CHAPTER 4

THE INVERSE SCATTERING TRANSFORM FOR THE MODIFIED INTERMEDIATE LONG WAVE EQUATION

Section 4.1: Introduction

The (mathematical) origin of the MILW equation,

$$V_t + \beta V_x\left(e^V - 1\right) + \frac{1}{\delta}V_x + V_x\mathbf{T}\left(V_x\right) + \mathbf{T}\left(V_{xx}\right) = 0, \tag{4.1.1}$$

was elucidated in Section 2.1 of this thesis, and we devoted the remainder of Chapter 2 to

Despite

the possible physical applications of the MILW equation, and the fact that this equation possesses an infinite number of conservation laws [112], a linear scattering problem [112], an auto BT [89,112] and multi-soliton solutions [89], the initial value problem for the MILW equation was not solved by the originators of this particular equation. Satsuma, Taha and Ablowitz in their seminal work on the integrability of the MILW equation remark

The scattering problem, and hence solutions by IST, is open for the MILW equation

developments in the area of integrable nonlinear singular integro-differential equations, the author mentions the MILW equation, but no reference is made to a solution scheme for the MILW equation. The research monograph by Ablowitz and Clarkson [11], which is amongst the most significant literature devoted to the IST, also contains references to the MILW equation, but these authors do not present a solution for its initial value problem, nor do they cite any literature dedicated to the solution of the initial value problem for the MILW equation. All the facts mentioned lead us to assert that the initial value problem for the MILW equation is open and that practical and theoretical issues merit the development of a solution scheme for this initial value problem.

In this chapter we will present the solution of the initial value problem for the MILW equation. We will assume that the known initial value, $V_0(x) \equiv V(x, 0)$, for the MILW equation is real-valued, sufficiently smooth and satisfies the boundary conditions

$$\frac{\partial^n V_0}{\partial x^n} \to 0 \text{ as } |x| \to \infty, \tag{4.1.2}$$

where n denotes a positive integer or zero; in the case $n = 0$, $\partial^n V_0 / \partial x^n \equiv V_0$. We will use the notation $V_{0,x}$ to denote $\partial V_0 / \partial x$.

The material in Chapter 3 forms a paradigm for the procedure that we will develop in this chapter to solve the initial value problem for equation (4.1.1). In analogy with the manner in which we commenced Section 3.1, we will begin this section with the Miura transformation between the ILW equation,

$$U_t + 2UU_x + \tfrac{1}{\delta} U_x + \mathbf{T}\left(U_{xx}\right) = 0, \tag{4.1.3}$$

and the MILW equation (4.1.1). According to equation (2.1.11), the relevant Miura transformation is

$$U = \tfrac{1}{2}\left\{\mathbf{T}\left(V_x\right) + \beta\left(e^V - 1\right) + iV_x\right\}, \tag{4.1.4}$$

where $U \equiv U(x,t)$ satisfies (4.1.3) and $V \equiv V(x,t)$ satisfies (4.1.1).

The explicit nonlinear transformation (4.1.4) maps a *real-valued* solution, $V(x,t)$, of (4.1.1) into a *complex-valued* solution, $U(x,t)$, of equation (4.1.3). In conformity with the

x and t that solves an appropriate nonlinear evolution equation. If we use the known initial value V_0 to evaluate the right hand side of (4.1.4), then we find that the complex-valued function

$$U(x,0) = \tfrac{1}{2}\left\{\mathbf{T}\left(V_{0,x}\right) + \beta\left(e^{V_0} - 1\right) + iV_{0,x}\right\} \tag{4.1.5}$$

is an initial value for the ILW equation (4.1.3). Henceforth, we will use the notation U_0 to denote $U(x,0)$. Equations (1.3.28) and (4.1.2) imply the boundary conditions

$$\frac{\partial^n U_0}{\partial x^n} \to 0 \text{ as } |x| \to \infty, \tag{4.1.6}$$

where n denotes a positive integer or zero.

Consider, now, the situation in which $U(x,t)$ is a complex-valued solution of equation (4.1.3) that has evolved from the *known* complex initial value (4.1.5). Equation (4.1.4) is valid for all $t \geq 0$ because $U(x,t)$ has evolved from a specialized form of (4.1.4), namely (4.1.5). The equation formed when we equate imaginary parts either side of (4.1.4) is

$$V_x(x,t) = 2\,\mathrm{Im}\left\{U(x,t)\right\}. \tag{4.1.7}$$

Equation (4.1.7) is significant because it reveals that the solution of the initial value problem for the MILW equation, with initial value V_0 and boundary conditions as specified by (4.1.2), can be generated from the solution of the initial value problem for the ILW equation with initial value (4.1.5) and boundary conditions (4.1.6). We can obtain the solution of the complexified initial value problem for the ILW equation by *extending* the Kodama-Ablowitz-Satsuma IST [63] for the real-valued ILW equation to incorporate the situation in which $U(x,t)$ emanates from the known complex-valued initial profile (4.1.5). Once the complexified initial value problem for $U(x,t)$ has been solved, we then proceed to deduce $V_x(x,t)$ from (4.1.7), after which $V(x,t)$ is determined by a quadrature with respect to x. The function of integration that will appear (as a consequence of the quadrature) in the formula for $V(x,t)$ is determined from the boundary conditions $V(x,t) \to 0$ (uniformly in t) as $|x| \to \infty$.

The remainder of this chapter is arranged into five sections. In Section 4.2 we present the Lax pair for the (complex-valued) ILW equation in a form that is suitable for application to the MILW equation. Section 4.3 is devoted to an analysis of the direct problem for the x-part of the Lax pair that appears in Section 4.2, namely once appropriate Jost functions are defined, the scattering data associated with our linear scattering problem are characterized. We will devote considerable effort in Section 4.3 to ascertaining the analytic properties (with respect to the spectral parameter) of the scattering data. An interesting feature of the scattering data is that the eigenvalues in the discrete spectrum are *not* restricted to the imaginary axis in the complex-plane of the spectral parameter. The reader who is familiar with the IST for the real-valued ILW equation [63] will know that

appropriate to recall that in Section 3.2 of this thesis we showed that the transition from the real-valued to the complex-valued KdV equation also induces (in the IST for the latter equation) the existence of discrete eigenvalues with a nonzero real-part. Section 4.4 is dedicated to the solution of the inverse problem that emerges as a natural consequence of our efforts to characterize the analytic properties of the scattering data. The time evolution of the scattering data is calculated in Section 4.5. In the last section of this chapter, Section 4.6, we demonstrate how the solution scheme annunciated in Sections 4.1-4.5 can be applied to calculate soliton solutions for the ILW and MILW equations. Throughout this

and the original papers on the application of the IST to the real-valued ILW equation [62,63,103].

Section 4.2: Linear System for the MILW Equation

Satsuma et al. [108] have shown that the (real-valued) ILW equation forms the compatibility condition for the following over-determined linear system of equations:

$$\psi_x^{+} = i(U-\mu)\psi^{+} - i\nu\psi^{-}; \tag{4.2.1}$$

$$\psi_t^{\pm} = -2\left(\mu + \frac{1}{2\delta}\right)\psi_x^{\pm} + i\psi_{xx}^{\pm} + \left(\pm U_x - i\mathbf{T}\left(U_x\right) + \Omega\right)\psi^{\pm}\ , \tag{4.2.2$^{\pm}$}$$

where $\psi^{\pm} \equiv \psi^{\pm}(x,t)$ and μ, ν and Ω represent arbitrary constants. The $\psi^{\pm}$ denote boundary values of functions that are analytic in the horizontal strips between $\mathrm{Im}\,(z) = 0$ and $\mathrm{Im}\,(z) = \pm\, 2\delta$ in the complex z-plane, $z = x + iy$ being the complex extension of the real variable x. The ILW equation,

$$U_t + 2UU_x + \frac{1}{\delta}U_x + \mathbf{T}\left(U_{xx}\right) = 0, \tag{4.2.3}$$

emerges from the compatibility condition

$$\psi_{xt}^{+} \equiv \psi_{tx}^{+}.$$

It is now appropriate that we remind the reader of several results that appeared in Section 1.3 of this thesis, each result being relevant to the subject matter in this section:

1) A sufficiently well behaved function, labelled as $\varphi(x)$ to facilitate our discussion, can be decomposed into the form

$$\varphi(x) = \psi^{+}(x) - \psi^{-}(x), \tag{4.2.4}$$

where $\varphi(x) \equiv \varphi(x,t)$, $\psi^{\pm}(x) \equiv \psi^{\pm}(x,t)$ and $\psi^{\pm}(x)$, which are not necessarily the same as the functions that appear in (4.2.1) and (4.2.2$^{\pm}$), denote boundary values of functions analytic in the horizontal strips between $\mathrm{Im}\,(z) = 0$ and $\mathrm{Im}\,(z) = \pm\, 2\delta$ in the complex z-plane. The solution of the RH boundary value problem (4.2.4) is

$$\psi^{\pm}(x) = \lim_{y \to 0^{\pm}} \frac{1}{4i\delta}\int_{-\infty}^{\infty} \coth\left(\frac{\pi}{2\delta}[\xi - (x+iy)]\right)\varphi(\xi)\,d\xi + \tilde{\alpha}, \tag{4.2.5$^{\pm}$}$$

which can also be written in the form

$$\psi^{\pm}(x) = \tfrac{1}{2}(\pm\mathbf{I} - i\mathbf{T})\varphi(x) + \tilde{\alpha}, \tag{4.2.6$^{\pm}$}$$

where $(\mathbf{I}\varphi)(x) \equiv \varphi(x)$ and $\tilde{\alpha}$ is a constant;

2) According to equation (1.3.54), the boundary values $\psi^{\pm}(x)$ are connected by the vertical periodicity condition

$$\psi^{-}(x) = \psi^{+}(x + 2i\delta), \tag{4.2.7}$$

where

$$\psi^{+}(x + 2i\delta) = \lim_{y \to 2\delta^{-}} \psi(x + iy);$$

3) By virtue of (1.3.62$^{\pm}$), the equations

$$\psi^{\pm}(x) = \tilde{\psi}(x \mp i\delta) \tag{4.2.8$^{\pm}$}$$

$\tilde{\psi}(x)$ to the boundary values $\psi^{\pm}(x)$. Equations (4.2.8$^{\pm}$) complete the review of the relevant results from Section 1.3 of this thesis.

