CHAPTER 5

THE INVERSE SCATTERING TRANSFORM

FOR THE

MODIFIED BENJAMIN-ONO EQUATION

Section 5.1: Introduction

The Modified Benjamin-Ono (MBO) equation,

$$Q_t + \alpha Q_x\left(e^Q - 1\right) + Q_x\mathbf{H}\left(Q_x\right) + \mathbf{H}\left(Q_{xx}\right) = 0, \tag{5.1.1}$$

has appeared in this thesis within the context of the deep water limit for the MILW equation. Some notation that is particularly relevant to the MBO equation is as follows: $Q \equiv Q(x, t)$, $-\infty < x < \infty$, $t \geq 0$, subscripts denote partial derivatives, α is a real parameter that is independent of x and t, and $\mathbf{H}$ denotes the Hilbert transform,

$$\left(\mathbf{H}f\right)(x) \underline{\underline{\text{def}}} \frac{1}{\pi}\left(\mathrm{P}\right)\int_{-\infty}^{\infty} \frac{f(\xi)}{\xi - x}\mathrm{d}\xi.$$

The conditions for the transition from the MILW equation to the MBO equation are set out in (2.1.14).

In this chapter we will present the solution of the initial value problem for (5.1.1), which has evolved from an initial value that has suitable asymptotic behaviour as $|x| \to \infty$. Remarks made in Section 4.1 about the MILW equation are also relevant to the MBO equation, particularly those remarks about the theoretical and practical issues that merit the development of a solution scheme for the MILW equation.

Less attention, when compared to the MILW equation, has been devoted to the MBO equation because researchers have been satisfied to derive results for the MBO equation by using the deep water limit of the appropriate results for the MILW equation. The first reference devoted exclusively to the MBO equation is Ref. 90 in which Nakamura derives multi-soliton and multi-periodic wave solutions for the MBO equation. Recently, Scoufis and Cosgrove [114] used the IST to solve the initial value problem for (5.1.1) with an arbitrary and sufficiently smooth real-valued initial value Q_0, where $Q_0 \equiv Q(x, 0)$. The authors of Ref. 114 used the boundary conditions

$$\frac{\partial^n Q_0}{\partial x^n} \to 0 \text{ as } |x| \to \infty,$$

where n denotes a positive integer or zero, and $\partial^n Q_0/\partial x^n \equiv Q_0$ whenever $n = 0$. In this chapter we reproduce an expanded version of Ref. 114. Conscious to avoid conflicts with notation used elsewhere in this thesis, we have found it necessary to significantly alter the notation used in Ref. 114.

In analogy with the procedures we have developed to solve the initial value problems for the MKdV and MILW equations, the scheme to solve the initial value problem for (5.1.1) relies on the Miura transformation that connects the Benjamin-Ono (BO) and MBO equations. According to equation (2.1.19) the relevant Miura transformation is

$$q = \frac{1}{2}\left\{\mathbf{H}\left(Q_x\right) + \alpha\left(e^Q - 1\right) + iQ_x\right\}, \tag{5.1.2}$$

where $q \equiv q(x,t)$ satisfies the (ordinary) BO equation,

$$q_t + 2qq_x + \mathbf{H}\left(q_{xx}\right) = 0. \tag{5.1.3}$$

Equating real and imaginary parts either side of (5.1.2) provides us with the following information:

$$\mathbf{H}\left(Q_x\right) + \alpha\left(e^Q - 1\right) = 2\,\mathrm{Re}\,\{q(x,t)\}; \tag{5.1.4}$$

$$Q_x = 2\,\mathrm{Im}\,\{q(x,t)\}. \tag{5.1.5}$$

Equation (5.1.2) maps real-valued solutions, $Q(x,t)$, of the MBO equation into complex-valued solutions, $q(x,t)$, of the BO equation. For example, the real-valued function Q_0 is mapped by (5.1.2) into the complex-valued function q_0, where

$$q_0 \ \underline{\underline{\mathrm{def}}}\ q(x,0) = \frac{1}{2}\left\{\mathbf{H}\left(Q_{0,x}\right) + \alpha\left(e^{Q_0} - 1\right) + iQ_{0,x}\right\} \tag{5.1.6}$$

and $Q_{0,x}$ denotes $\partial Q_0/\partial x$. It is clear from equation (5.1.6) that: **1)** q_0 is known whenever Q_0 is known; **2)** the boundary conditions for Q_0 unite with the deep water analogue of (1.3.28) to produce the boundary conditions

$$\frac{\partial^n q_0}{\partial x^n} \to 0 \ \text{as} \ |x| \to \infty. \tag{5.1.7}$$

The knowledge we have accumulated from Chapters 3 and 4 of this thesis motivates us to look for a solution of the initial value problem for (5.1.1) that emanates from the solution of the initial value problem for (5.1.3) with initial value (5.1.6) and boundary conditions (5.1.7). Fokas and Ablowitz [43] have solved the initial value problem for the real-valued BO equation by using a seminal application of the IST that involves the boundary conditions (5.1.7). Therefore, to obtain the solution of the initial value problem for the complexified version of (5.1.3) we will extend the work in Ref. 43 to incorporate evolutions from the initial value (5.1.6). We have found it necessary to retain the indicator, ε, and like elsewhere in this thesis $\varepsilon = \pm 1$ are the possible values assumed by the indicator. A suitable extension of the Fokas-Ablowitz IST for the BO equation will permit us to derive a formula for $q^{+\varepsilon}(x, t)$, where $q^{+\varepsilon}(x, t)$ denotes the boundary value of a function that admits analytic continuation (with respect to x) in the complex half-plane $\operatorname{sgn}(\operatorname{Im}(z)) = \varepsilon$; $z = x + iy$ refers to the complex extension of x. The reader who wishes to review the (generic) formulae for $q^{\pm}(x, t)$ can consult equations (1.3.65$^{\pm}$). Once a suitable formula for $q^{+\varepsilon}(x, t)$ is available, we will use the jump condition

$$q(x, t) = q^{+}(x, t) - q^{-}(x, t) \tag{5.1.8}$$

to solve for the physical variable, $q(x, t)$, in equation (5.1.3). After $q(x, t)$ has been determined we will then use this function in (5.1.5) to compute $Q_x(x, t)$, and thence $Q(x, t)$ by a quadrature with respect to x. The residual function of integration will be determined from the boundary conditions $Q(x, t) \to 0$ (uniformly in t) as $|x| \to \infty$. The last stage in the solution scheme for the MBO equation uses (5.1.4) to connect any parameter in $Q(x, t)$ that has emanated from the IST for the complex-valued BO equation to the parameter α in the MBO equation.

The remainder of this chapter is arranged into *five* sections. In the next section we present the linear problem for the BO equation in a form that is amenable to the solution scheme for the MBO equation. Suitable Jost functions that are parameterized by a real spectral parameter, denoted as λ, are defined in Section 5.2, and equations that characterize the scattering data are then derived. In the course of studying the direct problem for the complex-valued BO equation we noticed a *nongeneric* case that was overlooked by Fokas and Ablowitz in their IST for the real-valued BO equation [43]. The nongeneric case manifested itself when we sought to determine the asymptotics as $\lambda \to 0^{+}$ of the Jost functions and scattering data (continuous spectrum) to a higher precision than what appears in Refs 43 and 103. In order to accommodate the nongeneric case, which is associated with the vanishing of a certain integral, we have found it necessary to enlarge the list of scattering data presented by Fokas and Ablowitz in Ref. 43. Notwithstanding the amendments to the scattering data, the spectral parameter retains the essential features of the spectral parameter found in the IST for the real-valued BO equation [43]; the continuous spectrum remains real positive λ and the discrete spectrum still consists of a finite number of isolated points on the real axis $\lambda < 0$. We note that several researchers

derived by Scoufis and Cosgrove in Ref. 114. Section 5.3 is devoted to the solution of the inverse problem for the complex-valued BO equation. The solution of the inverse problem for the complex-valued BO equation, like the analogous problem for the real-valued case, requires the solution of a *nonlocal* Riemann-Hilbert Boundary Value Problem (RHBVP). Section 5.4 contains the time evolution of the scattering data for the complex-valued BO

produce the 1-soliton solution for the MBO equation, and thereby demonstrate how our calculations can be used to construct solutions for the MBO equation (5.1.1).

Section 5.2: The Direct Problem and Scattering Data

Nakamura [88] and Bock and Kruskal [18] have shown that the (real-valued) BO equation is a soliton-type equation, in particular that the BO equation can be expressed as the compatibility condition of an over-determined linear system of equations (Lax pair). The most direct route to the Lax pair for the BO equation is to determine the $\delta \to \infty$ limit of the Lax pair for the ILW equation, the latter Lax pair being contained in equations (4.3.12) and (4.3.13$^{\pm}$). A meaningful limit requires positive real λ, where λ is the spectral parameter in the IST for the (complex-valued) ILW equation. Scoufis and Cosgrove [114] have shown that the Lax pair for the complex-valued BO equation (in a form that can be used to solve the initial value problem for the MBO equation) is

$$i w_x^{+\varepsilon} + \varepsilon\lambda\left(w^{+\varepsilon} - w^{-\varepsilon}\right) = -\,\varepsilon q w^{+\varepsilon}, \tag{5.2.1}$$

$$w_t^{\pm\varepsilon} = i\varepsilon w_{xx}^{\pm\varepsilon} + 2\lambda w_x^{\pm\varepsilon} + \left[\pm q_x - i\varepsilon \mathbf{H}\left(q_x\right) + \nu\right] w^{\pm\varepsilon}. \tag{5.2.2$^{\pm}$}$$

A correct interpretation of (5.2.1) and (5.2.2$^{\pm}$) requires the following information:

- the indicator, denoted as ε, has the possible values $\varepsilon = \pm 1$;
- λ is the spectral parameter;
- $w^{\pm\varepsilon} \equiv w^{\pm\varepsilon}(x, t; \lambda)$ denotes (collectively) the boundary value of a function that is analytic in the complex half-plane $\operatorname{sgn}\left(\operatorname{Im}(z)\right) = \pm\varepsilon$, where $z = x + iy$ (with x and y real) defines the complex extension of x;
- $q \equiv q(x, t)$ denotes a complex-valued solution of (5.1.3), and the real and imaginary parts of this solution are connected to a solution of the MBO equation by equations (5.1.4) and (5.1.5), respectively;
- the compatibility condition $w_{xt}^{+\varepsilon} \equiv w_{tx}^{+\varepsilon}$ for the over-determined linear system comprising equations (5.2.1) and (5.2.2$^{\pm}$) yields the BO equation in the form (5.1.3);
- ν is independent of x and t, but its value is contingent on the boundary conditions for the eigenfunctions $w^{\pm\varepsilon}$.

suppress the explicit dependence of any quantity that depends on the temporal variable.

Unfortunately, the eigenfunctions $w^{+\varepsilon}$ and $w^{-\varepsilon}$ that appear in (5.2.1) and (5.2.2$^{\pm}$) are *not* connected by vertical periodicity similar to (4.2.17). Loss (as $\delta \to \infty$) of a formula that allows us to construct, say, $w^{-\varepsilon}$ from $w^{+\varepsilon}$ can be offset as follows: let $\Psi(x)$ denote a suitable complex-valued function of the real-variable *x*. Following Fokas and Ablowitz [43], we introduce into our work the notation $[\Psi]^{\pm\varepsilon}(x)$, where

$$[\Psi]^{\pm\varepsilon}(x) \underline{\underline{\mathrm{def}}} \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\Psi(\xi)}{\xi - \left(x \pm i\varepsilon 0^{+}\right)} \mathrm{d}\xi. \tag{5.2.3$^{\pm}$}$$

We will refer to $[\Psi]^{+\varepsilon}(x)$ (+ ε)-part of $\Psi(x)$ $[\Psi]^{-\varepsilon}(x)$ (– ε)-part of $\Psi(x)$ $[\Psi]^{+\varepsilon}(x)$ is the boundary value of a function analytic in the half-plane $\mathrm{sgn}\left(\mathrm{Im}\,(z)\right) = +\varepsilon$, and $[\Psi]^{-\varepsilon}(x)$ is the boundary value of a function analytic in the half-plane $\mathrm{sgn}\left(\mathrm{Im}\,(z)\right) = -\varepsilon$. The reader may express concern that (5.2.3$^{\pm}$) duplicates the notation used for $w^{+\varepsilon}$ and $w^{-\varepsilon}$. The square bracket notation will prove beneficial to our work because it allows us to reference the ($\pm\,\varepsilon$)-parts of functions that already have one of the superscripts + ε or – ε, for example

$$\left[q w^{+\varepsilon}\right]^{+\varepsilon}(x) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{q(\xi)\, w^{+\varepsilon}(\xi)}{\xi - \left(x + i\varepsilon 0^{+}\right)} \mathrm{d}\xi. \tag{5.2.4}$$

Equations (5.2.3$^{\pm}$) are the analogues of (4.2.15$^{\pm}$) in the deep water region. The $\delta \to \infty$ limit of (4.2.14) reveals that $[\Psi]^{\pm\varepsilon}(x)$ are connected by the jump condition

$$\Psi(x) = \varepsilon\left([\Psi]^{+\varepsilon}(x) - [\Psi]^{-\varepsilon}(x)\right). \tag{5.2.5}$$

Equations (5.2.3$^{\pm}$) and (5.2.5) allow us to retrieve a particular (analytic) part of suitable expressions, and thereby compensate us for the loss of information noted.

Bounded solutions (Jost functions) for (5.2.1) exist for all $\lambda > 0$ [43,63]. Fokas and Ablowitz [43] describe and execute the procedure for deriving linear (singular) integral equations that characterize the Jost functions mentioned, but their results apply when $\varepsilon = 1$ and q is a *real-valued* solution of the BO equation. Scoufis and Cosgrove [114] extended the Fokas-Ablowitz methodology by starting with (5.2.1), and then deriving appropriate integral equations for the Jost functions associated with the IST for the MBO equation, that is ε appears as a parameter in the equations derived by Scoufis and Cosgrove, and

these authors regard q as a *complex-valued* solution of the BO equation. As preparation for Chapter 7 (in which we will examine the deep water limit of the IST for the MBO equation) we will present a new derivation of the Scoufis-Cosgrove results, in particular those equations that identify the Jost functions for the system (5.2.1) and (5.2.2$^{\pm}$). Our method begins with the equations that characterize the Jost functions allied to the IST for the complex-valued ILW equation, and then considers the $\delta \to \infty$ limit of these equations. A crucial stage in our methodology is the determination of the $\delta \to \infty$
function that enters into the IST for the complex-valued ILW equation; the reader who

equations (4.3.50$^{\pm}$). The
for the real-valued ILW equation has been studied in Refs 63 and 103. The analysis that we will undertake extends the work in Refs 63 and 103 to incorporate particular nuances that are intrinsic to the functions defined by (4.3.50$^{\pm}$). All functions and constants introduced hereafter depend on ε, unless otherwise stated.

