Quantum Superalgebras at Roots of Unity and Topological Invariants of Three-manifolds

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Statement of Originality

To the best of my knowledge and belief, this thesis is original and my own work except as acknowledged in the text. This thesis has not been submitted, in whole or in part, for any other degree at this or any other university.

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Abstract

The general method of Reshetikhin and Turaev is followed to develop topological invariants of closed, connected, orientable 3-manifolds from a new class of algebras called pseudo-modular Hopf algebras. Pseudo-modular Hopf algebras are a class of $\mathbb{Z}_2$-graded ribbon Hopf algebras that generalise the concept of a modular Hopf algebra.

The quantum superalgebra $U_q(osp(1|2n))$ over $\mathbb{C}$ is considered with q a primitive N^{th} root of unity for all integers $N \geq 3$. For such a q, a certain left ideal $\mathcal{I}$ of $U_q(osp(1|2n))$ is also a two-sided Hopf ideal, and the quotient algebra $U_q^{(N)}(osp(1|2n)) = U_q(osp(1|2n))/\mathcal{I}$ is a $\mathbb{Z}_2$-graded ribbon Hopf algebra.

For all n and all $N \geq 3$, a finite collection of finite dimensional representations of $U_q^{(N)}(osp(1|2n))$ is defined. Each such representation of $U_q^{(N)}(osp(1|2n))$ is labelled by an integral dominant weight belonging to the truncated dominant Weyl chamber. Properties of these representations are considered: the quantum superdimension of each representation is calculated, each representation is shown to be self-dual, and more importantly, the decomposition of the tensor product of an arbitrary number of such representations is obtained for even N.

It is proved that the quotient algebra $U_q^{(N)}(osp(1|2n))$, together with the set of finite dimensional representations discussed above, form a pseudo-modular Hopf algebra when $N \geq 6$ is twice an odd number.

Using this pseudo-modular Hopf algebra, we construct a topological invariant of 3-manifolds. This invariant is shown to be different to the topological invariants of 3-manifolds arising from quantum $so(2n+1)$ at roots of unity.

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