

**The Effects of Uncertainty on
the Economics of Optimising Control**

A thesis submitted for the degree of
Doctor of Philosophy

by

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EMENDATIONS

- * Positive (+) line numbers are referenced from the top of the page, while negative line numbers (-) are referenced from the bottom.
- * An arrow (→) indicates that a change is to be made in the argument on the left to the one on the right.

<u>Page Number</u>	<u>Line Number</u>	<u>Corrections/Comments</u>
3	-9	The statement attributed to Horowitz (1991) should be viewed in light of an "ideal" world. If there are disturbances, noise and/or limited control action available, then feedback is necessary to achieve "good" performance.
4	+8	Skogestad, S. and P. Lundström, "Mu-optimal LV-Control of Distillation Columns", <u>Comp. and Chem. Eng.</u> , <u>14</u> (4/5), 401-413.
36	-3	finacial → financial
43	+1	The variables u , x and y should each be defined as a perturbation variable, that is, the deviation from the steady-state value.
45	Equ 2.19	A better representation for Σ is; $\Sigma = S \quad n = m$ $\Sigma = \begin{pmatrix} S \\ 0 \end{pmatrix} \quad n > m$ $\Sigma = (S \ 0) \quad n < m$
59	-2	avoided → avoid
95	-15	$3.435e-1 \rightarrow 3.435e-3$
139	+6	effect → affect
165	Equ 5.22	To clarify the concept of stability, each element in the transfer function matrix shown must be stable for the closed loop to be stable. In addition, the comments regarding the closed-loop stability depending only on the stability of G and C only refer to the IMC implementation. An alternative method for checking the closed-loop stability, to that presented of examining each element, is to consider the encirclements of the origin by the determinant of the return difference operator, the Nyquist Stability Criterion (Morari and Zafiriou, 1989, 10.2.2).
168	+11	Although not introduced at this point in the text, model uncertainty can limit stability and performance at frequencies lower than "noise".
169	Equ 5.35	$GckI \ll I \rightarrow \text{norm}(GckI) \ll I$. Similar corrections should be made to Equations 5.36, 5.37 and 5.38.

<u>Page Number</u>	<u>Line Number</u>	<u>Corrections/Comments</u>
171	Equ 5.39	Δy and $y \rightarrow \Delta y_i$ and y_i in the sense of Equ 5.43
171	Equ 5.40	Δu and $u \rightarrow \Delta u_i$ and u_i in the sense of Equ 5.44
179	+12	Stability is referred to in a robust sense. That is, although a high gain controller may perform well, its stability margin may be reduced.
192	-8	robust stability but only nominal performance $\rightarrow$ robust stability and nominal (but not robust) performance
316	Equ 6.52	$ck \rightarrow ck_2$

PREFACE

The conditions of candidature for the degree of Doctor of Philosophy at the University of Sydney require that "the candidate must state, generally in the preface, and specifically in the notes, the sources from which the information is derived, the extent to which the work of others has been made use of, and the portion of the work claimed as original".

The compartmental model used in Chapter 2 was developed by other workers. The application of this technique to the C₃ Splitter, the matching to rigorous and plant data, the optimisation studies, and open-loop and closed-loop analyses were all performed by the author. The cooling tower model was developed by the author.

The sensitivity analysis presented in Chapter 3 is based on the work of Fiacco (1983). The application of singular value analysis to determine the "worst" case is original. Although the reduced-order methods presented were arrived at independently by the author, others have presented similar results.

The concepts of "backoff" from constraints, and the linear versus nonlinear analysis are an extension and clarification of existing theory. The combination of these results into the Optimal Operation Algorithm is completely original. The results presented in Appendix II for the geometric calculation of input and output magnitudes, and of the "worst" model are an original extension to similar results presented by Morari and Zafirou (1989).

The flotation circuit modelling was originally presented by Chan (1988). However, modifications were made to these models as explained in the text. The multivariable application of the Optimal Operation Algorithm, and the inclusion of the uncertainty descriptions into the calculations, are new techniques developed by the author.

The following publications were derived entirely from work contained in this thesis:

Lear, J.B., G.W. Barton, and J.D. Perkins, "Optimising Control for an Industrial Distillation Column Using a Simplified Model", presented at 12th IMACS World Congress, Paris, France, May 1988.

Lear, J.B., G.W. Barton, and J.D. Perkins, "Optimising Control for an Industrial Distillation Column Using a Simplified Model", Modelling and Simulation of Systems, 341-345, P. Breedveld (Ed), J.C. Baltzer AG, Scientific Publishing Co., 1989. (*based on the IMACS conference proceedings*)

Lear, J.B., G.W. Barton, and J.D. Perkins, "Optimising Control for an Industrial Distillation Column Using a Simplified Model", Automatique Productique Informatique Industrielle, 23(3), 221-237, 1989. (*A more complete description of the work presented at IMACS*)

Lear, J.B., G.W. Barton, and J.D. Perkins, "Optimising Control As An Economic Design Tool", presented at PSE'91, August 1991, Quebec, Canada.

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I must also acknowledge the support of my family and friends who helped me *live* with this thesis for so long. Particular thanks go to my wife Anne and daughter Bronwyn for showing so much patience with a distracted husband and father.

Thanks too, to My many friends in the Systems' Group at Sydney Uni (John, Mark, Sheila, Chandra, Rita), other students (Boz, Frank, ...) and academic staff made the study bearable. Thanks also to the computer support staff (John, Synder, Warren, Tim) and other staff who were helpful.

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To

Anne and Bronwyn

SUMMARY

The ultimate in well operated chemical plants is one that continuously achieves the optimum of some measure of plant performance (e.g. profit), within the various limitations placed on plant operation. Although one technique for achieving this is well known, Optimising Control, its reported use is not widespread. One important requirement for the successful application of this technology, highlighted in this work, is the existence of quantitative descriptions of the uncertainty in the behaviour of the plant of likely disturbances.

It was also clear from the literature that there is a fundamental problem in process design when choosing between a highly profitable plant, and one which is inherently easy to control. Although indicators of both these two requirements are readily available (e.g. steady-state optimisation and the process condition number), it is impossible for the designer to express the two in the same units (e.g. dollars).

Initially, in order to examine the limitations of existing optimising control theory, a large industrial distillation column was used as a case study. Substantial economic benefits were identified for the optimal operation of this column. A simple optimising control scheme was designed using the technique of constraint control. The closed-loop controllers used to implement this form of optimising control were designed with the aid of open-loop singular value analysis. Unfortunately, due to the presence of model uncertainty and a lack of quantitative information on column disturbances, the achievable optimal operation could not be accurately predicted. This deficiency led to investigations into the role of process model uncertainty in limiting the performance of optimising control schemes.

One potential effect of uncertainty is to cause errors in the calculated values of optimal setpoints. New theory is presented for determining the worst case effect of uncertainty on such optimal setpoints. This is an extension of existing theory for analysing sensitivity in nonlinear programming, achieved through the application of singular value analysis. This theoretical advance enables those variables least affected by uncertainty to be selected as "good" candidates for setpoints in an optimising control scheme.

The effect of disturbances on an optimising control system can be broken down into two categories - linear and nonlinear. Linear effects result in an "averaging out" of the variation in the objective function, although there is still a net cost due to movement away from input and output constraints; this is necessary to ensure that constraints are not violated due to the impact of disturbances. Nonlinear effects result in a continuing deterioration of the objective function as the disturbance moves measurements away from their optimal values. That is, every movement away from the optimal values, no matter in which direction, leads to sub-optimal operation. In addition, the effects of disturbances on nonlinear systems are generally difficult to analyse, unlike linear systems where powerful frequency domain techniques are available. Therefore, to accurately predict the performance of an optimising control system in the face of disturbances, it is necessary to consider the case of a linear objective function, a linear steady-state model and a linear (or linearised) dynamic model. Although this may seem somewhat restrictive, the two examples presented demonstrate that these conditions are sufficient for useful comparative studies.

For these linear (and linearised) systems, the Optimal Operation Algorithm (developed as part of this study) can be used to calculate the optimal achievable operating point. For a given disturbance description, this procedure uses the expected maximum input and output magnitudes for the open-loop, partial and full closed-loop control cases, to modify input and output constraints within which the optimum operating point is to be found. Partial control was devised so as to provide a continuum between open-loop and full closed-loop control, by modifying the error sent to the controller. To provide an equitable basis for comparison, the "best" possible controller available (an H_{∞} controller) was used in the calculation of the input and output magnitudes. Using these modified constraints, the optimal operating point which the given design can achieve, despite disturbances, is found.

An extension to this basic algorithm was provided through the inclusion of various uncertainty descriptions into the algorithm. Thus, both steady-state and dynamic uncertainty can be specifically included when designing the controller, determining the input and output magnitudes, and when performing the steady-state optimisation. This modification significantly increases the accuracy of the predicted achievable optimum.

These principles were applied to two case studies. The first, an illustrative SISO example, clearly showed the costs associated with increasing amounts of uncertainty in the process design. The second case study was an extension of earlier research (Chan, 1988) into the problem of flowsheet synthesis. A number of alternative mineral flotation circuits were analysed to determine their achievable optimal performance under different assumptions of model uncertainty. The original research had found that several less conventional (and less controllable) circuits had significantly higher steady-state optima.

Somewhat surprisingly, when modelling uncertainty and disturbances were taken into account, optimal operation was frequently open-loop, due to the high cost of "backoff" from input constraints. However, whatever the uncertainty description, the conventional circuits performed the best. In an interesting variation (especially in the light of industry's desire for tight quality specifications), the objective function was modified to reflect the cost of excessive output movement. In this case, "good" closed-loop control significantly enhanced optimal operation. As found by Chan, the "best" circuits when closed-loop control is required, were those predicted using open-loop controllability indicators as the easiest to control.

Using the theory developed in this thesis, an important improvement in process analysis has been accomplished - the optimal achievable operation of a given design can be determined, without the need to perform dynamic simulations. The designer is able to incorporate as much process knowledge into this procedure as exists, through refinement of the various uncertainty and disturbance descriptions used. Importantly, the calculations involved are quite straightforward, especially when working in a modern process simulation and analysis environment. The availability of well supported (commercial) computer software packages such as SPEEDUP and MATLAB mean that these calculations are within reach of process designers in industry. The algorithm presented thus provides the missing link between the economics of process operation and the analysis of the inherent controllability of a process.

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CHAPTER 1

INTRODUCTION

In the past twenty five years, surprisingly little has changed in the objectives of Chemical Plant management, or in the ideas of Control Engineers as to how to achieve these objectives. In fact, it is illustrative to consider the "state of the art" in process control in the 1960s. In a review article by Rijnsdorp (1967), the use of off-line and on-line optimisation, adaptive control, feedforward, decoupling and nonlinear control are all discussed. Since then, significant advances have been made in the field of controller design. As examples, model-based control imbeds a model of the process in the controller, leading to simplified design, tuning and implementation (Garcia and Morari, 1982, Cutler and Ramaker, 1980), while H_{∞} controller techniques allow a further refinement by using knowledge of the uncertainty in the model to design controllers which will perform well despite model inaccuracy (Morari and Zafiriou, 1989). Also today, due to the significant advances in computer and associated technology, it is easier to put many of these control ideas into practice.

Increasingly, the reasons for seeking improvements through process control are driven by economics. The expenditure on the design, hardware and installation of process control systems must be covered by increased profits. There are a number of reasons for this financial drive;

- 1) The increased cost of capital means that extra production capacity or changes to products cannot always justify building new plants. Thus, every effort is made to improve existing facilities using techniques such as advanced process control.
- 2) The international drive for improved product quality means that plants have to be operated "better", often through improved control.
- 3) Tighter constraints on environmental impact can often only be met through the use of new technologies incorporating modern control schemes.

The first point of attack to achieve these new requirements, is at the process design stage. Currently, a process design is primarily chosen on its ability to meet product specifications profitably. The design operating point is chosen as the steady-state optimum for the plant calculated within the operational constraints defined by safety, product quality, environmental and mechanical requirements. Control of the plant is generally not considered at the design stage, but added once the best steady-state design is found. Clearly this can be unsatisfactory, as the optimal steady-state plant might be difficult to control.

Recently, advances have been made in the ability to check the controllability of a design at an early stage (Perkins, 1989). This open-loop analysis (that is, without any controllers in place) is performed on a linearised version of a dynamic model of the plant. A limitation of this approach, however, is that it is difficult to put a "dollar value" on the results. For instance, how does the designer choose between a highly profitable plant with "poor" controllability indicators and one less profitable which may be inherently more controllable.

To look at this problem, we must first consider why and how plants are controlled. Stephanopoulos (1984) identifies three needs that a control system should satisfy:

- (1) To suppress the influence of external disturbances.
- (2) To ensure the stability of the process.
- (3) To optimise the performance of the process.

The first requirement is to keep selected process variable measurements steady. This is achieved by manipulating process inputs in response to changes in the measurements caused by the external disturbances. The second requirement is not only necessary for safe operation, but is essential for the operation of certain processes, such as polymerisation reactors, where the operating point is not a stable steady-state. The third need is probably the least practised, but as will be discussed later often has the potential to significantly improve the profitability of the process.

Disturbances

Disturbances are caused by variations in uncontrolled plant inputs (e.g. feed rate and

temperature), and in the performance of the plant itself (e.g. catalyst degradation and heat exchanger fouling). These may be classified into "slow" and "fast" disturbances (Morari et al., 1980), based on how quickly they change (e.g. feed rate is generally classified as a fast, and catalyst degradation as a slow disturbance). As will be pointed out below, fast disturbances can be attenuated by a regulatory controller, while slow disturbances are most profitably dealt with by an optimising control scheme.

Reducing the impact of fast disturbances on a process can be important economically. Not only can excessive variation cause quality control problems, but as will be seen in Chapter 4, it can necessitate the movement of the operating point away from constraints, leading to suboptimal operation. Typically (Marlin et al., 1987), improvement in fast disturbance attenuation can reduce operating costs by 0.5 to 2.0 percent.

Much of the thrust of control research in the past twenty years has been to find improved methods of reducing the effect of fast disturbances (that is, improved regulatory control). Today, however, the classic Proportional-Integral-Derivative (PID) single loop controller is still the most commonly used regulatory controller. During the 1970s, quite sophisticated control algorithms were developed (Kwakernaak and Sivan, 1972), usually based on dynamic optimisation principles. However, these were rarely implemented in the process industries, as in practise they frequently performed quite poorly (Foss, 1973, Kestenbaum et al., 1976). Often, this failure to perform well was caused by differences (often referred to as "uncertainty") between the model used for controller design and the actual process.

The importance of uncertainty in the design of process control systems has been known for some considerable time. Horowitz (1991), in his review article, points out that if it were not for uncertainty in plant parameters and process disturbances, there would be "no inherent need for feedback even if the plant is unstable". Since the early 1960s, Horowitz and several other pioneers have been developing control theory to account for such uncertainty. However, it was during the 1980s that major advances were made in design techniques which achieve both controller performance and stability in the face of uncertainty. Zames (1981) presented the theoretical basis for designing controllers to achieve at least "worst-case" disturbance attenuation for a given uncertainty description. These were to become known as H_∞ controllers. Doyle (1982) extended this theory to

reduce the conservativeness of the uncertainty descriptions and to improve the performance aspects of the controller design.

Today, we are in the position where our ability to analyse control systems is quite sophisticated. Further, we are beginning to see the application of advanced controllers to achieve the level of control that the industrial designer requires. Most of these applications have been in the aerospace industry (e.g. Reichert, 1990), although researchers are beginning to investigate applications in the process industry (e.g. Skogestad and Lundström, 1990). Thus, the first two control requirements suggested by Stephanopoulos (1984), the suppression of external disturbances and ensuring the stability of the process, can readily be met today. The third, optimising the plant performance, is undoubtedly the least fulfilled goal at the present moment.

Optimising Control

The ultimate in well operated chemical plants is one that continuously achieves the optimum of some measure of plant performance (e.g. profit), within the various limitations placed on plant operation. A control system which achieves this aim is referred to as an optimising control scheme. In many cases, the major portion of the total benefits of advanced process control will come from some form of process optimisation. This may typically be equivalent to (Marlin et al., 1987) a reduction in operating costs of between 1.5 and 4 percent. This highlights the economic importance of being able to design and implement an optimising control scheme.

Figure 1.1 shows schematically the major components of, and information flow within, an optimising control scheme. Starting at the regulatory control system, the setpoints to be implemented are calculated by an optimisation program (optimiser). The job of the regulatory control system is to keep the plant operating at these setpoints.

The objective function for the optimisation process is some measure of plant performance which it is desirable to have optimised during operation (e.g. net profit). The constraints on the optimiser are the operational limitations within which the plant must operate and the equality constraints (equations) derived from the process model. These equations "tell" the optimiser how the plant behaves. Hence the model of the plant is a crucial component

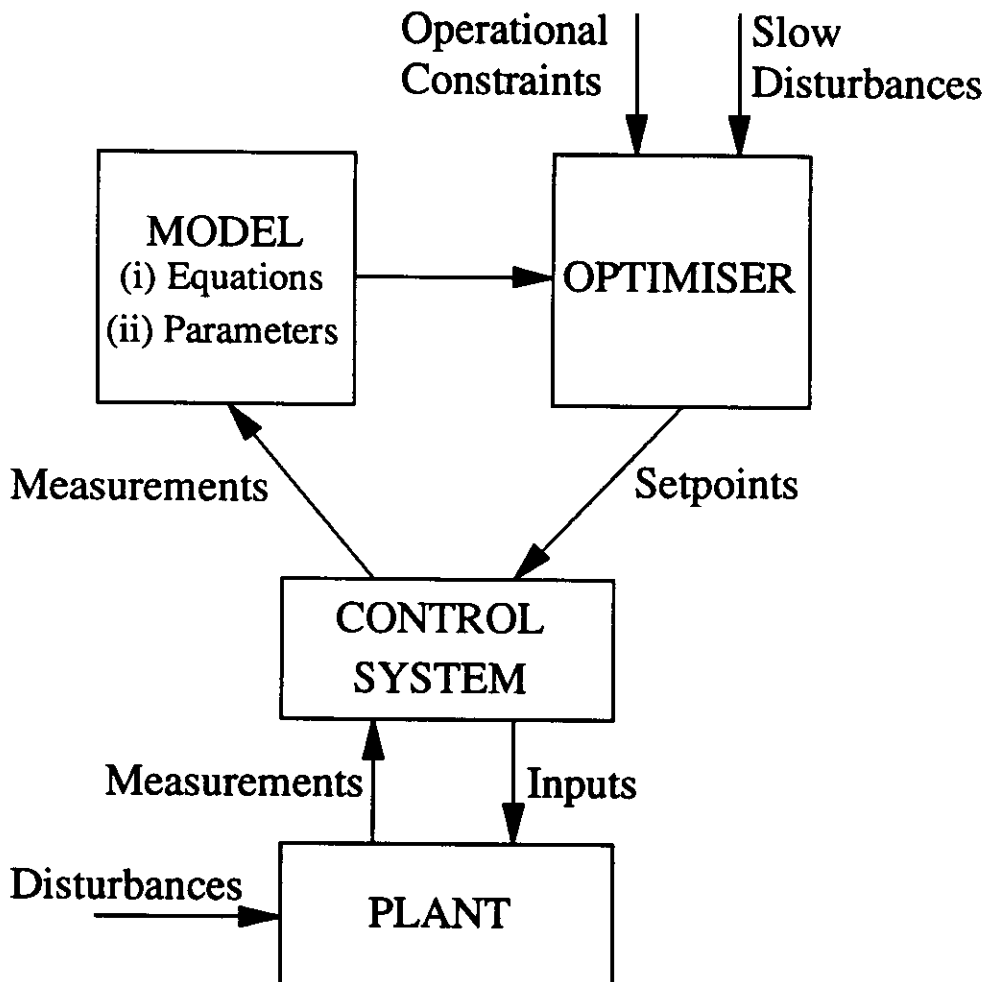


Figure 1.1 - Optimising Control Scheme

of the optimising control system. It must accurately represent the plant, otherwise the operating point which the control system is trying to achieve may be suboptimal, or even infeasible. To reduce the amount of plant-model mismatch, parameters in the model may be updated regularly using plant measurement information supplied by the control system.

The task of the optimiser is to deal with slow disturbances. These are changes that are essentially constant for a sufficiently long period for the plant to reach steady-state. Examples include changes in feedstock and the slow deterioration of catalyst activity. When such changes occur, a new steady-state optimal operating point is calculated and passed on as setpoints for the regulatory control system to implement. A more detailed review of Optimising Control is given in Section 2.1.

Uncertainty

A theme throughout the preceding discussion is the underlying importance of uncertainty. It is important in controller design and in ensuring "good" setpoints are calculated in on-line optimisation. Uncertainty occurs because the process descriptions used in such analyses are only models, and hence to some degree will always be inaccurate. Similarly, by their very nature, since the occurrence of most disturbances cannot be accurately predicted, they too are a source of uncertainty.

To properly analyse the economics of a process design, system uncertainty must somehow be included in the analysis. Referring back to Figure 1.1, uncertainty can be found in most parts of an optimising control scheme. Namely, the plant-model mismatch causing uncertainty in calculated setpoints, the accuracy to which setpoints are actually achieved, the accuracy of measurements and finally the poorly known manner in which unmeasured disturbances might enter the system.

As previously discussed, it is important to minimise the differences between the plant and the model so as to ensure that valid setpoints are calculated. To reduce this area of uncertainty, plant measurements can be used to update parameters in the model, keeping it suitably close to the behaviour of the plant. Further, by careful design of the optimising controller, it may be possible to choose setpoints which are more robust to plant-model mismatch (inherently more accurate).

The interaction between regulatory controller and optimising controller design is also important. Here, the last three areas of uncertainty in the optimising control system (that is, quality of control, measurements and disturbances) are accounted for. For example, there is a trade-off between choosing inherently accurate setpoints and those which can be controlled well, despite the impact of poorly known disturbances. Measurement accuracy can affect the system in two ways. Firstly, although the regulatory controller may be performing superbly, the measurement it is trying to control may be inaccurate. Secondly, measurement error could cause any model update procedure to incorrectly revise the model. Both these sources of uncertainty can cause the calculation of a suboptimal or even an infeasible set of operating conditions by the optimiser.

Thesis Objectives

It is generally accepted that significant economic benefits can be achieved through the use of a well designed optimising control system. However as discussed in this introduction, there are many aspects of such a system which are susceptible to inaccuracy. The overall aim of this thesis was to investigate the design and application of optimising control schemes, and to determine methods of analysing likely plant performance when such schemes are implemented.

Initially, existing techniques were applied to an industrial case study to investigate the applicability of these procedures. This investigation also enabled the magnitude of the benefits as well as the sources of potential problems in implementation to be assessed.

Next, the mathematics of optimising control were analysed to determine if there were any inherent properties to guide optimising controller design. This analysis included the important question of "robust" setpoint selection.

Finally, to improve the interaction between process design and process control at the design stage, the use of optimising control as a measure of the economic potential of alternative controlled processes was investigated. That is, how are the economics of plant operation affected by the ability to control the plant. Throughout these analyses, uncertainty in its various guises was included in the procedure.

CHAPTER 2

OPTIMISING CONTROL

It was stated in the Introduction that the ultimate in well operated chemical plants is one that continuously achieves the optimum of some measure of plant performance (e.g. profit). In this chapter, existing optimising control techniques are reviewed and their applicability to an industrial problem investigated. Finally, shortcomings with these approaches are discussed.

2.1 Existing Techniques

Most chemical plants are operated optimally in some manner. At one extreme, a profit oriented plant operator will keep making adjustments to the process so that it produces the maximum amount of "on-spec" material. His operating point at any time may include some conservativeness to allow for known poor control or the occurrence of unexpected disturbances. At the other extreme, the operator's "objective function" may more reflect his personal comfort. In this case, the degree of conservativeness will be much greater, so that despite what may happen to the plant, minimal intervention is required.

More formally, the approach of the first operator is a direct "hill-climbing" technique, where intuition is used to search for the optimum. Bhattacharya and Joseph (1982) describe four approaches to Optimising Control:

- (1) Perturbation methods.
- (2) Direct methods.
- (3) Indirect methods.
- (4) Dynamic model methods.

The first is the simplest. The plant input is adjusted until the "objective function" stops improving, and then adjusted back the other way until the optimum is found. It is suggested that this method is mainly applicable in single variable optimisation problems.

As the name implies, Direct Methods use the plant directly in the optimisation of the process. The main advantage of this technique is that no model of the process is required,

the plant itself being used to determine the optimum, thus avoiding the problems of plant-model mismatch. When adjusting a plant so as to find its optimum, there will be some plant inputs (or regulatory controller setpoints) available for manipulation. These are known as the degrees of freedom, the variables that are "free" to be altered independently. An example of variables which cannot be adjusted independently are the temperature and pressure of a vapour-liquid mixture at equilibrium. Direct on-line optimisation is a strategy whereby a calculated adjustment is made to these "free" inputs and the change in the objective function observed. The gradient information obtained from each change is used to predict a new change, and so on until some termination criteria is met. Arkun and Stephanopoulos (1980a) state that for chemical plants, the main problems with the Direct Method is the large number of setpoint changes required to determine the local behaviour of the objective function, as well as the long time taken, since steady-state must be reached after each setpoint change. As such, the method is really only suitable for processes with a relatively few degrees of freedom (few inputs to vary) and fast dynamics.

To avoid the waiting involved for a real chemical plant to come to steady-state before analysing the objective function, Indirect Methods of optimising control use a model of the plant to calculate the optimal operating point. The most critical aspect of this technique is how well the model matches the behaviour of the plant. If the model does not capture sufficient of the plant's "character", then not only might the calculated optimal operation be in fact suboptimal, but actual operating constraints may be exceeded if an attempt is made to implement the calculated operating conditions. To avoid these problems, an adaption scheme is normally used to update parameters within the model and ensure the optimum calculated is accurate.

Perhaps the simplest method for model based optimising control is that known as "Constraint Control". Maarleveld and Rijnsdorp (1970) discuss the central role that constraints play in the operation of optimising control. For the distillation column example they studied, the optimum was found to be achieved by operating with certain variables on their limiting constraint values. The method involves designing a regulatory control scheme to keep measurements of these variables at their limiting values, and hence keep the process at the optimum. Due to the importance of this technique in this present study, a more detailed description is given later in this section.

As suggested earlier, when using indirect optimising control, methods are available to reduce the effect of differences between the plant and the model. Findeisen et al. (1978) introduced a method which used measured plant values to improve the calculated optimal setpoints without actually updating the model. However, Arkun and Stephanopoulos (1980a) point out that although this method improves the calculated setpoints, it does not protect against the possibility of constraint violation. Also, since steady-state must be reached after each setpoint update has been made, the procedure can be very slow.

The problem of ensuring that constraints are not violated is indeed an important one. When a slow disturbance occurs (e.g. a feedstock change), the current operating point might exceed some constraint value(s) for the changed conditions. Given that this could lead to undesirable, or even unsafe operation of the plant, the optimising control system must ensure that when such a disturbance enters the system, or during the transition period from one operating point to another, feasible operation is always ensured. To overcome this problem, Arkun and Stephanopoulos (1980b) present a procedure for determining the best series of setpoint changes to initially find a feasible operating point and then move (always feasibly) to the new optimum, when such a disturbance occurs. The strategy includes the modification of the control structure as different constraints become active. They have extended this work (Arkun and Stephanopoulos, 1981) to systems of interconnected plants. In this case, each plant operates under local optimising control with a "coordinator" ensuring that the interconnected system is both feasible and optimal.

A major advance in the on-line estimation of the process model was made by Bamberg and Isermann (1978). Their approach is to identify a dynamic rather than a steady-state model, using either injected (deliberate) or existing movement in plant variables. The steady-state version of this dynamic model is then optimised to determine the optimal operating point. As the new operating point is approached, the model is checked and if necessary updated and a new optimum calculated. This method eliminates the time consuming need for repeatedly reaching steady-state before updating the model. A number of enhancements have been made to this procedure. Garcia and Morari (1981) extended the theory to integrated rather than individual plants, as well as using Internal Model Control (IMC) (Garcia and Morari, 1984) to implement the change to the new operating point. The primary advantage of using IMC is a simplified procedure of adjusting only a

single tuning parameter to achieve good control at each operating point. Lee and Lee (1985) used an updated dynamic model within an adaptive controller to give improved control for any difficult loop. Jang et al. (1987) extended the original approach by Bamberger and Isermann by also using the dynamic model to determine the optimal trajectory to the new optimum, using dynamic programming techniques. This is in contrast to previous approaches which used a conventional regulatory control scheme to move to the new operating point. Chen and Joseph (1987) found that as a result of the updating process, even a simple dynamic model produced a good optimising control scheme. An application to a real plant was reported by Hamer and Richenberg (1988). They found that on-line optimisation using this technique was more efficient than a procedure requiring repeated steady-state evaluations.

The brief review of existing optimising control techniques presented here indicates that many aspects of the theory are well developed. Methods for ensuring that constraints are not violated and for efficiently reducing the effects of plant-model mismatch are available. As noted above, details of an actual process implementation have been reported in the literature (Hamer and Richenberg, 1988). As may be expected in such a profitable area of process control, commercial exploitation has been closely following these developments. Latour (1979a, 1979b) reports on potential applications for optimising control in the hydrocarbon processing industry, while some details of an implementation on a major petrochemical plant are given by Nasi et al. (1983) and Sourander et al. (1984). The most profitable applications are usually on large process units, such as distillation and fluidised bed catalytic cracking within oil refineries. The improvements in these central units are in turn amplified because of their effects on the entire plant. Unfortunately, despite these commercial applications, optimising control is still an "add-on" rather than part of the actual design process.

Constraint Control

As discussed earlier, constraint control is a simple indirect method of optimising control. It is indirect in the calculation of the optimum but not in its implementation, where characteristics of the optimum (variables being at their constraint values) are directly implemented. One of the key advantages of this technique is the simplicity and robustness of its application, normally involving only a simple control structure.

A key concept in constraint control is that of "degrees of freedom" as applied to process control. In process design, the degrees of freedom are usually related to the equations and variables used in the model which describes the process. In this case, the degrees of freedom are equal to the number of variables in the model minus the number of equations. This difference is the number of variables which may be independently specified so as to give a unique solution to the equations, and hence a process design. These variables are known as the independent variables, while others in the system (which are calculated from these via the model equations) are the dependent variables.

For an operating chemical plant, it is not usually obvious how many degrees of freedom there are. Each vessel and piece of equipment does not have the number of equations and variables stamped on it. A better way of determining the degrees of freedom in this case is to calculate how many adjustable inputs there are to the plant. For the plant to operate, every input, which includes input flows and equipment parameters (e.g. stirrer speed), must be specified. This number is the degrees of freedom of the plant. However, some of these might not be available for adjustment by the operator. Typical examples here would be feeds from upstream plants and recommended operating conditions for particular equipment. After the removal of these, we are left with the operating degrees of freedom. To operate the plant, not all inputs may actually be available - a degree of freedom may instead be a setpoint of a controller. In effect this is replacing a plant input with a plant output (a measurement) as a variable for manipulation. This replacement does not alter the number of operating degrees of freedom.

The connection between the degrees of freedom and optimisation is quite straightforward. The optimisation process manipulates variables so as to find the optimum of the objective function within the specified constraints. The maximum number of variables that may be selected for optimisation is given by the degrees of freedom, and the variables selected for this process are known as the "free" variables. In the mathematical sense, once values for the "free" variables are found by the optimiser, then all other variables may be calculated via the model equations. In the operational sense, setting the plant inputs to these optimal values, means that the plant will run optimally.

The constraints that are applied to a model when it is optimised, come from safety, environmental, product quality and mechanical considerations (e.g. reactor exit

temperature $\leq 1000^{\circ}\text{C}$; column pressure ≤ 100 kPa). At the optimum some of these will be "active", that is, some variables will be at the limits imposed by the constraints (e.g. optimal reactor exit temperature = 1000°C , is an active constraint). The other constraints are then "inactive" (e.g. optimal column pressure = 50 kPa, is an inactive constraint). If the model, constraints and objective function are all linear functions, then at the optimum the number of active constraints will equal the number of degrees of freedom (Edgar and Himmelblau, 1988).

For many engineering systems, the number of active constraints at the optimum will equal the number of degrees of freedom, despite the nonlinearity of the system. It has actually been stated that "a well designed plant should operate at a constraint, or it is really over-designed" (Lee and Weekman, 1976). Although this seems a reasonable statement, implying that a well designed chemical plant has no excess capacity, in practice the accepted design procedure involves sizing equipment to slightly greater than design values to allow for inaccuracies, surge flows and the like. This means that the design operating point is not completely constrained. The practise of operating staff, however, is to gradually increase production of the plant over a period of time until constraints are found. Therefore, there will generally be some active constraints at the optimal operating point. Thus, in the general case some degrees of freedom may not be satisfied by active constraints but will be set to calculated optimal values.

Figure 2.1a illustrates using Maarleveld and Rijnsdorp's (1970) simple, two degrees of freedom distillation column example, how constraints can completely determine the optimum. The two axes represent the two degrees of freedom (Column pressure and Feed vaporisation) against which are plotted lines of constant objective function ($\$_1, \$_2, \$_3, \dots$) and the six constraints ;

- (1) $R_{\max}$ = Flooding constraint in the rectifying section.
- (2) $S_{\max}$ = Flooding constraint in the stripping section.
- (3) $H_{F \max}$ = Maximum feed preheat.
- (4) $H_{F \min}$ = Minimum feed preheat.
- (5) $C_{\max}$ = Maximum condenser cooling.
- (6) $H_{B \max}$ = Maximum reboil heat input.

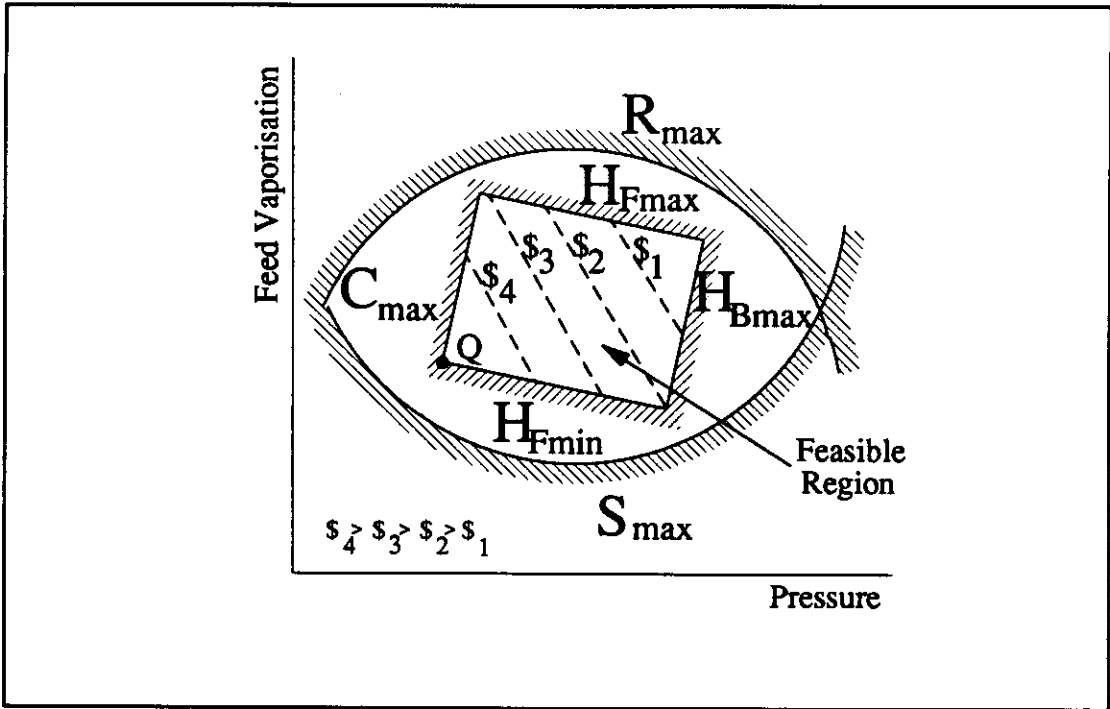


Figure 2.1a - Constraints and Objective Function (Low Throughput)

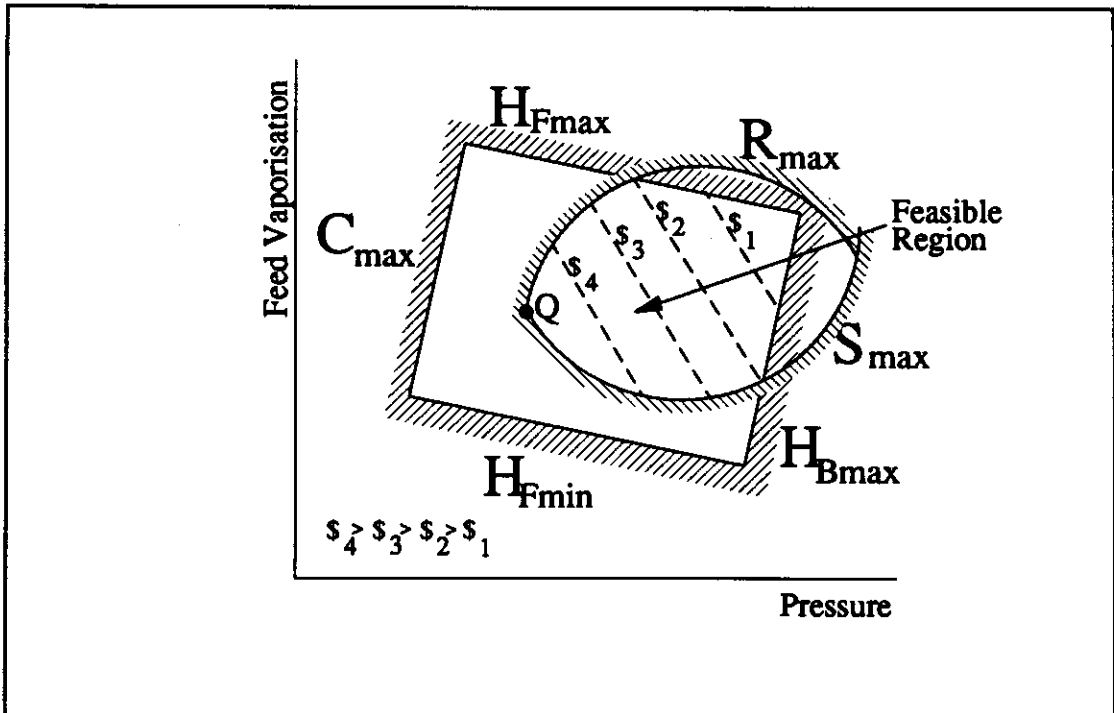


Figure 2.1b - Constraints and Objective Function (High Throughput)

The area within the constraints, that is where no constraints are violated, is known as the feasible region. For this simple example, it is clear that at low column throughput the optimal value of the objective function lies at the intersection of the $C_{\max}$ and $H_{F \min}$ constraints (that is, at point Q). Optimising control at this throughput is achieved by having the maximum cooling water flowrate ($C_{\max}$ active) and the minimum amount of feed preheat ($H_{F \min}$ active).

In some circumstances the active constraints defining the optimum remain unchanged despite disturbances entering the system. A control strategy which controls to these constraints will, thus, always ensure optimal operation. However, for the distillation column example, it was found that as throughput increased, the active constraints changed, as shown in Figure 2.1b. At this higher throughput the optimum was defined by the two flooding constraints. In this case, the optimum is achieved using a different control strategy - the cooling water flow and the amount of feed preheat are used to control the pressure drop (a measure of column flooding) in the rectifying and stripping sections.

It is the automatic ability to maintain the appropriate set of active constraints as slow disturbances occur, that Maarleveld and Rijnsdorp call "constraint control". In some circumstances, this will require a reconfiguration of the control system although for their distillation example, a control system which automatically achieves this without actually reconfiguring is presented.

Constraint control has the potential to achieve optimal operation with very simple control schemes for certain plants. Its limitations are that careful design is required to ensure that changes in the active constraint set are handled properly, and it is not appropriate for situations where there are more variables available for optimising control than active constraints. In that case, additional setpoints must be calculated by on-line optimisation, using a model which is updated using one of the schemes described earlier. An industrial distillation column example which uses a combination of constraint and feedforward control is developed in the next section.

2.2 Application to an Industrial Problem

Optimal operation of a chemical plant has the potential to significantly improve profitability. Part of the process of designing an optimising control system to achieve this improved profitability, is identifying the operational constraints on the plant. As noted previously, optimal operation frequently occurs when some plant measurements are controlled to values limited by various constraints.

The industrial distillation column studied in this section demonstrates the role of constraints within an optimising control scheme. It also has significant scope for optimisation, a moderate level of complexity (in terms of both the process model used and the number of degrees of freedom) and a description of likely slow disturbances is available. After introducing the problem and describing the modelling techniques used, an optimising control scheme is designed and the usefulness of some of the existing techniques is reviewed.

2.2.1 Details of the C₃ Splitter Distillation Column

Distillation is an extremely common unit operation in the process industry. It uses differences in vapour pressure to separate the components of a mixed feed stream. Usually, the higher vapour pressure components are removed at the top of the column, and lower vapour pressure components from the bottom. Sometimes, other component rich streams are extracted at intermediate points on the column.

Distillation can be an energy intensive operation. Large amounts of heat are required in the reboiler to boil the bottoms liquid and drive the vapour up through the column. This heat is then extracted at the top of the column by the condenser, liquefying the vapour coming from the top of the column, before sending it back down. The column contains a number of plates which ensure intimate mixing of vapour and liquid as the two phases pass up and down the column. The superficially wasteful net transfer of energy from the reboiler into cooling water in the condenser is frequently the target of energy minimisation programs.

It is often this consumption of energy that dominates the economics of a distillation column. In other cases, the increased value of the product stream(s), far outweighs the costs involved in the separation (heating, cooling and pumping costs). This latter scenario

is the case for the column studied here.

The case study column separates a five component feed stream to produce the high purity propylene (>99.5%) required in the manufacture of polypropylene (a common plastic). The column is part of ICI Australia's integrated petrochemical plant at Botany, Australia. Being part of an integrated plant means that its operation affects the upstream olefines plant supplying its feed (due to potential throughput limitations), as well as the downstream polypropylene plant consuming its main product. Optimal operation of this unit is important for the overall economics of the interconnected plants.

Although there are five components in the feed, the main separation performed by the column is propylene from propane. This is a difficult separation because the boiling points are quite close (-47.7°C and -42.1°C, respectively). Consequently, the column is large containing 120 plates, standing 60m tall and with a 3.6m diameter. Figure 2.2 is a schematic diagram of the column showing values of operating parameters at one calculated optimum.

Since the feed to the column is from another plant, its composition is subject to change, this being one of the main ("slow") disturbances to column operation. The vent flow from the top of the column, containing mainly light material such as ethane, is recycled back to the upstream plant. Vent flows greater than the recycle capacity are flared. The product propylene is sent to storage tanks for processing by the polypropylene plant and the bottom flow is sent back to the olefines plant for reprocessing. The other two important streams on Figure 2.2 are associated with heat flows. Reboil heat is supplied from low temperature waste heat available on site, while reflux condensation is provided by a dedicated cooling tower. The influence of the ambient wet bulb temperature on the cooling tower performance is the other major "slow" disturbance considered.

There are seven operational constraints for this column;

- (1) Column pressure ≤ 1.45 MPag (a mechanical limitation).
- (2) Cooling water temperature necessary for condensation $\geq$ Minimum available from the cooling tower.
- (3) Reboiler heat input ≤ 20 MW (upper limit of waste heat available).

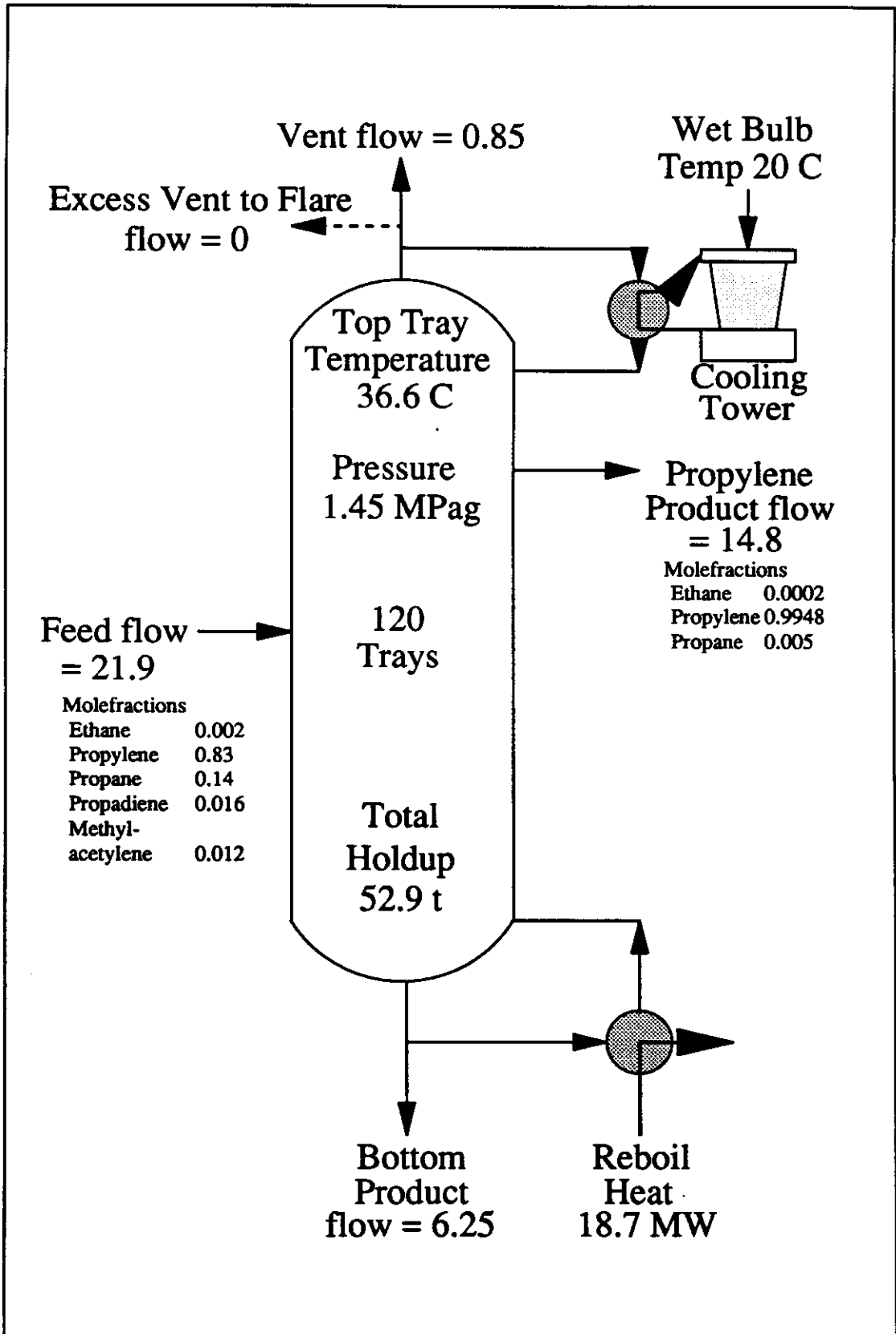


Figure 2.2 - Schematic Diagram of the C₃ Splitter Distillation Column

- (4) Maximum vapour rate at any point in the column ≤ 1.66 kmol/s (to prevent flooding).
- (5) Ethane in product propylene ≤ 450 ppm (quality specification).
- (6) Propane in product propylene ≤ 5000 ppm (quality specification).
- (7) Fraction of propadiene plus methylacetylene in bottom product ≤ 0.25 (to prevent polymerisation of these chemicals).

The second constraint is the most complex, as the cooling water temperature required in the condenser depends on the heat load. This affects the temperature of the water returned to the cooling tower which in turn affects the temperature of the cooled water available from the tower. The modelling of this interaction is explained in Section 2.2.3.

The degrees of freedom for this column can be determined by calculating how many inputs can be specified, as discussed earlier. For a specified (by the upstream plant) feed composition there are six inputs to be set - the feed, vent, propylene product, bottoms product and cooling water flowrates (or valve positions), and the heat input to the reboiler. However, not all flows can be set independently - one value of flow must "float" to maintain the material balance. This leaves five degrees of freedom for the column.

The availability of five degrees of freedom for optimisation of this column provides considerable scope for optimising control. However, at the same time it points to the need for an efficient method for accurately performing this optimisation, as the computational effort increases rapidly with the number of freed variables.

2.2.2 Modelling of Distillation Columns

To perform optimising control studies, a mathematical model needs to accurately predict both steady-state and dynamic process behaviour. However, the information produced by the model need not include every physical characteristic of the process, but only those necessary for the study. That is, the model used need only be as complex as the problem requires. Increasing the complexity of the model will never achieve an exact match with the process, so some compromise on model accuracy always needs to be made. After reviewing existing approaches to distillation column modelling, a simplified technique suitable for optimising control is presented.

Most approaches to modelling are based on some understanding of the physics of the process. If all known aspects have been included, then this is known as a rigorous model. For distillation, this means including equations to describe vapour-liquid equilibrium, heat balances, hydraulics, mass balances (both liquid and vapour phase), column pressure drops and plate efficiencies. Taken to an extreme, this could require using finite element analysis to develop such equations for a mesh of elements on each plate, for each plate in the column. This level of complexity is usually not warranted in modelling studies of distillation columns and frequently some major simplifications are made.

Doherty and Perkins (1982) give five levels of simplification for modelling distillation columns assuming the liquid on each plate is well-mixed. These levels are;

- (1) CHE. The dynamics of composition, holdup and enthalpy are modelled for each plate.
- (2) CH. The dynamics of composition and holdup are modelled for each plate. Enthalpy changes are assumed to occur instantly.
- (3) CE. The dynamics of composition and enthalpy are modelled for each plate. Changes in holdup are assumed to occur instantly.
- (4) C. Only plate composition dynamics are included. Both enthalpy and holdup changes are assumed to occur instantly.
- (5) CMO. Here only plate composition dynamics are included, but with the extra assumption of no changes in the internal flows from plate to plate (that is, constant molar overflow).

For the most complex of these (CHE), there are $(C+1)N$ differential equations arising from plate mass and energy balances plus a further $(C+1)N$ algebraic equations describing phase equilibria and hydrodynamic relationships (C = number of components, N = number of plates). Therefore, for the C_3 Splitter example, there would be 720 differential equations and a further 720 algebraic equations. In addition to these equations there is the need to calculate the necessary thermodynamic properties such as vapour-liquid equilibrium coefficients and enthalpies, and possibly even physical properties such as liquid and vapour viscosity. In fact it has been estimated that when it comes to solving these equations, up to 80% of the total computation time is spent determining thermodynamic and physical property information.

Using the CHE approach is thus computationally very expensive. The example given is a large distillation column, and it is only in the last ten years that rigorous steady-state simulation of such large columns has become common. In fact today, the dynamic simulation of large industrial columns is only rarely attempted. Unfortunately, even for the simplest column model described above (CMO), there are still $(N-1)C$ differential equations and $(N-1)C$ algebraic equations to be solved simultaneously, although the thermodynamic calculations are also considerably simplified. Consequently, to make the computations involved in optimising control feasible, another modelling approach is required.

One technique is the use of low-order models. These techniques use various simplifications to achieve useful approximations to the dynamic response of the "full" models. Benallou et al. (1986) presented a good review of the development of low-order models which they classified into three categories;

- (1) Linear low-order models.
- (2) Bilinear low-order models.
- (3) Nonlinear low-order models.

Three classes of linear low-order models were discussed. The first uses standard model reduction techniques to provide a mathematical approximation of the full-order model (Bonvin and Mellichamp, 1982). The second technique applies orthogonal collocation to a linearised model, whereby the plate variables throughout the column are approximated as continuous functions (Wong and Luus, 1980). The model uses the simplified continuous functions to predict variables at "fictitious" tray locations, which are an artefact of the method. Interpolation between these points is required to relate model outputs to the actual process. The third class of linear model uses transfer functions developed from step or impulse tests carried out on either a high-order model or the process itself.

Benallou et al. (1986) comment that bilinear models have considerable potential for producing good reduced order models. An example of this approach is presented by España and Landau (1978). However, this technique also introduces model variables with no physical significance.

An example of a nonlinear low-order model is that produced by orthogonal collocation applied directly to the nonlinear model (Cho and Joseph, 1983), rather than to a linearised model as described above. This technique, however, still suffers from the problems (results are only calculated for "fictitious" trays) associated with orthogonal collocation methods, as discussed before.

Compartmental Models

Based on their review, Benallou et al. (1986) suggested three desirable features of reduced order models which were not available in the then existing techniques;

- (1) The model should retain as much of the physical structure of the process as possible.
- (2) Differential (i.e. time varying) variables should have physical significance.
- (3) Model parameters should be related to process parameters.

As a result they presented a model based on work by Dahlgvist (1979), called the "compartmental" model. Here, groups of trays within a column are collected into compartments. The steady-state results from such a model agree very well with the full-order model, and the dynamics are qualitatively correct.

One tray within each compartment is selected as the "sensitive" stage whose composition is taken to be representative of the other trays. A dynamic component mass balance is performed over the compartment using the total mass of liquid on all the trays within the compartment as the holdup, and the composition of the compartment being assigned to the most "sensitive" tray. The separation performed by the compartment is calculated using a conventional steady-state "short-cut" technique (Ramalho and Tiller, 1962). Despite the promising characteristics of this approach, limitations have been reported by Gannavarapu (1991) including the arbitrary nature of allocating trays to compartments and a problem with the dynamic response known as inverse initial response. This latter problem, common in some reduced order models, involves the dynamic output initially moving in the wrong direction before reversing and continuing in the correct direction.

A similar but in some ways better approach was developed by Howell (1984), also based on the work of Dahlgvist (1979). It is actually a special case of the Benallou et al. (1986)

method, where the sensitive stage is always taken to be the lowest tray in the compartment. This method is often referred to as the Node-Section model. Groups of trays are collected into sections, which are joined by nodes which is where all input and output streams join the column. The column inventory, assigned to each node for the dynamic component balance, is taken as half that of each of the adjoining sections, plus that associated with any ancillary equipment (e.g. the reboiler and condenser). Figure 2.3 shows the Node-Section model layout for the C_3 splitter column. Figure 2.4 shows the details of a node and a section and the associated nomenclature.

For each node (j) a dynamic mass balance is written for $N-1$ of the components;

$$M_j \frac{dx_{ij}}{dt} = V_{j+1}y_{i,j+1} - V_jy_{i,j} + L_{j-1}x_{i,j-1} - L_jx_{i,j} + F_jx_{f,i,j} - P_jx_{p,i,j} \quad (2.1)$$

$$i = 1, \dots, N-1$$

The mole fraction for the N th component is calculated by difference (i.e. the sum of the mole fractions is equal to one). This allows an overall mass balance for the node plus the associated section to be written;

$$F_j + L_{j-1} + V_{j+1} = P_j + L_j + V_j \quad (2.2)$$

An enthalpy balance for the node plus the associated section, neglecting thermal dynamics is also included;

$$Q_j + hf_jF_j + h_{j-1}L_{j-1} + H_{j+1}V_{j+1} = hp_jP_j + h_jL_j + H_jV_j \quad (2.3)$$

Finally, the calculation of the separation performed in the section is included. Unlike the method described by Benallou et al. (1986) (who used a simplified steady-state separation calculation procedure), Howell solved the more accurate steady-state CMO equations (i.e. steady-state mass balances with the assumption of constant molar overflow, as described earlier) for the section assuming constant relative volatilities throughout the section. The relative volatility in node j ($\beta_{i,j}$) is defined as the ratio of the mole fraction in the vapour phase to that in the liquid phase for one component (i) as compared to a "key" component. This assumption means that for the trays within a given section, the vapour-liquid equilibrium conditions for a component are taken as the same on each tray. The

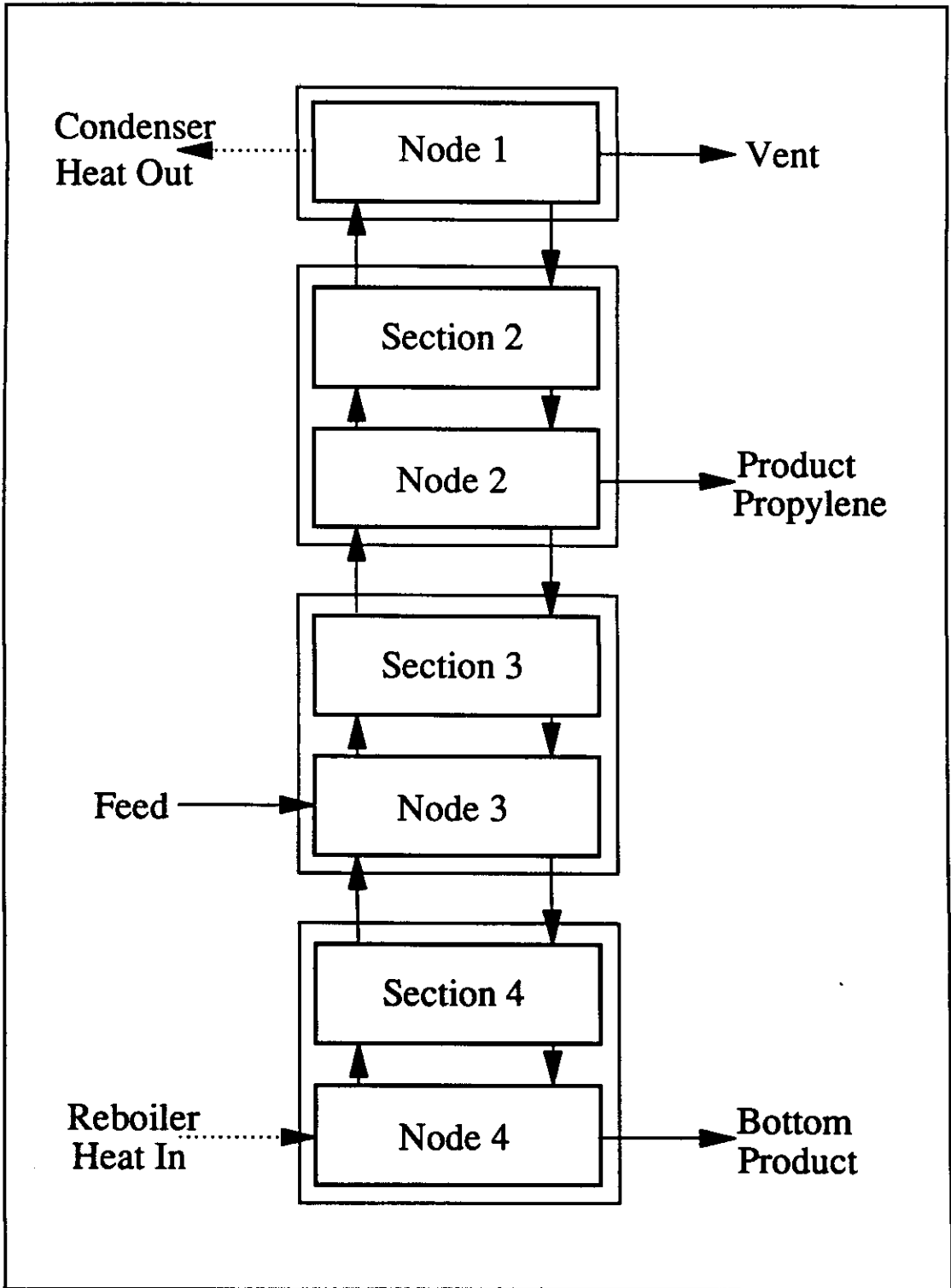


Figure 2.3 - Node-Section Arrangement for the C₃ Splitter Column

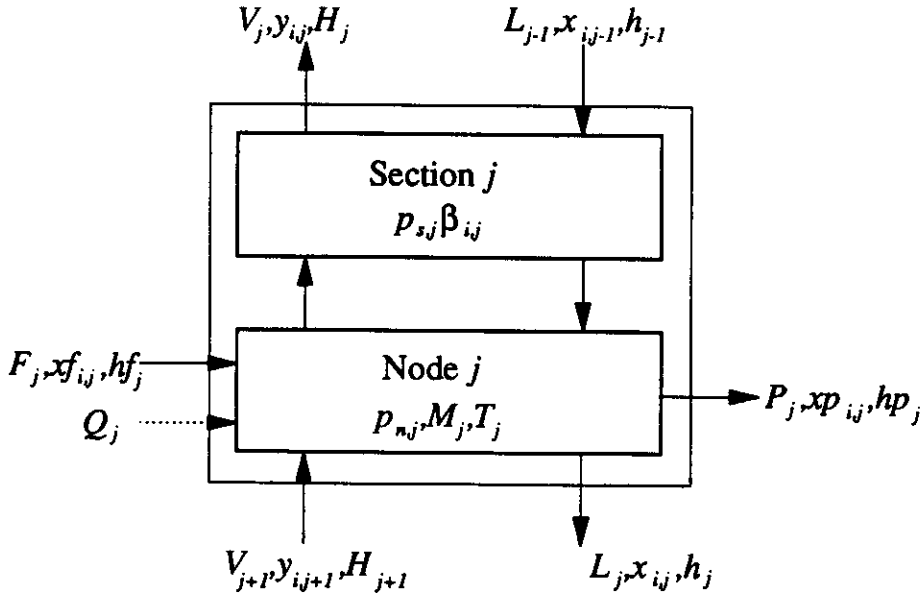


Figure 2.4 - Node-Section Details

Compartmental Model Nomenclature

- $\beta_{i,j}$ = Relative volatility of component i in section j .
- F_j = Molar flow of feed stream to node j .
- h_j = Molar enthalpy of liquid stream from node j .
- hf_j = Molar enthalpy of feed stream to node j .
- hp_j = Molar enthalpy of product stream from node j .
- H_j = Molar enthalpy of vapour stream from node j .
- L_j = Molar liquid flow from node j .
- M_j = Hold up of liquid associated with node j .
- P_j = Molar flow of product stream from node j .
- $p_{n,j}$ = Pressure in node j .
- $p_{s,j}$ = Pressure in section j .
- Q_j = External heat flow into node j .
- V_j = Molar vapour flow from node j .
- T_j = Temperature in node j .
- $x_{i,j}$ = Mole fraction of component i in liquid stream from node j .
- $x_{f,i,j}$ = Mole fraction of component i in feed stream into node j .
- $x_{p,i,j}$ = Mole fraction of component i in product stream from node j .
- $y_{i,j}$ = Mole fraction of component i in vapour stream from node j .

separation relationship for each section is written as an iterative procedure, where for known inlet and outlet liquid compositions, liquid and vapour flowrates, number of theoretical plates and the constant relative volatility for each component, the outlet vapour composition is calculated. The inlet vapour composition is not required for this calculation.

$$y_{i,j} = f(x_{i,j-1}, x_j, L_{j-1}, V_j, \beta_i) \quad i = 1 \dots N-1 \quad (2.4)$$

The temperature in the node is calculated as the bubble point temperature at which point the combined vapour pressure of the liquid mixture is equal to the column pressure.

Raoult's Law is used to calculate the equivalent vapour pressure of one component from the total pressure in the node, the relative volatilities and the liquid composition. This is then used to back-calculate the temperature from a temperature-vapour pressure relationship for that pure component. Further details of this calculation are given in Appendix I.

$$T_j = f(x_{i,j}, p_{n,j}, \beta_{i,j}) \quad i = 1, \dots, N-1 \quad (2.5)$$

The compartmental model is thus defined by Equations 2.1 to 2.5 for each node and section. The thermodynamic information required is the enthalpy of each stream, the relative volatilities for each non-key component in each section and a relationship giving temperature as a function of vapour pressure for one pure component.

As shown in Figure 2.3 the nodes and sections can be linked together to model a complete distillation column. Note that there will always be one more node than there are sections. In many cases, additional models may be required for ancillary equipment, such as reboilers and condensers. The next section describes the application of this model to the C_3 Splitter column, including the derivation of all necessary thermodynamic relationships and ancillary models.

2.2.3 Simulation Results for the C_3 Splitter

Both steady-state and dynamic models of the C_3 Splitter are required to design and analyse an optimising control scheme. In the previous section, the compartmental model was put forward as an efficient, yet accurate technique column modelling technique. An

important part of any distillation modelling is the methods used to calculate the necessary thermodynamic properties. For the limited range of operating conditions required of this simulation, some quite simple relationships were found.

Thermodynamics

The thermodynamic properties required for the compartmental model are;

- (1) The relative volatility of each component in each node-section pair.
- (2) The liquid and vapour enthalpy of each stream.
- (3) Temperature as a function of vapour pressure for one pure component.

Since the C₃ Splitter system contains predominantly simple hydrocarbons and is not usually operated far from a nominal operating point, it was felt that simplified relationships based on rigorous data over the operating range would be adequate for calculating thermodynamic properties. The results of numerous rigorous model, steady-state simulations for the column, performed using the ICI flowsheeting package FLOWPACK II (Berger and Perris, 1979), were available. These runs had been matched to plant rate trial data and so provided a good basis for determining the range of temperatures, pressures and compositions which are found in the column. Data was extracted from these rigorous model results for four important positions in the column, which are equivalent to the nodes used in the compartmental model. These positions are;

- (1) The top tray in the column.
- (2) The product propylene side-stream tray.
- (3) The feed tray.
- (4) The bottom tray (reboiler) of the column.

For the anticipated composition ranges at each of these points, the required thermodynamic properties were calculated rigorously for a typical range of temperatures (300 - 320K). The pressure used for these calculations was the corresponding bubble point pressure, as each tray was assumed to be at vapour-liquid equilibrium. The calculations were performed using ICI's proprietary physical properties package TABLES, using the Redlich-Kwong-Soave equation of state.

Although the physical properties calculated are functions of both pressure and temperature, these two latter variables are dependent on each other due to the vapour-liquid equilibrium that is assumed to exist throughout the column. Consequently in the simplified relationships, the physical properties are correlated to only temperature or pressure as appropriate. The relationships obtained are shown graphically in Figure 2.5.

As defined previously, the relative volatility (β_i) is the ratio of the mole fraction in the vapour phase to the mole fraction in the liquid phase for one component (i) divided by that ratio for the "key" component. The "key" component has no special significance but is selected for convenience. Here the heaviest component, methylacetylene, was selected as the "key". Good correlations were obtained for the relative volatility of each component as a linear function of pressure (p);

$$\begin{aligned}
 \beta_{\text{ethane}} &= -1.394p + 5.738 & r=0.9703 \\
 \beta_{\text{propylene}} &= -0.2392p + 1.884 & r=0.9876 \\
 \beta_{\text{propane}} &= -0.1522p + 1.579 & r=0.9637 \\
 \beta_{\text{propadiene}} &= -0.0355p + 1.134 & r=0.9825
 \end{aligned} \tag{2.6}$$

(r = correlation coefficient)

The rigorous thermodynamic calculations were performed at the bubble point pressure corresponding to the selected temperature. Unfortunately, this led to problems when calculating vapour enthalpies, as the vapour was of a different composition to that of the liquid, so the apparent heat of vaporisation included differences in the average heat of formation. To overcome this, the vapour enthalpy was recalculated at the corresponding dew point pressure (the pressure at which the first drop of liquid condenses). At this pressure, the vapour has the same composition as the liquid did at its bubble point. The difference between these vapour and liquid enthalpies was taken as approximating the heat of vaporisation for this composition, without any heat of formation effects. Using this data, a linear regression between the calculated heat of vaporisation of the mixture (ΔH_{vap}) and pressure (p) was found;

$$\Delta H_{\text{vap}} = -2.808p + 17.55 \text{ MJ/kmol} \quad r=0.9994 \tag{2.7}$$

For each selected position in the column, liquid enthalpy was plotted against temperature.

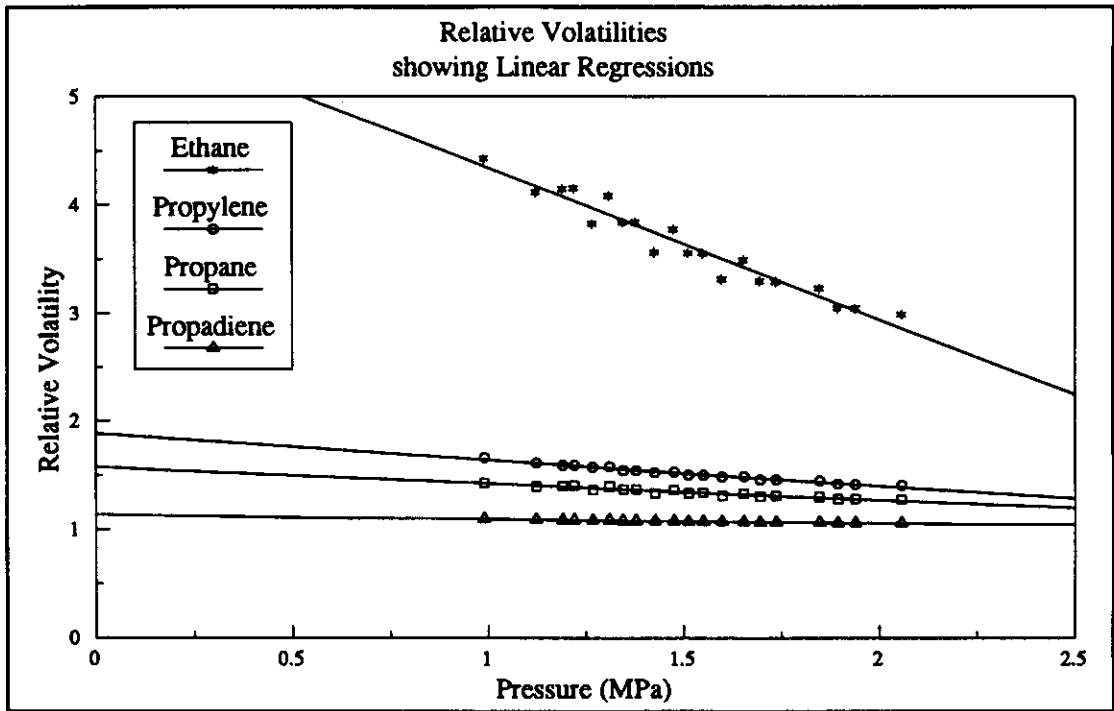


Figure 2.5a - Relative Volatilities Showing Linear Regressions

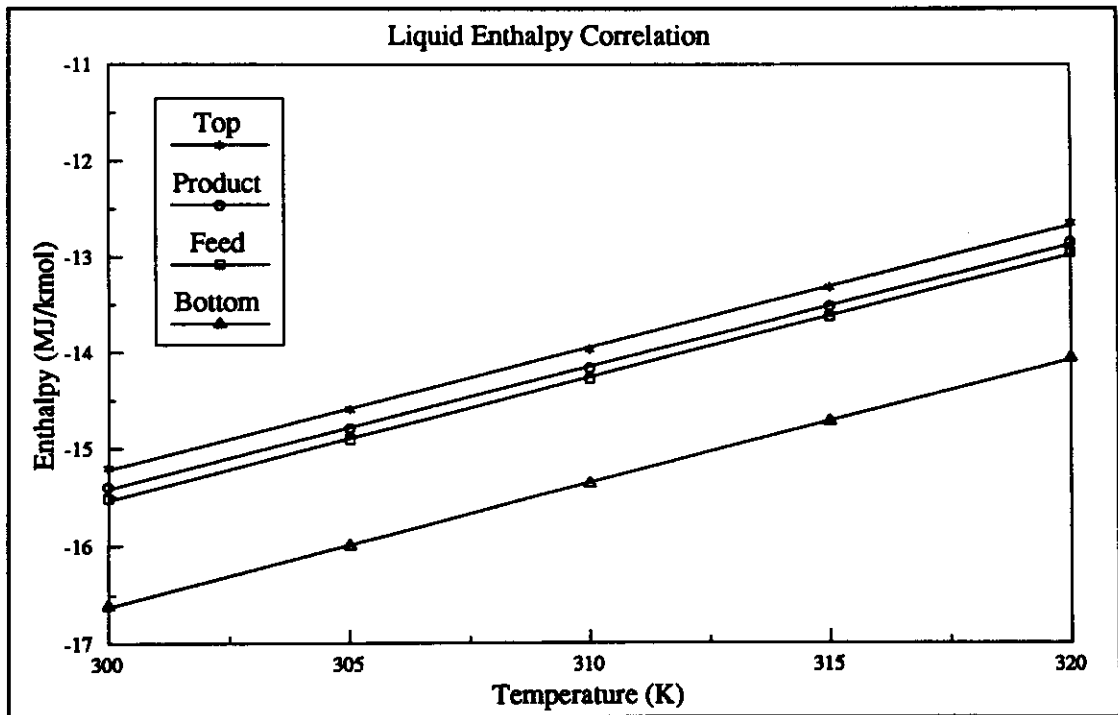


Figure 2.5b - Liquid Enthalpy Correlation

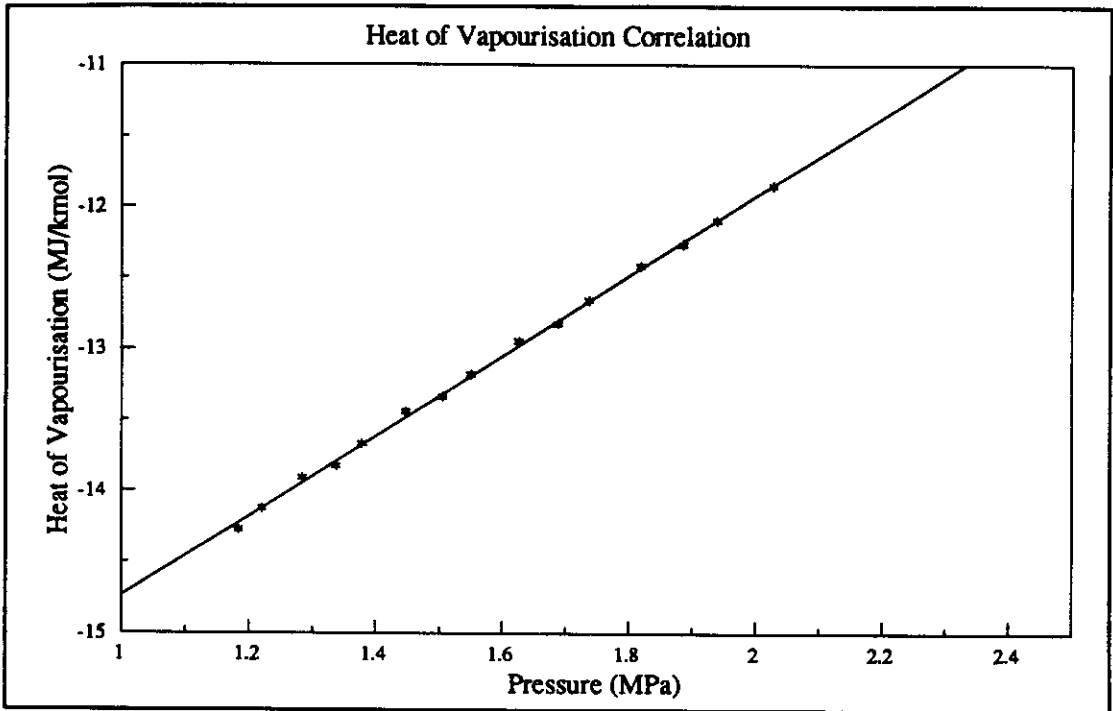


Figure 2.5c - Heat of Vapourisation Correlation

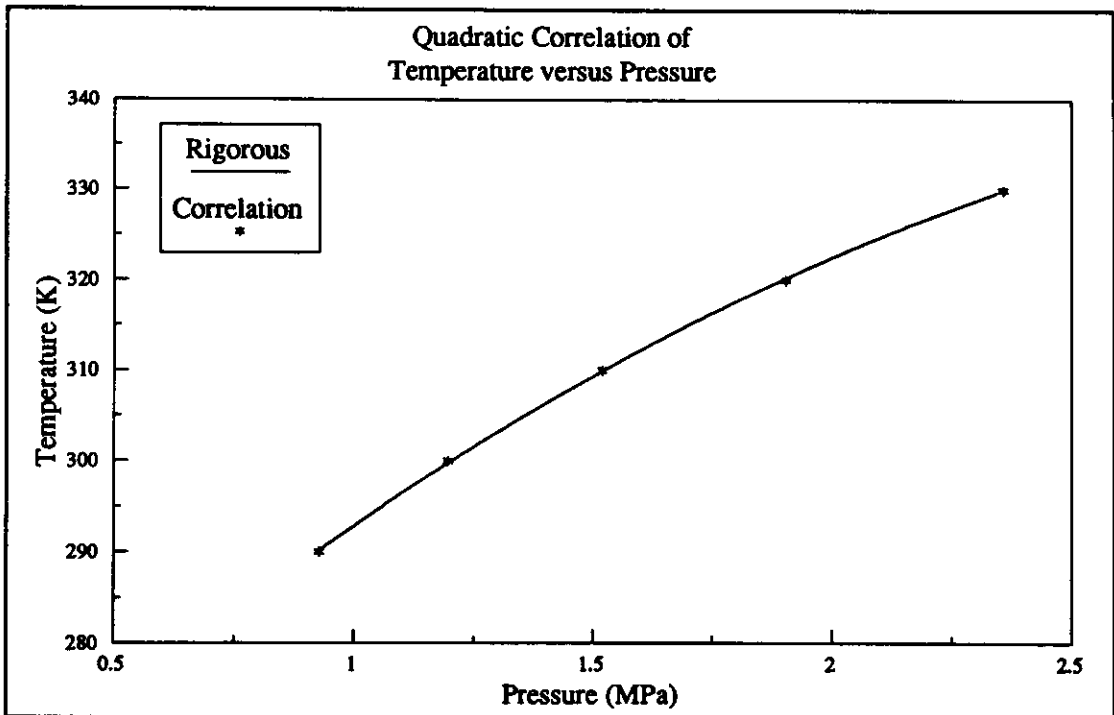


Figure 2.5d - Temperature versus Saturated Vapour Pressure (Propylene)

Again, excellent linear relationships were found, the correlation coefficient being better than 0.9997. The slope of each line is the specific heat (C_p) of the mixture at that position. The mean value of the slopes was used as an average value for the column, the maximum error brought about by such averaging being 0.3%.

$$C_p = 127.5 \text{ kJ/kmol}\cdot\text{K} \quad (2.8)$$

The liquid and vapour enthalpies for each stream were thus calculated using the liquid specific heat (Equation 2.8) and the correlation for the heat of vaporisation as a function of pressure (Equation 2.9).

Finally, temperatures in the model could be calculated using Raoult's Law, as discussed in the previous section (Equation 2.5) and detailed in Appendix I. This requires a function relating temperature to one pure component vapour pressure. The graph given in Figure 2.5 shows a quadratic function (Equation 2.9) fitted to data for propylene, giving the required temperature - vapour pressure relationship.

$$T = -6.6099p^2 + 49.499p + 249.95 \quad (2.9)$$

(T in K, p in MPa)

These approximate relationships (Equations 2.6, 2.7, 2.8, 2.9) provided the required set of thermodynamic properties to an acceptable level of accuracy, over the limited operating range under consideration, without imposing excessive computing overheads.

Ancillary Models

The compartmental model was used to model the C_3 Splitter distillation column. However, the complete system includes multiple, parallel reboilers and condensers, and a cooling tower. In this study, however, specific models were not incorporated for the reboiler and condenser on the C_3 Splitter, the heat flows associated with this equipment were simply incorporated into the appropriate node heat balances. In addition, the significant inventory of liquid in each of these ancillary units was allocated to the node to which they are connected, as hold-up in the dynamic mass balances. To a limited extent these simplifications excluded some of the dynamic effects. The simplifications made would not affect the steady-state results.

The cooling tower had been identified as a limiting factor by ICI personnel. Consequently to perform meaningful process optimisation, a model was needed which reflected the performance of the cooling tower.

A cooling tower operates by exposing warm process water to cooler air. The intimate mixing which occurs cools the water by not only warming the air, but also by increasing its humidity. It is the latter process, the result of water evaporation, which achieves the majority of the cooling effect. Consequently, there are many factors which affect the performance of a cooling tower. The efficiency of the unit is measured by how close the outlet air is to being completely humidified (that is, its wet bulb temperature equal to the dry bulb, or ambient, temperature). The inlet water temperature also affects how much heat may be removed - the hotter the water, the greater the average vapour pressure of water throughout the tower. It is the vapour pressure difference between completely saturated and ambient air which drives the evaporation process.

The required model had to calculate the outlet cooling water temperature for a given heat load and ambient air wet bulb temperature. Commercial cooling towers, like this one, are best described by empirical performance curves supplied by the manufacturer. Based on this information and experimental results on cooling tower performance, the following relationship was derived. It matched most of the experimental data by better than about 0.5°C.

$$T_{CW,min} = (1.025 - 1.88995 \times 10^{-2} Q_{cond}) T_{WB} + (1.5865 + 0.5268 Q_{cond})$$

T_{CW} = Temperature of the outlet cooling water
 Q_{cond} = Heat load on the cooling tower

(Temperature in °C, Q_{cond} in MW)

(2.10)

The use of the compartmental column model, in conjunction with the simplified thermodynamic relationships and the cooling tower model, thus enabled simulations of the C₃ Splitter to be performed.

Steady-State Simulation Results

Simulation of the column was performed by simultaneously solving the equations describing the system - that is, the compartmental model, the thermodynamics and the

cooling tower model. Steady-state solution of the ordinary differential equations, which are part of the compartmental model, was achieved by setting the time derivatives to zero, so that these equations become algebraic equations.

The entire simulation study (steady-state simulations, dynamic simulations and steady-state optimisation) was performed using the flowsheeting package SPEEDUP (Pantelides, 1988). A listing of the SPEEDUP input file, including the FORTRAN subroutines used in the procedures, is given in Appendix II.

An important feature of SPEEDUP is its equation-oriented structure. Most other simulation packages use what is known as a sequential modular approach. This consists of blocks of computer code representing physical units which are linked together and solved one at a time (hence, the term "sequential"). A limitation of this approach is the restriction placed on satisfying the degrees of freedom so as to allow solution. Certain variables must be specified so that each block of code can be executed. SPEEDUP on the other hand simultaneously solves the entire set of equations (that is, the entire flowsheet) once the degrees of freedom have been satisfied in a consistent manner. This flexibility allows different simulation requirements to be fulfilled using the same model. In particular, both performance simulations, where plant products are calculated for specified plant inputs and plant parameters (e.g. the response to feedstock changes), and design simulations, where plant parameters are calculated for specified plant feeds and product requirements (e.g. design calculation of reactor volumes), can be performed.

To check the accuracy of this simplified column model approach, a comparison was made with results from the rigorous (FLOWPACK) model discussed earlier. Since this study was aimed at eventually performing optimising control design, it was felt that any variables that were important in the objective function and/or constraints should be accurately modelled. These variables are;

- (1) Product flowrates.
- (2) Mole fractions of propylene and propane in the product propylene.
- (3) Column pressures.
- (4) Reboiler and condenser heat duties.
- (5) The temperature at the top of the column.

Variable group (1) is a key part of the objective function, groups (2) and (3) are known operational constraints, while (4) and (5) are necessary as a constraint was anticipated in the heat removal from the top of the column.

In addition to a detailed comparison at the nominal operating point, perturbations about this point were made to check that the simplified model changed in the correct way. It was found that at the nominal operating point, there were small discrepancies in some of the important variables. It was felt that this was most probably due to the use of constant relative volatilities throughout each section. To counter this effect, the steady-state separation was "fine-tuned" by adjusting the number of theoretical plates in each section. This substantially improved the accuracy throughout the column.

A final adjustment was necessary to improve the accuracy of the calculated temperature at the top of the column. Here the calculated temperature did not agree well with the rigorous calculations. This part of the column has the highest concentration of ethane, the lightest component, causing the greatest deviation from "ideal" thermodynamic laws, and from the simplified thermodynamic relationships. The latter can be observed in Figure 2.5 where the relative volatility of ethane has a greater "scatter" than the other components. To allow for these non-idealities a factor was included in the Raoult's law procedure used to predict temperatures from the node composition and pressure. The factor is equivalent to a liquid phase activity coefficient, which was set to 1.053. Details are given in Appendix I.

Table 2.1 shows a comparison of the results between the rigorous C_3 Splitter simulation and the simplified model, at the nominal operating point. The relative error was always less than five percent for all the important variables. One variable, the ethane content of the product propylene stream, had a much higher relative error, although the absolute size of the error is very small. Importantly, the time taken to execute the simplified model was only 20 CPU seconds on a DEC MicroVax II operating under the VMS operating system, compared to over 4 hours for the rigorous simulation on the same computer.

The final step in the validation procedure was a comparison of the dynamic response of the simplified model with plant data. The time constant of the real column is very long, about eight hours (i.e. the effect of an input change is not substantially observed until

Variable	Rigorous	Simplified	% Error
<u>Product Propylene</u>			
Flowrate	8.111×10^{-2}	8.075×10^{-2}	-0.45
x_{ethane}	8.4×10^{-5}	1.355×10^{-4}	61.33
$x_{\text{propylene}}$	9.955×10^{-1}	9.955×10^{-1}	0.00
x_{propane}	4.374×10^{-3}	4.364×10^{-3}	-0.23
$x_{\text{propadiene}}$	0	0	0
<u>Bottom Product</u>			
Flowrate	1.754×10^{-2}	1.754×10^{-2}	0.01
x_{ethane}	0	0	0
$x_{\text{propylene}}$	2.590×10^{-2}	2.472×10^{-2}	-4.57
x_{propane}	8.076×10^{-1}	8.086×10^{-1}	0.13
$x_{\text{propadiene}}$	9.496×10^{-2}	9.502×10^{-2}	0.06
<u>Temperatures</u>			
T_{top} (°C)	305.33	305.30	-0.01
$T_{\text{prod. prop.}}$ (°C)	307.30	308.04	0.24
T_{feed} (°C)	310.80	311.21	0.13
T_{bottom} (°C)	319.05	317.52	-0.48
<u>Pressures</u>			
p_{top} (MPa)	1.450	1.450	0
$p_{\text{prod. prop.}}$ (Mpa)	1.457	1.457	0
p_{feed} (Mpa)	1.500	1.501	0.07
p_{bottom} (Mpa)	1.526	1.526	0
<u>Heat Loads</u>			
Q_{cond} (MW)	16.417	16.867	2.74
Q_{reb} (MW)	16.649	16.972	1.94
<u>Theoretical Plates</u>			
Top - Prod. Prop.	10	10	
Prod. Prop. - Feed	66	72	
Feed - Bottom	38	37	

Table 2.1 - Rigorous versus Simplified Model Simulation Results

about eight hours after the change). Over such long periods, disturbances affect the operation of the column, requiring control actions to be taken to keep the product within quality specifications. As a result, no plant data corresponding to a given single change in column operating conditions was available. Consequently, a typical disturbance, a change in the feed composition of propane from 18% to 24% over a period of two hours, was simulated. The result of this simulation is shown in Figure 2.6. The calculated response was considered by operating staff to be consistent with observed plant performance. Again the SPEEDUP simulation ran very quickly, here at better than 100 times faster than real time. Additionally, more significant dynamic validations have been reported (Gannavarapu, 1991) on similar, simple hydrocarbon distillation systems using this node-section model.

The simplified C₃ Splitter model developed here gives the required steady-state accuracy and is also suitable for dynamic analysis. The first part of performing an optimising control design is to examine the optimum of the plant under different slow disturbance conditions. These optimisation results are reported in the next section.

2.2.4 Steady-State Optimisation Results

Optimisation of the column was also performed within the SPEEDUP framework using a feasible path optimiser (Han, 1977).

The objective function used in optimising the C₃ Splitter was net profit which may be expressed as;

$$\text{Profit} = a \times (\text{Product Propylene}) + b \times (\text{Bottom Product}) + c \times (\text{Vent}) - d \times (\text{Feed}) - e \times (\text{Excess Vent}) \quad (2.11)$$

The variables a, b, c, d, e are unit values (or costs) for the associated streams. This objective function contains the value of products (including recycled streams) as well as the cost of the feed and of disposing of any flared gas. Here the costs of operating the column (pumping, cooling and heating) are small compared to the value of the streams, and almost independent of operating conditions, so have not been included in the objective function. Although realistic, the profit figures presented do not indicate the exact financial returns of the column.

The operational inequality constraints are detailed in Section 2.2.1. However, the

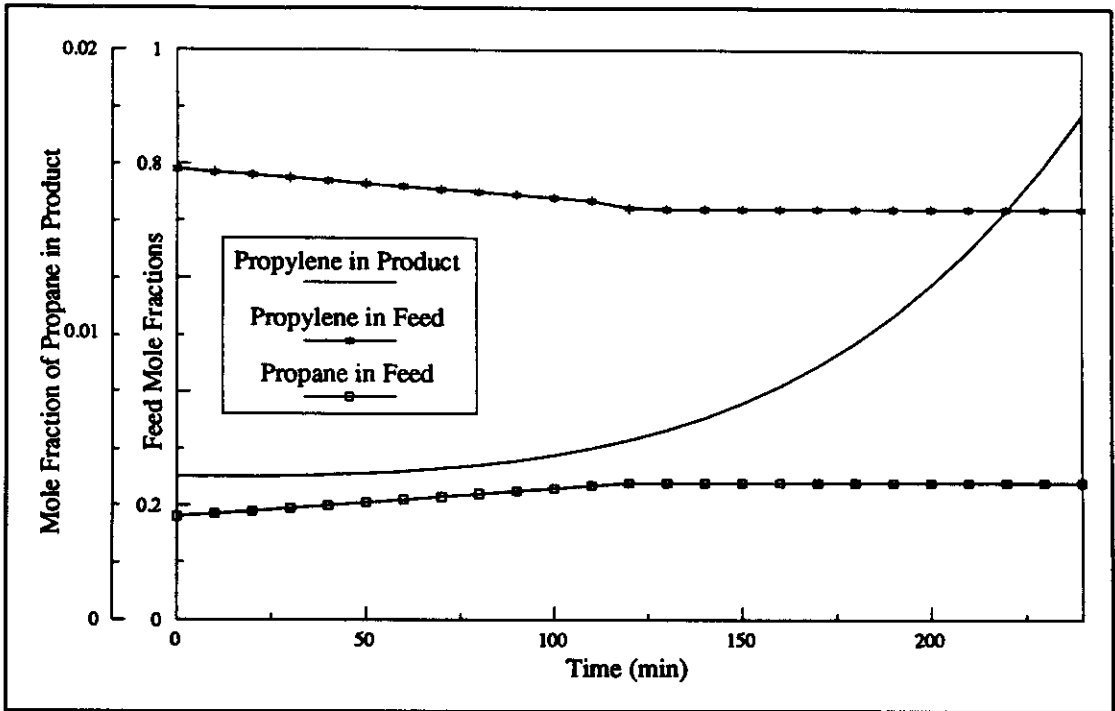


Figure 2.6 - Dynamic Response of Simplified Model

application of the cooling tower constraint requires some further explanation. The model of the cooling tower described in Section 2.2.3 predicts the cooling water outlet temperature, given the heat load and the ambient wet-bulb temperature. Incorporated into the column model is a calculation of the cooling water inlet temperature required to extract the heat of condensation (Q_{cond}). This equation uses experimentally determined heat transfer coefficients and makes the assumption of a 2°C approach temperature (i.e. the warmed cooling water does not get closer than 2°C to the assumed constant temperature vapour condensation). These two temperatures are considered in the second constraint, so that the required cooling water temperature is always less than or equal to the lowest temperature available from the cooling tower under the prevailing conditions.

To perform the optimisation, the following five "free" variables were selected;

- (1) Feed flowrate.
- (2) Propylene product flowrate.
- (3) Bottom product flowrate.
- (4) Reboiler heat input.
- (5) Column pressure.

Using these variables the optimum was found for a range of slow disturbance values (i.e. variables affecting column performance which change in a slow but persistent manner - here feed composition and wet-bulb temperature). The results are shown in Tables 2.2 and 2.3. The first table contains the optimal profit and the values of the five "free" variables at the optimum. The second table shows values of the Lagrange multipliers (μ_i) for the active constraints. These indicate how much the objective function (net profit) will change if the constraint is relaxed by one unit. For example, the case with 85% Propylene and 14°C wet-bulb temperature has a Lagrange multiplier of 0.774 for constraint (3), the maximum available reboiler heat. This means that if 1 MW more heat was available, the notional profit could be increased by \$0.774m/year. This simplified analysis does not, however, allow for limitations imposed by other constraints becoming active if this constraint were to be relaxed.

The slow disturbance values selected reflect the range actually experienced in these inputs. The feed composition varies from 85% propylene (12% propane), to 65%

$T_{\text{wet-bulb}}$	Feed Composition 85% Propylene 12% Propane	Feed Composition 75% Propylene 22% Propane	Feed Composition 65% Propylene 32% Propane
22.5°C	Profit = \$16.9m/yr Feed = 0.1414 Propylene = 0.1093 Bottom = 0.0251 Q_{reb} = 17.283 MW Pressure = 1.55 Mpa	Profit = \$11.6m/yr Feed = 0.1276 Propylene = 0.0869 Bottom = 0.0352 Q_{reb} = 16.706 MW Pressure = 1.55 Mpa	Profit = \$7.39m/yr Feed = 0.1161 Propylene = 0.0688 Bottom = 0.0431 Q_{reb} = 15.817 MW Pressure = 1.55 Mpa
20°C	Profit = \$19.0m/yr Feed = 0.1578 Propylene = 0.1225 Bottom = 0.0281 Q_{reb} = 19.256 MW Pressure = 1.55 Mpa	Profit = \$13.1m/yr Feed = 0.1429 Propylene = 0.0977 Bottom = 0.0395 Q_{reb} = 18.668 MW Pressure = 1.55 Mpa	Profit = \$8.36m/yr Feed = 0.1308 Propylene = 0.0779 Bottom = 0.0486 Q_{reb} = 17.763 MW Pressure = 1.55 Mpa
14°C	Profit = \$21.7m/yr Feed = 0.1656 Propylene = 0.1323 Bottom = 0.0299 Q_{reb} = 20.0 MW Pressure = 1.55 Mpa	Profit = \$15.3m/yr Feed = 0.1547 Propylene = 0.1084 Bottom = 0.0431 Q_{reb} = 20.0 MW Pressure = 1.55 Mpa	Profit = \$10.1m/yr Feed = 0.1488 Propylene = 0.0901 Bottom = 0.0556 Q_{reb} = 20.0 MW Pressure = 1.55 Mpa

Table 2.2 - Optimal Operating Points

$T_{\text{wet-bulb}}$	Feed Composition 85% Propylene 12% Propane	Feed Composition 75% Propylene 22% Propane	Feed Composition 65% Propylene 32% Propane
22.5°C	Profit = \$16.9m/yr $\mu_1 = 33.3$ $\mu_2 = 1.255$ $\mu_6 = 563$	Profit = \$11.6m/yr $\mu_1 = 22.2$ $\mu_2 = 0.863$ $\mu_6 = 357$	Profit = \$7.39m/yr $\mu_1 = 13.9$ $\mu_2 = 0.548$ $\mu_6 = 212$
20°C	Profit = \$19.0m/yr $\mu_1 = 31.1$ $\mu_2 = 1.198$ $\mu_6 = 636$	Profit = \$13.1m/yr $\mu_1 = 20.7$ $\mu_2 = 0.827$ $\mu_6 = 405$	Profit = \$8.36m/yr $\mu_1 = 12.9$ $\mu_2 = 0.528$ $\mu_6 = 242$
14°C	Profit = \$21.7m/yr $\mu_1 = 2.88$ $\mu_2 = 0.287$ $\mu_3 = 0.774$ $\mu_6 = 715$	Profit = \$15.3m/yr $\mu_1 = 3.13$ $\mu_2 = 0.268$ $\mu_3 = 0.473$ $\mu_6 = 470$	Profit = \$10.1m/yr $\mu_1 = 4.05$ $\mu_2 = 0.258$ $\mu_3 = 0.223$ $\mu_6 = 294$

Table 2.3 - Lagrange Multipliers

propylene (32% propane), according to changes in the operation of the upstream olefines plant. The remaining 3% of the feed was constant at 0.24% ethane, 1.38% propadiene and 1.38% methylacetylene. The three values selected for the wet-bulb temperature, 22.5°C, 20°C and 14°C are based on historical weather data for the site. They represent the wet-bulb temperatures which are exceeded for 1%, 10% and 50% of the year, respectively.

The results of the optimisation are quite interesting. Firstly, there is a large variation in the optimal notional profit, ranging from \$21.7m/year to \$7.4m/year. Clearly there is considerable scope to have the column operating optimally and thus take advantage of the existing conditions, rather than always being set up conservatively to handle the worst possible case.

A second feature of the optimisation results is that there are frequently less active constraints than "free" variables, and the number is dependent on the slow disturbance values. It is only at low wet-bulb temperatures when the cooling tower is able to handle large heat loads, that the limit on available reboiler heat input becomes important. As discussed earlier, it is often assumed that there will be an equal number of active constraints and "free" variables. This example shows clearly that this is not always the case. This fact, and the observation that the number of constraints changes with the slow disturbance values has important implications for the optimising controller design.

2.2.5 Optimising Control System Design for the C₃ Splitter

As defined earlier, an Optimising Control Scheme is one that makes adjustments to the operation of a chemical plant, so that it continuously achieves the optimum of some measure of plant performance, within the various limitations placed on plant operation. In Section 2.1 the concept of constraint control was discussed. In this technique use is made of the observation that at the optimum, a number of variables will be at values set by constraints and a control system which keeps these variables at their constraint values is said to be implementing "constraint control". In some cases, the "free" variables (degrees of freedom) of the plant cannot all be satisfied in this manner and the remainder must be set to values calculated by an optimiser. Normally, the optimiser will use a model of the process which is regularly updated using process measurements so as to minimise errors caused by plant-model mismatch.

This combination of constraint control and feedforward of calculated optimal values, would appear to be a feasible structure for optimising control of the C_3 Splitter. However, as noted earlier, a problem can occur with such an implementation if the active constraints change. The active constraint set for the C_3 Splitter is;

- (1) Maximum heat removal by the cooling tower.
- (2) Maximum column pressure.
- (3) Maximum allowable propane in the product propylene stream.

In addition, for cases where the wet-bulb temperature is low, a fourth constraint becomes active;

- (4) Maximum available reboiler heat input.

So although the number of constraints active is not constant, the first three are always active and therefore are candidates for constraint control. However, the first constraint is difficult to implement as an actual control loop, as it involves running the column so that the cooling tower is extracting as much heat as possible. Unfortunately there is not a direct measurement available for how much excess cooling tower capacity is available, so a control loop here would always require some sort of model. For instance, a model based on Equation 2.10 could be used to predict the outlet temperature based on measurements of the wet-bulb temperature and the heat load. The predicted temperature minus the actual outlet water temperature is an indication of excess capacity available. For simplicity, in this study the "free" variable that could be associated with this loop will be satisfied by another model based result, an optimal value calculated from the optimiser. That is, a control loop will not be directly used for this constraint, but it will be kept active indirectly by the use other optimal setpoints in the column.

Since the fourth constraint is not always active, it too is not a candidate for constraint control. This leaves the maximum column pressure and maximum propane in the product propylene stream as constraints to be controlled to by a constraint control scheme. The next step is to design a control scheme to achieve this. The input variables that are available for control purposes are;

- (1) Feed flow.
- (2) Vent flow.
- (3) Product propylene flow.
- (4) Bottom product flow.
- (5) Reboiler heat input.

For a constraint control scheme it is important that each control loop operates well, otherwise the constraints could be violated. To determine which would be the best inputs to use, several open-loop indicators (of potential closed-loop performance) are calculated in the next section. The performance of the closed-loops is then checked by dynamic simulation.

2.2.5.1 Open-Loop Indicators of Controllability

Recently significant advances have been made in the prediction of closed-loop performance on the basis of characteristics of an open-loop model. This enables a check to be made at the design stage on the likely closed-loop dynamic performance without requiring dynamic simulations. A full description of the various open-loop indicators available will be given in Chapter 5, however the use of one particular indicator, the process condition number (γ), will be made here to guide the selection of inputs for control of the C_3 Splitter.

Linear Theory and the Laplace Domain

Most of the open-loop controllability indicators are based on the use of linear process models. Although this may seem to be a limited approach, it has been found that the local behaviour of nonlinear systems is often well characterised by the use of a linearised model generated at the nominal operating point. Perkins (1989), in his review of the use of control considerations at the process design stage, cites a number of authors who have reported case studies to support the use of this linear theory.

A wealth of linear control theory is available in the literature, some of it developed as state-space theory (applied directly to the linearised ordinary differential equations) with other contributions for use in the frequency domain. Although the process condition number is a frequency domain tool, we will first show the relationship between a time domain model and the frequency domain.

The set of ordinary differential equations which are the linear model of the process (or the linearised version of a nonlinear model), can be represented as;

$$\frac{dx}{dt} = Ax(t) + Bu(t) \quad (2.12)$$

$$y(t) = Cx(t) + Du(t) \quad (2.13)$$

Here, $x(t)$ is a vector of state (differential) variables while $u(t)$ is a vector representing the manipulated input variables. Equation (2.13) defines the vector of output variables $y(t)$.

The matrices A , B , C and D contain the relevant coefficients in the equations. In general, these coefficients can be time varying, however they will be considered as time invariant here. Equations (2.12) and (2.13) are often referred to as the state-space or time domain representation of the system.

A complete description of the mathematics of Laplace transformations is outside the scope of this work. However, they offer a convenient manner for solving systems of ordinary differential equations. The first step in the solution procedure is the transformation of the variables and the model equations into the Laplace domain. This replaces the time variable (t) with the Laplace variable (s), and converts the ordinary differential equations into a set of algebraic equations. After manipulation, the inverse Laplace transform of the resultant equations is taken, the solution being the time domain behaviour of the system. The intermediate stage, the Laplace domain representation, is convenient mathematically for control system analysis as it can be used to show the direct relationship between the process inputs and outputs. Hence, it is often referred to as an input-output model. The Laplace domain equivalent to Equations 2.12 and 2.13 is;

$$y(s) = G(s)u(s) \quad (2.14)$$

A direct relationship between $G(s)$ and the A , B , C and D matrices of Equations 2.12 and 2.13 is given by;

$$G(s) = C(sI - A)^{-1}B + D \quad (2.15)$$

In Equation (2.14) $u(s)$ and $y(s)$ are the Laplace domain equivalents of the input and output variables respectively. $G(s)$ is known as the transfer function matrix, where each

element of the matrix (G_{ij}) can be expressed as the ratio of two polynomials in s .

$$G_{ij}(s) = \frac{(s+a_0)(s+a_1)\dots(s+a_m)}{(s+b_0)(s+b_1)\dots(s+b_n)} \quad m \leq n \quad (2.16)$$

Note that for real processes, there is a restriction that the order of the numerator is less than or equal to that of the denominator.

Important characteristics of the plant can be determined directly from this input-output form, as will be discussed in Chapter 5.

One reason why the Laplace domain is so useful in control theory is that it allows easy calculation of the effect of sine waves of given frequencies to be analysed. This is important, since through the use of Fourier transforms, any transient (describing the movement of a variable with time) can be characterised as the sum of a combination of sine waves. In addition, the frequency of a sine wave entering a linear system will remain unchanged on passing through the system. The combination of these two pieces of information means that the effect of any transient entering a linear system may be analysed by considering the individual sine wave components. Hence an understanding of how the system behaves over a range of frequencies is sufficient to predict its more general behaviour.

A more detailed analysis of transfer functions and frequency domain analysis is available elsewhere (e.g. Stephanopoulos, 1984). However, in summary, if a sine wave of frequency ω rad/s enters a linear system with transfer function $G(s)$, then the magnitude ratio and phase shift of the output sine wave can be calculated by substituting $s = i\omega$ (where $i = \sqrt{-1}$) into $G(s)$. This is shown for the single-input single-output case in Equations 2.17 and 2.18;

$$\begin{aligned} \text{Magnitude ratio} &= \frac{\text{Amplitude } y(t)}{\text{Amplitude } u(t)} = |G(i\omega)| \\ |G(i\omega)| &= \sqrt{[(\text{Real } G(i\omega))^2 + (\text{Imag } G(i\omega))^2]} \end{aligned} \quad (2.17)$$

$$\text{Phase Shift} = \arctan \left[\frac{\text{Imag}(G(i\omega))}{\text{Real}(G(i\omega))} \right] \quad (2.18)$$

Clearly then, the complex number (or complex matrix) $G(i\omega)$ can be used to analyse the dynamic behaviour of a linear system. It is the condition number of the matrix $G(i\omega)$, as a function of frequency, which will be used in the selection of inputs for the optimising control system of the C_3 Splitter.

Singular Value Decomposition

Although the mathematical principles of singular value decomposition (SVD) have been known for some time, it is only relatively recently that its application to control theory has provided one of the most powerful tools available to the modern practitioner. The basic principles and computational procedures are described by Klema and Laub (1980). SVD is basically a method of analysing a matrix to reveal *a priori* certain important properties. In particular, when considering a matrix as a map from one linear subspace (set of vectors) to another, singular value decomposition shows the directionality and amplifying properties of the matrix. Further analysis gives information about the numerical stability of the matrix, which turns out to be an important descriptor for control system analysis.

Equation 2.19 shows the singular value decomposition of a general non-square complex matrix A of size $m \times n$ ($A \in \mathbb{C}^{m \times n}$);

$$A = U \Sigma V^H \quad (2.19)$$

where $\Sigma = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}$

and $S = \text{diagonal}(\sigma_1, \dots, \sigma_r)$ with $\sigma_1 \geq \dots \geq \sigma_r > 0$.

The numbers $\sigma_1, \dots, \sigma_r$ and $\sigma_{r+1} = \dots = \sigma_n = 0$ are called the singular values of A , and are the positive square roots of the eigenvalues of $A^H A$ (where A^H is the complex conjugate transpose of the A matrix). The columns of U ($U \in \mathbb{C}^{m \times m}$) are called the left singular vectors of A and are the orthonormal eigenvectors of the matrix $A A^H$ while the columns of V ($V \in \mathbb{C}^{n \times n}$) are known as the right singular vectors of A and are the orthonormal

eigenvectors of the matrix $A^H A$.

A more physically oriented description can also be given for the singular value decomposition. Multiplication of a vector by the matrix A is equivalent to mapping that vector from one space of vectors into another space. In terms of the singular value decomposition, if A was to be multiplied by one of the right singular vectors (v_i), then the resultant would be in the direction of the corresponding left singular vector (u_i) (this may be viewed as a vector rotation), and the "size" (or length) of the input vector would be amplified by the amount of the corresponding singular value (σ_i). Here size is taken in the 2-norm sense;

$$\text{size}(x) = \|x\|_2 = \sqrt{x_1^2 + \dots + x_n^2} \quad (2.20)$$

Using this definition, the maximum ($\bar{\sigma}$) and minimum ($\underline{\sigma}$) singular values can be expressed as;

$$\bar{\sigma} = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \|A\|_2 \quad (2.21)$$

$$\underline{\sigma} = \min_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} \quad (2.22)$$

The concept expressed here is that the maximum singular value is the maximum amplification that A can achieve while the minimum singular value is the minimum amplification possible. This is expressed graphically in Figure 2.7, which shows how different vectors of unit length will be mapped for an example matrix A .

Each of the left and right singular vectors shown in Figure 2.7, in the singular value decomposition of A , are of unit length (U and V are orthonormal matrices). The right singular vectors are plotted as the "1st" and "2nd" vectors on the unit circle. When multiplied by A , each vector is rotated to the direction of the corresponding left singular vector, and its length is changed by a factor of the corresponding singular value. This is also shown in Figure 2.7, where the resultant vectors are shown in the second diagram.

Singular Value Decomposition of A

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0.4046 & 0.9145 \\ 0.9145 & -0.4046 \end{bmatrix} \begin{bmatrix} 5.465 & 0 \\ 0 & 0.366 \end{bmatrix} \begin{bmatrix} 0.5760 & -0.8174 \\ 0.8174 & 0.5760 \end{bmatrix}^H$$

Left
Singular
Vectors
Singular
Values
Right
Singular
Vectors

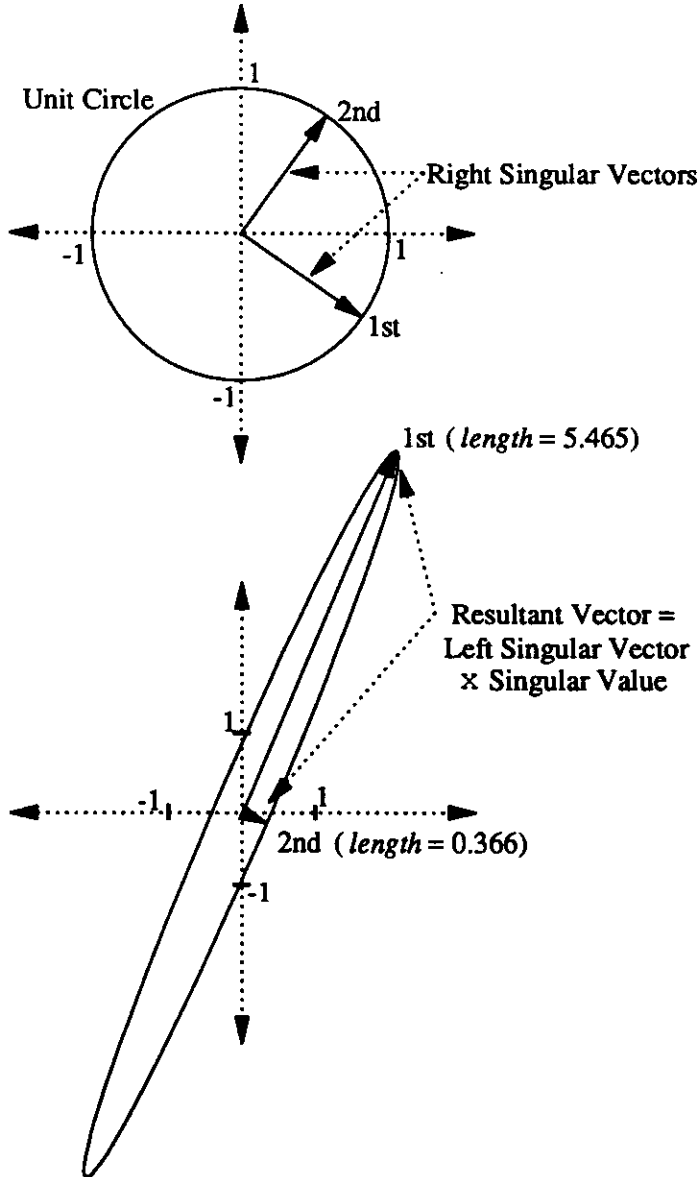


Figure 2.7 - Singular Value Decomposition

The unit circle has been stretched by A into an elongated ellipse.

Since any input vector can be expressed as the sum of multiples of the left singular vectors, then the resultant vector is the corresponding sum of the right singular vectors. Consequently, the singular values can be viewed as the range in amplifying power of the matrix A .

The ideas expressed here are applicable to analysing the dynamic performance of process plants. A singular value decomposition of the plant transfer function matrix ($G(s)$) evaluated at different frequencies, provides information about how adjustment of the inputs to the plant will affect the outputs. The singular values indicate how manipulated variable constraints could limit achievable performance. Given that the inputs to a plant are always constrained (a valve can only open and close so far), if the singular values are small, then only limited adjustments to the outputs can be made by manipulating the inputs. The minimum singular value can be used as an indicator of process's susceptibility to this problem (Morari, 1983), in which case it is desirable for a process to have a large minimum singular value.

Another important open-loop measure for analysing control systems is the process condition number (γ). This is defined as the "size" of a matrix, multiplied by the "size" of its inverse;

$$\gamma(A) = \|A\| \cdot \|A^{-1}\| \quad (2.23)$$

Here, the size of a matrix is defined as its norm;

$$\|A\| \triangleq \sup_{x \neq 0} \frac{|Ax|}{|x|} \quad (2.24)$$

The term "sup" is the *supremum*, or least upper bound, and for most purposes this can be taken as the maximum. Different vector norms are available for measuring the size of a vector depending on the particular requirement. The 2-norm was introduced in Equation (2.20). Definitions of some common vector norms are;

$$\begin{aligned}
l_1 \text{ norm: } \|x\|_1 &= \sum_i |x_i| \\
l_2 \text{ norm: } \|x\|_2 &= \sqrt{x_1^2 + \dots + x_n^2} \\
l_\infty \text{ norm: } \|x\|_\infty &= \max_i |x_i|
\end{aligned} \tag{2.25}$$

Depending on the particular vector norm used, different definitions of the matrix norm result;

$$\begin{aligned}
l_1 \text{ norm: } \|G\|_1 &= \max_j \sum_i |G_{ij}| \\
l_2 \text{ norm: } \|G\|_2 &= \bar{\sigma}(G) \\
l_\infty \text{ norm: } \|G\|_\infty &= \max_i \sum_j |G_{ij}|
\end{aligned} \tag{2.26}$$

If the 2-norm is used to measure these vector and matrix sizes (i.e. $\|x\|_2$ and $\|Ax\|_2$), then the "size" of the matrix is given by the maximum singular value, as described in Equation 2.21. In this special case, the condition number is the ratio of the largest to the smallest singular values, because of the property of singular values described in Equation 2.27 below;

$$\bar{\sigma}(A^{-1}) = \frac{1}{\underline{\sigma}(A)} = \frac{1}{\underline{\sigma}(A)} \tag{2.27}$$

$$\gamma_{2\text{-norm}} = \|A\|_2 \cdot \|A^{-1}\|_2 = \bar{\sigma} \cdot \frac{1}{\underline{\sigma}} \tag{2.28}$$

In numerical analysis, the condition number is frequently used as a measure of the susceptibility of matrix calculations to the effects of roundoff error in computations on a digital computer (Strang, 1980). In control theory, there are two consistent ways that the condition number of a matrix can be interpreted. Firstly, if the condition number is large, then this implies a large directionality in the plant. That is, if the inputs are adjusted in the direction of the right singular vector associated with the minimum singular value, then

there will be a particular (small) variation in the output variables. If, on the other hand, the input manipulation was in the direction of the maximum singular value, then the size of the output variation will be much larger. This large difference places a considerable constraint on the level of control achievable for large condition number processes (Perkins, 1989).

