Chapter 2

Theory of Radio Interferometry

In this chapter the key equations are presented and the framework for the following analysis is established. This starts with the development of the normal theory of a radio interferometer, and is followed by the special consideration of a source moving relative to the cosmic sources. There are several texts covering the theory of radio astronomy and interferometry, including Thompson et al. (1986), Goodman (1985), Born and Wolf (1970), and several lectures in Perley et al. (1989) and Taylor et al. (1999). The theoretical development in this chapter is based on previous work, with a particular emphasis on instrument response in the presence of both cosmic signals and interference.

2.1 Stationary Sources

Following Born and Wolf (1970), Thompson (1999), Romney (1999) and Bridle and Schwab (1999), consider a plane wave incident on an interferometer comprising two receivers. The wavefront is considered plane when it originates from an infinitely distant point source, relative to the physical scale of the interferometer. Suppose that the signal is monochromatic and has identical polarisation to the receivers. Suppose also that the signal propagates along a single path, and is undisturbed between the source and the receivers (multipathing of interference and scintillation of cosmic sources is ignored), and that the source is spatially incoherent (any correlated signal comes from a single direction).

While the contents of this chapter apply in general to both astronomical and non-astronomical RFI signals, the derivation starts with a description of the celestial coordinate system used in radio astronomy. Relative antenna positions are established in the \((u, v, w)\) coordinate system. This is defined with \(w\) pointing toward some predefined celestial position (the phase-tracking centre or phase-reference position), \(u\) pointing toward celestial East, and \(v\) normal to the \((u, w)\) plane in the sense of a right hand coordinate system as \((u, v, w)\). The vector describing the relative position of the two
Wavefronts approaching the antennae

Figure 2.1: Schematic of a two element interferometer. Antennae separated by $\vec{b}$ metres receive signals from the $\hat{s}$ direction $\tau_{jk} = \vec{b}, \hat{s} / c$ seconds apart.
In general, there will be a delay between the time the wavefront reaches the first antenna, \( k \), and the time it reaches the second antenna, \( j \). This geometric delay is illustrated in figure 2.1 and in free space is given by the dot product

\[
\tau_{jk} = \frac{\hat{b}_j \cdot \hat{s}}{c}. \tag{2.1}
\]

For astronomical wavefronts entering a flat atmosphere with an arbitrary vertical distribution of refractive index, the extra delay due to the refractive atmosphere is equal for both antennae, Thompson et al. (1986). So, as long as the array is small enough to ignore the curvature of the Earth, the horizontal distribution of the refractive index is uniform, and it is assumed that between the source and the Earth’s atmosphere it is a vacuum, the free space geometric delay given in (2.1) is correct. For terrestrial RFI this is not the case, since the geometric delay occurs in the refractive medium and (2.1) should include the atmosphere’s refractive index, \( n \),

\[
\tau'_{jk} = \frac{n \hat{b}_j \cdot \hat{s}}{c}. \tag{2.2}
\]

In the lower atmosphere \( \tau'_{jk} \) is longer than \( \tau_{jk} \) by about 0.03% (Thompson et al., 1986), and for long baselines and wide bandwidths (which will shortly be related to the inverse of a sample length) this small difference could be a significant factor. In subsequent discussions the geometric delay of astronomical sources will refer to (2.1) and RFI sources will refer to (2.2), however examples on simulated and sampled data throughout this thesis all use (2.1).

The source induces real-valued voltage fluctuations in the receivers. Since the polarisation state of the radiation is either unpolarised or matches that of the receivers, the voltages induced in the two receivers at time \( t \) by the source is

\[
\begin{align*}
\epsilon_j(t - \tau_{jk}) &= a(t - \tau_{jk}) \cos(2\pi \nu(t - \tau_{jk}) + \varphi(t - \tau_{jk})), \text{ and} \\
\epsilon_k(t) &= a(t) \cos(2\pi \nu t + \varphi(t)),
\end{align*}
\tag{2.3}
\]

where \( a(t) \) is the amplitude of the induced voltage at time \( t \), \( \varphi(t) \) is an arbitrary phase term (to allow other spectral components – to be considered shortly – to have different initial phases) and after reaching antenna \( k \) the wavefront takes \( \tau_{jk} \) seconds to reach antenna \( j \) (thus it is receiving an older version of signal \( \epsilon(t) \)).