Consider equations (4.3.50$^{\pm}$), but now expressed in the (convenient) forms

$$G^{+\,\varepsilon}_{\pm\,\varepsilon}(x,\eta;\lambda) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{e^{i(x-\eta)r+\varepsilon\delta r}}{\sinh\left(\delta r \mp i0^{+}\right)\left[\varepsilon\zeta_{+\,\varepsilon}\left(r \mp i0^{+}\right) - \zeta_{+}(\lambda)\right]}\mathrm{d}r, \tag{5.2.6$^{\pm}$}$$

where the equation for $G^{+\,\varepsilon}_{+\,\varepsilon}(x,\eta;\lambda)$ is assigned the label (5.2.6^{+}), and the equation for $G^{+\,\varepsilon}_{-\,\varepsilon}(x,\eta;\lambda)$ will be referred to as (5.2.6^{-}). Equations (5.2.6$^{\pm}$) can be derived from (4.3.50$^{\pm}$), once we parameterize $C_{\mp}$, the contour in (4.3.50$^{\pm}$), in the correct manner. We will assume that $\lambda > 0$ and that $\lambda = \mathrm{O}(1)$ as $\delta \to \infty$. Two results that will facilitate our calculation of the deep water limit for $G^{+\,\varepsilon}_{\pm\,\varepsilon}(x,\eta;\lambda)$ are as follows:

$$\zeta_{+\,\varepsilon}(x) = \frac{x}{2}\left(1 + \varepsilon\,\mathrm{sgn}\,(x)\right) - \frac{\varepsilon}{2\delta} + \mathrm{o}\left(e^{-\delta|x|}\right) \text{ as } \delta \to \infty \tag{5.2.7}$$

and

$$\lim_{\delta\to\infty} \frac{e^{\varepsilon\delta r}}{\sinh\left(\delta r \mp i0^{+}\right)} = \begin{cases} 2\theta(\varepsilon),\ r > 0 \\ -2\theta(-\varepsilon),\ r < 0. \end{cases} \tag{5.2.8}$$

We relegate (5.2.7) and (5.2.8) to the category of exercises for the reader, but we do wish to remind the reader that $\theta(\bullet)$ denotes the Heaviside step function:

$$\theta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0. \end{cases}$$

Now let us initiate the calculation of the deep water limit for $G_{\pm\,\varepsilon}^{+\,\varepsilon}(x, \eta; \lambda)$. First, split the integral within (5.2.6$^{\pm}$) into *two* integrals, the first over the interval $-\infty < r < 0$ and the second over the interval $0 < r < \infty$. Next, use (5.2.7) and (5.2.8) in each of the half-line integrals to obtain the result

$$\lim_{\delta \to \infty} G_{\pm\,\varepsilon}^{+\,\varepsilon}(x, \eta; \lambda) = \mathrm{I}_{\pm} - \mathrm{J}_{\pm}, \tag{5.2.9}$$

where

$$\mathrm{I}_{\pm} \underline{\underline{\mathrm{def}}} \frac{\theta(\varepsilon)}{\pi} \int_0^{\infty} \frac{e^{i(x-\eta)r}}{(\varepsilon + 1)\left(r \mp i0^{+}\right) - 2\lambda} \mathrm{d}r \tag{5.2.10a}$$

and

$$\mathrm{J}_{\pm} \underline{\underline{\mathrm{def}}} \frac{\theta(-\varepsilon)}{\pi} \int_{-\infty}^{0} \frac{e^{i(x-\eta)r}}{(\varepsilon - 1)\left(r \mp i0^{+}\right) - 2\lambda} \mathrm{d}r. \tag{5.2.10b}$$

We now wish to assert the following regarding $\mathrm{I}_{\pm}$ and $\mathrm{J}_{\pm}$:

$$\mathrm{I}_{\pm} = \frac{\theta(\varepsilon)}{2\pi} \int_0^{\infty} \frac{e^{i\varepsilon(x-\eta)r}}{r - \left(\lambda \pm i\varepsilon 0^{+}\right)} \mathrm{d}r \tag{5.2.11a}$$

and

$$\mathrm{J}_{\pm} = \frac{\theta(-\varepsilon)}{2\pi} \int_0^{\infty} \frac{e^{i\varepsilon(x-\eta)r}}{r - \left(\lambda \pm i\varepsilon 0^{+}\right)} \mathrm{d}r. \tag{5.2.11b}$$

The proof of (5.2.11a) and (5.2.11b) commences by focusing our attention on (5.2.10b), the equation that defines $\mathrm{J}_{\pm}$. Equation (5.2.10b) can be expressed in the form

$$\mathrm{J}_{\pm} = \frac{\theta(-\varepsilon)}{\pi} \int_0^{\infty} \frac{e^{-i(x-\eta)s}}{(\varepsilon - 1)\left(-s \mp i0^{+}\right) - 2\lambda} \mathrm{d}s, \tag{5.2.12}$$

provided one introduces the transformation $s = -r$ into (5.2.10b). Two crucial observations facilitate a connection between (5.2.11b) and (5.2.12):

1) without loss of generality we can replace the denominator in the integrand of (5.2.12) by the expression $2\left(s - \lambda \pm i0^{+}\right)$;

2) the exponential term in the integrand of (5.2.12) can without loss of precision be written in the form $\exp\left(i\varepsilon(x-\eta)s\right)$.

Both observations emanate from the fact that (5.2.12) vanishes whenever $\varepsilon = 1$ and $x - \eta \neq 0$. The reader who exploits the observations mentioned will require little effort to arrive at (5.2.11b). A repetition (with minor modifications) of the procedure used to derive (5.2.11b) from (5.2.10b) will show that (5.2.11a) is equivalent to (5.2.10a).

Now substitute (5.2.11a) and (5.2.11b) into (5.2.9) to obtain the result

$$\lim_{\delta\to\infty} G^{+\,\varepsilon}_{\pm\,\varepsilon}(x,\eta;\lambda) = \frac{1}{2\pi}\left(\theta(\varepsilon) - \theta(-\varepsilon)\right)\int_0^\infty \frac{e^{i\varepsilon(x-\eta)r}}{r - \left(\lambda \pm i\varepsilon 0^{+}\right)}\mathrm{d}r,$$

and because $\theta(\varepsilon) - \theta(-\varepsilon) = \mathrm{sgn}\,(\varepsilon)$, which evaluates to ε, we can state that the deep water limit of $G^{+\,\varepsilon}_{\pm\,\varepsilon}(x,\eta;\lambda)$ is

$$\lim_{\delta\to\infty} G^{+\,\varepsilon}_{\pm\,\varepsilon}(x,\eta;\lambda) = \varepsilon g^{+\,\varepsilon}_{\pm\,\varepsilon}(x,\eta;\lambda), \qquad (5.2.13^{\pm})$$

where

$$g^{+\,\varepsilon}_{\pm\,\varepsilon}(x,\eta;\lambda) \equiv \frac{1}{2\pi}\int_0^\infty \frac{e^{i\varepsilon(x-\eta)r}}{r - \left(\lambda \pm i\varepsilon 0^{+}\right)}\mathrm{d}r. \qquad (5.2.14^{\pm})$$

The notation we have employed, particularly in $(5.2.14^{\pm})$, requires clarification:

•

character in x, for example $g^{+\,\varepsilon}_{\pm\,\varepsilon}(x,\eta;\lambda)$ admits analytic continuation into $\mathrm{sgn}\,(\mathrm{Im}\,(z)) = +\,\varepsilon$;

• Let $\Lambda = \lambda + i\kappa$, where λ and κ are real numbers, define the complex extension of λ. Subscripts of the form $+\varepsilon$ or $-\varepsilon$ will emphasize the analytic character in λ of functions that admit continuation into either the first or fourth quadrants of the complex Λ-plane. For example, $g^{+\,\varepsilon}_{-\,\varepsilon}(x,\eta;\lambda)$ admits continuation into the *quadrant* defined by $\mathrm{Re}\,(\Lambda) > 0$ and $\mathrm{sgn}\,(\mathrm{Im}\,(\Lambda)) = -\,\varepsilon$ $(+\,\varepsilon)$ function in λ

$(-\,\varepsilon)$ function in λ

remarks about our notation apply only to quantities allied to the IST for the BO equation.

No doubt the reader has observed that

$$\frac{e^{i\varepsilon(x-\eta)r}}{r-\left(\lambda \pm i\varepsilon 0^{+}\right)} \notin L_1(0,\infty),$$

where $L_1(0,\infty)$ denotes the space of absolutely integrable functions over the interval $0 < r < \infty$. Concerns that the reader may have about the convergence of (5.2.14$^{\pm}$) are vanquished by the realization that the equation

$$g_{\pm\varepsilon}^{+\varepsilon} = i\varepsilon \operatorname{sgn}(x-\eta)\,\theta(\pm(x-\eta))e^{i\varepsilon(x-\eta)\lambda} + \frac{1}{2\pi}\int_0^{\infty} \frac{e^{-|x-\eta|r}}{r + i\varepsilon\lambda \operatorname{sgn}(x-\eta)}\mathrm{d}r\ , \qquad (5.2.15^{\pm})$$

where $g_{\pm\varepsilon}^{+\varepsilon} \equiv g_{\pm\varepsilon}^{+\varepsilon}(x,\eta;\lambda)$, provides a new perspective from which to view (5.2.14$^{\pm}$). Significant effort was invested by the author to derive (5.2.15$^{\pm}$), but in an expression of a preference to maintain continuity with the task at hand (IST for the MBO equation) we confine ourselves to offering the following two remarks:

1) equations (5.2.15$^{\pm}$) are derived from (5.2.14$^{\pm}$)
Residue Theorem in which the contour of integration is a quarter-circle in either the *first* or *fourth* quadrants of the plane formed by an analytic continuation of r off the real-axis;

2) the interested reader is referred to Ablowitz and Fokas [12, p. 231] for certain material that forms a paradigm for the derivation of (5.2.15$^{\pm}$) from (5.2.14$^{\pm}$).

In the light of the commentary made about (5.2.15$^{\pm}$), we can clearly see that $g_{\pm\varepsilon}^{+\varepsilon}(x,\eta;\lambda)$ exists and is well defined for all $\lambda > 0$.

One fact that we can infer from (5.2.15$^{\pm}$) is that whenever $\operatorname{Re}(\Lambda) > 0$ the nonuniformity

$$\lim_{\operatorname{Im}(\Lambda)\to 0^{+\varepsilon}} g_{+\varepsilon}^{+\varepsilon}(x,\eta;\Lambda) \neq \lim_{\operatorname{Im}(\Lambda)\to 0^{-\varepsilon}} g_{-\varepsilon}^{+\varepsilon}(x,\eta;\Lambda)$$

is present. If we allow $\Delta g^{+\varepsilon}(x,\eta;\lambda)$ to denote the jump in $g^{+\varepsilon}(x,\eta;\Lambda)$ across the positive real λ-axis, then from (5.2.15$^{\pm}$) we see that

$$g_{+\varepsilon}^{+\varepsilon}(x,\eta;\lambda) - g_{-\varepsilon}^{+\varepsilon}(x,\eta;\lambda) = i\varepsilon e^{i\varepsilon(x-\eta)\lambda}, \qquad (5.2.16)$$

where

$$\Delta g^{+\varepsilon}(x,\eta;\lambda) \ \underline{\underline{\mathrm{def}}}\ g_{+\varepsilon}^{+\varepsilon}(x,\eta;\lambda) - g_{-\varepsilon}^{+\varepsilon}(x,\eta;\lambda)$$

indicates the orientation of the jump across the positive real λ-axis.

Now that the (mathematical) infrastructure required for the derivation of equations that characterize the Jost functions has been assembled, we can proceed to derive the necessary equations. Let $m^{+\varepsilon}(x;\lambda)$ and $\bar{m}^{+\varepsilon}(x;\lambda)$ denote *left* Jost functions for (5.2.1), and let $n^{+\varepsilon}(x;\lambda)$ and $\bar{n}^{+\varepsilon}(x;\lambda)$ denote *right* Jost functions for (5.2.1). Each Jost function satisfies a particular version of the integro-differential equation

$$iw_x^{+\varepsilon} + \varepsilon\lambda w^{+\varepsilon} = -\left[qw^{+\varepsilon}\right]^{+\varepsilon}(x) + a_0, \tag{5.2.17}$$

where a_0 is a constant whose value is determined from a boundary condition for $w^{+\varepsilon}$, for example $w^{+\varepsilon} \to 1$ as $x \to \infty$ requires $a_0 = \varepsilon\lambda$. Equation (5.2.17) is the $(+\varepsilon)$-part (with respect to *x*) of (5.2.1), and $\left[qw^{+\varepsilon}\right]^{+\varepsilon}(x)$ is defined by (5.2.4).