The second way of viewing the condition number is as a measure of how uncertainty can affect the control of the system. A simple measure can be derived from the definition of the transfer function matrix (see Equation 2.14). It will be shown in Chapter 5 that "perfect" control can be achieved by calculating plant inputs using the inverse of the plant transfer function matrix (G^{-1}).

$$u(s) = G(s)^{-1}y(s) \quad (2.29)$$

Using this relationship (Equation 2.27) it can be shown (Perkins and Wong, 1985) that if there is an error in the transfer function (δG) used to calculate the input u , then the error in the input (δu) is bounded by;

$$\frac{|\delta u|}{|u|} \leq \gamma(G) \frac{|\delta G|}{|G|} \quad (2.30)$$

Scaling

There is, however, one potential problem with employing singular value analysis - it is not scale invariant. That is, the results depend upon the physical units used within the process model. If for instance, the analysis was performed for one model where temperatures were in degrees Fahrenheit, then the same plant modelled using temperatures in degrees Celsius would give different singular value analysis results.

Depending on the reason why the analysis is being performed, different solutions have been proposed to overcome the problem of scaling. Perkins (1989) suggests that for manipulated variable constraint analysis, inputs and outputs should be scaled so as to reflect equal importance (based on the work of Skogestad and Morari, 1987a; Johnston and Barton, 1987).

However, the use of the condition number to measure the possible effect of uncertainty

requires a different scaling approach. One proposed method (Perkins and Wong, 1985) is to scale the input variables (using a diagonal matrix S_1) and the output variables (using a diagonal matrix S_2) so as to minimise the condition number (γ_{min}). This is equivalent to scaling the transfer function;

$$\gamma_{min} = \min_{S_1, S_2} \{\gamma(S_2 G S_1^{-1})\} \quad (2.31)$$

This scaling puts all condition numbers on an equal footing, allowing a direct comparison between different process designs. Unfortunately, the problem of determining this minimally scaled condition number is not simple, and although the problem can be solved analytically for the l_1 and l_∞ norms (Bauer, 1963), it has not been solved for the l_2 norm which thus requires a numerical solution.

However, it has been found that by using the process of "G-balancing" (Ngyuen et al., 1988) a close approximation to the minimum value of $\|G\|_2$ can be found. This procedure involves alternatively scaling the rows and columns of the G matrix so that all rows have the same magnitude and all columns have the same magnitude. This approach has been proven to give the minimum 2-norm condition number for the 2x2 matrix case, and numerical evidence supports its use for larger systems.

In summary, the process condition number and the minimum singular value (both open-loop measures) have been found to provide a reliable indication of achievable closed-loop performance. However, care must be taken to correctly scale the transfer function matrix, so that the results obtained are meaningful.

C₃ Splitter Open-Loop Analysis

To apply singular value analysis based, open-loop measures to predict closed-loop performance, a linearised model of the process is required. For analysis of the C₃ Splitter, a linear model was produced from the nonlinear dynamic model described earlier. This was achieved using a software package, SPEECON (Wong, 1984), which uses the SPEEDUP database to calculate a linearised state-space model, relating the selected input and output variables, from the nonlinear model equations. The SPEECON package also produces various controllability measures, including those obtained from singular value

analysis.

The aim of the C₃ Splitter control system being designed here is to control column pressure and the product impurity level at their optimal constraint values, so that the column runs at its economic optimum. Five inputs are available for this task as outlined in the introduction to this section. These potential inputs are;

- (1) Feed flow.
- (2) Vent flow.
- (3) Product propylene flow.
- (4) Bottom product flow.
- (5) Reboiler heat input.

An analysis of the existing control scheme for the C₃ Splitter indicated that the bottoms product flow should be used to control the inventory in the bottom of the column. This leaves six possible alternative pairings using the four remaining inputs and the two outputs, column pressure and product impurity level. Figure 2.8 is a plot of the *G*-balanced condition number for the six cases over the frequency range 10^{-4} rad/min to 10^{-2} rad/min. Table 2.4 explains the key used in Figure 2.8.

KEY	Inputs Selected
QREB, D	Reboiler Heat Input, Product Propylene Flow
E, D	Vent Flow, Product Propylene Flow
QREB, F	Reboiler Heat Input, Feed Flow
E, F	Vent Flow, Feed Flow
D, F	Product Propylene Flow, Feed Flow
QREB, E	Reboiler Heat Input, Vent Flow

Table 2.4 - Definition of Key in Figure 2.8

From these results, it would be expected that a control scheme employing the reboiler

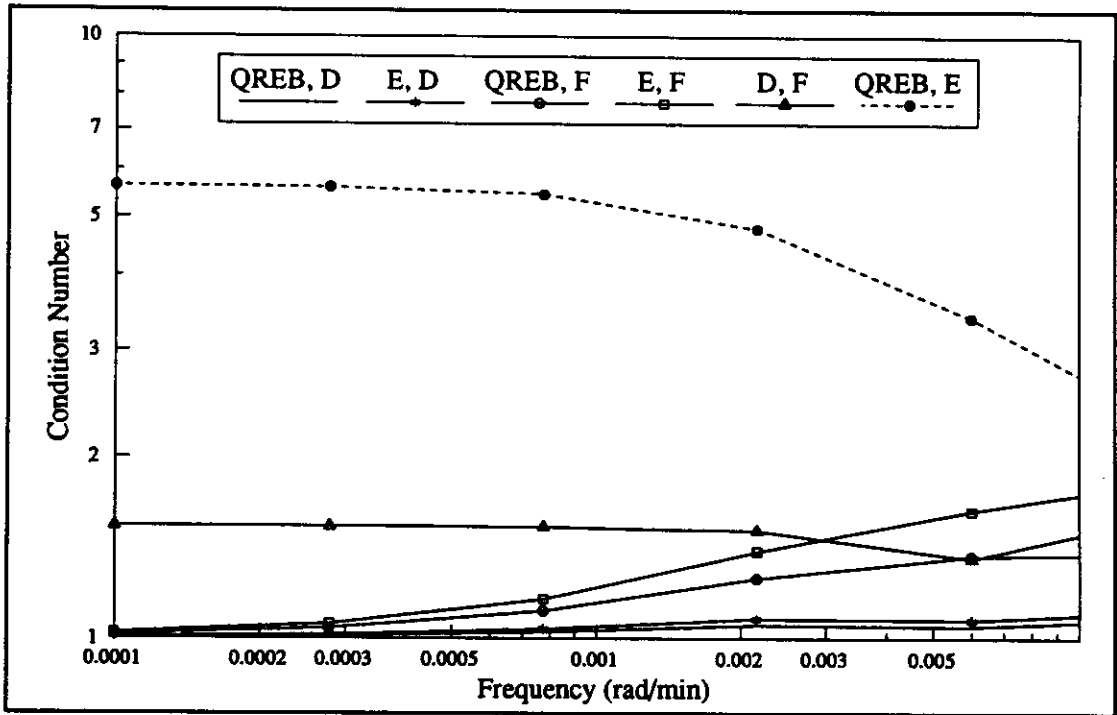


Figure 2.8 - Condition Number for Six Alternative Sets of Inputs and Outputs

heat input and the vent flow as inputs would perform significantly worse (having the highest condition number) than the other five alternatives. For example, two suitable inputs expected to achieve good closed-loop control would therefore be;

Case 1

<u>Inputs:</u>	Reboiler Heat Input	<u>Outputs:</u>	Pressure
	Propylene Product Flow		Propane in Propylene Product

with the remaining two inputs, Feed Flow and Vent Flow, being set to steady-state optimal values. By contrast, two inputs which would be expected to be less successful would be;

Case 2

<u>Inputs:</u>	Reboiler Heat Input	<u>Outputs:</u>	Pressure
	Vent Flow		Propane in Propylene Product

with the Feed Flow and Distillate Flow being again set to their steady-state optimal values.

It should be remembered that these predictions are based on open-loop measures of the inherent controllability of the plant as calculated using singular value analysis. To demonstrate the effectiveness of applying these open-loop indicators, the results of closed-loop dynamic simulations are given in the next section.

2.2.5.2 Closed-Loop Simulation Results

Closed-loop control of the C_3 Splitter is used to hold the column pressure and the propane level in the product propylene stream at their maximum allowable values. Two alternative sets of inputs were suggested above to control these two outputs (measurements). The simplest control scheme to achieve this is to use two single-input single-output controllers, where each controller has one input and one output. The use of multivariable controllers, where the controller adjusts several inputs simultaneously to control several outputs, will be discussed in Chapter 5.

To guide the selection of which input to use with which output, techniques such as the

Relative Gain Array (Bristol, 1966) are available. In general, these techniques aim to determine the input-output pairings that minimise the interaction between the loops, so that one control loop can be adjusted with the least impact on the other loops.

Relative Gain Array

The Relative Gain Array (RGA) was originally proposed by Bristol (1966) as a guide to the selection of "pairing" loops in multi-loop control schemes. Each element in the matrix array (λ_{ij}) is the ratio of two steady-state gains. The first is the open-loop (no loops closed) gain between the corresponding input and output. The second gain is between the same input and output variables, but this time with all the other loops "closed" (that is, the remaining outputs are held exactly at their setpoint values by manipulation of the other inputs, as if the other control loops were in place).

Other available measures of interaction include a dynamic extension to the RGA called the Relative Dynamic Gain Array (Tung and Edgar, 1981) and the Relative Disturbance Gain (Stanley, et al., 1985). The first of these identifies any different behaviour under dynamic operation leading to changes in loop interactions. The second measure includes the action of disturbances, so that in conjunction with the RGA, the best input-output pairings can be selected. Despite these refinements, however, the steady-state RGA remains the most popular tool in the selection of input-output pairings.

More recently, relationships have been found between the RGA and other measures of achievable control, such as the condition number (Skogestad and Morari, 1987b). For example, an exact relationship between the RGA and the condition number is given for two-input two-output processes, while bounds are given for larger systems. Interestingly enough, Bristol (1966) in his original paper, pointed out the relationship between the definition of the RGA and the condition number.

The steady-state RGA is simply calculated as;

$$\lambda_{ij} \triangleq G_{ij} \cdot G_{ji}^{-1} \quad (2.32)$$

This formula calculates each element of the RGA matrix (λ_{ij}) as the product of the (i,j) element of the steady-state transfer function matrix (G_{ij}) and the (j,i) element of the

inverse of the steady-state transfer function matrix (G_{ji}^{-1}).

Some key properties of the RGA are that any row or column adds up to one and the matrix is scale invariant. The general rule for selecting the best input-output pairing, those with the expected least control loop interaction, is to choose the pairs with RGA values closest to one. This means that the gain for the open-loop pairing is not affected by the presence of other controllers. In addition, it is suggested that pairings with large or negative numbers should be avoided (Bristol, 1966; Stephanopoulos, 1984).

For the C_3 Splitter, the RGA for the two cases is given below;

$$\begin{array}{l} \text{Case1:} \begin{array}{cc} \text{Propane} & P \\ \begin{bmatrix} 0.0017 & 0.9983 \\ 0.9983 & 0.0017 \end{bmatrix} & \begin{array}{l} Q_{REB} \\ D \end{array} \end{array} \\ \text{Case2:} \begin{array}{cc} \text{Propane} & P \\ \begin{bmatrix} 1.9526 & -0.9526 \\ -0.9526 & 1.9526 \end{bmatrix} & \begin{array}{l} Q_{REB} \\ E \end{array} \end{array} \end{array}$$

where the rows and columns of the matrices indicate the input and output variables (*Propane* - Propane in Product Propylene, *P* - Pressure, Q_{REB} - Reboiler Heat Input, *D* - Propylene Product Flow, *E* - Vent Flow).

Consequently, the multi-loop pairings chosen are;

Case 1

Reboiler Heat Input - Pressure
Propylene Product Flow - Propane in Propylene Product

Case 2

Reboiler Heat Input - Pressure
Vent Flow - Propane in Propylene Product

Note, that in agreement with the singular value analysis, the RGA predicts that the first case should be more controllable than the second case.

The controller chosen to be used with each of these pairings is the standard proportional integral (PI) controller. Although "better" controllers are available (see Chapters 5 and 6), this is by far the most commonly used controller in industry. The PI algorithm used to calculate the change in a plant input ($\Delta u(t)$) from the error between the setpoint and the measured output ($e(t)$), is given below;

$$\Delta u(t) = K_c [e(t) + \frac{1}{\tau_i} \int_0^t e(t) dt]$$

The main features of this algorithm are the simple action given by the first (proportional) term, and the zero steady-state offset due to the second (integral) term. There are two "tuning" constants to be set, the proportional gain (K_c) and the integral time (τ_i). There are many methods available for determining these "tuning constants" (Stephanopoulos, 1984), however for this example the "One-Quarter Decay Ratio" method was selected. This gives a closed-loop response to setpoint changes or disturbances which is slightly oscillatory, with the second peak being roughly one quarter the height of the first peak. This is generally considered to be a reasonably quick, yet stable response.

To calculate the tuning constants, approximate gains and time constants for the system are required. These were determined by applying small "step" changes to each of the three inputs, in turn, without controllers in place, and simulating the dynamic response using the SPEEDUP flowsheeting package. These open-loop results were then approximated as first-order processes (i.e. of the form $y(t) = Ke^{-t/\tau}$, or in the frequency domain $y(s) = K(\tau s + 1)^{-1}u(s)$), and the steady-state gain (K) and time constant (τ) for each input-output pairing calculated. Finally, the time constant of each closed-loop response was selected to be 0.6 that of the open-loop system (i.e. a faster closed-loop response). The tuning constants calculated using the one-quarter decay method are given in Table 2.5.

The performance of these two control schemes was examined via dynamic simulation, one example of which is shown in Figure 2.9. This shows the response to a "ramp-like" disturbance, decreasing the propane mole fraction in the column feed from 0.22 to 0.21 over a one hour period. It is clear that for Case 1 (that is, purity controlled by Product Propylene Flow D) the performance in the face of the disturbance is better than for Case 2 (where purity is controlled by Vent Flow E). This is in agreement with the predictions

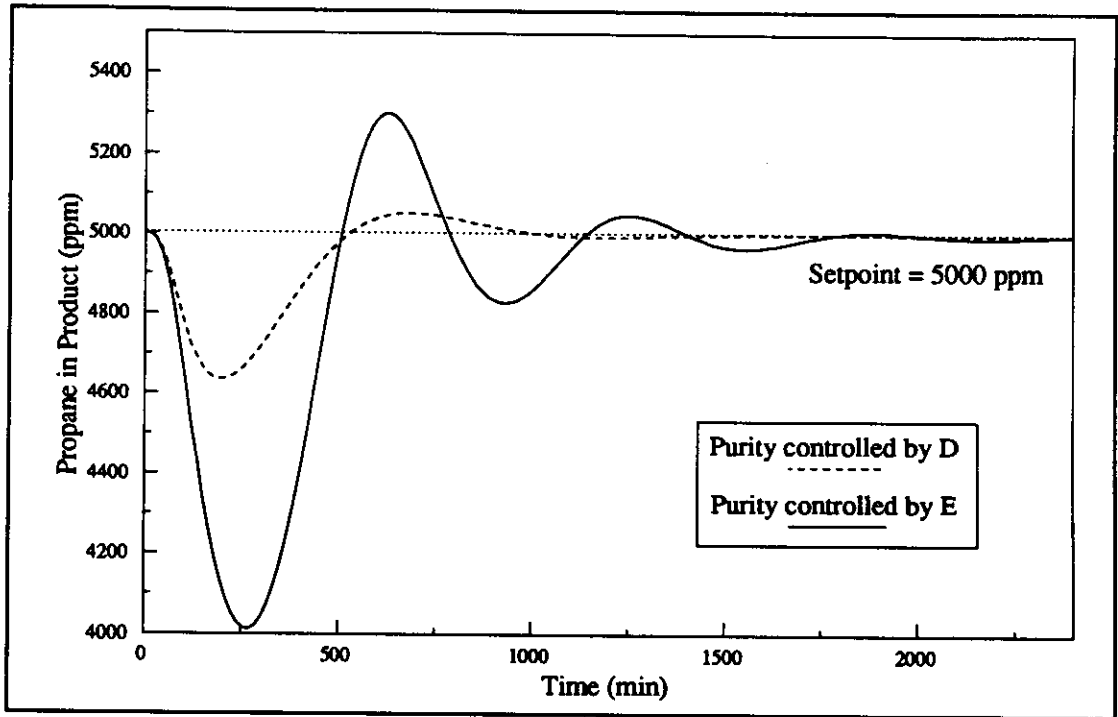


Figure 2.9 - Comparison of Control System Performance

Loop		Proportional Gain K_c	Integral Time τ_i (min)
Input	Output		
Reboiler Heat Input	Column Pressure	11.5	0.179
Product Propylene Flow	Propane in Product Propylene	32.8	23.2
Vent Flow	Propane in Product Propylene	25.6	52.1

Table 2.5 - C₃ Splitter Tuning Constants

of both the singular value analysis and the Relative Gain Array results.

Therefore, with the Case 1 pairings, an optimising control scheme could be implemented on this column, using an on-line optimiser to calculate the setpoints for the feed flow and vent flow controllers. However, there is a potential problem with the use of the reboiler heat input as a control input. As described earlier, under low wet-bulb temperature conditions, the constraint on the maximum available reboiler heat input becomes active. This would limit the usefulness of this variable for control purposes, as being constrained it could only decrease and not increase. This would require a change in the control of column pressure to another variable, say vent flow. Such a change necessitates the consideration of the trade off between the extra profit to be gained by running with maximum reboiler heat input, and having satisfactory control of the column. That is, one option is to increase the nominal reboiler heat input to a point less than the optimal value but with some capacity left for control. Another option is use the maximum (and optimal) amount of available reboiler heat input, but control the column pressure using the vent flow. If this second option gave worse control, then some caution may be required to avoided exceeding operating constraints. At present it is difficult to put the quality of control in economic terms, a technique to do this is presented in Chapters 5 and 6.

In summary, the C_3 Splitter distillation column provided the opportunity to investigate some aspects of the application of optimising control. In particular, the importance of constraints in the determination of the optimum, there being less active constraints than there were "free" variables (degrees of freedom). Those constraints which are active can be used to provide constraint control, which in conjunction with other calculated optimal values can be used to run the column optimally. However, as observed in this example, care must be taken in the design of the regulatory scheme to hold the process at the optimum operating conditions (or at least remain feasible) when the active constraint set changes. This highlights the need to be able to compare control system performance and the economics of running the column, on the same basis. This is necessary so that meaningful tradeoffs between control performance and process economics can be made.

2.3 Shortcomings With Previous Approaches to Optimising Control

The C_3 Splitter example demonstrates that it is possible to design an optimising control scheme for an industrial process using an existing approach. However, there are no guarantees that the system will actually perform close to the optimum without violating operating constraints. In addition, the design and implementation of an optimising control scheme is a non-trivial exercise that could well increase the design costs for the plant. These extra costs need to be met by the improved profitability of the design. However, as yet there are no indicators of achievable optimising control system performance which can be directly related to process economics. These issues are discussed in more detail in the following sections.

2.3.1 Allowing for Uncertainties in the System

Uncertainty in an optimising control system was discussed in Chapter 1 (see Figure 1.1). There we saw that almost every information flow in that diagram suffered to some extent from an uncertainty as to the "true" content. The net effect of these uncertainties is to place a cloud around the likely performance of the control system. A poorly known model, for example, may lead to an operating point being selected which if implemented actually causes some operational constraints to be exceeded. The various uncertainties in the information flow within an optimising control scheme will now be considered. For easy reference, Figure 1.1 is repeated as Figure 2.10 with areas of uncertainty highlighted.

The steady-state model equations are treated by the optimiser as equality constraints,

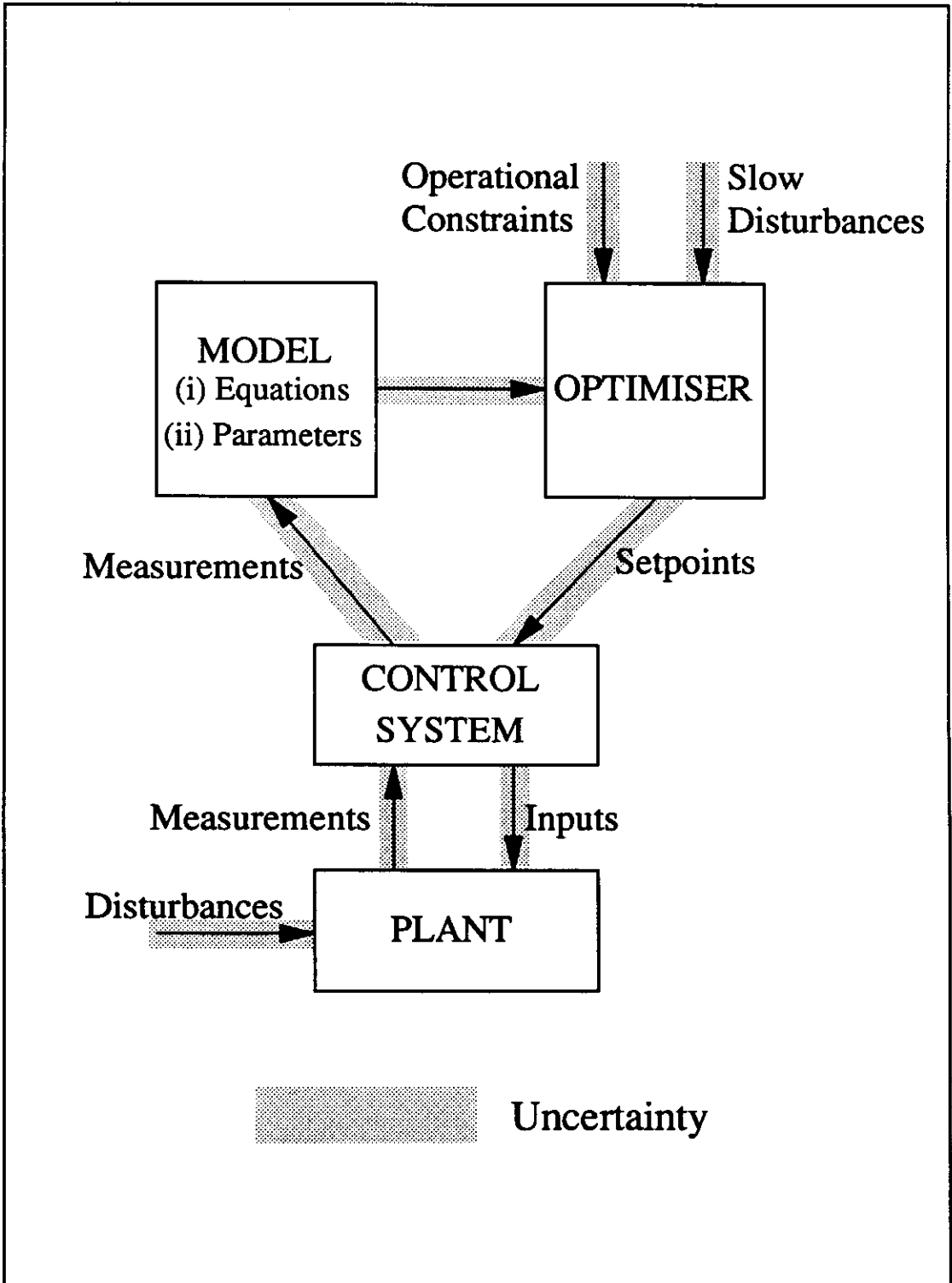


Figure 2.10 - Optimising Control Scheme Highlighting Sources of Uncertainty

which thus become an integral part of the optimal solution. As discussed in Section 2.1, there are a number of techniques available using process measurements to update the model, minimising the plant-model mismatch error, and hence the uncertainty.

It is the job of the design engineer to supply realistic values for operating constraints which should not be exceeded. If there is some uncertainty in the values supplied (e.g. $\text{Temperature} \leq 1000 \pm 10^\circ\text{C}$), then to operate safely the optimising control system must use the most conservative value in the range (i.e. $\text{Temperature} \leq 990^\circ\text{C}$). During the plant design process, an analysis needs to be made between the costs incurred in moving those constraints (e.g. the extra expense of a different material to allow higher furnace temperatures) versus the benefits (e.g. improved yield of desired products). The results of the plant optimisation are important in this analysis. For an active constraint, the Lagrange multiplier indicates the value to the optimum of changing the constraint value. If the constraint is not active, then there is no value in moving it.

The result of optimising the model equations within the operating constraints are optimal setpoints, which if implemented successfully by the regulatory control scheme will achieve optimal performance of the plant. The accuracy associated with these calculated setpoints is dependent mainly on the quality of the model and the operational constraints, as discussed above, rather than on the computational aspects of the optimisation process itself. However, it should be noted that the objective function is another potential source of uncertainty, where poorly known values for feeds and products, and/or poorly known operating costs could lead to the "wrong" optimum operating conditions.

The accuracy of the setpoints in some ways reflects the accumulation of all uncertainty in the system. No matter how well the regulatory control system achieves the setpoint values, if they are "wrong" then the plant will be running suboptimally. So there is a balance to be struck between choosing particular variables as setpoints because they can be calculated accurately, and choosing them on the basis of how well they can be controlled. As will be discussed in Chapter 3, due to the nature of the optimisation process, some variables are inherently less susceptible to uncertainty in the model equations and operational constraints. In Chapters 5 and 6, the combined effects of the quality of control achievable and the uncertainty in the process model will be analysed.

The next area where uncertainty needs to be considered is in the process measurements which are used in two parts of an optimising control scheme - in the regulatory controller and in updating the process model. The accuracy of measurements can be explicitly included in regulatory controller design as well as in the calculation of the likely performance of that controller, as will be described in Chapter 5. However, the effect of measurement accuracy on the parameter estimation used to reduce model error is a complex issue. Krishnan (1990) has proposed a technique for determining the "best" measurements to use to maximise the accuracy of the estimated model parameters. From an instrumentation point of view, the accuracy of most measurements is known reasonably well at the time of installation, however this may deteriorate with time. To counter this, data reconciliation techniques (e.g. Crowe, 1988) can be used to both improve the accuracy of measurements and to reject incorrect readings. This highlights the need to maintain plant instrumentation accuracy so as to achieve the most out of an optimising control scheme. This component of an optimising control scheme should not be disregarded in the overall analysis.

In a similar way to the effect measurement uncertainty has on the regulatory control system, any inaccuracy in achieving plant inputs can be included in the performance calculations at the design stage (i.e. limitations on the achievable control performance due to this uncertainty). In addition, if there is integral action in the controllers used, then at steady-state, input errors will not affect achieving the setpoints as the controller will adjust the inputs until the setpoints are reached.

The final source of uncertainty shown on Figure 2.10 is introduced through the disturbances which enter the plant. The effect of disturbances will be discussed in more detail in the next section.

The preceding analysis indicates that there are many avenues for uncertainty to enter the design of an optimising control scheme. Existing techniques allow for uncertainty in only two areas - changes in the active constraint set (e.g. Arkun and Stephanopoulos, 1980b) and correcting errors in the process model (e.g. Bamberger and Isermann, 1978). However, to ensure that such a control scheme is designed as well as possible, descriptions of uncertainty in other parts of the system should be included and accounted for.

2.3.2 Including Disturbance Information

The effect of disturbances on the operation of a process was introduced in Section 2.1, including the classification into slow and fast. "Slow" disturbances, such as catalyst deterioration, are handled by the steady-state optimiser whereas fast disturbances, such as variations in feed rate are controlled using a regulatory control scheme. Existing optimisation techniques can handle slow disturbances, determining the new steady-state operating point and following active constraints as they change. In addition, some techniques are available for incorporating disturbance information into regulatory controller design (Stanley et al., 1985; Skogestad and Morari, 1987a).

However, the effect of fast disturbances has not been included into any existing design procedure for optimising control schemes. It is illustrative to consider how fast disturbances can affect the operation of a plant. Disturbances at the regulatory level perturb the plant away from the desired steady-state. Thus, the point representing the desired steady-state is surrounded by a region within which the plant will operate as disturbances perturb it. This is illustrated in Figure 2.11, where the operating space for a two variable system is shown. For this simple system, constraints on the two variables are also shown, with the optimum being in the top right hand corner as indicated. However, the effect of the fast disturbances requires the operating point to be moved away from the optimum so that the constraints will not be exceeded. This approach is applicable to both open-loop and closed-loop operation. In the first case, the affect of the unattenuated disturbances will be to generate an operating region or "ball" for the outputs (measured variables). The situation is slightly more complex for the closed-loop case. Two operating regions need to be considered in this case, one for the plant inputs and another for the outputs. The size of the output "ball" may be very small if good control is achieved. However, the action of the controller will cause the inputs to move from their steady-state values, describing their own "ball" of operation. Thus, in this closed-loop case constraints on both the inputs and the outputs need to be considered.

Existing techniques for optimising control system design and analysis have not really used disturbance information effectively. In this work, the effect of disturbances on the objective function is discussed in Chapter 4, while the issue of the interaction between disturbances and constraints in determining the optimal operating point is the subject of Chapter 5.

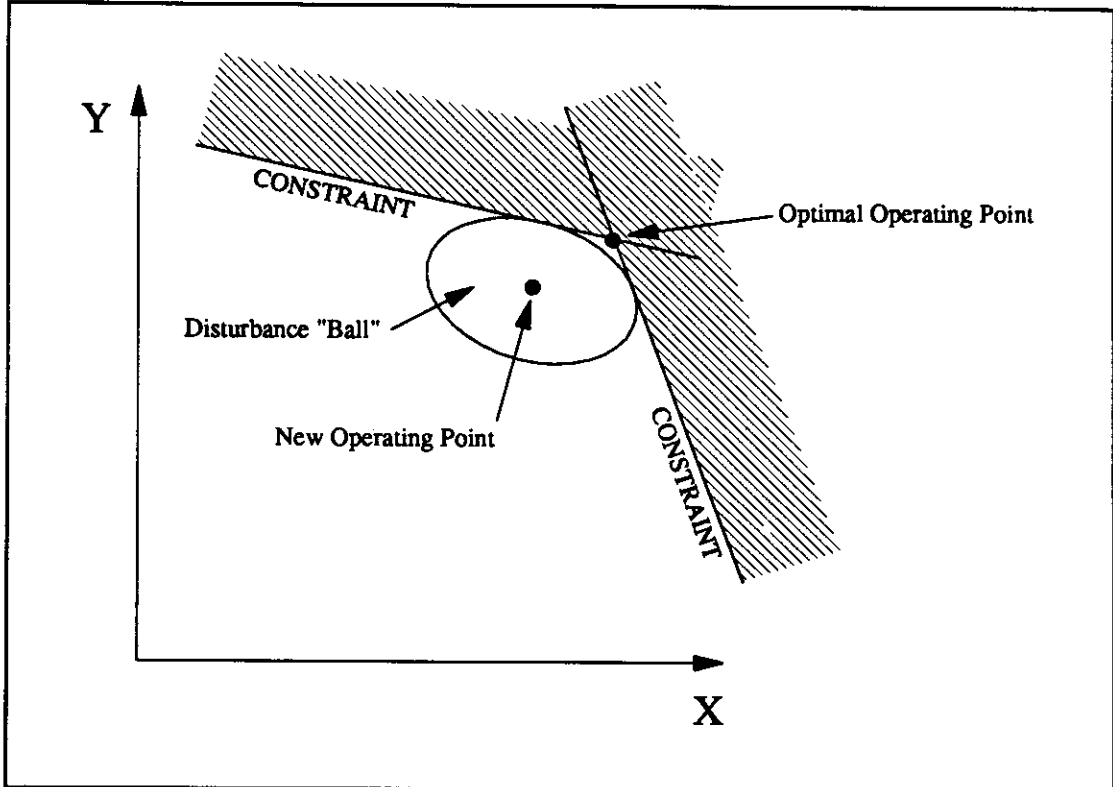


Figure 2.11 - Effect of Fast Disturbances in an Optimising Control Scheme

2.3.3 The Link Between Economics and Control

Initially, researchers responded to the need for better methods for the consideration of plant operability during the design stage through the use of open-loop indicators (such as those arising from singular value analysis, as described previously). It has been shown that these indicators (and others presented in Chapter 5) allow the controllability of a plant to be assessed at the design stage, without the need for time consuming dynamic simulations. Problems associated with plant dynamics discovered at the design stage can be rectified far more cheaply and satisfactorily than at commissioning time after the plant has been built.

However, a real problem exists for the designer when it comes to putting a dollar value on any open-loop indicator of dynamic behaviour that might be calculated. Looking back to the C₃ Splitter example, the condition number calculated for Case 2 was about five times greater than for Case 1. When a dynamic simulation of the response to a disturbance was performed with comparable controllers in place, the response of the Case 2 control system had a peak in the product impurity (propane) content about three times the size for the Case 1 control system. Unfortunately, despite this strong link between open-loop indicators and achievable performance, existing design techniques cannot attribute a cost to the quality of control performance.

To allow the direct comparison between plants to be made on an economic basis, the overall design process needs to be taken beyond steady-state optimisation, to calculate the achievable (dynamic) optimal performance. As will be developed in later chapters, existing optimising control theory can be extended to allow the achievable optimal operating point to be determined. The value of the objective function at this point is now an indicator of the optimum that can actually be realised for a given design. The algorithm to determine this operating point, finds the optimal input and output operating space "ball sizes" (by calculating the optimal amount of control required), so as to minimise the total cost of moving away from the steady-state optimal operating point. In addition, descriptions of the uncertainty applying to both the steady-state and dynamic aspects of the model are incorporated, so that it can be guaranteed that the plant will never provide worse performance than the calculated optimum.

Looking back to the schematic diagram representing optimising control presented in

Figure 2.10, existing techniques do not allow an optimising control system to be designed which has a guaranteed performance despite the uncertainties in the system. The following chapters further develop various aspects relevant to the incorporation of uncertainty analysis within optimising control schemes, culminating in the presentation of an "Optimal Operation Algorithm", which calculates an optimal performance indicator allowing a direct comparison to be made between different designs on an economic basis.

CHAPTER 3

UNCERTAINTY IN STEADY-STATE OPTIMISATION

Process plants are normally operated so that they are as close to steady-state as possible, that is, the measurements which describe the operation of the plant do not change with time. These measurements are kept constant through the use of closed-loop controllers making adjustments to the plant (e.g. changing feed and product flows). Hence the setpoints of the controllers, the desired values of the measurements, determine the plant's operating point and hence its financial performance. How well the setpoints are chosen can be the difference between running profitably and losing money.

The previous chapter described various techniques to determine setpoint values to run the plant optimally. Whether or not the plant actually achieves its optimum depends on a number of factors. Firstly, the sensitivity of the optimum to the setpoints is important. Depending on the nature of the optimum, the value of the objective function may or may not vary greatly with the actual measurement value achieved. Figure 3.1 shows two objective functions. In the first, an error in the setpoint will not cause a large change in the objective function, while in the second, the cost of the error is quite large. Setpoints chosen with the former characteristic will be more robust to inaccurately calculated setpoints and poor control.

The next criterion is the likely accuracy of the calculated optimal values. The setpoints are derived from an optimisation of the model of the plant, and as discussed earlier, since there is always some error in the model the setpoints will always be inaccurate to some extent. Clearly setpoints which are less sensitive to model errors are favourable, for even if a control system keeps measurements close to the setpoint values, if the setpoints are incorrect, then sub-optimal operation will result.

Finally, if the control system cannot keep the measurements at the optimal setpoints, then the plant will not be operating at its full potential. Consequently, the standard control system design problem of selecting setpoints (measurements) to achieve "good control" is an important part of optimising control system design. This is in addition to the requirements already imposed by optimising control system design on setpoint selection

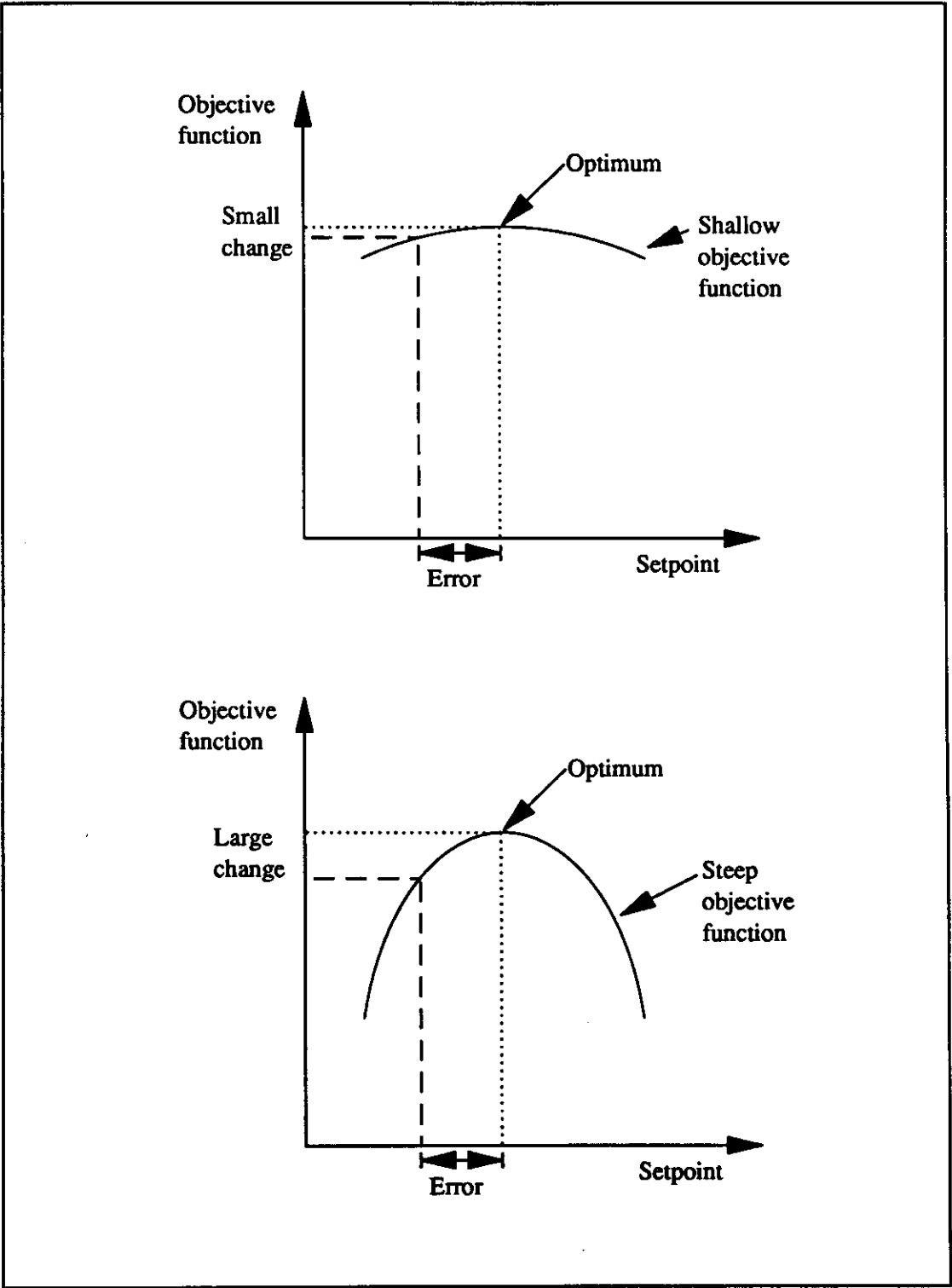


Figure 3.1 - Differing Objective Function Sensitivities

described above (i.e. setpoints to which the objective function is not excessively sensitive yet are inherently accurate). Traditionally, the choice of setpoints has been more dependent on the measurements the designer thought were important, rather than their usefulness in achieving optimal performance or tight process control. Lee and Morari (1991) present a brief review of existing methods for measurement selection and a new technique based on modern control theory. They commented that although the importance of measurement selection has been recognised, "there is an apparent lack of systematic selection criteria". The method they proposed ensures that setpoints are selected so that the designer's control performance criteria are met despite modelling uncertainties in the system.

Uncertainty within an optimising control scheme can cause the objective function to be less than the real steady-state optimal value. As discussed previously, this can be due to the use of setpoints to which the objective function is highly sensitive, inaccuracy in the calculated values used as setpoints, or the inability of the control system to achieve the calculated operating point. These factors highlight the importance of choosing the "best" setpoints to achieve the best performance from an optimising control system.

Two techniques for setpoint selection within optimising control system design are described in this chapter. The first, using assumptions on the likely errors in the system, is appealing due to its simplicity, but of limited usefulness in practice. The second, and more useful method, extends an existing optimisation sensitivity technique, to determine the inherent accuracy of calculated optimal variables in the presence of modelling uncertainty. These techniques are presented in Sections 3.1 and 3.2 respectively.

3.1 The Effect of Equal Relative Errors in the System

If there are errors in the setpoints used to control a process plant, then the plant will operate sub-optimally. As discussed above, it is possible that some measurements when used as setpoints will be more robust to error than others. To quantitatively investigate this, we first need to look at the objective function in terms of the chosen setpoints. This will directly show how setpoint errors can affect the objective function.

In a general optimisation problem, the objective function may involve variables found anywhere in the system (e.g. measurements, plant inputs, unmeasured model variables).

For example, a profit based objective function may be written as a function of feed and product flows. However, the setpoints used to control the plant must be measurements (e.g. pressures and temperatures) or variables calculated from measurements (so called secondary measurements, e.g. temperature used to infer composition in a distillation column). As will be discussed in the next section, it is possible using the model equations to rewrite an objective function which includes non-setpoint variables, in terms of the setpoints only. It should be noted that the control setpoints are in fact the independent variables of the system, "free" to be manipulated by the optimiser to achieve optimal performance.

This is illustrated in the following example, where the objective function (Φ) is written as a function of the feed (F) and product (P) flow rates, and the product concentration (C) - see Equation 3.1. The steady-state model equations provide a relationship between these three variables (Equations 3.2). Finally, there is an inequality constraint on the concentration (Equation 3.3).

$$\Phi = 10PC - 5F \quad (3.1)$$

$$F = P \quad (3.2)$$

$$P = 10C + 2$$

$$C \leq 1 \quad (3.3)$$

The optimum (maximum Φ) for this system is given by;

$$\Phi = 60$$

$$F = 12$$

$$P = 12$$

$$C = 1$$

All three variables in this system (F , P and C) are easily measured and so are potential setpoints. Initially, we will investigate the use of the product concentration (C) as a setpoint. Using Equations 3.2, the objective function can be expressed in terms of the variable C .

$$\begin{aligned}\Phi &= 10(10C + 2)C - 5(10C + 2) \\ &= 100C^2 - 30C - 10\end{aligned}\quad (3.4)$$

The next step in this analysis is to consider a linearised version of the objective function. The use of linear process models was discussed in Chapter 2, while details of the linearisation process are given later in this chapter. Basically, first derivative information is used to give an approximation of the nonlinear behaviour around the point of linearisation, here the optimum. The linearised objective function is given by;

$$\Phi_{LIN} = 170C - 110 \quad (3.5)$$

Taking a more general approach, the objective function can always be expressed as a linear function of a number of setpoints (S_i),

$$\Phi = \sum_i k_i S_i + c \quad (3.6)$$

where c is a constant.

Now, let the optimum be Φ^* at the setpoint values S^* (Note that S^* is a vector - the notation of **bold text** will be used to indicate vectors, while S_i is the i th element of S);

$$\Phi^* = \sum_i k_i S_i^* + c \quad (3.7)$$

If a fractional change δ_i is applied to each setpoint S_i , the change in the objective function $\Delta\Phi$ is given by;

$$\begin{aligned}\Delta\Phi &= \left(\sum_i (1 + \delta_i) k_i S_i + c \right) - \Phi^* \\ &= \left(\sum_i (1 + \delta_i) k_i S_i + c \right) - \left(\sum_i k_i S_i^* + c \right) \\ &= \sum_i \delta_i k_i S_i\end{aligned}\quad (3.8)$$

Now if the fractional change applied to each setpoint is the same (δ), then the change in the objective function is given by;

$$\Delta \Phi = \delta \sum_i k_i S_i \quad (3.9)$$

Errors in the setpoint values can be viewed as changes from their optimal values. Consequently, if we were to assume that there was the same relative error in each setpoint, then Equation 3.9 could be used as a method for calculating the sensitivity of the objective function to equal relative errors in the setpoints. The indicator of objective function sensitivity to setpoint errors, the linear sensitivity indicator (I) is, therefore;

$$I = \sum_i k_i S_i \quad (3.10)$$

It would be expected that a group of setpoints with a low I value would be less sensitive to errors in the setpoints than a group with a higher I value.

Applying this to the example described earlier, the indicator value for C can be determined from Equation 3.5 and is given by;

$$I = \sum_i k_i S_i = k_1 S_1 = 170C = 170$$

The use of C as a setpoint therefore has a linear sensitivity indicator of 170. An alternative setpoint is the product flow rate (P). Rewriting the objective function in terms of P only, linearising and calculating the linear sensitivity indicator gives;

$$\begin{aligned} \Phi &= P^2 - 7P \\ \Phi_{LN} &= 17P - 144 \\ I &= \sum_i k_i S_i \\ &= 17P = 17 \times 12 = 204 \end{aligned} \quad (3.11)$$

The linear sensitivity indicator for product flow rate (P) is therefore 204. This indicates that the objective function should be more sensitive to errors with P as a setpoint than if the concentration (C) was used as a setpoint. A simple check on this gives the following results for the percentage change in the objective function due to a 1% error (change) in each setpoint.

Nominal Case

$\Phi = 60$

$F = 12$

$P = 12$

$C = 1$

Concentration Setpoint

$C : 1 \rightarrow 0.99 \quad (1\% \text{ decrease})$

$\Phi = 58.31$

$F = 11.9$

$P = 11.9$

$C = 0.99$

$\Delta\Phi = -2.8\%$

Product Flow Setpoint

$P : 12 \rightarrow 11.88 \quad (1\% \text{ decrease})$

$\Phi = 57.94$

$F = 11.88$

$P = 11.88$

$C = 0.988$

$\Delta\Phi = -3.4\%$

These results confirm the sensitivity predictions that an error in the concentration as a setpoint causes a smaller change in the objective function than an equal relative error in the product flow.

This analysis can also be applied to more complex problems. In Section 3.2.1 a reactor example is detailed which has three measurements (F , P and T) from which two must be chosen as setpoints. For the three possible choices of pairs of setpoints, linearised objective functions written in terms of only the two candidate setpoints are presented as Equations 3.73, 3.74 and 3.75.

$$\text{Case 1: } (P, T) \quad \Phi_{LN} = -2000(39.2P - 0.3687T + 19.25)$$

$$\text{Case 2: } (F, T) \quad \Phi_{LN} = -2000F$$

$$\text{Case 3: } (F, P) \quad \Phi_{LN} = -2000F$$

Calculating the linear sensitivity indicator for each case at the nonlinear optimum (rather than linearly reduced optimum - see Table 3.6), gives the results shown in Table 3.1.

Here the application of the linear sensitivity algorithm is quite illustrative, in that for cases 2 and 3, since the objective function depends only on F , the sensitivity is the same for both cases and independent of errors in achieving the setpoint value of the second independent variable (T and P respectively). Looking at the result for case 1 suggests that if the relative error affecting P and T was the same as might occur in cases 2 and 3, then

Case	Candidate Setpoints	I
1	P, T	-201,304
2	F, T	-239,800
3	F, P	-239,800

Table 3.1 - Linear Sensitivity Indicator Values for the Reactor Example

the effect on the objective function would be less for case 1. This suggests that P and T would make the best setpoints.

This analysis shows that if the same error is assumed to occur for each setpoint, then an indicator of the sensitivity of the objective function to errors in that group of setpoints, independent of the actual value of the error, can be easily calculated. This allows a set of potential setpoints to be given an economic ranking. The "best" (i.e. the least sensitive to error) set can then be used in an optimising control scheme. Note that a similar analysis can be performed if the errors are not all the same, however, in that situation the sensitivity indicator is not independent of the size of the errors.

Setpoint errors can come from two sources. Firstly, during the optimisation process errors in the model parameters, constraints or objective function could lead to the wrong optimal values being calculated. Alternatively, since setpoints are the desired values for plant measurements, any difference between the setpoint and the measurement actually achieved can be viewed as an "error". This could be due to poor control, where the measured value simply does not reach the setpoint, or it could be due to errors in the measurements themselves. Measurement error would mean that despite the perfect operation of the control system (i.e. the measurement is equal to the setpoint) the actual value of the variable in the process would be wrong (e.g. setpoint = 100°C; measurement = 100°C; actual temperature in the reactor = 95°C).

The advantages of this linear equal relative error sensitivity analysis are that it is simple to calculate, and also, by analysing the components of the linearised objective function the potential setpoints to which the objective function will be most sensitive will have the

largest coefficient. Clearly there are also limitations with this method. The use of a linear objective function means that only an approximation to its sensitivity will be calculated. However, more important is the assumption that all the relative errors will be the same. This is extremely unlikely to occur in practice, as not only will the magnitude of the errors affecting the setpoints be different, but they may not even have the same sign. If we consider the case of measurement error, in a particular process a temperature may be accurately known (the measurement being say 0.1% too high), whereas a less accurately known concentration may be 3% low. This type of situation cannot be analysed using this simple approach. Therefore, a more rigorous method is required to examine the effect of uncertainty in steady-state optimisation. In particular, an indicator of how accurately setpoints are likely to be calculated, despite errors in model parameters, constraints and the objective function would be useful. Those setpoints least affected by uncertainty would be good candidates for use in an optimising control scheme. Such a technique is presented in the next section.

3.2 The Use of Conservative Steady-State Uncertainty Descriptions

In this chapter, methods to improve the actual implementation of an optimising control scheme by allowing operation closer to the true steady-state optimum are being investigated. In the previous section, a simple technique was presented to analyse the effect of "errors" in setpoints (due to the model, poor control or measurement inaccuracy), so as to enable those variables which least effect the objective function to be determined. However, the simplifying assumptions employed were found to be unrealistic, poorly describing the likely uncertainty in the system.

The key areas of uncertainty which can affect the calculation of the steady-state optimum, are inaccurate parameters within the model equations (which may be regarded as equality constraints), operational inequality constraints and objective function. It is possible that the error could also be in the structure of the equations (e.g. incorrect use of variables due to the incomplete modelling of species and phenomena) rather than in the parameters within the equations (e.g. reaction rate constants, heat transfer coefficients). However, in this analysis any structural errors will be ascribed to errors in the model parameters.

If a particular parameter is recognised as being poorly known, then it is possible to analyse the variation in the optimum due to the variation in that parameter. One method

of doing this is to actually perform the optimisation of the system for different values of parameter over its expected range (e.g. perform the optimisation for five values of a poorly known heat transfer coefficient - say 500, 510, 520, 530, 540 W/m²K). This procedure would provide a very exact description of the variation in "optimal" variable values due to uncertainty in the particular parameter. If this was the only uncertainty in the system, then variables which showed little variation due to this parameter could be selected for use as setpoints in an optimising control system.

However, when modelling complex process plants, the uncertainty in the model cannot usually be assigned to particular parameters in the equations. As will be discussed in Chapter 5, various techniques are available for describing model uncertainty with different amounts of conservatism. The use of a conservative uncertainty description means that all possible variations in the model have been included. Unfortunately, conservative uncertainty descriptions generally also include variations in the model which the plant may never actually exhibit. However, by determining the optimal operating point for all the models included in the conservative uncertainty description, the designer can be sure that the "worst case" has been allowed for in the design.

The following sections describe a technique for choosing setpoints in an optimising control system, so that for the "worst case" parameter variation within the model, the operational constraints and the objective function, the sensitivity of the objective function to setpoint errors will be minimised. To do this, an existing method for analysing the effect of parameter variation within optimisation problems is first described. This theory is then extended through the application of singular value analysis to predict the effect of "worst case" parameter variation on candidate setpoint variables.

3.2.1 Uncertainty Analysis in Nonlinear Programming

The optimal steady-state operating point for a process plant can be determined by solving an optimisation problem to minimise (or maximise) an objective function (Φ), within specified operational (inequality) constraints (g), subject to the relationships between variables described using a model (equality constraints (h)) of the process. In the most general form of this problem, the objective function and the model and constraint equations are all nonlinear functions of the variables in the problem (x).

$$\text{Minimise } \Phi(x) \quad (3.12)$$

$$\text{Subject to } g_i(x) \geq 0 \quad i = 1, \dots, m \quad (3.13)$$

$$h_j(x) = 0 \quad j = 1, 2, \dots, p \quad (3.14)$$

$$\text{where } x \in \mathbb{R}^n \quad \text{i.e. } x = (x_1, x_2, \dots, x_n)$$

Many techniques are available for solving this optimisation problem, such as Lagrange multipliers, quadratic programming and penalty function based methods. A review of such optimisation techniques is beyond the scope of this work and is available elsewhere (e.g. Edgar and Himmelblau, 1988; Gill et al., 1981). However, a detailed description of the method of Lagrange multipliers will be presented, for although it is not a widely used method for actually performing optimisation, the underlying theoretical construction will be used in the following uncertainty analysis.

The method of Lagrange multipliers involves the construction of a single function, the Lagrangian (L), into which is incorporated the objective function and the equality and inequality constraints. This function also introduces new variables, the Lagrange multipliers, one being associated with each inequality constraint (μ_i) and each equality constraint (ν_j). The Lagrangian is therefore a function of the model variables (x) and the Lagrange multipliers (μ, ν):

$$L(x, \mu, \nu) = \Phi(x) - \sum_{i=1}^m \mu_i g_i(x) + \sum_{j=1}^p \nu_j g_j(x) \quad (3.15)$$

It is important for this construction that the optimisation is written as a minimisation and the inequality constraints are written in the form of Equation 3.13 (i.e. $g(x) \geq 0$).

Maximisation problems can be changed into minimisation problems by taking the negative of the objective function. Similarly, inequality constraints can be changed into the form of Equation 3.13 by simple manipulation.

The optimum for the problem given by Equations 3.12, 3.13 and 3.14 is defined by the "Kuhn-Tucker" conditions for optimality. A full description of these can be found elsewhere (e.g. Edgar and Himmelblau, 1988), while a summary is presented below. The

use of the Lagrangian turns a difficult constrained nonlinear optimisation problem into a more easily solved unconstrained problem (i.e. a problem where the aim is to minimise an objective function not subject to any constraints). The minimum of the Lagrangian (L^* at x^*, μ^*, v^*) is in fact the minimum of the original problem (Φ^* at x^*). In addition, as will be discussed later, the Lagrange multipliers (μ, v) are important descriptors of the behaviour of the optimum.

Kuhn-Tucker Conditions for Optimality

The following are the *necessary conditions* for x^* to be a local minimum of $\Phi(x)$ subject to the inequality constraints ($g(x) \geq 0$) and equality constraints ($h(x) = 0$);

- 1) $\Phi(x)$, $g_i(x)$ and $h_j(x)$ are all twice differentiable at x^* .
- 2) The "second-order constraint qualification" holds. This requires that the gradients (with respect to x) of the active inequality constraints and all the equality constraints are linearly independent at x^* .
- 3) The Lagrange multipliers exist.
- 4) The constraints are all satisfied.

$$g_i(x^*) \geq 0$$

$$h_j(x^*) = 0$$

- 5) The Lagrange multipliers for all the inequality constraints are non-negative.

$$\mu_i^* \geq 0$$

- 6) The "strict complementary slackness" condition holds. That is, the Lagrange multipliers associated with the inactive constraints are equal to zero.

$$\mu_i^* g_j(x^*) = 0$$

- 7) The Lagrangian function is at a stationary point (i.e. the first derivative with respect to x is zero, this means that at x^* , L is neither increasing nor decreasing).

$$\nabla L_x(x^*, \mu^*, v^*) = 0.$$

- 8) A further condition, which is not actually a Kuhn-Tucker condition, is needed to guarantee that x^* is in fact a minimum (and not a maximum or a "saddle" point);

The Hessian of the Lagrangian, projected into the null-space of the active inequality constraints and equality constraints is positive definite.

For a general vector ξ such that;

$$\xi^T \nabla g_i(x^*) = 0 \quad (\text{for } i = \text{an active constraint})$$

$$\xi^T \nabla h_j(x^*) = 0 \quad (\text{for all } j)$$

$$\xi^T \nabla g_k(x^*) \geq 0 \quad (\text{for } k = \text{an inactive constraint})$$

this condition requires that

$$\xi^T \nabla^2 [L_x(x^*, \mu^*, v^*)] \xi > 0.$$

This last condition mathematically expresses the requirement that for the allowable movements in the model variables (x) (i.e. those movements satisfying all the constraints), the Lagrangian behaves like a minimum. Rigorous proofs of the validity of these conditions for defining a minimum are available elsewhere (e.g. Edgar and Himmelblau, 1988; Fiacco, 1983).

In order to calculate the optimum, these conditions can be summarised in a few important mathematical statements;

$$\nabla_x L(x, \mu, v) \equiv \nabla_x \Phi(x) - \sum_{i=1}^m \mu_i \nabla_{x_i} g_i(x) + \sum_{j=1}^p v_j \nabla_{x_j} h_j(x) = 0 \quad (3.16)$$

$$k = 1, \dots, n$$

$$\mu_i g_i(x) = 0, \quad i = 1, \dots, m \quad (3.17)$$

$$h_j(x) = 0, \quad j = 1, \dots, p \quad (3.18)$$

These $(n+m+p)$ equations are expressed in terms of $(n+m+p)$ unknown variables (x, μ, v) . The simultaneous solution of these equations yields the optimum, given that the other conditions (1-8) are also satisfied. As mentioned previously, this technique for finding the optimum is not normally used today, as it is computationally inefficient. However, other methods when implemented as computer software sometimes use Lagrange multiplier theory to provide termination criteria (i.e. checking whether they have found the optimum).

The Lagrange multipliers (μ, v) introduced in this solution procedure are more than just artefacts of the method, they can provide valuable information about the behaviour of the

optimum. In economic analysis, the Lagrange multipliers for the active inequality constraints are known as the "shadow prices" as they indicate how much the objective function can be changed by altering the constraint values (i.e. how much the objective function can be improved by making inequality constraints more or less strict). This was described in Section 2.2.4 for the C₃ Splitter, where increasing the maximum available reboiler heat input was shown to improve the objective function by an amount given by the Lagrange multiplier. The Lagrange multipliers associated with the equality constraints do not usually have the same sort of useful application.

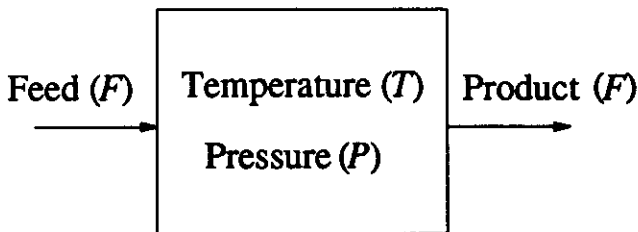
The relationship between the inequality constraint Lagrange multipliers (μ_i), the objective function (Φ) and an active constraint (g_i) can be written as;

$$\mu_i = \frac{\partial \Phi}{\partial g_i} \quad (3.19)$$

To demonstrate more fully the principles of the Lagrange multiplier approach and the Kuhn-Tucker optimality criteria, a simple example is developed below.

Reactor Example

The example system used is a simple reactor, as shown below. At steady-state, the feed flow rate (F) is the same as the product flow rate. The objective function is based on this flow. An equality constraint describes the reaction rate relationship between the feed/product flow rate (F), the reactor temperature (T) and the reactor pressure (P). The inequality constraint is a vapour pressure limitation involving the reactor pressure and temperature.



$$[F] = \text{tonne/day} \quad [P] = \text{MPa} \quad [T] = ^\circ\text{C}$$

The formal statement of the optimisation problem is;

$$\text{Minimise } \Phi(x) = -2000F \quad (3.20)$$

$$\text{Subject to } g_1(x) = -P - 0.075T^2 + 0.793T - 17.9 \geq 0 \quad (3.21)$$

$$h_1(x) = F - 12750\left(\frac{P}{T + 273}\right) = 0 \quad (3.22)$$

$$\text{where } x = (F, P, T)$$

To solve this optimisation problem, the Lagrangian (L) is defined as;

$$\begin{aligned} L = & -2000F - \mu(-P - 0.075T^2 + 0.793T - 17.9) \\ & + \nu\left(F - 12750\left(\frac{P}{T + 273}\right)\right) \end{aligned} \quad (3.23)$$

For this problem, Equations 3.16, 3.17 and 3.18 become;

$$\begin{aligned} \nabla_F L &= -2000 + \nu(1) = 0 \\ \nabla_P L &= -\mu(-1) + \nu\left(\frac{-12750}{T + 273}\right) = 0 \\ \nabla_T L &= -\mu(-0.015T + 0.793) + \nu\frac{12750P}{(T + 273)^2} = 0 \\ \mu g(x) &= \mu(-P - 0.075T^2 + 0.793T - 17.9) = 0 \\ h(x) &= F - 12750\left(\frac{P}{T + 273}\right) = 0 \end{aligned} \quad (3.24)$$

Solving these five equations simultaneously yields the minimum;

$$\begin{aligned} \Phi^* &= 239,800 \\ F^* &= 119.9 \\ P^* &= 3.059 \\ T^* &= 52.24 \\ \mu^* &= 78404 \\ \nu^* &= 2000 \end{aligned} \quad (3.25)$$

To prove that this is indeed the minimum, the eight optimality conditions are checked;

1) By inspection $\Phi(x)$, $g(x)$ and $h(x)$ are all twice differentiable at x^* .

2) The gradients of $g(x)$ and $h(x)$ are;

$$\nabla_x g(x) = \begin{pmatrix} 0 \\ -1 \\ 0.0094 \end{pmatrix}^T \quad \nabla_x h(x) = \begin{pmatrix} 1 \\ -39.2 \\ 0.3687 \end{pmatrix}^T \quad (3.26)$$

These two vectors are linearly independent.

3) The Lagrange multipliers exist;

$$\mu = 78404, \nu = 2000$$

4) All the constraints are satisfied;

$$g(x^*) = 0, h(x^*) = 0$$

5) The Lagrange multiplier for the active inequality constraint is non-negative;

$$\mu^* = 78404 \geq 0$$

6) "Strict Complementary Slackness" holds;

$$\mu^* g(x^*) = 78404 \times 0 = 0$$

7) By calculation the Lagrangian function is at a stationary point (i.e. the solution to Equation 3.24 is a stationary point).

8) First the vector ξ (the null-space) is calculated by finding a solution to the following equation;

$$\xi^T [\nabla g(x^*)^T, \nabla h(x^*)^T] = (0, 0)$$

$$\xi^T \begin{bmatrix} 0 & 1 \\ -1 & -39.2 \\ 0.0094 & 0.3687 \end{bmatrix} = (0, 0) \quad (3.27)$$

$$\xi = \begin{pmatrix} -0.5934 \\ -0.0076 \\ 0.8049 \end{pmatrix}$$

The Hessian of the Lagrangian ($\nabla^2 L_x$) is the matrix of second derivatives of the Lagrangian function (L) with respect to each of the model variables (x). That is,

$$\text{the } (i,j)\text{th element of the matrix } \nabla^2 L_x(x, \mu, v) = \frac{\partial}{\partial x_j} \left(\frac{\partial}{\partial x_i} (L(x, \mu, v)) \right) \quad (3.28)$$

Calculating this Hessian and applying the final check of Condition 8 gives;

$$\begin{aligned} \xi^T \nabla^2 [L_x(x^*, \mu^*, v^*)] \xi &= \begin{pmatrix} -0.5934 \\ -0.0076 \\ 0.8049 \end{pmatrix}^T \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 241.1 \\ 0 & 241.1 & 1171.5 \end{pmatrix} \begin{pmatrix} -0.5934 \\ -0.0076 \\ 0.8049 \end{pmatrix} \\ &= 756.0 > 0 \end{aligned} \quad (3.29)$$

The optimum presented thus fulfils all eight conditions and is therefore a local minimum of the problem.

Finally, the application of the calculated Lagrange multipliers is demonstrated.

Rearranging Equation 3.19 and taking a first-order Taylor series expansion gives;

$$\Delta \Phi \approx \mu_i \Delta g_i \quad (3.30)$$

Increasing the value of the inequality constraint (Equation 3.21) by 0.1, is equivalent to changing the equation so that "17.9" becomes "18.0". The actual change in the objective function (Actual $\Delta \Phi$) can be calculated by performing the optimisation again with the new constraint, while the predicted change in the objective function can be calculated as;

$$\begin{aligned} \Delta \Phi &\approx 78404 \times 0.1 \\ &\approx 7840.4 \end{aligned}$$

$$\text{Actual } \Delta \Phi = 7840.1$$

Some relevant features of this example are the nonlinearity of the equation set describing this system and the fact that there are less constraints (active inequality and equality) than variables. This ensures that the optimum is not simply defined by the intersection of equality constraints and active inequality constraints (like a linear system), but involves the curvature of the nonlinear functions. This will be important for applications of this simple example later in the chapter. Other aspects of nonlinear behaviour will be discussed in Chapter 4.

Uncertainty Analysis

The solution of large systems of linear equations is common in engineering, especially in structural analysis where the solution of tens of thousands of simultaneous linear equations is not uncommon. A standard part of this analysis is to check the accuracy of the solution, including the sensitivity to poorly known parameters in the system. It is only more recently that a similar procedure has been adopted for the solution of nonlinear programming (optimisation) problems. The key to this analysis is the calculation of how the solution to an optimisation problem changes due to variation in parameters (both constants and coefficients for variables) in the system. The change will usually be in both the objective function and in the value of the variables at the optimum, while the uncertain parameters may be in the objective function, the inequality constraints and in the equality constraints (i.e. the model equations).

A technique to perform this sensitivity analysis without requiring the reoptimisation of the system for each possible value of the uncertain parameters was developed by Fiacco (1983). Although the equations used in this method look complicated, the derivation and application is quite straightforward. The theoretical development is based on the Lagrange multiplier method for the solution of the optimisation problem. The basic principles are outlined below while a complete derivation of the sensitivity method is given in Fiacco (1983).

The parameters, which as pointed out above may appear in the objective function, the inequality constraints and the equality constraints are represented as a vector (ϵ). The objective function and the constraints thus become functions of the model variables (x) and the parameters (ϵ). The nonlinear programming problem thus becomes;

$$\begin{aligned}
 &\text{Minimise } \Phi(x, \epsilon) \\
 &\text{Subject to } \begin{aligned} &g_i(x, \epsilon) \geq 0 \quad i = 1, \dots, m \\ &h_j(x, \epsilon) = 0 \quad j = 1, \dots, p \end{aligned} \\
 &\text{where } x \in \mathcal{R}^n \quad \text{i.e. } x = (x_1, \dots, x_n) \\
 &\quad \epsilon \in \mathcal{R}^k \quad \text{i.e. } \epsilon = (\epsilon_1, \dots, \epsilon_k)
 \end{aligned} \tag{3.31}$$

Similarly, the Lagrangian also becomes a function of the parameters (ϵ);

$$L(x, \mu, v, \epsilon) = \Phi(x, \epsilon) - \sum_{i=1}^m \mu_i g_i(x, \epsilon) + \sum_{j=1}^p v_j h_j(x, \epsilon) \tag{3.32}$$

The solution to the optimisation problem (see Equations 3.16, 3.17, 3.18) is now a function of the parameters (ϵ) as well. Taking the total derivative of the equations defining the optimum (Equations 3.16, 3.17 and 3.18) with respect to the parameters (ϵ) and setting this value to zero (i.e. so they still define the optimum despite changes in ϵ), gives after rearrangement (see Fiacco 1983 for details) the following equation;

$$M(\epsilon) \nabla_{\epsilon} y(\epsilon) = N(\epsilon) \quad (3.33)$$

where

$$M = \begin{bmatrix} \nabla^2 L & \dots - \nabla_x g_i^T \dots & \dots \nabla_x h_j^T \dots \\ \vdots & \ddots & \\ \mu_i \nabla_x g_i & g_i & 0 \\ \vdots & & \\ \nabla_x h_j & 0 & 0 \\ \vdots & & \end{bmatrix} \quad (3.34)$$

$$y(\epsilon) = [x(\epsilon), \mu(\epsilon), v(\epsilon)]^T \quad (3.35)$$

$$N(\epsilon) = [-\nabla_x^2 L^T, \dots, -\mu_i \nabla_x g_i^T, \dots, -\nabla_x h_j^T, \dots]^T \quad (3.36)$$

This is the solution to the parameter sensitivity problem given by Fiacco (1983). The vector $\nabla_{\epsilon} y(\epsilon)$ is the first derivative of the variation in the optimal values of x , μ and v with respect to the parameters (ϵ). $\nabla_{\epsilon} y(\epsilon)$ can be calculated from Equation 3.33;

$$\nabla_{\epsilon} y(\epsilon) = M(\epsilon)^{-1} N(\epsilon) \quad (3.37)$$

Using this, a first-order Taylor series approximation can be made of the optimal values when a change in the parameters occurs;

$$y^*(\epsilon) \approx y^*(0) + \nabla_{\epsilon} y^*(0) \epsilon \quad (3.38)$$

At the original optimum, each parameter is taken to be zero ($\epsilon = 0$). Details are given in Fiacco (1983) of the use of higher order approximations.

The assumptions used in this technique are simply that the Kuhn-Tucker conditions for the optimum (including Condition 8 on the Hessian of the Lagrangian) are satisfied. These conditions are not severe, being met by most engineering applications. However, extensions of the theory for cases when these conditions are relaxed are also available (Fiacco, 1983).

One obvious limitation is the assumption of the invariance of the active constraint set (i.e. the same inequality constraints stay active after any parameter variation). Clearly it is not difficult to imagine for most optimisation problems, parameter variations which could cause the active constraint set to change. However, Fiacco (1983) proves that for small values of ϵ the active constraint set does not change. This limits the application of the theory to small variations in the parameters, but by considering differential variations the theory is applicable to uncertainty analysis.

One practical problem exists, however, with the application of this sensitivity technique. The calculation of the matrix M is complicated, involving among other things the Hessian of Lagrangian ($\nabla^2 L$ as defined in Equation 3.28). The production of this information analytically or even numerically, where the size of the Hessian is $n \times n$ (n being the number of model variables and a value above 1000 is not unusual even for modest engineering problems), is a difficult problem. In earlier work, Fiacco (1976) suggested that the information required for the matrix M was available as the result of performing a standard "penalty function" optimisation of the problem. However, Ganesh and Biegler (1987) pointed out that such optimisation techniques frequently use approximations to the Hessian of the Lagrangian, and as such are not suitable for performing sensitivity analysis. In their paper, Ganesh and Biegler (1987) provided an alternative technique for calculating the necessary M matrix. This procedure is discussed in Section 3.2.3.

To demonstrate the use of this sensitivity analysis, the variation of a parameter in the simple reactor system described previously is given. The parameter (ϵ) has been introduced into the equality constraint $h_1(x) = 0$;

$$h_1(x, \epsilon) = F - (1 + \epsilon)12750 \left(\frac{P}{T + 273} \right) = 0 \quad (3.39)$$

An analytical calculation of the Hessian and the constraint gradients gives the following M matrix;

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 241.1 & 1 & -39.2 \\ 0 & 241.1 & 1171.5 & -0.0094 & 0.3687 \\ 0 & -78404 & 736.96 & 0 & 0 \\ 1 & -39.2 & 0.3687 & 0 & 0 \end{bmatrix} \quad (3.40)$$

The vector N is calculated as;

$$N = \begin{pmatrix} 0 \\ 78404 \\ 241.06 \\ 0 \\ 119.92 \end{pmatrix} \quad (3.41)$$

Solving Equation 3.37 yields;

$$\nabla_{\epsilon} y = \begin{pmatrix} \nabla_{\epsilon} F \\ \nabla_{\epsilon} P \\ \nabla_{\epsilon} T \\ \nabla_{\epsilon} \mu \\ \nabla_{\epsilon} v \end{pmatrix} = \begin{pmatrix} 119.92 \\ 7.8176 \times 10^{-3} \\ 0.83166 \\ 78204 \\ 0 \end{pmatrix} \quad (3.42)$$

Using Equation 3.38, the change in the optimal value of the elements of y can be calculated. The predicted change in y was calculated for a value of $\epsilon = 0.01$. For comparison, an actual optimisation was also performed for this value of ϵ . The results are given in Table 3.2.

The predicted optimal changes for F , μ and v are in good agreement with those actually calculated from a rigorous optimisation. While the results for P and T are not as close, the magnitude of the changes are much smaller and the calculated optima (both with and without parameter variation) may not have been accurate enough to allow a fair comparison.

Variable	Predicted Change	Actual Change
ΔF^*	1.1992	1.1994
ΔP^*	7.8×10^{-5}	9×10^{-6}
ΔT^*	8.3×10^{-3}	1.8×10^{-5}
$\Delta \mu^*$	782.04	784.03
Δv^*	0	0

Table 3.2 - Predicted and Actual Optimisation Results

The theory developed by Fiacco (1983) would thus appear to be useful in calculating the change in the optimal value of variables due to the variation of specific parameters in the objective function, inequality constraints and equality constraints, without needing to actually perform the optimisation again. In the next section, a method of using this theory to analyse the "worst case" errors in calculated optimal setpoint values is presented.

3.2.2 The Application of Singular Value Analysis to Steady-State Uncertainty Analysis

Uncertainty in process plant models is not usually limited to specific parameters in the equations. Consequently, the method presented in the previous section to determine the sensitivity of potential optimal setpoint values may not capture the effect of all the uncertainty in the model. One way of ensuring that all the uncertainty has been included is to calculate the "worst case" sensitivity. That is, the maximum change in each variable due to some unknown (but upper bounded) variation in the model. Such a calculation may be performed using singular value analysis.

Singular value analysis was reviewed in Section 2.2.5.1. This procedure decomposes a matrix (A) into input directions (right singular vectors, v_i), amplification factors (singular values, σ_i) and output directions (left singular vectors, u_i). If A was multiplied by one of the right singular vectors (v_i), then the output would be in the direction of the corresponding left singular vector (u_i) (this may be viewed as a vector rotation), and the "size" (or length) of the input vector would be amplified by the amount of the corresponding singular value (σ_i). Multiplication of a general vector by the matrix A can

thus be viewed as taking the vector as an input, rotating it and amplifying its elements so as to produce an output vector.

In the sensitivity analysis proposed by Fiacco, Equation 3.37 gives the relationship between the variation caused by the parameters to the equations ($N(\epsilon)$), and the variation in the model variables and the Lagrange multipliers ($\nabla_y y(\epsilon)$). The matrix relating these variations is the inverse of the M matrix. This relationship can be viewed as an input variation (or uncertainty) in the equations ($N(\epsilon)$) causing an output change ($\nabla_y y(\epsilon)$) in the calculated variables. The inverse of the matrix M takes the uncertainty in the equations and calculates how it then appears in the variables.

To determine how uncertainty may cause a variation in the optimal value of the variables for an unknown "worst case" parameter variation, singular value analysis is applied to the inverse of the matrix M . Firstly, however, some simplifications can be made to the construction of M . The central block of M contains the value of the inequality constraints (g_i) at the optimum. For this analysis inactive inequality constraints can be disregarded, since the active constraint set is assumed to be constant. This leaves the central block of M containing the value of each active inequality constraint at the optimum, that is, zero. This then allows the Lagrange multipliers which multiply the inequality constraint gradients in M and in N to be removed. This leaves $\nabla_y y$ unchanged, but makes the calculation of M simpler (see Equation 3.44). The simplified version of M is designated as M' ;

$$M'(\epsilon) \nabla_y y(\epsilon) = N'(\epsilon) \quad (3.43)$$

where the simplified M matrix is given by;

$$M' = \begin{bmatrix} \nabla^2 L & \dots & -\nabla_x g_i^T \dots & \dots & \nabla_x h_j^T \dots \\ \vdots & & & & \\ \nabla_x g_i & & 0 & & 0 \\ \vdots & & & & \\ \vdots & & & & \\ \nabla_x h_j & & 0 & & 0 \\ \vdots & & & & \end{bmatrix} \quad (3.44)$$

and the simplified N matrix is given by;

$$N(\epsilon) = \left[-\nabla_{\epsilon}^2 L^T, \dots, -\nabla_{\epsilon} g_i^T, \dots, -\nabla_{\epsilon} h_j^T, \dots \right]^T \quad (3.45)$$

Using the singular value decomposition of M'^{-1} , an input uncertainty of "size" unity (that is, the same "size" as each right singular vector) in the direction of a particular right singular vector (v_i), will change the variables in the direction of the corresponding left singular vector (u_i) by a factor given by its singular value (σ_i).

For this particular normalised (i.e. "size" = 1) input uncertainty (v_i), we can therefore calculate, for any value of the parameter ϵ , the change in the optimal variable values using;

$$\nabla_{\epsilon} y(\epsilon) = u_i \sigma_i \quad (3.46)$$

In the conventional application of this sensitivity theory, the matrix N is calculated for known parameters (as in the previous reactor example). In this parameter uncertainty application, the parameter variations which will cause the greatest (worst) change in each of the variables is unknown. However, singular value analysis automatically considers all possible parameter variations, since any possible matrix N can be constructed from linear combinations of the right singular vectors (v_i). This is because of a property of singular value analysis, that the right (and left) singular vectors span the space of all possible vectors with their number of components.

Hence the possible range of output sensitivities is given by all linear combinations of the adjusted left singular vectors ($u_i \sigma_i$), as defined in Equation 3.46. The calculated variables (y_i) with the maximum sensitivity are, therefore, given by those elements in the adjusted left singular vectors with the greatest magnitude. The larger these values, the greater the possible change in the optimal value due to some unknown parameter variations. Finally, to allow a direct comparison to be made between each optimal variable sensitivity, with the aim of choosing low sensitivity variables as setpoints, a relative figure is obtained by dividing each element in the adjusted left singular vectors by the corresponding variable's unperturbed, optimal value. The figure thus calculated indicates the fractional (relative)

change that may occur in that variable due to unknown parameter variations.

The optimal sensitivity matrix S is therefore calculated by multiplying each left singular vector of the inverse of M' (u_i), by the corresponding singular value (σ_i), and then dividing each element in the resultant vectors ($u_i \sigma_i(j)$) by the optimal unperturbed value of the corresponding variable (y_j^*);

$$S_{j,i} = \text{abs} \left(\frac{\sigma_i U_{j,i}}{y_j^*} \right) \quad (3.47)$$

where U is the matrix of left singular vectors and y^* is the vector of unperturbed optimal variable values.

It should be noted from a computational point of view, that although the matrix M is invertible (by definition), it is not necessary to invert it for calculation of the optimal sensitivities. This is due to another property of singular value analysis which relates the singular vectors and singular values of a matrix to those of its inverse. For a square ($n \times n$) matrix A , the singular value properties (u_i , σ_i , v_i) are related to those of its inverse A^{-1} (u'_i , σ'_i , v'_i) by;

$$\begin{aligned} u'_i &= v_{n-i+1} \\ \sigma'_i &= \sigma_{n-i+1}^{-1} \\ v'_i &= u_{n-i+1} \end{aligned} \quad (3.48)$$

Reactor Example

The application of this optimal sensitivity analysis is now demonstrated using the simple reactor example described earlier. The y^* vector for the unperturbed system is;

$$y^* = \begin{pmatrix} F \\ P \\ T \\ \mu \\ v \end{pmatrix} = \begin{pmatrix} 119.9 \\ 3.059 \\ 52.24 \\ 78404 \\ 2000 \end{pmatrix}$$

The modified M matrix, M' , was determined analytically. To simplify displaying the numbers in M' , "e - notation" has been used to indicate the exponents of the conventional scientific notation.

$$M' = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2.411e+2 & 1 & -3.920e+1 \\ 0 & 2.411e+2 & 1.172e+3 & -9.400e-3 & 3.687e-1 \\ 0 & -1 & 9.400e-3 & 0 & 0 \\ 1 & -3.920e+1 & 3.687e-1 & 0 & 0 \end{bmatrix}$$

Singular value decomposition of $(M')^{-1}$, evaluated with the aid of Equation 3.48, gave the following results.

Left Singular Vectors (u_i is the i th column of U):

$$U = \begin{bmatrix} 4.831e-1 & 8.746e-1 & 4.061e-2 & 7.040e-3 & 4.877e-6 \\ 1.230e-2 & 2.222e-2 & -4.762e-1 & -8.573e-1 & -1.942e-1 \\ -2.525e-3 & -4.563e-3 & 9.954e-2 & 1.661e-1 & -9.810e-1 \\ 8.755e-1 & -4.826e-1 & -2.221e-2 & 1.242e-2 & -1.517e-4 \\ 6.798e-3 & -4.035e-2 & 8.725e-1 & -4.869e-1 & 5.946e-3 \end{bmatrix}$$

Singular Values:

$$\sigma_1 = 7.106e+1$$

$$\sigma_2 = 2.168e+1$$

$$\sigma_3 = 4.655e-2$$

$$\sigma_4 = 1.446e-2$$

$$\sigma_5 = 8.202e-4$$

Right Singular Vectors (v_i is the i th column of V):

$$V = \begin{bmatrix} 4.831e-1 & -8.746e-1 & 4.061e-2 & -7.040e-3 & 4.877e-6 \\ 1.230e-2 & -2.222e-2 & -4.762e-1 & 8.573e-1 & -1.942e-1 \\ -2.525e-3 & 4.563e-3 & 9.954e-2 & -1.661e-1 & -9.810e-1 \\ -8.755e-1 & -4.826e-1 & 2.221e-2 & 1.242e-2 & 1.517e-4 \\ 6.798e-3 & 4.035e-2 & 8.725e-1 & 4.869e-1 & 5.946e-3 \end{bmatrix}$$

After multiplying the left singular vectors by their singular values, dividing by the

unperturbed optimal values and taking the absolute values, the optimal sensitivity matrix is obtained. The largest elements are shown in bold.

$$S = \begin{bmatrix} \mathbf{2.863e-1} & 1.581e-1 & 1.576e-5 & 8.489e-7 & 3.336e-11 \\ \mathbf{2.856e-1} & 1.575e-1 & 7.246e-3 & 4.052e-3 & 5.206e-5 \\ 3.435e-3 & 1.894e-3 & 8.869e-5 & 4.614e-5 & 1.540e-5 \\ 7.935e-4 & 1.334e-4 & 1.318e-8 & 2.290e-9 & 1.587e-12 \\ 2.415e-4 & 4.373e-4 & 2.031e-5 & 3.520e-6 & 2.438e-9 \end{bmatrix}$$

To demonstrate the principle, the procedure for calculating the first column of S (s_1) will be shown in detail. The first singular value multiplied by the first left singular vector gives;

$$\begin{aligned} \sigma_1 u_1 &= 7.106e+1 \begin{pmatrix} 4.831e-1 \\ 1.230e-2 \\ -2.525e-3 \\ 8.755e-1 \\ 6.798e-3 \end{pmatrix} \\ &= \begin{pmatrix} 3.433e+1 \\ 8.740e-1 \\ -1.794e-1 \\ 6.221e+1 \\ 4.831e-1 \end{pmatrix} \end{aligned}$$

Dividing each element in this vector by the corresponding unperturbed optimal value and taking the absolute value gives s_i ;

$$s_1(i) = \text{abs}((\sigma_1 u_1) \div y^*) \quad (\text{where } \div \text{ is element by element division})$$

$$\begin{aligned} &= \begin{pmatrix} \text{abs}(3.433e+1 \div 1.199e+2) \\ \text{abs}(8.740e-1 \div 3.059e+0) \\ \text{abs}(-1.794e-1 \div 5.224e+1) \\ \text{abs}(6.221e+1 \div 7.840e+4) \\ \text{abs}(4.831e-1 \div 2.000e+3) \end{pmatrix} \\ &= \begin{pmatrix} 2.863e-1 \\ 2.856e-1 \\ 3.435e-3 \\ 7.935e-4 \\ 2.415e-4 \end{pmatrix} \end{aligned}$$

Each row in this matrix corresponds to the equivalent row in the y vector. Consequently,

only the first three rows (corresponding to F , P and T) are of interest in determining which variables are potential setpoints for an optimising control scheme, that is, which variables have a low sensitivity to parameter variations. Examining the matrix S , the largest numbers are in positions (1,1) and (2,1), corresponding to variables F and P . This means that there are possible parameter variations (corresponding to the first right singular vector) which could cause a relative sensitivity in F and P of 0.2863 and 0.2856, respectively. These values are significantly greater than the largest value for T , $3.435e-3$. This means that T should be a much better setpoint in an optimising control scheme, being less susceptible to parameter errors in the objective function, inequality and equality constraints when the reactor model is optimised to determine the optimal setpoint values. These results are summarised in Table 3.3.

Variable	Optimal Sensitivity
Feed (F)	$2.863e-1$
Pressure (P)	$2.856e-1$
Temperature (T)	$3.435e-1$

Table 3.3 - Optimal Sensitivity Results

The use of such singular value analysis can be extended to determine the likely source of the "worst" parameter variations. The maximum values of S for the example problem correspond to the first left singular vector. Consequently, the source of these errors is parameter variations in the direction of the first right singular vector. Examining this vector, the largest element has a value of $8.755e-1$. Remembering that the right singular vectors correspond to the input uncertainty, which in this case is the matrix N' , we can check which element in N' corresponds to the value $8.755e-1$. Looking at the definition of N' (Equation 3.45), this element corresponds to the $\nabla_{\mathbf{g}}$ element in N - parameter induced variation in the inequality constraint. This may be interpreted as meaning that poorly determined coefficients and/or variables in the inequality constraint (see Equation 3.21) could lead to the largest errors in calculating optimal values for F and P . When implementing an optimising control scheme for this reactor system, extra care should therefore be taken in determining accurate values in the inequality constraint.

If, in the general case, the largest values in S did not lie in the same column, then the source of the uncertainty is in the direction of more than one right singular vector, corresponding to the relevant columns in S . Examining each of these right singular vectors will indicate the potential sources of uncertainty, possibly from several different equations.

To demonstrate the effect of uncertainty in the reactor example, 0.1 was added to the inequality constraint and the optimum recalculated. In comparison, again looking at the first right singular vector, the element corresponding to $\nabla_e h$ in N (i.e. the variation in the equality constraint), is much smaller ($6.798e-3$) and so should cause less error in the calculated optimal values. A change was also made to the equality constraint, 0.1 added, and the optimum determined. The results for these two perturbations are given in Table 3.4 below.

Variable	Original Optimum	Inequality Constraint		Equality Constraint	
		Optimum	% Change	Optimum	% Change
F	119.9	123.8	3.3	120.0	0.08
P	3.059	3.158	4.9	3.059	0.0
T	52.24	52.22	-0.04	52.24	0.0
μ	78404	78408	0.005	78404	0.0
v	2000	2000	0.0	2000	0.0

Table 3.4 - The Effect of Uncertainty on the Optimum

The change introduced into the inequality constraint produced changes in the optimal values which agree well with the trends predicted by the singular value analysis, the largest changes being in F and P . Similarly, the change made to the equality constraint did not produce significant changes in any of the variables. In fact, again looking at the right singular vectors, the vector corresponding to a large change in the equality constraint (v_3) has a corresponding column in the S matrix (s_3) whose values are very small (the largest being $7.246e-3$), confirming that this change should have a minimal effect. It

should be stressed, that by using singular value analysis to find the largest possible change in the optimal values, the corresponding equivalent parameter variations may not actually occur in the implementation of the model. This procedure calculates the "worst case" sensitivity to parameter variations.

One potential problem with any application of singular value analysis is the effect of variable scaling. As discussed in Chapter 2, depending on the units used for the variables in the model, different predictions for the controllability of the process can result. For example, a process described using degrees Fahrenheit may appear more controllable than the same process modelled using degrees Celsius. Thus, the effect of scaling on the use of singular value analysis in determining optimal sensitivities needs to be checked. To examine this, the optimal sensitivity matrix was recalculated for the reactor problem with pressure (P) expressed in terms of kPa, rather than MPa as in the original problem. The optimal pressure is therefore 3059 kPa (3.059 MPa in the original problem). This unit change required changes in the equality and inequality constraints (compare below with Equations 3.21 and 3.22);

$$\begin{aligned} g_1(x) &= -P - 7.5T^2 + 793T - 17900 \geq 0 \\ h_1(x) &= F - 12.75 \left(\frac{P}{T + 273} \right) = 0 \end{aligned} \quad (3.49)$$

The M' matrix for these equations is;

$$M' = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2.411e-1 & 1 & -3.920e-2 \\ 0 & 2.411e-1 & 1.172e+3 & -9.400e+0 & 3.687e-1 \\ 0 & -1 & 9.400e+0 & 0 & 0 \\ 1 & -3.920e-2 & 3.687e-1 & 0 & 0 \end{bmatrix}$$

Singular value decomposition of M' gave the following results.

Left Singular Vectors:

$$U = \begin{bmatrix} -2.858e-1 & 6.540e-1 & -6.468e-1 & 2.688e-1 & 2.686e-7 \\ -6.587e-1 & 2.640e-1 & 2.913e-1 & -6.415e-1 & 1.989e-4 \\ -4.917e-3 & -2.148e-3 & 2.294e-3 & 5.518e-3 & 1.000e+0 \\ -6.400e-1 & -2.866e-1 & 2.676e-1 & 6.608e-1 & -8.023e-3 \\ -2.735e-1 & -6.484e-1 & -6.521e-1 & -2.821e-1 & 3.147e-4 \end{bmatrix}$$

Singular Values:

$$\sigma_1 = 1.045e+0$$

$$\sigma_2 = 1.009e+0$$

$$\sigma_3 = 9.919e-1$$

$$\sigma_4 = 9.530e-1$$

$$\sigma_5 = 8.536e-4$$

Right Singular Vectors:

$$V = \begin{bmatrix} -2.858e-1 & -6.540e-1 & -6.468e-1 & -2.688e-1 & 2.686e-7 \\ -6.587e-1 & -2.640e-1 & 2.913e-1 & 6.415e-1 & 1.989e-4 \\ -4.917e-3 & 2.148e-3 & 2.294e-3 & -5.518e-3 & 1.000e+0 \\ 6.400e-1 & -2.866e-1 & -2.676e-1 & 6.608e-1 & 8.023e-3 \\ -2.735e-1 & 6.484e-1 & -6.521e-1 & 2.821e-1 & 3.147e-4 \end{bmatrix}$$

The optimal sensitivity matrix is, thus, given (with the largest elements in bold) by;

$$S = \begin{bmatrix} 2.490e-3 & 5.502e-3 & \mathbf{5.351e-3} & 2.137e-3 & 1.912e-12 \\ \mathbf{2.250e-4} & 8.703e-5 & 9.446e-5 & 1.998e-4 & 5.550e-11 \\ 9.836e-5 & 4.147e-5 & 4.356e-5 & 1.007e-4 & 1.634e-5 \\ 8.529e-6 & 3.687e-6 & 3.385e-6 & 8.031e-6 & 8.734e-11 \\ 1.429e-4 & 3.270e-4 & 3.234e-4 & 1.344e-4 & 1.343e-10 \end{bmatrix}$$

This sensitivity matrix is quite different to that obtained for the case where MPa was used as the units for pressure (P). In fact, it appears that all the variables are much less sensitive to the "worst case" parameter variation. Notice that the maximum values of the sensitivities for F ($5.351e-3$) and P ($2.250e-4$) are two and three orders of magnitude smaller, respectively, than in the original case. Clearly, scaling the variables affects the

results obtained. However, this is not seen as a problem with the application of this method as the results obtained indicate the sensitivity to parameter variation of the optimal values for the given model, including the scaling used in that model. The results are, therefore, valid for the implementation of a particular model in an optimising control scheme. A different model (or different scaling) may give better results and so be preferable for use in the calculation of optimal values for the setpoints in an optimising control scheme. The result of this is that the sensitivity of optimal values to unknown parameter variations will depend on the equations being optimised, and that in a similar manner to other numerical analysis, simply changing the units of measurement could make the system more robust to "worst case" sensitivity.

The application of singular value analysis to the nonlinear programming sensitivity analysis developed by Fiacco (1983), extends the original procedure for determining the change in optimal values due to the variation of particular parameters, to determining the greatest change in the optimal value of each variable due to the "worst case" parameter variation. The method was shown to work for a simple reactor example, in which only a few variables and equations were involved. Potential problems exist with the application of this procedure to larger systems - in the determination of the M matrix, the singular value decomposition and the analysis of the results.

As discussed in Section 3.2.1, calculation of the M (or M') matrix can be difficult due to the nature of its contents (particularly the Hessian of the Lagrangian), although as mentioned earlier, there may be methods of determining the required information directly as a result of performing the original optimisation (Fiacco, 1976). Secondly, as can be seen from the structure of M' (Equation 3.44) this matrix can become very large ($n+m+p \times n+m+p$; for 1,000 model equations M' would have in excess of 1,000,000 elements) and sparse (i.e. many elements are identically zero by construction). This leads to inefficient computation of the singular value decomposition, for although techniques have been developed to take advantage of the sparse nature of matrices in other areas of mathematics, sparse singular value decomposition techniques are not readily available. Finally, when it comes to analysing the optimal sensitivity matrix, sensitivities are calculated for all variables in the model (plus all Lagrange multipliers). However, only some of these variables are candidate setpoints. For example, in the analysis of a distillation column, sensitivities would be produced for every variable calculated on every

tray, whereas only a few key variables are actually potential setpoints.

Consequently, to improve the potential application of this procedure it is desirable to reduce the size of the problem, and only consider a subset of all the variables in the system. Techniques for reducing the "order" of the problem are considered in the next section.

3.2.3 Reduced Order Analysis

The methods presented in this section for reducing the size of the sensitivity analysis problem are based on the use of the equality constraints to "eliminate" variables. This elimination technique is similar to that used in "Feasible Path" optimisation techniques where the apparent size of the problem is reduced by treating the model equations as a "black-box". The optimiser selects values for the black-box "input" variables (satisfying the steady-state degrees of freedom) and then using these values the model equations are solved. This model solution is then used by the optimiser to calculate the next step, and so on until the optimum is found. The optimiser works with a reduced order system - the variables within the model do not actually become part of the optimisation process. An advantage of this method of optimisation is that each step is "feasible", in that the variables used are a consistent solution to the model equations. By contrast, with infeasible path optimisation, all variables are part of the optimisation problem. Consequently, at any optimisation step the variables are not necessarily a valid solution for the model.

What is happening in feasible path optimisation is that the equality constraints (i.e. the model equations) are used to eliminate those variables not directly required by the optimiser. Specification by the optimiser of variables satisfying the steady-state degrees of freedom allows the calculation of other variables required within the optimisation process. This method of reducing the size of the system treats the model as external to the optimisation, and does not actually generate a new set of equations in terms of the reduced number of variables. In this study to determine the sensitivity of potential setpoints within an optimising control scheme, the equality constraints will be used to generate a new, smaller set of equations for the sensitivity analysis. Three techniques will be described - nonlinear elimination, linear elimination and linear Hessian reduction.

Before giving details of these techniques, the general principle will be reviewed. The original optimisation problem defined in Equations 3.12, 3.13 and 3.14 is written in terms of the "model" variables (x). The solution to the optimisation problem are the values of x which give the optimal value of the objective function while satisfying all the constraints. The optimisation process searches over all allowable (feasible) values of x to find the optimum. As discussed previously, this can be achieved by manipulating a number of variables equal to the steady-state degrees of freedom for the model. These variables are sometimes referred to as the "free" or independent variables. The term "independent" is used because values for these variables can be specified independently of each other. The calculated values of all other variables (the dependent variables) in the model are "dependent" on the values chosen for the independent variables.

The classification into independent and dependent variables is an important part of reduced order analysis. Sorting the model variables x into independent variables (x_I) and dependent variables (x_D), the optimisation problem can be rewritten as;

$$\begin{aligned}
 &\text{Minimise } \Phi(x_I, x_D) \\
 &\text{Subject to } \begin{aligned} &g_i(x_I, x_D) \geq 0 \quad i = 1, \dots, m \\ &h_j(x_I, x_D) = 0 \quad j = 1, \dots, p \end{aligned} \tag{3.50} \\
 &\text{where } x_I \in \mathbb{R}^{n-p} \quad \text{i.e. } x_I = (x_{I,1}, \dots, x_{I,n-p}) \\
 &\quad \quad x_D \in \mathbb{R}^p \quad \text{i.e. } x_D = (x_{D,1}, \dots, x_{D,p})
 \end{aligned}$$

This clearly shows that the degrees of freedom, or the number of independent variables, is given by the total number of variables (n) minus the number of equality constraints (p), that is, $n - p$. To eliminate the dependent variables (x_D), a function which expresses the dependent variables in terms of the independent variables (x_I) only (i.e. $x_D = f(x_I)$) is required.

This function can be determined from the equality constraints (i.e. the model equations), as these already contain the relationships between these two sets of variables, although generally expressed implicitly. The first reduction technique considered, nonlinear elimination, makes direct use of the equality constraints to form the reduced problem. The second approach, linear elimination, uses an approximation of any nonlinear equality constraints to eliminate dependent variables. The third method applies linear reduction to the M matrix for the full nonlinear system, to give an exact M matrix for the reduced

system.

Nonlinear Elimination

In theory, it is possible to rearrange the equality constraints ($h(x_I, x_D) = 0$) into an explicit form, giving the dependent variables as a function of the independent variables;

$$x_D = H(x_I) \quad (3.51)$$

The theory which guarantees the ability to perform this rearrangement is known as the Implicit Function Theory (Aoki, 1971). Unfortunately, in practice it is not always practical to find the required functions (Aoki, 1971). However, if such functions can be found, then substitution of each dependent variable's function into the objective function and inequality constraints, wherever that variable appears, eliminates the dependent variables from the optimisation problem. This leaves the optimisation problem in terms of the independent variables only;

$$\begin{aligned} &\text{Minimise } \Phi'(x_I) \\ &\text{Subject to } g'_i(x_I) \geq 0 \quad i = 1, \dots, m \\ &\text{where } x_I \in \mathcal{R}^{n-p} \text{ i.e. } x_I = (x_{I,1}, \dots, x_{I,n-p}) \end{aligned} \quad (3.52)$$

The optimal sensitivity analysis, described in the previous section, can then be applied to this reduced system. This elimination technique can involve a significant reduction in the size of the system - the size of the M matrix going from $(m+n+p \times m+n+p)$ to $(m+n-p \times m+n-p)$. For example, for 1020 variables (n), 1000 model equations (p) and 50 inequality constraints (m), this is a reduction in M from (2070×2070) to (70×70) . As discussed previously, apart from the computer memory storage and computational savings, the final results of the sensitivity analysis are only for the potential setpoints (e.g. only 20 variables using the above numbers).

To demonstrate the application of the nonlinear elimination technique, the reactor model example is used again. Restating the original problem;

$$\text{Minimise } \Phi(\mathbf{x}) = -2000F$$

$$\text{Subject to } g_1(\mathbf{x}) = -P - 0.075T^2 + 0.793T - 17.9 \geq 0$$

$$h_1(\mathbf{x}) = F - 12750 \left(\frac{P}{T + 273} \right) = 0$$

$$\text{where } \mathbf{x} = (F, P, T)$$

In this example, the only equality constraint is $h_1(\mathbf{x}) = 0$ which contains all three potential setpoint variables (F , P and T).

Fortunately, for this example the equality constraint can be easily rearranged to give each of these variables in terms of the other two variables which become the independent variables.

$$\text{Case 1: } F = 12750 \left(\frac{P}{T + 273} \right) \quad (3.53)$$

$$\text{Case 2: } P = \frac{F(T + 273)}{12750} \quad (3.54)$$

$$\text{Case 3: } T = \frac{12750P}{F} - 273 \quad (3.55)$$

Using these relationships the "new" optimisation problems become;

$$\text{Case 1 } (\mathbf{x}_I = (P, T)):$$

$$\text{Minimise } \Phi(\mathbf{x}_I) = -2.55 \times 10^7 \left(\frac{P}{T + 273} \right) \quad (3.56)$$

$$\text{subject to } g_1(\mathbf{x}_I) = -P - 0.075T^2 + 0.793T - 17.9 \geq 0$$

Case 2 ($x_i = (F, T)$):

$$\text{Minimise } \Phi(x_i) = -2000F$$

$$\text{subject to } g_1(x_i) = \frac{-F(T + 273)}{12750} - 0.075T^2 + 0.793 - 17.9 \geq 0 \quad (3.57)$$

Case 3 ($x_i = (F, P)$):

$$\text{Minimise } \Phi(x_i) = -2000F$$

$$\begin{aligned} \text{subject to } g_1(x_i) = & -P - 0.075 \left(\frac{12750P}{F} - 273 \right)^2 \\ & + 0.793 \left(\frac{12750P}{F} - 273 \right) - 17.9 \geq 0 \end{aligned} \quad (3.58)$$

The solution to each of these optimisation problems is, of course, the original optimum given in Equation 3.25. The modified sensitivity matrices, M' , were calculated to be;

$$\text{Case 1: } M' = \begin{bmatrix} 0 & 2.411e+2 & -1 \\ 2.411e+2 & 1.172e+3 & 9.400e-3 \\ 1 & -9.400e-3 & 0 \end{bmatrix} \quad (3.59)$$

$$\text{Case 2: } M' = \begin{bmatrix} 0 & 6.149 & -2.551e-2 \\ 6.149 & 1.176e+3 & -5.612e-6 \\ 2.551e-2 & 5.612e-6 & 0 \end{bmatrix} \quad (3.60)$$

$$\text{Case 3: } M' = \begin{bmatrix} 8.618e+3 & -3.385e+5 & -2.549e-2 \\ -3.385e+5 & 1.329e+7 & -5.701e-4 \\ 2.549e-2 & 5.701e-4 & 0 \end{bmatrix} \quad (3.61)$$

The optimal sensitivity matrices (see Equation 3.47) were then constructed for each case;

$$\text{Case 1: } S = \begin{bmatrix} 3.262e-1 & 6.774e-3 & 5.201e-5 \\ 3.923e-3 & 7.785e-5 & 1.540e-5 \\ 6.304e-4 & 5.512e-9 & 0 \end{bmatrix} \quad (3.62)$$

$$\text{Case 2: } S = \begin{bmatrix} 2.862e-1 & 1.579e-1 & 3.707e-8 \\ 3.436e-3 & 1.896e-3 & 1.628e-5 \\ 7.934e-4 & 1.333e-4 & 0 \end{bmatrix} \quad (3.63)$$

$$\text{Case 3: } S = \begin{bmatrix} 2.862e-1 & 1.580e-1 & 0 \\ 2.857e-1 & 1.577e-1 & 2.457e-8 \\ 7.935e-4 & 1.334e-4 & 0 \end{bmatrix} \quad (3.64)$$

These sensitivity matrices can be used to compare the sensitivity of the two candidate setpoints (i.e. the two model variables not eliminated) for each case. The full model and reduced order results are given in Table 3.5 below.

Variable	Maximum Sensitivity			
	Full	Case 1	Case 2	Case 3
<i>F</i>	2.863e-1		2.862e-1	2.862e-1
<i>P</i>	2.856e-1	3.262e-1		2.857e-1
<i>T</i>	3.435e-3	3.923e-3	3.436e-3	

P > *T**F* > *T**F* ≈ *P*

Table 3.5 - Nonlinear Elimination Sensitivities

A direct comparison between the sensitivities obtained for the full system and the reduced order sensitivities is complicated by the fact that for each of the reduced cases, there are only results for two variables. However, there are still two important observations that can be made from the reduced order results. Firstly, the ranking of the variables from the most sensitive to the least sensitive is the same for the full model analysis as for that obtained using the nonlinear elimination technique.

Most Sensitive *F* ≈ *P* > *T* Least Sensitive

That is, the optimal values of F and P are more susceptible to unknown parameter variations than T . The second observation is that the same values for the maximum sensitivity were obtained in for case 2 and 3 for the full and the reduced analysis.

The consistency of the two sets of results is due to the fundamental equivalence of the two approaches. The full optimisation problem eliminates the dependent variables through the solution of the equality constraints as part of the optimisation problem. Consequently, using the equality constraints to first eliminate the dependent variables is really solving the same problem as the full optimisation. Since the sensitivity analysis is derived from the equations defining the optimum (Equations 3.16, 3.17 and 3.18), the same sensitivity ranking is obtained because the nonlinear elimination reduced order problem contains the same equality constraint information as the full problem, just in a rearranged format.

The reason that not all of the maximum sensitivity values are the same for the full and reduced order systems, is the result of using singular value analysis to produce the worst case sensitivities. In fact, the differences can be explained by examining the left singular vectors of the full order M' matrix. First note that singular vectors are orthonormal, that is, they are linearly independent and of length one, where length is defined using the 2-norm as;

$$\text{length}(v_i) = \left(\sum_j v_{ji}^2 \right)^{1/2} \quad (3.65)$$

If we assume that the output direction of the sensitivity matrix for the reduced case is the same as the full model, then the corresponding elements of the left singular vectors for the full order and reduced order cases should be equivalent. This means we can remove the dependent variable and the equality constraint Lagrange multiplier components from the full order singular vector. Examining the first singular vector for the case where F is eliminated;

$$\text{Full Order } v_1 = \begin{pmatrix} 4.831e-1 \\ 1.234e-2 \\ -2.525e-3 \\ 8.755e-1 \\ 6.798e-3 \end{pmatrix} \Rightarrow \text{Reduced Order } v_1 = \begin{pmatrix} 1.234e-2 \\ -2.525e-3 \\ 8.755e-1 \end{pmatrix}$$

The resultant vector should then be scaled so that the length is still one.

$$\begin{pmatrix} 1.234e-2 \\ -2.525e-3 \\ 8.755e-1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1.409e-2 \\ -2.884e-3 \\ 9.999e-1 \end{pmatrix}$$

Multiplication of this vector by the original singular value, dividing by the optimal value for each variable and taking the absolute value should therefore give the "equivalent" sensitivity values.

$$\begin{pmatrix} 1.409e-2 \times \sigma_1 \div P^* \\ -2.884e-3 \times \sigma_1 \div T^* \\ 9.999e-1 \times \sigma_1 \div \mu^* \end{pmatrix} \Rightarrow \begin{pmatrix} 1.409e-2 \times 7.106e+1 \div 3.059e+0 \\ -2.884e-3 \times 7.106e+1 \div 5.224e+1 \\ 9.999e-1 \times 7.106e+1 \div 7.840e+4 \end{pmatrix} \Rightarrow \begin{pmatrix} 3.273e-1 \\ 3.923e-3 \\ 9.062e-4 \end{pmatrix}$$

Comparing these results with those obtained for the nonlinear elimination shows a good agreement. The predicted sensitivity for P , $3.273e-1$, compares well to the reduced order value, $3.262e-1$, and those obtained for T are the same. This confirms the consistency between the full order analysis and the nonlinear elimination, the difference in the values being due to the use of singular value analysis.

The results obtained using the method of nonlinear elimination to obtain a reduced system for optimal sensitivity analysis are in good agreement with those obtained from the full system. However, one limitation could be that the number of candidate setpoints which can be analysed at any one time must be equal to the number of variables in the problem minus the number of equality constraints (i.e. equal to the degrees of freedom). If more variables than this need to be analysed, then a number of cases must be considered, as in the reactor example, and the results combined to give an overall ranking for the candidate setpoints. However, the real limitation with this approach is the need to analytically perform the nonlinear elimination of variables using the equality constraints. As discussed earlier, this may not always be practical, and in any case for very large systems of equations the analytical elimination of variables can be computationally difficult. The next technique discussed uses the computationally simpler approach of linear elimination.

Linear Elimination

To overcome the problems associated with nonlinear elimination, linear approximations to the nonlinear equality constraints can be used in the elimination process.

Locke et al. (1983) presented an improved optimisation technique which reduced the computer storage required by the quadratic programming procedure proposed by Powell (1977). Both techniques use linear approximations of the equality constraints when calculating changes in the free variables so as to move towards the optimum. A relationship between changes in the independent variables and the required changes in the dependent variables so as to maintain the equality constraints is presented by Locke et al. (1983). This relationship can be derived by considering the total derivative of the equality constraints with respect to the dependent variables. If we require that the change in the value of the equality constraints, for a given change in the independent variables, be zero, then we can write;

$$\frac{d}{dx_D} h(x_I, x_D) = \frac{\partial h}{\partial x_I} + \frac{\partial h}{\partial x_D} \frac{\partial x_D}{\partial x_I} = 0 \quad (3.66)$$

Rearranging the above gives;

$$\frac{\partial x_D}{\partial x_I} = - \frac{\partial h}{\partial x_I} \left(\frac{\partial h}{\partial x_D} \right)^{-1} \quad (3.67)$$

Using Equation 3.67, a first-order Taylor's approximation to the change required in the dependent variables (Δx_D) for a given change in the independent variables (Δx_I) can be determined;

$$\begin{aligned} \Delta x_D &\approx \frac{\partial x_D}{\partial x_I} \times \Delta x_I \\ &\approx - \frac{\partial h}{\partial x_I} \left(\frac{\partial h}{\partial x_D} \right)^{-1} \times \Delta x_I \end{aligned} \quad (3.68)$$

Finally, to generate a linear function relating the independent (x_I) and dependent (x_D) variables the changes given by Equation 3.68 are added to the (known) values at the point of linearisation. That is, for known values of the independent and dependent variables (also used to determine the derivatives), Equation 3.68 can be used to calculate changes away from which point still satisfies (approximately) all equality constraints. Denoting values at the point of linearisation with an asterisk, we obtain the required relationship;

$$x_D \approx x_D^* - \frac{\partial h^*}{\partial x_I} \left(\frac{\partial h^*}{\partial x_D} \right)^{-1} \times (x_I - x_I^*) \quad (3.69)$$

This approximate relationship can be used to eliminate the dependent variables from the optimisation problem, allowing analysis of the reduced problem in terms of the independent variables only. Obviously from Equation 3.69, a requirement of this procedure is that the Jacobian matrix of first derivatives of the equality constraints with respect to the dependent variables must be invertible.

The linear elimination reduction procedure was applied to the reactor example. The linearised relationships between the dependent and independent variables determined at the optimum for the full order system (i.e. F^* , P^* , T^*) are;

$$\text{Case 1:} \quad F = 39.2P - 0.3687T + 19.25 \quad (3.70)$$

$$\text{Case 2:} \quad P = 0.02551F + 9.405 \times 10^{-3}T - 0.4910 \quad (3.71)$$

$$\text{Case 3:} \quad T = -2.712F + 106.3P + 52.24 \quad (3.72)$$

Substituting these into the original optimisation problem gives a reduced problem with only an objective function and inequality constraints. Despite the use of the linear elimination, these remaining elements can still be nonlinear;

Case 1 ($x_I = (P, T)$):

$$\text{Minimise } \Phi_{LN}(x_I) = -2000(39.2P - 0.3687T + 19.25) \quad (3.73)$$

$$\text{subject to } g_{LN}(x_I) = -P - 7.5 \times 10^{-3}T^2 + 0.793T - 17.9 \geq 0$$

Case 2 ($x_I = (F, T)$):

$$\text{Minimise } \Phi_{LN}(x_I) = -2000F \quad (3.74)$$

$$\text{subject to } g_{LN}(x_I) = -2.551 \times 10^{-2}F - 7.5 \times 10^{-3}T^2 + 0.7836T - 17.41 \geq 0$$

Case 3 ($x_I = (F, P)$):

$$\text{Minimise } \Phi_{LIN}(x_D) = -2000F$$

$$\text{subject to } g_{LIN}(x_D) = -5.516 \times 10^{-2} F^2 - 2.587 \times 10^{-2} F + 4.325 FP - 2.910 \times 10^{-2} P - 84.75 P^2 + 3.059 \geq 0 \quad (3.75)$$

For these linearly reduced systems, the M' matrices were calculated to be;

$$\text{Case 1: } M' = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1.176e+3 & -9.400e-3 \\ -1 & 9.400e-3 & 0 \end{bmatrix} \quad (3.76)$$

$$\text{Case 2: } M' = \begin{bmatrix} 0 & 0 & 2.551e-2 \\ 0 & 1.176e+3 & 0 \\ -2.551e-2 & 0 & 0 \end{bmatrix} \quad (3.77)$$

$$\text{Case 3: } M' = \begin{bmatrix} 8.648e+3 & -3.391e+5 & 1.992e-2 \\ -3.391e+5 & 1.329e+7 & -3.790e-2 \\ -1.992e-2 & 3.790e-2 & 0 \end{bmatrix} \quad (3.78)$$

The optimal sensitivity matrices were then constructed for each case. The maximum value for each independent variable is shown in bold.

$$\text{Case 1: } S = \begin{bmatrix} \mathbf{2.312e-1} & \mathbf{2.312e-1} & 0 \\ 1.083e-7 & 1.081e-7 & \mathbf{1.628e-5} \\ 9.019e-6 & 9.019e-6 & 0 \end{bmatrix} \quad (3.79)$$

$$\text{Case 2: } S = \begin{bmatrix} 0 & \mathbf{3.269e-1} & 0 \\ 0 & 0 & \mathbf{1.628e-5} \\ 5.000e-4 & 0 & 0 \end{bmatrix} \quad (3.80)$$

$$\text{Case 3: } S = \begin{bmatrix} \mathbf{4.401e-1} & 1.949e-3 & 0 \\ \mathbf{4.401e-1} & 1.949e-3 & 2.457e-8 \\ 1.520e-1 & 1.320e-8 & 0 \end{bmatrix} \quad (3.81)$$

As for the nonlinear elimination approach, each of these sensitivity matrices can be used to compare the sensitivity of the two independent variables (candidate setpoints) in each case. The results obtained are summarised in the Table 3.6 below.

Variable	Maximum Sensitivity			
	Full	Case 1	Case 2	Case 3
F	$2.863e-1$		$3.269e-1$	$4.401e-1$
P	$2.856e-1$	$2.312e-1$		$4.401e-1$
T	$3.435e-3$	$1.628e-5$	$1.628e-5$	

 $P > T$ $F > T$ $F \approx P$

Table 3.6 - Linear Elimination Sensitivities

The results here show the same trends as obtained for both the full order system and for nonlinear elimination. However, the actual values of the sensitivities are quite different. This is not surprising since quite significant changes (replacing nonlinear equations with linear approximations) have been made to the equation structure defining the optimum from which these sensitivities are derived. In fact, it is quite possible that a solution may not be found at all, as it is not difficult to conceive of an optimisation problem where the nonlinearity in the equality constraints defines the optimum. Replacing this nonlinearity with a linear approximation could cause the optimum to be no longer defined. For this example, however, the new optimisation problems (Equations 3.73, 3.74, 3.75) when solved gave solutions very close to the original nonlinear optimum as shown below in Table 3.7.