Consider now the more realistic case where the signals received at antennae \( j \) and \( k \) have bandwidth \( \Delta \nu \). At the receivers the induced voltages are filtered so that only a small band of frequencies, of width \( \Delta \nu \) and centred at \( \nu_0 \), is considered (all frequency components outside this band are assumed to have zero amplitude). Thus the voltages are band-limited to \( \Delta \nu \). This band of frequencies can be thought of as the addition of the infinite number of monochromatic signals in \( \Delta \nu \), with amplitudes \( A(\nu) \) and phases
\( \Phi(\nu) \) for \(|\nu - \nu_0| \leq \Delta \nu / 2\), which interfere to form the received signal (i.e., \( A(\nu) \) and \( \Phi(\nu) \) are the Fourier components of the signal). Astronomical signals are generally band-limited Gaussian random noise, Thompson et al. (1986), while some RFI sources with essentially only one discernible spectral component will induce a sinusoidal voltage sequence. The voltages in (2.3) are rewritten

\[
\epsilon(t) = \frac{1}{\Delta \nu} \int_{\nu_0 - \Delta \nu}^{\nu_0 + \Delta \nu} A(\nu) \cos(2\pi \nu t + \Phi(\nu)) \, d\nu. \tag{2.4}
\]

As an example of the effect of bandwidth, consider the case in which the amplitude terms of the monochromatic components of the integral in (2.4), \( A(\nu) \), are constant in time and all equal to \( A \). The resulting voltage sequence is

\[
\epsilon(t) = \frac{A}{\Delta \nu} \int_{\nu_0 - \Delta \nu}^{\nu_0 + \Delta \nu} \cos(2\pi \nu t + \Phi(\nu)) \, d\nu
\]

\[
= A \, \text{sinc}(\Delta \nu t) \cos(2\pi \nu_0 t + \Phi(\nu)),
\]

where \( \text{sinc}(x) = \sin(\pi x)/(\pi x) \). So for the case of a constant amplitude across the frequency band of interest, \( a(t) = A \, \text{sinc}(\Delta \nu t) \). This is a good approximation for many cases of interest in radio astronomy, in particular continuum observations of sources with slowly varying frequency spectrum structure, but also many spectral line observations, since one must often use very narrow frequency channels across which the source spectrum is approximately flat. Unless otherwise stated, it is assumed that for all sources the integral in (2.4) can be evaluated and written in the form of the voltages in (2.3). For instance a signal at the centre of the band which is essentially monochromatic (i.e. \( \Delta \nu \rightarrow 0 \)) can be written \( \epsilon(t) = A(\nu_0) \cos(2\pi \nu_0 t + \Phi(\nu_0)) \).

### 2.1.1 Complex Signals

It is convenient to consider the complex representations of the measured electric fields, which allows for a more straightforward interpretation of the Fourier theory used throughout this thesis (see, for example, Bracewell 2000). The voltages can then be obtained by taking the real part of the complex signals. Hence we have

\[
\begin{align*}
\epsilon_j(t - \tau_{jk}) & = a(t - \tau_{jk}) e^{i2\pi \nu_0(t - \tau_{jk})} e^{i\varphi(t - \tau_{jk})} = a(t - \tau_{jk}) e^{i2\pi \nu_0 t - i\varphi(\tau_{jk})}, \\
\epsilon_k(t) & = a(t) e^{i2\pi \nu_0 t + i\varphi(t)} = a(t) e^{i2\pi \nu_0 t},
\end{align*}
\]

where \( a(t) = a(t) e^{i\varphi(t)} \) describes the amplitude and phase of the voltage fluctuations. For astronomical signals \( a(t) \) is a complex Gaussian random variable drawn from a
band-limited process. As in (2.4) the field at time $t$ is generated by the addition of an infinite number of monochromatic signals, however these signals have complex amplitudes, $A(\nu)$ (here a slightly different band-limited rule is needed in that $A(\nu)$ is non-zero only when $|\nu \pm \nu_0| \leq \Delta \nu/2$, which contains negative frequencies):

$$a(t) = \int_{-\infty}^{\infty} A(\nu) e^{+i2\pi\nu t} d\nu,$$

where the complex amplitude is related to the real valued voltage amplitude and phase by $A(\nu) = \frac{1}{2} A(\nu) e^{i\phi(\nu)}$ for $\nu \geq 0$ and $A(-\nu) = A^*(\nu)$.

