overline{\mu},\sigma)\in p(\Lambda_N^+)\times\mathcal{W}$, the following component of the right hand side of (5.32) is invariant under the mapping $\overline{\lambda}\mapsto\sigma(\overline{\lambda})-\sigma(\overline{\mu})\in X_N$:

$$c\epsilon'(\sigma)\sum_{\overline{\lambda}\in X_N} q^{(\overline{\lambda},\overline{\lambda})}q^{2(\overline{\lambda}+\rho,\sigma(\overline{\mu}+\rho))}. \tag{5.35}$$

Applying this mapping to (5.35) gives us

$$\begin{aligned}
&c\epsilon'(\sigma)\sum_{\overline{\lambda}\in X_N} q^{\left(\sigma(\overline{\lambda})-\sigma(\overline{\mu}),\sigma(\overline{\lambda})-\sigma(\overline{\mu})\right)}q^{2\left(\sigma(\overline{\lambda})-\sigma(\overline{\mu})+\rho,\sigma(\overline{\mu}+\rho)\right)}\\
&\qquad = c\epsilon'(\sigma)\sum_{\overline{\lambda}\in X_N} q^{(\overline{\lambda},\overline{\lambda})-2(\overline{\lambda},\overline{\mu})+(\overline{\mu},\overline{\mu})}q^{2(\overline{\lambda},\overline{\mu}+\rho)-2(\overline{\mu},\overline{\mu}+\rho)}q^{2(\rho,\sigma(\overline{\mu}+\rho))}\\
&\qquad = c\epsilon'(\sigma)\sum_{\overline{\lambda}\in X_N} q^{(\overline{\lambda},\overline{\lambda}+2\rho)-(\overline{\mu},\overline{\mu}+2\rho)}q^{2(\rho,\sigma(\overline{\mu}+\rho))},
\end{aligned}$$

which allows us to rewrite (5.31) as

$$q^{-(\overline{\mu},\overline{\mu}+2\rho)}\sum_{\sigma\in\mathcal{W}}\epsilon'(\sigma)q^{2(\rho,\sigma(\overline{\mu}+\rho))} = c\sum_{\overline{\lambda}\in X_N} q^{(\overline{\lambda},\overline{\lambda}+2\rho)}q^{-(\overline{\mu},\overline{\mu}+2\rho)}\sum_{\sigma\in\mathcal{W}}\epsilon'(\sigma)q^{2(\rho,\sigma(\overline{\mu}+\rho))},$$

for each $\overline{\mu}\in p(\Lambda_N^+)$. As $Q(\overline{\mu})\neq 0$ for each $\overline{\mu}\in p(\Lambda_N^+)$, we easily obtain the following expression for c^{-1}:

$$c^{-1}=\sum_{\overline{\lambda}\in X_N} q^{(\overline{\lambda},\overline{\lambda}+2\rho)}, \tag{5.36}$$

which satisfies $|c^{-1}|=(2N)^{n/2}$, as is shown presently. We can rewrite the sum in (5.36) as

$$\sum_{\overline{\lambda}\in X_N} q^{(\overline{\lambda},\overline{\lambda}+2\rho)} = \prod_{k=0}^{n-1} G_+(N,2k+1) = \frac{\left[(1+i)\sqrt{N}\right]^n}{t^n}\left(\prod_{k=0}^{n-1}\frac{1}{q^{k(k+1)}}\right), \qquad t=\exp(\pi i/2N), \tag{5.37}$$

where $G_+(N,m)$ is a Gaussian sum defined in Appendix A. In proving (5.37), we used Lemma A.2.3 and the fact that $t^{(2k+1)^2} = tq^{k(k+1)}$. By inspection, the modulus of the far right hand side of (5.37) is $(2N)^{n/2}$, and we obtain the well defined expression:

$$x_{\overline{\lambda}} = \frac{q^{-(\overline{\lambda},2\rho)} t^n \left(\prod_{k=0}^{n-1} q^{k(k+1)}\right)}{\left[(1+i)\sqrt{N}\right]^n}. \tag{5.38}$$

Now consider the right hand side of (5.31):

$$\sum_{\overline{\lambda}\in X_N} x_{\overline{\lambda}} q^{(\overline{\lambda},\overline{\lambda}+2\rho)} S'_{\overline{\lambda},\overline{\mu}}. \tag{5.39}$$

Recall that we use N' to mean $N/2$. We will rewrite (5.39) as a sum over all the elements of $X_{N'} = X/N'X$ and then we will use certain results in [ZC96] to express (5.39) as a sum over all the elements of $\overline{\Lambda_N^+}$. We now consider some useful calculations. Let $\overline{\lambda}' = \overline{\lambda} + N'\epsilon_i$ for some $i \in \{1,2,\ldots,n\}$, then

$$q^{(\overline{\lambda}',\overline{\lambda}'+2\rho)} = q^{(\overline{\lambda}+N'\epsilon_i,\overline{\lambda}+N'\epsilon_i+2\rho)} = q^{(\overline{\lambda},\overline{\lambda}+2\rho)+N'(2\overline{\lambda}_i+2n-2i+1+N')}.$$

As $N \equiv 2 \pmod 4$, this equation leads to

$$q^{(\overline{\lambda}',\overline{\lambda}'+2\rho)} = q^{(\overline{\lambda},\overline{\lambda}+2\rho)}. \tag{5.40}$$

Thus

$$x_{\overline{\lambda}'} = \frac{q^{-(\overline{\lambda}+N'\epsilon_i,2\rho)} t^n \left(\prod_{k=0}^{n-1} q^{k(k+1)}\right)}{\left[(1+i)\sqrt{N}\right]^n} = \frac{-q^{-(\overline{\lambda},2\rho)} t^n \left(\prod_{k=0}^{n-1} q^{k(k+1)}\right)}{\left[(1+i)\sqrt{N}\right]^n} = -x_{\overline{\lambda}}, \tag{5.41}$$

and furthermore

$$\begin{aligned} S'_{\overline{\lambda}',\overline{\mu}} &= \sum_{\sigma\in\mathcal{W}} \epsilon'(\sigma) q^{2(\overline{\lambda}+N'\epsilon_i+\rho,\sigma(\overline{\mu}+\rho))} \\ &= \sum_{\sigma\in\mathcal{W}} \epsilon'(\sigma) q^{2(\overline{\lambda}+\rho,\sigma(\overline{\mu}+\rho))+N'(\epsilon_i,\sigma(2\overline{\mu}+2\rho))} = -S'_{\overline{\lambda},\overline{\mu}}. \end{aligned} \tag{5.42}$$

Using Eqs. (5.41)–(5.42), we obtain

$$x_{\overline{\lambda}'} q^{(\overline{\lambda}',\overline{\lambda}'+2\rho)} S'_{\overline{\lambda}',\overline{\mu}} = x_{\overline{\lambda}} q^{(\overline{\lambda},\overline{\lambda}+2\rho)} S'_{\overline{\lambda},\overline{\mu}}. \tag{5.43}$$

Note that $|X_N| = 2^n|X_{N'}|$, then (5.40) and (5.43) allow us to rewrite (5.39) as

$$\sum_{\overline{\lambda}\in X_N} x_{\overline{\lambda}} q^{(\overline{\lambda},\overline{\lambda}+2\rho)} S'_{\overline{\lambda},\overline{\mu}} = 2^n \sum_{\overline{\lambda}\in X_{N'}} x_{\overline{\lambda}} q^{(\overline{\lambda},\overline{\lambda}+2\rho)} S'_{\overline{\lambda},\overline{\mu}}. \tag{5.44}$$

We now aim to write the summation in the right hand side of (5.44) in terms of a summation over the elements of $\overline{\Lambda_N^+}$, and in doing this we will follow the general approach of [ZC96], which studied a similar problem for ordinary quantum groups at odd roots of unity. In particular, we consider the analogous problem for $U_q^{(N)}(so(2n+1))$. Consider the affine Weyl group $\mathcal{W}_M^{\mathfrak{g}}$ of a classical Lie algebra $\mathfrak{g}$, where $M \geq 3$ is an integer, generated by the maps $S_{\alpha,kM} : X \to X$, $\alpha \in \Phi_{\mathfrak{g}}^+$, $k \in \mathbb{Z}$, where $\Phi_{\mathfrak{g}}^+$ is the set of positive roots of $\mathfrak{g}$, and where the action of $S_{\alpha,kM}$ on $\mu \in H^*$ is defined by

$$S_{\alpha,kM} : \mu \mapsto \sigma_\alpha(\mu+\rho) - \rho + kM\alpha.$$

We derive the following from [Jan87, Sec. 6.2].

Remark 5.6.2. *Set $M \geq 3$ to be odd, then the affine Weyl group $\mathcal{W}_M^{\mathfrak{g}}$ of a classical Lie algebra $\mathfrak{g}$ acts on the chambers, that is the open connected components of*

$$X - \bigcup_{\alpha\in\Phi_{\mathfrak{g}}^+} \bigcup_{n\in\mathbb{Z}} \left\{ \mu \in X \middle| \, \frac{2(\mu+\rho,\alpha)}{(\alpha,\alpha)} = nM \right\},$$

transitively, with a fundamental domain

$$\left\{ \mu \in X \middle| \, 0 \leq \frac{2(\mu+\rho,\alpha)}{(\alpha,\alpha)} \leq M, \ \forall \alpha \in \Phi_{\mathfrak{g}}^+ \right\}. \tag{5.45}$$

Proposition 5.6.1. *Set $N \equiv 2 \pmod 4$, $N \geq 6$. The truncated Weyl alcove $\overline{\Lambda_N^+} \subset X$ given in Definition 4.2.1:*

$$\overline{\Lambda_N^+} = \left\{ \lambda \in X \middle| \, 0 \leq \frac{2(\lambda+\rho,\alpha)}{(\alpha,\alpha)} \leq N/2, \ \forall \alpha \in \overline{\Phi}_0^+ \cup \Phi_1^+ \right\}, \tag{5.46}$$

is identical to the fundamental domain of X under the action of the affine Weyl group $\mathcal{W}_{N/2}^{so(2n+1)}$ stated in (5.45).

Proof. The set of positive roots Φ^+ of $so(2n+1)$ is identical to the subset $\overline{\Phi}_0^+ \cup \Phi_1^+$ of positive roots of $osp(1|2n)$, and the expression for 2ρ in $so(2n+1)$ is given by

$$2\rho = \sum_{i<j} \left[(\epsilon_i - \epsilon_j) + (\epsilon_i + \epsilon_j) \right] + \sum_{k=1}^{n} \epsilon_k = \sum_{i=1}^{n} (2n-2i+1)\epsilon_i,$$

which is identical to the expression for 2ρ in $osp(1|2n)$. The result follows. □

Let $\beta \in H^*$ be arbitrary and let $\{s_\alpha | \ \alpha \in \Phi^+\}$ be a subset of elements of the Weyl group of $osp(1|2n)$. Now $s_{\epsilon_i}(\beta) = s_{2\epsilon_i}(\beta)$ for each $i = 1, \ldots, n$, and we identify $s_{2\epsilon_i}$ with s_{ϵ_i}, thus every element of $\mathcal{W}$ can be expressed as some ordered product of the elements of $\left\{ s_\alpha \in \mathcal{W} | \ \alpha \in \overline{\Phi}_0^+ \cup \Phi_1^+ \right\}$. As $\Phi_{so(2n+1)}^+ = \overline{\Phi}_0^+ \cup \Phi_1^+$, the Weyl groups of $osp(1|2n)$ and $so(2n+1)$ are identical.

We define the action of $S_{\alpha,kM} \in \mathcal{W}_M^{\mathfrak{g}}$ on an element of X_M by

$$S_{\alpha,kM}(\mu + MX) = S_{\alpha,kM}(\mu) + MX.$$

This coincides with the action of the Weyl group $\mathcal{W}^{\mathfrak{g}}$ on X_M defined by

$$\sigma(\mu + MX) = \sigma(\mu) + MX, \qquad \forall \sigma \in \mathcal{W}^{\mathfrak{g}},\ \mu \in X,$$

and we can deduce from this that the image of $\overline{\Lambda_N^+}$ under the canonical projection $p_{N/2} : X \to X_{N/2}$ furnishes a fundamental domain for X under the action of the Weyl group $\mathcal{W}$ [ZC96].

There is the following important result [ZC96]: for any $\lambda, \mu \in p_{N/2}(\Lambda_N^+) \subset X_{N/2}$ and any $\sigma, w \in \mathcal{W}^{so(2n+1)}$,

$$\sigma(\lambda + \rho) - \rho = w(\mu + \rho) - \rho \quad \text{iff} \quad \lambda = \mu \text{ and } \sigma = w. \tag{5.47}$$

This result also holds for any $\sigma, w \in \mathcal{W}^{osp(1|2n)}$ as $\mathcal{W}^{osp(1|2n)} = \mathcal{W}^{so(2n+1)}$.

In order to rewrite the right hand side of (5.44) in terms of a summation over the elements of $\overline{\Lambda_N^+}$ we will show that $S'_{\lambda,\mu} = 0$ if $\lambda \in \overline{\Lambda_N^+} \backslash \Lambda_N^+$ or if $\mu \in \overline{\Lambda_N^+} \backslash \Lambda_N^+$. The corresponding result is easily proved for $U_q^{(N/2)}(so(2n+1))$ [ZC96] but for $U_q^{(N)}(osp(1|2n))$ the proof is more intricate, principally due to the different properties of $\epsilon(\sigma)$ and $\epsilon'(\sigma)$ where σ is an element of $\mathcal{W}^{osp(1|2n)}$. Recall that $N \equiv 2 \pmod 4$ and that Λ_N^+ is defined by

$$\Lambda_N^+ = \left\{ \mu \in X \,\middle|\, 0 < \frac{2(\mu + \rho, \alpha)}{(\alpha, \alpha)} < N',\ \forall \alpha \in \overline{\Phi}_0^+ \cup \Phi_1^+ \right\},$$

which is similar to the definition of $\overline{\Lambda_N^+}$ in (5.46).

The following two important properties of $S'_{\lambda,\mu}$ are easily proved: for each $\lambda, \mu \in X$,

(i) $S'_{\lambda,\mu} = S'_{\mu,\lambda}$,

(ii) $S'_{s_\alpha(\lambda+\rho)-\rho,\mu} = \epsilon'(s_\alpha) S'_{\lambda,\mu}$ for any $s_\alpha \in \mathcal{W}$.

Lemma 5.6.3. *If $\lambda \in \overline{\Lambda_N^+} \backslash \Lambda_N^+$ or $\mu \in \overline{\Lambda_N^+} \backslash \Lambda_N^+$, $S'_{\lambda,\mu} = 0$.*

Proof. Define $hp_\alpha = \{\mu \in X |\ (\mu, \alpha) = 0\}$ for each $\alpha \in \overline{\Phi}_0^+ \cup \Phi_1^+$; hp_α is the subset of X invariant under the action of $s_\alpha \in \mathcal{W}$. For an element $\lambda \in \overline{\Lambda_N^+} \backslash \Lambda_N^+$, the definitions of $\overline{\Lambda_N^+}$ and Λ_N^+ imply that $\frac{2(\lambda+\rho,\alpha)}{(\alpha,\alpha)} = kN/2$ for some $k \in \mathbb{Z}_+$ and some $\alpha \in \overline{\Phi}_0^+ \cup \Phi_1^+$. For $\alpha = \epsilon_i \pm \epsilon_j$, where $1 \le i < j \le n$, we have $(\lambda + \rho, \alpha) = kN/2$, thus $(\lambda + \rho - kN\alpha/4, \alpha) = 0$, and $\lambda + \rho - kN\alpha/4 \in hp_\alpha$. Consequently, $s_\alpha(\lambda + \rho - kN\alpha/4) = \lambda + \rho - kN\alpha/4$, and $s_\alpha(\lambda + \rho) - \rho + kN\alpha/2 = \lambda$, and it follows that

$$S'_{\lambda,\mu} = S'_{s_\alpha(\lambda+\rho)-\rho+kN'\alpha,\mu} = S'_{s_\alpha(\lambda+\rho)-\rho,\mu} = \epsilon'(s_\alpha) S'_{\lambda,\mu},$$

which vanishes identically as $\epsilon'(s_\alpha) = -1$ for each $\alpha = \epsilon_i \pm \epsilon_j \in \overline{\Phi}_0^+$.

Now let $\lambda \in \overline{\Lambda_N^+}\backslash\Lambda_N^+$ and let $\alpha = \epsilon_i$, where $1 \leq i \leq n$, then $\frac{2(\lambda+\rho,\alpha)}{(\alpha,\alpha)} = kN/2$ for some $k \in \mathbb{Z}$. Consequently, we have $2(\lambda+\rho,\alpha) = kN/2$ which implies that $(\lambda+\rho-kN\alpha/4,\alpha) = 0$, and also that $\lambda+\rho-kN\alpha/4 \in hp_\alpha$. It follows that $s_\alpha(\lambda+\rho-kN\alpha/4) = \lambda+\rho-kN\alpha/4$, and that

$$s_\alpha(\lambda+\rho) - \rho + kN\alpha/2 = \lambda.$$

Consequently, we have

$$S'_{\lambda,\mu} = S'_{s_\alpha(\lambda+\rho)-\rho+kN'\alpha,\mu} = \epsilon'(s_\alpha)S'_{\lambda+kN'\alpha,\mu} = (-1)^k\epsilon'(s_\alpha)S'_{\lambda,\mu} = (-1)^k S'_{\lambda,\mu}, \tag{5.48}$$

where we have used the result $\epsilon'(s_\alpha) = 1$ as $\alpha \in \Phi_1^+$, and we have also used the following calculations:

$$\begin{aligned} S'_{\lambda+kN'\alpha,\mu} &= \sum_{\sigma\in\mathcal{W}} \epsilon'(\sigma) q^{2(\lambda+kN'\alpha+\rho,\sigma(\mu+\rho))} \\ &= \sum_{\sigma\in\mathcal{W}} \epsilon'(\sigma) q^{2(\lambda+\rho,\sigma(\mu+\rho))} q^{(kN'\alpha,\sigma(2\mu+2\rho))} = \begin{cases} S'_{\lambda,\mu}, & \text{if } k \text{ is even,} \\ -S'_{\lambda,\mu}, & \text{if } k \text{ is odd.} \end{cases} \end{aligned}$$

Here the last equality arises from the following calculation:

$$q^{(kN'\alpha,\sigma(2\mu+2\rho))} = \begin{cases} +1 & \text{if } k \text{ is even,} \\ -1 & \text{if } k \text{ is odd.} \end{cases}$$

If k is odd, $S'_{\lambda,\mu}$ vanishes identically by (5.48), and this completes the proof of the assertion that $S'_{\lambda,\mu} = 0$ if $\lambda \in \overline{\Lambda_N^+}\backslash\Lambda_N^+$. To show that k is indeed odd we note that the equation $(\lambda+\rho,\alpha) = kN/4$ implies that $\lambda_i + n - i + 1/2 = kN/4$, and thus we have $k \notin 2\mathbb{Z}$ as $\lambda \in X$. It follows then that $s_\alpha(\lambda+\rho) - \rho + kN\alpha/2 = \lambda$ for some odd k, and thus

$$S'_{\lambda,\mu} = -S'_{\lambda,\mu} = 0.$$

To complete the proof, we note that $S'_{\lambda,\mu} = S'_{\mu,\lambda}$, and thus $S'_{\lambda,\mu} = 0$ if $\mu \in \overline{\Lambda_N^+}\backslash\Lambda_N^+$. □

Corollary 5.6.1. *If $\overline{\lambda}$ or $\overline{\mu}$ belongs to $p_{N/2}\big(\overline{\Lambda_N^+}\backslash\Lambda_N^+\big) \subset X_{N/2}$, then $S'_{\overline{\lambda},\overline{\mu}} = 0$.*

Lemma 5.6.4. *The set $\{d_\lambda \in \mathbb{C}|\ \lambda \in \Lambda_N^+\}$ of constants defined by*

$$d_\lambda = d_0 sdim_q(V_\lambda), \qquad d_0 = \Omega Q(0), \tag{5.49}$$

with

$$\Omega = \frac{2^n t^n q^{n^3-n/2}}{\Big[(1+i)\sqrt{N}\Big]^n}, \tag{5.50}$$

$$Q(0) = \prod_{\alpha\in\overline{\Phi}_0^+} \big(q^{(\rho,\alpha)} - q^{-(\rho,\alpha)}\big) \prod_{\beta\in\Phi_1^+} \big(q^{(\rho,\beta)} + q^{-(\rho,\beta)}\big), \tag{5.51}$$

satisfies the relations (5.14).

Proof. Using (5.47) and Corollary 5.6.1, we can rewrite the right hand side of (5.44) as

$$\begin{aligned}2^n \sum_{\overline{\lambda}\in X_{N'}} x_{\overline{\lambda}} q^{(\overline{\lambda},\overline{\lambda}+2\rho)} S'_{\overline{\lambda},\overline{\mu}} &= 2^n \sum_{\lambda\in\Lambda_N^+}\sum_{\sigma\in\mathcal{W}} x_{\sigma(\lambda+\rho)-\rho} q^{(\sigma(\lambda+\rho)-\rho,\sigma(\lambda+\rho)+\rho)} S'_{\sigma(\lambda+\rho)-\rho,\overline{\mu}}\\ &= 2^n \sum_{\lambda\in\Lambda_N^+}\sum_{\sigma\in\mathcal{W}} x_{\sigma(\lambda+\rho)-\rho} q^{(\lambda,\lambda+2\rho)}\epsilon'(\sigma) S'_{\lambda,\overline{\mu}},\end{aligned}$$

and we consequently obtain from Eqs. (5.31) and (5.30) the following equation for d'_ν:

$$\sum_{\nu\in\Lambda_N^+} d'_\nu q^{(\nu,\nu+2\rho)} S'_{\nu,\mu} = 2^n \sum_{\lambda\in\Lambda_N^+}\sum_{\sigma\in\mathcal{W}} \epsilon'(\sigma) x_{\sigma(\lambda+\rho)-\rho} q^{(\lambda,\lambda+2\rho)} S'_{\lambda,\mu}, \quad \forall\mu\in\Lambda_N^+. \tag{5.52}$$

Eq. (5.52) is obviously satisfied by

$$d'_\lambda = 2^n \sum_{\sigma\in\mathcal{W}} \epsilon'(\sigma) x_{\sigma(\lambda+\rho)-\rho},$$

and we now evaluate d'_λ:

$$\begin{aligned} d'_\lambda &= 2^n \sum_{\sigma\in\mathcal{W}} \epsilon'(\sigma) q^{-(\sigma(\lambda+\rho)-\rho,2\rho)} / \sum_{\overline{\nu}\in X_N} q^{(\overline{\nu},\overline{\nu}+2\rho)}\\ &= 2^n q^{(2\rho,\rho)}\epsilon'(w_0) \sum_{\sigma\in\mathcal{W}} \epsilon'(\sigma) q^{-(w_0\sigma(\lambda+\rho),2\rho)} / \sum_{\overline{\nu}\in X_N} q^{(\overline{\nu},\overline{\nu}+2\rho)}\\ &= 2^n q^{(2\rho,\rho)} \sum_{\sigma\in\mathcal{W}} \epsilon'(\sigma) q^{(\sigma(\lambda+\rho),2\rho)} / \sum_{\overline{\nu}\in X_N} q^{(\overline{\nu},\overline{\nu}+2\rho)},\end{aligned}$$

where $w_0 = s_{\epsilon_1}s_{\epsilon_2}\cdots s_{\epsilon_n}$ is the longest element of $\mathcal{W}$; note that $\epsilon'(w_0)=1$. Therefore,

$$d'_\lambda = (-1)^{[\lambda]} \frac{2^n q^{(2\rho,\rho)} sdim_q(V_\lambda) Q(0)}{\sum_{\overline{\nu}\in X_N} q^{(\overline{\nu},\overline{\nu}+2\rho)}},$$

$$d_\lambda = \frac{2^n q^{(2\rho,\rho)} sdim_q(V_\lambda) Q(0)}{\sum_{\overline{\nu}\in X_N} q^{(\overline{\nu},\overline{\nu}+2\rho)}}.$$

(Recall that $d_\lambda = (-1)^{[\lambda]} d'_\lambda$.) Evaluating the Gaussian sums gives

$$d_\lambda = \frac{2^n q^{(2\rho,\rho)} sdim_q(V_\lambda) Q(0) t^n \left(\prod_{k=0}^{n-1} q^{k(k+1)}\right)}{\left[(1+i)\sqrt{N}\right]^n}, \tag{5.53}$$

which is non-zero for each $\lambda\in\Lambda_N^+$. Writing

$$d_\lambda = d_0 sdim_q(V_\lambda), \qquad d_0 = \Omega Q(0), \qquad \lambda\in\Lambda_N^+,$$

we then have

$$\begin{aligned}
\Omega &= \frac{2^n q^{(2\rho,\rho)} t^n \left(\prod_{k=0}^{n-1} q^{k(k+1)}\right)}{\left[(1+i)\sqrt{N}\right]^n} \\
&= \frac{2^n q^{(4n^3-n)/6} t^n q^{(n^3-n)/3}}{\left[(1+i)\sqrt{N}\right]^n} \\
&= \frac{2^n t^n q^{n^3-n/2}}{\left[(1+i)\sqrt{N}\right]^n}.
\end{aligned}$$

In this calculation we used the result $(2\rho, 2\rho) = (4n^3 - n)/3$. □

The constants d_λ are proportional to $sdim_q(V_\lambda)$ with constant of proportionality $d_0 \neq 0$ for each $\lambda \in \Lambda_N^+$. A similar phenomenon occurs for all other Reshetikhin-Turaev 3-manifold invariants constructed from quotients of quantum algebras and quantum superalgebras at even and odd roots of unity [RT91, TW93, Zh94, Zh95, ZC96, Zh97].

5.6.3 Proof of Axiom (VI)

We now show that z is as claimed.

Lemma 5.6.5. *Set* $z = \sum(L) = \sum_{\lambda \in \Lambda_N^+} d_\lambda q^{-(\lambda,\lambda+2\rho)} sdim_q(V_\lambda)$. *Then*

$$z = (-i)^n q^{2n^3-n} t^{2n}, \tag{5.54}$$

which clearly satisfies $|z| = 1$.

Proof. We calculate as follows:

$$\begin{aligned}
z &= \sum_{\lambda \in \Lambda_N^+} d_\lambda q^{-(\lambda,\lambda+2\rho)} sdim_q(V_\lambda) \\
&= \Omega Q(0) \sum_{\lambda \in \Lambda_N^+} q^{-(\lambda,\lambda+2\rho)} \big(sdim_q(V_\lambda)\big)^2 \\
&= \frac{\Omega}{Q(0)} \sum_{\lambda \in \Lambda_N^+} q^{-(\lambda,\lambda+2\rho)} \big(Q(\lambda)\big)^2,
\end{aligned} \tag{5.55}$$

where we have used (5.49) and the relation $\big(sdim_q(V_\lambda)\big)^2 = \big(S_{\lambda,0}/Q(0)\big)^2 = \big(Q(\lambda)/Q(0)\big)^2$.

Let us examine $q^{-(\lambda,\lambda+2\rho)}$ and $\big(Q(\lambda)\big)^2$ under the action of the map $\lambda \mapsto \sigma(\lambda+\rho)-\rho$, where $\sigma \in \mathcal{W}$. It is not difficult to show that for each $\sigma \in \mathcal{W}$ we have

$$q^{-(\sigma(\lambda+\rho)-\rho,\sigma(\lambda+\rho)-\rho+2\rho)} = q^{-(\lambda,\lambda+2\rho)}, \quad \text{and} \quad \big(Q(\sigma(\lambda+\rho)-\rho)\big)^2 = \big(Q(\lambda)\big)^2.$$

Now Eq. (5.40) states that

$$q^{-(\lambda+N'\epsilon_i,\lambda+N'\epsilon_i+2\rho)} = q^{-(\lambda,\lambda+2\rho)}.$$

Furthermore, by using the fact that $q^{(N'\epsilon_i,\sigma(2\rho))} = -1$ for each $i \in \{1,2,\ldots,n\}$, we obtain

$$\big(Q(\lambda + N'\epsilon_i)\big)^2 = \big(Q(\lambda)\big)^2.$$

Now $p_{N/2}\big(\overline{\Lambda_N^+}\big)$ is a fundamental domain for X under the action of the affine Weyl group $\mathcal{W}_{N'}$ and $Q(\lambda) = 0$ if $\lambda \in \overline{\Lambda_N^+}\backslash\Lambda_N^+$. The calculations in the previous paragraph imply that we can write the sum in (5.55) as a sum over the elements of $X_{N'}$, which considerably simplifies the calculations:

$$\sum_{\lambda\in\Lambda_N^+} q^{-(\lambda,\lambda+2\rho)}\big(Q(\lambda)\big)^2 = \frac{1}{|\mathcal{W}|}\sum_{\overline{\lambda}\in X_{N'}} q^{-(\overline{\lambda},\overline{\lambda}+2\rho)}\big(Q(\overline{\lambda})\big)^2, \tag{5.56}$$

where we note that $|\mathcal{W}| = 2^n n!$ [Hu72, p. 66, Table 1]. By using Eqs. (5.40) and (5.42) we can further rewrite the sum on the right hand side of (5.56):

$$\frac{1}{|\mathcal{W}|}\sum_{\overline{\lambda}\in X_{N'}} q^{-(\overline{\lambda},\overline{\lambda}+2\rho)}\big(Q(\overline{\lambda})\big)^2 = \frac{1}{2^n|\mathcal{W}|}\sum_{\overline{\lambda}\in X_N} q^{-(\overline{\lambda},\overline{\lambda}+2\rho)}\big(Q(\overline{\lambda})\big)^2, \tag{5.57}$$

which we will now evaluate. We firstly rewrite the right hand side of (5.57):

$$\begin{aligned}
&\frac{1}{2^n|\mathcal{W}|}\sum_{\overline{\lambda}\in X_N} q^{-(\overline{\lambda},\overline{\lambda}+2\rho)}\big(Q(\overline{\lambda})\big)^2 \\
&\quad= \frac{1}{2^n|\mathcal{W}|}\sum_{\overline{\lambda}\in X_N} q^{-(\overline{\lambda},\overline{\lambda}+2\rho)}\sum_{\sigma,w\in\mathcal{W}}\epsilon'(\sigma)\epsilon'(w)q^{2(\overline{\lambda}+\rho,\sigma(\rho)+w(\rho))} \\
&\quad= \frac{1}{2^n|\mathcal{W}|}\sum_{\sigma,w\in\mathcal{W}}\epsilon'(\sigma)\epsilon'(w)q^{2(\rho,\sigma(\rho)+w(\rho))}\sum_{\overline{\lambda}\in X_N} q^{-(\overline{\lambda},\overline{\lambda}+2\rho)}q^{(2\overline{\lambda},\sigma(\rho)+w(\rho))}.
\end{aligned} \tag{5.58}$$

We wish to disentangle the summation indices in the exponents of q in (5.58) so that we can factorise the summations into a sum over the elements of $\mathcal{W}\times\mathcal{W}$ and a sum over the elements of X_N, both of which we can calculate relatively easily. To do this, for each pair $(\sigma,w)\in\mathcal{W}\times\mathcal{W}$ in (5.58) and each $\overline{\lambda}\in X_N$ we apply the following map:

$$\overline{\lambda} \mapsto \overline{\lambda} + \sigma(\rho) + w(\rho) \in X_N. \tag{5.59}$$

Now

$$p\big(\lambda + \sigma(\rho) + w(\rho)\big) = p\big(\mu + \sigma(\rho) + w(\rho)\big) \quad \text{iff} \quad p(\lambda) = p(\mu), \quad \forall\lambda,\mu\in X,$$

thus the summation in (5.58) is unchanged under the mapping (5.59). Applying this map to the right hand side of (5.58) and equating the result with the left hand side of (5.56) gives

$$\sum_{\lambda\in\Lambda_N^+} q^{-(\lambda,\lambda+2\rho)}(Q(\lambda))^2 = \frac{q^{(2\rho,\rho)}}{2^n|\mathcal{W}|}\sum_{\sigma,w\in\mathcal{W}}\epsilon'(\sigma)\epsilon'(w)q^{2(\sigma(\rho),w(\rho))}\sum_{\overline{\lambda}\in X_N} q^{-(\overline{\lambda}+2\rho,\overline{\lambda})}. \tag{5.60}$$

It is not difficult to evaluate the right hand side of (5.60): note that

$$\sum_{\sigma,w\in\mathcal{W}}\epsilon'(\sigma)\epsilon'(w)q^{2(\sigma(\rho),w(\rho))} = |\mathcal{W}|Q(0).$$

This can be seen in the following way: firstly fix $\sigma_1\in\mathcal{W}$, then

$$\sum_{w\in\mathcal{W}}\epsilon'(w)q^{2(\sigma_1(\rho),w(\rho))} = \sum_{w\in\mathcal{W}}\epsilon'(\sigma_1)\epsilon'(w)q^{2(\rho,w(\rho))} = \epsilon'(\sigma_1)Q(0),$$

thus

$$\sum_{\sigma\in\mathcal{W}}\epsilon'(\sigma)\sum_{w\in\mathcal{W}}\epsilon'(w)q^{2(\sigma(\rho),w(\rho))} = \sum_{\sigma,w\in\mathcal{W}}\epsilon'(w)q^{2(\rho,w(\rho))} = \sum_{\sigma\in\mathcal{W}}Q(0) = |\mathcal{W}|Q(0).$$

Additionally, Lemma A.2.5 implies that

$$\sum_{\overline{\lambda}\in X_N} q^{-(\overline{\lambda}+2\rho,\overline{\lambda})} = \prod_{k=0}^{n-1}G_-(N,2k+1) = t^n\left[(1-i)\sqrt{N}\right]^n\left(\prod_{k=0}^{n-1}q^{k(k+1)}\right),$$

where $t=\exp(\pi i/2N)$. By combining these results we obtain

$$\begin{aligned}
z &= \frac{\Omega}{Q(0)}\sum_{\lambda\in\Lambda_N^+} q^{-(\lambda,\lambda+2\rho)}\big(Q(\lambda)\big)^2 \\
&= \frac{\Omega|\mathcal{W}|Q(0)q^{(2\rho,\rho)}t^n\left[(1-i)\sqrt{N}\right]^n\left(\prod_{k=0}^{n-1}q^{k(k+1)}\right)}{2^n|\mathcal{W}|Q(0)} \\
&= \frac{\Omega q^{(2\rho,\rho)}t^n\left[(1-i)\sqrt{N}\right]^n\left(\prod_{k=0}^{n-1}q^{k(k+1)}\right)}{2^n} \\
&= \frac{q^{(2\rho,2\rho)}t^{2n}\left[(1-i)\sqrt{N}\right]^n\left(\prod_{k=0}^{n-1}q^{k(k+1)}\right)^2}{\left[(1+i)\sqrt{N}\right]^n} \\
&= (-i)^n q^{2n^3-n}t^{2n},
\end{aligned}$$

and $|z|=1$. □

This completes the proof of Theorem 5.6.2.

5.7 Comparing the invariants from $U_q^{(N)}(osp(1|2n))$ and $U_q^{(N/2)}(so(2n+1))$

Our construction of 3-manifold invariants from $U_q^{(N)}(osp(1|2n))$ immediately gives rise to three important questions:

1. Are our topological invariants of 3-manifolds obtained from $U_q^{(N)}(osp(1|2n))$ *new* invariants?
2. Are the 3-manifold invariants obtained from $U_q^{(N)}(osp(1|2n))$ *complete* invariants, that is, do they distinguish non-homeomorphic 3-manifolds as well as telling us when two 3-manifolds are homeomorphic?
3. If the invariants are not complete, are they better than other invariants in distinguishing non-homeomorphic 3-manifolds?

These are difficult questions to answer. The first requires a comparison between our invariants and all other existing invariants, a positive answer to the second would solve the classification problem for closed, connected, orientable 3-manifolds and the third requires a theoretical investigation of the properties of the various invariants, or the calculation of various invariants for numerous 3-manifolds and directly comparing their performance in distinguishing non-homeomorphic 3-manifolds with the performance of our invariant. We do not know of any theorems that would allow such a theoretical investigation, and the numerical work needed to compare the performance of the various invariants is itself a non-trivial exercise, similar to that done to compare how well polynomial link invariants distinguish links that are not ambient isotopic (eg see [DeW99, Ch. 7]).

A fact touching on question 2 is that the Reshetikhin-Turaev method for constructing 3-manifold invariants does not ensure *completeness*: it does not necessarily distinguish non-homeomorphic 3-manifolds. For example, the topological invariants derived from $U_q^{(N)}(sl_2)$, where $N \geq 4$ satisfies $N \equiv 0 \pmod 4$, do not distinguish all non-homeomorphic 3-manifolds [KB93, Lic93, KL94].

Given the difficulty of answering these questions, we consider a more tractable problem. We will compare our invariants with the invariants derived from one quantum group at odd roots of unity, and ask the following question: are the invariants from $U_q^{(N)}(osp(1|2n))$ the same as those from $U_q^{(N/2)}(so(2n+1))$ when $N \geq 6$ satisfies $N \equiv 2 \pmod 4$? By *the same*, we mean that given a closed, connected, orientable 3-manifold M_L, we have

$$\mathcal{F}(M_L)_{U_q^{(N)}(osp(1|2n))} = \mathcal{F}(M_L)_{U_q^{(N/2)}(so(2n+1))}. \tag{5.61}$$

It is interesting to ask this question as the sets of integral weights in the truncated Weyl chambers Λ_N^+ of $U_q^{(N)}(osp(1|2n))$ and $U_q^{(N/2)}(so(2n+1))$ are the same, thus in calculating the topological invariant for any given 3-manifold M_L we sum over the same module labels. The reader may ask the obvious question why we are not comparing the invariants from quantum $osp(1|2n)$ and quantum $so(2n+1)$ at the same roots of unity. The reason is that the sets of integral weights in the truncated Weyl alcoves of quantum $osp(1|2n)$ and quantum $so(2n+1)$ are different when $N \equiv 2 \pmod 4$.

Our topological invariant has an S^3 normalisation:

$$\mathcal{F}(S^3)_{U_q^{(N)}(osp(1|2n))} = 1 = \mathcal{F}(S^3)_{U_q^{(N/2)}(so(2n+1))},$$

and therefore we choose a 3-manifold other than S^3 on which to compare the two families of invariants. For calculational ease we will determine the invariants associated with $S^2 \times S^1$, and we recall that we can obtain $S^2 \times S^1$ by performing surgery on an oriented unlink $L \subset S^3$ with *zero* framing. We now calculate $\mathcal{F}(S^2 \times S^1)$: the linking matrix of L is $A_L = (0)$, thus $\sigma(A_L) = 1$ and

$$\mathcal{F}(S^2 \times S^1) = z^{-1} \sum(L) = z^{-1} \sum_{\lambda \in \Lambda_N^+} d_\lambda sdim_q(V_\lambda),$$

where in calculating $\mathcal{F}(S^3)_{U_q^{(N/2)}(so(2n+1))}$ we take the quantum dimension of the irreducible $U_q^{(N/2)}(so(2n+1))$-module V_λ with highest weight λ instead of the quantum superdimension.

We firstly calculate $\mathcal{F}(S^2 \times S^1)_{U_q^{(N)}(osp(1|2n))}$. Let $N \geq 6$ satisfy $N \equiv 2 \pmod 4$ and let $q = \exp(2\pi i/N)$, then

$$\begin{aligned} \sum(L) &= \sum_{\lambda \in \Lambda_N^+} d_\lambda sdim_q(V_\lambda) = \Omega Q(0) \sum_{\lambda \in \Lambda_N^+} \left(sdim_q(V_\lambda)\right)^2 \\ &= \frac{\Omega}{Q(0)} \sum_{\lambda \in \Lambda_N^+} \left(Q(\lambda)\right)^2 = \frac{\Omega}{Q(0)|\mathcal{W}|} \sum_{\overline{\lambda} \in X_{N'}} \left(Q(\overline{\lambda})\right)^2 = \frac{\Omega}{2^n Q(0)|\mathcal{W}|} \sum_{\overline{\lambda} \in X_N} \left(Q(\overline{\lambda})\right)^2. \end{aligned} \tag{5.62}$$

Using the expressions for $\sum_{\overline{\lambda} \in X_N} \left(Q(\overline{\lambda})\right)^2$ (Lemma 5.7.1) and for Ω and z (Eqs. (5.50) and (5.54)):

$$\Omega = \frac{2^n q^{n^3 - n/2} t^n}{\left[(1+i)\sqrt{N}\right]^n}, \qquad z = (-i)^n q^{2n^3 - n} t^{2n}, \qquad t = \exp(\pi i/2N),$$

we have

$$\begin{aligned} \mathcal{F}(S^2 \times S^1)_{U_q^{(N)}(osp(1|2n))} &= z^{-1} \frac{(N/2)^{n/2} e^{-n\pi i/4} q^{n^3 - n/4}}{Q(0)} \\ &= \frac{(-i)^{-n} (N/2)^{n/2} e^{-n\pi i/4} q^{-3(\rho,\rho)}}{\prod_{\alpha \in \overline{\Phi}_0^+} \left(q^{(\alpha,\rho)} - q^{-(\alpha,\rho)}\right) \prod_{\beta \in \Phi_1^+} \left(q^{(\beta,\rho)} + q^{-(\beta,\rho)}\right)}. \end{aligned}$$

We now calculate $\mathcal{F}(S^2 \times S^1)_{U_q^{(N/2)}(so(2n+1))}$. With N as given in the calculation of $\mathcal{F}(S^2 \times S^1)_{U_q^{(N)}(osp(1|2n))}$ above, fix $\overline{N} = N/2$ and $\hat{q} = q^2$. From [ZC96, p. 635], we immediately have

$$\mathcal{F}(S^2\times S^1)_{U_q^{(N/2)}(so(2n+1))}=\frac{(-1)^{\left|\Phi^+_{so(2n+1)}\right|}(\hat{q})^{-(1+(\overline{N}+1)/2)(2\rho,\rho)}\left(G_1(\hat{q})\right)^n}{Q'_{\hat{q}}(0)},$$

where $G_k(\hat{q})$ is a Gaussian sum: $G_k(\hat{q})=\sum_{j=0}^{\overline{N}-1}(\hat{q})^{kj^2}$, and

$$Q'_{\hat{q}}(\mu)=\sum_{\sigma\in\mathcal{W}}\epsilon(\sigma)(\hat{q})^{(\sigma(2\mu+2\rho),\rho)}.$$

(Compare this to $Q(\mu)=\sum_{\sigma\in\mathcal{W}}\epsilon'(\sigma)q^{(\sigma(2\mu+2\rho),\rho)}$.) From [KL94, p. 150], we have

$$G_1(\hat{q})=\begin{cases}(\overline{N})^{1/2} & \text{if } \overline{N}\equiv 1 \pmod 4,\\ i(\overline{N})^{1/2} & \text{if } \overline{N}\equiv 3 \pmod 4.\end{cases}$$

As $\Phi^+_{so(2n+1)}=\{\epsilon_i,\epsilon_j\pm\epsilon_k|\ 1\le i\le n,\ 1\le j<k\le n\}$, we have $(-1)^{\left|\Phi^+_{so(2n+1)}\right|}=(-1)^n$, which leads to the following expression for $\mathcal{F}(S^2\times S^1)_{U_q^{(N/2)}(so(2n+1))}$:

$$\mathcal{F}(S^2\times S^1)_{U_q^{(N/2)}(so(2n+1))}=\frac{(-1)^n(-i)^{(2\rho,2\rho)}\,(N/2)^{n/2}\,\hat{q}^{-3(\rho,\rho)}}{\prod_{\alpha\in\overline{\Phi}_0^+\cup\Phi_1^+}\left(\hat{q}^{(\alpha,\rho)}-\hat{q}^{-(\alpha,\rho)}\right)}\times\begin{cases}1, & \text{if } \overline{N}\equiv 1 \pmod 4,\\ i^n, & \text{if } \overline{N}\equiv 3 \pmod 4.\end{cases}$$

Elementary algebra shows that $\mathcal{F}(S^2\times S^1)_{U_q^{(N)}(osp(1|2n))}=\mathcal{F}(S^2\times S^1)_{U_q^{(N/2)}(so(2n+1))}$ if and only if

$$e^{n\pi i/4}=i^{(2\rho,2\rho)}q^{3(\rho,\rho)}\prod_{\alpha\in\overline{\Phi}_0^+}\left(q^{(\alpha,\rho)}+q^{-(\alpha,\rho)}\right)\prod_{\beta\in\Phi_1^+}\left(q^{(\beta,\rho)}-q^{-(\beta,\rho)}\right)\times\begin{cases}i^{-n}, & \text{if } \overline{N}\equiv 1 \pmod 4,\\ (-1)^n, & \text{if } \overline{N}\equiv 3 \pmod 4.\end{cases}\tag{5.63}$$

Eq. (5.63) is never true for $n=1$, and for all odd $n\ge 3$ a necessary (and not sufficient) condition for it to be true is that $(n^3-n/4)\in\mathbb{Z}(k^2+k+1/4)$ where $N=2(2k+1)$. For each odd $n\ge 3$ we can easily choose a sufficiently large enough N so that $(n^3-n/4)\notin\mathbb{Z}(k^2+k+1/4)$, and this relation then holds true for all $N=2(2k'+1)$ where $k'>k$. Thus the invariants from $U_q^{(N)}(osp(1|2n))$ and $U_q^{(N/2)}(so(2n+1))$ are not the same.

Let us now prove the following result which has been used in the derivation of $\mathcal{F}(S^2\times S^1)_{U_q^{(N)}(osp(1|2n))}$.

Lemma 5.7.1. $\sum_{\overline{\lambda}\in X_N}\left(Q(\overline{\lambda})\right)^2=(2N)^n n!$

Proof. Recall that $Q(\overline{\lambda})$ is defined by $Q(\overline{\lambda})=\sum_{\sigma\in\mathcal{W}}\epsilon'(\sigma)q^{2(\sigma(\rho),\overline{\lambda}+\rho)}$ for each $\overline{\lambda}\in X_N$ Now we claim (i) and (ii) below, where $\sigma,w\in\mathcal{W}$:

(i) $\sum_{\overline{\lambda}\in X_N} q^{2(\sigma(\rho)+w(\rho),\overline{\lambda}+\rho)} = 0$, if $\sigma(\rho)+w(\rho)\neq 0$,

(ii) $\epsilon'(\sigma)\epsilon'(w)\sum_{\overline{\lambda}\in X_N} q^{2(\sigma(\rho)+w(\rho),\overline{\lambda}+\rho)} = N^n$, if $\sigma(\rho)+w(\rho)=0$.