An example of the problems that can be encountered with linear elimination can be seen by making a slight modification to the reactor example. Mathematically it is valid to swap the inequality and equality equations - although the resulting problem may have no real physical significance. Doing this the constraints become;

$$g_1(x) = F - 12750 \left(\frac{P}{T + 273} \right) \geq 0 \quad (3.82)$$

$$h_1(x) = -P - 0.075T^2 + 0.793T - 17.9 = 0 \quad (3.83)$$

Variable	Optima			
	Original	Case 1	Case 2	Case 3
Φ	-239,800	-239,800	-239,800	-252,400
F	119.9	119.9	119.9	126.2
P	3.059	3.059	3.058	3.221
T	52.24	52.24	52.24	52.27
μ	78,404	78,404	78,404	91,500

Table 3.7 - Linearised System Optima

Examining Case 2 for this example (i.e. eliminate P), the relationship between the dependent and independent variables is;

$$\text{Case 2:} \quad P = 9.4 \times 10^{-3}T + 2.568 \quad (3.84)$$

The optimisation problem thus becomes;

$$\begin{aligned} &\text{Minimise } \Phi_{LN}(x_D) = -2000F \\ &\text{subject to } g_{LN}(x_D) = F - 12750 \left(\frac{9.4 \times 10^{-3}T + 2.568}{T + 273} \right) \geq 0 \end{aligned} \quad (3.85)$$

Unfortunately, there is no solution to this optimisation problem. The replacement of the nonlinear information in the equality constraint (see Equation 3.83) with a linear approximation (Equation 3.84) means that an optimum no longer exists. Since an optimum is not defined, the sensitivity information cannot be calculated. It should be noted that applying the nonlinear elimination approach for this same problem determined the optimum correctly. This clearly demonstrates the type of problem that can be

encountered when trying to simplify sensitivity analysis using linear elimination techniques.

Linearly Reduced Hessian

An alternative method for producing a reduced M matrix for sensitivity analysis was proposed by Ganesh and Biegler (1987). Starting with the original M matrix developed by Fiacco (1983), they rewrote the problem in terms of dependent (x_D) and independent (x_I) variables. The form shown below in Equation 3.86 (M'') is similar to the M' matrix, in that the inactive inequality constraints and the redundant inequality constraint Lagrange multipliers have been removed. M'' appears slightly different to the matrix given in the paper by Ganesh and Biegler (1987), due to their different definition of the optimisation problem. That is, they used $g(x) \leq 0$ for the inequality constraints rather than $g(x) \geq 0$ as adopted here (and by Fiacco (1983)). The format of M'' is shown below;

$$M'' = \begin{bmatrix} \nabla_{x_I} L & \nabla_{x_D} L & -\nabla_{x_I} g^T & \nabla_{x_I} h^T \\ \nabla_{x_D x_I} L & \nabla_{x_D x_D} L & -\nabla_{x_D} g^T & \nabla_{x_D} h^T \\ \nabla_{x_I} g & \nabla_{x_D} g & 0 & 0 \\ \nabla_{x_I} h & \nabla_{x_D} h & 0 & 0 \end{bmatrix} \quad (3.86)$$

The corresponding matrices $\nabla_{\epsilon} y''$ and N'' are;

$$\nabla_{\epsilon} y'' = [\nabla_{\epsilon x_I}^T, \nabla_{\epsilon x_D}^T, \nabla_{\epsilon u}^T, \nabla_{\epsilon v}^T]^T \quad (3.87)$$

$$N'' = [-\nabla_{\epsilon x_I} L^T, -\nabla_{\epsilon x_D} L^T, -\nabla_{\epsilon} g^T, -\nabla_{\epsilon} h^T]^T \quad (3.88)$$

where;

$$M'' \nabla_{\epsilon} y'' = N'' \quad (3.89)$$

The relationship expressed in Equation 3.89 (and the original relationship Equation 3.33) is, in fact, a set of linear equations. Consequently, it is possible to rearrange elements in the matrices (M'' , $\nabla_{\epsilon} y''$ and N'') using row and column operations (e.g. swapping rows, swapping columns, adding multiples of rows to other rows, etc; all of which preserve the

fundamental relationships of the equations, see Strang (1980) for details). Ganesh and Biegler (1987) performed block Gaussian elimination on these matrices to obtain a structure which includes a reduced order sensitivity relationship. Block Gaussian elimination involves performing row and column operations on blocks of elements (e.g. each of the blocks shown in Equations 3.86, 3.87 and 3.33) rather than on individual matrix elements. After such elimination, the resultant system could be written as;

$$\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} I & 0 & E & 0 \\ 0 & I & F & G \\ 0 & 0 & H & J \\ 0 & 0 & J^T & 0 \end{bmatrix} \begin{bmatrix} \nabla_{\epsilon} x_D^T \\ \nabla_{\epsilon} v^T \\ \nabla_{\epsilon} x_I^T \\ \nabla_{\epsilon} \mu^T \end{bmatrix} \quad (3.90)$$

Examination of Equation 3.90 shows that the effects of the parameters on the independent variables ($\nabla_{\epsilon} x_I$) and the inequality constraint Lagrange multipliers ($\nabla_{\epsilon} \mu$) can be isolated from the rest of the system;

$$\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} H & J \\ J^T & 0 \end{bmatrix} \begin{bmatrix} \nabla_{\epsilon} x_I^T \\ \nabla_{\epsilon} \mu^T \end{bmatrix} \quad (3.91)$$

where;

$$\begin{aligned} C &= -\nabla_{x_I} L + \nabla_{x_D} L (\nabla_{x_D} h^T)^{-1} \nabla_{\epsilon} h + \nabla_{x_I} h (\nabla_{x_D} h)^{-1} \nabla_{x_D} L \\ &\quad - \nabla_{x_I} h (\nabla_{x_D} h)^{-1} \nabla_{x_D} L (\nabla_{x_D} h^T)^{-1} \nabla_{\epsilon} h \\ D &= \nabla_{\epsilon} g - \nabla_{x_D} g^T (\nabla_{x_D} h^T)^{-1} \nabla_{\epsilon} h \\ H &= \nabla_{x_I} L - \nabla_{x_D} L (\nabla_{x_D} h^T)^{-1} \nabla_{x_I} h^T \\ &\quad - \nabla_{x_I} h (\nabla_{x_D} h)^{-1} (\nabla_{x_D} L - \nabla_{x_D} L (\nabla_{x_D} h^T)^{-1} \nabla_{x_I} h^T) \\ J &= -\nabla_{x_I} g + \nabla_{x_I} h (\nabla_{x_D} h)^{-1} \nabla_{x_D} g \end{aligned} \quad (3.92)$$

Details of the other elements of Equation 3.90 (which are not required in the present development) can be found in Ganesh and Biegler (1987).

The goal of the work presented by Ganesh and Biegler was to determine a strategy for performing optimal sensitivity analysis without the need to construct the full matrix M . The procedure they presented uses Equation 3.91 to calculate the sensitivities for the reduced set of independent variables and inequality constraint Lagrange multipliers. Although analytical expressions for the matrices C , D , H and J are given above, the real power of the method they proposed is that these matrices can alternatively be determined by performing a number of evaluations of the Lagrangian and inequality and equality (model equations) constraints at different (perturbed) values of the independent variables. It is also possible to calculate the full M matrix using similar evaluations, however the strategy proposed by Ganesh and Biegler (1987) significantly reduces the amount of calculation required when compared to a similar perturbation strategy for evaluating the full M matrix.

For the purpose of calculating optimal sensitivities using singular value analysis, this reduced matrix M'' can be determined either analytically or by numerical evaluation. Either way, the same procedure of performing a singular value decomposition and modifying the left singular values to give the optimal sensitivity (as used previously, see Equation 3.47) can be used. The results obtained for the three cases with the reactor example are identical to those obtained for the nonlinear elimination, as shown in Table 3.8.

These results were obtained from information contained within the full M matrix rather than using perturbations (as suggested by Ganesh and Biegler), as the full M matrix was already available. The required derivative information was extracted from the full M matrix and used to calculate H and J , and hence the reduced order M matrices. The reason the two sets of results are the same is that the M'' matrix for the reduced Hessian method is identical to the M' matrix obtained by nonlinear elimination. In fact it can be proved analytically, using matrix algebra to eliminate the dependent variables from the relationships given in the original sensitivity equation (Equation 3.33), that the reduced M matrices (M' and M'') generated by the two methods are identical. Although the author has developed this proof, it is not presented here (due to the complexity and length), the results being available from Ganesh and Biegler.

The method proposed by Ganesh and Biegler (1987) provides an exact method for

Variable	Maximum Sensitivity			
	Full	Case 1	Case 2	Case 3
F	$2.863e-1$		$2.862e-1$ ($2.862e-1$)	$2.862e-1$ ($2.862e-1$)
P	$2.856e-1$	$3.262e-1$ ($3.262e-1$)		$2.857e-1$ ($2.857e-1$)
T	$3.435e-3$	$3.923e-3$ ($3.923e-3$)	$3.436e-3$ ($3.436e-3$)	

(nonlinear elimination results shown in parentheses)

Table 3.8 - Reduced Hessian and Nonlinear Elimination Sensitivity Results

calculating the sensitivity information for a reduced system in terms of the independent variables and inequality constraint Lagrange multipliers. The nonlinear elimination method proposed earlier, first reduced the size of the optimisation problem and then calculated the sensitivity information. The method proposed by Ganesh and Biegler operates differently, in that it can be thought of as either generating the full sensitivity information and then extracting the reduced system sensitivity values, or alternatively as a procedure for directly calculating the reduced sensitivity information by performing a number of evaluations of the equations in the optimisation problem.

3.3 Limitations in Using Conservative Steady-State Uncertainty Descriptions

Uncertainty in the steady-state behaviour of a process plant can cause problems in the design and implementation of an optimising control system. Without complete knowledge of how the plant performs at steady-state, the accurate prediction of the optimal operating point for given slow disturbances is not possible. Although measures can be taken to ensure that the model used for optimisation matches the plant as well as possible (e.g. online parameter estimation), there will always be some discrepancy between the plant and the model. In the best circumstances, the model errors will lead to only slightly suboptimal performance, while in the worst circumstances the implementation of the setpoints calculated using an inaccurate model could lead to real inequality constraints being violated.

One method of implementing optimising control which is not particularly susceptible to steady-state uncertainty is constraint control. As discussed in Chapter 2, the optimum of a process plant is frequently described by some variables being at their limiting values (i.e. active inequality constraints). If this condition is true (i.e. the same constraints are active) despite steady-state modelling errors, then controlling these variables to their limiting values will not only ensure that the optimum is maintained, but also gives a more direct control over the operational constraints. This can, therefore, be a robust method of implementing optimising control. However, as discussed previously, constraint control is not applicable in all circumstances, as sometimes not all the degrees of freedom can be satisfied by controlling measurements to their constraint values, others must be controlled to calculated optimal setpoints.

Another method that can be adopted to reduce the problems associated with steady-state uncertainty, is to choose setpoints for the optimising control scheme which will reduce the impact of the uncertainty. Benefits from using "good" setpoints include achieving an objective function closer to the true optimum, and less likelihood of violating operational inequality constraints. There are two criteria for choosing such setpoints;

(1) If a variable to which the optimum of the objective function is insensitive is used as a setpoint, then despite the variable being calculated inaccurately, or the control system not maintaining the measurement close to the setpoint, the objective function for the process will be close to its optimal value. In fact its use as a setpoint may be questioned since it has little effect on the objective function, but there may be other reasons that this measurement is important for control (e.g. tank level control).

(2) Variables whose optimal values do not change significantly due to modelling error are also good candidates for setpoints.

Two methods for selecting setpoints based on a knowledge of possible steady-state uncertainty were presented in this chapter. The first made the simple assumption that the same relative error would occur in each setpoint. For this assumption, a measure of the likely decrease in the objective function was derived, the linear sensitivity indicator (I), allowing groups of potential setpoints to be ranked according to their suitability for use in

an optimising control scheme. The second method presented extends existing nonlinear programming sensitivity analysis to determine the worst case error propagation from steady-state model error to the calculated optimal setpoint values.

The nonlinear programming sensitivity analysis used was based on work originally presented by Fiacco (1983) which allows calculation of the changes in the optimal values of variables in the optimisation problem due to changes in parameters within the objective function, inequality constraints and equality constraints (model equations). This is accomplished without actually performing the optimisation again. A singular value decomposition of the matrix relating these changes (M) provides a means of calculating the "worst case" sensitivity to unknown parameter variations. The optimal sensitivity matrix thus generated indicates for each variable in the optimisation problem the largest relative change which could occur in its value due to an unspecified combination of parameter variations within the problem. Those variables with a small optimal sensitivity should, therefore, be more robust to steady-state model errors, giving an objective function value closer to the true optimum despite this uncertainty.

Before reviewing reduced order analysis, it is worth comparing the results of applying the linear sensitivity analysis (presented in Section 3.1) and optimal sensitivity analysis (presented in Section 3.2.2) to the reactor example. Table 3.1 showed that for the assumption of equal relative errors occurring in each setpoint, the use of pressure (P) and temperature (T) have the least detrimental effect on the objective function if errors or poor control occurred. While in Table 3.3 the results from the optimal sensitivity analysis indicate that the feed (F) is the most susceptible variable to "worst case" parameter variations, with the pressure (P) almost as sensitive. Based on the optimal sensitivity results, the most robust setpoints would be the pressure (P) and temperature (T), in agreement with the linear sensitivity analysis. Although limitations with the linear sensitivity analysis have been discussed, in this case it agrees with the more rigorous optimal sensitivity analysis. From a theoretical point of view there should be no connection between the two techniques, as they analyse different effects. However the simple application of linear sensitivity analysis warrants further investigation.

One problem identified with the application of this optimal sensitivity calculation to "real" problems was the amount of information generated. For typical process plants, the

matrices generated could contain millions of elements, with sensitivity information being generated for each (of possibly thousands) variable in the problem, when only a handful may be potential setpoints. To reduce the size of the problem, a "reduced-order" approach was considered. To achieve this, the equality constraints are used to eliminate the dependent variables, leaving the optimisation problem in terms of independent variables only. Sensitivity analysis is then applied to this subset of variables.

Three approaches to eliminating the dependent variables were put forward. The first technique analytically rearranges the equality constraints to generate a series of functions describing the dependent variables in terms of the independent variables. These are then substituted into the objective function and inequality constraints to generate the reduced problem. This nonlinear elimination technique gives sensitivities equivalent to those generated by analysing the "full" optimisation problem. However, it is not always possible to rearrange the equality constraints as required.

The second approach is more attractive from a computational point of view - first derivatives are taken of the equality constraints and are used to generate an approximate (linear) relationship between the dependent and independent variables. Although this was shown to work in one example, the replacement of nonlinear equality constraints by linear approximations will not always give a well defined optimisation problem. Consequently, this simpler method is not generally applicable.

The third reduced order technique involves a linear reduction of the Hessian of the Lagrangian for the optimisation problem. Ganesh and Biegler (1987) had proposed a method for directly computing the sensitivity matrix for the reduced system consisting of the independent variables and the inequality constraint Lagrange multipliers (described by the matrix M''). This matrix is exactly equivalent to that generated by performing analytical nonlinear elimination. The M'' matrix can be found either by manipulation of the full M matrix, or by a perturbation analysis performed by evaluating the Lagrangian and constraints for different values of the independent and dependent variables. Although this approach is an improvement on both analytical nonlinear elimination and linear elimination, it is still computationally very involved.

A possible limitation with these reduced order techniques is that a fixed number of

variables are analysed in each reduced system. This number is equal to the degrees of freedom of the system, that is the total number of variables minus the number of model equations (equality constraints). In most circumstances the number of setpoints required to control a plant will be equal to the degrees of freedom. Consequently, a number of reduced order cases (with different independent variables) may have to be analysed, and a ranking order determined from the optimal sensitivities generated for each case. An alternative may be to perform a limited reduction, not removing all the equality constraints so that more variables are analysed in each reduced case. In theory it should not matter which equality constraints are not eliminated, since the analytical elimination technique and the linear Hessian reduction both retain all the original nonlinear information. This limited reduction approach was not followed up in this study.

The net result of the investigations discussed in this chapter would seem to be that while it is possible to calculate the inherent accuracy of the optimal setpoint values for use within an optimising control scheme, the calculations can be quite complex. In addition, even after the necessary analysis the results may still be of limited usefulness for several reasons. Firstly, the analysis produces the "worst case" error for each setpoint for some unknown parameter variation occurring somewhere in the objective function and/or the equality and inequality constraints. This is a conservative approach to analysing uncertainty - since the exact nature of the error in the model is unknown, the worst possible error is considered. This "worst case" analysis was achieved using singular value analysis of the sensitivity matrix (M , M' or M'' as appropriate), and combining the singular values and the left singular vectors to find the maximum possible variation in each variable. It is quite conceivable that the worst error considered in the model may never occur - for instance it may involve a discrepancy in a mass balance which will not happen. Using this analysis, a variable may be ruled out for use as a setpoint because of an unlikely model error, while a smaller but more likely error could still cause significant variation in the chosen setpoint.

Finally, the problem of integrating this setpoint uncertainty information into control system design still exists. Another criterion in choosing setpoints is the achievable performance of the closed-loop control system. A limitation of the optimal sensitivity analysis is that it does not indicate how to choose between setpoints which are accurately known but can only be poorly controlled, and other setpoints which although more

susceptible to model error when calculating their optimal value allow better closed-loop performance. Thus, although optimal sensitivity analysis is another step towards including knowledge of uncertainty in the process plant into the design of an optimising control system, other links between steady-state optimisation and measures of plant controllability are still required. In the next chapter, the dynamic characteristics of optimal operation are investigated. The final chapters of this thesis are concerned with how to incorporate both steady-state and dynamic uncertainties into an optimising control system and how such combined considerations may be used to select between alternative process flowsheets.

CHAPTER 4

THE EFFECT OF DISTURBANCES WITHIN OPTIMISING CONTROL

The optimal operation of process plants is primarily governed by two considerations - (i) the operational constraints within which the plant must operate, and (ii) how well it can be controlled in the face of external disturbances affecting its operation. An optimising control scheme can be implemented to keep a plant operating close to its optimum. This may involve calculating setpoints for a regulatory control scheme by optimising a steady-state model of the process which has previously been adjusted to match plant data. In Chapter 2 some aspects of existing optimising control system design were applied to an industrial case study. A limitation in this procedure was identified as the inability to determine how close to the optimum the plant can operate given the less than perfect knowledge of the process and the influence of external disturbances. The effect of steady-state uncertainty in the design of an optimising control scheme was analysed in Chapter 3. To minimise the effect of this uncertainty, it is possible to determine which variables will make the most robust setpoints - being least affected by "worst case" variation in the model.

Although the methods presented in Chapter 3 may be useful in guiding the design of an optimising control scheme, it does not provide a complete design procedure as it does not incorporate a measure of the likely performance of the plant under actual operating conditions. Such an indicator (of achievable optimal operation) could be used to assess the "best" performance possible from different processes at the design stage. Such an indicator would indirectly provide a link between steady-state economics and process controllability. In the remainder of this chapter, the central role played by disturbances in determining the achievable optimal operation will be analysed.

4.1 Linear and Nonlinear Systems

Disturbances have been defined as persistent variations in plant measurements (and, indeed, in unmeasured variables) due to the influence of external factors. Earlier, a distinction was drawn between "slow" and "fast" disturbances, depending on their frequency of occurrence. The effect of slow disturbances, occurring on a time scale much

slower than the dominant time constants of the system (e.g. changes in feedstock or catalyst activity), were to be handled by an optimising control scheme - the operating point (defined as a set of controller setpoints) of the plant being adjusted to match the current optimal operating policy. Fast disturbances, occurring at a rate more closely matching the response time of the plant (e.g. temperature of the feed and flow rate variations), were to be attenuated by the regulatory control scheme. This attenuation would be achieved by making changes to the plant inputs in response to observed measurement variations.

Slow disturbances and the operation of the optimising control system are obviously major influences on the economics of operating a plant. However, fast disturbances can also influence the profitability of the operation. This can occur in two ways;

- 1) "Backoff" from active constraints.
- 2) "Nonlinear" effects.

The first of these was introduced in Section 2.3.2, where the concept of moving the operating point away from the calculated steady-state optimum was introduced. This is necessary to prevent variables affected by disturbances violating constraints. The second point, nonlinear effects, can also be important economically but is more difficult to analyse and describe in simple physical terms.

To highlight some of the main concepts behind how fast disturbances can affect the objective function, it is first necessary to distinguish between linear and nonlinear systems. Previously, linear systems were defined as sets of equations consisting only of linear functions. For this analysis of the effect of fast disturbances, the key point is that the objective function is linear in terms of the variables affected by these disturbances. For example,

$$\Phi_{LN}(y) = ay_1 + by_2 + c \quad (4.1)$$

where a , b and c are constants and y_1 and y_2 are measurements, is in this context a linear system. Note that the process which determines y_1 and y_2 may be nonlinear - as shown later, it is the form of the objective function which determines the linearity of the system.

A nonlinear system includes nonlinear combinations of variables affected by the fast disturbances. For example,

$$\Phi_{NL}(y) = ay_1y_2 + by_2^2 \quad (4.2)$$

where a and b are constants and y_1 and y_2 are once again measurements.

To illustrate the difference in behaviour between linear and nonlinear systems, a simple numerical example will be presented. The aim is to show the difference in the effect on the objective function of a disturbance causing a single measured variable (y) to oscillate symmetrically about its steady-state with a magnitude of one. Obviously, in general disturbances and measurements will not oscillate symmetrically, but the simple variation considered here serves to illustrate some important concepts. Note that in this simple example, the objective functions have not been optimised, consequently restrictions imposed by being at the optimum (i.e., the impact of any active constraints) are not considered. These will be discussed later.

The linear and nonlinear objective functions are respectively;

$$\Phi_{LN} = y \quad (4.3)$$

$$\Phi_{NL} = y^2 \quad (4.4)$$

Figures 4.1a and 4.1b illustrate how these two objective functions are affected by a disturbance causing a variation in y .

Looking first at Equation 4.3, movement around the nominal operating point ($y = 0$) causes the objective function to fall to $\Phi = -1$, and then increase to $\Phi = 1$ as y moves over the range -1 to 1. Due to the assumed symmetric nature of these measurement oscillations, the net effect of the disturbance on the objective function will be zero. More specifically, the oscillation of the measurement is assumed to have an exactly equivalent period with a negative value to the time spent with positive values. A sinusoidal oscillation is an example of such an oscillation. Consequently, increases in the objective function will be exactly cancelled by equal but opposite decreases.

However, the effect of the same variation on the nonlinear objective function (see Figure

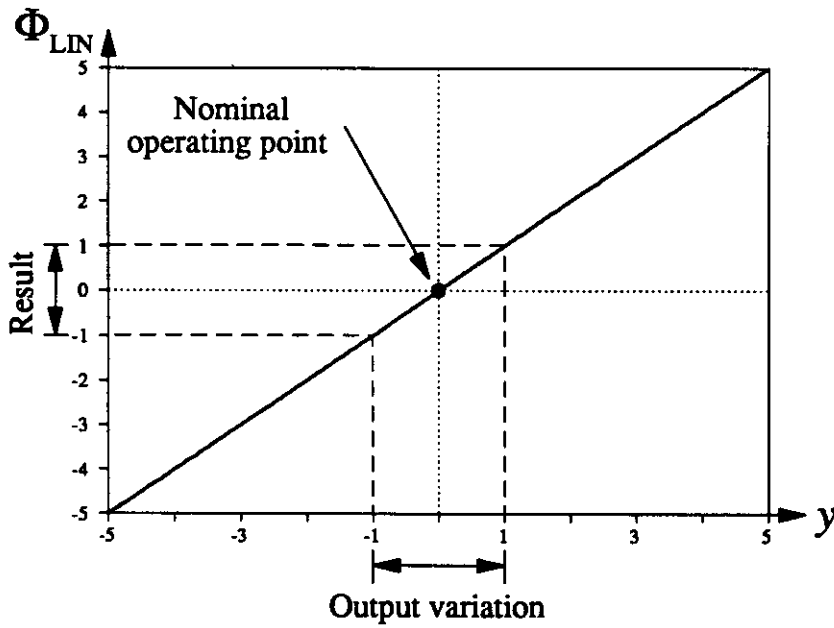


Figure 4.1a - Linear Objective Function

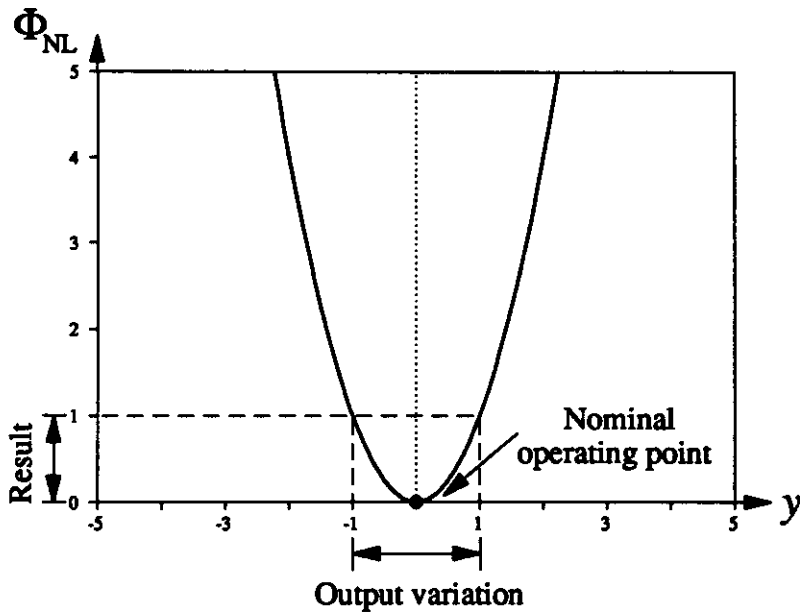


Figure 4.1b - Nonlinear Objective Function

4.1b) is quite different. As the disturbance causes the measured variable to deviate from the nominal operating point ($y = 0$), the objective function always increases.

Consequently, the net affect of a symmetric oscillation around the nominal operating point is an average objective function value higher than that obtained without the disturbance. If the goal of operating this simple process was to minimise the objective function, then the disturbance would cause the overall operation to be worse. This contrasts with the linear system where there is no net effect on the objective function due to the action of the disturbance.

Another way that disturbances can affect the objective function is due to "backoff". In Section 2.3, the concept of an operating "ball" was introduced. The effect of disturbances is to perturb the plant away from the nominal operating point, defining a region within which the plant may actually operate. To ensure that operational constraints are not violated, the nominal operating point must be sufficiently far from the constraints to ensure that the operating ball lies entirely within the feasible region. This means that despite disturbances causing the measured variables to change, all possible extremes will not violate any of the constraints. This movement of the operating point away from active constraints (and the optimum) is what is meant by the term "backoff".

Previously, it was pointed out that the optimum operating conditions for process plants frequently occurs when variables are at the limiting values imposed by operational constraints (e.g. maximum reactor temperature, or minimum column pressure). If the effect of disturbances is to require the operating point to be moved away from such active constraints, then in some cases this will result in a significant change in the objective function (and possibly a large change in process profitability). The cost of moving away from constraints can be calculated using the Lagrange multipliers. This relationship can be derived from the equations given in Chapter 3 which define the optimum (Equations 3.16, 3.17 and 3.18).

$$\nabla_x L(x, \mu, \nu) = \nabla_x \Phi(x) - \sum_{i=1}^m \mu_i \nabla_x g_i(x) + \sum_{j=1}^p \nu_j \nabla_x h_j(x) = 0 \quad (3.16)$$

$$k = 1, \dots, n$$

$$\mu_i g_i(x) = 0, \quad i = 1, \dots, m \quad (3.17)$$

$$h_j(x) = 0, \quad j = 1, \dots, p \quad (3.18)$$

where L is the Lagrangian (defined in equation 3.15)

Φ is the objective function

g is the vector of inequality constraints

h is the vector of equality constraints

x is the vector of process variables

μ is the vector of inequality constraint Lagrange multipliers

ν is the vector of equality constraint Lagrange multipliers

∇_x is the partial derivative with respect to x

For the simple case without equality constraints (in simple problems where measurements are the only variables involved in the optimisation problem, a process model, equality constraints, are not needed) Equation 3.16 becomes;

$$\begin{aligned} \nabla_x L &= \nabla_x \Phi - \mu^T \nabla_x g = 0 \\ &= \frac{d\Phi}{dx} - \mu^T \frac{dg}{dx} = 0 \end{aligned} \quad (4.5)$$

Rearranging this gives;

$$\frac{d\Phi}{dx} = \mu^T \frac{dg}{dx} \quad (4.6)$$

The analogous relationship for the more general case where there are also equality constraints is given by;

$$\frac{d\Phi}{dx_I} = \mu^T \left(\frac{\partial g}{\partial x_I} - \frac{\partial g}{\partial x_D} \left(\frac{\partial h}{\partial x_D} \right)^{-1} \frac{\partial h}{\partial x_I} \right) \quad (4.7)$$

In both Equation 4.6 and 4.7, the change in the objective function (Φ) due to a change in x (or in the independent variables x_I) is given as the amount that the change in the

variables affects the inequality constraint, multiplied by the Lagrange multiplier (μ). In the more general case, Equation 4.7, the changes that occur in the independent variables (x_i) also cause changes in the dependent variables (x_D). The total change in all the inequality constraints, due to the independent and dependent variables, is then multiplied by the Lagrange multipliers. Note that inactive inequality constraints will have Lagrange multipliers equal to zero, hence changes in their value will not contribute to the change in the objective function.

This Lagrange multiplier analysis gives the cost of moving the operating point away from the optimum so as to avoid violating operational constraints. However, disturbances can still affect the objective function at the "new" operating point (x_1'). To show the magnitude of these effects, a second-order Taylor series approximation can be made to the value of the objective function (Φ) as disturbances move the operating point (as reflected by changes in the independent variables Δx_i);

$$\Phi(x_i) \approx \Phi(x_i') + \left(\frac{d\Phi}{dx_i} \right)_{x_i'} (\Delta x_i) + \frac{1}{2} (\Delta x_i)^T \left(\frac{d^2\Phi}{dx_i^2} \right)_{x_i'} (\Delta x_i) \quad (4.8)$$

In this equation, the first and the second derivatives of the objective function must be evaluated at the new nominal operating point (x').

To illustrate more fully the difference between the linear and nonlinear effects that disturbances may have on an objective function, another simple numerical example will be used. This example involves two variables and one inequality constraint.

Example

$$\begin{aligned} \text{Minimise} \quad & \Phi = y_2^2 - y_1 \\ \text{such that} \quad & g(y) = 10 - y_1 \geq 0 \end{aligned} \quad (4.9)$$

The optimum to this problem is given by;

$$\begin{aligned} \Phi^* &= -10 \\ y_1^* &= 10 \\ y_2^* &= 0 \\ \mu &= 1 \end{aligned} \quad (4.10)$$

To calculate the cost of disturbances, the following derivative information is also required;

$$\begin{aligned}
 \frac{d\Phi}{dy_1} &= -1 \\
 \frac{d\Phi}{dy_2} &= 2y_2 \\
 \frac{d^2\Phi}{dy_1^2} &= 0 \\
 \frac{d^2\Phi}{dy_2^2} &= 2 \\
 \frac{dg}{dy_1} &= -1 \\
 \frac{dg}{dy_2} &= 0
 \end{aligned} \tag{4.11}$$

Without explicitly incorporating dynamics into our process model, the cost of disturbances can be examined by applying a sinusoidal variation of magnitude one to each variable in turn. Figures 4.2a and 4.2b show the objective function contours, the inequality constraint and the operating points for the cases when y_1 and y_2 are affected by disturbances, respectively.

The optimal value for y_1 lies on the active inequality constraint ($y_1^* = 10$). Consequently, a disturbance affecting y_1 only (y_2 remaining unperturbed at 0) means that the operating point will have to be moved away from the constraint by one unit to $y_1' = 9$ ($\Delta y_1 = -1$). Using Equation 4.6, the Lagrange multiplier can be used to calculate the cost of moving to this new operating point ($\Phi'(9,0)$);

$$\begin{aligned}
 \Delta\Phi &\approx \mu \frac{dg}{dy_1} \times \Delta y_1 \\
 &\approx 1 \times -1 \times -1 \\
 &\approx 1
 \end{aligned} \tag{4.12}$$

By evaluating the objective function at (y_1', y_2') , Equation 4.12 is found to give the exact change in the objective function for this example.

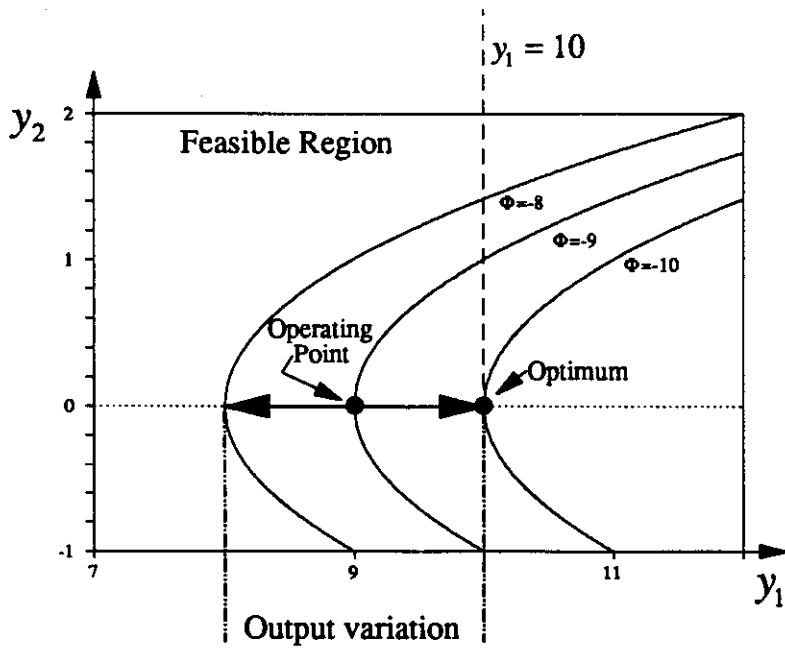


Figure 4.2a - The Effect of Disturbances on a Linear System

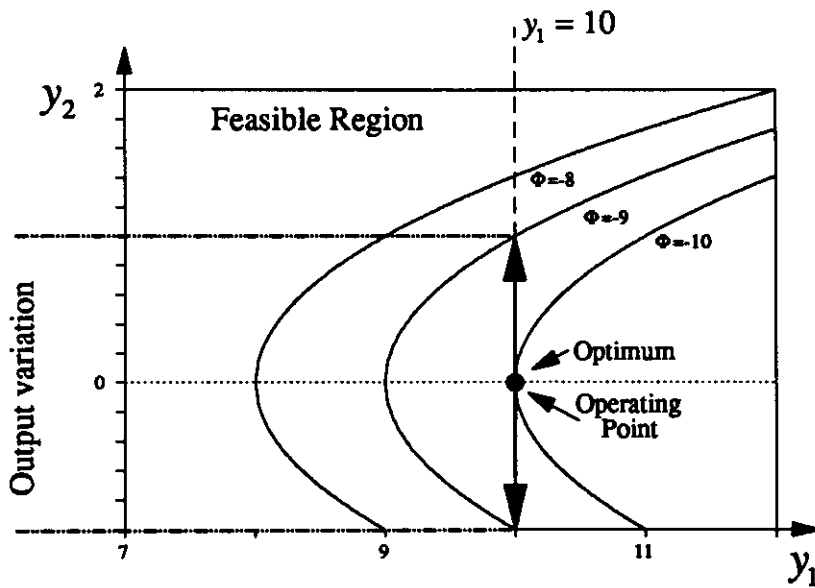


Figure 4.2b - The Effect of Disturbances on a Nonlinear System

The value of the objective function at the extremes of the movement of y_1 are $\Phi=-10$ and $\Phi=-8$. For a symmetric change in y_1 , these will balance out exactly leaving the average value as the nominal operating point value, $\Phi=-9$. This can be verified using the second-order Taylor series expansion given in Equation 4.8;

$$\begin{aligned}\Phi(y_1=10) &= \Phi(y_1=9) + (-1 \times 1) + (0 \times 1^2) \\ &= -9 - 1 \\ &= -10\end{aligned}\tag{4.13}$$

$$\begin{aligned}\Phi(y_1=8) &= \Phi(y_1=9) + (-1 \times -1) + (0 \times (-1)^2) \\ &= -9 + 1 \\ &= -8\end{aligned}\tag{4.14}$$

The use of the Taylor expansion clarifies the fact that no second-order (i.e. nonlinear) effects occur for disturbances affecting this variable - the second derivative of the objective function with respect to y_1 is zero.

By contrast, for disturbances affecting y_2 only, the objective function changes are due to second-order effects only. Since y_2 is not constrained by an active inequality constraint, the impact of any disturbance does not require the operating point to be moved away from the optimum ($y_1' = 10, y_2' = 0$).

In this case, the value of the objective function at the extremes of the movement in y_2 ($y_2 = -1, y_2 = 1$) is $\Phi = -9$. This means that as the disturbance causes y_2 to change, the value of the objective function will always be greater than the value (-10) of the original minimum. Once again, this point can be verified by examining the second-order Taylor expansion;

$$\begin{aligned}\Phi(y_2=-1) &= \Phi(y_2=0) + (0 \times -1) + (\frac{1}{2} \times 2 \times (-1)^2) \\ &= -10 + 1 \\ &= -9\end{aligned}\tag{4.15}$$

$$\begin{aligned}\Phi(y_2=1) &= \Phi(y_2=0) + (0 \times 1) + (\frac{1}{2} \times 2 \times 1^2) \\ &= -10 + 1 \\ &= -9\end{aligned}\tag{4.16}$$

Here we can see that the entire change in the objective function is due to second order (i.e. nonlinear) effects. The consequence of this is that the action of the disturbance is to only ever increase the objective function above the minimum at the nominal operating point.

Another way of viewing the effect of the disturbance is to plot the value of the objective function against time. The disturbance causes y_1 and y_2 to oscillate sinusoidally;

$$\begin{aligned} y_1 &= \sin\left(\frac{\pi}{5}t\right) \\ y_2 &= \sin\left(\frac{\pi}{5}t\right) \end{aligned} \quad (4.18)$$

Figures 4.3a and 4.3b show the value of the objective function when the disturbance affects y_1 and y_2 , respectively. The disturbance affecting y_1 causes a sinusoidal oscillation around the value of the objective function ($\Phi = -9$) at the new nominal operating point ($y_1 = 9, y_2 = 0$). By contrast, the disturbance affecting y_2 always causes the objective function to move away from the optimal value, always increasing its value.

It is quite clear from these results, that for this example, the disturbance affecting y_1 causes a greater average increase in the objective function than when the disturbance affects y_2 . More formally, this result can be quantified by integrating the objective function with respect to time (t), so as to calculate the total cost (Φ_{total}) for a given period.

$$\Phi_{\text{total}} = \int_0^t \Phi(t) dt \quad (4.19)$$

Integrating over the 100 time units shown in Figure 4.3, gives the following results for the two cases.

Disturbance affecting y_1 only

$$\begin{aligned} \Phi_{\text{total}} &= \int_0^{100} (y_2^2 - y_1) dt \\ &= \int_0^{100} (0 - 9 - \sin(\frac{\pi}{5}t)) dt \\ &= -900 \end{aligned} \quad (4.19)$$

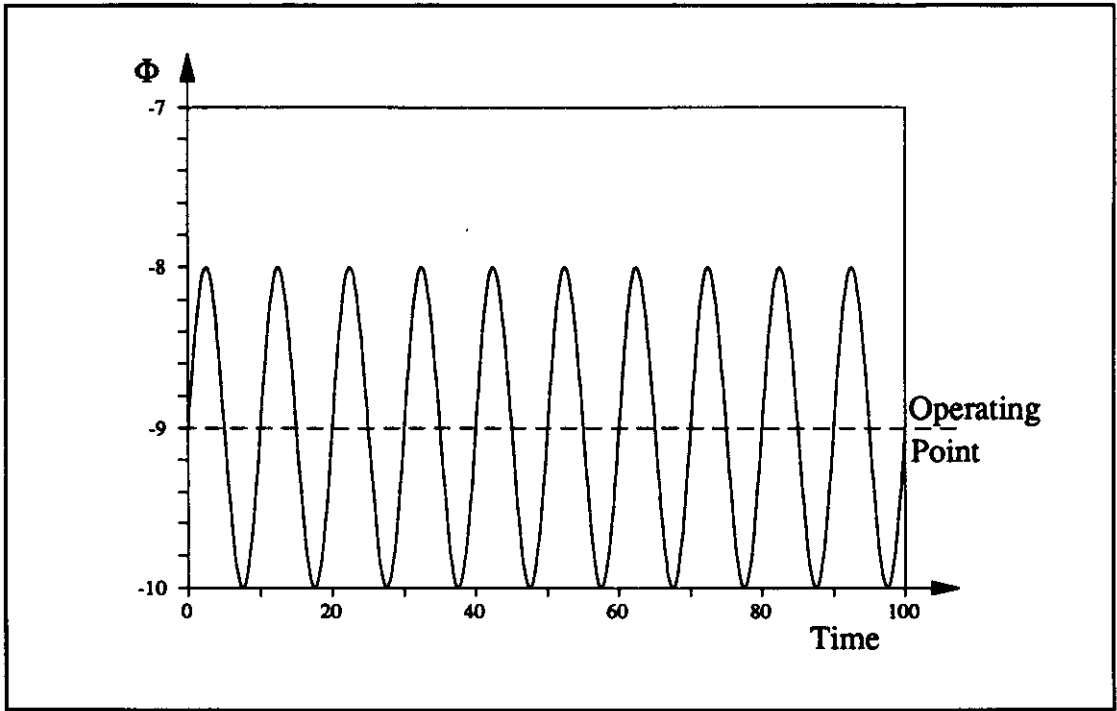


Figure 4.3a - Objective Function versus Time - Disturbance affecting y_1 only

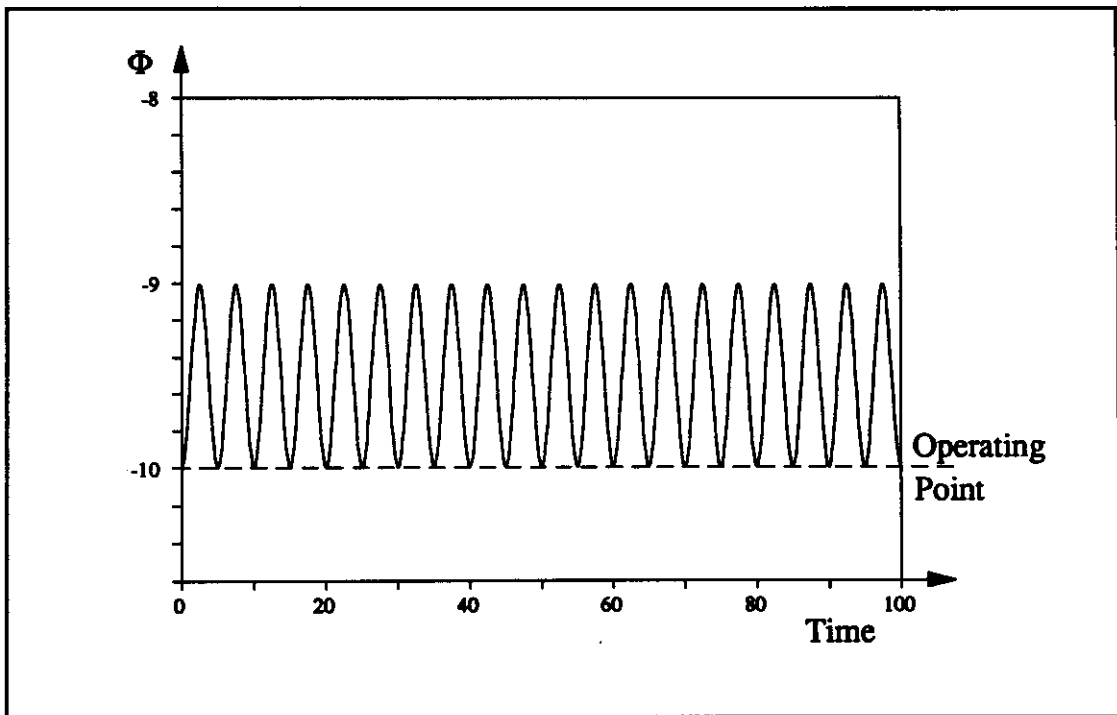


Figure 4.3b - Objective Function versus Time - Disturbance affecting y_2 only

Disturbance affecting y_2 only

$$\begin{aligned}
 \Phi_{\text{total}} &= \int_0^{100} (y_2^2 - y_1) dt \\
 &= \int_0^{100} \left(\left(\sin\left(\frac{\pi}{3} t\right) \right)^2 - 10 \right) dt \\
 &= -950
 \end{aligned} \tag{4.20}$$

These results are in agreement with Figures 4.3a and 4.3b. The overall effect of the disturbances is greater for the linear variable (y_1) due to backoff from the constraint $y_1 \leq 10$, than for the nonlinear variable (y_2) where the cost is due entirely to nonlinear effects.

This example demonstrates some of the features of linear and nonlinear systems when each is affected by fast disturbances. In the first case, when the disturbance affects y_1 only, changes in the objective function are linear. The only cost associated with such disturbances is that due to the initial movement in the operating point away from the optimum - the "backoff" necessary to avoid violating operational constraints. In the second case, when the disturbance affects y_2 only, then changes in the objective function are nonlinear. Here the operating point is at the original constrained optimum, but there is an increase in the objective function due to the nonlinear nature of the system.

In this illustrative example, one variable behaves linearly and the other nonlinearly, whereas in real process examples each variable could be associated with both linear and nonlinear terms. In such cases, the impact of disturbances could require the operating point to be moved away from all the active inequality constraints, while in addition, second-order effects could cause the average value of the objective function to be even further from the original optimal value.

The aim of this chapter is to develop the link between controllability (i.e. the dynamic movement of controlled variables under the influence of disturbances) and process economics. The example used in this section has shown that there are two distinct ways that disturbances can affect process economics - backoff from constraints and nonlinear effects. In the next chapter, process dynamics will be incorporated into the prediction of the best achievable plant performance in the face of both disturbances and model

uncertainty.

As this analysis is based on a linear (or linearised) model of the process, the following section deals briefly with several key concepts in the analysis of linear systems.

4.2 The Analysis of Linear Systems

To make a realistic prediction of the performance of a plant once it is actually built, the impact of disturbances on the plant's operation must be analysed. This prediction of operating performance can be used in the selection between different process designs. However, to make a fair comparison, not only the best, but also the worst possible performance must be considered. The actual operation of the plant will hopefully lie somewhere between the designer's optimistic best and worst possible predictions. In fact, it can be argued that the comparison of alternative process designs should be made on the basis of the worst possible performance only - in the knowledge that each plant should never operate any worse than that. Such an approach to the selection of plant design, selecting the "best worst" performance is a conservative approach. In some ways, it is analogous to the selection of optimising control setpoints on the basis of the worst case parameter variation, as proposed in Chapter 3.

The best possible performance is given by the optimal steady-state design, when there are no disturbances to perturb the operation away from the optimum. Calculation of the worst possible performance requires knowledge of the likely disturbances. Generally there is some uncertainty as to the exact nature of disturbances - their form (e.g. wave like or more like random noise), frequency and size. So from the set of all possible disturbances, the most detrimental effect on the performance of the plant needs to be found. In addition, if the steady-state and dynamic behaviour of the plant is poorly known, then the achievable performance calculation should be made at the worst conditions possible within these uncertain descriptions.

A method of quantifying the effect of the likely disturbances is clearly required. Such a determination of disturbance effects is dependent not only on the disturbances themselves, but also on the plant. For linear plants (i.e. those described by a set of linear model equations), it is possible to analyse the action of all disturbances using sine waves. This is achieved through the use of Fourier analysis, where any given function (i.e. a value of the

disturbance with time) can be decomposed into a set of sine waves. The summation of these sine waves gives the original wave. Although Fourier decomposition is also possible for the disturbances which enter nonlinear plants, it is not possible to characterise the general effect of disturbances purely on the basis of the behaviour of sine waves.

A simple explanation of why Fourier analysis can be used to analyse linear systems and not nonlinear systems, is that the amplification of the system (i.e. from process inputs to outputs) is independent of the input amplitude for linear systems, but not for nonlinear systems. This means that for a linear system, each of the component sine waves generated by a Fourier decomposition can be fed through the system separately and recombined at the output. The result of this process is exactly the same as sending the original wave into the linear system. Consequently, the behaviour of a linear system to a general input can be analysed using the component sine waves (i.e. the sine waves over the frequency range of the Fourier decomposition) of the original input. For a nonlinear system, the result of sending the original input and the component sine waves through the system is quite different.

This important difference can be illustrated with a simple example where an input (d), composed of two sine waves, is fed into a linear system (G_{LN}) and a nonlinear system (G_{NL}) to generate two outputs (y_{LN} , y_{NL}). The simple form of this input disturbance is similar to the result of a Fourier decomposition of a more complex function.

$$d = \sin(\pi t) + \sin(2\pi t) \quad (4.21)$$

The component ($d_1=\sin(\pi t)$, $d_2=\sin(2\pi t)$) and total input functions (d) are shown in Figure 4.4a.

The linear and nonlinear systems are given by;

$$\begin{aligned} y_{LN} &= G_{LN}(d) \\ &= d \end{aligned} \quad (4.22)$$

$$\begin{aligned} y_{NL} &= G_{NL}(d) \\ &= d^2 \end{aligned} \quad (4.23)$$

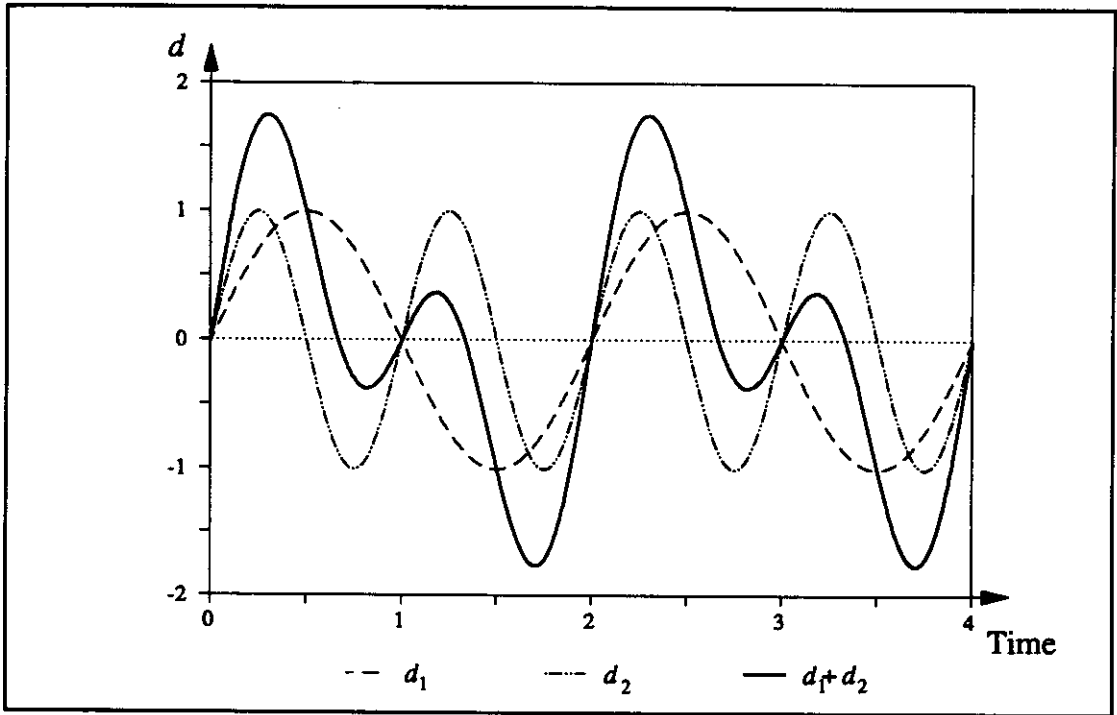


Figure 4.4a - Decomposition of "Sinusoidal" Disturbance

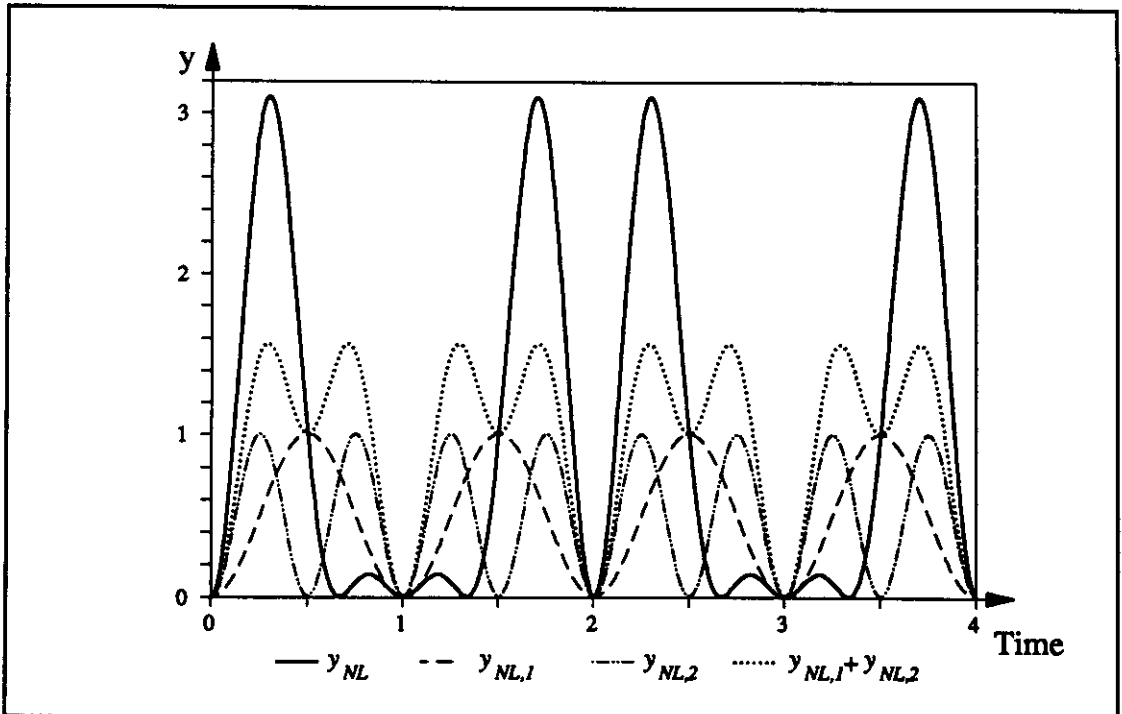


Figure 4.4b - Nonlinear System Behaviour

For the simple linear system, the output (y_{LIN}) is equal to the input (d). The same is true for the component waves when fed separately to the linear system;

$$\begin{aligned} y_{LIN} &= d \\ y_{LIN,1} &= d_1 \\ y_{LIN,2} &= d_2 \end{aligned} \tag{4.24}$$

So clearly, the sum of the component outputs is equal to the output generated by the original function;

$$\begin{aligned} y_{LIN,1} + y_{LIN,2} &= d_1 + d_2 \\ &= y_{LIN} \end{aligned} \tag{4.25}$$

Figure 4.4b shows the results of feeding the original function (d) and its two components (d_1, d_2) into the nonlinear system. Clearly, the result of feeding the original function is not equal to the sum of feeding the two components.

$$\begin{aligned} y_{NL} &= (d_1 + d_2)^2 \\ &= (\sin(\pi t) + \sin(2\pi t))^2 \\ y_{NL,1} + y_{NL,2} &= d_1^2 + d_2^2 \\ &= \sin^2(\pi t) + \sin^2(2\pi t) \\ y_{NL} &\neq y_{NL,1} + y_{NL,2} \end{aligned} \tag{4.26}$$

Consequently, there is no simple way to predict the behaviour of a nonlinear system to disturbances - every possible disturbance combination needs to be considered separately. By contrast, linear systems can be analysed by considering the response of the system to sine waves over a frequency range representative of the content (i.e. the Fourier decomposition) of the original disturbance. In Chapter 5 a method will be introduced for more explicitly describing those disturbances likely to occur. This is achieved by considering the set of all bounded inputs, such that the maximum magnitude at a given frequency is given by a function of frequency. The benefit of this approach is that the uncertain nature of the expected disturbances can be modelled by choosing a function which includes not only the nominal disturbance, but other "similar" disturbances.

Although Fourier type analysis (often referred to as frequency response analysis) is not directly applicable to nonlinear systems, linear based analysis can still be applied to such

systems. It has been found that over a limited range, many engineering systems can be approximated quite adequately by linear systems (Foss, 1973; Stephanopoulos, 1984). Consequently, linearised versions of nonlinear dynamic models can be used in the analysis of the dynamic behaviour of nonlinear systems for limited movements away from the point of linearisation.

It is therefore possible to use sine waves to characterise how disturbances effect a given (linearised) process design. To ensure that operational constraints are not violated, the effect of the "worst" disturbance combination needs to be found - a "bound" on the magnitude of the effect of disturbances is required. In many cases, such a bound corresponds to the behaviour of a system at steady-state, that is when the frequency is equal to zero. If this were true in general, then not only would linear analysis be simplified, but rigorous nonlinear analysis could be performed simply by evaluating the system behaviour at the extreme values of the disturbance only. Unfortunately this is not always the case, as may be demonstrated by the simple example of a second-order underdamped system. The differential equation representing the relationship between the disturbance (d) and measured variable (y) is;

$$\frac{d^2y}{dt^2} + 0.2 \frac{dy}{dt} + y = 2d \quad (4.27)$$

The response of the system to two different disturbances, each with the same amplitude, is shown in Figures 4.5a and 4.5b. The first graph shows the response to a low frequency disturbance (0.01 cycles per unit time), which approximates the steady-state behaviour. The second graph is for a disturbance at the resonant frequency of this second-order system (1 cycle per unit time). In both cases the disturbance was introduced to the system at steady-state ($y = 0$, $\dot{y} = 0$). The output of the system at the resonant frequency is significantly greater than the low frequency value. For this second-order system, the maximum effect of the disturbance does not occur at steady-state, but at a higher frequency. Although "resonant frequency" behaviour is well known for some simple systems (the resonant frequency can be determined from the model), the frequency at which maximum amplification occurs for a complex multiple-input multiple-output system cannot be determined as easily. Consequently the analysis must be performed over a suitable range of frequencies, to determine the maximum effect of disturbances.

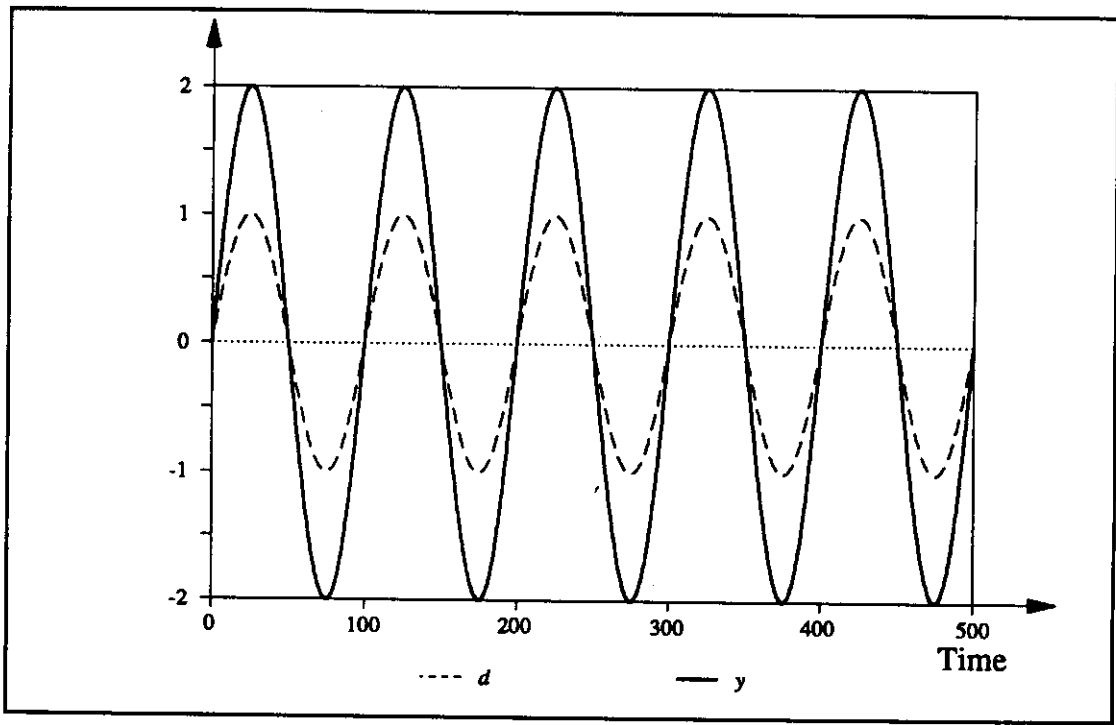


Figure 4.5a - Low Frequency Disturbance

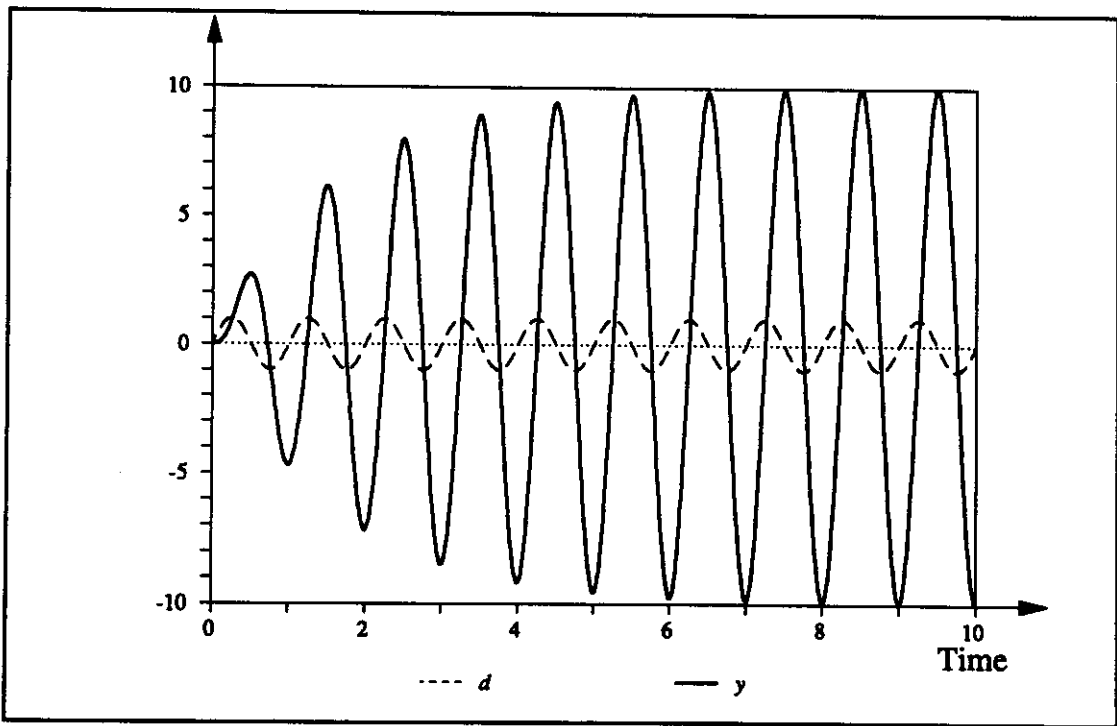


Figure 4.5b - High Frequency Disturbance

For linear systems, therefore, we are able to calculate exactly (and approximately for nonlinear systems via process model linearisation) the maximum magnitude of the variation in measured variables that disturbances will cause using frequency response analysis techniques. This information may then be used to calculate the necessary backoff from any active constraints to ensure that disturbances will not cause the constraints to be violated.

An alternative approach to using measures of the maximum effect of disturbances, is instead of moving the operating point, the inequality constraints in the original optimisation problem can be modified. A "buffer" added to these inequality constraints, equal to the necessary backoff, ensures that the optimal operating point calculated is sufficiently removed from the true inequality constraints to prevent their violation. In fact, this is a more rigorous approach, as constraints which formerly were inactive, may become active when the new optimisation is performed, thus requiring a backoff from these new active constraints. Consider the example used in Section 4.1, where the inequality constraint is;

$$10 - y_1 \geq 0 \quad (4.28)$$

In that example, the maximum variation in y_1 (due to any possible disturbance) was taken to be 1 (in general the maximum need not be at steady-state). An equivalent backoff buffer may be added to Equation 4.28 to give;

$$\begin{aligned} 10 - y_1 &\geq 1 \\ \Rightarrow 9 - y_1 &\geq 0 \\ \Rightarrow y_1 &\leq 9 \end{aligned} \quad (4.29)$$

The use of this modified constraint in the optimisation problem will ensure that operation at the calculated optimal operating point will not cause the inequality constraint to be violated due to disturbances affecting y_1 .

This method of using disturbance information to modify operational inequality constraints in the process optimisation problem, forms the basis for determining the achievable operating optimum. The optimal operating point, determined such that the disturbance induced operating "ball" around the nominal operating point lies completely within the

feasible region, is the best that the plant can achieve under actual operating conditions. This analysis (i.e. the backoff calculation) is exact for linear plants and approximate (through the use of a linearised model) for nonlinear plants. For nonlinear plants this procedure should still provide a good approximation to the best achievable operating point.

However, several problems still exist in the prediction of the best achievable optimal operating point for the nonlinear case. Earlier it was shown that for linear systems there is no net effect of disturbances on the objective function (other than backoff from constraints). That is, the average value of the objective function when symmetric disturbances enter the system is the same as the value at the nominal operating point. However, for nonlinear systems, there is an additional cost that causes the average value of the objective function to always be worse than the original optimal value. The only way of calculating, for a nonlinear system, the best achievable average objective function is to evaluate it for every likely disturbance.

An additional problem in this case, means that a linearised version of the nonlinear model is not useful in calculating the achievable performance. Although, as discussed, a linearised model can be used to approximately predict the magnitude of the effect of disturbances, it cannot be used to calculate the achievable performance (objective function) in the face of those disturbances. The reason for this is that the optimum of the linearised model will be fully constrained by active inequality constraints, so that the achievable performance calculation would involve the "backoff" cost from these constraints. Whereas the true nonlinear optimum, may, in the extreme, involve no active constraints - the optimum being determined by the curvature of the system. So the cost of disturbances would be solely due to nonlinear effects, not the backoff costs calculated using the linearised model.

Summary

In Chapters 5 and 6, the procedure of modifying constraints to allow calculation of the achievable optimal performance will be developed for both open and closed-loop linear systems (i.e. both with and without controllers). The algorithm to perform this analysis uses available descriptions of plant uncertainty in both the controller design and the calculation of the optimum. An operating point thus calculated ensures that for all

disturbances which could enter the system, the plant will operate optimally without violating operational constraints. The algorithm will be used in Chapter 6 to compare a set of four alternative flotation circuit designs, to determine which circuit can be operated most profitably in the face of disturbances. In Chapter 7, possibilities for extending the algorithm to a rigorous nonlinear analysis will be discussed.

CHAPTER 5

OPTIMAL OPERATION OF LINEAR SYSTEMS IN THE FACE OF BOTH DISTURBANCES AND UNCERTAINTIES

To enable a realistic comparison to be made between alternative process designs, an assessment must be made of how well they will perform once they are actually constructed, that is, in the face of disturbances and when the real plant begins to deviate from the design model. In the previous chapter, some of the underlying concepts of how disturbances can affect the performance of a plant were developed. Here, an algorithm will be presented which extends these concepts to allow the calculation of an operating point which is optimal, while ensuring that disturbances do not cause any violation of operational constraints.

In Chapter 6, the algorithm will be modified to allow for uncertain knowledge of the process, both when calculating the required constraint backoffs (dynamic uncertainty) and optimising the process model (steady-state uncertainty).

5.1 The Optimal Operation Algorithm

The optimal operating point, y^* , (i.e. the setpoint values for the process controllers) can be found by performing an optimisation to maximise (or minimise) an objective function (Φ) using a model of the process, expressed as equality constraints ($h = 0$), such that the operational inequality constraints ($g \geq 0$) are not exceeded in the face of disturbances. In the most general case, this may be expressed mathematically in terms of the plant inputs (u), the plant measurements, or setpoints, (y), disturbances (d) and other process model variables (z):

$$\begin{aligned}
 &\underset{y}{\text{Maximise}} \quad \Phi(u, y, d, z) \\
 &\text{such that} \quad g(u, y, d, z) \geq 0 \quad \dots (\text{in the face of disturbances, } d) \\
 &\quad \quad \quad h(u, y, d, z) = 0 \quad \dots (\text{process model})
 \end{aligned}
 \tag{5.1}$$

In this form the problem cannot be solved. A quantitative description of how disturbance dynamics affect the inequality constraints must be included. That is, what "backoff" is

required to ensure constraints are not violated due to the dynamic effect of disturbances on the input and output variables. To develop these relationships a simple single-input single-output example is presented in the next section to clarify the mechanics of constraint modification.

5.1.1 Constraint Modification

In the simple system to be considered, there is a single output y , and a single input u . These variables must lie within upper bounds (y_U, u_U) and lower bounds (y_L, u_L).

$$\begin{aligned} y_L &\leq y \leq y_U \\ u_L &\leq u \leq u_U \end{aligned} \quad (5.2)$$

This is shown diagrammatically in Figure 5.1a. The positions represented by y_0 and u_0 can be considered as the nominal operating point for this system. The relationship between changes in these steady-state values is therefore given by the steady-state model $G(0)$;

$$(y - y_0) = G(0)(u - u_0) \quad (5.3)$$

Taking the nominal operating point to be the origin (this can be done without any loss of generality), simplifies Equation 5.3 to;

$$y_0 = u_0 = 0 \Rightarrow y = G(0)u \quad (5.4)$$

If we now assume that this is a linear system, and optimise a linear objective function $\Phi(y, u)$, then the optimum will occur when one of the input or output constraints is active, that is, when either y or u (or both in some cases) lay on their upper or lower bound. This is shown in Figure 5.1b where the optimum is at $y = y^* = y_U$ and $u = u^* \neq u_L$.

Returning to the nominal operating point (y_0, u_0), we may now consider that a sinusoidal disturbance enters the system causing the output, y , to vary. The magnitude of the variation induced in y by the disturbance is d , as shown in Figure 5.2a. If we were to now optimise this system, we must ensure that y does not approach either of its bounds closer than d . This is shown in Figure 5.2b, where the optimal operating point is now at $y^* = y_U - d$. Also shown in the diagram are buffers which have been added to y_L and y_U . Operating points must lie in the region between the buffers to ensure that the output

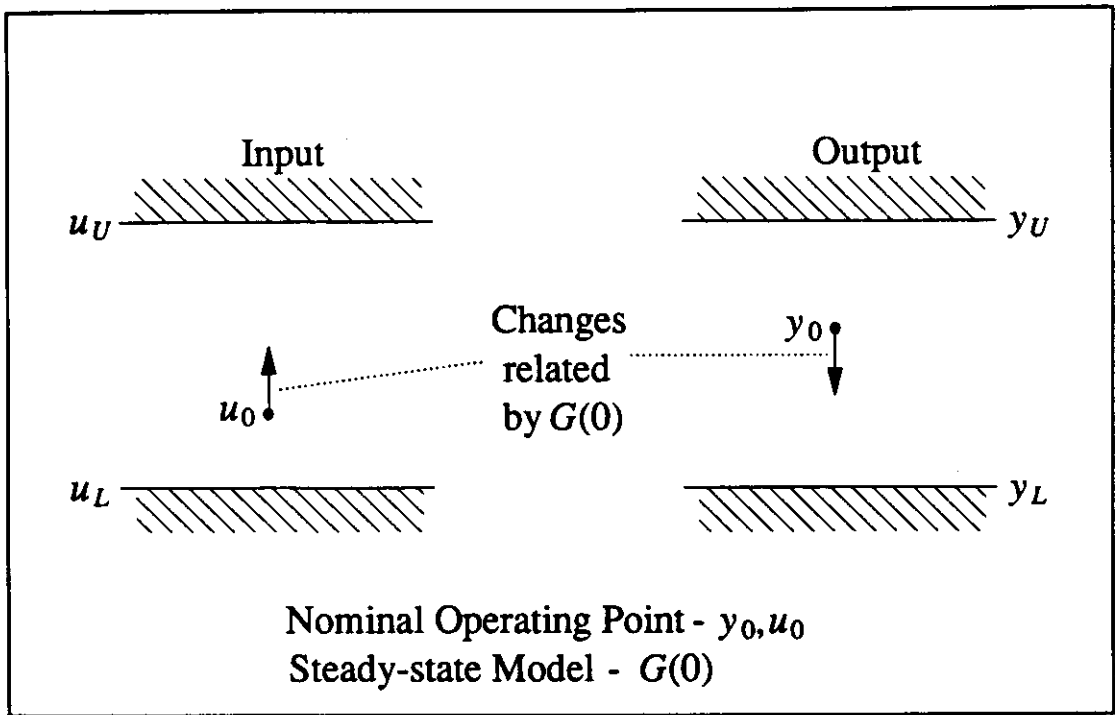


Figure 5.1a - Nominal Operating Point

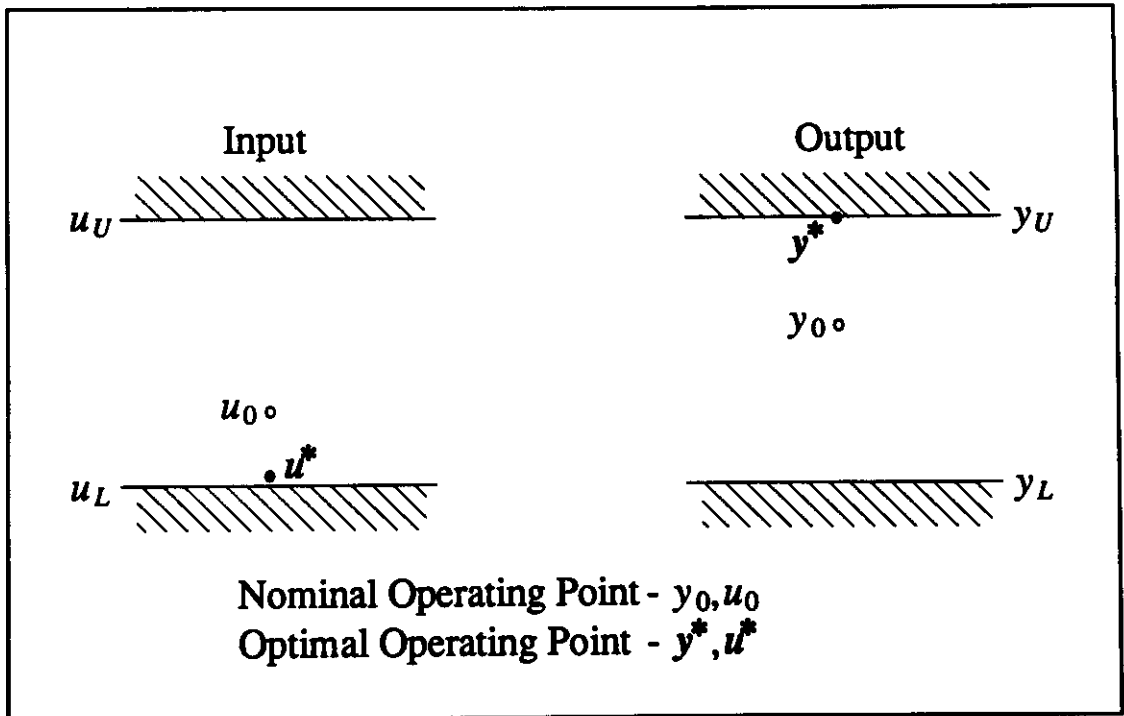


Figure 5.1b - Optimal Operating Point - No Disturbance

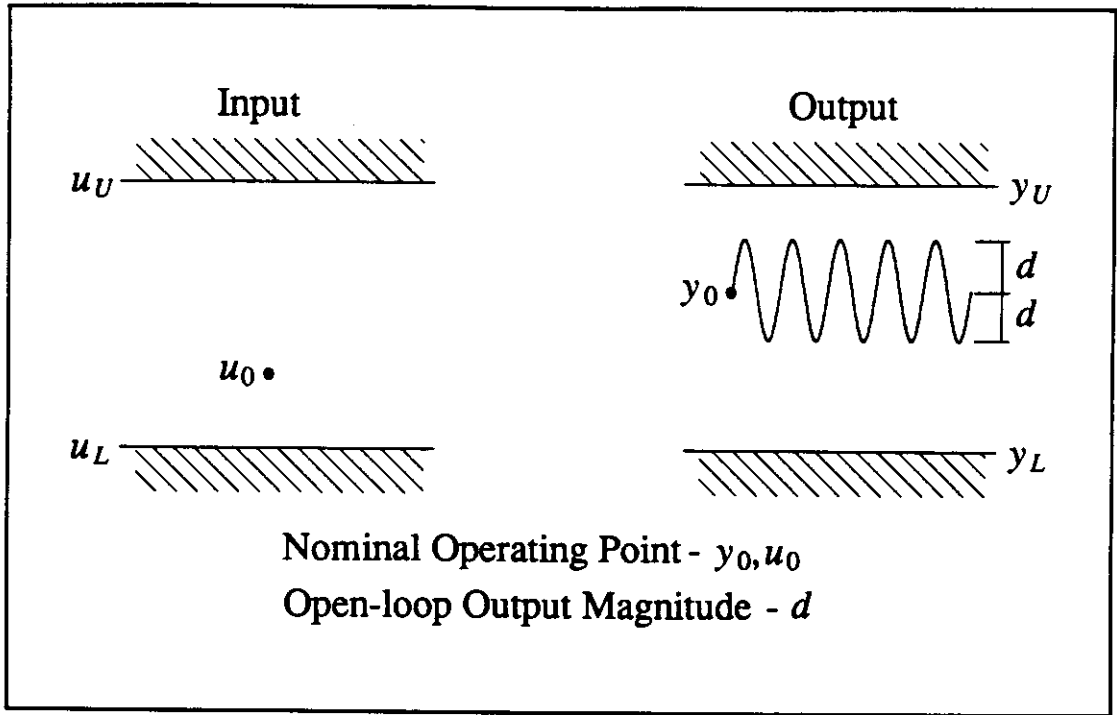


Figure 5.2a - Open-Loop Disturbance - Nominal Case

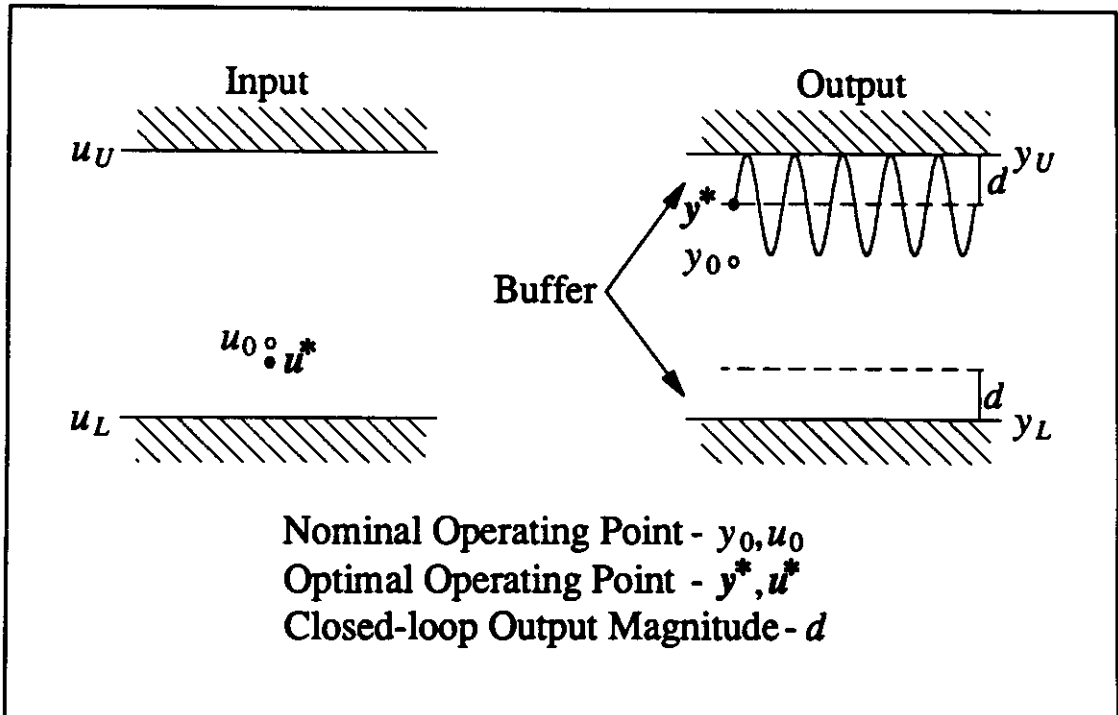


Figure 5.2b - Open-Loop Disturbance - Optimal Operating Point

bounds are not violated due to a disturbance. Note that for this open-loop situation, there is no variation in the input, u , due to the presence of the disturbance, although u has moved to u^* due to the change in y .

Another situation that can be considered is the closed-loop case, with a controller present to reduce the effect of the disturbance by manipulating the process input u . Depending on the process and the controller, there may still be some output variation despite the controller's manipulation of the input. However, in our search for the best performance that a process may ultimately achieve, we will consider the situation of "perfect" control. That is, some unknown controller has been implemented which ensures that there is no output variation when the disturbance occurs. More details of such controllers are discussed in Section 5.2. For the disturbance causing a variation in y of magnitude d discussed above, let the magnitude of the input manipulation necessary to achieve perfect control be u_d . This is shown in Figure 5.3a.

If this process under perfect control were to be optimised, we would now have to ensure that the value of u at the optimal operating point was at least u_d away from its bounds. This is shown in Figure 5.3b, where at the optimum, $u^* = u_L + u_d$. Also shown in the figure are the buffers of size u_d which must be added to the upper and lower bounds for u when performing the optimisation, so that the optimal value of u lies between these modified constraints.

For this simple system we now have two possible achievable optimal operating points. The first is when the system is open-loop which has an objective function value Φ_{OL} , and the second is for closed-loop perfect control, where the objective function is Φ_{CL} . Clearly the better of these two cases could define the achievable optimal performance for the process (Φ_A). Mathematically this could be described as;

$$\Phi_A = \text{Maximum}(\Phi_{OL}, \Phi_{CL}) \quad (5.5)$$

$$\text{where } \Phi_{OL} = \underset{y}{\text{Maximum}} \Phi(u, y) \quad (5.6)$$

$$\text{such that } \begin{matrix} y_L + d \leq y \leq y_U - d \\ u_L \leq u \leq u_U \end{matrix}$$

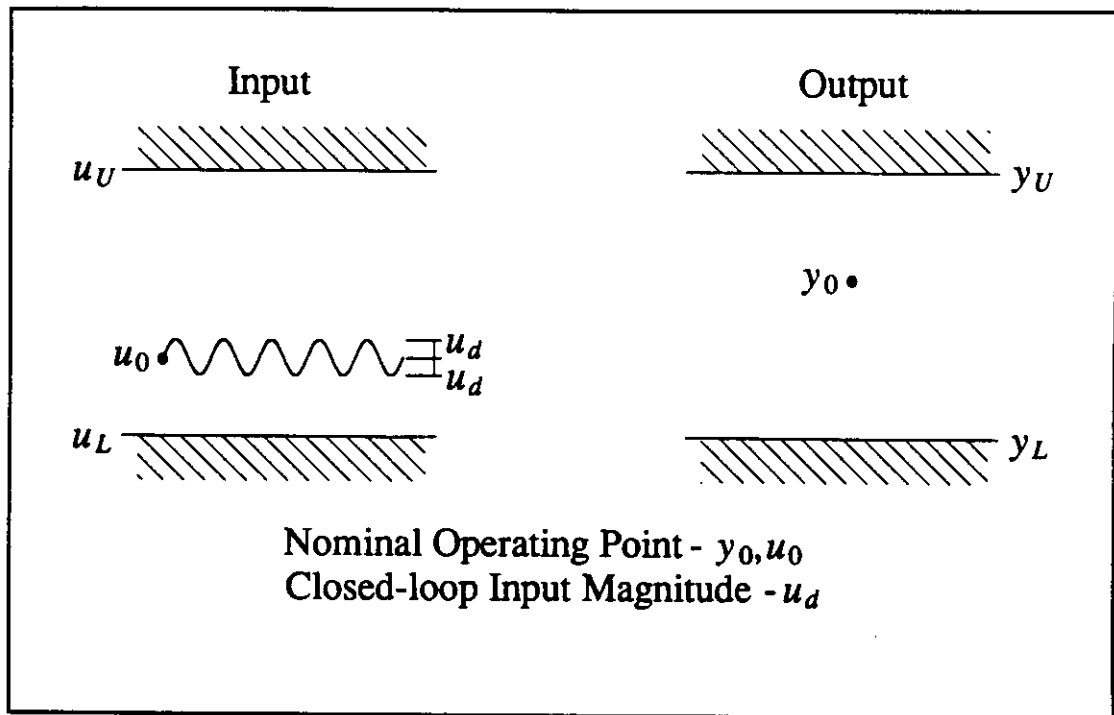


Figure 5.3a - Closed-Loop Disturbance - "Perfect" Controller

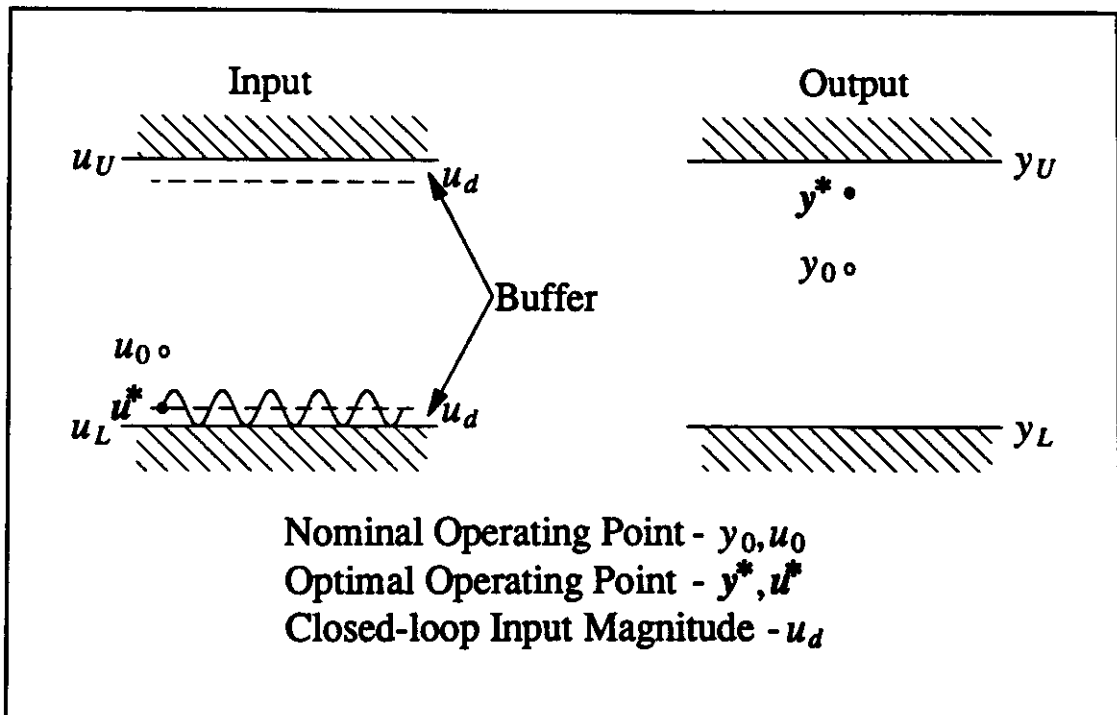


Figure 5.3b - Closed-Loop Disturbance - Optimal Operating Point

$$\begin{aligned}
 &\text{and} \quad \Phi_{CL} = \underset{y}{\text{Maximum}} \Phi(u, y) \\
 &\text{such that} \quad y_L \leq y \leq y_U \\
 &\quad \quad \quad u_L + u_d \leq u \leq u_U - u_d
 \end{aligned} \tag{5.7}$$

A useful extension can be made to the concepts presented here. That is the determination of the maximum disturbance that the system can handle. Looking first at the open-loop case, Figure 5.4a shows the maximum disturbance that the system can handle without exceeding the output constraints. Similarly, there will be a disturbance that under perfect closed-loop control will use all of the available input magnitude available as shown in Figure 5.4b. If the maximum disturbance to enter the plant is greater than the maximum that can be handled in the open or closed-loop cases, then feasible operation of the plant may not be possible.

Partial Control

However, a greater disturbance than in either the open or closed-loop cases may, in fact, be handled by combining the open and closed-loop performances. Consider a controller that is less than perfect, that is, despite manipulation of the input, some of the disturbance still appears at the output. If this controller were designed so that when a disturbance enters the system it uses the maximum available input magnitude, and then if this is insufficient it allows some of the disturbance to appear at the output, the capacity of the system to handle disturbances is increased. Looking at this another way, some of the disturbance is being attenuated by input manipulation and the rest is appearing at the output, achieving a larger total capacity for feasibly handling disturbances.

For this analysis, the procedure of designing a controller so that only part of the disturbance is attenuated is called "partial control". Details of the implementation and calculation procedures involving partial control are given in Section 5.2.1. However, it is useful now to introduce the partial control factor, k . This can be thought of as a scalar which modifies the error signal being sent to the controller.

$$\text{Open-Loop} \quad 0 \leq k \leq 1 \quad \text{Closed-Loop} \tag{5.8}$$

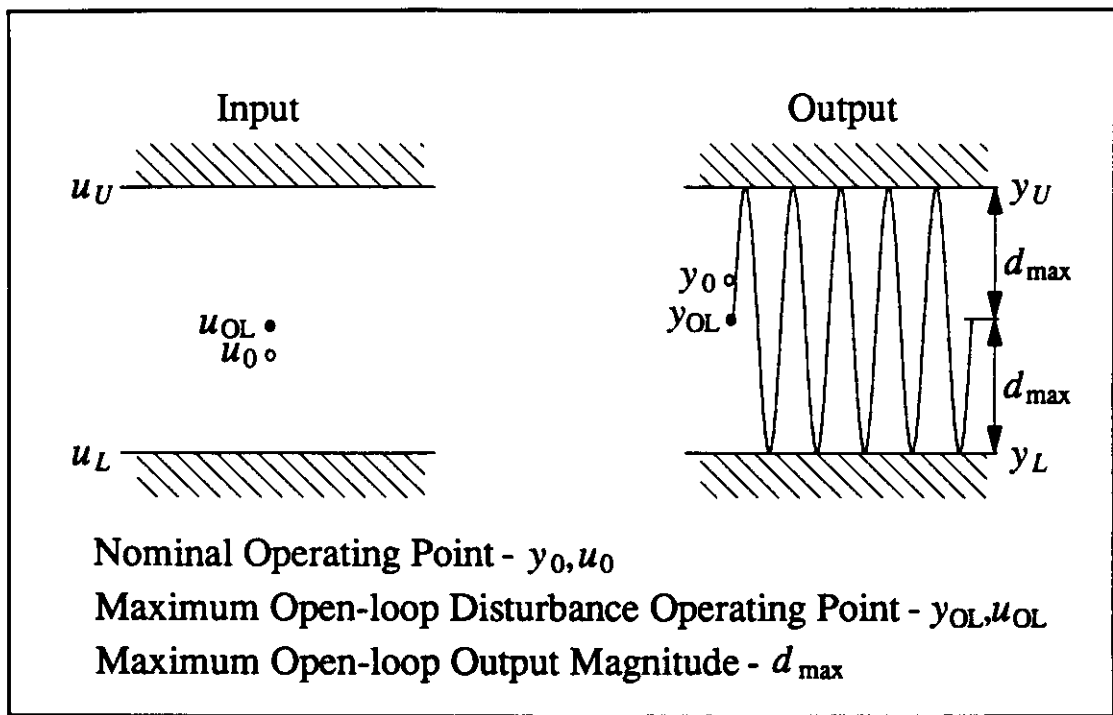


Figure 5.4a - Maximum Open-Loop Disturbance

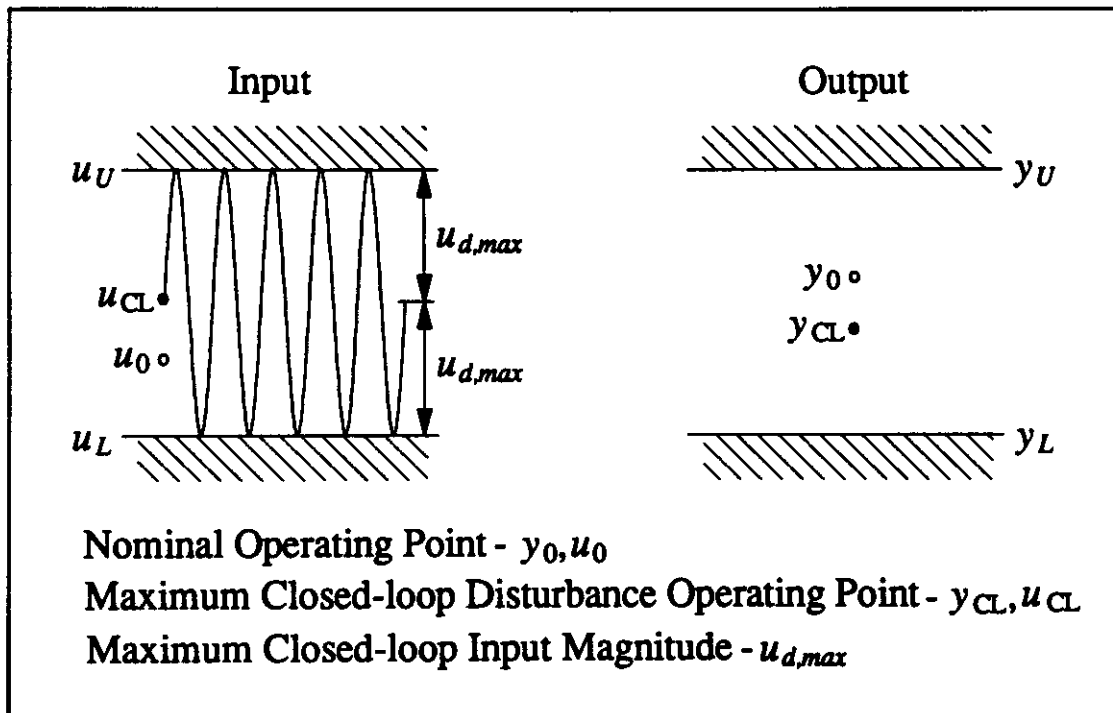


Figure 5.4b - Maximum Closed-Loop Disturbance

As shown in Equation 5.8, a value of $k = 0$ indicates that no error is being sent to the controller, which is the open-loop case. A value of $k = 1$ indicates that all the error is being sent to the controller, which is the normal closed-loop situation.

Without going into any detail at this stage, it should be noted that if the process to be controlled is open-loop unstable, that is bounded inputs (e.g. disturbances) can cause the plant to make unbounded output movements, then the lower limit on k will be greater than zero. This is because $k = 0$ is the unstable open-loop case, so some control is required to make the plant closed-loop stable (i.e. so that bounded inputs will not cause the plant to have unbounded outputs). Although not demonstrated in the examples presented here, this is an important limitation to the application of partial control.

The use of the partial control factor (k) is an important addition to the achievable optimal operation calculation. Firstly, not only does it allow the maximum permissible disturbance effect to be included in the problem, but it also improves the ease with which the balance between open and closed-loop control at the optimum may be found. In particular, we can now postulate two functions, $\Delta y(k)$ and $\Delta u(k)$ which give the magnitude of the disturbance appearing at the output (y) and the magnitude of the input required by the controller (u) as a function of the partial control factor k . (The delta has been used in the name of these functions to indicate that it is the movement of the variable away from the nominal steady-state values.) In general, we would expect that these two functions would show the following trends;

$\Delta y(k)$: As k increases from 0 to 1, $\Delta y(k)$ should decrease from the open-loop disturbance value (d), down to the minimum disturbance appearing at the output, zero for a perfect controller.

$\Delta u(k)$: As k increases from 0 to 1, $\Delta u(k)$ should increase from zero for the open-loop case to the maximum input requirement for closed-loop control.

Later, in Section 5.2.1, we will see that the shape of these functions is not always in agreement with simple intuition. For now, however, let us assume that they have been determined, methods for doing this will be described later.

It is now possible to set up an optimisation problem which will find the optimal achievable operating point, such that disturbances which enter the system will not cause the operational inequality constraints to be violated. The result of this optimisation will also indicate the value of the partial control factor, and thus whether it is optimal to operate the plant open-loop, closed-loop or with partial feedback control. Details of the solution to this problem are given in the next section.

5.1.2 Algorithm Implementation

Before detailing the optimisation problem, the results of Chapter 4 should be briefly reviewed to put this approach in perspective. It was found that the cost of disturbances can only be characterised for plants whose objective function behaves linearly, where the cost can be entirely attributed to the "backoff" from active constraints. Plants with nonlinear dynamic behaviour, but whose objective function is linear may also be analysed, as linearised dynamic behaviour can be used to calculate the magnitude of the effect of disturbances. The algorithm presented as Equation 5.9 below, gives the achievable optimal operating point for plants with a linear steady-state model and objective function, and for linear, or linearised, dynamics. The cost of movement away from the nominal optimum for this case is entirely due to the "backoff" from active constraints, to prevent their violation under all disturbance conditions.

The achievable optimal operating point (Φ_A) can be found by solving an optimisation problem which finds the setpoint values y^* , and the partial control factor k^* , which maximise the linear objective function Φ while satisfying the linear steady-state model $G(0)$, within the modified inequality constraints on the input and output variables. Mathematically, this may be expressed as;

Optimal Operation Algorithm

$$\begin{aligned}
 & \underset{y, k}{\text{maximise}} \quad \Phi(u, y) \\
 & y = G(0) u \\
 & u_L + \Delta u \leq u \leq u_U - \Delta u \\
 & y_L + \Delta y \leq y \leq y_U - \Delta y \\
 & 0 \leq k \leq 1 \\
 & \Delta u = f(k) : \Delta u(0)=0, \Delta u(1)=\max(\Delta u_{CL}) \\
 & \Delta y = g(k) : \Delta y(0)=\max(\Delta y_{OL}), \Delta y(1)=\max(\Delta y_{CL})
 \end{aligned} \tag{5.9}$$

The optimisation problem has been expressed in terms of input (u) and output (y) variables only. This is an extension of the single-input single-output example presented in Section 5.1.1 to the multivariable case. The constraint modification procedure is the same as developed previously, with "buffers" being added to each input and output constraint (see Figures 5.2b, 5.3b). The size of the buffer is different for each variable, as given by the vector functions $\Delta u(k)$ and $\Delta y(k)$, as defined in Section 5.1.1. Note, however, that the partial control factor (k) is still a scalar, affecting the performance of the error inputs to the "perfect" multivariable controller equally. At $k = 1$, the "perfect" controller will ensure that all inputs (u) are simultaneously manipulated to ensure that all outputs (y) are unaffected by the disturbance(s). Characteristics of the "perfect" multivariable controller will be discussed in Section 5.2.

Solving the optimisation problem, Equation 5.9, gives the optimal setpoint values, y^* , and the optimal partial control factor, k^* . Operation of the process with these setpoints and with a controller modified by k^* , ensures that despite the worst possible disturbance entering the system, the input and output constraints will not be violated and the objective function is optimal.

In order to use this method to determine the optimal achievable performance of a process design, information about the process model, the optimisation problem (i.e. the objective function and operational constraints), the disturbances and the model uncertainty are required so that the "perfect" controller can be designed, and the partial control relationships determined. The procedure for applying the algorithm is shown as a flowchart in Figure 5.5.

The first step shown in the flowchart is the determination of the required models, a linear steady-state model and a linear (or linearised) dynamic model. Next, to set up the basis for the optimisation problem, a linear objective function, written in terms of the input and output variables only, as well as inequality constraints for the input and output variables are required. The problem to be solved is written in a more limited form than the general problem posed in Equation 5.1, in that the only variables considered in the system are the input and output variables. In practice, both the objective function and the inequality constraints may include unmeasured variables (z) which are neither inputs nor outputs. In addition, the inequality constraints shown in Equation 5.9 are upper and lower bounds on

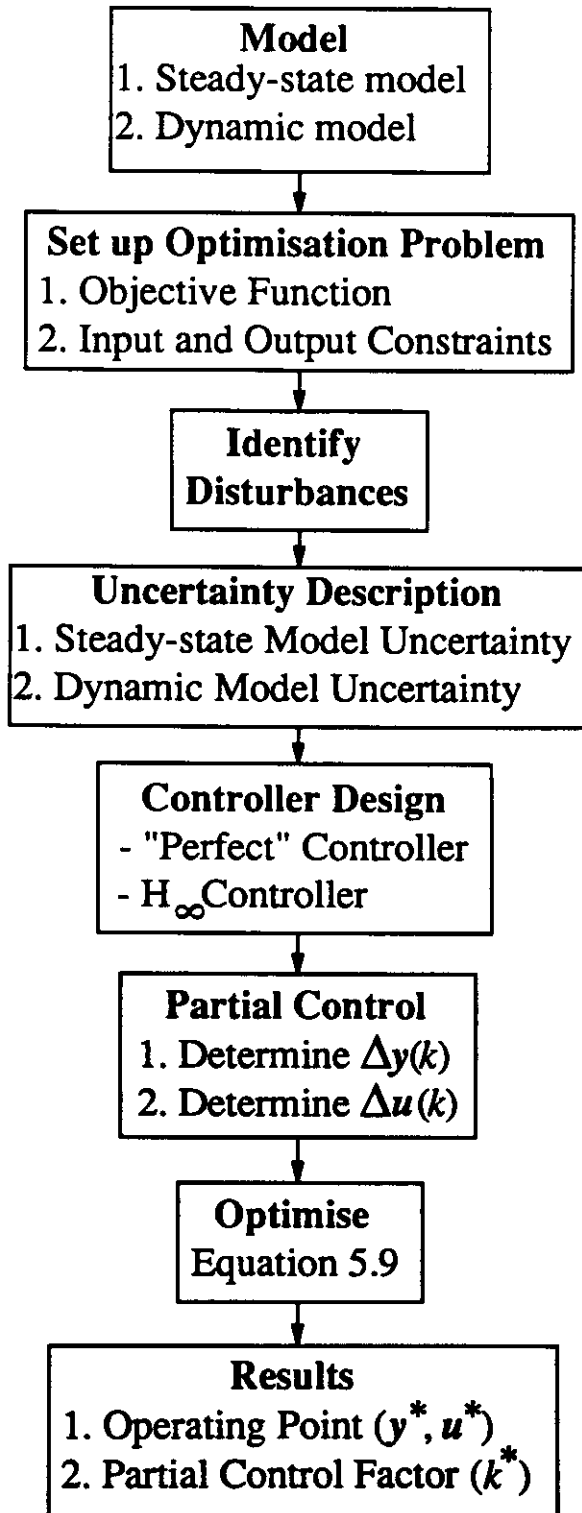


Figure 5.5 - Optimal Operation Algorithm Flowchart

each individual input and output variable, rather than mixed (linear) functions of the input and output variables. Examples of inequality constraints which do not fit these restrictions are;

$$\begin{aligned} z_1 &\geq 0 \\ u_1 + y_1 &\geq 0 \end{aligned} \tag{5.10}$$

Looking first at any unmeasured variables (z), as long as these variables only appear in the (linear) objective function and the (linear) steady-state model, then no modification is required to Equation 5.9. The unmeasured variables are simply included in the objective function and the steady-state model when the optimum is found.

However, using the algorithm for cases where there are non-standard inequality constraints (see Equation 5.10) is slightly more complex. The solution proposed here involves the use of "dummy" measurements, y_r . The approach is that any non-standard inequality constraint can be rewritten using these dummy measurements into a standard form. Using the above example;

$$\begin{aligned} y_{r,1} = z_1 &\quad \Rightarrow \quad y_{r,1} \geq 0 \\ y_{r,2} = u_1 + y_1 &\quad \Rightarrow \quad y_{r,2} \geq 0 \end{aligned} \tag{5.11}$$

These dummy measurements need not be used for control, but if they are required for constraint modification, the partial control functions, $\Delta y_r(k)$, can be generated from the new "dummy measurement" transfer functions in a similar manner to the expressions developed later for the controlled inputs and outputs. Although this technique is not used in the examples presented here, it has been included for completeness.

Bounded Disturbances

The next piece of information required for the application of the optimal operation algorithm, is a description of the disturbances likely to affect the process plant. Earlier the concept of sinusoidal disturbances entering a process was introduced. Here, using transfer functions, a simple method for describing a class of possible disturbances will be described. As presented, the method is most suitable for describing disturbances originating from a single source.

The disturbance description used allows the occurrence of sine waves of any phase (i.e. two waves of the same frequency are different in phase, if at any time they are at different points in their cycle), and whose magnitude is dependent on frequency.

$$\begin{aligned} d &= \text{disturbance} \\ &= m \sin(\omega t + \phi) \\ \phi &= \text{phase} : -2\pi \leq \phi \leq 2\pi \\ t &= \text{time} \end{aligned}$$

A convenient manner to describe this is to use a bounding function w ;

$$m = |d| \leq |w(s)|_{\omega} \quad (5.12)$$

The key to this description is the weighting function $w(s)$. This is in fact a transfer function (a function of the Laplace variable s), whose magnitude is a function of frequency. In Chapter 2, the magnitude of a transfer function at a frequency ω radians per unit time, was found by substituting $s = i\omega$ and calculating the magnitude of the resulting complex number, here $w(i\omega)$.

$$|w(s)|_{\omega} = \sqrt{[\text{Real}(w(i\omega))]^2 + [\text{Imag}(w(i\omega))]^2} \quad (5.13)$$

As an example, we can consider the disturbance weight given by;

$$w(s) = \frac{1}{100s + 1} \quad (5.14)$$

A graph of the magnitude of w as a function of frequency is shown in Figure 5.6. This particular weighting has a magnitude of 1 at steady-state (low frequencies) dropping down to 0.01 at a frequency of 1 radian per unit time. Using such a bounded disturbance description means that any sine wave whose magnitude is less than or equal to the magnitude of w is included in the analysis. The mathematics of using these weightings in the calculation of the maximum effect of a disturbance will be described in Section 5.2.1.

The next stage in the application of the optimal operation algorithm involves the design and analysis of the closed-loop control system. The first part of this analysis is the description of known (or expected) inaccuracies in the model of the process, both steady-

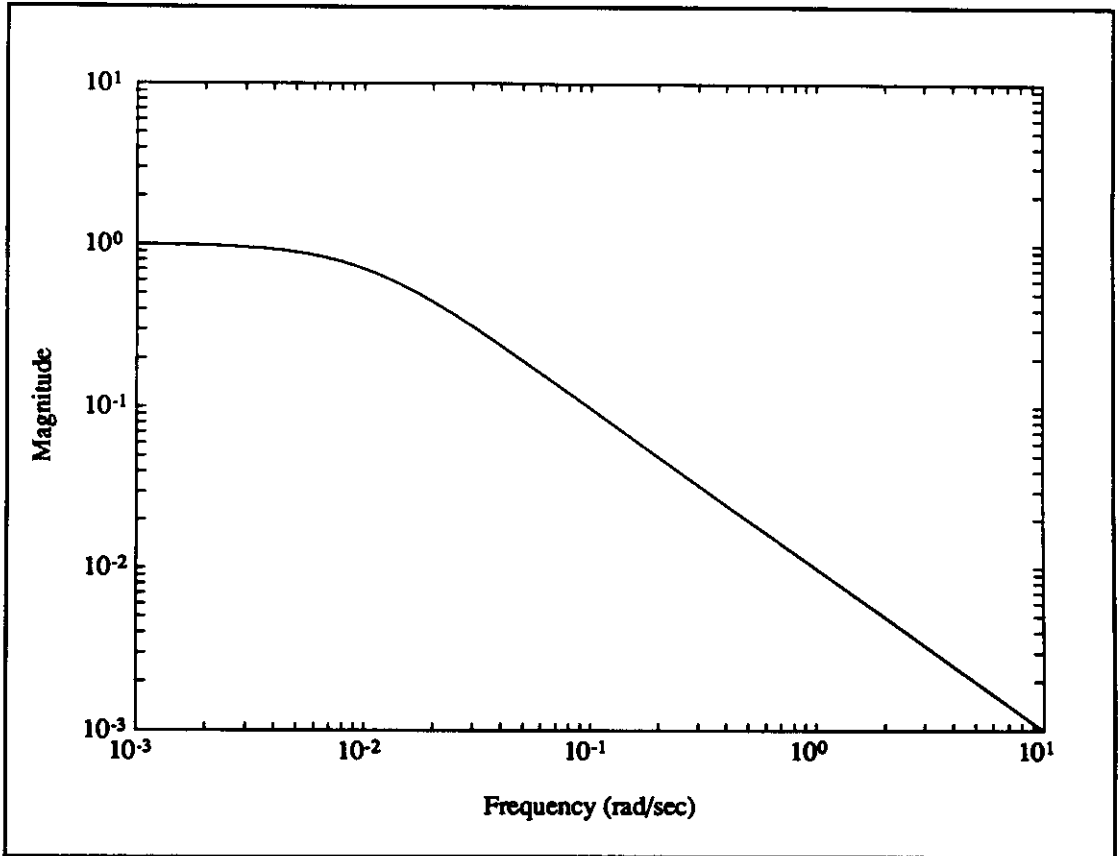


Figure 5.6 - Magnitude Bound for Example Disturbance

state and dynamic. Discrepancies between the actual process and the nominal model will always occur. This can be due to operation away from the point of linearisation for a nonlinear plant, or changes in the model parameters used to describe the process (e.g. catalyst activity). Using the dynamic model of the process, the description of the likely disturbances and of possible model uncertainties, a "perfect" controller can be designed and the partial control functions $\Delta y(k)$ and $\Delta u(k)$ can be found. This analysis is the subject of the next section.

5.2 "Perfect" Controller Design

In the previous section an optimisation problem was formulated, the solution to which is the optimal operating point at which a particular process design can operate without disturbances causing the operational inequality constraints to be violated. Part of the solution is the "amount" of control required at this operating point, described by the optimal partial control factor, k^* . For some designs the optimum may occur when there is no control (open-loop, $k = 0$), others may require full feedback closed-loop control ($k = 1$), while in some circumstances the optimum is achieved when only a fraction of the error is sent to the controller (partial control, $0 < k < 1$). When the partial control factor is at an intermediate value ($0 < k < 1$), then some control action is taken to attenuate the disturbance, with the "remainder" of the disturbance appearing as variation in the output variables.

To determine the best achievable operating point, the controller used for full (and partial) feedback closed-loop control needs to be the "best" available controller, in some sense. One possibility for this is the "perfect" controller, whereby all disturbances entering the system are completely attenuated. That is, the controller operates perfectly, with the measured variables never deviating from their setpoint values. From a theoretical point of view this is a desirable controller to use, especially in conjunction with the concept of partial control which allows a smooth graduation between the open-loop case with the entire disturbance effect appearing at the output, and no detectable disturbance effect under perfect closed-loop control.

5.2.1 Closed-Loop Performance Measures

In order to analyse the open and closed-loop performance of a process, so that the partial control functions ($\Delta y(k)$, $\Delta u(k)$) can be generated, some measures of the performance of

control systems need to be introduced.

This analysis starts from a block diagram layout of the information flow within a closed-loop control scheme, as shown in Figure 5.7. The major components of the closed-loop control scheme are the process plant (G) and the controller (C). For convenience, the models of these components are frequently represented as transfer functions, or transfer function matrices for the multivariable case. We should therefore write $G(s)$ and $C(s)$ for the plant and controller respectively. The arrows in Figure 5.7 represent signal flows, either in the time domain as occurs in the actual plant, or using frequency domain representations. The signal flows shown are;

d'	-	External disturbances entering the system.
d	-	Disturbances appearing at the plant output.
e	-	Error between setpoints and measurements.
n	-	Noise affecting the measurements.
r	-	Setpoint values.
u	-	Plant inputs (controller outputs).
y	-	Plant outputs or measurements.

These variables are shown in bold to indicate that they are vectors representing the general multivariable case. Again, for convenience, these are usually expressed in the frequency domain as functions of the Laplace variable s (e.g. $y(s)$). However, for simplicity the s is not usually indicated in the block diagram. The other element shown in Figure 5.7 is the transfer function matrix $G_d(s)$. This matrix represents the relationship between disturbances which enter the plant (d') and their effect (d) on the output measurements. In practice, disturbances do not usually directly affect plant measurements, but rather affect some (usually uncontrolled) plant input (e.g. feed composition). This input disturbance then propagates through the plant to appear as a variation in the plant measurements. It is this observed variation which can be used by the control system to make adjustment to other (manipulated) plant inputs to keep the plant steady.

An advantage of using Laplace transforms to represent the signals within the closed-loop, is that the transfer function matrices can be easily combined to show the relationship between different signals. The following are four fundamental relationships between the signals shown in Figure 5.7.