The fact that the voltage sequence, $a(t)$, must be band-limited can be seen from sampling theory (Bracewell, 2000). Consider the situation in which the continuous function $e(t)$ is sampled at regular time intervals separated by $\delta_t$ seconds. This can be expressed $III(t/\delta_t) e(t)$, where $III(x)$ is known as the shah function, with $III(x) = 0$ for all non-integer values of $x$, and $III(x)$ is a delta function at integer values of $x$, (Bracewell, 2000). The Fourier transform of the sampled function is $\delta_x III(\delta_x, \nu) * E(\nu)$, where $*$ denotes a convolution and $E(\nu)$ is the spectrum of the continuous function $e(t)$. Convolving a function $f$ with a series of delta functions is simply the process of replicating $f$ (appropriately scaled) at each of the delta functions and summing all the replicated functions together. So the spectrum of the sampled voltage sequence is the sum of an infinite number of versions of $E(\nu)$ each with a different central frequency given by $III(\delta_x, \nu)$. Since $e(t)$ is band-limited, all of the information in $E(\nu)$ is contained in a finite band of frequencies, and if the samples are sufficiently closely spaced so that $1/\delta_x$ exceeds the bandwidth then the replicated copies of the spectrum of $e(t)$ will not overlap. This means that one of the replicated copies of the spectrum can be isolated and the unsampled function, $e(t)$, obtained from the spectrum of the sampled function, $III(t/\delta_x)e(t)$, via an inverse Fourier transform (Bracewell, 2000).

So all of the information in $e(t)$ can be retained after sampling, as long as the replicating bandwidth, $1/\delta_x$, is wide enough. For a baseband spectrum, (where the full band has been moved down in frequency to run from $-\Delta \nu$ to $\Delta \nu$), which has a width of $2\Delta \nu$, $\delta_x = 1/(2\Delta \nu)$ is the critical sampling interval. This is known as the Nyquist rate, Bracewell (2000), and also holds for non-baseband spectra. See Thompson et al. (1986) and Bracewell (2000) for further discussions on sampling theory.

In general we want to measure the statistical variance associated with either $a(t)$ or $A(\nu)$, since this is related to the power received from the source. In synthesis telescopes, this power is measured as a function of the $u$ and $v$ coordinates, which under certain simplifying assumptions is the 2-dimensional Fourier transform of the ($l$, $m$) sky map of radio intensity in the frequency band $\Delta \nu$, Thompson et al. (1986). For an astronomical source with an intensity $I(\hat{s})$ W.m$^{-2}$Hz$^{-1}$sr$^{-1}$ (also known as surface brightness), the power received at an antenna with gain, $G(\hat{s})$ m$^2$ (which contains the effective antenna area, antennae efficiency as well as any electronic gain applied by the receiver), from the source element, $\Delta \Omega$ sr, over the bandwidth $\Delta \nu$ Hz is $G(\hat{s}) I(\hat{s}) \Delta \nu \Delta \Omega$ W, Thompson (1999). If the signal is statistically stationary in time (at least for the duration of the
averaging interval), the power is measured by the expectation value
\[ \langle \epsilon_j(t)\epsilon_k^*(t) \rangle = \langle a(t)a^*(t) \rangle = G(\tilde{s})I(\tilde{s})\Delta \nu \Delta \Omega, \]  
which can be approximated by a finite time average. For unpolarised sources, where only a half of the radiation is being measured by a single receiving system, the power term in (2.8) needs to be multiplied by a factor of 1/2.