We will prove these results momentarily. Using these results we calculate that:

$$\begin{aligned}
\sum_{\overline{\lambda}\in X_N}\left(Q(\overline{\lambda})\right)^2 &= \sum_{\overline{\lambda}\in X_N}\sum_{\sigma,w\in\mathcal{W}} \epsilon'(\sigma)\epsilon'(w)q^{2(\sigma(\rho)+w(\rho),\overline{\lambda}+\rho)}\\
&= \sum_{\overline{\lambda}\in X_N}\sum_{\substack{\sigma,w\in\mathcal{W}\\ \sigma(\rho)+w(\rho)\neq 0}} \epsilon'(\sigma)\epsilon'(w)q^{2(\sigma(\rho)+w(\rho),\overline{\lambda}+\rho)}\\
&\quad+\sum_{\overline{\lambda}\in X_N}\sum_{\substack{\sigma,w\in\mathcal{W}\\ \sigma(\rho)+w(\rho)=0}} \epsilon'(\sigma)\epsilon'(w)q^{2(\sigma(\rho)+w(\rho),\overline{\lambda}+\rho)}\\
&= \sum_{\substack{\sigma,w\in\mathcal{W}\\ w=-\sigma}} \epsilon'(\sigma)\epsilon'(w)\sum_{\overline{\lambda}\in X_N} q^{2(\sigma(\rho)+w(\rho),\overline{\lambda}+\rho)} \qquad (5.64)\\
&= |\mathcal{W}|N^n = (2N)^n\, n! \qquad (5.65)
\end{aligned}$$

where we obtain (5.64) from the fact that $\sigma(\rho)+w(\rho)=0$ if and only if $w=-\sigma$, and we obtain (5.65) from the fact that the order of $\mathcal{W}$ is $2^n n!$

Now we prove the claimed results (i) and (ii) above. We prove (i). Assume that $\sigma,w\in\mathcal{W}$ are such that $\sigma(\rho)+w(\rho)\neq 0$, and let us write $\sigma(\rho)+w(\rho)=\sum_{i=1}^n \mu_i\epsilon_i$. The properties of the reflections generated by the elements of $\mathcal{W}$ mean that $\mu_i\in\mathbb{Z}$ for each i.

By assumption, $\mu_i\neq 0$ for some $i=1,\ldots,n$. Fix such an i, then by considering the action of $\sigma,w\in\mathcal{W}$ on ρ, we have

$$2\leq |2\mu_i|\leq 4n-2. \qquad (5.66)$$

Now the assumption that Λ_N^+ is non-empty means that $N\geq 4n+2$. To see this, recall that Λ_N^+ is defined when $N\equiv 2 \pmod 4$ by $\Lambda_N^+=\{\lambda\in\mathcal{P}^+|\ 0\leq\lambda_1\leq N/4-n-1/2\}$. Then $q^{2\mu_i}\neq 1$ from Eq. (5.66). To complete the proof of (i), all we need is the following trivial calculation:

$$\begin{aligned}
\sum_{\overline{\lambda}\in X_N} q^{(\sigma(\rho)+w(\rho),2\overline{\lambda})} &= \sum_{\lambda_1,\ldots,\lambda_{i-1},\lambda_{i+1},\ldots,\lambda_n=0}^{N-1} q^{2\mu_1\lambda_1+\cdots+2\mu_{i-1}\lambda_{i-1}+2\mu_{i+1}\lambda_{i+1}+\cdots+2\mu_n\lambda_n}\sum_{\lambda_i=0}^{N-1} q^{2\mu_i\lambda_i}\\
&= 0.
\end{aligned}$$

We now prove (ii). Assume that $\sigma,w\in\mathcal{W}$ are such that $\sigma(\rho)+w(\rho)=0$, then $w=-\sigma=w_0\sigma$ where $w_0=\sigma_{\epsilon_1}\sigma_{\epsilon_2}\cdots\sigma_{\epsilon_n}$ is the longest element of $\mathcal{W}$, and we have $\epsilon'(w)=\epsilon'(w_0)\epsilon'(\sigma)=\epsilon'(\sigma)$. Finally,

$$\epsilon'(\sigma)\epsilon'(w)\sum_{\overline{\lambda}\in X_N} q^{2(\sigma(\rho)+w(\rho),\overline{\lambda}+\rho)} = \sum_{\overline{\lambda}\in X_N} 1 = N^n.$$

□

5.8 Some side results

5.8.1 An observation

We now discuss an observation of Turaev and Wenzl [TW93], who showed that in certain circumstances the Reshetikhin-Turaev 3-manifold invariants could be calculated by taking a weighted sum of the $F\big(\Gamma(L,\lambda)\big)$ where the link is cabled and each component of the cabled link is only ever coloured with the same module. Their observations equally apply to the 3-manifold invariants constructed from pseudo-modular Hopf algebras in this thesis. We briefly discuss this here and refer the reader to [TW93] for details and a proof.

Let A be a modular or pseudo-modular Hopf algebra with universal R-matrix R, and let $\{V_\lambda |\ \lambda \in I\}$ be the collection of A-modules used to construct the 3-manifold invariants. For each $\mu \in I$, let $\mathcal{C}_t(V_\mu)$ be the algebra over $\mathbb{C}$ generated by the elements

$$\left\{\check{\mathcal{R}}_i^{\pm 1} \in End_A\left(V_\mu\right)^{\otimes t} \mid 1 \leq i \leq t-1\right\}, \qquad \text{where}$$

$$\check{\mathcal{R}}_i^{\pm 1}(v_{j_1} \otimes \cdots \otimes v_{j_t}) = v_{j_1} \otimes \cdots \otimes v_{j_{i-1}} \otimes P \circ \left(R^{\pm 1}(v_{j_i} \otimes v_{j_{i+1}})\right) \otimes v_{j_{i+2}} \otimes \cdots \otimes v_{j_t}.$$

If each V_λ is isomorphic to $p_\lambda(V_\mu)^{\otimes t}$ for some idempotent $p_\lambda \in \mathcal{C}_t(V_\mu)$, for some μ in I, we say that V_μ is a *generating* module. For $U_q^{(N)}(osp(1|2n))$ at even roots of unity, the fundamental module V is generating. Turaev and Wenzl's observations only apply if the algebra has a generating module; we only consider such algebras below and denote the generating A-module by V.

The module V_λ is isomorphic to $p_\lambda(V^{\otimes t})$ for some idempotent $p_\lambda \in \mathcal{C}_t$ where we can write $p_\lambda = \sum_{j=1}^r c_j \mathscr{R}_j$, $c_j \in \mathbb{C}$. Here each $\mathscr{R}_j$ is an ordered product in the $\check{\mathcal{R}}_i^{\pm 1}$.

Now in calculating the 3-manifold invariants, one takes a weighted sum of the $F\big(\Gamma(L,\lambda)\big)$ where the sum is over all different possible colourings λ of $\Gamma(L)$. The fact that V_λ is isomorphic to $p_\lambda(V^{\otimes t})$ means that we can express $F\big(\Gamma(L,\lambda)\big)$ differently: we can write $F\big(\Gamma(L,\lambda)\big) = \sum_{j=1}^r c_j F\big(\Gamma(L'_j, V)\big)$ where L'_j is a link obtained from cabling L and V means that each component of $\Gamma(L)$ is coloured with the generating A-module V.

To see this, we consider a framed oriented knot L; the multicomponent case follows. Regard $\Gamma(L)$ as being coloured with $\lambda \in I$. For each $j = 1, \ldots, r$, let $\overline{L}'_j$ be a framed oriented link with t components obtained by cabling L. By referring to $\mathscr{R}_j$ we construct a new link L'_j. Of course, $\mathscr{R}_j = \check{\mathcal{R}}_{k_1}^{\epsilon_1} \check{\mathcal{R}}_{k_2}^{\epsilon_2} \cdots \check{\mathcal{R}}_{k_m}^{\epsilon_m}$, where $\epsilon_l \in \{-1,+1\}$ and $k_l \in \{1,2,\ldots,t-1\}$.

Now let $b_j = \sigma_{k_1}^{\epsilon_1} \sigma_{k_2}^{\epsilon_2} \cdots \sigma_{k_m}^{\epsilon_m}$ be an element of B_t, the Braid group on t strings, where σ^{+1} and σ^{-1} are the elements of B_2 in Figure 5.25, and $\sigma_k^{\pm 1} \in B_t$ acts as $\sigma^{\pm 1}$ on the k^{th} and $(k+1)^{st}$ components of the (t,t)-tangle (from the left) and leaves all other components unchanged. Now we construct the new t-component link L'_j by doing the following. Consider the intersection of $\overline{L}'_j$ with a 3-disk D^3 in such a way that all the components of $\overline{L}'_j$ in the intersection are parallel and oriented downwards. We obtain L'_j by replacing the oriented (t,t)-tangle in the intersection with an oriented (t,t)-tangle given by applying $b_j \in B_t$ to the t parallel components. We do this by firstly applying $\sigma_{k_1}^{\epsilon_1}$ to the top of the t components, and then inductively applying $\sigma_{k_{r+1}}^{\epsilon_{r+1}}$ below $\sigma_{k_r}^{\epsilon_r}$ for each $r = 1, \ldots, m-1$.

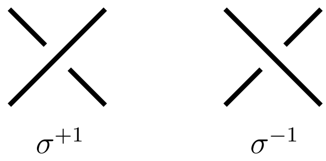


Figure 5.25: The elements σ^{+1}, σ^{-1} of B_2

After obtaining each L'_j, we have

$$F\big(\Gamma(L,\lambda)\big) = \sum_{j=1}^{r} c_j F\big(\Gamma(L'_j, V)\big),$$

which follows from the properties of F.

5.8.2 A further result

Theorem 5.8.1. *The algebra $U_q^{(N)}(osp(1|2n))$ and the set $\{V_\lambda |\ \lambda \in \Lambda_N^+\}$ of modules is not a pseudo-modular Hopf algebra when $N \geq 4$ satisfies $N \equiv 0 \pmod 4$.*

Proof. We will show that $U_q^{(N)}(osp(1|2n))$ and the set of modules do not satisfy Axiom (V) of a pseudo-modular Hopf algebra. To do this, we will show that the set (5.29) of equations is inconsistent if $N \equiv 0 \pmod 4$, $N \geq 4$.

We will show that for each $\mu \in \Lambda_N^+$ there exists some $\mu' \in \Lambda_N^+$, where $\mu' \neq \mu$, such that

(i) $q^{-(\mu',\mu'+2\rho)} = -q^{-(\mu,\mu+2\rho)}$, and

(ii) $S_{\lambda,\mu'}/Q(\mu') = S_{\lambda,\mu}/Q(\mu)$, and

we have the following pair of inconsistent equations:

$$q^{-(\mu,\mu+2\rho)} = \sum_{\lambda\in\Lambda_N^+} d_\lambda \chi_\lambda(v^{-1}) S_{\lambda,\mu}/Q(\mu), \tag{5.67}$$

$$q^{-(\mu',\mu'+2\rho)} = \sum_{\lambda\in\Lambda_N^+} d_\lambda \chi_\lambda(v^{-1}) S_{\lambda,\mu'}/Q(\mu'). \tag{5.68}$$

Let $N \geq 4$ satisfy $N \equiv 0 \pmod 4$. Let $\mu \in \Lambda_N^+$ and fix $s_{\epsilon_1} \in \mathcal{W}$ to be the element of $\mathcal{W}$ that reflects H^* in the subspace $hp_{\epsilon_1} = \{x \in H^* |\ (x, \epsilon_1) = 0\}$. The element s_{ϵ_1} acts on $\nu = \sum_{i=1}^n \nu_i \epsilon_i \in H^*$ via

$$s_{\epsilon_1} : \nu \mapsto -\nu_1\epsilon_1 + \sum_{i=2}^{n} \nu_i \epsilon_i.$$

Define the map $\sigma_{\epsilon_1,N'} : X \to X$ by

$$\sigma_{\epsilon_1,N'} : \mu \mapsto s_{\epsilon_1}(\mu+\rho) - \rho + N'\epsilon_1.$$

In the foregoing we will use Proposition 5.8.1, to which the reader is temporarily referred. Set $\mu' = \sigma_{\epsilon_1,N'}(\mu)$: the next two calculations will show that $q^{-(\mu',\mu'+2\rho)} = -q^{-(\mu,\mu+2\rho)}$. Now

$$\begin{aligned} q^{-(\mu+N'\epsilon_1,\mu+N'\epsilon_1+2\rho)} &= q^{-\left((\mu,\mu+2\rho)+N(\mu,\epsilon_1)+N'(2\rho,\epsilon_1)+(N')^2\right)} \\ &= q^{-((\mu,\mu+2\rho)+N'(2\rho,\epsilon_1))} \\ &= -q^{-(\mu,\mu+2\rho)}, \end{aligned}$$

and for any $w \in \mathcal{W}$ we have

$$q^{-(w(\mu+\rho)-\rho,w(\mu+\rho)-\rho+2\rho)} = q^{-(w(\mu+\rho),w(\mu+\rho))+(\rho,\rho)} = q^{-(\mu,\mu+2\rho)},$$

thus $q^{-(\mu',\mu'+2\rho)} = -q^{-(\mu,\mu+2\rho)}$.

The equality $S_{\lambda,\mu'}/Q(\mu') = S_{\lambda,\mu}/Q(\mu)$ results from the following properties of $S_{\lambda,\mu}$, which derive from direct calculations and from Lemma 5.6.1:

$$\begin{aligned} S_{\lambda,s_{\epsilon_1}(\mu+\rho)-\rho}/Q(s_{\epsilon_1}(\mu+\rho)-\rho) &= S_{\lambda,\mu}/Q(\mu), \\ S_{\lambda,\mu+N'\epsilon_1}/Q(\mu+N'\epsilon_1) &= S_{\lambda,\mu}/Q(\mu). \end{aligned}$$

The second relation follows from the equality $S_{\lambda,\mu+N'\epsilon_1} = -S_{\lambda,\mu}$.

It follows that for each $\mu \in \Lambda_N^+$ there exists some $\mu' \in \Lambda_N^+$ where $\mu' \neq \mu$, such that we have an inconsistent pair of equations (Eqs. (5.67) and (5.68)). As $\mu' = \sigma_{\epsilon_1,N'}(\mu)$ and $\mu = \sigma_{\epsilon_1,N'}(\mu')$, there are $|\Lambda_N^+|/2$ such pairs. □

Proposition 5.8.1. *Let $\sigma_{\epsilon_1,N'} : X \to X$ be a map given in Theorem 5.8.1 and let $\mu \in \Lambda_N^+$, then $\sigma_{\epsilon_1,N'}(\mu) = (N' - \mu_1 - 2n + 1)\epsilon_1 + \sum_{i=2}^n \mu_i\epsilon_i$. Furthermore,*

(i) $\sigma_{\epsilon_1,N'}\big(\sigma_{\epsilon_1,N'}(\mu)\big) = \mu$,

(ii) $\sigma_{\epsilon_1,N'}(\mu) \neq \mu$,

(iii) $\sigma_{\epsilon_1,N'}(\mu) \in \Lambda_N^+$.

Proof. The action of $\sigma_{\epsilon_1,N'}$ on μ is shown by trivial calculations. The proofs of (i)–(iii) are:

(i) $\sigma_{\epsilon_1,N'}\big(\sigma_{\epsilon_1,N'}(\mu)\big) = \sigma_{\epsilon_1,N'}\big((N' - \mu_1 - 2n + 1)\epsilon_1 + \sum_{i=2}^n \mu_i\epsilon_i\big) = \mu$.

(ii) Assume that $\sigma_{\epsilon_1,N'}(\mu) = \mu$, then $(N' - \mu_1 - 2n + 1)\epsilon_1 = \mu_1\epsilon_1$, which implies that $2\mu_1 = N' - 2n + 1$ and $\mu_1 \in \mathbb{Z} + 1/2$. However, this is not possible as $\mu \in \bigoplus_{i=1}^n \mathbb{Z}\epsilon_i$ and therefore $\sigma_{\epsilon_1,N'}(\mu) \neq \mu$.

(iii) The definition of Λ_N^+ implies that $\mu' \in \Lambda_N^+$ if and only if $0 \leq \mu_1' + \mu_2' < N' - 2n + 2$. Now as $\mu \in \Lambda_N^+$, the components of μ satisfy $-1 < \mu_1 - \mu_2 \leq N' - 2n + 1$ which we can rewrite as

$$0 \leq N' - \mu_1 - 2n + 1 + \mu_2 < N' - 2n + 2. \tag{5.69}$$

The statement of the proposition tells us that $\sigma_{\epsilon_1,N'}(\mu) = (N' - \mu_1 - 2n + 1)\epsilon_1 + \sum_{i=2}^n \mu_i\epsilon_i$, that is $\mu_1' = N' - \mu_1 - 2n + 1$ and $\mu_2' = \mu_2$. This allows us to rewrite (5.69) as $0 \leq \mu_1' + \mu_2' < N' - 2n + 2$, which is precisely the condition under which $\sigma_{\epsilon_1,N'}(\mu)$ is an element of Λ_N^+.

□

Appendix A

Gaussian binomial identities and Gaussian sums

In this Appendix we give certain identities involving the Gaussian binomial coefficients and we also investigate Gaussian sums. The pseudo-Gaussian binomial coefficients are closely related to the Gaussian binomial coefficients by $(n)_q = [n]^{q^{-1}}$, but we consider them separately to aid comprehension.

A.1 Gaussian binomial identities

In this section we give certain identities involving the Gaussian and pseudo-Gaussian binomial coefficients. Lemmas A.1.1 and A.1.2 are given without proof; they are easily proved by induction.

Lemma A.1.1. *The Gaussian binomial coefficients satisfy the following relations, where $i, n \in \mathbb{Z}_+$ and $i \leq n$:*

(i) $$\left[\begin{array}{c} n+1 \\ i \end{array}\right]^q = \left[\begin{array}{c} n \\ i \end{array}\right]^q + q^{n+1-i} \left[\begin{array}{c} n \\ i-1 \end{array}\right]^q,$$

(ii) $$\left[\begin{array}{c} n+1 \\ i \end{array}\right]^q = \left[\begin{array}{c} n \\ i-1 \end{array}\right]^q + q^{i} \left[\begin{array}{c} n \\ i \end{array}\right]^q.$$

Lemma A.1.2. *The pseudo-Gaussian binomial coefficients satisfy the following relations, where $i, n \in \mathbb{Z}_+$ and $i \leq n$:*

(i) $$\left(\begin{array}{c} n+1 \\ i \end{array}\right)_q = \left(\begin{array}{c} n \\ i \end{array}\right)_q + (-q)^{n+1-i} \left(\begin{array}{c} n \\ i-1 \end{array}\right)_q,$$

(ii) $$\left(\begin{array}{c} n+1 \\ i \end{array}\right)_q = \left(\begin{array}{c} n \\ i-1 \end{array}\right)_q + (-q)^{i} \left(\begin{array}{c} n \\ i \end{array}\right)_q.$$

A.2 Gaussian sums

In this section we consider some properties of Gaussian sums. Here we fix $q = \exp(2\pi i/N)$ where $N \geq 3$ is some integer, and we fix $G_+(N,m)$ and $G_-(N,m)$ to mean the following Gaussian sums:

$$G_+(N,m) = \sum_{n=0}^{N-1} q^{n(n+m)}, \qquad G_-(N,m) = \sum_{n=0}^{N-1} q^{-n(n+m)}.$$

Lemma A.2.1. *Let $N \equiv 2 \pmod 4$ where $N \in \mathbb{Z}_+$, then $G_+(N,0) = 0$.*

Proof. As $q^{(j+N/2)^2} = q^{j^2+N^2/4} = -q^{j^2}$, for each $j = 0, 1, \ldots, N/2-1$, we have $G_+(N,0) = \sum_{k=0}^{N-1} q^{k^2} = 0$. □

Lemma A.2.2. *Let $N \equiv 0 \pmod 4$ where $N \in \mathbb{Z}_+$, then $G_+(N,0) = (1+i)\sqrt{N}$.*

Proof. See [KL94, Sect. 12.8]. □

Lemma A.2.3. *Let $m \geq 1$ be an odd integer, $N \equiv 2 \pmod 4$ where $N \in \mathbb{Z}_+$, and let $t = \exp(\pi i/2N)$, then*

$$G_+(N,m) = \frac{(1+i)\sqrt{N}}{t^{m^2}}.$$

Proof. Observe that

$$\sum_{n=0}^{N-1} q^{n(n+m)} = \frac{1}{2}\sum_{n=0}^{2N-1} q^{n(n+m)},$$

and that $q^{n(n+m)} = t^{(2n+m)^2}/t^{m^2}$, then

$$\begin{aligned} G_+(N,m) &= \frac{1}{2t^{m^2}} \sum_{n=0}^{2N-1} t^{(2n+m)^2} = \frac{1}{2t^{m^2}} \left(t^{m^2} + t^{(2+m)^2} + t^{(4+m)^2} + \cdots + t^{(4N-2+m)^2} \right) \\ &= \frac{1}{2t^{m^2}} \left(t^{1^2} + t^{3^2} + t^{5^2} + \cdots + t^{(4N-1)^2} \right) = \frac{1}{2t^{m^2}} \big(G_+(4N,0) - 2G_+(N,0) \big) \\ &= \frac{(1+i)\sqrt{N}}{t^{m^2}}, \end{aligned}$$

which follows from the observation that $G_+(N,0) = 0$ as $N \equiv 2 \pmod 4$. □

Lemmas A.2.4 and A.2.5 are proved by noting that $G_-(N,m)$ is the complex conjugate of $G_+(N,m)$.

Lemma A.2.4.

(i) Let $N \equiv 2 \pmod 4$ where $N \in \mathbb{Z}_+$, then $G_-(N,0) = 0$,

(ii) Let $N \equiv 0 \pmod 4$ where $N \in \mathbb{Z}_+$, then $G_-(N,0) = (1-i)\sqrt{N}$.

Lemma A.2.5. *Let $m \geq 1$ be an odd integer, let $N \equiv 2 \pmod 4$ where $N \in \mathbb{Z}_+$, and let $t = \exp(\pi i/2N)$, then $G_-(N,m) = t^{m^2}(1-i)\sqrt{N}$.*

Appendix B

The q-binomial theorem and generalisations

The q-binomial theorem is given in Lemmas B.0.6 and B.0.7 without proof; each of these lemmas is easily proved. The lemmas are identical in the sense that applying the map $q \mapsto -q$ to Lemma B.0.6 gives Lemma B.0.7. We state both lemmas as their properties differ when q is a primitive root of unity.

Lemma B.0.6. *Let a and b be elements of an associative algebra over $\mathbb{C}$ satisfying the relation $ba = qab$, where $0 \neq q \in \mathbb{C}$. Then $(a+b)^n = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}^q a^i b^{n-i}$ for all $n \in \mathbb{N}$.*

Lemma B.0.7. *Let a and b be elements of an associative algebra over $\mathbb{C}$ satisfying the relation $ba = -qab$ where $0 \neq q \in \mathbb{C}$. Then $(a+b)^n = \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix}_q a^i b^{n-i}$ for all $n \in \mathbb{N}$.*

Now we investigate these lemmas when q is a primitive root of unity. Set $q = \exp(2\pi i/N)$ for some integer $N \geq 3$ and let a and b be elements of an associative algebra over $\mathbb{C}$ satisfying $ab = q^2 ba$, then we have

$$(a+b)^{N'} = \sum_{i=0}^{N'} \begin{bmatrix} N' \\ i \end{bmatrix}^{q^2} b^i a^{N'-i} = a^{N'} + b^{N'},$$

as

$$\begin{bmatrix} N' \\ i \end{bmatrix}^{q^2} = \begin{cases} 1, & i \in \{0, N'\}, \\ 0, & \text{otherwise,} \end{cases}$$

which follows from the fact that $[i]^{q^2} = 0$ if and only if $i = kN'$ for some $k \in \mathbb{Z}$.

Now let a and b be elements of an associative algebra over $\mathbb{C}$ satisfying $ab = -qba$, then

$$(a+b)^{\overline{N}} = \sum_{i=0}^{\overline{N}} \begin{pmatrix} \overline{N} \\ i \end{pmatrix}_q b^i a^{\overline{N}-i} = a^{\overline{N}} + b^{\overline{N}},$$

as

$$\begin{pmatrix} \overline{N} \\ i \end{pmatrix}_q = \begin{cases} 1, & i \in \{0, \overline{N}\}, \\ 0, & \text{otherwise}, \end{cases}$$

which follows from the fact that $(i)_q = 0$ if and only if $i = k\overline{N}$ for some $k \in \mathbb{Z}$.

We give two further generalisations of Pascal's binomial theorem below, each of which is a generalisation of the q-binomial theorem. We are unaware of these generalisations appearing in the literature, so we present them with the relevant proofs. Note that we obtain the q-binomial theorem in Lemma B.0.8 if we fix $c = 0$, and we obtain the q-multinomial theorem if we artificially fix $\xi = 0$ in Lemma B.0.9.

Lemma B.0.8. *Let a, b and c be elements of an associative algebra over $\mathbb{C}$ satisfying*

$$ab = -qba + c, \qquad ac = q^2 ca, \qquad cb = q^2 bc,$$

where $0 \neq q \in \mathbb{C}$ and $q^2 \neq 1$, then

$$(a+b)^n = \sum_{\substack{\alpha,\beta,\gamma \in \mathbb{Z}_+ \\ \alpha+2\beta+\gamma=n}} \frac{(n)_q!}{(\alpha)_q!(\gamma)_q!(2)_q(4)_q \cdots (2\beta)_q} \, b^\alpha c^\beta a^\gamma, \qquad n \in \mathbb{N}.$$

Proof. By using the algebra relations we can inductively prove that

$$a^n b = (-q)^n b a^n + (-q)^{n-1}(n)_q c a^{n-1}, \qquad n \in \mathbb{N},$$

which we can use to obtain the following relations, where we use $\cdot$ to denote the algebra multiplication and let α, β, γ be non-negative integers:

$$\begin{aligned} b^\alpha c^\beta a^\gamma \cdot a &= b^\alpha c^\beta a^{\gamma+1}, \\ b^\alpha c^\beta a^\gamma \cdot b &= (-q)^{\gamma+2\beta} \, b^{\alpha+1} c^\beta a^\gamma + (-q)^{\gamma-1}(\gamma)_q \, b^\alpha c^{\beta+1} a^{\gamma-1}. \end{aligned}$$

We can prove that $\alpha + 2\beta + \gamma = n$ if $b^\alpha c^\beta a^\gamma$ is a component in $(a+b)^n$, thus we have

$$(a+b)^n = \sum_{\substack{\alpha,\beta,\gamma \in \mathbb{Z}_+ \\ \alpha+2\beta+\gamma=n}} \theta(\alpha,\beta,\gamma) \, b^\alpha c^\beta a^\gamma, \qquad n \in \mathbb{N}, \tag{B.1}$$

for some set of coefficients $\{\theta(\alpha,\beta,\gamma) \in \mathbb{C} | \; \alpha,\beta,\gamma \in \mathbb{Z}_+\}$.

From (B.1) and the algebra relations, the coefficients $\theta(\alpha,\beta,\gamma)$ satisfy the recursion relation

$$\theta(\alpha,\beta,\gamma) = \theta(\alpha,\beta,\gamma-1) + (-q)^{\gamma+2\beta}\theta(\alpha-1,\beta,\gamma) + (-q)^\gamma(\gamma+1)_q\theta(\alpha,\beta-1,\gamma+1) \tag{B.2}$$

and the boundary conditions $\theta(1,0,0) = \theta(0,0,1) = 1$. In (B.2) we fix $\theta(\alpha,\beta,\gamma) = 0$ if any of α, β or γ are negative. To complete the proof we just need to show that the

$$\theta(\alpha,\beta,\gamma) = \frac{(\alpha+2\beta+\gamma)_q!}{(\alpha)_q!(\gamma)_q!(2)_q(4)_q\cdots(2\beta)_q}, \tag{B.3}$$

furnish a solution to the recurrence relation that also satisfy the boundary conditions. It is easy to see that the $\theta(\alpha, \beta, \gamma)$ satisfy the boundary conditions, and substituting them into the right hand side of (B.2) gives

$$\begin{aligned} & \frac{(\alpha+2\beta+\gamma-1)_q!}{(\alpha)_q!(\gamma-1)_q!(2)_q(4)_q\cdots(2\beta)_q} + (-q)^{\gamma+2\beta}\frac{(\alpha+2\beta+\gamma-1)_q!}{(\alpha-1)_q!(\gamma)_q!(2)_q(4)_q\cdots(2\beta)_q} \\ & \quad +(-q)^{\gamma}(\gamma+1)_q\frac{(\alpha+2\beta+\gamma-1)_q!}{(\alpha)_q!(\gamma+1)_q!(2)_q(4)_q\cdots(2\beta-2)_q} \\ = & \ \frac{(\alpha+2\beta+\gamma-1)_q!}{(\alpha)_q!(\gamma)_q!(2)_q(4)_q\cdots(2\beta)_q}\left((\gamma)_q + (-q)^{\gamma+2\beta}(\alpha)_q + (-q)^{\gamma}(2\beta)_q\right) \\ = & \ \frac{(\alpha+2\beta+\gamma)_q!}{(\alpha)_q!(\gamma)_q!(2)_q(4)_q\cdots(2\beta)_q}, \end{aligned}$$

as required. □

Lemma B.0.9. *Let a, b and c be elements of an associative algebra over $\mathbb{C}$ satisfying*

$$ac = q^2 ca + \xi b^2, \qquad ab = q^2 ba, \qquad bc = q^2 cb,$$

where $0 \neq q \in \mathbb{C}$, $q^2 \neq 1$ and $\xi = -(1+q)^2/(q-q^{-1})$, then

$$(a+b+c)^n = \sum_{\substack{\alpha,\beta,\gamma\in\mathbb{Z}_+ \\ \alpha+\beta+\gamma=n}} \frac{[n]^{q^2}!\ \phi_\beta}{[\alpha]^{q^2}![\beta]^{q^2}![\gamma]^{q^2}!}\ c^\alpha b^\beta a^\gamma, \qquad n \in \mathbb{N},$$

where $\phi_\beta \in \mathbb{C}$ is recursively defined by

$$\phi_0 = 1, \qquad \phi_1 = 1, \qquad \phi_\beta = \phi_{\beta-1} + \xi[\beta-1]^{q^2}\phi_{\beta-2}, \qquad \beta \in \mathbb{N}\backslash\{1\}.$$

Proof. By using the algebra relations we can inductively prove that

$$a^n c = q^{2n} c a^n + \xi q^{2(n-1)}[n]^{q^2} b^2 a^{n-1}, \qquad n \in \mathbb{N},$$

and by using this we obtain the following relations, where we use $\cdot$ to denote the algebra multiplication, and let α, β, γ be non-negative integers:

$$\begin{aligned} c^\alpha b^\beta a^\gamma \cdot a &= c^\alpha b^\beta a^{\gamma+1} \\ c^\alpha b^\beta a^\gamma \cdot b &= q^{2\gamma} c^\alpha b^{\beta+1} a^\gamma \\ c^\alpha b^\beta a^\gamma \cdot c &= q^{2\gamma+2\beta} c^{\alpha+1} b^\beta a^\gamma + \xi q^{2(\gamma-1)}[\gamma]^{q^2} c^\alpha b^{\beta+2} a^{\gamma-1}. \end{aligned}$$

We can prove that $\alpha+\beta+\gamma = n$ if $c^\alpha b^\beta a^\gamma$ is a component in $(a+b+c)^n$, thus

$$(a+b+c)^n = \sum_{\substack{\alpha,\beta,\gamma\in\mathbb{Z}_+ \\ \alpha+\beta+\gamma=n}} \theta(\alpha,\beta,\gamma)\ c^\alpha b^\beta a^\gamma, \qquad n \in \mathbb{N}, \tag{B.4}$$

for some collection of coefficients $\{\theta(\alpha,\beta,\gamma)\in\mathbb{C}|\ \alpha,\beta,\gamma\in\mathbb{Z}_+\}$.

From (B.4) and the algebra relations, the coefficients $\theta(\alpha,\beta,\gamma)$ satisfy the recursion relation

$$\begin{aligned}\theta(\alpha,\beta,\gamma) &= \theta(\alpha,\beta,\gamma-1)+q^{2\gamma}\theta(\alpha,\beta-1,\gamma)+q^{2\gamma+2\beta}\theta(\alpha-1,\beta,\gamma)\\ &\quad+\xi q^{2\gamma}[\gamma+1]^{q^2}\theta(\alpha,\beta-2,\gamma+1)\end{aligned} \tag{B.5}$$

and the boundary conditions $\theta(1,0,0)=\theta(0,1,0)=\theta(0,0,1)=1$. Here we fix $\theta(\alpha,\beta,\gamma)=0$ if any of α,β,γ are negative. To complete the proof all we need do is show that

$$\theta(\alpha,\beta,\gamma)=\frac{[\alpha+\beta+\gamma]^{q^2}!\ \phi_\beta}{[\alpha]^{q^2}![\beta]^{q^2}![\gamma]^{q^2}!}, \tag{B.6}$$

solves the recurrence relation and satisfies the boundary conditions, where ϕ_β is itself recursively defined (as stated in the lemma). Clearly, (B.6) satisfies the boundary conditions, and substituting (B.6) into the right hand side of (B.5) gives

$$\begin{aligned}&\frac{[\alpha+\beta+\gamma-1]^{q^2}!\ \phi_\beta}{[\alpha]^{q^2}![\beta]^{q^2}![\gamma-1]^{q^2}!}+q^{2\gamma}\frac{[\alpha+\beta+\gamma-1]^{q^2}!\ \phi_{\beta-1}}{[\alpha]^{q^2}![\beta-1]^{q^2}![\gamma]^{q^2}!}\\ &\quad+q^{2\gamma+2\beta}\frac{[\alpha+\beta+\gamma-1]^{q^2}!\ \phi_\beta}{[\alpha-1]^{q^2}![\beta]^{q^2}![\gamma]^{q^2}!}+\xi q^{2\gamma}[\gamma+1]^{q^2}\frac{[\alpha+\beta+\gamma-1]^{q^2}!\ \phi_{\beta-2}}{[\alpha]^{q^2}![\beta-2]^{q^2}![\gamma+1]^{q^2}!}\\ &=\frac{[\alpha+\beta+\gamma-1]^{q^2}!}{[\alpha]^{q^2}![\beta]^{q^2}![\gamma]^{q^2}!}\left([\gamma]^{q^2}\phi_\beta+q^{2\gamma}[\beta]^{q^2}\phi_{\beta-1}+q^{2\gamma+2\beta}[\alpha]^{q^2}\phi_\beta+\xi q^{2\gamma}[\beta]^{q^2}[\beta-1]^{q^2}\phi_{\beta-2}\right)\\ &=\frac{[\alpha+\beta+\gamma-1]^{q^2}!}{[\alpha]^{q^2}![\beta]^{q^2}![\gamma]^{q^2}!}\left([\gamma]^{q^2}\phi_\beta+q^{2\gamma+2\beta}[\alpha]^{q^2}\phi_\beta+q^{2\gamma}[\beta]^{q^2}\left[\phi_{\beta-1}+\xi[\beta-1]^{q^2}\phi_{\beta-2}\right]\right)\end{aligned} \tag{B.7}$$

By writing $\phi_\beta=\phi_{\beta-1}+\xi[\beta-1]^{q^2}\phi_{\beta-2}$ for each $\beta\in\mathbb{N}\backslash\{1\}$, we can rewrite (B.7) as

$$\frac{[\alpha+\beta+\gamma-1]^{q^2}!}{[\alpha]^{q^2}![\beta]^{q^2}![\gamma]^{q^2}!}\left([\gamma]^{q^2}\phi_\beta+q^{2\gamma+2\beta}[\alpha]^{q^2}\phi_\beta+q^{2\gamma}[\beta]^{q^2}\phi_\beta\right)=\frac{[\alpha+\beta+\gamma]^{q^2}!\ \phi_\beta}{[\alpha]^{q^2}![\beta]^{q^2}![\gamma]^{q^2}!},$$

which proves the lemma. □

We obtain an explicit expression for the ϕ_β appearing in Lemma B.0.9 below.

Lemma B.0.10. *Let $0\neq q\in\mathbb{C}$ satisfy $q^2\neq 1$ and let $\phi_\beta\in\mathbb{C}$ be recursively defined by*

$$\phi_0=1,\qquad \phi_1=1,\qquad \phi_\beta=\phi_{\beta-1}+\xi[\beta-1]^{q^2}\phi_{\beta-2},\qquad \beta\in\mathbb{N}\backslash\{1\},$$

where $\xi=-(1+q)^2/(q-q^{-1})$. Then ϕ_β is given by

$$\phi_0=1,\qquad \phi_1=1,\qquad \phi_{2i}=(1-q)^{-i}\Psi_{2i},\qquad \phi_{2i+1}=[2i+1]^q\phi_{2i},$$

for each $i\in\mathbb{N}$, where

$$\Psi_{2i}=\frac{[4]^q}{[2]^q}[3]^q\frac{[8]^q}{[4]^q}[5]^q\frac{[12]^q}{[6]^q}[7]^q\cdots[2i-1]^q\frac{[4i]^q}{[2i]^q}.$$

Proof. We firstly calculate ϕ_2:

$$\phi_2 = 1 + \xi[1]^{q^2} = (1+q^2)/(1-q) = (1-q)^{-1}[4]^q/[2]^q,$$

thus the claimed solution for ϕ_β is true for $\beta = 0, 1, 2$. Assume that ϕ_{2i} is as given in the lemma for some $i \in \mathbb{N}$, then we calculate that

$$\begin{aligned}
\phi_{2i+1} &= \phi_{2i} - \frac{(1+q)^2}{q-q^{-1}}[2i]^{q^2}\phi_{2i-1} \\
&= [2i-1]^q \frac{1+q}{q-q^{-1}} \frac{[4i]^q}{[2i]^q}\left(-q^{-1} - [2i]^q\right)\phi_{2i-2} \\
&= (1-q)^{-1}[2i-1]^q \frac{[4i]^q}{[2i]^q}[2i+1]^q \phi_{2i-2} \\
&= [2i+1]^q \phi_{2i},
\end{aligned}$$

as required, and we have

$$\begin{aligned}
\phi_{2i+2} &= [2i+1]^q \phi_{2i} - \frac{(1+q)^2}{q-q^{-1}}[2i+1]^{q^2}\phi_{2i} \\
&= \phi_{2i}\left([2i+1]^q - \frac{(1+q)}{q-q^{-1}}[4i+2]^q\right) \qquad \text{(B.8)} \\
&= \phi_{2i}(q-q^{-1})^{-1}\left(\frac{-q^{-1}(1+q)[2i+1]^q[4i+4]^q}{[2i+2]^q}\right) \\
&= (1-q)^{-1}\frac{[2i+1]^q[4i+4]^q}{[2i+2]^q}\phi_{2i}.
\end{aligned}$$

Here we used $(1+q)\,[2i+1]^{q^2} = [4i+2]^q$ to obtain (B.8).

□

We now examine the two generalisations of the binomial theorem when q is a primitive root of unity. Set $q = \exp(2\pi i/N)$ where $N \geq 3$ is an integer, and let a, b and c be elements of an associative algebra over $\mathbb{C}$ satisfying

$$ab = -qba + c, \qquad ac = q^2 ca, \qquad cb = q^2 bc.$$

Then Lemma B.0.8 implies that

$$(a+b)^{\overline{N}} = \begin{cases} a^{2N} + b^{2N} + (1)_q(3)_q(5)_q \cdots (2N-1)_q c^N, & N \equiv 1, 3 \pmod 4, \\ a^N + b^N + (1)_q(3)_q(5)_q \cdots (N-1)_q c^{N/2}, & N \equiv 0 \pmod 4, \\ a^{N/2} + b^{N/2}, & N \equiv 2 \pmod 4. \end{cases}$$

Now redefine a, b and c to be elements of an associative algebra over $\mathbb{C}$ satisfying

$$ac = q^2 ca + \xi b^2, \qquad ab = q^2 ba, \qquad bc = q^2 cb,$$

where $\xi = -(1+q)^2/(q-q^{-1})$. Then Lemmas B.0.9–B.0.10 imply that

$$(a+b+c)^{N'} = a^{N'} + \phi_{N'} b^{N'} + c^{N'},$$

where

$$\phi_{N'} = \begin{cases} (1-q)^{-(N-1)/2}[N]^q \Psi_{N-1} & = \ 0, \ \text{if } N \equiv 1,3 \pmod 4, \\ (1-q)^{-N/4}\Psi_{N/2} & = \ 0, \ \text{if } N \equiv 0 \pmod 4, \\ (1-q)^{-(N/2-1)/2}[N/2]^q \Psi_{N/2-1} & \neq \ 0, \ \text{if } N \equiv 2 \pmod 4. \end{cases}$$

Using this result for $\phi_{N'}$ we have

$$(a+b+c)^{N'} = \begin{cases} a^N + c^N, & \text{if } N \equiv 1,3 \pmod 4, \\ a^{N/2} + c^{N/2}, & \text{if } N \equiv 0 \pmod 4, \\ a^{N/2} + \phi_{N/2} b^{N/2} + c^{N/2}, & \text{if } N \equiv 2 \pmod 4. \end{cases}$$

Appendix C

The Weyl supercharacter formula

In this appendix we recall the Weyl supercharacter of a finite dimensional irreducible $U(osp(1|2n))$-module with integral dominant highest weight [Kac78a, Kac78b].

For each $\Lambda \in H^*$ let $D(\Lambda) \subset H^*$ be defined by

$$D(\Lambda) = \left\{ \Lambda - \sum_{\alpha \in \Phi^+} n_\alpha \alpha \,\middle|\, n_\alpha \in \mathbb{Z}_+ \right\}.$$

Let E be the algebra of functions on H^* that vanish outside the union of finitely many sets of the form $D(\Lambda)$. The convolution of two elements $f, g \in E$ is defined by $f \cdot g(\lambda) = \sum_{\mu \in H^*} f(\lambda - \mu) g(\mu)$. This sum is well-defined as only a finite number of terms in the sum are non-zero. The algebra E is a commutative algebra with respect to convolution.

Let $e^\lambda \in E$ be a function defined by

$$e^\lambda(\nu) = \delta_{\lambda\nu}.$$

The convolution of two such functions e^λ, $e^\mu \in E$ is

$$e^\lambda \cdot e^\mu(\nu) = \sum_{\gamma \in H^*} e^\lambda(\nu - \gamma) e^\mu(\gamma) = \sum_{\gamma \in H^*} \delta_{\lambda, \nu-\gamma} \delta_{\mu,\gamma} = \sum_{\mu \in H^*} \delta_{\lambda, \nu-\mu} = e^{\lambda+\mu}(\nu).$$

The element e^0 is the unit of E. Any function $f \in E$ can be expressed as a sum $f = \sum_{\lambda \in H^*} f(\lambda) e^\lambda$.

Let $\mathcal{W}^{\mathfrak{g}}$ be the Weyl group of the Lie (super)algebra $\mathfrak{g}$ and let $\Phi_{\mathfrak{g}}$ be the set of roots of $\mathfrak{g}$. The Weyl group $\mathcal{W}^{\mathfrak{g}}$ is generated by the elements $\{s_\alpha | \, \alpha \in \Phi_{\mathfrak{g}}\}$, where s_α is the map $s_\alpha : H^* \to H^*$ defined by

$$s_\alpha(\lambda) = \lambda - \frac{2(\alpha, \lambda)\alpha}{(\alpha, \alpha)},$$

where $(\cdot, \cdot) : H^* \times H^* \to \mathbb{C}$ is the non-degenerate bilinear form defined by $(\epsilon_i, \epsilon_j) = \delta_{ij}$.

Let E' be the set of rational expressions in the elements of E. By definition, the Weyl supercharacter of a finite dimensional irreducible $U(osp(1|2n))$-module V_Λ with integral dominant highest weight $\Lambda \in H^*$ is

$$sch_\Lambda = \sum_\lambda (-1)^{[\lambda]} m(\lambda) e^\lambda,$$

where the sum is over all weight spaces of V_Λ, $[\lambda]$ is the grading of the vectors of V_Λ in the weight space λ, and $m(\lambda)$ is the multiplicity of the weight space λ. Define a homomorphism $\epsilon' : \mathcal{W} \to \{-1, +1\}$ by:

$$\epsilon'(\sigma) = \begin{cases} -1, & \text{if the number of reflections in the expression of } \sigma \text{ with respect to} \\ & \text{the elements of } \overline{\Phi}_0^+ \text{ is odd,} \\ +1, & \text{otherwise.} \end{cases}$$

(Recall that $\overline{\Phi}_0^+$ is the set of roots $\{\epsilon_i \pm \epsilon_j | \ 1 \le i < j \le n\}$.) Then the supercharacter of V_Λ is [Kac78b]:

$$sch_\Lambda = (-1)^{[\Lambda]} (L')^{-1} \sum_{\sigma \in \mathcal{W}} \epsilon'(\sigma) e^{\sigma(\Lambda+\rho)},$$

where we write $[\Lambda]$ to mean the grading of the highest weight vector of V_Λ, and

$$L' = \frac{\prod_{\alpha \in \Phi_0^+} \left(e^{\alpha/2} - e^{-\alpha/2}\right)}{\prod_{\beta \in \Phi_1^+} \left(e^{\beta/2} - e^{-\beta/2}\right)}.$$

The expression for sch_Λ dramatically simplifies for $U(osp(1|2n))$. From the root system of $osp(1|2n)$, we have

$$L' = \prod_{\alpha \in \overline{\Phi}_0^+} \left(e^{\alpha/2} - e^{-\alpha/2}\right) \prod_{\beta \in \Phi_1^+} \left(e^{\beta/2} + e^{-\beta/2}\right).$$

Now the supercharacter of the trivial (one-dimensional) $U(osp(1|2n))$-module V_0 is e^0 (as the grading of the highest weight vector of V_0 is even), thus

$$e^0 = \frac{\sum_{\sigma \in \mathcal{W}} \epsilon'(\sigma) e^{\sigma(\rho)}}{\prod_{\alpha \in \overline{\Phi}_0^+} (e^{\alpha/2} - e^{-\alpha/2}) \prod_{\beta \in \Phi_1^+} (e^{\beta/2} + e^{-\beta/2})},$$

which yields a variant of Weyl's denominator formula for $U(osp(1|2n))$:

$$\sum_{\sigma \in \mathcal{W}} \epsilon'(\sigma) e^{\sigma(\rho)} = \prod_{\alpha \in \overline{\Phi}_0^+} (e^{\alpha/2} - e^{-\alpha/2}) \prod_{\beta \in \Phi_1^+} (e^{\beta/2} + e^{-\beta/2}).$$

We thus obtain the following expression for sch_Λ:

$$sch_\Lambda = (-1)^{[\Lambda]} \frac{\sum_{\sigma \in \mathcal{W}} \epsilon'(\sigma) e^{\sigma(\Lambda+\rho)}}{\sum_{\sigma \in \mathcal{W}} \epsilon'(\sigma) e^{\sigma(\rho)}}.$$

Appendix D

Hopf ideal of $U_q(osp(1|2n))$ at roots of unity

In this appendix we prove that the left ideal $\mathcal{I} \subset U_q(osp(1|2n))$ given in Chapter 4, where $q = \exp(2\pi i/N)$ and $N \geq 3$ is an integer, is a two-sided Hopf ideal. We have not seen the results in this appendix appearing in the literature, and we present them in full, together with all relevant proofs, for completeness.

The calculations in this appendix are often quite involved. In particular, the calculations showing that $\mathcal{I}$ is a two-sided co-ideal in which we obtain expressions for powers of the co-multiplication of each root vector in $U_q(osp(1|2n))$, are very intricate. We do these particular calculations by using the generalisations of the binomial theorem in Appendix B. Note that in this appendix we always fix $q = \exp(2\pi i/N)$ where $N \geq 3$ is an integer, and we use $\mathfrak{g}$ to denote $osp(1|2n)$.

