CHAPTER 6

THE SHALLOW WATER LIMIT OF THE

INVERSE SCATTERING TRANSFORM

FOR THE

MODIFIED INTERMEDIATE LONG WAVE

EQUATION

Section 6.1: Introduction

In Section 2.1 of this thesis we demonstrated that the ILW equation,

$$U_t + \frac{1}{\delta}U_x + 2UU_x + \mathbf{T}\left(U_{xx}\right) = 0, \qquad (6.1.1)$$

reduces in the limit $\delta \to 0^+$ to the KdV equation,

$$u_t + 6uu_x + u_{xxx} = 0. \qquad (6.1.2)$$

The requirements, see (2.1.15) and (2.1.16), for the transition from (6.1.1) to (6.1.2) are

$$\left.\begin{aligned} U &= \frac{\delta}{4}u + O\left(\delta^2\right), \\ x' &= \frac{x}{2}, \\ t' &= \frac{\delta}{24}t. \end{aligned}\right\} \qquad (6.1.3)$$

Equation (6.1.2) is retrieved from (6.1.1) when we: substitute (6.1.3) into (6.1.1), invoke (1.3.26), omit primes, and neglect $O(\delta)$ terms.

Kodama et al. [63] were the first (and only) researchers to investigate the shallow water limit of the IST for the (real-valued) ILW equation. However, the authors of Ref. 63 present an incomplete set of results, verifying only that the shallow water limit of the

limit $\delta \to 0^+$

of any theory developed for the ILW equation.

In this chapter we will present a complete set of results for the limit $\delta \to 0^+$ of

the complex-valued ILW equation coincide in the limit $\delta \to 0^+$ with their counterparts in the IST for the complex-valued KdV equation. The prominence of the complex-valued ILW equation in the solution scheme for the MILW equation establishes a link between this chapter and the contents of Chapters 3 and 4. Therefore, a demonstration that the IST for the complex-valued ILW equation reduces correctly in the shallow water limit to the IST for the complex-valued KdV equation provides independent verification of the correctness of our solution schemes for the MKdV and MILW equations.

Although the conditions stipulated in (6.1.3) are adequate for the transition from (6.1.1) to (6.1.2) they are not of sufficient precision to complete the calculations in the remainder of this chapter. Henceforth, we will assume that in the shallow water region the following conditions are satisfied:

$$\left.\begin{aligned} U &= \frac{\delta}{4}u + \delta^2 u_1 + \mathrm{O}\left(\delta^3\right), \\ x' &= \frac{x}{2}, \\ t' &= \frac{\delta}{24}t. \end{aligned}\right\} \tag{6.1.4}$$

The function $u_1 \equiv u_1(x,t)$ is nonzero, but otherwise arbitrary. We will adhere in this chapter to the standard convention of omitting primes from any expression or equation that has been derived by an application of (6.1.4), or any other transformation that involves primes.

The remainder of this chapter is partitioned into *two* sections. Section 6.2 contains the limit as $\delta \to 0^+$ of the Jost functions and direct problem for the complex-valued ILW equation. Section 6.3, which is the last section in this chapter, is devoted to the shallow water limit of the inverse problem for the complex-valued ILW equation. The requirements enumerated in (6.1.4) for the transition from the complex-valued ILW equation to the complex-valued KdV equation are unique to this thesis. Therefore, we will elect *not* to complete the existing work [63] on the limit $\delta \to 0^+$

proceed to derive the necessary results en masse.

Section 6.2: The Limit as $\delta \to 0^+$ of the Jost Functions and Direct Problem for the Complex-valued ILW Equation

Satsuma et al. [108] have shown that the linear problem for the ILW equation contracts as $\delta \to 0^+$ to the linear problem for the KdV equation. Therefore, we will omit the proof that: **1)** the shallow water limit of (4.3.12) is (3.2.9); **2)** the shallow water limit of the system (4.3.13$^\pm$) is the single equation (3.2.10). The reader who wishes to reproduce the calculations found in Ref. 108 requires the information

$$\lambda = k + \mathrm{O}\left(\delta^2\right) \text{ as } \delta \to 0^+. \tag{6.2.1}$$