Kodama, Ablowitz and Satsuma [63] have used an adroit implementation of the IST to solve the initial value problem for the (real-valued) ILW equation. The authors of Ref. 63 derive an explicit formula for $U^{+}(x,t)$ in terms of quantities that can be calculated from the known initial value $U(x,0)$. Equation (4.2.6^{+}) furnishes us with the information

$$U^{+}(x,t) = \tfrac{1}{2}(\mathbf{I} - i\mathbf{T})U(x,t).$$

Kodama, Ablowitz and Satsuma [63] solve for the physical variable, $U(x,t)$, by using the formula

$$U(x,t) = U^{+}(x,t) + \left\{U^{+}(x,t)\right\}^{*}, \tag{4.2.9}$$

where * designates the complex conjugate.

In our work, $U(x,t)$ is intrinsically a complex-valued function. Once an appropriate IST has been implemented to solve the complexified initial value problem for the ILW equation, the complex-valued character of $U(x,t)$ prohibits us from using (4.2.9) to construct the physical variable. The situation in which $U(x,t)$ is a complex-valued function requires the derivation of *separate* formulae for $U^{+}(x,t)$ and $U^{-}(x,t)$. Once suitable formulae for $U^{+}(x,t)$ and $U^{-}(x,t)$ are available, appropriate versions of equations (4.2.6$^{\pm}$) can be employed to determine the physical variable, as follows:

$$U(x,t) = U^{+}(x,t) - U^{-}(x,t). \tag{4.2.10}$$

The known function $U(x,t)$ is then used as the input to (4.1.7), after which the procedure that was elaborated in Section 4.1 of this thesis is used to determine $V(x,t)$ of the initial value problem for the MILW equation.

Separate equations that each express $U^{+}(x,t)$ and $U^{-}(x,t)$ in terms of appropriate scattering data can be derived through the introduction of an *indicator* into the linear scattering problem (4.2.1) and (4.2.2$^{\pm}$). In conformity with the notation used in Chapter 3 of this thesis, we will use ε to denote the indicator. The permissible values of the indicator are

$$\varepsilon = \pm\, 1. \tag{4.2.11}$$

We wish to emphasize that ε *does not* belong to the scattering data, but the scattering data will depend parametrically on ε. The reader is invited to review Section 3.2 of this thesis as a guide to understanding how the indicator will arise throughout the remainder of this chapter.

Equations (4.2.1) and (4.2.2$^{\pm}$) must be modified to incorporate the parameter ε. The modification of equations (4.2.1) and (4.2.2$^{\pm}$) that incorporates the parameter ε is

$$\psi_x^{+\,\varepsilon} = i\varepsilon(U - \mu)\psi^{+\,\varepsilon} - i\varepsilon\nu\psi^{-\,\varepsilon} \tag{4.2.12}$$

and

$$\psi_t^{\pm\,\varepsilon} = -\,2\left(\mu + \frac{1}{2\delta}\right)\psi_x^{\pm\,\varepsilon} + i\varepsilon\psi_{xx}^{\pm\,\varepsilon} + \left(\pm\, U_x - i\varepsilon\mathbf{T}(U_x) + \Omega\right)\psi^{\pm\,\varepsilon}, \tag{4.2.13$^{\pm}$}$$

respectively. In equations (4.2.12) and (4.2.13$^{\pm}$), the symbols μ, ν and Ω retain the meanings that have already been assigned to them, namely they are still arbitrary constants. The new symbols $\psi^{\pm\,\varepsilon} \equiv \psi^{\pm\,\varepsilon}(x,t)$ denote the boundary values of functions that are analytic in the horizontal strips between $\mathrm{Im}\,(z) = 0$ and $\mathrm{Im}\,(z) = \pm\, 2\varepsilon\delta$ in the complex z-plane. Specifically,

$$\psi^{\pm\,\varepsilon}(x) = \lim_{y \to 0^{+}} \psi(x + i\varepsilon y).$$

The ILW equation is produced from the compatibility condition

$$\psi_{xt}^{+\,\varepsilon} \equiv \psi_{tx}^{+\,\varepsilon}.$$

Equations (4.2.12) and (4.2.13$^{\pm}$) were derived after we conducted exhaustive testing of the above compatibility condition in which undetermined coefficients were initially placed in the linear system (4.2.12) and (4.2.13$^{\pm}$). An important feature of equations (4.2.12) and (4.2.13$^{\pm}$) is that $U(x,t)$ now defines a complex-valued solution of the ILW equation; the real and imaginary parts of $U(x,t)$ are specified by equation (4.1.4).

Equations (4.2.4)-(4.2.8$^{\pm}$) with ε incorporated are as follows:

1) the function $\varphi(x)$ admits the decomposition

$$\varphi(x) = \varepsilon\left(\psi^{+\,\varepsilon}(x) - \psi^{-\,\varepsilon}(x)\right), \tag{4.2.14}$$

where the $(\pm\,\varepsilon)$-parts of $\varphi(x)$ are determined by the functionals

$$\psi^{\pm\,\varepsilon}(x) = \lim_{y\to 0^+}\frac{1}{4i\delta}\int_{-\infty}^{\infty}\coth\left(\frac{\pi}{2\delta}[\xi-(x\pm i\varepsilon y)]\right)\varphi(\xi)\,\mathrm{d}\xi + \tilde{\alpha}, \tag{4.2.15$^{\pm}$}$$

or when expressed in terms of $\mathbf{T}$, the boundary values of the sectionally holomorphic function $\psi(z)$ are

$$\psi^{\pm\,\varepsilon}(x) = \frac{1}{2}(\pm\,\varepsilon\mathbf{I} - i\mathbf{T})\varphi(x) + \tilde{\alpha}; \tag{4.2.16$^{\pm}$}$$

2) the connection between the boundary values $\psi^{+\,\varepsilon}(x)$ and $\psi^{-\,\varepsilon}(x)$ is

$$\psi^{-\,\varepsilon}(x) = \psi^{+\,\varepsilon}(x + 2i\varepsilon\delta); \tag{4.2.17}$$

3) the simple and elegant formulae

$$\psi^{\pm\,\varepsilon}(x) = \tilde{\psi}\left(x \mp i\varepsilon\delta\right) \tag{4.2.18$^{\pm}$}$$

$\tilde{\psi}(x)$ to the boundary values $\psi^{\pm\,\varepsilon}(x)$, the latter being defined by (4.2.15$^{\pm}$).

The inclusion of ε in the linear scattering problem for the complex-valued ILW equation opens to us the possibility of deriving a formula for $U^{+\,\varepsilon}(x,t)$, where, according to (4.2.16$^{\pm}$),

$$U^{+\,\varepsilon}(x,t) = \frac{1}{2}(\varepsilon\mathbf{I} - i\mathbf{T})U(x,t) + \tilde{\alpha}. \tag{4.2.19}$$

Once a formula for $U^{+\,\varepsilon}(x,t)$ is available we can activate the procedure outlined in Section 4.1 of this thesis to build the solution of the initial value problem for the MILW equation.

We will complete this section with a concatenation of all the stages in the scheme to solve the initial value problem for the MILW equation (4.1.1):

Stage I: Use the known real-valued initial value, V_0, for the MILW equation to form the complex-valued function defined by equation (4.1.5). The boundary conditions imposed on the solution of the MILW equation are specified by (4.1.2).

Stage II: Transform, through the *direct problem* for equation (4.2.12), the initial value (4.1.5) into appropriate scattering data; t and ε will enter parametrically throughout the analysis of the direct problem for (4.2.12).

Stage III: Solve a suitable *inverse problem* to derive a formula for $U^{+\varepsilon}(x,t)$ in terms of quantities (scattering data) that can be calculated form the known initial value (4.1.5). The boundary value $U^{+\varepsilon}(x,t)$ is related to the physical variable that solves the ILW equation by equation (4.2.19).

Stage IV: Determine the *time evolution* of the scattering data by an appropriate use of equations ($4.2.13^{\pm}$), supplemented by the boundary conditions (4.1.6).

Stage V: Use *separate* equations for $U^{+}(x,t)$ and $U^{-}(x,t)$ in equation (4.2.10) to construct the physical variable, $U(x,t)$, that solves the complex-valued ILW equation.

Stage VI: Solve for the physical variable, $V(x,t)$, in the MILW equation (4.1.1) by a suitable application of (4.1.7), a single quadrature (with respect to x) and the boundary conditions (4.1.2).

Stage VII: Use the equation

$$\operatorname{Re}\{U(x,t)\} = \tfrac{1}{2}\left\{\mathbf{T}(V_x) + \beta\left(e^{V} - 1\right)\right\} \tag{4.2.20}$$

to connect all parameters that originate from the IST (for the complex-valued ILW equation) to the fundamental parameter β in the MILW equation (4.1.1). Equation (4.2.20) is formed when we equate *real* parts either side of (4.1.4).