D.1 Preliminaries

Recall that the q-bracket $[\cdot,\cdot]_q$ is defined for homogeneous $x, y \in U_q(\mathfrak{g})$ with weights $wt(x)$, $wt(y)$ respectively, by

$$[x,y]_q = xy - (-1)^{[x][y]} q^{(wt(x),wt(y))} yx.$$

We obtain the graded commutator if we formally fix $q = 1$ in the q-bracket. The q-bracket satisfies the following useful identities:

$$\begin{aligned}
[x,yz]_q &= [x,y]_q\, z + (-1)^{[x][y]} q^{(wt(x),wt(y))} y\, [x,z]_q\,, && \text{(D.1)}\\
[xy,z]_q &= x\,[y,z]_q + (-1)^{[y][z]} q^{(wt(y),wt(z))}\, [x,z]_q\, y, && \text{(D.2)}\\
[x,y^n]_q &= \sum_{i=0}^{n-1} (-1)^{i[x][y]} q^{i(wt(x),wt(y))} y^i\, [x,y]_q\, y^{n-1-i}, \quad n \in \mathbb{N}, && \text{(D.3)}\\
[x^n,y]_q &= \sum_{i=0}^{n-1} (-1)^{i[x][y]} q^{i(wt(x),wt(y))} x^{n-1-i}\, [x,y]_q\, x^i, \quad n \in \mathbb{N}. && \text{(D.4)}
\end{aligned}$$

Note that we can obtain (D.2) (resp. (D.4)) from (D.1) (resp. (D.3)) by using the obvious symmetry properties of the q-bracket. We will extensively use Eqs. (D.1)–(D.4) in this appendix.

D.2 Root vectors in $U_q(osp(1|2n))$

We will extensively use the following theorem due to Khoroshkin and Tolstoy in this appendix [KT91, Prop. 3.3], which we will refer to as *Khoroshkin and Tolstoy's proposition.*

Theorem D.2.1. *Let $\mathcal{N}(\phi)$ be a normal order of the elements of ϕ and let the root vectors $e_\gamma \in U_q(\mathfrak{g})$, $\gamma \in \phi$, be constructed with respect to $\mathcal{N}(\phi)$ following Subsection 3.3.2. Let $\alpha, \beta \in \phi$ satisfy $\alpha \prec \beta$ with respect to $\mathcal{N}(\phi)$, then*

$$[e_\alpha, e_\beta]_q = \sum_{\alpha \prec \gamma_1 \prec \cdots \prec \gamma_t \prec \beta} C_{(\gamma_1,k_1,\ldots,\gamma_t,k_t)} (e_{\gamma_1})^{k_1} (e_{\gamma_2})^{k_2} \cdots (e_{\gamma_t})^{k_t},$$

where $\gamma_1, \ldots, \gamma_t \in \phi$, $\sum_{i=1}^t k_i\gamma_i = \alpha + \beta$ and the coefficients $C_{(\gamma_1,k_1,\ldots,\gamma_t,k_t)}$ are complex constants.

An important consequence of this theorem is that $[e_\alpha, e_\beta]_q = 0$ if there does not exist any set of elements $\gamma_1, \ldots, \gamma_t \in \phi$ satisfying $\alpha \prec \gamma_1 \prec \ldots \prec \gamma_t \prec \beta$ and $\sum_{i=1}^t k_i\gamma_i = \alpha + \beta$ for any set of constants $k_i \in \mathbb{N}$. A further useful result is that $[\overline{e}_\beta, \overline{e}_\gamma]_q = 0$ if $[e_\gamma, e_\beta]_q = 0$.

Henceforth in this Appendix we fix the normal order $\mathcal{N}(\phi)$ and the root vectors in $U_q(\mathfrak{g})$ to be as we defined in Subsection 4.1.1.

We now prove some very useful identities.

Proposition D.2.1.

(i) For all $1 \le i < j \le n$, $\left[e_i, e_{\alpha_{i+1}+\cdots+\alpha_j}\right]_q = e_{\alpha_i+\cdots+\alpha_j}$.

(ii) For each $i = 1, \ldots, n-2$, $\left[e_i, e_{\alpha_{i+1}+\cdots+2\alpha_n}\right]_q = e_{\alpha_i+\cdots+2\alpha_n}$.

(iii) For each $i = 1, \ldots, n-2$ and each $j = i+2, \ldots, n-1$,

$$\left[e_i, e_{\alpha_{i+1}+\cdots+2\alpha_j+\cdots+2\alpha_n}\right]_q = e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}.$$

Proof. We prove (i). Firstly, for each $i = 1, \ldots, n-2$,

$$\begin{aligned}\left[e_i, e_{\alpha_{i+1}+\alpha_{i+2}}\right]_q &= [e_i, e_{i+1}]_q e_{i+2} + q^{-1} e_{i+1} [e_i, e_{i+2}]_q - q^{-1} [e_i, e_{i+2}]_q e_{i+1} - q^{-1} e_{i+2} [e_i, e_{i+1}]_q \\ &= e_{\alpha_i+\alpha_{i+1}} e_{i+2} - q^{-1} e_{i+2} e_{\alpha_i+\alpha_{i+1}} = e_{\alpha_i+\alpha_{i+1}+\alpha_{i+2}}.\end{aligned}$$

Keeping i fixed, assume that $\left[e_i, e_{\alpha_{i+1}+\cdots+\alpha_j}\right]_q = e_{\alpha_i+\cdots+\alpha_j}$ for some $j = i+2, \ldots, n-1$, then

$$\begin{aligned}\left[e_i, e_{\alpha_{i+1}+\cdots+\alpha_{j+1}}\right]_q &= \left[e_i, e_{\alpha_{i+1}+\cdots+\alpha_j}\right]_q e_{j+1} + q^{-1} e_{\alpha_{i+1}+\cdots+\alpha_j} [e_i, e_{j+1}]_q \\ &\quad - q^{-1} [e_i, e_{j+1}]_q e_{\alpha_{i+1}+\cdots+\alpha_j} - q^{-1} e_{j+1} \left[e_i, e_{\alpha_{i+1}+\cdots+\alpha_j}\right]_q \\ &= e_{\alpha_i+\cdots+\alpha_j} e_{j+1} - q^{-1} e_{j+1} e_{\alpha_i+\cdots+\alpha_j} = e_{\alpha_i+\cdots+\alpha_{j+1}},\end{aligned}$$

as $[e_i, e_{j+1}]_q = 0$.

We now prove (ii). A simple calculation shows that $\left[e_{n-2}, e_{\alpha_{n-1}+2\alpha_n}\right]_q = e_{\alpha_{n-2}+\alpha_{n-1}+2\alpha_n}$. Assume that $\left[e_i, e_{\alpha_{i+1}+\cdots+2\alpha_n}\right]_q = e_{\alpha_i+\cdots+2\alpha_n}$ for some $i = 2, \ldots, n-2$, then

$$\begin{aligned}
&[e_{i-1}, e_{\alpha_i+\cdots+2\alpha_n}]_q \\
&\quad = [e_{i-1}, e_{\alpha_i+\cdots+\alpha_n}]_q e_n + q^{-1} e_{\alpha_i+\cdots+\alpha_n} [e_{i-1}, e_n]_q + [e_{i-1}, e_n]_q e_{\alpha_i+\cdots+\alpha_n} + e_n [e_{i-1}, e_{\alpha_i+\cdots+\alpha_n}]_q \\
&\quad = e_{\alpha_{i-1}+\cdots+\alpha_n} e_n + e_n e_{\alpha_{i-1}+\cdots+\alpha_n} = e_{\alpha_{i-1}+\cdots+2\alpha_n},
\end{aligned}$$

as $[e_{i-1}, e_n]_q = 0$.

We now prove (iii). A simple calculation shows that $\left[e_i, e_{\alpha_{i+1}+\cdots+2\alpha_{n-1}+2\alpha_n}\right]_q = e_{\alpha_i+\cdots+2\alpha_{n-1}+2\alpha_n}$ for each $i = 1, \ldots, n-3$. Assume that $\left[e_i, e_{\alpha_{i+1}+\cdots+2\alpha_j+\cdots+2\alpha_n}\right]_q = e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}$ for some $i = 1, \ldots, n-3$ and some $j = i+3, \ldots, n-1$, then

$$\begin{aligned}
&\left[e_i, e_{\alpha_{i+1}+\cdots+2\alpha_{j-1}+\cdots+2\alpha_n}\right]_q \\
&\quad = \left[e_i, e_{\alpha_{i+1}+\cdots+2\alpha_j+\cdots+2\alpha_n}\right]_q e_{j-1} + q^{-1} e_{\alpha_{i+1}+\cdots+2\alpha_j+\cdots+2\alpha_n} [e_i, e_{j-1}]_q \\
&\qquad -q^{-1} [e_i, e_{j-1}]_q e_{\alpha_{i+1}+\cdots+2\alpha_j+\cdots+2\alpha_n} - q^{-1} e_{j-1} \left[e_i, e_{\alpha_{i+1}+\cdots+2\alpha_j+\cdots+2\alpha_n}\right]_q \\
&\quad = e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n} e_{j-1} - q^{-1} e_{j-1} e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n} = e_{\alpha_i+\cdots+2\alpha_{j-1}+\cdots+2\alpha_n},
\end{aligned}$$

as $[e_i, e_{j-1}]_q = 0$. □

D.3 The left ideal $\mathcal{I} \subset U_q(osp(1|2n))$

From Chapter 4, $\mathcal{I} \subset U_q(osp(1|2n))$ is the left ideal generated by the elements of the set

$$I = \left\{(e_\gamma)^{N'}, (e_\beta)^{\overline{N}}, (\overline{e}_\gamma)^{N'}, (\overline{e}_\beta)^{\overline{N}}, (f_\gamma)^{N'}, (f_\beta)^{\overline{N}}, (\overline{f}_\gamma)^{N'}, (\overline{f}_\beta)^{\overline{N}}, (J_i)^{\pm N} - 1 | \ 1 \le i \le n\right\}, \tag{D.5}$$

where γ (resp. β) ranges over all the even (resp. odd) elements of ϕ. Recall that the even (resp. odd) elements of ϕ are $\{\epsilon_j \pm \epsilon_k | \ 1 \le j < k \le n\}$ (resp. $\{\epsilon_i | \ 1 \le i \le n\}$). It is convenient to introduce a convention in this section and in Sections D.4–D.5 that γ (resp. β) means an even (resp. odd) element of ϕ, and η means any element of ϕ.

We now define a graded antiautomorphism $\omega : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ to map between e_η and f_η. Slightly generalising the definition of a similar map in [Zh92a], we define ω by

$$\omega : e_i \mapsto f_i, \qquad f_i \mapsto e_i, \qquad K_i^{\pm 1} \mapsto K_i^{\mp 1}, \qquad c \mapsto \overline{c},$$

where $\overline{c}$ is the complex conjugate of $c \in \mathbb{C}$. By definition,

$$\omega(xy) = (-1)^{[x][y]} \omega(y)\omega(x), \qquad \forall x, y \in U_q(\mathfrak{g}), \tag{D.6}$$

and it is easy to see that $[\omega(x)] = [x]$ for each $x \in U_q(\mathfrak{g})$.

Proposition D.3.1. *The graded antiautomorphism ω is an involution.*

Proof. Firstly $\omega^2(x) = x$ for each generator $x \in U_q(\mathfrak{g})$. Now assume that $\omega^2(y) = y$ and $\omega^2(z) = z$ for two elements y and z of $U_q(\mathfrak{g})$, then

$$\omega^2(yz) = (-1)^{[y][z]}\omega\big(\omega(z)\omega(y)\big) = \omega^2(y)\omega^2(z) = yz, \qquad \text{and we also have}$$

$$\omega^2\left(c_1 y + c_2 z\right) = \omega\big(\overline{c}_1\omega(y) + \overline{c}_2\omega(z)\big) = c_1 y + c_2 z, \qquad \forall c_1, c_2 \in \mathbb{C}.$$

□

Proposition D.3.2. *The graded antiautomorphism* ω *commutes with the antipode, ie*

$$S \circ \omega = \omega \circ S.$$

Proof. Recall that the action of the antipode on the generators of $U_q(\mathfrak{g})$ is

$$S : e_i \mapsto -e_i K_i^{-1}, \qquad f_i \mapsto -K_i f_i, \qquad K_i^{\pm 1} \mapsto K_i^{\mp 1},$$

and that $[S(x)] = [x]$ for each $x \in U_q(\mathfrak{g})$. A direct calculation shows that $S\big(\omega(x)\big) = \omega\big(S(x)\big)$ for each generator $x \in U_q(\mathfrak{g})$. Now assume that there exist elements y and z in $U_q(\mathfrak{g})$ such that $S\big(\omega(y)\big) = \omega\big(S(y)\big)$ and $S\big(\omega(z)\big) = \omega\big(S(z)\big)$, then an elementary calculation shows that $S\big(\omega(yz)\big) = \omega\big(S(yz)\big)$ and that $S\big(\omega(c_1 y + c_2 z)\big) = \omega\big(S(c_1 y + c_2 z)\big)$ for all complex constants c_1 and c_2.

□

Proposition D.3.3. *The graded antiautomorphism* ω *satisfies the relation*

$$(\omega \otimes \omega) \circ \Delta' = \Delta \circ \omega. \tag{D.7}$$

Proof. A direct calculation shows that (D.7) is true for each generator of $U_q(\mathfrak{g})$, and the result then follows by a straightforward calculation. □

Recall that if e_μ is defined by $e_\mu = [e_\eta, e_i]_q$, then f_μ is defined by $f_\mu = [f_i, f_\eta]_{q^{-1}}$. The graded antiautomorphism ω maps between e_μ and f_μ as follows.

Proposition D.3.4. *For all* $1 \leq i < j \leq n$, *we have*

(i) $\omega(e_{\alpha_i + \cdots + \alpha_j}) = f_{\alpha_i + \cdots + \alpha_j}$,

(ii) $\omega(e_{\alpha_i + \cdots + 2\alpha_j + \cdots + 2\alpha_n}) = -f_{\alpha_i + \cdots + 2\alpha_j + \cdots + 2\alpha_n}$,

(iii) $\omega(\overline{e}_{\alpha_i + \cdots + \alpha_j}) = \overline{f}_{\alpha_i + \cdots + \alpha_j}$,

(iv) $\omega(\overline{e}_{\alpha_i + \cdots + 2\alpha_j + \cdots + 2\alpha_n}) = -\overline{f}_{\alpha_i + \cdots + 2\alpha_j + \cdots + 2\alpha_n}$.

Proof. By definition, $\omega(e_i) = f_i$ for each simple root α_i. We now prove (i): assume that $\omega(e_{\alpha_i+\cdots+\alpha_j}) = f_{\alpha_i+\cdots+\alpha_j}$ for some $j = i, \ldots, n-1$, then

$$\omega(e_{\alpha_i+\cdots+\alpha_{j+1}}) = f_{j+1}f_{\alpha_i+\cdots+\alpha_j} - qf_{\alpha_i+\cdots+\alpha_j}f_{j+1} = f_{\alpha_i+\cdots+\alpha_{j+1}}.$$

We now prove (ii): for each $i = 1, \ldots, n-1$,

$$\omega(e_{\alpha_i+\cdots+2\alpha_n}) = -\left(f_n f_{\alpha_i+\cdots+\alpha_n} + f_{\alpha_i+\cdots+\alpha_n}f_n\right) = -f_{\alpha_i+\cdots+2\alpha_n}.$$

Now assume that $\omega(e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}) = -f_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}$ for some $j = i+2, \ldots, n$, then

$$\begin{aligned}\omega(e_{\alpha_i+\cdots+2\alpha_{j-1}+\cdots+2\alpha_n}) &= -\left(f_{j-1}f_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n} - qf_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}f_{j-1}\right)\\ &= -f_{\alpha_i+\cdots+2\alpha_{j-1}+\cdots+2\alpha_n}.\end{aligned}$$

The proof of (iii) (resp. (iv)) is almost identical to the proof of (i) (resp. (ii)). □

Proposition D.3.5. *The action of the antipode on e_η, for each $\eta \in \phi$, is $S(e_\eta) = c_\eta \overline{e}_\eta K_\eta^{-1}$ for some scalar $c_\eta \neq 0$.*

Proof. Fixing $\overline{e}_i = e_i$, the proposition is trivially true for all simple roots α_i with $c_{\alpha_i} = -1$. Assume now that the proposition is true for some $\eta \in \phi$ and define $e_\mu = [e_\eta, e_i]_q$ where $\mu = \eta + \alpha_i \in \phi$, then $\overline{e}_\mu = [e_i, \overline{e}_\eta]_q$, and

$$\begin{aligned}S(e_\mu) &= S(e_\eta e_i) - (-1)^{[e_\eta][e_i]}q^{(\eta,\alpha_i)}S(e_i e_\eta)\\ &= -(-1)^{[e_\eta][e_i]}q^{-(\eta,\alpha_i)}c_\eta\left(e_i\overline{e}_\eta - (-1)^{[e_\eta][e_i]}q^{(\eta,\alpha_i)}\overline{e}_\eta e_i\right)K_\mu^{-1}\\ &= c_\mu \overline{e}_\mu K_\mu^{-1},\end{aligned}$$

where $c_\mu = -(-1)^{[e_\eta][e_i]}q^{-(\eta,\alpha_i)}c_\eta$. This formula determines c_μ recursively. □

Proposition D.3.6. *The action of the antipode on $\overline{e}_\eta$, for each $\eta \in \phi$, is $S(\overline{e}_\eta) = d_\eta e_\eta K_\eta^{-1}$ for some scalar $d_\eta \neq 0$.*

Proof. The proposition is trivially true for all simple roots α_i with $d_{\alpha_i} = -1$. Assume now that the proposition is true for some $\eta \in \phi$ and define $\overline{e}_\mu = [e_i, \overline{e}_\eta]_q$ where $\mu = \eta + \alpha_i \in \phi$, then $e_\mu = [e_\eta, e_i]_q$, and

$$\begin{aligned}S(\overline{e}_\mu) &= S(e_i\overline{e}_\eta) - (-1)^{[e_\eta][e_i]}q^{(\eta,\alpha_i)}S(\overline{e}_\eta e_i)\\ &= -(-1)^{[e_i][\overline{e}_\eta]}q^{-(\eta,\alpha_i)}d_\eta\left(e_\eta e_i - (-1)^{[e_i][e_\eta]}q^{(\eta,\alpha_i)}e_i e_\eta\right)K_\mu^{-1}\\ &= d_\mu e_\mu K_\mu^{-1},\end{aligned}$$

where $d_\mu = -(-1)^{[e_\eta][e_i]}q^{-(\eta,\alpha_i)}d_\eta$. This formula determines d_μ recursively.

□

Proposition D.3.7. *The action of the antipode on f_η, for each $\eta \in \phi$, is $S(f_\eta) = \overline{c}_\eta K_\eta \overline{f}_\eta$ for some scalar $\overline{c}_\eta \neq 0$.*

Proof. As the antipode commutes with the graded antiautomorphism ω, for each $1 \le i < j \le n$ we have

$$S(f_{\alpha_i+\cdots+\alpha_j}) = S\big(\omega(e_{\alpha_i+\cdots+\alpha_j})\big) = \omega\big(c_{\alpha_i+\cdots+\alpha_j}\overline{e}_{\alpha_i+\cdots+\alpha_j}K^{-1}_{\alpha_i+\cdots+\alpha_j}\big) = \overline{c}_{\alpha_i+\cdots+\alpha_j}K_{\alpha_i+\cdots+\alpha_j}\overline{f}_{\alpha_i+\cdots+\alpha_j},$$

where $\overline{c}_{\alpha_i+\cdots+\alpha_j} \neq 0$ is a scalar, and we have used Proposition D.3.5. An almost identical calculation proves the corresponding result when $\eta = \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_n$. □

Proposition D.3.8. *The action of the antipode on $\overline{f}_\eta$, for each $\eta \in \phi$, is $S(\overline{f}_\eta) = \overline{d}_\eta K_\eta f_\eta$ for some scalar $\overline{d}_\eta \neq 0$.*

Proof. The fact that the antipode commutes with the graded antiautomorphism ω means that for each $1 \le i < j \le n$ we have

$$S(\overline{f}_{\alpha_i+\cdots+\alpha_j}) = S\big(\omega(\overline{e}_{\alpha_i+\cdots+\alpha_j})\big) = \omega\big(d_{\alpha_i+\cdots+\alpha_j}e_{\alpha_i+\cdots+\alpha_j}K^{-1}_{\alpha_i+\cdots+\alpha_j}\big) = \overline{d}_{\alpha_i+\cdots+\alpha_j}K_{\alpha_i+\cdots+\alpha_j}f_{\alpha_i+\cdots+\alpha_j},$$

where $\overline{d}_{\alpha_i+\cdots+\alpha_j} \neq 0$ is a scalar, and we have used Proposition D.3.6. An almost identical calculation proves the corresponding result when $\eta = \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_n$. □

D.4 The left ideal $\mathcal{I}$ is a two-sided ideal

In this section we prove that $\mathcal{I}$ is a two-sided ideal of $U_q(osp(1|2n))$. To do this, we show that each element of the set I from (D.5) commutes or anticommutes with each generator of $U_q(\mathfrak{g})$. We firstly show that $(e_\gamma)^{N'}$ and $(e_\beta)^{\overline{N}}$ have this property for each $\gamma, \beta \in \phi$, then it is not difficult to show that $(\overline{e}_\gamma)^{N'}, (\overline{e}_\beta)^{\overline{N}}, (f_\gamma)^{N'}, (f_\beta)^{\overline{N}}, (\overline{f}_\gamma)^{N'}$ and $(\overline{f}_\beta)^{\overline{N}}$ also have this property using the antipode and ω.

Trivially, $(K_i)^{\pm N}$ is central in $U_q(\mathfrak{g})$ and thus so is $(J_i)^{\pm N} - 1$. We now show that $(e_i)^{N'}$ and $(e_n)^{\overline{N}}$ commute or anticommute with each generator of $U_q(\mathfrak{g})$ for each $i = 1, \ldots, n-1$.

D.4.1 $(e_i)^{N'}$ and $(e_n)^{\overline{N}}$

Set $1 \le i \le n-1$ and $1 \le j \le n$, then trivially $(e_i)^{N'}$ and $K_j^{\pm 1}$ (anti)commute. We now show that $(e_i)^{N'}f_j = f_j(e_i)^{N'}$. The $U_q(\mathfrak{g})$ relations state that e_j and f_i commute for all $j \neq i$, thus we need only consider the relations between e_i and f_i. The quadruple $\{e_i, f_i, K_i^{\pm 1}\}$ generates a $U_q(sl_2)$ subalgebra of $U_q(osp(1|2n))$, and for each $t \in \mathbb{N}$,

$$\begin{aligned}(e_i)^t f_i &= f_i(e_i)^t \\ &\quad + \frac{1}{q-q^{-1}}\Big[\big(1+q^{-2}+\cdots+q^{-2(t-1)}\big)K_i - \big(1+q^2+\cdots+q^{2(t-1)}\big)K_i^{-1}\Big](e_i)^{t-1} \\ &= f_i(e_i)^t + [t]^{q^2}\left(\frac{q^{-2t+2}K_i - K_i^{-1}}{q-q^{-1}}\right)(e_i)^{t-1}.\end{aligned}$$

As $[N']^{q^2} = 0$, $(e_i)^{N'}f_i = f_i(e_i)^{N'}$.

We now compute the relations between $(e_i)^{N'}$ and e_j for all $i \neq j$. If $|i-j| > 1$, the relevant Serre relation tells us that e_i and e_j commute, so assume that $j = i \pm 1$. The Serre relation is

$$(e_i)^2 e_{i\pm1} - (q+q^{-1}) e_i e_{i\pm1} e_i + e_{i\pm1}(e_i)^2 = 0. \tag{D.8}$$

By repeatedly using (D.8), we have

$$\begin{aligned}
(e_i)^t e_{i\pm1} &= (q+q^{-1})(e_i)^{t-1} e_{i\pm1} e_i - (e_i)^{t-2} e_{i\pm1}(e_i)^2 \\
&= \left[(q+q^{-1})^2 - 1\right](e_i)^{t-2} e_{i\pm1}(e_i)^2 - (q+q^{-1})(e_i)^{t-3} e_{i\pm1}(e_i)^3 \\
&= a_r (e_i)^{t-1-r} e_{i\pm1}(e_i)^{1+r} + b_r (e_i)^{t-2-r} e_{i\pm1}(e_i)^{2+r},
\end{aligned}$$

for all $0 \le r \le t-2$, where $a_r, b_r \in \mathbb{C}$ satisfy the recurrence relations

$$\begin{aligned}
a_{r+1} &= (q+q^{-1}) a_r + b_r, \\
b_{r+1} &= -a_r,
\end{aligned}$$

where $a_0 = q+q^{-1}$ and $b_0 = -1$. The solution for a_r and b_r is

$$\begin{pmatrix} a_r \\ b_r \end{pmatrix} = \frac{q^{-1-r}}{q^{-1}-q}\begin{pmatrix} q^{-1} \\ -1 \end{pmatrix} + \frac{q^{1+r}}{q-q^{-1}}\begin{pmatrix} q \\ -1 \end{pmatrix},$$

and by setting $r = t-2$, we obtain

$$(e_i)^t e_{i\pm1} = \left(\frac{q^t - q^{-t}}{q-q^{-1}}\right) e_i e_{i\pm1}(e_i)^{t-1} + \left(\frac{-q^{t-1}+q^{1-t}}{q-q^{-1}}\right) e_{i\pm1}(e_i)^t.$$

Setting $t = N'$, we have

$$\begin{aligned}
(e_i)^N e_{i\pm1} &= e_{i\pm1}(e_i)^N, && \text{if } N \text{ is odd}, \\
(e_i)^{N/2} e_{i\pm1} &= -e_{i\pm1}(e_i)^{N/2}, && \text{if } N \text{ is even}.
\end{aligned}$$

We now consider $(e_n)^{\overline{N}}$: fix $1 \le i \le n$. Trivially, $(e_n)^{\overline{N}}$ and $K_i^{\pm1}$ (anti)commute. The $U_q(\mathfrak{g})$ relations state that e_n and f_i commute for all $i < n$, thus we consider the relations between e_n and f_n. The quadruple $\{e_n, f_n, K_n^{\pm1}\}$ generates a $U_q(osp(1|2))$ subalgebra of $U_q(osp(1|2n))$, and for each $t \in \mathbb{N}$,

$$\begin{aligned}
(e_n)^t f_n &= (-1)^t f(e_n)^t \\
&\quad + (e_n)^{t-1}\frac{1}{q-q^{-1}}\Big[(1-q+\cdots+(-q)^{n-1})K_n \\
&\qquad\qquad - (1-q^{-1}+\cdots+(-q)^{-(n-1)})K_n^{-1}\Big] \\
&= (-1)^t f(e_n)^t + (e_n)^{t-1}(n)_q\left(\frac{K_n - (-q)^{1-n}K_n^{-1}}{q-q^{-1}}\right).
\end{aligned}$$

As $(\overline{N})_q = 0$, $(e_n)^{\overline{N}} f_n = (-1)^{\overline{N}} f_n (e_n)^{\overline{N}}$.

We now compute the relations between $(e_n)^{\overline{N}}$ and e_i for all $i < n$. If $n - i > 1$, the relevant Serre relation tells us that e_n and e_i commute, so fix $i = n-1$. The Serre relation is

$$(e_n)^3 e_{n-1} - (q - 1 + q^{-1})(e_n)^2 e_{n-1} e_n - (q - 1 + q^{-1}) e_n e_{n-1} (e_n)^2 + e_{n-1}(e_n)^3 = 0. \quad \text{(D.9)}$$

By repeatedly using (D.9), we have

$$(e_n)^t e_{n-1} = a_r (e_n)^{t-1-r} e_{n-1} (e_n)^{1+r} + b_r (e_n)^{t-2-r} e_{n-1} (e_n)^{2+r} + c_r (e_n)^{t-3-r} e_{n-1} (e_n)^{3+r}, \quad \text{(D.10)}$$

for each $0 \leq r \leq t-3$, where $a_r, b_r, c_r \in \mathbb{C}$ satisfy the recurrence relations

$$\begin{aligned} a_r &= (q - 1 + q^{-1}) a_{r-1} + b_{r-1}, \\ b_r &= (q - 1 + q^{-1}) a_{r-1} + c_{r-1}, \\ c_r &= -a_{r-1}, \end{aligned}$$

where $a_0 = b_0 = (q - 1 + q^{-1})$ and $c_0 = -1$. The solution for a_r, b_r, c_r is

$$\begin{pmatrix} a_r \\ b_r \\ c_r \end{pmatrix} = d_1 q^{-r} \begin{pmatrix} -q^{-1} \\ 1 - q^{-1} \\ 1 \end{pmatrix} + d_2 (-1)^r \begin{pmatrix} 1 \\ -q - q^{-1} \\ 1 \end{pmatrix} + d_3 q^r \begin{pmatrix} q \\ q - 1 \\ -1 \end{pmatrix},$$

where

$$d_1 = \frac{1}{(q^2-1)(q+1)}, \quad d_2 = \frac{-q}{(q+1)^2}, \quad d_3 = \frac{q^3}{(q^2-1)(q+1)}.$$

By setting $r = t - 3$, we obtain

$$\begin{aligned} a_{t-3} &= \frac{-q^{2-t}(1+q) + (-1)^t q (q^2-1) + q^{t+1}(q+1)}{(q^2-1)(1+q)^2}, \\ b_{t-3} &= \frac{q^{3-t}(1-q^{-1})(1+q) + (-1)^{t-1} q (q + q^{-1})(q^2-1) + q^t (q-1)(q+1)}{(q^2-1)(1+q)^2}, \\ c_{t-3} &= \frac{q^{3-t}(1+q) + (-1)^t q (q^2-1) - q^t (1+q)}{(q^2-1)(1+q)^2}. \end{aligned}$$

Fixing $t = \overline{N}$, we have $a_{\overline{N}-3} = 0$, $b_{\overline{N}-3} = 0$ and $c_{\overline{N}-3} = (-1)^{\overline{N}}$, thus

$$(e_n)^{\overline{N}} e_{n-1} = (-1)^{\overline{N}} e_{n-1} (e_n)^{\overline{N}}.$$

D.4.2 Relations between $(e_\gamma)^{N'}$, $(e_\beta)^{\overline{N}}$ and e_i

We now prove that $(e_\gamma)^{N'}$ and $(e_\beta)^{\overline{N}}$ (anti)commute with each generator of $U_q(\mathfrak{g})$ for each non-simple root $\gamma, \beta \in \phi$. These calculations are simplified by noting that if we can show that $[e_\gamma, e_i] = 0$ or that $[e_\gamma, e_i]_q = 0$, then $(e_\gamma)^{N'}$ automatically (anti)commutes with e_i, and if we can show that $[e_\beta, e_i] = 0$ or that $[e_\beta, e_i]_q = 0$, then $(e_\beta)^{\overline{N}}$ automatically (anti)commutes with e_i. In the following we write μ to mean any element of ϕ.

We will determine the relations between e_i and $(e_\gamma)^{N'}$, $(e_\beta)^{\overline{N}}$, for a fixed i, by breaking the problem into 4 sub-problems:

(i) $\alpha_{i+2} \prec \mu$,

(ii) $\alpha_i \prec \mu \prec \alpha_{i+1}$,

(iii) $\alpha_{i+1} \prec \mu \prec \alpha_{i+2}$,

(iv) $\mu \prec \alpha_i$.

Case 1: $\alpha_{i+2} \prec \mu$**.**

Here $\mu = \alpha_{i+2} + \cdots + \alpha_j$ or $\mu = \alpha_{i+2} + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_n$ for some $j = i+3, \ldots, n$. From Khoroshkin and Tolstoy's proposition, $[e_i, e_\mu]_q = 0$, which also follows from the following Serre relation: $e_i e_j = e_j e_i$ if $|i - j| > 1$.

Case 2: $\alpha_i \prec \mu \prec \alpha_{i+1}$**.**

Here $\mu = \alpha_i + \cdots + \alpha_j$ or $\mu = \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_n$ for some $j = i+1, \ldots, n$. From Khoroshkin and Tolstoy's proposition, we have

$$\begin{aligned} \left[e_i, e_{\alpha_i+\cdots+\alpha_j}\right]_q &= 0, \qquad i < j \le n, \\ \left[e_i, e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}\right]_q &= 0, \qquad i+2 \le j \le n, \qquad \text{and} \end{aligned}$$

$$\begin{aligned} &\left[e_i, e_{\alpha_i+2\alpha_{i+1}+\cdots+2\alpha_n}\right]_q \\ &\quad = \sum_{k=1}^{n-i-1} C_k e_{\alpha_i+\cdots+\alpha_{i+k}} e_{\alpha_i+\cdots+2\alpha_{i+k+1}+\cdots+2\alpha_n} + C_{n-i}\left(e_{\alpha_i+\cdots+\alpha_n}\right)^2, \ C_k \in \mathbb{C}. \end{aligned}$$

The first two identities dispose of much of this case. To deal with the remaining problem, we claim that the right hand side of Eq. (D.11) below vanishes identically:

$$\begin{aligned} &\left[e_i, \left(e_{\alpha_i+2\alpha_{i+1}+\cdots+2\alpha_n}\right)^{N'}\right]_q \\ &\quad = \sum_{m=0}^{N'-1} \left(e_{\alpha_i+2\alpha_{i+1}+\cdots+2\alpha_n}\right)^m \left[e_i, e_{\alpha_i+2\alpha_{i+1}+\cdots+2\alpha_n}\right]_q \left(e_{\alpha_i+2\alpha_{i+1}+\cdots+2\alpha_n}\right)^{N'-1-m}. \end{aligned} \tag{D.11}$$

To show this, we use the identities:

$$\begin{aligned} \left[e_{\alpha_i+\cdots+\alpha_k}, e_{\alpha_i+2\alpha_{i+1}+\cdots+2\alpha_n}\right]_q &= 0, \quad i+1 \le k \le n, \\ \left[e_{\alpha_i+\cdots+2\alpha_k+\cdots+2\alpha_n}, e_{\alpha_i+2\alpha_{i+1}+\cdots+2\alpha_n}\right]_q &= 0, \quad i+2 \le k \le n, \end{aligned}$$

which can be rewritten, respectively, as

$$\begin{aligned} e_{\alpha_i+\cdots+\alpha_k} e_{\alpha_i+2\alpha_{i+1}+\cdots+2\alpha_n} &= q e_{\alpha_i+2\alpha_{i+1}+\cdots+2\alpha_n} e_{\alpha_i+\cdots+\alpha_k}, \\ e_{\alpha_i+\cdots+2\alpha_k+\cdots+2\alpha_n} e_{\alpha_i+2\alpha_{i+1}+\cdots+2\alpha_n} &= q e_{\alpha_i+2\alpha_{i+1}+\cdots+2\alpha_n} e_{\alpha_i+\cdots+2\alpha_k+\cdots+2\alpha_n}. \end{aligned}$$

Using these, we rewrite the right hand side of (D.11) as

$$\sum_{m=0}^{N'-1} q^{-2m} \left[e_i, e_{\alpha_i+2\alpha_{i+1}+\cdots+2\alpha_n}\right]_q \left(e_{\alpha_i+2\alpha_{i+1}+\cdots+2\alpha_n}\right)^{N'-1} = 0,$$

as $\sum_{m=0}^{N'-1} q^{-2m} = 0$. Thus $\left(e_{\alpha_i+2\alpha_{i+1}+\cdots+2\alpha_n}\right)^{N'}$ (anti)commutes with e_i.

Case 3: $\alpha_{i+1} \prec \mu \prec \alpha_{i+2}$**.**

This is the most difficult of the four cases. The identity: $\left[e_{\alpha_i+\cdots+\alpha_j}, e_{\alpha_{i+1}+\cdots+\alpha_j}\right]_q = 0$, for $j < n$, will be useful here. Note that we can rewrite this identity as $e_{\alpha_i+\cdots+\alpha_j} e_{\alpha_{i+1}+\cdots+\alpha_j} = q e_{\alpha_{i+1}+\cdots+\alpha_j} e_{\alpha_i+\cdots+\alpha_j}$. To prove this case we break it into a number of sub-cases, each of which we consider in the following proposition.

Proposition D.4.1. *We have*

(i) $\left[e_i, \left(e_{\alpha_{i+1}+\cdots+\alpha_j}\right)^{N'}\right]_q = 0$*, for each* $j = i+2, \ldots, n-1$*,*

(ii) $\left[e_i, \left(e_{\alpha_{i+1}+\cdots+\alpha_n}\right)^{\overline{N}}\right]_q = 0$*,*

(iii) $\left[e_i, \left(e_{\alpha_{i+1}+\cdots+2\alpha_j+\cdots+2\alpha_n}\right)^{N'}\right]_q = 0$*, for each* $j = i+1, \ldots, n$*.*

Proof. We use Eqs. (D.3)–(D.4) in this proof. We firstly prove (i):

$$\left[e_i, \left(e_{\alpha_{i+1}+\cdots+\alpha_j}\right)^{N'}\right]_q = \sum_{k=0}^{N'-1} q^{-2k} e_{\alpha_i+\cdots+\alpha_j} \left(e_{\alpha_{i+1}+\cdots+\alpha_j}\right)^{N'-1} = 0.$$

We prove (ii):

$$\begin{aligned}
&\left[e_i, \left(e_{\alpha_{i+1}+\cdots+\alpha_n}\right)^{\overline{N}}\right]_q \\
&= \sum_{k=0}^{\overline{N}-1} q^{-k} \left(e_{\alpha_{i+1}+\cdots+\alpha_n}\right)^k e_{\alpha_i+\cdots+\alpha_n} \left(e_{\alpha_{i+1}+\cdots+\alpha_n}\right)^{\overline{N}-1-k} \\
&= e_{\alpha_i+\cdots+\alpha_n} \left(e_{\alpha_{i+1}+\cdots+\alpha_n}\right)^{\overline{N}-1} + \sum_{k=1}^{\overline{N}-1} q^{-k} \left(e_{\alpha_{i+1}+\cdots+\alpha_n}\right)^k e_{\alpha_i+\cdots+\alpha_n} \left(e_{\alpha_{i+1}+\cdots+\alpha_n}\right)^{\overline{N}-1-k} \\
&= e_{\alpha_i+\cdots+\alpha_n} \left(e_{\alpha_{i+1}+\cdots+\alpha_n}\right)^{\overline{N}-1} + \sum_{k=1}^{\overline{N}-1} (-q)^{-k} e_{\alpha_i+\cdots+\alpha_n} \left(e_{\alpha_{i+1}+\cdots+\alpha_n}\right)^{\overline{N}-1} \\
&\quad + \sum_{k=1}^{\overline{N}-1} q^{1-2k} (k)_q \left[e_{\alpha_i+\cdots+\alpha_n}, e_{\alpha_{i+1}+\cdots+\alpha_n}\right]_q \left(e_{\alpha_{i+1}+\cdots+\alpha_n}\right)^{\overline{N}-2},
\end{aligned} \tag{D.12}$$

where we have used the following calculation:

$$\begin{aligned}
& e_{\alpha_i+\cdots+\alpha_n}\left(e_{\alpha_{i+1}+\cdots+\alpha_n}\right)^k - (-1)^k\left(e_{\alpha_{i+1}+\cdots+\alpha_n}\right)^k e_{\alpha_i+\cdots+\alpha_n} \\
&= \left[e_{\alpha_i+\cdots+\alpha_n}, \left(e_{\alpha_{i+1}+\cdots+\alpha_n}\right)^k\right]_q \\
&= \sum_{m=0}^{k-1}(-1)^m\left(e_{\alpha_{i+1}+\cdots+\alpha_n}\right)^m\left[e_{\alpha_i+\cdots+\alpha_n}, e_{\alpha_{i+1}+\cdots+\alpha_n}\right]_q\left(e_{\alpha_{i+1}+\cdots+\alpha_n}\right)^{k-1-m} \\
&= \sum_{m=0}^{k-1}(-1)^m q^{-m}\left[e_{\alpha_i+\cdots+\alpha_n}, e_{\alpha_{i+1}+\cdots+\alpha_n}\right]_q\left(e_{\alpha_{i+1}+\cdots+\alpha_n}\right)^{k-1} \\
&= (-q)^{-k+1}(k)_q\left[e_{\alpha_i+\cdots+\alpha_n}, e_{\alpha_{i+1}+\cdots+\alpha_n}\right]_q\left(e_{\alpha_{i+1}+\cdots+\alpha_n}\right)^{k-1}.
\end{aligned}$$

Here we used the following calculation: from Khoroshkin and Tolstoy's proposition, we have

$$\begin{aligned}
& \left[e_{\alpha_i+\cdots+\alpha_n}, e_{\alpha_{i+1}+\cdots+\alpha_n}\right]_q \\
&\quad = C_{i+1}e_{\alpha_i+2\alpha_{i+1}+\cdots+2\alpha_n} + \sum_{p=i+2}^{n} C_p e_{\alpha_i+\cdots+2\alpha_p+\cdots+2\alpha_n}e_{\alpha_{i+1}+\cdots+\alpha_{p-1}}, \ C_p \in \mathbb{C}, \quad \text{(D.13)}
\end{aligned}$$

and we also have

$$\begin{aligned}
\left[e_{\alpha_{i+1}+\cdots+\alpha_{p-1}}, e_{\alpha_{i+1}+\cdots+\alpha_n}\right]_q &= 0, \quad p = i+2, \ldots, n, \\
\left[e_{\alpha_i+\cdots+2\alpha_p+\cdots+2\alpha_n}, e_{\alpha_{i+1}+\cdots+\alpha_n}\right]_q &= 0, \quad p = i+2, \ldots, n, \\
\left[e_{\alpha_i+2\alpha_{i+1}+\cdots+2\alpha_n}, e_{\alpha_{i+1}+\cdots+\alpha_n}\right]_q &= 0.
\end{aligned}$$

We can now re-write (D.12) as

$$\begin{aligned}
& (-q)^{1-\overline{N}}(\overline{N})_q e_{\alpha_i+\cdots+\alpha_n}\left(e_{\alpha_{i+1}+\cdots+\alpha_n}\right)^{\overline{N}-1} \\
&\quad + \sum_{k=1}^{\overline{N}-1} q^{1-2k}(k)_q\left[e_{\alpha_i+\cdots+\alpha_n}, e_{\alpha_{i+1}+\cdots+\alpha_n}\right]_q\left(e_{\alpha_{i+1}+\cdots+\alpha_n}\right)^{\overline{N}-2} \qquad \text{(D.14)} \\
&= \begin{cases}
q^3(2N-1)_q\,[N]^{q^2}\left[e_{\alpha_i+\cdots+\alpha_n}, e_{\alpha_{i+1}+\cdots+\alpha_n}\right]_q\left(e_{\alpha_{i+1}+\cdots+\alpha_n}\right)^{\overline{N}-2}, & N \equiv 1, 3 \pmod 4, \\
q^3(N-1)_q\,[N/2]^{q^2}\left[e_{\alpha_i+\cdots+\alpha_n}, e_{\alpha_{i+1}+\cdots+\alpha_n}\right]_q\left(e_{\alpha_{i+1}+\cdots+\alpha_n}\right)^{\overline{N}-2}, & N \equiv 0 \pmod 4, \\
q^3(N/2)_q\,[(N-2)/4]^{q^2}\left[e_{\alpha_i+\cdots+\alpha_n}, e_{\alpha_{i+1}+\cdots+\alpha_n}\right]_q\left(e_{\alpha_{i+1}+\cdots+\alpha_n}\right)^{\overline{N}-2}, & N \equiv 2 \pmod 4,
\end{cases} \\
&= 0, \qquad \text{(D.15)}
\end{aligned}$$

as the first term in (D.14) vanishes from $(\overline{N})_q = 0$, and the second term in (D.14) vanishes from Proposition D.9.1 as $[N]^{q^2} = 0$ for an odd integer N, $[N/2]^{q^2} = 0$ if $N \equiv 0 \pmod 4$, and $(N/2)_q = 0$ if $N \equiv 2 \pmod 4$.

We prove (iii): for each $j = i+1, \dots, n$,

$$\begin{aligned}
&\left[e_i, \left(e_{\alpha_{i+1}+\cdots+2\alpha_j+\cdots+2\alpha_n}\right)^{N'}\right]_q \\
&= \sum_{k=0}^{N'-1} q^{-k} \left(e_{\alpha_{i+1}+\cdots+2\alpha_j+\cdots+2\alpha_n}\right)^k e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n} \left(e_{\alpha_{i+1}+\cdots+2\alpha_j+\cdots+2\alpha_n}\right)^{N'-1-k} \\
&= \sum_{k=0}^{N'-1} q^{-2k} e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n} \left(e_{\alpha_{i+1}+\cdots+2\alpha_j+\cdots+2\alpha_n}\right)^{N'-1} = 0,
\end{aligned}$$

where we have used the identity $\left[e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}, e_{\alpha_{i+1}+\cdots+2\alpha_j+\cdots+2\alpha_n}\right]_q = 0.$ □

Case 4: $\mu \prec \alpha_i$.

We deal with this case by breaking it into a number of subcases in the following proposition.

Proposition D.4.2. *We have*

(i) $\left[e_{\alpha_j+\cdots+\alpha_k}, e_i\right]_q = 0$, *for all* $1 \le j < k \le i-2$,

(ii) $\left[\left(e_{\alpha_j+\cdots+\alpha_{i-1}}\right)^{N'}, e_i\right]_q = 0$, *for each* $j = 1, \dots, i-2$,

(iii) $\left[e_{\alpha_j+\cdots+\alpha_i}, e_i\right]_q = 0$, *for all* $1 \le j < i \le n-1$,

(iv) $\left[\left(e_{\alpha_j+\cdots+\alpha_n}\right)^{\overline{N}}, e_n\right]_q = 0$, *for each* $j = 1, \dots, n-1$,

(v) $\left[e_{\alpha_j+\cdots+\alpha_k}, e_i\right]_q = 0$, *for all* $1 \le j < i < k \le n$,

(vi) $\left[e_{\alpha_j+\cdots+2\alpha_k+\cdots+2\alpha_n}, e_i\right]_q = 0$, *for all* $1 \le j < k \le i \le n$,

(vii) $\left[\left(e_{\alpha_j+\cdots+2\alpha_{i+1}+\cdots+2\alpha_n}\right)^{N'}, e_i\right]_q = 0$, *for each* $j = 1, \dots, i-1$,

(viii) $\left[e_{\alpha_j+\cdots+2\alpha_k+\cdots+2\alpha_n}, e_i\right]_q = 0$, *for all* $1 \le j < i \le k-2$, *where* $k \le n$.