In (2.8) the voltages are aligned in time, while in general they are being evaluated at different times, \( t - \tau_{jk} \) and \( t \), as in (2.6). If the signals were monochromatic, measuring the power or cross-correlation function, \( \langle \epsilon(t)\epsilon^*(t) \rangle \), would give a part of a cosine with an amplitude of \( G(\tilde{s})I(\tilde{s})\Delta \nu \Delta \Omega \) and a phase shift given by the value of the geometric delay, \( \tau_{jk} \) (the cross-correlation function evaluated over \( \tau_{jk} \) is often referred to as the fringe pattern of \( \epsilon \)), Thompson (1999). However, since there is non-zero bandwidth, the geometric delay results in the monochromatic components of the frequency band being phase shifted by different amounts, which in turn leads to a decorrelation of the fringe pattern amplitude due to adding together of many cosines all with different wavelengths. Thus only at \( \tau_{jk} = 0 \) (the centre of the so-called white light fringe), where all of the fringes are in phase, can a true estimate of \( G(\tilde{s})I(\tilde{s})\Delta \nu \Delta \Omega \) be calculated. While this may seem limiting, measuring the fringe pattern offers a mechanism for probing the shape of the frequency spectrum, and is discussed in the next section.

### 2.1.2 Aligning Voltages and the Cross-Correlation Function

Measurement of the cross-correlation will only give the total power from the source when the voltages have been aligned in time, so that the wavefront measured by one receiver is directly correlated with the same wavefront measured at the other receiver. The voltages are naturally aligned in time when the geometric delay is zero, or they can be manually aligned by inserting delays into the signal paths after the receivers. The obvious choice from (2.6) is to insert a compensating delay into the signal path of antenna \( k \), the instrumental delay, \( \tau_i \), which is equal to the geometric delay, \( \tau_{jk} \). It was also mentioned at the end of the last section that the fringe pattern is intimately linked with the frequency spectrum of \( \epsilon \) in the band \( \Delta \nu \). The different monochromatic components of the signal have their own fringe patterns with different wavelengths, and the combined fringe pattern is smeared out, a process known as bandwidth smearing. So one can correlate the voltages for a set of delays, often called lags, to measure the smeared fringe pattern, which is then used to determine the frequency structure (Thompson et al., 1986).

In practice, the delaying and correlating are usually carried out after the radio frequency (RF) band has been shifted to a more manageable intermediate frequency (IF) band centred at frequency \( \nu_{IF} \). Applying the compensating delay at a frequency other than at RF, however, leads to phase shifts which need to be corrected. The frequency shift, called downconversion when the IF band is at a lower frequency, is implemented by
mixing the signals with a sinusoid (known as the local oscillator signal) of frequency $\nu_{LO} = \nu_0 \pm \nu_{IF}$. The IF band centred at $\nu_{IF}$ can arise from two separate RF bands, $\nu_0 = \nu_{LO} \mp \nu_{IF}$, one on the lower, and one on the upper side of $\nu_{LO}$. These are called the lower and upper sidebands respectively. For a single sideband system (which is considered throughout the thesis), one of the RF bands is filtered out prior to mixing.

The instrumental delay is applied at IF, but the original geometric delay occurred at RF, which leads to an extra phase term in the cross-correlation function (D’Addario, 1989). To counter the extra term a compensating phase shift, $\phi_{LO}$ is added to the local oscillator signal during the downconversion of signal $k$, which is known as fringe rotation or fringe stopping. Following D’Addario (1989) and Romney (1999), the IF signals are

\[
\textbf{v}_j(t - \tau_{jk}) = \textbf{c}_j(t - \tau_{jk}) e^{-i2\pi \nu_{LO} t} = \textbf{a}(t - \tau_{jk}) e^{+i2\pi \nu_0(t-\tau_{jk})} e^{-i2\pi(\nu_0 \pm \nu_{IF})t} = \textbf{a}(t - \tau_{jk}) e^{-i2\pi \nu_{IF} \tau_{jk}} e^{i2\pi \nu_{IF} t},
\]