Equation (6.2.1) connects λ (the real spectral parameter in the IST for the complex-valued ILW equation) to k (the real spectral parameter in the IST for the complex-valued KdV equation). A byproduct of (6.2.1) is that the quantity $\zeta_{+\varepsilon}(\lambda)$, whose definition is provided by (4.3.14), admits the representations

$$\zeta_{+\varepsilon}(\lambda) = \frac{k}{2} + \varepsilon\delta\frac{k^2}{6} + \mathrm{O}\left(\delta^2\right) \text{ as } \delta \to 0^+ \tag{6.2.2a}$$

and

$$\frac{1}{\zeta_{+\varepsilon}(\lambda)} = \frac{2}{k} - \frac{2\varepsilon}{3k}\delta + \mathrm{O}\left(\delta^2\right) \text{ as } \delta \to 0^+. \tag{6.2.2b}$$

equation is defined by equation (4.3.21) together with (4.3.23). Our first task in this section is to ascertain the behaviour as $\delta \to 0^+$ of the function denoted as $\hat{g}(r;\lambda)$ and defined by (4.3.23). The cosech (δr) term in (4.3.23) places a simple pole in $\hat{g}(r;\lambda)$ at $\delta = 0$, whereas all the other components of (4.3.23) are regular at $\delta = 0$. Using (6.2.2a) in (4.3.23), and the observations about the singularity profile of (4.3.23) we find that

$$\hat{g}(r;\lambda) = \frac{\delta^{-1}}{r(\varepsilon r - k)} + \frac{2\varepsilon r - k}{3r(\varepsilon r - k)} + \mathrm{O}(\delta), \tag{6.2.3}$$

where $0 < \delta << 1$ and $\varepsilon = \pm 1$. Substituting (6.2.3) into (4.3.21) we obtain

$$G^{+\varepsilon}(x,\eta;\lambda) = \frac{1}{2\pi\delta}\int_C \frac{e^{i(x-\eta)r}}{r(\varepsilon r - k)}\mathrm{d}r + \mathrm{O}(1) \text{ as } \delta \to 0^+, \tag{6.2.4}$$

where C denotes a suitable straight line from $-\infty$ to ∞. We are now required to connect the dominant term in (6.2.4) with the function defined by (3.2.15) and (3.2.16), the

complex-valued KdV equation. The necessary connection can be visualized through the change of variables

$$x = 2x', \ \eta = 2\xi \text{ and } p = 2r. \tag{6.2.5}$$

The equation

$$G^{+\varepsilon}(x,\eta;\lambda) = \frac{\varepsilon}{\pi\delta}\int_C \frac{e^{i(x-\xi)p}}{p(p-2\varepsilon k)}\mathrm{d}p + \mathrm{O}(1) \text{ as } \delta \to 0^+ \tag{6.2.6}$$

arises from a straightforward substitution of (6.2.5) into (6.2.4). A relationship between $G^{+\varepsilon}(x,\eta;\lambda)$ and $G(x,\xi;k)$

KdV equation, is forged when we compare (6.2.6) to the composite of equations (3.2.15) and (3.2.16); upon making such a comparison we find that

$$G^{+\varepsilon}(x,\eta;\lambda) = \frac{2\varepsilon}{\delta}G(x,\xi;k) + \mathrm{O}(1) \text{ as } \delta \to 0^+. \tag{6.2.7}$$

Equation (6.2.7) discloses that $G^{+\varepsilon}(x,\eta;\lambda)$ is singular as $\delta \to 0^+$, and this type of asymptotic profile is not unexpected because from (6.1.4) it is possible to deduce that

$$U(\eta)G^{+\varepsilon}(x,\eta;\lambda) = \mathrm{O}(1) \text{ as } \delta \to 0^+,$$

from which we conclude that all components of (4.3.19) are $\mathrm{O}(1)$ as $\delta \to 0^+$.

Equations (4.3.47) and (4.3.48) provide series representations for the two versions

Counterparts of

(4.3.47) and (4.3.48) in the IST for the complex-valued KdV equation are provided by (3.2.17) and (3.2.18), but we should mention that the latter two equations are in *closed form*. A valuable exercise is to show that the series representations coincide (as $\delta \to 0^+$) with the closed forms mentioned. Successful completion of such an exercise would demonstrate that the analytic character (with respect to the spectral parameter) of the

shallow water region. Let us begin work on the exercise we have set ourselves by defining a new entity $\tilde{\mathrm{T}}(r_n)$, where n denotes some positive integer, through the equation

$$\tilde{T}(r_n) \underline{\underline{\text{def}}} \frac{e^{i(x-\eta)r_n}}{r_n - \varepsilon\zeta_+(\lambda)},$$

where r_n is a complex solution of (4.3.28), and $\text{Im}\left(r_n\right) > 0$. We now assert that

$$\lim_{\delta \to 0^+} \theta(x - \eta)\tilde{T}(r_n) = 0 \tag{6.2.8a}$$

and

$$\lim_{\delta \to 0^+} \theta(\eta - x)\tilde{T}(r_n^*) = 0, \tag{6.2.8b}$$

where both limits are attained *uniformly* with respect to n, $\theta(\bullet)$ denotes the Heaviside step function, and r_n^* is the complex conjugate of r_n