Proof. The proof of (i) follows from the Serre relation stating that e_r and e_t commute if $|r-t| > 1$. We prove (ii):

$$\left[\left(e_{\alpha_j+\cdots+\alpha_{i-1}}\right)^{N'}, e_i\right]_q = \sum_{k=0}^{N'-1} q^{N'-1-2k} e_{\alpha_j+\cdots+\alpha_i} \left(e_{\alpha_j+\cdots+\alpha_{i-1}}\right)^{N'-1} = 0.$$

A trivial calculation proves (iii), and we now prove (iv):

$$\left[\left(e_{\alpha_j+\cdots+\alpha_n}\right)^{\overline{N}}, e_n\right]_q = \sum_{k=0}^{\overline{N}-1} (-1)^k q^{\overline{N}-1-k} e_{\alpha_j+\cdots+2\alpha_n} \left(e_{\alpha_j+\cdots+\alpha_n}\right)^{\overline{N}-1} = 0.$$

The proofs of (v), (vi) and (viii) are trivial, and to complete the proof we prove (vii):

$$\left[\left(e_{\alpha_j+\cdots+2\alpha_{i+1}+\cdots+2\alpha_n}\right)^{N'}, e_i\right]_q = \sum_{k=0}^{N'-1} q^{N'-1-2k} e_{\alpha_j+\cdots+2\alpha_i+\cdots+2\alpha_n}\left(e_{\alpha_j+\cdots+2\alpha_{i+1}+\cdots+2\alpha_n}\right)^{N'-1} = 0.$$

□

D.4.3 The relations between $(e_\gamma)^{N'}$, $(e_\beta)^{\overline{N}}$ and f_i

To complete the proof that $(e_\gamma)^{N'}$ and $(e_\beta)^{\overline{N}}$ (anti)commute with each generator of $U_q(\mathfrak{g})$, we now prove that $(e_\gamma)^{N'}$ and $(e_\beta)^{\overline{N}}$ each (anti)commute with f_i for each $i = 1, \ldots, n$. By writing $\mu = \sum_{i=1}^{n} \mu_i \alpha_i$ and examining the definition of f_μ and the $U_q(\mathfrak{g})$ relations, it is apparent that e_μ commutes with f_j if $\mu_j = 0$, which gives a partial solution to the problem. To complete the consideration of this problem, we prove Propositions D.4.3 and D.4.4.

Proposition D.4.3. *We have*

(i) $\left[e_{\alpha_i+\cdots+\alpha_j}, f_i\right] = -q^{-1}K_i^{-1}e_{\alpha_{i+1}+\cdots+\alpha_j}$, *for all* $1 \le i < j \le n$,

(ii) $\left[e_{\alpha_i+\cdots+\alpha_j}, f_j\right] = K_j e_{\alpha_i+\cdots+\alpha_{j-1}}$, *for all* $1 \le i < j \le n$,

(iii) $\left[e_{\alpha_i+\cdots+\alpha_j}, f_k\right] = 0$, *for all* $1 \le i < k < j \le n$,

(iv) $[e_{\alpha_i+\cdots+2\alpha_k+\cdots+2\alpha_n}, f_j] = 0$, *for all* $1 \le i < j < k \le n$,

(v) $\left[e_{\alpha_{n-1}+2\alpha_n}, f_{n-1}\right] = -q^{-1}(1+q^{-1})K_{n-1}^{-1}(e_n)^2$,

(vi) $[e_{\alpha_i+\cdots+2\alpha_n}, f_n] = -K_n e_{\alpha_i+\cdots+\alpha_n}$, *for each* $i = 1, \ldots, n-2$,

(vii) $\left[e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}, f_j\right] = K_j e_{\alpha_i+\cdots+2\alpha_{j+1}+\cdots+2\alpha_n}$, *for each* $j = i+1, \ldots, n-1$,

(viii) $\left[e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}, f_n\right] = 0$, *for each* $j = i+1, \ldots, n-1$,

(ix) $\left[e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}, f_k\right] = 0$, *for each* $k = i+2, \ldots, n-1$, *and* $j = i+1, \ldots, k-1$,

(x) $\left[e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}, f_i\right] = -q^{-1}K_i^{-1}e_{\alpha_{i+1}+\cdots+2\alpha_j+\cdots+2\alpha_n}$, *for each* $j = i+2, \ldots, n$,

(xi) $\left[e_{\alpha_i+2\alpha_{i+1}+\cdots+2\alpha_n}, f_i\right] = -q^{-1}K_i^{-1}e_{\alpha_{i+1}+2\alpha_{i+2}+\cdots+2\alpha_n}e_{i+1} + q^{-3}K_i^{-1}e_{i+1}e_{\alpha_{i+1}+2\alpha_{i+2}+\cdots+2\alpha_n}$, *for each* $i = 1, \ldots, n-2$,

Proof. We prove (i): firstly, for each $i = 1, \ldots, n-1$,

$$\begin{aligned}\left[e_{\alpha_i+\alpha_{i+1}}, f_i\right] &= \left[e_i e_{i+1} - q^{-1}e_{i+1}e_i, f_i\right] \\ &= e_i\left[e_{i+1}, f_i\right] + \left[e_i, f_i\right]e_{i+1} - q^{-1}e_{i+1}\left[e_i, f_i\right] - q^{-1}\left[e_{i+1}, f_i\right]e_i = -q^{-1}K_i^{-1}e_{i+1}.\end{aligned}$$

Now for each $j = i+2, \ldots, n$,

$$\begin{aligned}
\left[e_{\alpha_i+\cdots+\alpha_j}, f_i\right] &= \left[e_i e_{\alpha_{i+1}+\cdots+\alpha_j} - q^{-1} e_{\alpha_{i+1}+\cdots+\alpha_j} e_i, f_i\right] \\
&= e_i \left[e_{\alpha_{i+1}+\cdots+\alpha_j}, f_i\right] + \left[e_i, f_i\right] e_{\alpha_{i+1}+\cdots+\alpha_j} \\
&\quad -q^{-1} e_{\alpha_{i+1}+\cdots+\alpha_j} \left[e_i, f_i\right] - q^{-1} \left[e_{\alpha_{i+1}+\cdots+\alpha_j}, f_i\right] e_i = -q^{-1} K_i^{-1} e_{\alpha_{i+1}+\cdots+\alpha_j},
\end{aligned}$$

as $\left[e_{\alpha_{i+1}+\cdots+\alpha_j}, f_i\right] = 0$.

We prove (ii) using a similar approach to the proof of (i). Firstly, for each $j = 2, \ldots, n$,

$$\begin{aligned}
\left[e_{\alpha_{j-1}+\alpha_j}, f_j\right] &= \left[e_{j-1} e_j - q^{-1} e_j e_{j-1}, f_j\right] \\
&= e_{j-1} \left[e_j, f_j\right] + (-1)^{[f_j]} \left[e_{j-1}, f_j\right] e_j - q^{-1} e_j \left[e_{j-1}, f_j\right] - q^{-1} \left[e_j, f_j\right] e_{j-1} \\
&= K_j e_{j-1}.
\end{aligned}$$

For each $i = 1, \ldots, j-2$,

$$\begin{aligned}
\left[e_{\alpha_i+\cdots+\alpha_j}, f_j\right] &= \left[e_{\alpha_i+\cdots+\alpha_{j-1}} e_j - q^{-1} e_j e_{\alpha_i+\cdots+\alpha_{j-1}}, f_j\right] \\
&= e_{\alpha_i+\cdots+\alpha_{j-1}} \left[e_j, f_j\right] + (-1)^{[f_j]} \left[e_{\alpha_i+\cdots+\alpha_{j-1}}, f_j\right] e_j \\
&\quad -q^{-1} e_j \left[e_{\alpha_i+\cdots+\alpha_{j-1}}, f_j\right] - q^{-1} \left[e_j, f_j\right] e_{\alpha_i+\cdots+\alpha_{j-1}} = K_j e_{\alpha_i+\cdots+\alpha_{j-1}},
\end{aligned}$$

as $\left[e_{\alpha_i+\cdots+\alpha_{j-1}}, f_j\right] = 0$. We now prove (iii) using (ii) and induction. For each $k = 2, \ldots, n-1$,

$$\begin{aligned}
\left[e_{\alpha_{k-1}+\alpha_k+\alpha_{k+1}}, f_k\right] &= \left[e_{\alpha_{k-1}+\alpha_k} e_{k+1} - q^{-1} e_{k+1} e_{\alpha_{k-1}+\alpha_k}, f_k\right] \\
&= e_{\alpha_{k-1}+\alpha_k} \left[e_{k+1}, f_k\right] + \left[e_{\alpha_{k-1}+\alpha_k}, f_k\right] e_{k+1} \\
&\quad -q^{-1} e_{k+1} \left[e_{\alpha_{k-1}+\alpha_k}, f_k\right] - q^{-1} \left[e_{k+1}, f_k\right] e_{\alpha_{k-1}+\alpha_k} \\
&= K_k e_{k-1} e_{k+1} - K_k e_{k+1} e_{k-1} = 0.
\end{aligned}$$

Keeping k fixed, assume that $\left[e_{\alpha_i+\cdots+\alpha_{k+1}}, f_k\right] = 0$ for some $i = 2, \ldots, k-1$, then

$$\begin{aligned}
\left[e_{\alpha_{i-1}+\cdots+\alpha_{k+1}}, f_k\right] &= \left[e_{i-1} e_{\alpha_i+\cdots+\alpha_{k+1}} - q^{-1} e_{\alpha_i+\cdots+\alpha_{k+1}} e_{i-1}, f_k\right] \\
&= e_{i-1} \left[e_{\alpha_i+\cdots+\alpha_{k+1}}, f_k\right] + \left[e_{i-1}, f_k\right] e_{\alpha_i+\cdots+\alpha_{k+1}} \\
&\quad -q^{-1} e_{\alpha_i+\cdots+\alpha_{k+1}} \left[e_{i-1}, f_k\right] - q^{-1} \left[e_{\alpha_i+\cdots+\alpha_{k+1}}, f_k\right] e_{i-1} = 0,
\end{aligned}$$

as $\left[e_{\alpha_i+\cdots+\alpha_{k+1}}, f_k\right] = \left[e_{i-1}, f_k\right] = 0$. Now assume that $\left[e_{\alpha_i+\cdots+\alpha_j}, f_k\right] = 0$ for some $i = 1, \ldots, k-1$, and some $j = k+1, \ldots, n-1$, then

$$\left[e_{\alpha_i+\cdots+\alpha_{j+1}}, f_k\right] = \left[e_{\alpha_i+\cdots+\alpha_j} e_{j+1} - q^{-1} e_{j+1} e_{\alpha_i+\cdots+\alpha_j}, f_k\right] = 0,$$

as $\left[e_{\alpha_i+\cdots+\alpha_j}, f_k\right] = \left[e_{j+1}, f_k\right] = 0$, which completes the proof of (iii).

We now prove (iv). For each $j = i+1, \ldots, n-1$,

$$\left[e_{\alpha_i+\cdots+2\alpha_n}, f_j\right] = \left[e_{\alpha_i+\cdots+\alpha_n} e_n + e_n e_{\alpha_i+\cdots+\alpha_n}, f_j\right] = 0,$$

as $[e_{\alpha_i+\cdots+\alpha_n}, f_j] = 0$ from (iii) and $[e_n, f_j] = 0$. Keeping j fixed, assume that $[e_{\alpha_i+\cdots+2\alpha_k+\cdots+2\alpha_n}, f_j] = 0$ for some $k = j+2, \ldots, n$, then

$$\left[e_{\alpha_i+\cdots+2\alpha_{k-1}+\cdots+2\alpha_n}, f_j\right] = \left[e_{\alpha_i+\cdots+2\alpha_k+\cdots+2\alpha_n} e_{k-1} - q^{-1} e_{k-1} e_{\alpha_i+\cdots+2\alpha_k+\cdots+2\alpha_n}, f_j\right] = 0,$$

as $[e_{\alpha_i+\cdots+2\alpha_k+\cdots+2\alpha_n}, f_j] = [e_{k-1}, f_j] = 0$.

We prove (v) using (i):

$$\begin{aligned}\left[e_{\alpha_{n-1}+2\alpha_n}, f_{n-1}\right] &= \left[e_{\alpha_{n-1}+\alpha_n} e_n + e_n e_{\alpha_{n-1}+\alpha_n}, f_{n-1}\right] \\ &= e_{\alpha_{n-1}+\alpha_n}\left[e_n, f_{n-1}\right] + \left[e_{\alpha_{n-1}+\alpha_n}, f_{n-1}\right] e_n \\ &\quad + e_n\left[e_{\alpha_{n-1}+\alpha_n}, f_{n-1}\right] + \left[e_n, f_{n-1}\right] e_{\alpha_{n-1}+\alpha_n} \\ &= -q^{-1}(1+q^{-1}) K_{n-1}^{-1}(e_n)^2.\end{aligned}$$

We prove (vi) using (ii):

$$\begin{aligned}\left[e_{\alpha_i+\cdots+2\alpha_n}, f_n\right] &= \left[e_{\alpha_i+\cdots+\alpha_n} e_n + e_n e_{\alpha_i+\cdots+\alpha_n}, f_n\right] \\ &= e_{\alpha_i+\cdots+\alpha_n}\left[e_n, f_n\right] - \left[e_{\alpha_i+\cdots+\alpha_n}, f_n\right] e_n \\ &\quad + e_n\left[e_{\alpha_i+\cdots+\alpha_n}, f_n\right] - \left[e_n, f_n\right] e_{\alpha_i+\cdots+\alpha_n} \\ &= -K_n e_{\alpha_i+\cdots+\alpha_{n-1}} e_n + q^{-1} K_n e_n e_{\alpha_i+\cdots+\alpha_{n-1}} = -K_n e_{\alpha_i+\cdots+\alpha_n},\end{aligned}$$

as $[e_n, f_n]$ commutes with $e_{\alpha_i+\cdots+\alpha_n}$.

We prove (vii) using (iv). For each $j = i+1, \ldots, n-1$,

$$\begin{aligned}\left[e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}, f_j\right] &= \left[e_{\alpha_i+\cdots+2\alpha_{j+1}+\cdots+2\alpha_n} e_j - q^{-1} e_j e_{\alpha_i+\cdots+2\alpha_{j+1}+\cdots+2\alpha_n}, f_j\right] \\ &= e_{\alpha_i+\cdots+2\alpha_{j+1}+\cdots+2\alpha_n}\left[e_j, f_j\right] + \left[e_{\alpha_i+\cdots+2\alpha_{j+1}+\cdots+2\alpha_n}, f_j\right] e_j \\ &\quad - q^{-1} e_j\left[e_{\alpha_i+\cdots+2\alpha_{j+1}+\cdots+2\alpha_n}, f_j\right] - q^{-1}\left[e_j, f_j\right] e_{\alpha_i+\cdots+2\alpha_{j+1}+\cdots+2\alpha_n} \\ &= K_j e_{\alpha_i+\cdots+2\alpha_{j+1}+\cdots+2\alpha_n}.\end{aligned}$$

We now prove (viii) using (vi). Firstly, for each $i = 1, \ldots, n-2$,

$$\begin{aligned}\left[e_{\alpha_i+\cdots+2\alpha_{n-1}+2\alpha_n}, f_n\right] &= \left[e_{\alpha_i+\cdots+2\alpha_n} e_{n-1} - q^{-1} e_{n-1} e_{\alpha_i+\cdots+2\alpha_n}, f_n\right] \\ &= e_{\alpha_i+\cdots+2\alpha_n}\left[e_{n-1}, f_n\right] + \left[e_{\alpha_i+\cdots+2\alpha_n}, f_n\right] e_{n-1} \\ &\quad - q^{-1} e_{n-1}\left[e_{\alpha_i+\cdots+2\alpha_n}, f_n\right] - q^{-1}\left[e_{n-1}, f_n\right] e_{\alpha_i+\cdots+2\alpha_n} \\ &= K_n(e_{n-1} e_{\alpha_i+\cdots+\alpha_n} - e_{\alpha_i+\cdots+\alpha_n} e_{n-1}) = 0,\end{aligned}$$

as e_{n-1} commutes with $e_{\alpha_i+\cdots+\alpha_n}$. Assume that $\left[e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}, f_n\right] = 0$ for some $j = i+2, \ldots, n-1$, then

$$\left[e_{\alpha_i+\cdots+2\alpha_{j-1}+\cdots+2\alpha_n}, f_n\right] = \left[e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n} e_{j-1} - q^{-1} e_{j-1} e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}, f_n\right] = 0,$$

as $\left[e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}, f_n\right] = [e_{j-1}, f_n] = 0$, which proves (viii).

We now prove (ix). Recall from (vii) that

$$[e_{\alpha_i+\cdots+2\alpha_k+\cdots+2\alpha_n}, f_k] = K_k e_{\alpha_i+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n},$$

for each $k = i+1, \ldots, n-1$. Now for each $k = i+2, \ldots, n-1$,

$$\begin{aligned}\left[e_{\alpha_i+\cdots+2\alpha_{k-1}+\cdots+2\alpha_n}, f_k\right] &= \left[e_{\alpha_i+\cdots+2\alpha_k+\cdots+2\alpha_n} e_{k-1} - q^{-1} e_{k-1} e_{\alpha_i+\cdots+2\alpha_k+\cdots+2\alpha_n}, f_k\right] \\ &= e_{\alpha_i+\cdots+2\alpha_k+\cdots+2\alpha_n}\left[e_{k-1}, f_k\right] + \left[e_{\alpha_i+\cdots+2\alpha_k+\cdots+2\alpha_n}, f_k\right] e_{k-1} \\ &\quad -q^{-1} e_{k-1}\left[e_{\alpha_i+\cdots+2\alpha_k+\cdots+2\alpha_n}, f_k\right] - q^{-1}\left[e_{k-1}, f_k\right] e_{\alpha_i+\cdots+2\alpha_k+\cdots+2\alpha_n} \\ &= K_k e_{\alpha_i+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n} e_{k-1} - K_k e_{k-1} e_{\alpha_i+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n} = 0,\end{aligned}$$

as e_{k-1} commutes with $e_{\alpha_i+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}$. We now do the inductive step: assume that

$$\left[e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}, f_k\right] = 0,$$

for some $j = i+2, \ldots, k-1$, then

$$\left[e_{\alpha_i+\cdots+2\alpha_{j-1}+\cdots+2\alpha_n}, f_k\right] = \left[e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n} e_{j-1} - q^{-1} e_{j-1} e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}, f_k\right] = 0,$$

as $\left[e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}, f_k\right] = [e_{j-1}, f_k] = 0$, which proves (ix).

We now prove (x). Firstly, for each $i = 1, \ldots, n-2$ (the $i = n-1$ case is dealt with in (v)), we have

$$\begin{aligned}\left[e_{\alpha_i+\cdots+2\alpha_n}, f_i\right] &= \left[e_{\alpha_i+\cdots+\alpha_n} e_n + e_n e_{\alpha_i+\cdots+\alpha_n}, f_i\right] \\ &= e_{\alpha_i+\cdots+\alpha_n}\left[e_n, f_i\right] + \left[e_{\alpha_i+\cdots+\alpha_n}, f_i\right] e_n + e_n\left[e_{\alpha_i+\cdots+\alpha_n}, f_i\right] + \left[e_n, f_i\right] e_{\alpha_i+\cdots+\alpha_n} \\ &= -q^{-1} K_i^{-1}\left(e_{\alpha_{i+1}+\cdots+\alpha_n} e_n + e_n e_{\alpha_{i+1}+\cdots+\alpha_n}\right) = -q^{-1} K_i^{-1} e_{\alpha_{i+1}+\cdots+2\alpha_n}.\end{aligned}$$

If $i < n-2$,

$$\begin{aligned}&\left[e_{\alpha_i+\cdots+2\alpha_{n-1}+2\alpha_n}, f_i\right] \\ &\quad = \left[e_{\alpha_i+\cdots+2\alpha_n} e_{n-1} - q^{-1} e_{n-1} e_{\alpha_i+\cdots+2\alpha_n}, f_i\right] \\ &\quad = e_{\alpha_i+\cdots+2\alpha_n}\left[e_{n-1}, f_i\right] + \left[e_{\alpha_i+\cdots+2\alpha_n}, f_i\right] e_{n-1} - q^{-1} e_{n-1}\left[e_{\alpha_i+\cdots+2\alpha_n}, f_i\right] - q^{-1}\left[e_{n-1}, f_i\right] e_{\alpha_i+\cdots+2\alpha_n} \\ &\quad = -q^{-1} K_i^{-1}\left(e_{\alpha_{i+1}+\cdots+2\alpha_n} e_{n-1} - q^{-1} e_{n-1} e_{\alpha_{i+1}+\cdots+2\alpha_n}\right) = -q^{-1} K_i^{-1} e_{\alpha_{i+1}+\cdots+2\alpha_{n-1}+2\alpha_n}.\end{aligned}$$

Let us assume that $\left[e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}, f_i\right] = -q^{-1} K_i^{-1} e_{\alpha_{i+1}+\cdots+2\alpha_j+\cdots+2\alpha_n}$ for some $j = i+3, \ldots, n-1$, then

$$\begin{aligned}\left[e_{\alpha_i+\cdots+2\alpha_{j-1}+\cdots+2\alpha_n}, f_i\right] &= \left[e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n} e_{j-1} - q^{-1} e_{j-1} e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}, f_i\right] \\ &= e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}\left[e_{j-1}, f_i\right] + \left[e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}, f_i\right] e_{j-1} \\ &\quad -q^{-1} e_{j-1}\left[e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}, f_i\right] - q^{-1}\left[e_{j-1}, f_i\right] e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n} \\ &= -q^{-1} K_i^{-1}\left(e_{\alpha_{i+1}+\cdots+2\alpha_j+\cdots+2\alpha_n} e_{j-1} - q^{-1} e_{j-1} e_{\alpha_{i+1}+\cdots+2\alpha_j+\cdots+2\alpha_n}\right) \\ &= -q^{-1} K_i^{-1} e_{\alpha_{i+1}+\cdots+2\alpha_{j-1}+\cdots+2\alpha_n},\end{aligned}$$

which proves (x).

We now prove (xi). For each $i = 1, \ldots, n-2$,

$$\begin{aligned}\left[e_{\alpha_i+2\alpha_{i+1}+\cdots+2\alpha_n}, f_i\right] &= \left[e_{\alpha_i+\alpha_{i+1}+2\alpha_{i+2}+\cdots+2\alpha_n} e_{i+1} - q^{-1} e_{i+1} e_{\alpha_i+\alpha_{i+1}+2\alpha_{i+2}+\cdots+2\alpha_n}, f_i\right] \\ &= e_{\alpha_i+\alpha_{i+1}+2\alpha_{i+2}+\cdots+2\alpha_n}\left[e_{i+1}, f_i\right] + \left[e_{\alpha_i+\alpha_{i+1}+2\alpha_{i+2}+\cdots+2\alpha_n}, f_i\right] e_{i+1} \\ &\quad -q^{-1} e_{i+1}\left[e_{\alpha_i+\alpha_{i+1}+2\alpha_{i+2}+\cdots+2\alpha_n}, f_i\right] - q^{-1}\left[e_{i+1}, f_i\right] e_{\alpha_i+\alpha_{i+1}+2\alpha_{i+2}+\cdots+2\alpha_n} \\ &= -q^{-1} K_i^{-1} e_{\alpha_{i+1}+2\alpha_{i+2}+\cdots+2\alpha_n} e_{i+1} + q^{-3} K_i^{-1} e_{i+1} e_{\alpha_{i+1}+2\alpha_{i+2}+\cdots+2\alpha_n}.\end{aligned}$$

□

The following proposition completes the proof that $(e_\gamma)^{N'}$ and $(e_\beta)^{\overline{N}}$ (anti)commute with f_i for all $\gamma, \beta \in \phi$.

Proposition D.4.4. *There are the identities:*

(i) $\left[\left(e_{\alpha_i+\cdots+\alpha_j}\right)^{N'}, f_i\right] = 0$, *for all* $1 \le i < j \le n-1$,

(ii) $\left[\left(e_{\alpha_i+\cdots+\alpha_n}\right)^{\overline{N}}, f_i\right] = 0$, *for each* $i = 1, \dots, n-1$,

(iii) $\left[\left(e_{\alpha_i+\cdots+\alpha_j}\right)^{N'}, f_j\right] = 0$, *for all* $1 \le i < j \le n-1$,

(iv) $\left[\left(e_{\alpha_i+\cdots+\alpha_n}\right)^{\overline{N}}, f_n\right] = 0$, *for each* $i = 1, \dots, n-1$,

(v) $\left[\left(e_{\alpha_{n-1}+2\alpha_n}\right)^{N'}, f_{n-1}\right] = 0$,

(vi) $\left[\left(e_{\alpha_i+\cdots+2\alpha_n}\right)^{N'}, f_n\right] = 0$, *for each* $i = 1, \dots, n-2$,

(vii) $\left[\left(e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}\right)^{N'}, f_j\right] = 0$, *for each* $j = i+1, \dots, n-1$,

(viii) $\left[\left(e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}\right)^{N'}, f_i\right] = 0$, *for each* $j = i+2, \dots, n$,

(ix) $\left[\left(e_{\alpha_i+2\alpha_{i+1}+\cdots+2\alpha_n}\right)^{N'}, f_i\right] = 0$, *for each* $i = 1, \dots, n-2$.

Proof. We prove (i): for all $1 \le i < j \le n-1$,

$$\left[\left(e_{\alpha_i+\cdots+\alpha_j}\right)^{N'}, f_i\right] = -q^{-1}K_i^{-1}\sum_{t=0}^{N'-1} q^{2N'-2-2t} e_{\alpha_{i+1}+\cdots+\alpha_j}\left(e_{\alpha_i+\cdots+\alpha_j}\right)^{N'-1} = 0.$$

We prove (ii). For each $i = 1, \dots, n-1$,

$$\begin{aligned}
\left[\left(e_{\alpha_i+\cdots+\alpha_n}\right)^{\overline{N}}, f_i\right] &= -q^{\overline{N}-2}K_i^{-1}\left(e_{\alpha_i+\cdots+\alpha_n}\right)^{\overline{N}-1} e_{\alpha_{i+1}+\cdots+\alpha_n} \\
&\quad -q^{\overline{N}-2}K_i^{-1}\sum_{t=1}^{\overline{N}-1} q^{-t}\left(e_{\alpha_i+\cdots+\alpha_n}\right)^{\overline{N}-1-t} e_{\alpha_{i+1}+\cdots+\alpha_n}\left(e_{\alpha_i+\cdots+\alpha_n}\right)^t \\
&= -q^{\overline{N}-2}K_i^{-1}\sum_{t=0}^{\overline{N}-1}(-q)^{-t}\left(e_{\alpha_i+\cdots+\alpha_n}\right)^{\overline{N}-1} e_{\alpha_{i+1}+\cdots+\alpha_n} \\
&\quad -q^{\overline{N}-2}K_i^{-1}\sum_{t=1}^{\overline{N}-1} q^{1-2t}(t)_q\left(e_{\alpha_i+\cdots+\alpha_n}\right)^{\overline{N}-2} e_{\alpha_{i+1}+\cdots+\alpha_n} = 0,
\end{aligned}$$

which vanishes for the same reasons that (D.15) vanishes. Here we used the following result:

$$\begin{aligned}
&\left(e_{\alpha_i+\cdots+\alpha_n}\right)^t e_{\alpha_{i+1}+\cdots+\alpha_n} - (-1)^t e_{\alpha_{i+1}+\cdots+\alpha_n}\left(e_{\alpha_i+\cdots+\alpha_n}\right)^t \\
&\quad = \left[\left(e_{\alpha_i+\cdots+\alpha_n}\right)^t, e_{\alpha_{i+1}+\cdots+\alpha_n}\right]_q \\
&\quad = \sum_{s=0}^{t-1}(-1)^s q^{-s}\left(e_{\alpha_i+\cdots+\alpha_n}\right)^{t-1}\left[e_{\alpha_i+\cdots+\alpha_n}, e_{\alpha_{i+1}+\cdots+\alpha_n}\right]_q \\
&\quad = (-q)^{1-t}(t)_q\left(e_{\alpha_i+\cdots+\alpha_n}\right)^{t-1}\left[e_{\alpha_i+\cdots+\alpha_n}, e_{\alpha_{i+1}+\cdots+\alpha_n}\right]_q,
\end{aligned}$$

in which we have used Eq. (D.13) and the following results:

$$\begin{aligned}
\left[e_{\alpha_i+\cdots+\alpha_n}, e_{\alpha_i+2\alpha_{i+1}+\cdots+2\alpha_n}\right]_q &= 0, \\
\left[e_{\alpha_i+\cdots+\alpha_n}, e_{\alpha_i+\cdots+2\alpha_p+\cdots+2\alpha_n}\right]_q &= 0, \quad p = i+2,\ldots,n, \\
\left[e_{\alpha_i+\cdots+\alpha_n}, e_{\alpha_{i+1}+\cdots+\alpha_{p-1}}\right]_q &= 0, \quad p = i+2,\ldots,n.
\end{aligned}$$

We prove (iii). For all $1 \le i < j \le n-1$,

$$\left[\left(e_{\alpha_i+\cdots+\alpha_j}\right)^{N'}, f_j\right] = K_j \sum_{t=0}^{N'-1} q^{-N'+1+2t}\left(e_{\alpha_i+\cdots+\alpha_j}\right)^{N'-1} e_{\alpha_i+\cdots+\alpha_{j-1}} = 0.$$

We prove (iv): for each $i = 1,\ldots,n-1$,

$$\left[\left(e_{\alpha_i+\cdots+\alpha_n}\right)^{\overline{N}}, f_n\right] = K_n \sum_{t=0}^{\overline{N}-1}(-q)^t\left(e_{\alpha_i+\cdots+\alpha_n}\right)^{\overline{N}-1} e_{\alpha_i+\cdots+\alpha_{n-1}} = 0.$$

We prove (v):

$$\left[\left(e_{\alpha_{n-1}+2\alpha_n}\right)^{N'}, f_{n-1}\right] = -q^{-1}(1+q^{-1})K_{n-1}^{-1}\sum_{t=0}^{N'-1} q^{-2t}\left(e_{\alpha_{n-1}+2\alpha_n}\right)^{N'-1}(e_n)^2 = 0.$$

We prove (vi): for each $i = 1,\ldots,n-2$,

$$\left[\left(e_{\alpha_i+\cdots+2\alpha_n}\right)^{N'}, f_n\right] = -K_n \sum_{t=0}^{N'-1} q^{-N'+1+2t}\left(e_{\alpha_i+\cdots+2\alpha_n}\right)^{N'-1} e_{\alpha_i+\cdots+\alpha_n} = 0.$$

We prove (vii): for each $j = i+1,\ldots,n-1$,

$$\begin{aligned}
&\left[\left(e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}\right)^{N'}, f_j\right] \\
&= K_j \sum_{t=0}^{N'-1} q^{-N'+1+2t}\left(e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}\right)^{N'-1} e_{\alpha_i+\cdots+2\alpha_{j+1}+\cdots+2\alpha_n} = 0.
\end{aligned}$$

We prove (viii): for each $j = i+2, \ldots, n$,

$$\begin{aligned}&\left[\left(e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}\right)^{N'}, f_i\right]\\ &= -q^{-1}K_i^{-1}\sum_{t=0}^{N'-1} q^{N'-1-2t}\left(e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}\right)^{N'-1} e_{\alpha_{i+1}+\cdots+2\alpha_j+\cdots+2\alpha_n} = 0.\end{aligned}$$

We prove (ix): for each $i = 1, \ldots, n-2$,

$$\begin{aligned}&\left[\left(e_{\alpha_i+2\alpha_{i+1}+\cdots+2\alpha_n}\right)^{N'}, f_i\right]\\ &= -q^{-1}K_i^{-1}\sum_{t=0}^{N'-1} q^{-2t}\left(e_{\alpha_i+2\alpha_{i+1}+\cdots+2\alpha_n}\right)^{N'-1} e_{\alpha_{i+1}+2\alpha_{i+2}+\cdots+2\alpha_n} e_{i+1}\\ &\quad +q^{-3}K_i^{-1}\sum_{t=0}^{N'-1} q^{-2t}\left(e_{\alpha_i+2\alpha_{i+1}+\cdots+2\alpha_n}\right)^{N'-1} e_{i+1} e_{\alpha_{i+1}+2\alpha_{i+2}+\cdots+2\alpha_n} = 0,\end{aligned}$$

where we have used $\left[e_{\alpha_i+2\alpha_{i+1}+\cdots+2\alpha_n}, e_{i+1}\right]_q = \left[e_{\alpha_i+2\alpha_{i+1}+\cdots+2\alpha_n}, e_{\alpha_{i+1}+2\alpha_{i+2}+\cdots+2\alpha_n}\right]_q = 0.$ □

D.4.4 The remaining elements of I

We have proved that $(e_\gamma)^{N'}$ and $(e_\beta)^{\overline{N}}$ (anti)commute with each generator of $U_q(\mathfrak{g})$ for each positive root $\gamma, \beta \in \phi$. In this subsection we use the antipode S and the graded antiautomorphism ω to prove that each of $(f_\gamma)^{N'}$, $(f_\beta)^{\overline{N}}$, $(\overline{e}_\gamma)^{N'}$, $(\overline{e}_\beta)^{\overline{N}}$, $(\overline{f}_\gamma)^{N'}$ and $(\overline{f}_\beta)^{\overline{N}}$ also (anti)commute with each generator of $U_q(\mathfrak{g})$ for each positive root $\gamma, \beta \in \phi$.

As ω is a graded antiautomorphism, $\omega(x^m) = (-1)^{m(m-1)[x]/2}\big(\omega(x)\big)^m$ for each $m \in \mathbb{N}$. Now for each generator $x \in U_q(\mathfrak{g})$, we have

$$\left(e_{\alpha_i+\cdots+\alpha_j}\right)^{N'} x = \pm x\left(e_{\alpha_i+\cdots+\alpha_j}\right)^{N'},$$

and applying ω to this equation shows that $\left(f_{\alpha_i+\cdots+\alpha_j}\right)^{N'}$ also (anti)commutes with x. Using almost identical arguments, we can show that each of $(f_{\alpha_i+\cdots+\alpha_n})^{\overline{N}}$ and $\left(f_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}\right)^{N'}$ also (anti)commute with each generator of $U_q(\mathfrak{g})$ for each $j = i+1, \ldots, n$.

Recall that S is a graded antiautomorphism, that $S(e_\eta) = c_\eta \overline{e}_\eta K_\eta^{-1}$ where $0 \neq c_\eta \in \mathbb{C}$, and that $S(f_\eta) = c_\eta K_\eta \overline{f}_\eta$ where $0 \neq c_\eta \in \mathbb{C}$, for each $\eta \in \phi$. It is then almost trivial to prove that the remaining elements of I (anti)commute with each generator of $U_q(\mathfrak{g})$. This completes the proof that the left ideal $\mathcal{I} \subset U_q(\mathfrak{g})$ is a two-sided ideal.

D.5 The co-multiplication of e_μ, $\mu \in \phi$

We now prove that the two-sided ideal $\mathcal{I}$ is a two-sided co-ideal, that is, that

$$\Delta(x) \in \mathcal{I} \otimes U_q(\mathfrak{g}) + U_q(\mathfrak{g}) \otimes \mathcal{I}, \qquad \forall x \in \mathcal{I}.$$

To prove that $\mathcal{I}$ is a two-sided co-ideal, it suffices to show that

$$\Delta(x) \in \mathcal{I} \otimes U_q(\mathfrak{g}) + U_q(\mathfrak{g}) \otimes \mathcal{I}, \qquad \forall x \in I,$$

which we will show in a straightforward way. We will firstly prove that

$$\Delta(e_\gamma)^{N'} \in \mathcal{I} \otimes U_q(\mathfrak{g}) + U_q(\mathfrak{g}) \otimes \mathcal{I}, \tag{D.16}$$

$$\Delta(e_\beta)^{\overline{N}} \in \mathcal{I} \otimes U_q(\mathfrak{g}) + U_q(\mathfrak{g}) \otimes \mathcal{I}, \tag{D.17}$$

for each $\gamma, \beta \in \phi$, and then by using the antipode S, the graded antiautomorphism ω, the relations

$$\Delta \circ S = (S \otimes S) \circ \Delta', \qquad \Delta \circ \omega = (\omega \otimes \omega) \circ \Delta',$$

and Propositions D.3.4–D.3.8, we will prove the appropriate results for all other elements of I except $(J_i^{\pm N} - 1)$: trivially, $\Delta(J_i^{\pm N} - 1) \in \mathcal{I} \otimes U_q(\mathfrak{g}) + U_q(\mathfrak{g}) \otimes \mathcal{I}$.

We now explicitly determine the components of $\Delta(e_\mu)$ before calculating powers of $\Delta(e_\mu)$ for each $\mu \in \phi$. In succeeding sections, we will calculate the commutation relations between the components of $\Delta(e_\mu)$, and then by using these commutation relations, the q-binomial theorem and the two generalisations of the binomial theorem in Appendix B, we will prove relations (D.16)–(D.17) for all γ and β in ϕ.

The co-multiplication of e_{α_i} for each simple root α_i is given in the definition of $U_q(\mathfrak{g})$ and will not be further considered.

It is not a trivial matter to calculate the co-multiplication of e_μ for each non-simple root $\mu \in \phi$, as the method for constructing e_μ does not 'commute' with the co-multiplication. (A similar situation occurs for quantum algebras [CP94].) We will directly calculate $\Delta(e_\mu)$: it is not difficult to calculate $\Delta(e_{\alpha_i+\cdots+\alpha_j})$ for all $1 \le i < j \le n$, nor is it difficult to calculate $\Delta(e_{\alpha_i+\cdots+2\alpha_n})$. However, it is much more difficult to calculate $\Delta(e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n})$ for each $j = i+1, \ldots, n-1$.

After calculating $\Delta(e_\mu)$, we can calculate $\Delta(\overline{e}_\mu)$, $\Delta(f_\mu)$ and $\Delta(\overline{f}_\mu)$ by applying the antipode S and the graded antiautomorphism ω to $\Delta(e_\mu)$ as in the following equations. In these equations the proportionality sign means that the left hand side is proportional to the right hand side with a non-zero scalar constant of proportionality:

$$\begin{array}{rcccl}
\Delta(f_\mu) & \propto & \Delta\big(\omega(e_\mu)\big) & \propto & (\omega \otimes \omega) \circ \Delta'(e_\mu), \\
\Delta(\overline{e}_\mu) & \propto & \Delta\big(S(e_\mu)\big)\Delta(K_\mu) & \propto & [(S \otimes S) \circ \Delta'(e_\mu)]\, \Delta(K_\mu), \\
\Delta(\overline{f}_\mu) & \propto & \Delta\big(\omega(\overline{e}_\mu)\big) & \propto & (\omega \otimes \omega) \circ \Delta'(\overline{e}_\mu).
\end{array}$$

D.5.1 $\Delta(e_{\alpha_i+\cdots+\alpha_j})$, $1 \le i < j \le n$

Lemma D.5.1. *For each $j = i+1, \ldots, n$, the co-multiplication of $e_{\alpha_i+\cdots+\alpha_j}$ is*

$$\begin{aligned}
\Delta(e_{\alpha_i+\cdots+\alpha_j}) &= e_{\alpha_i+\cdots+\alpha_j} \otimes K_{\alpha_i+\cdots+\alpha_j} + 1 \otimes e_{\alpha_i+\cdots+\alpha_j} \\
&\quad + (q - q^{-1}) \sum_{k=i+1}^{j} e_{\alpha_k+\cdots+\alpha_j} \otimes K_{\alpha_k+\cdots+\alpha_j} e_{\alpha_i+\cdots+\alpha_{k-1}}.
\end{aligned} \tag{D.18}$$

Proof. Obviously, $\Delta(e_{\alpha_i+\cdots+\alpha_j}) = \Delta(e_{\alpha_i+\cdots+\alpha_{j-1}})\Delta(e_j) - q^{-1}\Delta(e_j)\Delta(e_{\alpha_i+\cdots+\alpha_{j-1}})$. We will prove the lemma inductively. Firstly, we calculate $\Delta(e_{\alpha_i+\alpha_{i+1}})$ for each $i = 1, \ldots, n-1$:

$$\begin{aligned}
\Delta(e_{\alpha_i+\alpha_{i+1}}) &= (e_i \otimes K_i + 1 \otimes e_i)(e_{i+1} \otimes K_{i+1} + 1 \otimes e_{i+1}) \\
&\quad -q^{-1}(e_{i+1} \otimes K_{i+1} + 1 \otimes e_{i+1})(e_i \otimes K_i + 1 \otimes e_i) \\
&= e_i e_{i+1} \otimes K_i K_{i+1} - q^{-1} e_{i+1} e_i \otimes K_{i+1} K_i + 1 \otimes e_i e_{i+1} - q^{-1}\left(1 \otimes e_{i+1} e_i\right) \\
&\quad + e_i \otimes K_i e_{i+1} - q^{-1}\left(e_{i+1} \otimes K_{i+1} e_i\right) + e_{i+1} \otimes e_i K_{i+1} - q^{-1}\left(e_i \otimes e_{i+1} K_i\right) \\
&= e_{\alpha_i+\alpha_{i+1}} \otimes K_{\alpha_i+\alpha_{i+1}} + 1 \otimes e_{\alpha_i+\alpha_{i+1}} + \left(q - q^{-1}\right)\left(e_{i+1} \otimes K_{i+1} e_i\right).
\end{aligned}$$

Now assume that (D.18) is true for some $j \geq i+1$, then we need to calculate

$$\Delta(e_{\alpha_i+\cdots+\alpha_{j+1}}) = \Delta(e_{\alpha_i+\cdots+\alpha_j})\Delta(e_{j+1}) - q^{-1}\Delta(e_{j+1})\Delta(e_{\alpha_i+\cdots+\alpha_j}).$$

Firstly,

$$\begin{aligned}
&\left(e_{\alpha_i+\cdots+\alpha_j} \otimes K_{\alpha_i+\cdots+\alpha_j} + 1 \otimes e_{\alpha_i+\cdots+\alpha_j}\right)\left(e_{j+1} \otimes K_{j+1} + 1 \otimes e_{j+1}\right) \\
&\qquad -q^{-1}\left(e_{j+1} \otimes K_{j+1} + 1 \otimes e_{j+1}\right)\left(e_{\alpha_i+\cdots+\alpha_j} \otimes K_{\alpha_i+\cdots+\alpha_j} + 1 \otimes e_{\alpha_i+\cdots+\alpha_j}\right) \\
&= e_{\alpha_i+\cdots+\alpha_{j+1}} \otimes K_{\alpha_i+\cdots+\alpha_{j+1}} + 1 \otimes e_{\alpha_i+\cdots+\alpha_{j+1}} + (q-q^{-1}) e_{j+1} \otimes K_{j+1} e_{\alpha_i+\cdots\alpha_j},
\end{aligned}$$

and to determine the remaining components of $\Delta(e_{\alpha_i+\cdots+\alpha_{j+1}})$ we calculate that

$$\begin{aligned}
&\left(e_{\alpha_k+\cdots+\alpha_j} \otimes K_{\alpha_k+\cdots+\alpha_j} e_{\alpha_i+\cdots+\alpha_{k-1}}\right)\left(e_{j+1} \otimes K_{j+1} + 1 \otimes e_{j+1}\right) \\
&\qquad -q^{-1}\left(e_{j+1} \otimes K_{j+1} + 1 \otimes e_{j+1}\right)\left(e_{\alpha_k+\cdots+\alpha_j} \otimes K_{\alpha_k+\cdots+\alpha_j} e_{\alpha_i+\cdots+\alpha_{k-1}}\right) \\
&= e_{\alpha_k+\cdots+\alpha_{j+1}} \otimes K_{\alpha_k+\cdots+\alpha_{j+1}} e_{\alpha_i+\cdots+\alpha_{k-1}}.
\end{aligned}$$

□

D.5.2 $\Delta(e_{\alpha_i+\cdots+2\alpha_n})$, $i = 1, \ldots, n-1$

Lemma D.5.2. *For each $i = 1, 2, \ldots, n-1$, the co-multiplication of $e_{\alpha_i+\cdots+2\alpha_n}$ is*

$$\begin{aligned}
\Delta(e_{\alpha_i+\cdots+2\alpha_n}) &= e_{\alpha_i+\cdots+2\alpha_n} \otimes K_{\alpha_i+\cdots+2\alpha_n} + 1 \otimes e_{\alpha_i+\cdots+2\alpha_n} \\
&\quad + (q-q^{-1}) \sum_{k=i+1}^{n-1} e_{\alpha_k+\cdots+2\alpha_n} \otimes K_{\alpha_k+\cdots+2\alpha_n} e_{\alpha_i+\cdots+\alpha_{k-1}} \\
&\quad + (q-q^{-1})\Big((1+q)(e_n)^2 \otimes (K_n)^2 e_{\alpha_i+\cdots+\alpha_{n-1}} + e_n \otimes K_n e_{\alpha_i+\cdots+\alpha_n}\Big).
\end{aligned}$$

Proof. Recall that $e_{\alpha_i+\cdots+2\alpha_n} = e_{\alpha_i+\cdots+\alpha_n} e_n + e_n e_{\alpha_i+\cdots+\alpha_n}$. Now from Lemma D.5.1,

$$\begin{aligned}
\Delta(e_{\alpha_i+\cdots+\alpha_n}) &= e_{\alpha_i+\cdots+\alpha_n} \otimes K_{\alpha_i+\cdots+\alpha_n} + 1 \otimes e_{\alpha_i+\cdots+\alpha_n} \\
&\quad + (q-q^{-1}) \sum_{k=i+1}^{n} e_{\alpha_k+\cdots+\alpha_n} \otimes K_{\alpha_k+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_{k-1}}, \qquad \text{(D.19)}
\end{aligned}$$

and we determine $\Delta(e_{\alpha_i+\cdots+2\alpha_n})$ from the following calculations. Firstly,

$$\begin{aligned}
&\big(e_{\alpha_i+\cdots+\alpha_n}\otimes K_{\alpha_i+\cdots+\alpha_n}+1\otimes e_{\alpha_i+\cdots+\alpha_n}\big)\Delta(e_n)+\Delta(e_n)\big(e_{\alpha_i+\cdots+\alpha_n}\otimes K_{\alpha_i+\cdots+\alpha_n}+1\otimes e_{\alpha_i+\cdots+\alpha_n}\big)\\
&\quad= e_{\alpha_i+\cdots+\alpha_n}e_n\otimes K_{\alpha_i+\cdots+2\alpha_n}+e_ne_{\alpha_i+\cdots+\alpha_n}\otimes K_{\alpha_i+\cdots+2\alpha_n}+1\otimes e_{\alpha_i+\cdots+\alpha_n}e_n+1\otimes e_ne_{\alpha_i+\cdots+\alpha_n}\\
&\quad= e_{\alpha_i+\cdots+2\alpha_n}\otimes K_{\alpha_i+\cdots+2\alpha_n}+1\otimes e_{\alpha_i+\cdots+2\alpha_n},
\end{aligned}$$

and furthermore,

$$\begin{aligned}
&\sum_{k=i+1}^{n}\Big[e_{\alpha_k+\cdots+\alpha_n}\otimes K_{\alpha_k+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_{k-1}}\Delta(e_n)+\Delta(e_n)e_{\alpha_k+\cdots+\alpha_n}\otimes K_{\alpha_k+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_{k-1}}\Big]\\
&\quad=q(e_n)^2\otimes(K_n)^2e_{\alpha_i+\cdots+\alpha_{n-1}}+(e_n)^2\otimes(K_n)^2e_{\alpha_i+\cdots+\alpha_{n-1}}\\
&\qquad+e_n\otimes K_ne_{\alpha_i+\cdots+\alpha_{n-1}}e_n-q^{-1}e_n\otimes K_ne_ne_{\alpha_i+\cdots+\alpha_{n-1}}\\
&\qquad+\sum_{k=i+1}^{n-1}\Big[e_{\alpha_k+\cdots+\alpha_n}e_n\otimes K_{\alpha_k+\cdots+2\alpha_n}e_{\alpha_i+\cdots+\alpha_{k-1}}+e_ne_{\alpha_k+\cdots+\alpha_n}\otimes K_{\alpha_k+\cdots+2\alpha_n}e_{\alpha_i+\cdots+\alpha_{k-1}}\\
&\qquad\qquad+e_{\alpha_k+\cdots+\alpha_n}\otimes K_{\alpha_k+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_{k-1}}e_n-e_{\alpha_k+\cdots+\alpha_n}\otimes K_{\alpha_k+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_{k-1}}e_n\Big]\\
&\quad=\sum_{k=i+1}^{n-1}e_{\alpha_k+\cdots+2\alpha_n}e_n\otimes K_{\alpha_k+\cdots+2\alpha_n}e_{\alpha_i+\cdots+\alpha_{k-1}}\\
&\qquad+(1+q)(e_n)^2\otimes(K_n)^2e_{\alpha_i+\cdots+\alpha_{n-1}}+e_n\otimes K_ne_{\alpha_i+\cdots+\alpha_n}.
\end{aligned}$$

□

D.5.3 $\Delta(e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n})$, $1\le i<j<n$

Lemma D.5.3. *For all $1\le i<j<n$, the co-multiplication of $e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}$ is*

$$\Delta(e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n})=D_0+(q-q^{-1})\sum_{k=i+1}^{n}D_k+(q-q^{-1})\overline{D}+(q-q^{-1})\sum_{l=j}^{n-1}F_l+F_{j-1},$$

where

$$\begin{aligned}
D_0&=e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}\otimes K_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n},\\
D_k&=e_{\alpha_k+\cdots+2\alpha_j+\cdots+2\alpha_n}\otimes K_{\alpha_k+\cdots+2\alpha_j+\cdots+2\alpha_n}e_{\alpha_i+\cdots+\alpha_{k-1}}, && k=i+1,\dots,j-1,\\
\overline{D}&=\big(qe_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_j-q^{-1}e_je_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}\big)\\
&\qquad\otimes(K_j\cdots K_n)^2e_{\alpha_i+\cdots+\alpha_{j-1}},\\
D_k&=\overline{e}_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}\otimes K_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}e_{\alpha_i+\cdots+\alpha_k}, && k=j,\dots,n-1,\\
D_n&=\overline{e}_{\alpha_j+\cdots+\alpha_n}\otimes K_{\alpha_j+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_n},\\
F_p&=\overline{e}_{\alpha_j+\cdots+\alpha_p}\otimes K_{\alpha_j+\cdots+\alpha_p}e_{\alpha_i+\cdots+2\alpha_{p+1}+\cdots+2\alpha_n}, && p=j,\dots,n-1,\\
F_{j-1}&=1\otimes e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}.
\end{aligned}$$

Proof. Firstly, recall that

$$e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n} = e_{\alpha_i+\cdots+2\alpha_{j+1}+\cdots+2\alpha_n}e_j - q^{-1}e_je_{\alpha_i+\cdots+2\alpha_{j+1}+\cdots+2\alpha_n}, \qquad j = i+1,\dots,n-1.$$

In proving this lemma, we firstly determine $\Delta(e_{\alpha_i+\cdots+2\alpha_{n-1}+2\alpha_n})$, and then prove the remaining cases by induction.