Section 4.3: Jost Functions and the Direct Problem

The direct problem for the complex-valued ILW equation necessitates that we use (4.2.12) to map the initial value for the ILW equation,

$$U_0 = \frac{1}{2}\left\{\mathbf{T}\left(V_{0,x}\right) + \beta\left(e^{V_0} - 1\right) + iV_{0,x}\right\},$$

into appropriate scattering data. An examination of the direct problem for (4.2.12) requires that we introduce into our work the *real* spectral parameter, denoted as λ, through the boundary conditions

$$\tilde{\psi}(x;\lambda) \to \exp\left(\pm\frac{i\varepsilon\lambda x}{2}\right) \text{ as } |x| \to \infty. \tag{4.3.1}$$

$\tilde{\psi}(x)$ is connected to the functions (boundary values) $\psi^{\pm\varepsilon}(x)$ that appear in (4.2.12) by means of the equations ($4.2.18^{\pm}$). In complete detail we should write $\tilde{\psi}(x,t;\lambda)$ instead of $\tilde{\psi}(x;\lambda)$, but in this section and its sequel we will elect to suppress (in our notation) the explicit dependence of any quantity on the temporal variable, *t*.

The boundary conditions (4.3.1) induce specific parameterizations (in terms of λ) for the quantities μ and ν in equation (4.2.12). We can ascertain the precise dependence of μ and ν on λ by isolating the following two cases:

Case 1: $\tilde{\psi}(x;\lambda) \to \exp\left(\frac{i\varepsilon\lambda x}{2}\right)$ as $|x| \to \infty$.

From ($4.2.18^{\pm}$) we infer that

$$\psi^{\pm\varepsilon}(x;\lambda) \to \exp\left(\frac{i\varepsilon\lambda x \pm \delta\lambda}{2}\right) \text{ as } |x| \to \infty. \tag{4.3.2}$$

In the limit $|x| \to \infty$, equation (4.2.12) reduces to the autonomous equation

$$\psi_x^{+\varepsilon} = -i\varepsilon\left(\mu\psi^{+\varepsilon} + \nu\psi^{-\varepsilon}\right). \tag{4.3.3}$$

Substituting (4.3.2) into (4.3.3) we find that

$$\lambda = -2\left(\mu + \nu e^{-\delta\lambda}\right) \tag{4.3.4}$$

is one relationship that connects μ, ν and λ.

Case 2: $\tilde{\psi}(x;\lambda) \to \exp\left(-\frac{i\varepsilon\lambda x}{2}\right)$ as $\left|x\right| \to \infty$.

In this particular case, the analogue of (4.3.4) is

$$\lambda = 2\left(\mu + \nu e^{\delta\lambda}\right), \tag{4.3.5}$$

which can be derived by an application of the discrete point transformation $\lambda \to -\lambda$ in (4.3.4).

We now have a set of equations that involve μ, ν and λ. Solving equations (4.3.4) and (4.3.5) we obtain the following parameterizations:

$$\mu = -\frac{\lambda}{2}\coth\,(\delta\lambda); \tag{4.3.6}$$

$$\nu = \frac{\lambda}{2}\operatorname{cosech}\,(\delta\lambda). \tag{4.3.7}$$

It is now an opportune moment to define the Jost functions. Let $m^{+\varepsilon}(x;\lambda)$ and $\bar{m}^{+\varepsilon}(x;\lambda)$ denote *left* Jost functions for (4.2.12), and let $n^{+\varepsilon}(x;\lambda)$ and $\bar{n}^{+\varepsilon}(x;\lambda)$ denote *right* Jost functions for (4.2.12). The Jost functions satisfy the differential-difference equation (4.2.12), are $(+\varepsilon)$ functions in x, and are distinguished from each other by their large-x asymptotics:

$$\left.\begin{aligned} m^{+\varepsilon}(x;\lambda) &\to \exp\left(-\frac{i\varepsilon\lambda(x-i\varepsilon\delta)}{2}\right) \\ \bar{m}^{+\varepsilon}(x;\lambda) &\to \exp\left(\frac{i\varepsilon\lambda(x-i\varepsilon\delta)}{2}\right) \end{aligned}\right\} \text{ as } x \to -\infty \tag{4.3.8}$$

and

$$\left.\begin{aligned} n^{+\varepsilon}(x;\lambda) &\to \exp\left(\frac{i\varepsilon\lambda(x-i\varepsilon\delta)}{2}\right) \\ \bar{n}^{+\varepsilon}(x;\lambda) &\to \exp\left(-\frac{i\varepsilon\lambda(x-i\varepsilon\delta)}{2}\right) \end{aligned}\right\} \text{ as } x \to \infty. \tag{4.3.9}$$

The following remarks will clarify several issues related to the Jost functions:

1) $\bar{m}^{+\varepsilon}$ is *not* the complex conjugate of $m^{+\varepsilon}$, and $\bar{n}^{+\varepsilon}$ is *not* the complex conjugate of $n^{+\varepsilon}$. We will use the notation m^* to denote the complex conjugate of m, and n^* for the complex conjugate of n.

2) $(\pm\,\varepsilon)$ function in x

continuation off the real x-axis into the horizontal strips between $\operatorname{Im}(z) = 0$ and $\operatorname{Im}(z) = \pm\, 2\varepsilon\delta$ in the complex z-plane, where $z = x + iy$ is the complex extension of x;

3) No loss of generality arises from the decision to use Jost functions that are $(+\,\varepsilon)$ functions in x because the $(-\,\varepsilon)$ version of each Jost function can be constructed by recourse to (4.2.17);

4) We have arrived at the asymptotics contained in equations (4.3.8) and (4.3.9) by using (4.2.18$^+$) and (4.3.1).

Our analysis of the direct problem for (4.2.12) will proceed in an efficient manner if we work with the modified eigenfunctions $W^{\pm\varepsilon}(x;\lambda)$, where

$$W^{\pm\varepsilon}(x;\lambda) \underline{\underline{\text{def}}}\ \psi^{\pm\varepsilon}(x;\lambda)\exp\left(\frac{i\varepsilon\lambda x \pm \delta\lambda}{2}\right). \qquad (4.3.10^{\pm})$$

An important feature of (4.3.10$^{\pm}$) is that it preserves the periodicity condition (4.2.17):

$$W^{-\varepsilon}(x;\lambda) = W^{+\varepsilon}(x + 2i\varepsilon\delta;\lambda). \qquad (4.3.11)$$

We are now required to express (4.2.12) and (4.2.13$^{\pm}$) in terms of $W^{\pm\varepsilon}(x;\lambda)$. Using (4.3.10$^{\pm}$) to eliminate $\psi^{\pm\varepsilon}(x;\lambda)$ from equations (4.2.12) and (4.2.13$^{\pm}$) we obtain the following:

$$iW_x^{+\varepsilon} + \varepsilon\left(\zeta_+(\lambda) + \frac{1}{2\delta}\right)\left(W^{+\varepsilon} - W^{-\varepsilon}\right) = -\,\varepsilon U W^{+\varepsilon}; \qquad (4.3.12)$$

$$W_t^{\pm\varepsilon} = i\varepsilon W_{xx}^{\pm\varepsilon} + 2\zeta_+(\lambda)W_x^{\pm\varepsilon} + \left[\pm\, U_x - i\varepsilon\mathbf{T}(U_x) + \Gamma\right]W^{\pm\varepsilon}, \qquad (4.3.13^{\pm})$$

where

$$\zeta_{+\varepsilon}(\lambda) \underline{\underline{\text{def}}}\ \frac{\lambda}{2} + \varepsilon\frac{\lambda}{2}\coth(\delta\lambda) - \frac{\varepsilon}{2\delta} \qquad (4.3.14)$$

and

$$\Gamma \underline{\underline{\text{def}}}\ \Omega + \frac{i\varepsilon}{4}\lambda^2 - i\varepsilon\lambda\zeta_+(\lambda).$$

The asymptotic data contained in equations (4.3.8) and (4.3.9) require alterations to incorporate the transition embodied in $(4.3.10^{+})$. The amended asymptotics of the Jost functions are:

$$\left.\begin{aligned} &M^{+\varepsilon}(x;\lambda) \to 1 \\ &\bar{M}^{+\varepsilon}(x;\lambda) \to \exp\left(i\varepsilon\lambda(x - i\varepsilon\delta)\right) \end{aligned}\right\} \text{ as } x \to -\infty \tag{4.3.15}$$

and

$$\left.\begin{aligned} &N^{+\varepsilon}(x;\lambda) \to \exp\left(i\varepsilon\lambda(x - i\varepsilon\delta)\right) \\ &\bar{N}^{+\varepsilon}(x;\lambda) \to 1 \end{aligned}\right\} \text{ as } x \to \infty, \tag{4.3.16}$$

where

$$M^{+\varepsilon}(x;\lambda) = m^{+\varepsilon}(x;\lambda)\exp\left(\frac{i\varepsilon\lambda(x - i\varepsilon\delta)}{2}\right)$$

and so forth for the remaining Jost functions. Each of the Jost functions in (4.3.15) and (4.3.16) is a $(+\,\varepsilon)$ function in x and each Jost function satisfies (4.3.12).