The Jost functions (lower-case) affiliated to the complex-valued BO equation can be retrieved from the Jost functions (upper-case) bound to the complex-valued ILW equation in the following manner [103]:

$$m^{+\varepsilon}(x;\lambda) = \lim_{\delta\to\infty} M^{+\varepsilon}(x;\lambda); \tag{5.2.18a}$$

$$\bar{m}^{+\varepsilon}(x;\lambda) = \lim_{\delta\to\infty} \bar{M}^{+\varepsilon}(x;\lambda)e^{-\delta\lambda}; \tag{5.2.18b}$$

$$n^{+\varepsilon}(x;\lambda) = \lim_{\delta\to\infty} N^{+\varepsilon}(x;\lambda)e^{-\delta\lambda}; \tag{5.2.18c}$$

$$\bar{n}^{+\varepsilon}(x;\lambda) = \lim_{\delta\to\infty} \bar{N}^{+\varepsilon}(x;\lambda). \tag{5.2.18d}$$

Explicit asymptotics of the Jost functions allied to the complex-valued BO equation can be deduced by reference to equations (4.3.15), (4.3.16) and (5.2.18a-d). The reader will find that the boundary conditions for the respective Jost functions are as follows:

$$\left.\begin{aligned} m^{+\varepsilon}(x;\lambda) &\to 1 \\ \bar{m}^{+\varepsilon}(x;\lambda) &\to e^{i\varepsilon\lambda x} \end{aligned}\right\} \text{ as } x \to -\infty \tag{5.2.19}$$

and

$$\left.\begin{aligned} n^{+\varepsilon}(x;\lambda) &\to e^{i\varepsilon\lambda x} \\ \bar{n}^{+\varepsilon}(x;\lambda) &\to 1 \end{aligned}\right\} \text{ as } x \to \infty. \tag{5.2.20}$$

Equations (5.2.19) and (5.2.20) are consistent with the observation that the large-x behaviour of (5.2.17) is

$$iw_x^{+\varepsilon} + \varepsilon\lambda w^{+\varepsilon} \sim a_0 \text{ as } |x| \to \infty,$$

and this behaviour implies that

$$w^{+\varepsilon} = \frac{\varepsilon a_0}{\lambda} + b_0 e^{i\varepsilon\lambda x} \text{ as } |x| \to \infty,$$

where b_0 is a constant of integration; the most recent equation justifies our selection of $\left\{1, e^{i\varepsilon\lambda x}\right\}$ as a basis for the large-x behaviour of $w^{+\varepsilon}$.

In Section 4.3 of this thesis we showed that (4.3.51) and (4.3.52) are the specific equations that characterize the Jost functions present in the IST for the complex-valued ILW equation. The deep water limit

$$\lim_{\delta \to \infty} U(\eta) = q(\eta)$$

and the aggregate of (4.3.51), (4.3.52), (5.2.13$^{\pm}$) and (5.2.18a-d) allows us to conclude that

$$\begin{pmatrix} m^{+\varepsilon}(x;\lambda) \\ \bar{m}^{+\varepsilon}(x;\lambda) \end{pmatrix} = \begin{pmatrix} 1 \\ e^{i\varepsilon\lambda x} \end{pmatrix} + \int_{-\infty}^{\infty} g_{+\varepsilon}^{+\varepsilon}(x;\eta;\lambda)\, q(\eta) \begin{pmatrix} m^{+\varepsilon}(\eta;\lambda) \\ \bar{m}^{+\varepsilon}(\eta;\lambda) \end{pmatrix} \mathrm{d}\eta \qquad (5.2.21)$$

and

$$\begin{pmatrix} n^{+\varepsilon}(x;\lambda) \\ \bar{n}^{+\varepsilon}(x;\lambda) \end{pmatrix} = \begin{pmatrix} e^{i\varepsilon\lambda x} \\ 1 \end{pmatrix} + \int_{-\infty}^{\infty} g_{-\varepsilon}^{+\varepsilon}(x;\eta;\lambda)\, q(\eta) \begin{pmatrix} n^{+\varepsilon}(\eta;\lambda) \\ \bar{n}^{+\varepsilon}(\eta;\lambda) \end{pmatrix} \mathrm{d}\eta \qquad (5.2.22)$$

are the integral equations that characterize $m^{+\varepsilon}(x;\lambda)$ and so forth. A cursory survey of (5.2.21) and (5.2.22) will show the reader that these equations are examples of (inhomogeneous) Fredholm integral equations of the second-kind. Comparable equations to (5.2.21) and (5.2.22) [see (4.3.51) and (4.3.52)] in the IST for the complex-valued ILW equation are (inhomogeneous) Volterra integral equations of the second-kind. At a suitable time we will return to (5.2.21) and (5.2.22) with the intention to extract from these equations the analytic character (with respect to λ) of the eigenfunctions $m^{+\varepsilon}(x;\lambda)$ and so forth.

Equations (5.2.21) and (5.2.22) provide a portrait of the Jost functions, and when these equations are used appropriately with (5.2.16) they also deliver to us an *identity* that involves the eigenfunctions $m^{+\varepsilon}(x;\lambda)$, $n^{+\varepsilon}(x;\lambda)$ and $\bar{n}^{+\varepsilon}(x;\lambda)$. The identity in question is

$$m^{+\varepsilon}(x;\lambda) = \bar{n}^{+\varepsilon}(x;\lambda) + \rho(\lambda)n^{+\varepsilon}(x;\lambda), \tag{5.2.23}$$

where λ is a positive real number and

$$\rho(\lambda) \underline{\underline{\mathrm{def}}}\; i\varepsilon\int_{-\infty}^{\infty} q(\eta)\, m^{+\varepsilon}(\eta;\lambda)\, e^{-i\varepsilon\lambda\eta}\,\mathrm{d}\eta. \tag{5.2.24}$$

The entity $\rho(\lambda)$ depends on t, is a member of the *continuous spectrum* and must be included in the list of scattering data. Equation (5.2.23) is the analogue of (4.3.53), the latter equation being central to the development of a Riemann-Hilbert Boundary Value Problem (RHBVP) that facilitated the solution of the initial value problem for the MILW equation. In due course we will embed (5.2.23) within the framework of the RHBVP, but for the moment we turn our attention to the proof of (5.2.23). By virtue of equations (5.2.21) and (5.2.22) we can state that the difference $m^{+\varepsilon}(x;\lambda) - \bar{n}^{+\varepsilon}(x;\lambda)$ is constrained by the equation

$$m^{+\varepsilon}(x;\lambda) - \bar{n}^{+\varepsilon}(x;\lambda) = \int_{-\infty}^{\infty} g^{+\varepsilon}_{+\varepsilon}(x,\eta;\lambda)\, q(\eta)\, m^{+\varepsilon}(\eta;\lambda)\,\mathrm{d}\eta$$

$$- \int_{-\infty}^{\infty} g^{+\varepsilon}_{-\varepsilon}(x,\eta;\lambda)\, q(\eta)\, \bar{n}^{+\varepsilon}(\eta;\lambda)\,\mathrm{d}\eta. \tag{5.2.25}$$

Using (5.2.16) to eliminate $g^{+\varepsilon}_{+\varepsilon}(x,\eta;\lambda)$ from (5.2.25) we obtain

$$m^{+\varepsilon}(x;\lambda) - \bar{n}^{+\varepsilon}(x;\lambda) = \rho(\lambda)e^{i\varepsilon\lambda x} + \int_{-\infty}^{\infty} g^{+\varepsilon}_{-\varepsilon}(x,\eta;\lambda)\, q(\eta)\left\{m^{+\varepsilon}(\eta;\lambda) - \bar{n}^{+\varepsilon}(\eta;\lambda)\right\}\mathrm{d}\eta,$$

which provides an integral equation satisfied by the difference $m^{+\varepsilon}(x;\lambda) - \bar{n}^{+\varepsilon}(x;\lambda)$. The *exponential* component of the forcing term in the most recent equation can be eliminated by reference to (5.2.22), in particular by use of the equation for $n^{+\varepsilon}(x;\lambda)$. The *homogeneous* equation

$$\Phi(x;\lambda) = \int_{-\infty}^{\infty} g^{+\varepsilon}_{-\varepsilon}(x,\eta;\lambda)\, q(\eta)\, \Phi(\eta;\lambda)\,\mathrm{d}\eta, \tag{5.2.26}$$

where

$$\Phi(x;\lambda) \underline{\underline{\mathrm{def}}}\; m^{+\varepsilon}(x;\lambda) - \bar{n}^{+\varepsilon}(x;\lambda) - \rho(\lambda)n^{+\varepsilon}(x;\lambda),$$

is produced when we complete the procedure advocated. Equation (5.2.26) is an example of a linear homogeneous integral equation of the

trivial solution whenever λ is a positive real number; our assumption applies irrespective of whether $q(\eta)$ is real-valued or complex-valued. In the light of the assumption made about (5.2.26) we see that $\Phi(x;\lambda) \equiv 0$, from which we conclude that (5.2.23) is correct.

Our intention is to eventually interpret (5.2.23) as some type of RHBVP in a sector of the complex Λ-plane. A connection between the eigenfunctions $\bar{n}^{+\varepsilon}(x;\lambda)$ and $n^{+\varepsilon}(x;\lambda)$ must be established before our plan for (5.2.23) can succeed. The identity

$$\frac{\partial}{\partial\lambda}\left\{n^{+\varepsilon}(x;\lambda)\right\} - i\varepsilon x n^{+\varepsilon}(x;\lambda) = f(\lambda)\bar{n}^{+\varepsilon}(x;\lambda), \tag{5.2.27}$$

where

$$f(\lambda) \underline{\underline{\mathrm{def}}} -\frac{1}{2\pi\lambda}\int_{-\infty}^{\infty} q(\eta)n^{+\varepsilon}(\eta;\lambda)\mathrm{d}\eta, \tag{5.2.28}$$

is the requisite connection between $\bar{n}^{+\varepsilon}(x;\lambda)$ and $n^{+\varepsilon}(x;\lambda)$. The function $f(\lambda)$ is a member of the continuous spectrum and depends on t. Equation (5.2.27) is definitely a nontrivial result, which merits any time and effort required for its substantiation. A lemma is required before we can embark on the proof of (5.2.27). The necessary lemma (which we will prove) is

$$\frac{\partial}{\partial\lambda} g^{+\varepsilon}_{\pm\varepsilon}(x,\eta;\lambda) = i\varepsilon(x-\eta)g^{+\varepsilon}_{\pm\varepsilon}(x,\eta;\lambda) - \frac{1}{2\pi\lambda}, \tag{5.2.29$^{\pm}$}$$

where $g^{+\varepsilon}_{\pm\varepsilon}(x,\eta;\lambda)$ is defined by (5.2.15$^{\pm}$). Let us agree to use the notation

$$\Upsilon \underline{\underline{\mathrm{def}}} \frac{1}{2\pi}\int_0^{\infty} \Omega(r,\lambda)e^{-|x-\eta|r}\mathrm{d}r, \tag{5.2.30}$$

where

$$\Omega(r,\lambda) \underline{\underline{\mathrm{def}}} \frac{1}{r + i\varepsilon\lambda\,\mathrm{sgn}\,(x-\eta)}, \tag{5.2.31}$$

to refer to the nonlocal term in (5.2.15$^{\pm}$). An interesting symmetry, which can be verified by direct use of (5.2.31), is

$$\frac{\partial}{\partial\lambda}\Omega(r,\lambda) = i\varepsilon\,\mathrm{sgn}\,(x-\eta)\frac{\partial}{\partial r}\Omega(r,\lambda). \tag{5.2.32}$$

Differentiating with respect to λ the function defined by (5.2.30) delivers to us the equation

$$\frac{\partial\Upsilon}{\partial\lambda} = \frac{1}{2\pi}\int_0^{\infty}\frac{\partial}{\partial\lambda}\left\{\Omega(r,\lambda)\right\}e^{-|x-\eta|r}\mathrm{d}r,$$

and on account of (5.2.32) the equation for $\partial\Upsilon/\partial\lambda$ can be expressed in the form

$$\frac{\partial \Upsilon}{\partial \lambda} = \frac{i\varepsilon}{2\pi} \operatorname{sgn}\left(x - \eta\right) \int_0^{\infty} \frac{\partial}{\partial r}\left\{\Omega\left(r, \lambda\right)\right\} e^{-|x-\eta|r} \mathrm{d}r. \tag{5.2.33}$$

A single integration by parts when applied to (5.2.33) yields

$$\frac{\partial \Upsilon}{\partial \lambda} = -\frac{1}{2\pi\lambda} + i\varepsilon\left(x - \eta\right)\Upsilon. \tag{5.2.34}$$

The reader who attempts to reproduce (5.2.34) will require equations for $\Omega\left(0, \lambda\right)$ and $\Omega\left(\infty, \lambda\right)$, each of which can be derived from (5.2.31); the notation $\Omega\left(\infty, \lambda\right)$ denotes the limit of $\Omega\left(r, \lambda\right)$ as $r \to \infty$. We remind the reader that Υ is defined by the aggregate of equations (5.2.30) and (5.2.31). Differentiating both sides of (5.2.15$^{\pm}$) with respect to λ, and then using the identity (5.2.34) to simplify the derivative of the (common) nonlocal term we arrive at (5.2.29$^{\pm}$).

Now that we have proved (5.2.29$^{\pm}$), let us start work on the proof of (5.2.27). Consider (5.2.22), in particular the integral equation that characterizes $n^{+\varepsilon}\left(x; \lambda\right)$. The equation under present consideration can be adapted to read

$$n^{+\varepsilon}\left(x; \lambda\right) e^{-i\varepsilon\lambda x} = 1 + \int_{-\infty}^{\infty} g_{-\varepsilon}^{+\varepsilon}\left(x, \eta; \lambda\right) e^{-i\varepsilon\lambda(x-\eta)} q\left(\eta\right) n^{+\varepsilon}\left(\eta; \lambda\right) e^{-i\varepsilon\lambda\eta} \mathrm{d}\eta, \tag{5.2.35}$$

which is simply the integral equation that characterizes the modified eigenfunction $n^{+\varepsilon}\left(x; \lambda\right) e^{-i\varepsilon\lambda x}$. Our decision to use a modified eigenfunction is predicated on the observation that the derivative of $n^{+\varepsilon}\left(x; \lambda\right)$ with respect to λ is easier to access from (5.2.35) than from (5.2.22). Differentiating both sides of (5.2.35) with respect to λ, and then using (5.2.29^{-}) to eliminate the $\partial g_{-\varepsilon}^{+\varepsilon}(x, \eta; \lambda) / \partial\lambda$ term from the ensuing equation we find that

$$\frac{\partial}{\partial \lambda} n^{+\varepsilon}\left(x; \lambda\right) e^{-i\varepsilon\lambda x} = f(\lambda) e^{-i\varepsilon\lambda x} + \int_{-\infty}^{\infty} g_{-\varepsilon}^{+\varepsilon}\left(x, \eta; \lambda\right) q\left(\eta\right) \frac{\partial}{\partial \lambda}\left\{n^{+\varepsilon}\left(\eta; \lambda\right) e^{-i\varepsilon\lambda\eta}\right\} \mathrm{d}\eta, \tag{5.2.36}$$

where $f\left(\lambda\right)$ is defined by (5.2.28). We now aim to move in a direction that allows us to cast (5.2.36) in a form compatible with (5.2.26), so that we can again employ the vanishing lemma hypothesis that is associated with the latter equation. Equation (5.2.22), but written in the form

$$\bar{n}^{+\varepsilon}\left(x; \lambda\right) - \int_{-\infty}^{\infty} g_{-\varepsilon}^{+\varepsilon}\left(x, \eta; \lambda\right) q\left(\eta\right) \bar{n}^{+\varepsilon}\left(\eta; \lambda\right) \mathrm{d}\eta = 1, \tag{5.2.37}$$

can be used to achieve the desired transformation of (5.2.36). Equations (5.2.36) and (5.2.37) imply a particular version of (5.2.26) in which

$$\Phi(x;\lambda)e^{i\varepsilon\lambda x} \equiv \frac{\partial}{\partial\lambda}\left\{n^{+\,\varepsilon}(x;\lambda)\right\} - i\varepsilon x n^{+\,\varepsilon}(x;\lambda) - f(\lambda)\bar{n}^{+\,\varepsilon}(x;\lambda),$$

and because of our assumption that $\Phi(x;\lambda) \equiv 0$ for all $\lambda > 0$ we arrive at the identity (5.2.27).