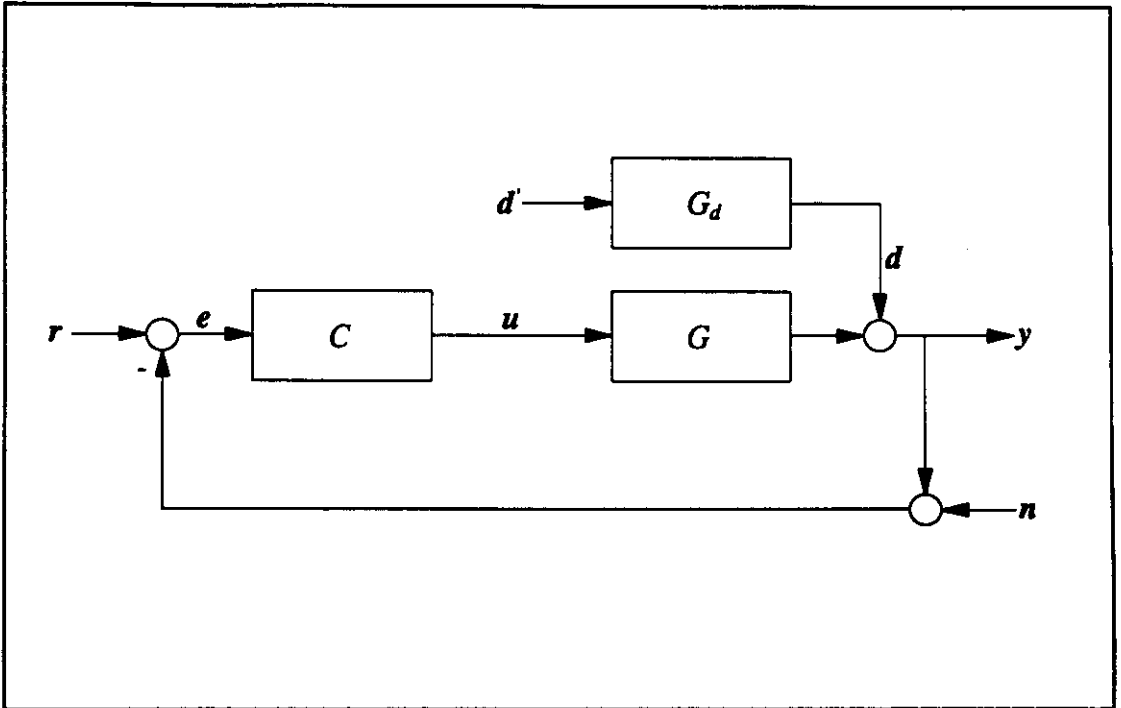


Figure 5.7 - Closed-Loop Feedback Block Diagram

$$d(s) = G_d(s) d'(s) \quad (5.15)$$

$$y(s) = G(s)u(s) + d(s) \quad (5.16)$$

$$u(s) = C(s)e(s) \quad (5.17)$$

$$e(s) = r(s) - (y(s) + n(s)) \quad (5.18)$$

Open-Loop Analysis

Before determining the required measures of performance for the closed-loop system, we should first investigate the important properties of the process itself, starting with internal stability. Quoting the definition given by Morari and Zafiriou (1989);

Internal Stability

A control system is internally stable if bounded signals injected at any point of the control system generate bounded responses at any other point.

If this were not the case, then it would be possible that small inputs to (or errors in) some parts of the closed-loop system could cause some signals to become unbounded, and hence the process to become unstable.

For the open-loop plant, stability is completely dependent on the process itself. In Chapter 2, state-space (Equations 2.12, 2.13) and frequency domain (Equations 2.14, 2.15, 2.16) descriptions were presented for the relationship between the process inputs (u) and the process outputs (y). Characteristics of the process can be determined by analysing either the state-space or the frequency domain representation. Details of these characteristics can be found in many standard process control texts (e.g. Stephanopoulos, 1984; Morari and Zafiriou, 1989). A summary of some of the more important features will be given here.

Poles

The stability of the open-loop plant can be determined from the "poles" (π) of the process model. These are defined as the eigenvalues of the A matrix in the state-space representation. The relationship between these values and the stability of the system is most easily demonstrated by their appearance in the Laplace inversion of the transfer function into the time domain, where the eigenvalues become exponents of the value e

(e.g. $y = e^{\pi t}$). Consequently, a system is only stable if the eigenvalues have strictly negative real parts (i.e. $\text{real}(\pi_i) < 0$). Such values lie in the open, that is not on the imaginary axis, left half of the complex plane. Such negative values lead to decaying exponential terms in the time domain solution.

In the SISO case, the poles may also be found as the roots of the denominator of the transfer function. This definition is not as convenient as the former (i.e. state-space representation) for the multivariable MIMO case where the denominator may not always be the same function for each element in the transfer function matrix. In that situation, manipulation of the elements in the transfer function matrix is required to determine the poles for the system. However, in the SISO case the transfer function can be most informative.

Consider the following example, where the state-space representation is given by;

$$\begin{aligned}\frac{dx}{dt} &= x + u \\ y &= x\end{aligned}\tag{5.19}$$

$$A = 1, B = 1, C = 1, D = 0$$

Using the relationship between the state-space model and the transfer function given by Equation 2.16 gives;

$$\begin{aligned}G(s) &= C(sI - A)^{-1}B + D \\ &= 1(s - 1)^{-1}1 \\ &= \frac{1}{(s - 1)}\end{aligned}\tag{5.20}$$

For this example, the eigenvalue of A is 1, and the root of the denominator of the transfer function $(s - 1) = 0$ is also 1. Clearly this is not strictly negative, leading to a term in the time domain solution of e^{+1t} . This is, thus, an unstable process model.

Zeros

Another important characteristic of open-loop processes are the "zeros" (ζ). Looking at the SISO case first, the zeros are the roots of the numerator of the transfer function.

When the Laplace variable s takes on one of these values ($s = \zeta_i$), the transfer function has the value zero. It is difficult to give an exact physical meaning to this situation, however as explained below, it has been shown that the presence of right half plane zeros (that is, zeros whose real part is non-negative) can cause control problems.

One problem associated with right half plane zeros is inverse response. When this occurs, the change in a measured variable due to an input change is initially in the opposite direction to the final change. This causes problems for conventional feedback controllers which may respond to the initial opposite response, exacerbating the measurement error. Generally, the closer a right half plane zero is to the imaginary axis, the greater the limitation on the achievable control. A thorough discussion on the effects of right half plane zeros can be found in Holt and Morari (1985) and Morari et al. (1987).

An example of a right half plane zero is found in the following transfer function;

$$G(s) = \frac{s - 1}{s + 1} \quad (5.21)$$

$$\Rightarrow \text{Zero} = \zeta = \text{root of } (s - 1) = 0$$

This example has a right half plane zero at the value $s = 1$.

The issue of right half plane zeros for the multivariable case is more complex. The definition given by Morari and Zafiriou (1989) is;

" ζ is a zero of $G(s)$ if the rank of $G(\zeta)$ is less than the normal rank of $G(s)$."

This is similar to the SISO case, where the value of the transfer function becomes zero at a system zero. In the MIMO case, part of the transfer function matrix becomes zero, reducing its rank. Morari and Zafiriou (1989) describe a technique for determining the zeros of a transfer function matrix, while Laub and Moore (1978) give a procedure for the calculation of zeros from the state-space representation.

Closed-Loop Stability

The preceding text described stability and other control issues for the open-loop process. To analyse the stability of the closed-loop, we need to make an addition to the closed-

loop block diagram given in Figure 5.7. In Figure 5.8 another input signal has been added, u' , which is an external signal added to the input calculated by the controller, u , to give the total input to the process. For this system, there are only two different inputs from a stability analysis point of view, u' and r . The other signal flows shown in Figure 5.8 which are subject to potential external inputs (just as u is affected by u') are d and e . However, for stability analysis these are equivalent to r . Similarly, the two independent outputs are y and u (u is generally referred to as an input, but for closed-loop analysis, the output of the controller can be measured, and so is regarded as an output). The system is *internally stable* if any bounded input injected into u' or r , only generates bounded outputs at the two independent outputs y and u .

By block diagram algebra, it is possible to determine the transfer functions between these independent inputs and outputs, as shown in Equation 5.22.

$$\begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} GC(I + GC)^{-1} & (I + GC)^{-1}G \\ C(I + GC)^{-1} & -C(I + GC)^{-1}G \end{pmatrix} \begin{pmatrix} r \\ u' \end{pmatrix} \quad (5.22)$$

If each element in this transfer function matrix is stable in the open-loop sense, then the closed-loop MIMO system is also stable. From this analysis, it can be shown that if both the process (G) and the controller (C) are stable, then the closed-loop system is also stable.

An alternative method for checking the stability of the closed-loop system is to use the Nyquist Stability Criterion, a derivation of which is given in Morari and Zafiriou (1989).

Closed-Loop Performance

The two most important relationships between inputs and outputs for the closed-loop system are the sensitivity function and the complementary sensitivity function. To see how these are related to the closed-loop performance, the following relationships, derived from Figure 5.7 and Equations 5.16, 5.17, 5.18, are important.

$$\begin{aligned} y &= Gu + d \\ &= GCe + d \\ &= GC(r - y - n) + d \\ \Rightarrow (I + GC)y &= GC(r - n) + d \\ \therefore y &= (I + GC)^{-1}GC(r - n) + (I + GC)^{-1}d \end{aligned} \quad (5.23)$$

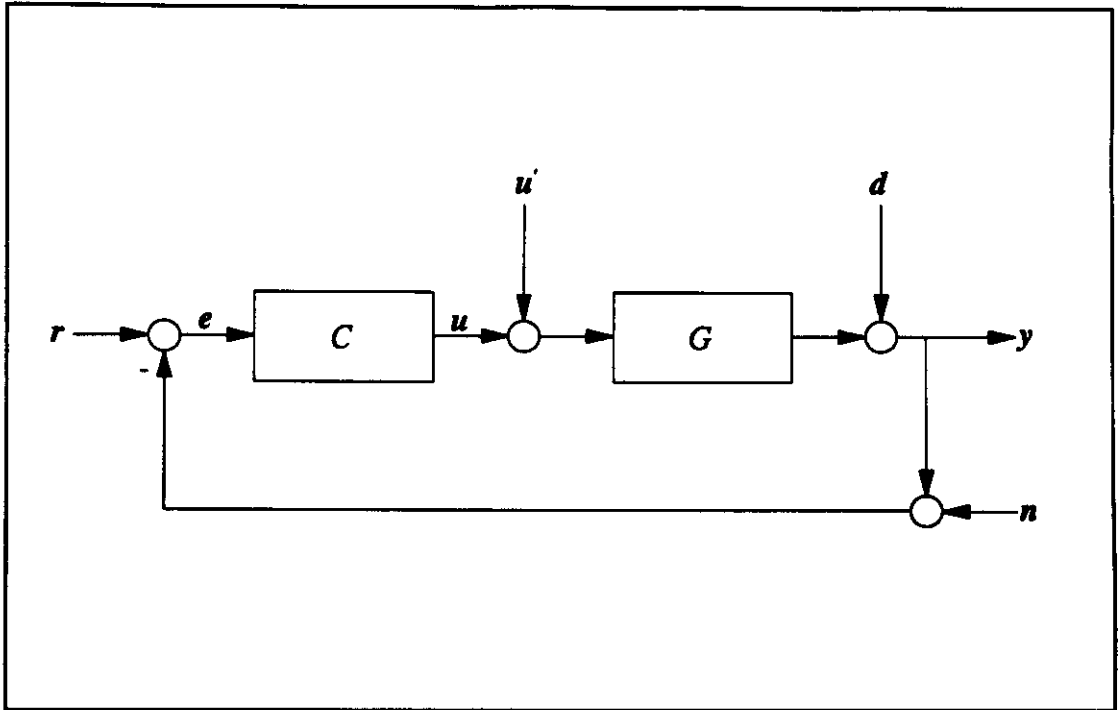


Figure 5.8 - Closed-Loop Block Diagram For Stability Analysis

$$\begin{aligned}
 e &= r - y - n \\
 &= r - d - GCe - n \\
 \Rightarrow (I + GC)e &= r - d - n \\
 \therefore e &= (I + GC)^{-1}(r - d - n)
 \end{aligned} \tag{5.24}$$

These relationships are applicable to both the SISO and MIMO case. The sensitivity (E) and complementary sensitivity functions (H), or matrices in the MIMO case, appear in the above relationships. They are defined as;

$$E = (I + GC)^{-1} \tag{5.25}$$

$$H = (I + GC)^{-1}GC \tag{5.26}$$

The sensitivity function (E) relates disturbances (d) to the measurements (y), and the setpoints (r), the disturbances (d) and measurement noise (n) to the error (e). From a performance point of view (disturbance rejection and good setpoint tracking), it is therefore desirable to make E as small as possible.

Unfortunately, for real physical systems, the term GC , which is a function of the Laplace variable s , tends to 0 as s tends to infinity.

$$\lim_{s \rightarrow \infty} [GC(s)] = \lim_{\omega \rightarrow \infty} [GC(i\omega)] = 0 \tag{5.27}$$

The consequence of this for the sensitivity function is;

$$\lim_{\omega \rightarrow \infty} |E(i\omega)| = \lim_{\omega \rightarrow \infty} |(I + GC)^{-1}| = I \tag{5.28}$$

Therefore for real systems, the most desirable value for E (i.e. zero) can only be achieved over a limited frequency range.

The complementary sensitivity function derives its name from the following relationship.

$$E + H = I \quad (5.29)$$

This function relates the measurements (y) to the setpoint values (r), hence it should be close to one (or I for the MIMO case) to ensure good setpoint tracking. However, the complementary sensitivity function also relates measurement noise (n) to the measurements (y) and from this point of view it should be made as small as possible. Using the limit given in Equation 5.27, it is clear that H will tend to zero at high frequencies, limiting good setpoint tracking to a finite frequency range.

This problem of striking a balance between closed-loop performance and noise rejection is fundamental to controller design. A compromise must be struck between good performance ($E = 0, H = I$) and good noise rejection ($H = 0, E = I$). Fortunately, in process control applications measurement noise is not a major concern (Morari and Zafiriou, 1989), so that controller design usually tends towards achieving good performance rather than noise rejection.

Partial Control

In Section 5.1 the concept of partial control was developed. Here through the use of the sensitivity function, we will develop expressions for the partial control functions $\Delta y(k)$ and $\Delta u(k)$.

Using the transfer function relationships given in Equation 5.23, for the case where there are no setpoint changes or measurement noise, just disturbance rejection (performance), the closed-loop outputs (y) due to disturbances appearing at the outputs (d) are given by;

$$\begin{aligned} y &= (I + GC)^{-1}d \\ &= Ed \end{aligned} \quad (5.30)$$

Similarly, the controller outputs (plant inputs, u) are found by multiplying this by the controller transfer function matrix C .

$$\begin{aligned} u &= C(I + GC)^{-1}d \\ &= CE d \end{aligned} \quad (5.31)$$

Partial control is achieved by sending only a fraction of the error between the setpoint and the measurement to the controller. In effect the controller actually in the closed loop, C' , is the original controller, C , scaled by the partial control factor, kI , where I is the identity matrix used to equally scale all error signals in the multivariable case.

$$C' = kIC \quad (5.32)$$

In a more general sense, it may be possible to scale each error signal differently, and hence achieve a more "optimal" design. However, the goal of this analysis is to find a "first pass" balance between open and closed-loop in the search for the optimum achievable operating point. Consequently, the additional complexity of using different scalings was not felt to be warranted.

The inclusion of the partial control factor in the closed-loop diagram is shown in Figure 5.9. For this system, the relationships between the disturbance at the output (d), the measurement (y) and the controller output (u) become;

$$y = (I + GCkI)^{-1}d \quad (5.33)$$

$$u = CkI(I + GCkI)^{-1}d \quad (5.34)$$

From these relationships, it is possible to predict how k may affect the measurement and the controller outputs for a given disturbance. Looking first at the measurements, if the term $GCkI$ is "small" then Equation 5.33 becomes;

$$\begin{aligned} y &= (I + GCkI)^{-1}d \\ \Rightarrow y &\approx Id \quad (\text{if } GCkI \ll I) \end{aligned} \quad (5.35)$$

This means that the magnitude of the disturbance measured at (or reaching) the output is independent of k , and is in fact much the same as in the open-loop case (as might be expected for a small controller term). Alternatively, if $GCkI$ is "large" then Equation 5.33 becomes;

$$\begin{aligned} y &= (I + GCkI)^{-1}d \\ \Rightarrow y &\approx \frac{1}{k}I(GC)^{-1}d \quad (\text{if } GCkI \gg I) \end{aligned} \quad (5.36)$$

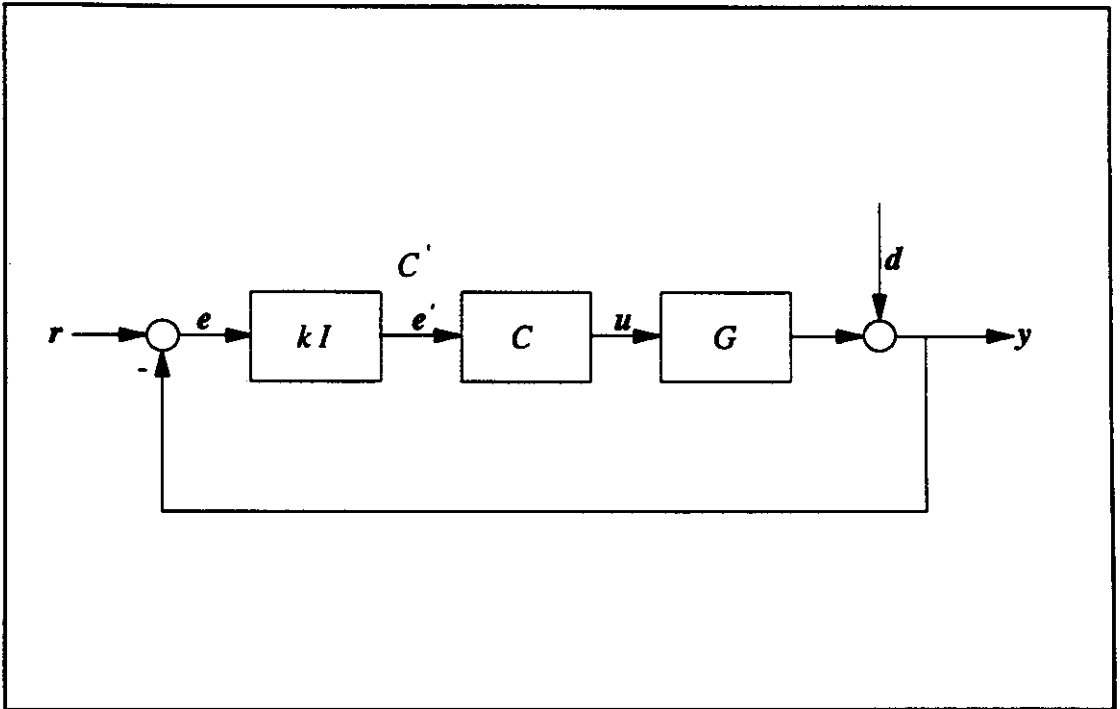


Figure 5.9 - Closed-Loop Block Diagram showing the Partial Controller C'

In this case, the disturbance appearing at the output is inversely proportional to k . Although this change is in the expected direction, intuitively the relationship may be expected to be linear.

For the calculation of the controller output, when $GCKI$ is "small", Equation 5.34 becomes;

$$\begin{aligned} u &= CKI(I + GCKI)^{-1}d \\ \Rightarrow u &\approx kICd \quad (\text{if } GCKI \ll 1) \end{aligned} \quad (5.37)$$

This means that the controller output is proportional to the partial control factor k . This relationship is in agreement with simple intuitive expectations. If $GCKI$ is "large", then Equation 5.34 becomes;

$$\begin{aligned} u &= CKI(I + GCKI)^{-1}d \\ \Rightarrow u &\approx Gd \quad (\text{if } GCKI \gg 1) \end{aligned} \quad (5.38)$$

For this situation, the inputs to the plant generated by the controller are independent of k . This is the least expected result, that under certain circumstances, as the controller is effectively "turned down", the magnitude of the controller output remains unchanged.

The partial control functions, which are required for the optimal operation algorithm, can be generated using Equations 5.33 and 5.34. However, it is difficult to predict what form they may take (as shown by the differing limiting trends described above), making it difficult to derive any generalisations about the outcome of employing the optimal operation algorithm.

An extension is required to these formulae to determine the maximum effect that any disturbance may have on the system. In this light, $\Delta y(k)$ and $\Delta u(k)$ can be defined as;

$$\Delta y(k) = \text{Maximum } [y = (I + GCKI)^{-1}d] \quad (5.39)$$

for any d

$$\Delta u(k) = \text{Maximum } [u = CKI(I + GCKI)^{-1}d] \quad (5.40)$$

for any d

Using the definition of bounded disturbances given in Equation 5.12, these maxima may be found using frequency analysis. In the examples considered later in this work in Section 5.3 and Chapter 6, the possible disturbances are treated separately as single variable inputs to each variable in the closed loop system, hence each element in the partial control functions can be treated as a single-input single-output system. In the more general case, where more than one simultaneous disturbance occurs, a method would be required to find the maximum effect for the simultaneous occurrence of the disturbances.

For the SISO like cases considered in this thesis, the magnitude of y_i for a particular disturbance is therefore given by;

$$\begin{aligned} |y_i| &= |((I + GCkI)^{-1})_i d| \\ &= |((I + GCkI)^{-1})_i| |d| \end{aligned} \quad (5.41)$$

Using the disturbance description given in Equation 5.12 gives;

$$\begin{aligned} |d| &\leq |w| \\ \therefore |y_i| &\leq |((I + GCkI)^{-1})_i| |w| \\ &\leq |((I + GCkI)^{-1})_i w| \end{aligned} \quad (5.42)$$

The partial control function $\Delta y(k)$ is therefore given by;

$$\Delta y_i(k) = \text{Maximum}_{\text{for any } \omega} |((I + GCkI)^{-1})_i w| \quad (5.43)$$

A similar development gives the result for $\Delta u(k)$;

$$\Delta u_i(k) = \text{Maximum}_{\text{for any } \omega} |(CkI(I + GCkI)^{-1})_i w| \quad (5.44)$$

For a given controller design and disturbance description, the partial control functions can be generated using the technique described above. It should be noted, that if the description of the disturbance entering the system is not in terms of its effect at the output, but rather first passes through a disturbance transfer function (e.g. d' passes through G_d in Figure 5.7), then the w term in the above expressions should be replaced by $G_d w$, where w is the bounding function describing d' . Methods for determining the most appropriate controller will be discussed initially in the context of Internal Model Control.

Internal Model Control

In order to study "perfect" control, it is convenient to introduce the concept of Internal Model Control (IMC). This is an alternative to the conventional feedback control structure, as shown in Figure 5.7. Internal Model Control embeds a model of the plant ($\tilde{G}$) into the closed-loop control structure. This technique was first developed in the 1950's, with the term Internal Model Control being coined somewhat later by Garcia and Morari (1982). Apart from the analytical advantages that we will see here, it is claimed that IMC gives an improved insight into plant control, as well as simplified stability and controller tuning procedures.

A brief description of some of the more important characteristics of IMC will be given. Much has been written about the properties of IMC - Morari and Zafiriou (1989) provide references to the wealth of literature available.

Figure 5.10 shows a simplified IMC block diagram - note that measurement noise has not been included. The principle of operation is that the difference between the actual measurement (y) and that predicted by a model ($\tilde{G}$) of the process ($\hat{y}$), which has been fed the same inputs as the process itself (u), drives the controller (Q), rather than the error between the setpoint and measurement as used in classic feedback control.

There is, in fact, a direct relationship between the IMC and "classic" feedback control structures. A slight rearrangement of the IMC structure given in Figure 5.10, gives the arrangement shown in Figure 5.11a, which is of the classic feedback form, where the controller C is given by the combination of Q and $\tilde{G}$ in the shaded box. The relationship between the classic feedback controller C and the IMC controller Q is;

$$C = Q(I - \tilde{G}Q)^{-1} \quad (5.45)$$

Similarly, starting with the classic feedback structure, if a model of the plant ($\tilde{G}$) is added around the plant (G) and then subtracted from around the feedback controller (C), see Figure 5.11b, introducing no net change to the plant inputs (u) and outputs (y), then the combination of C and $\tilde{G}$ shown in the shaded box, is equivalent to the IMC controller Q . Thus we have the relationship,

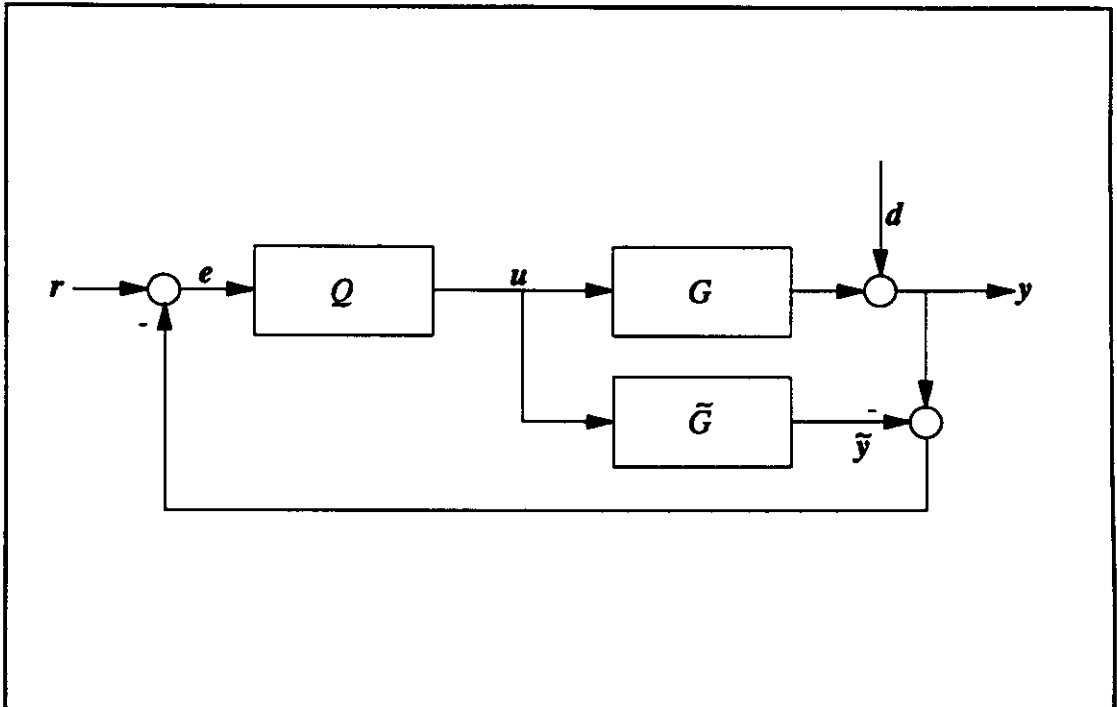


Figure 5.10 - Internal Model Control Block Diagram

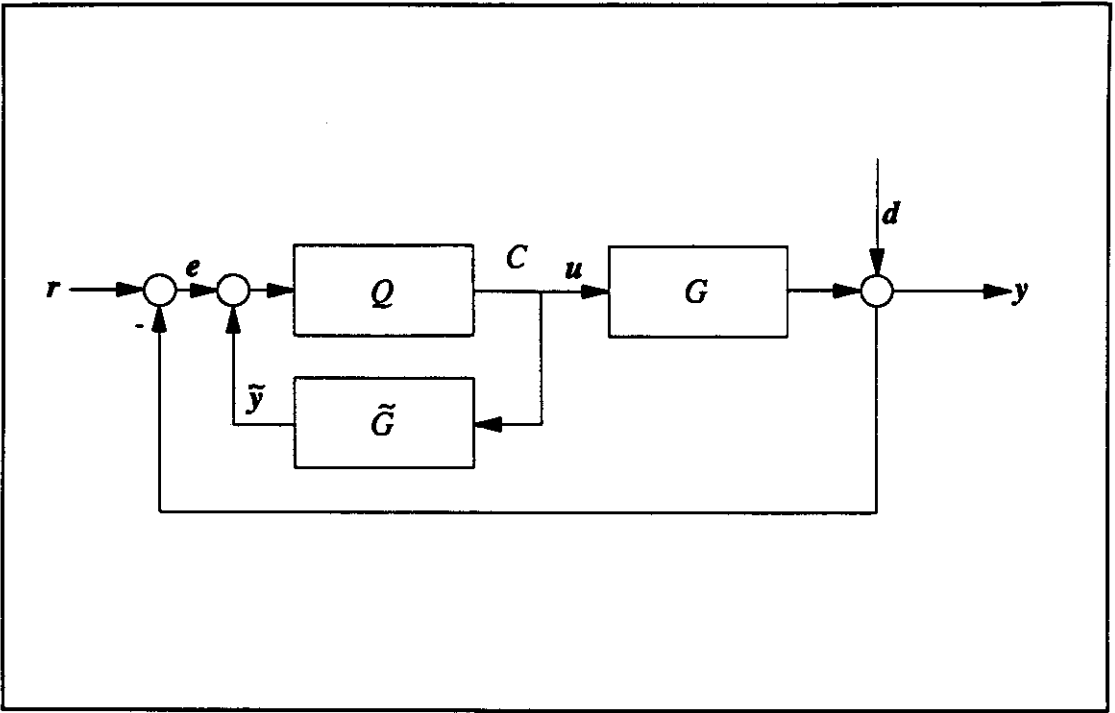


Figure 5.11a - Classic Feedback Block Diagram Showing its Relationship to IMC

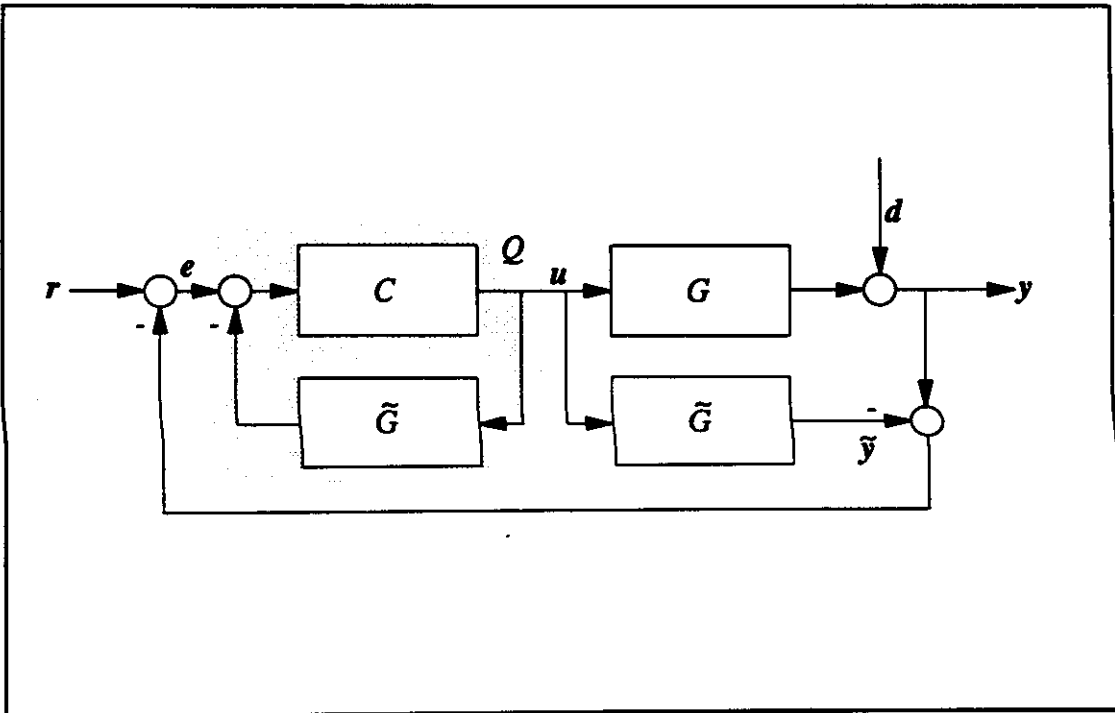


Figure 5.11b - IMC Block Diagram Showing its Relationship to Classic Feedback Control

$$Q = C(I + \tilde{G}C)^{-1} \quad (5.46)$$

which is equivalent to Equation 5.45.

Looking at the internal stability of the IMC configuration, there are now three independent inputs (u_1 , u_2 , r) and three independent outputs (y , u , $\bar{y}$), as shown in Figure 5.12. The relationships between these inputs and outputs for the case where the model is perfect (i.e. $G = \tilde{G}$) is given by;

$$\begin{pmatrix} y \\ u \\ \bar{y} \end{pmatrix} = \begin{pmatrix} GQ & (I - GQ)G & G \\ Q & -QG & 0 \\ GQ & -GQG & G \end{pmatrix} \begin{pmatrix} r \\ u_1 \\ u_2 \end{pmatrix} \quad (5.47)$$

This system will only be stable when both the plant and the IMC controller are stable. A consequence of this is that IMC cannot be used to stabilise a process which is open-loop unstable (a characteristic which classic feedback control can achieve). Looking at the IMC block diagram (Figure 5.12), when the model is the same as the process, then the only signal being sent back to the controller is the disturbance (d). Consequently, if G is unstable, then no action is taken by the control system to stabilise the system. Although it is not possible to implement control for an open-loop unstable system using IMC, it is in fact possible to design an IMC controller for an unstable process and implement it as the equivalent classic controller (Morari and Zafiriou, 1989). It may seem contradictory that IMC can be used for analysis but not for implementation, however, the guarantee that control will not be implemented using IMC, means that the limitations of the IMC theory are no longer applicable, and it may in fact be used for analysis of the closed-loop system.

The sensitivity function (E) and the complementary sensitivity function (H) can also be determined for the IMC structure. Although the general case ($G \neq \tilde{G}$) formulae look complex, for the nominal case where the model is the same as the process ($G = \tilde{G}$), they simplify significantly.

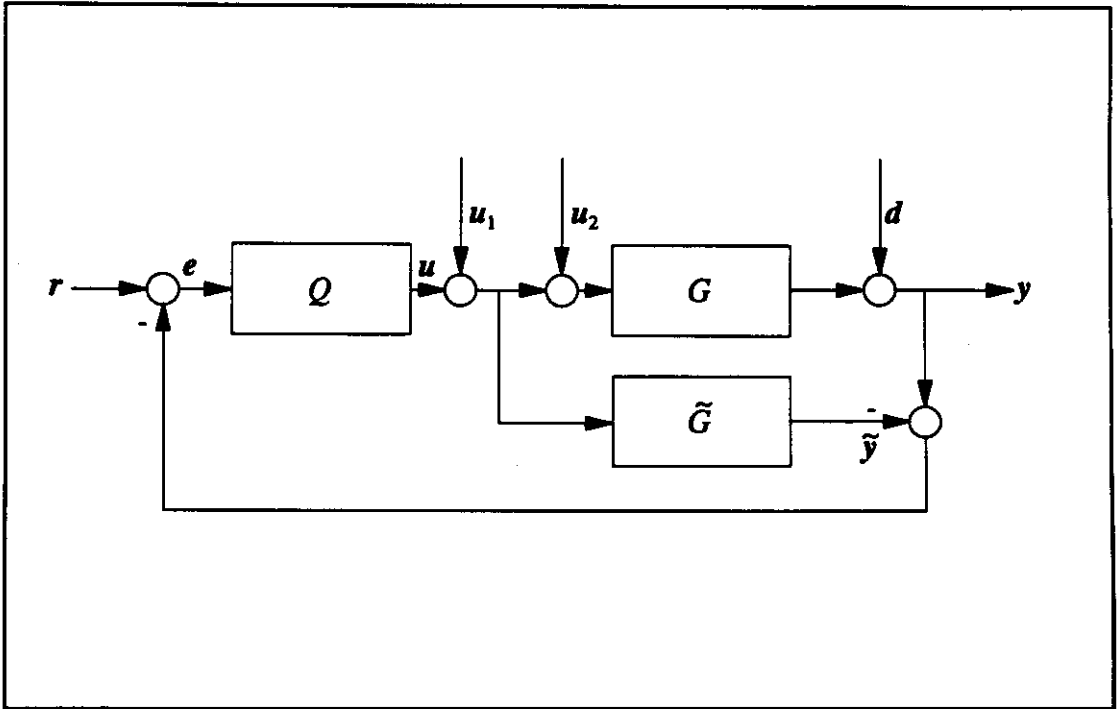


Figure 5.12 - IMC Block Diagram for Stability Analysis

Sensitivity Function

$$\begin{aligned}
 E &= (I - \tilde{G}Q)(I + (G - \tilde{G})Q)^{-1} \\
 \Rightarrow E &= I - GQ \quad (\text{if } \tilde{G} = G)
 \end{aligned}
 \tag{5.48}$$

Complementary Sensitivity Function

$$\begin{aligned}
 H &= GQ(I + (G - \tilde{G})Q)^{-1} \\
 \Rightarrow H &= GQ \quad (\text{if } \tilde{G} = G)
 \end{aligned}
 \tag{5.49}$$

These definitions of the sensitivity function and the complementary sensitivity function, for the case where the model is the same as the process, greatly simplify the controller design procedure, as the IMC controller (Q) is a simple function within the two expressions (cf. classic feedback control, Equations 5.25 and 5.26).

Perfect Control

The sensitivity function relates the disturbances appearing at the output (d) to the final closed-loop measurements (y). A "perfect" controller should ensure that despite any disturbance entering the system, the output measurements should not change. This requires that the sensitivity function should be zero. A similar argument can be made on the basis of achieving perfect setpoint tracking, leading to the requirement that the complementary sensitivity function should be one (or I for the multivariable case). Looking at the IMC definition of the sensitivity function, for the case where the model is perfect (Equation 5.48), we can rearrange this to give the "perfect" IMC controller.

$$\begin{aligned}
 E &= I - GQ \\
 \Rightarrow Q &= G^{-1} \quad (\text{if } E = 0)
 \end{aligned}
 \tag{5.50}$$

In fact, even if the model does not match the plant exactly, as long as the controller is the inverse of the process model ($Q = \tilde{G}^{-1}$), there will be perfect regulatory control. This simple form for the perfect controller could give the impression that perfect control can be readily achieved. However, earlier it was pointed out that it is not possible to have perfect control for all frequencies due to the behaviour of real systems (see Equation 5.28). In addition, it will be shown later that model uncertainty can limit the achievable

performance, if we are to ensure closed-loop stability. Another simple limitation that will also be discussed later is the inability to invert all process models.

One final limitation to the attainment of perfect control can be seen by determining the feedback controller equivalent to the IMC perfect controller. Using Equation 5.45 gives the equivalent perfect feedback controller.

$$\begin{aligned}
 C &= Q(I - \tilde{G}Q)^{-1} \\
 \Rightarrow C &= \tilde{G}^{-1}(I - \tilde{G}\tilde{G}^{-1})^{-1} \quad (\text{when } Q = \tilde{G}^{-1}) \\
 &= \tilde{G}^{-1}(0)^{-1} \\
 &= \infty I
 \end{aligned} \tag{5.51}$$

Thus, the feedback controller equivalent to the perfect IMC controller has infinite gain. This is in agreement with the general observation that high gain controllers have better performance, although at the expense of stability.

Consequently, from a theoretical point of view it is feasible to design a "perfect" controller. However, as will be shown in Section 5.2.3, such perfect controllers generally have to be modified to ensure stability in the face of uncertainty. Techniques for describing uncertainty in process models will be discussed in the next section.

5.2.2 Model Uncertainty Descriptions

Much of the analysis presented so far is based on a model of the process being studied. It has also been pointed out that there is usually uncertainty associated with the model, that is, it may not exactly describe the behaviour of the process under all circumstances. In particular, the type of model normally used to study dynamic behaviour, linear time invariant (i.e. the parameters within the linear model do not change with time), is susceptible to a number of sources of uncertainty. Firstly, linear models derived by linearising a nonlinear dynamic model are most accurate near the point of linearisation, so any deviation of the operation away from that point increases the model inaccuracy. Secondly, there may be changes in process parameters (e.g. catalyst activity) or in variables which cannot be measured, these will both lead to a poorer model. Finally, one other source of model error is due to aspects of the process that have been (maybe deliberately) ignored, for example high-order dynamics.

In some instances, quite exact parametric descriptions of the model uncertainty may be available. If the model is entirely based on well known mechanisms, then particular parameters within that model could be known to vary, each over a particular range. Alternatively, if the model was developed by performing tests on a real process, then there may be statistically based ranges over which the calculated parameters could vary. Unfortunately, however, this type of uncertainty description is not in a convenient form for use in controller design.

The method that will be adopted to describe model uncertainty is based in the frequency domain. The basic principles of frequency analysis have been introduced earlier (see Section 2.2.5.1). However, the relevant aspects will be briefly reviewed here.

A transfer function $G(s)$ is usually expressed as the ratio of two polynomials (possibly with an exponential term to model dead-time). In the general multiple-input multiple-output case, the input-output relationship is a matrix of transfer functions, a typical element relating the i th output to the j th input being shown below.

$$G_{i,j}(s) = \frac{(s+a_0)(s+a_1)\dots(s+a_m)e^{-\tau s}}{(s+b_0)(s+b_1)\dots(s+b_n)} = \frac{y_i(s)}{u_j(s)} \quad (5.52)$$

This function relates Laplace domain descriptions of the inputs and the outputs of the model. Of particular importance is the ease with which the effect of sine waves on the system can be analysed using transfer functions. For the single-input single-output case, the steady-state response to an input sine wave (i.e. ignoring initial transients) can be found by making the substitution $s = i\omega$ (where ω is the sine wave frequency in radians per unit time) into the transfer function. The amplitude ratio (i.e. the ratio of the amplitude of the output to input sine waves) is given as the modulus of the complex number $G(i\omega)$, and the phase shift (i.e. the phase of the output sine wave minus the phase of the input sine wave) is given by the argument of $G(i\omega)$.

$$\text{Magnitude ratio} = \frac{\text{Amplitude } y(t)}{\text{Amplitude } u(t)} = |G(i\omega)| \quad (5.53)$$

$$|G(i\omega)| = \sqrt{[\text{Real } G(i\omega)]^2 + [\text{Imag } G(i\omega)]^2}$$

$$\text{Phase shift} = \arctan \left[\frac{\text{Imag}(G(i\omega))}{\text{Real}(G(i\omega))} \right] \quad (5.54)$$

An illustrative method for describing the frequency domain behaviour of a transfer function is through the use of a Nyquist plot. This two-dimensional graph plots the real part (on the x -axis) against the imaginary part (on the y -axis) of the transfer function evaluated at a particular frequency. The transfer function is then plotted for all frequencies over a given range. A SISO example of this is shown in Figure 5.13, where the plot begins on the real axis at the value of the steady-state transfer function ($G(0)$, $\omega = 0$), and ends at the origin where the value of the transfer function tends to zero at high frequencies. Also shown on the graph are two important SISO dynamic measures, the amplitude ratio and phase shift, which are the distance from the origin and the clockwise angle to the positive real axis, respectively.

An example of the usefulness of this plot is a simple check for stability of SISO systems. The Bode stability criterion states that if the amplitude ratio exceeds a value of unity when the phase lag is π radians, then the system will be unstable, although there are limitations to this theory in complex cases (Stephanopoulos, 1984). By drawing the unit circle, centre the origin, onto the Nyquist plot the stability of the system's transfer function can be checked. The more general Nyquist Stability Criterion (Stephanopoulos, 1984) also uses Nyquist plots to determine closed-loop stability.

Single-Input Single-Output Uncertainty Descriptions

Nyquist plots can also be used to indicate uncertainty in a transfer function. Initially we will investigate the SISO case, extending this to the MIMO case later.

At a given frequency (ω), the nominal value of the model transfer function is represented by the point $\tilde{G}(i\omega)$. Uncertainty in the true value of the transfer function representing the process, means that there will be a region around this point in which the actual transfer function will lie. Extending this representation over a given frequency range will map out a region on the Nyquist plane in which the actual transfer function will lie. Depending on the available description of the uncertainty, the shape of these regions on the Nyquist plane will be different. Figure 5.14a shows an example for two frequencies, where there is uncertainty in the gain and phase shift of the transfer function.

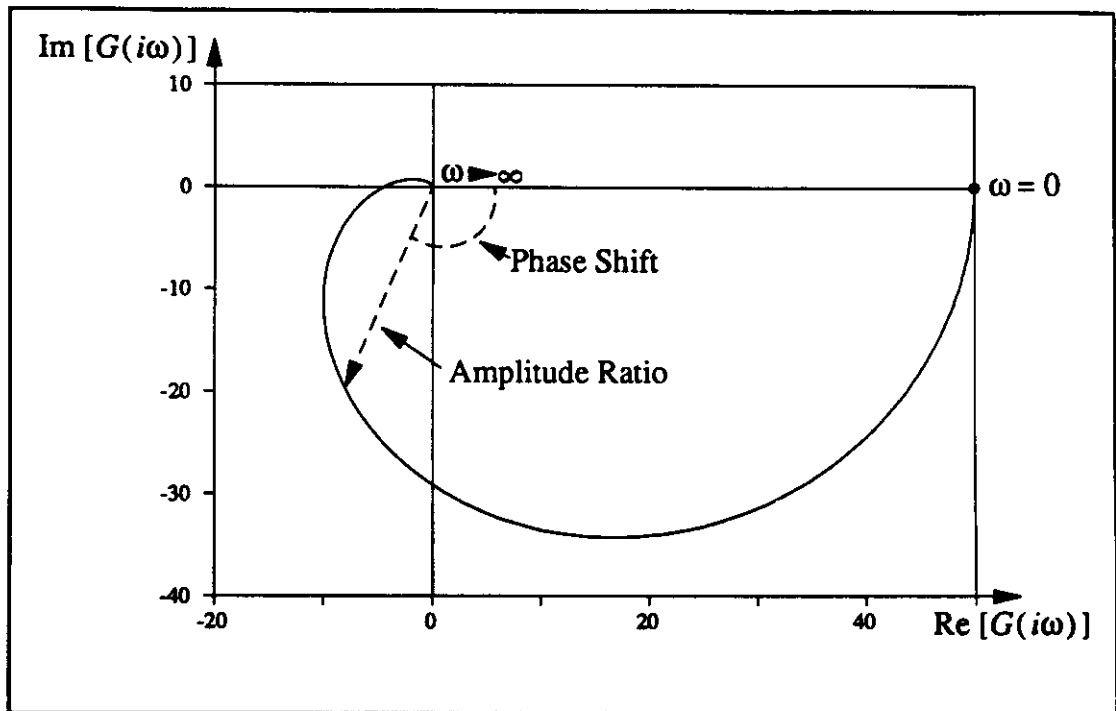


Figure 5.13 - Typical Nyquist Plot

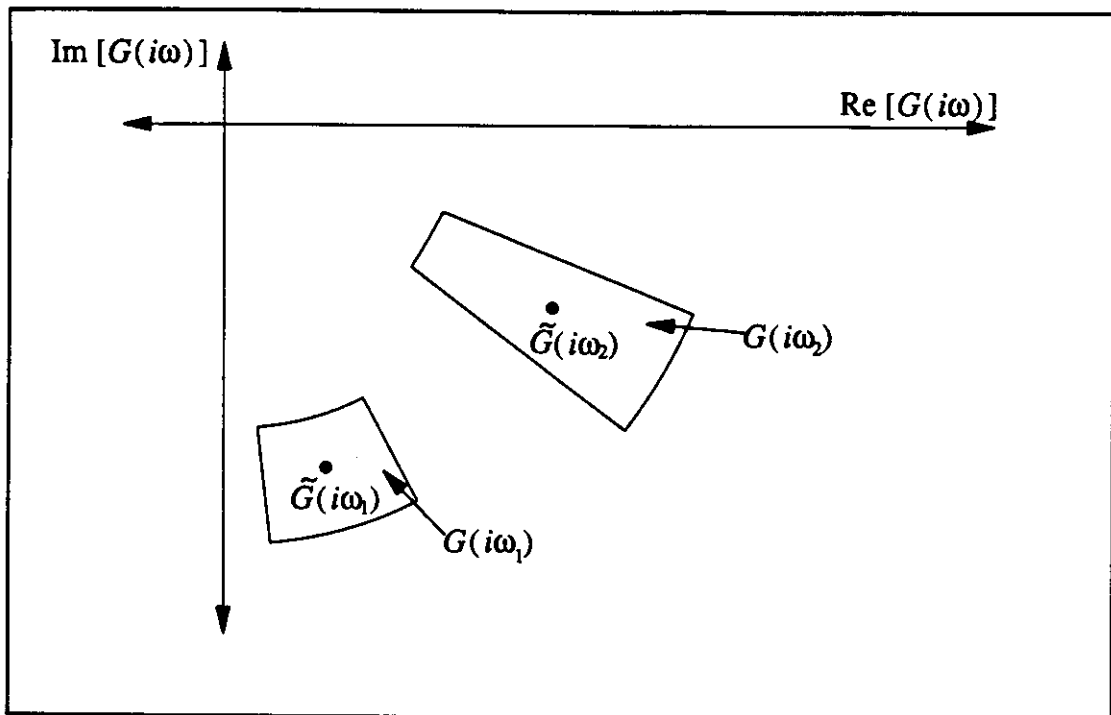


Figure 5.14a - Uncertainty Regions on the Nyquist Plane

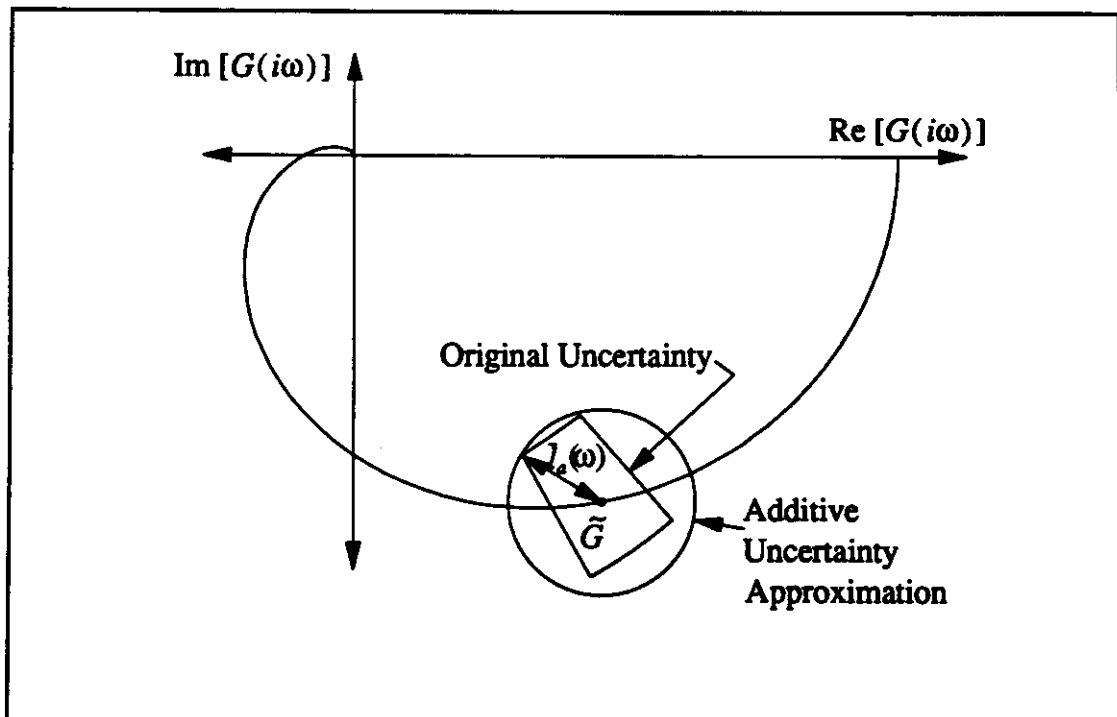


Figure 5.14b - Disk Approximation to Uncertainty Region

In Section 5.3, an example involving uncertainty in the parameters in the model will be given. However, to perform calculations on the behaviour of uncertain systems, descriptions based on experimental or parametric uncertainty are generally not convenient. As shown in Figure 5.14b, one more general approach is for the uncertainty region to be approximated by a series of disks. Depending on the original uncertainty description, using disks may be a very conservative approximation technique, adding considerably more uncertainty than may actually occur.

For the disk shown in Figure 5.14b, with a radius $\bar{l}_a$, the family of all possible plants, Π , can be described mathematically as;

$$\Pi = \{ G : |G(i\omega) - \bar{G}(i\omega)| \leq \bar{l}_a(\omega) \} \quad (5.55)$$

As before, $\bar{G}(i\omega)$ is the nominal plant, now at the centre of the disk, and any plant within the family Π , satisfies the following;

$$G(i\omega) = \bar{G}(i\omega) + l_a(i\omega) \quad (5.56)$$

where

$$|l_a(i\omega)| \leq \bar{l}_a(\omega) \quad (5.57)$$

Equation 5.56 is an additive uncertainty description, where the bound on the uncertainty, Equation 5.57, is a function of frequency. This description allows the flexibility of having a smaller uncertainty at low frequencies where the transfer function is better known, and a larger value at high frequencies where the uncertainty is generally greater.

Another method for describing the uncertainty is to use a multiplicative description, limiting the relative error by an upper bound. The family of plants Π in this case is given by;

$$\Pi = \left\{ G : \frac{|G(i\omega) - \bar{G}(i\omega)|}{|\bar{G}(i\omega)|} \leq \bar{l}_m(\omega) \right\} \quad (5.58)$$

In this case, each member of Π satisfies;

$$G(i\omega) = \tilde{G}(i\omega)(1 + l_m(i\omega)) \quad (5.59)$$

where

$$|l_m(i\omega)| \leq \bar{l}_m(\omega) \quad (5.60)$$

A plot of a typical multiplicative uncertainty bound ($\bar{l}_m$) is given in Figure 5.15.

For the SISO case, a simple relationship between these two uncertainty descriptions is available;

$$\begin{aligned} l_m(i\omega) &= \frac{l_a(i\omega)}{\tilde{G}(i\omega)} \\ \bar{l}_m(\omega) &= \frac{\bar{l}_a(\omega)}{|\tilde{G}(i\omega)|} \end{aligned} \quad (5.61)$$

Both the additive and multiplicative uncertainty descriptions can be shown on a block diagram. Figure 5.16a shows the additive uncertainty l_a as a transfer function added to the nominal plant $\tilde{G}$, while Figure 5.16b shows the effect of a multiplicative uncertainty description on the input to the nominal plant $\tilde{G}$. For the SISO case, multiplicative uncertainty on the input is equivalent to the same uncertainty on the output.

Multiple-Input Multiple-Output Uncertainty Descriptions

The preceding analysis provides a useful description of uncertainty for SISO systems which can be extended to the MIMO case. In the method presented here, an unstructured perturbation away from the nominal model $\tilde{G}(s)$ will be employed as an uncertainty description. As will be shown, this allows a range of matrices, with the same dimensions as $\tilde{G}(s)$, but all within given criteria, to describe the range of possible values the actual transfer function matrix may take.

This approach is in contrast to a structured description of the uncertainty, where a more accurate representation of the uncertainty in particular elements of the model transfer function matrix can be analysed. To describe a limited uncertainty such as this using an unstructured approach, requires calculation of a perturbation matrix, with the same dimensions as $\tilde{G}(s)$, but with sufficient scope to completely cover the smaller (structured)

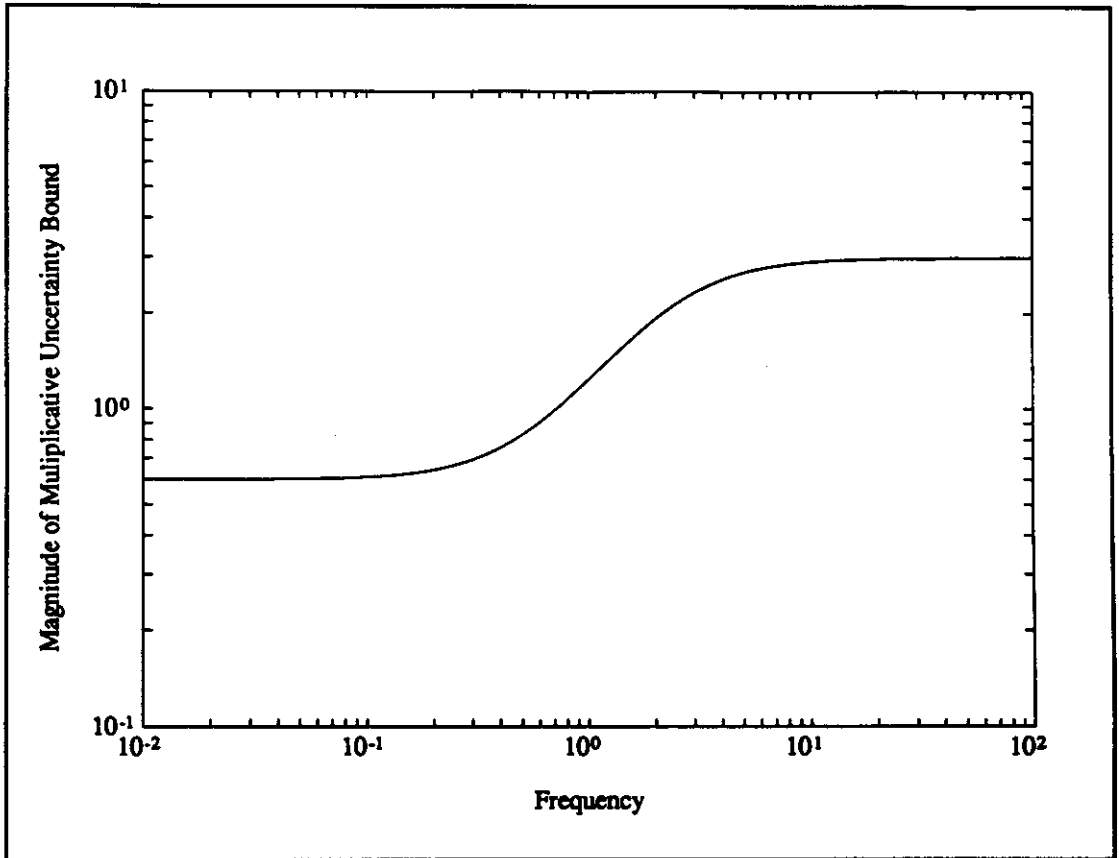


Figure 5.15 - Multiplicative Uncertainty Bound versus Frequency

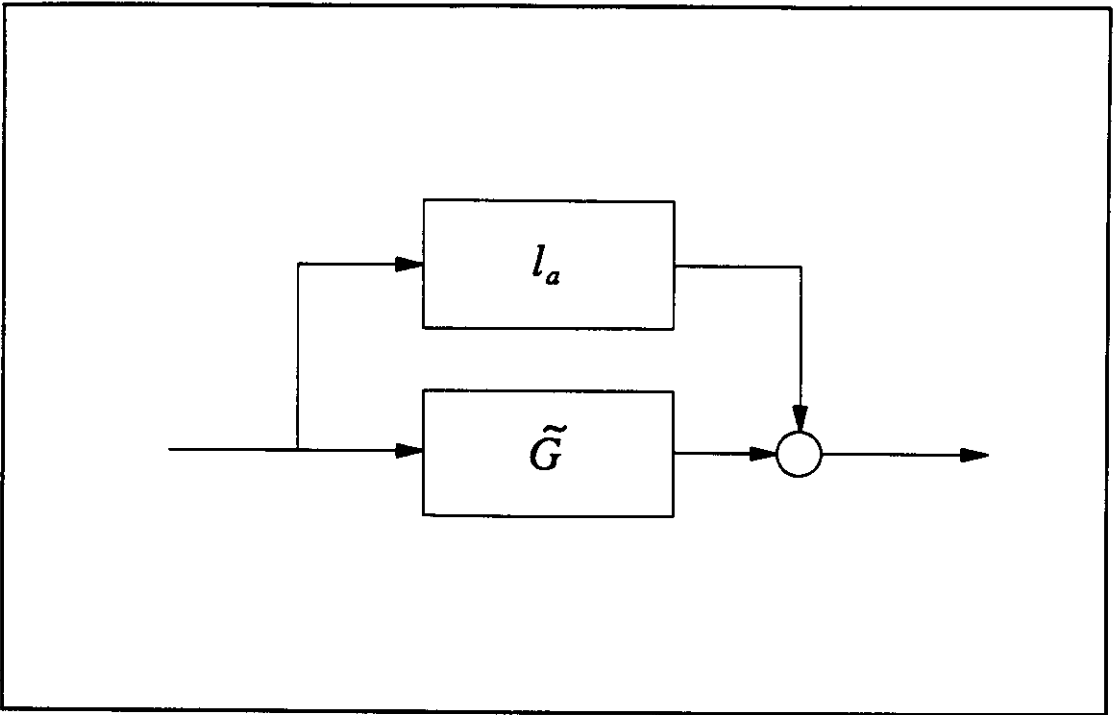


Figure 5.16a - Block Diagram for SISO Additive Uncertainty

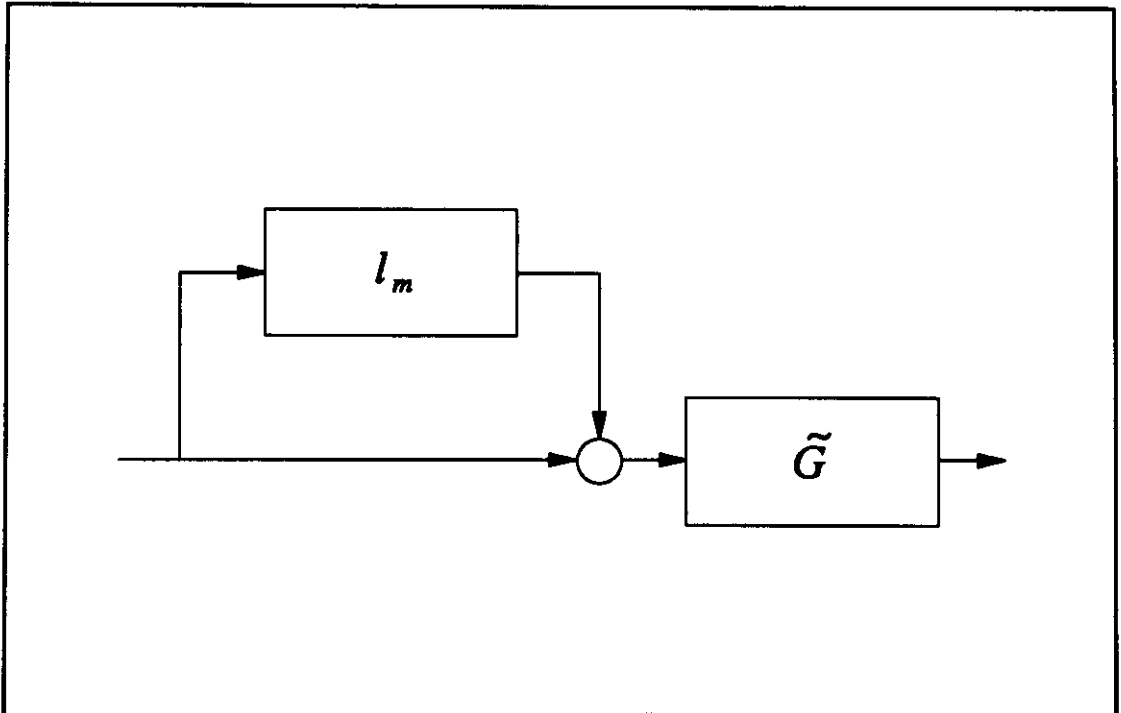


Figure 5.16b - Block Diagram for SISO Multiplicative Uncertainty

perturbation. Clearly this can be a very conservative technique for describing well known and limited uncertainties. However, application techniques for structured uncertainty are still not fully developed and readily available as computer software, so only unstructured techniques will be used in this analysis.

The use of unstructured uncertainty will allow us to describe uncertainty in the MIMO process transfer function model in a similar manner to that described for SISO systems. Figure 5.17 shows block diagrams for three common techniques for describing unstructured MIMO uncertainty. In each case the "uncertainty" matrix (L_A , L_I , L_O) has the same dimensions as the nominal process model $\tilde{G}$. Notice that for the MIMO case, a distinction has been drawn between multiplicative uncertainty on the input to and on the output from the process. The mathematical reason for this difference will be given later. The input uncertainty can be thought of as actuator uncertainty with output uncertainty as measurement uncertainty.

As in the SISO case, frequency based descriptions are used to describe the uncertainty. The mathematical relationships equivalent to the block diagrams given in Figure 5.17 are;

Additive Uncertainty:

$$G = \tilde{G} + L_A \quad L_A = G - \tilde{G} \quad (5.62)$$

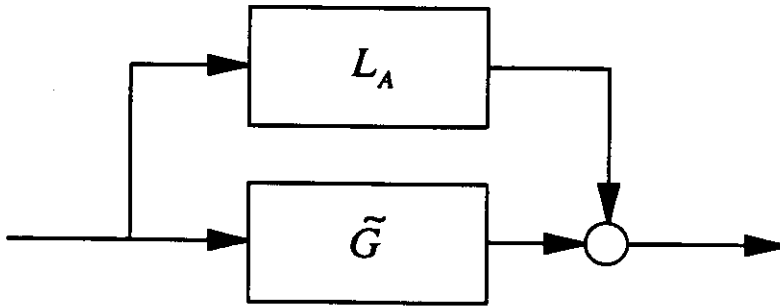
Multiplicative Input Uncertainty:

$$G = \tilde{G}(I + L_I) \quad L_I = \tilde{G}^{-1}(G - \tilde{G}) \quad (5.63)$$

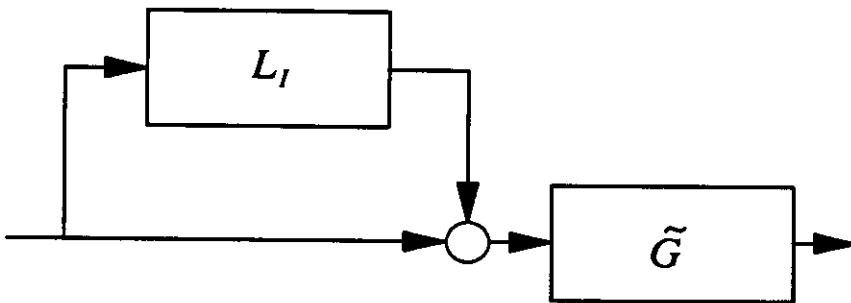
Multiplicative Output Uncertainty:

$$G = (I + L_O)\tilde{G} \quad L_O = (G - \tilde{G})\tilde{G}^{-1} \quad (5.64)$$

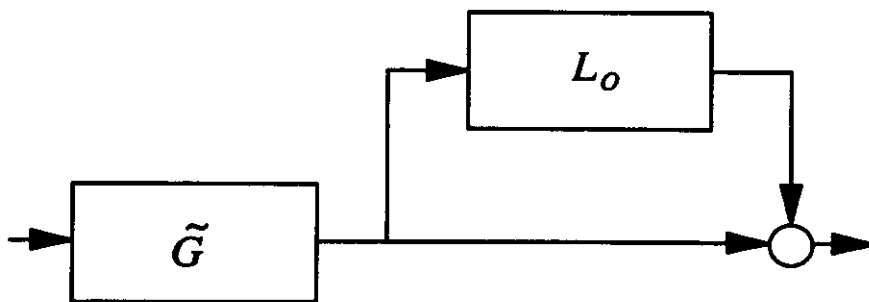
The bound on the uncertainty in these equations is written as a frequency dependent function describing the maximum magnitude of the allowable uncertainty. To be consistent with H_∞ controller design theory (presented in Section 5.2.3) the selected measure of the magnitude of the uncertainty is the spectral norm or maximum singular



a) Additive Uncertainty



b) Multiplicative Input Uncertainty



c) Multiplicative Output Uncertainty

Figure 5.17 - Block Diagrams for MIMO Uncertainties

value of the uncertainty matrix.

Additive Uncertainty:

$$\bar{\sigma}(L_A) \leq \bar{l}_A(\omega) \quad (5.65)$$

Multiplicative Input Uncertainty:

$$\bar{\sigma}(L_I) \leq \bar{l}_I(\omega) \quad (5.66)$$

Multiplicative Output Uncertainty:

$$\bar{\sigma}(L_O) \leq \bar{l}_O(\omega) \quad (5.67)$$

For each of these uncertainty descriptions, the upper bound on the uncertainty is therefore expressed as an upper limit on the maximum singular value of the uncertainty.

Using these equations, the relationship between the multiplicative input and multiplicative output uncertainty descriptions can be found (Morari and Zafiriou, 1989).

$$\begin{aligned} \text{Equations 5.63, 5.64} &\Rightarrow L_O = \tilde{G}L_I\tilde{G}^{-1} \\ \Rightarrow \bar{\sigma}(L_O) &= \bar{\sigma}(\tilde{G}L_I\tilde{G}^{-1}) \leq \bar{\sigma}(\tilde{G})\bar{\sigma}(\tilde{G}^{-1})\bar{\sigma}(L_I) \leq \gamma(\tilde{G})\bar{l}_I \\ &\Rightarrow \bar{l}_O \leq \gamma(\tilde{G})\bar{l}_I \end{aligned} \quad (5.68)$$

γ is the process condition number introduced previously in Chapter 2. The importance of this relationship is that it demonstrates the need to use the uncertainty description most appropriate to the source of the uncertainty. For example, since the process condition number is always greater than or equal to one, uncertainty which might be described with a small upper bound at the input, will involve a larger upper bound when described at the output.

Using the maximum singular value as the upper bound for the uncertainty description is clearly a conservative way of describing the uncertainty in the system, as many matrices may be constructed (with quite different directionalities, for example) which fit the

maximum singular value criteria but which do not correspond to any feasible plant. However, as will be shown in the next section, this description does allow us to design controllers which will be stable for all plants within the uncertainty description and whose performance for the nominal model is within the designer's specifications.

5.2.3 H_∞ Controllers

In Section 5.1 the concept of "perfect" control was introduced as part of the process to determine the best achievable operating point for a given process design. To allow a fair comparison between different designs, the best possible controller should be used, hence the use of perfect control. Unfortunately, perfect control cannot always be achieved due to certain characteristics of the process and its model;

- 1) The presence of time delays.
- 2) The presence of right half plane zeros.
- 3) Limits on available process input manipulation.
- 4) Model uncertainty.

In Section 5.2.1, the perfect controller for the IMC structure was given in Equation 5.50 as the inverse of the process model. Characteristics 1 and 2 above both prevent the inversion of the process model, the first requiring the prediction of future events, and the second leading to an unstable controller. Characteristic 3 is important because perfect controllers can require very large movements in the process inputs, which are not physically achievable for most process plants.

Therefore the H_∞ design procedure was adopted as a consistent method for designing the "best" controller for a process. Although other standard, and hence equitable, design techniques are available for SISO systems (e.g. Ziegler Nichols and BLT, Luyben, 1990), such standard techniques are not readily available for the multivariable case. Consequently, the H_∞ procedure was adopted for both the SISO and MIMO cases.

Simply put, the objective of the H_∞ design procedure is to minimise the sensitivity function (Equation 5.25), or the maximum possible amplification in the MIMO case, so as to achieve the best possible disturbance rejection. In addition, the controller must ensure stability and suitable performance for all processes which may occur within the given

uncertainty description. These requirements can be formalised as;

Nominal Stability

The closed-loop system with the process model at its nominal value ($\tilde{G}$) must be stable (see Equations 5.22 and 5.47).

Nominal Performance

The performance of the closed-loop system, when the process model is at its nominal value ($\tilde{G}$), as measured by the sensitivity function, must always be within the designer's specification.

Robust Stability

The closed-loop system, for all possible process models included in the uncertainty description, must be stable.

Robust Performance

The performance of the closed-loop system, for all possible process models included in the uncertainty description, must always be within the designer's specification.

These requirements are increasingly demanding, with the nominal case being a subset of the robust case. The use of the H_∞ design procedure allows each of these criteria to be achieved at different stages in the design. For the SISO case, designing for robust performance will also ensure robust stability. However, for the MIMO case, the procedure presented in this thesis will give robust stability but only nominal performance. Procedures for designing to also ensure robust performance are currently being developed, making use of the structured singular value analysis (μ -Analysis) of Doyle (1982).

H_∞ Controller Derivation

Initially the derivation of H_∞ concepts will be presented for the SISO case. Following the major SISO results, the equivalent MIMO formulae will be presented and explained.

To derive the definition of an H_∞ controller, we need to refine the concept of bounded disturbances previously presented in Section 5.1.2. There, a set of disturbances was

considered as being a group of sine waves whose magnitude was bounded by a given transfer function. The H_∞ controller is designed to achieve designated stability and performance characteristics for any possible input v (not just sine waves) which the designer specifies. However, mathematically it is desirable that the description of all possible inputs (disturbances) takes on a particular form. The form chosen is still quite general, it includes all bounded inputs v' , such that the integral of the magnitude squared, from time zero to infinity is less than or equal to one.

Set of bounded inputs (e.g. all inputs with a 2-norm bounded by unity)

$$V' = \left\{ v' : \|v'\|_2^2 \equiv \int_0^\infty v'^2(t) dt \leq 1 \right\} \quad (5.69)$$

It is therefore necessary to find a transformation between the actual process inputs, v , and those convenient mathematically (as introduced above). The relationship between these is given by the weighting function w_1 .

$$v = w_1 v' \quad (5.70)$$

Physically, we can think of this transformation as passing the bounded input v' through the weight w_1 to give the physical input v . This is shown in Figure 5.18a. It should be noted that the bounded input v' has been introduced to aid in the solution of the mathematics associated with H_∞ controller design, rather than as a description of real inputs.

Before giving an example of the application of the bounded input weighting, it is illustrative to examine these principles when applied to a specific input (this will also be useful later when looking at H_2 -optimal controller design). An example of a specific input is the impulse function;

Specific Input (e.g. the impulse function)

$$v'(s) = 1 \quad (5.71)$$

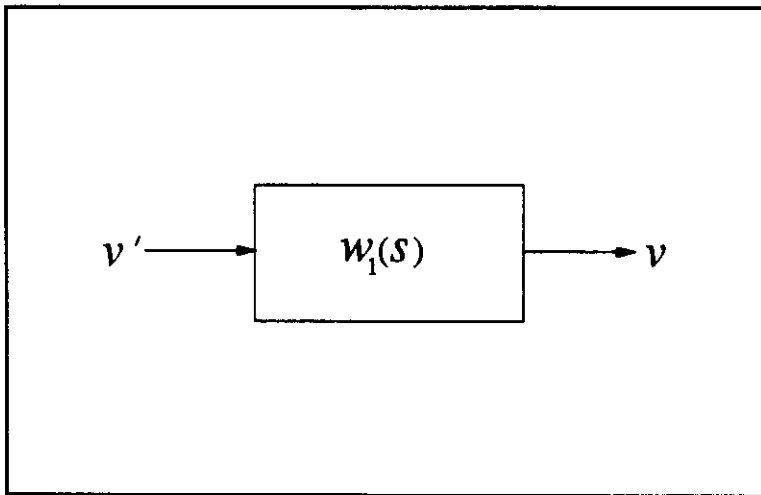


Figure 5.18a - Weighting of Normalised Input v' by w_1 to give Plant Input v

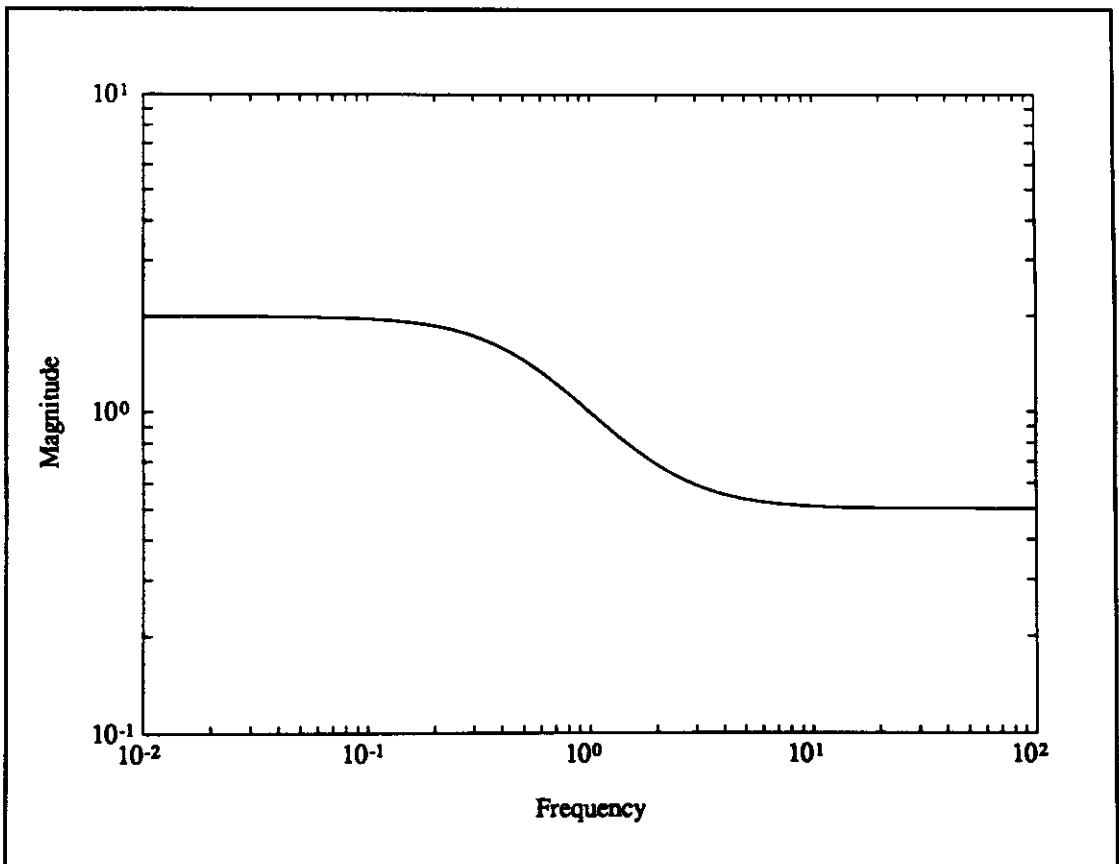


Figure 5.18b - Typical Weight w_1

Consider that the impulse function is to be used as the mathematically convenient input v' , while the actual plant input is a step function (for which the Laplace transform is s^{-1}). Then it can be shown that the weight w_1 for this transformation is an integrator as shown below.

$$\begin{aligned} v &= w_1 v' = w_1 \times 1 = w_1 \\ \therefore \text{ if } v &= \frac{1}{s} \Rightarrow w_1 = \frac{1}{s} \end{aligned} \quad (5.72)$$

To develop a similar example for the bounded input case, it is necessary to express the bounded inputs in the frequency domain, rather than the time domain as defined in Equation 5.69. Parseval's theorem gives the result below;

$$V' = \left\{ v' : \|v'\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |v'(i\omega)|^2 d\omega \leq 1 \right\} \quad (5.73)$$

An example of the use of the weighting function applied to bounded inputs is given by Morari and Zafiriou (1989). If the plant inputs v are known to be step like, then it can be shown that the use of the weight and bounded input described below, not only provides a description of step inputs, but also other inputs similar to steps, e.g. a step modified by a lead or lag element.

$$\begin{aligned} w_1(s) &= \frac{s + \beta}{s\sqrt{2\beta}} \quad \beta > 0 \\ v' &= \frac{\sqrt{2\beta}}{s + \beta} \end{aligned} \quad (5.74)$$

Note: $v = w_1 v' = \frac{1}{s}$ as specified
and v' satisfies Equation 5.73

Although the use of this input description technique may seem like a mathematical abstraction, it will be shown that the correct choice of weighting function for use within a controller design procedure, will ensure that the entire class of expected inputs can be handled. In addition, rather than the exact mathematical definition of the weighting

function defined above, w_1 can in fact be used by the designer to "tune" the controller, specifying in a qualitative manner the importance of inputs at different frequencies. A typical weight is shown in Figure 5.18b, where low frequency inputs are expected to have greater amplitudes than high frequency inputs.

Before defining the H_∞ controller, it is illustrative to look at the H_2 -optimal controller. For many years this has been one of the more powerful controllers available, although as discussed in the Introduction, not without its problems. The controller is optimal, in that it is designed to minimise an error term, the integral square error (ISE) for a specific input.

$$\text{ISE} = \|e\|_2^2 = \int_0^\infty e^2(t) dt \quad (5.75)$$

Using Parseval's Theorem, the H_2 -optimal controller design problem becomes one of finding the controller C that minimises the ISE for one particular input v .

$$\min_C \|e\|_2^2 = \min_C \frac{1}{2\pi} \int_{-\infty}^{\infty} |e(i\omega)|^2 d\omega \quad (5.76)$$

Substituting the product of the sensitivity function E (Equation 5.24) and the chosen impulse input v (Equation 5.72, $v = w_1$) for the error e ($e = Ev = Ew_1$), gives;

$$\min_C \|e\|_2^2 = \min_C \frac{1}{2\pi} \int_{-\infty}^{\infty} |E(i\omega)w_1(i\omega)|^2 d\omega \quad (5.77)$$

This can be interpreted as minimising the average value of the sensitivity function weighted by w_1 . In mathematical terms, this controller minimises the 2-norm of the sensitivity function weighted for a particular input.

In comparison to the H_2 -optimal approach, the H_∞ -optimal controller is designed not for a particular input but for a set of possible inputs, as described below. Here, Equation 5.73 has been rearranged to define the set of all possible plant inputs V in terms of w_1 , so as to satisfy the definition of bounded inputs given in Equation 5.69.

$$V = \left\{ v: \|v'\|_2^2 = \left\| \frac{v}{w_1} \right\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{v(i\omega)}{w_1(i\omega)} \right|^2 d\omega \leq 1 \right\} \quad (5.78)$$

The objective of the H_∞ design procedure is to minimise the "worst" error (rather than the average value as in the H_2 -optimal case);

$$\min_C \sup_{v \in V} \|e\|_2 \quad (5.79)$$

Here the mathematical expression "sup" is the limit superior or the least upper bound. The error (e) can be substituted as the sensitivity function (E) multiplied by a general input (v), that is, $e = Ev$. Using the definition of the normalised inputs given in Equation 5.70, we can also substitute $w_1 v'$ for v . The H_∞ -optimal design problem therefore becomes;

$$\min_C \sup_{v' \in V'} \|E w_1 v'\|_2 \quad (5.80)$$

To simplify the design procedure, it is desirable to calculate an upper bound for the error term ($E w_1 v'$) above. Substituting for the normalised bounded inputs, using Equation 5.73, allows this to be calculated.

$$\begin{aligned} \sup_{v' \in V'} \|E w_1 v'\|_2^2 &= \sup_{v' \in V'} \frac{1}{2\pi} \int_{-\infty}^{\infty} |E w_1 v'|^2 d\omega \\ &\leq \sup_{\omega} |E w_1|^2 \sup_{v' \in V'} \frac{1}{2\pi} \int_{-\infty}^{\infty} |v'|^2 d\omega \end{aligned} \quad (5.81)$$

Then using the definition of the normalised upper bound, Equation 5.78, it is possible to show that in the limit (Morari and Zafiriou, 1989) the term involving v' on the right hand side of Equation 5.81 tends to 1, and thus;

$$\sup_{v' \in V'} \|E w_1 v'\|_2^2 = \sup_{\omega} |E w_1|^2 \quad (5.82)$$

Defining the ∞ -norm of the weighted sensitivity function as,

$$\|E w_1\|_{\infty} \equiv \sup_{\omega} |E w_1| \quad (5.83)$$

allows the H_{∞} -optimal design problem to be written as;

$$\min_C \|E w_1\|_{\infty} = \min_C \left[\sup_{\omega} |E w_1(i\omega)| \right] \quad (5.84)$$

In summary, the H_{∞} -optimal controller C , is the controller which minimises the least upper bound (maximum) of the sensitivity function weighted by w_1 . This can be compared to the definition of the H_2 -optimal controller which seeks to minimise the average value of the weighted sensitivity function.

As mentioned previously, the use of the weighting function w_1 can be extended beyond the original definition of the normalised bounded inputs (Equation 5.78). In a more general sense, w_1 can be used as a tuning tool, allowing the designer to place a specification on the upper bound of the sensitivity function E . In this case, the H_{∞} design problem becomes one of finding a controller which ensures that the specified bound on the sensitivity function is not exceeded.

$$\begin{aligned} |E(i\omega)| &< |w_1(i\omega)|^{-1} \quad \forall \omega \\ \Rightarrow |E(i\omega)| |w_1(i\omega)| &< 1 \quad \forall \omega \\ \Rightarrow \|E w_1\|_{\infty} &< 1 \end{aligned} \quad (5.85)$$

This relationship, which is similar in form to the H_{∞} -optimal design problem, can be used as the H_{∞} design criterion for nominal performance of the closed-loop system. Figure 5.19 shows a typical bound (w_1^{-1}) which the designer may use so as to limit the sensitivity function. The slope of one down to a magnitude of $-\infty$ at zero frequency is typical of integral action. The high frequency limit is often greater than one, meaning that at high

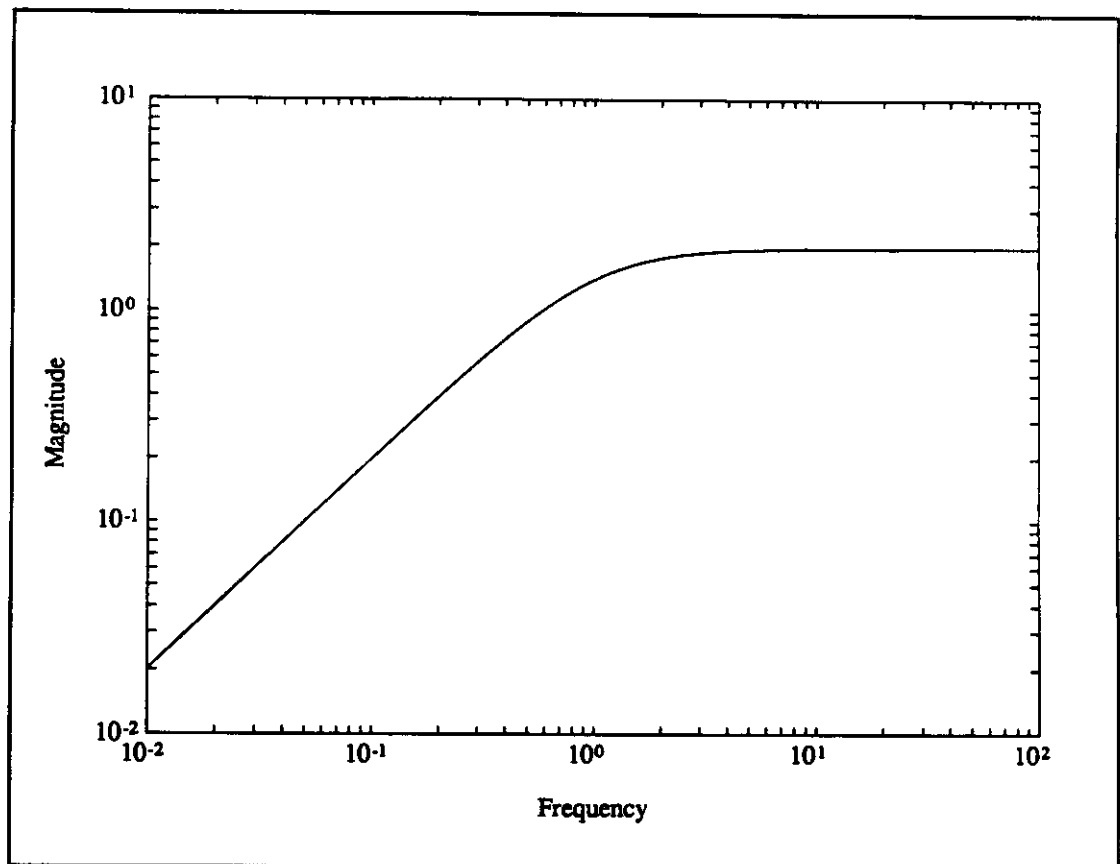


Figure 5.19 - Typical form of w_1^{-1}

frequencies (hopefully where disturbances are smaller) there may in fact be some disturbance amplification. Although this is undesirable, it is a characteristic of the design procedure, however for typically low noise process systems this is not a significant problem. Thus the H_∞ framework allows the designer to explicitly express his requirements for the sensitivity function including both the bandwidth (in simple terms, the frequency at which thirty percent of the disturbance is rejected) and limits on disturbance amplification.

MIMO H_∞ Derivation

One difference between the SISO development presented above and the analogous MIMO results, is in the definition of the bounded inputs (see Equation 5.69). For the MIMO case, where there are multiple inputs, the weighting matrix $W_1(s)$, which multiplies the vector of normalised inputs v' , is now a full matrix whose dimensions are compatible with the vector of inputs to the plant v . Although in the examples presented in Chapter 6, W_1 is a diagonal matrix, it need not be so in the general case. Indeed Morari and Zafiriou (1989) recommend using the disturbance transfer function G_d (see Figure 5.7), which in general is a non-diagonal "full" matrix.

Another difference involves the weighting of the errors. In the general MIMO case, each element in the error vector e may differ in importance, over different frequencies. The use of an output scaling matrix, $W_2(s)$, whose diagonal elements represent frequency dependent weightings for each element in e can incorporate this requirement into the design. Figure 5.20 shows how W_1 and W_2 can be used to scale the normalised input vector v' to produce the scaled error vector e' .

Using these weighting matrices, the H_∞ -optimal controller design problem becomes;

$$\min_C \left[\max_{v \in V} \|e'\|_2 \right] = \min_C \left[\max_{v' \in V'} \|W_2 E W_1 v'\|_2 \right] \quad (5.86)$$

That is, find the controller C , which minimises the size (measured as the 2-norm) of the maximum weighted error e' , for any possible disturbance v . This is equivalent to the SISO result given in Equation 5.80.

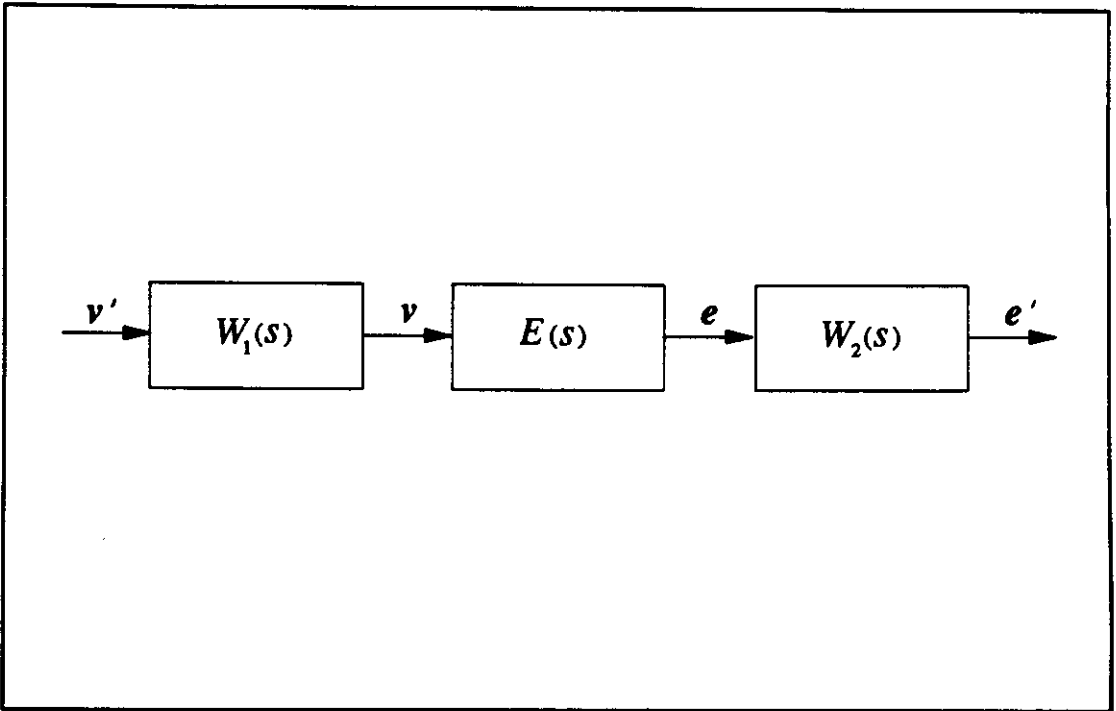


Figure 5.20 - MIMO Input and Output Weights on the Sensitivity Function

For this analysis, the correct measure for the "size" of the weighted sensitivity function (W_2EW_1) is the spectral radius, or the maximum singular value ($\bar{\sigma}$). Following similar arguments to those used in Equations 5.81 to 5.85, the H_∞ -optimal controller design problem can be written in the MIMO case as;

$$\min_C \|W_2EW_1\|_\infty = \min_C \left[\sup_{\omega} \bar{\sigma}(W_2EW_1(i\omega)) \right] \quad (5.87)$$

Consequently, the H_∞ nominal performance specification (similar to Equation 5.85) for the MIMO case is;

$$\|W_2EW_1\|_\infty < 1 \quad (5.88)$$

The goal of this controller design is to achieve the best possible disturbance rejection. This is measured by the sensitivity function, which relates the input disturbance to the amount appearing at the output for the closed-loop. So, the smaller the (weighted) sensitivity, the better the performance of the controller.

SISO Design Criteria

Using H_∞ techniques, mathematical statements may be formulated to check that a given controller design meets the robust stability and robust performance criteria defined earlier. For the SISO case, robust stability can be checked using the Nyquist stability criterion. A complete description of the theory behind this technique can be found elsewhere (e.g. Kuo, 1982). In simple terms, stability for the SISO system is determined by counting the number of encirclements of the point (-1,0) on the Nyquist plane made by the transfer function GC . Given that the system is closed-loop stable for the nominal model ($\tilde{G}$), to ensure robust stability, the region in the Nyquist plane within which the family of all uncertain plants Π lies (i.e. all GC) must not include the point (-1,0) at any frequency. This is shown geometrically in Figure 5.21 for one frequency using a multiplicative uncertainty description (see Equations 5.58 to 5.61).

To ensure robust stability, that is the disk does not include (-1,0), the geometry in Figure 5.21 can be used to generate the following mathematical requirement;

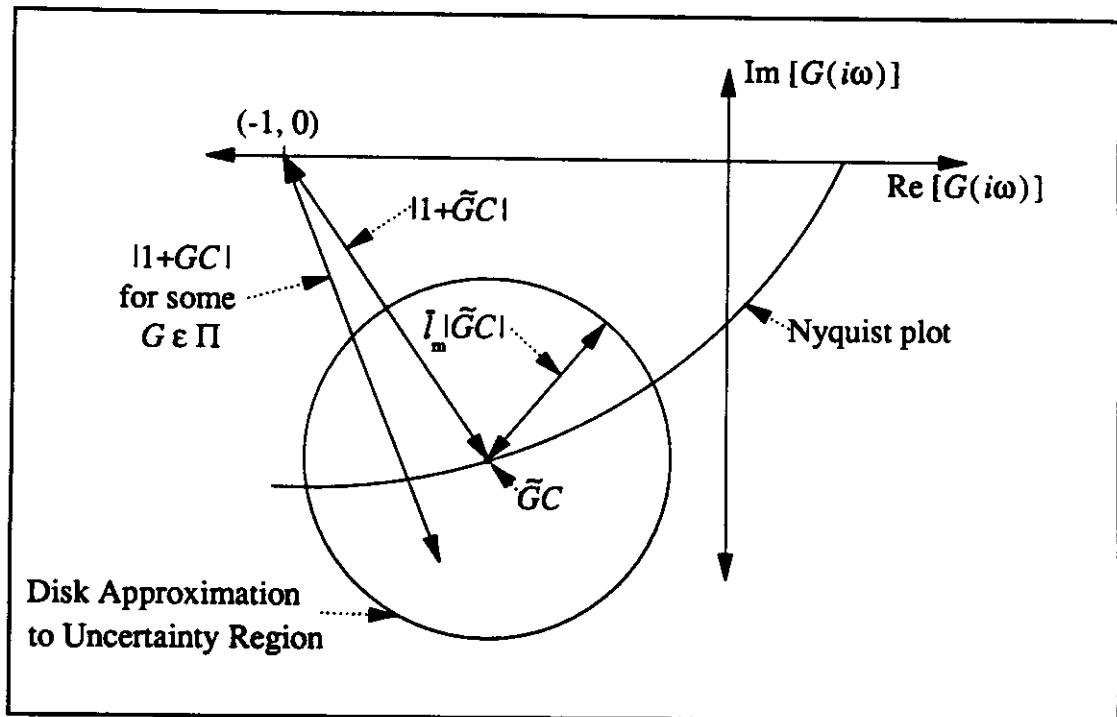


Figure 5.21 - Nyquist Plot for Robust Stability Analysis

$$|1 + \tilde{G}C(i\omega)| > |\tilde{G}C(i\omega)|\bar{l}_m(\omega) \quad \forall \omega \quad (5.89)$$

Rearranging this gives;

$$\begin{aligned} \left| \frac{\tilde{G}C(i\omega)}{1 + \tilde{G}C(i\omega)} \right| \bar{l}_m(\omega) &= |\tilde{H}(i\omega)|\bar{l}_m(\omega) \quad \forall \omega \\ &= |\tilde{H}(i\omega)\bar{l}_m(i\omega)| < 1 \quad \forall \omega \end{aligned} \quad (5.90)$$

where $\tilde{H} = \tilde{G}C(1 + \tilde{G}C)^{-1}$
 = nominal complementary sensitivity function

Formalising this gives the following definition of SISO robust stability (Morari and Zafiriou, 1989);

Robust Stability

If all plants G contained within the family of plants Π given by the multiplicative uncertainty description (Equation 5.58) have the same number of RHP poles, and a particular controller C stabilises the nominal plant $\tilde{G}$, then robust stability is guaranteed if the following bound is met;

$$\|\tilde{H}\bar{l}_m\|_\infty = \sup_{\omega} |\tilde{H}\bar{l}_m(\omega)| < 1 \quad (5.91)$$

To ensure robust performance requires that the nominal performance criterion (Equation 5.85) is satisfied for all plants G in the family of plants Π given by the multiplicative uncertainty description (see Equations 5.58 to 5.61).

$$\|Ew_1\|_\infty = \sup_{\omega} |Ew_1(i\omega)| < 1 \quad \forall G \in \Pi \quad (5.92)$$

From the geometry shown in Figure 5.21, a relationship can be developed between the sensitivity function (H) for any G in Π , and the nominal sensitivity function ($\tilde{E}$), the nominal complimentary sensitivity function ($\tilde{H}$) and the multiplicative uncertainty bound ($\bar{l}_m$).

$$\begin{aligned}
 |1 + GC| &\geq |1 + \bar{G}C| - |\bar{G}C| \bar{l}_m \quad \forall G \in \Pi \\
 \Rightarrow |E| = \left| \frac{1}{1 + GC} \right| &\leq \frac{|\bar{E}|}{1 - |\bar{H}| \bar{l}_m} \quad \forall G \in \Pi
 \end{aligned} \tag{5.93}$$

Substituting this relationship into Equation 5.92 gives;

$$\frac{|\bar{E}w_1|}{1 - |\bar{H}| \bar{l}_m} < 1 \quad \forall \omega \tag{5.94}$$

Rearranging this then gives;

$$|\bar{H} \bar{l}_m| + |\bar{E}w_1| < 1 \quad \forall \omega \tag{5.95}$$

Formalising this gives the following definition of SISO robust performance (Morari and Zafiriou, 1989);

Robust Performance

If all plants G contained within the family of plants Π given by the multiplicative uncertainty description (Equation 5.58) have the same number of RHP poles, the closed-loop performance specification (Equation 5.92) is met if and only if the nominal system is closed-loop stable and the condition on the sensitivity function and the complimentary sensitivity function expressed in Equation 5.95 is met.

Actually, this is just the robust stability requirement (Equation 5.91) with an extra (tighter) margin to ensure that the performance of the worst plant (which might otherwise be only just stable and hence performing badly) is within the designer's requirement.

Consequently, designing to ensure robust performance also gives robust stability.

MIMO Design Criteria

The equivalent criteria for robust stability and robust performance for multivariable systems are more complex than for the SISO case. The analysis is performed using the $M - \Delta$ formulation, whereby each of the uncertainty descriptions for the MIMO system shown in Figure 5.17 are rearranged into the form shown in Figure 5.22a. The M matrix contains the closed-loop while the Δ matrix contains the uncertainty as a normalised

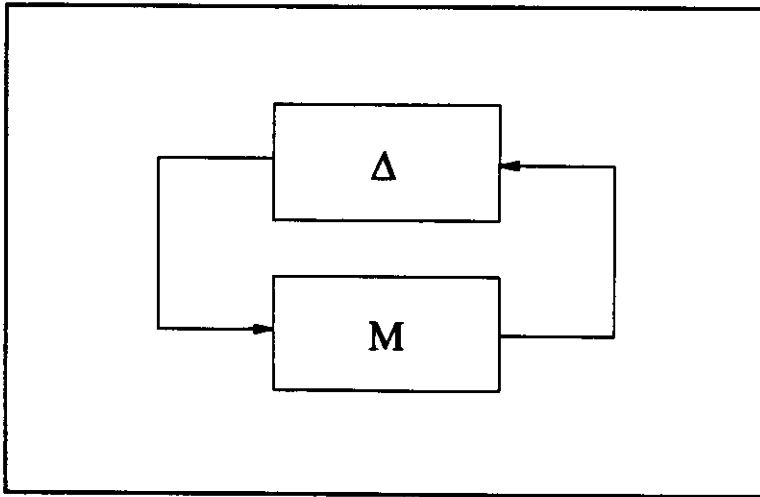


Figure 5.22a - $M - \Delta$ Configuration

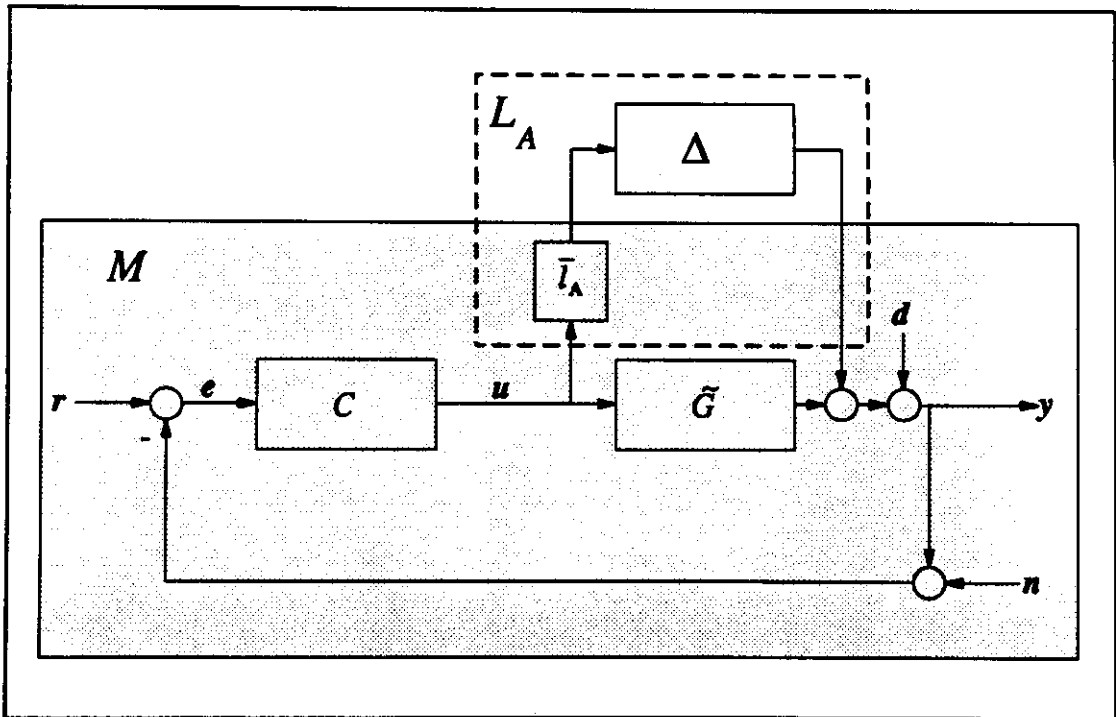


Figure 5.22b - $M - \Delta$ Configuration for Additive Uncertainty Description

perturbation, which can be multiplied by a scalar weight (within M) to give the required uncertainty matrix.

For example, looking at additive uncertainty (described by the matrix L_A , see Equation 5.62), the upper bound on the allowable uncertainty is given by the maximum singular value of L_A . We can write L_A in terms of a Δ perturbation and the scalar bound $\bar{l}_A$;

$$\begin{aligned} \text{If } \bar{\sigma}(L_A) &\leq \bar{l}_A(\omega) \\ \Rightarrow L_A(s) &= \bar{l}_A(s)\Delta(s) \quad \text{where } \bar{\sigma}(\Delta(i\omega)) \leq 1 \quad \forall \omega \end{aligned} \quad (5.96)$$

This shows that the additive uncertain can be described using the scalar value $\bar{l}_A$, and a general multivariable uncertainty Δ , the only limit on which is a maximum singular value less than or equal to one. Similar descriptions can be written for the multiplicative input and the multiplicative output uncertainty descriptions.

The $M - \Delta$ arrangement may initially appear to be an awkward technique for describing multivariable uncertainty. However, the benefit lies in its usefulness in mathematical terms. Figure 5.22b shows how the additive uncertainty description can be rearranged into the $M - \Delta$ formulation. The normal closed loop system lies inside the shaded box, with the uncertainty (L_A) to be added to the process model $\tilde{G}$, contained partly within and partly outside the box. The additive uncertainty L_A is broken down into the scalar part $\bar{l}_A$, and the multivariable part Δ . The scalar component becomes part of the M matrix. For stability analysis, the required inputs and outputs to M are those leading to and from the uncertainty. For the additive uncertainty case, the matrix M is given by;

$$M = -C(I + C\tilde{G})^{-1}\bar{l}_A \quad (5.97)$$

Using the $M - \Delta$ formulation, it can be shown (Morari and Zafiriou, 1989) that if the closed-loop system is nominally stable, then the system is stable for all normalised perturbations Δ if;

$$\|M\|_\infty < 1 \quad (5.98)$$

Therefore for the additive uncertainty description, robust stability is guaranteed when;

Additive Uncertainty

$$\|M\|_{\infty} = \|C(I + C\tilde{G})^{-1}\tilde{l}_A\|_{\infty} = \|C\tilde{E}\tilde{l}_A\|_{\infty} < 1 \quad (5.99)$$

Similar results can be obtained for the multiplicative input and the multiplicative output uncertainty descriptions.

Multiplicative Input Uncertainty

$$\|M\|_{\infty} = \|(I + C\tilde{G})^{-1}C\tilde{G}\tilde{l}_I\|_{\infty} = \|\tilde{H}_I\tilde{l}_I\|_{\infty} < 1 \quad (5.100)$$

Multiplicative Output Uncertainty

$$\|M\|_{\infty} = \|\tilde{G}C(I + \tilde{G}C)^{-1}\tilde{l}_O\|_{\infty} = \|\tilde{H}_O\tilde{l}_O\|_{\infty} < 1 \quad (5.101)$$

These bounds are, in fact, quite similar to those developed for the SISO case (Equation 5.91).

The uncertainty descriptions used in the preceding analysis of robust stability are frequently poor descriptions of the uncertainty which may actually occur in the closed-loop system. For example, uncertainty may occur at both the inputs and the outputs of the plant. This would best be described using both a multiplicative input and a multiplicative output description. However, using the above theory, both of these uncertainties would have to be "lumped" together into the same block. Such a technique will not describe either uncertainty well, and will increase the conservativeness of the description significantly (that is, including many more plants in the uncertainty description than may feasibly ever occur).

An improved method for describing the uncertainty is the "structured uncertainty" description, introduced by Doyle (1982). In simple terms, the analysis allows several of the Δ blocks shown in Figure 5.22a to be incorporated into the analysis. Robust stability for these descriptions is performed using the "structured singular value", also introduced by Doyle (1982). The mathematics for performing this analysis, however, is still in its infancy, and only recently have procedures for finding reasonable estimates of the

structured singular value become available (Packard, 1988). Consequently, structured techniques were not used in the examples presented in this work.

The calculation of the MIMO robust performance criterion is another example of the use of structured singular value techniques, whereby it is actually possible to perform robust controller synthesis, μ -Synthesis (Doyle, 1987). This design procedure gives a controller which not only ensures closed-loop robust stability, but also guarantees the performance the designer specifies for all perturbations within the given uncertainty description. For the MIMO examples presented here, robust performance was not sought, as not only is the procedure not easily achievable, but as will be shown in Chapter 6, it is not actually required for optimal operation analysis.

In summary, criteria have been derived to check that the closed-loop system meets the designer's requirements for both the SISO and MIMO cases.

SISO System

Nominal Performance:

$$\begin{aligned} |E(i\omega)| &< |w_1(i\omega)|^{-1} \quad \forall \omega \\ \Rightarrow |E(i\omega)| |w_1(i\omega)| &< 1 \quad \forall \omega \\ \Rightarrow \|Ew_1\|_{\infty} &< 1 \end{aligned} \quad (5.85)$$

Robust Stability:

$$\|\bar{H}\bar{L}_m\|_{\infty} = \sup_{\omega} |\bar{H}\bar{L}_m(\omega)| < 1 \quad (5.91)$$

Robust Performance:

$$|\bar{H}\bar{L}_m| + |\bar{E}w_1| < 1 \quad \forall \omega \quad (5.95)$$

MIMO System

Nominal Performance:

$$\|W_2 E W_1\|_\infty < 1 \quad (5.88)$$

Robust Stability:

$$\|M\|_\infty < 1 \quad (5.98)$$

where M depends on the uncertainty description used.

These criteria may now be used in the design of robust controllers. The first procedure presented uses the IMC formulation to allow a controller to be designed which fulfils the necessary requirements. The IMC design procedure described in Section 5.2.3.1 is based on that presented by Morari and Zafiriou (1989). This will be applied to a SISO example in Section 5.3. The IMC method was chosen for this example as it is easy to apply to the SISO case as well as being quite illustrative of robust controller design.

Robust controller design for the MIMO case was performed using software developed by Chiang and Safanov (1988) within the MATLAB (MathWorks) framework. As will be discussed in Section 5.2.3.2, this design procedure is based on the two-Ricatti formulac of Glover and Doyle (Doyle et al., 1989). This will be applied to a MIMO example in Chapter 6. The IMC procedure was not used for the MIMO example as the application is more difficult than for the SISO case, the MATLAB software providing a more efficient design procedure.

It should be noted that neither of these procedures are actually H_∞ -optimal controller design techniques, which aim to minimise the performance measures, but are H_∞ procedures for meeting the designer's specifications without necessarily doing any better.

5.2.3.1 Internal Model Control Approach

The principle of Internal Model Control (IMC) was introduced in Section 5.2.1 in the context of closed-loop performance measures. IMC uses the difference between the measured outputs and those calculated by a model of the plant ($\tilde{G}$), as the signal to a controller (Q). The robust IMC controller design procedure consists of two steps.

1. A controller ($\tilde{Q}$) is designed to achieve the nominal performance.

2. A filter (F) is added to the controller to achieve robust stability and robust performance.

The general layout of the closed-loop IMC system, with the controller Q , consisting of the nominal controller $\tilde{Q}$ and the filter F , is shown in Figure 5.23.

$$Q = \tilde{Q}F \quad (5.102)$$

Morari and Zafiriou (1989) present a detailed derivation and procedure for performing the IMC robust design. They break the general problem down into three sub-sections;

1. SISO open-loop stable systems.
2. SISO open-loop unstable systems.
3. MIMO systems.

In each case, the controller chosen to achieve nominal performance can be determined using H_2 -optimal control principles for a specified input. For the simplest case, an open-loop stable SISO system where the process does not have any time delays or right half plane zeros (RHPZ), the controller which gives nominal performance is simply the inverse of the nominal process model.

$$\tilde{Q} = \tilde{G}^{-1} \quad (5.103)$$

As discussed in Section 5.2.1, such a controller ensures that the sensitivity function (E) is zero, giving perfect disturbance rejection. In general, controllers designed to achieve nominal performance tend to be too "aggressive", causing problems (both in stability and performance actually achieved) when the process deviates from the nominal model. In this design procedure, a filter is used to modify the nominal controller, "slowing it down" and ensuring that robust stability and robust performance are achieved.

For the other two cases (other SISO systems and MIMO processes), the nominal controller is usually based on an inverted model of the process which has been modified by removing terms arising from the RHPZ and time delay components of the model.

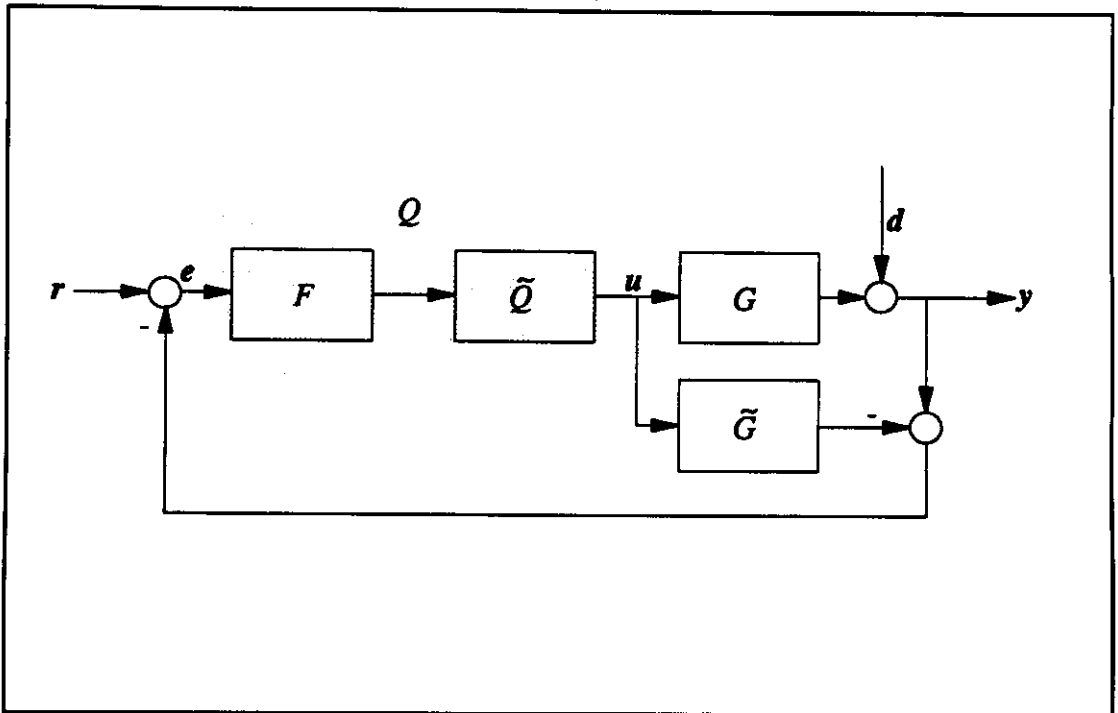


Figure 5.23 - IMC Configuration Showing Controller and Filter $Q = \tilde{Q}F$

The form of the filter used depends on the properties of the open-loop system and on the desired setpoint tracking properties (e.g. zero steady-state offset). For the simplest case (as above), a typical filter is;

$$F(s) = \frac{1}{(\psi s + 1)^n} \quad (5.104)$$

where n is chosen to ensure that Q is proper (order of numerator $\leq$ order of denominator). The time constant of the filter (ψ) is chosen to meet the robust stability and robust performance requirements.