(2.9)

for the downconverted signal at antenna $j$, and

\[
\textbf{v}_k(t) = \textbf{c}_k(t) e^{-i2\pi \nu_{LO} t + i\phi_{LO}} = \textbf{a}(t) e^{+i2\pi \nu_0 t} e^{-i2\pi(\nu_0 \pm \nu_{IF})t + i\phi_{LO}} = \textbf{a}(t) e^{i2\pi \nu_{IF} t + i\phi_{LO}},
\]

(2.10)

for the downconverted signal at antenna $k$, which can now be delayed by $\tau_i$

\[
\textbf{v}_k(t - \tau_i) = \textbf{a}(t - \tau_i) e^{\pm i2\pi \nu_{IF} \tau_i \mp i2\pi \nu_{IF} t + i\phi_{LO}}.
\]

(2.11)

Now that the signals at the two antennae have been downconverted and aligned they are in a position to be digitised and cross-correlated (multiplied together and averaged) for some accumulation time, $\tau_A$, to approximate (2.8). There is additional noise in the voltages from sources like the antenna surroundings and the low noise amplifiers used to amplify the source voltages, called receiver noise since the receivers contribute most of this noise at centimetre wavelengths. The receiver noise is assumed to have a Gaussian distribution which is uncorrelated for different receivers so it will average towards zero during the cross-correlation to reveal any cross-correlated power from $s$. Assuming that the geometric delay, $\tau_{jk}$, is constant over the time average, and setting $\Delta \tau = \tau_{jk} - \tau_i$, the output of this correlator for time $t_0$ is the real part of

\[
P_{jk}(\Delta \tau) = \frac{1}{\tau_A} \int_{t_0 - \tau_A/2}^{t_0 + \tau_A/2} \textbf{v}_j(t - \tau_{jk}) \textbf{v}_k^*(t - \tau_i) dt,
\]

(2.12)

which in the limit is the cross-correlation function of the signal received at antennae $j$ and $k$. Provided that the duration of the time average is much greater than the
sample length, $1/(2\Delta \nu)$, $p_{jk}(\tau_{jk})$ can be used to approximate the true cross-correlation function given by the expectation value

$$ p_{jk}(\Delta \tau) = \langle v_j(t - \tau_{jk})v_k^*(t - \tau_i) \rangle $$

$$ = \langle a(t - \tau_{jk})a^*(t - \tau_i) \rangle e^{-i2\pi\nu_0\tau_{jk} + i2\pi\nu_i\tau - \phi_{LO}} $$

$$ = s_{jk}(\Delta \tau) e^{-i2\pi\nu_0\tau_{jk} + i2\pi\nu_i\tau - \phi_{LO}}, $$

(2.13)

where $s_{jk}(\Delta \tau) = \langle a(t - \tau_{jk})a^*(t - \tau_i) \rangle$ is a function proportional to the amplified intensity from the source, c.f. (2.8), which also contains the bandwidth smearing factor that envelopes the fringe pattern (the fringe envelope). There are other terms in the second and third lines of (2.13) due to correlations with receiver noise which have been omitted since their expectation values are zero. In practice though, correlations are given by a discrete version of (2.12) and are not infinite in duration so the noise terms never actually become zero, and set a noise limit to observations. If the local oscillator phase, $\phi_{LO}$, is set equal to $-2\pi\nu_{LO}\tau_{jk}$ then $p_{jk}(\Delta \tau)$ becomes

$$ p_{jk}(\Delta \tau) = s_{jk}(\Delta \tau) e^{-i2\pi\nu_0\tau_{jk} + i2\pi\nu_i\tau - i2\pi\nu_{LO}\tau_{jk}} $$

$$ = s_{jk}(\Delta \tau) e^{-i2\pi\nu_0\tau_{jk} + i2\pi\nu_i\tau - i2\pi(\nu_0 - \nu_i)\tau_{jk}} $$

$$ = s_{jk}(\Delta \tau) e^{\pm i2\pi\nu_i\Delta \tau}. $$

(2.14)