As mentioned in Section 5.1 of this thesis, we need to discuss the asymptotics as $\lambda \to 0^+$ of the Jost functions and scattering data (continuous spectrum) to a higher precision than what appears in the extant literature [43,103]. The asymptotic formulae that we will derive contribute significantly to the solution scheme for the inverse problem associated with the complex-valued ILW equation. Our starting point is $g^{+\,\varepsilon}_{-\,\varepsilon}(x,\eta;\lambda)$ [see (5.2.15$^-$) for the relevant equation] because the behaviour as $\lambda \to 0^+$ of this function will influence either explicitly or in a secondary manner how the Jost functions and scattering data behave as $\lambda \to 0^+$. Once we have assembled the asymptotic profile of $g^{+\,\varepsilon}_{-\,\varepsilon}(x,\eta;\lambda)$ for small positive values of λ, we can use (5.2.16) to resolve the asymptotics of $g^{+\,\varepsilon}_{+\,\varepsilon}(x,\eta;\lambda)$ as $\lambda \to 0^+$. In particular, (5.2.16) constrains $g^{+\,\varepsilon}_{+\,\varepsilon}(x,\eta;\lambda)$ and $g^{+\,\varepsilon}_{-\,\varepsilon}(x,\eta;\lambda)$ so that

$$g^{+\,\varepsilon}_{+\,\varepsilon}(x,\eta;\lambda) = g^{+\,\varepsilon}_{-\,\varepsilon}(x,\eta;\lambda) + i\varepsilon + \mathrm{O}(\lambda) \quad \text{as } \lambda \to 0^+, \tag{5.2.38}$$

and it is therefore sufficient for us to ascertain how $g^{+\,\varepsilon}_{-\,\varepsilon}(x,\eta;\lambda)$ behaves as $\lambda \to 0^+$. We

$g^{+\,\varepsilon}_{\pm\,\varepsilon}(x,\eta;\lambda)$ is defined by (5.2.15$^\pm$).

In the course of this section we have assigned the label Υ to the nonlocal term within (5.2.15$^-$). According to the assemblage of equations (5.2.30) and (5.2.31) we have

$$\Upsilon = \frac{1}{2\pi}\int_0^\infty \frac{e^{-|x-\eta|r}}{r + i\varepsilon\lambda\,\mathrm{sgn}\,(x-\eta)}\mathrm{d}r.$$

A convenient form in which to express Υ in is

$$\Upsilon = \frac{1}{2\pi}\int_0^\infty \frac{\partial}{\partial r}\left\{\ln\left(r + i\varepsilon\lambda\,\mathrm{sgn}\,(x-\eta)\right)\right\}e^{-|x-\eta|r}\mathrm{d}r$$

because such an expression is amenable to (a single) integration by parts, which upon completion leaves us with the equation

$$\Upsilon = -\frac{1}{2\pi}\ln\left(i\varepsilon\lambda\,\mathrm{sgn}\,(x-\eta)\right)$$
$$+\frac{|x-\eta|}{2\pi}\int_0^\infty \ln\left(r + i\varepsilon\lambda\,\mathrm{sgn}\,(x-\eta)\right)e^{-|x-\eta|r}\mathrm{d}r. \quad (5.2.39)$$

A technical issue that we must now resolve is our selection of the *branch cut* for $\ln(z)$. Our branch cut for $\ln(z)$ starts at $z = 0$ and continues indefinitely along the $\mathrm{Re}(z) < 0$ axis. Given that $\lambda > 0$, we can now state that

$$\ln\left(i\varepsilon\lambda\,\mathrm{sgn}\,(x-\eta)\right) = \ln(\lambda) + i\varepsilon\frac{\pi}{2}\mathrm{sgn}\,(x-\eta) \quad (5.2.40a)$$

and

$$\ln\left(r + i\varepsilon\lambda\,\mathrm{sgn}\,(x-\eta)\right) = \ln(r) + \ln\left(1 + \frac{i\varepsilon\lambda}{r}\mathrm{sgn}\,(x-\eta)\right). \quad (5.2.40b)$$

Substituting (5.2.40a) and (5.2.40b) into (5.2.39) we obtain

$$\Upsilon = -\frac{\ln(\lambda)}{2\pi} - \frac{i\varepsilon}{4}\mathrm{sgn}\,(x-\eta) + \Delta_1 + \Delta_2, \quad (5.2.41)$$

where

$$\Delta_1 \underset{=}{\mathrm{def}} \frac{|x-\eta|}{2\pi}\int_0^\infty \ln(r)e^{-|x-\eta|r}\mathrm{d}r \quad (5.2.42)$$

and

$$\Delta_2 \underset{=}{\mathrm{def}} \frac{|x-\eta|}{2\pi}\int_0^\infty \ln\left(1 + \frac{i\varepsilon\lambda}{r}\mathrm{sgn}\,(x-\eta)\right)e^{-|x-\eta|r}\mathrm{d}r. \quad (5.2.43)$$

We now need to ascertain how Δ_1 and Δ_2 behave as $\lambda \to 0^+$. Let us first consider Δ_1. Introduce into (5.2.42) the change of variable $p = |x-\eta|r$. When we scale the integrand of Δ_1 we find that

$$\Delta_1 = \frac{1}{2\pi}\int_0^\infty \ln(p)e^{-p}\mathrm{d}p - \frac{\Gamma(1)}{2\pi}\ln|x-\eta|,$$

which can be written in the form

$$\Delta_1 = -\frac{\gamma + \ln|x-\eta|}{2\pi}, \quad (5.2.44)$$

where γ p.252] through the quadrature

$$-\gamma = \int_0^\infty \ln(p)e^{-p}\mathrm{d}p.$$

Fortunately, we can ignore the contribution from Δ_2 in the limit $\lambda \to 0^+$ because the integrand of (5.2.43) decays to zero as $\lambda \to 0^+$. Armed with (5.2.44) and the knowledge that $\Delta_2 \to 0$ as $\lambda \to 0^+$, we can declare that the equation

$$\Upsilon = -\frac{\ln(\lambda)}{2\pi} - \frac{1}{2\pi}\{\gamma + \ln(i\varepsilon(x-\eta))\} + (\text{decaying terms})$$

captures the behaviour of (5.2.41) as $\lambda \to 0^+$, or when we use (5.2.30) and (5.2.31) to write the full expression for Υ we have

$$\frac{1}{2\pi}\int_0^\infty \frac{e^{-|x-\eta|r}\,\mathrm{d}r}{r + i\varepsilon\lambda\,\mathrm{sgn}(x-\eta)} \sim -\frac{\ln(\lambda)}{2\pi} - \frac{1}{2\pi}\{\gamma + \ln(i\varepsilon(x-\eta))\} \text{ as } \lambda \to 0^+. \tag{5.2.45}$$

The reader who wishes to verify (5.2.45) should be cognisant of the following identity:

$$\frac{1}{2\pi}\ln(i\varepsilon(x-\eta)) = \frac{i\varepsilon}{4}\mathrm{sgn}(x-\eta) + \frac{1}{2\pi}\ln|x-\eta| \;.$$

Let $k_0(x)$ be defined through the equation

$$k_0(x) \underset{=}{\underline{\mathrm{def}}} -\frac{1}{2\pi}\{\gamma + \ln(i\varepsilon x)\} + i\varepsilon\,\mathrm{sgn}(x)\theta(-x) \;. \tag{5.2.46}$$

Our decision to define the function $k_0(x)$ is vindicated by the fact that

$$g^{+\,\varepsilon}_{-\,\varepsilon}(x,\eta;\lambda) = -\frac{\ln(\lambda)}{2\pi} + k_0(x-\eta) + (\text{decaying terms}) \text{ as } \lambda \to 0^+, \tag{5.2.47}$$

which can be verified by use of ($5.2.15^-$) and (5.2.45). As mentioned when we began our calculation to determine the asymptotics of $g^{+\,\varepsilon}_{\pm\,\varepsilon}(x,\eta;\lambda)$ as $\lambda \to 0^+$, we can use (5.2.38) to deduce from (5.2.47) the small-λ asymptotics of $g^{+\,\varepsilon}_{+\,\varepsilon}(x,\eta;\lambda)$:

$$g^{+\,\varepsilon}_{+\,\varepsilon}(x,\eta;\lambda) = -\frac{\ln(\lambda)}{2\pi} + k_0(x-\eta) + i\varepsilon + (\text{decaying terms}) \text{ as } \lambda \to 0^+. \tag{5.2.48}$$

Equations (5.2.47) and (5.2.48) agree at leading order (secular term) with the asymptotics presented in Refs 43 and 103, but the equations we have derived contain the small-λ asymptotics of $g^{+\,\varepsilon}_{\pm\,\varepsilon}(x,\eta;\lambda)$ to a higher precision than in these works.

Given that we now know the asymptotic behaviour as $\lambda \to 0^+$ of $g^{+\,\varepsilon}_{\pm\,\varepsilon}(x,\eta;\lambda)$ we can proceed to compute the small-λ asymptotics of the Jost functions and scattering data. First, let us define the function $p(x)$ through the (linear) Fredholm integral equation

$$p(x) = 1 + \int_{-\infty}^{\infty} k_0(x-\eta)\,q(\eta)\,p(\eta)\,d\eta, \tag{5.2.49}$$

and then define p_0 by means of the inner product

$$p_0 \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty} q(\eta)\,p(\eta)\,d\eta. \tag{5.2.50}$$

It is an opportune moment for us to recall that $q(\eta)$ satisfies the complex-valued BO equation (5.1.3), and $k_0(x)$ is defined by (5.2.46). The constant p_0 depends on ε, and must be added to the list of scattering data because it allows us to distinguish the *generic case* ($p_0 \neq 0$) from the *nongeneric case* ($p_0 = 0$). The distinction between generic and nongeneric potentials will be particularly relevant when we consider (in the next section) the inverse problem for the complex-valued BO equation. Fokas and Ablowitz consider only the generic case in their work [43]. The inclusion [114] of p_0 into our work provides scope for the first unified treatment of generic and nongeneric potentials associated with the IST for the BO equation. In terms of $p(x)$ and p_0 we find that the Jost functions display the following asymptotics as $\lambda \to 0^+$:

$$\left.\begin{matrix} m^{+\varepsilon}(x;\lambda) \\ \bar{m}^{+\varepsilon}(x;\lambda) \end{matrix}\right\} = \frac{p(x)}{1 + [\ln(\lambda) - 2i\varepsilon\pi]p_0} + \begin{cases} O(\lambda) \\ O(\lambda/\ln(\lambda)); \end{cases} \tag{5.2.51}$$

$$\left.\begin{matrix} n^{+\varepsilon}(x;\lambda) \\ \bar{n}^{+\varepsilon}(x;\lambda) \end{matrix}\right\} = \frac{p(x)}{1 + p_0\ln(\lambda)} + \begin{cases} O(\lambda/\ln(\lambda)) \\ O(\lambda). \end{cases} \tag{5.2.52}$$

Equation (5.2.51) can be derived by use of (5.2.48) in (5.2.21), whereas the derivation of (5.2.52) requires the reader to use the formula (5.2.47) in (5.2.22).

Sharpened small-λ asymptotics of $\rho(\lambda)$ and $f(\lambda)$ are also within our reach, where $\rho(\lambda)$ and $f(\lambda)$ are defined by equations (5.2.24) and (5.2.28), respectively. Using the equations that define $\rho(\lambda)$ and $f(\lambda)$ we find

$$\rho(\lambda) = \frac{2i\varepsilon\pi p_0}{1 + [\ln(\lambda) - 2i\varepsilon\pi]p_0} + O(\lambda/\ln\lambda) \text{ as } \lambda \to 0^+ \tag{5.2.53}$$

and

$$f(\lambda) = -\frac{p_0}{\lambda[1 + p_0\ln(\lambda)]} + O(1/\ln\lambda) \text{ as } \lambda \to 0^+, \tag{5.2.54}$$

where p_0 is defined by (5.2.50).

The O-bounds that are included in equations (5.2.51)-(5.2.54) are valid for $p_0 \neq 0$. Clearly, the asymptotic character of the Jost functions and of $\rho(\lambda)$ and $f(\lambda)$ changes significantly when $p_0 = 0$. Incidentally, the case $p_0 = 0$ occurs simultaneously for both choices of ε. In the case $p_0 = 0$ (the nongeneric case) the four Jost functions each have the leading term $p(x, t)$ for small-λ, and therefore these functions do *not* tend to zero as $\lambda \to 0^+$. The reader familiar with the IST for the BO equation [43] will undoubtedly agree with the author of this thesis who regards the existence of eigenfunctions that do not decay to zero as $\lambda \to 0^+$ to be a significant discovery. Recently, Kaup and Matsuno [61] have improved the formulae (5.2.51)-(5.2.54). We refer the reader to Section 5 of Ref. 61 for the Kaup-Matsuno results.