$\delta \to 0^+$) of the infinite series within (4.3.47) and

$$r_n = \frac{r_0^{(n)}}{\delta} + r_1^{(n)} + \delta r_2^{(n)} + O\left(\delta^2\right), \tag{6.2.9}$$

where $0 < \delta << 1$ and the $r_j^{(n)}$, for $j = 0, 1, 2, \ldots$, are complex numbers independent of δ. The reader can generate (recursively) equations for the $r_j^{(n)}$ by substituting (6.2.9) into (4.3.28), and then balancing powers of δ either side of the ensuing equation. For example, the reader will find that the equation for $r_0^{(n)}$ is

$$\varepsilon r_0^{(n)} + r_0^{(n)}\coth\left(r_0^{(n)}\right) = 1,$$

and this particular equation is produced when we balance terms that contain δ^{-1}. A corollary to (6.2.9) is that

$$\text{Im}\left(r_0^{(n)}\right) > 0, \tag{6.2.10}$$

which is implied by the estimate contained in (4.3.26).

Let us now focus on (6.2.8a). Equation (6.2.8a) is trivial when $x - \eta < 0$, so in our examination of (6.2.8a) we will assume that $x - \eta > 0$. If we decompose the complex quantity r_n into its real and imaginary parts by means of the equation

$$r_n = \text{Re}\left(r^{(n)}\right) + i\,\text{Im}\left(r^{(n)}\right),$$

then we find that the *modulus* of $\tilde{T}(r_n)$ is

$$\left|\tilde{T}(r_n)\right| = \frac{e^{-(x-\eta)\,\text{Im}\left(r^{(n)}\right)}}{\left|\text{Re}\left(r^{(n)}\right) - \varepsilon\zeta_+(\lambda) + i\,\text{Im}\left(r^{(n)}\right)\right|}.$$

Equation (6.2.9) and the inequality (6.2.10) can be used (in tandem) to show that $\left|\tilde{\mathrm{T}}(r_n)\right|$ is exponentially small as $\delta \to 0^+$, and from this behaviour we deduce that (6.2.8a) is correct. We omit the proof of (6.2.8b) because it is similar to the proof of (6.2.8a).

Now that we have proved (6.2.8a) and (6.2.8b), the reader can use these results, supplemented by (6.2.2b) and (6.2.5), in equations (4.3.47) and (4.3.48) to show that

$$G^{+\,\varepsilon}_{\pm\,\varepsilon}(x,\eta;\lambda) = \frac{2\varepsilon}{\delta} G_{\pm\,\varepsilon}(x,\xi;k) + \mathrm{O}(1) \text{ as } \delta \to 0^+, \tag{6.2.11}$$

where $G_{-\,\varepsilon}(x,\xi;k)$ and $G_{+\,\varepsilon}(x,\xi;k)$ are defined by (3.2.17) and (3.2.18), respectively. We note that (6.2.11) sharpens (6.2.7).

Each of the Jost functions in the IST for the complex-valued ILW equation satisfies a particular version of the integral equation (4.3.19). The analogue of (4.3.19) in the IST for the complex-valued KdV equation is (3.2.13). We now wish to show that in the limit $\delta \to 0^+$ equation (4.3.19) coincides with (3.2.13). First, expand the $W^{+\,\varepsilon}(x;\lambda)$ and $W_0^{+\,\varepsilon}(x;\lambda)$ terms in (4.3.19) into series involving ascending powers of δ, and thereby obtain

$$\left.\begin{matrix} W^{+\,\varepsilon}(x;\lambda) \\ \\ W_0^{+\,\varepsilon}(x;\lambda) \end{matrix}\right\} = \left.\begin{matrix} W(x;k) \\ \\ W_0(x;k) \end{matrix}\right\} + \mathrm{O}(\delta) \text{ as } \delta \to 0^+. \tag{6.2.12}$$

Substituting (6.1.4), (6.2.5), (6.2.7) and (6.2.12) into the nonlocal term of (4.3.19) produces (after we perform some algebraic manipulations) the equation

$$\varepsilon\int_{-\infty}^{\infty} G^{+\,\varepsilon}(x,\eta;\lambda)U(\eta)W^{+\,\varepsilon}(\eta;\lambda)\,\mathrm{d}\eta = \int_{-\infty}^{\infty} G(x,\xi;k)u(\xi)W(\xi;k)\,\mathrm{d}\xi + \mathrm{O}(\delta), \tag{6.2.13}$$

valid only in the shallow water region. Equations (4.3.19), (6.2.12) and (6.2.13) when considered simultaneously imply the equation

$$W(x;k) = W_0(x;k) + \int_{-\infty}^{\infty} G(x,\xi;k)u(\xi)W(\xi;k)\,\mathrm{d}\xi, \tag{6.2.14}$$

and this particular equation is identical to (3.2.13). The appearance of (6.2.14) shows that in the limit $\delta \to 0^+$ equation (4.3.19) contracts to (3.2.13).