For the case where the open-loop SISO process is unstable, the filter must be chosen to ensure that the closed-loop remains stable. For the MIMO case, the filter may be a full transfer function matrix. Typically, however, it has a diagonal form.

In practice, the IMC robust controller design procedure, for the open-loop stable SISO case, becomes one of choosing the smallest filter time constant ψ which satisfies the robust performance (and hence also robust stability) criterion.

$$\begin{aligned} Q &= \tilde{Q}F \\ &= \tilde{G}^{-1} \left[\frac{1}{\psi s + 1} \right] \end{aligned} \quad (5.105)$$

where ψ = minimum value such that

$$|\tilde{H}\tilde{I}_m| + |\tilde{E}w_1| < 1 \quad \forall \omega$$

5.2.3.2 Two-Ricatti Formulae

The software produced by Chiang and Safanov (1988) for performing H_∞ controller design is suitable for designing MIMO (and hence also SISO) controllers. The procedure uses an unstructured description of uncertainty and a simple performance criterion, usually expressed as a scalar or diagonal matrix, to produce a controller which gives nominal performance and robust stability. There are two steps in the design procedure, the first involves the formation of an augmented plant ($P(s)$) which incorporates the performance weighting (W_1^{-1}) and the uncertainty bound (W_3^{-1}). The second step is the design of the controller which fulfils the design requirements for the augmented plant.

The nominal performance requirement is equivalent to that presented as Equation 5.88, with the modification that only an output weight (W_1) on the sensitivity function has been used, instead of both an input and an output weight. (Note that the W_1 used here, is equivalent to the W_2 used in the Equation 5.88. This is to be consistent with Chiang and Safanov's nomenclature.) The use of only one weighting matrix could lead to some limitation in the flexibility of the design procedure, however, it is sufficient for this analysis. The mathematical requirement for nominal performance can therefore be expressed as;

$$\begin{aligned}\bar{\sigma}(E(i\omega)) &\leq |W_1^{-1}(i\omega)| \quad \forall \omega \\ \Rightarrow \|W_1 E\|_{\infty} &\leq 1\end{aligned}\tag{5.106}$$

In the example presented in Chapter 6, uncertainty is described using the multiplicative output uncertainty description. Again, the robust stability requirement imposed using this description is equivalent to one derived earlier (Equation 5.101),

$$\begin{aligned}\bar{\sigma}(H(i\omega)) &\leq |W_3^{-1}(i\omega)| \quad \forall \omega \\ \Rightarrow \|W_3 H\|_{\infty} &\leq 1\end{aligned}\tag{5.107}$$

Here W_3 gives the upper bound on the uncertainty, equivalent to $\bar{l}_o$ used in Equation 5.101. The final results expressed in Equations 5.101 and 5.107 show a different order of multiplication, however, since W_3 and $\bar{l}_o$ are both scalars (or scalars times the identity matrix) the order of multiplication is not important.

Both of these requirements (Equations 5.106 and 5.107) are satisfied simultaneously by satisfying an infinity norm bound on the matrix T_{y1u1} , which is constructed by augmenting the matrices $W_1 E$ and $W_3 H$. Using the singular value relationship,

$$\max\{\bar{\sigma}(A), \bar{\sigma}(B)\} \leq \bar{\sigma}([A \ B])\tag{5.108}$$

it is possible (Chiang and Safanov, 1988) to show that if,

$$T_{y1u1} = \begin{bmatrix} W_1 E \\ W_3 H \end{bmatrix}\tag{5.109a}$$

then satisfying the bound

$$\|T_{y|u}\|_{\infty} \leq 1 \quad (5.109b)$$

ensures both nominal performance and robust stability.

The "plant augmentation" implied by Equation 5.109 involves adding the weights W_1 and W_3 to the error and measurement respectively. The design problem then becomes one of finding the controller such that the closed-loop system is internally stable and the infinity norm is less than or equal to one (Equation 5.109b).

The procedure adopted by Chiang and Safanov (1988) for finding the controller which satisfies this infinity norm bound, employs the two-Ricatti formulae attributed to Glover and Doyle. The mathematics behind this procedure is extremely complex, however a tutorial paper which develops the H_{∞} theory in parallel with the H_2 theory is available (Doyle et al., 1989). The controller developed is only one of an infinite set of controllers which satisfy the H_{∞} criteria. The result is, in fact, a parameterisation of all controllers which fit the designer's criteria. The mathematics of the design procedure will not be explained here, but may be found, for example, in Doyle et al.(1989), Doyle (1984) and Limebeer (1989).

In the following section, an illustrative SISO example will be developed which will be used to demonstrate aspects of the calculation of the optimal achievable performance as described in Section 5.1, as well as aspects of H_{∞} controller design. In Chapter 6, a MIMO example will be presented which will be used to examine how the Optimal Operation Algorithm, now incorporating a MIMO H_{∞} controller design, can be used to choose between a set of alternative process flowsheets on the basis of their optimal economic performance.

5.3 Illustrative Single-Input Single-Output Example

To illustrate some of the features of the calculations involved in determining the optimal achievable performance as discussed in Section 5.1, a single-input single-output (SISO) example will be developed. The goal of determining how well a particular design can operate, in the face of disturbances, is to give a better indication at the design stage of the achievable economic performance. The traditional design procedure has used the optimal

steady-state performance as an economic indicator, neglecting the effects of plant dynamics and disturbances. As described in Section 5.1, by modifying the input and output constraints to account for the input manipulation required by the controller, or the unattenuated disturbances appearing at the output, the Optimal Operation Algorithm can be used to determine both the optimal operating point, and the amount of control necessary, to ensure that disturbances do not cause the input and output constraints to be violated.

A key component of the Optimal Operation Algorithm is the concept of partial control, whereby only a fraction of the measured error is sent to the controller. From an optimisation perspective, this allows the optimiser to select the amount of control required (open-loop, partial control or full closed-loop). To simplify the presentation of this example, only the two extremes of control, open and closed-loop, will be considered. The full implementation of the Optimal Operation Algorithm will be demonstrated in Chapter 6.

In this example of a design problem, the designer is to choose not between two different process designs, but between the use of two different plant inputs in a conventional SISO control scheme. The input which is not to be used for control will be set to an optimal value. The example will demonstrate that the process economics can be considerably influenced by the different characteristics of the two inputs.

The two inputs, u_1 and u_2 , both affect the state variable x , which in turn is also the measured output y . The state variable is also affected by a disturbance d' at the input to the plant. These relationships are shown in the following state-space representation.

$$\begin{aligned}\frac{dx}{dt} &= 3u_1 + 4u_2 + d' - x \\ y &= x\end{aligned}\tag{5.110}$$

An equivalent transfer function relationship is given by Equations 5.111 and 5.112. Here the disturbance is given as d , the value which appears at the output, as shown in Figure 5.24.

$$y(s) = \frac{3}{s+1}u_1(s) + \frac{4}{s+1}u_2(s) + d(s) \quad (5.111)$$

$$d(s) = \frac{1}{s+1}d'(s) \quad (5.112)$$

The disturbance expected to enter the system (d') has a zero mean value, and a frequency content described using the bounded weight technique discussed in Section 5.1 (Equations 5.12, 5.13, 5.14). The particular weight chosen for this example is given by the following transfer function;

$$|d'| \leq \left| \frac{1}{100s+1} \right| \quad (5.113)$$

As shown previously in Figure 5.6, this allows a maximum magnitude of 1 at low frequencies, dropping down to 0.01 at a frequency of 1 radian per unit time. The frequency range that will be considered throughout this analysis is 10^{-3} to 10 radians per unit time, as this covers almost the steady-state ($\omega = 0$) through to 10 times the frequency corresponding to the system time constant, 1 time unit.

The objective of operating the process described by these equations, is to maximise the following (linear) objective function,

$$\max_{u_1, u_2} \Phi = -5u_1 - 6u_2 + 7y \quad (5.114)$$

subject to the steady-state model (the equality constraint) and the input and output inequality constraints,

$$\begin{aligned} y &= 3u_1 + 4u_2 + d' \\ y &\leq 10 \\ u_1 &\geq 0 \\ u_2 &\geq 0 \end{aligned} \quad (5.115)$$

Optimisation of this system without allowing for the impact of the disturbance (i.e. when $d' = 0$) gives the following result.

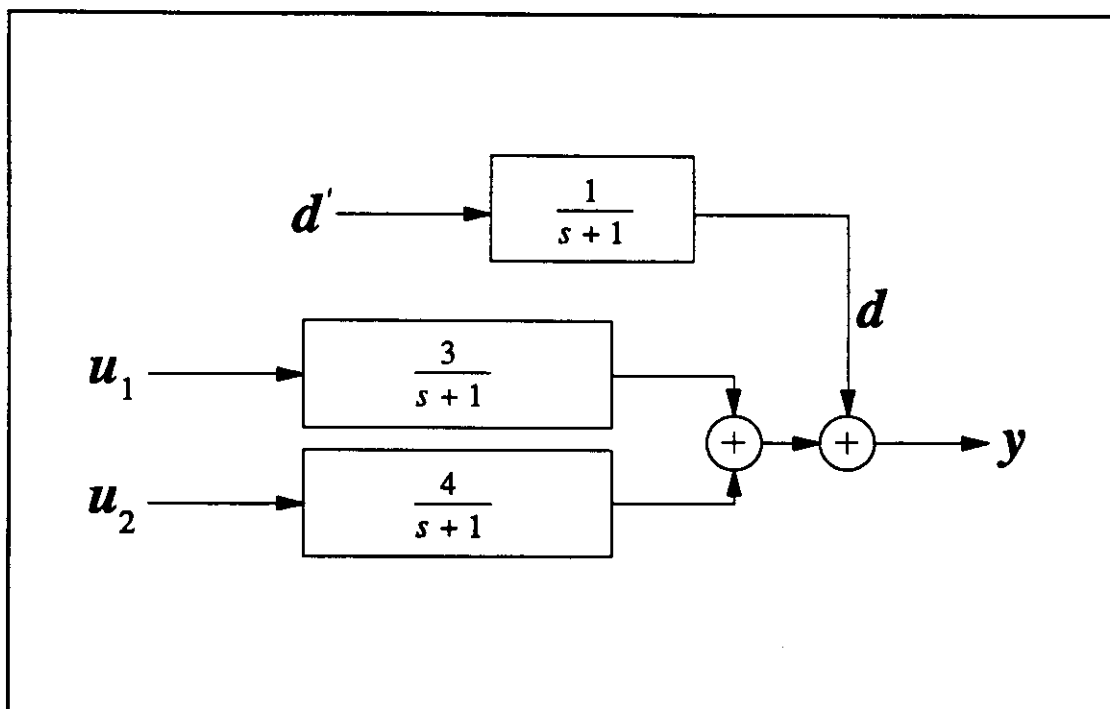


Figure 5.24 - SISO Example Open-Loop Transfer Functions

$$\begin{aligned}
 \Phi &= 55 \\
 y &= 10 \\
 u_1 &= 0 \\
 u_2 &= 2.5
 \end{aligned}
 \tag{5.116}$$

There are two degrees of freedom (u_1 and u_2) for this linear system, consequently at the optimum there are two active constraints, $u_1 = 0$ and $y = 10$.

In the following section, we will analyse how disturbances, and taking measures to reduce their impact, can effect the optimum.

5.3.1 Optimal Operation Results

A number of different approaches will be adopted in determining the optimal manner in which to operate the example plant. First, the possibility of running the plant open-loop (no control) will be investigated. Then the closed-loop case will be analysed using three types of controller, perfect control, "real" control and H_∞ control. For each of the closed-loop methods, two cases will be considered;

Case 1

Input u_1 Output y

Case 2

Input u_2 Output y

No Control

Using the optimal operation principles explained in Section 5.1, the maximum open-loop magnitude of the disturbance appearing at the output of the plant (Δy_{ol}) needs to be evaluated, so that a modification can be made to the output constraints. An optimum generated using these modified output constraints will ensure that implementing the optimal operating point thus calculated, will not allow the action of the disturbance to cause output constraints to be violated. Using the bounded description of the disturbance appearing at the plant input (d') and the transfer function relating this to the disturbance appearing at the plant output gives the following;

$$\begin{aligned}
 |d(s)| &\leq \left| \frac{1}{s+1} \frac{1}{100s+1} \right| \\
 \Rightarrow \Delta y_{OL} &= \max_{\omega} \left| \frac{1}{s+1} \frac{1}{100s+1} \right|
 \end{aligned}
 \tag{5.117}$$

Figure 5.25 shows the magnitude of d as a function of frequency. The maximum value of this is 1, therefore, the modified output constraint becomes;

$$\begin{aligned}
 y &\leq y_U - \Delta y_{OL} \\
 \Rightarrow y &\leq 10 - 1 \\
 &\leq 9
 \end{aligned}
 \tag{5.118}$$

The result of performing the optimisation again but with this constraint is;

$$\begin{aligned}
 \Phi &= 49.5 \\
 y &= 9 \\
 \mu_1 &= 0 \\
 \mu_2 &= 2.25
 \end{aligned}
 \tag{5.119}$$

The change in the objective function is;

$$\begin{aligned}
 \Delta \Phi &= \Phi(\text{With disturbance}) - \Phi(\text{No disturbance}) \\
 &= 49.5 - 55 \\
 &= -5.5
 \end{aligned}
 \tag{5.120}$$

Therefore, ensuring that the disturbance does not cause the constraints to be violated, means changing the operating point so that the mean value of the objective function decreases by 5.5 units to 49.5. In this situation, the disturbance is "costing money".

Perfect Control

The extreme opposite to the open-loop calculations performed above, is when there is perfect closed-loop control. This means that despite the impact of the disturbance d , the measurement (y) remains constant, due to manipulation of one of the inputs, μ_1 or μ_2 . In Section 5.2.1, the perfect controller using the IMC configuration was shown to be the

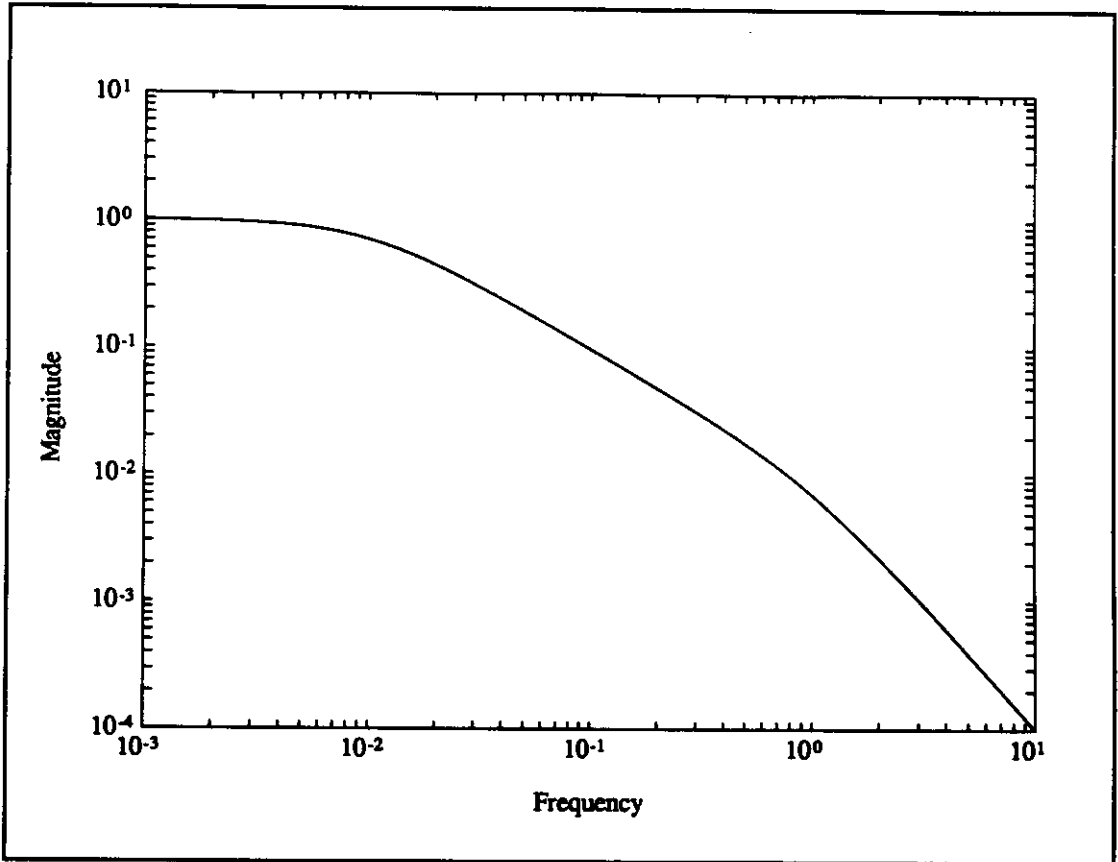


Figure 5.25 - Magnitude of d versus Frequency

inverse of the plant model $\tilde{G}^{-1}$ (Equation 5.50). This ensures that the sensitivity function (Equation 5.48) is zero even if there is plant-model mismatch.

$$\begin{array}{ll} \text{Case 1} & Q = \tilde{G}^{-1} \\ (u_1 - y) & = \frac{s + 1}{3} \end{array} \qquad \begin{array}{ll} \text{Case 2} & Q = \tilde{G}^{-1} \\ (u_2 - y) & = \frac{s + 1}{4} \end{array} \quad (5.121)$$

To calculate the optimal achievable operating point for perfect control, the output constraints do not require modification as there is no variation at the output. However, there will be input manipulation to achieve this, consequently the input constraints must be modified to ensure that the optimal operating point allows sufficient input variance without violating the input constraints. The modification will be by the maximum amount of input manipulation required by the controller, Δu_{CL} . For the normal feedback structure, the relationship between the disturbance appearing at the output (d) and the input to the process was given by Equation 5.31. Substitution of the equivalent IMC controller Q (Equation 5.45) into this relationship gives;

$$\begin{aligned} u &= C(1 + GC)^{-1}d, \quad C = Q(1 - \tilde{G}Q)^{-1} \\ \Rightarrow u &= Q(1 + (G - \tilde{G})Q)^{-1}d \end{aligned} \quad (5.122)$$

For the case where the model is known perfectly ($\tilde{G} = G$), this simplifies to;

$$u = Qd \quad (5.123)$$

The maximum magnitude of the input required may, therefore, be determined using the transfer function description of the bounded input entering the plant (Equation 5.113).

$$\begin{aligned} \therefore |u(s)| &\leq \left| Q(s) \frac{1}{s + 1} \frac{1}{100s + 1} \right| \\ \Rightarrow \Delta u_{CL} &= \max_{\omega} \left| Q(s) \frac{1}{s + 1} \frac{1}{100s + 1} \right| \end{aligned} \quad (5.124)$$

The magnitude of the input, as a function of frequency, for the two controllers given in

Equation 5.121 is shown in Figure 5.26. The maximum values are;

$$\begin{array}{ll} \text{Case 1} & \Delta u_{CL} = 0.3333 \\ (u_1 - y) & \Delta y_{CL} = 0 \end{array} \qquad \begin{array}{ll} \text{Case 2} & \Delta u_{CL} = 0.25 \\ (u_2 - y) & \Delta y_{CL} = 0 \end{array} \quad (5.125)$$

Note that for perfect control, the maximum magnitude at the output, Δy_{CL} , is zero.

The modified input constraints are therefore;

$$\begin{array}{ll} \text{Case 1} & u \geq u_L + \Delta u_{CL} \\ (u_1 - y) & \end{array} \qquad \begin{array}{ll} \text{Case 2} & u \geq u_L + \Delta u_{CL} \\ (u_2 - y) & \end{array} \quad (5.126)$$

$$\Rightarrow u_1 \geq 0.3333 \qquad \Rightarrow u_2 \geq 0.25$$

In each case, the constraint corresponding to the input not being used for control need not be modified.

The result of performing the modified optimisations were;

$$\begin{array}{ll} \text{Case 1} & \Phi = 54.8333 \\ (u_1 - y) & y = 10 \\ & u_1 = 0.3333 \\ & u_2 = 2.25 \\ & \Delta \Phi = -0.1667 \end{array} \qquad \begin{array}{ll} \text{Case 2} & \Phi = 55 \\ (u_2 - y) & y = 10 \\ & u_1 = 0 \\ & u_2 = 2.5 \\ & \Delta \Phi = 0 \end{array} \quad (5.127)$$

Therefore, when using a perfect controller, Case 2 (u_2 as the input for the SISO controller) is more profitable than Case 1. This is because at the nominal optimum (Equation 5.116), u_2 was sufficiently removed from its lower bound, that it did not have to move to allow for its use for control. The result is that there is no cost associated with the use of u_2 as the control input.

"Real" Controllers

The perfect controller used in the previous analysis gives a limit to how well the process

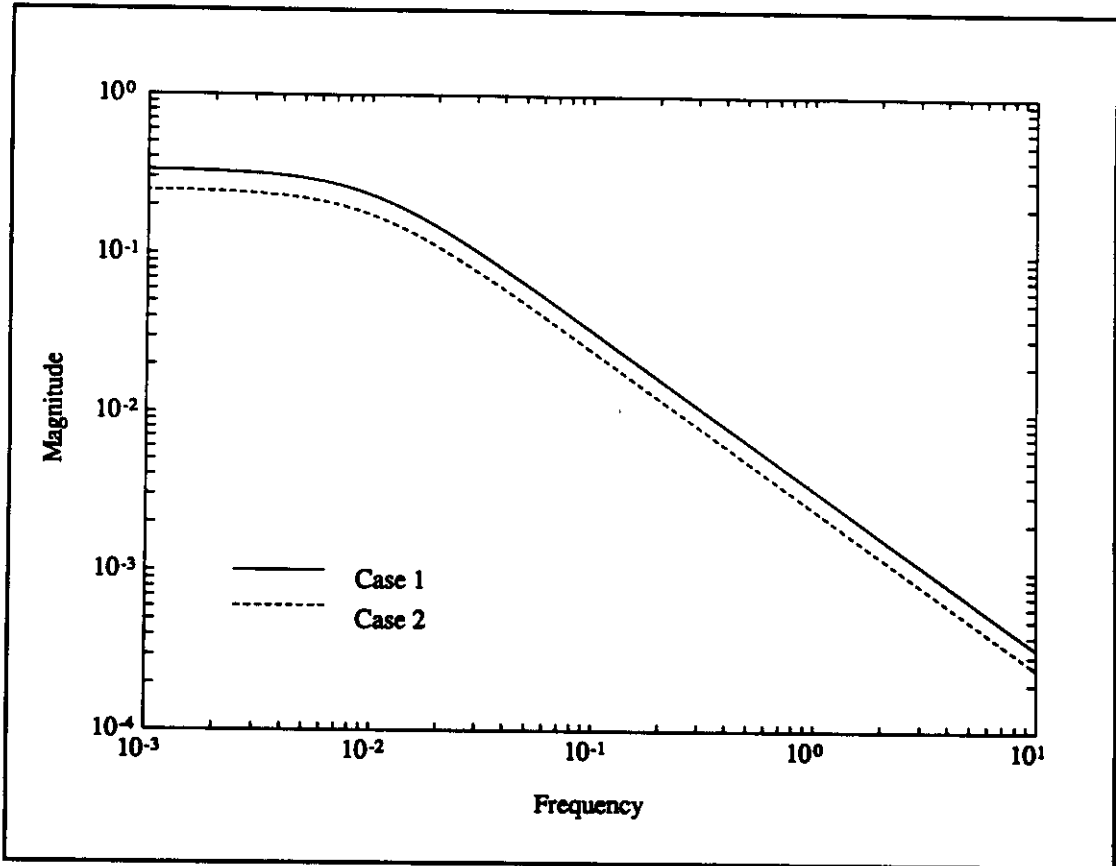


Figure 5.26 - Input Magnitude (u_1 , u_2) for Perfect Control

can be controlled in theory. However, the controllers developed cannot be implemented in practice because they contain "pure differentiators". That is, the order of the numerator is greater than the order of the denominator in the transfer function, requiring the time domain equivalent to take exact derivatives of measurements with respect to time. In real control systems, it is not possible to use the exact value of such derivatives (Luyben, 1990). For example, if a step was input to such a controller, then the derivative term would be infinite, which is not possible in real control systems. One method of overcoming this is to "slow" the controller down by placing a filter on the input, in a similar manner to the IMC robust controller design procedure described in Section 5.2.3.1. The use of a simple filter will ensure that the order of the numerator is not greater than that of the denominator.

Without selecting the filter to meet H_∞ requirements, the following filter (F) was chosen so that the perfect controller could be implemented as a "real" controller. The value chosen for the filter time constant ($\psi = 0.01$), was selected as being small enough to give "almost perfect" control, yet not so small as to cause numerical difficulties. As will be seen later, this filter is indeed much "faster" than that chosen using H_∞ criteria.

$$F = \frac{1}{0.01s + 1} \quad (5.128)$$

The real controllers used are therefore;

$$\begin{array}{ll} \text{Case 1 } Q = \tilde{G}^{-1}F & \text{Case 2 } Q = \tilde{G}^{-1}F \\ (u_1 - y) & (u_2 - y) \end{array} \quad (5.129)$$

$$= \frac{s + 1}{3} \frac{1}{0.01s + 1} \quad = \frac{s + 1}{4} \frac{1}{0.01s + 1}$$

These controllers are no longer perfect controllers, consequently the sensitivity function is not zero at all frequencies, being zero at low frequencies and tending to one at high frequencies, as shown in Figure 5.27. Consequently, there will be some disturbance unattenuated at the output despite the action of the controller in manipulating an input. The value of the disturbance appearing at the output can be found using Equation 5.30 and either substituting for the IMC controller (Equation 5.45) or using the IMC sensitivity function (E , Equation 5.48). The IMC result is shown for the simplified case where the

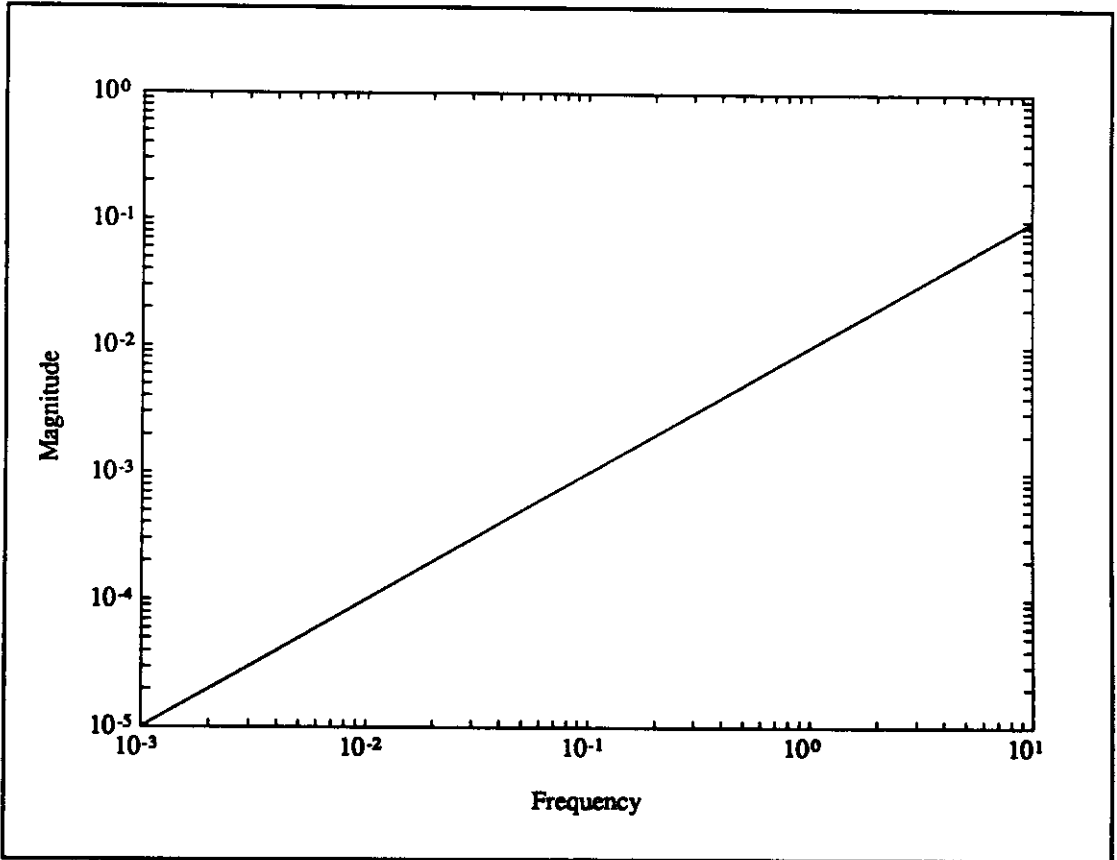


Figure 5.27 - Sensitivity Function E for Real Controller (Cases 1 and 2)

process model is perfectly known ($\tilde{G} = G$).

$$\begin{aligned}
 y &= Ed \\
 &= (I - \tilde{G}Q)d \\
 &= (I - \tilde{G}\tilde{G}^{-1}F)d \\
 &= (I - F)d
 \end{aligned} \tag{5.130}$$

This may now be used in conjunction with the description of the bounded input to the plant, to calculate the maximum magnitude of the output variation due to the disturbance (Δy_{CL}).

$$\begin{aligned}
 \therefore |y(s)| &\leq \left| E(s) \frac{1}{s+1} \frac{1}{100s+1} \right| \\
 \Rightarrow \Delta y_{CL} &= \max_{\omega} \left| E(s) \frac{1}{s+1} \frac{1}{100s+1} \right|
 \end{aligned} \tag{5.131}$$

Using the filter given in Equation 5.128, values for the maximum input variation magnitude (Δu_{CL}), calculated using Equations 5.123 and 5.124, and the maximum output variation magnitude (Δy_{CL}) were determined for each case in this example. Plots of the magnitudes versus frequency are given in Figures 5.28a and 5.28b. The values determined were;

$$\begin{array}{ll}
 \text{Case 1} & \Delta u_{CL} = 0.3333 \\
 (u_1 - y) & \Delta y_{CL} = 0.0001 \\
 \text{Case 2} & \Delta u_{CL} = 0.25 \\
 (u_2 - y) & \Delta y_{CL} = 0.0001
 \end{array} \tag{5.132}$$

As expected, the use of a real controller means that there is some variation in the output due to the disturbance. For these controllers the variation is, however, very small since the filter time constant was chosen to give "almost perfect" control. A larger time constant (slower filter) gives more output variation, as will be seen for the H_{∞} controllers. Due to imperfect control, and the consequent output variation, the output constraints, as well as the input constraints, have to be modified before optimising to find the optimal achievable operating point. Note that the values of Δu_{CL} are unaffected by using a real controller, as these maxima occurred at low frequencies and were unaffected by the use of the filter. The modified constraints are;

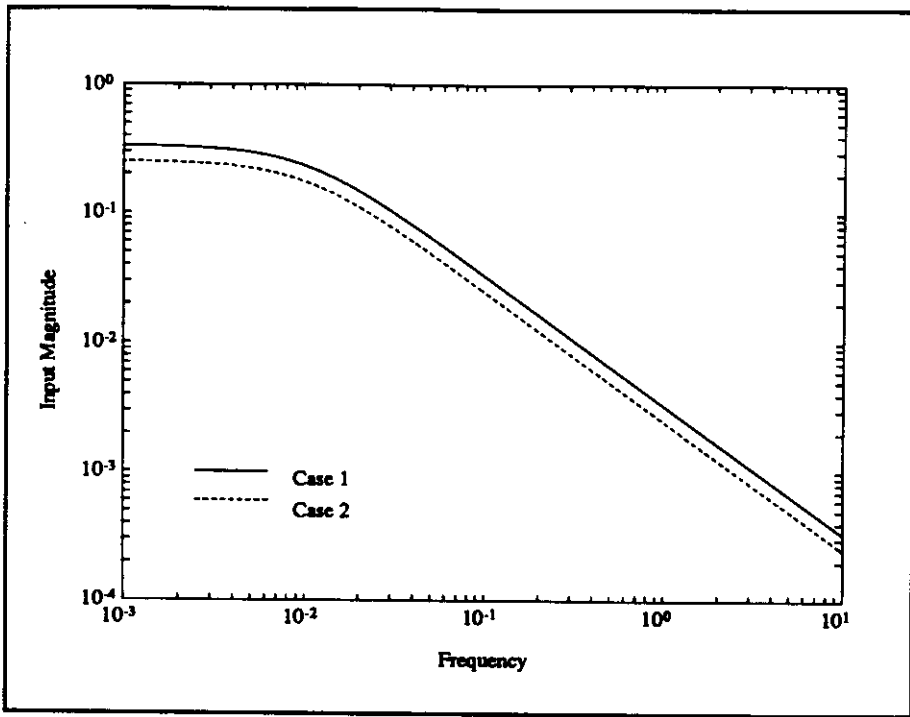


Figure 5.28a - Real Controller Input Magnitudes

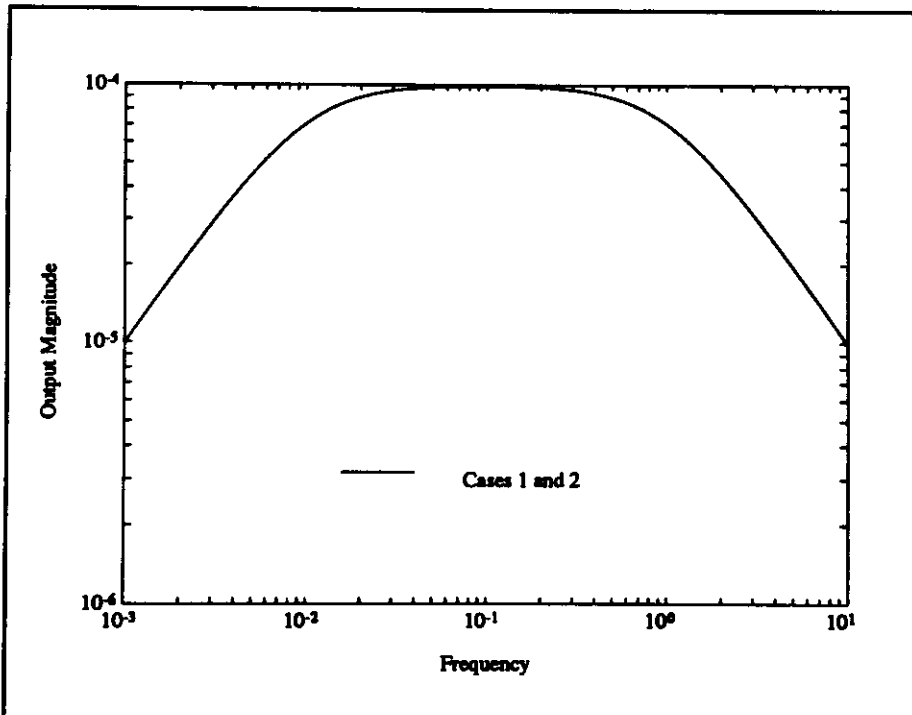


Figure 5.28b - Real Controller Output Magnitudes

$$\begin{array}{ll}
\text{Case 1} & \text{Case 2} \\
(u_1 - y) & (u_2 - y) \\
u \geq u_L + \Delta u_{CL} & u \geq u_L + \Delta u_{CL} \\
\Rightarrow u_1 \geq 0.3333 & \Rightarrow u_2 \geq 0.25 \\
y \leq y_U - \Delta y_{CL} & y \leq y_U - \Delta y_{CL} \\
\Rightarrow y \leq 9.9999 & \Rightarrow y \leq 9.9999
\end{array} \quad (5.133)$$

Performing the optimisation again with these modified constraints gives the following results;

$$\begin{array}{ll}
\text{Case 1} & \text{Case 2} \\
(u_1 - y) & (u_2 - y) \\
\Phi = 54.8328 & \Phi = 54.9995 \\
y = 9.9999 & y = 9.9999 \\
u_1 = 0.3333 & u_1 = 0 \\
u_2 = 2.24998 & u_2 = 2.49998 \\
\Delta \Phi = -0.1672 & \Delta \Phi = -0.0005
\end{array} \quad (5.134)$$

These results are slightly different to those obtained using a perfect controller. In particular, the output y lies on the modified constraint value (9.9999) reflecting the small amount of output variation which may occur. Consequently, although Case 2 is still superior to Case 1, the objective function value is less than in the nominal case. Here we see that the use of a real (rather than a perfect controller) means that the disturbance costs us money in both cases.

H_∞ Controller

To further increase the realism of the example, parametric uncertainty in the transfer function dynamic model was considered. Uncertainty in both the gain and time constants were included using the α and β terms in the following equation.

$$y(s) = \frac{3(1 + \alpha)}{(1 + \beta)s + 1} u_1(s) + \frac{4(1 + \alpha)}{(1 + \beta)s + 1} u_2(s) \quad (5.135)$$

where $|\alpha| \leq 0.6, \quad |\beta| \leq 0.6$

These uncertain parameters mean that the value of the transfer function at any given frequency is imprecisely known, so that the real plant is one member of a family of possible plants, as discussed in Section 5.2.2. To design a controller that will give robust stability and robust performance, the parametric uncertainty must be approximated using an additive or multiplicative unstructured uncertainty description. For this example, a multiplicative uncertainty description (Equations 5.58, 5.59, 5.60) was chosen. To find the upper bound on the multiplicative uncertainty $\bar{l}_m(\omega)$, the relative error l_m was calculated for the extreme variations of the α and β parameters. For Case 1 ($u_1 - y$) this gave l_m to be;

$$\begin{aligned}
 l_m &= \frac{G(s) - \bar{G}(s)}{\bar{G}(s)} \\
 &= \frac{\frac{3(1 + \alpha)}{(1 + \beta)s + 1} - \frac{3}{s + 1}}{\frac{3}{s + 1}} \\
 &= \frac{(\alpha - \beta)s + \alpha}{(1 + \beta)s + 1}
 \end{aligned} \tag{5.136}$$

Notice that the gain element, 3, from the original transfer function does not appear in the general formula for l_m . Consequently, Equation 5.136 is, in fact, valid for both cases ($u_1 - y$ and $u_2 - y$). The four combinations of extreme variation in the parameters give the following expressions for l_m :

$$\begin{aligned}
 \text{(i)} \quad \alpha &= 0.6 \quad \beta = 0.6 \quad l_m = \frac{0.6}{1.6s + 1} \\
 \text{(ii)} \quad \alpha &= 0.6 \quad \beta = -0.6 \quad l_m = \frac{1.2s + 0.6}{0.4s + 1} \\
 \text{(iii)} \quad \alpha &= -0.6 \quad \beta = 0.6 \quad l_m = \frac{-1.2s - 0.6}{1.6s + 1} \\
 \text{(iv)} \quad \alpha &= -0.6 \quad \beta = -0.6 \quad l_m = \frac{-0.6}{0.4s + 1}
 \end{aligned} \tag{5.137}$$

Plots of these four cases are shown in Figure 5.29, where it is clear that the case with the largest magnitude is case (ii), $\alpha = 0.6$, $\beta = -0.6$. Therefore, the multiplicative uncertainty

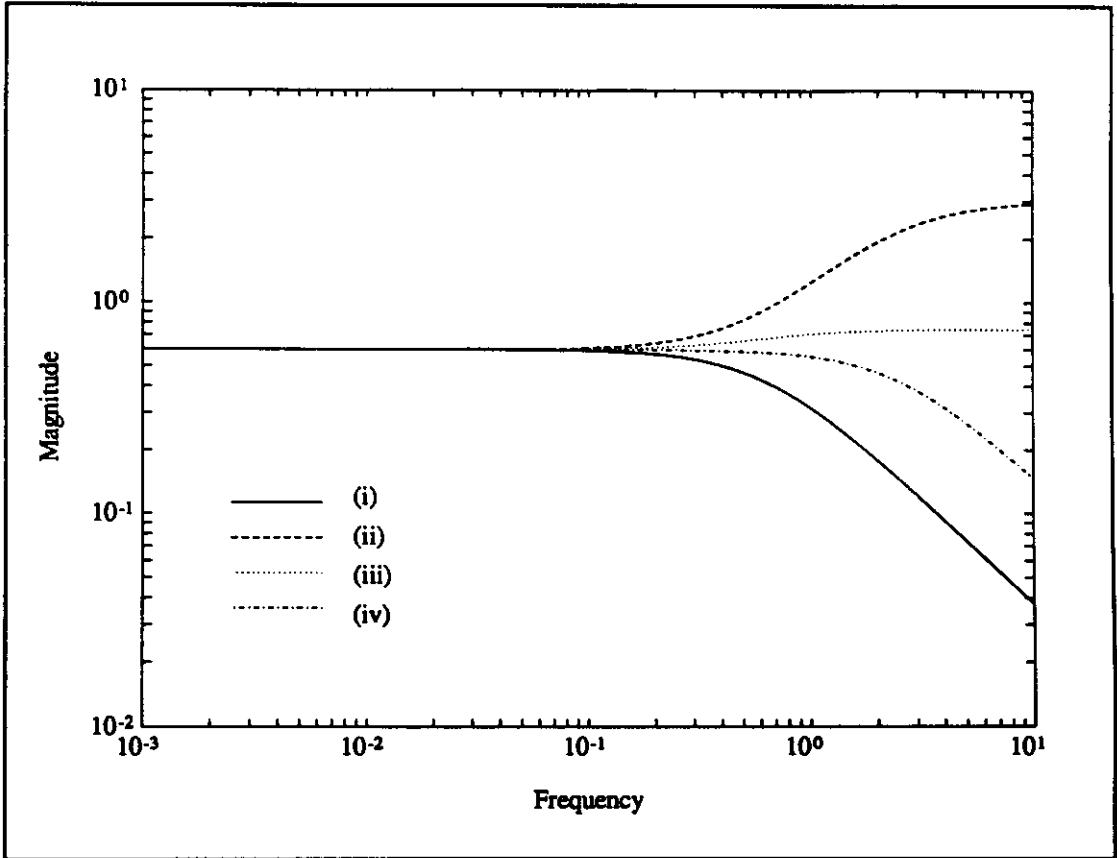


Figure 5.29 - Multiplicative Uncertainty for Four Parameter Extremes

bound, $\bar{l}_m(\omega)$, must have a magnitude equal to case (ii). This gives the following frequency domain representation;

$$\bar{l}_m(\omega) = \left| \frac{1.2i\omega + 1}{0.4i\omega + 1} \right| \quad (5.138)$$

The procedure adopted for the design an H_∞ controller, to achieve robust stability and robust performance for this example, was the IMC technique as described in Section 5.2.3.1. This involves first designing an "optimal" controller, and then modifying this by adding a filter to give the required robust characteristics. For the stable, minimum phase (i.e. no right half plane zeros) process modelled here, the optimal controller is the process model inverse ($\tilde{G}^{-1}$), and the required filter is of the same form as previously used for the real controller. However, now the filter time constant (ψ) must be chosen to meet the robust performance (and hence also robust stability) design criterion, given in Equation 5.95.

$$|\tilde{H}\bar{l}_m| + |\tilde{E}w_1| < 1 \quad \forall \omega \quad (5.95)$$

Knowledge of the disturbance characteristics are useful when determining the sensitivity function weighting, as it allows the sensitivity function to be made small where disturbances are large, and larger at frequencies where disturbances are less of a concern. Noting that the inverse of w_1 , is an upper limit on the sensitivity function leads to the use of the magnitude of the disturbance appearing at the output for the sensitivity function weighting, as is the case here.

$$w_1 = \frac{1}{s+1} \frac{1}{100s+1} \quad (5.139)$$

Substituting the sensitivity function and complementary sensitivity function with their IMC controller relationships (for the case where the model is perfectly known, $G = \tilde{G}$, these are Equations 5.48 and 5.49), and using the process model inverse as the optimal controller and the filter described above, gives the bound;

$$|\tilde{G}\tilde{G}^{-1}F\bar{l}_m| + |(1 - \tilde{G}\tilde{G}^{-1}F)w_1| < 1 \quad (5.140)$$

$$\left| \frac{1}{\psi s + 1} \frac{1.2s}{0.4s + 1} \right| + \left| \left(1 - \frac{1}{\psi s + 1} \right) \frac{1}{s + 1} \frac{1}{100s + 1} \right| < 1$$

By trial and error, a filter time constant $\psi = 0.9$ was found to just satisfy this robust performance bound. Figure 5.30 shows the robust performance bound for a number of filter time constants. As indicated by the trend in this figure, the filter constant used earlier for the real controller ($\psi = 0.01$) would not have given robust performance.

To determine the optimal achievable performance, the maximum input magnitude (Δu_{CL}) and the maximum magnitude of the unattenuated disturbance at the output (Δy_{CL}) are required. However, for this case where the model of the process is uncertain, the calculation is not as straight forward as that presented for the perfect and real controller cases. In fact, to determine the maximum possible values, all plants within the multiplicative uncertainty model (i.e. $G \in \Pi$) need to be considered, over all possible frequencies. For the SISO case, however, it is possible to determine analytically these two maxima. The derivations of these formulae are given in Appendix III;

$$\begin{aligned}\Delta u_{CL} &= \max_{G \in \Pi, \omega} |u| \\ &= \max_{\omega} \left(\frac{|\tilde{G}^{-1} F d|}{(1 - |F| \bar{l}_m)} \right)\end{aligned}\quad (5.141)$$

$$\begin{aligned}\Delta y_{CL} &= \max_{G \in \Pi, \omega} |y| \\ &= \max_{\omega} \left(\frac{|(1-F)d|}{(1 - |F| \bar{l}_m)} \right)\end{aligned}\quad (5.142)$$

Figures 5.31a and 5.31b are plots of the magnitudes derived from these equations. The values calculated for Δu_{CL} and Δy_{CL} from these plots are;

$$\begin{array}{ll}\frac{\text{Case 1}}{(u_1 - y)} \Delta u_{CL} = 0.8333 & \frac{\text{Case 2}}{(u_2 - y)} \Delta u_{CL} = 0.625 \\ \Delta y_{CL} = 0.1005 & \Delta y_{CL} = 0.1005\end{array}\quad (5.143)$$

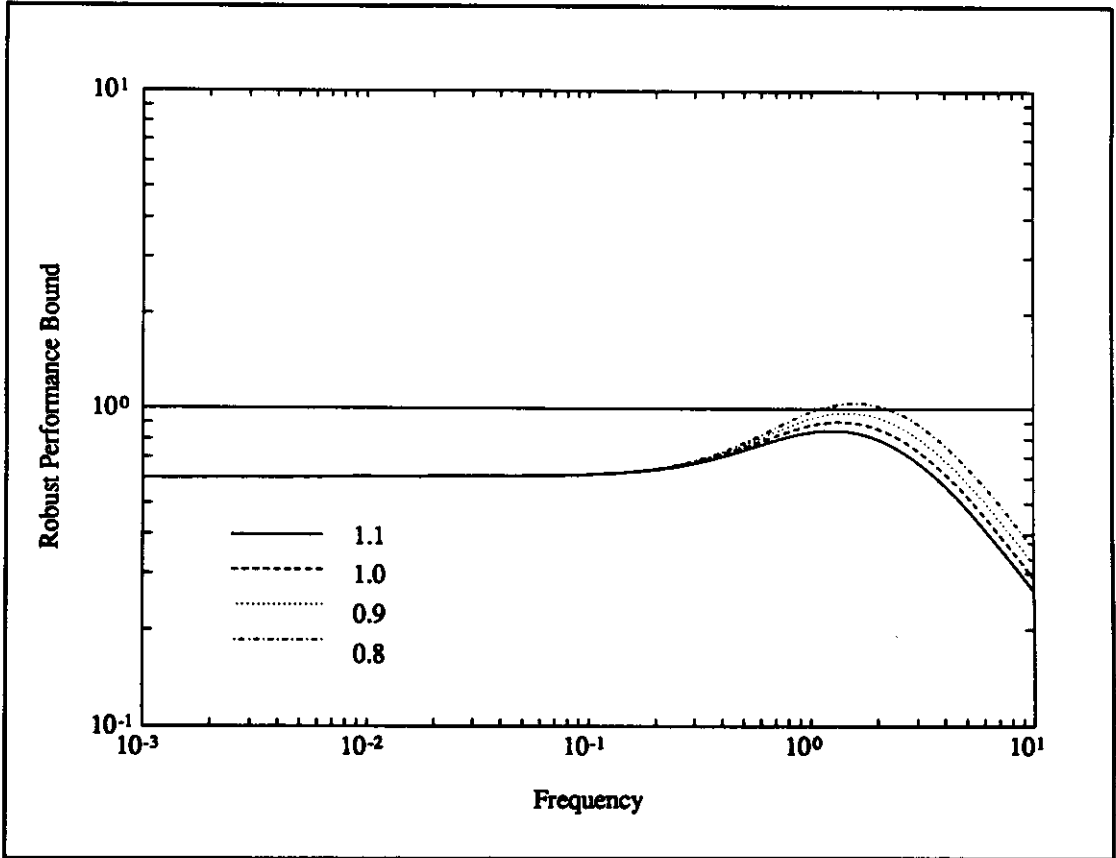


Figure 5.30 - Robust Performance Bound (Different filter time constants)

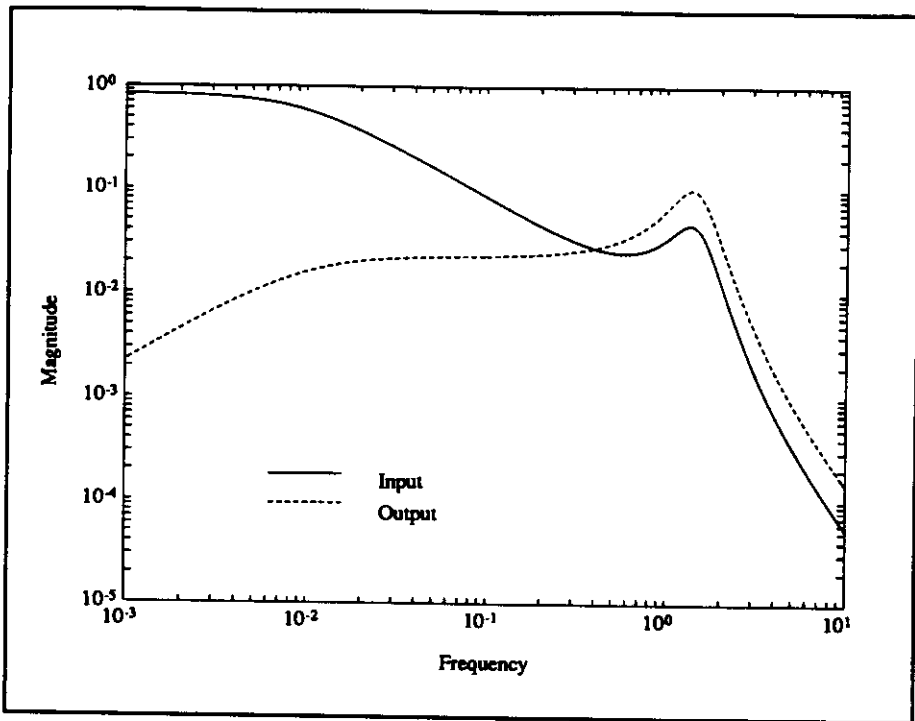


Figure 5.31a - H_∞ Controller, Case 1 Input and Output Magnitudes

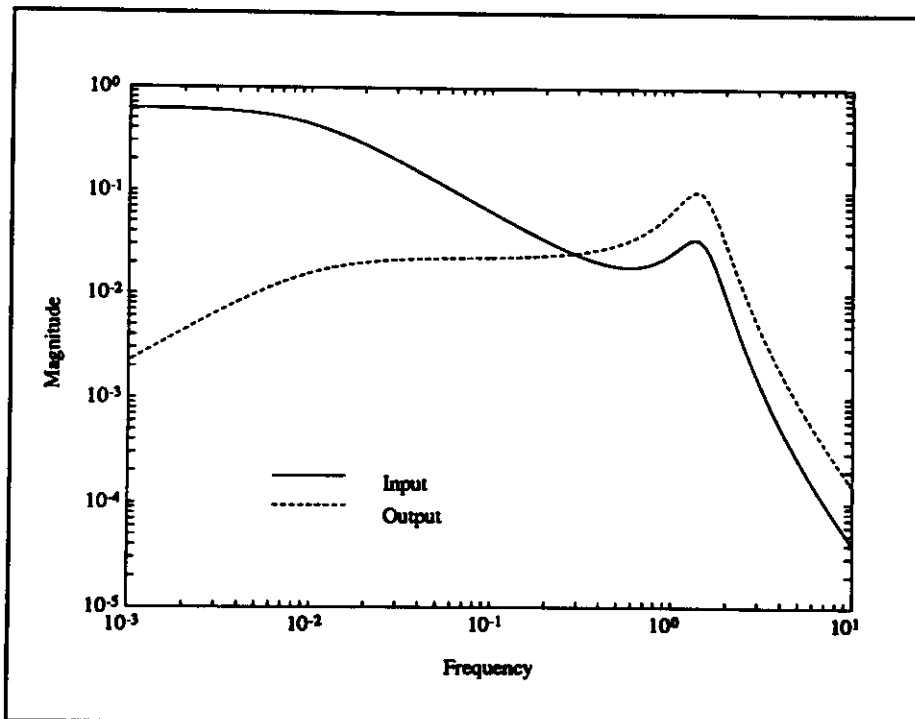


Figure 5.31b - H_∞ Controller, Case 2 Input and Output Magnitudes

Again both the input and output constraints require modification;

$$\begin{array}{ll}
 \begin{array}{l} \text{Case 1} \\ (u_1 - y) \end{array} & \begin{array}{l} u \geq u_L + \Delta u_{CL} \\ \Rightarrow u_1 \geq 0.8333 \\ \\ y \leq y_U - \Delta y_{CL} \\ \Rightarrow y \leq 9.8995 \end{array} & \begin{array}{l} \text{Case 2} \\ (u_2 - y) \end{array} & \begin{array}{l} u \geq u_L + \Delta u_{CL} \\ \Rightarrow u_2 \geq 0.6250 \\ \\ y \leq y_U - \Delta y_{CL} \\ \Rightarrow y \leq 9.8995 \end{array} & (5.144)
 \end{array}$$

Optimisation using these modified constraints yields the following results;

$$\begin{array}{ll}
 \begin{array}{l} \text{Case 1} \\ (u_1 - y) \end{array} & \begin{array}{l} \Phi = 54.031 \\ y = 9.8995 \\ u_1 = 0.8333 \\ u_2 = 1.8513 \\ \Delta \Phi = -0.969 \end{array} & \begin{array}{l} \text{Case 2} \\ (u_2 - y) \end{array} & \begin{array}{l} \Phi = 54.447 \\ y = 9.8995 \\ u_1 = 0 \\ u_2 = 2.4763 \\ \Delta \Phi = -0.553 \end{array} & (5.145)
 \end{array}$$

These results indicate that model uncertainty, and the consequent need to use an H_∞ controller to ensure robust stability and performance, significantly changes the economics of the process. Although Case 2 is still the more profitable, the mean objective function value is much less than for the previous controlled cases.

To summarise the operation of the Optimal Operation Algorithm as shown by these results, the maximum possible input and output manipulations were determined for the given disturbance description and controller design. These values were used to modify the relevant input and output constraints. The optimum thus determined has two active inequality constraints. In both cases the modified output constraint is active, and also the constraint on the first input. Note, however, that for Case 2 the input constraint is unmodified, as this variable was not used for control. The results for all controllers and cases are shown in Table 5.1.

The trend shown in these results is that as more "realism" is incorporated into the design procedure, the controller becomes less effective, with greater amounts of the disturbance

Controller	Case	u_1	u_2	y	Φ	$\Delta\Phi$
Base Case		0	2.5	10	55	0
None		0	2.25	9	49.5	-5.5
Perfect	1	0.3333	2.25	10	54.833	-0.1667
Perfect	2	0	2.5	10	55	0
Real	1	0.3333	2.24998	9.9999	54.8328	-0.1672
Real	2	0	2.49998	9.9999	54.9995	-0.0005
H_∞	1	0.8333	1.8513	9.8995	54.031	-0.969
H_∞	2	0	2.4763	9.8995	54.447	-0.553

Table 5.1 - SISO Example Optimal Operation Results

remaining unattenuated at the output. In addition, for the H_∞ controller the maximum necessary input magnitude is much greater than for the other cases. The results for the H_∞ controller cases illustrate the effect of considering the control of plants different from the nominal model. Not only does stability dictate a "slower" controller, but there are also some plants contained in the uncertainty description which are more difficult to control. It should be noted, however, that the underlying result, that u_2 is a more economic input to use for control (Case 2), is shown in all cases despite the use of different controllers. In addition, for this example, a considerable saving is made by using closed-loop control compared to the open-loop case.

Before discussing more general aspects of the application of this technique for the analysis of achievable performance at the design stage, it is worth examining the use of the unstructured multiplicative uncertainty description in the H_∞ controller design and subsequent input and output magnitude analysis. It has been mentioned several times previously, that the multiplicative (and other approximate) uncertainty description techniques can be very conservative, including in the analysis many plants (possibly significantly "worse" plants) that were not in the original uncertainty description. Figures 5.32a and 5.32b are Nyquist plots showing for two frequencies, $\omega = 0.1$ and 1 radians per

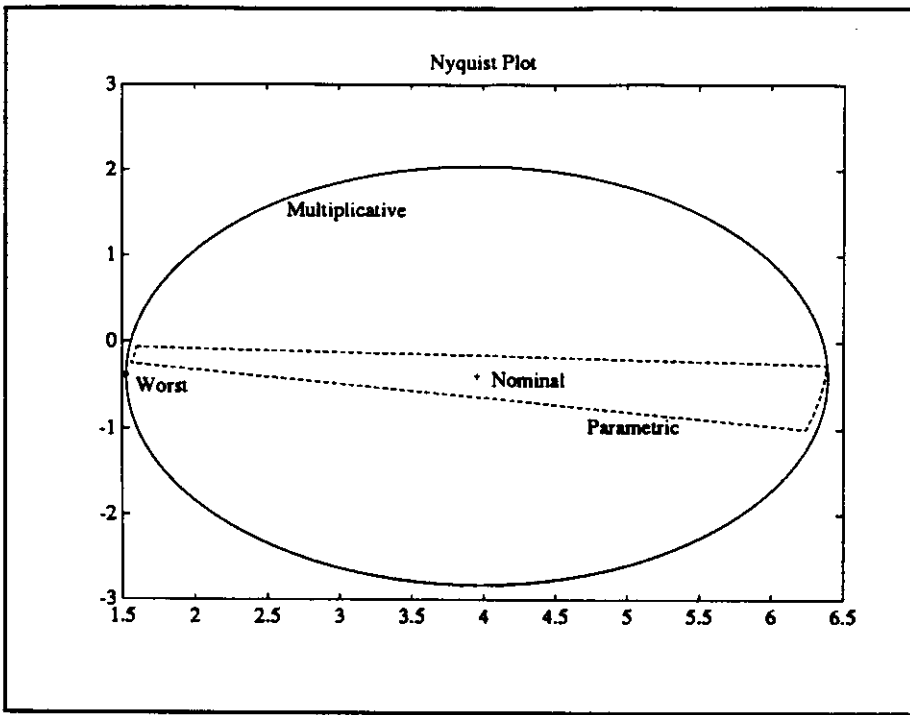


Figure 5.32a - Uncertainty Regions, frequency = 0.1 radians per unit time

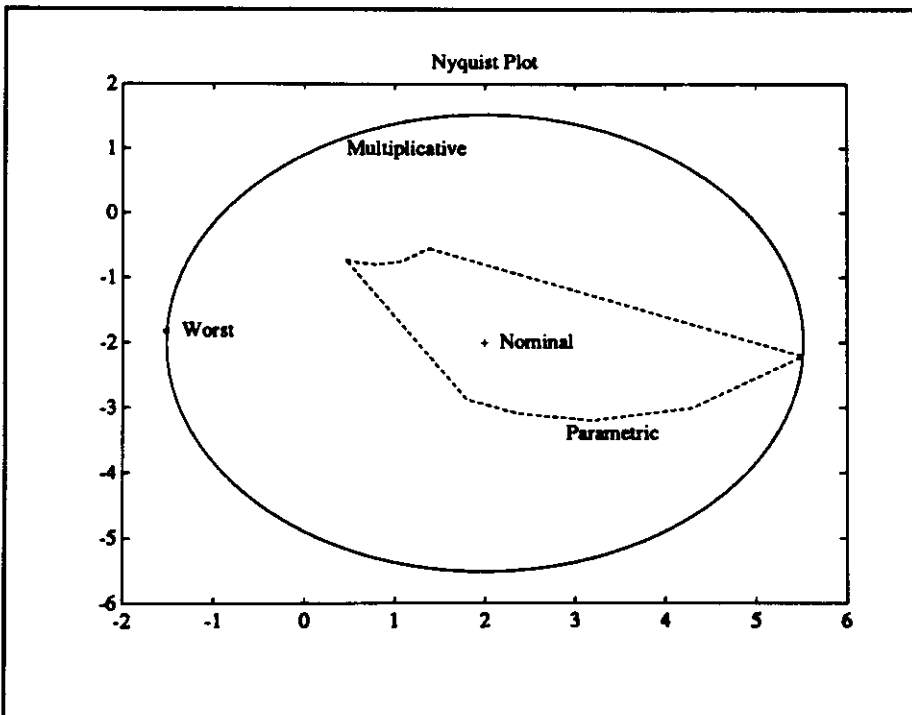


Figure 5.32b - Uncertainty Regions, frequency = 1 radian per unit time

unit time respectively, the nominal plant, the parameter (α and β) based uncertainty region and the multiplicative uncertainty region. It is clear that in both cases, the multiplicative uncertainty region is much larger than the region describing plants based on known parameter variations in the model. Also shown on the plots are the "worst" plants at each frequency, that is, the plants corresponding to the largest output magnitude for the given disturbance description. The technique used to determine the "worst" plant is given in Appendix III. At the lower frequency, the worst plant is actually quite close to the parametric uncertainty region. However, at the higher frequency, the worst plant was well away from those which could possibly occur. This analysis reinforces the conservative nature of the unstructured uncertainty descriptions used here. It may be possible to reduce the level of conservativeness by using structured uncertainty descriptions, introduced earlier in the chapter, however this has not been investigated.

Summary

In this chapter, a method for estimating the best performance that a plant can achieve within its input and output constraints, and for a given description of likely disturbances, was proposed. The technique modifies input and output constraints to ensure that the optimal operating point is sufficiently removed from these bounds, so that input manipulation and output variation due to the impact of disturbances do not cause the constraints to be violated. In the example presented in Section 5.3, four types of control were analysed to determine the optimal operating point for each scenario. It was found that for this example, closed-loop control improved the economic performance of the plant. However, as the controller was made more realistic, leaving some disturbance to appear unattenuated at the output, the value of the objective function fell. Consequently, the analysis which included a description of the likely uncertainty in the model, predicted the worst controlled performance. The use of H_∞ techniques does, however, allow a more complete analysis to be performed by incorporating more of the designer's knowledge (i.e. an uncertainty description) into the process design. It was shown that approximate unstructured uncertainty descriptions, such as the multiplicative technique, can be quite conservative, including in the analysis plants which are unlikely to occur. However, it should be realised that at the early stages of the process design procedure, detailed uncertainty descriptions would almost certainly be unavailable.

In the next chapter, the achievable performance analysis is extended to the MIMO case.

In addition, the Optimal Operation Algorithm is extended to include greater use of uncertainty descriptions more applicable for MIMO systems. The final extension of the algorithm presented, is to place a penalty on excessive output variance, incorporating the requirements of the recent drive for improved quality in the process industry.

CHAPTER 6

EXTENSION TO MULTIPLE-INPUT MULTIPLE-OUTPUT SYSTEMS

Real process plants are generally multivariable in nature, with many plant inputs requiring adjustment to maintain numerous plant measurements at their desired values. Once again, optimal operation of these processes requires that the nominal operating point be selected so that operational constraints are not violated by the action of disturbances. In Chapter 5, an algorithm was presented for calculating such an operating point, making use of perfect and H_∞ control as the "best" control that can be achieved. Subsequently, input and output constraints were modified, to allow sufficient room for the effect of disturbances. As presented, the algorithm is suitable for plants with a linear steady-state model and objective function, whose dynamics can be described by a linear (or linearised) model.

This approach to the evaluation of process economics at the design stage can be used in a number of ways. Firstly, as was demonstrated in Chapter 5, different control schemes can be ranked to see how their performance affects plant economics. Secondly, as will be examined here, alternative process designs may be evaluated to determine which design could achieve the best economics.

In this chapter, as well as presenting a MIMO alternative flowsheet selection example, the optimal operation algorithm will be extended to include greater use of estimates of the uncertainty associated with the models used, both in a dynamic and a steady-state sense. Finally, an output variance penalty is applied to incorporate the costs associated with this type of variation.

To demonstrate the progression in the inclusion of control performance at the design stage, the example developed is based on previous work by Chan (1988). In that study, the steady-state performance of different flotation circuits was compared with open-loop indicators of inherent controllability.

6.1 An Example of the Assessment of Alternative Process Flowsheets

The work by Chan (1988) considered techniques for optimal flowsheet synthesis. In one example, a fixed total cell volume was optimally allocated to three cells in each of forty nine possible arrangements of a flotation circuit, so as to maximise the recovery of valuable material in the product stream. For eleven of the more conventional designs, which achieved the minimum required concentration of valuable material in the product, open-loop controllability indicators were calculated. These showed that the two circuits with the highest recoveries were likely to be more difficult to control than other circuits (with lower recoveries). Initially, details of the flotation circuit modelling will be presented, followed by a summary of Chan's (1988) results.

6.1.1 Flotation Circuit Modelling

As mined, valuable minerals are generally found in a mixture with other minerals and worthless material (gangue). To allow further processing (e.g. refining), techniques such as froth flotation are used to separate the valuable material from the gangue. This exploits differing surface properties such that the valuable material is more hydrophobic than the gangue. Various chemicals are added to enhance this effect and to help form a stable layer of froth when air is bubbled through the mixture. The froth formed contains the valuable material. The tailings stream of solids extracted from the bottom of the cell, contains a lower concentration of valuable material. This is shown in Figure 6.1a.

In practice, a number of these cells are connected together to form a bank, with the combined concentrate (froth) being the product, and the tailings being withdrawn from the final cell. In this work, each bank is modelled as a single cell.

Figure 6.1b shows details of the streams entering and leaving a cell, as well as nomenclature for other important variables in the system. Dynamic mass balances can be written for the valuable (Z) and gangue (G) material in the cell.

$$\frac{dM_G}{dt} = G_f - G_c - G_t \quad (6.1)$$

$$\frac{dM_Z}{dt} = Z_f - Z_c - Z_t \quad (6.2)$$

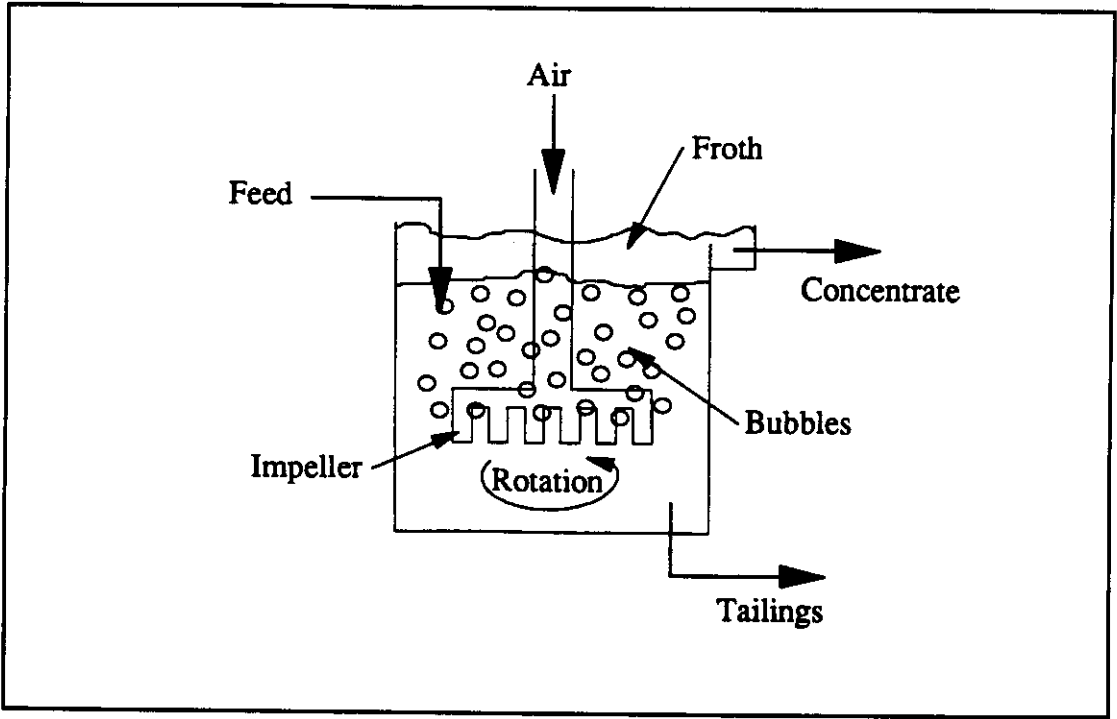


Figure 6.1a - Layout of Flotation Cell

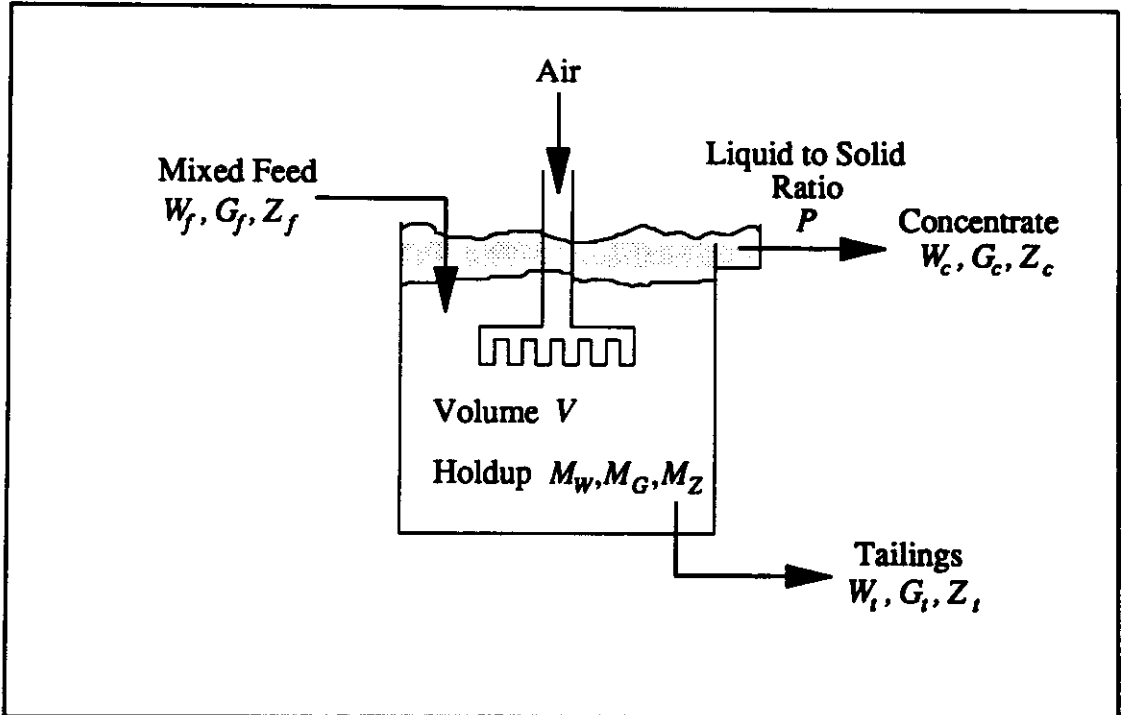


Figure 6.1b - Flotation Cell Model Nomenclature

Perfect level control is assumed, that is, constant cell volume. Hence the volume of each component, valuable (Z), gangue (G) and water (W) must sum to the total volume (V). ρ_i is the density of component i .

$$\frac{M_w}{\rho_w} + \frac{M_G}{\rho_G} + \frac{M_Z}{\rho_Z} = V \quad (6.3)$$

A volumetric flow balance may also be written, that is, the volume flowing in the feed stream is equal to the sum of the volumes flowing out in the product streams;

$$\left[\frac{W_f}{\rho_w} + \frac{G_f}{\rho_G} + \frac{Z_f}{\rho_Z} \right] = \left[P(Z_c + G_c) + \frac{G_c}{\rho_G} + \frac{Z_c}{\rho_Z} \right] + \left[\frac{W_t}{\rho_w} + \frac{G_t}{\rho_G} + \frac{Z_t}{\rho_Z} \right] \quad (6.4)$$

In the above equation; P is the ratio of liquid to solid in the concentrate stream, thus the term $P(Z_c + G_c)$ is the volume of water in the concentrate stream.

The cell is assumed to be well-mixed, hence the concentration in the tailings stream is equal to that in the bulk of the cell;

$$\frac{Z_t}{(W_t + G_t + Z_t)} = \frac{M_Z}{(M_w + M_G + M_Z)} \quad (6.5)$$

$$\frac{G_t}{(W_t + G_t + Z_t)} = \frac{M_G}{(M_w + M_G + M_Z)} \quad (6.6)$$

The actual kinetics of froth flotation are quite complex. For example, the behaviour of different particle sizes can be quite different. However, it has been found (Lynch et al., 1981) that when the separation is substantially by true flotation (rather than by simple entrainment in the air bubbles), the overall recovery of valuable material can be adequately modelled using first-order kinetics.

$$Z_c = k_a K_Z M_Z \quad (6.7)$$

$$G_c = k_a K_G M_G \quad (6.8)$$

where K_Z and K_G are the flotation rate constants for valuable and gangue respectively. The

factor k_a is the "air factor", which is used to model in a simple fashion, the effect of air flowrate.

The last equation used in the model, fixes the water balance in the cell by specifying the liquid to solid ratio (P) in the froth overflow.

$$W_c = (P \rho_w)(Z_c + G_c) \quad (6.9)$$

Although simple, these equations form the basis of a realistic approach to flotation modelling. The next section describes the use of these models by Chan (1988) in his analysis of alternative flowsheets.

6.1.2 Previous Flowsheet Synthesis Results

The main thrust of Chan's work, was the investigation of flowsheet synthesis for separation processes, of which the flotation circuit was one example. For this case, a given separation of valuable material from the ore was specified, to be achieved using three banks of flotation cells. The synthesis technique produced forty nine possible interconnections of these banks (e.g. the tailings of one bank being recirculated back to the previous bank), without stream splitting. One restriction placed on the design problem was a maximum of 54 m³ total volume to be allocated amongst the three banks (this can be thought of as a simple limit on capital investment). An optimisation was then carried out to determine the optimal volume allocation such that the final concentrate contained at least sixty percent valuable (on a dry basis), with the maximum recovery of valuable in the concentrate stream. Other problem specifications are given below.

Feed:

Water	=	1022	kg/min
Gangue	=	1056	kg/min
Valuable	=	144	kg/min

Physical Properties:

	Water	Gangue	Valuable
Density (kg/m ³)	1000	3100	4000
Rate constant (h ⁻¹)	-	0.8	12.0
Mass fraction in concentrate (P)	0.46	-	-
Air factor (k_a) = 1 (at steady-state)			

Of the forty nine flowsheets, eleven were selected as being of a "conventional" nature and capable of meeting the sixty percent grade specification. The best four of these, Circuits 1, 2, 4 and 8 (in Chan's numbering system), will be further examined in this work. The layouts, optimal recoveries and volumes for these four circuits are shown in Figure 6.2. Of particular interest is the fact that the best two circuits, Circuit 8 with a recovery of 94.7% and Circuit 4 with 94.4% are unusual arrangements but exhibit significantly better recoveries than the more conventional Circuit 2 (recovery of 93.6%) and Circuit 1 (recovery of 93.3%).

The next step in Chan's analysis, was to compare these two groups of flowsheets, the high recovery circuits (4 and 8), with the more conventional arrangement, lower recovery circuits (1 and 2) from a controllability perspective through the use of open-loop indicators. In particular, characteristics of the higher recovery circuits which might prevent them from achieving these results in practice were sought. To allow control, two manipulated variables were selected, the water flow in the feed stream (W_f) and the air factor (k_a). It should be noted that for this example the air factor affects all banks equally. The measured variables were selected as the final concentrate grade ($Grade = Z_c / (Z_c + G_c)$) and the percentage recovery of valuable in the concentrate stream ($Recovery = 100 \times Z_c / Z_f \%$).

The use of open-loop indicators has been discussed elsewhere in this work (Chapters 2 and 5), however, those used by Chan will be briefly reviewed below.

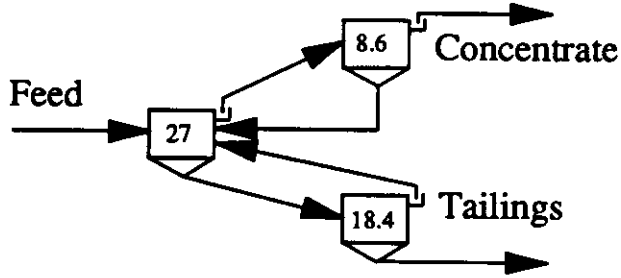
κ - This is the condition number of the input-output transfer function matrix, evaluated at a given frequency, subject to the l_1 norm. A low value indicates "good" controllability.

γ - This is the condition number of the input-output transfer function matrix, evaluated at a given frequency, subject to the l_2 norm. Again, a low value is desirable for "good" controllability.

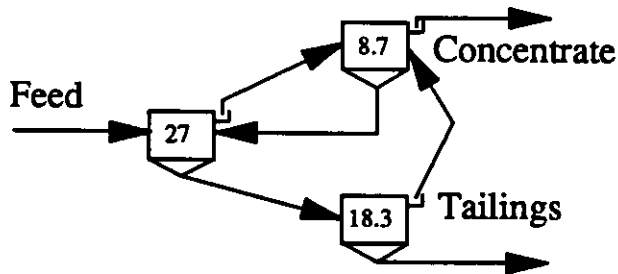
σ - This is the minimum singular value of the input-output transfer function matrix evaluated at a given frequency. A large value is desirable here to ensure that the process has sufficient "gain" to avoid input saturation.

Circuit 1

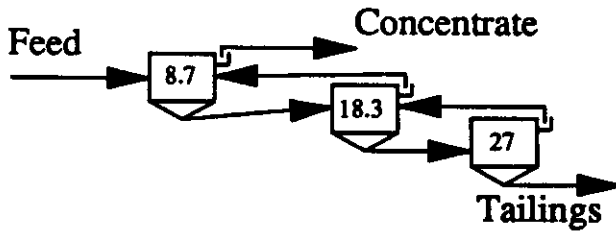
Recovery
= 93.3%

Circuit 2

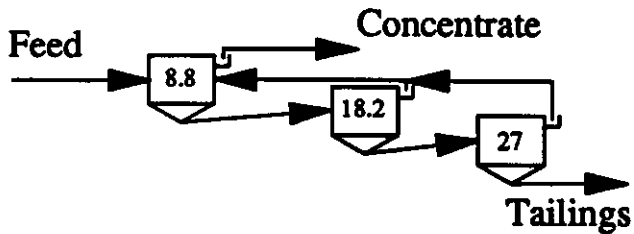
Recovery
= 93.6%

Circuit 4

Recovery
= 94.4%

Circuit 8

Recovery
= 94.7%



 X = Volume in m³

Figure 6.2 - Four Flotation Circuits Showing Optimal Volumes and Recoveries

θ - This is the interaction angle proposed by Lau et al. (1985). For multivariable systems, it is desirable that manipulation of one input does not affect all outputs equally. A value of 0 for θ indicates that there is no interaction. This measure is a function of frequency, based on the input-output transfer function matrix.

RHP Zeros - The presence of these in the input-output transfer function matrix is known to be a limitation on the quality of achievable control. The closer the position is to the origin, the greater the limitation.

λ_{12} - This is the (1,2) element of the Relative Gain Array. The value given here is the steady-state result. A value close to one, indicating little interaction, is desirable.

Complete plots of these measures as a function of frequency are available in Chan and Barton et al. (1991). A summary of the steady-state results (frequency ≤ 0.001 rad/min) is given in Table 6.1.

Circuit	Recovery (%)	κ	γ	σ	θ	RHP Zeros	λ_{12}
8	94.7	20.6	39	0.0117	34°	0.00286	5.73
4	94.4	60	90	0.0055	40°	0.00105	16.37
2	93.6	1.62	6.08	0.0666	15°	None	0.89
1	93.3	1.65	4.92	0.0819	17°	None	0.88

Table 6.1 - Chan's (1988) Open-Loop Indicators

It is clear from these results that once again the circuits fall into two categories. Those which should be amenable to good control (Circuits 1 and 2) and those which may be more difficult to control (Circuits 4 and 8). It is illustrative that all of the open-loop indicators are in agreement. The "good" condition numbers are about one-tenth the size of the "bad" condition numbers, while the minimum singular values for Circuits 1 and 2 are about five times the size for Circuits 4 and 8. Similarly, the interaction angle for the poor

circuits is double that of the better two, while the relative gain array indicates that good multi-loop control should be possible for Circuits 1 and 2, but quite difficult for Circuits 4 and 8. Finally, the right half plane zeros of Circuits 4 and 8 are close to the origin, also indicating likely control difficulties.

To validate the predictions of the open-loop analysis, Chan next designed multi-loop controllers for each circuit. The relative gain array indicated that the best input-output pairings for each circuit were;

Loop 1

Input: *Air*

Output: *Grade*

Loop 2

Input: *Water*

Output: *Recovery*

Using a simple and consistent tuning procedure, Chan determined proportional plus integral controller tuning constants for each loop, in each circuit. The closed-loop systems were then subjected to a five percent step increase in the solids feed ($G_f + Z_p$) to the circuit. Circuits 1 and 2 behaved well for this disturbance, however, Circuits 4 and 8 performed poorly. It was noted that better control for these latter circuits may have been possible using either a more sophisticated control scheme, or using different manipulated variables and/or measurements. However, the general approach adopted here still provides a consistent basis for analysing the trade-off problem of steady-state economics versus control.

Summarising the flotation circuit example of Chan, a number of flowsheets were synthesised, and the available volume optimally allocated amongst three flotation banks so as to maximise valuable material recovery, while meeting a grade specification. The two circuits with the highest recovery, Circuits 4 and 8, were shown to be difficult to control in the face of likely disturbances. However, the two circuits with the next highest recoveries, Circuits 1 and 2, which were of a more conventional nature, proved to be amenable to high quality control. This example clearly demonstrates the problem outlined in the preceding chapters - the choice between good steady-state economics and high quality control. For Chan's results, however, it is not at all obvious how much the poor

control of Circuits 4 and 8 will cost in terms of recovery. Without extensive dynamic simulation, it is not possible to determine which flowsheet could be operated most economically in the face of likely disturbances.

In the following section, the Optimal Operation Algorithm (see Section 5.1.2) will be applied to Chan's alternative flotation circuit problem, to determine which circuit has the greatest potential to be operated in an optimal economic fashion.

6.2 No Model Uncertainty

The simplest case of the flotation circuit analysis to be undertaken in this chapter, is for the situation when no account is taken of the uncertainty in the knowledge of the process model. In subsequent sections, allowance will be made for possible variation and inaccuracies in both the steady-state and dynamic models used in determining the optimal achievable operating point.

As discussed previously, one of the more important recent advances in control theory has been the ability to include an increased amount of process knowledge into controller design through uncertainty descriptions. This allows the design of high performance, robust controllers.

Initially, however, the assumption will be made that there is perfect knowledge of the process model, as this will aid in the description of the application of the optimal operation algorithm. In the following subsections, details will be given of the modelling techniques used, disturbance descriptions, open-loop output magnitudes, H_∞ controller design, closed-loop input and output magnitudes, and finally the optimal operation results.

6.2.1 Modelling Techniques

The equations given in Section 6.1 for describing a single flotation cell, were used within the flowsheet simulation package SPEEDUP (Pantelides, 1988) to model the flotation circuits shown in Figure 6.2. In addition to the flotation bank models, the necessary interconnections and recycles involved in each circuit were included in the SPEEDUP model description. An example listing of the SPEEDUP input file for Flotation Circuit 1 is given in APPENDIX IV.

The use of SPEEDUP was described previously in Chapter 2, for the example of a distillation column. In this application, use will be made of the nonlinear steady-state optimisation and model linearisation facilities.

The SPEEDUP models were validated by performing the same optimisation problem as set by Chan.

$$\begin{array}{l} \text{Maximise } \textit{Recovery} \\ V_1, V_2, V_3 \end{array}$$

$$\text{such that } \textit{Grade} \geq 60\% \quad (6.10)$$

$$V_1 + V_2 + V_3 = 54 \text{ m}^3$$

$$(\textit{Recovery}, \textit{Grade}) = f(V_1, V_2, V_3) \dots\dots \text{Steady-State Model}$$

The steady-state model used here is the full nonlinear SPEEDUP model, where for the freed variables V_1 , V_2 and V_3 , the *Grade* and *Recovery* are calculated. In each case, the same distribution of volumes and optimal recoveries were obtained.

Referring to the Optimal Operation Algorithm (Equation 5.9), the objective function and steady-state model used are linear and are of the form where the control outputs (*Grade*, *Recovery*) are written as a function of the control inputs (*Air*, *Water*).

$$(\textit{Recovery}, \textit{Grade}) = f(\textit{Air}, \textit{Water}) \quad (6.11)$$

The SPEEDUP model therefore required some modification (rearrangement and linearisation) for use within the Optimal Operation Algorithm, although some possible extensions to the linear model limitation will be discussed in Chapter 7. Writing the steady-state model in the form of Equation 6.11 shows that the optimisation problem has greater scope than that originally used by Chan, in that *Air* and *Water* can also be freed when searching for the optimum. In fact, as part of the Optimal Operation Algorithm, optimal values are found for the plant inputs within upper and lower bounds. For the flotation circuits, the limits used for air and water were;

$$\begin{aligned}
 0.5 &\leq \textit{Air} \leq 2.0 \\
 0.8 &\leq \textit{Water} \leq 2.0
 \end{aligned}
 \tag{6.12}$$

The flowsheet synthesis optimisation problem should, therefore, really be to manipulate the bank volumes, feed water and air so as to maximise the recovery of valuable material. However, to maintain some consistency with Chan's results, the original optimal volumes shown in Figure 6.2 are used in the remainder of this work. The new optimisation problem, with fixed volumes but the full nonlinear steady-state model is;

$$\begin{aligned}
 &\text{Maximise } \textit{Recovery} \\
 &\textit{Air, Water} \\
 &\text{such that } \textit{Grade} \geq 60\% \\
 &0.5 \leq \textit{Air} \leq 2.0 \\
 &0.8 \leq \textit{Water} \leq 2.0 \\
 &(\textit{Recovery}, \textit{Grade}) = f(\textit{Air}, \textit{Water}) \dots\dots \text{Steady-State Model}
 \end{aligned}
 \tag{6.13}$$

The optimal recoveries obtained for this optimisation problem are greater than those obtained by Chan with his fixed air and water flows. In addition, the results for the four circuits are now much closer together. The design problem is now somewhat different, as instead of choosing between two high and two slightly lower recovery circuits, there are now four almost equal recovery circuits to select from. The revised optimisation results, and those of Chan are shown in Table 6.2. Note that for convenience, the units of *Water* flow have been changed to tonne/min.

As stated previously, a linearised steady-state model is required for determining the optimal operating point using the Optimal Operation Algorithm. The point of linearisation was chosen to be the optima given above. For each circuit, the linearisation software developed by Wong (1984) was used to extract a state-space representation of the linearised model from the full nonlinear model contained within SPEEDUP. Using this information, the steady-state gain matrices were used to write a linear model for each circuit.

Circuit	Chan (1988), Equation 6.10			Revised, Equation 6.13		
	Air	Water	Recovery	Air	Water	Recovery
1	1	1.022	93.3	1.017	0.8	94.873
2	1	1.022	93.6	1.014	0.8	94.953
4	1	1.022	94.4	0.853	0.8	94.592
8	1	1.022	94.7	0.888	0.8	95.040

Table 6.2 - Chan (1988) *cf.* Equation 6.13 Optimisation

Circuit 1

$$\text{Recovery} = 8.597 \times \text{Air} - 6.065 \times \text{Water} + 90.892$$

$$\text{Grade} = -23.426 \times \text{Air} - 1.678 \times \text{Water} + 85.167$$

Circuit 2

$$\text{Recovery} = 8.034 \times \text{Air} - 5.109 \times \text{Water} + 90.893$$

$$\text{Grade} = -23.663 \times \text{Air} - 1.412 \times \text{Water} + 85.125$$

Circuit 4

$$\text{Recovery} = 16.020 \times \text{Air} - 11.366 \times \text{Water} + 90.020$$

$$\text{Grade} = -26.260 \times \text{Air} + 17.447 \times \text{Water} + 68.442$$

Circuit 8

$$\text{Recovery} = 12.101 \times \text{Air} - 7.547 \times \text{Water} + 90.331$$

$$\text{Grade} = -25.331 \times \text{Air} + 12.308 \times \text{Water} + 72.648$$

As a check, the revised optimisation problem, Equation 6.13, was solved using these linear models in place of the full nonlinear model. As expected, the optima obtained were the same as those given in Table 6.2. This is because these two freed variable problems (varying *Air* and *Water*), are constrained by two active constraints at the optimum (minimum *Grade* and minimum *Water*), and are still constrained in the same way with the linear model.

One small modification was made to the dynamics of Chan's flotation circuit model. Early H_{∞} controller design proved difficult due to the open-loop state-space matrices having a non-zero "D" matrix. Mathematically, this was due to the air input directly affecting an output without "passing through" a state variable. Physically, this would not be the case, with some "lag" or even dead-time associated with air adjustments. Therefore, the model was modified using the following assumptions;

- 1) All bubbles have the same rise velocity and hence rise time.
- 2) The bank (of one cell) can be modelled as a CSTR with the same concentration of bubbles throughout.

A first-order relationship was then assumed between the input air factor (the variable actually adjusted by the control system) $k_{a,IN}$ and the effect of the air factor k_a .

$$\tau_{RT} \frac{dk_a}{dt} = k_{a,IN} - k_a \quad (6.14)$$

τ_{RT} , the rise time, is the time taken for a bubble to move from where it enters the cell to the froth. The value used here for τ_{RT} is one minute. As will be shown, this addition did not change the open-loop controllability ranking of the flotation circuits.

The analysis of the dynamics of these circuits requires the selection of a suitable frequency range. To determine this, the time constants for each bank in Circuit 1, at the optimum given in Table 6.2, were calculated as the cell volume divided by the volumetric flow through the bank.

Bank	Volume (m ³)	Time Constant
Rougher	27	14.53 minutes
Cleaner	8.6	14.41 minutes
Scavenger	18.4	14.59 minutes

Table 6.3 - Circuit 1 - Time Constants

Taking the inverse of the average (14.5 minutes) gives a frequency of 0.069 cycles/min,

or 0.43 radians/minute. A "rule of thumb" for frequency analysis suggests taking a factor of 100 either side of this average for investigation. For convenience, the frequency range 0.001 rad/min to 10 rad/min was selected for subsequent investigations.

Four open-loop controllability indicators were checked for each circuit - the minimum scaled l_1 condition number (κ_{MIN}), the sum of the absolute values of the frequency dependent RGA ($\|A\|_1$), the presence and position of right half plane zeros and the steady-state relative gain array. κ_{MIN} is the process condition number (κ), minimised by optimally scaling the input-output transfer function. As discussed in Chapter 2, this provides a scaling independent value, which should be close to one for good control. The frequency dependent RGA was included as Skogestad and Morari (1988) recommend that this value is easier to compute than condition numbers, but is just as informative. Here the lower the value, the easier it should be to achieve good control. These indicators were calculated at the optima given in Table 6.2, using the linear models described previously.

Figure 6.3a is a comparative plot of κ_{MIN} versus frequency for the four circuits. Clearly, as found by Chan (1988), Circuits 4 and 8 should be much more difficult to control than Circuits 1 and 2 whose open-loop indicators are quite low over this frequency range. As expected, Figure 6.3b shows that the frequency dependent RGA gives much the same information as the minimum scaled condition number. Table 6.4 gives the (1,2) element of the relative gain array, for comparison with Chan's results shown in Table 6.2, and any right half plane zeros. The values calculated agree well with those given by Chan and are in agreement with the condition number rankings shown in Figure 6.3. Note that the right half plane zeros present are extremely close to the origin, and as discussed earlier, the closer a zero is to the origin, the more likely it is that control will be difficult.

Circuit	RHPZ	RGA(1,2)
1	None	0.908
2	None	0.914
4	0.00091	15.74
8	0.00297	4.526

Table 6.4 - Open-Loop Indicators for Reoptimised Flotation Circuits

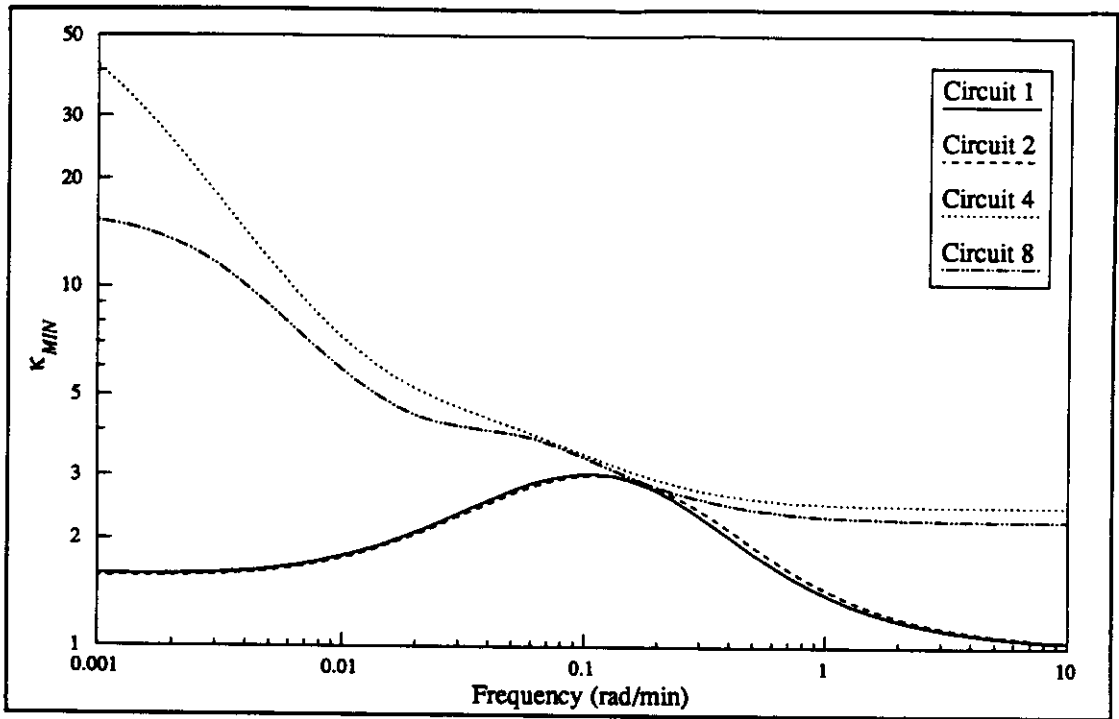


Figure 6.3a - Minimum Scaled I_{∞} Condition Number

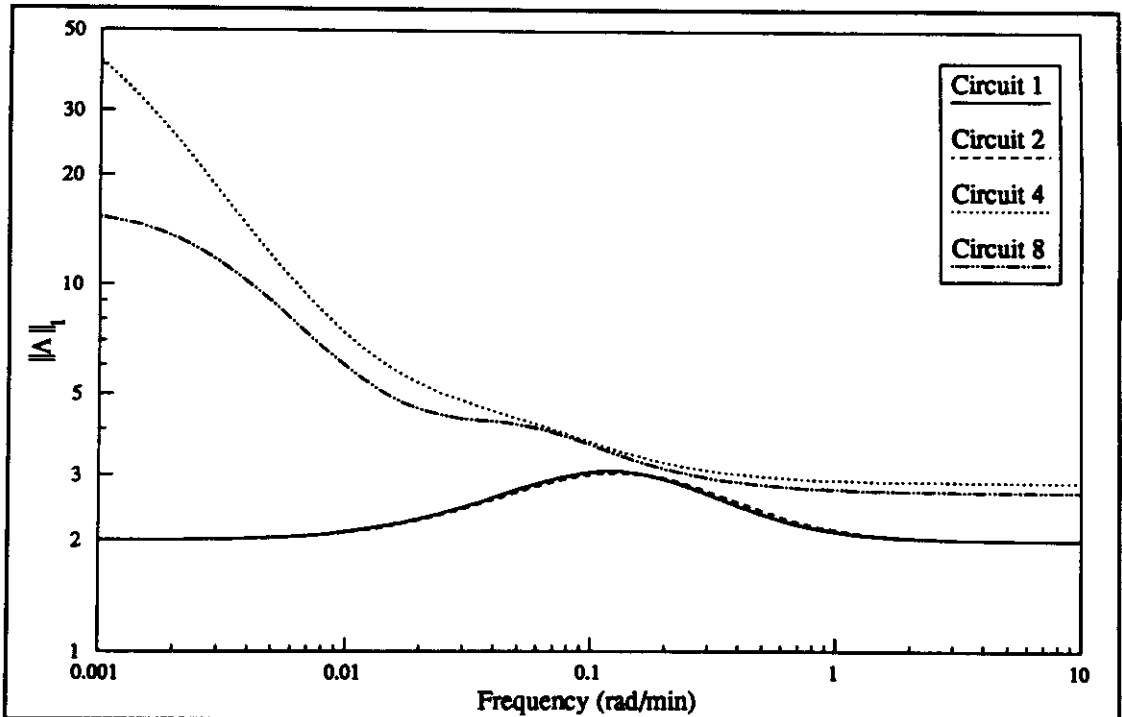


Figure 6.3b - $\|A\|_1$, Frequency Dependent RGA

Using the nonlinear models originally developed by Chan, a linear model has been developed for use in the optimal operation algorithm and to perform open-loop controllability analysis. The four flotation circuits have been shown to have quite different potentials for high quality control, yet similar steady-state optima. To check how well these plants may operate under real conditions, the impact of disturbances now needs to be assessed.

6.2.2 Open-Loop Operation

The operation of a flotation circuit is affected by the upstream plants supplying the feed. Typically this might include a grinding circuit, required to produce the small particle size necessary for successful flotation separation. Unfortunately, the rate at which product is produced by a grinding circuit can change in a somewhat unpredictable manner. In addition, natural variation in the ore body fed to the grinding circuit, means that the distribution of valuable and gangue material is also variable. These two variables, feed rate and feed composition (or grade), are therefore used as the two major disturbances affecting the flotation circuits being analysed here.

To allow the effect of these disturbances to be quantified, a description of their magnitude and frequency content is required. In a similar way to the example given in Chapter 5, a transfer function description of the likely magnitude of the disturbances will be used. The frequency content of the disturbances at their entry to the plant (d') is given by Equation 6.15.

$$|d'| \leq d_0 \left| \frac{1}{(10s + 1)^2} \right| \quad (6.15)$$

d_0 is the steady-state value of the disturbance. Initially, this will be taken as five percent of the steady-state value;

$$d_{0,f} = 60 \text{ kg/min}$$

$$d_{0,g} = 0.006$$

The two values given above are for disturbances in the solids feed rate ($d_{0,f}$) and the feed grade ($d_{0,g}$). Grade is defined as the mass fraction of valuable material in the solid component of the feed. In Section 6.1, the steady-state values were given as 1200 kg/min and 0.12, respectively. The magnitudes of d'_f and d'_g are shown as functions of frequency in Figure 6.4.

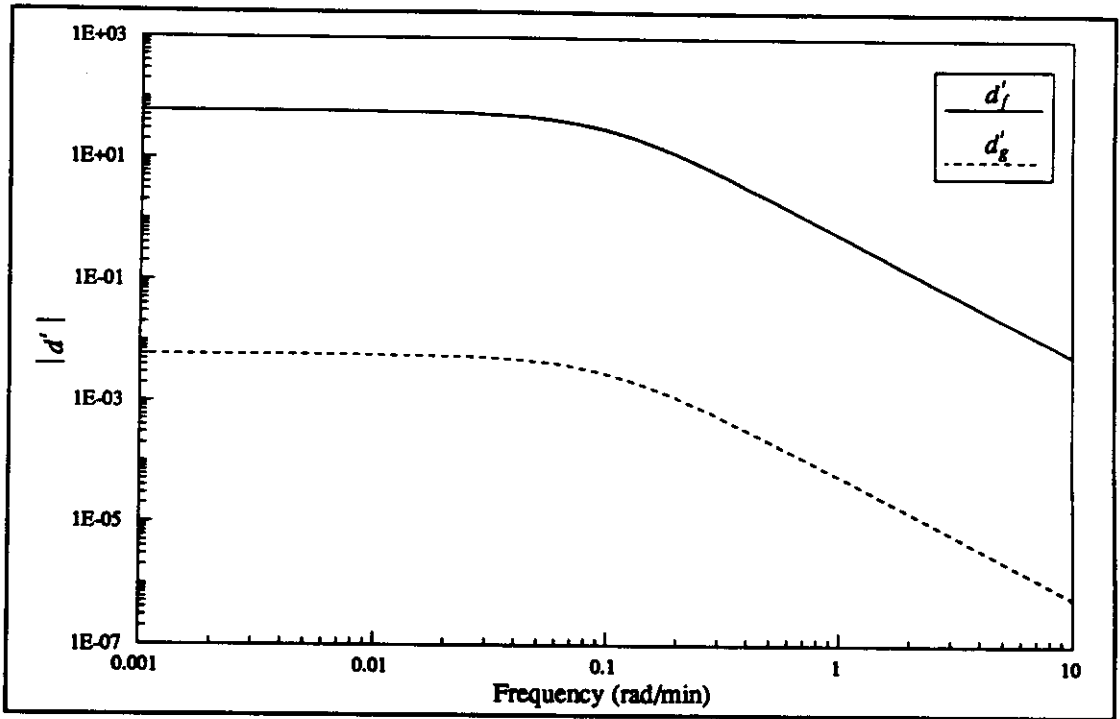


Figure 6.4 - Disturbance Magnitudes

The determination of how each of these disturbances affects the output variables,

Recovery (y_R) and *Grade* (y_G), requires knowledge of the disturbance transfer function

matrices $G_{d,f}$ and $G_{d,g}$. As shown in Equations 6.16 and 6.17, these relate the feed rate and feed grade disturbances to the outputs, in the Laplace domain.

$$\begin{bmatrix} \text{Recovery} \\ \text{Grade} \end{bmatrix} = \begin{bmatrix} y_{R,f,OL} \\ y_{G,f,OL} \end{bmatrix} = G_{d,f} d_f \quad (6.16)$$

$$\begin{bmatrix} \text{Recovery} \\ \text{Grade} \end{bmatrix} = \begin{bmatrix} y_{R,g,OL} \\ y_{G,g,OL} \end{bmatrix} = G_{d,g} d_g \quad (6.17)$$

Where $y_{R,f,OL}$ is the *Recovery* output (R), for the feed rate disturbance (f) when operating open-loop (OL). Subscript G , refers to the *Grade* output, and g to the feed grade disturbance.

We may therefore calculate the open-loop magnitude of the variation in the output variables due to these disturbances by combining Equations 6.15, 6.16 and 6.17.

$$\begin{aligned} |y_{R,f,OL}| &= |G_{d,R,f} d_f| = d_{0,f} \left| G_{d,R,f} \frac{1}{(10s + 1)^2} \right| \\ |y_{G,f,OL}| &= |G_{d,G,f} d_f| = d_{0,f} \left| G_{d,G,f} \frac{1}{(10s + 1)^2} \right| \\ |y_{R,g,OL}| &= |G_{d,R,g} d_g| + d_{0,g} \left| G_{d,R,f} \frac{1}{(10s + 1)^2} \right| \\ |y_{G,g,OL}| &= |G_{d,G,g} d_g| = d_{0,g} \left| G_{d,G,f} \frac{1}{(10s + 1)^2} \right| \end{aligned} \quad (6.18)$$

here $G_{d,R,f}$ and $G_{d,G,f}$ are the *Recovery* (R) and *Grade* (R) components of the feed rate (f) disturbance transfer function matrix $G_{d,f}$.

Plots of these magnitudes versus frequency for the four flotation circuits are given in Figure 6.5. The important information to be read from Figure 6.5 is the maximum value of the open-loop magnitude. Table 6.5 lists these maximum values for each disturbance acting on each flotation circuit.

The open-loop magnitudes described above indicate the possible size of variations in the

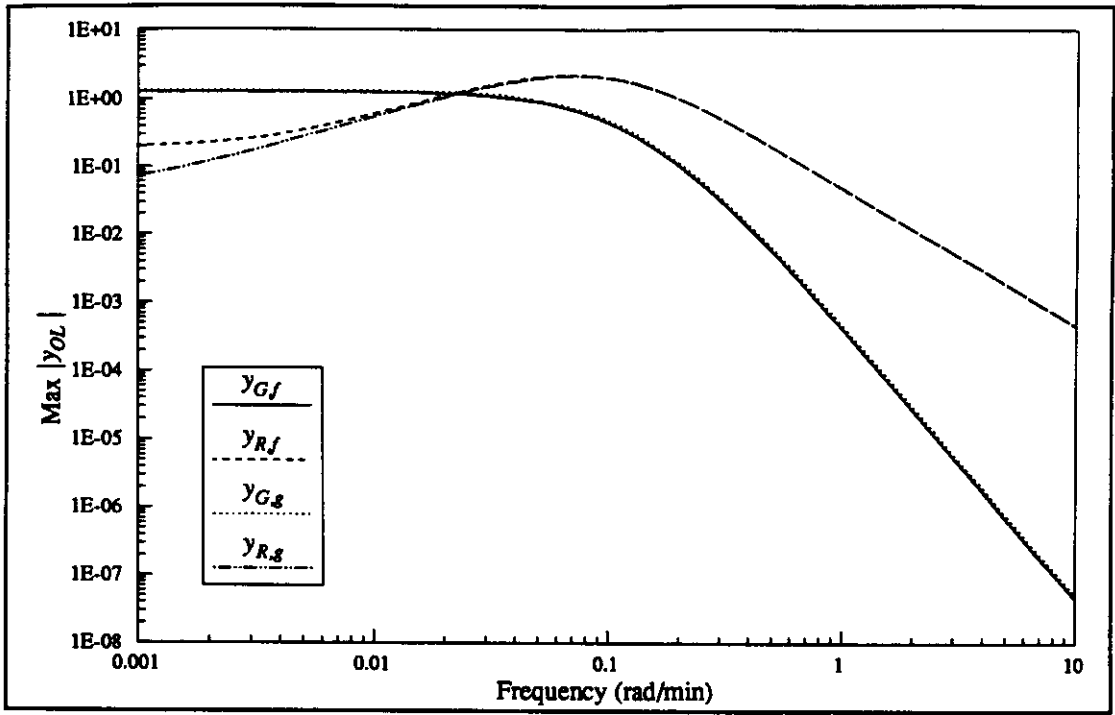


Figure 6.5a - Open-Loop Magnitudes Circuit 1

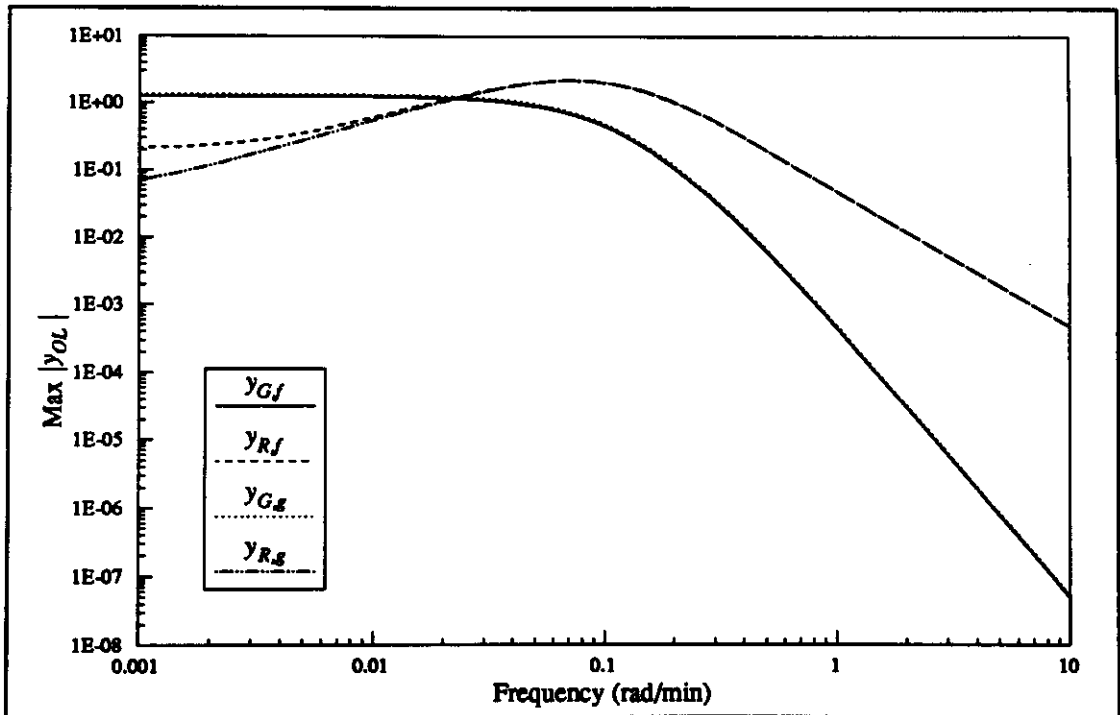


Figure 6.5b - Open-Loop Magnitudes Circuit 2

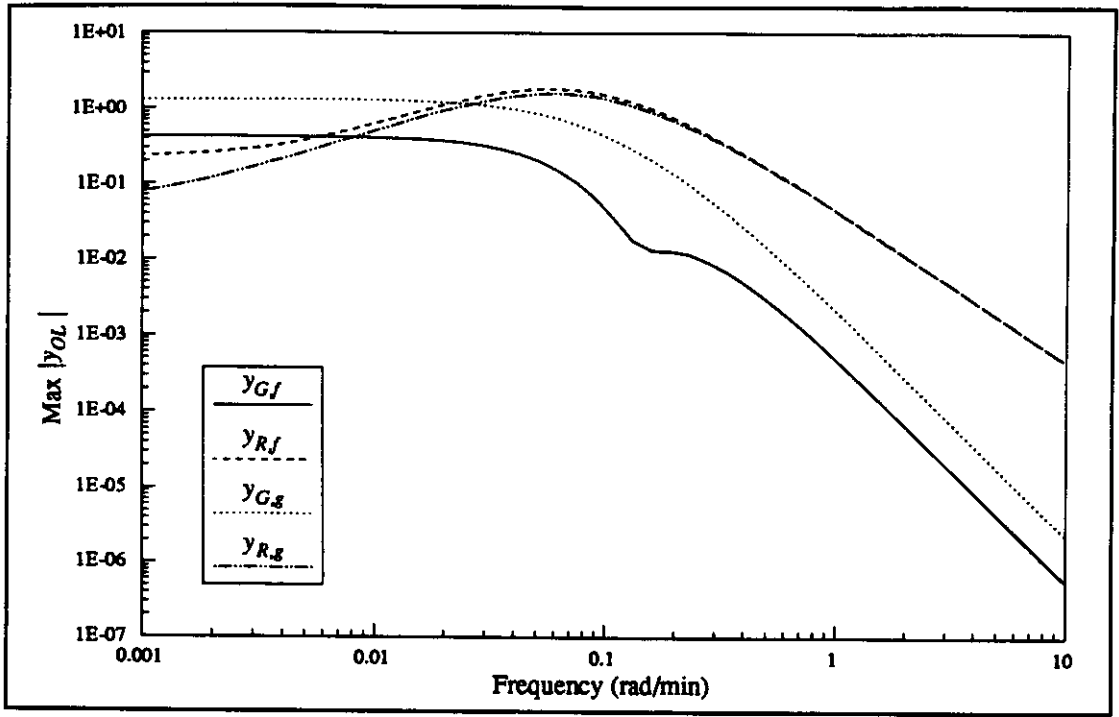


Figure 6.5c - Open-Loop Magnitudes Circuit 4

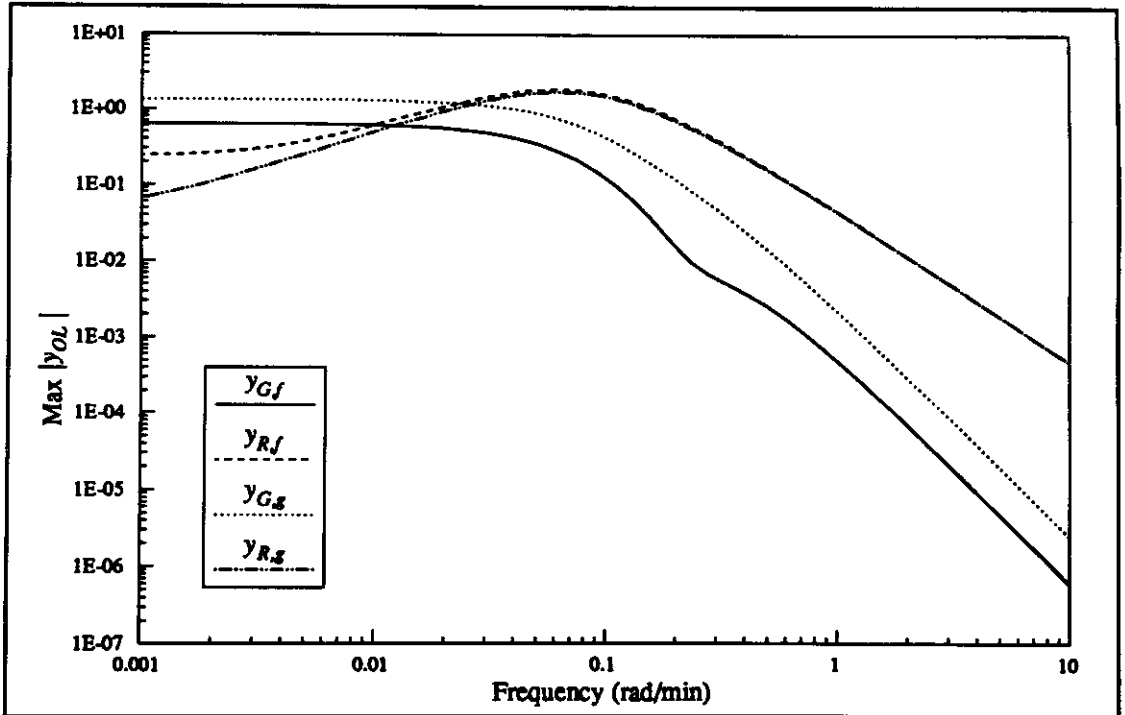


Figure 6.5d - Open-Loop Magnitudes Circuit 8

Circuit	$ y_{R,f,OL} $	$ y_{R,g,OL} $	$ y_{G,f,OL} $	$ y_{G,g,OL} $
1	2.216	2.167	1.257	1.326
2	2.159	2.180	1.256	1.324
4	1.852	1.613	0.422	1.323
8	1.849	1.711	0.633	1.326

Table 6.5 - Maximum Open-Loop Magnitudes

output variables due to feed rate and feed grade disturbances. If we look at Circuit 1, for example, it is possible that the disturbances being considered here could cause a variation in the *Recovery* of up to ± 2.216 percentage points, that is, 92.657 to 97.089 percent recovery of the valuable material. This occurs for a feed rate disturbance, and looking at Figure 6.5a, it occurs at a frequency of about 0.1 rad/min.