Discrete (bound states) for the eigenvalue problem (5.2.1) are also possible. The mechanism that produces the current discrete eigenvalues differs significantly from the mechanism that drives the production of discrete eigenvalues for the complex-valued ILW equation. In particular, bound states for the BO equation manifest themselves as solutions of the *homogeneous* version of the Fredholm integral equations (5.2.21) and (5.2.22). Bound states of (5.2.1) will be denoted as $\Phi_j^{+\varepsilon}(x)$, where $j = 1, 2, \ldots, \ell$ and ℓ is some finite positive integer. Each discrete eigenfunction is associated with a discrete eigenvalue. We will use the symbol λ_j to denote the discrete eigenvalues, where the discrete index j assumes each of the values $j = 1, 2, \ldots, \ell$. The discrete eigenvalues λ_j are negative real constants that are independent of ε and t. Coifman and Wickerhauser have shown [see Theorem 7.1 in Ref. 30] that for *complex-valued* solutions of the BO equation the discrete eigenvalues of the associated spectral problem cannot accumulate at the origin. All the information we have about the discrete spectrum for (5.2.1) leads us to declare that the following homogeneous Fredholm integral equation characterizes $\Phi_j^{+\varepsilon}(x)$:

$$\Phi_j^{+\varepsilon}(x) = \int_{-\infty}^{\infty} h^{+\varepsilon}(x, \eta; \lambda_j)\, q(\eta)\, \Phi_j^{+\varepsilon}(\eta)\, \mathrm{d}\eta, \tag{5.2.55}$$

where

$$h^{+\varepsilon}(x, \eta; \lambda_j) \equiv g^{+\varepsilon}_{\pm\varepsilon}(x, \eta; \lambda_j) = \frac{1}{2\pi}\int_0^{\infty} \frac{e^{i\varepsilon(x-\eta)r}}{r - \lambda_j}\, \mathrm{d}r. \tag{5.2.56}$$

The boundary conditions (as $|x| \to \infty$) satisfied by the discrete eigenfunctions require clarification, and we will now invest some effort to elucidate this behaviour. First, the equation

$$i\varepsilon(x-\eta)h^{+\varepsilon}\left(x,\eta;\lambda_j\right) = \frac{1}{2\pi\lambda_j} + \frac{1}{2\pi}\int_0^{\infty}\frac{e^{i\varepsilon(x-\eta)r}}{\left(r-\lambda_j\right)^2}\mathrm{d}r \tag{5.2.57}$$

is derived from (5.2.26) by a single integration by parts. The Riemann-Lebesgue lemma provides us with a reason as to why we can neglect as $|x| \to \infty$ the nonlocal term in (5.2.57). Therefore, from (5.2.57) we infer that

$$h^{+\varepsilon}\left(x,\eta;\lambda_j\right) = \frac{1}{2\pi i\varepsilon\lambda_j x} + \mathrm{O}\left(\frac{1}{x^2}\right) \text{ as } |x| \to \infty. \tag{5.2.58}$$

Substituting (5.2.58) into (5.2.55) we procure the desired information:

$$\lim_{|x|\to\infty} x\Phi_j^{+\varepsilon}(x) = k_j, \tag{5.2.59}$$

where

$$k_j \underline{\underline{\text{def}}} \frac{1}{2\pi i\varepsilon\lambda_j}\int_{-\infty}^{\infty} q(\eta)\Phi_j^{+\varepsilon}(\eta)\mathrm{d}\eta. \tag{5.2.60}$$

Without loss of generality we can make the assignment $k_j = 1$ because if $\Phi_j^{+\varepsilon}(x)$ solves (5.2.55), then so does $k_j\Phi_j^{+\varepsilon}(x)$. As a direct consequence of our normalization, we deduce from (5.2.59) that

$$\lim_{|x|\to\infty} x\Phi_j^{+\varepsilon}(x) = 1, \tag{5.2.61}$$

and from (5.2.60) we find that

$$\lambda_j = \frac{\varepsilon}{2\pi i}\int_{-\infty}^{\infty} q(\eta)\Phi_j^{+\varepsilon}(\eta)\mathrm{d}\eta, \tag{5.2.62}$$

where $j = 1, 2, \ldots, \ell$. The derivation of equation (5.2.62) is a convenient place to close this section.

Section 5.3: The Inverse Problem and its Solution

In this section we formulate and solve the inverse problem associated with the complex-valued BO equation. The formulation and solution of the inverse problem in question is inextricably connected to the interpretation of (5.2.23) as a Riemann-Hilbert Boundary Value Problem (RHBVP) in the complex Λ-plane (the complex extension with respect to the $\lambda > 0$ axis). Before we can interpret (5.2.23) as a RHBVP we require detailed knowledge of the analyticity with respect to λ of all functions in (5.2.23), namely the functions $m^{+\varepsilon}(x;\lambda)$, $n^{+\varepsilon}(x;\lambda)$, $\bar{n}^{+\varepsilon}(x;\lambda)$ and $\rho(\lambda)$. Initially, we will work towards procuring the information we require.

Little effort is required to conclude that $n^{+\varepsilon}(x;\lambda)$ and $\rho(\lambda)$ *cannot* be analytically continued off the $\mathrm{Im}(\Lambda) = 0$ axis. A cursory glance at the equations that define $n^{+\varepsilon}(x;\lambda)$ and $\rho(\lambda)$ [see (5.2.22) and (5.2.24), respectively] is sufficient to discover a conspicuous exponential term in each of these equations, and it is the presence of such a term in each of these equations that prohibits continuation off the $\mathrm{Im}(\Lambda) = 0$ axis.

Let us now turn our attention to $m^{+\varepsilon}(x;\lambda)$ and $\bar{n}^{+\varepsilon}(x;\lambda)$. We assert that the following is true:

$$\left.\begin{matrix} m^{+\varepsilon}(x;\lambda) \\ \\ \bar{n}^{+\varepsilon}(x;\lambda) \end{matrix}\right\} = 1 - \frac{\varepsilon}{\lambda}[q]^{+\varepsilon}(x) + O\left(\frac{1}{\lambda^2}\right) \text{ as } \lambda \to \infty, \tag{5.3.1}$$

where $[q]^{+\varepsilon}(x)$ is defined by (5.2.4), and λ grows without bound along the ray $\lambda > 0$. In due course we will demonstrate the relevance of (5.3.1) to the analytic properties of $m^{+\varepsilon}(x;\lambda)$ and $\bar{n}^{+\varepsilon}(x;\lambda)$, but for the moment we will content ourselves with a proof of (5.3.1). The proof of (5.3.1) relies on the large-λ asymptotics of $g^{+\varepsilon}_{\pm\varepsilon}(x,\eta;\lambda)$, where $g^{+\varepsilon}_{\pm\varepsilon}(x,\eta;\lambda)$ is defined by (5.2.15$^{\pm}$). Standard asymptotic analysis of (5.2.15$^{\pm}$) for large-λ delivers the result

$$g^{+\varepsilon}_{\pm\varepsilon}(x,\eta;\lambda) = i\varepsilon\,\mathrm{sgn}(x-\eta)\theta(\pm(x-\eta))e^{i\varepsilon(x-\eta)\lambda} - \frac{i\varepsilon}{2\pi\lambda(x-\eta)} + \frac{1}{2\pi\lambda^2(x-\eta)^2} + O\left(\frac{1}{\lambda^3}\right), \tag{5.3.2}$$

where $\lambda >> 1$. Equation (5.3.2) has a nonuniformity along the line $x - \eta = 0$ in the (x,η)

Equations (5.2.21), (5.2.22), (5.3.2) and the Riemann-Lebesgue lemma combine to produce the equation

$$\left.\begin{array}{l} m^{+\varepsilon}(x;\lambda) \\ \\ \bar{n}^{+\varepsilon}(x;\lambda) \end{array}\right\} = 1 + O\left(\frac{1}{\lambda}\right) \text{ as } \lambda \to \infty. \tag{5.3.3}$$

In accordance with (5.3.3), let

$$\left.\begin{array}{l} m^{+\varepsilon}(x;\lambda) \\ \\ \bar{n}^{+\varepsilon}(x;\lambda) \end{array}\right\} = 1 + \frac{1}{\lambda}\varphi^{+\varepsilon}(x) + O\left(\frac{1}{\lambda^2}\right) \text{ as } \lambda \to \infty, \tag{5.3.4}$$

where $\varphi^{+\varepsilon}(x)$ is at present unknown. We can determine $\varphi^{+\varepsilon}(x)$ from [see (5.2.17)] the equation

$$i w_x^{+\varepsilon} + \varepsilon\lambda w^{+\varepsilon} = -\left[q w^{+\varepsilon}\right]^{+\varepsilon}(x) + \varepsilon\lambda, \tag{5.3.5}$$

which is the (integro-differential) equation satisfied by $m^{+\varepsilon}(x;\lambda)$ and $\bar{n}^{+\varepsilon}(x;\lambda)$. Substituting (5.3.4) into (5.3.5), and then matching $O(1)$ terms we find (as claimed) that $\varphi^{+\varepsilon}(x) = -\varepsilon[q]^{+\varepsilon}(x)$. Incidentally, we could have arrived at (5.3.1) by starting with the assumption

$$w^{+\varepsilon}(x;\lambda) = \varphi_0^{+\varepsilon}(x) + \frac{1}{\lambda}\varphi_1^{+\varepsilon}(x) + O\left(\frac{1}{\lambda^2}\right) \text{ as } \lambda \to \infty,$$

substituting this equation into (5.3.5), and then solving for $\varphi_0^{+\varepsilon}(x)$ and $\varphi_1^{+\varepsilon}(x)$. However, the author of this thesis considers the method we have employed to derive (5.3.1) supplies a rigorous justification of our conclusions.

The kernel of the integral equation that characterizes $m^{+\varepsilon}(x;\lambda)$ [see (5.2.21)] is a $(+\varepsilon)$ function in λ, whereas the kernel of the integral equation that characterizes $\bar{n}^{+\varepsilon}(x;\lambda)$ [see (5.2.22)] is a $(-\varepsilon)$ function in λ. Both the Jost functions $m^{+\varepsilon}(x;\lambda)$ and $\bar{n}^{+\varepsilon}(x;\lambda)$

$m^{+\varepsilon}(x;\lambda)$ and $\bar{n}^{+\varepsilon}(x;\lambda)$ provides the following two valuable pieces of information:

1) $m^{+\varepsilon}(x;\lambda)$ is a $(+\varepsilon)$ function in the Λ-plane, except for a finite number of poles at $\lambda = \lambda_j$, where $j = 1, 2, \ldots, \ell$;

2) $\bar{n}^{+\varepsilon}(x;\lambda)$ is a $(-\varepsilon)$ function in the Λ-plane, except for a finite number of poles at $\lambda = \lambda_j$, where $j = 1, 2, \ldots, \ell$.

We reiterate that λ_j denotes a discrete eigenvalue of (5.2.1).

Collating the information we possess about $m^{+\varepsilon}(x;\lambda)$ entitles us to write the equation

$$m^{+\varepsilon}(x;\lambda) = 1 + \sum_{j=1}^{\ell} \frac{R_j^{+\varepsilon}(x)}{\lambda - \lambda_j} + \tilde{m}_{+\varepsilon}^{+\varepsilon}(x;\lambda), \qquad (5.3.6)$$

where $R_j^{+\varepsilon}(x)$ is at present unknown and $\tilde{m}_{+\varepsilon}^{+\varepsilon}(x;\lambda)$ is a $(+\varepsilon)$ function in λ. Equation (5.3.1) is now relevant because it allows us to conclude that $\tilde{m}_{+\varepsilon}^{+\varepsilon}(x;\lambda) \to 0$ as $\lambda \to \infty$. In order to arrive at (5.3.6) we have assumed that each discrete eigenvalue is a *simple pole* of the resolvent kernel [60] constructed from the second-kind Fredholm integral equation (5.2.21). Pelinovsky and Sulem [96] have recently shown that for real-valued solutions of the BO equation, no coalescence (to produce multiple order poles in $m^{+\varepsilon}(x;\lambda)$ at λ_j) amongst the discrete eigenvalues is possible. Given that multiple order poles are possible in the reflection coefficient for the complex-valued ILW equation (see Section 4.3 of this thesis) we anticipate that whenever $q(x,t)$ satisfies the complex-valued BO equation the discrete spectrum of (5.2.1) can also contain eigenvalues that are multiple order poles of the resolvent kernel associated with (5.2.21). We do not consider in this thesis the issue of multiple order poles in $m^{+\varepsilon}(x;\lambda)$ at $\lambda = \lambda_j$.

Let us now determine the residue $R_j^{+\varepsilon}(x)$ in terms of the scattering data. Substituting (5.3.6) into (5.2.21), and then calculating the limit $\lambda \to \lambda_j$ either side of the resultant equation we obtain

$$R_j^{+\varepsilon}(x) = \int_{-\infty}^{\infty} h^{+\varepsilon}\left(x,\eta;\lambda_j\right) q(\eta) R_j^{+\varepsilon}(\eta)\, \mathrm{d}\eta. \qquad (5.3.7)$$

The proportionality

$$R_j^{+\varepsilon}(x) \propto \Phi_j^{+\varepsilon}(x)$$

eventuates from a comparison of (5.3.7) to (5.2.55), the latter equation being the equation that characterizes the discrete eigenfunction $\Phi_j^{+\varepsilon}(x)$. We will remove the proportionality sign between $R_j^{+\varepsilon}(x)$ and $\Phi_j^{+\varepsilon}(x)$ through the assignment $R_j^{+\varepsilon}(x) = -i\varepsilon\Phi_j^{+\varepsilon}(x)$, and therefore (5.3.6) now becomes

$$m^{+\varepsilon}(x;\lambda) = 1 - i\varepsilon \sum_{j=1}^{\ell} \frac{\Phi_j^{+\varepsilon}(x)}{\lambda - \lambda_j} + \tilde{m}_{+\varepsilon}^{+\varepsilon}(x;\lambda). \qquad (5.3.8)$$

The equation

$$\bar{n}^{+\varepsilon}(x;\lambda) = 1 - i\varepsilon \sum_{j=1}^{\ell} \frac{\Phi_j^{+\varepsilon}(x)}{\lambda - \lambda_j} + \tilde{\bar{n}}_{-\varepsilon}^{+\varepsilon}(x;\lambda), \tag{5.3.9}$$

where $\tilde{\bar{n}}_{-\varepsilon}^{+\varepsilon}(x;\lambda)$ is a $(-\varepsilon)$ function in λ such that $\tilde{\bar{n}}_{-\varepsilon}^{+\varepsilon}(x;\lambda) \to 0$ as $\lambda \to \infty$, can be derived from an application of a similar procedure to the one used to derive (5.3.8).