Equation (6.2.14) makes it possible for us to now connect individual eigenfunctions in the IST for the complex-valued ILW equation to particular eigenfunctions within the IST for the complex-valued KdV equation. Consistency with equations (6.2.11) and (6.2.14) necessitates the following identities:

$$\lim_{\delta \to 0^+} M^{+\varepsilon}(x;\lambda) = M(x;k); \tag{6.2.15a}$$

$$\lim_{\delta \to 0^+} \overline{M}^{+\varepsilon}(x;\lambda) = \overline{M}(x;k); \tag{6.2.15b}$$

$$\lim_{\delta \to 0^+} N^{+\varepsilon}(x;\lambda) = N(x;k); \tag{6.2.15c}$$

$$\lim_{\delta \to 0^+} \overline{N}^{+\varepsilon}(x;\lambda) = \overline{N}(x;k). \tag{6.2.15d}$$

Equations (6.2.15a-d) require supplementation with the following comments:

1) the Jost functions for the complex-valued KdV equation are those particular functions that exhibit parametric dependence on k;

2) we have decided not to show the analytic character (with respect to λ) of the Jost functions that originate from the IST for the complex-valued ILW equation because we know from (6.2.11) that this character is preserved as $\delta \to 0^+$;

3)

comprehension for the reader.

recapitulate the essential elements of (4.3.53):

$$M^{+\varepsilon}_{+\varepsilon}(x;\lambda) = A(\lambda)\,\overline{N}^{+\varepsilon}_{-\varepsilon}(x;\lambda) + B(\lambda)\,N^{+\varepsilon}(x;\lambda), \tag{6.2.16}$$

where $A(\lambda)$ and $B(\lambda)$ are defined by (4.3.57) and (4.3.58), respectively. The shallow water limits of the various eigenfunctions in (6.2.16) are provided by the appropriate identities in the list (6.2.15a-d), so we will direct our attention to ascertaining the behaviours as $\delta \to 0^+$ of $A(\lambda)$ and $B(\lambda)$. In the shallow water region ($0 < \delta << 1$) we have

$$A(\lambda) = a(k) + O(\delta) \tag{6.2.17}$$

and

$$B(\lambda) = b(k) + O(\delta), \tag{6.2.18}$$

where $a(k)$ and $b(k)$ appear in the IST for the complex-valued KdV equation and are defined by (3.2.23) and (3.2.24), respectively. There are no significant or intricate manipulations involved in the derivation of (6.2.17) or (6.2.18), so it is sufficient for us to mention that these equations are derived by appropriate uses of (6.1.4), (6.2.1), (6.2.2b), (6.2.5) and (6.2.15a) in the equations for $A(\lambda)$ and $B(\lambda)$. An immediate consequence from the aggregate of equations (6.2.15a), (6.2.15c), (6.2.15d), (6.2.17) and (6.2.18) is that the transition $\delta \to 0^+$ instigates a metamorphosis of (6.2.16) into the equation

$$M(x;k) = a(k)\bar{N}(x;k) + b(k)N(x;k), \tag{6.2.19}$$

and this particular equation is identical to (3.2.22).

Our attention now transfers to the discrete spectrum (*bound states*) for the complex-valued KdV and ILW equations. We briefly review some relevant notation and facts about the bound states in question:

- a bound state (denoted as λ_j) for the complex-valued ILW equation is a solution of equation (4.3.67);
- a bound state (denoted as k_j) for the complex-valued KdV equation is a solution of equation (3.2.30);
- $j = 1, 2, \ldots, q$ (where q denotes some finite positive integer);
- λ_j and k_j are always complex and both of these bound states can display nonzero real parts;
- $\operatorname{sgn}\left(\operatorname{Im}(\lambda_j)\right) = \varepsilon = \operatorname{sgn}\left(\operatorname{Im}(k_j)\right)$.

In Section 4.3 of this thesis [see (4.3.83)] we derived the following equation:

$$i\int_{-\infty}^{\infty} V_x \left| m^{+\varepsilon}(x;\lambda_j)\right|^2 \mathrm{d}x = \left\{\nu(\lambda_j) - \nu^*(\lambda_j)\right\} \left\| m(x;\lambda_j)\right\|^2$$
$$- \left\{\mu^*(\lambda_j) - \mu(\lambda_j)\right\} \left\| m^{+\varepsilon}(x;\lambda_j)\right\|^2, \tag{6.2.20}$$

where

$$\mu(\lambda) \underline{\underline{\mathrm{def}}} -\frac{\lambda}{2}\coth(\delta\lambda), \tag{6.2.21}$$

$$\nu(\lambda) \underline{\underline{\mathrm{def}}} \frac{\lambda}{2}\operatorname{cosech}(\delta\lambda) \tag{6.2.22}$$

and

$$\left\| f(x;\lambda_j)\right\| \underline{\underline{\mathrm{def}}} \left(\int_{-\infty}^{\infty} \left| f(x;\lambda_j)\right|^2 \mathrm{d}x\right)^{1/2}.$$