An important consequence of the transition to modified eigenfunctions is that our spectral parameter is now $\zeta_{+}(\lambda)$. For future reference, we will record that $\zeta_{+\varepsilon}(\lambda)$ appears in the following identities:

$$\zeta_{+\varepsilon}(\lambda) + \zeta_{-\varepsilon}(\lambda) = \lambda; \tag{4.3.17a}$$

$$\zeta_{+\varepsilon}(-\lambda) = -\,\zeta_{-\varepsilon}(\lambda); \tag{4.3.17b}$$

$$\left(\zeta_{+\varepsilon}(\lambda) + \frac{\varepsilon}{2\delta}\right)e^{-2\varepsilon\delta\lambda} = \frac{\varepsilon}{2\delta} - \zeta_{-\varepsilon}(\lambda). \tag{4.3.17c}$$

Equations (4.3.17a) and (4.3.17b) are consequences of (4.3.14) and the equation

$$\zeta_{-\varepsilon}(\lambda) = \frac{\lambda}{2} - \varepsilon\frac{\lambda}{2}\coth(\delta\lambda) + \frac{\varepsilon}{2\delta}.$$

Equation (4.3.17c) is not immediately obvious, so we now take steps to present its derivation. Starting with equation (4.3.14), but expressed in the form

$$\zeta_{+\varepsilon}(\lambda) + \frac{\varepsilon}{2\delta} = \frac{\lambda}{2} + \varepsilon\frac{\lambda}{2}\coth(\delta\lambda),$$

we find that

$$\zeta_{+\varepsilon}\left(\lambda\right) + \frac{\varepsilon}{2\delta} = \frac{\lambda}{2}\,\frac{e^{\varepsilon\delta\lambda}}{\sinh\left(\varepsilon\delta\lambda\right)} \tag{4.3.18}$$

because $\varepsilon\coth\left(\delta\lambda\right) = \coth\left(\varepsilon\delta\lambda\right)$. Multiplying *both* sides of (4.3.18) by $e^{-2\varepsilon\delta\lambda}$ and then using the identity

$$e^{-\varepsilon\delta\lambda} = \cosh\left(\varepsilon\delta\lambda\right) - \sinh\left(\varepsilon\delta\lambda\right)$$

in the new version of (4.3.18) will lead us (with little effort) to (4.3.17c).

The Jost functions possess a latent analytic character when viewed as functions of either λ or $\zeta_+\left(\lambda\right)$, and these analytic properties lead to the solution of the initial value problem for the MILW equation. Unless otherwise stated, in the remainder of this chapter any reference to analyticity will be with respect to the spectral parameter. We can ascertain the analytic character of the Jost functions by converting (4.3.12) into an appropriate linear integral equation. The linear second-kind integral equation

$$W^{+\varepsilon}\left(x;\lambda\right) = W_0^{+\varepsilon}\left(x;\lambda\right) + \varepsilon\int_{-\infty}^{\infty} G^{+\varepsilon}\left(x,\eta;\lambda\right)U\left(\eta\right)W^{+\varepsilon}\left(\eta;\lambda\right)\mathrm{d}\eta \tag{4.3.19}$$

constitutes the general solution of equation (4.3.12). The following remarks are designed to clarify certain aspects of equation (4.3.19):

1) The function $W_0^{+\varepsilon}\left(x;\lambda\right)$ solves the homogeneous version of (4.3.12), namely

$$i\frac{\partial W_0^{+\varepsilon}}{\partial x} + \varepsilon\left(\zeta_+(\lambda) + \frac{1}{2\delta}\right)\left(W_0^{+\varepsilon} - W_0^{-\varepsilon}\right) = 0;$$

2) $G^{+\varepsilon}\left(x,\eta;\lambda\right)$, which is a $\left(+\varepsilon\right)$ function in x, satisfies the equation

$$iG_x^{+\varepsilon} + \varepsilon\left(\zeta_+(\lambda) + \frac{1}{2\delta}\right)\left(G^{+\varepsilon} - G^{-\varepsilon}\right) = -\,\delta\left(x-\eta\right), \tag{4.3.20}$$

where $G^{\pm\varepsilon} \equiv G^{\pm\varepsilon}\left(x,\eta;\lambda\right)$ and $\delta(\bullet)$ denotes the Dirac delta function;

3) The integral equation that characterizes $W^{-\varepsilon}\left(x;\lambda\right)$ can be constructed from (4.3.19) with the assistance of (4.3.11).

Let us assume that the solution of (4.3.20) can be expressed in the form

$$G^{+\varepsilon}(x,\eta;\lambda) = \frac{1}{2\pi}\int_C \hat{g}(r;\lambda)e^{i(x-\eta)r}\mathrm{d}r, \tag{4.3.21}$$

where C denotes a suitable horizontal line of infinite extent in the complex r-plane, $G^{-\varepsilon}(x,\eta;\lambda) = G^{+\varepsilon}(x+2i\varepsilon\delta,\eta;\lambda)$ and $\hat{g}(r;\lambda)$ is an unknown function that is independent of x and η. Substituting (4.3.21) into (4.3.20), and then ensuring that the resulting expression is consistent with the integral representation [12, p. 270]

$$\delta(x-\eta) = \frac{1}{2\pi}\int_C e^{i(x-\eta)r}\mathrm{d}r$$

we obtain an equation for $\hat{g}(r;\lambda)$,

$$\hat{g}(r;\lambda) = \frac{1}{r-\varepsilon\left[\zeta_+(\lambda)+(2\delta)^{-1}\right]\left[1-e^{-2\varepsilon\delta r}\right]}. \tag{4.3.22}$$

Equation (4.3.22) can also be expressed in the form

$$\hat{g}(r;\lambda) = \frac{\operatorname{cosech}(\delta r)e^{\varepsilon\delta r}}{2\left[\varepsilon\zeta_{+\varepsilon}(r)-\zeta_+(\lambda)\right]} \tag{4.3.23}$$

because of the identity

$$1-e^{-2\varepsilon\delta r} = 2\varepsilon\sinh(\delta r)e^{-\varepsilon\delta r}.$$

The function $\hat{g}(r;\lambda)$ has an intricate pole structure when viewed in the complex r-plane. We begin the task of elucidating the pole structure of $\hat{g}(r;\lambda)$ with the observation that

$$\{\hat{g}(r;\lambda)\}^* = \hat{g}(r^*;\lambda), \tag{4.3.24}$$

which means that if $r = r(\lambda;\varepsilon)$ is a pole of (4.3.23), then the complex conjugate $r = r^*(\lambda;\varepsilon)$ is also a pole of (4.3.23). The reflection property (4.3.24) allows us, without loss of generality, to focus our attention on those poles of (4.3.23) for which $\operatorname{Im}(r) > 0$. Hence, let $r = r_n(\lambda;\varepsilon)$, where $\operatorname{Im}(r_n) > 0$ and n denotes a positive integer, be a complex-valued solution of the equation

$$\varepsilon\zeta_{+\varepsilon}(r) - \zeta_+(\lambda) = 0. \tag{4.3.25}$$

The following three remarks are now timely:

1) The multi-valued function $\zeta_{+\varepsilon}^{-1}(\bullet)$ will induce infinitely many complex-valued solutions of (4.3.25);

2) The points at $r = 0$ and $r = \varepsilon\lambda$ are the only singularities of $\hat{g}(r;\lambda)$ that are located on the Im $(r) = 0$ axis. In conformity with the notation used in Ref. 63, we will use the notation $r_{-1} = 0$ and $r_0 = \varepsilon\lambda$ to designate those poles of $\hat{g}(r;\lambda)$ that reside on the Im $(r) = 0$ axis;

3) Generically the r_n, where $n \geq -1$, constitute *simple* poles of $\hat{g}(r;\lambda)$. Double poles, which are produced from a coalescence of two simple poles, can also occur. For example, $r = 0$ is a double pole of $\hat{g}(r;\lambda)$ whenever $\lambda = 0$. In the remainder of this chapter we will assume that any pole of $\hat{g}(r;\lambda)$ is simple.

If the spectral parameter λ is real, then $\zeta_+(\lambda)$ is also real-valued, and in turn we have that Re $(r_n) \neq 0$ for all $n \geq 0$. Kodama et al. [63] provide the estimate

$$\frac{\pi}{2\delta}(2n-1) < \mathrm{Im}\,(r_n) < \frac{\pi}{2\delta}(2n+3)$$

for the imaginary part of r_n, where $n \geq 0$. We will sharpen the estimate provided by the authors of Ref. 63 through the assertion that for $n \geq 0$ the improved estimate is

$$\frac{\pi}{2\delta}(2n-1) < \mathrm{Im}\,(r_n) < \frac{\pi}{2\delta}(2n+1). \tag{4.3.26}$$

A detailed proof of (4.3.26) requires a considerable amount of algebra. For the benefit of the reader who wishes to pursue the derivation of (4.3.26) we provide the following outline:

- Let

$$F(r;\lambda) \underline{\underline{\mathrm{def}}}\ \varepsilon\zeta_{+\varepsilon}(r) - \zeta_+(\lambda),$$

where λ is a real parameter, define the complex-valued function $F(r;\lambda)$. The function $F(r;\lambda)$ is proportional to the *denominator* of $\hat{g}(r;\lambda)$, and therefore zeroes of $F(r;\lambda)$ coincide with the poles of $\hat{g}(r;\lambda)$;

- Let

$$r = \sigma + i\tau,$$

where σ and τ are real numbers, define the complex number r. The partitioning of $F(r;\lambda)$ into its real and imaginary parts provides us with the following equations:

$$\mathrm{Re}\,\{F(r;\lambda)\} = \frac{\tau\sin(2\delta\tau) + \sigma\sinh(2\delta\sigma)}{4\sin^2(\delta\tau) + 4\sinh^2(\delta\sigma)} + \varepsilon\frac{\sigma}{2} - \zeta_+(\lambda) - \frac{1}{2\delta}$$

and

$$\mathrm{Im}\,\{F(r;\lambda)\} = \frac{\tau\sinh(2\delta\sigma) - \sigma\sin(2\delta\tau)}{4\sin^2(\delta\tau) + 4\sinh^2(\delta\sigma)} + \varepsilon\frac{\tau}{2};$$