In order to determine the necessary "backoff" from constraints in the Optimal Operation Algorithm, the maximum magnitudes for each output, from any likely disturbance, are required. These are shown in bold in Table 6.5. By taking the maximum of the two disturbance sources, sufficient backoff will be included for *either* of the disturbances to occur, but not both simultaneously. The use of *either* has been made for simplicity in this example.

The next step in determining the optimal operating point is the determination of the "best" controller for each circuit and the calculation of the maximum input and output variations for the feed rate and feed grade disturbances.

6.2.3 Closed-Loop Operation

In order to determine how well a plant might operate under real conditions, the best possible controller needs to be determined for that plant, so that fair comparisons can be made between designs. As discussed in Chapter 5, one approach would be to use the "perfect" controller, totally rejecting all disturbances that might occur. However, due to various plant characteristics such as dead-time and right half plane zeros, this is not always possible. So the "best" controller chosen here is the H_∞ multivariable controller.

This allows a specification to be placed on the sensitivity function, that is, a specification is made as to how well disturbances are rejected. The tighter the specification, the closer the controller is to "perfect". As will be demonstrated in later sections, the use of an H_∞ controller also allows uncertainty descriptions to be included in the controller design.

Section 5.2.3 introduced the theoretical basis of H_∞ controllers. This allows a controller to be designed which ensures that the design specification is not exceeded for any part of the frequency spectra considered. The design tool used to produce the H_∞ controllers was the Robust Control Toolbox of Chiang and Safanov (1988) for MATLAB.

In practice, this procedure requires the definition of two Laplace domain weighting functions; the performance weight (W_1^{-1}) and the uncertainty bound (W_3^{-1}). Equation 5.106 shows that the maximum singular value of the closed-loop sensitivity function (E) is limited by the performance weight. The sensitivity function is a measure of disturbance attenuation, so limiting its maximum singular value to the value of W_1^{-1} , ensures that the greatest amplification a disturbance can undergo (as measured by the 2-norm of the vector) is no greater than the specified value. The designer chooses W_1^{-1} to give the disturbance rejection desired.

Model uncertainty is incorporated into the controller design through the use of the bounding function W_3^{-1} . As shown in Equation 5.107, this places an upper bound on the maximum singular value of the complementary sensitivity function. This is equivalent to ensuring robust stability for a multiplicative output uncertainty description (see Equation 5.101).

The H_∞ controller produced using this technique is limited by the bound on the sensitivity function, the bound on the complementary sensitivity function, or both. For the example being considered in this section, allowance for uncertainty in the process model is not required, so a very large value was chosen for W_3^{-1} , allowing the complementary sensitivity function to become very large before becoming a limitation on controller performance. The particular function chosen is given below.

$$W_3^{-1} = \frac{s + 10}{0.01s} \quad (6.19)$$

This is equivalent to saying that there is very little uncertainty in our knowledge of the process model. The size of this bound is demonstrated in Figure 6.6, where it is shown in comparison to a "real" uncertainty bound used later in Section 6.3 (for bubble rise time uncertainty). The use of this function (Equation 6.19) ensured that the H_∞ controllers designed were only limited by the performance specification.

The design of the H_∞ controllers is driven by the desire to find the best possible controller for the circuit under consideration. To provide maximum disturbance rejection, we therefore require the sensitivity function to be as small as possible. A value of zero at all frequencies would mean perfect control had been achieved, however, such a controller cannot generally be achieved. More realistically, the designer selects the "shape" for W_1^{-1} (e.g. shape = $f(s)$), and then iteratively determines a scalar multiplier γ , effectively tightening the W_1^{-1} (i.e. $W_1^{-1} = \gamma f(s)$) specification until Equation 5.106 is just satisfied. This method places a great deal of flexibility in the hands of the designer, but requires some experience in determining suitable W_1^{-1} weightings which will allow the required control performance to be achieved in practice.

To this end, many different weighting functions were tried in determining a set which was felt to represent the best that could be achieved for each circuit. The functions for Circuits 1 and 2 were quite similar, as were those for Circuits 4 and 8. The magnitudes of the weightings used for each circuit are shown in Figure 6.7, together with the magnitudes of the maximum ($\bar{\sigma}$) and minimum ($\underline{\sigma}$) singular values of the sensitivity function. Notice that at some point over the frequency range shown, the maximum singular value approaches or equals (but never exceeds) the weighting function. The actual weighting functions are given below.

$$\text{Circuit 1} \quad W_1^{-1} = \frac{1.1s}{s + 100} \quad (6.20)$$

$$\text{Circuit 2} \quad W_1^{-1} = \frac{0.2s}{s + 100} \quad (6.21)$$

$$\text{Circuit 4} \quad W_1^{-1} = 0.18 \frac{5s^2 + 10s + 0.01}{s^2 + 100s + 0.01} \quad (6.22)$$

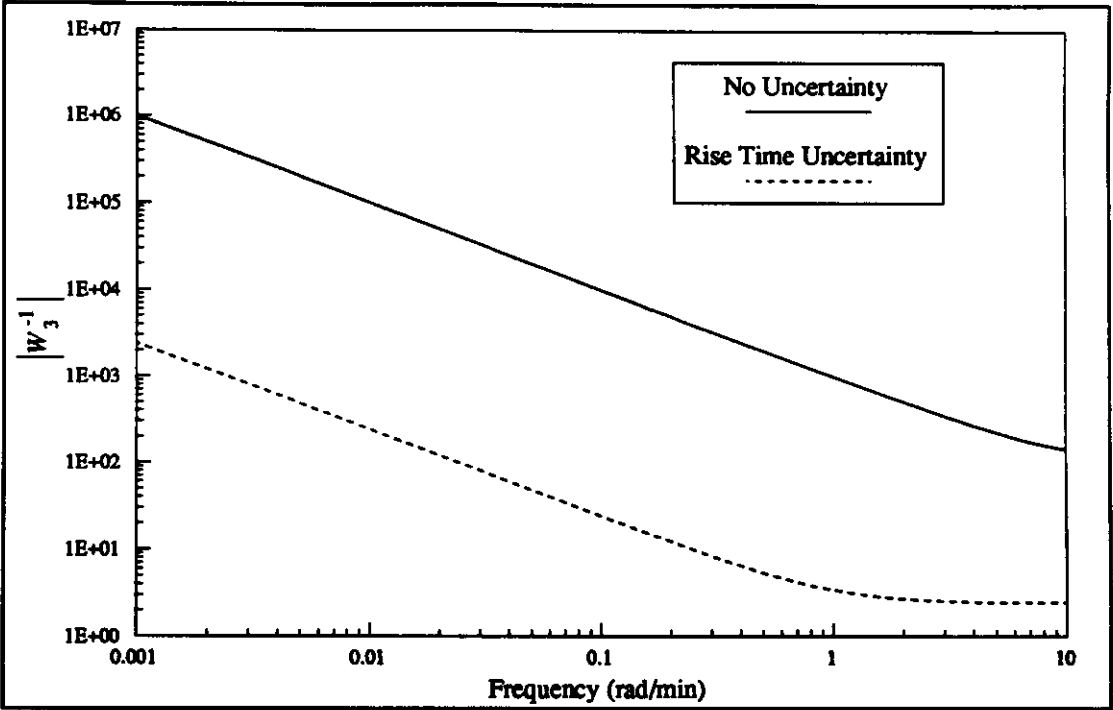
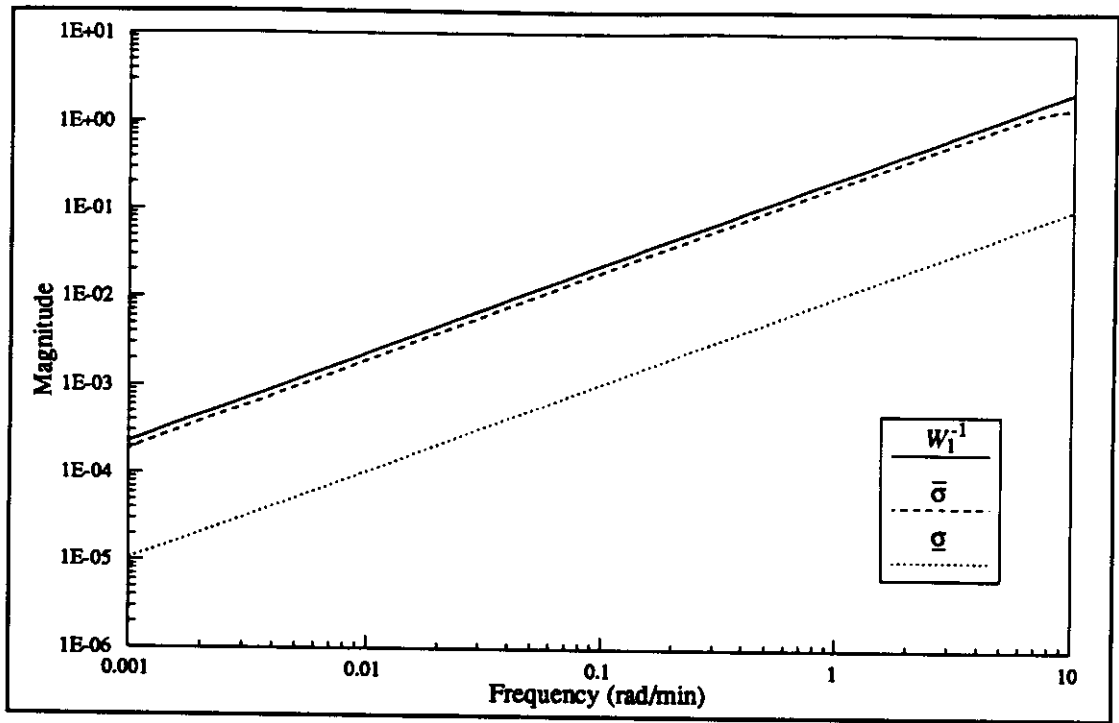
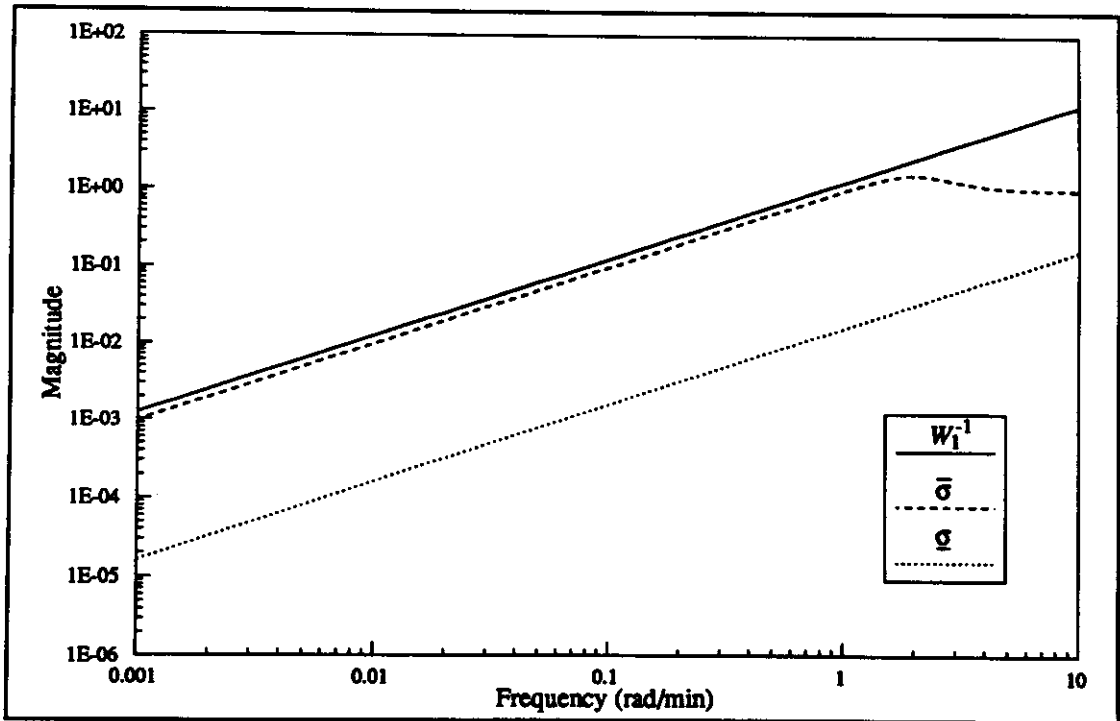
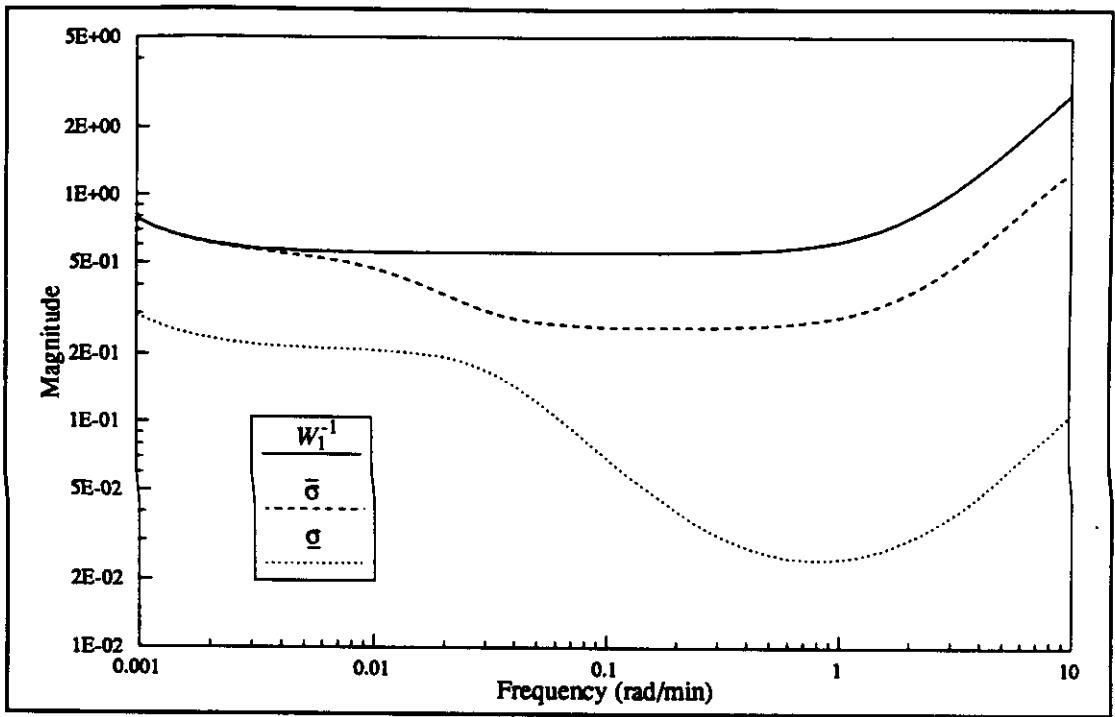
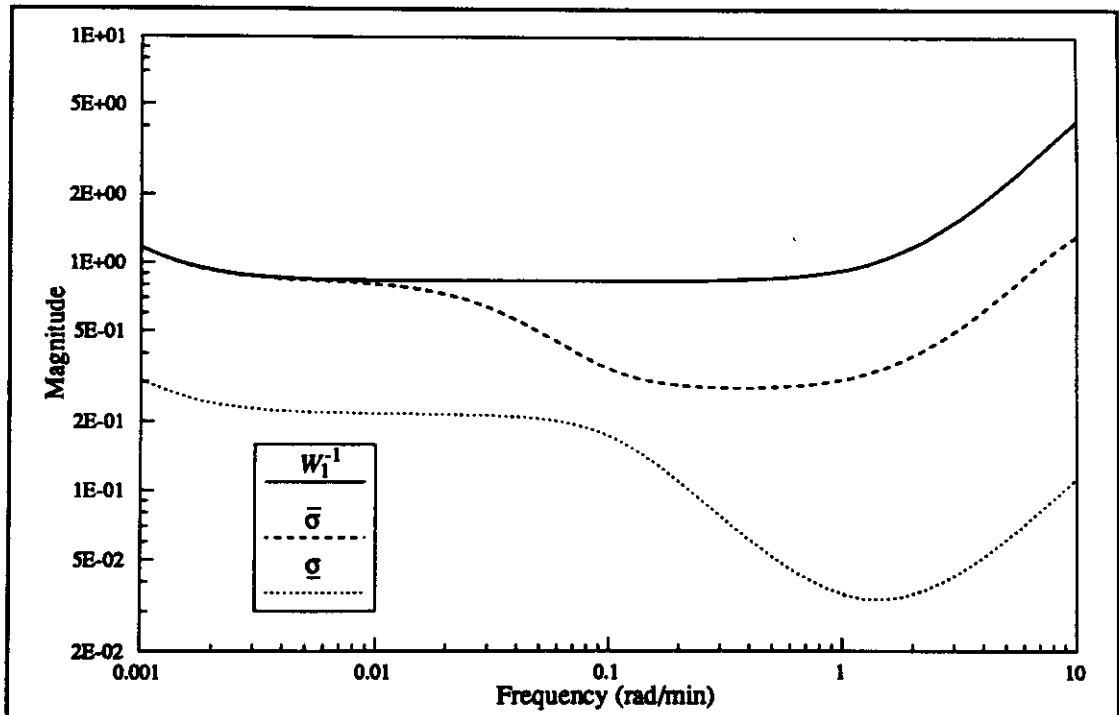


Figure 6.6 - No Uncertainty *cf.* Rise Time Uncertainty

Figure 6.7a - Circuit 1 H_∞ Controller DesignFigure 6.7b - Circuit 2 H_∞ Controller Design

Figure 6.7c - Circuit 4 H_∞ Controller DesignFigure 6.7d - Circuit 8 H_∞ Controller Design

Circuit 8

$$W_1^{-1} = 0.12 \frac{5s^2 + 10s + 0.01}{s^2 + 100s + 0.01} \quad (6.23)$$

It is generally desirable for the sensitivity function to approach zero at low frequencies, as this is equivalent to integral action, giving zero error at steady-state. This was readily achieved for Circuits 1 and 2, however for Circuits 4 and 8 the low frequency value of the sensitivity function is non-zero, indicating that some steady-state offset could be expected. This is in agreement with the observations of Chan (1988), who found that with a multi-loop control scheme there was a steady-state offset in the *Recovery* after a disturbance.

At high frequencies, it was found necessary to have the weighting function for each circuit greater than one, allowing a certain amount of high frequency (noise) amplification. However, this is at the upper limit of the frequency range of interest, and is likely to be due to the simple forms used for W_1^{-1} .

The application of the Optimal Operation Algorithm requires a function describing the maximum input and output magnitudes achieved using these H_∞ controllers for the given disturbance description, as a function of the partial control factor, k . The partial control factor was introduced in Section 5.2.1 as a means of determining "how much control" is required at the optimal operating point. Section 5.2.1 developed the transfer function relationships between the amount of disturbance "appearing at the output" (d), and the inputs (u , Equation 5.34) and outputs (y , Equation 5.33). Substituting Equation 5.15, the relationship between disturbances at the plant input (d') and the disturbance transfer function matrix (G_d), for the equivalent disturbance at the output (d), gives the following.

$$y = (I + GCkI)^{-1}G_d d' \quad (6.24)$$

$$u = CkI(I + GCkI)^{-1}G_d d' \quad (6.25)$$

Again following the developments in Chapter 5, we must really interpret these equations as relationships between perturbation variables, or the magnitude of the variations away from the steady-state. For the disturbance description d' (Equation 6.15), we require the maximum value that these perturbations may take as a function of k . In particular, we

require a function for each output and for each input, the greater of the feed rate and feed grade disturbances. This requires eight functions to be evaluated for each circuit.

$$\Delta y_{i,j,CL}(k) = \text{Maximum for any } \omega \left| \left((I + GCkI)^{-1} \right)_i G_{aj} d_j \right| \quad (6.26)$$

$$\Delta u_{l,j,CL}(k) = \text{Maximum for any } \omega \left| \left(CkI(I + GCkI)^{-1} \right)_i G_{aj} d_j \right| \quad (6.27)$$

for $i = R(ecovery), G(rade)$ - outputs

$j = f(eed), g(rade)$ - disturbances

$l = A(ir), W(ater)$ - inputs

These maxima were determined by generating plots of the magnitude of each variable, for various k values over the frequency range 0.001 to 10 rad/min. Figure 6.8 shows for a feed grade disturbance, plots of the magnitude of the variation in final concentrate grade for Circuit 1, and in the recovery of valuable material for Circuit 8. In each case a plot is shown for $k = 0$ (open-loop), 10^{-6} , 0.001, 0.1, 0.5, 0.75, 1 (full-control). Circuit 1 demonstrates what might be expected to be the "normal" trend, with a decline in magnitude from open-loop through to full-control. However, Circuit 8 has two opposing trends. At intermediate to high frequencies (approximately ≥ 0.1 rad/min) the "normal" trend is observed, with full-control ($k = 1$) having a lower magnitude than open-loop ($k = 0$). However, at lower frequencies (approximately ≤ 0.001 rad/min) the open-loop case actually has a greater magnitude than the full-control case. This can be explained as follows. Looking back to Figure 6.7, the plots of the sensitivity function for each controller, the maximum singular value for Circuit 8 at these low frequencies was less than one. It would therefore be expected that there should not be any disturbance amplification ($|closed-loop| \leq |open-loop|$). However, this is a consequence of the use of the 2-norm for measuring vector size in the controller design. Although the 2-norm of the output vector y is less for the closed-loop than for the open-loop case, it is not necessarily so for the individual components.

The information actually required for the Optimal Operation Algorithm is the maximum value of these magnitudes (see Equations 6.26 and 6.27) for each k value. These were extracted from the complete set of data used to generate Figure 6.8. A log-log plot of the maximum magnitudes versus k is most instructive, as shown in Figure 6.9 for both the

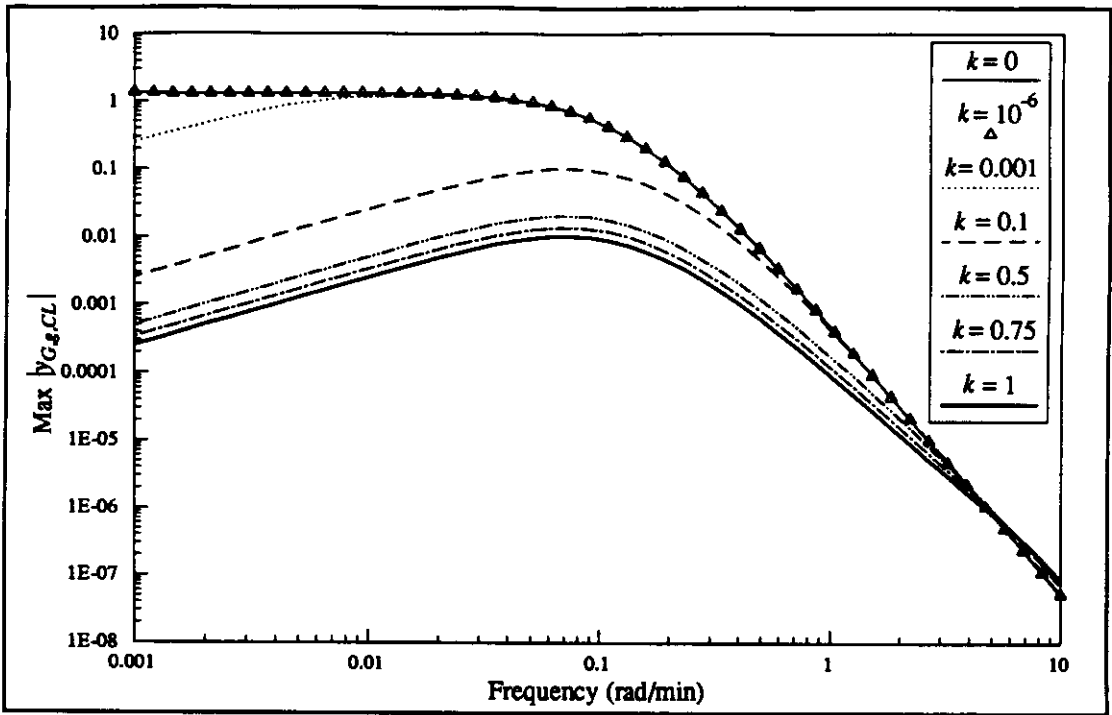


Figure 6.8a - Circuit 1 $|y_{G_s}|$ Closed-Loop, Various k Values

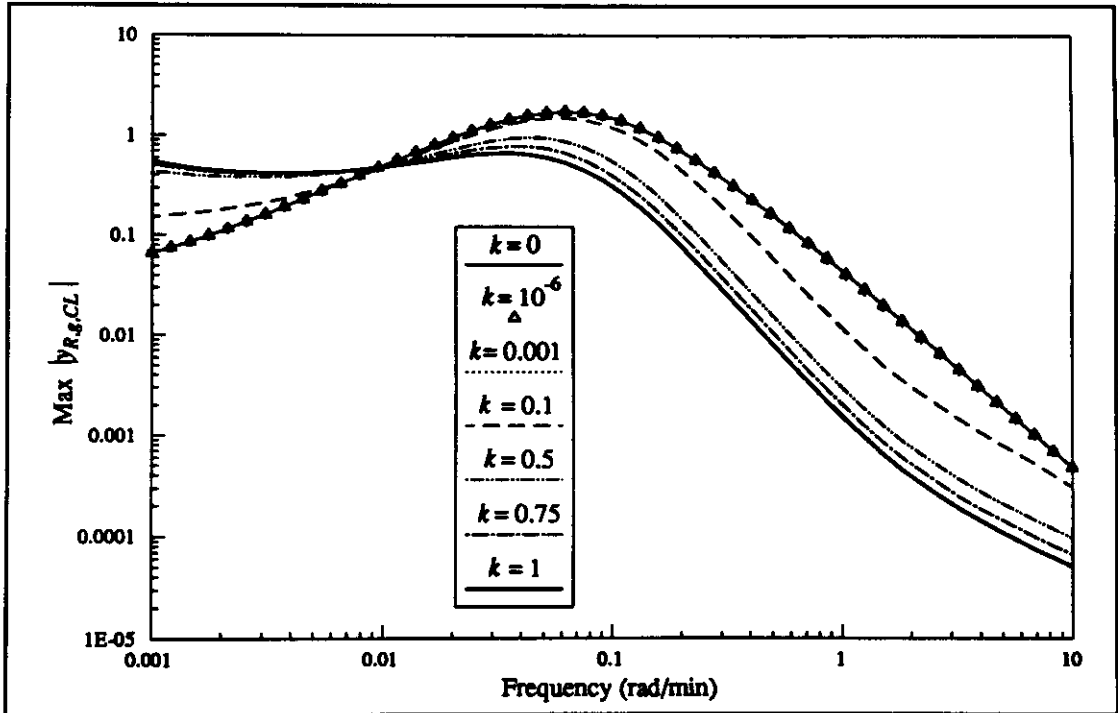


Figure 6.8b - Circuit 8 $|y_{R_s}|$ Closed-Loop, Various k Values

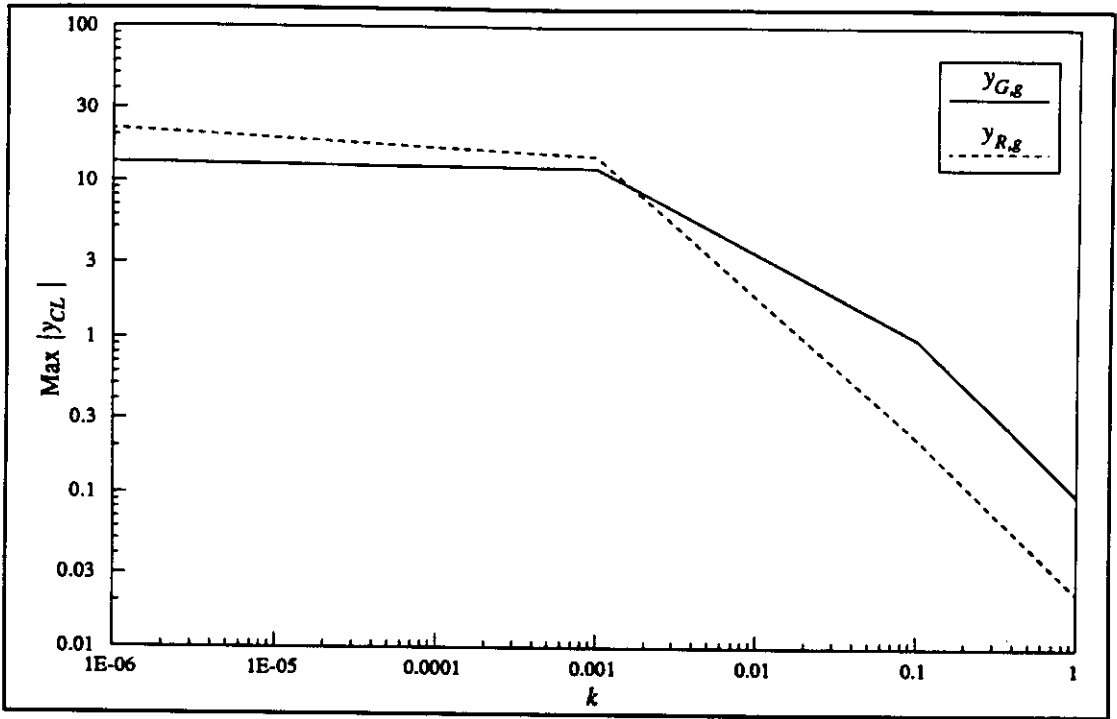


Figure 6.9a - Circuit 1 Max $|y_{G,s}|$, $|y_{R,s}|$ Closed-Loop, Various k Values

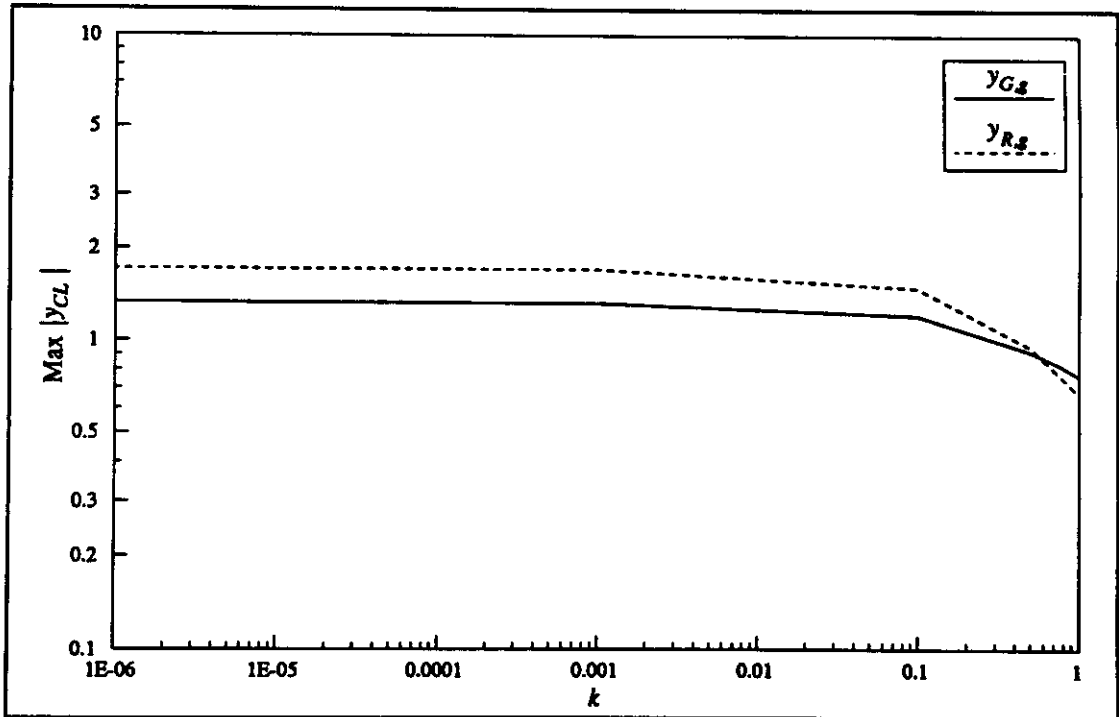


Figure 6.9b - Circuit 8 Max $|y_{G,s}|$, $|y_{R,s}|$ Closed-Loop, Various k Values

Grade and *Recovery* outputs, for the feed grade disturbance on Circuits 1 and 8. Again, Circuit 1 shows "normal" trends, the magnitude gradually declining with k . At larger k values, there is almost a linear relationship between $\log(|y_i|)$ and $\log(k)$. This was predicted in Chapter 5, where for "large" $GCKI$ (see Equation 5.36), $|y|$ is inversely proportional to k . Circuit 8, however, exhibits slightly different behaviour, with the magnitudes almost constant, below $k \approx 0.1$. Again this was predicted in Chapter 5 (see Equation 5.35) for the case of a "small" $GCKI$ term.

The variation in the maximum input magnitudes also demonstrated interesting trends. These are shown in Figure 6.10, for Circuits 1 and 8 for a feed grade disturbance. Looking first at Circuit 1, the maximum *Air* magnitude is fairly constant at large k values in agreement with the prediction of Equation 5.38 ($|u| \neq f(k)$), but falls off at lower k values. However, the water showed a more unusual variation, indicating a greater magnitude for $k = 10^{-3}$ than for the full-control case. That is, the input manipulation increased as the controller was "turned down". Circuit 8 followed an expected trend, with input magnitudes steadily declining with k (see Equation 5.37, $|u| \propto k \rightarrow \log|u| \propto \log(k)$).

The predictions of maximum magnitudes described above include some quite unexpected observations, such as input and output magnitudes increasing, or staying constant, as the partial control factor (that is, the fraction of the error sent to the controller) is decreased. To verify that these magnitude predictions were correct, linear dynamic simulations were performed for these process models and controllers, using sine waves at the frequencies indicated on the magnitude plots. In all cases, the simulations verified the predictions obtained from the sensitivity function analysis.

For completeness, the full-control ($k = 1$) closed-loop magnitudes, for both inputs and outputs are given in Tables 6.6a and 6.6b, respectively (Table 6.5 gave the corresponding open-loop ($k = 0$) results). For each circuit, the maximum magnitude for each variable (i.e. due to the feed grade or feed rate disturbance) is shown in bold.

In the above calculation procedure, possible disturbances in the feed rate and feed grade to the various flotation circuits have been considered. Using the "best" controller that could be designed for each circuit, the maximum input and output magnitudes have been

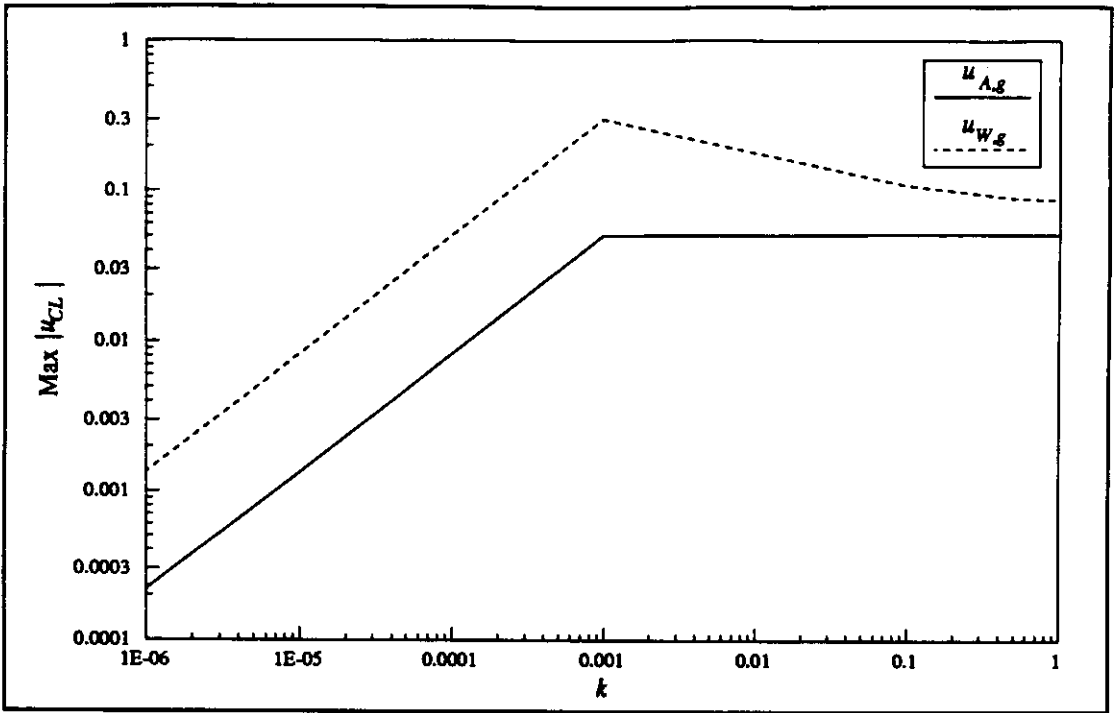


Figure 6.10a - Circuit 1 Maximum Closed-Loop Input Magnitudes, Various k Values

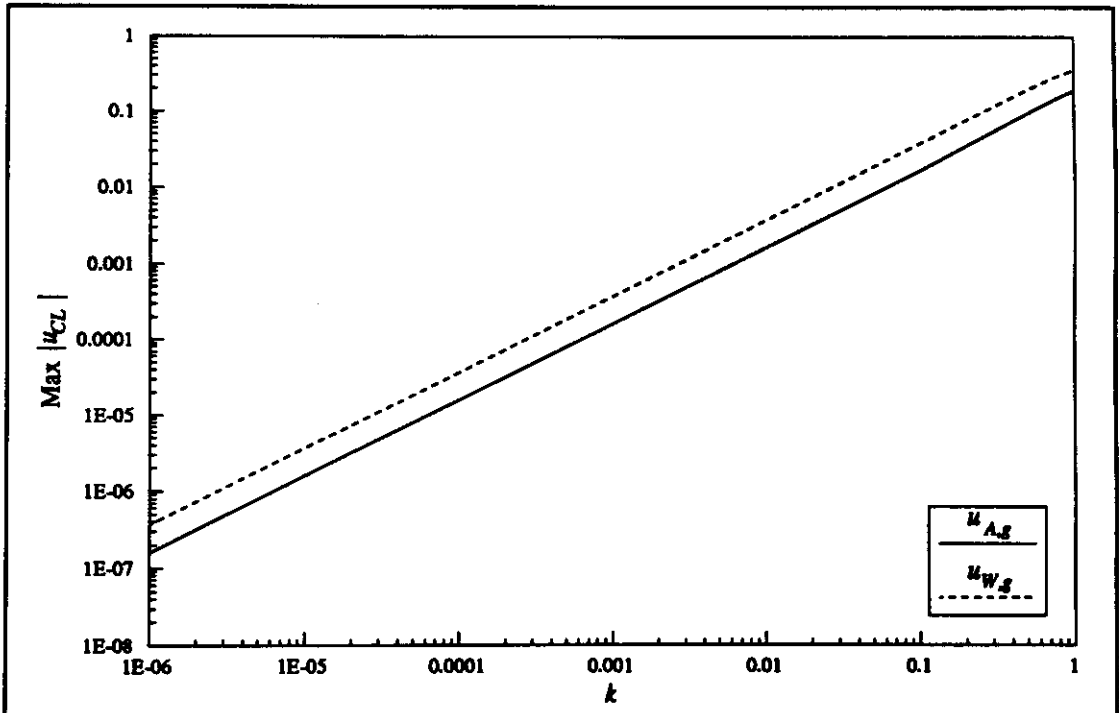


Figure 6.10b - Circuit 8 Maximum Closed-Loop Input Magnitudes, Various k Values

Circuit	$ y_{Rf,CL} $	$ y_{R,g,CL} $	$ y_{Gf,CL} $	$ y_{G,g,CL} $
1	2.304×10^{-3}	2.303×10^{-3}	9.305×10^{-3}	1.001×10^{-2}
2	2.523×10^{-3}	2.561×10^{-3}	4.794×10^{-2}	5.156×10^{-2}
4	0.3161	0.3231	0.1357	0.6060
8	0.7283	0.6673	0.2786	0.7677

Table 6.6a - Closed-Loop Magnitudes, Outputs

Circuit	$ u_{Af,CL} $	$ u_{A,g,CL} $	$ u_{Wf,CL} $	$ u_{W,g,CL} $
1	5.087×10^{-2}	5.086×10^{-2}	4.000×10^{-2}	8.694×10^{-2}
2	5.071×10^{-2}	5.071×10^{-2}	8.770×10^{-2}	1.495×10^{-1}
4	3.405×10^{-2}	0.3846	2.960×10^{-2}	0.5650
8	3.590×10^{-2}	0.1862	6.666×10^{-2}	0.3584

Table 6.6b - Closed-Loop Magnitudes, Inputs

determined for a range of values for the partial control factor, k . These magnitudes provide the final set of data necessary for determination of the optimal operating point.

6.2.4 Optimal Operating Point

In order to determine the optimal operating point (setpoints and input values) for a given process, the likely variation in input and output variables due to the impact of disturbances is required. This is so that the operating point is selected to ensure that input and output constraints will not be violated. The optimal operation of the process may be open-loop or closed-loop, or with some form of "partial" control. In Chapter 5, an optimisation problem was described (Equation 5.9) which formally included input and output constraint modification to ensure disturbances did not cause these constraints to be violated. This has been rewritten in terms of the flotation circuit problem, as Equation 6.28 below.

Maximise *Recovery*

$$u_{A,0}, u_{W,0}$$

such that

$$y_{G,0} \geq 60\% + \Delta y_G$$

$$0.5 + \Delta u_A \leq u_{A,0} \leq 2.0 - \Delta u_A$$

$$0.8 + \Delta u_W \leq u_{W,0} \leq 2.0 - \Delta u_W$$

(6.28)

$$(y_{R,0}, y_{G,0}) = f(u_{A,0}, u_{W,0}) \dots\dots \text{Steady-State Model}$$

$$0 \leq k \leq 1$$

$$\left. \begin{aligned} \Delta y_G &= f_1(k) \\ \Delta u_A &= f_2(k) \\ \Delta u_W &= f_3(k) \end{aligned} \right\} \text{Partial Control Functions}$$

where $y_{G,0}$, $u_{A,0}$ and $u_{W,0}$ are the optimal steady-state values.

This optimisation problem was solved for each circuit using the optimisation facilities within SPEEDUP. Initially, simple linear approximations were made to the partial control functions (that is, the input and output magnitudes as a function of k : f_1, f_2, f_3) to provide a first estimate of the optimal operating point, and in particular the value for the partial control factor k . For the five percent disturbance level being analysed, all optima lay at extreme values of k . Circuits 1, 2 and 8 being open-loop ($k = 0$), while circuit 4 was under full-control. The optimal operating points determined are shown in Table 6.7.

These first results are for a disturbance magnitude of $\pm 5\%$ of the nominal value for the solids feed rate and feed grade. In order to determine how the optimal operation may change with the size of the disturbance, variations of 10%, 20%, 40% and 50% were also considered. The associated input and output magnitudes, for both the open and closed-loop cases, were calculated by "scaling up" using the disturbance size from the values determined for the 5% case. This was possible because linear control theory was used in this analysis. However, this also means that as the disturbances becomes larger, the linear approximations become poorer.

Using the optimisation procedure outlined above, with approximate partial control functions, the optimal operating points were determined for each circuit. However, for the larger disturbances, the optimal partial control factor for Circuit 4 was found to lie

Circuit 1	<i>Recovery</i>	<i>Grade</i>	<i>Air</i>	<i>Water</i>	<i>k</i>
$d_0 = 0\%$	94.873	60	1.017	0.8	-
$d_0 = 5\%$	94.386	61.326	0.960	0.8	0
$d_0 = 10\%$	93.900	62.652	0.904	0.8	0
$d_0 = 20\%$	92.926	65.305	0.791	0.8	0
$d_0 = 40\%$	90.979	70.610	0.564	0.8	0
Circuit 2	<i>Recovery</i>	<i>Grade</i>	<i>Air</i>	<i>Water</i>	<i>k</i>
$d_0 = 0\%$	94.953	60	1.014	0.8	-
$d_0 = 5\%$	94.503	61.324	0.958	0.8	0
$d_0 = 10\%$	94.054	62.649	0.902	0.8	0
$d_0 = 20\%$	93.154	65.298	0.790	0.8	0
$d_0 = 40\%$	91.355	70.596	0.566	0.8	0
Circuit 4	<i>Recovery</i>	<i>Grade</i>	<i>Air</i>	<i>Water</i>	<i>k</i>
$d_0 = 0\%$	94.592	60	0.853	0.8	-
$d_0 = 5\%$	93.814	60.606	1.205	1.365	1
$d_0 = 10\%$	93.078	61.932	1.089	1.265	0.343
$d_0 = 20\%$	91.557	64.264	1.089	1.400	0.210
$d_0 = 40\%$	88.082	70.582	0.5	0.875	0
$d_0 = 50\%$	86.299	73.323	0.5	1.032	0
Circuit 8	<i>Recovery</i>	<i>Grade</i>	<i>Air</i>	<i>Water</i>	<i>k</i>
$d_0 = 0\%$	95.040	60	0.888	0.8	-
$d_0 = 5\%$	94.407	61.326	0.832	0.8	0
$d_0 = 10\%$	93.773	62.651	0.779	0.8	0
$d_0 = 20\%$	92.507	65.302	0.675	0.8	0
$d_0 = 40\%$	89.855	70.605	0.5	0.871	0

Table 6.7 - No Uncertainty Optimal Operation Results

between 0 and 1. This therefore required a more accurate representation of the partial control function for Circuit 4. Suitable logarithmic functions were found which closely matched the input and output magnitudes over the required range ($0.1 \leq k \leq 1$). Table 6.8 lists the simple and exponential functions used for Circuit 4.

Variable	Simple Function	Exponential Function
<i>Grade</i>	$-0.7169 \times k + 1.3228$	$0.6491 \times e^{-0.2979k}$
<i>Air</i>	$0.3848 \times k$	$0.4120 \times e^{0.8932k}$
<i>Water</i>	$0.5652 \times k$	$0.6080 \times e^{0.8819k}$

Table 6.8 - Simple and Exponential Partial Control Functions, Circuit 4

The results for all disturbance levels are shown in Table 6.7, including the original optimal operating points ($d_0 = 0\%$). A k value is not applicable for the original results, as with no disturbances, open-loop and closed-loop perform the same.

The maximum disturbance level (calculated for the largest of the two possible disturbance sources, feed grade or feed rate), that can be handled is 50% for Circuit 4 and 40% for the other circuits. Feasible operation could not be sustained for the other circuits at the 50% disturbance level.

The optimal recoveries are plotted in Figure 6.11. Clearly, at low disturbance levels all circuits can achieve similar recoveries, with Circuit 4 being slightly lower than the other three. Interestingly, all circuits except Circuit 4 achieve their best recovery when operated open-loop. That is, the movement away from the optimal steady-state input values, required to achieve some form of control, is more expensive than operating at a higher average grade, to ensure that the grade constraint ($\geq 60\%$) is not exceeded. Circuit 4, however, achieves its highest recovery when operated closed-loop at the 5% disturbance level, moving through points of partial control at 10% and 20%, before switching to open-loop operation at higher disturbance values.

At the 40% disturbance level, the recoveries for Circuits 1 and 2 are more than 1% better

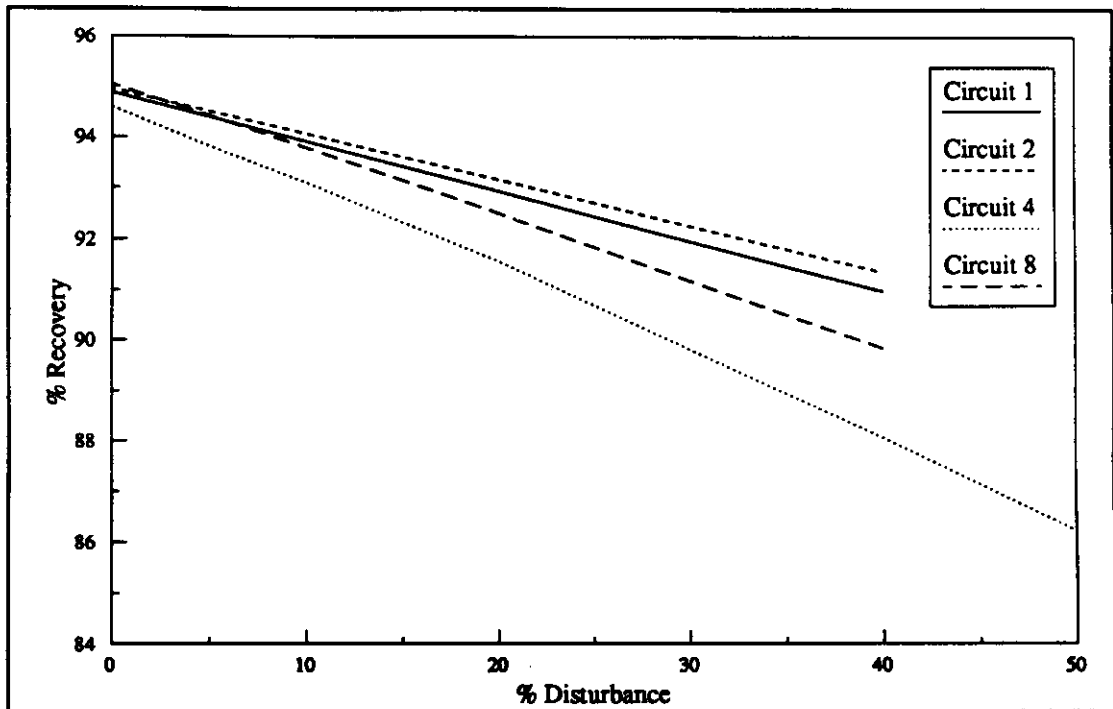


Figure 6.11 - Optimal Recoveries, No Uncertainty

than for Circuits 4 and 8. If the disturbance level is expected to be high, as it frequently is in the mineral processing industry, then these more conventional designs are likely to perform better than the higher steady-state recovery Circuits 4 and 8. Although Circuits 1 and 2 were also the more "controllable" circuits, the better performance at high disturbance levels is achieved under open-loop conditions, and so is probably due to the disturbance and process gains rather than the inherent circuit controllability.

The use of the Optimal Operation Algorithm has allowed each of the circuits to be analysed in terms of the dynamic environment they may encounter once in operation. The controller designed for each circuit was chosen as the "best" that could be achieved, to put all circuits on an equal footing. However, the experience gained designing these controllers resulted in the development of a technique for improving the circuit performance even further.

6.2.5 Improved Performance Controllers

The H_{∞} controllers used in the preceding investigation of achievable optimal performance were designed using the software of Chiang and Safanov (1988). The application of this software was described in detail in Section 5.2.3.2. As discussed earlier in this Chapter, the key to the performance specification is the weighting matrix W_1^{-1} . As part of the mathematics of the controller design, W_1 is multiplied by the sensitivity function matrix E . As used here, W_1 is in fact a transfer function multiplied by the identity matrix of the correct dimensions (here 2×2). It is therefore possible within W_1 to apply a different scalar weighting to each of the outputs, emphasising the importance of the control of one output over the other.

For the flotation circuit example, control of the *Grade* is more important than control of the *Recovery*, as there is no output constraint associated with *Recovery*. Consequently, a scaling of 10^{-3} was applied to the sensitivity function matrix output associated with *Recovery*, making control of *Grade* 1000 times more important than *Recovery*. Using this technique, new H_{∞} controllers were designed for each circuit. The use of this scaling made it possible to use the same performance weighting function for all four circuits.

$$W_1^{-1} = \gamma \frac{0.05s + 0.0001}{0.01s + 1} \quad (6.29)$$

Following the same procedure as was used for the unweighted controller design, γ (Equation 6.29) was increased until the H_∞ controller design criteria were just satisfied. The values found for each circuit are shown in Table 6.9 below. Note that this weighting function does not actually provide integral action ($|W_1^{-1}| \rightarrow 0$ as $s \rightarrow 0$), however the low frequency value tends to 10^{-4} , which is equivalent to almost complete disturbance rejection (very small steady-state offset). The weighting function and singular value plots for the Circuit 1 controller are shown in Figure 6.12 as representative of all four circuits. The line for the minimum singular value (σ) is coincident with W_1^{-1} for most of the plot.

Circuit	1	2	4	8
γ	2.5	12.5	10	10

Table 6.9 - Improved Controller Design, γ Values

Note that this is quite different from the plots obtained for the equally weighted controllers (Figure 6.7). Here, the maximum singular value is actually greater than the weighting function for most of the frequency range. This is because the singular values plotted are actually for the unweighted sensitivity function matrix. If the same scaling matrix used in the controller design (i.e. with 10^{-3} weighting on the *Recovery* output) was applied to the sensitivity function matrix, the singular value plots would lie below the W_1^{-1} line as required.

For the same feed rate and feed grade disturbances considered previously for the equally weighted controller, the open-loop output variation magnitudes are the same as those given in Table 6.5. The closed-loop input and output magnitudes were recalculated at various k values, so that the partial control functions could be determined. The maximum values for the full-control case are given in Table 6.10 for comparison with the previous results shown in Table 6.6. Once again, the bold figures indicate for each output variable, for each circuit, the maximum magnitude due to either disturbance source.

Two trends are noticeable when comparing these results with those from the unweighted controller. Firstly, as expected, the magnitude of the *Recovery* variation is much greater than for the equally weighted controller. In fact, Circuits 4 and 8 have more variation

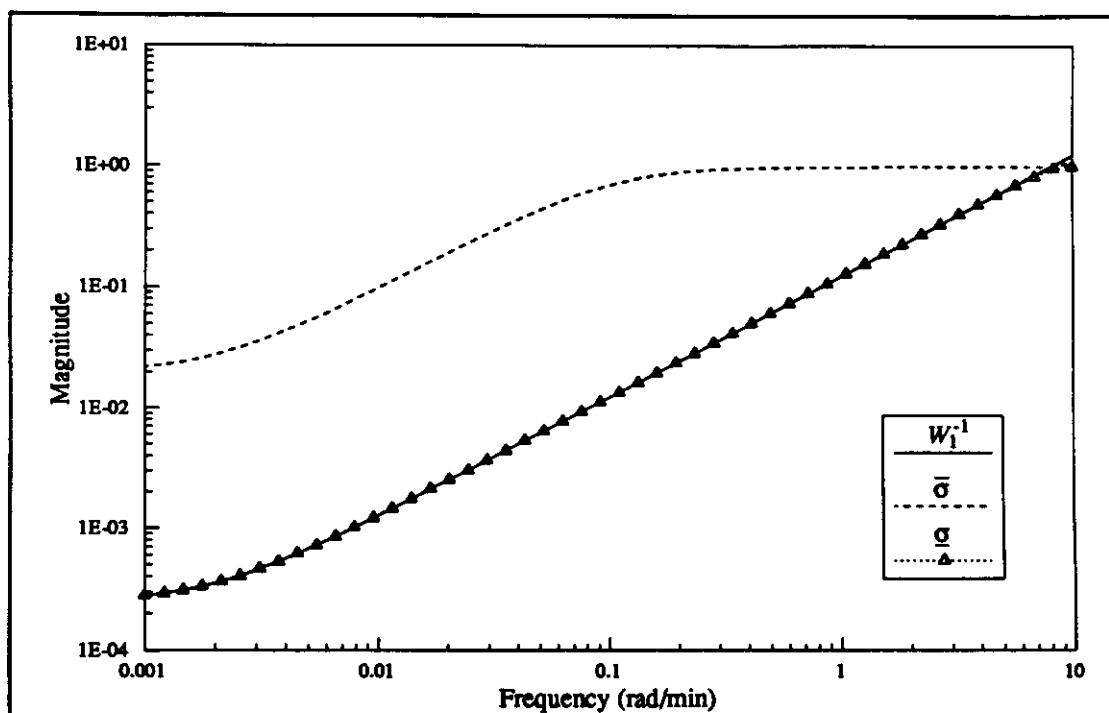


Figure 6.12 - Weighted Controller Design, Circuit 1

Circuit	$ y_{R,f,CL} $	$ y_{R,g,CL} $	$ y_{G,f,CL} $	$ y_{G,g,CL} $
1	1.432	1.425	6.088×10^{-3}	6.558×10^{-3}
2	1.417	1.444	3.034×10^{-2}	3.262×10^{-2}
4	3.959	3.518	2.632×10^{-3}	9.221×10^{-3}
8	2.880	2.752	3.305×10^{-3}	8.250×10^{-3}

Table 6.10a - Closed-Loop Magnitudes (Outputs), Weighted Controller

Circuit	$ \mu_{A,f,CL} $	$ \mu_{A,g,CL} $	$ \mu_{W,f,CL} $	$ \mu_{W,g,CL} $
1	5.082×10^{-2}	5.090×10^{-2}	2.224×10^{-1}	2.329×10^{-1}
2	5.084×10^{-2}	5.101×10^{-2}	4.482×10^{-1}	4.589×10^{-1}
4	6.877×10^{-2}	1.405×10^{-1}	1.237×10^{-1}	2.028×10^{-1}
8	4.173×10^{-2}	7.227×10^{-2}	1.260×10^{-1}	2.015×10^{-1}

Table 6.10b - Closed-Loop Magnitudes (Inputs), Weighted Controller

than the open-loop case. Secondly, the *Grade* magnitudes are now smaller than for the previous controller, considerably so for Circuits 4 and 8. By disregarding control of *Recovery*, the *Grade* control has been significantly improved.

As expected for these different controllers, quite different optimal operation results were obtained as shown in Table 6.11. For this example, the 50% disturbance value was not computed.

These results are shown graphically in Figure 6.13. The use of these weighted controllers means that at low disturbance levels the economics of operation are improved through the use of closed-loop control (for all but Circuit 2). Up to a 10% disturbance level, all circuits are essentially equally profitable, with the difference in the recovery of valuable material being less than 0.3%. In fact, Circuit 2 can achieve this level of recovery without any control. At the 20% disturbance level, Circuit 8 performs the best, with partial control

Circuit 1	<i>Recovery</i>	<i>Grade</i>	<i>Air</i>	<i>Water</i>	<i>k</i>
$d_0 = 0\%$	94.873	60	1.017	0.8	-
$d_0 = 5\%$	94.531	60.007	1.013	0.851	1
$d_0 = 10\%$	94.118	60.013	1.009	0.902	1
$d_0 = 20\%$	92.926	65.305	0.791	0.8	0
$d_0 = 40\%$	90.979	70.610	0.564	0.8	0
Circuit 2	<i>Recovery</i>	<i>Grade</i>	<i>Air</i>	<i>Water</i>	<i>k</i>
$d_0 = 0\%$	94.953	60	1.014	0.8	-
$d_0 = 5\%$	94.503	61.324	0.958	0.8	0
$d_0 = 10\%$	94.054	62.649	0.902	0.8	0
$d_0 = 20\%$	93.154	65.298	0.790	0.8	0
$d_0 = 40\%$	91.355	70.596	0.566	0.8	0
Circuit 4	<i>Recovery</i>	<i>Grade</i>	<i>Air</i>	<i>Water</i>	<i>k</i>
$d_0 = 0\%$	94.592	60	0.853	0.8	-
$d_0 = 5\%$	94.440	60.009	0.987	1.003	1
$d_0 = 10\%$	94.288	60.018	1.122	1.206	1
$d_0 = 20\%$	91.364	65.291	0.652	0.800	0
$d_0 = 40\%$	88.082	70.582	0.5	0.875	0
Circuit 8	<i>Recovery</i>	<i>Grade</i>	<i>Air</i>	<i>Water</i>	<i>k</i>
$d_0 = 0\%$	95.040	60	0.888	0.8	-
$d_0 = 5\%$	94.700	60.008	0.986	1.002	1
$d_0 = 10\%$	94.361	60.017	1.084	1.203	1
$d_0 = 20\%$	93.797	60.508	1.160	1.400	0.06
$d_0 = 40\%$	89.855	70.605	0.5	0.871	0

Table 6.11 - No Uncertainty Optimal Operation Results (Weighted Controller)

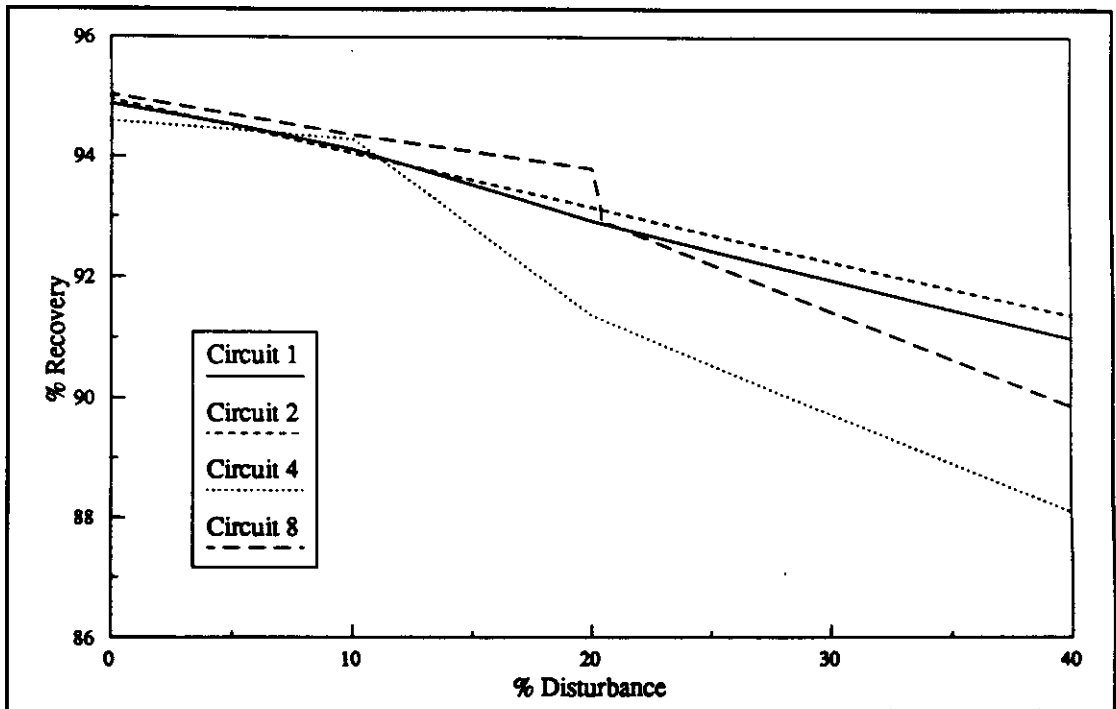


Figure 6.13 - Optimal Recoveries, No Uncertainty, Weighted Controllers

enabling approximately 0.5% more recovery than the other circuits. However, at the highest disturbance levels, all circuits are once again optimal when open-loop, with Circuits 1 and 2 having the highest recoveries.

The partial control of Circuit 8 at the 20% disturbance level is quite interesting. Notice that the nominal value for the *Water* is 1.4, which is the midpoint of the upper and lower bounds on water flow ($0.8 \leq \text{Water} \leq 2$), so that the variation in water flow, as a control action, will take it to its allowable extremes. The Optimal Operation Algorithm has therefore found the partial control factor which maximises the use of the available inputs.

The results presented in this section demonstrate that it is now possible to calculate *a priori*, the achievable economic potential of a process design under the conditions it is likely to encounter once built. The results obtained depend on the quality of the controller designed - it being necessary to use the "best" possible controller (or the best that is likely to be implemented) to allow a fair comparison between designs.

To further improve the "realism" of the analysis, model uncertainty should also be taken into account. In the next section, descriptions of uncertainty in the dynamic behaviour of the process will be explicitly included in the determination of the achievable optimal operating point.

6.3 Dynamic Uncertainty

The previous section dealt with the calculation of the optimal operating point of a process plant when it is assumed that there is no uncertainty in the knowledge of the process model. However, as discussed in Chapter 5, one of the recent advances in controller design is the ability to explicitly include descriptions of model uncertainty at the design stage. These techniques allow a controller to be designed which will ensure stability, and in some cases performance guarantees, for all plants which lie within the uncertainty description.

In this section, estimates of uncertainty in the process model will be used at the controller design stage and in the calculation of the input and output magnitudes due to the impact of a disturbance. This takes the procedure of determining the optimal operating point closer to that which may be achieved by the "real" plant.

6.3.1 Dynamic Uncertainty and Controller Design

Model uncertainty can arise from many sources - changes in process parameters, structural model inadequacy and linearisation errors. These will cause the outputs from the model to differ from those of the "real" plant, for the same inputs. For this work, uncertainty has been divided into two categories, steady-state and dynamic uncertainty. In essence, dynamic uncertainty is a subset of the steady-state case, where the uncertainty only affects the non steady-state, or dynamic behaviour of the model.

Process variation also causes the input and output changes in response to a given disturbance to vary, depending upon where in the uncertainty description the real plant lies. In Section 6.3.2, a procedure will be presented for determining the appropriate input and output magnitudes for use within the Optimal Operation Algorithm.

If the steady-state behaviour is poorly known, then determination of the steady-state optimum becomes a more complex issue. For example, if for a given set of inputs, the outputs and hence the objective function are poorly known, the concept of the maximum (or minimum) of the objective function becomes less clear. It could be the greatest maximum, or the least maximum. Calculation of the optimal operating point when steady-state uncertainty is present is the subject of Section 6.4.

There may also be uncertainty in the way disturbances affect the process. Many disturbances enter the process in a similar way to the process inputs themselves. Consequently, the variation in the outputs due to a disturbance can depend on the deviation of the process from the nominal model. This has a flow-on effect to the calculation of input and output magnitudes, because these are determined in response to the disturbances. Therefore, the effect of disturbance uncertainty must also be incorporated into the calculation of the input and output magnitudes.

A mathematical treatment of uncertainty descriptions was given in Section 5.2.2. For the multivariable case, three uncertainty descriptions were defined, Additive (Equation 5.62), Multiplicative Input (Equation 5.63) and Multiplicative Output (Equation 5.64). Various methods are available for estimating model uncertainty, based on "plant" trials, through known variations in model parameters, or via other design input data.

For this example of dynamic uncertainty, variation in only one model parameter, the bubble rise time, has been chosen to study the effect of uncertainty on the optimal performance of the four flotation circuits. The bubble rise time, τ_{RT} , was introduced in Section 6.2.1 (Equation 6.14) as the first-order lag between changes in the air input to the circuits and the air factor's effect on the separation. This parameter will vary depending on bubble size and agitation conditions. In reality this may not be the dominant uncertainty, however, here it forms the basis of an illustrative example.

The nominal value of the bubble rise time is 1 minute, however it will be assumed that it could vary over the range 0.9 to 1.1 minutes. Since this parameter only affects how quickly changes in the air to the circuit affect the separation, its variance would not be expected to change the steady-state performance of the circuit. This was verified by evaluating the steady-state transfer function matrix at the nominal optimum for each circuit at the allowable extremes of τ_{RT} . In each case, the result was unchanged.

The first step in calculating the uncertainty description was the evaluation of the input-output transfer function matrix over the frequency range 0.001 to 10 rad/min. These were calculated using the linearised state-space representation produced for each circuit, for $\tau_{RT} = 0.9$ and 1.1, using SPEEDUP. At each frequency, the additive (L_A) and multiplicative output (L_O) errors were calculated.

$$L_{A,j,\omega,RT} = G_{j,RT}(i\omega) - G_{j,RT=1}(i\omega) \quad (6.30)$$

$$L_{O,j,\omega,RT} = (G_{j,RT}(i\omega) - G_{j,RT=1}(i\omega)) G_{j,RT=1}^{-1}(i\omega) \quad (6.31)$$

for $j = \text{Circuit 1, 2, 4 and 8}$
 $\omega = 0.001 \text{ to } 10 \text{ rad/min}$
 $RT : \tau_{RT} = 0.9, 1.1$

For use in controller design calculations, these "exact" uncertainty matrices need to be described using a bound on the maximum singular value of L_A or L_O (see Equations 5.65 and 5.67). Therefore, the greater of the two maximum singular values calculated using $\tau_{RT} = 0.9$ and 1.1 for L_A , were found at each frequency. Similar maxima were also found for L_O . Representative plots of these values are given in Figure 6.14, for Circuit 1.

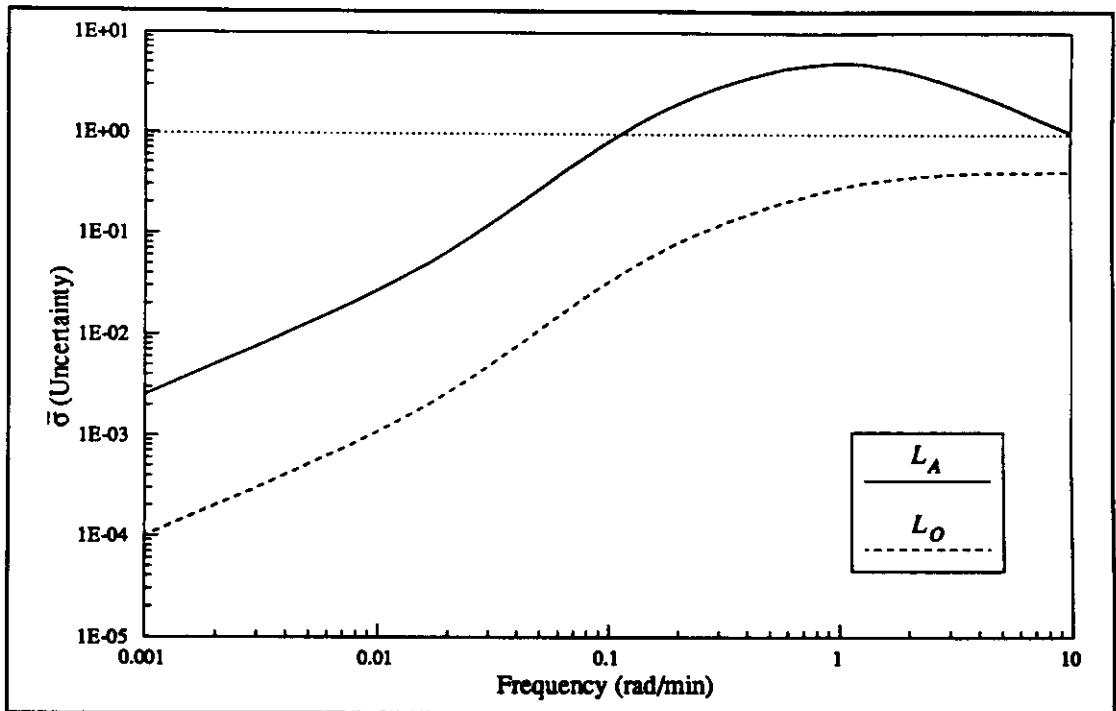


Figure 6.14 - L_A cf. L_O , 10% Rise Time Uncertainty, Circuit 1

The H_∞ controller design procedure of Chiang and Safanov (1988) can be used for either additive or multiplicative output uncertainty. For this example, the multiplicative output description was chosen (i.e. using the maximum singular value of L_o described above, the greater of the $\tau_{RT} = 0.9$ and 1.1 values at each frequency), as in practice it proved to be easier to "fit" for use in controller design.

The bounding transfer function, $W_3(s)$ (see Equation 5.107), is used as a scalar value multiplying the identity matrix of the correct dimensions. The appropriate W_3 transfer functions were determined iteratively, so that when plotted their magnitude provided a bound on the maximum singular value of L_o . Figure 6.15 shows the magnitude of these functions, and the singular values that they bound. The bounding functions are given below.

$$\text{Circuit 1} \quad W_3 = \frac{0.4142s}{(s + 1)} \quad (6.32)$$

$$\text{Circuit 2} \quad W_3 = \frac{0.4170s}{(s + 1)} \quad (6.33)$$

$$\text{Circuit 4} \quad W_3 = \frac{2s}{(0.001s + 1)} \quad (6.34)$$

$$\text{Circuit 8} \quad W_3 = \frac{1.6s}{(0.001s + 1)} \quad (6.35)$$

The performance specification placed on the design of the controllers for the rise time uncertainty, made use of the unequal weighting technique presented in Section 6.2.4. Again, this means that good control of the *Recovery* will be sacrificed to improve the disturbance rejection on the *Grade* output. With the same weighting value of 0.001, the same "shape" of performance transfer function weighting could be used for each circuit as had been used for the no uncertainty case, (Section 6.2.5, Equation 6.29).

$$W_1^{-1} = \gamma \frac{0.05s + 0.0001}{0.01s + 1} \quad (6.29)$$

However, with "real" uncertainty included, the γ values found during the controller design

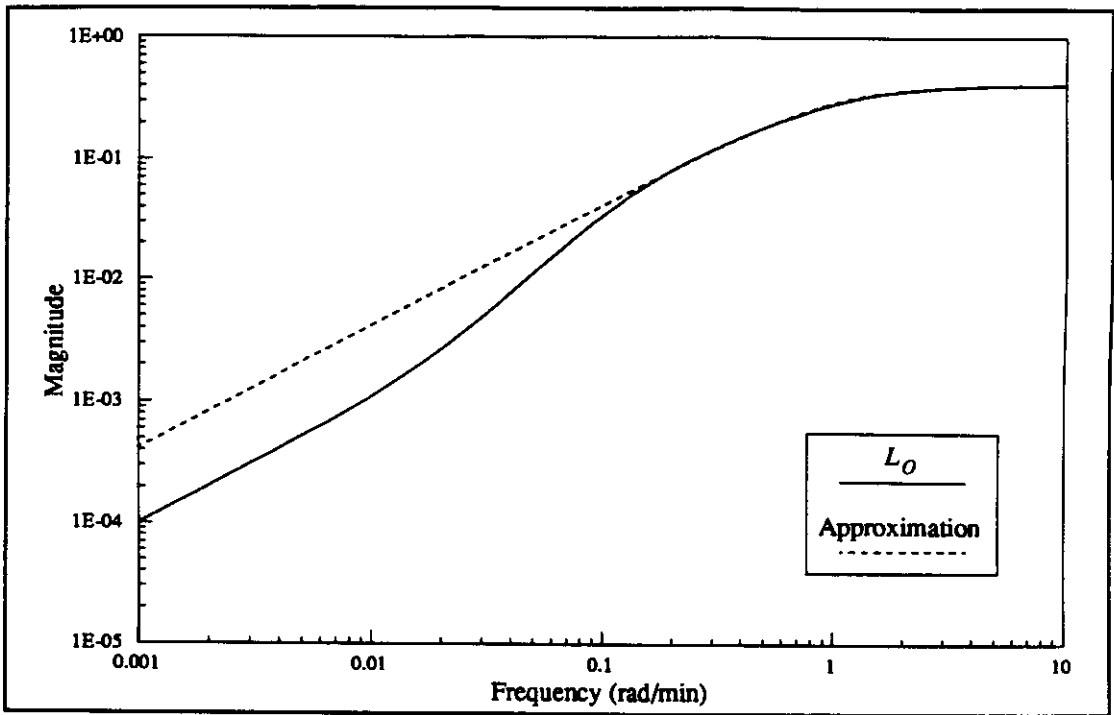


Figure 6.15a - L_o and W_3 Comparison, 10% Rise Time Uncertainty, Circuit 1

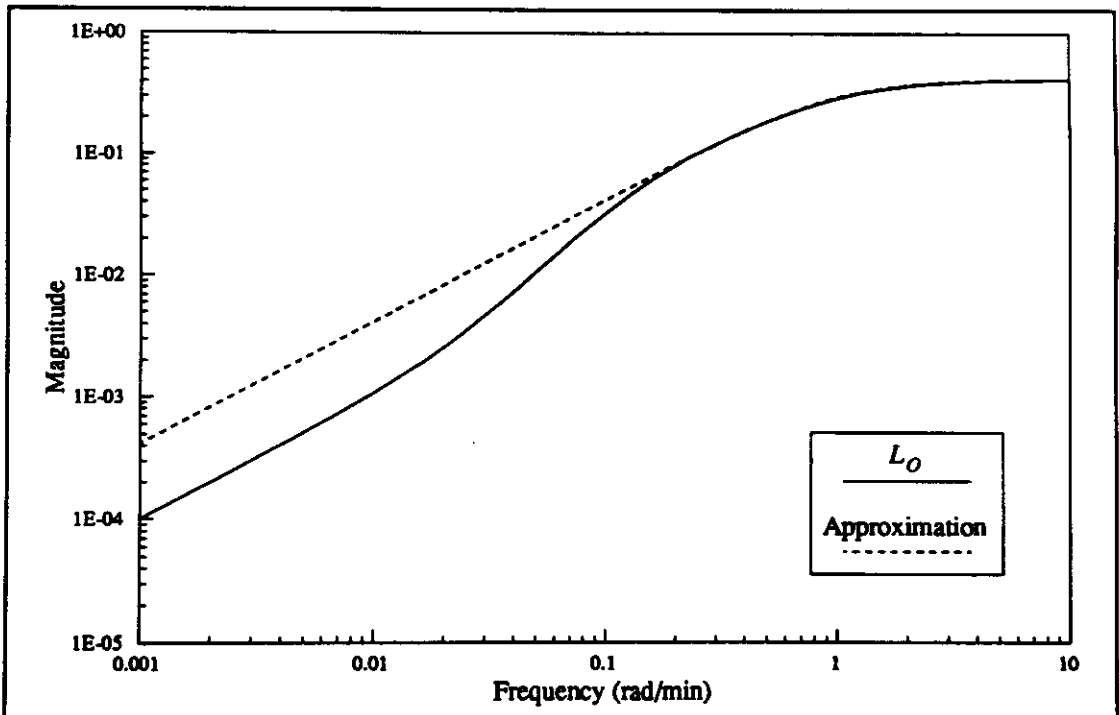


Figure 6.15b - L_o and W_3 Comparison, 10% Rise Time Uncertainty, Circuit 2

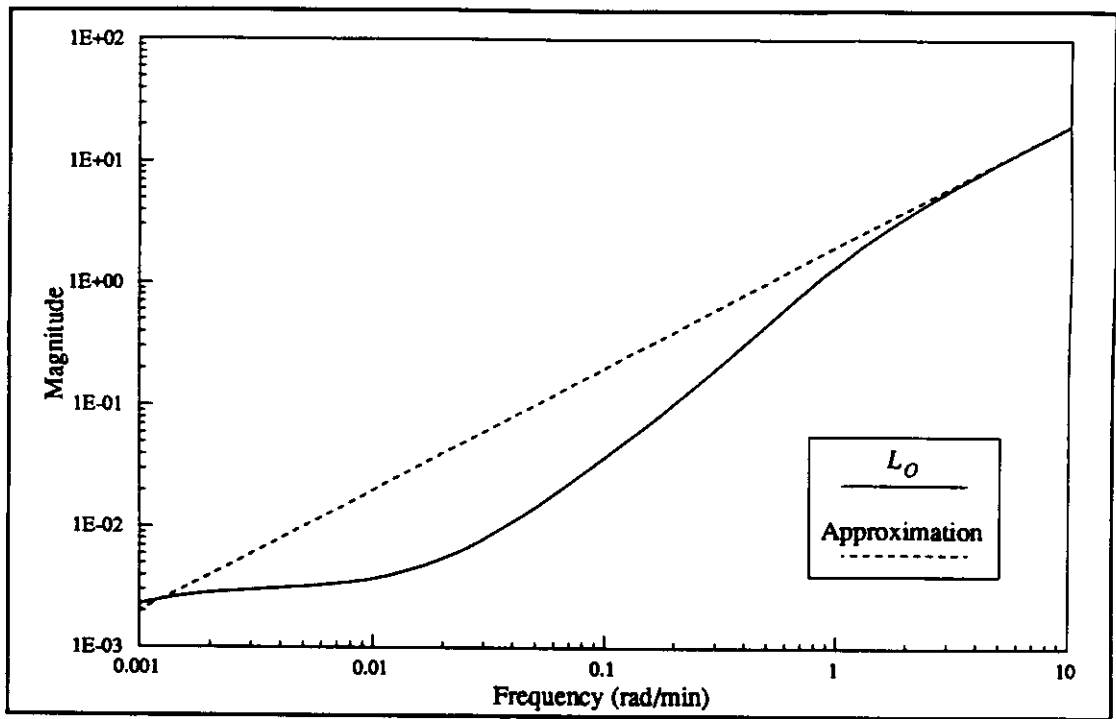


Figure 6.15c - L_O and W_3 Comparison, 10% Rise Time Uncertainty, Circuit 4

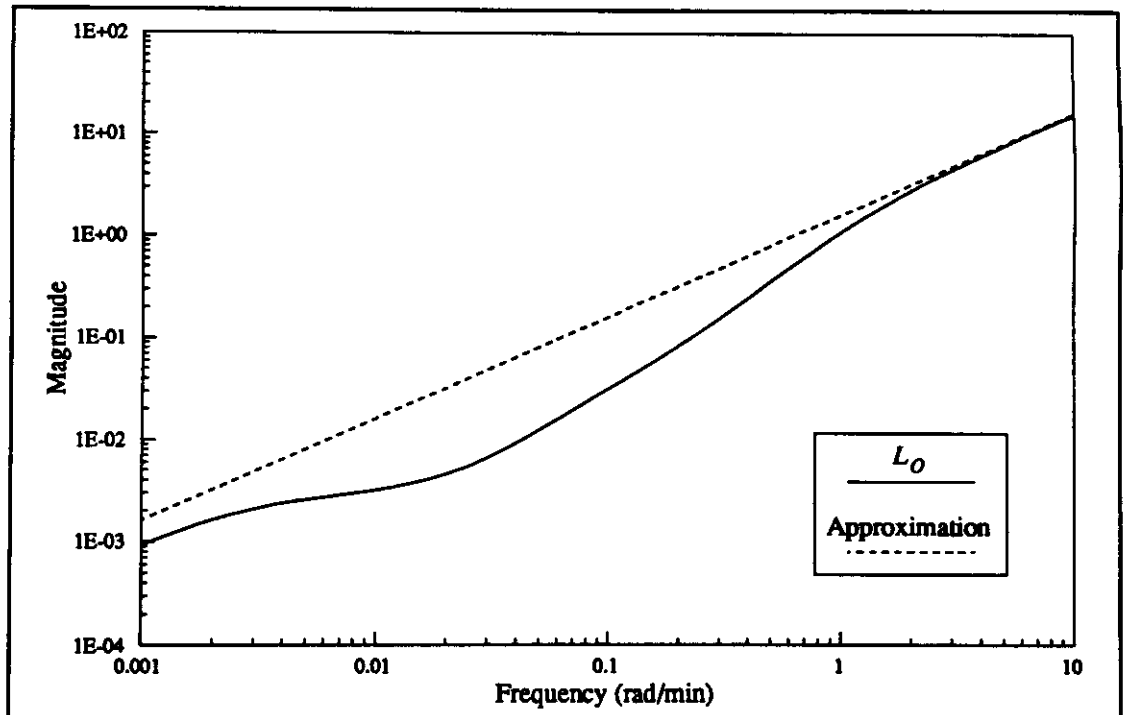


Figure 6.15d - L_O and W_3 Comparison, 10% Rise Time Uncertainty, Circuit 8

are generally larger than for the no uncertainty case, as shown in Table 6.12. This indicates that as measured by the 2-norm, the performance achieved with rise time uncertainty was not as good as for the no uncertainty case. That is, there will be less disturbance rejection.

Circuit	1	2	4	8
γ (Rise Time)	2.86	12.5	40.0	118
γ (No Uncertainty)	2.5	12.5	10	10

Table 6.12 - Rise Time Uncertainty Controller Design, γ Values

By incorporating "real" uncertainty, the controller design could now be limited by either the performance specification or the uncertainty bound. Limitations on the latter can be observed by plotting the singular values of the complementary sensitivity function (Equation 5.107). Figure 6.16 shows plots of the singular values of the sensitivity function (E) and the complementary sensitivity function (H), with their corresponding weightings, for each circuit. Again, because of the unequal weightings on the performance specification, one singular value of the sensitivity function exceeds W_1^{-1} . In all cases, the H_∞ controller has been designed to be limited by both performance and uncertainty, that is, achieving the best performance possible for the uncertainty description.

In the next section, these controllers will be used to determine the maximum possible input and output magnitudes that could occur, for the given disturbances for all plants contained within the uncertainty description.

6.3.2 Input-Output Magnitudes

Disturbances in the feed rate and feed grade result in open-loop variation in the outputs, and in both input and output variation for the closed-loop case. It would be expected that the uncertainty being considered here, bubble rise time, should not change the way these disturbances affect the open-loop process outputs. That is, that the open-loop output magnitudes should be independent of the bubble rise time. This was verified by evaluating the disturbance transfer functions (G_d), for the extremes of the uncertainty ($\tau_{RT} = 0.9, 1.1$ minutes) over the frequency range being studied. The relative error from the

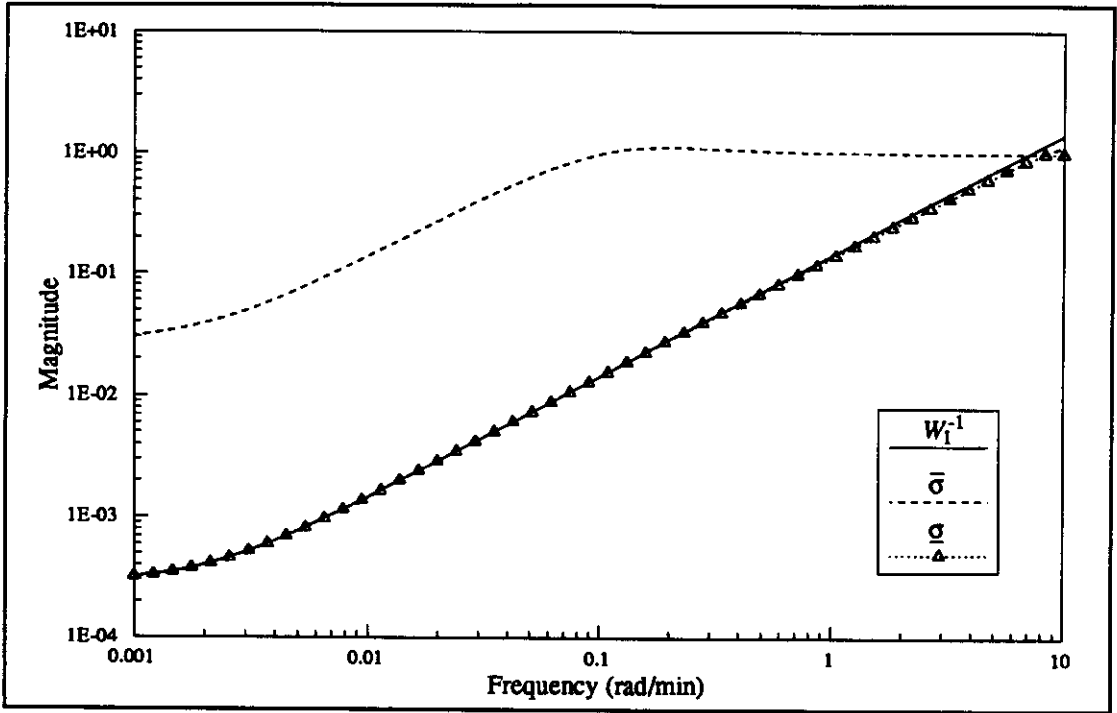


Figure 6.16a - Circuit 1, H_∞ Controller Design, Sensitivity Function

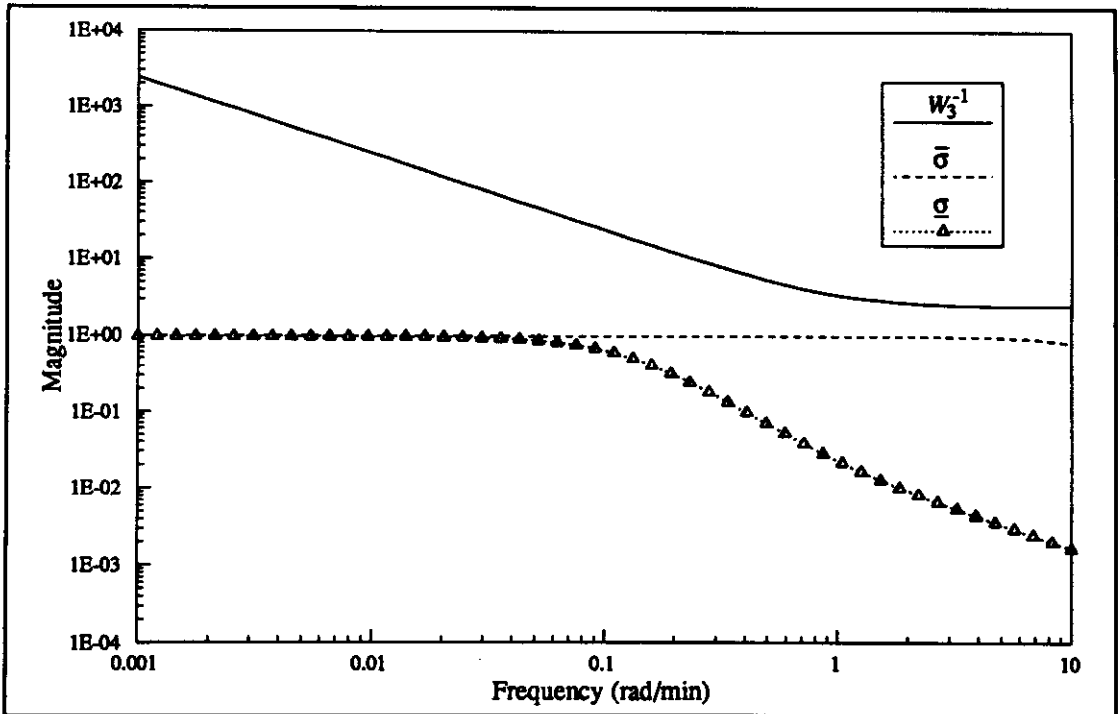


Figure 6.16b - Circuit 1, H_∞ Controller Design, Complementary Sensitivity Function

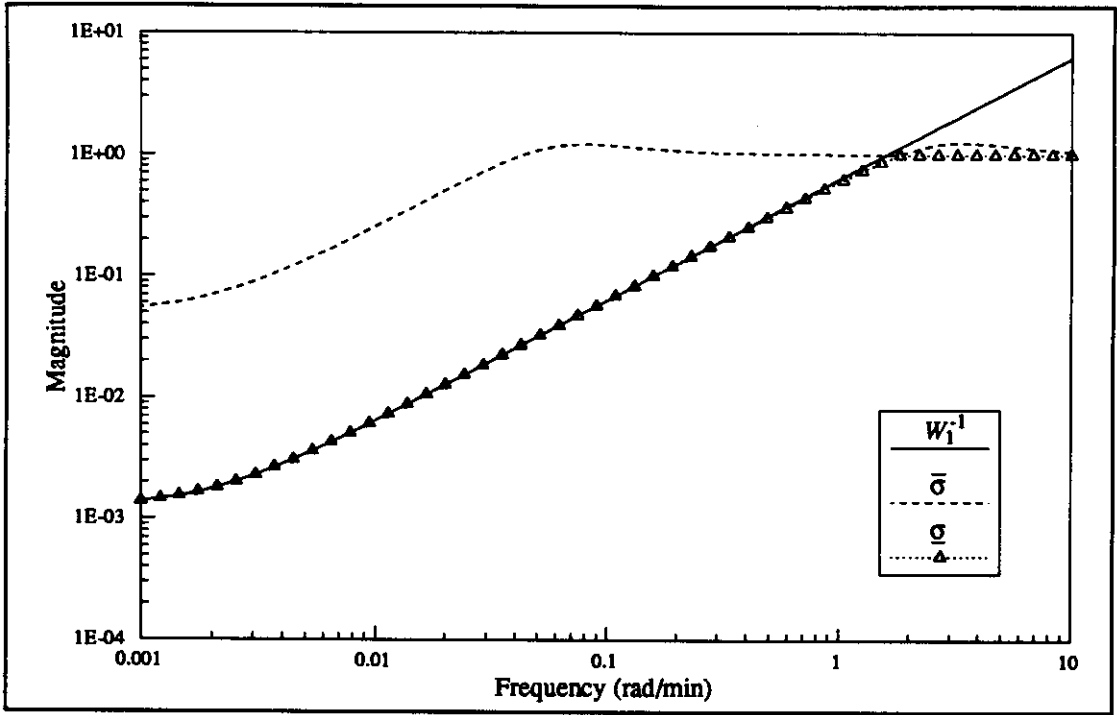


Figure 6.16c - Circuit 2, H_∞ Controller Design, Sensitivity Function

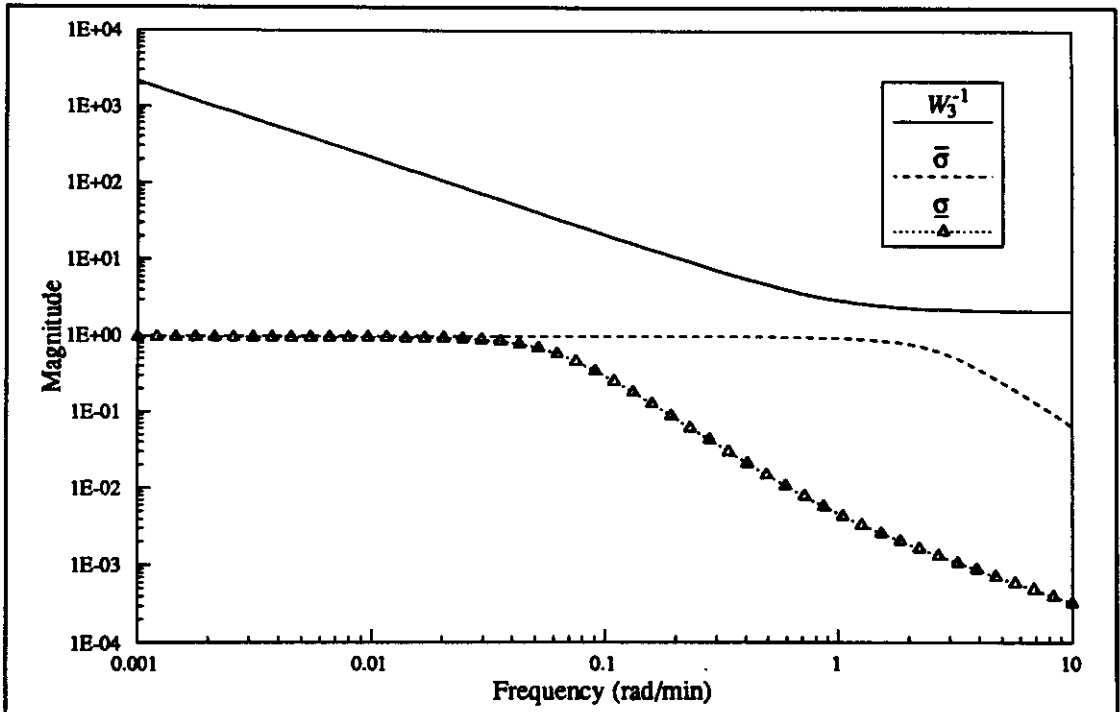


Figure 6.16d - Circuit 2, H_∞ Controller Design, Complementary Sensitivity Function

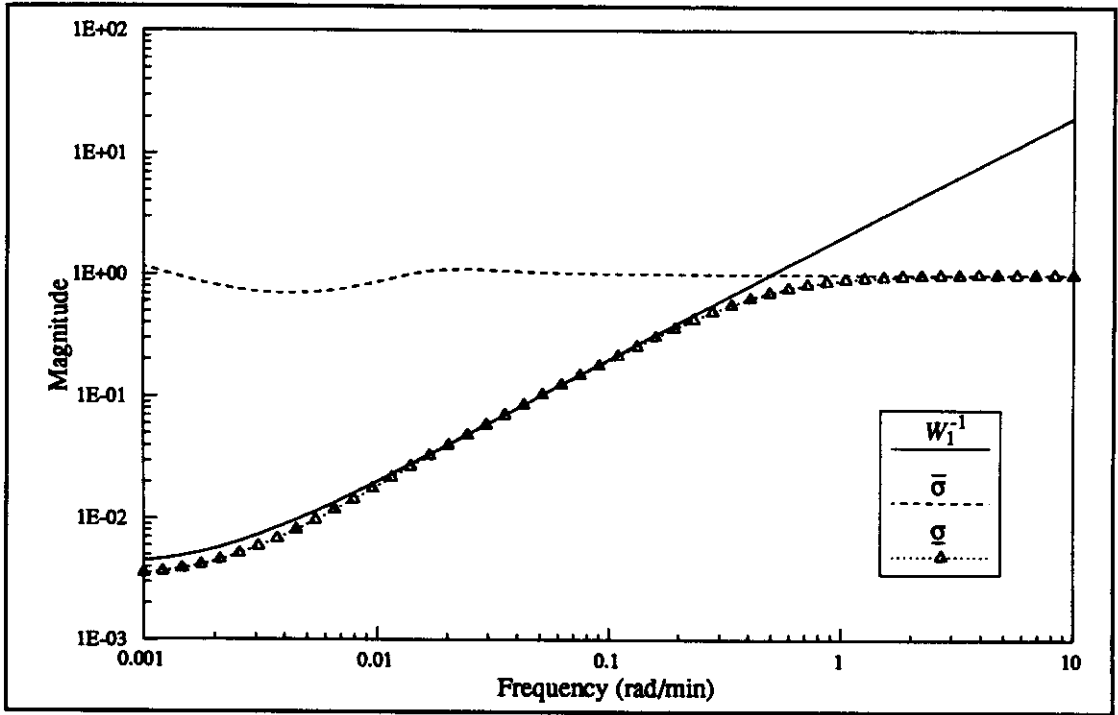


Figure 6.16e - Circuit 4, H_∞ Controller Design, Sensitivity Function

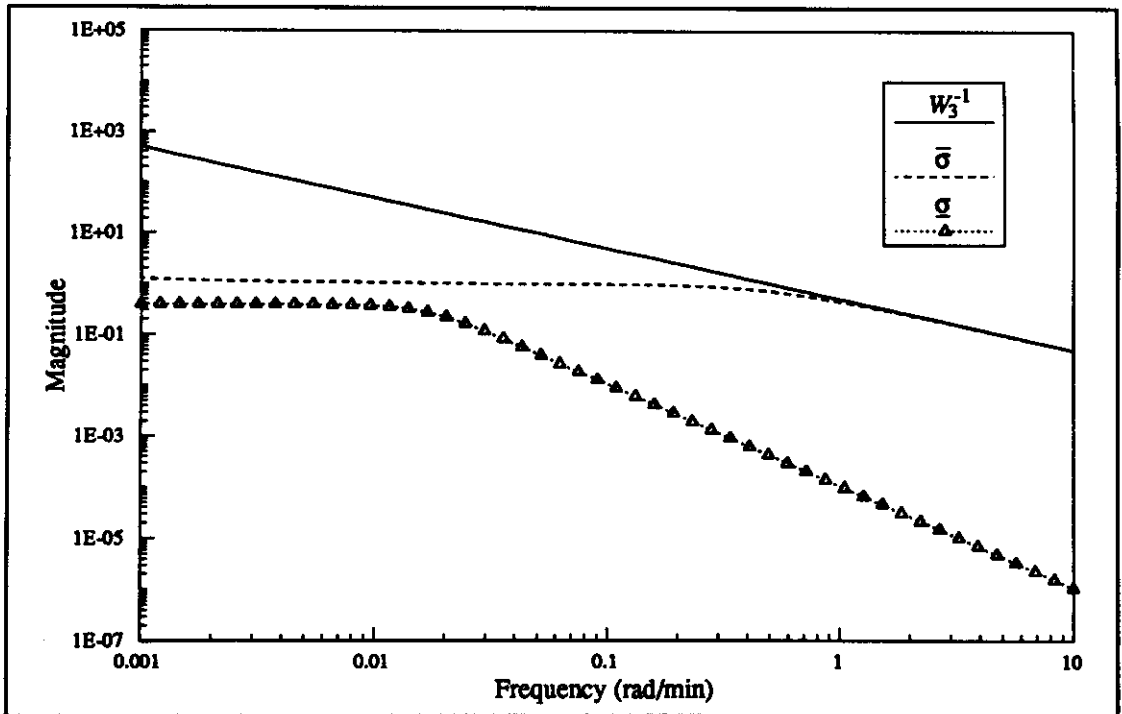


Figure 6.16f - Circuit 4, H_∞ Controller Design, Complementary Sensitivity Function

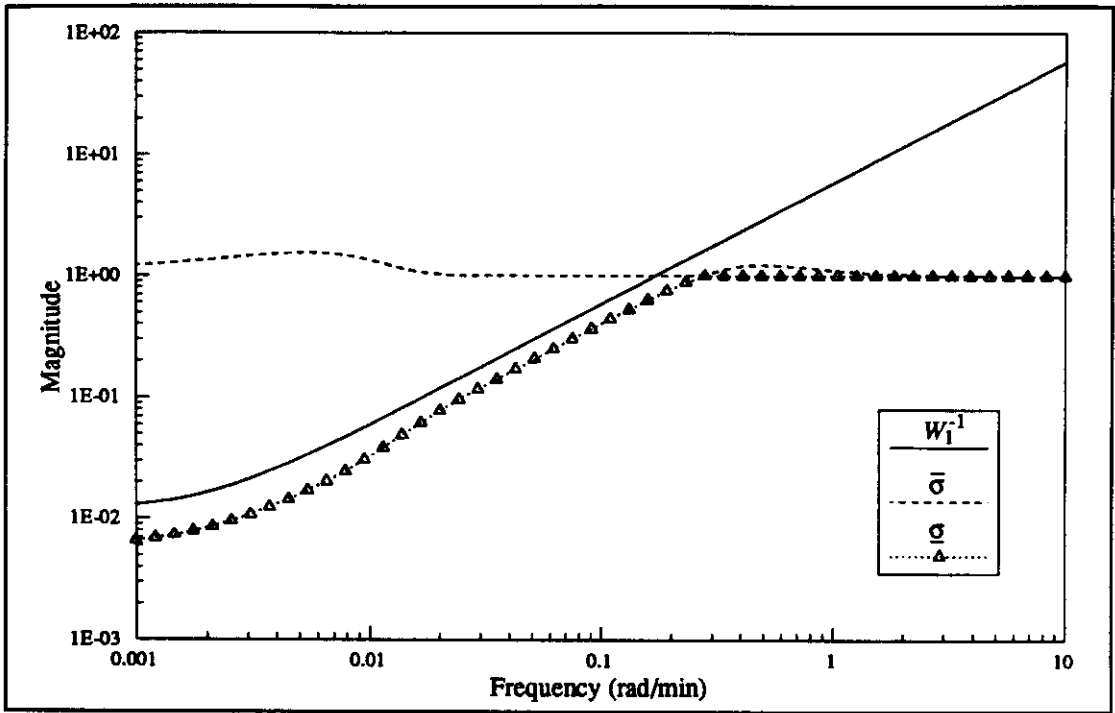


Figure 6.16g - Circuit 8, H_∞ Controller Design, Sensitivity Function

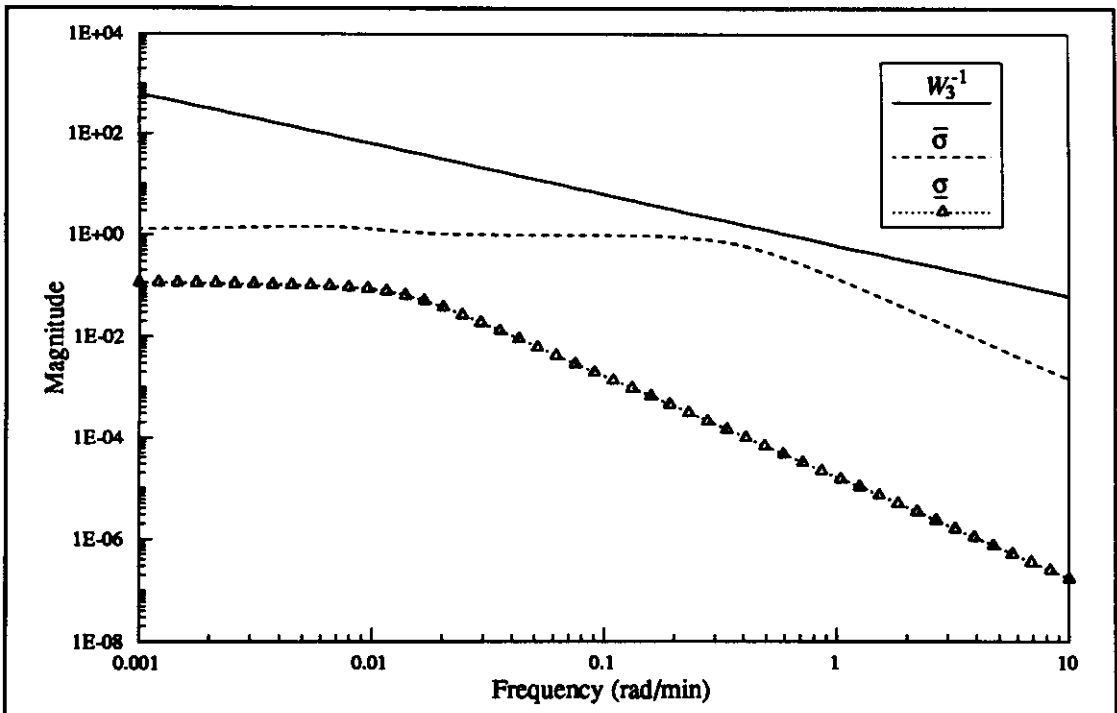


Figure 6.16h - Circuit 8, H_∞ Controller Design, Complementary Sensitivity Function

nominal transfer function, on an element by element basis, was less than 10^{-12} .

The invariance of the open-loop magnitudes with uncertainty, led to a simpler analysis than the more general case, where the open-loop magnitudes could vary with uncertainty, as occurs in Section 6.4. Therefore, the open-loop magnitudes appropriate for use in this section are those given previously, in Table 6.5.

This still leaves the evaluation of the closed-loop magnitudes - a more complex task than when there is no uncertainty. In this case, the process could take on any value in the uncertainty description, and hence the response to control actions will be less predictable.

As discussed previously, the H_∞ controller design procedure guarantees robust stability, but only nominal performance. This means that the specification placed on the sensitivity function is only met for the nominal plant. If the controller designed could ensure robust performance (i.e. for all models within the uncertainty description), then this could aid in the determination of the maximum closed-loop output magnitudes. However, the achievement of robust performance requires a controller based on μ -Synthesis principles (Section 5.2.3; Doyle, 1987) and as will be shown below, even then the singular value measurements associated with robust performance controllers can lead to very poor estimates of the magnitudes of the individual elements of the input and output vectors, as required for the Optimal Operation Algorithm.

The robust performance, singular value based, magnitude analysis will be demonstrated for a 2×2 example, although the results are applicable for all sizes of system. The closed-loop outputs y are related to the disturbances appearing at the output (d) by the sensitivity function matrix E .

$$y = E d \quad (6.36)$$

A singular value decomposition of this equation gives;

$$y = [u_1 \ u_2] \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} [v_1 \ v_2]^H d \quad (6.37)$$

Taking the upper limit of the product of the right singular vectors and the disturbance components (i.e. $|v_{1,j}| \leq 1$, etc.) gives an upper bound on the magnitude of y .

$$|y| \leq [u_1 \ u_2] \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \sum_{i=1}^{n-2} |d_i| \quad (6.38)$$

Further, assuming that all singular values are equal to the maximum ($\bar{\sigma}$) gives,

$$|y| \leq [u_1 \ u_2] \begin{bmatrix} \bar{\sigma} \Sigma |d_i| \\ \bar{\sigma} \Sigma |d_i| \end{bmatrix} \quad (6.39)$$

Similarly, since the size of the elements of the left singular vectors cannot be ascertained *a priori* (i.e. $|u_{1,i}| \leq 1$, etc.), we find an upper limit on the output components, $|y_i|$ as,

$$|y_i| \leq n \bar{\sigma} \Sigma |d_i| \quad (6.41)$$

This is saying that the upper bound on the magnitude of each element in the output vector is the product of the size of the output vector (number of elements, n), the maximum singular value of the sensitivity function and the sum of the absolute values of the elements of the disturbance vector. Clearly, this is generally a very conservative estimate of how large the magnitude of the y_i components of the output vector may be. Therefore, even if a robust performance controller design had been performed, this "short cut" (singular value) calculation of the individual input and output magnitudes would be of limited usefulness in the Optimal Operation Algorithm context.

An alternative method has therefore been adopted for these calculations. An optimisation problem (shown as Equation 6.42) was set up to find the maximum individual input and output magnitudes within the upper bound imposed by the uncertainty constraint.

This optimisation problem, to find the elements of L_o (i.e. g_d), was solved using the optimisation routines within the MATLAB package at different frequencies over the range being considered, for both disturbances. The solution is thus the exact value of the maximum magnitude that an output element could achieve for any process model lying within the uncertainty description. A similar procedure was used to calculate the maximum input magnitudes. In a similar fashion to the no uncertainty case, these calculations were performed for a range of values of the partial control factor (k).

Table 6.13 gives the maximum *Grade* and input magnitudes for the 5% disturbance case,

$$\begin{aligned}
 & \text{Maximise } |y_j| \\
 & \quad g_k \\
 \text{where} \quad & |y_j(i\omega)| = \left| (EG_d d(i\omega))_j \right| \\
 & E = (I + GC)^{-1} \\
 & G = (I + L_o)\tilde{G} \\
 & L_o = \begin{bmatrix} g_1 + g_2 i, & g_3 + g_4 i \\ g_5 + g_6 i, & g_7 + g_8 i \end{bmatrix} \\
 & \bar{\sigma}(L_o) \leq W_3(i\omega)
 \end{aligned} \tag{6.42}$$

for the "full" control ($k = 1$) case. Comparison with Table 6.10 shows that generally these maximum magnitudes are greater than for the no uncertainty case. This is the expected result, as variance in the process leads to some models which will not be controlled as well as others, and situations where greater input magnitudes are necessary. Figure 6.17 is a comparison between the rise time and no uncertainty cases, for the *Grade* and input magnitudes with "full" control ($k = 1$) for Circuits 1 and 8. The frequency dependent results shown in Figure 6.17 display the same trend as the maximum magnitudes given in Table 6.13. The bold elements in Table 6.13 are the maximum magnitudes possible from either disturbance for each variable.

A careful comparison between Table 6.10 and Table 6.13 shows that there are a few cases where the no uncertainty case had slightly larger magnitudes than the case with rise time uncertainty. The tables show that these cases are the Circuit 2 *Grade* magnitudes (feed rate disturbance, < 0.5% greater), and the Circuit 4 input magnitudes for the feed rate disturbance ($\approx 30\%$ greater). In addition, the frequency plots indicate a number of similar occasions, although there are reasonable explanations for these apparent discrepancies.