The fringe envelope, $s_{jk}(\Delta \tau)$, arises from the bandwidth smearing mentioned above (different parts of the frequency spectrum have different fringe patterns, which add together and smear out as $\Delta \tau$ increases). The Wiener-Khinchin relation (see, for example, Bracewell 2000) says that the fringe envelope is the Fourier transform of the cross-power spectrum, $S_{jk}(\nu) = \langle A_j(\nu)A_k^*(\nu) \rangle$, which can be written

$$ s_{jk}(\tau) = \langle a(t - \tau)a^*(t) \rangle = \int_{-\infty}^{\infty} S_{jk}(\nu) \ e^{+i2\pi\nu\tau}d\nu $$

$$ = \int_{-\infty}^{\infty} s_{jk}(\tau) e^{-i2\pi\nu\tau} d\nu. $$

(2.15)

This Fourier relationship is the reason $s_{jk}(\Delta \tau)$ is known as the bandwidth pattern of $p_{jk}(\tau)$, and can be seen if the cross-correlated power is expressed as an integration over frequency with the substitution, $\nu' = \nu - \nu_i$,

$$ p_{jk}(\Delta \tau) = \int_{\nu_i - \Delta\nu/2}^{\nu_i + \Delta\nu/2} A_j(\nu)A_k^*(\nu) \ e^{\pm i2\pi\nu\Delta \tau} d\nu $$

$$ \Rightarrow s_{jk}(\Delta \tau) e^{\pm i2\pi\nu_i\Delta \tau} = e^{\pm i2\pi\nu_i\Delta \tau} \int_{-\infty}^{\infty} S_{jk}(\nu') e^{\pm i2\pi\nu'\Delta \tau} d\nu', $$

(2.16)
where the integral can be extended to infinity because $A_j(\nu)$ and $A_k(\nu)$ are band-limited and thus non-zero only in the range $\nu_{\nu'} - \Delta \nu/2 \leq \nu \leq \nu_{\nu'} + \Delta \nu/2$.

The Wiener-Khinchin relation of (2.15) is fundamental in radio interferometry. The cross-power spectrum of a source can be measured from the fringe pattern at a range of $\Delta \tau$ values, and then Fourier transformed. Alternatively, voltage time series may be Fourier transformed into much smaller frequency bands, then correlated separately with $\Delta \tau = 0$ (D'Addario, 1989). The latter correlation method, known as an FX correlator, is likely to be used in the next era of radio array telescopes – which will have very large numbers of receivers – due to the increase in capability and decrease in price of electronic components. The former correlation method, known as an XF or lag correlator, is currently used at large radio astronomy facilities, and is briefly reviewed in the next section. One should note that, even though the two types of correlators are technically different – see, for example, D’Addario (1989) or Romney (1999) – it is assumed throughout the thesis that for a single source the resulting power spectrum is identical for both correlators. There are advantages of FX correlators for some combinations of multiple sources, for example a wide-band cosmic source with narrow-band interference, where the interference is only present in a small proportion of the correlations.

### 2.1.3 Lag Correlators

The cross-correlation function of a signal can be estimated by calculating $p_{jk}$ for a set of delays, $\Delta \tau$, called lags. By the Wiener-Khinchin relation, the cross-power spectrum can then be determined via an inverse Fourier transform. A lag correlator measures the real part of $p_{jk}(\Delta \tau)$ by inserting a set of $N$ instrumental delays, Romney (1999). The delays are spaced by a single sample length, $\tau_i(n) = \tau_{plc} + n\delta_\tau$; $n \in [-\frac{N}{2}, \frac{N}{2} - 1]$, where $\tau_{plc}$ is the delay of the phase-tracking centre which may or may not be equal to $\tau_{jk}$. Recalling that the sample spacing required for Nyquist sampling is $\delta_\tau = 1/(2\Delta \nu)$ seconds, the delay can be written $\Delta \tau(n) = \tau_{jk} - \tau_i(n) = (\tau_{jk} - \tau_{plc}) - n\delta_\tau$.

