CONCLUSION

Our objectives at the commencement of this thesis were to derive the solutions of the initial value problems for the Modified Intermediate Long Wave (MILW) and Modified Benjamin-Ono (MBO) equations. Complete solutions of the initial value problems for the MILW and MBO equations have now been derived.

In the course of writing this thesis the author has observed (and in scattered locations throughout this thesis has commented on) several interesting problems that emerge from the body of results contained in this thesis. We close this thesis by discussing two interesting (open) problems that merit attention. In order for there to exist a clear demarcation between the open problems that we will discuss, we shall use the subheadings *Problem 1* and *Problem 2*

be assigned the labels (C1), (C2) and so forth.

Problem 1

A legitimate way to interpret the MILW equation,

$$V_t + \beta V_x\left(e^V - 1\right) + \frac{1}{\delta} V_x + V_x \mathbf{T}\left(V_x\right) + \mathbf{T}\left(V_{xx}\right) = 0, \tag{C1}$$

is to view this equation as a generalization of the Modified Korteweg-de Vries (MKdV) equation

$$v_t + 6v^2 v_x + v_{xxx} = 0. \tag{C2}$$

One of the early physical applications [and prior to the relevance of (C2) as a mathematical construction in Ref. 46] of (C2) was as a model for nonlinear hydromagnetic waves in a cold collisionless plasma; see T. Kakutani and H. Ono, *Journal of the Physical Society of Japan*, vol. 26, pp. 1305-1318 (1969). The MKdV equation is now considered a significant, robust model in hydrodynamics [35,93]. The MILW equation, and indeed the MBO equation,

$$Q_t + \alpha Q_x\left(e^Q - 1\right) + Q_x \mathbf{H}\left(Q_x\right) + \mathbf{H}\left(Q_{xx}\right) = 0, \tag{C3}$$

remain purely synthetic equations that have *not* been derived as a model of any physical system. A natural extension from the physical applications of the MKdV equation is to derive the MILW equation from the appropriate set of hydrodynamical equations and boundary conditions. We anticipate that a derivation of the MILW equation in a

nonlinear waves because the exponential term in (C1) provides a strong nonlinear interaction.

Problem 2

The work in Chapter 3 of this thesis formed a conceptual bridge to the solution schemes for the MILW and MBO equations. At the centre of Chapter 3 is a solution of the initial value problem for equation (C2). Our solution of (C2) was constructed from the solution of the initial value problem for the complex-valued KdV equation

$$u_t + 6uu_x + u_{xxx} = 0. \tag{C4}$$

Integrable (2+1)-dimensional generalizations of the KdV and MKdV equations have been derived, but such generalizations are *not* unique [64]. A well known, and well studied, (2+1)-dimensional integrable generalization of the KdV equation is the Kadomstev-Petviashvili equation [59]

$$\frac{\partial}{\partial x}\{\psi_t + 6\psi\psi_x + \psi_{xxx}\} = -3\sigma^2\psi_{yy}, \tag{C5}$$

where $-\infty < y < \infty$, $\psi \equiv \psi(x, y, t)$ and σ is a parameter (not necessarily real). Kadomstev and Petviashvili [59] first derived (C5) in order to examine the stability of a one-dimensional soliton against transverse perturbations. We will adhere to the well established convention of referring to the version of (C5) in which $\sigma = i$

equation and to the case $\sigma = 1$ σ, which in physical applications influences the sign of the dispersive medium that supports wave propagation, significantly impacts on the implementation of the IST. Seminal works dedicated to the application of the IST to the KP equation can be found in Refs 8, 37, 41, 69 and 124.

A (2+1)-dimensional integrable generalization of the MKdV equation is the Modified Kadomstev-Petviashvili equation [see Section 3.6, Ref. 64]

$$\varphi_t - 6\sigma^2\varphi^2\varphi_x - 6\sigma^2\varphi_x\partial_x^{-1}(\varphi_y) - 3\sigma^2\partial_x^{-1}(\varphi_{yy}) + \varphi_{xxx} = 0, \tag{C6}$$

where $\varphi \equiv \varphi(x, y, t)$, $\sigma = i$ or $\sigma = 1$ and

$$\left(\partial_x^{-1} f\right)(x, y) \underline{\underline{\mathrm{def}}} \int_{-\infty}^{x} f(\eta, y)\,\mathrm{d}\eta.$$

In a direct analogy with the KP equation, two versions of the MKP equation exist. Equation (C6) with $\sigma = i$ is the MKP-I equation, and equation (C6) with $\sigma = 1$ is the MKP-II equation. The IST of (C6) is examined in Refs 38 and 64.

Dubrovsky and Konopelchenko [38] have shown that the explicit nonlinear transformation

$$\psi = \sigma^2 \partial_x^{-1}\left(\varphi_y\right) - \sigma^2 \varphi^2 + \sigma\varphi_x \tag{C7}$$

maps solutions of (C6) into solutions of (C5). Equation (C7) is the (2+1)-dimensional extension of the Miura transformation [86].

Consider the equation

$$\psi = \varphi^2 - \partial_x^{-1}\left(\varphi_y\right) + i\varphi_x, \tag{C8}$$

which is simply (C7) with $\sigma = i$. Equation (C8) maps *real-valued* solutions of the MKP-I equation into *complex-valued* solutions of the KP-I equation. Equating imaginary parts either side of equation (C8) we obtain

$$\varphi_x(x, y, t) = \mathrm{Im}\,\{\psi(x, y, t)\}, \tag{C9}$$

complex-valued KdV equation as a means of solution for equation (C2). From (C9) we deduce that the procedure used to solve (C2) can in principle be extended to solve the MKP-I equation. Fokas and Ablowitz [41] produced a beautiful and elegant implementation of the IST to solve the initial value problem for the KP-I equation. A complexification of the Fokas-Ablowitz IST for the KP-I equation underpins the solution scheme for the MKP-I equation that we have suggested. Dubrovsky and Konopelchenko [38] overlooked complexification of the KP-I equation as a means to solve the initial value problem for the MKP-I equation; these two practitioners of the IST chose to use the Lax pair for the MKP-I equation. In light of the success we achieved in solving equation (C2) by using a complexification of (C4), the solution of the MKP-I equation derived from the solution of the complex-valued KP-I equation should be pursued with enthusiasm.

Unfortunately, the complexification procedure we have described cannot be applied to solve the MKP-II equation. The inability of our method to cope with the KP-II equation originates from the fact that (C7) with $\sigma = 1$ transforms *real-valued* solutions of the MKP-II equation into *real-valued* solutions of the KP-II equation. The reader may recall that in Section 3.1 of this thesis we noted that the defocusing form of the MKdV equation [see (3.1.8)] could not be solved by a complexification of the KdV equation. The

Miura transformation [see (3.1.9)] to couple real-valued solutions of the defocusing MKdV equation with complex-valued solutions of the KdV equation.