A smaller number of frequencies were evaluated for an uncertain case (15 of 50 points) than for a no uncertainty case, to reduce calculation time for these more complex optimisation problems. This meant that the true maximum for the frequency range may not have occurred near any of the frequencies used. The frequency plots given for Circuit 8 (Figure 6.17d) are illustrative of this. Also, notice that at frequencies above about 0.015 rad/min, the no uncertainty inputs are greater than for the rise time uncertainty case. This

Circuit	$ y_{Gf,CL} $	$ y_{Gs,CL} $
1	7.195×10^{-3}	7.724×10^{-3}
2	3.028×10^{-2}	3.243×10^{-2}
4	2.309×10^{-2}	1.064×10^{-1}
8	7.798×10^{-2}	2.035×10^{-1}

Table 6.13a - Closed-Loop Magnitudes (Outputs), Rise Time Uncertainty

Circuit	$ \mu_{Af,CL} $	$ \mu_{As,CL} $	$ \mu_{Wf,CL} $	$ \mu_{Ws,CL} $
1	5.221×10^{-2}	5.254×10^{-2}	3.357×10^{-1}	3.489×10^{-1}
2	5.414×10^{-2}	5.498×10^{-2}	8.215×10^{-1}	8.405×10^{-1}
4	5.238×10^{-2}	3.453×10^{-1}	9.760×10^{-2}	5.752×10^{-1}
8	7.798×10^{-2}	2.035×10^{-1}	1.903×10^{-1}	3.906×10^{-1}

Table 6.13b - Closed-Loop Magnitudes (Inputs), Rise Time Uncertainty

is due to the controller being "less aggressive" at higher frequencies, where the uncertainty is greater. This is felt to be the likely cause of the apparent discrepancy in the Circuit 4 results. The small discrepancy in the Circuit 2 *Grade* (possibly due to the fewer points evaluated) will not greatly affect the results as the error is small, while it is likely that the Circuit 4 results are valid.

These closed-loop maxima were then used to determine the optimal achievable operation for the four circuits when the bubble rise time is uncertain.

6.3.3 Optimal Plant Performance

The optimal recoveries presented in Section 6.2.4 and 6.2.5 for the case where no uncertainty was considered, were calculated using the disturbance which caused the greatest input and output magnitudes, that is, either feed rate or feed grade. These results were therefore the "best worst" recoveries. That approach has been extended in this

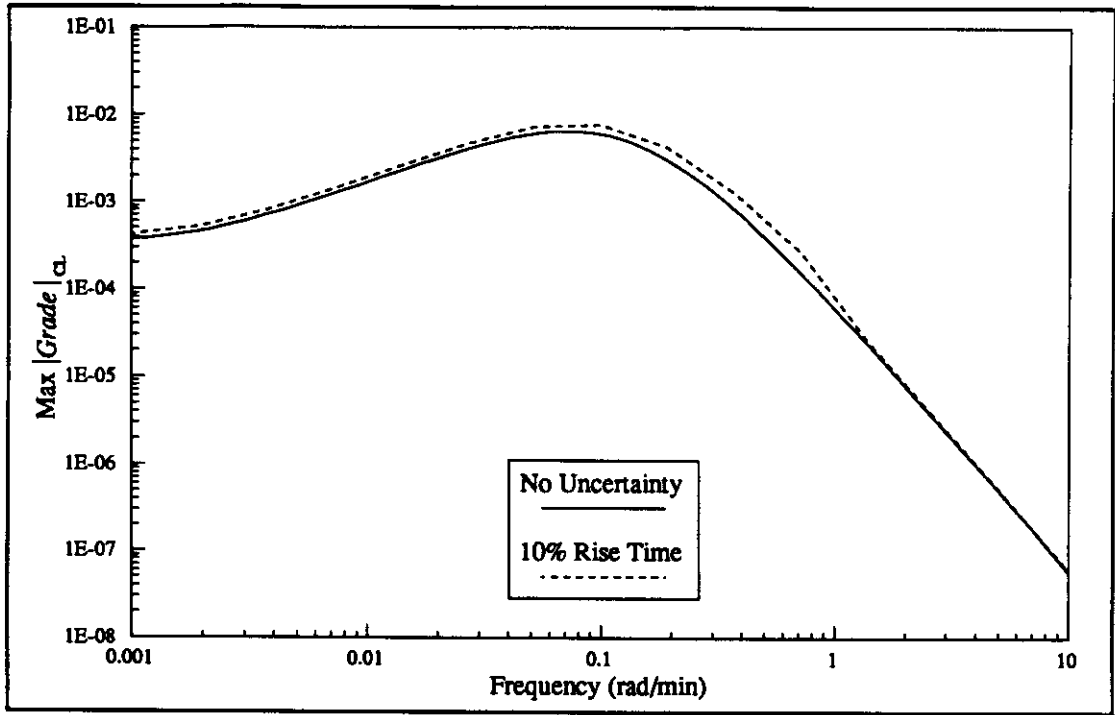


Figure 6.17a - Circuit 1, $|Grade|$, No Uncertainty *cf.* 10% Rise Time Uncertainty

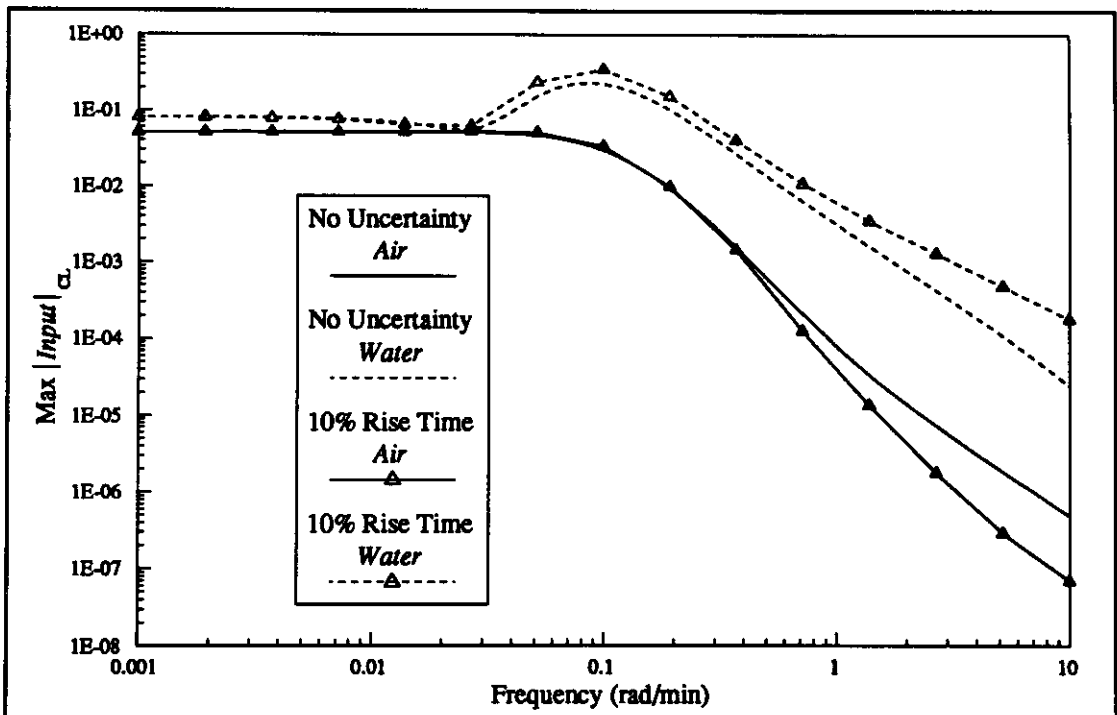


Figure 6.17b - Circuit 1, $|Inputs|$, No Uncertainty *cf.* 10% Rise Time Uncertainty

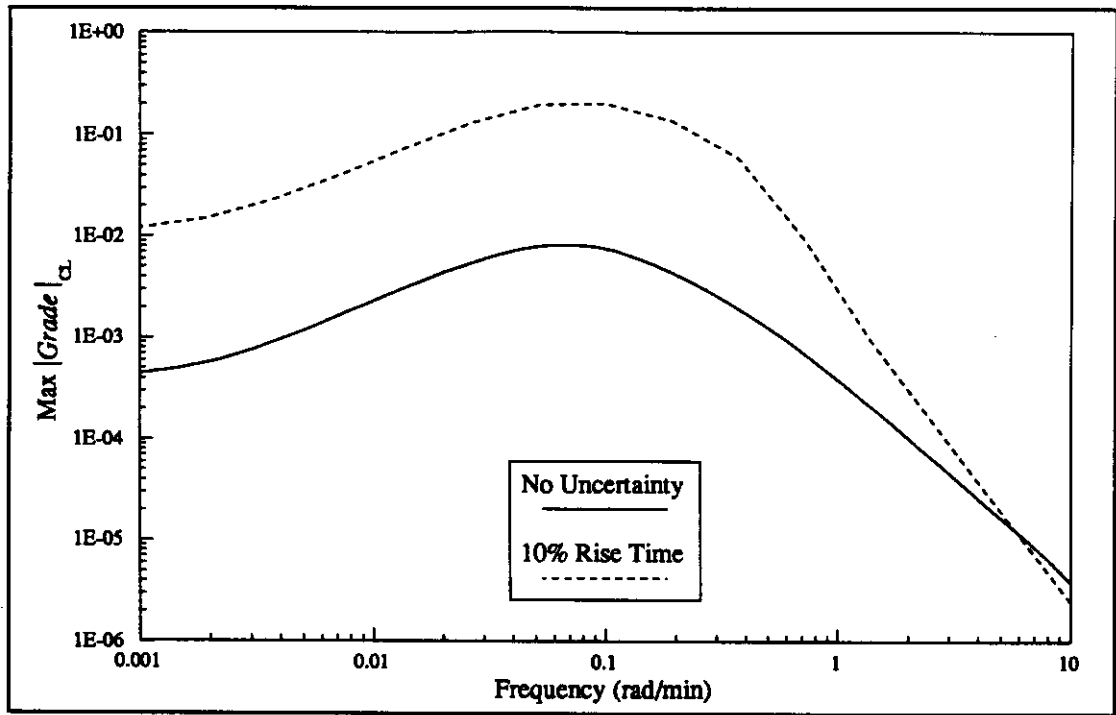


Figure 6.17c - Circuit 8, $|Grade|$, No Uncertainty *cf.* 10% Rise Time Uncertainty

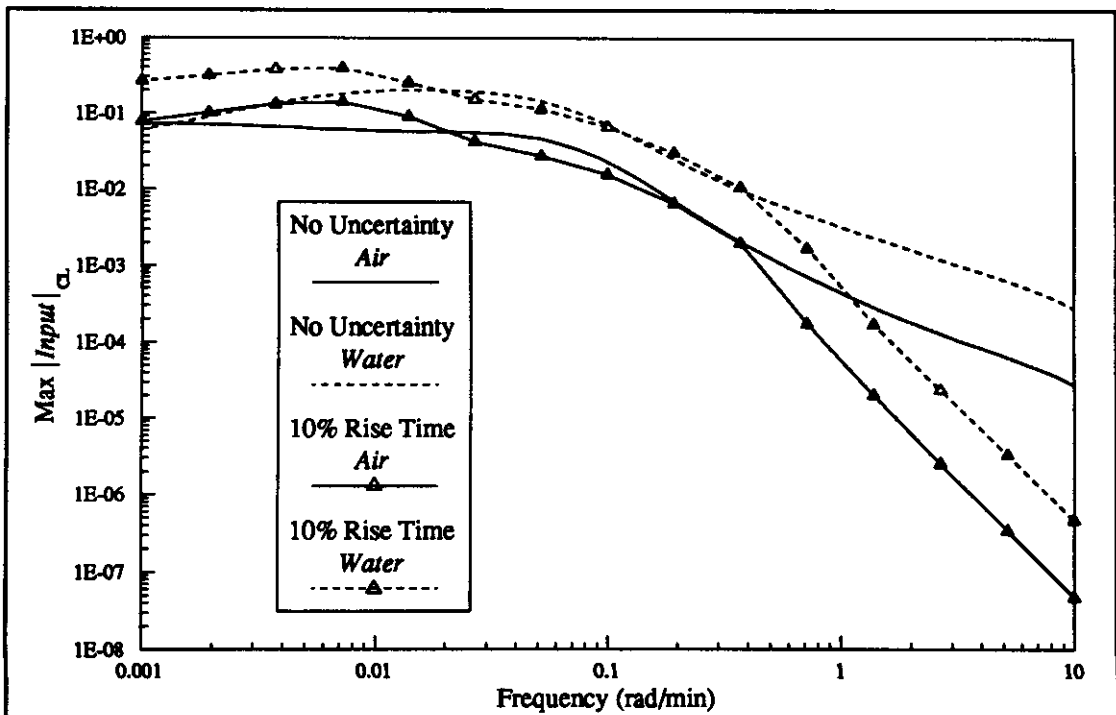


Figure 6.17d - Circuit 8, $|Inputs|$, No Uncertainty *cf.* 10% Rise Time Uncertainty

section by considering the maximum input and output magnitudes that could occur for any plant in the uncertainty description derived from the bubble rise time uncertainty. The optimal recoveries are therefore the best performance that will be achieved by the worst uncertain plant. Using this figure for comparison between the different designs guarantees a lower limit on the performance of each circuit. In practice, the chosen circuit should never perform worse than these results indicate.

For this example, where the uncertainty only affects the dynamics of the process, the Optimal Operation Algorithm (Equation 6.28) is unchanged. The only difference is in the calculation of the partial control functions, which are determined using the optimisation procedure outlined in Equation 6.42. As for the no uncertainty case, the initial calculation of the optimal performance was performed using simple approximations to the partial control functions. These results clearly indicated that the optima for Circuits 1, 2 and 8 lay at the extremes of k , whereas Circuit 4 was possibly optimal at an intermediate value. Therefore, more accurate approximations to the partial control functions were used for the final calculations for Circuit 4. For k over the range 0.01 to 1, the cubic form shown in Equation 6.43 was used (with the coefficients given in Table 6.14) to represent the partial control functions, that is, input and output magnitudes as a function of the partial control factor.

$$\log(\Delta\psi) = a\xi^3 + b\xi^2 + c\xi + d \quad (6.43)$$

where $\xi = \log(k)$
 $\psi = \text{Grade, Air, Water}$

ψ	a	b	c	d
<i>Grade</i>	1.0070×10^{-2}	-4.7622×10^{-2}	-9.5211×10^{-1}	-2.2423
<i>Air</i>	1.5671×10^{-2}	1.3968×10^{-1}	5.6415×10^{-1}	-1.0613
<i>Water</i>	1.4675×10^{-2}	1.2079×10^{-1}	4.5832×10^{-1}	-5.5149×10^{-1}

Table 6.14 - Circuit 4, Partial Control Function Coefficients

Once again, the optimal operating point was evaluated for four disturbance levels, 5, 10, 20 and 40%. The results are given in Table 6.15, where the no uncertainty results are also

included for comparison. The optimal recoveries are plotted in Figure 6.18 against the size of the disturbance.

The key result to note here is that (unlike the no uncertainty case) there is only one occasion where the optimal operation is not open-loop. That is for the five percent disturbance level for Circuit 4, where a partial control factor of 0.52 is optimal. This reduction in the ability to control the plant when uncertainty is considered is to be expected. For the no uncertainty case, when the high performance weighted controller was used, some control was found to be optimal in some cases. Here, however, when process uncertainty causes the input and output magnitudes to generally become greater, control is once again "too expensive". That is, input manipulation requiring "backoff" from input constraints, can lead to a significant reduction in the objective function.

In the next section, the effect of bubble rise time uncertainty will be investigated in more detail by performing the analysis over a wider range of uncertainties.

6.3.4 Effect of Uncertainty Level on Circuit 8

In the previous calculations, a 10% variation in bubble rise time was considered as an uncertain parameter in the process model. The corresponding uncertainty description was then used in the design of an H_{∞} controller, and in the calculation of the maximum input and output magnitudes likely to be encountered for the possible disturbances. It was observed that when this uncertainty was explicitly included in the optimal operation calculation, the amount of control action that was optimal was less than in the no uncertainty case.

By performing the optimal operation analysis for other levels of bubble rise time uncertainty, the effect of gradually introducing uncertainty can be analysed. The optimal operation was therefore also determined for 2 and 5% uncertainty levels for Circuit 8. Following the previous procedure, the maximum singular value of the multiplicative output uncertainty was calculated for each of the new uncertainty levels. Transfer functions were fitted to these plots and used in H_{∞} controller designs. The uncertainty magnitudes and the controller design results are shown in Figures 6.19 and 6.20, respectively. The transfer functions fitted to the uncertainty magnitudes are given in

No Uncertainty

Circuit 1	Recovery	Grade	Air	Water	k	Recovery	k
$d_0 = 0\%$	94.873	60	1.017	0.8	0	94.873	-
$d_0 = 5\%$	94.386	61.326	0.961	0.8	0	94.531	1
$d_0 = 10\%$	93.900	62.652	0.904	0.8	0	94.118	1
$d_0 = 20\%$	92.926	65.305	0.791	0.8	0	92.926	0
$d_0 = 40\%$	90.979	70.610	0.564	0.8	0	90.979	0
Circuit 2	Recovery	Grade	Air	Water	k	Recovery	k
$d_0 = 0\%$	94.953	60	1.014	0.8	0	94.503	-
$d_0 = 5\%$	94.503	61.324	0.958	0.8	0	94.054	0
$d_0 = 10\%$	94.054	62.649	0.902	0.8	0	93.154	0
$d_0 = 20\%$	93.154	65.298	0.790	0.8	0	93.154	0
$d_0 = 40\%$	91.355	70.596	0.566	0.8	0	91.355	0
Circuit 4	Recovery	Grade	Air	Water	k	Recovery	k
$d_0 = 0\%$	94.592	60	0.853	0.8	0	94.592	-
$d_0 = 5\%$	94.151	60.193	1.143	1.248	0.52	94.440	1
$d_0 = 10\%$	92.978	62.646	0.752	0.8	0	94.288	1
$d_0 = 20\%$	91.364	65.291	0.652	0.8	0	91.364	0
$d_0 = 40\%$	88.082	70.582	0.5	0.875	0	88.082	0
Circuit 8	Recovery	Grade	Air	Water	k	Recovery	k
$d_0 = 0\%$	95.040	60	0.888	0.8	0	95.040	-
$d_0 = 5\%$	94.407	61.326	0.836	0.8	0	94.700	1
$d_0 = 10\%$	93.773	62.651	0.784	0.8	0	94.361	1
$d_0 = 20\%$	92.507	65.302	0.679	0.8	0	93.797	0.006
$d_0 = 40\%$	89.855	70.605	0.5	0.871	0	89.855	0

Table 6.15 - Rise Time Uncertainty Optimal Operation Results

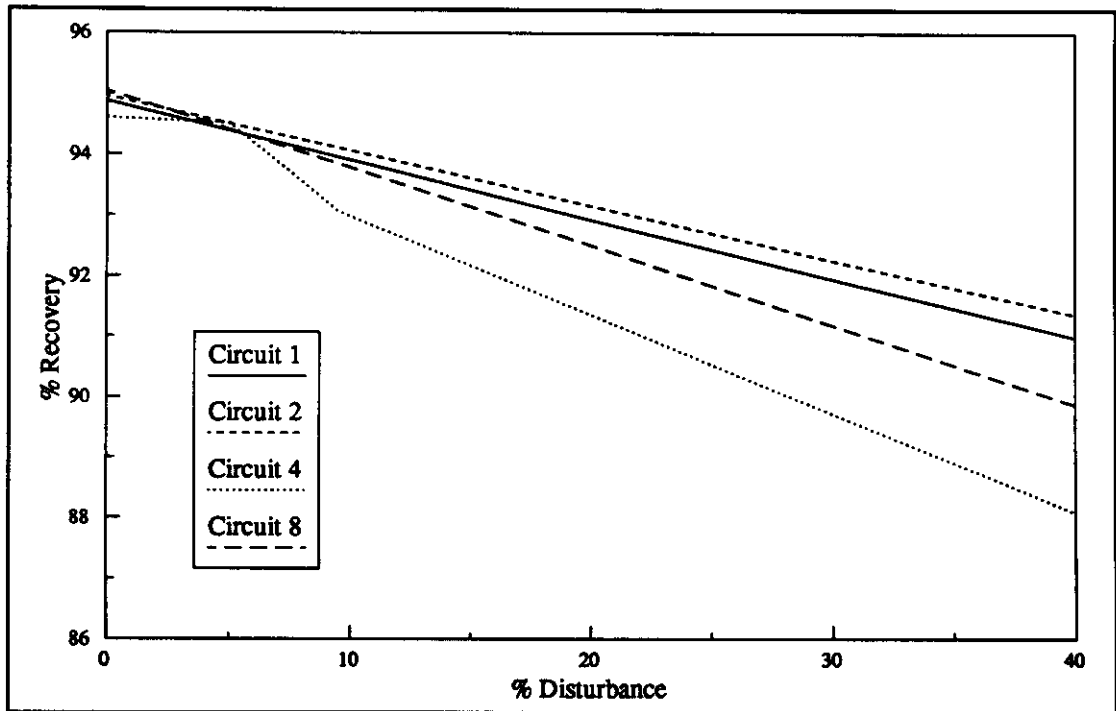


Figure 6.18 - Optimal Recoveries, 10% Rise Time Uncertainty

Equations 6.44 and 6.45.

$$\text{2\% Uncertainty} \quad W_3 = \frac{0.29s}{(0.001s + 1)} \quad (6.44)$$

$$\text{5\% Uncertainty} \quad W_3 = \frac{0.75s}{(0.001s + 1)} \quad (6.45)$$

The maximum closed-loop input and output magnitudes were then calculated using the optimisation procedure given in Equation 6.42 for a range of k values. Good approximations were found for these partial control functions for use in the Optimal Operation Algorithm. Disturbances over the range 5 to 40% were considered, with extra points evaluated where the partial control factor was found to change quickly with the disturbance level. The optimal recoveries are summarised in Table 6.16 and are plotted in Figure 6.21.

Before discussing the trends shown as the uncertainty increases, a note must be made about a detail of the calculation procedure. Careful analysis of the values given in this table for the no uncertainty results indicates a difference between the numeric values here, and those reported in earlier tables (6.10, 6.13). This is due to the extra care used in this portion of the analysis, where accurate representations of the partial control functions were used at all stages of the calculation procedure. It was not necessary to find such functions in the earlier work, as despite using only approximate functions, the values obtained for the optimal recoveries were quite close to those shown here, and thus did not affect the ranking of the circuits.

Figure 6.21 illustrates well the effect of uncertainty on the achievable optimal recovery for Circuit 8. The highest recoveries are obtained when there is no uncertainty, and this high level can be maintained to an almost 20% disturbance level. At a 2% uncertainty level, the optimal recovery is slightly less, due to the slightly poorer control. At 5% uncertainty, the optimal recovery is significantly lower, and "useful" control can only be maintained up to about a 15% disturbance level. At the 10% uncertainty level, the quality of control is so affected that the optimal operation of the circuit is always open-loop. In summary, as might be expected the best performance of this circuit can only be achieved when the process model is well known, and even then for only moderate disturbance

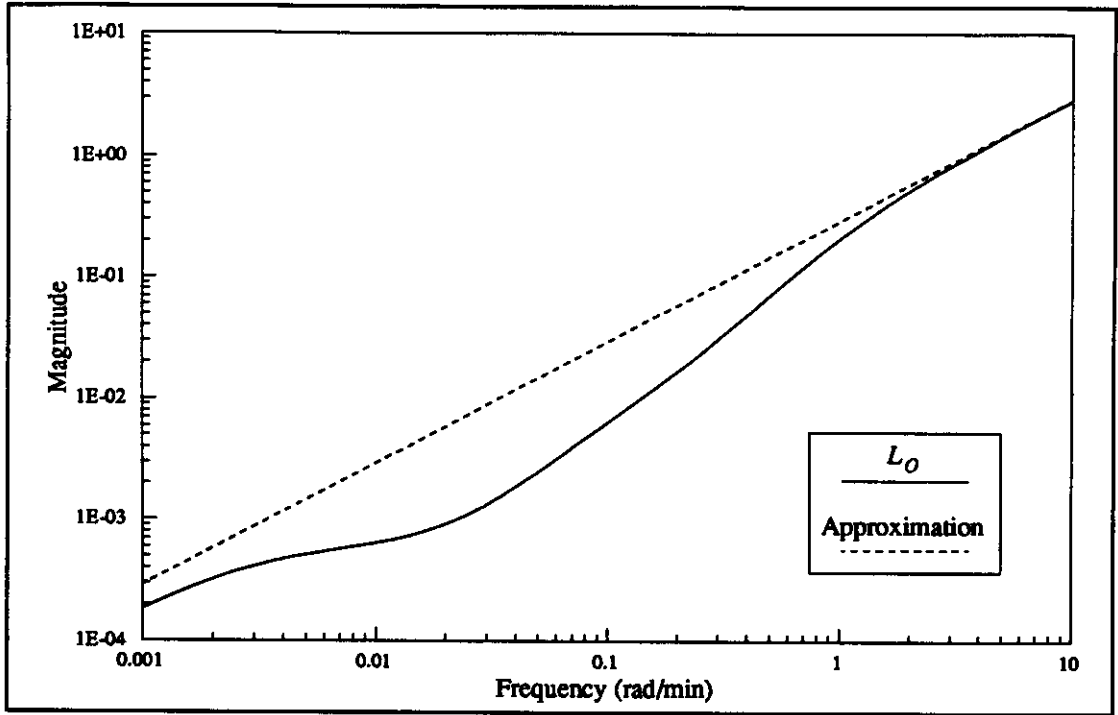


Figure 6.19a - L_O and W_3 Comparison, 2% Rise Time Uncertainty, Circuit 8

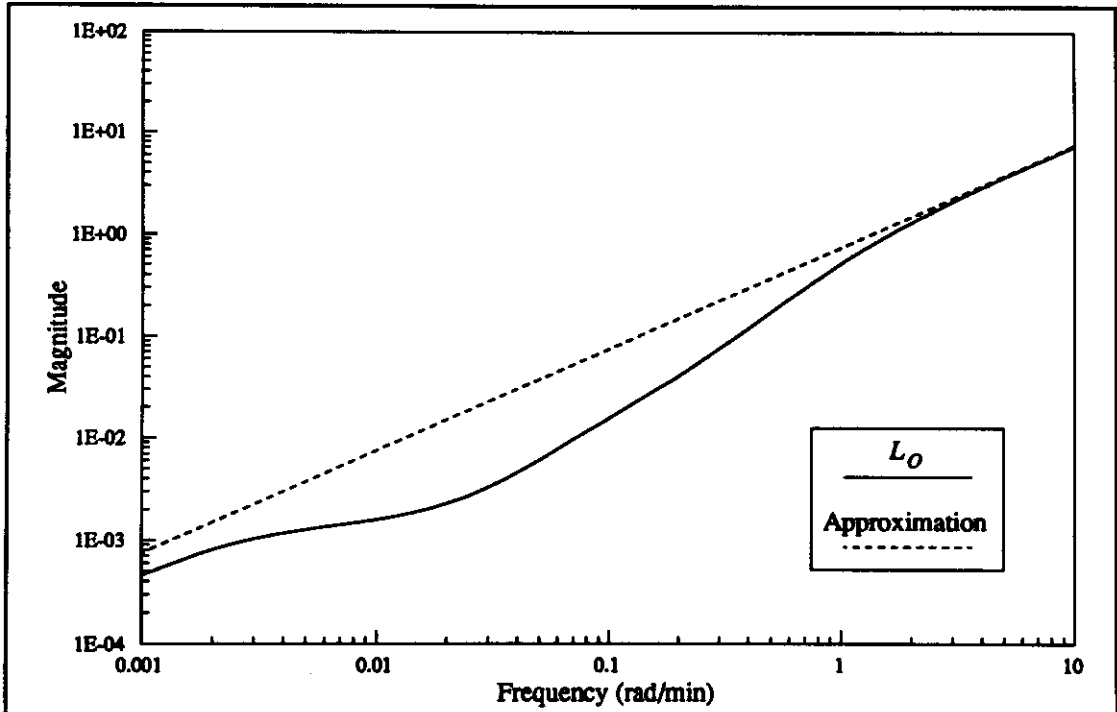


Figure 6.19b - L_O and W_3 Comparison, 5% Rise Time Uncertainty, Circuit 8

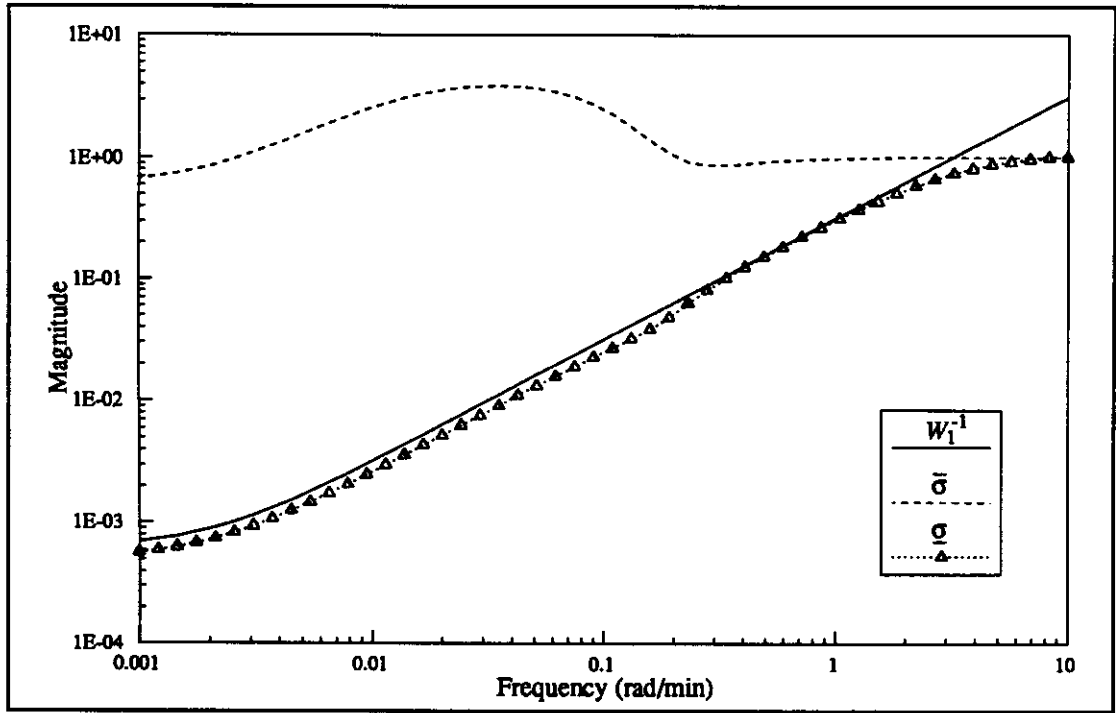


Figure 6.20a - Circuit 8, 2% Rise Time Uncertainty, Sensitivity Function

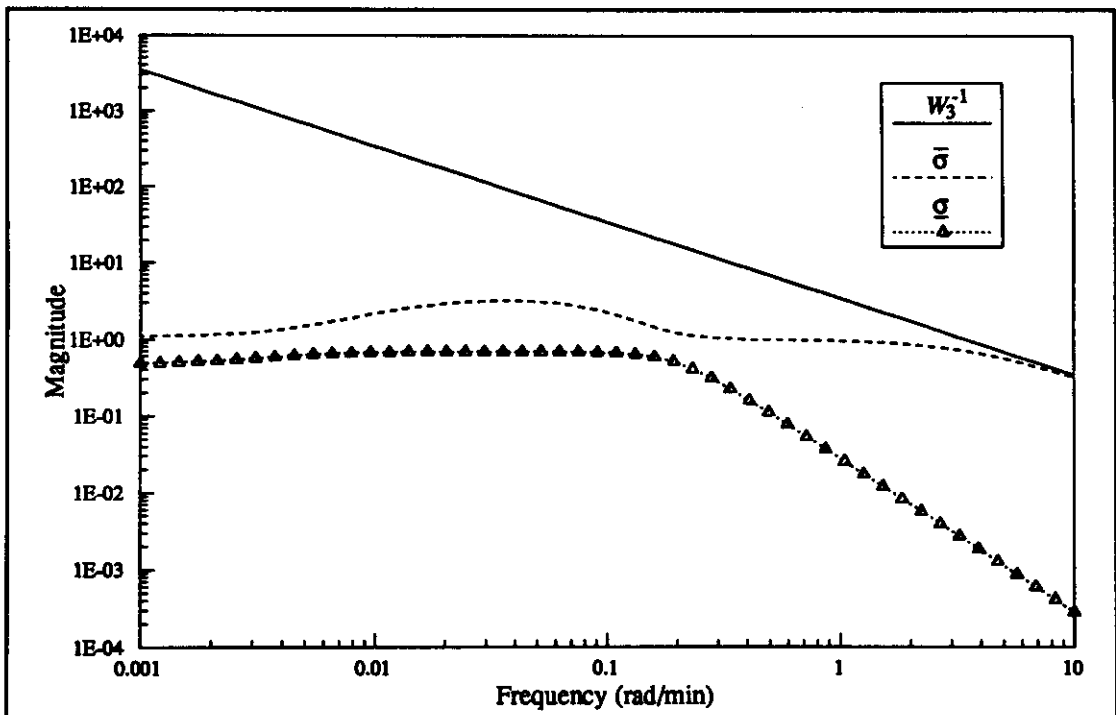


Figure 6.20b - Circuit 8, 2% Rise Time Uncertainty, Complementary Sensitivity Function

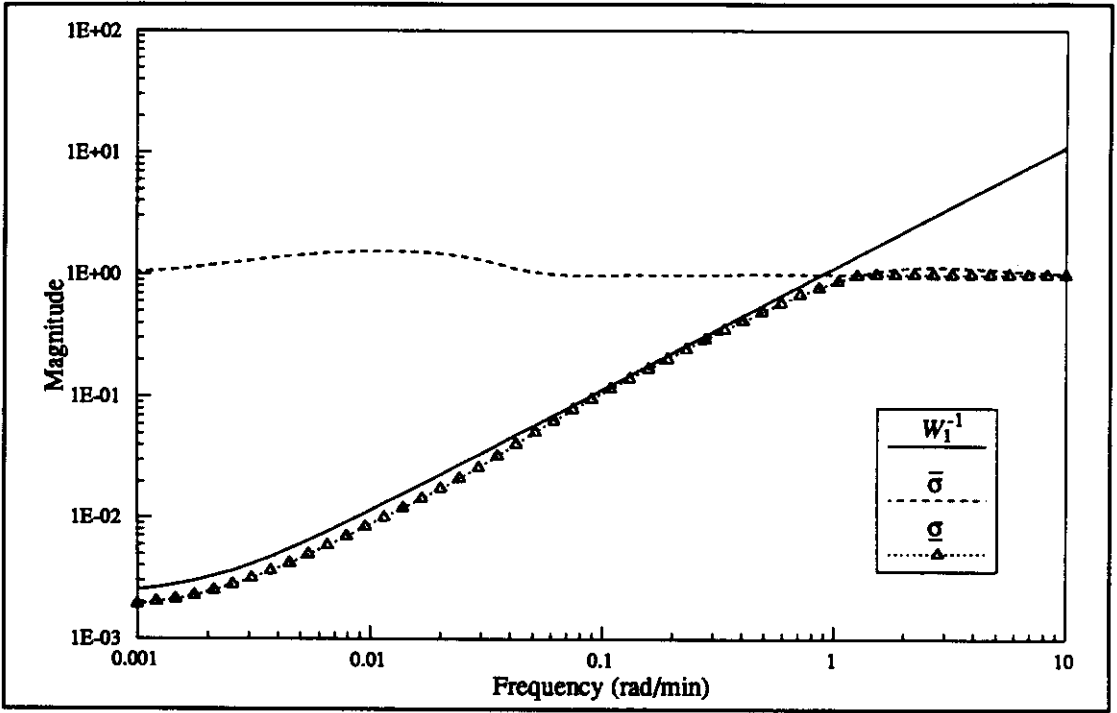


Figure 6.20c - Circuit 8, 5% Rise Time Uncertainty, Sensitivity Function

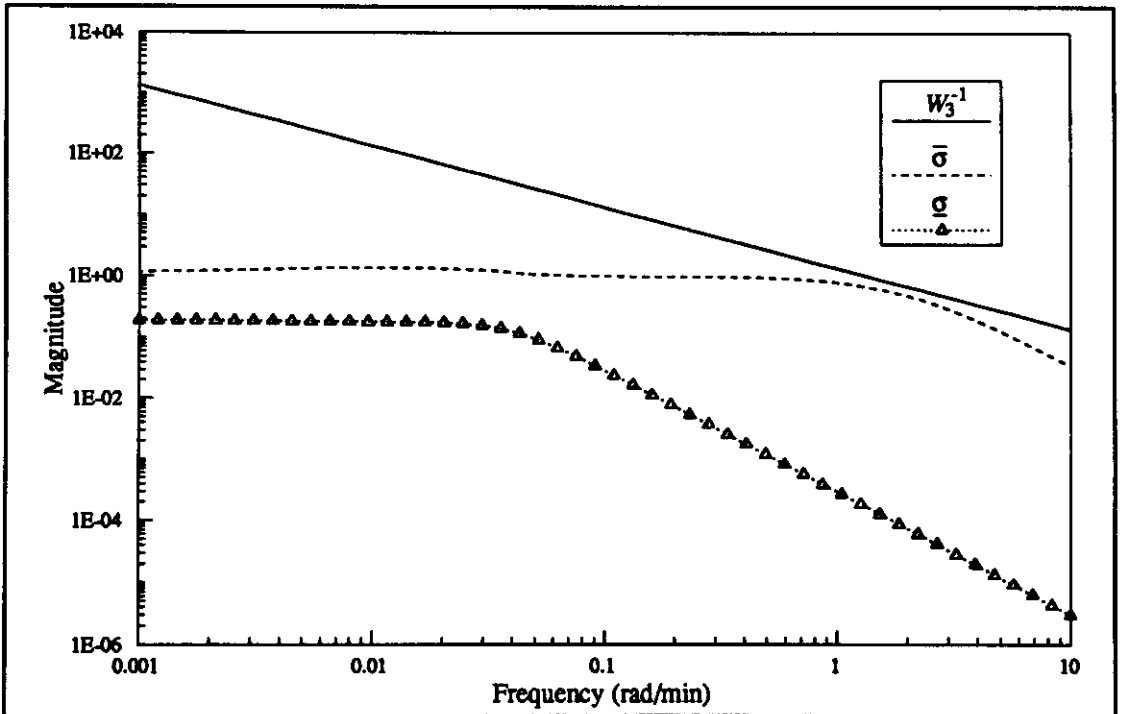


Figure 6.20d - Circuit 8, 5% Rise Time Uncertainty, Complementary Sensitivity Function

Recovery	$d_0 = 0\%$	$d_0 = 5\%$	$d_0 = 10\%$	$d_0 = 20\%$	$d_0 = 40\%$
$RT = 0\%$	95.040	94.760	94.480	93.797	89.855
$RT = 2\%$	95.040	94.736	94.432	93.497	89.855
$RT = 5\%$	95.040	94.620	94.200	92.507	89.855
$RT = 10\%$	95.040	94.407	93.773	92.507	89.870
Grade	$d_0 = 0\%$	$d_0 = 5\%$	$d_0 = 10\%$	$d_0 = 20\%$	$d_0 = 40\%$
$RT = 0\%$	60	60.035	60.070	60.508	70.605
$RT = 2\%$	60	60.059	60.117	61.136	70.605
$RT = 5\%$	60	60.052	60.104	62.076	70.605
$RT = 10\%$	60	61.326	62.651	65.302	70.605
Air	$d_0 = 0\%$	$d_0 = 5\%$	$d_0 = 10\%$	$d_0 = 20\%$	$d_0 = 40\%$
$RT = 0\%$	0.888	0.964	1.039	1.160	0.5
$RT = 2\%$	0.888	0.967	1.045	1.135	1.068
$RT = 5\%$	0.888	1.001	1.114	0.675	0.5
$RT = 10\%$	0.888	0.836	0.784	0.679	0.5
Water	$d_0 = 0\%$	$d_0 = 5\%$	$d_0 = 10\%$	$d_0 = 20\%$	$d_0 = 40\%$
$RT = 0\%$	0.8	0.958	1.116	1.400	0.871
$RT = 2\%$	0.8	0.966	1.131	1.400	0.871
$RT = 5\%$	0.8	1.037	1.273	0.8	0.871
$RT = 10\%$	0.8	0.8	0.8	0.8	0.862
k	$d_0 = 0\%$	$d_0 = 5\%$	$d_0 = 10\%$	$d_0 = 20\%$	$d_0 = 40\%$
$RT = 0\%$	-	0.23	0.23	0.061	0
$RT = 2\%$	-	0.251	0.251	0.049	0
$RT = 5\%$	-	1	1	0	0
$RT = 10\%$	-	0	0	0	0

Table 6.16 - Circuit 8, 0 to 10% Bubble Rise Time Uncertainty

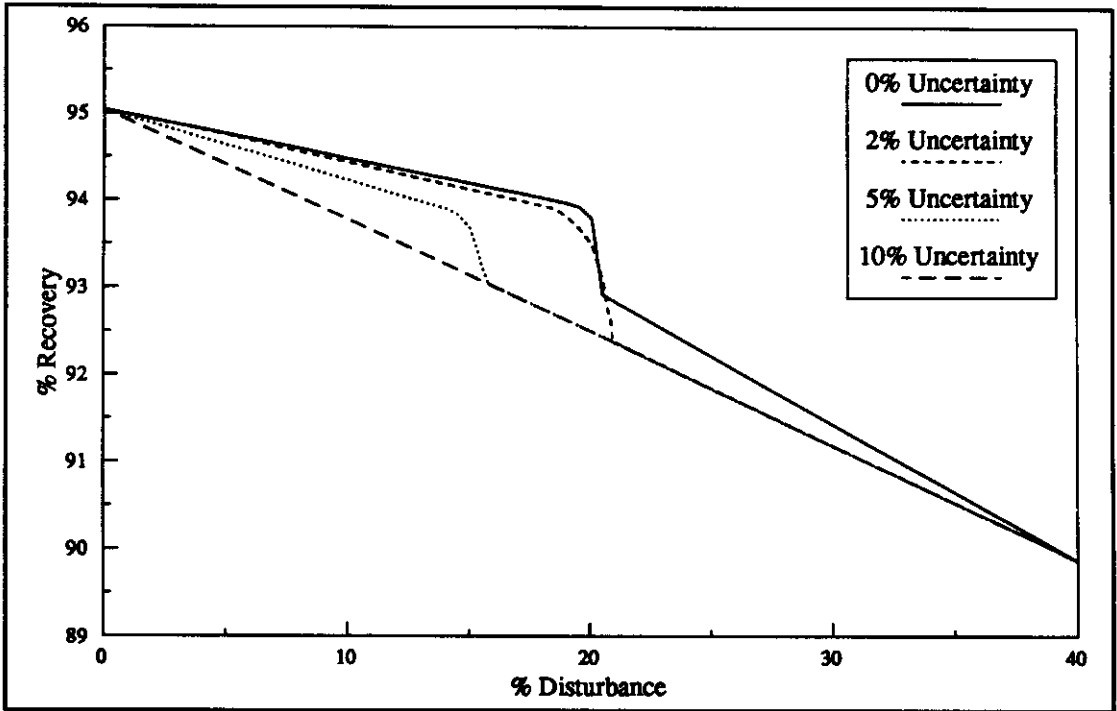


Figure 6.21 - Optimal Recovery, 0 - 10% Rise Time Uncertainty, Circuit 8

levels.

In this section, the expected effect of process model uncertainty on the achievable optimal recovery has been clearly demonstrated. Uncertainty descriptions were derived by calculating the maximum singular value of the multiplicative output uncertainty in the process transfer functions. These were used to calculate an H_∞ controller, and in the calculation of the maximum possible input and output magnitudes for the disturbances being considered. In the next section, even more "realism" will be added to this procedure for determining the optimal achievable performance for a process design. A method for including steady-state uncertainty, likely to be encountered in almost all design studies, will be presented.

6.4 Steady-State and Dynamic Uncertainty

In the previous section, the effect of dynamic uncertainty was introduced into the problem of determining the optimal achievable operation of a process design. Frequently, uncertainty in the process model will also change the steady-state behaviour of the model. This is a more difficult problem than that analysed in Section 6.3, as the steady-state model is the "key" to the optimisation of a process.

6.4.1 Uncertainty Analysis

For the flotation circuit example, a major source of uncertainty in the model is the flotation rate constants, K_Z and K_G (Equations 6.7 and 6.8). In Section 6.1.1, the use of first-order kinetics to model the flotation of the valuable and gangue material was described. It is the difference between the "floatability" of these two materials which causes the separation. Consequently, if the rate constants describing this process change, then it is to be expected that the composition of the product streams would change. The values for these rate constants depend on such properties as the effectiveness of the added chemicals, the particle size distribution and impurities in the feed ore.

Steady-State Uncertainty

Extending the concepts of uncertainty used previously, the steady-state model can be described as having a nominal value $\tilde{M}$, and a current value M , which lies within some uncertainty region. Therefore, the use of the steady-state model as the equality constraints in an optimisation problem needs some interpretation in view of this variation. Recall the

nominal optimisation problem (Equation 6.13) used to determine the optimal recovery for each circuit without disturbances. This may now be rewritten (as Equation 6.46) to incorporate the variation in the steady-state model.

$$\begin{aligned}
 & \text{Maximise } \textit{Recovery} \\
 & \textit{Air, Water} \\
 & \text{such that } \textit{Grade} \geq 60\% \\
 & 0.5 \leq \textit{Air} \leq 2.0 \\
 & 0.8 \leq \textit{Water} \leq 2.0 \\
 & (\textit{Recovery}, \textit{Grade}) = M(\textit{Air}, \textit{Water}) \dots\dots \text{Steady-State Model} \\
 & \text{for } M \in \text{Some Uncertainty Region}
 \end{aligned} \tag{6.46}$$

This naturally leads to three possible optima - the *Best* maximum recovery, the *Nominal* maximum recovery and the *Worst* maximum recovery, depending on which steady-state model is used. The variation in the steady-state model is due entirely in this case to the uncertainty in the rate constants. We can therefore define the following nominal values and uncertainties.

$$\text{Nominal: } \tilde{K}_Z = 0.2 \quad \tilde{K}_G = 0.013333 \tag{6.47}$$

$$\text{Uncertain: } K_Z \in U_Z \quad K_G \in U_G$$

where U_Z and U_G are uncertain regions around $\tilde{K}_Z$ and $\tilde{K}_G$ respectively.

The three optimisations mentioned before now become;

$$\begin{aligned}
 \text{Best: } & \text{Maximise } \textit{Recovery} \\
 & \textit{Air, Water} \\
 & K_Z \in U_Z \\
 & K_G \in U_G
 \end{aligned} \tag{6.48}$$

$$\begin{aligned}
 \text{Nominal: } & \text{Maximise } \textit{Recovery} \\
 & \textit{Air, Water} \\
 & K_Z = \tilde{K}_Z \\
 & K_G = \tilde{K}_G
 \end{aligned} \tag{6.49}$$

$$\text{Worst:} \quad \begin{array}{l} \text{Minimise} \\ K_z \in U_z \\ K_G \in U_G \end{array} \left\{ \begin{array}{l} \text{Maximise Recovery} \\ \text{Air, Water} \end{array} \right\} \quad (6.50)$$

These three optimisations will be used in Section 6.4.3 to determine possible optimal operating points for the flotation circuits. These optima will be determined using the Optimal Operation Algorithm, as last stated in Equation 6.28, modified as required by Equations 6.48, 6.49 and 6.50.

The optimisation within the Optimal Operation Algorithm requires a linearised model, based on the steady-state transfer function. For the no uncertainty case, these circuit models were given in Section 6.2.1, however a method for incorporating the change in the model elements with rate constant variation is required here.

The actual variation in flotation rate constants considered here is $\pm 10\%$;

$$0.18 \leq K_z \leq 0.22$$

$$0.012 \leq K_G \leq 0.014666$$

The variation is considered to be uncorrelated, so that the variation in one rate constant does not determine in any way the variation in the other. To analyse the changes in the steady-state transfer function matrices, the flotation circuits were reoptimised at the extremes of the rate constant variations. As an example, the optimal recoveries for the five cases (including no variation in rate constant) are shown in Table 6.17 for Circuit 4.

From this data, it can be seen that the greatest variation in optimal recovery occurs when the rate constants are at opposite extremes, that is, K_z up 10% and K_G down 10%, and vice versa. For the other two cases, where the ratio of the rate constants (and hence the relative ease of separation) remains unchanged, the optimal recovery remains virtually the same. This led to the adoption of a scheme for *parameterising* the steady-state transfer function elements as a function of a variable k_2 , which indicates the amount of variation

Change in K_z	Change in K_G	Recovery
+10%	-10%	96.964
+10%	+10%	94.593
0%	0%	94.592
-10%	-10%	94.592
-10%	+10%	90.181

Table 6.17 - Circuit 4, Optimal Recoveries for K Variation

in the rate constants between the two extremes.

$$\begin{aligned}
 K_z &= (1 - k_2 \times 0.1) \bar{K}_z \\
 K_G &= (1 + k_2 \times 0.1) \bar{K}_G \\
 -1 &\leq k_2 \leq 1
 \end{aligned}
 \tag{6.51}$$

The steady-state transfer function was evaluated for each circuit for $k_2 = -1, -0.5, 0, 0.5, 1$, representing the transition between the two extremes of the rate constants. It is realised that this is only an approximate representation of the likely effect of possible variations in $G(0)$. Although it includes the greatest change in recovery (the objective function in the optimisation), other variations in $G(0)$ are possible which may influence optimal operation via their effect on steady-state changes in the *Air* and *Water* values, and their "backoff" from constraints. However, the parameterisation used is sufficient for this example.

Cubic polynomials were fitted to the data values for each element in $G(0)$, for each circuit. The actual data points and the polynomial curves for Circuit 4 are shown in Figure 6.22, while Table 6.18 gives the polynomial coefficients used in Equation 6.52.

$$G_{ij}(0) = ak_2^3 + bk_2^2 + ck + d \tag{6.52}$$

Section 6.1.3 will describe how these polynomials were used within the Optimal Operation Algorithm.

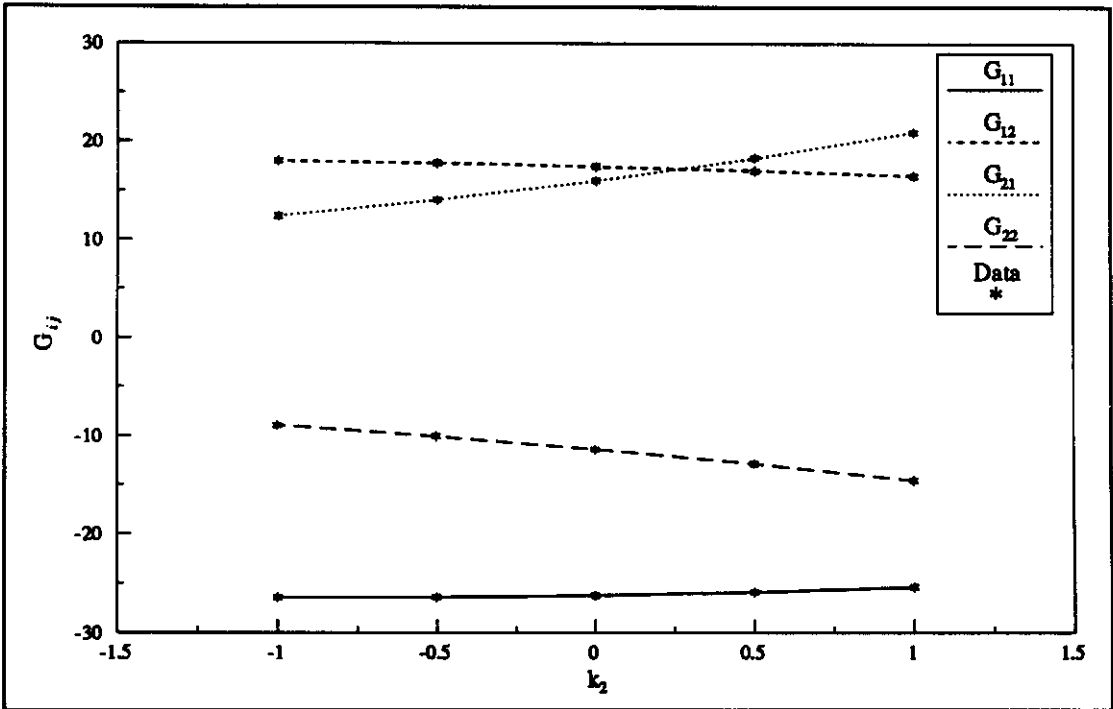


Figure 6.22 - Circuit 4, $G(0)$ Parameterisation

<i>i</i>	<i>j</i>	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
<i>Recovery</i>	<i>Air</i>	0.06202	0.6443	4.232	16.020
<i>Recovery</i>	<i>Water</i>	-0.03267	-0.3900	-2.784	-11.366
<i>Grade</i>	<i>Air</i>	0.02599	0.3139	0.5074	-26.261
<i>Grade</i>	<i>Water</i>	-0.01339	-0.2123	-0.7201	17.447

Table 6.18 - Circuit 4 $G(0)$ Parameterisation Coefficients

Disturbance Transfer Function

Unlike the rise time uncertainty case studied in Section 6.3, the variation in flotation rate constants changes the way that the feed rate and feed grade disturbances affect the *Grade* and *Recovery* outputs. Using a similar calculation procedure to that adopted in Section 6.3.1 to determine the uncertainty in the process transfer function, the maximum additive error (L_d) in each element of the disturbance transfer function (G_d) was calculated at a number of frequencies. To be consistent with the approach adopted for the steady-state transfer function parameterisation, the maximum variation in the disturbance transfer function was checked at $k_2 = -1$ and 1 .

$$L_{d,l,m,\omega} = \max_{k_2} |G_{d,l,m,k_2}(i\omega) - G_{d,l,m,k_2=0}(i\omega)|$$

for $l = R(ecovery), G(rade)$

$m = \text{Feed Rate, Feed Grade}$

$\omega = 0.001 \text{ to } 10 \text{ rad/min}$

$k_2 = -1, 1$

(6.53)

In Section 6.3.1, a transfer function was fitted to the uncertainty associated with the process model, however, due to the calculation procedure used here for the disturbance transfer function (described later in Section 6.4.2), only the maximum magnitudes (given by Equation 6.53) at the selected frequencies are required. As an example, Figure 6.23 shows these maximum magnitudes of the additive uncertainty for the Circuit 4 disturbance transfer function.

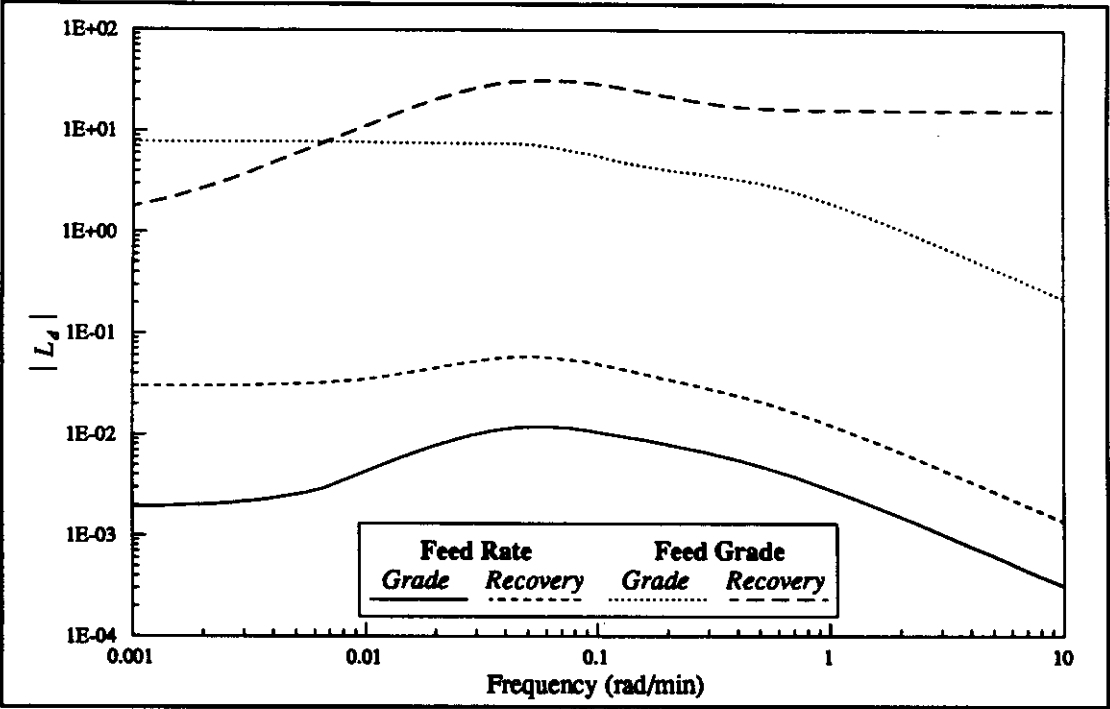


Figure 6.23 - Circuit 4, Additive Disturbance Uncertainty

Process Model Uncertainty

The upper bound (W_3) on the multiplicative output uncertainty (L_O) was next calculated using the same method as used for the rise time uncertainty (Equation 6.31). The extremes in the variation of the process transfer function were again taken to be at $k_2 = -1$ and 1, to be consistent with the previous steady-state analysis. Hence the greater of the maximum singular values of the multiplicative output error matrices calculated for $k_2 = -1$ and $k_2 = 1$, was used for L_O .

The maximum singular value of L_O , and the magnitude of W_3 are shown in Figure 6.24 for each circuit. For comparison, the maximum singular values are also shown for the corresponding circuit rise time uncertainty. The shape of the rate constant uncertainty curves is quite different from that obtained for the rise time variation. In particular, the non-zero low frequency value is consistent with the observed variation in the steady-state process transfer function. Circuits 1 and 2 have a slightly higher level of uncertainty with rate constant variation, while Circuits 4 and 8 have considerably less high frequency uncertainty than for the rise time uncertainty case. For frequencies in the range 0.01 to 0.1 rad/min, the transfer functions used to describe W_3 overstate the maximum values obtained from the actual rate constant variation, although the error is small. The bounding transfer functions are given below for the four circuits.

$$\text{Circuit 1} \quad W_3 = \frac{(1.3701s + 0.1751)}{(3s + 1)} \quad (6.54)$$

$$\text{Circuit 2} \quad W_3 = \frac{(3.3915s + 0.1579)}{(1.9s + 1)} \quad (6.55)$$

$$\text{Circuit 4} \quad W_3 = \frac{(311.4s + 0.6427)}{(1000s + 1)} \quad (6.56)$$

$$\text{Circuit 8} \quad W_3 = \frac{(67.585s + 0.4781)}{(350s + 1)} \quad (6.57)$$

This section has described the source of the flotation rate constant uncertainty, and has quantified its effect on the steady-state process transfer function, the disturbance transfer function and the process model. This information can now be used to determine the input and output magnitudes invoked by the feed rate and feed grade disturbances, so that the

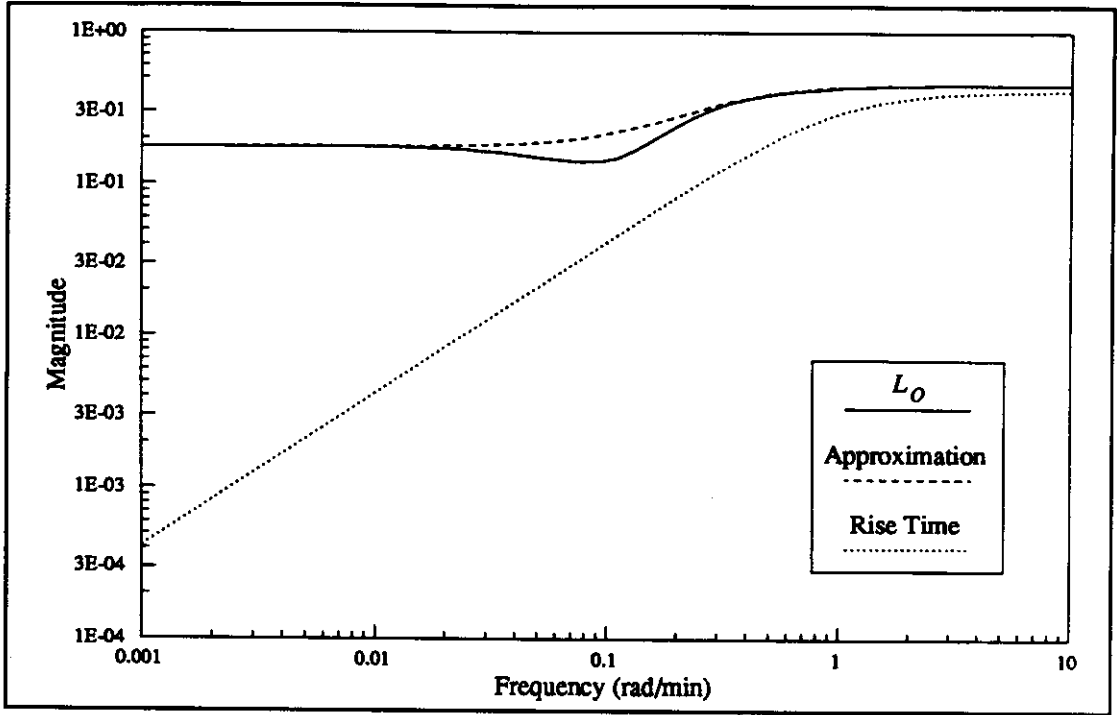


Figure 6.24a - L_O and W_3 Comparison, Rate Constant Uncertainty, Circuit 1

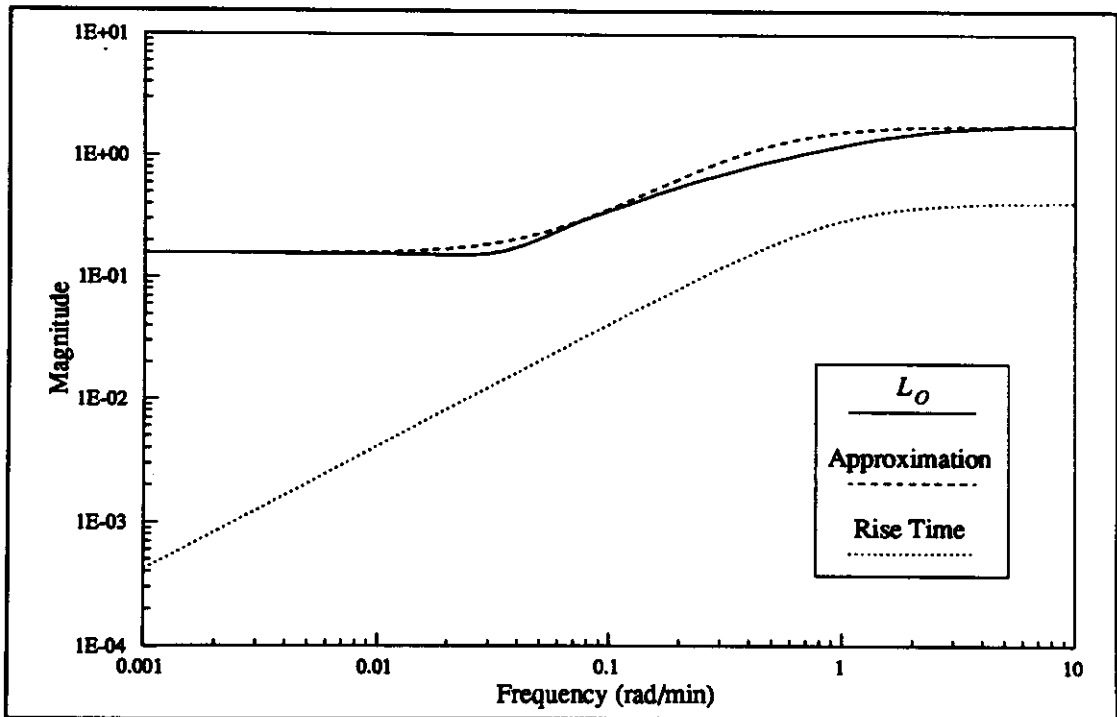


Figure 6.24b - L_O and W_3 Comparison, Rate Constant Uncertainty, Circuit 2

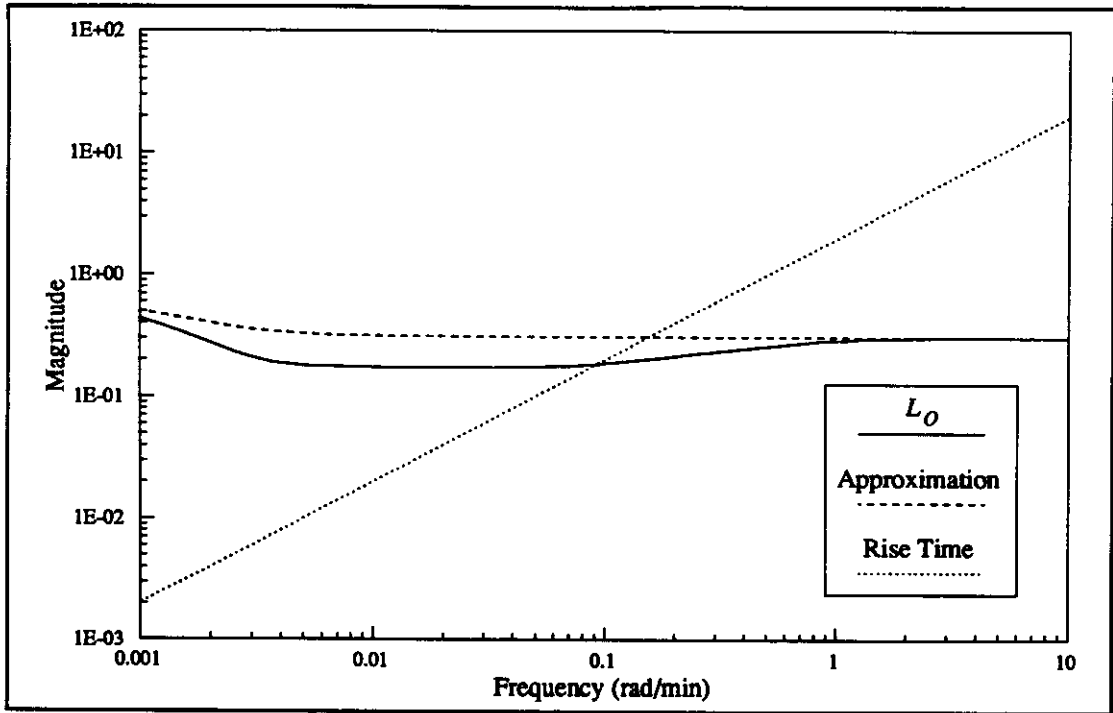


Figure 6.24c - L_O and W_3 Comparison, Rate Constant Uncertainty, Circuit 4

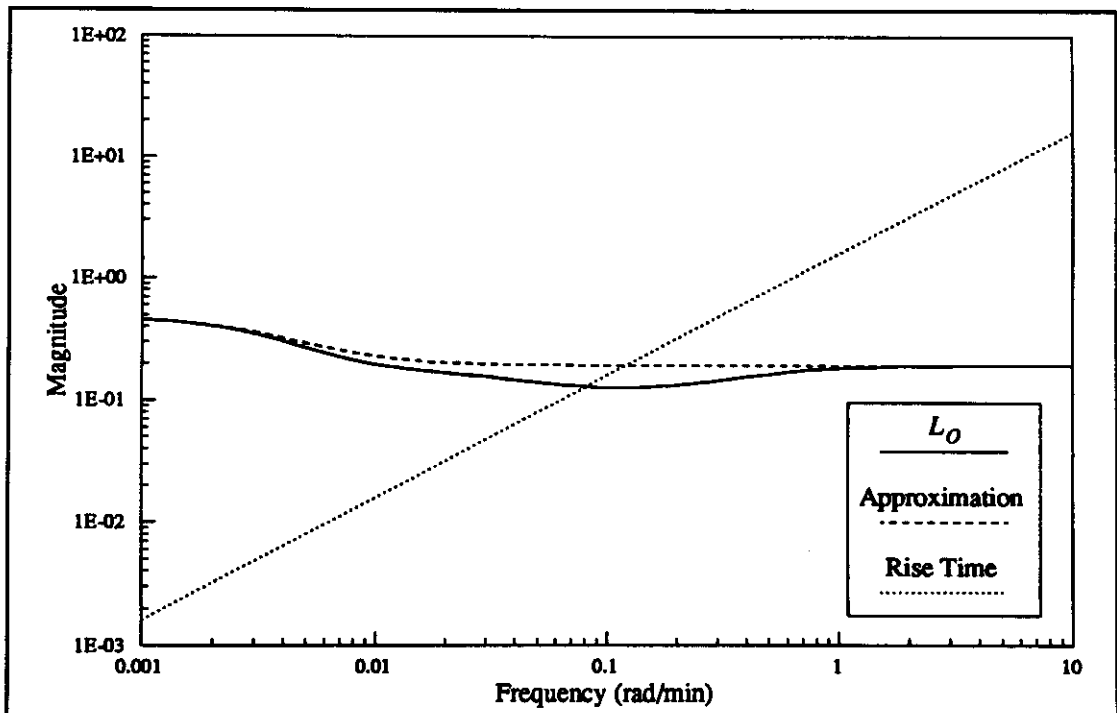


Figure 6.24d - L_O and W_3 Comparison, Rate Constant Uncertainty, Circuit 8

Optimal Operation Algorithm can be used to find the maximum achievable recovery.

6.4.2 Input-Output Magnitudes

To calculate the optimal balance between "backoff" from output constraints (open-loop operation), and "backoff" from both input and output constraints (closed-loop operation), the maximum magnitudes of the input and output variables in response to possible feed disturbances are required. In addition, the possibility of "partial control" type operation must be considered, by determining these magnitudes for the case when only part of the error is fed back to the controller ($0 < k < 1$).

Open-Loop Calculation

Due to the variation of the disturbance transfer function (G_d) with the flotation rate constants (K_z and K_G), the open-loop magnitudes will not be the same as those for the no uncertainty and rise time uncertainty cases. The rate constant variation is incorporated through the use of k_2 , giving the extremes of possible K_z and K_G values. The open-loop magnitudes were determined using the optimisation procedure outlined in Equation 6.58.

$$\begin{aligned}
 \Delta y_{l,m,OL} &= \text{Maximum}_{\omega, k_2} |y_{l,m,OL}| \\
 &= \text{Max}_{\omega} (\text{Max}_{k_2} |y_{l,m,OL}|) \\
 &= \text{Max}_{\omega} (\text{Max}_{k_2} |G_{d,l,m} \frac{d_{0,m}}{(10s + 1)^2}|) \\
 &= \text{Max}_{\omega} (\text{Max}_{L_{A,l,m,\omega}} |G_{d,l,m} \frac{d_{0,m}}{(10s + 1)^2}|)
 \end{aligned}
 \tag{6.58}$$

such that $|L_{A,l,m,\omega}| \leq L_{d,l,m,\omega}$

$$\begin{aligned}
 \text{where } G_{d,l,m}(i\omega) &= G_{d,l,m,k_2=0}(i\omega) + L_{A,l,m,\omega} \\
 l &= R(\text{ecovery}), G(\text{rade}) \\
 m &= \text{Feed Rate, Feed Grade} \\
 k_2 &= -1, 1
 \end{aligned}$$

In this procedure, the exact values of $L_{d,l,m,\omega}$ were available from Equation 6.53, since

Equation 6.58 was evaluated at the same frequencies as Equation 6.53, eliminating the need for a function to fit the previously determined data points. The optimisations were performed within the MATLAB framework, combining the frequency and optimisation routines into the one program.

The maximum open-loop *Grade* magnitudes are given in Table 6.19, as well as the values for the no uncertainty case for comparison. Recovery magnitudes are not shown as they are not required in the Optimal Operation Algorithm, since the only output constraint is on the *Grade*. The maximum of the two disturbances, the feed grade disturbance (shown in bold), is only slightly greater with the rate constant variation, while the maximum feed rate magnitudes are discernibly larger. Figure 6.25 shows the open-loop *Grade* magnitudes for Circuit 4 as a function of frequency, with the no uncertainty values for comparison.

Circuit	Rate Constant Uncertainty		No Uncertainty	
	$ y_{G,f,OL} $	$ y_{G,s,OL} $	$ y_{G,f,OL} $	$ y_{G,s,OL} $
1	1.2860	1.3263	1.2585	1.3262
2	1.2826	1.3246	1.2562	1.3244
4	0.43389	1.3229	0.42189	1.3228
8	0.66122	1.3257	0.63275	1.3256

Table 6.19 - Rate Constant Uncertainty Open-Loop Magnitudes

Controller Design

Using the W_3 upper bounds on the flotation rate constant uncertainty, H_∞ controllers were designed for each of the four circuits with an 0.001 weighting on the *Recovery* output. The performance weighting (Equation 6.29) used for the rise time uncertainty controller design was also used here. The γ values are given in Table 6.20.

$$W_1^{-1} = \gamma \frac{0.05s + 0.0001}{0.01s + 1} \quad (6.29)$$

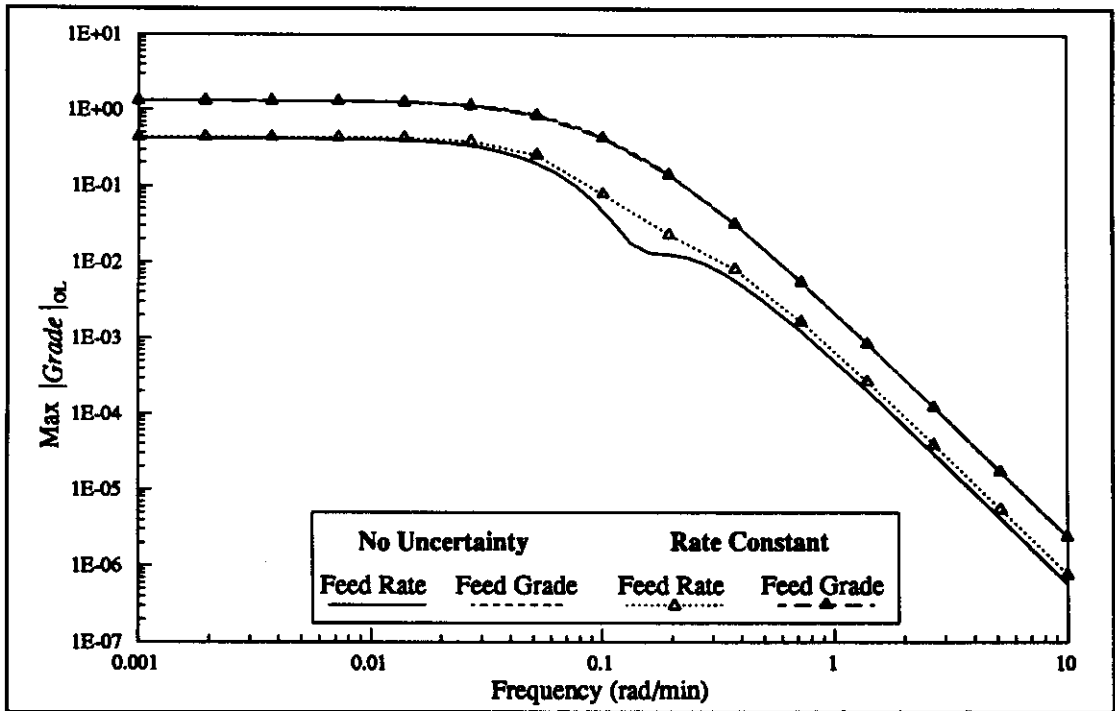


Figure 6.25 - $|Grade|_{OL}$, Circuit 4, No Uncertainty *cf.* Rate Constant Uncertainty

Circuit	Rate Constant Uncertainty	Rise Time Uncertainty
1	2.86	2.86
2	40.0	12.5
4	10.5	40.0
8	6.67	118

Table 6.20 - Uncertain Controller Design, Comparison of γ Values

Interestingly, the smaller the γ value the tighter the performance specification, so that with rate constant uncertainty Circuit 2 now has the worst performance, not Circuits 4 and 8. Table 6.20 also shows the γ values for the rise time uncertainty controllers. Circuit 1 has the same γ value as the rise time case, indicating that the controller design is being limited by performance rather than uncertainty. The improved performance of Circuits 4 and 8 must be due to the rate constant uncertainty having a less severe impact on the controller design than the rise time uncertainty. Figure 6.26 shows plots of the performance specification and singular values of the sensitivity function (E), as well as the uncertainty bound and singular values of the complementary sensitivity function (H).

In order to investigate the importance of finding the controller design specification which produces the best outcome from the Optimal Operation Algorithm, an additional controller performance specification was used for Circuit 4. This was obtained in an early "iteration" of the design procedure, while searching for the most appropriate transfer function. By using the function given below, the benefit achieved through the refinement of the controller design can be observed by comparing the results obtained with this specification, with those of the original Circuit 4 controller.

$$W_1^{-1} = \frac{0.2857(5s + 1)}{(s + 1)} \quad (6.59)$$

This performance specification does not include integral action ($E \rightarrow 0$ as $\omega \rightarrow 0$), but only provides disturbance attenuation by a factor of 0.2857 at low frequencies. In contrast, the other controller used for Circuit 4 achieves a disturbance attenuation of 1.05×10^{-3} at low frequencies. Plots of the performance specification and singular values of

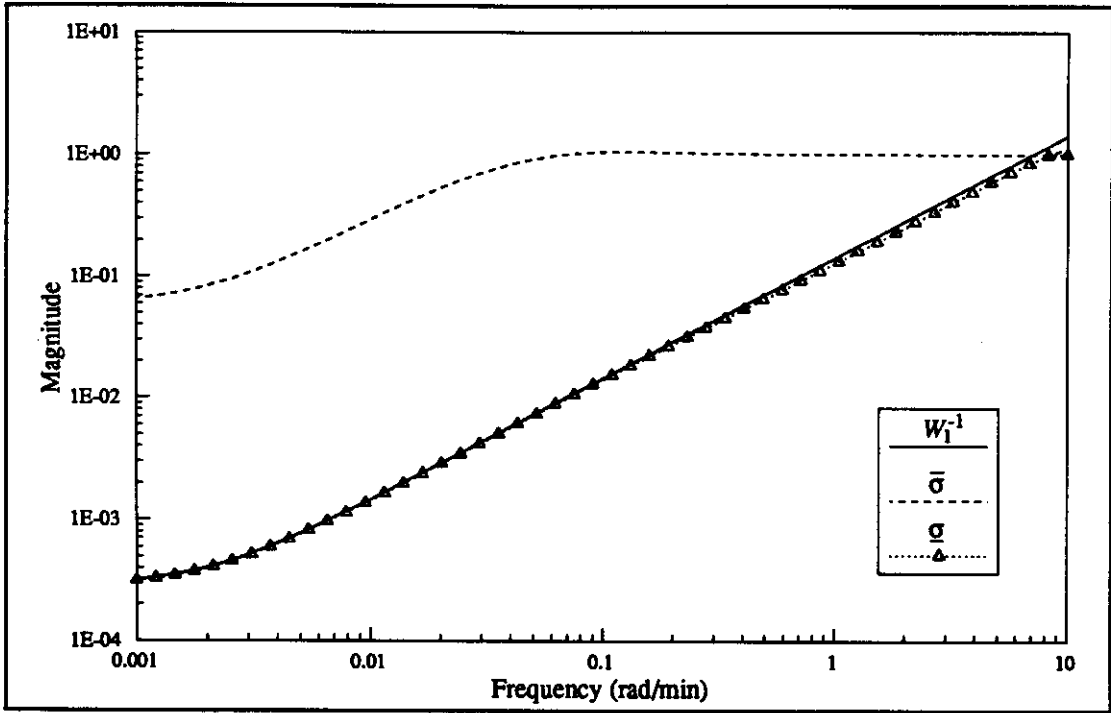


Figure 6.26a - Circuit 1, Rate Constant Uncertainty, H_∞ Controller Design, E

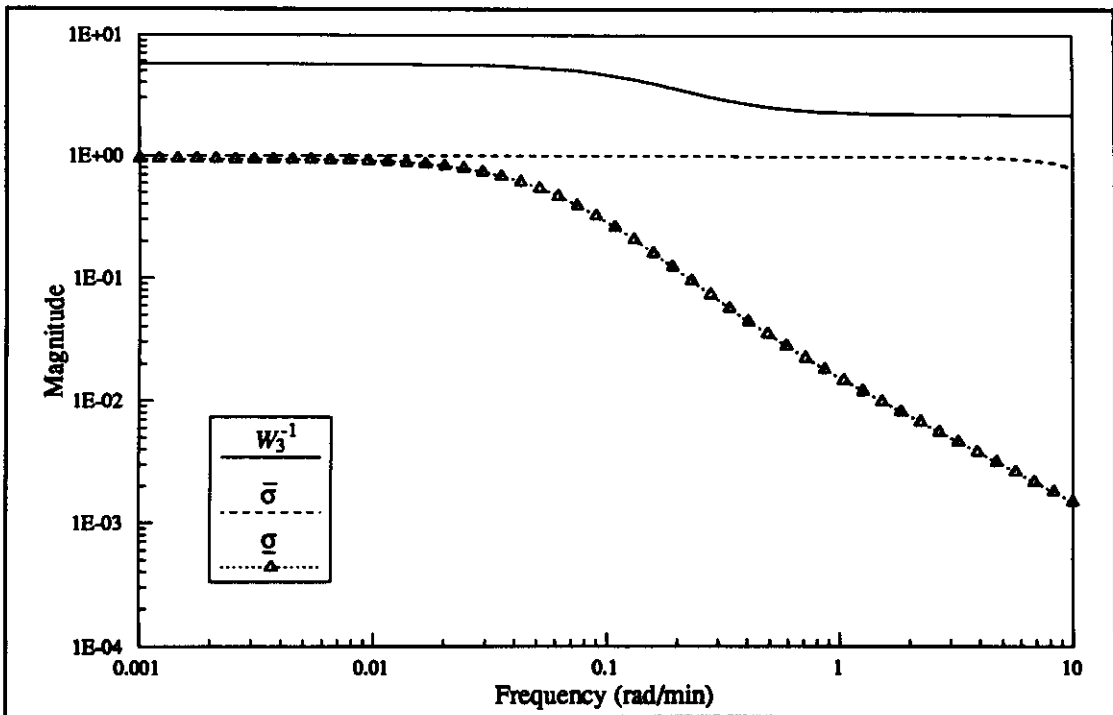


Figure 6.26b - Circuit 1, Rate Constant Uncertainty, H_∞ Controller Design, H

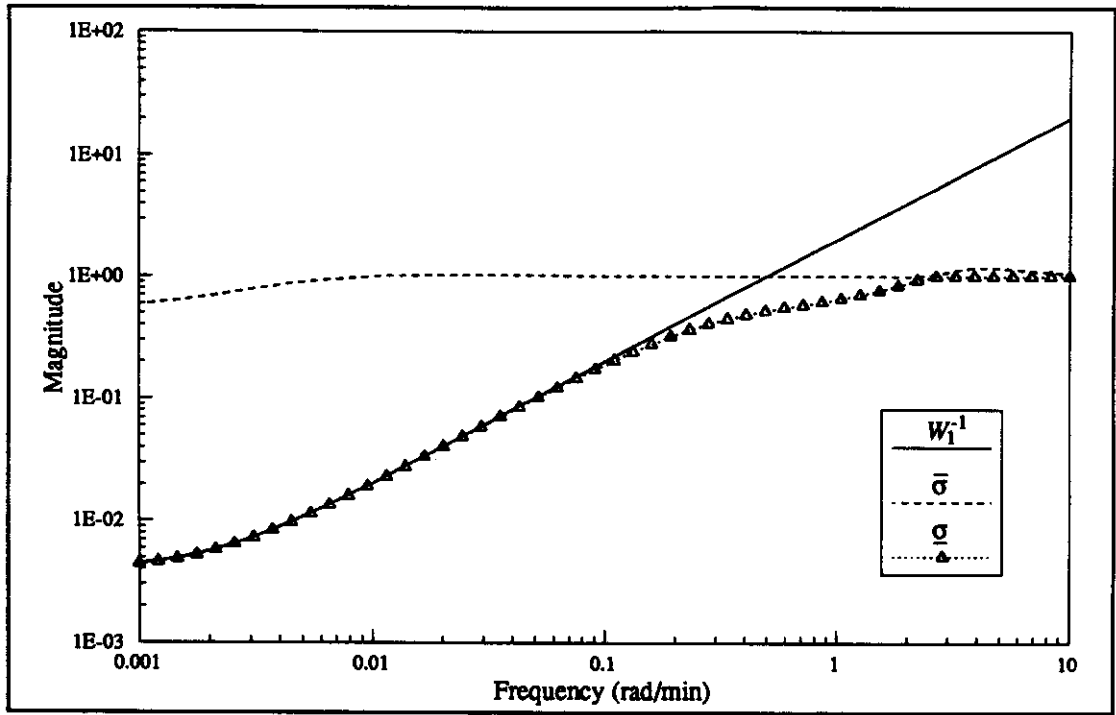


Figure 6.26c - Circuit 2, Rate Constant Uncertainty, H_∞ Controller Design, E

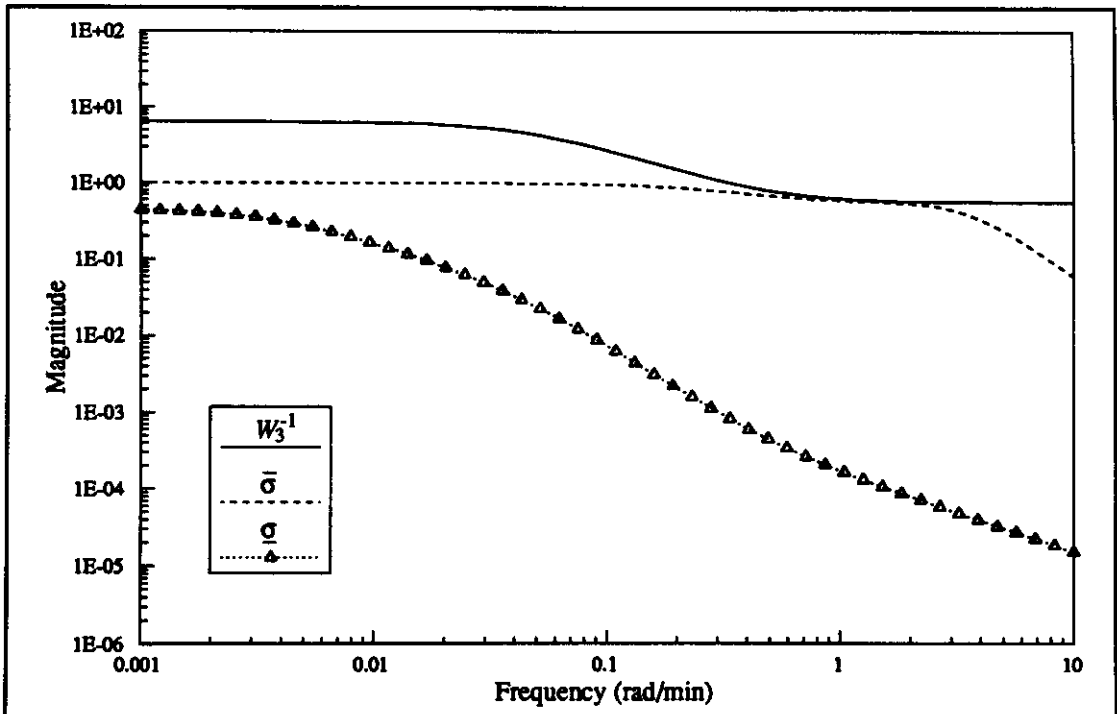


Figure 6.26d - Circuit 2, Rate Constant Uncertainty, H_∞ Controller Design, H

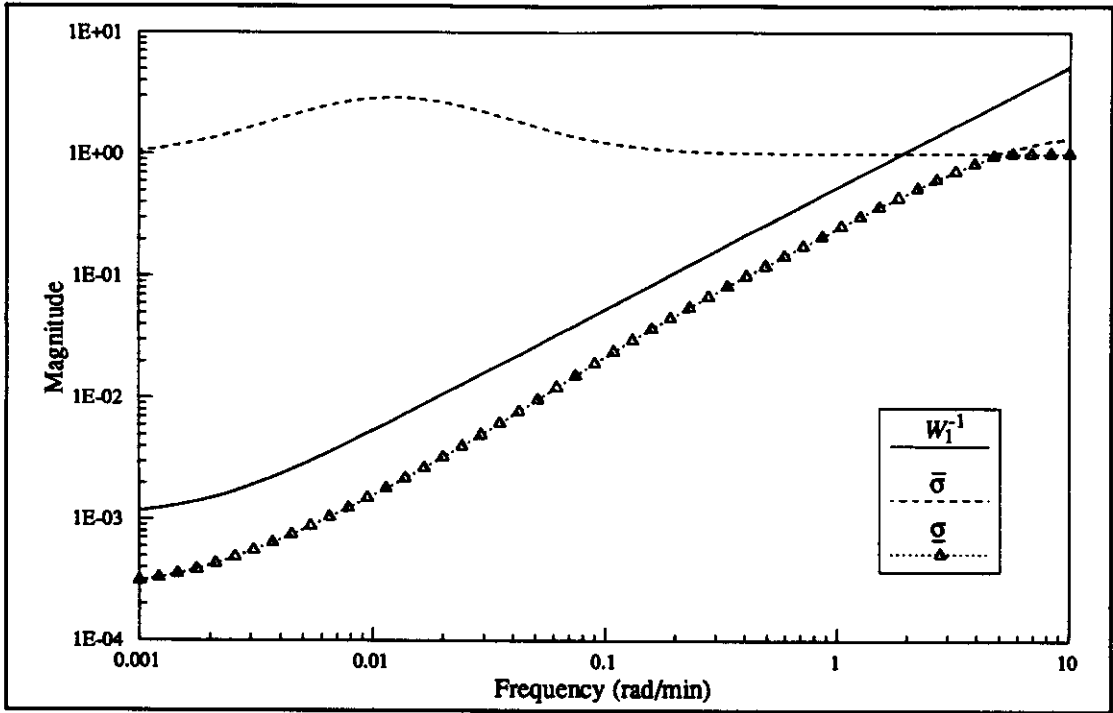


Figure 6.26e - Circuit 4, Rate Constant Uncertainty, H_∞ Controller Design, E

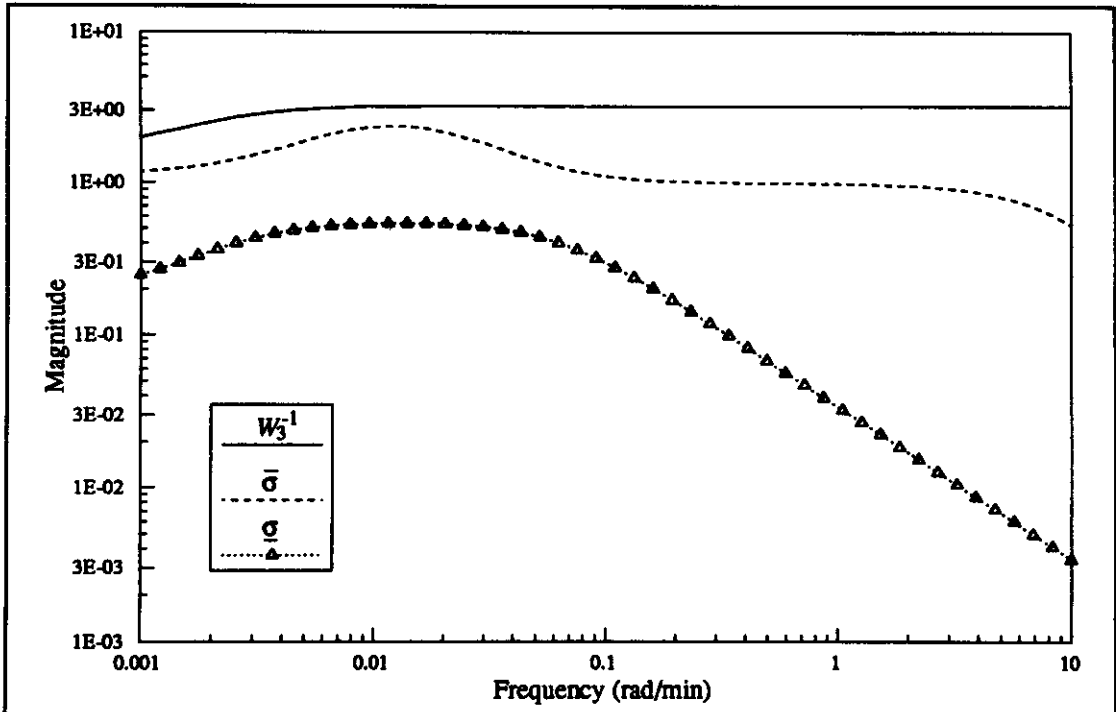


Figure 6.26f - Circuit 4, Rate Constant Uncertainty, H_∞ Controller Design, H

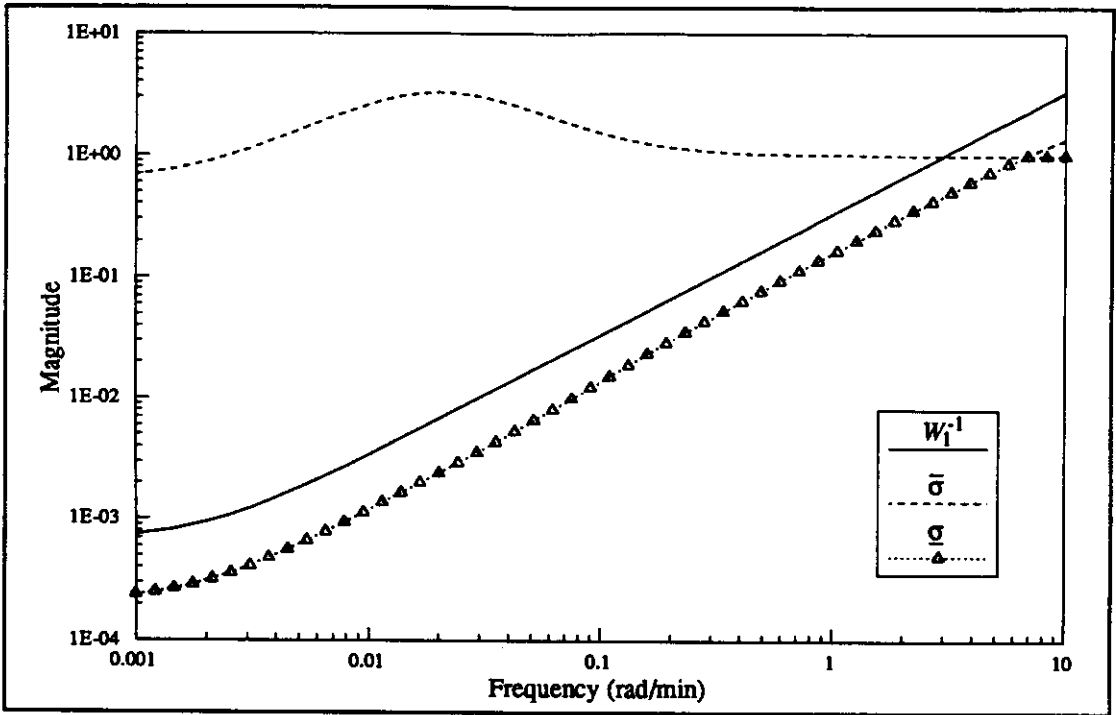


Figure 6.26g - Circuit 8, Rate Constant Uncertainty, H_∞ Controller Design, E

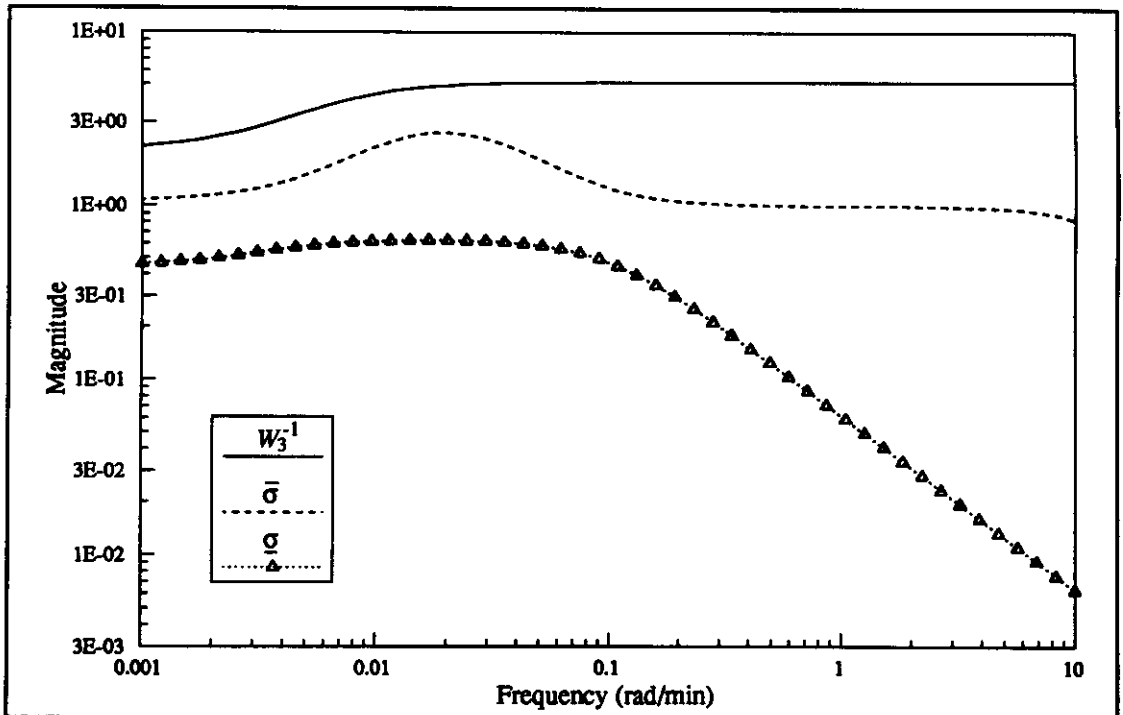


Figure 6.26h - Circuit 8, Rate Constant Uncertainty, H_∞ Controller Design, H

the sensitivity function, as well as the uncertainty bound and singular values of the complementary sensitivity function for this alternative controller are shown in Figure 6.27. Note that in the following text, this controller will be referred to as Circuit 4a.

Closed-Loop Magnitudes

The method for calculating the closed-loop output magnitudes for the feed rate and feed grade disturbances, using the controllers described above, is similar to that used for the open-loop case (Equation 6.58). The difference is that the optimisation (to maximise the possible magnitudes) searches over all possible frequencies, disturbance transfer functions and process transfer functions. The latter are constrained by the upper bound (W_3) imposed on the maximum singular value of the multiplicative output uncertainty (L_0), determined in Section 6.4.1. This procedure is outlined in Equation 6.60.

$$\begin{aligned}
 \Delta y_{l,m,CL} &= \text{Maximum}_{\omega, k_2, L_0} |y_{l,m,CL}| \\
 &= \text{Max}_{\omega} (\text{Max}_{k_2, L_0} |y_{l,m,CL}|) \\
 &= \text{Max}_{\omega} (\text{Max}_{k_2, L_0} |E_l d_m|) \\
 &= \text{Max}_{\omega} \left(\text{Max}_{L_{A,l,m,\omega}, L_0} \left| (I + (I + L_0)\bar{G})_l^{-1} G_{d,m} \frac{d_{0,m}}{(10s + 1)^2} \right| \right)
 \end{aligned}
 \tag{6.60}$$

such that $|L_{A,l,m,\omega}| \leq L_{l,m,\omega}$

$\bar{\sigma}(L_0) \leq |W_3(i\omega)|$

$$\text{where } G_{d,m}(i\omega) = G_{d,m,k_2=0}(i\omega) + \begin{bmatrix} L_{A,R,m,\omega} \\ L_{A,G,m,\omega} \end{bmatrix}$$

$l = R(ecovery), G(rade)$

$m = \text{Feed Rate, Feed Grade}$

$k_2 = -1, 1$

The terms E_l and $(I + (I + L_0)\bar{G})_l^{-1}$ refer to the *Recovery* and *Grade* rows of these matrices, respectively, as the optimisation needs to be performed separately for each

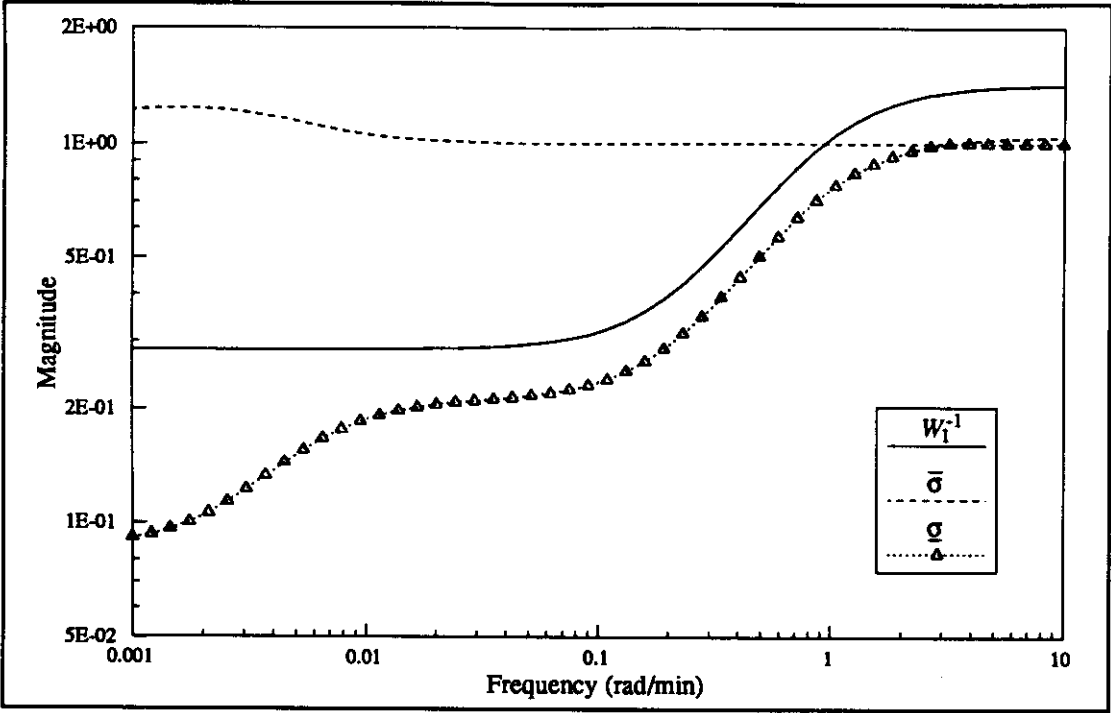


Figure 6.27a - Circuit 4a, Rate Constant Uncertainty, H_∞ Controller Design, E

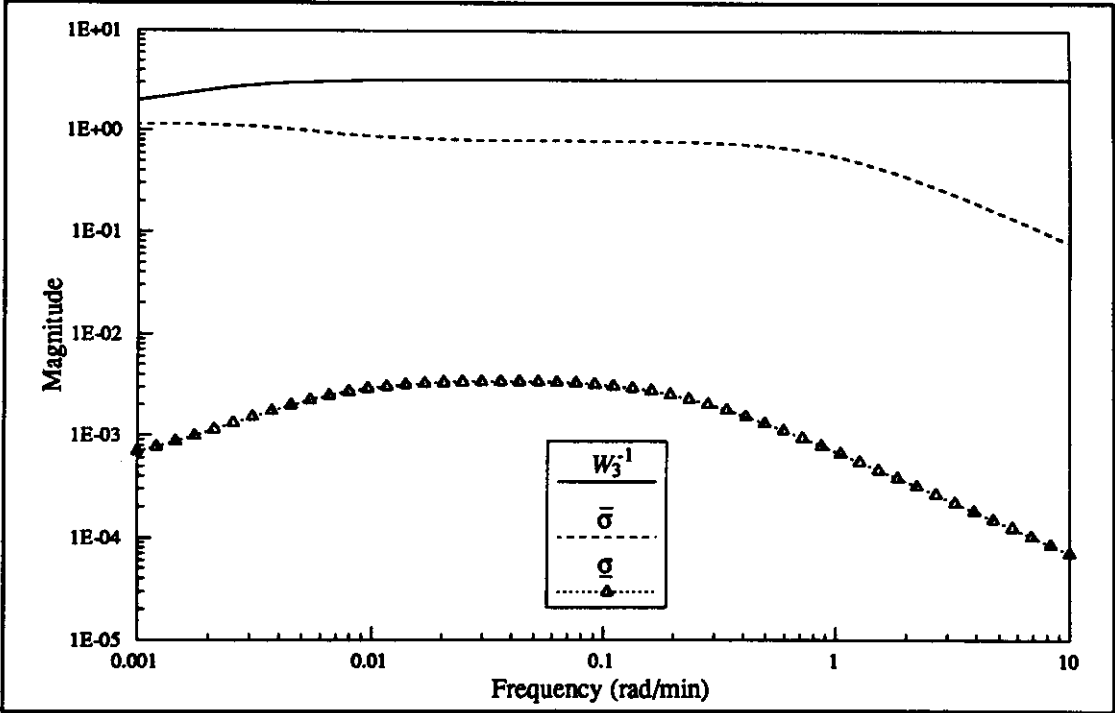


Figure 6.27b - Circuit 4a, Rate Constant Uncertainty, H_∞ Controller Design, H

output. In contrast to the open-loop case, the full G_d matrix is required, hence the more complex problem formulation to include the additive uncertainty at each frequency for each element in G_d . The optimisations were performed within MATLAB using a similar procedure to that described for the rise time uncertainty (Equation 6.42), that is, the individual elements of L_o and L_A were found which maximised the objective function.

Not explicitly shown in the above procedure for determining closed-loop magnitudes, is the use of the partial control factor k . As before, this was included as part of the controller transfer function, as a scalar multiplying the inputs to the controller. The magnitudes were determined for various k values as required. The input magnitudes were calculated using a similar procedure to that given in Equation 6.60, except that the sensitivity function (E) is replaced by CE , where C is the controller transfer function matrix.

Table 6.21 shows the maximum closed-loop magnitudes for the five cases (Circuits 1, 2, 4, 4a, 8), with the maximum value (for either of the disturbances) shown in bold.

Comparison with Table 6.13, the rise time uncertainty closed-loop magnitudes, shows that generally greater values are encountered with a 10% rate constant uncertainty than with a 10% rise time uncertainty.

In particular, larger output magnitudes were obtained for Circuits 1 and 2 which achieved either comparable or worse controller performances than in the rise time uncertainty case. The output magnitudes for Circuits 4 and 8 are lower than the corresponding rise time uncertainty values, due to the better controller performance achieved. These values are, however, still greater than the no uncertainty magnitudes, shown in Table 6.10. The input magnitudes are larger because of the generally better disturbance rejection achieved, and the greater demands placed on the controller by the disturbance variation. The magnitudes for the two controllers designed for Circuit 4 show that the "4a" controller has greater output magnitudes and less input manipulation than the conventional controller. This is expected, because the design specification for the "4a" controller required less disturbance rejection than the "4" controller.

Figure 6.28 shows the closed-loop magnitudes for Circuit 8 for the rate constant, rise time and no uncertainty cases. The maximum output value (i.e. *Grade*) for the rate constant

Circuit	$ y_{Gf,CL} $	$ y_{Gg,CL} $
1	9.842×10^{-3}	9.432×10^{-3}
2	1.451×10^{-1}	1.543×10^{-1}
4	1.472×10^{-2}	2.219×10^{-2}
4a	1.196×10^{-1}	3.655×10^{-1}
8	1.075×10^{-2}	1.380×10^{-2}

Table 6.21a - Closed-Loop Magnitudes (Outputs), Flotation Rate Constant Uncertainty

Circuit	$ \mu_{Af,CL} $	$ \mu_{Ag,CL} $	$ \mu_{Wf,CL} $	$ \mu_{Wg,CL} $
1	7.069×10^{-2}	6.681×10^{-2}	5.412×10^{-1}	5.392×10^{-1}
2	6.032×10^{-2}	6.233×10^{-2}	1.150	1.222
4	3.159×10^{-1}	6.584×10^{-1}	4.716×10^{-1}	1.115
4a	8.546×10^{-2}	2.701×10^{-1}	1.177×10^{-1}	5.597×10^{-1}
8	2.468×10^{-1}	3.323×10^{-1}	5.397×10^{-1}	7.324×10^{-1}

Table 6.21b - Closed-Loop Magnitudes (Inputs), Flotation Rate Constant Uncertainty

case is greater than that for no uncertainty, but less than that for rise time uncertainty. The maximum magnitude for each of the inputs is greatest for the rate constant case, then rise time and no uncertainty. Figure 6.29 shows the input and output magnitudes for the Circuit 4 and 4a controllers. These show the same trend as observed with the maximum magnitudes (see Table 6.21), with the higher performance controller "4" having lower output magnitudes, but requiring greater input magnitudes to achieve this.

The open and closed-loop magnitudes determined in this section can now be used in calculating the appropriate "backoff" from constraints at the optimal operating point. In the next section, steady-state uncertainty is incorporated into the Optimal Operation Algorithm, to produce the *Best*, *Nominal* and *Worst* achievable optimal recoveries.