We now obtain two very useful results. Set $k = i+1,\dots,n-2$ and $k < j < n$, and in writing $e_{\alpha_i+\cdots+\alpha_{k-1}}$ in the calculations below we will always consider this to be identically equal to 1 if $k-1 < i$. Furthermore, if $k-1 = i$, then in writing $\alpha_i + \cdots + \alpha_{k-1}$ we will always take this to mean α_i. The first useful result is

$$\begin{aligned}
&\left(e_{\alpha_k+\cdots+2\alpha_{j+1}+\cdots+2\alpha_n} \otimes K_{\alpha_k+\cdots+2\alpha_{j+1}+\cdots+2\alpha_n}e_{\alpha_i+\cdots+\alpha_{k-1}}\right)\Delta(e_j)\\
&\quad -q^{-1}\Delta(e_j)\left(e_{\alpha_k+\cdots+2\alpha_{j+1}+\cdots+2\alpha_n} \otimes K_{\alpha_k+\cdots+2\alpha_{j+1}+\cdots+2\alpha_n}e_{\alpha_i+\cdots+\alpha_{k-1}}\right)\\
=\;& e_{\alpha_k+\cdots+2\alpha_{j+1}+\cdots+2\alpha_n}e_j \otimes K_{\alpha_k+\cdots+2\alpha_j+\cdots+2\alpha_n}e_{\alpha_i+\cdots+\alpha_{k-1}}\\
&-q^{-1}e_je_{\alpha_k+\cdots+2\alpha_{j+1}+\cdots+2\alpha_n} \otimes K_{\alpha_k+\cdots+2\alpha_j+\cdots+2\alpha_n}e_{\alpha_i+\cdots+\alpha_{k-1}}\\
&+e_{\alpha_k+\cdots+2\alpha_{j+1}+\cdots+2\alpha_n} \otimes K_{\alpha_k+\cdots+2\alpha_{j+1}+\cdots+2\alpha_n}e_{\alpha_i+\cdots+\alpha_{k-1}}e_j\\
&-e_{\alpha_k+\cdots+2\alpha_{j+1}+\cdots+2\alpha_n} \otimes K_{\alpha_k+\cdots+2\alpha_{j+1}+\cdots+2\alpha_n}e_je_{\alpha_i+\cdots+\alpha_{k-1}}\\
=\;& e_{\alpha_k+\cdots+2\alpha_j+\cdots+2\alpha_n} \otimes K_{\alpha_k+\cdots+2\alpha_j+\cdots+2\alpha_n}e_{\alpha_i+\cdots+\alpha_{k-1}},
\end{aligned} \tag{D.20}$$

as e_j commutes with $e_{\alpha_i+\cdots+\alpha_{k-1}}$. The second useful result is as follows: set $i < j < n$, then

$$\begin{aligned}
&\left(1 \otimes e_{\alpha_i+\cdots+2\alpha_{j+1}+\cdots+2\alpha_n}\right)\Delta(e_j) - q^{-1}\Delta(e_j)\left(1 \otimes e_{\alpha_i+\cdots+2\alpha_{j+1}+\cdots+2\alpha_n}\right)\\
&\quad = (q-q^{-1})\left(e_j \otimes K_je_{\alpha_i+\cdots+2\alpha_{j+1}+\cdots+2\alpha_n}\right) + 1 \otimes e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}.
\end{aligned} \tag{D.21}$$

These results imply that

$$\begin{aligned}
&\Big[e_{\alpha_i+\cdots+2\alpha_n} \otimes K_{\alpha_i+\cdots+2\alpha_n} + 1 \otimes e_{\alpha_i+\cdots+2\alpha_n}\\
&\qquad +(q-q^{-1})\sum_{k=i+1}^{n-2} e_{\alpha_k+\cdots+2\alpha_n} \otimes K_{\alpha_k+\cdots+2\alpha_n}e_{\alpha_i+\cdots+\alpha_{k-1}}\Big]\Delta(e_{n-1})\\
&-q^{-1}\Delta(e_{n-1})\Big[e_{\alpha_i+\cdots+2\alpha_n} \otimes K_{\alpha_i+\cdots+2\alpha_n} + 1 \otimes e_{\alpha_i+\cdots+2\alpha_n}\\
&\qquad\qquad +(q-q^{-1})\sum_{k=i+1}^{n-2} e_{\alpha_k+\cdots+2\alpha_n} \otimes K_{\alpha_k+\cdots+2\alpha_n}e_{\alpha_i+\cdots+\alpha_{k-1}}\Big]\\
&\quad = e_{\alpha_i+\cdots+2\alpha_{n-2}+2\alpha_n} \otimes K_{\alpha_i+\cdots+2\alpha_{n-2}+2\alpha_n} + 1 \otimes e_{\alpha_i+\cdots+2\alpha_{n-2}+2\alpha_n}\\
&\qquad +(q-q^{-1})\sum_{k=i+1}^{n-2} e_{\alpha_k+\cdots+2\alpha_{n-1}+2\alpha_n} \otimes K_{\alpha_k+\cdots+2\alpha_{n-1}+2\alpha_n}e_{\alpha_i+\cdots+\alpha_{k-1}}\\
&\qquad +(q-q^{-1})\left(e_{n-1} \otimes K_{n-1}e_{\alpha_i+\cdots+2\alpha_n}\right).
\end{aligned}$$

To complete the proof, we need only simplify

$$\Big[e_{\alpha_{n-1}+2\alpha_n} \otimes K_{\alpha_{n-1}+2\alpha_n} e_{\alpha_i+\cdots+\alpha_{n-2}} \\ +(1+q)(e_n)^2 \otimes (K_n)^2 e_{\alpha_i+\cdots+\alpha_{n-1}} + e_n \otimes K_n e_{\alpha_i+\cdots+\alpha_n}\Big]\Delta(e_{n-1}) \\ -q^{-1}\Delta(e_{n-1})\Big[e_{\alpha_{n-1}+2\alpha_n} \otimes K_{\alpha_{n-1}+2\alpha_n} e_{\alpha_i+\cdots+\alpha_{n-2}} \\ +(1+q)(e_n)^2 \otimes (K_n)^2 e_{\alpha_i+\cdots+\alpha_{n-1}} + e_n \otimes K_n e_{\alpha_i+\cdots+\alpha_n}\Big]. \quad \text{(D.22)}$$

To simplify (D.22) we perform the following calculations:

$$\begin{aligned} &\left(e_{\alpha_{n-1}+2\alpha_n} \otimes K_{\alpha_{n-1}+2\alpha_n} e_{\alpha_i+\cdots+\alpha_{n-2}}\right)\Delta(e_{n-1}) \\ &\qquad -q^{-1}\Delta(e_{n-1})\left(e_{\alpha_{n-1}+2\alpha_n} \otimes K_{\alpha_{n-1}+2\alpha_n} e_{\alpha_i+\cdots+\alpha_{n-2}}\right) \\ &= \left(q e_{\alpha_{n-1}+2\alpha_n} e_{n-1} - q^{-1} e_{n-1} e_{\alpha_{n-1}+2\alpha_n}\right) \otimes (K_{\alpha_{n-1}+\alpha_n})^2 e_{\alpha_i+\cdots+\alpha_{n-2}} \\ &\quad + e_{\alpha_{n-1}+2\alpha_n} \otimes K_{\alpha_{n-1}+2\alpha_n} e_{\alpha_i+\cdots+\alpha_{n-1}}, \end{aligned}$$

and

$$\begin{aligned} &\Big((e_n)^2 \otimes (K_n)^2 e_{\alpha_i+\cdots+\alpha_{n-1}}\Big)\Delta(e_{n-1}) - q^{-1}\Delta(e_{n-1})\Big((e_n)^2 \otimes (K_n)^2 e_{\alpha_i+\cdots+\alpha_{n-1}}\Big) \\ &= q^{-1}(e_n)^2 e_{n-1} \otimes K_{n-1}(K_n)^2 e_{\alpha_i+\cdots+\alpha_{n-1}} - q^{-1} e_{n-1}(e_n)^2 \otimes K_{n-1}(K_n)^2 e_{\alpha_i+\cdots+\alpha_{n-1}} \\ &\quad + (e_n)^2 \otimes (K_n)^2 e_{\alpha_i+\cdots+\alpha_{n-1}} e_{n-1} - q(e_n)^2 \otimes (K_n)^2 e_{n-1} e_{\alpha_i+\cdots+\alpha_{n-1}} \\ &= q^{-1}\Big((e_n)^2 e_{n-1} - e_{n-1}(e_n)^2\Big) \otimes K_{n-1}(K_n)^2 e_{\alpha_i+\cdots+\alpha_{n-1}}, \end{aligned}$$

where we have used the fact that $[e_{\alpha_i+\cdots+\alpha_{n-1}}, e_{n-1}]_q = 0$. Furthermore,

$$\begin{aligned} &(e_n \otimes K_n e_{\alpha_i+\cdots+\alpha_n})\,\Delta(e_{n-1}) - q^{-1}\Delta(e_{n-1})\,(e_n \otimes K_n e_{\alpha_i+\cdots+\alpha_n}) \\ &= e_n e_{n-1} \otimes K_{n-1} K_n e_{\alpha_i+\cdots+\alpha_n} - q^{-1} e_{n-1} e_n \otimes K_{n-1} K_n e_{\alpha_i+\cdots+\alpha_n} \\ &\quad + e_n \otimes K_n e_{\alpha_i+\cdots+\alpha_n} e_{n-1} - q^{-1} e_n \otimes e_{n-1} K_n e_{\alpha_i+\cdots+\alpha_n} \\ &= \overline{e}_{\alpha_{n-1}+\alpha_n} \otimes K_{n-1} K_n e_{\alpha_i+\cdots+\alpha_n}, \end{aligned}$$

where we have used the fact that $[e_{\alpha_i+\cdots+\alpha_n}, e_{n-1}]_q = 0$.

Combining these results, we can rewrite (D.22) as

$$\begin{aligned} &\left(q e_{\alpha_{n-1}+2\alpha_n} e_{n-1} - q^{-1} e_{n-1} e_{\alpha_{n-1}+2\alpha_n}\right) \otimes (K_{n-1})^2 (K_n)^2 e_{\alpha_i+\cdots+\alpha_{n-2}} \\ &\quad + \Big[e_{\alpha_{n-1}+2\alpha_n} + (1+q^{-1})\left((e_n)^2 e_{n-1} - e_{n-1}(e_n)^2\right)\Big] \otimes K_{\alpha_{n-1}+2\alpha_n} e_{\alpha_i+\cdots+\alpha_{n-1}} \\ &\qquad + \overline{e}_{\alpha_{n-1}+\alpha_n} \otimes K_{n-1} K_n e_{\alpha_i+\cdots+\alpha_n} \\ &= \left(q e_{\alpha_{n-1}+2\alpha_n} e_{n-1} - q^{-1} e_{n-1} e_{\alpha_{n-1}+2\alpha_n}\right) \otimes (K_{n-1})^2 (K_n)^2 e_{\alpha_i+\cdots+\alpha_{n-2}} \\ &\quad + \overline{e}_{\alpha_{n-1}+2\alpha_n} \otimes K_{\alpha_{n-1}+2\alpha_n} e_{\alpha_i+\cdots+\alpha_{n-1}} + \overline{e}_{\alpha_{n-1}+\alpha_n} \otimes K_{n-1} K_n e_{\alpha_i+\cdots+\alpha_n}, \end{aligned}$$

where we have used the elementary result that

$$\overline{e}_{\alpha_{n-1}+2\alpha_n} = e_{\alpha_{n-1}+2\alpha_n} + (1+q^{-1})\left[(e_n)^2 e_{n-1} - e_{n-1}(e_n)^2\right].$$

Combining these calculations gives us

$$\begin{aligned}
&\Delta(e_{\alpha_i+\cdots+2\alpha_{n-1}+2\alpha_n}) \\
&= e_{\alpha_i+\cdots+2\alpha_{n-1}+2\alpha_n}\otimes K_{\alpha_i+\cdots+2\alpha_{n-1}+2\alpha_n} \\
&\quad +(q-q^{-1})\left[\sum_{k=i+1}^{n-2} e_{\alpha_k+\cdots+2\alpha_{n-1}+2\alpha_n}\otimes K_{\alpha_k+\cdots+2\alpha_{n-1}+2\alpha_n}e_{\alpha_i+\cdots+\alpha_{k-1}}\right] \\
&\quad +(q-q^{-1})\Big[\left(qe_{\alpha_{n-1}+2\alpha_n}e_{n-1}-q^{-1}e_{n-1}e_{\alpha_{n-1}+2\alpha_n}\right)\otimes(K_{n-1}K_n)^2e_{\alpha_i+\cdots+\alpha_{n-2}}\Big] \\
&\quad +(q-q^{-1})\Big[\overline{e}_{\alpha_{n-1}+2\alpha_n}\otimes K_{\alpha_{n-1}+2\alpha_n}e_{\alpha_i+\cdots+\alpha_{n-1}}\Big] \\
&\quad +(q-q^{-1})\Big[\overline{e}_{\alpha_{n-1}+\alpha_n}\otimes K_{\alpha_{n-1}+\alpha_n}e_{\alpha_i+\cdots+\alpha_n}\Big] \\
&\quad +(q-q^{-1})\Big[e_{n-1}\otimes K_{n-1}e_{\alpha_i+\cdots+2\alpha_n}\Big] \\
&\quad +1\otimes e_{\alpha_i+\cdots+2\alpha_{n-1}+2\alpha_n},
\end{aligned}$$

which proves the lemma for $\Delta(e_{\alpha_i+\cdots+2\alpha_{n-1}+2\alpha_n})$.

Now assume that the lemma is true for $\Delta(e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n})$ for some $j=i+2,\ldots,n-1$, then Eqs. (D.20)–(D.21) allow us to write

$$\begin{aligned}
&\left[D_0+(q-q^{-1})\sum_{k=i+1}^{j-2}D_k+F_{j-1}\right]\Delta(e_{j-1})-q^{-1}\Delta(e_{j-1})\left[D_0+(q-q^{-1})\sum_{k=i+1}^{j-2}D_k+F_{j-1}\right] \\
&= e_{\alpha_i+\cdots+2\alpha_{j-1}+\cdots+2\alpha_n}\otimes K_{\alpha_i+\cdots+2\alpha_{j-1}+\cdots+2\alpha_n} \\
&\quad +(q-q^{-1})\left(\sum_{k=i+1}^{j-2}e_{\alpha_k+\cdots+2\alpha_{j-1}+\cdots+2\alpha_n}\otimes K_{\alpha_k+\cdots+2\alpha_{j-1}+\cdots+2\alpha_n}e_{\alpha_i+\cdots+\alpha_{k-1}}\right) \\
&\quad +(q-q^{-1})\left(e_{j-1}\otimes K_{j-1}e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}\right) \\
&\quad +1\otimes e_{\alpha_i+\cdots+2\alpha_{j-1}+\cdots+2\alpha_n}.
\end{aligned}$$

We now perform another calculation that will greatly assist the proof of this lemma. Let φ be any element of ϕ satisfying $\alpha_j\preceq\varphi\prec\alpha_{j+1}$ with respect to $\mathcal{N}(\phi)$, then

$$\begin{aligned}
&\left(\overline{e}_\varphi\otimes K_\varphi e_{(\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n-\varphi)}\right)\Delta(e_{j-1})-q^{-1}\Delta(e_{j-1})\left(\overline{e}_\varphi\otimes K_\varphi e_{(\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n-\varphi)}\right) \\
&= \overline{e}_\varphi e_{j-1}\otimes K_\varphi e_{(\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n-\varphi)}K_{j-1}+\overline{e}_\varphi\otimes K_\varphi e_{(\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n-\varphi)}e_{j-1} \\
&\quad -q^{-1}e_{j-1}\overline{e}_\varphi\otimes K_{j-1}K_\varphi e_{(\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n-\varphi)}e_{j-1}-q^{-1}\overline{e}_\varphi\otimes e_{j-1}K_\varphi e_{(\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n-\varphi)} \\
&= \left[\overline{e}_\varphi,e_{j-1}\right]_q\otimes K_{j-1}K_\varphi e_{(\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n-\varphi)} \\
&\quad +\overline{e}_\varphi\otimes K_\varphi e_{(\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n-\varphi)}e_{j-1}-\overline{e}_\varphi\otimes K_\varphi e_{j-1}e_{(\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n-\varphi)} \\
&= \overline{e}_{\alpha_{j-1}+\varphi}\otimes K_{j-1}K_\varphi e_{(\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n-\varphi)},
\end{aligned}$$

where we have used the fact that $\left[e_{(\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n-\varphi)},e_{j-1}\right]_q=0$ (note that this can be rewritten as $e_{(\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n-\varphi)}e_{j-1} = e_{j-1}e_{(\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n-\varphi)})$. This allows us to write

$$\begin{aligned}
&\left[(q-q^{-1})\sum_{k=j}^{n} D_k + (q-q^{-1})\sum_{l=j}^{n-1} F_l\right]\Delta(e_{j-1})\\
&\qquad -q^{-1}\Delta(e_{j-1})\left[(q-q^{-1})\sum_{k=j}^{n} D_k + (q-q^{-1})\sum_{l=j}^{n-1} F_l\right]\\
&= (q-q^{-1})\left(\sum_{k=j}^{n-1}\overline{e}_{\alpha_{j-1}+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}\otimes K_{\alpha_{j-1}+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}e_{\alpha_i+\cdots+\alpha_k}\right)\\
&\qquad +(q-q^{-1})\left(\overline{e}_{\alpha_{j-1}+\cdots+2\alpha_n}\otimes K_{\alpha_{j-1}+\cdots+\alpha_n}e_{\alpha_i+\cdots+2\alpha_n}\right)\\
&\qquad +(q-q^{-1})\sum_{l=j}^{n-1}\left(\overline{e}_{\alpha_{j-1}+\cdots+\alpha_l}\otimes K_{\alpha_{j-1}+\cdots+\alpha_l}e_{\alpha_i+\cdots+2\alpha_{l+1}+\cdots+2\alpha_n}\right),
\end{aligned}$$

and to complete the proof of the lemma we need only calculate

$$\left(\overline{D}+D_{j-1}\right)\Delta(e_{j-1}) - q^{-1}\Delta(e_{j-1})\left(\overline{D}+D_{j-1}\right).$$

Now

$$\begin{aligned}
&D_{j-1}\Delta(e_{j-1}) - q^{-1}\Delta(e_{j-1})D_{j-1}\\
&= \left(e_{\alpha_{j-1}+2\alpha_j+\cdots+2\alpha_n}\otimes K_{\alpha_{j-1}+2\alpha_j+\cdots+2\alpha_n}e_{\alpha_i+\cdots+\alpha_{j-2}}\right)\Delta(e_{j-1})\\
&\quad -q^{-1}\Delta(e_{j-1})\left(e_{\alpha_{j-1}+2\alpha_j+\cdots+2\alpha_n}\otimes K_{\alpha_{j-1}+2\alpha_j+\cdots+2\alpha_n}e_{\alpha_i+\cdots+\alpha_{j-2}}\right) \qquad \text{(D.23)}\\
&= qe_{\alpha_{j-1}+2\alpha_j+\cdots+2\alpha_n}e_{j-1}\otimes(K_{\alpha_{j-1}+\cdots+\alpha_n})^2 e_{\alpha_i+\cdots+\alpha_{j-2}}\\
&\quad -q^{-1}e_{j-1}e_{\alpha_{j-1}+2\alpha_j+\cdots+2\alpha_n}e_{j-1}\otimes(K_{\alpha_{j-1}+\cdots+\alpha_n})^2 e_{\alpha_i+\cdots+\alpha_{j-2}}\\
&\quad +e_{\alpha_{j-1}+2\alpha_j+\cdots+2\alpha_n}\otimes K_{\alpha_{j-1}+2\alpha_j+\cdots+2\alpha_n}e_{\alpha_i+\cdots+\alpha_{j-2}}e_{j-1}\\
&\quad -q^{-1}e_{\alpha_{j-1}+2\alpha_j+\cdots+2\alpha_n}\otimes K_{\alpha_{j-1}+2\alpha_j+\cdots+2\alpha_n}e_{j-1}e_{\alpha_i+\cdots+\alpha_{j-2}}\\
&= \left(qe_{\alpha_{j-1}+2\alpha_j+\cdots+2\alpha_n}e_{j-1} - q^{-1}e_{j-1}e_{\alpha_{j-1}+2\alpha_j+\cdots+2\alpha_n}\right)\otimes(K_{\alpha_{j-1}+\cdots+\alpha_n})^2 e_{\alpha_i+\cdots+\alpha_{j-2}}\\
&\quad +e_{\alpha_{j-1}+2\alpha_j+\cdots+2\alpha_n}\otimes K_{\alpha_{j-1}+2\alpha_j+\cdots+2\alpha_n}e_{\alpha_i+\cdots+\alpha_{j-1}}, \qquad \text{(D.24)}
\end{aligned}$$

and

$$\begin{aligned}
&\overline{D}\Delta(e_{j-1}) - q^{-1}\Delta(e_{j-1})\overline{D}\\
&= \Big[e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_j e_{j-1} - q^{-2}e_j e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_{j-1}\\
&\qquad -e_{j-1}e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_j + q^{-2}e_{j-1}e_j e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}\Big]\\
&\qquad\qquad \otimes K_{\alpha_{j-1}+2\alpha_j+\cdots+2\alpha_n}e_{\alpha_i+\cdots+\alpha_{j-1}}. \qquad \text{(D.25)}
\end{aligned}$$

We combine (D.25) and the second line of (D.24) using the following calculation:

$$\begin{aligned}
& e_{\alpha_{j-1}+2\alpha_j+\cdots+2\alpha_n} + e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n} e_j e_{j-1} - e_{j-1} e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n} e_j \\
& \quad -q^{-2} e_j e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n} e_{j-1} + q^{-2} e_{j-1} e_j e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n} \\
= & \quad -q^{-1} e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n} e_{j-1} e_j - q^{-1} e_j e_{j-1} e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n} \\
& \quad + e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n} e_j e_{j-1} + q^{-2} e_{j-1} e_j e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n} \\
= & \quad e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n} \overline{e}_{\alpha_{j-1}+\alpha_j} - q^{-1} \overline{e}_{\alpha_{j-1}+\alpha_j} e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n} \\
= & \quad [e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}, \overline{e}_{\alpha_{j-1}+\alpha_j}]_q \\
= & \quad \overline{e}_{\alpha_{j-1}+2\alpha_j+\cdots+2\alpha_n},
\end{aligned}$$

where the last equality arises from Proposition D.9.8. It follows that

$$\begin{aligned}
& \left(\overline{D} + D_{j-1}\right) \Delta(e_{j-1}) - q^{-1} \Delta(e_{j-1}) \left(\overline{D} + D_{j-1}\right) \\
= & \left(q e_{\alpha_{j-1}+2\alpha_j+\cdots+2\alpha_n} e_{j-1} - q^{-1} e_{j-1} e_{\alpha_{j-1}+2\alpha_j+\cdots+2\alpha_n}\right) \otimes (K_{\alpha_{j-1}+\cdots+\alpha_n})^2 e_{\alpha_i+\cdots+\alpha_{j-2}} \\
& + \overline{e}_{\alpha_{j-1}+2\alpha_j+\cdots+2\alpha_n} \otimes K_{\alpha_{j-1}+2\alpha_j+\cdots+2\alpha_n} e_{\alpha_i+\cdots+\alpha_{j-1}}.
\end{aligned}$$

□

D.6 The commutation relations between the components of $\Delta(e_\mu)$

The commutation relations between the components of $\Delta(e_\mu)$ are, in general, quite complicated, and the algebraic structures they give rise to come in three families depending on the nature of μ.

In this section we prove the following results. The algebra underlying the components of $\Delta(e_i)$, where $i = 1, \ldots, n$, is the q-binomial theorem. The algebra underlying the components of $\Delta(e_{\alpha_i+\cdots+\alpha_j})$, where $i < j < n$, is the q-multinomial theorem. The algebra underlying the components of $\Delta(e_{\alpha_i+\cdots+\alpha_n})$ is more complicated: it is a combination of the q-binomial theorem and the associative algebra given in Lemma B.0.8. The algebra underlying the components of $\Delta(e_{\alpha_i+\cdots+2\alpha_j+\cdots+\alpha_n})$ is a combination of the q-binomial theorem and the associative algebra given in Lemma B.0.9.

The q-binomial theorem and the two generalisations of the binomial theorem given in Appendix B allow us to compute any desired power of $\Delta(e_\mu)$.

D.6.1 $\Delta(e_i)$, $i = 1, \ldots, n$

Here $\Delta(e_i) = e_i \otimes K_i + 1 \otimes e_i$, and elementary calculations show that

$$(e_i \otimes K_i)(1 \otimes e_i) = q^2 (1 \otimes e_i)(e_i \otimes K_i), \quad i < n,$$

$$(e_n \otimes K_n)(1 \otimes e_n) = -q (1 \otimes e_n)(e_n \otimes K_n).$$

D.6.2 $\Delta(e_{\alpha_i+\dots+\alpha_j})$, $1 \leq i < j < n$

For each $j = i+1, \dots, n-1$, we write

$$\Delta(e_{\alpha_i+\dots+\alpha_j}) = D_i + (q - q^{-1}) \sum_{k=i+1}^{j} D_k + D_\infty, \qquad \text{where}$$

$$\begin{aligned} D_i &= e_{\alpha_i+\dots+\alpha_j} \otimes K_{\alpha_i+\dots+\alpha_j}, \\ D_k &= e_{\alpha_k+\dots+\alpha_j} \otimes K_{\alpha_k+\dots+\alpha_j} e_{\alpha_i+\dots+\alpha_{k-1}}, \\ D_\infty &= 1 \otimes e_{\alpha_i+\dots+\alpha_j}. \end{aligned}$$

We will show that the commutation relations between the components of $\Delta(e_{\alpha_i+\dots+\alpha_j})$ are

$$D_r D_s = q^2 D_s D_r, \qquad \forall r < s.$$

For each $k = i+1, \dots, j$, the relations between D_i and D_k are

$$\begin{aligned} D_i D_k &= q e_{\alpha_k+\dots+\alpha_j} e_{\alpha_i+\dots+\alpha_j} \otimes K_{\alpha_i+\dots+\alpha_j} K_{\alpha_k+\dots+\alpha_j} e_{\alpha_i+\dots+\alpha_{k-1}} \\ &= q^2 e_{\alpha_k+\dots+\alpha_j} e_{\alpha_i+\dots+\alpha_j} \otimes K_{\alpha_k+\dots+\alpha_j} e_{\alpha_i+\dots+\alpha_{k-1}} K_{\alpha_i+\dots+\alpha_j} = q^2 D_k D_i, \end{aligned}$$

where we have used $\left[e_{\alpha_i+\dots+\alpha_j}, e_{\alpha_k+\dots+\alpha_j}\right]_q = 0$. The relation between D_i and D_∞ is

$$D_i D_\infty = q^2 e_{\alpha_i+\dots+\alpha_j} \otimes e_{\alpha_i+\dots+\alpha_j} K_{\alpha_i+\dots+\alpha_j} = q^2 D_\infty D_i.$$

Now for each $k = i+1, \dots, j-1$, and each l satisfying $k < l \leq j$, the relation between D_k and D_l is

$$\begin{aligned} D_k D_l &= q e_{\alpha_l+\dots+\alpha_j} e_{\alpha_k+\dots+\alpha_j} \otimes K_{\alpha_k+\dots+\alpha_j} e_{\alpha_i+\dots+\alpha_{k-1}} K_{\alpha_l+\dots+\alpha_j} e_{\alpha_i+\dots+\alpha_{l-1}} \\ &= q^2 e_{\alpha_l+\dots+\alpha_j} e_{\alpha_k+\dots+\alpha_j} \otimes K_{\alpha_l+\dots+\alpha_j} K_{\alpha_k+\dots+\alpha_j} e_{\alpha_i+\dots+\alpha_{l-1}} e_{\alpha_i+\dots+\alpha_{k-1}} \\ &= q^2 D_l D_k. \end{aligned}$$

Finally, the relation between D_k and D_∞, for each $k = i+1, \dots, j$, is

$$\begin{aligned} D_k D_\infty &= e_{\alpha_k+\dots+\alpha_j} \otimes K_{\alpha_k+\dots+\alpha_j} e_{\alpha_i+\dots+\alpha_{k-1}} e_{\alpha_i+\dots+\alpha_j} \\ &= q^2 e_{\alpha_k+\dots+\alpha_j} \otimes e_{\alpha_i+\dots+\alpha_j} K_{\alpha_k+\dots+\alpha_j} e_{\alpha_i+\dots+\alpha_{k-1}} = q^2 D_\infty D_k. \end{aligned}$$

These calculations give the flavour of many of the calculations in this section.

D.6.3 $\Delta\left(e_{\alpha_i+\dots+\alpha_n}\right)$, $i = 1, \dots, n-1$

This case is more difficult than the previous one. We write

$$\Delta(e_{\alpha_i+\dots+\alpha_n}) = D_i + \sum_{k=i+1}^{n} D_k + D_\infty, \qquad \text{where}$$

$$\begin{aligned}
D_i &= e_{\alpha_i+\cdots+\alpha_n} \otimes K_{\alpha_i+\cdots+\alpha_n}, \\
D_k &= (q-q^{-1}) \left(e_{\alpha_k+\cdots+\alpha_n} \otimes K_{\alpha_k+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_{k-1}} \right), \qquad k = i+1, \ldots, n, \\
D_\infty &= 1 \otimes e_{\alpha_i+\cdots+\alpha_n}.
\end{aligned}$$

We will show that the commutation relations between the components of $\Delta(e_{\alpha_i+\cdots+\alpha_n})$ are (D.26)–(D.32) below, where we fix $i \le p, r, s, t \le n$ throughout these equations, $r < t$ in (D.26)–(D.31), $r < s < t$ in (D.29), and $p < r < t$ in (D.30)–(D.31):

$$\begin{aligned}
D_r D_t &= -q D_t D_r + E_{r,t}, && \text{(D.26)} \\
D_r E_{r,t} &= q^2 E_{r,t} D_r, && \text{(D.27)} \\
E_{r,t} D_t &= q^2 D_t E_{r,t}, && \text{(D.28)} \\
D_s E_{r,t} &= E_{r,t} D_s, && \text{(D.29)} \\
D_p E_{r,t} &= q^2 E_{r,t} D_p + F_{p,r,t}, && \text{(D.30)} \\
0 &= F_{p,r,t} + (1-q^2) E_{p,t} D_r, && \text{(D.31)} \\
D_r D_\infty &= -q D_\infty D_r. && \text{(D.32)}
\end{aligned}$$

Note that relations (D.26)–(D.28) imply the following relations

$$D_p E_{r,t} - q^2 E_{r,t} D_p = E_{p,r} D_t - q^2 D_t E_{p,r}, \qquad p < r < t, \tag{D.33}$$

$$E_{p,r} E_{p,t} = q^2 E_{p,t} E_{p,r}, \qquad E_{p,t} E_{r,t} = q^2 E_{r,t} E_{p,t}, \qquad p < r < t. \tag{D.34}$$

The following proposition shows that if the components of $\Delta(e_{\alpha_i+\cdots+\alpha_n})$ satisfy (D.26)–(D.32), then we can use Lemma B.0.8 to give an expansion of $(D_i + D_{i+1} + \cdots + D_n)^m$ for each $m \in \mathbb{N}$. Note that relations (D.35)–(D.37) in Proposition D.6.1 are precisely the relations between the elements a, b, c, respectively, in the associative algebra given in Lemma B.0.8.

Proposition D.6.1. *For each $k = i+1, \ldots, n$, and each j satisfying $i \le j < k$, we have*

$$\begin{aligned}
&(D_j + D_{j+1} + \cdots + D_{k-1}) D_k \\
&\qquad = -q D_k (D_j + D_{j+1} + \cdots + D_{k-1}) + (E_{j,k} + E_{j+1,k} + \cdots + E_{k-1,k}),
\end{aligned} \tag{D.35}$$

where

$$\left(\sum_{b=j}^{k-1} E_{b,k} \right) D_k = q^2 D_k \left(\sum_{b=j}^{k-1} E_{b,k} \right), \tag{D.36}$$

and

$$\left(\sum_{a=j}^{k-1} D_a \right) \left(\sum_{b=j}^{k-1} E_{b,k} \right) = q^2 \left(\sum_{b=j}^{k-1} E_{b,k} \right) \left(\sum_{a=j}^{k-1} D_a \right). \tag{D.37}$$

Proof. Relation (D.35) follows from (D.26) and (D.36) follows from (D.28). The proof of (D.37) is only slightly more difficult: firstly, the elements D_{k-1} and D_k satisfy the relations

$$D_{k-1} D_k = -q D_k D_{k-1} + E_{k-1,k}, \quad D_{k-1} E_{k-1,k} = q^2 E_{k-1,k} D_{k-1}, \quad E_{k-1,k} D_k = q^2 D_k E_{k-1,k},$$

which proves (D.37) for $j = k-1$. Now assume that (D.37) is true for some $j \leq k-1$, we will prove that (D.37) is true for $j-1$ if $j-1 \geq i$:

$$\begin{aligned}
&\left(\sum_{a=j-1}^{k-1} D_a\right)\left(\sum_{b=j-1}^{k-1} E_{b,k}\right) \\
&= \left[D_{j-1} + \left(\sum_{a=j}^{k-1} D_a\right)\right]\left[E_{j-1,k} + \left(\sum_{b=j}^{k-1} E_{b,k}\right)\right] \\
&= D_{j-1}E_{j-1,k} + \left(\sum_{a=j}^{k-1} D_a\right)\left(\sum_{b=j}^{k-1} E_{b,k}\right) + D_{j-1}\left(\sum_{c=j}^{k-1} E_{c,k}\right) + \left(\sum_{d=j}^{k-1} D_d\right)E_{j-1,k} \\
&= q^2 E_{j-1,k}D_{j-1} + q^2\left(\sum_{b=j}^{k-1} E_{b,k}\right)\left(\sum_{a=j}^{k-1} D_a\right) + \sum_{c=j}^{k-1}\left(q^2 E_{c,k}D_{j-1} + F_{j-1,c,k}\right) \\
&\quad + q^2 E_{j-1,k}\left(\sum_{d=j}^{k-1} D_d\right) + (1-q^2)E_{j-1,k}\left(\sum_{e=j}^{k-1} D_e\right) \\
&= q^2\left(\sum_{c=j-1}^{k-1} E_{c,k}\right)D_{j-1} + q^2 E_{j-1,k}\left(\sum_{a=j}^{k-1} D_a\right) + q^2\left(\sum_{b=j}^{k-1} E_{b,k}\right)\left(\sum_{a=j}^{k-1} D_a\right) \qquad \text{(D.38)} \\
&= q^2\left(\sum_{b=j-1}^{k-1} E_{b,k}\right)\left(\sum_{a=j-1}^{k-1} D_a\right),
\end{aligned}$$

where (D.38) follows from (D.31), completing the induction. □

We now show that the components of $\Delta(e_{\alpha_i+\cdots+\alpha_n})$ satisfy the claimed commutation relations. We firstly prove relation (D.32):

$$\begin{aligned}
D_i D_\infty &= e_{\alpha_i+\cdots+\alpha_n} \otimes K_{\alpha_i+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_n} \\
&= q e_{\alpha_i+\cdots+\alpha_n} \otimes e_{\alpha_i+\cdots+\alpha_n} K_{\alpha_i+\cdots+\alpha_n} = -q D_\infty D_i,
\end{aligned}$$

and for each $k = i+1, \ldots, n$, the relation between D_k and D_∞ is

$$\begin{aligned}
D_k D_\infty &= (q-q^{-1})\left(e_{\alpha_k+\cdots+\alpha_n} \otimes K_{\alpha_k+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_{k-1}} e_{\alpha_i+\cdots+\alpha_n}\right) \\
&= q(q-q^{-1})\left(e_{\alpha_k+\cdots+\alpha_n} \otimes e_{\alpha_i+\cdots+\alpha_n} K_{\alpha_k+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_{k-1}}\right) = -q D_\infty D_k,
\end{aligned}$$

proving relation (D.32).

We now prove relation (D.26). For each $k = i+1, \ldots, n$, the relation between D_i and

D_k is

$$\begin{aligned} D_iD_k &= (q-q^{-1})\left(e_{\alpha_i+\cdots+\alpha_n}e_{\alpha_k+\cdots+\alpha_n}\otimes K_{\alpha_i+\cdots+\alpha_n}K_{\alpha_k+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_{k-1}}\right)\\ &= (q-q^{-1})\left(-e_{\alpha_k+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_n}+[e_{\alpha_i+\cdots+\alpha_n},e_{\alpha_k+\cdots+\alpha_n}]_q\right)\\ &\qquad\qquad\otimes K_{\alpha_i+\cdots+\alpha_n}K_{\alpha_k+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_{k-1}}\\ &= (q-q^{-1})\Big(-qe_{\alpha_k+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_n}\otimes K_{\alpha_k+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_{k-1}}K_{\alpha_i+\cdots+\alpha_n} \qquad \text{(D.39)}\\ &\qquad +[e_{\alpha_i+\cdots+\alpha_n},e_{\alpha_k+\cdots+\alpha_n}]_q\otimes K_{\alpha_i+\cdots+\alpha_n}K_{\alpha_k+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_{k-1}}\Big)\\ &= -qD_kD_i+E_{i,k}, \end{aligned}$$

where we have used the relation $[e_{\alpha_i+\cdots+\alpha_n},e_{\alpha_k+\cdots+\alpha_n}]_q=0$ to obtain (D.39), and where we set

$$E_{i,k}=(q-q^{-1})\left([e_{\alpha_i+\cdots+\alpha_n},e_{\alpha_k+\cdots+\alpha_n}]_q\otimes K_{\alpha_i+\cdots+\alpha_n}K_{\alpha_k+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_{k-1}}\right).$$

For all k,p satisfying $i+1\le k<p\le n$, the relation between D_k and D_p is

$$\begin{aligned} D_kD_p &= (q-q^{-1})^2\left(e_{\alpha_k+\cdots+\alpha_n}e_{\alpha_p+\cdots+\alpha_n}\otimes K_{\alpha_k+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_{k-1}}K_{\alpha_p+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_{p-1}}\right)\\ &= (q-q^{-1})^2\left(-e_{\alpha_p+\cdots+\alpha_n}e_{\alpha_k+\cdots+\alpha_n}+\left[e_{\alpha_k+\cdots+\alpha_n},e_{\alpha_p+\cdots+\alpha_n}\right]_q\right)\\ &\qquad\qquad\otimes K_{\alpha_k+\cdots+\alpha_n}K_{\alpha_p+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_{k-1}}e_{\alpha_i+\cdots+\alpha_{p-1}}\\ &= -q(q-q^{-1})^2\left(e_{\alpha_p+\cdots+\alpha_n}e_{\alpha_k+\cdots+\alpha_n}\otimes K_{\alpha_k+\cdots+\alpha_n}K_{\alpha_p+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_{p-1}}e_{\alpha_i+\cdots+\alpha_{k-1}}\right)\\ &\quad +(q-q^{-1})^2\left(\left[e_{\alpha_k+\cdots+\alpha_n},e_{\alpha_p+\cdots+\alpha_n}\right]_q\otimes K_{\alpha_k+\cdots+\alpha_n}K_{\alpha_p+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_{k-1}}e_{\alpha_i+\cdots+\alpha_{p-1}}\right) \qquad \text{(D.40)}\\ &= -qD_pD_k+E_{k,p}, \end{aligned}$$

where we have used the relation $\left[e_{\alpha_i+\cdots+\alpha_{k-1}},e_{\alpha_i+\cdots+\alpha_{p-1}}\right]_q=0$ to obtain (D.40), and we set

$$E_{k,p}=(q-q^{-1})^2\left(\left[e_{\alpha_k+\cdots+\alpha_n},e_{\alpha_p+\cdots+\alpha_n}\right]_q\otimes K_{\alpha_k+\cdots+\alpha_n}K_{\alpha_p+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_{k-1}}e_{\alpha_i+\cdots+\alpha_{p-1}}\right).$$

It follows that the components of $\Delta(e_{\alpha_i+\cdots+\alpha_n})$ satisfy relation (D.26).

We now consider the relations between the D_k and $E_{k,p}$: we will show that D_k and $E_{k,p}$ satisfy relation (D.27) for all $k<p$. For calculational ease we will write $E_{i,p}$ and $E_{k,p}$ as the following sums:

$$E_{i,p}=\sum_{j=p}^{n}E^j_{i,p}, \qquad E_{k,p}=\sum_{j=p}^{n}E^j_{k,p}, \qquad k>i,$$

where the components of these sums are given in Proposition D.9.2:

$$\begin{aligned}
E^p_{i,p} &= (q-q^{-1})(-q)^{n-p}\left(e_{\alpha_i+\cdots+2\alpha_p+\cdots+2\alpha_n}\otimes K_{\alpha_i+\cdots+\alpha_n}K_{\alpha_p+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_{p-1}}\right),\\
E^j_{i,p} &= (q-q^{-1})^2(-q)^{n-j}\big(e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}e_{\alpha_p+\cdots+\alpha_{j-1}}\\
&\qquad \otimes K_{\alpha_i+\cdots+\alpha_n}K_{\alpha_p+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_{p-1}}\big), \qquad j>p,\\
E^p_{k,p} &= (q-q^{-1})^2(-q)^{n-p}\left(e_{\alpha_k+\cdots+2\alpha_p+\cdots+2\alpha_n}\otimes K_{\alpha_k+\cdots+\alpha_n}K_{\alpha_p+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_{k-1}}e_{\alpha_i+\cdots+\alpha_{p-1}}\right),\\
E^j_{k,p} &= (q-q^{-1})^3(-q)^{n-j}\big(e_{\alpha_k+\cdots+2\alpha_j+\cdots+2\alpha_n}e_{\alpha_p+\cdots+\alpha_{j-1}}\\
&\qquad \otimes K_{\alpha_k+\cdots+\alpha_n}K_{\alpha_p+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_{k-1}}e_{\alpha_i+\cdots+\alpha_{p-1}}\big). \qquad j>p.
\end{aligned}$$

The following identities allow us to determine the relations between D_i and $E_{i,p}$, and between D_k and $E_{k,p}$, quite easily:

$$\left[e_{\alpha_i+\cdots+\alpha_n}, e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}\right]_q = 0, \qquad 1\le i\le n-1,\quad i+1\le j\le n, \tag{D.41}$$

$$\left[e_{\alpha_i+\cdots+\alpha_n}, e_{\alpha_p+\cdots+\alpha_{j-1}}\right]_q = 0, \qquad 1\le i\le n-2,\quad i<p<j-1<n. \tag{D.42}$$

We can rewrite (D.41) and (D.42), respectively, as

$$\begin{aligned}
e_{\alpha_i+\cdots+\alpha_n}e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n} &= q e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}e_{\alpha_i+\cdots+\alpha_n},\\
e_{\alpha_i+\cdots+\alpha_n}e_{\alpha_p+\cdots+\alpha_{j-1}} &= e_{\alpha_p+\cdots+\alpha_{j-1}}e_{\alpha_i+\cdots+\alpha_n}.
\end{aligned}$$

Using these identities we compute the relations between D_i and $E_{i,p}$ where $i=1,\dots,n-1$ and $p>i$:

$$\begin{aligned}
&D_iE^p_{i,p}\\
&\quad= (q-q^{-1})(-q)^{n-p}\\
&\qquad\times\left(e_{\alpha_i+\cdots+\alpha_n}e_{\alpha_i+\cdots+2\alpha_p+\cdots+2\alpha_n}\otimes K_{\alpha_i+\cdots+\alpha_n}K_{\alpha_i+\cdots+\alpha_n}K_{\alpha_p+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_{p-1}}\right)\\
&\quad= q^2(q-q^{-1})(-q)^{n-p}\\
&\qquad\times\left(e_{\alpha_i+\cdots+2\alpha_p+\cdots+2\alpha_n}e_{\alpha_i+\cdots+\alpha_n}\otimes K_{\alpha_i+\cdots+\alpha_n}K_{\alpha_p+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_{p-1}}K_{\alpha_i+\cdots+\alpha_n}\right)\\
&\quad= q^2E^p_{i,p}D_i,
\end{aligned}$$

and for $p>i$ and $j>p$ we have

$$\begin{aligned}
&D_iE^j_{i,p}\\
&\quad= (q-q^{-1})^2(-q)^{n-j}\\
&\qquad\times\left(e_{\alpha_i+\cdots+\alpha_n}e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}e_{\alpha_p+\cdots+\alpha_{j-1}}\otimes K_{\alpha_i+\cdots+\alpha_n}K_{\alpha_i+\cdots+\alpha_n}K_{\alpha_p+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_{p-1}}\right)\\
&\quad= q^2(q-q^{-1})^2(-q)^{n-j}\\
&\qquad\times\left(e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}e_{\alpha_p+\cdots+\alpha_{j-1}}e_{\alpha_i+\cdots+\alpha_n}\otimes K_{\alpha_i+\cdots+\alpha_n}K_{\alpha_p+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_{p-1}}K_{\alpha_i+\cdots+\alpha_n}\right)\\
&\quad= q^2E^j_{i,p}D_i.
\end{aligned}$$

These calculations show that $D_iE_{i,p}=q^2E_{i,p}D_i$ for $p=i+1,\dots,n$. Now we determine the relations between D_k and the components of $E_{k,p}$. For $k=i+1,\dots,n-1$ and $p>k$

we have

$$\begin{aligned}
&D_k E^p_{k,p}\\
&= (q-q^{-1})^3(-q)^{n-p}\\
&\quad\times\big(e_{\alpha_k+\cdots+\alpha_n}e_{\alpha_k+\cdots+2\alpha_p+\cdots+2\alpha_n}\\
&\qquad\otimes K_{\alpha_k+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_{k-1}}K_{\alpha_k+\cdots+\alpha_n}K_{\alpha_p+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_{k-1}}e_{\alpha_i+\cdots+\alpha_{p-1}}\big)\\
&= q^2(q-q^{-1})^3(-q)^{n-p}\\
&\quad\times\big(e_{\alpha_k+\cdots+2\alpha_p+\cdots+2\alpha_n}e_{\alpha_k+\cdots+\alpha_n}\\
&\qquad\otimes K_{\alpha_k+\cdots+\alpha_n}K_{\alpha_p+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_{k-1}}e_{\alpha_i+\cdots+\alpha_{p-1}}K_{\alpha_k+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_{k-1}}\big)\\
&= q^2E^p_{k,p}D_k,
\end{aligned}$$

and for $j>p$,

$$\begin{aligned}
&D_k E^j_{k,p}\\
&= (q-q^{-1})^4(-q)^{n-j}\\
&\quad\times\big(e_{\alpha_k+\cdots+\alpha_n}e_{\alpha_k+\cdots+2\alpha_j+\cdots+2\alpha_n}e_{\alpha_p+\cdots+\alpha_{j-1}}\\
&\qquad\otimes K_{\alpha_k+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_{k-1}}K_{\alpha_k+\cdots+\alpha_n}K_{\alpha_p+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_{k-1}}e_{\alpha_i+\cdots+\alpha_{p-1}}\big)\\
&= q^2(q-q^{-1})^4(-q)^{n-j}\\
&\quad\times\big(e_{\alpha_k+\cdots+2\alpha_j+\cdots+2\alpha_n}e_{\alpha_p+\cdots+\alpha_{j-1}}e_{\alpha_k+\cdots+\alpha_n}\\
&\qquad\otimes K_{\alpha_k+\cdots+\alpha_n}K_{\alpha_p+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_{k-1}}e_{\alpha_i+\cdots+\alpha_{p-1}}K_{\alpha_k+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_{k-1}}\big)\\
&= q^2E^j_{k,p}D_k.
\end{aligned}$$

This shows that $D_kE_{k,p}=q^2E_{k,p}D_k$ for $p=k+1,\dots,n$. Together with $D_iE_{i,p}=q^2E_{i,p}D_i$, this proves that D_k and $E_{k,p}$ satisfy relation (D.27).