- The function $F(r;\lambda)$ is analytic inside the contour C shown in Fig. 3, except for the *simple* pole at $\delta r = i\pi n$, where n denotes a positive integer;

- Evaluate $\mathrm{Re}\,\{F(r;\lambda)\}$ and $\mathrm{Im}\,\{F(r;\lambda)\}$ along each of the straight line segments shown in Fig. 3. For example, on C_1, where $2\delta r = 2\delta\sigma + i\pi(2n-1)$ and $-R < \sigma < R$, the reader will find that

$$\mathrm{Re}\,\{F(r;\lambda)\}\,|_{C_1} = \frac{\sigma\sinh(2\delta\sigma)}{4 + 4\sinh^2(\delta\sigma)} + \varepsilon\frac{\sigma}{2} - \frac{\lambda}{2} - \frac{\lambda}{2}\coth(\delta\lambda)$$

and

$$\mathrm{Im}\,\{F(r;\lambda)\}\,|_{C_1} = \frac{\pi(2n-1)\sinh(2\delta\sigma)}{8\delta + 8\delta\sinh^2(\delta\sigma)} + \frac{\varepsilon\pi}{4\delta}(2n-1).$$

- Use the limits

$$\lim_{\sigma \to \pm\infty} \frac{\sinh(2\delta\sigma)}{1 + \sinh^2(\delta\sigma)} = \pm 2$$

to evaluate $\arg\{F(r;\lambda)\}$ at each vertex of the rectangular contour C.

- The reader will find that

$$\Delta_C[\arg\{F(r;\lambda)\}] = 0,$$

where $\Delta_C[\arg\{F(r;\lambda)\}]$ denotes the change in $\arg\{F(r;\lambda)\}$ as we traverse the contour C in an anticlockwise direction. According to the *Argument Principle* [12, p. 259], $F(r;\lambda)$ has one simple zero inside C. Therefore, $\hat{g}(r;\lambda)$ has one simple pole inside the interval specified by (4.3.26). It is now an opportune moment to leave the proof of (4.3.26), and continue our analysis of the direct problem.

Although not essential to the remaining calculations in this chapter,

$$\frac{\mathrm{d}r_n}{\mathrm{d}\zeta_+} = \frac{r_n}{2\varepsilon\delta\left\{\zeta_+(\lambda) + (2\delta)^{-1}\right\}\left\{r_n - \varepsilon\zeta_+(\lambda)\right\}} \tag{4.3.27}$$

is the differential equation satisfied by r_n and r_n^* for $n \geq 0$. We will derive equation (4.3.27) because we wish to correct a specific statement made by Kodama et al. in Ref. 63. According to (4.3.25), r_n satisfies the algebraic equation

$$\zeta_+(\lambda) = \varepsilon\zeta_{+\,\varepsilon}\left(r_n\right), \tag{4.3.28}$$

where

$$\zeta_{+\,\varepsilon}\left(r_n\right) = \frac{r_n}{2} + \frac{\varepsilon r_n}{2}\coth\left(\delta r_n\right) - \frac{\varepsilon}{2\delta}. \tag{4.3.29}$$

Differentiate both sides of (4.3.28) with respect to r_n, and thereby arrive at the equation

$$\frac{\mathrm{d}\zeta_+}{\mathrm{d}r_n} = \frac{\varepsilon}{2}\left\{1 + \varepsilon\coth\left(\delta r_n\right) - \varepsilon\delta r_n \operatorname{cosech}^2\left(\delta r_n\right)\right\}. \tag{4.3.30}$$

The hyperbolic terms in the right hand side of (4.3.30) are connected by the familiar identity

$$\operatorname{cosech}^2\left(\delta r_n\right) = \coth^2\left(\delta r_n\right) - 1. \tag{4.3.31}$$

Substituting (4.3.31) into (4.3.30) will instigate a sequence of results that enables us to ultimately conclude that

$$\frac{\mathrm{d}\zeta_+}{\mathrm{d}r_n} = \frac{\varepsilon}{2}\left\{1 + \varepsilon\coth\left(\delta r_n\right)\right\}\left\{1 + \varepsilon\delta r_n\left[1 - \varepsilon\coth\left(\delta r_n\right)\right]\right\}. \tag{4.3.32}$$

Two identities that facilitate our retrieval of (4.3.27) from (4.3.32) are

$$1 + \varepsilon\coth\left(\delta r_n\right) = \frac{2\varepsilon}{r_n}\left\{\zeta_+(\lambda) + (2\delta)^{-1}\right\} \tag{4.3.33a}$$

and

$$1 + \varepsilon\delta r_n\left\{1 - \varepsilon\coth\left(\delta r_n\right)\right\} = 2\varepsilon\delta\left\{r_n - \varepsilon\zeta_+(\lambda)\right\}. \tag{4.3.33b}$$

Equations (4.3.33a) and (4.3.33b) are derived by appropriate rearrangements of (4.3.28) and (4.3.29). Substituting (4.3.33a) and (4.3.33b) into (4.3.32), and then effecting the reciprocal of both sides of the resultant equation lead us to the desired result: equation (4.3.27).

Kodama et al. [63] have used (4.3.27) to deduce that r_0, r_n and r_n^*, where $n \geq 1$, each possess a (fixed) *logarithmic branch* point at $\zeta_+(\lambda) = -(2\delta)^{-1}$. After completing a local analysis of the differential equation (4.3.27) we have determined that in the neighbourhood of $\zeta_+(\lambda) = -(2\delta)^{-1}$, the solution of (4.3.27) admits the representation

$$r_n \sim \tilde{a}\left[\zeta_+(\lambda) + (2\delta)^{-1}\right] - 2\varepsilon\delta\tilde{a}(\tilde{a} - \varepsilon)\left[\zeta_+(\lambda) + (2\delta)^{-1}\right]^2, \tag{4.3.34}$$

where $\tilde{a}$ is an arbitrary (nonzero) constant and $n \geq 0$. The series (4.3.34) clearly demonstrates that r_0, r_n and r_n^*, where $n \geq 1$, each possess a (fixed) *simple zero* at $\zeta_+(\lambda) = -(2\delta)^{-1}$.

Let us now return to complete the evaluation of $G^{+\varepsilon}(x, \eta; \lambda)$. The integral

$$G^{+\varepsilon}(x, \eta; \lambda) = \frac{1}{4\pi}\int_C \frac{\operatorname{cosech}(\delta r)e^{i(x-\eta)r + \varepsilon\delta r}}{\varepsilon\zeta_{+\varepsilon}(r) - \zeta_+(\lambda)}\mathrm{d}r, \tag{4.3.35}$$

which materializes after we substitute (4.3.23) into the integrand of (4.3.21).

Our objective is to use the analytic character (with respect to λ) of $G^{+\varepsilon}(x, \eta; \lambda)$ to deduce the analytic properties of the eigenfunctions $M(x;\lambda)$, $N(x;\lambda)$, $\overline{M}(x;\lambda)$ and $\overline{N}(x;\lambda)$. Evaluating the contour integral (4.3.35) by the *Residue Theorem* provides us with the most efficient means by which we can elucidate the analytic character of the eigenfunctions. Several technical issues that concern the integrand of (4.3.35) need to be addressed before we can implement the Residue Theorem. The first matter that merits our attention is the behaviour of the expression

$$\frac{\operatorname{cosech}(\delta r)e^{\varepsilon\delta r}}{\varepsilon\zeta_{+\varepsilon}(r) - \zeta_+(\lambda)}$$

as $r \to \pm\infty$. We will now show that

$$\lim_{r \to \pm\infty} \frac{\operatorname{cosech}(\delta r)e^{\varepsilon\delta r}}{\varepsilon\zeta_{+\varepsilon}(r) - \zeta_+(\lambda)} = 0. \tag{4.3.36}$$

Starting with

$$\lim_{x \to \pm\infty} \coth(x) = \pm 1$$

we can derive the following two limit laws:

$$\lim_{r \to \pm\infty} \frac{1}{\varepsilon\zeta_{+\varepsilon}(r) - \zeta_{+}(\lambda)} = -\frac{2\delta\theta(\mp\varepsilon)}{2\delta\zeta_{+}(\lambda) + 1};$$

$$\lim_{r \to \pm\infty} \operatorname{cosech}(\delta r) e^{\varepsilon\delta r} = \pm 2\theta(\pm\varepsilon),$$

where $\theta(\bullet)$ denotes the Heaviside step function. Elementary limit laws yield the equation

$$\lim_{r \to \pm\infty} \frac{\operatorname{cosech}(\delta r) e^{\varepsilon\delta r}}{\varepsilon\zeta_{+\varepsilon}(r) - \zeta_{+}(\lambda)} = \mp \frac{4\delta\theta(\pm\varepsilon)\theta(\mp\varepsilon)}{2\delta\zeta_{+}(\lambda) + 1},$$

which in turn implies (4.3.36) because $\theta(\pm\varepsilon)\theta(\mp\varepsilon) \equiv 0$. Equation (4.3.36) justifies why we can neglect contributions from the (horizontal) endpoints of any contour used in the evaluation of (4.3.35) by the Residue Theorem.