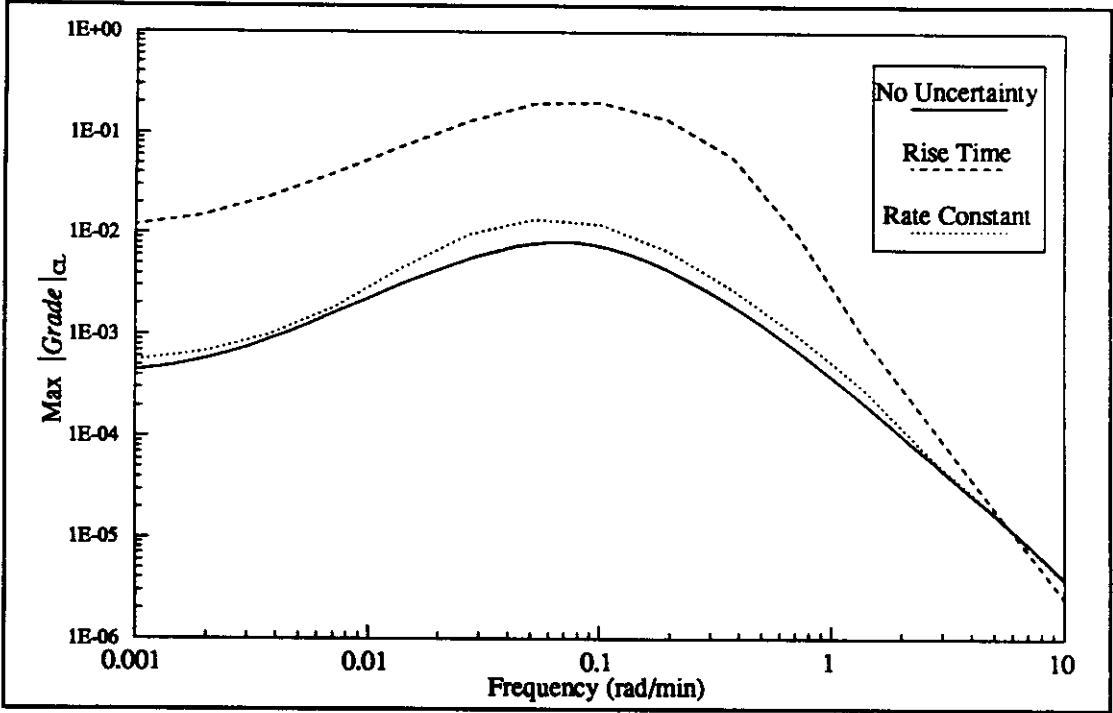


Figure 6.28a - $|Grade|_{CL}$, Uncertainty Comparisons, Circuit 8

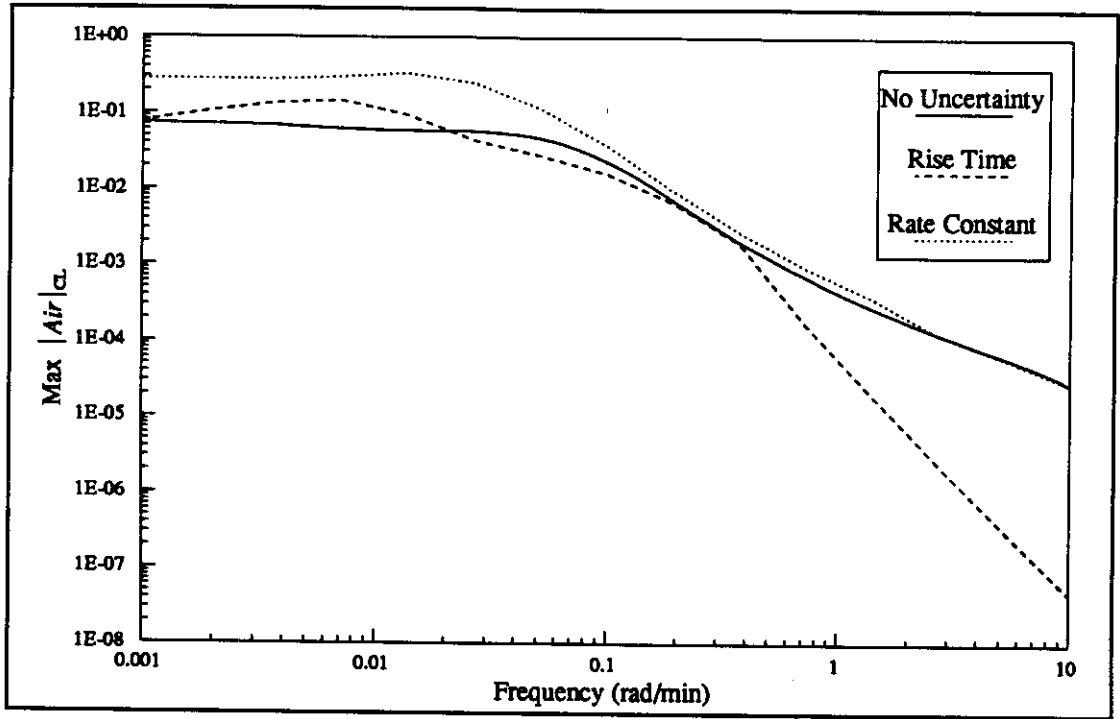


Figure 6.28b - $|Air|_{CL}$, Uncertainty Comparisons, Circuit 8

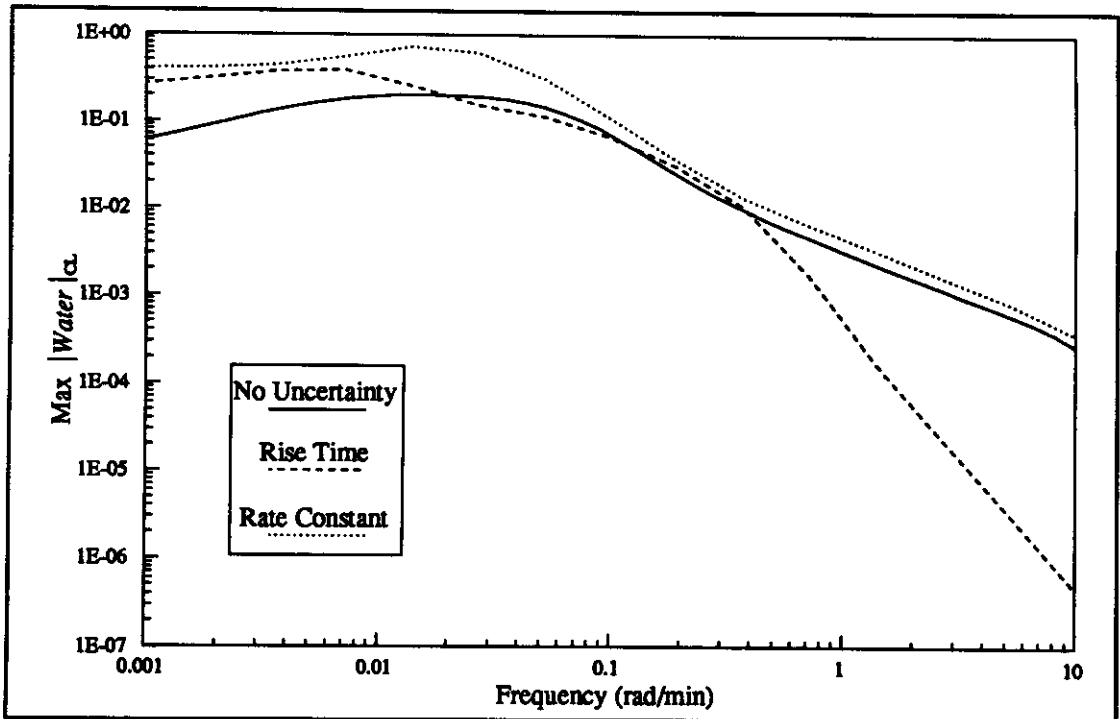


Figure 6.28c - $|Water|_{CL}$, Uncertainty Comparisons, Circuit 8

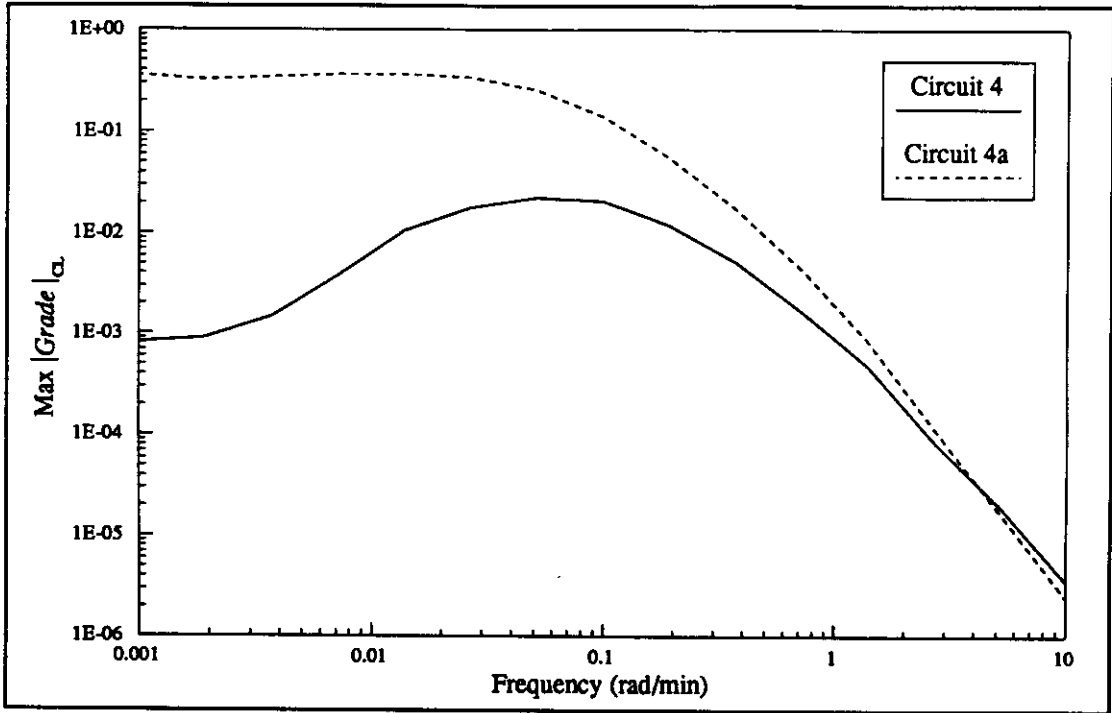


Figure 6.29a - $|Grade|_{CL}$, Rate Constant Uncertainty, Circuit 4 *cf.* 4a

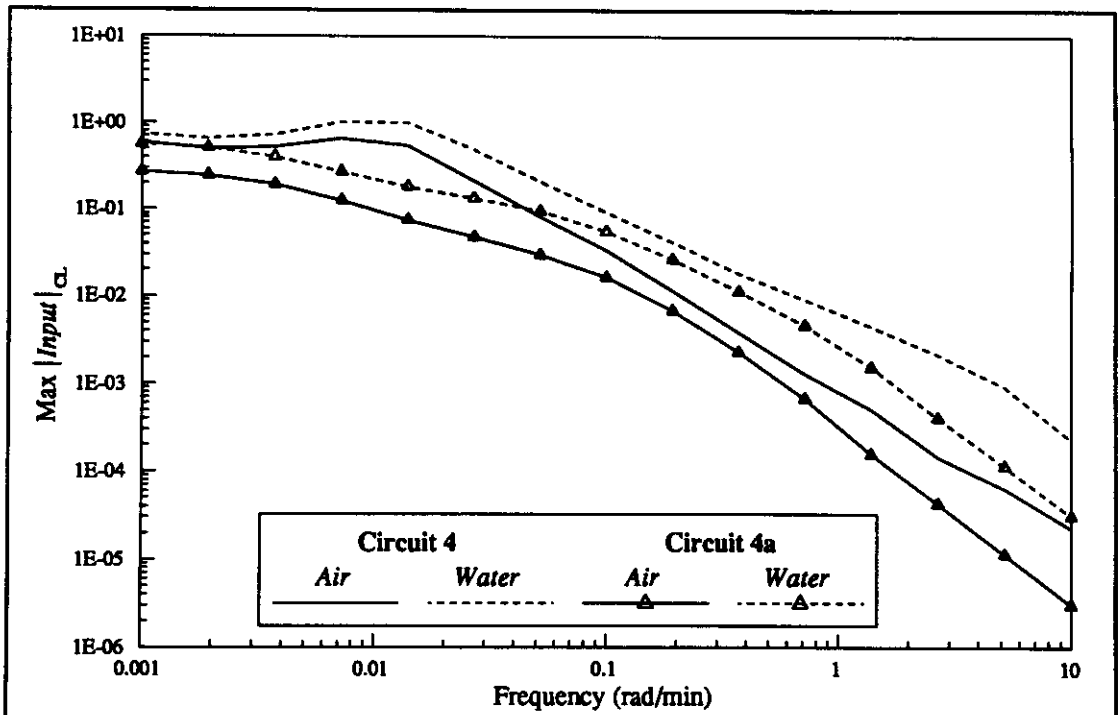


Figure 6.29b - $|Inputs|_{CL}$, Rate Constant Uncertainty, Circuit 4 *cf.* 4a

6.4.3 Optimal Plant Performance

In Section 6.4.1 the effect that uncertainty in the steady-state transfer function ($G(0)$) has in determining the optimum was discussed. Three optima were defined, the *Best* optimal recovery, where $G(0)$, the *Air* and *Water* were all varied to determine the highest *Recovery* that could be achieved, the *Nominal* optimum, where $G(0)$ was set to its nominal value ($k_2 = 0$), and the *Worst* optimum, where $G(0)$ was varied to give the lowest maximum *Recovery* obtained by varying *Air* and *Water*. To determine these achievable optima, the "backoff" from constraints, governed by the largest input and output magnitudes due to the disturbances under consideration, need to be considered. In order to calculate these optima, the basic Optimal Operation Algorithm (Equation 6.28) was modified as detailed below.

The appropriate Objective Functions are based on those given in Equations 6.48, 6.49 and 6.50. These were modified to use the k_2 parameter to describe the $G(0)$ variation, as shown in Equations 6.62, 6.63 and 6.64.

Objective Function (see 6.62, 6.63, 6.64)

such that

$$y_{G,0} \geq 60\% + \Delta y_G$$

$$0.5 + \Delta u_A \leq u_{A,0} \leq 2.0 - \Delta u_A$$

$$0.8 + \Delta u_W \leq u_{W,0} \leq 2.0 - \Delta u_W$$

$$(y_{R,0}, y_{G,0}) = M(u_{A,0}, u_{W,0}, k_2) \dots \text{Steady-State Model} \quad (6.61)$$

$$0 \leq k \leq 1$$

$$-1 \leq k_2 \leq 1$$

$$\left. \begin{aligned} \Delta y_G &= f_1(k) \\ \Delta u_A &= f_2(k) \\ \Delta u_W &= f_3(k) \end{aligned} \right\} \text{Partial Control Functions}$$

Objective Functions

$$\begin{array}{lll} \text{(from 6.48)} & \text{Best:} & \text{Maximise Recovery} \\ & & \text{Air, Water, } k_2 \end{array} \quad (6.62)$$

$$\begin{array}{ll}
 \text{(from 6.49)} & \text{Nominal: } \begin{array}{l} \text{Maximise } \text{Recovery} \\ \text{Air, Water} \\ (k_2 = 0) \end{array} \quad (6.63)
 \end{array}$$

$$\begin{array}{ll}
 \text{(from 6.50)} & \text{Worst: } \begin{array}{l} \text{Minimise} \\ k_2 \end{array} \left\{ \begin{array}{l} \text{Maximise } \text{Recovery} \\ \text{Air, Water} \end{array} \right\} \quad (6.64)
 \end{array}$$

Once again the calculations were performed within the MATLAB framework. The "Worst" case involved the maximisation optimisation to be called from within the minimisation routine. Table 6.22 lists the results for the optimal achievable performance for each circuit, for each optima, for 0%, 5%, 10%, 20% and 40% disturbance levels. The results are also displayed in Figure 6.30, where the range of optima for each circuit is grouped together at a given disturbance level, in the same order as that shown for the 40% disturbance level.

The first observation to be made about these results is that, except for Circuit 4a at the 5% disturbance level, all circuits are optimal when operated open-loop. The investigation of the effect of different controller design specifications, Circuits 4 *cf* 4a, shows that a "lower performance" controller (4a) can achieve slightly better optimal operation than a conventional controller (4) with partial feedback of the error signal ($k < 1$). That is to say, given the opportunity to have partial control, Circuit 4 achieved its best result open-loop - a "detuned" high performance controller does not necessarily behave the same as a "lower performance" controller. In any case, the improvement in recovery of Circuit 4 over Circuit 4a at less than 0.3% is only marginal. Consequently, with all other cases being optimal with open-loop operation, the results obtained are very much the same as encountered with rise time uncertainty. Clearly then, for these circuits, moderate amounts of uncertainty appearing in the process models lead to the optimal operation being open-loop. Uncertainty has reduced the economics of operation. One other observation is that the "Best" optima all have $k_2 = -1$ and the "Worst" optima all have $k_2 = 1$ indicating a consistent trend in the effect of steady-state uncertainty.

Looking at Figure 6.30, it can be seen that the range of possible optimal recovery increases with increasing disturbance level. This is to be expected, since at higher

d_0	Optima	Recovery	Grade	Air	Water	k	k_2
0%		94.873	60	1.017	0.8	0	-
5%	Best	94.465	61.326	0.960	0.8	0	-1
	Nominal	94.386	61.326	0.961	0.8	0	0
	Worst	94.276	61.326	0.961	0.8	0	1
10%	Best	94.057	62.653	0.903	0.8	0	-1
	Nominal	93.900	62.653	0.904	0.8	0	0
	Worst	93.679	62.653	0.904	0.8	0	1
20%	Best	93.241	65.305	0.788	0.8	0	-1
	Nominal	92.926	65.305	0.791	0.8	0	0
	Worst	92.484	65.305	0.791	0.8	0	1
40%	Best	91.610	70.610	0.558	0.8	0	-1
	Nominal	90.979	70.610	0.564	0.8	0	0
	Worst	90.096	70.610	0.565	0.8	0	1

Table 6.22a - Circuit 1, Flotation Rate Constant Uncertainty, Optimal Operation Results

d_0	Optima	Recovery	Grade	Air	Water	k	k_2
0%		94.953	60	1.014	0.8	0	-
5%	Best	94.578	61.325	0.957	0.8	0	-1
	Nominal	94.504	61.325	0.958	0.8	0	0
	Worst	94.399	61.325	0.958	0.8	0	1
10%	Best	94.203	62.649	0.901	0.8	0	-1
	Nominal	94.054	62.649	0.902	0.8	0	0
	Worst	93.846	62.649	0.903	0.8	0	1
20%	Best	93.452	65.298	0.787	0.8	0	-1
	Nominal	93.154	65.298	0.790	0.8	0	0
	Worst	92.738	65.298	0.791	0.8	0	1
40%	Best	91.950	70.597	0.560	0.8	0	-1
	Nominal	91.355	70.597	0.566	0.8	0	0
	Worst	90.522	70.597	0.568	0.8	0	1

Table 6.22b - Circuit 2, Flotation Rate Constant Uncertainty, Optimal Operation Results

d_0	Optima	Recovery	Grade	Air	Water	k	k_2
0%		94.592	60	0.853	0.8	0	-
5%	Best	93.974	61.323	0.803	0.8	0	-1
	Nominal	93.785	61.323	0.803	0.8	0	0
	Worst	93.501	61.323	0.801	0.8	0	1
10%	Best	93.356	62.646	0.753	0.8	0	-1
	Nominal	92.978	62.646	0.752	0.8	0	0
	Worst	92.410	62.646	0.749	0.8	0	1
20%	Best	92.120	65.292	0.653	0.8	0	-1
	Nominal	91.364	65.292	0.651	0.8	0	0
	Worst	90.228	65.292	0.645	0.8	0	1
40%	Best	89.611	70.583	0.5	0.869	0	-1
	Nominal	88.081	70.583	0.5	0.875	0	0
	Worst	85.770	70.583	0.5	0.898	0	1

Table 6.22c - Circuit 4, Flotation Rate Constant Uncertainty, Optimal Operation Results

d_0	Optima	Recovery	Grade	Air	Water	k	k_2
0%		94.592	60	0.853	0.8	0	-
5%	Best	94.118	60.365	1.218	1.358	1	-1
	Nominal	93.967	60.365	1.210	1.358	1	0
	Worst	93.754	60.365	1.201	1.358	1	1
10%	Best	93.356	62.646	0.753	0.8	0	-1
	Nominal	92.978	62.646	0.752	0.8	0	0
	Worst	92.410	62.646	0.749	0.8	0	1
20%	Best	92.120	65.292	0.653	0.8	0	-1
	Nominal	91.364	65.292	0.651	0.8	0	0
	Worst	90.228	65.292	0.645	0.8	0	1
40%	Best	89.611	70.583	0.5	0.869	0	-1
	Nominal	88.081	70.583	0.5	0.875	0	0
	Worst	85.770	70.583	0.5	0.898	0	1

Table 6.22d - Circuit 4a, Flotation Rate Constant Uncertainty, Optimal Operation Results

d_0	Optima	Recovery	Grade	Air	Water	k	k_2
0%		95.040	60	0.888	0.8	0	-
5%	Best	94.549	61.326	0.836	0.8	0	-1
	Nominal	94.407	61.326	0.836	0.8	0	0
	Worst	94.201	61.326	0.835	0.8	0	1
10%	Best	94.057	62.651	0.784	0.8	0	-1
	Nominal	93.773	62.651	0.784	0.8	0	0
	Worst	93.362	62.651	0.781	0.8	0	1
20%	Best	93.075	65.303	0.679	0.8	0	-1
	Nominal	95.507	65.303	0.679	0.8	0	0
	Worst	91.684	65.303	0.675	0.8	0	1
40%	Best	91.036	70.606	0.5	0.859	0	-1
	Nominal	89.869	70.606	0.5	0.862	0	0
	Worst	88.143	70.606	0.5	0.883	0	1

Table 6.22e - Circuit 8, Flotation Rate Constant Uncertainty, Optimal Operation Results

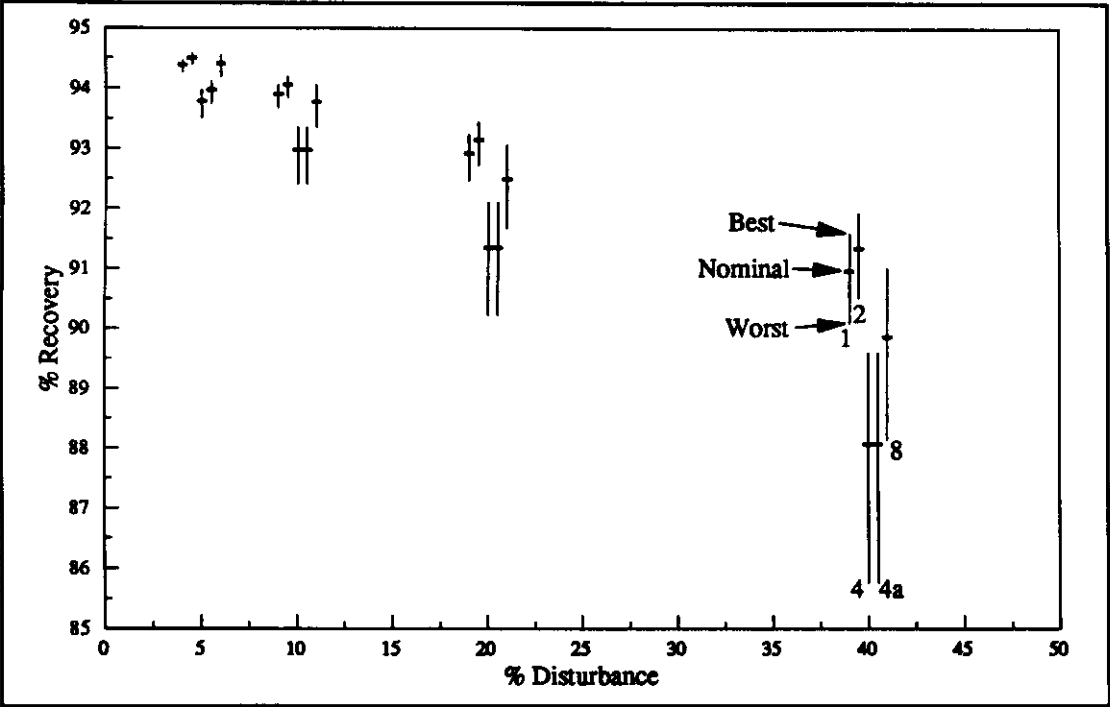


Figure 6.30 - Optimal Operation Results, Flotation Rate Constant Uncertainty

disturbance levels, the "backoffs" necessary are proportionally higher. It is also interesting to note that the recovery ranges for Circuits 4 and 8 are greater than those for Circuits 1 and 2. This implies that the appropriate gains for these former circuits are more susceptible to error, which is in agreement with the higher steady-state condition number observed in Section 6.1. The condition number is a measure of the susceptibility of a matrix to errors, or changes, in its elements.

Up to this point, many of the designs have been shown to be optimal when operated open-loop. Explanations have been offered for this observation, based on the relative cost of "backoff" from input and output constraints. However, if this analysis was being performed with a view to actually constructing one of these circuits, then given recent industrial trends, it is unlikely that the design constructed would be operated open-loop. This suggests two ways to interpret the open-loop results. Firstly, it could be an indication of input saturation, that is, that the nominal (undisturbed) optimal value for one or more inputs is too close to constraints. For example, if the minimum allowable values for *Air* and *Water* were reduced, then it may be the case that optimal operation could occur closed-loop while still allowing sufficient room for "backoff" from these constraints. The second interpretation is that if the reason that an open-loop circuit is unacceptable is excessive output variation, then the objective function of the optimisation did not capture all the requirements of the designer. An example of how such a modification to the Optimal Operation Algorithm can be achieved is given in Section 6.5.

This section completes the inclusion of uncertainty descriptions into the determination of the achievable optimal operation. Both the steady-state and dynamic aspects of process model uncertainty can be explicitly accounted for when determining how well a given design may operate. In addition, uncertainty in the disturbance transfer function can be analysed and the relevant variation included in the analysis. The result of applying this technique is a measure of how well a given design will perform under the worst circumstances, or for the case of steady-state uncertainty, a range of circumstances. This analysis is applicable at the process design stage, where badly performing designs may be discarded, and better prospects more fully investigated.

Such additional analysis would include considering the use of a more practical controller, rather than the "best possible" H_{∞} design for those cases where closed-loop control is

optimal. In fact, if it was desired, other controllers could be used at all stages of the design process. For example, multi-loop controllers, possibly multiple SISO Proportional plus Integral (PI) controllers, can be designed and incorporated into the sensitivity function matrix, and hence the calculation of input and output magnitudes. The sensitivity function matrix for multiple SISO PI controllers is a diagonal matrix with the PI controller transfer functions as the diagonal elements. The reason that H_∞ controllers were used in this work is that they provide a consistent basis for comparison between designs, although consistent approaches can also be adopted for the PI controllers. One advantage of the H_∞ approach is the ability to include uncertainty descriptions into the controller design.

The Optimal Operation Algorithm can also be used to highlight at an early stage in the design process, sources of potential control problems (e.g. input saturation) which may cause a given design to operate poorly, with regard to disturbance attenuation or robustness in the face of process uncertainty.

In the next section, a different aspect of optimal operation will be analysed, product quality. In all the studies to date, the only aspect of product quality that has been important has been that a given constraint ($Grade \geq 60\%$) is never exceeded. However, this still allows the instantaneous value to vary over an arbitrarily wide range. Today, this sort of product variation is often unacceptable for quality focused operations.

6.5 Cost Penalised Output Variation

The disturbances which affect the operation of these flotation circuits are changes in the feed rate and feed grade. In Section 6.2.2, these disturbances were described using an upper bound on magnitude in the frequency domain. For any frequency, this bound limits the magnitude of the disturbance. Although the actual disturbances may not be sine waves of a given frequency, any other wave form may be broken down into its constituent sine waves, and hence included in the bounding function. "Real" disturbances will generally be time varying in nature, causing oscillation in the process inputs and outputs. This variation in product quality, here the concentration of valuable material (or *Grade*), can affect the suitability of the product for further processing. This problem will be addressed within the framework of the Optimal Operation Algorithm in this section.

6.5.1 Penalising Variation

To illustrate the effect of output variation, the results obtained for Circuit 4 in Section 6.4.3 (flotation rate constant uncertainty) will be analysed. Two different controllers were designed for Circuit 4, a conventional H_{∞} controller and a lower performance H_{∞} controller referred to as Circuit 4a. Table 6.22c shows that at the 5% disturbance level, Circuit 4 is operated optimally when open-loop, with a mean Nominal *Recovery* of 93.785% at an average *Grade* of 61.323%. Table 6.22d shows that Circuit 4a operates optimally with full feedback ($k = 1$) at a mean Nominal *Recovery* of 93.967% and an average *Grade* of 60.365%.

This means that for Circuit 4, the instantaneous *Grade* value could lie anywhere in the range 60 to 62.646%, while for Circuit 4a the range is only 60 to 60.730%. If the frequency of the disturbance is very low, then it is possible that the *Grade* of the product could stay close to these extremes for long periods. Clearly, the open-loop case (Circuit 4) could produce large amounts of concentrate at 62% *Grade* while at other times it would be at 60%. However, the closed-loop case (Circuit 4a) will have much less variation in the final product. This marked difference in variation could significantly affect downstream processing, as for example the higher *Grade* material may be more difficult to process further. It could be argued therefore, that there should be an upper limit placed on the *Grade* to restrict the amount of "difficult to process material" produced. However, manufacturing experience shows that it is often the unexpected change in quality, rather than the actual level of the quality parameter that causes processing difficulty.

To reflect the "costs" associated with the variance in product properties, it is possible to modify the objective function used in determining the optimal recovery. The approach adopted is to use a "penalty" function, so that as the variation in the *Grade* increases, a penalty is placed on the *Recovery*. This causes the optimiser to find a balance between high *Recovery* and large variance. In practice, it should be possible to determine an exact function relating the true cost of the variation, to the value of the product. Here a representative function has been chosen to indicate the behaviour of the process. The three points shown in Table 6.23 were chosen as defining the penalty function.

Effectively, if the *Grade* varies by 2% (e.g. 60% Average to 62% Maximum) then the calculated *Recovery* is multiplied by 0.95 (e.g. 90% $\rightarrow$ 85.5%). Equation 6.65, shown also

Variation in <i>Grade</i> (Percentage Points)	Factor Applied to <i>Recovery</i>
0	0
2	0.95
5	0.8

Table 6.23 - Penalty Function Data Points

in Figure 6.31, fits these three points. The quadratic shape of the function causes the penalty to increase rapidly as the *Grade* variation (Δy_G) increases.

$$\text{Penalty Factor} = 1 - 0.005 \times \Delta y_G^2 - 0.015 \times \Delta y_G \quad (6.65)$$

To demonstrate the application of this technique, the simplest case previously studied (no uncertainty, unweighted controller, see Section 6.2), was used. Equation 6.66 shows the Optimal Operation Algorithm including the penalised objective function.

$$\text{Maximise } \{ \text{Recovery} \times (1 - 0.005 \times \Delta y_G^2 - 0.015 \times \Delta y_G) \}$$

$$u_{A,0}, u_{w,0}$$

such that

$$y_{G,0} \geq 60\% + \Delta y_G$$

$$0.5 + \Delta u_A \leq u_{A,0} \leq 2.0 - \Delta u_A$$

$$0.8 + \Delta u_w \leq u_{w,0} \leq 2.0 - \Delta u_w$$

(6.66)

$$(y_{R,0}, y_{G,0}) = f(u_{A,0}, u_{w,0}) \dots\dots \text{Steady-State Model}$$

$$0 \leq k \leq 1$$

$$\Delta y_G = f_1(k)$$

$$\Delta u_A = f_2(k)$$

$$\Delta u_w = f_3(k)$$

} Partial Control Functions

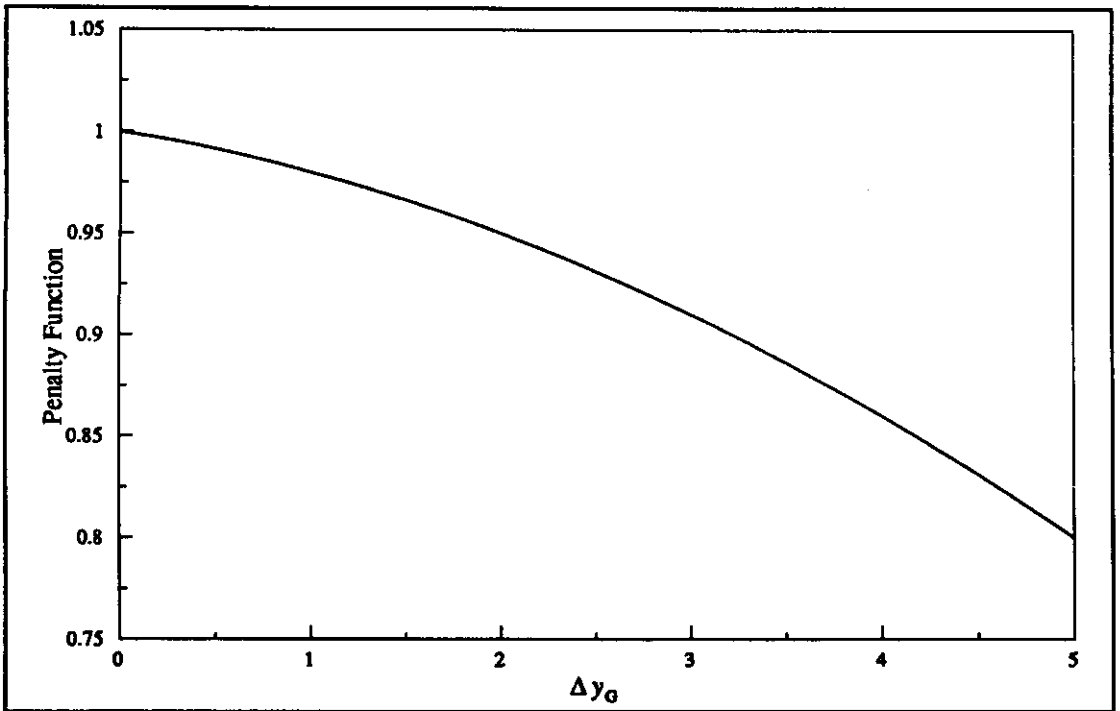


Figure 6.31 - Output Variation Penalty Function

The input and output magnitudes, together with the partial control functions determined in Section 6.2 were used to determine the achievable optimal *Recovery*.

6.5.2 Optimal Plant Performance

Not surprisingly, the optimal results obtained with the penalty applied to output variation are quite different to previous results. In particular, at lower disturbance levels, optimal operation occurs with full feedback closed-loop control, minimising the magnitude of the disturbance appearing at the output, and hence reducing the penalty. At high disturbance levels, where there is insufficient input magnitude available, the optimal operation is once again open-loop.

The optimal operating points are given in Table 6.24. Table 6.25 gives two comparisons, the first between the optimal and the open-loop objective functions, and the second between the original "unpenalised" optimal *Recovery* and the penalised value determined here. Figure 6.32 shows the objective function versus disturbance level, and Figure 6.33 the penalty optimised *Recovery*.

As indicated in Table 6.24, optimal operation is achieved with full ($k = 1$) closed-loop control for most cases. In particular, Circuits 1 and 2 are optimal at $k = 1$ until the point where there is insufficient input magnitude available, then they "switch" to open-loop optimal. By contrast, Circuits 4 and 8 benefit from some control down to very low levels of the partial control factor ($k \rightarrow 0.1$), neither actually becoming open-loop. The disparity may be due to a fundamental difference in the circuits; generating partial control functions which give greater benefits to Circuits 4 and 8 for reducing the amount of error fed back to the controller, than for Circuits 1 and 2.

Table 6.25 shows for comparison the value of the penalised objective function when optimised at open-loop ($k = 0$). This clearly shows the benefit of control ($k \neq 0$) in reducing the cost associated with output variation. Also shown are the optimal *Recoveries* with and without the output variation penalty. Although control improves the objective function, it is at the expense of less recovered material, the decrease usually being quite small ($< 1.5\%$). Figures 6.32 and 6.33 are quite revealing about the nature of the performance of the circuits when optimised so as to restrict output variation. If the objective function truly represents the value of the product (i.e. includes the true cost of

Circuit 1	Objective	Recovery	Grade	Air	Water	k
$d_0 = 0\%$	94.873	94.873	60	1.017	0.8	-
$d_0 = 5\%$	94.274	94.288	60.010	1.011	0.887	1
$d_0 = 10\%$	93.676	93.704	60.020	1.004	0.974	1
$d_0 = 20\%$	92.479	92.535	60.040	0.991	1.148	1
$d_0 = 40\%$	25.295	90.979	70.610	0.564	0.8	0
Circuit 2	Objective	Recovery	Grade	Air	Water	k
$d_0 = 0\%$	94.953	94.953	60	1.014	0.8	-
$d_0 = 5\%$	94.027	94.101	60.052	1.003	0.949	1
$d_0 = 10\%$	93.100	93.249	60.103	0.992	1.099	1
$d_0 = 20\%$	91.242	91.544	60.206	0.970	1.397	1
$d_0 = 40\%$	25.555	91.355	70.596	0.566	0.8	0
Circuit 4	Objective	Recovery	Grade	Air	Water	k
$d_0 = 0\%$	94.592	94.592	60	0.853	0.8	-
$d_0 = 5\%$	92.787	93.813	60.607	1.206	1.367	1
$d_0 = 10\%$	89.169	93.082	61.765	1.184	1.4	0.425
$d_0 = 20\%$	77.174	91.536	64.299	1.088	1.4	0.200
$d_0 = 40\%$	35.861	88.356	69.503	0.894	1.407	0.100
Circuit 8	Objective	Recovery	Grade	Air	Water	k
$d_0 = 0\%$	95.040	95.040	60	0.888	0.8	-
$d_0 = 5\%$	92.713	94.074	60.768	1.033	1.160	1
$d_0 = 10\%$	89.742	93.261	61.630	1.116	1.4	0.816
$d_0 = 20\%$	79.734	92.173	63.907	1.026	1.4	0.391
$d_0 = 40\%$	42.576	89.804	68.865	0.830	1.4	0.195

Table 6.24 - Output Penalty Optimal Operation Results

Disturbance	Penalised Objective		Optimal Recovery	
	Optimal	Open-Loop	Penalised	No Penalty
Circuit 1				
$d_0 = 0\%$	94.873	94.873	94.873	94.873
$d_0 = 5\%$	94.274	91.326	94.288	94.386
$d_0 = 10\%$	93.676	86.861	93.704	93.900
$d_0 = 20\%$	92.479	72.457	92.535	92.926
$d_0 = 40\%$	25.295	25.295	90.979	90.979
Circuit 2				
$d_0 = 0\%$	94.953	94.953	94.953	94.953
$d_0 = 5\%$	94.027	91.797	94.101	94.503
$d_0 = 10\%$	93.100	87.017	93.249	94.054
$d_0 = 20\%$	91.242	72.679	91.544	93.154
$d_0 = 40\%$	25.555	25.555	91.355	91.355
Circuit 4				
$d_0 = 0\%$	94.592	94.592	94.592	94.592
$d_0 = 5\%$	92.787	91.104	93.813	93.814
$d_0 = 10\%$	89.169	86.035	93.082	93.078
$d_0 = 20\%$	77.174	71.324	91.536	91.557
$d_0 = 40\%$	35.861	24.783	88.356	88.082
Circuit 8				
$d_0 = 0\%$	95.040	95.040	95.040	95.040
$d_0 = 5\%$	92.713	91.700	94.074	94.407
$d_0 = 10\%$	89.742	86.749	93.261	93.773
$d_0 = 20\%$	79.734	72.145	92.173	92.507
$d_0 = 40\%$	42.576	25.042	89.804	89.855

Table 6.25 - Output Penalty Optimal Operation Results Comparison

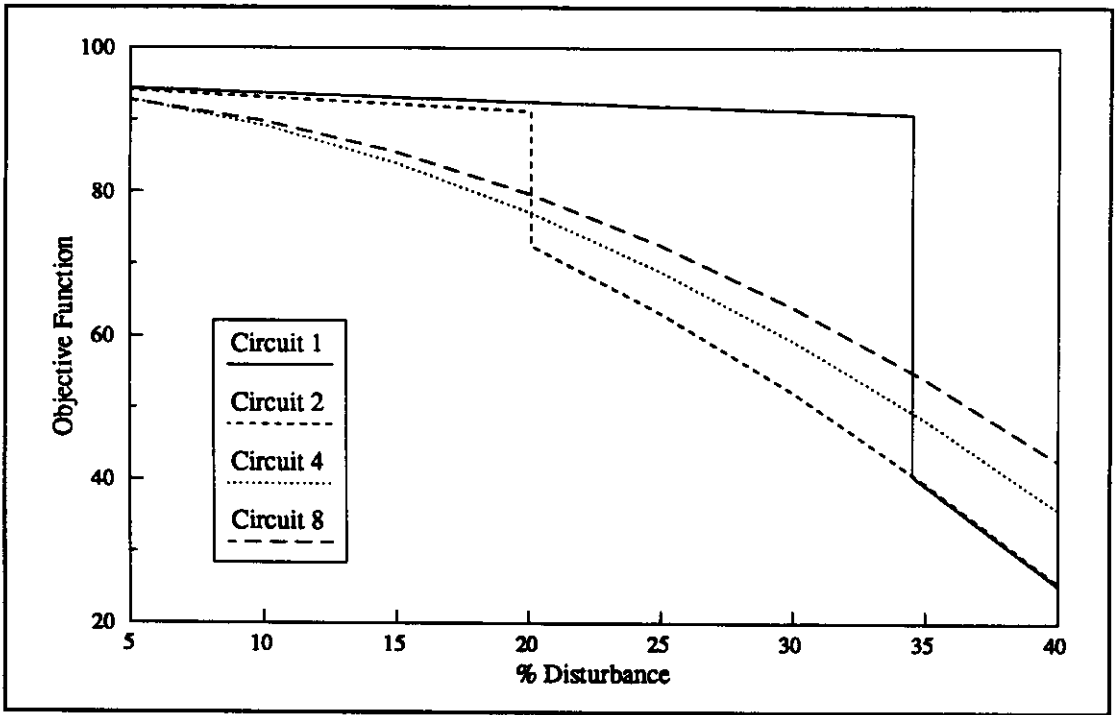


Figure 6.32 - Penalised Output Variance, Objective Function

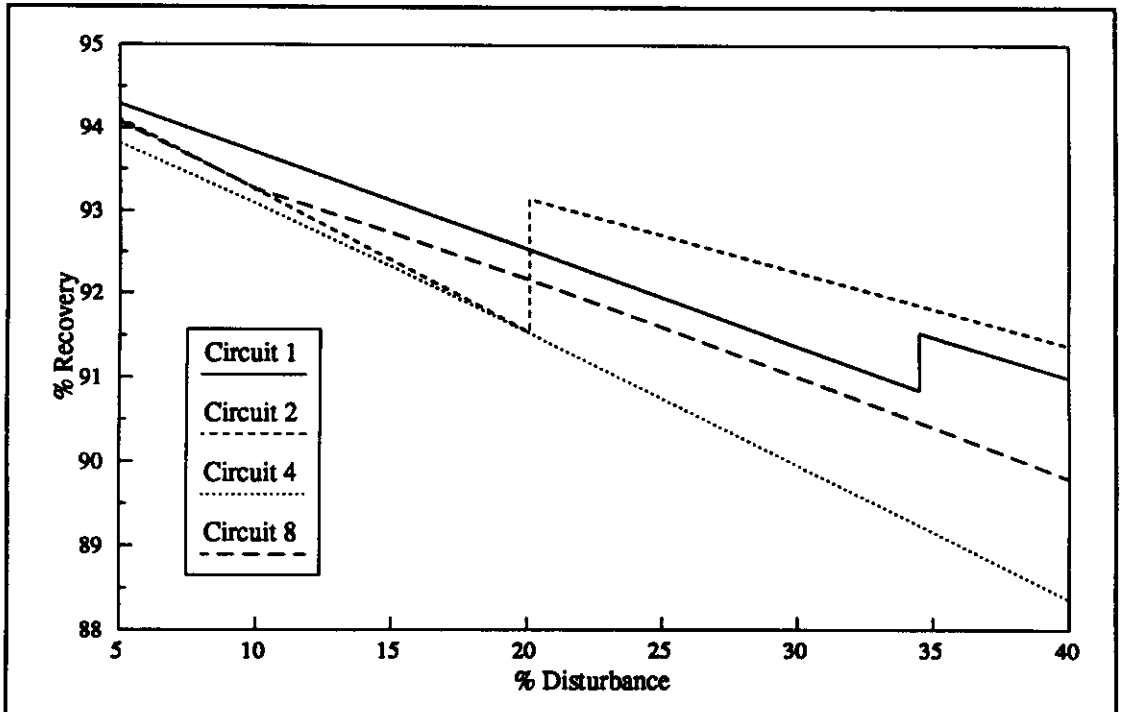


Figure 6.33 - Penalised Output Variance, Optimal *Recovery*

variation), then Circuit 1 is clearly the best performer, up to quite high levels of disturbance. Circuit 2 behaves similarly, although only to the 20% disturbance level. The higher levels of the objective function are due to the greater ease with which the outputs of these two circuits can be controlled. This was identified previously in Section 6.1, where the open-loop controllability measures indicated that Circuits 1 and 2 should be easier to control than Circuits 4 and 8. However, beyond the point where control is possible, Circuits 4 and 8 give higher recoveries than Circuits 1 and 2, as was the case for the original no uncertainty analysis.

The mean *Recovery* values shown in Figure 6.33 show trends consistent with Figure 6.32. Interestingly, at low disturbance levels the *Recovery* of Circuit 2 is comparable with Circuits 4 and 8, however the larger output variance associated with these latter circuits reduces the objective function as shown in Figure 6.32. Circuits 1 and 2 both show step increases in the mean *Recovery* where optimal operation "switches" from closed-loop to open-loop. This indicates that higher recoveries could be obtained operating open-loop, however when this is balanced against the cost of the output variance, the overall performance (as measured by the objective function) suffers. Circuits 4 and 8 have a smooth decrease in the partial control factor as the disturbance level increases, so that a step change in the objective function and *Recovery* are not observed.

This section has demonstrated that the techniques developed in this chapter for determining the achievable optimal operation for a process design can be "tailored" to suit the required design or performance analysis. In particular, process and disturbance model uncertainty, as well as objective function modifications (e.g. to take into account quality variation), have been incorporated into the proposed procedure, the Optimal Operation Algorithm. In the next section, results from the application of the algorithm to the flotation circuit example will be summarised, and the general applicability of the procedure reviewed.

6.6 Comparison of Results

The aim of this chapter was to apply the Optimal Operation Algorithm, developed in Chapter 5, to a more substantial, multivariable case study - the alternative flowsheet selection for a flotation circuit problem. In addition, extensions of the algorithm to incorporate process and disturbance model uncertainty descriptions, and objective function

modifications such as an output variance penalty, have been developed and applied. The result in each case has been an estimate of the likely *Recovery* that will be achieved in the face of the expected feed rate and feed grade disturbances. For each of these optimal operating points, the amount of control, the value of the partial control factor (k), was also determined. Surprisingly, for each of these flotation circuits the optimum was frequently open-loop. In such cases, the "cost" associated with the necessary "backoff" from input constraints was too high to allow the input manipulation required by closed-loop control.

The problem originally proposed by Chan (1988) was to find the optimal flotation circuit for fixed inputs, and a fixed total cell volume. Chan found that the two most attractive circuits from a steady-state point of view (4 and 8), were of an unusual configuration, and when their "controllability" was checked, there were indications that they may be more difficult to control than the next best, more conventional circuits (1 and 2). When the manipulated inputs (*Air* and *Water*) were included in the optimisation as additional freed variables, these four circuits obtained almost equal *Recovery*. Given the choice between potential control problems, Circuits 4 and 8, and the more easily controllable Circuits 1 and 2 for the same *Recovery*, most designers would choose the latter. However, as was found in this chapter, when the effects of likely disturbances, process and disturbance model uncertainty, and different objective functions are included, the choice becomes more complex.

These extensions to the basic flowsheet selection problem were performed using the Optimal Operation Algorithm (Equation 6.28, 6.61, 6.66). Table 6.26 summarises the optimal operation results at the 5% disturbance level, listing the optimal recoveries and the partial control factor. The original undisturbed ($d_0 = 0\%$) optima are also given for comparison. Thus, the first observation to make is that in all cases, the presence of disturbances reduces the economic potential of each circuit by lowering the achievable *Recovery*.

The first case studied (Section 6.2) included the effect of disturbances on the economics, but assumed no process uncertainty. The original ("standard") H_∞ controller, designed to control both *Recovery* and *Grade*, achieved optimal operation for Circuits 1, 2 and 8 when open-loop, with a significant reduction in the average *Recovery*. Circuit 4 operated

Circuit Variable	$d_0 = 0\%$	No Uncertainty		Rise Time	Rate Constant	Output Variance
		Standard	Weighted			
1. R k	94.873	94.386	94.531	94.386	94.386	94.288
	-	0	1	0	0	1
2. R k	94.953	94.503	94.503	94.503	94.504	94.101
	-	0	0	0	0	1
4. R (4a. R) k (4a. k)	94.592	93.184	94.440	94.151	93.785 (93.967)	93.813
	-	1	1	0.52	0 (1)	1
8. R k	95.040	94.407	94.700	94.407	94.407	92.713
	-	0	1	0	0	1

Table 6.26 - Comparison of Results, 5% Disturbance Level

optimally with some control up to the 40% disturbance level. Overall, in this case, Circuits 1 and 2 could be operated better than 4 and 8, especially at higher disturbance levels.

Since only the *Grade* output is involved in determining the optimal operation, it is not as important to control the *Recovery* output to the same extent. The "high performance" controllers developed in Section 6.2.5 had a weighting of 0.001 applied to the *Recovery* portion of the sensitivity function weight, effectively achieving little control of this variable, and using both *Air* and *Water* to control *Grade*. The closed-loop magnitudes reflected this, with larger values obtained for *Recovery* and substantially better control for *Grade*. When optimised, the result was more use of control action at low disturbance levels, which improved optimal recoveries. This allowed the performance of Circuit 8 to improve enough to be the best (by almost 1% extra *Recovery*) up to the 20% disturbance level.

In Section 6.3, the variation in the bubble rise time introduced dynamic uncertainty into the calculation of the achievable optimal recovery. Specifically, this uncertainty in the process model affected both the controller design and the calculation of the maximum input and output magnitudes for closed-loop operation. The rise time variation was modelled as a multiplicative output uncertainty (L_o), by matching the maximum singular value of the relative model error to the W_3 transfer function. (W_3 is used in the H_∞ controller design as an upper bound on the complementary sensitivity function, to guarantee robust stability in the face of uncertainty.) Although this uncertainty did not affect the open-loop magnitudes, the closed-loop values increased significantly. The net result was that control became "too expensive", and the optimal operation of all circuits (except Circuit 4 at the 5% disturbance level) was open-loop. In this case at high disturbance levels, Circuits 1 and 2 were capable of achieving slightly higher recoveries than Circuit 8 and substantially more than Circuit 4. At low levels, all circuits achieved similar results.

Also studied in this section was the effect of varying the uncertainty level. Circuit 8 was analysed for bubble rise time uncertainties of 0, 2, 5, and 10%. As the uncertainty level rose, the economic performance of the circuit fell. The addition of 2% uncertainty only caused a small change from the original no uncertainty case, with some control being useful up to the 20% disturbance level. 5% uncertainty had a greater effect, dropping recovery slightly from the no uncertainty value, and only using control action to about the 15% disturbance level. As reported above, 10% uncertainty set the lower limit on performance, with all circuits operating open-loop at all disturbance levels.

A more realistic type of uncertainty, due to flotation rate constant variation, was studied in Section 6.4. This uncertainty affected both the steady-state and dynamic characteristics of the circuits, causing variations in both the process and disturbance transfer functions. This meant that both the open-loop and closed-loop magnitudes were subject to the effects of variation in the rate constants. Interestingly, this uncertainty meant that better controllers could be designed for Circuits 4 and 8, leaving Circuit 2 with the worst performance. The steady-state uncertainty also caused changes in the steady-state model used in the optimisation. To incorporate this, three optima (maximum recoveries) were determined, the "Best", the "Nominal" and the "Worst". This uncertainty also led to optimal operation being open-loop, giving much the same results as obtained for no

uncertainty (unweighted controllers) and for bubble rise time variation.

The other noteworthy aspect of these results was that the spread between the "Best" and "Worst" optima for any given circuit was greatest for Circuits 4 and 8. This is consistent with the larger condition number for these circuits, indicating a greater susceptibility to errors in the model.

The final modification made to the Optimal Operation Algorithm was the addition of a penalty to the objective function to include the costs associated with output variation, here *Grade*. This modified algorithm was applied to the no uncertainty unweighted controller case. The use of the penalty function significantly changed the optimal operation results. To avoid the reduction in the objective function due to output variation, the optima were now substantially closed-loop. Indeed, Circuit 1 stands out as achieving the highest objective function up to large disturbance levels (35%), beyond which there was insufficient input manipulation available to maintain control. Similarly, Circuit 2 performed well up to a 20% disturbance level. The performance of Circuits 4 and 8 was improved by some partial control over the entire disturbance range.

In determining the likely achievable performance of a given process design (or control scheme), the effects of disturbances, process and disturbance model uncertainty, and the true operational objective of the given design are all required. Armed with this information, the Optimal Operation Algorithm can be applied to determine the optimal operating point (and hence objective function value) that will ensure input and output constraints will not be violated due to the impact of disturbances. Also calculated is a measure of the quality of control required, from open-loop ($k = 0$) to full closed-loop ($k = 1$). If deficiencies in the achievable optimal performance of a given design can be traced back to design limitations, then this information may be used to modify a given design, or to choose an alternative.

This chapter has demonstrated that it is possible to bring together process economics and controllability, to give a measure of overall performance in economic terms. The explicit inclusion of model uncertainty can increase the designer's confidence in the reliability of the results produced.

CHAPTER 7

CONCLUSIONS AND RECOMMENDATIONS

Significant economic benefits can be achieved through the use of optimising control. However, the uncertainties usually encountered in process design mean that operation at the nominal optimum may not be achieved. In addition, the impact of disturbances usually requires operation away from otherwise optimal process limits. The work described in this thesis has investigated existing optimising control theory and practice, and incorporated recent mathematical and control theory developments into an overall approach to aid in the design and evaluation of optimising control schemes.

7.1 Conclusions

The review conducted here of existing optimising control theory and applications, indicated that although a theoretical basis exists for designing such systems, reported applications are limited in number. Elaborate schemes have been proposed in the literature to calculate optimal setpoints as plant parameters change, however the most practical approach for industry is still "constraint control". This is almost a "common-sense" approach that has been practised for some time now to obtain optimal operation by running at process limits, giving optimal operation without actually designing an optimising control scheme. Constraint control formed the conceptual basis for the optimal operation analysis presented in Chapters 5 and 6.

The distillation modelling work contained in Chapter 2 confirmed the usefulness of the "compartmental" simplified modelling technique. In addition, the value of open-loop controllability indicators, such as those based on singular value analysis, in predicting closed-loop performance was demonstrated. A simple constraint control scheme was readily identified for the C_3 Splitter column, however this did not take into account how model uncertainty and process disturbances could affect the performance achieved once this control scheme was implemented.

Optimising control can be implemented via the use of off-line or on-line calculated optimal setpoints, however, there will always be some concern over the accuracy of these

setpoint values. In Chapter 3, some aspects of the effect of this uncertainty on the achievable performance were discussed. The most thorough appraisal was provided by a singular value extension to existing nonlinear programming sensitivity analysis. This produces a ranking of the susceptibility of candidate setpoints to "worst case" parameter uncertainty. The application of this theory proved to be quite complex, although some reduced-order methods were presented which could improve the usefulness of this technique. Simpler methods were also presented (for the selection of setpoints to which the objective function is insensitive, and an equal relative error indicator), both of which proved to be of limited use.

While the theory developed in Chapter 3 can be used to predict which setpoints may be the best to use, it does not really address the problem of determining the expected economic return once the plant is operational. In Chapter 4, the impact of disturbances were categorised into those having linear and nonlinear effects. Linear behaviour in response to disturbances shows an "averaging" effect, with the only overall cost being due to movement away from constraints. However, the effect of nonlinear behaviour is always detrimental to the objective function. In addition, it was shown that disturbances acting on nonlinear systems cannot be characterised as easily as those acting on linear systems, in effect requiring each possible disturbance to be individually analysed. The result of these investigations into the fundamental nature of systems under optimising control, showed that it is only really possible to accurately predict the performance of systems where the objective function is linear, the steady-state model is linear and a linear (or linearised) dynamic model is used. As demonstrated in two case studies, even with these limitations the theory is adequate for many applications, including performance comparisons between candidate designs.

Within these above restrictions, it is possible to determine how well a given design may operate faced with expected disturbances. The method presented is called the Optimal Operation Algorithm. The steady-state optimisation part of this algorithm, uses a linear steady-state model and constraints on the input and output variables, modified to reflect the amount of input manipulation and output variation expected. Additionally, the amount of control (allowed to vary between open-loop and closed-loop) required to optimally operate the given design is determined. To place candidate designs on an equal footing, the best possible controller is used when determining the input and output magnitudes

induced by the disturbances. For some cases, this is a "perfect" controller, theoretically capable of completely rejecting all disturbances. Other examples use a multivariable H_∞ controller as the best that can be achieved.

A considerable extension to this basic optimisation problem was achieved through the inclusion of various model uncertainty descriptions. Performing the calculation without allowing for any model uncertainty produces the most optimistic prediction of the optimal operation. Model uncertainty was introduced into a number of components of the overall calculation (i) the disturbance and process transfer functions used in determining the open-loop and closed-loop magnitudes, (ii) the H_∞ controller design, and (iii) the steady-state process model, encountered when evaluating the objective function.

The effect of uncertainty on two examples was investigated. The first looked at an illustrative SISO problem, for which the performance of two alternative manipulated variables was checked. The second example looked at the more complex case of alternative flowsheet selection using flotation circuits. Both examples demonstrated that uncertainty in all its forms reduces the economic potential of a given design. From the control perspective, process model uncertainty limits the performance of the controller that can be designed while ensuring robust stability (and possibly, robust performance). In addition, when determining the input and output magnitudes, the variation in the disturbance and process transfer functions means that the quality of control is reduced, increasing output magnitudes while at the same time potentially demanding larger input magnitudes. This not only leads to control becoming less effective, but costing more in terms of input manipulation. The use of these uncertainty descriptions takes the prediction of likely optimal performance closer to that which may be realised in practice. However, the method used is somewhat conservative (as shown explicitly for the SISO example), potentially including a wider range of uncertain models than may actually occur - to the detriment of the calculated optima.

Despite these limitations, the Optimal Operation Algorithm provides a flexible framework within which the process designer can include as much (uncertainty description) information as is available, to determine the optimal performance in terms of a meaningful (e.g. modified objective function) indicator.

The alternative flowsheet synthesis (flotation circuit) example demonstrated how the achievable optimal operating point can be determined. The results are an interesting extension of the work by Chan (1988). For the various uncertainty descriptions used, the effect of disturbances on these circuits reduced the performance below that predicted by Chan. More surprisingly, the more conventional, more controllable Circuits 1 and 2 were predicted to achieve higher recoveries than the less conventional (and less controllable) Circuits 4 and 8, which Chan had predicted to be the best steady-state performers.

Surprisingly perhaps, the optimum for most of these circuits was generally open-loop - input variable manipulation proving to be too expensive in terms of the backoff necessary from input constraints. It is an interesting observation that despite recent trends in industrial applications to commonly install distributed control systems on all new process plant, open-loop operation can still prove to be more profitable, unless the design has been properly analysed.

The concept of partial control introduced into the optimal operation design procedure proved to be a useful modification to the conventional feedback controller. By using this continuous variation between open-loop and closed-loop, the relationship between the changes in the size of the input and output magnitudes, and hence their "cost", was able to be included in the problem description as a constraint modification. Importantly, the analysis of the effect of the partial control factor on these magnitudes shows that "unexpected" trends (such as no reduction in input magnitudes) may be observed when "detuning" a controller. Although partial control as such is unlikely to be implemented on an operational plant, controller detuning (an equivalent procedure) is common.

Two aspects of the effect of controller design were examined. The "high performance" controller used throughout much of Chapter 6 concentrated the control effort on only one output variable, the product *Grade*. This demonstrated that it may be preferable to only expend control effort on those output variables for which there are important operational constraints. The other facet of controller design explored was the benefit associated with careful selection of the performance weighting. In particular for the rate constant uncertainty case, two controllers were designed for Circuit 4. The lower performance controller (requiring less input manipulation), in fact, showed the potential for achieving the higher optimal recovery, although the improvement, at less than 0.3%, is not

particularly significant.

The example presented of objective function modification (here via a quality based penalty function) showed that the evaluation of a given design must encompass those aspects important to the final operation. Usually there will be hard constraints associated with input variables which must not be violated. In addition, there will also be some output variables with similar constraints, however it may be equally important to minimise the variability in some output variables. It is in this latter case that the type of objective function modification demonstrated is useful. Typically this will then lead to the requirement for "good" control, to improve the economics by reducing the variability component of the objective function.

The good agreement observed between the open-loop controllability measures (e.g. process condition number) and the quality of control achieved illustrates the usefulness of these simple to compute measures.

The computational component of these investigations showed that linear control theory is still a powerful tool - more so today than ever before because of the greater availability of computing resources, and recent theoretical advances in areas such as uncertainty analysis and H_∞ controller design. In addition, modern software tools such as SPEEDUP and MATLAB make the complex computations required for this type of research quite possible.

In summary then, the application of an optimising control scheme to a given process design can ensure that the process will operate optimally, within the limitations imposed by model uncertainty and disturbances. This thesis has shown how the inherent controllability of a process may be expressed in economic terms, by incorporating the operating objective, the operational constraints and disturbances, and process uncertainty descriptions through the type of analysis presented in the Optimal Operation Algorithm.

7.2 Recommendations

The benefits indicated in Chapter 2 to be obtained by running the C_3 Splitter optimally, indicate that a constraint control based, optimising control scheme should be applied to

this column. The controllability analysis and closed-loop simulations indicate that the column pressure should be controlled by adjusting the reboiler heat input, and the propane level in the propylene product stream should be controlled by adjusting the product propylene flow. The implementation of such a scheme is currently being performed by the author, as part of an advanced process control project being undertaken by ICI Australia Engineering.

The manifestation of uncertainty as inaccurate, "optimal" setpoint values was the subject of Chapter 3. The two simple methods presented (one based on equal relative error and the other on the selection of setpoints to which the objective function is insensitive), are both worthy of further investigation. If simple procedures can be found for exploiting these techniques, then quick measures of setpoint suitability would be available. Also presented in Chapter 3 was a singular value analysis based extension of the nonlinear programming sensitivity method. The associated calculations are complex, however a technique of eliminating some equations and hence producing sensitivity results for a selection of candidate setpoints was suggested. This technique is also worthy of further investigation.

However, before techniques such as the Optimal Operation Algorithm can be applied, the most suitable setpoints for the optimising control scheme must be selected. This need underlies the importance of the concepts discussed above. Once the setpoints have been selected, then the Optimal Operation Algorithm could benefit from the inclusion of these estimates of potential errors as extra sources of uncertainty.

Chapter 4 introduced the differences between the linear and nonlinear effects of disturbances. Any nonlinearity in the objective function causes the costs associated with the disturbances to be underestimated. However, it is realised that some indicator is needed which shows the relative importance of linear and nonlinear effects, so that those situations where the nonlinear components are significant can be recognised. The restriction on the linearity of the steady-state model is for two reasons, (i) to prevent the propagation of nonlinear behaviour into the objective function, and (ii) the need to ensure that the optimum is completely defined by active inequality constraints. However, an approximate analysis could be performed using a nonlinear steady-state model if the optimum is completely defined by active inequality constraints, indicating only limited

nonlinear behaviour. With this approach, the input and output magnitudes would still be calculated using a linearised dynamic model. For the flotation case example, this would entail performing the optimisation with the full nonlinear model arranged to give the *Recovery* and *Grade* values as output results for specified *Air* and *Grade* values. The feasibility of this hybrid (part linear, part nonlinear) approach should be investigated.

A number of other aspects of the Optimal Operation Algorithm are worthy of further investigation. As discussed in Chapter 5, the form of the algorithm presented only provides for inequality constraints on control inputs and outputs, and only these variables may be included in the objective function. It was proposed in Chapter 5, that other variables could be included by making them "dummy" measurements, and their appropriate variation due to disturbances calculated. This suggestion needs to be formalised and tested. Another limitation identified is that only single source disturbances could be analysed. In the flotation circuit example, disturbances in the feed grade or the feed rate were considered, but not both together. In reality, disturbances will occur simultaneously, and although the effect of the two could simply be added, this is not necessarily the best approach. The most appropriate method for handling multiple disturbances needs to be found.

The methods used for including uncertainty, presented in Chapters 5 and 6, were shown to be quite conservative, allowing for many plants in the analysis which may never actually eventuate. Recent theory uses a "structured" approach to including uncertainty, in a manner closely resembling its actual occurrence in the process. It would be advantageous to make use of this less conservative approach when calculating the input and output magnitudes required for application of the Optimal Operation Algorithm.

Recommendations for future work arising from the flotation circuit example include the use of the steady-state condition number as an indicator of the variation in the optimum and a re-examination of the control structure. The control structure used in Chapter 6 is the same as that used by Chan, to allow comparison with those earlier results. However, when viewed from an optimal operation perspective, the chosen output variables are not necessarily the most appropriate. Indeed, the "high performance" controllers concentrated on controlling the *Grade* at the expense of *Recovery*. An investigation should therefore be made of the most appropriate input and output variables, and also of whether a

multivariable controller is warranted at all. It may actually be the case that an appropriately designed multiloop controller may give equally good performance. As discussed in Chapter 6, multiloop controllers can be incorporated into the Optimal Operation Algorithm through calculation of the appropriate controller transfer function (C), and the sensitivity function (E). However, the ubiquitous question of input and output pairing, and the loss of performance when compared to multivariable control still needs to be answered. Although by using the Optimal Operation Algorithm, performance evaluation may be carried out on economic terms.

Process model uncertainty is an essential element in the economic analysis of the optimal operation of process plant. The link between process design and process control has been shown to be optimising control. The process designer's desire to produce an optimal design, and the process control engineer's requirement to fulfil the operating objective, naturally leads to the use of optimising control. These investigations into the incorporation of optimising control at the design stage, have produced a single economic indicator of optimal operation and inherent controllability.

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APPENDIX I**USE OF RAOULT'S LAW FOR BUBBLE POINT CALCULATION
IN THE C₃ SPLITTER SIMULATION**

In the SPEEDUP simulation of the C₃ Splitter, temperatures were determined in each node using a modified Raoult's Law procedure. The derivation of the modifications used are presented here. The nomenclature used is given at the end of this appendix.

Raoult's Law can be found in any standard thermodynamic text (e.g. Smith and Van Ness, 1982);

$$y_i = \frac{x_i P_i^{sat}}{P} \quad (\text{II.1})$$

This may be rearranged in terms of the K value for component i ;

$$K_i = \frac{y_i}{x_i} = \frac{P_i^{sat}}{P} \quad (\text{II.2})$$

The modified form introduces the liquid activity coefficient γ_i ;

$$K_i = \frac{\gamma_i P_i^{sat}}{P} \quad (\text{II.3})$$

Within the simulation, this equation is used to back calculate the equivalent saturated vapour pressure of component i , using the value of K_i . As shown later, K_i can be calculated from the known relative volatilities A_i in the node. This saturated vapour pressure is then used to determine the temperature, using the temperature-saturated vapour pressure relationship for one component, here propylene.

Again following standard thermodynamic theory, at equilibrium the fugacity in the vapour phase is equal to the fugacity in the liquid,

$$\hat{f}_i^v = \hat{f}_i^l \quad (\text{II.4})$$

The vapour phase fugacity may be expressed in terms of the mole fraction, the activity coefficient and the pressure.

$$\hat{f}_i^v = y_i \hat{\phi}_i P \quad (\text{II.5})$$

While the liquid phase fugacity is expressed in terms of the mole fraction, the liquid activity coefficient and the standard state fugacity.

$$\hat{f}_i^l = x_i \gamma_i f_i^o \quad (\text{II.6})$$

Now the standard state fugacity is taken in the Lewis-Randell sense, that is of the pure liquid at the same temperature and pressure. If we then assume that the fugacity of the pure liquid is the same as that at the same temperature but at the saturated vapour pressure, we can write,

$$\begin{aligned} f_i^o &= f_i \\ &= f_i^{sat} \\ &= \phi_i^{sat} P_i^{sat} \end{aligned} \quad (\text{II.7})$$

Substituting this into Equation II.6 and equating with Equation II.5 gives,

$$y_i \hat{\phi}_i P = x_i \gamma_i P_i^{sat} \phi_i^{sat} \quad (\text{II.8})$$

Assuming that the fugacity coefficient for the vapour phase calculation is equal to the saturated vapour value (but not necessarily equal to 1) gives,

$$y_i P = x_i \gamma_i P_i^{sat} \quad (\text{II.9})$$

Writing this in terms of the K value,

$$\frac{y_i}{x_i} = K_i = \frac{\gamma_i P_i^{sat}}{P} \quad (\text{II.10})$$

$$\therefore P_i^{sat} = \frac{K_i P}{\gamma_i} \quad (\text{II.11})$$

In the above calculations the K value for one component (propylene) was used. However, within the simulation, only the relative volatility A_i is known, for each non-key component. The relative volatility is defined as the ratio of the K value of one component to that of a "key" component, here taken to be methylacetylene.

$$A_i = \frac{K_i}{K_{key}} \quad (\text{II.12})$$

The equation given below shows how $K_{\text{PROPYLENE}}$ can be calculated from the available information. Note that in the summations shown (Σ), the sum is only taken over the non-key components.

$$\begin{aligned} \frac{A_{\text{PROPYLENE}}}{1 + \sum (x_i (A_i - 1))} &= \frac{A_{\text{PROPYLENE}}}{1 + \sum \left(\frac{x_i K_i}{K_{key}} \right) - \sum x_i} \\ &= \frac{A_{\text{PROPYLENE}}}{1 + \sum \left(\frac{y_i}{K_{key}} \right) - (1 - x_{key})} \\ &= \frac{A_{\text{PROPYLENE}}}{\frac{1}{K_{key}} (1 - y_{key}) + x_{key}} \\ &= \frac{K_{key} A_{\text{PROPYLENE}}}{1 - y_{key} + K_{key} x_{key}} \\ &= \frac{K_{key} \frac{K_{\text{PROPYLENE}}}{K_{key}}}{1 - y_{key} + y_{key}} \\ &= K_{\text{PROPYLENE}} \end{aligned} \quad (\text{II.13})$$

Nomenclature

f_i	=	Fugacity of pure component i
f_i°	=	Fugacity of component i in the standard state

$\hat{f}_i^v$	=	Fugacity of component i in the mixture in the vapour phase
$\hat{f}_i^l$	=	Fugacity of component i in the mixture in the liquid phase
K_i	=	Relative volatility of component i
P	=	Pressure
P_i^{sat}	=	Saturated vapour pressure at system temperature
x_i	=	Mole fraction of component i in the liquid phase
y_i	=	Mole fraction of component i in the vapour phase
$\hat{\phi}_i$	=	Vapour phase fugacity coefficient for component i in the mixture
$\hat{\phi}_i^{sat}$	=	Vapour phase fugacity coefficient for pure component i at saturated conditions

APPENDIX II**SPEEDUP LISTING FOR THE C₃ SPLITTER**

```

TITLE
  Distillation Column
****
DECLARE
  TYPE
    FLOW=1:0:5000 UNIT="AMOUNT/TIME",
    MOLES=1:0:10 UNIT="AMOUNT * 1E3",
    ENTHALPY=10:-50:50 UNIT="ENERGY/AMOUNT",
    HEAT=20:0:50 UNIT="ENERGY/SEC",
    TEMPERATURE=300:250:350 UNIT="K",
    PRESSURE=1.3:0.5:3.0 UNIT="MPa",
    SPEC_HEAT=100:0:1000 UNIT="ENERGY/AMOUNT/K",
    NONNE=1:-500:2000,
    MOLEFRACTION=0.2:0:1,
    HTCOEF = 2:0:1.E6 UNIT="ENEY/SEC/K"
    DENSITY=0.6:0.1:50.0 UNIT="AMOUNT/VOLUME"
  STREAM MAINSTREAM
    TYPE FLOW, MOLEFRACTION(4)
  ****
MODEL COLUMN
  TYPE
#1=ETHANE, 2=PROPYLENE, 3=PROPANE, 4=PROPADIENE, 5=METHYLACETYLENE#
  E, VP, LP, D, VQ, LQ, FV, FL, VR, LR, B, CWFLOW, VENT, YMAX, FLARE, VMAX AS FLOW,
  YE, YP, XP, YQ, XQ, XD, YF, XF, YR, XR, XB, XBB AS ARRAY(4) OF MOLEFRACTION,
  M1, M2, M3, M4 AS MOLES,
  HVP, HLP, HVF, HLF, HVR, HLR, HVQ, HLQ, HD, HLB AS ENTHALPY,
  LAMBDAP, LAMBDAQ, LAMBDAR, LAMBDAS AS ENTHALPY,
  QCOND, QREB AS HEAT,
  T1, T2, TF, T3, T4, TCWIN, TCWMIN, TWBULB AS TEMPERATURE,
  P1, P2, P3, P4, PP, PQ, PF, PR AS PRESSURE,
  CP, CWHEAT AS SPEC HEAT,
  ANT AS ARRAY(3) OF NONNE
  AP, AQ, AR, AN1, AN2, AN3, AN4, A1, A2 AS ARRAY(4) OF NONNE,
  RECOV, TRAYSP, TRAYSQ, TRAYSR, RRATIO, AL1, AL2, PROFIT AS NONNE,
  GAMMA1, GAMMA2, GAMMA3, GAMMA4, KP2, KP3, KP4 AS NONNE,
  UA AS HTCOEF
  RHOV, RHOVS AS DENSITY
  STREAM
    INPUT 1 IS FV, YF
    INPUT 2 IS FL, XF
    OUTPUT 1 IS E, YE
    OUTPUT 2 IS D, XD
    OUTPUT 3 IS B, XB
  EQUATION
#NODE 1#
  RHOVS=RHOV;
  $RHOV*600.0#M3#=-VP-E-LP;
  M1*$XP=(VP*YP-E*YE-LP*XP)*1.E-3;
  QCOND*60.0=HVP*VP-E*HVP-LP*HLP;
  YE=YP;
#SECTION P#
# VP=VQ;#
#NODE 2#
  LP+VQ=D+LQ+VP;
  M2*$XQ=(LP*XP-D*XD-LQ*XQ+VQ*YQ-VP*YP)*1.E-3;
  HLP*LP+HVQ*VQ=HD*D+HLQ*LQ+HVP*VP;

```

```

XD=XQ;
#NODE 3#
FV+VR+FL+LQ=VQ+LR;
M3*$XR=(LQ*XQ+FL*XF+VR*YR+FV*YF-VQ*YQ-LR*XR)*1.E-3;
FV*HVF+FL*HLF+VR*HVR+LQ*HLQ=VQ*HVQ+LR*HLR;
#NODE 4#
LR=VR+B;
XBB=XB;
M4*$XBB=(LR*XR-VR*YR-B*XB)*1.E-3;
QREB*60.0=VR*HVR+B*HLB-LR*HLR;
#PRESSURES#
P2=KP2*P1;
P3=KP3*P1;
P4=KP4*P1;
#ENTHALPY CALCULATION#
LAMBDA P=AL1*P1+AL2;
LAMBDA F=AL1*PF+AL2;
LAMBDA Q=AL1*P2+AL2;
LAMBDA R=AL1*P3+AL2;
# ZERO ENTHALPY AT 300K FOR LIQUID#
HLP=T1*CP-38.25;
HVP=HLP+LAMBDA P;
HLF=TF*CP-38.25;
HVF=HLF+LAMBDA F;
HLQ=T2*CP-38.25;
HD=HLQ;
HVQ=HLQ+LAMBDA Q;
HLR=T3*CP-38.25;
HVR=HLR+LAMBDA R;
HLB=T4*CP-38.25;
#VLE#
PP=(P1+P2)/2.0;
PQ=(P2+P3)/2.0;
PR=(P3+P4)/2.0;
AP=A1*PP+A2;
AQ=A1*PQ+A2;
AR=A1*PR+A2;
AN1=A1*P1+A2;
AN2=A1*P2+A2;
AN3=A1*P3+A2;
AN4=A1*P4+A2;
#CONDENSER OPERATION#
TCWIN=T1-QCOND*(1.0+UA/2.0/CWFLOW/CWHEAT)/UA;
TCWMIN=(1.025-1.88995e-2*QCOND)*(TWBULB-273.15)
+(1.5865+0.5268*QCOND)+273.15;
#OTHER#
P1=RHOF*T1*8.314E-3;
RECOV*(FV*YF(2)+FL*XF(2))=XD(2)*D;
RRATIO*D=LQ;
PROFIT*1.9013=14.286*D+3.863*B+13.095*VENT
-8.1395*(FV+FL)-0.476*FLARE;
# MAXIMUM VAPOUR RATE #
VMAX = FUNCTION VAPMAX(VP,VQ,VR);
PROCEDURE
#BUBBLE POINTS AT NODES#
(T1) BUBJBL(XP,AN1,ANT,P1,GAMMA1)
(T2) BUBJBL(XQ,AN2,ANT,P2,GAMMA2)
(T3) BUBJBL(XR,AN3,ANT,P3,GAMMA3)
(T4) BUBJBL(XB,AN4,ANT,P4,GAMMA4)
# COLUMN SECTIONS#
(YP) SECTION(XP,XQ,LP,VP,TRAYSP,AP)
(YQ) SECTION(XQ,XR,LQ,VQ,TRAYSQ,AQ)
(YR) SECTION(XR,XB,LR,VR,TRAYSR,AR)
# FLARE CALCULATION#
(VENT,FLARE) FLARE(E,YMAX)
****
PROCEDURE BUBJBL:BUBJBL

```

```

INPUT
  MOLEFRACTION(4), NONNE(4), NONNE(3), PRESSURE, NONNE
OUTPUT
  TEMPERATURE
****
PROCEDURE FLARE
  INPUT FLOW, FLOW
  OUTPUT FLOW, FLOW
****
PROCEDURE SECTION:COLJBL
  INPUT
    MOLEFRACTION(4), MOLEFRACTION(4), FLOW, FLOW, NONNE, NONNE(4)
  OUTPUT
    MOLEFRACTION(4)
****
FUNCTION VAPMAX:VAPMAX
  INPUT
    FLOW, FLOW, FLOW
  OUTPUT
    FLOW
****
UNIT      COLUMN
****
FLWSHEET
FEED 1 INPUT 1 COLUMN
FEED 2 INPUT 2 COLUMN
OUTPUT 1 COLUMN PRODUCT 1
PRODUCT 2 OUTPUT 2 COLUMN
PRODUCT 3 OUTPUT 3 COLUMN
****
OPERATION
SET
  WITHIN COLUMN
    QREB=20,
    B=1.997,
    P1=1.55,
    PF=1.501,
    KP2=1.005,
    KP3=1.035,
    KP4=1.053,
    M1=0.1381,
    M2=0.2976,
    M3=0.4095,
    M4=0.4143,
    ANT=<-6.6099, 49.499, 249.95>,
    CP=0.1275,
    A1=<-1.394, -0.2392, -0.1522, -0.0355>,
    A2=<5.738, 1.884, 1.579, 1.134>,
    D=7.493,
    FV=0.,
    YF=<0.0, 0.0, 0.0, 0.0>,
    FL=9.715,
    XF=<0.0024, 0.8246, 0.14379, 0.016484>,
    TF=302.67,
    TRAYSP=10,
    TRAYSQ=72,
    TRAYSR=37,
    AL1=-2.808,
    AL2=17.55,
    CWFLOW=30.56,
    TWBULB=288.15,
    CWHEAT=0.0756,
    UA=2.37,
    YMAX=.13889,
    GAMMA1=1.053,
    GAMMA2=1.0,
    GAMMA3=1.0,

```

```

      GAMMA4=1.0
PRESET
  WITHIN COLUMN
    RHOV=0.6,
    P2=1.55775,
    P3=1.60425,
    P4=1.63215,
    YP=<0.4E-6,0.942,1.2E-3,0>,
    XP=<0.4E-6,0.942,1.2E-3,0>,
    YQ=<4.E-6,0.995,5.E-3,0.>,
    XQ=<2.E-6,0.995,5.E-3,0.>,
    YR=<0.2E-6,0.776,0.22,4.E-3>,
    XR=<0.2E-6,0.76,0.23,6.E-3>,
    XBB=<0.,0.116,0.784,0.06>,
    XB=<0.,0.116,0.784,0.06>,
    XD=<2.E-6,0.995,,0.>,
    E=0.34164,
    T1=309.7,
    T2=311.0,
    T3=313.7,
    T4=319.1,
    VP=84.1,
    LP=83.8,
    VQ=84.1,
    LQ=77.9,
    VR=85.7,
    LR=88.1,
    HVP=14.4,
    HLP=1.24,
    HVF=15.592,
    HLF=0.340,
    HVR=14.79,
    HLR=1.75,
    HVQ=14.583,
    HLQ=1.407,
    HLB=2.435,
    QCOND=18.43,
    TCWIN=297.979,
    TCWMIN=297.979
FREE
  COLUMN.FL,COLUMN.D,COLUMN.B,COLUMN.QREB,COLUMN.P1
****
GLOBAL
  MAXIMIZE COLUMN.PROFIT
  CONSTRAINT
    COLUMN.TCWIN >= COLUMN.TCWMIN;
    COLUMN.XD(1) <= 4.5E-4;
    COLUMN.XD(2) <= 5.E-3;
    COLUMN.QREB <= 20;
    COLUMN.VMAX <= 99.6;
    COLUMN.P1 <= 1.55;
    COLUMN.XB(1) + COLUMN.XB(2) + COLUMN.XB(3) >= 0.75;
****

```

The following subroutines were used as procedures within the SPEEDUP model for determination of the bubble point temperature and for the iterative *section* calculation.

```

      SUBROUTINE BUBBLE(X,NX,A,NA,ANT,NANT,PRESS,GAMMA,T)
C
C THIS SUBROUTINE COMPUTES THE BUBBLE POINT OF
C A MIXTURE OF NX+1 COMPONENTS FOR WHICH THE ANTOINE DATA IS
C KNOWN FOR THE SECOND COMPONENT AND CONST. RELATIVE VOLATILITY

```

C IS KNOWN FOR REMAINING COMPONENTS.

C

C AUTHOR: J.M.HOWELL

C

C REVISED: J.B.LEAR 5-1-87

C

IMPLICIT DOUBLE PRECISION (A-H,O-Z)

DOUBLE PRECISION X(NX),A(NA),ANT(NANT)

C

SUM=0

DO 10 I=1,NX

10 SUM=SUM+X(I)*(A(I)-1.0D0)

VALUEK=1.0D0/(SUM+1.0D0)

PVAP=GAMMA*VALUEK*A(2)*PRESS

C

C THIS QUADRATIC EQUATION USES MPa AND DEG K

C

T=(ANT(1)*PVAP+ANT(2))*PVAP+ANT(3)

RETURN

END

SUBROUTINE COLBIN(XT,XB,L,V,NT,A,Y,IF)

C

C TEMPORARY INTERFACE COLBIN FOR BINARY COLUMNS

C *****

C

C CALLS COLUMN WITH SPEEDUP COMPATIBLE ARGUMENTS FOR NX=1

C

C+IBM

IMPLICIT DOUBLE PRECISION (A-H,O-Z)

DOUBLE PRECISION L,V,NT

C-IBM

CALL COLUMN (XT,1,XB,1,L,V,NT,A,1,Y,1,IF)

RETURN

END

SUBROUTINE COLUMN (XT,NX,XB,NXB,L,V,NT,A,NA,Y,NY,IFAIL)

C

C SUBROUTINE NAME - COLUMN - COLUMN SECTION PROCEDURE FOR SPEEDUP

C *****

C

C AUTHOR - J.M.HOWELL DATE - MARCH 1983

C MODIFIED FOR VAPOUR MOLFRACTION OUTPUT - J.M.H. DECEMBER 1983

C

C DESCRIPTION - THIS PROCEDURE SIMULATES THE STEADY STATE BEHAVIOUR

C OF A COLUMN SECTION CONSISTING OF NT THEORETICAL TRAYS WITH

C CONSTANT RELATIVE VOLATILITY AND EQUAL MOLAR OVERFLOW.

C IF A NON-INTEGGER NUMBER OF PLATES ARE SPECIFIED, OR NT<1,

C LINEAR INTERPOLATION IS USED TO FIND Y.

C

C INPUTS - XT(NX) COMPOSITION VECTOR OF TOP NODE

C XB(NXB) COMPOSITION VECTOR OF BOTTOM NODE

C L,V LIQUID AND VAPOUR MOLAR FLOWRATES

C NT NUMBER OF THEORETICAL TRAYS IN SECTION

C A(NA) ARRAY OF RELATIVE VOLATILITIES TO LEAST VOL.

COMP.

C MOST VOLATILE COMPONENTS FIRST

C OUTPUTS - Y(NY) VAPOUR COMPOSITION VECTOR ON TOP TRAY

C

C-----

C

C DEFINE ARRAYS AND DATA STRUCTURE

C

C+IBM

IMPLICIT DOUBLE PRECISION (A-H,O-Z)

DOUBLE PRECISION XT(NX),XB(NXB),L,V,NT,A(NA),Y(NY)

DOUBLE PRECISION X(6),MINFLO

```

      PARAMETER (MINFLO = 1D-10)
      PARAMETER (ZERO = 0.0D0, POINT1 = 0.1D0, POINT9 = 0.9D0,
+              ONE = 1.0D0, TWO = 2.0D0, TEN = 10.0D0)
C-IBM
      COMMON /CSCOM/ VL,XW(6,203),B(203)
C
C INITIALISATION - CHECK DIMENSIONS
C
      NP=MAX (NT,ONE)
      NC=NX+1
      IF (NC.GT.6.OR.NP.GT.200) THEN
          IFAIL=-5
          PRINT *, ' COLUMN-F- PROBLEM TOO LARGE'
          RETURN
      ENDIF
      IF (NXB.NE.NX.OR.NA.NE.NX.OR.NY.NE.NX) THEN
          IFAIL=-6
          PRINT *, ' COLUMN-F- INCOMPATIBLE DIMENSIONS'
          RETURN
      ENDIF
C
C CHECK THAT VAPOUR AND LIQUID FLOWS ARE POSITIVE
C IF EITHER FLOW IS ZERO, PASS INPUT DIRECTLY TO OTHER END
C
      IF (V.LT.MINFLO) THEN
          CALL TRAY1 (XT,A,NC,Y)
          RETURN
      ENDIF
      IF (L.LT.MINFLO) THEN
          CALL TRAY1 (XB,A,NC,Y)
          RETURN
      ENDIF
      VL=V/L
C
C SOLVE SYSTEM FOR BASIC NUMBER OF TRAYS
C
      CALL CSCONV (XT,XB,NX,NP+2,A,Y,IFAIL)
      DO 10 I=1,NX
          Y(I)=A(I)*B(2)*XW(I,2)/VL
10      CONTINUE
C
C CHECK FOR NON-INTEGGER NUMER OF TRAYS
C AND COMPUTE ADDITIONAL POINT FOR INTERPOLATION IF REQUIRED
C
      IF (NT.NE.DBLE(NP)) THEN
          IF (NT.LT.ONE) THEN
              CALL TRAY1 (XB,A,NC,XW(1,2))
              DO 40 I=1,NX
                  Y(I)=Y(I)*NT+XW(I,2)*(ONE-NT)
40              CONTINUE
          ELSE
C
C SOLVE SYSTEM FOR ONE EXTRA TRAY
C
              CALL CSCONV(XT,XB,NX,NP+3,A,Y,IFAIL)
C
C INTERPOLATE RESULT FOR NT>2
C
              DO 70 I=1,NX
                  Y(I)=Y(I)*(ONE-NT+NP)+A(I)*B(2)*XW(I,2)*(NT-NP)/VL
70              CONTINUE
          ENDIF
      ENDIF
      RETURN
      END
      SUBROUTINE CSCONV (XT,XB,NX,NP,A,Y,IFAIL)
C

```



```

C SUBROUTINE NAME - CSCONV - PART OF COLUMN SUITE
C *****
C
C AUTHOR - J.M.HOWELL    DATE - MARCH 1983
C
C DESCRIPTION - CALLED FROM COLUMN TO CONVERGE THE TRAY COMPOSITIONS
C TO SATISFY THE MASS BALANCE AND MOLEFRACTION SUM EQUATIONS.
C THESE ARE BASED ON EQUAL MOLAR OVERFLOW AND CONSTANT RELATIVE
C VOLATILITY ASSUMPTIONS.CSSOLV IS CALLED REPEATEDLY UNTIL
C THE ERROR IN MOLE FRACTION SUM IS ACCEPTABLY SMALL.
C
C MOD TO USE PREVIOUS VAPOUR COMPOSITION TO PRESET BETA
C J.M.H. JUNE 1984
C INPUTS -   XT(NX)    COMPOSITION OF TOP NODE
C           XB(NX)    COMPOSITION OF BOTTOM NODE
C           NP        NUMBER OF THEORETICAL PLATES INC. BOTH NODES
C           A(NX)     RELATIVE VOLATILITY VECTOR
C           Y(NX)     PREVIOUS VAPOUR COMPOSITION SOLUTION
C
C OUTPUTS    IFAIL     ERROR FLAG SET -VE IF CONVERGENCE FAILS
C           X(NC,NP)  INDIVIDUAL TRAY COMPOSTIONS
C           B(NP)     BETA PARAMETER FOR ALL TRAYS
C
C WORKSPACE XS(NP)    MOL FRACTION SUM FOR EACH TRAY
C           YTEMP(NC)  HOLDS INTERMEDIATE VAPOUR COMPOSITIONS
C
C+IBM
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DOUBLE PRECISION XT(NX),XB(NX),A(NX),Y(NX),YTEMP(6)
      DOUBLE PRECISION XS(203), MINTOL
      PARAMETER (MINTOL = 1.0D-8)
      PARAMETER (ZERO = 0.0D0, POINT1 = 0.1D0, POINT9 = 0.9D0,
+             ONE = 1.0D0, TWO = 2.0D0, TEN = 10.0D0)
C-IBM
      COMMON /CSCOM/ VL,X(6,203),B(203)
C
C COMPUTE NODE COMPOSITIONS FOR NC'TH COMPONENT
C
      NC=NX+1
      X(NC,1)=ONE
      X(NC,NP)=ONE
      DO 10 I=1,NX
         X(I,1)=XT(I)
         X(NC,1)=X(NC,1)-XT(I)
         X(I,NP)=XB(I)
         X(NC,NP)=X(NC,NP)-XB(I)
10    CONTINUE
C
C SET UP BOUNDARY BETA VALUES
C
      B(1)=VL*CSBETA(A,X(1,1),NC)
      B(NP)=VL*CSBETA(A,X(1,NP),NC)
C
C LOAD TEMPORARY VAPOUR COMPOSITION VECTOR
C
      YTEMP(NC)=ONE
      DO 12 I=1,NX
         YTEMP(I)=Y(I)
         YTEMP(NC)=YTEMP(NC)-Y(I)
12    CONTINUE
C
C CHECK RESULT FOR SENSIBLE STARTING VALUES
C ALSO RETURN HERE IF BACK SUBSTITUTION GIVES NEGATIVE BETA
C
      13 IF (YTEMP(NC).LT.ONE.AND.YTEMP(NC).GT.ZERO) THEN
C
C CONSTRUCT BETA AND TRAY COMPOSITION PROFILE BY BACK SUBSTITUTION

```

C STARTING AT TOP TRAY AND WORKING DOWNWARDS

```

C
      DO 18 J=2,NP-1
        BTEMP=YTEMP(NC)
        DO 14 I=1,NX
          BTEMP=BTEMP+YTEMP(I)/A(I)
14      CONTINUE
        B(J)=BTEMP*VL
        YTEMP(NC)=ONE
        IF (B(J).LT.ZERO) GOTO 13
        DO 16 I=1,NX
          X(I,J)=YTEMP(I)/A(I)/BTEMP
          YTEMP(I)=YTEMP(I)+(X(I,J)-X(I,J-1))/VL
          YTEMP(NC)=YTEMP(NC)-YTEMP(I)
16      CONTINUE
18      CONTINUE

```

C
C COMPUTE FIRST GUESS FOR BETA BY LINEAR INTERPOLATION
C OF BETA FOR EACH NODE
C

```

      ELSE
        DO 20 J=2,NP-1
          B(J)=(B(1)*(NP-J)+B(NP)*(J-1))/(NP-1)
20      CONTINUE
      ENDIF

```

C
C MAIN LOOP - COMPUTE SIGMA X USING LAST PREDICTED BETA
C RETURN WHEN MOLEFRACTION SUM CONVERGES
C

```

      DO 50 K=1,150
C
C EVALUATE MASS BALANCE FOR COMPONENTS 1...NC-1
C AND THEN FOR COMPONENT NC (WITH A(NC)=ONE)
C

```

```

      DO 30 I=1,NX
        CALL CSSOLV(A(I),NP,I)
30      CONTINUE
      CALL CSSOLV(ONE,NP,NX+1)

```

C
C COMPUTE MF SUM AND ERROR TOL.
C

```

      TOL=ZERO
      DO 70 J=2,NP-1
        XS(J)=ZERO
        DO 80 I=1,NC
          XS(J)=XS(J)+X(I,J)
80      CONTINUE
        TOL=TOL+ABS(XS(J)-ONE)
70      CONTINUE

```

C
C PREDICT NEW BETA
C ONE FOR EACH TRAY
C

```

      DO 40 J=2,NP-1
        B(J)=XS(J)*VL*CSBETA(A,X(1,J),NC)
40      CONTINUE

```

C
C CONVERGENCE TEST - RETURN WHEN MF SUM = ONE
C NOTE THIS TEST MUST COME AFTER BETA IS UPDATED AS BETA
C IS USED IN 'COLUMN'
C

```

      IF (TOL.LT.(NP-2)*MINTOL) RETURN
C++DEBUG

```

C
C LOOP FINISH AND CAUTION MESSAGE
C

50 CONTINUE

```

WRITE (6,(' COLUMN-F- CONVERGENCE FAILS ,TOL='',G10.5)') TOL
IFAIL=-4
RETURN
END

```

```

C
C SUBROUTINE NAME - TRAY1 - SINGLE TRAY FLASH CALCULATION
C *****
C
C AUTHOR - J.M.HOWELL    DATE - DECEMBER 1983
C
C DESCRIPTION - THIS SUBROUTINE COMPUTES THE VAPOUR COMPOSITION
C FOR A SINGLE DISTILLATION COLUMN STAGE ( 1 THEORETICAL TRAY )
C OF KNOWN LIQUID COMPOSITION
C
C INPUTS - X              LIQUID COMPOSITION NC-1 COMPS.
C           A              RELATIVE VOLATILITIES NC-1 COMPS.
C           NC             TOTAL NO OF COMPONENTS
C OUTPUTS- Y              VAPOUR COMPOSITION NC-1 COMPS.
C

```

```

      SUBROUTINE TRAY1 ( X,A,NC,Y )

```

```

C+IBM
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DOUBLE PRECISION X(NC),Y(NC),A(NC)
      DOUBLE PRECISION VL,XW(6)
      PARAMETER (ZERO = 0.0D0, POINT1 = 0.1D0, POINT9 = 0.9D0,
+               ONE   = 1.0D0, TWO    = 2.0D0, TEN     =10.0D0)

```

```

C-IBM
      COMMON /CSCOM/ VL,XW
C
C COMPUTE K VALUES FOR ALL NC COMPONENTS AS A(I)*BETA
C

```

```

      XW(NC)=ONE
      DO 10 I=1,NC-1
        XW(I)=X(I)
        XW(NC)=XW(NC)-X(I)
10     CONTINUE
      B=CSBETA(A,XW(1),NC)
C
C COMPUTE INDIVIDUAL COMPOSITIONS AS X(I)*KVALUE(I)
C
      DO 20 I=1,NC-1
        Y(I)=X(I)*A(I)*B
20     CONTINUE
      RETURN
      END
      DOUBLE PRECISION FUNCTION CSBETA (A,X,NC)

```

```

C
C FUNCTION NAME - CSBETA - PART OF 'COLUMN' SUITE
C *****
C

```

```

C AUTHOR - J.M.HOWELL    DATE - MARCH 1983
C
C DESCRIPTION - CALLED BY CSCONV TO COMPUTE PARAMETER BETA
C TO INITIALISE CONVERGENCE LOOP
C

```

```

C INPUTS - A(NC-1)  RELATIVE VOLATILITIES
C           X(NC)    MOLE FRACTIONS
C
C OUTPUT - CSBETA  =1./SIGMA(A*X)
C

```

```

C+IBM
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DOUBLE PRECISION A(NC),X(NC)
      PARAMETER (ZERO = 0.0D0, POINT1 = 0.1D0, POINT9 = 0.9D0,
+               ONE   = 1.0D0, TWO    = 2.0D0, TEN     =10.0D0)
C-IBM

```

```

      SUM=X(NC)
      DO 10 I=1,NC-1
        SUM=SUM+A(I)*X(I)
10    CONTINUE
      CSBETA=ONE/SUM
      RETURN
      END
      SUBROUTINE CSSOLV (A,NP,I)
C
C SUBROUTINE NAME - CSSOLV - PART OF 'COLUMN' SUITE
C *****
C
C AUTHOR - J.M.HOWELL    DATE - MARCH 1983
C
C DESCRIPTION - CALLED FROM CCONV TO
C SOLVE SYSTEM OF COMPONENT MASS BALANCE EQUATIONS
C  $X(J-1) - (1+AB)*X(J) + AB*X(J+1) = 0$ 
C COMPUTE LEADING DIAGONAL OF LU DECOMPOSITION MATRIX IN G(NP)
C COMPUTE MODIFIED CONSTANT VECTOR LB IN H(NP)
C FOR EACH COMPONENT AND SOLVE.
C
C+IBM
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DOUBLE PRECISION G(203),H(203)
      PARAMETER (ZERO = 0.0D0, POINT1 = 0.1D0, POINT9 = 0.9D0,
+               ONE   = 1.0D0, TWO    = 2.0D0, TEN    =10.0D0)
C-IBM
      COMMON /CSCOM/ VL,X(6,203),B(203)
C
C FIRST ELEMENT OF G
C
      G(1)=ZERO
      H(1)=X(I,1)
C
C RECURSION TO FIND REMAINING ELEMENTS OF G
C AND H
C
      DO 40 J=2,NP-1
        G(J)=ONE/(A*B(J)*(ONE-G(J-1))+ONE)
        H(J)=H(J-1)*G(J)
40    CONTINUE
C
C BACK SUBSTITUTE TO FIND X
C
C RECURSION TO FIND REMAINING X
C
      DO 50 JJ=2,NP-1
        J=(NP-1)+2-JJ
        X(I,J)=H(J)+G(J)*A*B(J+1)*X(I,J+1)
50    CONTINUE
      RETURN
      END

```

APPENDIX III

ANALYTICAL DERIVATION OF MAXIMUM INPUT AND OUTPUT MAGNITUDES FOR THE SISO CASE

Section 5.3.1 uses the maximum input and output magnitudes for an uncertain SISO system, to determine the optimal achievable operating point. The formulae used are presented as Equations 5.141 and 5.142, for the case where an IMC controller (Q) is used in the presence of multiplicative uncertainty (l_m). The nomenclature used in this derivation of these equations is the same as that used in Chapter 5.

Equation 5.122 gives the input variable magnitude Δu in terms of the IMC controller Q , the uncertain process transfer function G , the perfect process model $\tilde{G}$ and the disturbance appearing at the output d .

$$\Delta u = Q(1 + (G - \tilde{G})Q)^{-1}d \quad (\text{III.1})$$

For the SISO case this can be rearranged,

$$\Delta u = \frac{Q}{1 + Q(G - \tilde{G})}d \quad (\text{III.2})$$

We require the maximum input magnitude for any possible process transfer function G and frequency ω .

$$\text{Max}_{G, \omega} |\Delta u| = \text{Max}_{G, \omega} \left| \frac{Q}{1 + Q(G - \tilde{G})}d \right| \quad (\text{III.3})$$

Substituting G with the multiplicative uncertainty definition, Equation 5.59 and 5.60.

$$G(i\omega) = \tilde{G}(i\omega)(1 + l_m(i\omega)) \quad (\text{5.59})$$

where

$$|l_m(i\omega)| \leq \bar{l}_m \quad (\text{5.60})$$

$$\text{Max}_{G, \omega} |\Delta u| = \text{Max}_{l_m, \omega} \left| \frac{Q}{1 + Q l_m \bar{G}} d \right| \quad (\text{III.4})$$

The IMC controller Q is given by Equation 5.129 ($Q = \tilde{G}^{-1}F$), where F is a filter,

$$\text{Max}_{G, \omega} |\Delta u| = \text{Max}_{l_m, \omega} \left| \left(\frac{\tilde{G}^{-1}F}{1 + F l_m} d \right) \right| \quad (\text{III.5})$$

The expression in the parentheses in Equation III.5, when evaluated at a given frequency is the ratio of two complex numbers. The magnitude of such a ratio is equal to the ratio of two magnitudes.

$$\left| \frac{a + bi}{c + di} \right| = \frac{|a + bi|}{|c + di|}$$

Therefore,

$$\text{Max}_{G, \omega} |\Delta u| = \text{Max}_{l_m, \omega} \left(\frac{|\tilde{G}^{-1}F d|}{|1 + F l_m|} \right) \quad (\text{III.6})$$

At any given frequency, the values of $\tilde{G}$, d and F are known, the only variable being l_m . This therefore fixes the value of the numerator on the right hand side of Equation III.6. The expression can therefore be maximised at a given frequency, by minimising the value of the denominator, $|1 + F l_m|$. The minimum value of this term can be found geometrically as shown in Figure III.1. This nyquist plot (real component plotted on the x -axis, imaginary component on the y -axis), shows the vector representation of F , $F l_m$, $1 + 0i$ and $1 + F l_m$.

The only restriction on l_m is an upper bound on the magnitude. Therefore, the magnitude of $F l_m$ is limited by,

$$\begin{aligned} |F l_m| &= |F| |l_m| \\ &\leq |F| \bar{l}_m \end{aligned} \quad (\text{III.7})$$

In addition, the argument (Arg) of $F l_m$, or the rotational position, can be shown to be

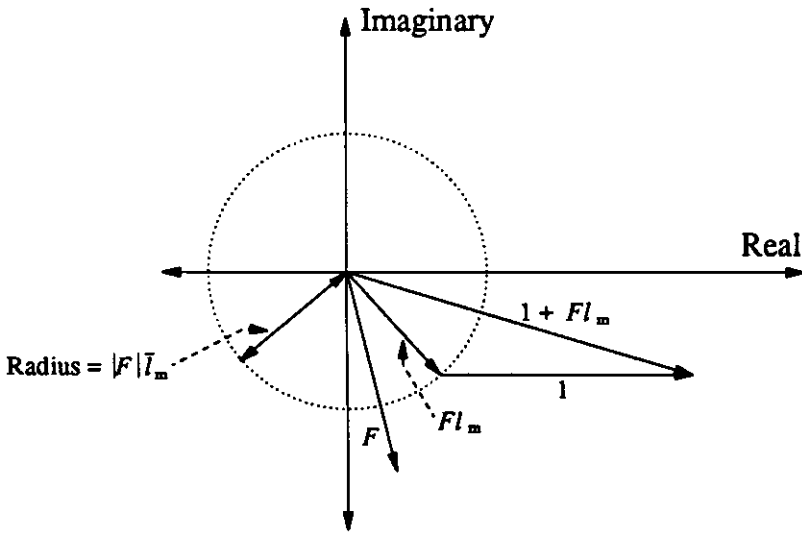


Figure III.1 - SISO Multiplicative Uncertainty Magnitudes

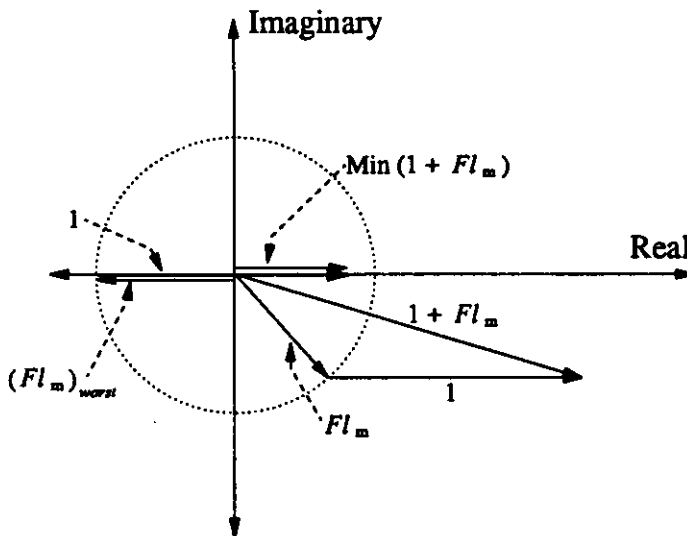


Figure III.2 - Multiplicative Uncertainty, Maximum Magnitudes

unrestricted.

$$\begin{aligned} \text{Arg}(Fl_m) &= \text{Arg}(F) + \text{Arg}(l_m) \\ \text{but } -2\pi &\leq \text{Arg}(l_m) \leq 2\pi \end{aligned} \quad (\text{III.8})$$

Therefore, the vector Fl_m could lie anywhere within the circle centre the origin and radius $|F|\bar{l}$. However, a limitation on the size of $|F|\bar{l}$ is given by the Robust Performance specification, Equation 5.140. In effect this requires the magnitude of $|F|\bar{l}$ to be less than 1.

Looking at Figure III.2, the minimum value for $1+Fl_m$ is clearly $1-|F|\bar{l}$. This may now be substituted into Equation III.6 to give Equation 5.141.

$$\begin{aligned} \Delta u_{CL} &= \max_{G \in \Pi, \omega} |u| \\ &= \max_{\omega} \left(\frac{|\tilde{G}^{-1}Fd|}{(1 - |F|\bar{l}_m)} \right) \end{aligned} \quad (5.141)$$

The other magnitude required is the maximum output magnitude. Combining Equations 5.30 and 5.48 gives the output variable magnitude Δy in terms of the IMC controller Q , the uncertain process transfer function G , the perfect process model $\tilde{G}$ and the disturbance appearing at the output d .

$$\Delta y = (I - \tilde{G}Q)(I + (G - \tilde{G})Q)^{-1}d \quad (\text{III.9})$$

Rearranging and substituting as above gives,

$$\text{Max}_{G, \omega} |\Delta y| = \text{Max}_{l_m, \omega} \left(\frac{|(1 - F)d|}{|1 + Fl_m|} \right) \quad (\text{III.10})$$

Once again this is maximised by minimising the denominator, giving the result presented as Equation 5.142.

$$\begin{aligned}
 \Delta y_{CL} &= \max_{G \in \Pi, \omega} |y| \\
 &= \max_{\omega} \left(\frac{|(1-F)d|}{(1 - |F|\bar{l}_m)} \right)
 \end{aligned} \tag{5.142}$$

The other analytical result used in Section 5.3.1, is the actual value of G which maximises the input and output magnitudes. Looking at Figure III.2, the vector which causes this maximisation is $-|F|\bar{l} + 0i$. We may therefore generate the following equality,

$$-|F|\bar{l}_m = Fl_m \tag{III.11}$$

Rearranging gives,

$$l_m = -|F|\bar{l}_m \frac{1}{F} \tag{III.12}$$

This may be substituted into the definition of the multiplicative uncertainty (Equation 5.59) to give the value of the "worst" process transfer function as a function of frequency, $G_{worst}(i\omega)$.

$$G_{worst}(i\omega) = \left(1 - |F(i\omega)|\bar{l}_m(i\omega) \frac{1}{F(i\omega)} \right) \tilde{G}(i\omega) \tag{III.13}$$

APPENDIX IV**SPEEDUP LISTING FOR FLOTATION CIRCUIT 1**

```

TITLE
  FLOTATION CIRCUIT 1
****
DECLARE
TYPE
  DELTA = 0.0 : -1.0 : 1.0 ,
  HOLDUP = 100.0 : 0.1 : 5000.0  UNIT = "10 KG" ,
  FLOWRATE = 10.0 : 0.0 : 1000.0  UNIT = "10 KG/MIN" ,
  DENSITY = 100.0 : 10.0 : 5000.0  UNIT = "10 KG/CU M" ,
  FRACTION = 0.1 : 0.0 : 1.0 ,
  FACTOR = 1.0 : 0.0 : 100.0 ,
  VOLUME = 10.0 : 0.0 : 100.0  UNIT = "CU M" ,
  RATE = 0.1 : 0.001 : 10.0  UNIT = "PER MIN" ,
  VOL_FLOW = 0.1 : 0.0 : 10.0  UNIT = "CU M/MIN" ,
  PERCENT = 10.0 : 0.0 : 100.0  UNIT = "PERCENT" ,
  TIME_UNITS = 1 : 0.0 : 1000  UNIT = "MINUTES"
STREAM MAINSTREAM
  SET NOCOMP = 2
  TYPE FLOWRATE, FLOWRATE(NOCOMP)
STREAM ASSAY
  TYPE PERCENT, PERCENT
STREAM AIR
  TYPE FACTOR
****
MODEL AIR_SUPPLY
SET NOCOMP
TYPE
  AIR_OUT AS FACTOR
  DEL_AIR AS DELTA
STREAM
  OUTPUT AIR_OUT
EQUATION
  AIR_OUT = 1.0 + DEL_AIR ;
****
MODEL DISTRIBUTOR
SET NOCOMP
TYPE
  AIR_1, AIR_2, AIR_3, AIR_IN AS FACTOR
STREAM
  INPUT    AIR_IN
  OUTPUT 1 AIR_1
  OUTPUT 2 AIR_2
  OUTPUT 3 AIR_3
EQUATION
  AIR_1 = AIR_IN ;
  AIR_2 = AIR_IN ;
  AIR_3 = AIR_IN ;
****
MODEL FEEDER
SET NOCOMP
TYPE
  WATER_OUT, SOLID_TOTAL, VALUE_IN AS FLOWRATE,
  SOLID_OUT AS ARRAY (NOCOMP) OF FLOWRATE,
  DEN_WATER AS DENSITY,
  WATER_VOL AS VOL_FLOW,
  STEP_CHANGE, GRADE AS FRACTION

```

```

STREAM
  OUTPUT WATER_OUT, SOLID_OUT
  CONNECTION 1002 IS VALUE_IN
EQUATION
  SIGMA (SOLID_OUT) = (1.0 + STEP_CHANGE) * SOLID_TOTAL ;
  SOLID_OUT(2) = GRADE * SIGMA (SOLID_OUT) ;
  WATER_OUT = DEN_WATER * WATER_VOL ;
  VALUE_IN = SOLID_OUT(2) ;
****

MODEL FLOTATION_CELL
#
# MODEL OF FLOTATION CELL, SOLIDS FLOAT WITH 1ST ORDER KINETICS #
# WATER / SOLID FRACTION IS ASSUMED CONSTANT IN THE CONCENTRATE #
# PULP VOLUME IS ASSUMED CONSTANT (PERFECT LEVEL CONTROL)      #
#                                                                #
SET NOCOMP
TYPE
  MASS_WATER AS HOLDUP,
  MASS_SOLID AS ARRAY (NOCOMP) OF HOLDUP,
  FEED_WATER, CONC_WATER, TAIL_WATER AS FLOWRATE,
  FEED_SOLID, CONC_SOLID, TAIL_SOLID AS ARRAY (NOCOMP) OF FLOWRATE,
  DEN_WATER AS DENSITY,
  DEN_SOLID AS ARRAY (NOCOMP) OF DENSITY,
  WAT_SOL, ADD_W_S, T_LS_RATIO AS FRACTION,
  PULP_VOL AS VOLUME,
  K_SOLID AS ARRAY (NOCOMP) OF RATE,
  K_AIR, K_AIR_IN AS FACTOR,
  HOLD_VOL AS ARRAY (NOCOMP) OF VOLUME,
  FEED_VOL, CONC_VOL, TAIL_VOL AS ARRAY (NOCOMP) OF VOL_FLOW
  RISE_T AS TIME_UNITS
STREAM
  INPUT 1 FEED_WATER, FEED_SOLID
  INPUT 2 K_AIR_IN
  OUTPUT 1 CONC_WATER, CONC_SOLID
  OUTPUT 2 TAIL_WATER, TAIL_SOLID
EQUATION
  HOLD_VOL = MASS_SOLID / DEN_SOLID ;
  MASS_WATER / DEN_WATER + SIGMA (HOLD_VOL) = PULP_VOL ;
  FEED_VOL = FEED_SOLID / DEN_SOLID ;
  CONC_VOL = CONC_SOLID / DEN_SOLID ;
  TAIL_VOL = TAIL_SOLID / DEN_SOLID ;
  SIGMA (FEED_VOL) + FEED_WATER / DEN_WATER =
    (WAT_SOL / DEN_WATER) * SIGMA (CONC_SOLID) + SIGMA (CONC_VOL)
    + SIGMA (TAIL_VOL) + TAIL_WATER / DEN_WATER ;
  TAIL_SOLID * (SIGMA (MASS_SOLID) + MASS_WATER) =
    MASS_SOLID * (SIGMA (TAIL_SOLID) + TAIL_WATER) ;
  CONC_SOLID = K_AIR * K_SOLID * MASS_SOLID ;
  CONC_WATER = ADD_W_S * SIGMA (CONC_SOLID) ;
  SIGMA (TAIL_SOLID) * T_LS_RATIO = TAIL_WATER ;
#
# DYNAMIC EQUATIONS FOLLOW - SOLIDS BALANCES
#
# $MASS_SOLID = FEED_SOLID - CONC_SOLID - TAIL_SOLID ;
#
# AIR DYNAMICS
#
  RISE_T * $K_AIR = K_AIR_IN - K_AIR ;
****
MODEL MEASURE
SET NOCOMP
TYPE
  FEED_WATER, VALUE_IN AS FLOWRATE,
  FEED_SOLID AS ARRAY (NOCOMP) OF FLOWRATE,
  GRADE, RECOVERY AS PERCENT
STREAM

```

```

INPUT FEED_WATER, FEED_SOLID
OUTPUT GRADE, RECOVERY
CONNECTION 1003 IS VALUE_IN
EQUATION
  GRADE * SIGMA (FEED_SOLID) = FEED_SOLID(2) * 100.0 ;
  RECOVERY * VALUE_IN = FEED_SOLID(2) * 100.0 ;
****
MODEL MIX_THREE
SET NOCOMP
TYPE
  WATER_IN1, WATER_IN2, WATER_IN3, WATER_OUT AS FLOWRATE,
  SOLID_IN1, SOLID_IN2, SOLID_IN3, SOLID_OUT AS
  ARRAY (NOCOMP) OF FLOWRATE
STREAM
  INPUT 1  WATER_IN1, SOLID_IN1
  INPUT 2  WATER_IN2, SOLID_IN2
  INPUT 3  WATER_IN3, SOLID_IN3
  OUTPUT   WATER_OUT, SOLID_OUT
EQUATION
  WATER_OUT = WATER_IN1 + WATER_IN2 + WATER_IN3 ;
  SOLID_OUT = SOLID_IN1 + SOLID_IN2 + SOLID_IN3 ;
****
UNIT AIR_SUPPLY
****
UNIT CLEANER_FLOTATION_CELL
****
UNIT DISTRIBUTOR
****
UNIT FEEDER
****
UNIT MEASURE
****
UNIT MIXER_MIX_THREE
****
UNIT ROUGHER_FLOTATION_CELL
****
UNIT SCAVENGER_FLOTATION_CELL
****
FLWSHEET
  OUTPUT FEEDER INPUT 1 MIXER
  OUTPUT MIXER INPUT 1 ROUGHER
  OUTPUT 1 ROUGHER INPUT 1 CLEANER
  OUTPUT 1 CLEANER INPUT MEASURE
  OUTPUT MEASURE PRODUCT 1 TYPE ASSAY
  OUTPUT 2 CLEANER INPUT 2 MIXER
  OUTPUT 2 ROUGHER INPUT 1 SCAVENGER
  OUTPUT 1 SCAVENGER INPUT 3 MIXER
  OUTPUT 2 SCAVENGER PRODUCT 2
  OUTPUT AIR_SUPPLY INPUT DISTRIBUTOR TYPE AIR
  OUTPUT 1 DISTRIBUTOR INPUT 2 ROUGHER TYPE AIR
  OUTPUT 2 DISTRIBUTOR INPUT 2 CLEANER TYPE AIR
  OUTPUT 3 DISTRIBUTOR INPUT 2 SCAVENGER TYPE AIR
  CONNECTION 1002 OF FEEDER IS CONNECTION 1003 TO MEASURE
****
OPERATION
SET
  WITHIN FEEDER
    SOLID_TOTAL = 120.0, GRADE = 0.121,
    DEN_WATER = 100.0, STEP_CHANGE = IF T<1.0 THEN 0.0 ELSE 0.05
  ENDIF,
  WATER_VOL = 0.8
  WITHIN ROUGHER
    DEN_WATER = 100.0, DEN_SOLID = <310.0, 400.0>,
    WAT_SOL = 0.8519, ADD_W_S = 0.8519, PULP_VOL = 27.0,
    K_SOLID = <0.013333, 0.2>, RISE_T = 10.0,
  WITHIN CLEANER
    DEN_WATER = 100.0, DEN_SOLID = <310.0, 400.0>,

```

```

WAT_SOL = 0.8519, ADD_W_S = 0.8519, PULP_VOL = 8.6262,
K_SOLID = <0.013333, 0.2>, RISE_T = 10.0,
WITHIN SCAVENGER
DEN_WATER = 100.0, DEN_SOLID = <310.0, 400.0>,
WAT_SOL = 0.8519, ADD_W_S = 0.8519, PULP_VOL = 18.3738,
K_SOLID = <0.013333, 0.2>, RISE_T = 10.0,
WITHIN AIR_SUPPLY
#DEL_AIR = 0.0# AIR_OUT = 1.017
PRESET
  WITHIN FEEDER
    VALUE_IN = 14.4
FREE
  FEEDER.WATER_VOL, AIR_SUPPLY.DEL_AIR
****
GLOBAL
  MAXIMIZE
    MEASURE.RECOVERY
  CONSTRAINT
    FEEDER.WATER_VOL > 0.5;
    FEEDER.WATER_VOL <2.0;
    AIR_SUPPLY.AIR_OUT>0.5;
    AIR_SUPPLY.AIR_OUT<2.0;
    MEASURE.GRADE >= 60.0;
****

```



14 FEB 1993

This thesis has been
accepted for the award
of the degree in the
Faculty of Engineering