The mathematical information required to interpret (5.2.23) as a RHBVP in the complex Λ-plane is now accessible. Before we proceed to derive the solution of the

relevant facts:

- $m^{+\varepsilon}(x;\lambda)$ is a $(+\varepsilon)$ function in the Λ-plane, except for a finite number of simple poles at $\lambda = \lambda_j$, where $j = 1, 2, \ldots, \ell$;
- $\bar{n}^{+\varepsilon}(x;\lambda)$ is a $(-\varepsilon)$ function in the Λ-plane, except for a finite number of simple poles at $\lambda = \lambda_j$, where $j = 1, 2, \ldots, \ell$;
- $n^{+\varepsilon}(x;\lambda)$ and $\rho(\lambda)$ cannot be analytically continued off the $\lambda > 0$ axis;
- equation (5.2.27) contains the necessary connection between the eigenfunctions $n^{+\varepsilon}(x;\lambda)$ and $\bar{n}^{+\varepsilon}(x;\lambda)$.

Our efforts will now be aimed towards the solution of the RHBVP (5.2.23). First, the equation

$$\rho(\lambda)n^{+\varepsilon}(x;\lambda) = \varepsilon\left\{\hat{q}_{+\varepsilon}^{+\varepsilon}(x;\lambda) - \hat{q}_{-\varepsilon}^{+\varepsilon}(x;\lambda)\right\}, \tag{5.3.10}$$

where

$$\hat{q}_{\pm\varepsilon}^{+\varepsilon}(x;\lambda) \;\underline{\underline{\mathrm{def}}}\; \frac{1}{2\pi i}\int_0^{\infty} \frac{\rho(\mu)n^{+\varepsilon}(x;\mu)}{\mu - \left(\lambda \pm i\varepsilon 0^{+}\right)}\,\mathrm{d}\mu, \tag{5.3.11}$$

is the fragmentation of $\rho(\lambda)n^{+\varepsilon}(x;\lambda)$ into its $(\pm\varepsilon)$-parts. Equations (5.3.10) and (5.3.11) can be retrieved from the deep water limit of (4.4.9) and (4.4.10), respectively. What is most interesting about (5.3.10) is that it expresses $\rho(\lambda)n^{+\varepsilon}(x;\lambda)$ in terms of functions that have a jump across the $\lambda > 0$ axis. Interestingly, on the $\lambda > 0$ axis we also have the junction condition

$$m^{+\varepsilon}(x;\lambda) - \bar{n}^{+\varepsilon}(x;\lambda) = \tilde{m}_{+\varepsilon}^{+\varepsilon}(x;\lambda) - \tilde{\bar{n}}_{-\varepsilon}^{+\varepsilon}(x;\lambda), \tag{5.3.12}$$

and this particular identity can be derived from equations (5.3.8) and (5.3.9). Equations (5.2.23), (5.3.10) and (5.3.12) may be combined to produce the identity

$$\tilde{m}^{+\varepsilon}_{+\varepsilon}(x;\lambda) - \tilde{n}^{+\varepsilon}_{-\varepsilon}(x;\lambda) = \varepsilon\hat{q}^{+\varepsilon}_{+\varepsilon}(x;\lambda) - \varepsilon\hat{q}^{+\varepsilon}_{-\varepsilon}(x;\lambda),$$

from which we infer that

$$\tilde{n}^{+\varepsilon}_{-\varepsilon}(x;\lambda) = \varepsilon\hat{q}^{+\varepsilon}_{-\varepsilon}(x;\lambda) + \tilde{\alpha},$$

where $\tilde{\alpha}$ is a constant. The boundary conditions

$$\left.\begin{array}{l}\tilde{n}^{+\varepsilon}_{-\varepsilon}(x;\lambda)\\ \\ \hat{q}^{+\varepsilon}_{-\varepsilon}(x;\lambda)\end{array}\right\} \to 0 \text{ as } \lambda \to \infty$$

allow us to conclude that $\tilde{\alpha} = 0$, and therefore we arrive at the equation

$$\tilde{n}^{+\varepsilon}_{-\varepsilon}(x;\lambda) = \varepsilon\hat{q}^{+\varepsilon}_{-\varepsilon}(x;\lambda). \tag{5.3.13}$$

Substituting (5.3.13) into (5.3.9) we obtain

$$\bar{n}^{+\varepsilon}(x;\lambda) = 1 - i\varepsilon\sum_{j=1}^{\ell}\frac{\Phi_j^{+\varepsilon}(x)}{\lambda - \lambda_j} + \frac{\varepsilon}{2\pi i}\int_0^\infty \frac{\rho(\mu)n^{+\varepsilon}(x;\mu)}{\mu - \left(\lambda - i\varepsilon 0^+\right)}\mathrm{d}\mu, \tag{5.3.14}$$

where we have decided to write the formula [see (5.3.11)] for $\hat{q}^{+\varepsilon}_{-\varepsilon}(x;\lambda)$ in complete detail.

An interesting and useful result can be derived from the large-λ asymptotics of (5.3.14). Specifically, we will derive a reconstruction formula for the potential $q(x)$ solution of the complex-valued BO equation (5.1.3) at a fixed t. Consider the equation

$$\bar{n}^{+\varepsilon}(x;\lambda) = 1 - \frac{\varepsilon}{\lambda}\left\{i\sum_{j=1}^{\ell}\Phi_j^{+\varepsilon}(x) + \frac{1}{2\pi i}\int_0^\infty \rho(\mu)n^{+\varepsilon}(x;\mu)\mathrm{d}\mu\right\} + \mathrm{O}\left(\frac{1}{\lambda^2}\right), \tag{5.3.15}$$

which expresses the large-λ asymptotics of (5.3.14) to a sufficient precision. At the beginning of this section [see (5.3.1)] we showed that $\bar{n}^{+\varepsilon}(x;\lambda)$ possesses the canonical normalization. Matching the $\mathrm{O}(1/\lambda)$ terms in equations (5.3.1) and (5.3.15) we obtain

$$[q]^{+\varepsilon}(x) = i\sum_{j=1}^{\ell}\Phi_j^{+\varepsilon}(x) + \frac{1}{2\pi i}\int_0^\infty \rho(\mu)n^{+\varepsilon}(x;\mu)\mathrm{d}\mu, \tag{5.3.16}$$

where

$$[q]^{+\varepsilon}(x) = \frac{\varepsilon}{2}q(x) - \frac{i}{2}(\mathbf{H}q)(x). \tag{5.3.17}$$

Perhaps the simplest way to interpret (5.3.17) is to view this particular equation as the $\delta \to \infty$ limit of (4.2.16$^+$). We now have a reconstruction formula for $q(x)$ in terms of $\Phi_j^{+\varepsilon}(x)$, $\rho(\lambda)$ and $n^{+\varepsilon}(x;\lambda)$ because from (5.3.17)

$$q(x) = [q]^{+\varepsilon}(x)\Big|_{\varepsilon=1} - [q]^{+\varepsilon}(x)\Big|_{\varepsilon=-1}, \tag{5.3.18}$$

where $[q]^{+\varepsilon}(x)$ is determined from (5.3.16) for *both* choices of ε.

The important issue that we must now focus upon is to relate $\Phi_j^{+\varepsilon}(x)$ to quantities (scattering data) that can be calculated from the initial value (5.1.6). Scoufis and Cosgrove [114] use the identity

$$\lim_{\lambda\to\lambda_j}\left\{\bar{n}^{+\varepsilon}(x;\lambda) + i\varepsilon\frac{\Phi_j^{+\varepsilon}(x)}{\lambda-\lambda_j}\right\} = \left(x+\gamma_j\right)\Phi_j^{+\varepsilon}(x), \tag{5.3.19}$$

where $j = 1, 2, \ldots, \ell$, to construct the desired connection. The γ_j are complex, depend on ε, and are members (together with the λ_j) of the discrete spectrum. Fokas and Ablowitz [43] provide sufficient detail for the reader who wishes to pursue the derivation of equation (5.3.19), and we therefore omit the proof of this particular equation.

Let $\lambda = \lambda_k$, where $k = 1, 2, \ldots, \ell$, denote one of the discrete eigenvalues. Rearrange (5.3.14) into the form

$$\bar{n}^{+\varepsilon}(x;\lambda) + i\varepsilon\frac{\Phi_k^{+\varepsilon}(x)}{\lambda-\lambda_k} = 1 - i\varepsilon\sum_{\substack{j=1\\(j\neq k)}}^{\ell}\frac{\Phi_j^{+\varepsilon}(x)}{\lambda-\lambda_j} + \frac{\varepsilon}{2\pi i}\int_0^{\infty}\frac{\rho(\mu)n^{+\varepsilon}(x;\mu)}{\mu-\left(\lambda-i\varepsilon 0^+\right)}\mathrm{d}\mu. \tag{5.3.20}$$

Equation (5.3.20) serves to isolate the discrete eigenfunction associated with the discrete eigenvalue λ_k. The limit $\lambda\to\lambda_k$ throughout (5.3.20), followed by use of (5.3.19) returns the *system* of equations

$$\left(x+\gamma_k\right)\Phi_k^{+\varepsilon}(x) = 1 - i\varepsilon\sum_{\substack{j=1\\(j\neq k)}}^{\ell}\frac{\Phi_j^{+\varepsilon}(x)}{\lambda_k-\lambda_j} + \frac{\varepsilon}{2\pi i}\int_0^{\infty}\frac{\rho(\mu)n^{+\varepsilon}(x;\mu)}{\mu-\lambda_k}\mathrm{d}\mu. \tag{5.3.21}$$

Equation (5.3.21) relates $\Phi_j^{+\varepsilon}(x)$ to $\rho(\lambda)$, $n^{+\varepsilon}(x;\lambda)$, λ_j and γ_j, where $j = 1, 2, \ldots, \ell$.

A survey of what we have achieved in this section up to the derivation of (5.3.21) will serve as a valuable guide. Initially we derived equation (5.3.16), which expresses $[q]^{+\varepsilon}(x)$ in terms of $\rho(\lambda)$, $n^{+\varepsilon}(x;\lambda)$ and $\Phi_j^{+\varepsilon}(x)$. Equation (5.3.21) provides the necessary system of equations that relates $\Phi_j^{+\varepsilon}(x)$ to $\rho(\lambda)$, $n^{+\varepsilon}(x;\lambda)$, λ_j and γ_j. Clearly, what emerges from our brief survey is the necessity to derive an equation that connects the eigenfunction $n^{+\varepsilon}(x;\lambda)$ to the scattering data

$$f(\lambda),\ \rho(\lambda),\ \lambda_j \text{ and } \gamma_j,$$

where $j = 1, 2, \ldots, \ell$

$n^{+\varepsilon}(x;\lambda)$ is a Fredholm integral equation of the second-kind. The precise equation that relates $n^{+\varepsilon}(x;\lambda)$ to the scattering data is as follows [114]:

$$n^{+\varepsilon}(x;\lambda) = \tilde{v}(x;\lambda) - \sum_{j=1}^{\ell} w(x;\lambda;\lambda_j)\Phi_j^{+\varepsilon}(x) + \frac{1}{2\pi}\int_0^{\infty} w(x;\lambda;\mu)\rho(\mu)n^{+\varepsilon}(x;\mu)\,d\mu, \quad (5.3.22)$$

where

$$\tilde{v}(x;\lambda) \underline{\underline{\text{def}}}\ e^{i\varepsilon\lambda x}\int_0^{\lambda}\left\{f_s(\nu) + f(\nu)e^{-i\varepsilon\nu x}\right\}d\nu, \quad (5.3.23a)$$

$$w(x;\lambda;\mu) \underline{\underline{\text{def}}}\ i\varepsilon e^{i\varepsilon\lambda x}\int_0^{\lambda}\left\{\frac{f(\nu)e^{-i\varepsilon\nu x}}{\nu - (\mu + i\varepsilon 0^+)} - \frac{1}{\mu}f_s(\nu)\right\}d\nu, \quad (5.3.23b)$$

$$f_s(\nu) \underline{\underline{\text{def}}} \begin{cases} 1/(\nu\ln(\nu)), & p_0 \neq 0 \\ 0, & p_0 = 0, \end{cases} \quad (5.3.23c)$$

$f(\lambda)$ is defined by (5.2.28), and p_0 is defined by (5.2.50). The derivation of equation (5.3.22) and its associated components, namely (5.3.23a-c), can be achieved as follows: substitute (5.3.14) into (5.2.27), and then integrate with respect to λ. Considerable care must be used during the integration because of the presence in the integrand of a term that becomes singular as $\lambda \to 0^+$. Indeed, the function $f_s(\nu)$ serves to cancel the singular term. We have found it necessary to adjust the definition of the function $f_s(\nu)$ used in Ref. 43. Specifically, our extended definition of $f_s(\nu)$ includes the generic ($p_0 \neq 0$) and nongeneric ($p_0 = 0$) potentials in a unified treatment.

Equations (5.3.16), (5.3.21) and (5.3.22) express in terms of the scattering data,

$$f(\lambda),\ \rho(\lambda),\ \lambda_j \text{ and } \gamma_j,$$

the solution of the inverse problem. In the next section of this thesis we present the equations for the evolution of the scattering data.

Section 5.4: The Evolution of the Scattering Data

At the conclusion of Section 5.3 we noted that the scattering data required to

$$f(\lambda,t),\ \rho(\lambda,t),\ \left\{\lambda_j(t),\ \gamma_j(t)\right\}_{j=1}^{j=\ell} \text{ and } p_0.$$

The notation $\{\bullet\}_{j=1}^{j=\ell}$ refers to a finite set in which there exists a one-to-one mapping between the ordered pair (λ_j, γ_j) and the integers $j = 1, 2, \ldots, \ell$. Explicit dependence of any variable on *t* will be shown throughout this section. This section is devoted to the time evolution of the scattering data for the complex-valued BO equation.