We used (6.2.20) to prove that bound states with $\mathrm{Re}\left(\lambda_j\right) \neq 0$ are indeed possible in the IST for the complex-valued ILW equation. The analogue of (6.2.20) in the IST for the complex-valued KdV equation is (3.2.35). Our immediate aim is to show that (3.2.35) is the shallow water limit of (6.2.20). The equation

$$\lambda_j = k_j + \mathrm{O}\left(\delta^2\right) \text{ as } \delta \to 0^+ \tag{6.2.23}$$

is the discrete analogue of (6.2.1), and is indispensable for the progression of our proof. Equations (6.2.21), (6.2.22) and (6.2.23) guide us towards the Laurent series

$$\mu\left(\lambda_j\right) = -\frac{1}{2\delta} - \frac{\delta}{6}k_j^2 + \mathrm{O}\left(\delta^3\right) \text{ as } \delta \to 0^+ \tag{6.2.24}$$

and

$$\nu\left(\lambda_j\right) = \frac{1}{2\delta} - \frac{\delta}{12}k_j^2 + \mathrm{O}\left(\delta^3\right) \text{ as } \delta \to 0^+. \tag{6.2.25}$$

It is an easy exercise for the reader to derive the equations

$$\mu^*\left(\lambda_j\right) - \mu\left(\lambda_j\right) = -\left\{\left(k_j^2\right)^* - k_j^2\right\}\frac{\delta}{6} + \mathrm{O}\left(\delta^3\right) \text{ as } \delta \to 0^+ \tag{6.2.26}$$

and

$$\nu\left(\lambda_j\right) - \nu^*\left(\lambda_j\right) = \left\{\left(k_j^2\right)^* - k_j^2\right\}\frac{\delta}{12} + \mathrm{O}\left(\delta^3\right) \text{ as } \delta \to 0^+ \tag{6.2.27}$$

from equations (6.2.24) and (6.2.25). Using the change of variable $x = 2x'$ in the integrals that define the nonlocal terms $\left\| m\left(x;\lambda_j\right) \right\|^2$ and $\left\| m^{+\varepsilon}\left(x;\lambda_j\right) \right\|^2$, and then performing some algebraic simplifications in the resultant equations we are able to conclude that

$$\lim_{\delta \to 0^+} \left\{ \begin{matrix} \left\| m\left(x;\lambda_j\right) \right\|^2 \\ \\ \left\| m^{+\varepsilon}\left(x;\lambda_j\right) \right\|^2 \end{matrix} \right. = 2\left\| m\left(x;k_j\right) \right\|^2. \tag{6.2.28}$$

The result of particular relevance to us is that

$$\left\{ \begin{matrix} \text{Right Hand Side} \\ \text{of (6.2.20)} \end{matrix} \right\} = \frac{\delta}{2}\left[\left(k_j^2\right)^* - k_j^2\right]\left\| m\left(x;k_j\right) \right\|^2 + \mathrm{O}\left(\delta^3\right) \text{ as } \delta \to 0^+, \tag{6.2.29}$$

which the reader can verify by making appropriate use of (6.2.26), (6.2.27) and (6.2.28) in the *right hand side* of (6.2.20). A pathway to the behaviour in the shallow water region of the *left hand side* of (6.2.20) has been laid by (2.1.15), the equation that sets out how the transition MILW → MKdV is achieved. Substituting (2.1.15) and (6.2.23) into the left hand side of (6.2.20) we obtain

$$\left\{\begin{array}{c}\text{Left Hand Side}\\ \text{of (6.2.20)}\end{array}\right\} = i\delta\int_{-\infty}^{\infty} v_x \left| m\left(x;k_j\right)\right|^2 \mathrm{d}x + \mathrm{O}\left(\delta^3\right) \text{ as } \delta \to 0^+. \tag{6.2.30}$$

Forcing the $\mathrm{O}(\delta)$ terms in (6.2.29) and (6.2.30) to match leaves us with (3.2.35), the last equation being the desired outcome.

In Section 4.3 of this thesis we corrected a particular statement made in Ref. 63 about the normalizing coefficient linked to the IST for the *real-valued* ILW equation. Our correction to the work in Ref. 63 was based on the equation [see (4.3.91)]

$$\left\{2\delta\nu\left(\lambda_j\right) - e^{-\delta\lambda_j}\right\}\frac{\dot{A}\left(\lambda_j\right)}{B\left(\lambda_j\right)} = i\varepsilon\dot{\mu}\left(\lambda_j\right)\left\| n^{+\varepsilon}\left(x;\lambda_j\right)\right\|^2 + i\varepsilon\dot{\nu}\left(\lambda_j\right)\left\| n\left(x;\lambda_j\right)\right\|^2, \tag{6.2.31}$$

where:

