

AN APPLICATION OF THE  
INVERSE SCATTERING TRANSFORM  
**TO SOME NONLINEAR SINGULAR**  
INTEGRO-DIFFERENTIAL EQUATIONS

A thesis submitted by George Scoufis in fulfilment of the  
requirements associated with candidature for the degree

University of Sydney

July, 1999

# ABSTRACT

The quest to model wave propagation in various physical systems has produced a large set of diverse nonlinear equations. Nonlinear singular integro-differential equations rank amongst the intricate nonlinear wave equations available to study the classical problem of wave propagation in physical systems. Integro-differential equations are characterized by the simultaneous presence of integration and differentiation in a single equation.

Substantial interest exists in nonlinear wave equations that are amenable to the Inverse Scattering Transform (IST). The IST is an adroit mathematical technique that delivers analytical solutions of a certain type of nonlinear equation: soliton equation. Initial value problems of numerous physically significant nonlinear equations have now been solved through elegant and novel implementations of the IST.

The prototype nonlinear singular integro-differential equation receptive to the IST is the Intermediate Long Wave (ILW) equation, which models one-dimensional weakly nonlinear internal wave propagation in a density stratified fluid of finite total depth. In the deep water limit the ILW equation bifurcates into a physically significant nonlinear singular

water limit of the ILW equation is the famous Korteweg-de Vries (KdV) equation. Both the KdV and BO equations have been solved by dissimilar implementations of the IST.

The Modified Korteweg-de Vries (MKdV) equation is a nonlinear partial differential equation, which was significant in the historical development of the IST. Solutions of the MKdV equation are mapped by an explicit nonlinear transformation

Miura transformation manifested the intimate connection between solutions of the KdV equation.

versions of the ILW and BO equations. Solutions of each modified nonlinear singular integro-differential equation should be mapped by an analogue of the original Miura

cases of the ILW equation, the modified version of the ILW equation should reduce to the MKdV equation in the shallow water limit and to the modified version of the BO equation in the deep water limit.

The Modified Intermediate Long Wave (MILW) and Modified Benjamin-Ono (MBO) equations are the two nonlinear singular integro-differential equations that display all the required attributes. Several researchers have shown that the MILW and MBO equations exhibit the signature characteristic of soliton equations. Despite the significance of the MILW and MBO equations to soliton theory, and the possible physical applications of the MILW and MBO equations, the initial value problems for these equations have not been solved. In this thesis we use the IST to solve the initial value problems for the MILW and MBO equations on the real-line. The only restrictions that we place on the initial values for the MILW and MBO equations are that they be real-valued, sufficiently smooth and decay to zero as the absolute value of the spatial variable approaches large values.

# TABLE OF CONTENTS

	Page
<b>CHAPTER 1: THE SIGNIFICANCE OF THE MODIFIED INTERMEDIATE LONG WAVE AND MODIFIED BENJAMIN-ONO EQUATIONS TO SOLITON THEORY.....</b>	1
<b>SECTION:</b>	
<b>1.1</b> Introduction.....	2
<b>1.2</b> A Survey of Integrable Nonlinear Singular Integro-differential Equations..	9
<b>1.3</b> Properties of the Operators <b>T</b> and <b>H</b> .....	24
 <b>CHAPTER 2: DERIVATIONS OF THE MODIFIED INTERMEDIATE LONG WAVE AND MODIFIED BENJAMIN-ONO EQUATIONS.....</b>	 40
<b>SECTION:</b>	
<b>2.1</b> The Origin of the MILW and MBO Equations.....	41
<b>2.2</b> Bilinear Form of the MILW Equation.....	47
 <b>CHAPTER 3: THE INVERSE SCATTERING TRANSFORM FOR THE MODIFIED KORTEWEG-de VRIES EQUATION: A NEW PERSPECTIVE.....</b>	 53
<b>SECTION:</b>	
<b>3.1</b> Introduction.....	54
<b>3.2</b> The Direct Problem.....	58
<b>3.3</b> The Inverse Problem.....	69
<b>3.4</b> The Time Evolution of the Scattering Data.....	75
<b>3.5</b> 1-Soliton Solution for the MKdV Equation.....	82

## Table of Contents

	Page
<b>CHAPTER 4: THE INVERSE SCATTERING TRANSFORM FOR THE MODIFIED INTERMEDIATE LONG WAVE EQUATION.....</b>	<b>85</b>
<b>SECTION:</b>	
<b>4.1</b> Introduction.....	86
<b>4.2</b> Linear System for the MILW Equation.....	89
<b>4.3</b> Jost Functions and the Direct Problem.....	94
<b>4.4</b> The Inverse Problem and its Solution.....	121
<b>4.5</b> The Evolution of the Scattering Data.....	128
<b>4.6</b> Soliton Solutions for the ILW and MILW Equations.....	130
<b>CHAPTER 5: THE INVERSE SCATTERING TRANSFORM FOR THE MODIFIED BENJAMIN-ONO EQUATION.....</b>	<b>136</b>
<b>SECTION:</b>	
<b>5.1</b> Introduction.....	137
<b>5.2</b> The Direct Problem and Scattering Data.....	141
<b>5.3</b> The Inverse Problem and its Solution.....	158
<b>5.4</b> The Evolution of the Scattering Data.....	166
<b>5.5</b> The 1-Soliton Solution for the MBO Equation.....	170
<b>CHAPTER 6: THE SHALLOW WATER LIMIT OF THE INVERSE SCATTERING TRANSFORM FOR THE MODIFIED INTERMEDIATE LONG WAVE EQUATION.....</b>	<b>177</b>
<b>SECTION:</b>	
<b>6.1</b> Introduction.....	178
<b>6.2</b> The Limit as $\delta \rightarrow 0^+$ of the Jost Functions and Direct Problem for the Complex-valued ILW Equation.....	180
<b>6.3</b> Shallow Water Limit of the Inverse Problem for the Complex-valued ILW Equation.....	189

## Table of Contents

	Page
<b>CHAPTER 7: THE DEEP WATER LIMIT OF THE INVERSE SCATTERING TRANSFORM FOR THE MODIFIED INTERMEDIATE LONG WAVE EQUATION: SOME COMMENTS.....</b>	193
<b>SECTION:</b>	
<b>7.1</b> Introduction.....	194
<b>7.2</b> The Limit as $\delta \rightarrow \infty$ of the Direct Problem for the Complex-valued ILW Equation.....	196
<b>CONCLUSION:.....</b>	205
<b>APPENDIX A:.....</b>	209
Figure 1.....	210
Figure 2.....	211
Figure 3.....	212
Figure 4.....	213
Figure 5.....	214
<b>REFERENCES.....</b>	215