We now show that D_k and $E_{k,p}$ satisfy relation (D.28). The identity

$$\big[e_{\alpha_i+\cdots+2\alpha_p+\cdots+2\alpha_n},e_{\alpha_p+\cdots+\alpha_n}\big]_q=0,\qquad i+1\le p\le n,$$

which we can rewrite as $e_{\alpha_i+\cdots+2\alpha_p+\cdots+2\alpha_n}e_{\alpha_p+\cdots+\alpha_n}=qe_{\alpha_p+\cdots+\alpha_n}e_{\alpha_i+\cdots+2\alpha_p+\cdots+2\alpha_n}$, will be useful. The following calculations are similar to those immediately above. For each $k=i+1,\dots,n-1$ and $p>k$,

$$\begin{aligned}
E^p_{i,p}D_p &= (q-q^{-1})^2(-q)^{n-p}\\
&\quad\times\big(e_{\alpha_i+\cdots+2\alpha_p+\cdots+2\alpha_n}e_{\alpha_p+\cdots+\alpha_n}\\
&\qquad\otimes K_{\alpha_i+\cdots+\alpha_n}K_{\alpha_p+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_{p-1}}K_{\alpha_p+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_{p-1}}\big)\\
&= q(q-q^{-1})^2(-q)^{n-p}\\
&\quad\times\big(e_{\alpha_p+\cdots+\alpha_n}e_{\alpha_i+\cdots+2\alpha_p+\cdots+2\alpha_n}\\
&\qquad\otimes K_{\alpha_i+\cdots+\alpha_n}K_{\alpha_p+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_{p-1}}K_{\alpha_p+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_{p-1}}\big)\\
&= q^2(q-q^{-1})^2(-q)^{n-p}\\
&\quad\times\big(e_{\alpha_p+\cdots+\alpha_n}e_{\alpha_i+\cdots+2\alpha_p+\cdots+2\alpha_n}\\
&\qquad\otimes K_{\alpha_p+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_{p-1}}K_{\alpha_i+\cdots+\alpha_n}K_{\alpha_p+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_{p-1}}\big)\\
&= q^2D_pE^p_{i,p}.
\end{aligned}$$

For each $j = p+1, \ldots, n$, we have $\left[e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}, e_{\alpha_p+\cdots+\alpha_n}\right]_q = 0$, and

$$\begin{aligned}
E^j_{i,p} D_p &= (q-q^{-1})^3(-q)^{n-j} \\
&\quad \times e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n} e_{\alpha_p+\cdots+\alpha_{j-1}} e_{\alpha_p+\cdots+\alpha_n} \\
&\qquad \otimes K_{\alpha_i+\cdots+\alpha_n} K_{\alpha_p+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_{p-1}} K_{\alpha_p+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_{p-1}} \\
&= q(q-q^{-1})^3(-q)^{n-j} \\
&\quad \times e_{\alpha_p+\cdots+\alpha_n} e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n} e_{\alpha_p+\cdots+\alpha_{j-1}} \\
&\qquad \otimes K_{\alpha_i+\cdots+\alpha_n} K_{\alpha_p+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_{p-1}} K_{\alpha_p+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_{p-1}} \\
&= q^2(q-q^{-1})^3(-q)^{n-j} \\
&\quad \times e_{\alpha_p+\cdots+\alpha_n} e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n} e_{\alpha_p+\cdots+\alpha_{j-1}} \\
&\qquad \otimes K_{\alpha_p+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_{p-1}} K_{\alpha_i+\cdots+\alpha_n} K_{\alpha_p+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_{p-1}} \\
&= q^2 D_p E^j_{i,p}.
\end{aligned}$$

For each $k = i+1, \ldots, n-1$ and $p > k$,

$$\begin{aligned}
E^p_{k,p} D_p &= (q-q^{-1})^3(-q)^{n-p} \\
&\quad \times e_{\alpha_k+\cdots+2\alpha_p+\cdots+2\alpha_n} e_{\alpha_p+\cdots+\alpha_n} \\
&\qquad \otimes K_{\alpha_k+\cdots+\alpha_n} K_{\alpha_p+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_{k-1}} e_{\alpha_i+\cdots+\alpha_{p-1}} K_{\alpha_p+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_{p-1}} \\
&= q(q-q^{-1})^3(-q)^{n-p} \\
&\quad \times e_{\alpha_p+\cdots+\alpha_n} e_{\alpha_k+\cdots+2\alpha_p+\cdots+2\alpha_n} \\
&\qquad \otimes K_{\alpha_k+\cdots+\alpha_n} K_{\alpha_p+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_{k-1}} e_{\alpha_i+\cdots+\alpha_{p-1}} K_{\alpha_p+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_{p-1}} \\
&= q^2(q-q^{-1})^3(-q)^{n-p} \\
&\quad \times e_{\alpha_p+\cdots+\alpha_n} e_{\alpha_k+\cdots+2\alpha_p+\cdots+2\alpha_n} \\
&\qquad \otimes K_{\alpha_p+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_{p-1}} K_{\alpha_k+\cdots+\alpha_n} K_{\alpha_p+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_{k-1}} e_{\alpha_i+\cdots+\alpha_{p-1}} \\
&= q^2 D_p E^p_{k,p},
\end{aligned}$$

and for $j = p+1, \ldots, n$,

$$\begin{aligned}
E^j_{k,p} D_p &= (q-q^{-1})^4(-q)^{n-j} \\
&\quad \times e_{\alpha_k+\cdots+2\alpha_j+\cdots+2\alpha_n} e_{\alpha_p+\cdots+\alpha_{j-1}} e_{\alpha_p+\cdots+\alpha_n} \\
&\qquad \otimes K_{\alpha_k+\cdots+\alpha_n} K_{\alpha_p+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_{k-1}} e_{\alpha_i+\cdots+\alpha_{p-1}} K_{\alpha_p+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_{p-1}} \\
&= q(q-q^{-1})^4(-q)^{n-j} \\
&\quad \times e_{\alpha_p+\cdots+\alpha_n} e_{\alpha_k+\cdots+2\alpha_j+\cdots+2\alpha_n} e_{\alpha_p+\cdots+\alpha_{j-1}} \\
&\qquad \otimes K_{\alpha_k+\cdots+\alpha_n} K_{\alpha_p+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_{k-1}} e_{\alpha_i+\cdots+\alpha_{p-1}} K_{\alpha_p+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_{p-1}} \\
&= q^2(q-q^{-1})^4(-q)^{n-j} \\
&\quad \times e_{\alpha_p+\cdots+\alpha_n} e_{\alpha_k+\cdots+2\alpha_j+\cdots+2\alpha_n} e_{\alpha_p+\cdots+\alpha_{j-1}} \\
&\qquad \otimes K_{\alpha_p+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_{p-1}} K_{\alpha_k+\cdots+\alpha_n} K_{\alpha_p+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_{k-1}} e_{\alpha_i+\cdots+\alpha_{p-1}} \\
&= q^2 D_p E^j_{k,p}.
\end{aligned}$$

These calculations show that $E_{k,p}D_p = q^2 D_p E_{k,p}$ for each $k = i, \ldots, n-1$, and $p > k$, proving (D.28).

We now show that $D_s E_{k,p} = E_{k,p} D_s$ for all $i \le k < s < p \le n$; and in doing so we will use the following identity: for all $i \le k < s < p \le n$ we have

$$\left[e_{\alpha_k+\cdots+2\alpha_p+\cdots+2\alpha_n}, e_{\alpha_s+\cdots+\alpha_n}\right]_q = 0,$$

which we can rewrite as $e_{\alpha_k+\cdots+2\alpha_p+\cdots+2\alpha_n} e_{\alpha_s+\cdots+\alpha_n} = e_{\alpha_s+\cdots+\alpha_n} e_{\alpha_k+\cdots+2\alpha_p+\cdots+2\alpha_n}$. The relation between D_s and $E^p_{i,p}$ is

$$\begin{aligned}
D_s E^p_{i,p} &= (q-q^{-1})^2(-q)^{n-p} e_{\alpha_s+\cdots+\alpha_n} e_{\alpha_i+\cdots+2\alpha_p+\cdots+2\alpha_n} \\
&\quad \otimes K_{\alpha_s+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_{s-1}} K_{\alpha_i+\cdots+\alpha_n} K_{\alpha_p+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_{p-1}} \\
&= (q-q^{-1})^2(-q)^{n-p} e_{\alpha_i+\cdots+2\alpha_p+\cdots+2\alpha_n} e_{\alpha_s+\cdots+\alpha_n} \\
&\quad \otimes K_{\alpha_i+\cdots+\alpha_n} K_{\alpha_p+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_{p-1}} K_{\alpha_s+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_{s-1}} \\
&= E^p_{i,p} D_s.
\end{aligned}$$

For each $j = p+1, \ldots, n$,

$$\begin{aligned}
D_s E^j_{i,p} &= (q-q^{-1})^3(-q)^{n-j} \\
&\quad \times e_{\alpha_s+\cdots+\alpha_n} e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n} e_{\alpha_p+\cdots+\alpha_{j-1}} \\
&\qquad \otimes K_{\alpha_s+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_{s-1}} K_{\alpha_i+\cdots+\alpha_n} K_{\alpha_p+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_{p-1}} \\
&= (q-q^{-1})^3(-q)^{n-j} \\
&\quad \times e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n} e_{\alpha_p+\cdots+\alpha_{j-1}} e_{\alpha_s+\cdots+\alpha_n} \\
&\qquad \otimes K_{\alpha_i+\cdots+\alpha_n} K_{\alpha_p+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_{p-1}} K_{\alpha_s+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_{s-1}} \\
&\qquad \times \left(e_{\alpha_s+\cdots+\alpha_n} \otimes K_{\alpha_s+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_{s-1}}\right) \\
&= E^j_{i,p} D_s.
\end{aligned}$$

For $k = i+1, \ldots, n-1$ and $k < s < p$,

$$\begin{aligned}
D_s E^p_{k,p} &= (q-q^{-1})^3(-q)^{n-p} \\
&\quad \times e_{\alpha_s+\cdots+\alpha_n} e_{\alpha_k+\cdots+2\alpha_p+\cdots+2\alpha_n} \\
&\qquad \otimes K_{\alpha_s+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_{s-1}} K_{\alpha_k+\cdots+\alpha_n} K_{\alpha_p+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_{k-1}} e_{\alpha_i+\cdots+\alpha_{p-1}} \\
&= (q-q^{-1})^3(-q)^{n-p} \\
&\quad \times e_{\alpha_k+\cdots+2\alpha_p+\cdots+2\alpha_n} e_{\alpha_s+\cdots+\alpha_n} \\
&\qquad \otimes K_{\alpha_k+\cdots+\alpha_n} K_{\alpha_p+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_{k-1}} e_{\alpha_i+\cdots+\alpha_{p-1}} K_{\alpha_s+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_{s-1}} \\
&= E^p_{k,p} D_s,
\end{aligned}$$

and for $j > p$,

$$\begin{aligned} D_s E^j_{k,p} &= (q-q^{-1})^4(-q)^{n-j} \\ &\qquad \times e_{\alpha_s+\cdots+\alpha_n} e_{\alpha_k+\cdots+2\alpha_j+\cdots+2\alpha_n} e_{\alpha_p+\cdots+\alpha_{j-1}} \\ &\qquad\qquad \otimes K_{\alpha_s+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_{s-1}} K_{\alpha_k+\cdots+\alpha_n} K_{\alpha_p+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_{k-1}} e_{\alpha_i+\cdots+\alpha_{p-1}} \\ &= (q-q^{-1})^4(-q)^{n-j} \\ &\qquad \times e_{\alpha_k+\cdots+2\alpha_j+\cdots+2\alpha_n} e_{\alpha_p+\cdots+\alpha_{j-1}} e_{\alpha_s+\cdots+\alpha_n} \\ &\qquad\qquad \otimes K_{\alpha_k+\cdots+\alpha_n} K_{\alpha_p+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_{k-1}} e_{\alpha_i+\cdots+\alpha_{p-1}} K_{\alpha_s+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_{s-1}} \\ &= E^j_{k,p} D_s. \end{aligned}$$

These calculations show that $D_s E_{k,p} = E_{k,p} D_s$ for all $i \le k < s < p \le n$, proving relation (D.29).

For all $i \le p < r < t \le n$, $F_{p,r,t}$ is defined by $F_{p,r,t} = D_p E_{r,t} - q^2 E_{r,t} D_p$, and we calculate it from its alternative definition in relation (D.33): $F_{p,r,t} = E_{p,r} D_t - q^2 D_t E_{p,r}$. We firstly calculate $F_{i,r,t}$:

$$\begin{aligned} F_{i,r,t} &= E_{i,r} D_t - q^2 D_t E_{i,r} \\ &= (q-q^{-1})^3 \Bigg[\sum_{j=r+1}^{n} (-q)^{n-j} e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n} e_{\alpha_r+\cdots+\alpha_{j-1}} e_{\alpha_t+\cdots+\alpha_n} \\ &\qquad - \sum_{j=r+1}^{t-1} (-q)^{n-j} e_{\alpha_t+\cdots+\alpha_n} e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n} e_{\alpha_r+\cdots+\alpha_{j-1}} \\ &\qquad - (-q)^{n-t} e_{\alpha_t+\cdots+\alpha_n} e_{\alpha_i+\cdots+2\alpha_t+\cdots+2\alpha_n} e_{\alpha_r+\cdots+\alpha_{t-1}} \\ &\qquad - \sum_{j=t+1}^{n} (-q)^{n-j} e_{\alpha_t+\cdots+\alpha_n} e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n} e_{\alpha_r+\cdots+\alpha_{j-1}} \Bigg] \\ &\qquad\qquad \otimes K_{\alpha_i+\cdots+\alpha_n} K_{\alpha_r+\cdots+\alpha_n} K_{\alpha_t+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_{r-1}} e_{\alpha_i+\cdots+\alpha_{t-1}} \\ &= (q-q^{-1})^3 \Bigg[\sum_{j=t}^{n} (-q)^{n-j} e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n} e_{\alpha_r+\cdots+\alpha_{j-1}} e_{\alpha_t+\cdots+\alpha_n} \\ &\qquad - (-q)^{n-t} q^{-1} e_{\alpha_i+\cdots+2\alpha_t+\cdots+2\alpha_n} e_{\alpha_t+\cdots+\alpha_n} e_{\alpha_r+\cdots+\alpha_{t-1}} \\ &\qquad - \sum_{j=t+1}^{n} (-q)^{n-j} e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n} e_{\alpha_t+\cdots+\alpha_n} e_{\alpha_r+\cdots+\alpha_{j-1}} \Bigg] \\ &\qquad\qquad \otimes K_{\alpha_i+\cdots+\alpha_n} K_{\alpha_r+\cdots+\alpha_n} K_{\alpha_t+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_{r-1}} e_{\alpha_i+\cdots+\alpha_{t-1}} \\ &= (q-q^{-1})^3 \Bigg[(-q)^{n-t} e_{\alpha_i+\cdots+2\alpha_t+\cdots+2\alpha_n} e_{\alpha_r+\cdots+\alpha_n} \\ &\qquad + \sum_{j=t+1}^{n} (q-q^{-1})(-q)^{n-j} e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n} e_{\alpha_r+\cdots+\alpha_n} e_{\alpha_t+\cdots+\alpha_{j-1}} \Bigg] \\ &\qquad\qquad \otimes K_{\alpha_i+\cdots+\alpha_n} K_{\alpha_r+\cdots+\alpha_n} K_{\alpha_t+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_{r-1}} e_{\alpha_i+\cdots+\alpha_{t-1}}. \end{aligned}$$

In obtaining the last equality we used the two identities

$$\begin{aligned} q^{-1}e_{\alpha_t+\cdots+\alpha_n}e_{\alpha_r+\cdots+\alpha_{t-1}} &= -e_{\alpha_r+\cdots+\alpha_n}+e_{\alpha_r+\cdots+\alpha_{t-1}}e_{\alpha_t+\cdots+\alpha_n}, \\ e_{\alpha_r+\cdots+\alpha_{j-1}}e_{\alpha_t+\cdots+\alpha_n}-e_{\alpha_t+\cdots+\alpha_n}e_{\alpha_r+\cdots+\alpha_{j-1}} &= (q-q^{-1})e_{\alpha_r+\cdots+\alpha_n}e_{\alpha_t+\cdots+\alpha_{j-1}}, \qquad t\le j-1, \end{aligned}$$

which arise from Propositions D.9.3 and D.9.4 respectively.

Now we will show that $F_{i,r,t}+(1-q^2)E_{i,t}D_r=0$. Note that

$$\begin{aligned} (1-q^2)E_{i,t}D_r &= (1-q^2)(q-q^{-1})^2 \\ &\quad\times\Bigg[(-q)^{n-t}e_{\alpha_i+\cdots+2\alpha_t+\cdots+2\alpha_n}e_{\alpha_r+\cdots+\alpha_n} \\ &\qquad+(q-q^{-1})\sum_{j=t+1}^{n}(-q)^{n-j}e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}e_{\alpha_t+\cdots+\alpha_{j-1}}e_{\alpha_r+\cdots+\alpha_n}\Bigg] \\ &\qquad\otimes K_{\alpha_i+\cdots+\alpha_n}K_{\alpha_t+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_{t-1}}K_{\alpha_r+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_{r-1}}. \end{aligned}$$

An elementary calculation shows that $F_{i,r,t}+(1-q^2)E_{i,t}D_r=0$ after manipulating the expansion of $E_{i,t}D_r$ using the following identities:

$$\begin{aligned} e_{\alpha_t+\cdots+\alpha_{j-1}}e_{\alpha_r+\cdots+\alpha_n} &= e_{\alpha_r+\cdots+\alpha_n}e_{\alpha_t+\cdots+\alpha_{j-1}}, \\ e_{\alpha_i+\cdots+\alpha_{t-1}}K_{\alpha_r+\cdots+\alpha_n} &= K_{\alpha_r+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_{t-1}}, \\ e_{\alpha_i+\cdots+\alpha_{t-1}}e_{\alpha_i+\cdots+\alpha_{r-1}} &= q^{-1}e_{\alpha_i+\cdots+\alpha_{r-1}}e_{\alpha_i+\cdots+\alpha_{t-1}}. \end{aligned}$$

Set $i<p<r<t\le n$. Using similar calculations as those above gives

$$\begin{aligned} F_{p,r,t} &= E_{p,r}D_t-q^2D_tE_{p,r} \\ &= (q-q^{-1})^4 \\ &\quad\times\Bigg[(-q)^{n-t}e_{\alpha_p+\cdots+2\alpha_t+\cdots+2\alpha_n}e_{\alpha_r+\cdots+\alpha_n} \\ &\qquad+(q-q^{-1})\sum_{j=t+1}^{n}(-q)^{n-j}e_{\alpha_p+\cdots+2\alpha_j+\cdots+2\alpha_n}e_{\alpha_r+\cdots+\alpha_n}e_{\alpha_t+\cdots+\alpha_{j-1}}\Bigg] \\ &\qquad\otimes K_{\alpha_t+\cdots+\alpha_n}K_{\alpha_p+\cdots+\alpha_n}K_{\alpha_r+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_{p-1}}e_{\alpha_i+\cdots+\alpha_{r-1}}e_{\alpha_i+\cdots+\alpha_{t-1}}, \end{aligned}$$

and by using the following identities

$$\begin{aligned} e_{\alpha_t+\cdots+\alpha_{j-1}}e_{\alpha_r+\cdots+\alpha_n} &= e_{\alpha_r+\cdots+\alpha_n}e_{\alpha_t+\cdots+\alpha_{j-1}}, \\ e_{\alpha_i+\cdots+\alpha_{t-1}}e_{\alpha_i+\cdots+\alpha_{r-1}} &= q^{-1}e_{\alpha_i+\cdots+\alpha_{r-1}}e_{\alpha_i+\cdots+\alpha_{t-1}}, \\ e_{\alpha_i+\cdots+\alpha_{p-1}}e_{\alpha_i+\cdots+\alpha_{t-1}}K_{\alpha_r+\cdots+\alpha_n} &= K_{\alpha_r+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_{p-1}}e_{\alpha_i+\cdots+\alpha_{t-1}}, \end{aligned}$$

one can easily show that

$$F_{p,r,t}+(1-q^2)E_{p,t}D_r=0,$$

proving relation (D.31).

We have shown that the components $D_i, D_{i+1}, \ldots, D_n, D_\infty$ of $\Delta(e_{\alpha_i+\cdots+\alpha_n})$ satisfy all the claimed relations, thus we can calculate $\big(\Delta(e_{\alpha_i+\cdots+\alpha_n})\big)^m$ for each $m\in\mathbb{N}$.

D.6.4 $\Delta(e_{\alpha_i+\cdots+2\alpha_n})$, $i=1,\ldots,n-1$

We now determine the commutation relations between the components of $\Delta(e_{\alpha_i+\cdots+2\alpha_n})$. We write $\Delta(e_{\alpha_i+\cdots+2\alpha_n})$ as

$$\Delta(e_{\alpha_i+\cdots+2\alpha_n}) = \sum_{k=i}^{n-1} D_k + D_n + D_0 + D_\infty, \qquad \text{where}$$

$$\begin{aligned}
D_i &= e_{\alpha_i+\cdots+2\alpha_n} \otimes K_{\alpha_i+\cdots+2\alpha_n}, \\
D_k &= (q-q^{-1})\left(e_{\alpha_k+\cdots+2\alpha_n} \otimes K_{\alpha_k+\cdots+2\alpha_n} e_{\alpha_i+\cdots+\alpha_{k-1}}\right), \qquad k=i+1,\ldots,n-1, \\
D_n &= (q-q^{-1})(1+q)(e_n)^2 \otimes (K_n)^2 e_{\alpha_i+\cdots+\alpha_{n-1}} \\
D_0 &= (q-q^{-1})\left(e_n \otimes K_n e_{\alpha_i+\cdots+\alpha_n}\right), \\
D_\infty &= 1 \otimes e_{\alpha_i+\cdots+2\alpha_n}.
\end{aligned}$$

We claim that the commutation relations between these components are

$$\begin{aligned}
D_i D_k &= q^2 D_k D_i, && k=0,i+1,\ldots,n,\infty, \\
D_j D_k &= q^2 D_k D_j, && i+1 \le j < k \le n-1, \\
D_j D_m &= q^2 D_m D_j, && i+1 \le j \le n-1, \quad m=0,n,\infty, \\
D_n D_0 &= q^2 D_0 D_n, && \\
D_n D_\infty &= q^2 D_\infty D_n + \xi(D_0)^2, && \xi = -(1+q)^2/(q-q^{-1}), \\
D_0 D_\infty &= q^2 D_\infty D_0. &&
\end{aligned}$$

The following easily proved identities will assist in proving these commutation relations:

$$\begin{aligned}
\left[e_{\alpha_i+\cdots+2\alpha_n}, e_{\alpha_k+\cdots+2\alpha_n}\right]_q &= 0, && 1 \le i < k \le n-1, \\
\left[e_{\alpha_i+\cdots+2\alpha_n}, e_n\right]_q &= 0, && 1 \le i \le n-1, \\
\left[e_{\alpha_i+\cdots+\alpha_k}, e_{\alpha_i+\cdots+2\alpha_n}\right]_q &= 0, && 1 \le i < k \le n-2, \\
\left[e_{\alpha_i+\cdots+\alpha_j}, e_{\alpha_i+\cdots+\alpha_k}\right]_q &= 0, && 1 \le i < j < k \le n, \\
\left[e_{\alpha_i+\cdots+\alpha_{n-1}}, e_{\alpha_i+\cdots+2\alpha_n}\right]_q &= (1+q)(e_{\alpha_i+\cdots+\alpha_n})^2. &&
\end{aligned}$$

We now prove that the components of $\Delta(e_{\alpha_i+\cdots+2\alpha_n})$ do satisfy the claimed commutation relations. For each $k=i+1,\ldots,n-1$ the relation between D_i and D_k is

$$\begin{aligned}
D_i D_k &= (q-q^{-1})\left(e_{\alpha_i+\cdots+2\alpha_n} e_{\alpha_k+\cdots+2\alpha_n} \otimes K_{\alpha_i+\cdots+2\alpha_n} K_{\alpha_k+\cdots+2\alpha_n} e_{\alpha_i+\cdots+\alpha_{k-1}}\right) \\
&= q^2(q-q^{-1})\left(e_{\alpha_k+\cdots+2\alpha_n} e_{\alpha_i+\cdots+2\alpha_n} \otimes K_{\alpha_k+\cdots+2\alpha_n} e_{\alpha_i+\cdots+\alpha_{k-1}} K_{\alpha_i+\cdots+2\alpha_n}\right) = q^2 D_k D_i.
\end{aligned}$$

Similar (although not identical) calculations show the following:

- $D_i D_n = q^2 D_n D_i$,
- $D_i D_0 = q^2 D_0 D_i$,
- $D_i D_\infty = q^2 D_\infty D_i$,

- $D_kD_j = q^2D_jD_k$ for each $k = i+1, \ldots, n-2$ and each $j = k+1, \ldots, n-1$,
- $D_kD_n = q^2D_nD_k$ for each $k = i+1, \ldots, n-1$,
- $D_kD_0 = q^2D_0D_k$ for each $k = i+1, \ldots, n-1$,
- $D_kD_\infty = q^2D_\infty D_k$ for each $k = i+1, \ldots, n-1$,
- $D_nD_0 = q^2D_0D_n$,
- $D_0D_\infty = q^2D_\infty D_0$,

and to complete the proof, the relation between D_n and D_∞ is

$$\begin{aligned} D_nD_\infty &= (q-q^{-1})(1+q)\left((e_n)^2 \otimes (K_n)^2 e_{\alpha_i+\cdots+\alpha_{n-1}} e_{\alpha_i+\cdots+2\alpha_n}\right) \\ &= (q-q^{-1})(1+q)\left((e_n)^2 \otimes (K_n)^2\left[e_{\alpha_i+\cdots+2\alpha_n} e_{\alpha_i+\cdots+\alpha_{n-1}} + (1+q)(e_{\alpha_i+\cdots+\alpha_n})^2\right]\right) \\ &= q^2(q-q^{-1})(1+q)\left((e_n)^2 \otimes e_{\alpha_i+\cdots+2\alpha_n}(K_n)^2 e_{\alpha_i+\cdots+\alpha_{n-1}}\right) \\ &\quad +(q-q^{-1})(1+q)^2\left((e_n)^2 \otimes (K_n)^2(e_{\alpha_i+\cdots+\alpha_n})^2\right) = q^2D_\infty D_n + \xi(D_0)^2, \end{aligned}$$

where $\xi = -(1+q)^2/(q-q^{-1})$.

D.6.5 $\Delta(e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n})$, $1 \le i < j \le n-1$

We write the co-multiplication of $e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}$ for all $1 \le i < j \le n-1$ as

$$\Delta(e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}) = D_0 + \sum_{k=i+1}^{j-1} D_k + \overline{D} + \sum_{k=j}^{n} D_k + \sum_{p=j}^{n-1} F_p + F_{j-1},$$

where

$$\begin{aligned} D_0 &= e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n} \otimes K_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}, \\ D_k &= (q-q^{-1})\left(e_{\alpha_k+\cdots+2\alpha_j+\cdots+2\alpha_n} \otimes K_{\alpha_k+\cdots+2\alpha_j+\cdots+2\alpha_n} e_{\alpha_i+\cdots+\alpha_{k-1}}\right), \qquad k = i+1, \ldots, j-1, \\ \overline{D} &= (q-q^{-1})\left(q e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n} e_j - q^{-1} e_j e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}\right) \\ &\qquad\qquad \otimes (K_{\alpha_j+\cdots+\alpha_n})^2 e_{\alpha_i+\cdots+\alpha_{j-1}}, \\ D_k &= (q-q^{-1})\left(\overline{e}_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n} \otimes K_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n} e_{\alpha_i+\cdots+\alpha_k}\right), \quad k = j, \ldots, n-1, \\ D_n &= (q-q^{-1})\left(\overline{e}_{\alpha_j+\cdots+\alpha_n} \otimes K_{\alpha_j+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_n}\right), \\ F_p &= (q-q^{-1})\left(\overline{e}_{\alpha_j+\cdots+\alpha_p} \otimes K_{\alpha_j+\cdots+\alpha_p} e_{\alpha_i+\cdots+2\alpha_{p+1}+\cdots+2\alpha_n}\right), \qquad p = j, \ldots, n-1, \\ F_{j-1} &= 1 \otimes e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}, \end{aligned}$$

noting the slightly different normalisations compared to those used in the previous section. We claim that the commutation relations between the components of $\Delta(e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n})$ are

$$\begin{aligned} &(\overline{D} + D_j + D_{j+1} + \cdots + D_{n-1})(F_{n-1} + F_{n-2} + \cdots + F_{j-1}) \\ &\quad = q^2(F_{n-1} + F_{n-2} + \cdots + F_{j-1})(\overline{D} + D_j + D_{j+1} + \cdots + D_{n-1}) + \xi(D_n)^2, \end{aligned} \tag{D.43}$$

where $\xi = -(1+q)^2/(q-q^{-1})$, and

$$\begin{array}{rcll} D_0 D_k &=& q^2 D_k D_0, & k=i+1,\dots,j-1,\\ D_0\overline{D} &=& q^2\overline{D}D_0, & \\ D_0 D_k &=& q^2 D_k D_0, & k=j,\dots,n,\\ D_0 F_p &=& q^2 F_p D_0, & p=j-1,\dots,n-1,\\ D_k D_p &=& q^2 D_p D_k, & i+1\le k<p\le j-1,\\ D_k\overline{D} &=& q^2\overline{D}D_k, & k=i+1,\dots,j-1,\\ D_k D_p &=& q^2 D_p D_k, & k=i+1,\dots,j-1,\ p=j,\dots,n,\\ D_k F_p &=& q^2 F_p D_k, & k=i+1,\dots,j-1,\ p=j-1,\dots,n-1,\\ \overline{D}D_k &=& q^2 D_k\overline{D}, & k=j,\dots,n,\\ \overline{D}F_p &=& q^2 F_p\overline{D}, & p=j,\dots,n-1,\\ D_k D_p &=& q^2 D_p D_k, & j\le k<p\le n,\\ D_k F_p &=& q^2 F_p D_k, & k=j,\dots,n,\ p=j-1,\dots,n-1,\ k\ne p,\\ F_k F_p &=& q^2 F_p F_k, & n-1\ge k>l\ge j-1. \end{array}$$

If the components of $\Delta(e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n})$ do satisfy these commutation relations then the following relations

$$\begin{aligned} &(\overline{D}+D_j+D_{j+1}+\cdots+D_{n-1})(F_{n-1}+F_{n-2}+\cdots+F_{j-1})\\ &\qquad = q^2(F_{n-1}+F_{n-2}+\cdots+F_{j-1})(\overline{D}+D_j+D_{j+1}+\cdots+D_{n-1})+\xi(D_n)^2,\\ &(\overline{D}+D_j+D_{j+1}+\cdots+D_{n-1})D_n = q^2 D_n(\overline{D}+D_j+D_{j+1}+\cdots+D_{n-1}),\\ &D_n(F_{n-1}+F_{n-2}+\cdots+F_{j-1}) = q^2(F_{n-1}+F_{n-2}+\cdots+F_{j-1})D_n, \end{aligned}$$

show that $(\overline{D}+D_j+D_{j+1}+\cdots+D_{n-1})$, D_n and $(F_{n-1}+F_{n-2}+\cdots+F_{j-1})$ satisfy the same relations as do the elements a, b and c, respectively, in Lemma B.0.9. This means that we can immediately use the q-binomial theorem and Lemma B.0.9 (one of the generalisations of the Binomial theorem) to obtain an expression for $\big(\Delta(e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n})\big)^m$ for each $m\in\mathbb{N}$.

We now prove that the components of $\Delta(e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n})$ do satisfy the claimed commutation relations. We firstly consider the easier relations, and then we consider the more complicated calculations needed to prove (D.43). In showing this last relation we must consider the most complicated commutation relations in this problem, namely those between $\overline{D}$ and F_{j-1}, and between D_k and F_k for each $k=j,\dots,n-1$.

We firstly consider some relations that we will extensively use. For each $p=j,\dots,n$,

$$\left[e_{\alpha_k+\cdots+2\alpha_j+\cdots+2\alpha_n}, e_p\right]_q = 0, \tag{D.44}$$

which we can rewrite as

$$\begin{array}{rcll} e_{\alpha_k+\cdots+2\alpha_j+\cdots+2\alpha_n}e_j &=& q e_j e_{\alpha_k+\cdots+2\alpha_j+\cdots+2\alpha_n}, & \\ e_{\alpha_k+\cdots+2\alpha_j+\cdots+2\alpha_n}e_m &=& e_m e_{\alpha_k+\cdots+2\alpha_j+\cdots+2\alpha_n}, & \text{if } m=j+1,\dots,n. \end{array}$$

Equation (D.44) implies the following result: for each $\gamma\in\phi$ satisfying $\alpha_j\preceq\gamma\prec\alpha_{j+1}$ we have

$$\left[e_{\alpha_k+\cdots+2\alpha_j+\cdots+2\alpha_n}, \overline{e}_\gamma\right]_q = 0,$$

which we can rewrite as $e_{\alpha_k+\cdots+2\alpha_j+\cdots+2\alpha_n}\overline{e}_\gamma = q\overline{e}_\gamma e_{\alpha_k+\cdots+2\alpha_j+\cdots+2\alpha_n}$. For all $i<k$ we have

$$\left[e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}, e_{\alpha_k+\cdots+2\alpha_j+\cdots+2\alpha_n}\right]_q = 0,$$

which we can rewrite as

$$e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}e_{\alpha_k+\cdots+2\alpha_j+\cdots+2\alpha_n} = qe_{\alpha_k+\cdots+2\alpha_j+\cdots+2\alpha_n}e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}.$$

A further useful identity is $\left[e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}, e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}\right]_q = 0$, which we can rewrite as

$$e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n} = qe_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}.$$

We now prove that the components of $\Delta(e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n})$ do satisfy the claimed commutation relations. We will firstly prove that $D_0D_k = q^2D_kD_0$ for each $k=i+1,\dots,j-1$. For each such k,

$$\begin{aligned}
D_0D_k &= (q-q^{-1})e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}e_{\alpha_k+\cdots+2\alpha_j+\cdots+2\alpha_n}\\
&\qquad\otimes K_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}K_{\alpha_k+\cdots+2\alpha_j+\cdots+2\alpha_n}e_{\alpha_i+\cdots+\alpha_{k-1}}\\
&= q^2(q-q^{-1})e_{\alpha_k+\cdots+2\alpha_j+\cdots+2\alpha_n}e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}\\
&\qquad\otimes K_{\alpha_k+\cdots+2\alpha_j+\cdots+2\alpha_n}e_{\alpha_i+\cdots+\alpha_{k-1}}K_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}\\
&= q^2D_kD_0.
\end{aligned}$$

We now prove that $D_0\overline{D} = q^2\overline{D}D_0$:

$$\begin{aligned}
D_0\overline{D} &= (q-q^{-1})\left(e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}\otimes K_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}\right)\\
&\quad\times\left(qe_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_j - q^{-1}e_je_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}\right)\otimes(K_{\alpha_j+\cdots+\alpha_n})^2e_{\alpha_i+\cdots+\alpha_{j-1}}\\
&= q^2(q-q^{-1})\left(qe_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_j - q^{-1}e_je_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}\right)e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}\\
&\quad\otimes(K_{\alpha_j+\cdots+\alpha_n})^2e_{\alpha_i+\cdots+\alpha_{j-1}}K_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n} = q^2\overline{D}D_0.
\end{aligned}$$

In a similar way, we can prove the following:

- $D_0D_k = q^2D_kD_0$ for each $k=j,\dots,n-1$,
- $D_0D_n = q^2D_nD_0$,
- $D_0F_k = q^2F_kD_0$ for each $k=n-1,\dots,j$,
- $D_0F_{j-1} = q^2F_{j-1}D_0$.

We now prove that $D_kD_p = q^2D_pD_k$ for each $k=i+1,\dots,j-2$ and each $p=k+1,\dots,j-1$:

$$\begin{aligned}
D_kD_p &= (q-q^{-1})^2e_{\alpha_k+\cdots+2\alpha_j+\cdots+2\alpha_n}e_{\alpha_p+\cdots+2\alpha_j+\cdots+2\alpha_n}\\
&\qquad\otimes K_{\alpha_k+\cdots+2\alpha_j+\cdots+2\alpha_n}e_{\alpha_i+\cdots+\alpha_{k-1}}K_{\alpha_p+\cdots+2\alpha_j+\cdots+2\alpha_n}e_{\alpha_i+\cdots+\alpha_{p-1}}\\
&= q(q-q^{-1})^2e_{\alpha_p+\cdots+2\alpha_j+\cdots+2\alpha_n}e_{\alpha_k+\cdots+2\alpha_j+\cdots+2\alpha_n}\\
&\qquad\otimes K_{\alpha_k+\cdots+2\alpha_j+\cdots+2\alpha_n}K_{\alpha_p+\cdots+2\alpha_j+\cdots+2\alpha_n}e_{\alpha_i+\cdots+\alpha_{k-1}}e_{\alpha_i+\cdots+\alpha_{p-1}}\\
&= q^2(q-q^{-1})^2e_{\alpha_p+\cdots+2\alpha_j+\cdots+2\alpha_n}e_{\alpha_k+\cdots+2\alpha_j+\cdots+2\alpha_n}\\
&\qquad\otimes K_{\alpha_p+\cdots+2\alpha_j+\cdots+2\alpha_n}e_{\alpha_i+\cdots+\alpha_{p-1}}K_{\alpha_k+\cdots+2\alpha_j+\cdots+2\alpha_n}e_{\alpha_i+\cdots+\alpha_{k-1}}\\
&= q^2D_pD_k.
\end{aligned}$$

We now prove that $D_k\overline{D} = q^2\overline{D}D_k$ for each $k = i+1, \ldots, j-1$:

$$\begin{aligned}
D_k\overline{D} &= q^2(q-q^{-1})^2\left(qe_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_j - q^{-1}e_je_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}\right)e_{\alpha_k+\cdots+2\alpha_j+\cdots+2\alpha_n}\\
&\qquad \otimes K_{\alpha_k+\cdots+2\alpha_j+\cdots+2\alpha_n}(K_{\alpha_j+\cdots+\alpha_n})^2 e_{\alpha_i+\cdots+\alpha_{k-1}}e_{\alpha_i+\cdots+\alpha_{j-1}}\\
&= q^3(q-q^{-1})^2\left(qe_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_j - q^{-1}e_je_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}\right)e_{\alpha_k+\cdots+2\alpha_j+\cdots+2\alpha_n}\\
&\qquad \otimes K_{\alpha_k+\cdots+2\alpha_j+\cdots+2\alpha_n}(K_{\alpha_j+\cdots+\alpha_n})^2 e_{\alpha_i+\cdots+\alpha_{j-1}}e_{\alpha_i+\cdots+\alpha_{k-1}}\\
&= q^2(q-q^{-1})^2\left(qe_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_j - q^{-1}e_je_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}\right)e_{\alpha_k+\cdots+2\alpha_j+\cdots+2\alpha_n}\\
&\qquad \otimes (K_{\alpha_j+\cdots+\alpha_n})^2 e_{\alpha_i+\cdots+\alpha_{j-1}}K_{\alpha_k+\cdots+2\alpha_j+\cdots+2\alpha_n}e_{\alpha_i+\cdots+\alpha_{k-1}}\\
&= q^2\overline{D}D_k.
\end{aligned}$$

In a similar way, we can show the following:

- $D_kD_p = q^2D_pD_k$ for each $k = i+1, \ldots, j-1$ and each $p = j, \ldots, n-1$,
- $D_kD_n = q^2D_nD_k$ for each $k = i+1, \ldots, j-1$,
- $D_kF_p = q^2F_pD_k$ for each $k = i+1, \ldots, j-1$ and each $p = j, \ldots, n-1$,
- $D_kF_{j-1} = q^2F_{j-1}D_k$ for each $k = i+1, \ldots, j-1$.

We now prove that $\overline{D}F_k = q^2F_k\overline{D}$ for all $k = j, \ldots, n-1$:

$$\begin{aligned}
&\overline{D}F_k\\
&= (q-q^{-1})^2\overline{e}_{\alpha_j+\cdots+\alpha_k}\left(qe_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_j - q^{-1}e_je_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}\right)\\
&\qquad \otimes (K_{\alpha_j+\cdots+\alpha_n})^2 e_{\alpha_i+\cdots+\alpha_{j-1}}K_{\alpha_j+\cdots+\alpha_k}e_{\alpha_i+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}\\
&= q(q-q^{-1})^2\overline{e}_{\alpha_j+\cdots+\alpha_k}\left(qe_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_j - q^{-1}e_je_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}\right)\\
&\qquad \otimes K_{\alpha_j+\cdots+\alpha_k}(K_{\alpha_j+\cdots+\alpha_n})^2 e_{\alpha_i+\cdots+\alpha_{j-1}}e_{\alpha_i+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}\\
&= q^2(q-q^{-1})^2\overline{e}_{\alpha_j+\cdots+\alpha_k}\left(qe_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_j - q^{-1}e_je_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}\right)\\
&\qquad \otimes K_{\alpha_j+\cdots+\alpha_k}(K_{\alpha_j+\cdots+\alpha_n})^2 e_{\alpha_i+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}e_{\alpha_i+\cdots+\alpha_{j-1}}\\
&= q^2(q-q^{-1})^2\overline{e}_{\alpha_j+\cdots+\alpha_k}\left(qe_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_j - q^{-1}e_je_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}\right)\\
&\qquad \otimes K_{\alpha_j+\cdots+\alpha_k}e_{\alpha_i+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}(K_{\alpha_j+\cdots+\alpha_n})^2 e_{\alpha_i+\cdots+\alpha_{j-1}} = q^2F_k\overline{D}.
\end{aligned}$$

In this calculation we used the fact that the element

$$\left(qe_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_j - q^{-1}e_je_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}\right)$$

commutes with e_k for each $k = j, \ldots, n$ (see Proposition D.9.7). We can also show the following:

- $D_kD_p = q^2D_pD_k$ for all $j \leq k < p \leq n-1$,
- $D_kD_n = q^2D_nD_k$ for each $k = j, \ldots, n-1$.

We now prove that $D_kF_p = q^2F_pD_k$ for all $k = j, \dots, n-1$ and all $p = j, \dots, n-1$ satisfying $k \neq p$:

$$\begin{aligned} D_kF_p &= (q-q^{-1})^2\big(\overline{e}_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}\overline{e}_{\alpha_j+\cdots+\alpha_p} \\ &\qquad \otimes K_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}e_{\alpha_i+\cdots+\alpha_k}K_{\alpha_j+\cdots+\alpha_p}e_{\alpha_i+\cdots+2\alpha_{p+1}+\cdots+2\alpha_n}\big) \\ &= q(q-q^{-1})^2\overline{e}_{\alpha_j+\cdots+\alpha_p}\overline{e}_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n} \\ &\qquad \otimes K_{\alpha_j+\cdots+\alpha_p}K_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}e_{\alpha_i+\cdots+\alpha_k}e_{\alpha_i+\cdots+2\alpha_{p+1}+\cdots+2\alpha_n} \\ &= q^2(q-q^{-1})^2\overline{e}_{\alpha_j+\cdots+\alpha_p}\overline{e}_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n} \\ &\qquad \otimes K_{\alpha_j+\cdots+\alpha_p}e_{\alpha_i+\cdots+2\alpha_{p+1}+\cdots+2\alpha_n}K_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}e_{\alpha_i+\cdots+\alpha_k} \\ &= q^2F_pD_k, \end{aligned}$$

where we used the relations

$$\big[e_{\alpha_j+\cdots+\alpha_p}, e_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}\big]_q = 0, \quad \big[\overline{e}_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}, \overline{e}_{\alpha_j+\cdots+\alpha_p}\big]_q = 0, \qquad k \neq p.$$

We can also show the following are true:

- $\overline{D}D_k = q^2D_k\overline{D}$ for each $k = j, \dots, n-1$,
- $\overline{D}D_n = q^2D_n\overline{D}$,
- $D_kF_{j-1} = q^2F_{j-1}D_k$ for each $k = j, \dots, n-1$.