- $\lambda_j = i\varepsilon\hat{\lambda}_j$ and $\hat{\lambda}_j$ is such that [63] $0 < \delta\hat{\lambda}_j < \pi$;
- $\mu(\lambda)$ and $\nu(\lambda)$ are defined by (6.2.21) and (6.2.22), respectively;
- λ, for

example

$$\dot{\mu}\left(\lambda_j\right) \underline{\underline{\text{def}}} \left.\frac{\mathrm{d}\mu(\lambda)}{\mathrm{d}\lambda}\right|_{\lambda=\lambda_j}.$$

The analogue of (6.2.31) in the IST for the *real-valued* KdV equation is [see (3.2.42)]

$$\int_{-\infty}^{\infty} n^2\left(x;k_j\right)\mathrm{d}x = i\varepsilon\frac{\dot{a}\left(k_j\right)}{b\left(k_j\right)}, \tag{6.2.32}$$

where k_j denotes a discrete eigenvalue for the real-valued KdV equation. Our plan is to show that (6.2.31) → (6.2.32) as $\delta \to 0^+$

validity of the particular amendment we have made to the work in Ref. 63. Although (6.2.23) has been used in calculations that involve the complex-valued ILW and KdV equations, the proof we are about to present makes use of (6.2.23). However, the reader should understand that for the duration of our proof λ_j and k_j are allied to the real-valued ILW and KdV equations, respectively.

Consider the *left hand side* of (6.2.31). Appropriate use of (6.2.17), (6.2.18), (6.2.23) and (6.2.25) in the left hand side of (6.2.31) furnishes us with the result

$$\left\{ \begin{matrix} \text{Left Hand Side} \\ \text{of (6.2.31)} \end{matrix} \right\} = \delta k_j \frac{\dot{a}(k_j)}{b(k_j)} + \mathrm{O}(\delta^2) \text{ as } \delta \to 0^+. \tag{6.2.33}$$

The series

$$\dot{\mu}(\lambda_j) = -\frac{\delta}{3} k_j + \mathrm{O}(\delta^3) \text{ as } \delta \to 0^+ \tag{6.2.34}$$

and

$$\dot{\nu}(\lambda_j) = -\frac{\delta}{6} k_j + \mathrm{O}(\delta^3) \text{ as } \delta \to 0^+ \tag{6.2.35}$$

provide an insight into how the *right hand side* of (6.2.31) behaves in the shallow water region. Verification by the reader of (6.2.34), for example, entails use of (6.2.21) to compute the necessary derivative, and then recourse to (6.2.23). The shallow water limits of $\left\| n^{+\varepsilon}(x;\lambda_j) \right\|^2$ and $\left\| n(x;\lambda_j) \right\|^2$ can be deduced from (6.2.28). Therefore, it follows from (6.2.34) and (6.2.35) that

$$\left\{ \begin{matrix} \text{Right Hand Side} \\ \text{of (6.2.31)} \end{matrix} \right\} = -i\varepsilon\delta k_j \left\| n(x;k_j) \right\|^2 + \mathrm{O}(\delta^3) \text{ as } \delta \to 0^+. \tag{6.2.36}$$

Matching the $\mathrm{O}(\delta)$ terms in equations (6.2.33) and (6.2.36) leads us to the equation

$$\int_{-\infty}^{\infty} n(x;k_j) n^*(x;k_j) \mathrm{d}x = i\varepsilon \frac{\dot{a}(k_j)}{b(k_j)}, \tag{6.2.37}$$

where we have chosen to write $\left\| n(x;k_j) \right\|^2$ in full. At the conclusion of Section 3.2 we noted that $n(x;k_j)$ is real-valued whenever $u(x,t)$ is a real-valued solution of the KdV equation. The reality of $n(x;k_j)$ means that (6.2.37) coincides with (6.2.32). Therefore, we have now shown that the Jost functions and all components of the direct problem for the complex-valued ILW equation reduce correctly in the shallow water limit to their counterparts in the IST for the complex-valued KdV equation. In the next section we turn our attention to the shallow water limit of the inverse problem (and affiliated quantities) for the complex-valued ILW equation.