The signal from a source at $\hat{s}$ will contribute $s_{jk}(\Delta \tau)$ to the cross-correlation. This will be some function centred at $n\delta_\tau = \tau_{jk} - \tau_{plc}$. For a single point source with a constant amplitude, $A$, across $\Delta \nu$, the bandwidth function goes as

$$s_{jk}(\Delta \tau) = A \text{sinc}(\Delta \nu((\tau_{jk} - \tau_{plc}) - n\delta_\tau)).$$

(2.17)

If $(\tau_{jk} - \tau_{plc})/\delta_\tau$ is larger than the maximum lag at $N/2$, the central part of $s_{jk}(\Delta \tau)$ will lie beyond the edge of lag space, and most of the power will not be measured. This same lack causes complete bandwidth smearing for frequency channel widths of $\Delta \nu/N$ (that is, if the cross-power spectrum was formed by first transforming the voltage sequences to the frequency domain and then multiplying and accumulating each of the $N$ spectral channels separately). In that situation $s_{jk}(\Delta \tau) = A \text{sinc}\left(\frac{\Delta \nu}{N}(\tau_{jk} - \tau_{plc})\right) = \ldots$
\[ A \text{sinc}\left(\frac{2}{\delta \tau}(\tau_{jk} - \tau_{\text{ptc}})\right), \] which equals zero when \((\tau_{jk} - \tau_{\text{ptc}})/\delta \tau = N/2\).

If \(F^{(\nu)}_{jk}\) represents the proportion of the correlated signal remaining relative to the maximum at the white light fringe, \(\Delta \tau = 0\), (the superscript \(\nu\) in parentheses is used to show that the decorrelation results from a variable frequency), the bandwidth smearing relation is

\[ p_{jk}(\Delta \tau) = F^{(\nu)}_{jk}(0). \quad (2.18) \]

For a constant amplitude across the frequency band, a rectangle function which has the sinc function as a Fourier transform, the proportion of the signal remaining is

\[ F^{(\nu)}_{jk} = \text{sinc}(\Delta \nu \Delta \tau). \quad (2.19) \]

The cross-correlation function \(p_{jk}(\Delta \tau)\), with envelope \(s_{jk}(\Delta \tau)\), dies away quickly with \(\Delta \tau\), the offset from the the geometric delay, as the interferometers response is moved away from the white light fringe.

Since \(\tau_{jk}\) is slowly moving as the Earth rotates, it is often reasonable to assume that during the integration in (2.12), over \(\tau_A\) seconds, the geometric delay of the source, \(\tau_{jk}\), is approximately constant. The implications arising when this assumption does not hold are now considered.

### 2.2 Non-Stationary Sources

If the geometric delay is changing, for example if the source is moving so that \(\hat{s}\) is not constant, or if the baseline vector \(\vec{b}\) is changing due to Earth’s rotation, then the centre of the bandpass function will change position during the time average, and the measured power will be smeared. It also means that the visibility measurement will be smeared across the \(uv\)-plane. This is known as time-average smearing and the resulting loss of correlated power is known as fringe rotation decorrelation, since the source is moving through the fringe pattern. For astronomical sources, the inserted delay and fringe rotation have been applied to ensure that \(\hat{s}\) is always constant, and the integration length is made short enough to make sure that any change in \(\vec{b}\) causes minimal decorrelation. However in some cases, particularly for RFI, the effect of a varying geometric delay is substantial. The final part of this chapter is devoted to an overview of the effect.

Since astronomy source positions are constant relative to the phase-tracking centre, time-average smearing is usually described using the time derivative of \(\vec{b}\) (changes in the \((u, v)\) coordinates due to the rotation of the Earth). A loss in signal strength occurs because the power is being smeared across the \(uv\)-plane. An analogous procedure is to consider a source moving relative to the celestial sphere. In this case changes in the
\((l, m)\) coordinates of the source cause the decorrelation (which in general should not be as damaging to a synthesis image of the sky since it will only affect the power of the moving source, not the entire image).