We now prove that $D_nF_k = q^2F_kD_n$ for each $k = n-1, \dots, j$:

$$\begin{aligned} D_nF_k &= (q-q^{-1})^2\overline{e}_{\alpha_j+\cdots+\alpha_n}\overline{e}_{\alpha_j+\cdots+\alpha_k} \otimes K_{\alpha_j+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_n}K_{\alpha_j+\cdots+\alpha_k}e_{\alpha_i+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n} \\ &= q(q-q^{-1})^2\overline{e}_{\alpha_j+\cdots+\alpha_k}\overline{e}_{\alpha_j+\cdots+\alpha_n} \otimes K_{\alpha_j+\cdots+\alpha_k}K_{\alpha_j+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_n}e_{\alpha_i+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n} \\ &= q^2(q-q^{-1})^2\overline{e}_{\alpha_j+\cdots+\alpha_k}\overline{e}_{\alpha_j+\cdots+\alpha_n} \otimes K_{\alpha_j+\cdots+\alpha_k}e_{\alpha_i+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}K_{\alpha_j+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_n} \\ &= q^2F_kD_n, \end{aligned}$$

as $\big[e_{\alpha_i+\cdots+\alpha_n}, e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}\big]_q = 0$, and the relation between D_n and F_{j-1} is

$$D_nF_{j-1} = q^2(q-q^{-1})\overline{e}_{\alpha_j+\cdots+\alpha_n} \otimes e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}K_{\alpha_j+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_n} = q^2F_{j-1}D_n.$$

We now prove that $F_kF_p = q^2F_pF_k$ for each $k = n-1, \dots, j+1$ and each $p = k-1, \dots, j$:

$$\begin{aligned} F_kF_p &= (q-q^{-1})^2\overline{e}_{\alpha_j+\cdots+\alpha_k}\overline{e}_{\alpha_j+\cdots+\alpha_p} \\ &\qquad \otimes K_{\alpha_j+\cdots+\alpha_k}e_{\alpha_i+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}K_{\alpha_j+\cdots+\alpha_p}e_{\alpha_i+\cdots+2\alpha_{p+1}+\cdots+2\alpha_n} \\ &= q^2(q-q^{-1})^2\overline{e}_{\alpha_j+\cdots+\alpha_p}\overline{e}_{\alpha_j+\cdots+\alpha_k} \\ &\qquad \otimes K_{\alpha_j+\cdots+\alpha_p}K_{\alpha_j+\cdots+\alpha_k}e_{\alpha_i+\cdots+2\alpha_{p+1}+\cdots+2\alpha_n}e_{\alpha_i+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n} \\ &= q^2F_pF_k, \end{aligned}$$

where we have used the identity $\big[e_{\alpha_i+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}, e_{\alpha_i+\cdots+2\alpha_{p+1}+\cdots+2\alpha_n}\big]_q = 0$. In addition, for each $k = n-1, \dots, j$,

$$F_kF_{j-1} = q^2(q-q^{-1})\overline{e}_{\alpha_j+\cdots+\alpha_k} \otimes e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}K_{\alpha_j+\cdots+\alpha_k}e_{\alpha_i+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n} = q^2F_{j-1}F_k.$$

This completes the proof of the easier commutation relations. We now prove the more complicated relations. The relation between $\overline{D}$ and F_{j-1} is:

$$\begin{aligned}
&\overline{D}F_{j-1}\\
&= (q-q^{-1})\left(qe_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_j - q^{-1}e_je_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}\right)\\
&\qquad \otimes(K_{\alpha_j+\cdots+\alpha_n})^2 e_{\alpha_i+\cdots+\alpha_{j-1}}e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}\\
&= (q-q^{-1})\left(qe_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_j - q^{-1}e_je_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}\right)\\
&\qquad \otimes(K_{\alpha_j+\cdots+\alpha_n})^2 \left(e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}e_{\alpha_i+\cdots+\alpha_{j-1}} + \left[e_{\alpha_i+\cdots+\alpha_{j-1}}, e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}\right]_q\right)\\
&= q^2(q-q^{-1})\left(qe_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_j - q^{-1}e_je_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}\right)\\
&\qquad \otimes e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}(K_{\alpha_j+\cdots+\alpha_n})^2 e_{\alpha_i+\cdots+\alpha_{j-1}}\\
&\quad +(q-q^{-1})\left(qe_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_j - q^{-1}e_je_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}\right)\\
&\qquad \otimes(K_{\alpha_j+\cdots+\alpha_n})^2 \left[e_{\alpha_i+\cdots+\alpha_{j-1}}, e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}\right]_q\\
&= q^2F_{j-1}\overline{D} + (q-q^{-1})\left(qe_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_j - q^{-1}e_je_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}\right)\\
&\qquad\qquad \otimes(K_{\alpha_j+\cdots+\alpha_n})^2 \left[e_{\alpha_i+\cdots+\alpha_{j-1}}, e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}\right]_q.
\end{aligned}$$

The relation between D_k and F_k for each $k = j, \ldots, n-1$ is:

$$\begin{aligned}
D_kF_k &= (q-q^{-1})^2\overline{e}_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}\overline{e}_{\alpha_j+\cdots+\alpha_k}\\
&\qquad \otimes K_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}e_{\alpha_i+\cdots+\alpha_k}K_{\alpha_j+\cdots+\alpha_k}e_{\alpha_i+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}\\
&= (q-q^{-1})^2\overline{e}_{\alpha_j+\cdots+\alpha_k}\overline{e}_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}\\
&\qquad \otimes K_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}e_{\alpha_i+\cdots+\alpha_k}K_{\alpha_j+\cdots+\alpha_k}e_{\alpha_i+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}\\
&\quad +(q-q^{-1})^2\left[\overline{e}_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}\overline{e}_{\alpha_j+\cdots+\alpha_k}\right]_q\\
&\qquad \otimes K_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}e_{\alpha_i+\cdots+\alpha_k}K_{\alpha_j+\cdots+\alpha_k}e_{\alpha_i+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}\\
&= q^{-1}(q-q^{-1})^2\overline{e}_{\alpha_j+\cdots+\alpha_k}\overline{e}_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}\\
&\qquad \otimes(K_{\alpha_j+\cdots+\alpha_n})^2e_{\alpha_i+\cdots+\alpha_k}e_{\alpha_i+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}\\
&\quad +q^{-1}(q-q^{-1})^2\left[\overline{e}_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}\overline{e}_{\alpha_j+\cdots+\alpha_k}\right]_q\\
&\qquad \otimes(K_{\alpha_j+\cdots+\alpha_n})^2e_{\alpha_i+\cdots+\alpha_k}e_{\alpha_i+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}\\
&= (q-q^{-1})^2\overline{e}_{\alpha_j+\cdots+\alpha_k}\overline{e}_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}\\
&\qquad \otimes K_{\alpha_j+\cdots+\alpha_k}e_{\alpha_i+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}K_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}e_{\alpha_i+\cdots+\alpha_k}\\
&\quad +q^{-1}(q-q^{-1})^2\overline{e}_{\alpha_j+\cdots+\alpha_k}\overline{e}_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}\\
&\qquad \otimes(K_{\alpha_j+\cdots+\alpha_n})^2\left[e_{\alpha_i+\cdots+\alpha_k}, e_{\alpha_i+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}\right]_q\\
&\quad +q^{-1}(q-q^{-1})^2\left[\overline{e}_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}, \overline{e}_{\alpha_j+\cdots+\alpha_k}\right]_q\\
&\qquad \otimes(K_{\alpha_j+\cdots+\alpha_n})^2e_{\alpha_i+\cdots+\alpha_k}e_{\alpha_i+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}\\
&= F_kD_k + q^{-1}(q-q^{-1})^2\overline{e}_{\alpha_j+\cdots+\alpha_k}\overline{e}_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}\\
&\qquad\qquad \otimes(K_{\alpha_j+\cdots+\alpha_n})^2\left[e_{\alpha_i+\cdots+\alpha_k}, e_{\alpha_i+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}\right]_q\\
&\qquad +q^{-1}(q-q^{-1})^2\left[\overline{e}_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}, \overline{e}_{\alpha_j+\cdots+\alpha_k}\right]_q\\
&\qquad\qquad \otimes(K_{\alpha_j+\cdots+\alpha_n})^2e_{\alpha_i+\cdots+\alpha_k}e_{\alpha_i+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}.
\end{aligned}$$

We can rewrite this relation as

$$\begin{aligned}
D_kF_k &= q^2F_kD_k + q^{-1}(1-q^2)(q-q^{-1})^2\overline{e}_{\alpha_j+\cdots+\alpha_k}\overline{e}_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n} \\
&\qquad \otimes(K_{\alpha_j+\cdots+\alpha_n})^2 e_{\alpha_i+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}e_{\alpha_i+\cdots+\alpha_k} \\
&\quad +q^{-1}(q-q^{-1})^2\overline{e}_{\alpha_j+\cdots+\alpha_k}\overline{e}_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n} \\
&\qquad \otimes(K_{\alpha_j+\cdots+\alpha_n})^2\left[e_{\alpha_i+\cdots+\alpha_k}, e_{\alpha_i+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}\right]_q \\
&\quad +q^{-1}(q-q^{-1})^2\left[\overline{e}_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}, \overline{e}_{\alpha_j+\cdots+\alpha_k}\right]_q \\
&\qquad \otimes(K_{\alpha_j+\cdots+\alpha_n})^2 e_{\alpha_i+\cdots+\alpha_k}e_{\alpha_i+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n} \\
&= q^2F_kD_k + (q-q^{-1})^2\Big[(q^{-1}-q)\overline{e}_{\alpha_j+\cdots+\alpha_k}\overline{e}_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n} \\
&\qquad +q^{-1}\left[\overline{e}_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}, \overline{e}_{\alpha_j+\cdots+\alpha_k}\right]_q\Big] \\
&\qquad \otimes(K_{\alpha_j+\cdots+\alpha_n})^2 e_{\alpha_i+\cdots+\alpha_k}e_{\alpha_i+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n} \\
&\quad +q(q-q^{-1})^2\overline{e}_{\alpha_j+\cdots+\alpha_k}\overline{e}_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n} \\
&\qquad \otimes(K_{\alpha_j+\cdots+\alpha_n})^2\left[e_{\alpha_i+\cdots+\alpha_k}, e_{\alpha_i+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}\right]_q .
\end{aligned}$$

From these calculations we have the following relation:

$$\begin{aligned}
&\big(\overline{D}+D_j+D_{j+1}+\cdots+D_{n-1}\big)\big(F_{n-1}+F_{n-2}+\cdots+F_{j-1}\big) \\
&\quad = q^2\big(F_{n-1}+F_{n-2}+\cdots+F_{j-1}\big)\big(\overline{D}+D_j+D_{j+1}+\cdots+D_{n-1}\big) \\
&\qquad +(q-q^{-1})\left(qe_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_j - q^{-1}e_je_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}\right) \\
&\qquad\qquad \otimes(K_{\alpha_j+\cdots+\alpha_n})^2\left[e_{\alpha_i+\cdots+\alpha_{j-1}}, e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}\right]_q && \text{(D.45)} \\
&\qquad +(q-q^{-1})^2 && \text{(D.46)} \\
&\qquad \times\sum_{k=j}^{n-1}\Bigg(\Big[(q^{-1}-q)\overline{e}_{\alpha_j+\cdots+\alpha_k}\overline{e}_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n} \\
&\qquad\qquad +q^{-1}\left[\overline{e}_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}, \overline{e}_{\alpha_j+\cdots+\alpha_k}\right]_q\Big] \\
&\qquad\qquad \otimes(K_{\alpha_j+\cdots+\alpha_n})^2 e_{\alpha_i+\cdots+\alpha_k}e_{\alpha_i+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n} && \text{(D.47)} \\
&\qquad\quad +q\overline{e}_{\alpha_j+\cdots+\alpha_k}\overline{e}_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n} \\
&\qquad\qquad \otimes(K_{\alpha_j+\cdots+\alpha_n})^2\left[e_{\alpha_i+\cdots+\alpha_k}, e_{\alpha_i+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}\right]_q\Bigg). && \text{(D.48)}
\end{aligned}$$

To simplify this expression we will expand out the terms in the second tensor power and sum the terms accordingly. The easiest part of this calculation is as follows: the component in the expansion for which the second tensor power contains a term of the form

$e_{\alpha_i+\cdots+\alpha_j}e_{\alpha_i+\cdots+2\alpha_{j+1}+\cdots+2\alpha_n}$, is

$$\begin{aligned}
&(q-q^{-1})^2\Big[qe_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_j - q^{-1}e_je_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n} \\
&\qquad +q^{-1}\overline{e}_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_j - qe_j\overline{e}_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}\Big] \\
&\qquad\qquad \otimes(K_{\alpha_j+\cdots+\alpha_n})^2e_{\alpha_i+\cdots+\alpha_j}e_{\alpha_i+\cdots+2\alpha_{j+1}+\cdots+2\alpha_n} \qquad \text{(D.49)}\\
&= 0,
\end{aligned}$$

where (D.49) arises from (D.45) and the $k=j$ term in (D.47), and Proposition D.9.9 then implies that (D.49) vanishes.

To simplify the remaining terms in the above expression we need to know the expansions of

$$\left[e_{\alpha_i+\cdots+\alpha_{j-1}}, e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}\right]_q \quad \text{and} \quad \left[\overline{e}_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}, \overline{e}_{\alpha_j+\cdots+\alpha_k}\right]_q$$

which we have detailed in Propositions D.9.5 and D.9.6 respectively. Using these propositions, we can determine the component in the expansion of the above expression for which the second tensor power contains a term of the form $e_{\alpha_i+\cdots+\alpha_{j+p}}e_{\alpha_i+\cdots+2\alpha_{j+p+1}+\cdots+2\alpha_n}$ for each $p=1,\ldots,n-j-1$. This component is

$$\begin{aligned}
&\Big[(-q)^p(q-q^{-1})^2\left(qe_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_j - q^{-1}e_je_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}\right) \\
&\quad +\sum_{k=j}^{j+p-1}(-q)^{j+p-k-1}q(q-q^{-1})^3\overline{e}_{\alpha_j+\cdots+\alpha_k}\overline{e}_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n} \\
&\quad +(q-q^{-1})^2\left(-q\overline{e}_{\alpha_j+\cdots+\alpha_{j+p}}\overline{e}_{\alpha_j+\cdots+2\alpha_{j+p+1}+\cdots+2\alpha_n} + q^{-1}\overline{e}_{\alpha_j+\cdots+2\alpha_{j+p+1}+\cdots+2\alpha_n}\overline{e}_{\alpha_j+\cdots+\alpha_{j+p}}\right)\Big] \\
&\qquad\qquad \otimes(K_{\alpha_j+\cdots+\alpha_n})^2e_{\alpha_i+\cdots+\alpha_{j+p}}e_{\alpha_i+\cdots+2\alpha_{j+p+1}+\cdots+2\alpha_n}.
\end{aligned}$$

The element of $U_q(\mathfrak{g})$ in the first tensor power of this component is

$$\begin{aligned}
&(q-q^{-1})^2\Big[(-q)^pqe_j\overline{e}_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n} - q^{-1}(-q)^p\overline{e}_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_j \\
&\qquad +(-q)^{p-1}q(q-q^{-1})e_j\overline{e}_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n} \\
&\qquad +\sum_{k=j+1}^{j+p-1}(-q)^{j+p-k-1}q(q-q^{-1})\overline{e}_{\alpha_j+\cdots+\alpha_k}\overline{e}_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n} \\
&\qquad -q\overline{e}_{\alpha_j+\cdots+\alpha_{j+p}}\overline{e}_{\alpha_j+\cdots+2\alpha_{j+p+1}+\cdots+2\alpha_n} + q^{-1}\overline{e}_{\alpha_j+\cdots+2\alpha_{j+p+1}+\cdots+2\alpha_n}\overline{e}_{\alpha_j+\cdots+\alpha_{j+p}}\Big],
\end{aligned}$$

which we can rewrite as

$$\begin{aligned}
&(q-q^{-1})^2\Big[(-1)^pq^{p-1}e_j\overline{e}_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n} - (-1)^pq^{p-1}\overline{e}_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_j \qquad \text{(D.50)}\\
&\qquad +\sum_{k=j+1}^{j+p-1}(-q)^{j+p-k-1}q(q-q^{-1})\overline{e}_{\alpha_j+\cdots+\alpha_k}\overline{e}_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n} \\
&\qquad -q\overline{e}_{\alpha_j+\cdots+\alpha_{j+p}}\overline{e}_{\alpha_j+\cdots+2\alpha_{j+p+1}+\cdots+2\alpha_n} + q^{-1}\overline{e}_{\alpha_j+\cdots+2\alpha_{j+p+1}+\cdots+2\alpha_n}\overline{e}_{\alpha_j+\cdots+\alpha_{j+p}}\Big].
\end{aligned}$$

Now we calculate that

$$\begin{aligned}
&(-1)^p q^{p-1}\left(e_j\overline{e}_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}-\overline{e}_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_j\right)\\
&\quad= (-q)^{p-1}\left[\overline{e}_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n},e_j\right]_q\\
&\quad= (-q)^{p-1}\left[e_{j+1}\overline{e}_{\alpha_j+\alpha_{j+1}+2\alpha_{j+2}+\cdots+2\alpha_n}-q^{-1}\overline{e}_{\alpha_j+\alpha_{j+1}+2\alpha_{j+2}+\cdots+2\alpha_n}e_{j+1},e_j\right]_q\\
&\quad= (-q)^{p-1}\left(q\left[e_{j+1},e_j\right]_q\overline{e}_{\alpha_j+\alpha_{j+1}+2\alpha_{j+2}+\cdots+2\alpha_n}-q^{-1}\overline{e}_{\alpha_j+\alpha_{j+1}+2\alpha_{j+2}+\cdots+2\alpha_n}\left[e_{j+1},e_j\right]_q\right),
\end{aligned}$$

and substituting this into (D.50) gives

$$\begin{aligned}
&(q-q^{-1})^2\\
&\qquad\times\Big[(-q)^{p-1}\left(q\overline{e}_{\alpha_j+\alpha_{j+1}}\overline{e}_{\alpha_j+\alpha_{j+1}+2\alpha_{j+2}+\cdots+2\alpha_n}-q^{-1}\overline{e}_{\alpha_j+\alpha_{j+1}+2\alpha_{j+2}+\cdots+2\alpha_n}\overline{e}_{\alpha_j+\alpha_{j+1}}\right)\\
&\qquad\quad+(-q)^{j+p-j-2}q(q-q^{-1})\overline{e}_{\alpha_j+\alpha_{j+1}}\overline{e}_{\alpha_j+\alpha_{j+1}+2\alpha_{j+2}+\cdots+2\alpha_n}\\
&\qquad\quad+\sum_{k=j+2}^{j+p-1}(-q)^{j+p-k-1}(q^2-1)\overline{e}_{\alpha_j+\cdots+\alpha_k}\overline{e}_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}\\
&\qquad\quad-q\overline{e}_{\alpha_j+\cdots+\alpha_{j+p}}\overline{e}_{\alpha_j+\cdots+2\alpha_{j+p+1}+\cdots+2\alpha_n}+q^{-1}\overline{e}_{\alpha_j+\cdots+2\alpha_{j+p+1}+\cdots+2\alpha_n}\overline{e}_{\alpha_j+\cdots+\alpha_{j+p}}\Big]\\
&\quad= (q-q^{-1})^2 &\text{(D.51)}\\
&\qquad\times\Big[(-q)^{p-2}\left[\overline{e}_{\alpha_j+\alpha_{j+1}+2\alpha_{j+2}+\cdots+2\alpha_n},\overline{e}_{\alpha_j+\alpha_{j+1}}\right]_q &\text{(D.52)}\\
&\qquad\quad+\sum_{k=j+2}^{j+p-1}(-q)^{j+p-k-1}(q^2-1)\overline{e}_{\alpha_j+\cdots+\alpha_k}\overline{e}_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n} &\text{(D.53)}\\
&\qquad\quad-q\overline{e}_{\alpha_j+\cdots+\alpha_{j+p}}\overline{e}_{\alpha_j+\cdots+2\alpha_{j+p+1}+\cdots+2\alpha_n}+q^{-1}\overline{e}_{\alpha_j+\cdots+2\alpha_{j+p+1}+\cdots+2\alpha_n}\overline{e}_{\alpha_j+\cdots+\alpha_{j+p}}\Big]. &\text{(D.54)}
\end{aligned}$$

We can dramatically simplify this expression by using the following calculation:

$$\begin{aligned}
&(-q)^{p-m}\left[\overline{e}_{\alpha_j+\cdots+2\alpha_{j+m}+\cdots+2\alpha_n},\overline{e}_{\alpha_j+\cdots+\alpha_{j+m-1}}\right]_q\\
&\qquad+(-q)^{p-m-1}(q^2-1)\overline{e}_{\alpha_j+\cdots+\alpha_{j+m}}\overline{e}_{\alpha_j+\cdots+2\alpha_{j+m+1}+\cdots+2\alpha_n}\\
&\quad= (-q)^{p-m}\left[e_{j+m}\overline{e}_{\alpha_j+\cdots+2\alpha_{j+m+1}+\cdots+2\alpha_n}-q^{-1}\overline{e}_{\alpha_j+\cdots+2\alpha_{j+m+1}+\cdots+2\alpha_n}e_{j+m},\overline{e}_{\alpha_j+\cdots+\alpha_{j+m-1}}\right]_q\\
&\qquad+(-q)^{p-m-1}(q^2-1)\overline{e}_{\alpha_j+\cdots+\alpha_{j+m}}\overline{e}_{\alpha_j+\cdots+2\alpha_{j+m+1}+\cdots+2\alpha_n}\\
&\quad= (-q)^{p-m}\left(q\overline{e}_{\alpha_j+\cdots+\alpha_{j+m}}\overline{e}_{\alpha_j+\cdots+2\alpha_{j+m+1}+\cdots+2\alpha_n}-q^{-1}\overline{e}_{\alpha_j+\cdots+2\alpha_{j+m+1}+\cdots+2\alpha_n}\overline{e}_{\alpha_j+\cdots+\alpha_{j+m}}\right)\\
&\qquad+(-q)^{p-m-1}(q^2-1)\overline{e}_{\alpha_j+\cdots+\alpha_{j+m}}\overline{e}_{\alpha_j+\cdots+2\alpha_{j+m+1}+\cdots+2\alpha_n}\\
&\quad= (-q)^{p-m-1}\left[\overline{e}_{\alpha_j+\cdots+2\alpha_{j+m+1}+\cdots+2\alpha_n},\overline{e}_{\alpha_j+\cdots+\alpha_{j+m}}\right]_q. &\text{(D.55)}
\end{aligned}$$

By repeatedly using this identity in (D.52)–(D.53), we can rewrite (D.51)–(D.54) as the following expression for each $t = 2, \ldots, p-1$:

$$\begin{aligned}
&(q-q^{-1})^2 \\
&\quad\times\Big[(-q)^{p-t}\left[\overline{e}_{\alpha_j+\cdots+2\alpha_{j+t}+\cdots+2\alpha_n}, \overline{e}_{\alpha_j+\cdots+\alpha_{j+t-1}}\right]_q \\
&\qquad+\sum_{k=j+t}^{j+p-1}(-q)^{j+p-k-1}(q^2-1)\overline{e}_{\alpha_j+\cdots+\alpha_k}\overline{e}_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n} \\
&\qquad-q\overline{e}_{\alpha_j+\cdots+\alpha_{j+p}}\overline{e}_{\alpha_j+\cdots+2\alpha_{j+p+1}+\cdots+2\alpha_n}+q^{-1}\overline{e}_{\alpha_j+\cdots+2\alpha_{j+p+1}+\cdots+2\alpha_n}\overline{e}_{\alpha_j+\cdots+\alpha_{j+p}}\Big].
\end{aligned} \tag{D.56}$$

Substituting $t = p-1$ into (D.56) and using (D.55), we can rewrite (D.51)–(D.54) as

$$\begin{aligned}
&(q-q^{-1})^2\Big[\left[\overline{e}_{\alpha_j+\cdots+2\alpha_{j+p}+\cdots+2\alpha_n}, \overline{e}_{\alpha_j+\cdots+\alpha_{j+p-1}}\right]_q \\
&\qquad-q\overline{e}_{\alpha_j+\cdots+\alpha_{j+p}}\overline{e}_{\alpha_j+\cdots+2\alpha_{j+p+1}+\cdots+2\alpha_n}+q^{-1}\overline{e}_{\alpha_j+\cdots+2\alpha_{j+p+1}+\cdots+2\alpha_n}\overline{e}_{\alpha_j+\cdots+\alpha_{j+p}}\Big] \\
&= 0,
\end{aligned}$$

where we have used the following result:

$$\begin{aligned}
&\left[\overline{e}_{\alpha_j+\cdots+2\alpha_{j+p}+\cdots+2\alpha_n}, \overline{e}_{\alpha_j+\cdots+\alpha_{j+p-1}}\right]_q \\
&= \left[e_{j+p}\overline{e}_{\alpha_j+\cdots+2\alpha_{j+p+1}+\cdots+2\alpha_n} - q^{-1}\overline{e}_{\alpha_j+\cdots+2\alpha_{j+p+1}+\cdots+2\alpha_n}e_{j+p}, \overline{e}_{\alpha_j+\cdots+\alpha_{j+p-1}}\right]_q \\
&= q\overline{e}_{\alpha_j+\cdots+\alpha_{j+p}}\overline{e}_{\alpha_j+\cdots+2\alpha_{j+p+1}+\cdots+2\alpha_n} - q^{-1}\overline{e}_{\alpha_j+\cdots+2\alpha_{j+p+1}+\cdots+2\alpha_n}\overline{e}_{\alpha_j+\cdots+\alpha_{j+p}}.
\end{aligned}$$

This substantially simplifies the problem. All we now have to do is to determine the component for which the second tensor power contains a term of the form $(e_{\alpha_j+\cdots+\alpha_n})^2$. This is not difficult to do: the first tensor power of this component is

$$\begin{aligned}
&(-q)^{n-j}(1+q)(q-q^{-1})\left(qe_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_j - q^{-1}e_je_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}\right) \\
&\qquad\sum_{k=j}^{n-1}(-q)^{n-k-1}q(1+q)(q-q^{-1})^2\overline{e}_{\alpha_j+\cdots+\alpha_k}\overline{e}_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}.
\end{aligned} \tag{D.57}$$

Now by repeatedly using (D.55), we have

$$\begin{aligned}
&(-q)^{n-j}\left(qe_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_j - q^{-1}e_je_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}\right) \\
&\qquad+\sum_{k=j}^{n-1}(-q)^{n-k-1}(q^2-1)\overline{e}_{\alpha_j+\cdots+\alpha_k}\overline{e}_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n} \\
&= \left[\overline{e}_{\alpha_j+\cdots+2\alpha_n}, \overline{e}_{\alpha_j+\cdots+\alpha_{n-1}}\right]_q \\
&= (1+q)\left(\overline{e}_{\alpha_j+\cdots+\alpha_n}\right)^2,
\end{aligned}$$

and thus (D.57) is

$$(1+q)^2(q-q^{-1})\left(\overline{e}_{\alpha_j+\cdots+\alpha_n}\right)^2 = \xi(D_n)^2, \qquad \xi = -(1+q)^2/(q-q^{-1}),$$

as required.

D.7 $\mathcal{I}$ is a two-sided co-ideal

In this section we prove that $\mathcal{I}$ is a two-sided co-ideal. We firstly note the almost trivial result that $\epsilon(x) = 0$ for each $x \in I$ and thus $\epsilon(x) = 0$ for all $x \in \mathcal{I}$. We now deal with the more substantial problem: we will prove that

$$\Delta(x) \in \mathcal{I} \otimes U_q(\mathfrak{g}) + U_q(\mathfrak{g}) \otimes \mathcal{I},$$

for each $x \in I$. Firstly,

$$\Delta(J_i^{\pm N} - 1) = J_i^{\pm N} \otimes J_i^{\pm N} - 1 \otimes 1 = J_i^{\pm N} \otimes (J_i^{\pm N} - 1) + (J_i^{\pm N} - 1) \otimes 1,$$

which is an element of $\mathcal{I} \otimes U_q(\mathfrak{g}) + U_q(\mathfrak{g}) \otimes \mathcal{I}$. The remaining problems are harder and we break them down into a number of subcases. Initially we will show that $(e_\gamma)^{N'}$ and $(e_\beta)^{\overline{N}}$ are elements of $\mathcal{I} \otimes U_q(\mathfrak{g}) + U_q(\mathfrak{g}) \otimes \mathcal{I}$; then we will use the antipode S and the graded antiautomorphism ω to prove the remaining cases.

In writing down the components of $\Delta(e_\mu)$ we will use the notation used in Section D.5, although we may use slightly different normalisations. Any alternative normalisations will not significantly affect the calculations.

Case 1. $(e_i)^{N'}$, $(e_n)^{\overline{N}}$, $1 \leq i < n$

The components of $\Delta(e_i)$ satisfy the q-binomial theorem for each i, thus we have the following results from Appendix B:

$$\begin{aligned}
\big(\Delta(e_i)\big)^{N'} &= (e_i \otimes K_i)^{N'} + (1 \otimes e_i)^{N'} = (e_i)^{N'} \otimes (K_i)^{N'} + 1 \otimes (e_i)^{N'}, \quad i < n, \\
\big(\Delta(e_n)\big)^{\overline{N}} &= (e_n \otimes K_n)^{\overline{N}} + (1 \otimes e_n)^{\overline{N}} = (e_n)^{\overline{N}} \otimes (K_n)^{\overline{N}} + 1 \otimes (e_n)^{\overline{N}}.
\end{aligned}$$

As $(e_i)^{N'} \in \mathcal{I}$ and $(e_n)^{\overline{N}} \in \mathcal{I}$, $\big(\Delta(e_i)\big)^{N'}$ and $\big(\Delta(e_n)\big)^{\overline{N}}$ are elements of $\mathcal{I} \otimes U_q(\mathfrak{g}) + U_q(\mathfrak{g}) \otimes \mathcal{I}$.

Case 2. $(e_{\alpha_i + \cdots + \alpha_j})^{N'}$, $1 \leq i < j < n$

By writing $\Delta(e_{\alpha_i+\cdots+\alpha_j}) = \sum_{k=i}^{j} D_k + D_\infty$ and noting the relations $D_r D_s = q^2 D_s D_r$ for all $r < s$ we can use the q-multinomial theorem to immediately obtain

$$\big(\Delta(e_{\alpha_i+\cdots+\alpha_j})\big)^{N'} = \sum_{k=i}^{j} (D_k)^{N'} + (D_\infty)^{N'}.$$

As $(e_{\alpha_k+\cdots+\alpha_j})^{N'} \in I$ for all $1 \leq k \leq j < n$, we have that $(D_k)^{N'}$ and $(D_\infty)^{N'}$ belong to $\mathcal{I} \otimes U_q(\mathfrak{g}) + U_q(\mathfrak{g}) \otimes \mathcal{I}$ for all $i \leq k \leq j$, and thus

$$\big(\Delta(e_{\alpha_i+\cdots+\alpha_j})\big)^{N'} \in \mathcal{I} \otimes U_q(\mathfrak{g}) + U_q(\mathfrak{g}) \otimes \mathcal{I}.$$

Case 3. $(e_{\alpha_i+\cdots+\alpha_n})^{\overline{N}}$**,** $i = 1, \ldots, n-1$

Recall that $\Delta(e_{\alpha_i+\cdots+\alpha_n}) = \sum_{k=i}^{n} D_k + D_\infty$. We claim that

$$\left(\Delta\left(e_{\alpha_i+\cdots+\alpha_n}\right)\right)^{\overline{N}} = (D_\infty)^{\overline{N}} + \begin{cases} \displaystyle\sum_{k=i}^{n}(D_k)^{2N} + (1)_q(3)_q\cdots(2N-1)_q\left[\sum_{i\le r<t\le n}(E_{r,t})^N\right], & \text{if } N \equiv 1,3 \pmod 4, \\ \displaystyle\sum_{k=i}^{n}(D_k)^{N} + (1)_q(3)_q\cdots(N-1)_q\left[\sum_{i\le r<t\le n}(E_{r,t})^{N/2}\right], & \text{if } N \equiv 0 \pmod 4, \\ \displaystyle\sum_{k=i}^{n}(D_k)^{N/2}, & \text{if } N \equiv 2 \pmod 4. \end{cases}$$

Firstly, from the relation $D_k D_\infty = -qD_\infty D_k$ for all $i \le k < \infty$, we have

$$\left(\sum_{k=i}^{n} D_k + D_\infty\right)^{\overline{N}} = (D_\infty)^{\overline{N}} + \left(\sum_{k=i}^{n} D_k\right)^{\overline{N}},$$

and we now use induction to obtain an expression for $\left(\sum_{k=i}^{n} D_k\right)^{\overline{N}}$. The elements D_i and D_{i+1} satisfy the relations

$$D_iD_{i+1} = -qD_{i+1}D_i + E_{i,i+1}, \quad D_iE_{i,i+1} = q^2E_{i,i+1}D_i, \quad E_{i,i+1}D_{i+1} = q^2D_{i+1}E_{i,i+1},$$

and therefore

$$\left(D_i + D_{i+1}\right)^{\overline{N}} = \begin{cases} (D_i)^{2N} + (D_{i+1})^{2N} + (1)_q(3)_q\cdots(2N-1)_q(E_{i,i+1})^N, & \text{if } N \equiv 1,3 \pmod 4, \\ (D_i)^{N} + (D_{i+1})^{N} + (1)_q(3)_q\cdots(N-1)_q(E_{i,i+1})^{N/2}, & \text{if } N \equiv 0 \pmod 4, \\ (D_i)^{N/2} + (D_{i+1})^{N/2}, & \text{if } N \equiv 2 \pmod 4. \end{cases}$$

Using induction, it is quite simple to prove the following expansion for each $j = i+1, \ldots, n$:

$$\left(\sum_{k=i}^{j} D_k\right)^{\overline{N}} = \begin{cases} \displaystyle\sum_{k=i}^{j}(D_k)^{2N} + (1)_q(3)_q\cdots(2N-1)_q\left[\sum_{i\le r<t\le j}(E_{r,t})^N\right], & \text{if } N \equiv 1,3 \pmod 4, \\ \displaystyle\sum_{k=i}^{j}(D_k)^{N} + (1)_q(3)_q\cdots(N-1)_q\left[\sum_{i\le r<t\le j}(E_{r,t})^{N/2}\right], & \text{if } N \equiv 0 \pmod 4, \\ \displaystyle\sum_{k=i}^{j}(D_k)^{N/2}, & \text{if } N \equiv 2 \pmod 4, \end{cases}$$

where for the inductive step we use the relations

$$\begin{aligned}
&(D_i + D_{i+1} + \cdots + D_j)D_{j+1} \\
&\quad = -qD_{j+1}(D_i + D_{i+1} + \cdots + D_j) + (E_{i,j+1} + E_{i+1,j+1} + \cdots + E_{j,j+1}), \\
&(D_i + D_{i+1} + \cdots + D_j)(E_{i,j+1} + E_{i+1,j+1} + \cdots + E_{j,j+1}) \\
&\quad = q^2(E_{i,j+1} + E_{i+1,j+1} + \cdots + E_{j,j+1})(D_i + D_{i+1} + \cdots + D_j), \\
&(E_{i,j+1} + E_{i+1,j+1} + \cdots + E_{j,j+1})D_{j+1} \\
&\quad = q^2 D_{j+1}(E_{i,j+1} + E_{i+1,j+1} + \cdots + E_{j,j+1}).
\end{aligned}$$

This proves the claimed expression for $\big(\Delta(e_{\alpha_i+\cdots+\alpha_n})\big)^{\overline{N}}$.

Now $(e_{\alpha_k+\cdots+\alpha_n})^{\overline{N}} \in \mathcal{I}$ for each $k = i, \ldots, n$ and thus

$$(D_k)^{\overline{N}} \in \mathcal{I} \otimes U_q(\mathfrak{g}) + U_q(\mathfrak{g}) \otimes \mathcal{I}, \qquad i \le k \le \infty.$$

It remains to show that $(E_{r,t})^{N'} \in \mathcal{I} \otimes U_q(\mathfrak{g}) + U_q(\mathfrak{g}) \otimes \mathcal{I}$ when $N \equiv 0, 1, 3 \pmod 4$ for all $i \le r < t \le n$. To show this, we note that the second tensor power of $E_{i,p}$ is

$$K_{\alpha_i+\cdots+\alpha_n} K_{\alpha_p+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_{p-1}}, \qquad p = i+1, \ldots, n,$$

and the fact that $\big(e_{\alpha_i+\cdots+\alpha_{p-1}}\big)^{N'} \in \mathcal{I}$ means that

$$(E_{i,p})^{N'} \in \mathcal{I} \otimes U_q(\mathfrak{g}) + U_q(\mathfrak{g}) \otimes \mathcal{I}, \qquad p = i+1, \ldots, n.$$

The second tensor power of $E_{k,p}$ for each $k = i+1, \ldots, n-1$ and each $p = k+1, \ldots n$ is

$$K_{\alpha_k+\cdots+\alpha_n} K_{\alpha_p+\cdots+\alpha_n} e_{\alpha_i+\cdots+\alpha_{k-1}} e_{\alpha_i+\cdots+\alpha_{p-1}}.$$

Using the fact that $\big[e_{\alpha_i+\cdots+\alpha_{k-1}}, e_{\alpha_i+\cdots+\alpha_{p-1}}\big]_q = 0$ and that $\big(e_{\alpha_i+\cdots+\alpha_{p-1}}\big)^{N'} \in \mathcal{I}$, it is quite easy to see that

$$(E_{k,p})^{N'} \in \mathcal{I} \otimes U_q(\mathfrak{g}) + U_q(\mathfrak{g}) \otimes \mathcal{I},$$

which completes the proof of this case.

Case 4. $(e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n})^{N'}$**,** $1 \le i < j \le n$

We firstly consider $\big(\Delta(e_{\alpha_i+\cdots+2\alpha_n})\big)^{N'}$. Writing $\Delta(e_{\alpha_i+\cdots+2\alpha_n}) = \sum_{k=i}^{n-1} D_k + D_n + D_0 + D_\infty$, the q-binomial theorem and one of the generalisations of the binomial theorem immediately gives

$$\begin{aligned}
&\big(\Delta(e_{\alpha_i+\cdots+2\alpha_n})\big)^{N'} \\
&\quad = \sum_{k=i}^{n-1} (D_k)^{N'} + (D_n + D_0 + D_\infty)^{N'} \\
&\quad = \sum_{k=i}^{n-1} (D_k)^{N'} + \begin{cases} (D_n)^{N'} + (D_\infty)^{N'}, & \text{if } N \equiv 0, 1, 3 \pmod 4, \\ (D_n)^{N'} + (D_\infty)^{N'} + \phi_{N/2}(D_0)^{N'} & \text{if } N \equiv 2 \pmod 4, \end{cases}
\end{aligned}$$

where $\phi_{N/2} \neq 0$. Now

$$(e_{\alpha_k+\cdots+2\alpha_n})^{N'} \in \mathcal{I}, \qquad k = i, \ldots, n-1,$$

and the fact that $(e_n)^{\overline{N}} \in \mathcal{I}$ means that $(e_n)^{2N'} \in \mathcal{I}$. By using these facts and examining some simple calculations it follows that

$$(D_k)^{N'}, (D_\infty)^{N'} \in \mathcal{I} \otimes U_q(\mathfrak{g}) + U_q(\mathfrak{g}) \otimes \mathcal{I}, \qquad k = i, \ldots, n.$$

It remains to show that $(D_0)^{N'} \in \mathcal{I} \otimes U_q(\mathfrak{g}) + U_q(\mathfrak{g}) \otimes \mathcal{I}$ when $N \equiv 2 \pmod 4$; this follows from $(e_n)^{\overline{N}} \in \mathcal{I}$.

We now consider the more general problem: write

$$\Delta(e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n}) = D_0 + \sum_{k=i+1}^{n} D_k + \overline{D} + \sum_{l=j-1}^{n-1} F_l.$$

Using the q-binomial theorem we immediately obtain

$$\begin{aligned}\big(\Delta(e_{\alpha_i+\cdots+2\alpha_j+\cdots+2\alpha_n})\big)^{N'} &= (D_0)^{N'} + \sum_{k=i+1}^{j-1} (D_k)^{N'} \\ &\quad + \big(\overline{D} + D_j + \cdots + D_{n-1} + D_n + F_{n-1} + F_{n-2} + \cdots + F_{j-1}\big)^{N'}.\end{aligned} \tag{D.58}$$

We can expand out the last component of the right hand side of this expression by using the following relations:

$$\begin{aligned}&(\overline{D} + D_j + \cdots + D_{n-1})(F_{n-1} + F_{n-2} + \cdots + F_{j-1}) \\ &\quad = q^2(F_{n-1} + F_{n-2} + \cdots + F_{j-1})(\overline{D} + D_j + \cdots + D_{n-1}) + \xi(D_n)^2,\\ &(\overline{D} + D_j + \cdots + D_{n-1})D_n = q^2 D_n(\overline{D} + D_j + \cdots + D_{n-1}),\\ &D_n(F_{n-1} + F_{n-2} + \cdots + F_{j-1}) = q^2(F_{n-1} + F_{n-2} + \cdots + F_{j-1})D_n,\end{aligned}$$

where $\xi = -(1+q)^2/(q-q^{-1})$. Consequently,

$$\begin{aligned}&\big(\overline{D} + D_j + \cdots + D_{n-1} + D_n + F_{n-1} + F_{n-2} + \cdots + F_{j-1}\big)^{N'}\\ &= \begin{cases} (\overline{D} + D_j + \cdots + D_{n-1})^{N'} + (F_{n-1} + F_{n-2} + \cdots + F_{j-1})^{N'}, & \text{if } N \equiv 0,1,3 \pmod 4,\\ (\overline{D} + D_j + \cdots + D_{n-1})^{N'} + (F_{n-1} + F_{n-2} + \cdots + F_{j-1})^{N'} \\ \qquad + \phi_{N/2}(D_n)^{N'}, & \text{if } N \equiv 2 \pmod 4,\end{cases}\end{aligned}$$

where $\phi_{N/2} \neq 0$. The q-binomial theorem then implies that

$$(\overline{D} + D_j + \cdots + D_{n-1})^{N'} = (\overline{D})^{N'} + \sum_{k=j}^{n-1} (D_k)^{N'}, \tag{D.59}$$

$$(F_{n-1}+F_{n-2}+\cdots+F_{j-1})^{N'}=\sum_{k=j-1}^{n-1}(F_k)^{N'}. \tag{D.60}$$

Now the facts that $(e_{\alpha_k+\cdots+2\alpha_j+\cdots+2\alpha_n})^{N'}\in\mathcal{I}$ for each $k=i,\ldots,n-1$ and each $j=k+1,\ldots,n$, and that $(e_{\alpha_i+\cdots+\alpha_k})^{N'}\in\mathcal{I}$ for each $k=i+1,\ldots,n-1$, mean that each component of the right hand sides of (D.58)–(D.60) belongs to $\mathcal{I}\otimes U_q(\mathfrak{g})+U_q(\mathfrak{g})\otimes\mathcal{I}$.

It remains to show that the same is true for $(D_n)^{N'}$ when $N\equiv 2 \pmod 4$. To see this, note that $(\overline{e}_{\alpha_j+\cdots+\alpha_n})^{\overline{N}}\in\mathcal{I}$ and that $\overline{N}=N/2$ for $N\equiv 2 \pmod 4$, thus $(D_n)^{N'}\in\mathcal{I}\otimes U_q(\mathfrak{g})+U_q(\mathfrak{g})\otimes\mathcal{I}$ when $N\equiv 2 \pmod 4$. This completes the proof of this case.

The remaining elements of I

We have shown that $\Delta(e_\gamma)^{N'}$ and $\Delta(e_\beta)^{\overline{N}}$ are elements of $\mathcal{I}\otimes U_q(\mathfrak{g})+U_q(\mathfrak{g})\otimes\mathcal{I}$. We now prove that the same is true for the remaining elements of $\mathcal{I}$ in the following calculations, in which we write the proportionality sign to mean that the left hand side is proportional to the right hand side with a non-zero scalar constant of proportionality. The cases we do not consider here are almost identically proved:

$$\Delta(f_\gamma)^{N'}\propto\Delta\left(\omega\left(e_\gamma^{N'}\right)\right)\propto(\omega\otimes\omega)\circ\Delta'(e_\gamma)^{N'},$$

$$\Delta(\overline{e}_\gamma)^{N'}\propto\Delta\left(S\left(e_\gamma^{N'}\right)\right)\Delta(K_\gamma)^{N'}\propto\left[(S\otimes S)\circ\Delta'(e_\gamma)^{N'}\right]\Delta(K_\gamma)^{N'},$$

$$\Delta(\overline{f}_\gamma)^{N'}\propto\Delta\left(\omega\left(\overline{e}_\gamma^{N'}\right)\right)\propto(\omega\otimes\omega)\circ\Delta'(\overline{e}_\gamma)^{N'}.$$

Each of these expressions is an element of $\mathcal{I}\otimes U_q(\mathfrak{g})+U_q(\mathfrak{g})\otimes\mathcal{I}$ from Proposition D.8.1, thus $\mathcal{I}$ is a two-sided co-ideal.

D.8 $\mathcal{I}$ is a two-sided Hopf ideal

We have proved that $\mathcal{I}$ is a two-sided ideal and a two-sided co-ideal of $U_q(osp(1|2n))$. To prove that $\mathcal{I}$ is a two-sided Hopf ideal all we need do is prove that $S(x)\in\mathcal{I}$ for each $x\in\mathcal{I}$, and to show this it suffices to show that $S(x)\in\mathcal{I}$ for each $x\in I$.

Proposition D.8.1. *For each $x\in I$, $\omega(x)\in\mathcal{I}$ and $S(x)\in\mathcal{I}$.*

Proof. Firstly note that $\omega(J_i^{\pm N}-1)=(J_i^{\mp N}-1)\in\mathcal{I}$ and $S(J_i^{\pm N}-1)=(J_i^{\mp N}-1)\in\mathcal{I}$. The rest of the proof follows from Propositions D.3.4–D.3.8 and the facts that ω is an involution and that ω and S are graded antiautomorphisms. □

D.9 Technical results

In this section we prove technical results used previously in this appendix.

Proposition D.9.1. *For each $j \in \mathbb{Z}_+$,*

$$\sum_{k=1}^{2j} q^{-2k}(k)_q = q^{-4j}(2j+1)_q[j]^{q^2}, \qquad \sum_{k=1}^{2j+1} q^{-2k}(k)_q = q^{-2(2j+1)}(2j+1)_q[j+1]^{q^2}.$$

Proof. By direct calculation we have

- $q^{-2}(1)_q = q^{-2}$,
- $\sum_{k=1}^{2} q^{-2k}(k)_q = q^{-4}(3)_q$,
- $\sum_{k=1}^{3} q^{-2k}(k)_q = q^{-6}(3)_q[2]^{q^2}$,
- $\sum_{k=1}^{4} q^{-2k}(k)_q = q^{-8}(5)_q[2]^{q^2}$.