We now address the computation of the residues at the (isolated) singularities in the integrand of (4.3.35). The function

$$I(r) \underline{\underline{\text{def}}} \frac{1}{4\pi} \frac{\operatorname{cosech}(\delta r) e^{i(x-\eta)r + \varepsilon\delta r}}{\varepsilon\zeta_{+\varepsilon}(r) - \zeta_{+}(\lambda)}, \tag{4.3.37}$$

which constitutes the integrand of (4.3.35), is the instrument that will facilitate the computation of the necessary residues. Each of the points $r = r_{-1}$, $r = r_n$ and $r = r_n^*$, where n denotes a positive integer or zero, is a simple pole of $I(r)$. For the benefit of the reader we recapitulate the relevant notation: $r_{-1} = 0$; $r_0 = \varepsilon\lambda$, r_n and r_n^* (where n is a positive integer) are complex-valued solutions for (4.3.25). Our calculation of the residues distinguishes the following cases:

- The $\operatorname{cosech}(\delta r)$ term in (4.3.37) is the source of the simple pole at $r = r_{-1}$. Hence, the residue at $r = r_{-1}$ is

$$\operatorname{Res}(r = r_{-1}) = \lim_{r \to 0} rI(r),$$

and because

$$\zeta_{+\varepsilon}(0) \underline{\underline{\text{def}}} \lim_{r \to 0} \zeta_{+\varepsilon}(r) = 0$$

we find that

$$\operatorname{Res}(r = r_{-1}) = -\frac{1}{4\pi\zeta_{+}(\lambda)} \lim_{r \to 0} \frac{r}{\sinh(\delta r)}. \tag{4.3.38}$$

$r = r_{-1}$:

$$\operatorname{Res}\left(r = r_{-1}\right) = -\frac{1}{4\pi\delta\zeta_{+}(\lambda)}. \tag{4.3.39}$$

- Residues at $r = r_n$, where $n \geq 0$, emanate from the simple pole in (4.3.37) that is guaranteed by (4.3.25). The residue at $r = r_n$ in the function defined by (4.3.37) is

$$\operatorname{Res}\left(r = r_n\right) = \frac{e^{i(x-\eta)r_n + \varepsilon\delta r_n}}{4\pi\sinh\left(\delta r_n\right)} \lim_{r \to r_n} \frac{r - r_n}{\varepsilon\zeta_{+\varepsilon}(r) - \zeta_{+}(\lambda)}.$$

applied to the indefinite form in the formula that determines Res $\left(r = r_n\right)$ we find that

$$\operatorname{Res}\left(r = r_n\right) = \frac{e^{i(x-\eta)r_n + \varepsilon\delta r_n}}{4\pi\varepsilon\zeta'_{+\varepsilon}\left(r_n\right)\sinh\left(\delta r_n\right)}, \tag{4.3.40}$$

where

$$\zeta'_{+\varepsilon}\left(r_n\right) \;\underline{\underline{\mathrm{def}}}\; \left.\frac{\mathrm{d}\zeta_{+\varepsilon}(r)}{\mathrm{d}r}\right|_{r = r_n}.$$

A repetition (with minor modifications) of the procedure used to derive (4.3.27), but now applied to the function defined by equation (4.3.29) yields

$$\zeta'_{+\varepsilon}\left(r_n\right) = \frac{\delta\left[r_n - \varepsilon\zeta_{+}(\lambda)\right]e^{\varepsilon\delta r_n}}{\sinh\left(\delta r_n\right)}. \tag{4.3.41}$$

In view of (4.3.41), equation (4.3.40) simplifies to

$$\operatorname{Res}\left(r = r_n\right) = \frac{\varepsilon e^{i(x-\eta)r_n}}{4\pi\delta\left[r_n - \varepsilon\zeta_{+}(\lambda)\right]}. \tag{4.3.42}$$

A final remark about the residues of (4.3.37) is that (4.3.42) simplifies to

$$\operatorname{Res}\left(r = r_0\right) = \frac{\varepsilon e^{i\varepsilon(x-\eta)\lambda}}{4\pi\delta\zeta_{-}(\lambda)} \tag{4.3.43}$$

at $n = 0$. Henceforth we will use (4.3.42) as the value of Res $\left(r = r_n\right)$ for $n \geq 1$. Formally replacing r_n with r_n^* in (4.3.42) provides us with the equation

$$\operatorname{Res}\left(r = r_n^*\right) = \frac{\varepsilon e^{i(x-\eta)r_n^*}}{4\pi\delta\left[r_n^* - \varepsilon\zeta_{+}(\lambda)\right]}. \tag{4.3.44}$$

Now that certain technical issues concerning the integral in (4.3.35) have been resolved, we can resume our quest to evaluate this integral by the Residue Theorem. Define the function $J\left(x, \eta; \lambda\right)$ as follows:

$$J(x,\eta;\lambda)\ \underline{\underline{\text{def}}}\ \frac{1}{4\pi}\oint_{\Gamma}\frac{\operatorname{cosech}(\delta r)\,e^{i(x-\eta)r+\varepsilon\delta r}}{\varepsilon\zeta_{+\varepsilon}(r)-\zeta_{+}(\lambda)}\,\mathrm{d}r, \tag{4.3.45}$$

where Γ denotes some semicircular contour that contains C [the contour associated with(4.3.21)] as its base. The real quantities $r_{-1}=0$ and $r_0=\varepsilon\lambda$ are singularities of the integrand in (4.3.45). Singularities in the integrand of (4.3.45) that reside on the axis $\operatorname{Im}(r)=0$ necessitate that we evaluate (4.3.45) by distinguishing the following two choices for the contour Γ:

Choice 1: Γ as depicted in Fig. 4

Applying the Residue Theorem to (4.3.45) we find that

$$G^{+\varepsilon}(x,\eta;\lambda)=2\pi i\left\{\theta(x-\eta)\sum_{n=1}^{\infty}\operatorname{Res}(r=r_n)-\theta(\eta-x)\sum_{n=-1}^{\infty}\operatorname{Res}(r=r_n^{*})\right\}. \tag{4.3.46}$$

For this particular case the contour C is the straight line segment $-\infty+i0^{+}$ to $\infty+i0^{+}$ in the complex r-plane; we will use the notation C_{+} to designate this contour. Substituting (4.3.39), (4.3.42), (4.3.43) and (4.3.44) into their appropriate positions within equation (4.3.46) will yield us the infinite series

$$G^{+\varepsilon}_{-\varepsilon}(x,\eta;\lambda)=\frac{i}{2\delta}\theta(\eta-x)\left\{\frac{1}{\zeta_{+}(\lambda)}-\frac{e^{i\varepsilon(x-\eta)\lambda}}{\zeta_{-}(\lambda)}-\varepsilon\sum_{n=1}^{\infty}\frac{e^{i(x-\eta)r_n^{*}}}{r_n^{*}-\varepsilon\zeta_{+}(\lambda)}\right\}$$

$$+\frac{i\varepsilon}{2\delta}\theta(x-\eta)\sum_{n=1}^{\infty}\frac{e^{i(x-\eta)r_n}}{r_n-\varepsilon\zeta_{+}(\lambda)}. \tag{4.3.47}$$

The function defined by (4.3.47) is a $(-\varepsilon)$ function with respect to λ, that is when viewed as a function of λ it admits analytic continuation into the half-plane $\operatorname{sgn}(\operatorname{Im}(\lambda))=-\varepsilon$, where $\lambda=\lambda_R+i\lambda_I$ (with λ_R and λ_I real) defines the complex extension of λ. In order to x and λ, we have (and will continue

subscripts: *superscripts*

the analytic character with respect to x, whereas *subscripts* will indicate the analytic character with respect to λ.

Choice 2: Γ as depicted in Fig. 5

In this situation the contour C coincides with the straight line segment $-\infty - i0^{+}$ to $\infty - i0^{+}$ in the complex r-plane; we will use the notation C_{-} to refer to this particular contour. Repeating the procedure described in Case 1, but suitably modified to take account of the present version of the contour Γ, will deliver to us the equation

$$G^{+\varepsilon}_{+\varepsilon}(x,\eta;\lambda) = \frac{i}{2\delta}\theta(x-\eta)\left\{-\frac{1}{\zeta_{+}(\lambda)} + \frac{e^{i\varepsilon(x-\eta)\lambda}}{\zeta_{-}(\lambda)} + \varepsilon\sum_{n=1}^{\infty}\frac{e^{i(x-\eta)r_n}}{r_n - \varepsilon\zeta_{+}(\lambda)}\right\}$$
$$-\frac{i\varepsilon}{2\delta}\theta(\eta - x)\sum_{n=1}^{\infty}\frac{e^{i(x-\eta)r_n^{*}}}{r_n^{*} - \varepsilon\zeta_{+}(\lambda)}. \qquad (4.3.48)$$

The function defined by the infinite series (4.3.48) is a $(+\varepsilon)$ function in the variable λ. The

function.