When dealing with interfering sources, both \(\tilde{b}\) and \(\tilde{s}\) may be changing, so changes in the geometric delay (which encompasses both forms of motion) will be considered. If \(p'_{jk}(\Delta \tau)\) is the measured power, where the prime, ', is used to indicate that it may be decorrelated due to a variable geometric delay, then (2.13) is rewritten as the integral

\[
p'_{jk}(\Delta \tau) = s_{jk}(\Delta \tau) \int_{t_0 - \frac{T_A}{2}}^{t_0 + \frac{T_A}{2}} e^{- \frac{2\pi \nu_0 \tau_{jk} (t - t_0)}{1 + \frac{2\pi \nu_1 \nu_1'}{\nu_0}} - i \phi_{LO}} dt.
\]

If the delay for the source at time \(t_0\) is \(\tau_{jk_0}\) seconds, and we substitute \(\tau'_{jk} = \tau_{jk} - \tau_{jk_0}\), assume that the rate of change of \(\tau_{jk}\) with time is approximately constant over the time integration and equal to \(\Delta \tau_{jk}/\tau_A\) seconds of delay per second, and use the chain rule, \(\frac{dp'_{jk}}{dt} = \frac{dp'_{jk}}{d\tau_{jk}} \frac{d\tau_{jk}}{dt}\), then (2.20) can be written

\[
p'_{jk}(\Delta \tau) = s_{jk}(\Delta \tau) e^{- \frac{2\pi \nu_0 \tau_{jk_0}}{1 + \frac{2\pi \nu_1 \nu_1'}{\nu_0}} - i \phi_{LO}} \int_{-\frac{\Delta \tau_{jk}}{\tau_{jk}}}^{\frac{\Delta \tau_{jk}}{\tau_{jk}}} e^{- \frac{2\pi \nu_0 \tau'_{jk}}{\tau_{jk}}} d\tau'_{jk}.
\]

(2.21)

The integral can be written as the Fourier transform of the sinusoid at frequency \(\nu_0\) multiplied by the rectangle function, \(\Pi(\tau'_{jk}/\Delta \tau_{jk})\), which is unity inside the time interval and zero elsewhere, \(\Pi(x) = \begin{cases} 1; & |x| \leq 1/2, \\ \{0; & |x| > 1/2, \end{cases}\)

\[
p'_{jk}(\Delta \tau) = p_{jk}(\Delta \tau) \frac{1}{\Delta \tau_{jk}} \int_{-\infty}^{+\infty} \Pi(\tau'_{jk}/\Delta \tau_{jk}) e^{- \frac{2\pi \nu_0 \tau'_{jk}}{\Delta \tau_{jk}}} d\tau'_{jk}.
\]

(2.22)

The Fourier transform of the rectangle function above with respect to \(\tau'_{jk}\) is

\[
\mathcal{F}{\mathcal{T}} \left\{ \Pi(\tau'_{jk}/\Delta \tau_{jk}) \right\} = \Delta \tau_{jk} \text{sinc}(\nu_0 \Delta \tau_{jk}) = \Delta \tau_{jk} \text{sinc}(\Delta \tau_{jk} c/\lambda_0).
\]

(2.23)

where \(\text{sinc}(x) = \sin(\pi x)/(\pi x)\). Applying the Fourier transform from (2.23) to (2.22) shows that the proportion of correlated power remaining after fringe rotation decorrelation with a constantly varying geometric delay is the sinc of the total number of interference fringes to cycle during the integration, that is

\[
p'_{jk}(\Delta \tau) = \text{sinc}(\nu_0 \Delta \tau_{jk}) p_{jk}(\Delta \tau).
\]

(2.24)

If the source and baseline are both stationary relative to the celestial sphere and the phase-tracking centre during the integration, then there will be no decorrelation since
sinc(0) = 1. If the source moves so that exactly one fringe period occurs during the integration, then the signal should cancel itself out, since sinc(1) = 0. The proportion of the correlator output remaining after decorrelation is the ratio of $p'_j k(\Delta \tau)$ to $p_{jk}(\Delta \tau)$, which for baseline $j k$ will denoted $F_{jk}^{(\tau)}$, where the superscript $\tau$ in parentheses is used to show that the decorrelation results from a variable delay. For a constantly varying geometric delay of $\Delta \tau_{jk}$ seconds per integration, we have

$$F_{jk}^{(\tau)} = \text{sinc}(\nu_0 \Delta \tau_{jk}).$$

(2.25)