Section 6.3: Shallow Water Limit of the Inverse Problem for the Complex-valued ILW Equation

Equations (4.5.5)-(4.5.7) contain the solution of the inverse problem for the complex-valued ILW equation. The solution generated from (4.5.5)-(4.5.7) is valid for $t \geq 0$. Counterparts to (4.5.5)-(4.5.7) in the IST for the complex-valued KdV equation are supplied by equations (3.4.23)-(3.4.25). Our endeavour to show that all components of the IST for the complex-valued ILW equation reduce (as $\delta \to 0^+$) to their corresponding entities in the IST for the complex-valued KdV equation necessitates that we now verify the limits

$$\begin{pmatrix} (4.5.5) \\ (4.5.6) \\ (4.5.7) \end{pmatrix} \to \begin{pmatrix} (3.4.23) \\ (3.4.24) \\ (3.4.25) \end{pmatrix} \text{ as } \delta \to 0^+. \tag{6.3.1}$$

After an inspection of (4.5.5)-(4.5.7), the reader will inevitably conclude that several of the quantities within these equations have shallow water asymptotics that are known from Section 6.2. Before we can verify (6.3.1) we need to determine: **1)** how the function $\vartheta(\lambda, t)$ defined by (4.5.4) behaves as $\delta \to 0^+$; **2)** the shallow water limit of $U^{+\,\varepsilon}(x, t)$, where $U^{+\,\varepsilon}(x, t)$ is defined by (4.4.18). In the limit $\delta \to 0^+$ we find that

$$\vartheta(\lambda, t) = 8k^3 t + O\left(\delta^2\right), \tag{6.3.2}$$

which manifests itself when we substitute [see (6.1.4)] $t' = \delta t/24$ and (6.2.1) into (4.5.4), and then allow for the transition $\delta \to 0^+$. Let us now ascertain how $U^{+\,\varepsilon}(x, t)$ behaves as $\delta \to 0^+$; during this calculation we will suppress explicit t-dependence in our notation. By virtue of (4.4.18) we know that

$$U^{+\,\varepsilon}(x) = \frac{1}{4i\delta}\int_{-\infty}^{\infty} \coth\left(\frac{\pi}{2\delta}\left[\eta - \left(x + i\varepsilon 0^+\right)\right]\right) U(\eta)\,\mathrm{d}\eta. \tag{6.3.3}$$

Scale the x and η variables in (6.3.3) by the factors shown in (6.2.5), and thereby obtain the equation

$$U^{+\,\varepsilon}\left(x'\right) = \frac{1}{2i\delta}\int_{-\infty}^{\infty} \coth\left(\frac{\pi}{\delta}\left[\xi - \left(x' + i\frac{\varepsilon}{2}0^+\right)\right]\right) U(\xi)\,\mathrm{d}\xi. \tag{6.3.4}$$

The $\coth(\bullet)$ term in the integrand of (6.3.4) dictates how $U^{+\varepsilon}(x')$ behaves in the shallow water region because from (6.1.4) we see that $U(\xi)/\delta = \mathrm{O}(1)$ as $\delta \to 0^+$. The equation

$$U^{+\varepsilon}(x') = \frac{1}{2i\delta}\left(\int_{-\infty}^{x'} + \int_{x'}^{\infty}\right)\coth\left(\frac{\pi}{\delta}\left[\xi - \left(x' + i\frac{\varepsilon}{2}0^+\right)\right]\right)U(\xi)\,\mathrm{d}\xi \tag{6.3.5}$$

allows us to cater for the asymmetrical behaviour of $\coth(\bullet)$ as $(\bullet) \to \pm\infty$. Substituting [see (6.1.4)] $U = \delta u/4 + \mathrm{O}(\delta^2)$ into (6.3.5), and then considering the limit $\delta \to 0^+$ we obtain

$$U^{+\varepsilon}(x') = \frac{i}{8}\left(\int_{-\infty}^{x'} + \int_{\infty}^{x'}\right)u(\xi)\,\mathrm{d}\xi + \mathrm{O}(\delta),$$

upon which we can use the Fundamental Theorem of Calculus to show that

$$\frac{\partial}{\partial x}\left\{U^{+\varepsilon}(x,t)\right\} = \frac{i}{4}u(x,t) + \mathrm{O}(\delta) \ \text{as} \ \delta \to 0^+. \tag{6.3.6}$$

The reader now has all the information necessary to verify (6.3.1), and consequently we will leave this particular calculation as an exercise.

We conclude this section (and chapter) with an example: the transition as $\delta \to 0^+$ of the 1-soliton solution for the *real-valued* ILW equation into the 1-soliton solution for the *real-valued* KdV equation. Although the example we provide involves a real-valued $\delta \to 0^+$ is not sensitive to the distinction between real-valued and complex-valued solutions of the ILW equation. The reader may recall that we have already demonstrated that the 1-soliton solution [see (2.2.19) or (4.6.17)] for the MILW equation reduces correctly as $\delta \to 0^+$ to the 1-soliton solution [see (2.2.7)] for the MKdV equation. In Section 4.6 of this thesis [see (4.6.8)] we showed that our amended IST for the real-valued ILW equation delivers the following 1-soliton solution:

$$U(x,t) = \frac{\hat{\lambda}_1 \sin\left(\hat{\lambda}_1[\delta + \theta_1]\right)}{\cos\left(\hat{\lambda}_1[\delta + \theta_1]\right) + \cosh\left(\hat{\lambda}_1[x - \phi t + \hat{x}_0]\right)}, \tag{6.3.7}$$