According to equations (4.3.47) and (4.3.48), the jump, denoted as $\Delta G^{+\varepsilon}(x,\eta;\lambda)$, in the sectionally analytic function $G^{+\varepsilon}(x,\eta;\lambda)$ across the axis Im $(\lambda) = 0$ is

$$\Delta G^{+\varepsilon}(x,\eta;\lambda) \equiv G^{+\varepsilon}_{+\varepsilon}(x,\eta;\lambda) - G^{+\varepsilon}_{-\varepsilon}(x,\eta;\lambda) = \frac{i}{2\delta}\left(\frac{e^{i\varepsilon(x-\eta)\lambda}}{\zeta_{-}(\lambda)} - \frac{1}{\zeta_{+}(\lambda)}\right), \qquad (4.3.49)$$

where

$$G^{+\varepsilon}_{\pm\varepsilon}(x,\eta;\lambda) = \lim_{\mathrm{Im}\,(\lambda)\to 0^{\pm\varepsilon}} G^{+\varepsilon}(x,\eta;\lambda).$$

Armed with the experience gained from the derivation of equations (4.3.47) and (4.3.48) we can state that the boundary values $G^{+\varepsilon}_{\pm\varepsilon}(x,\eta;\lambda)$ admit the integral representations

$$G^{+\varepsilon}_{\pm\varepsilon}(x,\eta;\lambda) = \frac{1}{4\pi}\int_{C_{\mp}}\frac{\mathrm{cosech}\,(\delta r)e^{i(x-\eta)r+\varepsilon\delta r}}{\varepsilon\zeta_{+\varepsilon}(r) - \zeta_{+}(\lambda)}\mathrm{d}r, \qquad (4.3.50^{\pm})$$

where $C_{\mp}$ runs from $-\infty \mp i0^{+}$ to $\infty \mp i0^{+}$ in the complex r-plane.

Consistency with the boundary conditions (4.3.15) and (4.3.16) provides a means by which we can allocate $G^{+\varepsilon}_{\pm\varepsilon}(x,\eta;\lambda)$ to the kernel of the integral equation (4.3.19). Preservation of the boundary condition (4.3.15) requires the *left* Jost functions, $M^{+\varepsilon}(x;\lambda)$ and $\bar{M}^{+\varepsilon}(x;\lambda)$, to each

function is $G^{+\varepsilon}_{+\varepsilon}(x,\eta;\lambda)$, whereas conformity with (4.3.16) necessitates that the *right* Jost functions, $N^{+\varepsilon}(x;\lambda)$ and $\bar{N}^{+\varepsilon}(x;\lambda)$, each satisfy an appropriate version of (4.3.19) in $G^{+\varepsilon}_{-\varepsilon}(x,\eta;\lambda)$. Adhering to the prescription described for the

integral equations:

$$\begin{pmatrix} M^{+\varepsilon}(x;\lambda) \\ \bar{M}^{+\varepsilon}(x;\lambda) \end{pmatrix} = \begin{pmatrix} 1 \\ e^{i\varepsilon\lambda(x-i\varepsilon\delta)} \end{pmatrix} + \varepsilon\int_{-\infty}^{\infty} G^{+\varepsilon}_{+\varepsilon}(x,\eta;\lambda)U(\eta)\begin{pmatrix} M^{+\varepsilon}(\eta;\lambda) \\ \bar{M}^{+\varepsilon}(\eta;\lambda) \end{pmatrix}\mathrm{d}\eta; \tag{4.3.51}$$

$$\begin{pmatrix} N^{+\varepsilon}(x;\lambda) \\ \bar{N}^{+\varepsilon}(x;\lambda) \end{pmatrix} = \begin{pmatrix} e^{i\varepsilon\lambda(x-i\varepsilon\delta)} \\ 1 \end{pmatrix} + \varepsilon\int_{-\infty}^{\infty} G^{+\varepsilon}_{-\varepsilon}(x,\eta;\lambda)U(\eta)\begin{pmatrix} N^{+\varepsilon}(\eta;\lambda) \\ \bar{N}^{+\varepsilon}(\eta;\lambda) \end{pmatrix}\mathrm{d}\eta. \tag{4.3.52}$$

Equations (4.3.51) and (4.3.52) are examples of (inhomogeneous) Volterra integral equations of the second-kind. Our purpose for deriving (4.3.51) and (4.3.52) was to use these equations to ascertain the analytic character (with respect to λ) of the eigenfunctions $M^{+\varepsilon}(x;\lambda)$, $\bar{M}^{+\varepsilon}(x;\lambda)$, $N^{+\varepsilon}(x;\lambda)$ and $\bar{N}^{+\varepsilon}(x;\lambda)$

us to extract the following information from (4.3.51) and (4.3.52):

- $M^{+\varepsilon}(x;\lambda)$ is a $(+\varepsilon)$ function in λ;
- $\bar{N}^{+\varepsilon}(x;\lambda)$ is a $(-\varepsilon)$ function in λ.

$\bar{M}^{+\varepsilon}(x;\lambda)$ and $N^{+\varepsilon}(x;\lambda)$ prohibits us from analytically continuing these eigenfunctions off the Im (λ) = 0 axis. Henceforth, we adopt the notation $M^{+\varepsilon}_{+\varepsilon}(x;\lambda)$ for the eigenfunction $M^{+\varepsilon}(x;\lambda)$, and $\bar{N}^{+\varepsilon}_{-\varepsilon}(x;\lambda)$ now denotes the eigenfunction $\bar{N}^{+\varepsilon}(x;\lambda)$. Subscripts will not be attached to $\bar{M}^{+\varepsilon}(x;\lambda)$ and $N^{+\varepsilon}(x;\lambda)$ because these eigenfunctions *cannot* be analytically continued off the Im (λ) = 0 axis.

Superficially, it would appear that the Jost functions are disparate entities. In fact, the eigenfunctions $M^{+\varepsilon}_{+\varepsilon}(x;\lambda)$, $N^{+\varepsilon}(x;\lambda)$ and $\bar{N}^{+\varepsilon}_{-\varepsilon}(x;\lambda)$ are connected by an identity, which we now proceed to derive. Motivated by the form of equation (3.2.22), we make the

$$M^{+\varepsilon}_{+\varepsilon}(x;\lambda) = A(\lambda)\bar{N}^{+\varepsilon}_{-\varepsilon}(x;\lambda) + B(\lambda)N^{+\varepsilon}(x;\lambda) \tag{4.3.53}$$

for our *scattering equation*, where $A(\lambda)$ and $B(\lambda)$ must be added to the scattering data.

Formulae for $A(\lambda)$ and $B(\lambda)$ are required because such formulae allow us to append the analytic character of $A(\lambda)$ and $B(\lambda)$ to the existing database of analytic properties. Consider the equation

$$G^{+\varepsilon}_{+\varepsilon}(x,\eta;\lambda) = G^{+\varepsilon}_{-\varepsilon}(x,\eta;\lambda) + \frac{i}{2\delta}\left(\frac{e^{i\varepsilon(x-\eta)\lambda}}{\zeta_-(\lambda)} - \frac{1}{\zeta_+(\lambda)}\right), \tag{4.3.54}$$

which is formed after a minor rearrangement of (4.3.49). Define the function $\tilde{M}(\eta;\lambda)$ through the equation

$$\tilde{M}(\eta;\lambda) \ \underline{\underline{\text{def}}}\ U(\eta)M^{+\varepsilon}_{+\varepsilon}(\eta;\lambda).$$

The equation

$$M^{+\varepsilon}_{+\varepsilon}(x;\lambda) = 1 - \frac{i\varepsilon}{2\delta\zeta_+(\lambda)}\int_{-\infty}^{\infty}\tilde{M}(\eta;\lambda)\,\mathrm{d}\eta + \varepsilon\int_{-\infty}^{\infty}G^{+\varepsilon}_{-\varepsilon}(x,\eta;\lambda)\tilde{M}(\eta;\lambda)\,\mathrm{d}\eta$$

$$+\frac{i\varepsilon e^{i\varepsilon\lambda x}}{2\delta\zeta_-(\lambda)}\int_{-\infty}^{\infty}\tilde{M}(\eta;\lambda)e^{-i\varepsilon\eta\lambda}\,\mathrm{d}\eta \tag{4.3.55}$$

results when we substitute (4.3.54) into the integrand of (4.3.51), in particular the integral equation for $M^{+\varepsilon}_{+\varepsilon}(x;\lambda)$, and then integrate term-by-term. Using the equations labelled

$$M^{+\varepsilon}_{+\varepsilon}(x;\lambda) = A(\lambda) + B(\lambda)e^{i\varepsilon\lambda x+\delta\lambda}$$

$$+\varepsilon\int_{-\infty}^{\infty}G^{+\varepsilon}_{-\varepsilon}(x,\eta;\lambda)U(\eta)\left\{A(\lambda)\bar{N}^{+\varepsilon}_{-\varepsilon}(\eta;\lambda) + B(\lambda)N^{+\varepsilon}(\eta;\lambda)\right\}\mathrm{d}\eta,$$

which simplifies to

$$M^{+\varepsilon}_{+\varepsilon}(x;\lambda) = A(\lambda) + B(\lambda)e^{i\varepsilon\lambda x+\delta\lambda} + \varepsilon\int_{-\infty}^{\infty}G^{+\varepsilon}_{-\varepsilon}(x,\eta;\lambda)\tilde{M}(\eta;\lambda)\,\mathrm{d}\eta \tag{4.3.56}$$

because of (4.3.53). Equating (4.3.55) and (4.3.56), and then matching the coefficients of $e^{i\varepsilon\lambda x}$ in the resultant identity we obtain the required formulae:

$$A(\lambda) = 1 - \frac{i\varepsilon}{2\delta\zeta_+(\lambda)}\int_{-\infty}^{\infty}U(\eta)M^{+\varepsilon}_{+\varepsilon}(\eta;\lambda)\,\mathrm{d}\eta; \tag{4.3.57}$$

$$B(\lambda) = \frac{i\varepsilon}{2\delta\zeta_-(\lambda)}\int_{-\infty}^{\infty}U(\eta)M^{+\varepsilon}_{+\varepsilon}(\eta;\lambda)e^{-i\varepsilon\lambda(\eta-i\varepsilon\delta)}\,\mathrm{d}\eta. \tag{4.3.58}$$

From (4.3.57) we deduce that $A(\lambda)$ is a $(+\varepsilon)$ function in the variable λ because $A(\lambda)$ inherits the analytic character of $M^{+\varepsilon}_{+\varepsilon}(x;\lambda)$. Analytic continuation of $B(\lambda)$ is *not* possible because (4.3.58) contains the term $e^{-i\varepsilon\lambda\eta}$ in its integrand, where η covers the interval $-\infty < \eta < \infty$.