Assume that the proposition is true for $\sum_{k=1}^{2j} q^{-2k}(k)_q$, then

$$\begin{aligned}\sum_{k=1}^{2j+1} q^{-2k}(k)_q &= q^{-4j}(2j+1)_q[j]^{q^2} + q^{-4j-2}(2j+1)_q \\ &= q^{-4j-2}(2j+1)_q(q^2+q^4+\cdots+q^{2j}) + q^{-4j-2}(2j+1)_q \\ &= q^{-4j-2}(2j+1)_q[j+1]^{q^2},\end{aligned}$$

as required. Now assume that the proposition is true for $\sum_{k=1}^{2j+1} q^{-2k}(k)_q$, then

$$\begin{aligned}\sum_{k=1}^{2j+2} q^{-2k}(k)_q &= q^{-4j-2}(2j+1)_q[j+1]^{q^2} + q^{-4j-4}(2j+2)_q \\ &= q^{-4j-4}\big((2j+1)_q(q^2+q^4+\cdots+q^{2j+2}) + (2j+2)_q\big) \\ &= q^{-4j-4}\big((2j+1)_q\,[j+1]^{q^2} + (-q)^{2j+1} + (-q)^{2j+2} + \cdots + (-q)^{4j+2}\big) \\ &= q^{-4j-4}\big((2j+1)_q\,[j+1]^{q^2} + \big((-q)^{2j+1} + (-q)^{2j+2}\big)\,[j+1]^{q^2}\,\big) \\ &= q^{-4j-4}(2j+3)_q\,[j+1]^{q^2},\end{aligned}$$

completing the proof. ∎

Proposition D.9.2. *For all k, p satisfying $i \le k < p \le n$,*

$$\begin{aligned}\left[e_{\alpha_k+\cdots+\alpha_n}, e_{\alpha_p+\cdots+\alpha_n}\right]_q &= (-q)^{n-p} e_{\alpha_k+\cdots+2\alpha_p+\cdots+2\alpha_n} \\ &\quad + (q-q^{-1})\left[\sum_{j=1}^{n-p}(-q)^{j-1} e_{\alpha_k+\cdots+2\alpha_{n-j+1}+\cdots+2\alpha_n} e_{\alpha_p+\cdots+\alpha_{n-j}}\right].\end{aligned} \tag{D.61}$$

Proof. The relation $[e_{\alpha_k+\cdots+\alpha_n}, e_n]_q = e_{\alpha_k+\cdots+2\alpha_n}$ proves the proposition for $p = n$ and all k. Assume that $k < n-1$, then

$$\begin{aligned}\left[e_{\alpha_k+\cdots+\alpha_n}, e_{\alpha_{n-1}+\alpha_n}\right]_q &= \left[e_{\alpha_k+\cdots+\alpha_n}, e_{n-1}e_n - q^{-1}e_ne_{n-1}\right]_q \\ &= \left[e_{\alpha_k+\cdots+\alpha_n}, e_{n-1}\right]_q e_n + e_{n-1}\left[e_{\alpha_k+\cdots+\alpha_n}, e_n\right]_q \\ &\quad -q^{-1}\left[e_{\alpha_k+\cdots+\alpha_n}, e_n\right]_q e_{n-1} + q^{-1}e_n\left[e_{\alpha_k+\cdots+\alpha_n}, e_{n-1}\right]_q \\ &= e_{n-1}e_{\alpha_k+\cdots+2\alpha_n} - q^{-1}e_{\alpha_k+\cdots+2\alpha_n}e_{n-1},\end{aligned}$$

as $[e_{\alpha_k+\cdots+\alpha_n}, e_{n-1}]_q = 0$. By re-writing the relation $[e_{\alpha_k+\cdots+2\alpha_n}, e_{n-1}]_q = e_{\alpha_k+\cdots+2\alpha_{n-1}+2\alpha_n}$ as $e_{n-1}e_{\alpha_k+\cdots+2\alpha_n} = qe_{\alpha_k+\cdots+2\alpha_n}e_{n-1} - qe_{\alpha_k+\cdots+2\alpha_{n-1}+2\alpha_n}$, we obtain

$$\left[e_{\alpha_k+\cdots+\alpha_n}, e_{\alpha_{n-1}+\alpha_n}\right]_q = -qe_{\alpha_k+\cdots+2\alpha_{n-1}+2\alpha_n} + (q-q^{-1})e_{\alpha_k+\cdots+2\alpha_n}e_{n-1},$$

proving the proposition for $p = n-1$. Now assume that the proposition is true for some $p = k+2, \ldots, n-1$, then

$$\begin{aligned}&\left[e_{\alpha_k+\cdots+\alpha_n}, e_{\alpha_{p-1}+\cdots+\alpha_n}\right]_q \\ &= \left[e_{\alpha_k+\cdots+\alpha_n}, e_{p-1}e_{\alpha_p+\cdots+\alpha_n} - q^{-1}e_{\alpha_p+\cdots+\alpha_n}e_{p-1}\right]_q \\ &= \left[e_{\alpha_k+\cdots+\alpha_n}, e_{p-1}\right]_q e_{\alpha_p+\cdots+\alpha_n} + e_{p-1}\left[e_{\alpha_k+\cdots+\alpha_n}, e_{\alpha_p+\cdots+\alpha_n}\right]_q \\ &\quad -q^{-1}\left[e_{\alpha_k+\cdots+\alpha_n}, e_{\alpha_p+\cdots+\alpha_n}\right]_q e_{p-1} + q^{-1}e_{\alpha_p+\cdots+\alpha_n}\left[e_{\alpha_k+\cdots+\alpha_n}, e_{p-1}\right]_q \\ &= e_{p-1}\left[e_{\alpha_k+\cdots+\alpha_n}, e_{\alpha_p+\cdots+\alpha_n}\right]_q - q^{-1}\left[e_{\alpha_k+\cdots+\alpha_n}, e_{\alpha_p+\cdots+\alpha_n}\right]_q e_{p-1},\end{aligned} \tag{D.62}$$

as $[e_{\alpha_k+\cdots+\alpha_n}, e_{p-1}]_q = 0$. Substituting (D.61) into (D.62) gives

$$\begin{aligned}&(-q)^{n-p}e_{p-1}e_{\alpha_k+\cdots+2\alpha_p+\cdots+2\alpha_n} \\ &+(q-q^{-1})\left[\sum_{j=1}^{n-p}(-q)^{j-1}e_{p-1}e_{\alpha_k+\cdots+2\alpha_{n-j+1}+\cdots+2\alpha_n}e_{\alpha_p+\cdots+\alpha_{n-j}}\right] \\ &-q^{-1}(-q)^{n-p}e_{\alpha_k+\cdots+2\alpha_p+\cdots+2\alpha_n}e_{p-1} \\ &-q^{-1}(q-q^{-1})\left[\sum_{j=1}^{n-p}(-q)^{j-1}e_{\alpha_k+\cdots+2\alpha_{n-j+1}+\cdots+2\alpha_n}e_{\alpha_p+\cdots+\alpha_{n-j}}e_{p-1}\right] \\ =\ &-q(-q)^{n-p}e_{\alpha_k+\cdots+2\alpha_{p-1}+\cdots+2\alpha_n} + (q-q^{-1})(-q)^{n-p}e_{\alpha_k+\cdots+2\alpha_p+\cdots+2\alpha_n}e_{p-1} \\ &+(q-q^{-1})\left[\sum_{j=1}^{n-p}(-q)^{j-1}e_{\alpha_k+\cdots+2\alpha_{n-j+1}+\cdots+2\alpha_n}e_{p-1}e_{\alpha_p+\cdots+\alpha_{n-j}}\right. \\ &\qquad\left. -q^{-1}(-q)^{j-1}e_{\alpha_k+\cdots+2\alpha_{n-j+1}+\cdots+2\alpha_n}e_{\alpha_p+\cdots+\alpha_{n-j}}e_{p-1}\right] \\ =\ &(-q)^{n-p+1}e_{\alpha_k+\cdots+2\alpha_{p-1}+\cdots+2\alpha_n} \\ &+(q-q^{-1})\left[\sum_{j=1}^{n-p+1}(-q)^{j-1}e_{\alpha_k+\cdots+2\alpha_{n-j+1}+\cdots+2\alpha_n}e_{\alpha_{p-1}+\cdots+\alpha_{n-j}}\right],\end{aligned}$$

completing the induction.

□

Proposition D.9.3. *For all* $1 \leq i < t \leq n$,

$$\left[e_{\alpha_i+\cdots+\alpha_{t-1}}, e_{\alpha_t+\cdots+\alpha_n}\right]_q = e_{\alpha_i+\cdots+\alpha_n}.$$

Proof. Firstly $\left[e_{\alpha_i+\cdots+\alpha_{n-1}}, e_n\right]_q = e_{\alpha_i+\cdots+\alpha_n}$. Now assume that $\left[e_{\alpha_i+\cdots+\alpha_{t-1}}, e_{\alpha_t+\cdots+\alpha_n}\right]_q = e_{\alpha_i+\cdots+\alpha_n}$ for some $i+2 \leq t \leq n$, then

$$\begin{aligned}
\left[e_{\alpha_i+\cdots+\alpha_{t-2}}, e_{\alpha_{t-1}+\cdots+\alpha_n}\right]_q &= \left[e_{\alpha_i+\cdots+\alpha_{t-2}}, e_{t-1}e_{\alpha_t+\cdots+\alpha_n} - q^{-1}e_{\alpha_t+\cdots+\alpha_n}e_{t-1}\right]_q \\
&= \left[e_{\alpha_i+\cdots+\alpha_{t-2}}, e_{t-1}\right]_q e_{\alpha_t+\cdots+\alpha_n} + q^{-1}e_{t-1}\left[e_{\alpha_i+\cdots+\alpha_{t-2}}, e_{\alpha_t+\cdots+\alpha_n}\right]_q \\
&\quad -q^{-1}\left[e_{\alpha_i+\cdots+\alpha_{t-2}}, e_{\alpha_t+\cdots+\alpha_n}\right]_q e_{t-1} - q^{-1}e_{\alpha_t+\cdots+\alpha_n}\left[e_{\alpha_i+\cdots+\alpha_{t-2}}, e_{t-1}\right]_q \\
&= e_{\alpha_i+\cdots+\alpha_{t-1}}e_{\alpha_t+\cdots+\alpha_n} - q^{-1}e_{\alpha_t+\cdots+\alpha_n}e_{\alpha_i+\cdots+\alpha_{t-1}} \\
&= \left[e_{\alpha_i+\cdots+\alpha_{t-1}}, e_{\alpha_t+\cdots+\alpha_n}\right]_q = e_{\alpha_i+\cdots+\alpha_n},
\end{aligned}$$

as $\left[e_{\alpha_i+\cdots+\alpha_{t-2}}, e_{\alpha_t+\cdots+\alpha_n}\right]_q = 0$.

□

Proposition D.9.4. *For all* $1 \leq i < t \leq j-1 < n$,

$$\left[e_{\alpha_i+\cdots+\alpha_{j-1}}, e_{\alpha_t+\cdots+\alpha_n}\right]_q = (q-q^{-1})e_{\alpha_i+\cdots+\alpha_n}e_{\alpha_t+\cdots+\alpha_{j-1}}.$$

Proof. Firstly $\left[e_{\alpha_i+\cdots+\alpha_{j-1}}, e_{\alpha_j+\cdots+\alpha_n}\right]_q = e_{\alpha_i+\cdots+\alpha_n}$, and

$$\begin{aligned}
\left[e_{\alpha_i+\cdots+\alpha_{j-1}}, e_{\alpha_{j-1}+\cdots+\alpha_n}\right]_q &= \left[e_{\alpha_i+\cdots+\alpha_{j-1}}, e_{j-1}e_{\alpha_j+\cdots+\alpha_n} - q^{-1}e_{\alpha_j+\cdots+\alpha_n}e_{j-1}\right]_q \\
&= \left[e_{\alpha_i+\cdots+\alpha_{j-1}}, e_{j-1}\right]_q e_{\alpha_j+\cdots+\alpha_n} + qe_{j-1}\left[e_{\alpha_i+\cdots+\alpha_{j-1}}, e_{\alpha_j+\cdots+\alpha_n}\right]_q \\
&\quad -q^{-1}\left[e_{\alpha_i+\cdots+\alpha_{j-1}}, e_{\alpha_j+\cdots+\alpha_n}\right]_q e_{j-1} - q^{-2}e_{\alpha_j+\cdots+\alpha_n}\left[e_{\alpha_i+\cdots+\alpha_{j-1}}, e_{j-1}\right]_q \\
&= qe_{j-1}e_{\alpha_i+\cdots+\alpha_n} - q^{-1}e_{\alpha_i+\cdots+\alpha_n}e_{j-1} = (q-q^{-1})e_{\alpha_i+\cdots+\alpha_n}e_{j-1},
\end{aligned}$$

as $e_{j-1}e_{\alpha_i+\cdots+\alpha_n} = e_{\alpha_i+\cdots+\alpha_n}e_{j-1}$. Now assume that

$$\left[e_{\alpha_i+\cdots+\alpha_{j-1}}, e_{\alpha_t+\cdots+\alpha_n}\right]_q = (q-q^{-1})e_{\alpha_i+\cdots+\alpha_n}e_{\alpha_t+\cdots+\alpha_{j-1}},$$

for some $i+2 \leq t \leq j-1$, then

$$\begin{aligned}
&\left[e_{\alpha_i+\cdots+\alpha_{j-1}}, e_{\alpha_{t-1}+\cdots+\alpha_n}\right]_q \\
&= \left[e_{\alpha_i+\cdots+\alpha_{j-1}}, e_{t-1}e_{\alpha_t+\cdots+\alpha_n} - q^{-1}e_{\alpha_t+\cdots+\alpha_n}e_{t-1}\right]_q \\
&= \left[e_{\alpha_i+\cdots+\alpha_{j-1}}, e_{t-1}\right]_q e_{\alpha_t+\cdots+\alpha_n} + e_{t-1}\left[e_{\alpha_i+\cdots+\alpha_{j-1}}, e_{\alpha_t+\cdots+\alpha_n}\right]_q \\
&\quad -q^{-1}\left[e_{\alpha_i+\cdots+\alpha_{j-1}}, e_{\alpha_t+\cdots+\alpha_n}\right]_q e_{t-1} - q^{-1}e_{\alpha_t+\cdots+\alpha_n}\left[e_{\alpha_i+\cdots+\alpha_{j-1}}, e_{t-1}\right]_q \\
&= e_{t-1}\left[e_{\alpha_i+\cdots+\alpha_{j-1}}, e_{\alpha_t+\cdots+\alpha_n}\right]_q - q^{-1}\left[e_{\alpha_i+\cdots+\alpha_{j-1}}, e_{\alpha_t+\cdots+\alpha_n}\right]_q e_{t-1} \\
&= (q-q^{-1})e_{t-1}e_{\alpha_i+\cdots+\alpha_n}e_{\alpha_t+\cdots+\alpha_{j-1}} - q^{-1}(q-q^{-1})e_{\alpha_i+\cdots+\alpha_n}e_{\alpha_t+\cdots+\alpha_{j-1}}e_{t-1} \\
&= (q-q^{-1})e_{\alpha_i+\cdots+\alpha_n}e_{\alpha_{t-1}+\cdots+\alpha_{j-1}},
\end{aligned}$$

as $\left[e_{\alpha_i+\cdots+\alpha_{j-1}}, e_{t-1}\right]_q = 0$ and e_{t-1} commutes with $e_{\alpha_i+\cdots+\alpha_n}$. □

We can easily prove Propositions D.9.5 and D.9.6 inductively using elementary calculations.

Proposition D.9.5. *For all $1 \leq i < j < n$,*

$$\begin{aligned}\left[e_{\alpha_i+\cdots+\alpha_j}, e_{\alpha_i+\cdots+2\alpha_{j+1}+\cdots+2\alpha_n}\right]_q &= (-q)^{n-j-1}(1+q)(e_{\alpha_i+\cdots+\alpha_n})^2 \\ &\quad +(q-q^{-1})\sum_{p=1}^{n-j-1}(-q)^{p-1}e_{\alpha_i+\cdots+\alpha_{j+p}}e_{\alpha_i+\cdots+2\alpha_{j+p+1}+\cdots+2\alpha_n}.\end{aligned}$$

Proposition D.9.6. *For all $1 \leq i < j < n$,*

$$\begin{aligned}\left[\overline{e}_{\alpha_i+\cdots+2\alpha_{j+1}+\cdots+2\alpha_n}, \overline{e}_{\alpha_i+\cdots+\alpha_j},\right]_q &= (-q)^{n-j-1}(1+q)(\overline{e}_{\alpha_i+\cdots+\alpha_n})^2 \\ &\quad +(q-q^{-1})\sum_{p=1}^{n-j-1}(-q)^{p-1}\overline{e}_{\alpha_i+\cdots+2\alpha_{j+p+1}+\cdots+2\alpha_n}\overline{e}_{\alpha_i+\cdots+\alpha_{j+p}}.\end{aligned}$$

Proposition D.9.7. *The element $\left(qe_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_j - q^{-1}e_je_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}\right)$ commutes with e_k for each $k = 1, \ldots, n$.*

Proof. Firstly, the Serre relations state that e_k commutes with e_i if $|k-i| > 1$. For each $k = j+2, \ldots, n$, we have the relation

$$\left[e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}, e_k\right]_q = 0,$$

which states that each such e_k commutes with $e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}$.

Now we will show that e_{j+1} commutes with $\left(qe_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_j - q^{-1}e_je_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}\right)$. We calculate that

$$\begin{aligned}&\left(qe_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_j - q^{-1}e_je_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}\right)e_{j+1} \\ &\quad = qe_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_je_{j+1} - e_je_{j+1}e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n} \\ &\quad = qe_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_{\alpha_j+\alpha_{j+1}} + e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_{j+1}e_j \\ &\qquad -e_{\alpha_j+\alpha_{j+1}}e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n} - q^{-1}e_{j+1}e_je_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n} \\ &\quad = e_{j+1}\left(qe_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_j - q^{-1}e_je_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}\right),\end{aligned}$$

where we have used the relations

$$\left[e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}, e_{j+1}\right]_q = 0, \qquad \left[e_{\alpha_j+\alpha_{j+1}}, e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}\right]_q = 0.$$

To complete the proof we will show that

$$\begin{aligned}&e_j\left(qe_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_j - q^{-1}e_je_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}\right) \\ &\qquad = \left(qe_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_j - q^{-1}e_je_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}\right)e_j,\end{aligned}$$

and to prove this we note that

$$\begin{aligned}\left[e_j, e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}\right]_q &= e_j e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n} - e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n} e_j \\ &= \sum_{k=j}^{n-1} C_k e_{\alpha_j+\cdots+\alpha_k} e_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n} + C_n (e_{\alpha_j+\cdots+\alpha_n})^2, \; C_k \in \mathbb{C},\end{aligned}$$

and that $\left[e_j, e_{\alpha_j+\cdots+\alpha_k}\right]_q = 0$ and $\left[e_j, e_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}\right]_q = 0$ for each $k = j+1, \ldots, n-1$, and thus

$$e_j e_{\alpha_j+\cdots+\alpha_k} = q e_{\alpha_j+\cdots+\alpha_k} e_j,$$

$$e_j e_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n} = q e_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n} e_j.$$

Consequently,

$$\begin{aligned}& e_j \left(q e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n} e_j - q^{-1} e_j e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}\right) \\ &= q \left(e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n} (e_j)^2 + \sum_{k=j}^{n-1} C_k e_{\alpha_j+\cdots+\alpha_k} e_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n} e_j + C_n (e_{\alpha_j+\cdots+\alpha_n})^2 e_j \right) \\ &\quad - q^{-1} e_j \left(e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n} e_j + \sum_{k=j}^{n-1} C_k e_{\alpha_j+\cdots+\alpha_k} e_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n} + C_n (e_{\alpha_j+\cdots+\alpha_n})^2 \right) \\ &= \left(q e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n} e_j - q^{-1} e_j e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}\right) e_j.\end{aligned}$$

□

Proposition D.9.8. *For each* $j = 2, 3, \ldots, n-1$,

$$[e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}, \overline{e}_{\alpha_{j-1}+\alpha_j}]_q = \overline{e}_{\alpha_{j-1}+2\alpha_j+\cdots+2\alpha_n}.$$

Proof. The proof of this proposition is lengthy, and we prove in in a number of stages. Firstly consider $\left[e_{\alpha_j+\alpha_{j+1}}, \overline{e}_{\alpha_{j-1}+\alpha_j}\right]_q$ for each $j = 2, 3, \ldots, n-1$:

$$\begin{aligned}\left[e_{\alpha_j+\alpha_{j+1}}, \overline{e}_{\alpha_{j-1}+\alpha_j}\right]_q &= \left[e_j e_{j+1} - q^{-1} e_{j+1} e_j, \overline{e}_{\alpha_{j-1}+\alpha_j}\right]_q \\ &= e_j \overline{e}_{\alpha_{j-1}+\alpha_j+\alpha_{j+1}} + q^{-1} \left[e_j, [e_j, e_{j-1}]_q\right]_q e_{j+1} \\ &\qquad - q^{-1} e_{j+1} \left[e_j, [e_j, e_{j-1}]_q\right]_q - \overline{e}_{\alpha_{j-1}+\alpha_j+\alpha_{j+1}} e_j \\ &= e_j \overline{e}_{\alpha_{j-1}+\alpha_j+\alpha_{j+1}} - \overline{e}_{\alpha_{j-1}+\alpha_j+\alpha_{j+1}} e_j = 0,\end{aligned}$$

as $\left[e_j, [e_j, e_{j-1}]_q\right]_q = 0$ is just a restatement of the Serre relation $(ad_q e_j)^2(e_{j-1}) = 0$ for $j < n$, and $\left[e_{\alpha_{j-1}+\alpha_j+\alpha_{j+1}}, e_j\right]_q = 0$ implies $\left[e_j, \overline{e}_{\alpha_{j-1}+\alpha_j+\alpha_{j+1}}\right]_q = 0$.

Now we will show that

$$\left[e_{\alpha_j+\cdots+\alpha_k}, \overline{e}_{\alpha_{j-1}+\alpha_j}\right]_q = 0,$$

for each $k = j+1, \ldots, n$. The calculation immediately above states that this is true for $k = j+1$, now assume that it is true for some $k \geq j+1$, then

$$\begin{aligned}\left[e_{\alpha_j+\cdots+\alpha_{k+1}}, \overline{e}_{\alpha_{j-1}+\alpha_j}\right]_q &= \left[e_{\alpha_j+\cdots+\alpha_k} e_{k+1} - q^{-1} e_{k+1} e_{\alpha_j+\cdots+\alpha_k}, \overline{e}_{\alpha_{j-1}+\alpha_j}\right]_q \\ &= e_{\alpha_j+\cdots+\alpha_k}\left[e_{k+1}, \overline{e}_{\alpha_{j-1}+\alpha_j}\right]_q + \left[e_{\alpha_j+\cdots+\alpha_k}, \overline{e}_{\alpha_{j-1}+\alpha_j}\right]_q e_{k+1} \\ &\quad - q^{-1} e_{k+1}\left[e_{\alpha_j+\cdots+\alpha_k}, \overline{e}_{\alpha_{j-1}+\alpha_j}\right]_q - q^{-1}\left[e_{k+1}, \overline{e}_{\alpha_{j-1}+\alpha_j}\right]_q e_{\alpha_j+\cdots+\alpha_k} \\ &= 0,\end{aligned}$$

as $\left[e_{k+1}, \overline{e}_{\alpha_{j-1}+\alpha_j}\right]_q = 0$ and $\left[e_{\alpha_j+\cdots+\alpha_k}, \overline{e}_{\alpha_{j-1}+\alpha_j}\right]_q = 0$ by assumption.

Now for each $j < n-1$, we claim that

$$\left[e_{\alpha_j+\cdots+2\alpha_k+\cdots+2\alpha_n}, \overline{e}_{\alpha_{j-1}+\alpha_j}\right]_q = 0,$$

for each $k = j+2, \ldots, n$. We firstly show that this is true for $k = n$:

$$\begin{aligned}\left[e_{\alpha_j+\cdots+2\alpha_n}, \overline{e}_{\alpha_{j-1}+\alpha_j}\right]_q &= \left[e_{\alpha_j+\cdots+\alpha_n} e_n + e_n e_{\alpha_j+\cdots+\alpha_n}, \overline{e}_{\alpha_{j-1}+\alpha_j}\right]_q \\ &= e_{\alpha_j+\cdots+\alpha_n}\left[e_n, \overline{e}_{\alpha_{j-1}+\alpha_j}\right]_q + \left[e_{\alpha_j+\cdots+\alpha_n}, \overline{e}_{\alpha_{j-1}+\alpha_j}\right]_q e_n \\ &\quad + e_n\left[e_{\alpha_j+\cdots+\alpha_n}, \overline{e}_{\alpha_{j-1}+\alpha_j}\right]_q + \left[e_n, \overline{e}_{\alpha_{j-1}+\alpha_j}\right]_q e_{\alpha_j+\cdots+\alpha_n} \\ &= 0,\end{aligned}$$

as the preceding calculation implies that $\left[e_{\alpha_j+\cdots+\alpha_n}, \overline{e}_{\alpha_{j-1}+\alpha_j}\right]_q = 0$ and $\left[e_n, \overline{e}_{\alpha_{j-1}+\alpha_j}\right]_q = 0$ from the Serre relations. Now assume that

$$\left[e_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}, \overline{e}_{\alpha_{j-1}+\alpha_j}\right]_q = 0,$$

for some $k+1 = j+3, \ldots, n$, then

$$\begin{aligned}&\left[e_{\alpha_j+\cdots+2\alpha_k+\cdots+2\alpha_n}, \overline{e}_{\alpha_{j-1}+\alpha_j}\right]_q \\ &= \left[e_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n} e_k - q^{-1} e_k e_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}, \overline{e}_{\alpha_{j-1}+\alpha_j}\right]_q \\ &= e_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}\left[e_k, \overline{e}_{\alpha_{j-1}+\alpha_j}\right]_q + \left[e_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}, \overline{e}_{\alpha_{j-1}+\alpha_j}\right]_q e_k \\ &\quad - q^{-1} e_k\left[e_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}, \overline{e}_{\alpha_{j-1}+\alpha_j}\right]_q - q^{-1}\left[e_k, \overline{e}_{\alpha_{j-1}+\alpha_j}\right]_q e_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n} \\ &= 0,\end{aligned}$$

as $\left[e_k, \overline{e}_{\alpha_{j-1}+\alpha_j}\right]_q = 0$ and $\left[e_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}, \overline{e}_{\alpha_{j-1}+\alpha_j}\right]_q = 0$ by assumption.

Now consider $\left[e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}, \overline{e}_{\alpha_{j-1}+\alpha_j}\right]_q$:

$$\begin{aligned}&\left[e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}, \overline{e}_{\alpha_{j-1}+\alpha_j}\right]_q \\ &= \left[e_{\alpha_j+\alpha_{j+1}+2\alpha_{j+2}+\cdots+2\alpha_n} e_{j+1} - q^{-1} e_{j+1} e_{\alpha_j+\alpha_{j+1}+2\alpha_{j+2}+\cdots+2\alpha_n}, \overline{e}_{\alpha_{j-1}+\alpha_j}\right]_q \\ &= e_{\alpha_j+\alpha_{j+1}+2\alpha_{j+2}+\cdots+2\alpha_n}\left[e_{j+1}, \overline{e}_{\alpha_{j-1}+\alpha_j}\right]_q + q^{-1}\left[e_{\alpha_j+\alpha_{j+1}+2\alpha_{j+2}+\cdots+2\alpha_n}, \overline{e}_{\alpha_{j-1}+\alpha_j}\right]_q e_{j+1} \\ &\quad - q^{-1} e_{j+1}\left[e_{\alpha_j+\alpha_{j+1}+2\alpha_{j+2}+\cdots+2\alpha_n}, \overline{e}_{\alpha_{j-1}+\alpha_j}\right]_q - q^{-1}\left[e_{j+1}, \overline{e}_{\alpha_{j-1}+\alpha_j}\right]_q e_{\alpha_j+\alpha_{j+1}+2\alpha_{j+2}+\cdots+2\alpha_n} \\ &= e_{\alpha_j+\alpha_{j+1}+2\alpha_{j+2}+\cdots+2\alpha_n}\overline{e}_{\alpha_{j-1}+\alpha_j+\alpha_{j+1}} - q^{-1}\overline{e}_{\alpha_{j-1}+\alpha_j+\alpha_{j+1}} e_{\alpha_j+\alpha_{j+1}+2\alpha_{j+2}+\cdots+2\alpha_n} \\ &= \left[e_{\alpha_j+\alpha_{j+1}+2\alpha_{j+2}+\cdots+2\alpha_n}, \overline{e}_{\alpha_{j-1}+\alpha_j+\alpha_{j+1}}\right]_q,\end{aligned}$$

as $\left[e_{\alpha_j+\alpha_{j+1}+2\alpha_{j+2}+\cdots+2\alpha_n}, \overline{e}_{\alpha_{j-1}+\alpha_j}\right]_q = 0$ from the preceding calculation.

Now we claim that

$$\left[e_{\alpha_j+\cdots+2\alpha_k+\cdots+2\alpha_n}, \overline{e}_{\alpha_{j-1}+\cdots+\alpha_{k-1}}\right]_q = \left[e_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}, \overline{e}_{\alpha_{j-1}+\cdots+\alpha_k}\right]_q,$$

for each $k = j+1, \ldots, n$. This is true for $k = j+1$ from the preceding calculation, now assume that it is true for some $k \geq j+1$, then

$$\begin{aligned}
&\left[e_{\alpha_j+\cdots+2\alpha_k+\cdots+2\alpha_n}, \overline{e}_{\alpha_{j-1}+\cdots+\alpha_{k-1}}\right]_q \\
&= \left[e_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n} e_k - q^{-1} e_k e_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}, \overline{e}_{\alpha_{j-1}+\cdots+\alpha_{k-1}}\right]_q \\
&= e_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n} \left[e_k, \overline{e}_{\alpha_{j-1}+\cdots+\alpha_{k-1}}\right]_q + q^{-1} \left[e_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}, \overline{e}_{\alpha_{j-1}+\cdots+\alpha_{k-1}}\right]_q e_k \\
&\quad -q^{-1} e_k \left[e_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}, \overline{e}_{\alpha_{j-1}+\cdots+\alpha_{k-1}}\right]_q - q^{-1} \left[e_k, \overline{e}_{\alpha_{j-1}+\cdots+\alpha_{k-1}}\right]_q e_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n} \\
&= e_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n} \left[e_k, \overline{e}_{\alpha_{j-1}+\cdots+\alpha_{k-1}}\right]_q - q^{-1} \left[e_k, \overline{e}_{\alpha_{j-1}+\cdots+\alpha_{k-1}}\right]_q e_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n} \\
&= \left[e_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}, \overline{e}_{\alpha_{j-1}+\cdots+\alpha_k}\right]_q,
\end{aligned}$$

where we have used the result $\left[e_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}, \overline{e}_{\alpha_{j-1}+\cdots+\alpha_{k-1}}\right]_q = 0$ which we will now prove. To prove this last result recall that $\left[e_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}, \overline{e}_{\alpha_{j-1}+\alpha_j}\right]_q = 0$. Now assume that $\left[e_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}, \overline{e}_{\alpha_{j-1}+\cdots+\alpha_m}\right]_q = 0$ for some m satisfying $j \leq m \leq k-2$, then

$$\begin{aligned}
&\left[e_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}, \overline{e}_{\alpha_{j-1}+\cdots+\alpha_{m+1}}\right]_q \\
&= \left[e_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}, e_{m+1}\overline{e}_{\alpha_{j-1}+\cdots+\alpha_m} - q^{-1}\overline{e}_{\alpha_{j-1}+\cdots+\alpha_m} e_{m+1}\right]_q \\
&= \left[e_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}, e_{m+1}\right]_q \overline{e}_{\alpha_{j-1}+\cdots+\alpha_m} + e_{m+1} \left[e_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}, \overline{e}_{\alpha_{j-1}+\cdots+\alpha_m}\right]_q \\
&\quad -q^{-1} \left[e_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}, \overline{e}_{\alpha_{j-1}+\cdots+\alpha_m}\right]_q e_{m+1} - q^{-1}\overline{e}_{\alpha_{j-1}+\cdots+\alpha_m} \left[e_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}, e_{m+1}\right]_q \\
&= 0,
\end{aligned}$$

as $\left[e_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}, \overline{e}_{\alpha_{j-1}+\cdots+\alpha_m}\right]_q = 0$ by assumption and $\left[e_{\alpha_j+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}, e_{m+1}\right]_q = 0$. We have thus shown that

$$[e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}, \overline{e}_{\alpha_{j-1}+\alpha_j}]_q = [e_{\alpha_j+\cdots+2\alpha_n}, \overline{e}_{\alpha_{j-1}+\cdots+\alpha_{n-1}}]_q.$$

Now

$$\begin{aligned}
&[e_{\alpha_j+\cdots+2\alpha_n}, \overline{e}_{\alpha_{j-1}+\cdots+\alpha_{n-1}}]_q \\
&\quad = [e_{\alpha_j+\cdots+\alpha_n} e_n + e_n e_{\alpha_j+\cdots+\alpha_n}, \overline{e}_{\alpha_{j-1}+\cdots+\alpha_{n-1}}]_q \\
&\quad = e_{\alpha_j+\cdots+\alpha_n}[e_n, \overline{e}_{\alpha_{j-1}+\cdots+\alpha_{n-1}}]_q + q^{-1}[e_{\alpha_j+\cdots+\alpha_n}, \overline{e}_{\alpha_{j-1}+\cdots+\alpha_{n-1}}]_q e_n \\
&\qquad + e_n[e_{\alpha_j+\cdots+\alpha_n}, \overline{e}_{\alpha_{j-1}+\cdots+\alpha_{n-1}}]_q + [e_n, \overline{e}_{\alpha_{j-1}+\cdots+\alpha_{n-1}}]_q e_{\alpha_j+\cdots+\alpha_n} \\
&\quad = e_{\alpha_j+\cdots+\alpha_n}[e_n, \overline{e}_{\alpha_{j-1}+\cdots+\alpha_{n-1}}]_q + [e_n, \overline{e}_{\alpha_{j-1}+\cdots+\alpha_{n-1}}]_q e_{\alpha_j+\cdots+\alpha_n} \\
&\quad = [e_{\alpha_j+\cdots+\alpha_n}, \overline{e}_{\alpha_{j-1}+\cdots+\alpha_n}]_q,
\end{aligned}$$

where we have used the result $[e_{\alpha_j+\cdots+\alpha_n}, \overline{e}_{\alpha_{j-1}+\cdots+\alpha_{n-1}}]_q = 0$ which we now prove. Recall that $\left[e_{\alpha_j+\cdots+\alpha_n}, \overline{e}_{\alpha_{j-1}+\alpha_j}\right]_q = 0$ for each $j = 2, \ldots, n-1$. Assume that $\left[e_{\alpha_j+\cdots+\alpha_n}, \overline{e}_{\alpha_{j-1}+\cdots+\alpha_k}\right]_q = 0$ for some $k = j, \ldots, n-2$, then

$$\begin{aligned}
&\left[e_{\alpha_j+\cdots+\alpha_n}, \overline{e}_{\alpha_{j-1}+\cdots+\alpha_{k+1}}\right]_q \\
&\quad = \left[e_{\alpha_j+\cdots+\alpha_n}, e_{k+1}\overline{e}_{\alpha_{j-1}+\cdots+\alpha_k} - q^{-1}\overline{e}_{\alpha_{j-1}+\cdots+\alpha_k}e_{k+1}\right]_q \\
&\quad = \left[e_{\alpha_j+\cdots+\alpha_n}, e_{k+1}\right]_q \overline{e}_{\alpha_{j-1}+\cdots+\alpha_k} + e_{k+1}\left[e_{\alpha_j+\cdots+\alpha_n}, \overline{e}_{\alpha_{j-1}+\cdots+\alpha_k}\right]_q \\
&\qquad -q^{-1}\left[e_{\alpha_j+\cdots+\alpha_n}, \overline{e}_{\alpha_{j-1}+\cdots+\alpha_k}\right]_q e_{k+1} - q^{-1}\overline{e}_{\alpha_{j-1}+\cdots+\alpha_k}\left[e_{\alpha_j+\cdots+\alpha_n}, e_{k+1}\right]_q \\
&\quad = 0,
\end{aligned}$$

as $\left[e_{\alpha_j+\cdots+\alpha_n}, e_{k+1}\right]_q = 0$ and $\left[e_{\alpha_j+\cdots+\alpha_n}, \overline{e}_{\alpha_{j-1}+\cdots+\alpha_k}\right]_q = 0$ by assumption. An implication of this is $[e_{\alpha_j+\cdots+\alpha_n}, \overline{e}_{\alpha_{j-1}+\cdots+\alpha_{n-1}}]_q = 0$. It follows that

$$[e_{\alpha_j+\cdots+2\alpha_n}, \overline{e}_{\alpha_{j-1}+\cdots+\alpha_{n-1}}]_q = [e_{\alpha_j+\cdots+\alpha_n}, \overline{e}_{\alpha_{j-1}+\cdots+\alpha_n}]_q.$$

Now

$$\begin{aligned}
&[e_{\alpha_j+\cdots+\alpha_n}, \overline{e}_{\alpha_{j-1}+\cdots+\alpha_n}]_q \\
&\quad = [e_{\alpha_j+\cdots+\alpha_{n-1}}e_n - q^{-1}e_n e_{\alpha_j+\cdots+\alpha_{n-1}}, \overline{e}_{\alpha_{j-1}+\cdots+\alpha_n}]_q \\
&\quad = e_{\alpha_j+\cdots+\alpha_{n-1}}[e_n, \overline{e}_{\alpha_{j-1}+\cdots+\alpha_n}]_q - [e_{\alpha_j+\cdots+\alpha_{n-1}}, \overline{e}_{\alpha_{j-1}+\cdots+\alpha_n}]_q e_n \\
&\qquad -q^{-1}e_n[e_{\alpha_j+\cdots+\alpha_{n-1}}, \overline{e}_{\alpha_{j-1}+\cdots+\alpha_n}]_q - q^{-1}[e_n, \overline{e}_{\alpha_{j-1}+\cdots+\alpha_n}]_q e_{\alpha_j+\cdots+\alpha_{n-1}} \\
&\quad = e_{\alpha_j+\cdots+\alpha_{n-1}}\overline{e}_{\alpha_{j-1}+\cdots+2\alpha_n} - q^{-1}\overline{e}_{\alpha_{j-1}+\cdots+2\alpha_n}e_{\alpha_j+\cdots+\alpha_{n-1}} \\
&\quad = [e_{\alpha_j+\cdots+\alpha_{n-1}}, \overline{e}_{\alpha_{j-1}+\cdots+2\alpha_n}]_q,
\end{aligned}$$

as $[e_{\alpha_j+\cdots+\alpha_{n-1}}, \overline{e}_{\alpha_{j-1}+\cdots+\alpha_n}]_q = 0$, and

$$\begin{aligned}
&[e_{\alpha_j+\cdots+\alpha_{n-1}}, \overline{e}_{\alpha_{j-1}+\cdots+2\alpha_n}]_q \\
&\quad = [e_{\alpha_j+\cdots+\alpha_{n-2}}e_{n-1} - q^{-1}e_{n-1}e_{\alpha_j+\cdots+\alpha_{n-2}}, \overline{e}_{\alpha_{j-1}+\cdots+2\alpha_n}]_q \\
&\quad = e_{\alpha_j+\cdots+\alpha_{n-2}}[e_{n-1}, \overline{e}_{\alpha_{j-1}+\cdots+2\alpha_n}]_q + q^{-1}[e_{\alpha_j+\cdots+\alpha_{n-2}}, \overline{e}_{\alpha_{j-1}+\cdots+2\alpha_n}]_q e_{n-1} \\
&\qquad -q^{-1}e_{n-1}[e_{\alpha_j+\cdots+\alpha_{n-2}}, \overline{e}_{\alpha_{j-1}+\cdots+2\alpha_n}]_q - q^{-1}[e_{n-1}, \overline{e}_{\alpha_{j-1}+\cdots+2\alpha_n}]_q e_{\alpha_j+\cdots+\alpha_{n-2}} \\
&\quad = e_{\alpha_j+\cdots+\alpha_{n-2}}[e_{n-1}, \overline{e}_{\alpha_{j-1}+\cdots+2\alpha_n}]_q - q^{-1}[e_{n-1}, \overline{e}_{\alpha_{j-1}+\cdots+2\alpha_n}]_q e_{\alpha_j+\cdots+\alpha_{n-2}} \\
&\quad = [e_{\alpha_j+\cdots+\alpha_{n-2}}, \overline{e}_{\alpha_{j-1}+\cdots+2\alpha_{n-1}+2\alpha_n}]_q,
\end{aligned}$$

as $[e_{\alpha_j+\cdots+\alpha_{n-2}}, \overline{e}_{\alpha_{j-1}+\cdots+2\alpha_n}]_q = 0$.

We now claim that

$$[e_{\alpha_j+\cdots+\alpha_{k+1}}, \overline{e}_{\alpha_{j-1}+\cdots+2\alpha_{k+2}+\cdots+2\alpha_n}]_q = [e_{\alpha_j+\cdots+\alpha_k}, \overline{e}_{\alpha_{j-1}+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}]_q,$$

for each $k+1 = j+2, \ldots, n-1$. This is true for $k+1 = n-1$, and assume that it is true for some $k+1 = j+3, \dots, n-1$, then

$$\begin{aligned}
&[e_{\alpha_j+\cdots+\alpha_k}, \overline{e}_{\alpha_{j-1}+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}]_q \\
&= [e_{\alpha_j+\cdots+\alpha_{k-1}}e_k - q^{-1}e_k e_{\alpha_j+\cdots+\alpha_{k-1}}, \overline{e}_{\alpha_{j-1}+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}]_q \\
&= e_{\alpha_j+\cdots+\alpha_{k-1}}[e_k, \overline{e}_{\alpha_{j-1}+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}]_q + q^{-1}[e_{\alpha_j+\cdots+\alpha_{k-1}}, \overline{e}_{\alpha_{j-1}+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}]_q e_k \\
&\quad -q^{-1}e_k[e_{\alpha_j+\cdots+\alpha_{k-1}}, \overline{e}_{\alpha_{j-1}+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}]_q - q^{-1}[e_k, \overline{e}_{\alpha_{j-1}+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}]_q e_{\alpha_j+\cdots+\alpha_{k-1}} \\
&= e_{\alpha_j+\cdots+\alpha_{k-1}}[e_k, \overline{e}_{\alpha_{j-1}+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}]_q - q^{-1}[e_k, \overline{e}_{\alpha_{j-1}+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}]_q e_{\alpha_j+\cdots+\alpha_{k-1}} \\
&= [e_{\alpha_j+\cdots+\alpha_{k-1}}, \overline{e}_{\alpha_{j-1}+\cdots+2\alpha_k+\cdots+2\alpha_n}]_q,
\end{aligned}$$

as $[e_{\alpha_j+\cdots+\alpha_{k-1}}, \overline{e}_{\alpha_{j-1}+\cdots+2\alpha_{k+1}+\cdots+2\alpha_n}]_q = 0$.

To complete the proof it is a simple matter to show the following results using calculations almost identical to those immediately above:

$$[e_{\alpha_j+\alpha_{j+1}}, \overline{e}_{\alpha_{j-1}+\alpha_j+\alpha_{j+1}+2\alpha_{j+2}+\cdots+2\alpha_n}]_q = [e_{\alpha_j}, \overline{e}_{\alpha_{j-1}+\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}]_q,$$
$$[e_{\alpha_j}, \overline{e}_{\alpha_{j-1}+\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}]_q = \overline{e}_{\alpha_{j-1}+2\alpha_j+\cdots+2\alpha_n}.$$

□

Proposition D.9.9.

$$\begin{aligned}
&q e_{j-1}\overline{e}_{\alpha_{j-1}+2\alpha_j+\cdots+2\alpha_n} - q^{-1}\overline{e}_{\alpha_{j-1}+2\alpha_j+\cdots+2\alpha_n}e_{j-1} \\
&\qquad = q e_{\alpha_{j-1}+2\alpha_j+\cdots+2\alpha_n}e_{j-1} - q^{-1}e_{j-1}e_{\alpha_{j-1}+2\alpha_j+\cdots+2\alpha_n}.
\end{aligned} \tag{D.63}$$

Proof. Proposition D.9.8 implies the following result for each $j = 2, \dots, n-1$:

$$\overline{e}_{\alpha_{j-1}+2\alpha_j+\cdots+2\alpha_n} = \left[e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}, \overline{e}_{\alpha_{j-1}+\alpha_j}\right]_q,$$

which we can use to rewrite the left hand side of (D.63):

$$\begin{aligned}
&q e_{j-1}\overline{e}_{\alpha_{j-1}+2\alpha_j+\cdots+2\alpha_n} - q^{-1}\overline{e}_{\alpha_{j-1}+2\alpha_j+\cdots+2\alpha_n}e_{j-1} \\
&= q e_{j-1}e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}\overline{e}_{\alpha_{j-1}+\alpha_j} - e_{j-1}\overline{e}_{\alpha_{j-1}+\alpha_j}e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n} \\
&\quad -q^{-1}e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}\overline{e}_{\alpha_{j-1}+\alpha_j}e_{j-1} + q^{-2}\overline{e}_{\alpha_{j-1}+\alpha_j}e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_{j-1} \\
&= q e_{j-1}e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}\overline{e}_{\alpha_{j-1}+\alpha_j} - e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_{j-1}\overline{e}_{\alpha_{j-1}+\alpha_j} \\
&\quad -q^{-1}\overline{e}_{\alpha_{j-1}+\alpha_j}e_{j-1}e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n} + q^{-2}\overline{e}_{\alpha_{j-1}+\alpha_j}e_{\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_{j-1} \qquad \text{(D.64)} \\
&= q e_{\alpha_{j-1}+\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}\overline{e}_{\alpha_{j-1}+\alpha_j} - q^{-1}\overline{e}_{\alpha_{j-1}+\alpha_j}e_{\alpha_{j-1}+\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n} \\
&= q e_{\alpha_{j-1}+\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_j e_{j-1} - e_{\alpha_{j-1}+\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_{j-1}e_j \\
&\quad -q^{-1}e_j e_{j-1}e_{\alpha_{j-1}+\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n} + q^{-2}e_{j-1}e_j e_{\alpha_{j-1}+\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n} \\
&= q e_{\alpha_{j-1}+\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_j e_{j-1} - q^{-1}e_{j-1}e_{\alpha_{j-1}+\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_j \qquad \text{(D.65)} \\
&\quad -e_j e_{\alpha_{j-1}+\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_{j-1} + q^{-2}e_{j-1}e_j e_{\alpha_{j-1}+\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n} \\
&= q e_{\alpha_{j-1}+2\alpha_j+\cdots+2\alpha_n}e_{j-1} - q^{-1}e_{j-1}e_{\alpha_{j-1}+2\alpha_j+\cdots+2\alpha_n}.
\end{aligned}$$

We used the relation $\overline{e}_{\alpha_{j-1}+\alpha_j}e_{j-1} = q e_{j-1}\overline{e}_{\alpha_{j-1}+\alpha_j}$ to obtain (D.64) and the relation $e_{j-1}e_{\alpha_{j-1}+\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n} = q e_{\alpha_{j-1}+\alpha_j+2\alpha_{j+1}+\cdots+2\alpha_n}e_{j-1}$ to obtain (D.65).

□

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