The equations that govern the time evolution of the scattering data associated with the IST for the complex-valued BO equation are as follows [114]:

$$f(\lambda,t) = f(\lambda,0)e^{-i\varepsilon\lambda^2 t}; \tag{5.4.1}$$

$$\rho(\lambda,t) = \rho(\lambda,0)e^{i\varepsilon\lambda^2 t}; \tag{5.4.2}$$

$$\lambda_j(t) = \lambda_j(0); \tag{5.4.3}$$

$$\gamma_j(t) = 2\lambda_j t + \gamma_j(0); \tag{5.4.4}$$

$$p_0 = \text{constant},$$

where $f(\lambda,0)$, $\rho(\lambda,0)$, $\lambda_j(0)$, $\gamma_j(0)$ and p_0 are all determined from the *known* initial value (5.1.6).

In this section we present only the derivation of (5.4.1), and content ourselves to merely quote the equations that command the time evolution of $\rho(\lambda,t)$, $\lambda_j(t)$ and $\gamma_j(t)$. Equations (5.4.2), (5.4.3) and (5.4.4) can be derived by suitable modifications to the procedures contained in the monograph by Ablowitz and Clarkson [11], particularly the

We now wish to present a novel derivation of (5.4.1). Surprisingly, our derivation of (5.4.1) begins with (5.3.14). Our interest now focuses on the asymptotics as $\lambda \to 0^+$ of (5.3.14). Let us recall that in Section 5.2 of this thesis [see (5.2.52)] we showed that

$$\lim_{\lambda \to 0^+} \bar{n}^{+\varepsilon}(x,t;\lambda) = \begin{cases} 0, & p_0 \neq 0 \\ p(x,t), & p_0 = 0, \end{cases}$$

where $p(x,t)$ solves the Fredholm integral equation (5.2.49), and the complex constant p_0 is defined by (5.2.50). Armed with the small-λ asymptotics of $\bar{n}^{+\varepsilon}(x,t;\lambda)$, it is now possible to show that the limit $\lambda \to 0^+$ either side of (5.3.14) leads to the equation

$$1 + i\varepsilon \sum_{j=1}^{\ell} \frac{1}{\lambda_j} \Phi_j^{+\varepsilon}(x,t) + \frac{\varepsilon}{2\pi i} \int_0^\infty \frac{\rho(\mu,t) n^{+\varepsilon}(x,t;\mu)}{\mu + i\varepsilon 0^+} \mathrm{d}\mu = \begin{cases} 0, & p_0 \neq 0 \\ p(x,t), & p_0 = 0. \end{cases} \tag{5.4.5}$$

Incidentally, the reader who pursued the derivation of equation (5.3.22) should have already encountered equation (5.4.5), the latter equation being indispensable in the cancellation of a singularity that appears during a integration that eventually leads to (5.3.22). The equation

$$\int_{-\infty}^{\infty} q(x,t)\,\mathrm{d}x + i\varepsilon \sum_{j=1}^{\ell} \frac{1}{\lambda_j} \int_{-\infty}^{\infty} q(x,t)\Phi_j^{+\varepsilon}(x,t)\,\mathrm{d}x$$
$$+ \frac{\varepsilon}{2\pi i} \int_{-\infty}^{\infty} q(x,t)\,\mathrm{d}x \int_0^\infty \frac{\rho(\mu,t) n^{+\varepsilon}(x,t;\mu)}{\mu + i\varepsilon 0^+} \mathrm{d}\mu = 0 \tag{5.4.6}$$

appears after we multiply both sides of (5.4.5) by $q(x,t)$, integrate each term in the resultant equation over the interval $-\infty < x < \infty$, and finally absorb the generic and nongeneric cases into a single case by reference to (5.2.50). Equation (5.4.6) simplifies to the equation

$$\int_{-\infty}^{\infty} q(x,t)\,\mathrm{d}x = 2\pi\ell - i\varepsilon \int_0^\infty \rho(\mu,t) f(\mu,t)\,\mathrm{d}\mu \tag{5.4.7}$$

because of equations (5.2.28) and (5.2.62). Equation (5.4.7) relates the number, denoted as ℓ, of bound states to the solution of the BO equation (5.1.3) and the functions $\rho(\lambda,t)$ and $f(\lambda,t)$ in the continuous spectrum.

Let us suspend our derivation of (5.4.1) to make some remarks about (5.4.7). Pure soliton solutions of the BO equation (5.1.3) appear whenever $\rho(\lambda,t) \equiv 0$ because a zero reflection coefficient reduces the reconstruction formula (5.3.16) to a sum over discrete terms. In the case of pure soliton potentials, equation (5.4.7) contracts to the simple and elegant formula

$$\int_{-\infty}^{\infty} q(x,t)\,\mathrm{d}x = 2\pi\ell. \tag{5.4.8}$$

Equation (5.4.8) is the number density function for the BO equation that was first derived by Matsuno [73,74], and only recently verified by Miloh et al. [85] using a numerical

number of conserved densities displayed by the BO equation. The derivation of (5.4.8) that

is applicable to pure soliton solutions of the real-valued and complex-valued versions of the BO equation. Indeed, when $q(x,t)$ is a pure soliton solution of the complex-valued BO equation (5.1.3) we can use (5.4.8) to derive a number density function for the MBO equation (5.1.1). The derivation of the number density function for the MBO equation (5.1.1) hinges on [see (2.1.19)] the Miura transformation

$$q(x,t) = \frac{1}{2}\left\{\mathbf{H}\big(Q_x(x,t)\big) + \alpha\left(e^{Q(x,t)} - 1\right) + iQ_x(x,t)\right\}, \tag{5.4.9}$$

which maps a real-valued solution, $Q(x,t)$, of the MBO equation into a complex-valued solution, $q(x,t)$, of the BO equation. Substituting (5.4.9) into (5.4.8) we obtain

$$\int_{-\infty}^{\infty} \left(e^{Q(x,t)} - 1\right)\mathrm{d}x = \frac{4\pi\ell}{\alpha}, \tag{5.4.10}$$

where $Q(x,t)$ denotes the pure ℓ-soliton solution of the MBO equation. Equation (5.4.10) sharpens a result first derived by Satsuma et al. [112] who showed that

$$\int_{-\infty}^{\infty} \left(e^{Q(x,t)} - 1\right)\mathrm{d}x = \text{constant}$$

for *all* solutions of the MBO equation.

Our mathematical interlude has finished and we now return to complete the derivation of equation (5.4.1). Differentiating both sides of (5.4.7) with respect to t we obtain

$$\int_{-\infty}^{\infty} q_t(x,t)\,\mathrm{d}x = -i\varepsilon\int_0^{\infty} \left\{ f(\mu,t)\rho_t(\mu,t) + \rho(\mu,t)\, f_t(\mu,t)\right\}\mathrm{d}\mu,$$

and because $q(x,t)$ satisfies equation (5.1.3) the differentiated version of (5.4.7) becomes

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial x}\left\{q^2(x,t) + \mathbf{H}\big(q_x(x,t)\big)\right\} \mathrm{d}x$$
$$= i\varepsilon\int_0^{\infty} \left\{ f(\mu,t)\rho_t(\mu,t) + \rho(\mu,t)\, f_t(\mu,t)\right\} \mathrm{d}\mu. \quad (5.4.11)$$

The left hand side of (5.4.11) is simply

$$\lim_{x\to\infty}\left\{q^2(x,t) + \mathbf{H}\big(q_x(x,t)\big)\right\} - \lim_{x\to-\infty}\left\{q^2(x,t) + \mathbf{H}\big(q_x(x,t)\big)\right\},$$

which evaluates to zero because of the boundary conditions (5.1.7). A zero left hand side in (5.4.11) justifies the validity of the identity

$$f(\mu,t)\rho_t(\mu,t) + \rho(\mu,t)\, f_t(\mu,t) \equiv 0. \quad (5.4.12)$$

Equation (5.4.12) is trivial for the case of pure soliton solutions, so we will consider the interesting case: $\rho(\mu,t) \neq 0$, where $0 < \mu < \infty$. Although we do not include the proof of (5.4.2) in this section, the reader can rest in the certitude that the author of this thesis has used $(5.2.2^+)$ to derive (5.4.2). Therefore, we will treat (5.4.2) as if it were at our disposal. Using (5.4.2) in (5.4.12) we find that

$$\left[f_t(\mu,t) + i\varepsilon\mu^2 f(\mu,t)\right]\rho(\mu,t) = 0,$$

and because we have (temporarily) excluded reflectionless potentials from our analysis we are left with the equality

$$f_t(\mu,t) + i\varepsilon\mu^2 f(\mu,t) = 0. \quad (5.4.13)$$

It is an elementary exercise for the reader to integrate the separable differential equation (5.4.13) and arrive at (5.4.1). The completion of the proof for equation (5.4.1) coincides with the close of this section.

Section 5.5: The 1-Soliton Solution for the MBO Equation

Our work in this chapter has culminated in the solution of the initial value problem for the MBO equation,

$$Q_t + \alpha Q_x\left(e^Q - 1\right) + Q_x\mathbf{H}\left(Q_x\right) + \mathbf{H}\left(Q_{xx}\right) = 0. \tag{5.5.1}$$

We remind the reader of some generic notation that has been used throughout this thesis: $Q_t = \partial Q/\partial t$ and so forth; $\mathbf{H}(\bullet)$ denotes the Hilbert transform (with respect to x) and is defined by the equation

$$(\mathbf{H}f)(x) \,\underline{\underline{\mathrm{def}}}\, \tfrac{1}{\pi}(\mathrm{P})\int_{-\infty}^{\infty} \frac{f(\xi)}{\xi - x}\mathrm{d}\xi. \tag{5.5.2}$$

As promised in Section 1.1, in this section we demonstrate how the work in Sections 5.1-5.4 of this thesis can be used to derive the 1-soliton solution for equation (5.5.1).

The 1-soliton solution for the MBO equation is derived from the 1-soliton solution for the complex-valued BO equation, the latter equation having the form

$$q_t + 2qq_x + \mathbf{H}\left(q_{xx}\right) = 0. \tag{5.5.3}$$

Pure soliton solutions for (5.5.3) are produced when the reconstruction formula (5.3.16) involves only a sum over a finite number of discrete terms. Therefore, pure soliton solutions of (5.5.3) are produced whenever $\rho(\lambda, t) \equiv 0$, where the definition of $\rho(\lambda, t)$ is presented in equation (5.2.24). Equation (5.3.21) characterizes completely the ℓ discrete eigenfunctions associated with the pure ℓ-soliton solution of (5.5.3), ℓ being the number of bound states. Our intention is to derive the pure 1-soliton solution for the complex-valued BO equation (5.5.3). From (5.3.21) we find that

$$\Phi_1^{+\varepsilon}(x, t) = \frac{1}{x + \gamma_1(t; \varepsilon)} \tag{5.5.4}$$

is the 1-soliton discrete eigenfunction. The notation we have used for γ_1 is designed to show that γ_1 depends on the choice of ε, where $\varepsilon = \pm 1$ are the only two admissible values for the parameter ε. The time evolution of $\gamma_1(t; \varepsilon)$ is governed by equation (5.4.4), and when we substitute this equation into (5.5.4) we obtain

$$\Phi_1^{+\varepsilon}(x, t) = \frac{1}{x - \kappa t + \gamma_1(0; \varepsilon)}, \tag{5.5.5}$$

where $\kappa = -2\lambda_1$ and $\gamma_1(0; \varepsilon)$ is a complex constant that is dependent on ε. The parameter

λ_1 (the solitary discrete eigenvalue) is a negative constant independent of ε, and therefore κ is a *positive* constant independent of ε. Our decision to work with κ instead of λ_1 is based solely on the ease with which subsequent calculations in this section proceed when κ is used instead of λ_1. Substituting (5.5.5) into (5.3.16) we obtain

$$[q]^{+\varepsilon}(x,t) = \frac{i}{x - \kappa t + \gamma_1(0;\varepsilon)}. \tag{5.5.6}$$

The physical variable, $q(x,t)$, that solves the complex-valued BO equation is constructed from $[q]^{+\varepsilon}(x,t)$ with the assistance of (5.3.18). Separate equations for $[q]^{+}(x,t)$ and $[q]^{-}(x,t)$ are now available because (5.5.6) caters for such a scenario. Substituting (5.5.6) into (5.3.18) we find that the pure 1-soliton solution for the complex-valued BO equation (5.5.3) has the structure

$$q(x,t) = \frac{i\left(\gamma_1^- - \gamma_1^+\right)}{\left(x - \kappa t + \gamma_1^+\right)\left(x - \kappa t + \gamma_1^-\right)}, \tag{5.5.7}$$

where $\gamma_1^{\pm} = \gamma_1(0; \varepsilon = \pm 1)$.

A subtle, but beautiful result, that was derived in Section 5.4 of this thesis now plays a significant role in our work. The result we are referring to is (5.4.8), which constrains the total mass of fluid carried by the 1-soliton solution to be

$$\int_{-\infty}^{\infty} q(x,t)\,\mathrm{d}x = 2\pi. \tag{5.5.8}$$

Equation (5.5.8) induces some type of restriction (or restrictions) on the complex constants $\gamma_1^{\pm}$ because when (5.5.7) and (5.5.8) are considered in tandem we have

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\left(x - \kappa t + \gamma_1^+\right)\left(x - \kappa t + \gamma_1^-\right)} = \frac{2\pi i}{\gamma_1^+ - \gamma_1^-}. \tag{5.5.9}$$

The integrand of (5.5.9) is a rational function (in x) with simple poles at $x = \kappa t - \gamma_1^{\pm}$ that do not reside on the real axis of the complex z-plane; $z = x + iy$ defines the complex extension of x. Contour integration can be applied to (5.5.9) to show that

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\left(x - \kappa t + \gamma_1^+\right)\left(x - \kappa t + \gamma_1^-\right)} = \begin{cases} \dfrac{2\pi i}{\gamma_1^+ - \gamma_1^-}, \; \textit{if } \operatorname{Im}\left(\gamma_1^+\right) > 0 \textit{ and } \operatorname{Im}\left(\gamma_1^-\right) < 0 \\[2em] \dfrac{2\pi i}{\gamma_1^- - \gamma_1^+}, \; \textit{if } \operatorname{Im}\left(\gamma_1^+\right) < 0 \textit{ and } \operatorname{Im}\left(\gamma_1^-\right) > 0. \end{cases} \tag{5.5.10}$$

Consistency of (5.5.10) with (5.5.9) demands that $\mathrm{Im}\left(\gamma_1^+\right) > 0$ and $\mathrm{Im}\left(\gamma_1^-\right) < 0$.