where

$$\phi = \frac{1}{\delta} - \hat{\lambda}_1 \cot\left(\delta\hat{\lambda}_1\right),$$

$$\hat{\lambda}_1 > 0,$$

$$\theta_1 \neq 0,$$

and

$$\hat{x}_0 = \frac{\ln\left(\hat{\lambda}_1 / R_1\right)}{\hat{\lambda}_1}, \tag{6.3.8}$$

where $R_1 > 0$. From (6.2.23) we know that $\hat{\lambda}_1 \equiv |\lambda_1| = O(1)$ as $\delta \to 0^+$. Let us assume that $\theta_1 = O\left(\delta^2\right)$ as $\delta \to 0^+$ and $\hat{x}_0 = O(1)$ as $\delta \to 0^+$. Substituting [see (6.2.23)] $\hat{\lambda}_1 = \hat{k}_1 + O\left(\delta^2\right)$ and [see (6.1.4)] $x' = x/2$ and $t' = \delta t/24$ into (6.3.7) we find that

$$U(x,t) = \frac{\delta \hat{k}_1^2}{\cosh\left(2\hat{k}_1\left[x - 4\hat{k}_1^2 t + \frac{1}{2}\hat{x}_0\right]\right) + 1} + O\left(\delta^2\right) \text{ as } \delta \to 0^+,$$

which by virtue of the identity $\cosh(2\varpi) + 1 = 2\cosh^2(\varpi)$ can be expressed as

$$U(x,t) = \frac{\delta}{2}\hat{k}_1^2 \operatorname{sech}^2\left(\hat{k}_1\left[x - 4\hat{k}_1^2 t + \frac{1}{2}\hat{x}_0\right]\right) + O\left(\delta^2\right). \tag{6.3.9}$$

Matching the $O(\delta)$ term in (6.3.9) with [see (6.1.4)] the $O(\delta)$ term from the series $U = \delta u/4 + O\left(\delta^2\right)$ we obtain

$$u(x,t) = 2\hat{k}_1^2 \operatorname{sech}^2\left(\hat{k}_1\left[x - 4\hat{k}_1^2 t + \frac{1}{2}\hat{x}_0\right]\right), \tag{6.3.10}$$

which is the famous 1-soliton solution for the (real-valued) KdV equation [12, p. 615].

We will (to achieve a degree of completeness) go beyond (6.3.10) by providing a concrete example from our work that verifies (6.3.6). In Section 4.6 of this thesis [see (4.6.6)] we showed that

$$U^+(x,t) = -\frac{i\hat{\lambda}_1 R_1 e^{i\hat{\lambda}_1(\delta + \theta_1)}}{\hat{\lambda}_1 e^{\hat{\lambda}_1(x - \phi t)} + R_1 e^{i\hat{\lambda}_1(\delta + \theta_1)}}, \tag{6.3.11}$$

in the case of the pure 1-soliton solution for the (real-valued) ILW equation. Applying to (6.3.11) a slightly altered version of the procedure used to derive (6.3.9) will yield us

$$U^+(x,t) = -\frac{i\hat{k}_1 R_1 e^{8\hat{k}_1^3 t}}{\hat{k}_1 e^{2\hat{k}_1 x} + R_1 e^{8\hat{k}_1^3 t}} + O(\delta) \text{ as } \delta \to 0^+. \tag{6.3.12}$$

Considering our objective is to verify (6.3.6), let us now differentiate both sides of (6.3.12) with respect to *x*, and as a result bring about the equation

$$\frac{\partial}{\partial x}\left\{U^{+}(x,t)\right\} = \frac{2i\hat{k}_1^3 R_1 e^{2\hat{k}_1 x + 8\hat{k}_1^3 t}}{\left(\hat{k}_1 e^{2\hat{k}_1 x} + R_1 e^{8\hat{k}_1^3 t}\right)^2} + \mathrm{O}(\delta) \text{ as } \delta \to 0^{+}. \qquad (6.3.13)$$

Some astute algebraic manipulations performed on the leading term in (6.3.13), coupled with the use of (6.3.8) to eliminate the ratio $\hat{k}_1/R_1$ from the $\mathrm{O}(1)$ term in (6.3.13) produces

$$\frac{\partial}{\partial x}\left\{U^{+}(x,t)\right\} = \frac{i}{2}\hat{k}_1^2 \operatorname{sech}^2\left(\hat{k}_1\left[x - 4\hat{k}_1^2 t + \frac{1}{2}\hat{x}_0\right]\right) + \mathrm{O}(\delta) \text{ as } \delta \to 0^{+}. \qquad (6.3.14)$$

From (6.3.10) we see that the leading term in (6.3.14) is such that we have conformity with (6.3.6). The appearance of (6.3.14) and the conclusion associated with this equation is a signal that we should now close this chapter.