The function defined by (5.5.7) is the unrefined form of the 1-soliton solution for the complex-valued BO equation (5.5.3). At present there are five parameters in (5.5.7), namely κ, $\mathrm{Re}\left(\gamma_1^+\right)$, $\mathrm{Im}\left(\gamma_1^+\right)$, $\mathrm{Re}\left(\gamma_1^-\right)$ and $\mathrm{Im}\left(\gamma_1^-\right)$. It is unlikely that a 1-soliton solution for a (1+1) dimensional nonlinear evolution equation can support five arbitrary parameters. We anticipate a reduction in the number of parameters through interrelationships amongst some (possibly all) of the five parameters enumerated. We can visualize the connection (or connections) between the parameters in (5.5.7) by substituting (5.5.7) into (5.5.3). No doubt the reader is aware that substitution into (5.5.3) requires calculation of the specific Hilbert transform $\mathbf{H}\left(q_{xx}\right)$, where $q \equiv q(x,t)$ is defined by (5.5.7). A sketch of the procedure involved in the computation of $\mathbf{H}\left(q_{xx}\right)$

- substitute (5.5.7) into (5.5.2) to derive the equation

$$\mathbf{H}(q(x,t)) = \frac{i}{\pi}(\mathrm{P})\int_{-\infty}^{\infty} \frac{\left(\gamma_1^- - \gamma_1^+\right)\mathrm{d}\xi}{\left(\xi - x\right)\left(\xi - \kappa t + \gamma_1^+\right)\left(\xi - \kappa t + \gamma_1^-\right)}; \tag{5.5.11}$$

- evaluate (5.5.11) by recourse to the contour integral

$$\oint_C \frac{\mathrm{d}\omega}{\left(\omega - x\right)\left(\omega - \kappa t + \gamma_1^+\right)\left(\omega - \kappa t + \gamma_1^-\right)}, \tag{5.5.12}$$

where $\omega = \xi + i\eta$ defines the complex extension of ξ, and C is a semi-circle that has a semi-circular indentation on the ξ-axis at $\omega = x$;

- the information $\mathrm{Im}\left(\gamma_1^+\right) > 0$ and $\mathrm{Im}\left(\gamma_1^-\right) < 0$ that we have obtained from (5.5.8) allows us to evaluate (5.5.12) by means of the Residue Theorem;

- the final result from an evaluation of (5.5.12) by the Residue Theorem is

$$\mathbf{H}(q(x,t)) = iq(x,t) - \frac{2}{x - \kappa t + \gamma_1^-}; \tag{5.5.13}$$

- apply the identity (1.3.38) to compute $\mathbf{H}\left(q_x\right)$ from (5.5.13), and thence reapply the identity (1.3.38) to compute $\mathbf{H}\left(q_{xx}\right)$ from $\mathbf{H}\left(q_x\right)$.

Substituting (5.5.7) and (5.5.13) into (5.5.3) we find that

$$i\kappa\left(x - \kappa t + \gamma_1^-\right)^2 - i\kappa\left(x - \kappa t + \gamma_1^+\right)^2 - 2\left(x - \kappa t + \gamma_1^+\right) - 2\left(x - \kappa t + \gamma_1^-\right) \equiv 0 \tag{5.5.14}$$

is the condition that must be satisfied for (5.5.7) to be a solution of (5.5.3). Embedded within (5.5.14) is a set of equations for the real and imaginary parts of $\gamma_1^{\pm}$ that we now plan to extricate. Let $\gamma_1^{+} = a + ib$ and $\gamma_1^{-} = c + id$, where a, b, c and d are real numbers, define the complex constants $\gamma_1^{\pm}$. We have already proved in this section that $b > 0$ and $d < 0$. Substituting $\gamma_1^{+} = a + ib$ and $\gamma_1^{-} = c + id$ into (5.5.14), noting that κ is a positive constant, and then partitioning into real and imaginary parts we obtain

$$\textbf{Re:} \qquad 2(b\kappa - d\kappa - 2)x + 2\kappa(d\kappa - b\kappa + 2)t - 2(cd\kappa - ab\kappa + c + a)$$

and

$$\textbf{Im:} \qquad 2\kappa(c - a)x - 2\kappa^2(c - a)t + \kappa\left(c^2 + b^2 - a^2 - d^2\right) - 2(b + d).$$

The identity (5.5.14) compels a, b, c and d to be constrained by the equations

$$b\kappa - d\kappa - 2 = 0, \tag{5.5.15a}$$

$$cd\kappa - ab\kappa + c + a = 0, \tag{5.5.15b}$$

$$c - a = 0 \tag{5.5.15c}$$

and

$$\kappa\left(c^2 + b^2 - a^2 - d^2\right) - 2(b + d) = 0. \tag{5.5.15d}$$

Manipulating the system of equations (5.5.15a-d) we find the reduced set of constraints

$$a = c \tag{5.5.16a}$$

and

$$\kappa(b - d) = 2. \tag{5.5.16b}$$

It is not surprising that $a = c$ because the translation $x \to x + \mathit{constant}$ can be applied to (5.5.7) to absorb either Re $\left(\gamma_1^{+}\right)$ or Re $\left(\gamma_1^{-}\right)$.

Let us temporarily interrupt our derivation of the 1-soliton solution for the MBO equation to examine the consequences of a particular solution of equation (5.5.16b). First, decompose the complex-valued function defined by (5.5.7) into the form

$$q(x,t) = q_R(x,t) + iq_I(x,t), \tag{5.5.17a}$$

where

$$q_R(x,t) \underline{\underline{\mathrm{def}}} \frac{b}{(x - \kappa t + a)^2 + b^2} - \frac{d}{(x - \kappa t + c)^2 + d^2}, \tag{5.5.17b}$$

$$q_I(x,t) \underline{\underline{\mathrm{def}}} \frac{x - \kappa t + a}{(x - \kappa t + a)^2 + b^2} - \frac{x - \kappa t + c}{(x - \kappa t + c)^2 + d^2}, \tag{5.5.17c}$$

$a = \mathrm{Re}\left(\gamma_1^+\right)$, $b = \mathrm{Im}\left(\gamma_1^+\right)$, $c = \mathrm{Re}\left(\gamma_1^-\right)$ and $d = \mathrm{Im}\left(\gamma_1^-\right)$. Consider the situation in which $a = x_0$ and $b = -d$, where x_0 is an arbitrary real constant. We note that the equality $b = -d$ is consistent with the constraints $\mathrm{Im}\left(\gamma_1^+\right) > 0$ and $\mathrm{Im}\left(\gamma_1^-\right) < 0$. According to (5.5.16a), the selection $a = x_0$ binds us to accept the equality $c = x_0$. Also, from (5.5.16b) we see that $b = -d$ leads to the particular solutions $b = 1/\kappa$ and $d = -1/\kappa$. Substituting $a = x_0$, $b = 1/\kappa$, $c = x_0$ and $d = -1/\kappa$ into equations (5.5.17b) and (5.5.17c), and then substituting each resultant expression into (5.5.17a) we obtain the *real-valued* function

$$q(x,t) = \frac{2\kappa}{1 + \kappa^2\left(\quad - \kappa \quad + x \quad\right)^2}. \tag{5.5.18}$$

Equation (5.5.18) was first derived by Benjamin [15], and is the rational (algebraic) 1-soliton solution for the real-valued BO equation (5.5.3). Two reasons have motivated the (mathematical) detour that led us to (5.5.18), and these reasons are:

1) demonstrate the consistency of our results with the theory for the real-valued BO equation;

2) elucidate any inadmissible selections of the parameters a, b, c and d direction we have found that the pairing $b = \quad / \kappa$ and $d = -1/\kappa$ is incompatible with the requirement of a complex-valued solution for the BO equation.

We now return to the task of computing the 1-soliton solution for the MBO equation. Let $\mathrm{Im}\left\{q(x,t)\right\}$ denote the imaginary part of the complex-valued function defined by (5.5.7). Equations (5.5.17a-c) support the conclusion

$$\mathrm{Im}\left\{q(x,t)\right\} = \frac{x - \kappa t + a}{(x - \kappa t + a)^2 + b^2} - \frac{x - \kappa t + c}{(x - \kappa t + c)^2 + d^2},$$

which we prefer to write in the form

$$\mathrm{Im}\left\{q(x,t)\right\} = \frac{1}{2}\frac{\partial}{\partial x}\left\{\ln\left[\frac{(x - \kappa t + a)^2 + b^2}{(x - \kappa t + c)^2 + d^2}\right]\right\}. \tag{5.5.19}$$

The mapping from $\mathrm{Im}\left\{q(x,t)\right\}$ to the physical variable that solves the MBO equation (5.5.1) is expressed by (5.1.5). Substituting (5.5.19) into (5.1.5) we obtain

$$Q_x(x,t) = \frac{\partial}{\partial x}\left\{\ln\left[\frac{(x - \kappa t + a)^2 + b^2}{(x - \kappa t + c)^2 + d^2}\right]\right\}. \tag{5.5.20}$$

A quadrature with respect to x when applied to (5.5.20) yields

$$Q(x,t) = \ln\left(\frac{(x-\kappa t+a)^2+b^2}{(x-\kappa t+c)^2+d^2}\right)+\hat{Q}(t), \tag{5.5.21}$$

where $\hat{Q}(t)$ denotes a function of integration. Compatibility of (5.5.21) with the boundary conditions $Q(x,t)\to 0$ as $|x|\to\infty$ necessitates that $\hat{Q}(t)\equiv 0$, and therefore

$$Q(x,t) = \ln\left(\frac{(x-\kappa t+a)^2+b^2}{(x-\kappa t+c)^2+d^2}\right). \tag{5.5.22}$$

Equations (5.5.16a) and (5.5.16b) should now be used to reduce the number of parameters present in (5.5.22). Let $a = x_0$ and $c = x_0$, where x_0 is an arbitrary real constant. Note that without loss of generality we can make the assignment $x_0 = 0$ because the MBO equation is invariant under the group $x \to x + constant$. Also, let us agree to eliminate b from (5.5.22) by using [see (5.5.16b)] $b = d + 2/\kappa$. Substituting $a = x_0$, $b = d + 2/\kappa$ and $c = x_0$ into (5.5.22) we obtain

$$Q(x,t) = \ln\left(\frac{(x-\kappa t+x_0)^2+(d+2/\kappa)^2}{(x-\kappa t+x_0)^2+d^2}\right). \tag{5.5.23}$$

$d = \gamma - 1/\kappa$, where $0 < |\gamma\kappa| < 1$. The restrictions on γ require explanation: $\gamma \neq 0$ is necessary to maintain a complex-valued solution of the BO equation; $|\gamma\kappa| < 1$ is essential for *simultaneous* consistency with the requirements $b > 0$ and $d < 0$. Substituting $d = \gamma - 1/\kappa$ into (5.5.23) we obtain

$$Q(x,t) = \ln\left(\frac{(x-\kappa t+x_0)^2+\gamma^2+\frac{1}{\kappa^2}+2\frac{\gamma}{\kappa}}{(x-\kappa t+x_0)^2+\gamma^2+\frac{1}{\kappa^2}-2\frac{\gamma}{\kappa}}\right),$$

which through an application of the identity $\tanh^{-1}(\vartheta) = \frac{1}{2}\ln\left(\frac{1+\vartheta}{1-\vartheta}\right)$ can be written in the form

$$Q(x,t) = 2\tanh^{-1}\left(\frac{2\frac{\gamma}{\kappa}}{(x-\kappa t+x_0)^2+\gamma^2+\frac{1}{\kappa^2}}\right). \tag{5.5.24}$$

One last piece of information is required before we can claim that (5.5.24) is the 1-soliton solution for the MBO equation. A formula that relates the parameters γ and κ in (5.5.24) to the fundamental parameter α that appears in the MBO equation (5.5.1) is indispensable. Two routes can be taken to derive the necessary formula: (5.1.4) or (5.4.10). Computationally, equation (5.4.10) is the most efficient of the options available, so we will pursue this option. Equation (5.4.10) specializes to

$$\int_{-\infty}^{\infty}\left(e^{Q(x,\,t)}-1\right)\mathrm{d}x=\frac{4\pi}{\alpha} \tag{5.5.25}$$

in the case of the pure 1-soliton solution for the MBO equation. Substituting (5.5.24) into the *left hand side* of (5.5.25) we obtain

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{\left(x-\kappa t+x_0\right)^2+\left(\frac{1}{\kappa}-\gamma\right)^2}=\frac{\pi\kappa}{\alpha\gamma}. \tag{5.5.26}$$

Contour integration can be applied to evaluate the integral in (5.5.26), but the most expedient method is to use the transformation $\eta = x-\kappa t+x_0$ to convert (5.5.26) into a standard integral. Using the procedure advocated and the observation that $1-\gamma\kappa > 0$ we obtain the (surprisingly simple) formula

$$\gamma=\frac{1}{\alpha+\kappa}. \tag{5.5.27}$$

Finally, substituting (5.5.27) into (5.5.24) we arrive at the 1-soliton solution for the MBO equation (5.5.1):

$$Q(x,t)=2\tanh^{-1}\left(\frac{\frac{2}{(\alpha+\kappa)\kappa}}{\left(x-\kappa t+x_0\right)^2+\frac{1}{\kappa^2}+\frac{1}{(\alpha+\kappa)^2}}\right). \tag{5.5.28}$$

Equation (5.5.28) was first derived by Nakamura [see equation (12) in Ref. 90] using

following result useful:

$$\mathbf{H}(Q_x)=\frac{\frac{2}{\alpha+\kappa}+\frac{2}{\kappa}}{\left(x-\kappa t+x_0\right)^2+\left(\frac{1}{\alpha+\kappa}+\frac{1}{\kappa}\right)^2}+\frac{\frac{2}{\alpha+\kappa}-\frac{2}{\kappa}}{\left(x-\kappa t+x_0\right)^2+\left(\frac{1}{\alpha+\kappa}-\frac{1}{\kappa}\right)^2}.$$