For future reference, we will now show that $M^{+\varepsilon}_{+\varepsilon}(x;\lambda)$ and $A(\lambda)$ each possess the *canonical normalization* as $\lambda \rightarrow \infty$ (along the real λ-axis). The large-λ asymptotics of $M^{+\varepsilon}_{+\varepsilon}(x;\lambda)$ and $A(\lambda)$ are contingent on the results

$$\frac{1}{\zeta_{+}(\lambda)} = \frac{1}{\lambda} + \frac{1}{2\delta\lambda^2} + \mathrm{O}\left(\frac{1}{\lambda^3}\right) \text{ as } \lambda \rightarrow \infty \tag{4.3.59}$$

and

$$\lim_{\lambda \rightarrow \infty} \frac{1}{\zeta_{-}(\lambda)} = 2\delta, \tag{4.3.60}$$

each of which can be derived by an appropriate use of (4.3.14). Substituting (4.3.59) and (4.3.60) into (4.3.48) we find that

$$G^{+\varepsilon}_{+\varepsilon}(x,\eta;\lambda) \sim i\theta(x-\eta)e^{i\varepsilon(x-\eta)\lambda} + \mathrm{O}\left(\frac{1}{\lambda}\right) \text{ as } \lambda \rightarrow \infty. \tag{4.3.61}$$

Equations (4.3.51) and (4.3.61) when combined with a use of the Riemann-Lebesgue lemma lead us to the conclusion

$$M^{+\varepsilon}_{+\varepsilon}(x;\lambda) = 1 + \mathrm{O}\left(\frac{1}{\lambda}\right) \text{ as } \lambda \rightarrow \infty. \tag{4.3.62}$$

A straightforward use of (4.3.59) and (4.3.62) in (4.3.57) yields the equation

$$A(\lambda) = 1 - \frac{i\varepsilon}{2\delta\lambda}\int_{-\infty}^{\infty} U(\eta)\,\mathrm{d}\eta + \mathrm{O}\left(\frac{1}{\lambda^2}\right) \text{ as } \lambda \rightarrow \infty. \tag{4.3.63}$$

Equations (4.3.62) and (4.3.63) adequately demonstrate the correctness of the claim made about $M^{+\varepsilon}_{+\varepsilon}(x;\lambda)$ and $A(\lambda)$. An inspection of (4.3.58) will show the reader that

$$B(\lambda) = \mathrm{o}(1) \text{ as } \lambda \rightarrow \infty. \tag{4.3.64}$$

Our calculations in this section have thus far dealt exclusively with the continuous spectrum for the linear eigenvalue problem (4.2.12). Discrete (complex) eigenvalues for the spectral problem (4.2.12) are possible. We will use the notation λ_j, where $j = 1, 2, \ldots, q$

and sgn $\left(\mathrm{Im}\,(\lambda_j)\right) = \varepsilon$, to denote the discrete eigenvalues of (4.2.12). Discrete eigenvalues of (4.2.12) induce exponentially localized eigenfunctions that are characterized by the boundary conditions

$$m^{+\varepsilon}\left(x;\lambda_j\right) \to 0 \text{ as } x \to -\infty \tag{4.3.65}$$

and

$$n^{+\varepsilon}\left(x;\lambda_j\right) \to 0 \text{ as } x \to \infty. \tag{4.3.66}$$

The equation

$$A\left(\lambda_j\right) = 0, \tag{4.3.67}$$

where $A(\lambda)$ is defined by (4.3.57), codifies the determination of all the discrete eigenvalues. From (4.3.67) we see that (4.3.53) simplifies to

$$M^{+\varepsilon}_{+\varepsilon}\left(x;\lambda_j\right) = B\left(\lambda_j\right)N^{+\varepsilon}\left(x;\lambda_j\right), \tag{4.3.68}$$

whenever $\lambda = \lambda_j$.

The transformation ($4.3.10^{+}$) allows us the flexibility of working either with the original (lower-case) Jost functions or the modified (upper-case) Jost functions. Hence, equations (4.3.53) and (4.3.68) can also be expressed as

$$m^{+\varepsilon}(x;\lambda) = A(\lambda)\bar{n}^{+\varepsilon}(x;\lambda) + B(\lambda)n^{+\varepsilon}(x;\lambda) \tag{4.3.69}$$

and

$$m^{+\varepsilon}\left(x;\lambda_j\right) = B\left(\lambda_j\right)n^{+\varepsilon}\left(x;\lambda_j\right), \tag{4.3.70}$$

respectively; we have elected not to show the analytic character (with respect to λ) of the Jost functions in (4.3.69) and (4.3.70) because such information is not required for our intended uses of these equations. The boundary condition

$$m^{+\varepsilon}\left(x;\lambda_j\right) \to 0 \text{ as } x \to \infty \tag{4.3.71}$$

is an immediate consequence of (4.3.66) and (4.3.70).

Kodama et al. [63] have shown that all the discrete eigenvalues associated with the IST for the real-valued ILW equation reside on the imaginary axis of the complex λ-plane; λ λ in which Re $(\lambda) = \lambda_R$ and Im $(\lambda) = \lambda_I$. In Section 4.1 of this thesis we mentioned that the discrete eigenvalues in the IST for the complex-valued ILW equation are *not* restricted to the imaginary axis in the complex λ-plane. It is now appropriate that we prove the claim made about the increase in the mobility of the discrete eigenvalues, namely that discrete eigenvalues (in the IST for the complex-valued ILW equation) with $\mathrm{Re}\left(\lambda_j\right) \neq 0$ are indeed possible. Our proof begins with the differential-difference equation satisfied by the discrete eigenfunction $m^{+\varepsilon}\left(x;\lambda_j\right)$:

$$m_x^{+\varepsilon}\left(x;\lambda_j\right) = i\varepsilon\left\{U - \mu\left(\lambda_j\right)\right\}m^{+\varepsilon}\left(x;\lambda_j\right) - i\varepsilon\nu\left(\lambda_j\right)m^{-\varepsilon}\left(x;\lambda_j\right) \tag{4.3.72}$$

where, according to equations (4.3.6) and (4.3.7),

$$\mu\left(\lambda_j\right) = -\tfrac{1}{2}\lambda_j \coth\left(\delta\lambda_j\right) \tag{4.3.73}$$

and

$$\nu\left(\lambda_j\right) = \tfrac{1}{2}\lambda_j \,\mathrm{cosech}\left(\delta\lambda_j\right). \tag{4.3.74}$$

The boundary conditions (4.3.65) and (4.3.71) supplement equation (4.3.72). The equation

$$\begin{aligned}\left\{m_x^{+\varepsilon}\left(x;\lambda_j\right)\right\}^* = -\,i\varepsilon\left\{U^* - \mu^*\left(\lambda_j\right)\right\}\left\{m^{+\varepsilon}\left(x;\lambda_j\right)\right\}^* \\ + i\varepsilon\nu^*\left(\lambda_j\right)\left\{m^{-\varepsilon}\left(\ ;\lambda\ \right)\right\}^*,\end{aligned} \tag{4.3.75}$$

where

$$U^* = \frac{\ }{2}\left\{\ \ (V\ ) + \ \ \left(e^V\ \ 1\right) - \ \ V_x\right\}$$

is the complex conjugate of (4.3.72). We remind the reader that V denotes a (real-valued) solution of the MILW equation (4.1.1).

Equations (4.3.72) and (4.3.75) can be used to derive the *nonlinear* equation

$$\begin{aligned}\frac{\partial}{\partial x}\left\{\left|m^{+\varepsilon}\left(x;\lambda_j\right)\right|^2\right\} = -\,\varepsilon V_x\left|m^{+\varepsilon}\left(x;\lambda_j\right)\right|^2 + i\varepsilon\left\{\mu^*\left(\lambda_j\right) - \mu\left(\lambda_j\right)\right\}\left|m^{+\varepsilon}\left(x;\lambda_j\right)\right|^2 \\ + i\varepsilon\nu^*\left(\lambda_j\right)m^{+\varepsilon}\left(x;\lambda_j\right)\left\{m^{-\varepsilon}\left(x;\lambda_j\right)\right\}^* \\ - i\varepsilon\nu\left(\lambda_j\right)m^{-}\ \left(x;\ \ _j\right)\left\{m\ \ ^{\varepsilon}\left(x\ \lambda_j\right)\right\}\ .\end{aligned} \tag{4.3.76}$$

The reader who wishes to derive equation (4.3.76) should multiply (4.3.72) by $\left\{m^{+\varepsilon}\left(x;\lambda_j\right)\right\}^*$, then multiply (4.3.75) by $m^{+\varepsilon}\left(x;\lambda_j\right)$, and (finally) subtract one of the resultant nonlinear equations from the other nonlinear equation.

The boundary conditions (4.1.2), (4.3.65) and (4.3.71) create an environment in which we can integrate *separately* each term in (4.3.76) over the interval $-\infty < x \quad \infty$. Integrating both sides of (4.3.76) over the interval $-\infty < x < \infty$, and then taking account of the boundary conditions mentioned, we obtain the following equation:

$$i\int_{-\infty}^{\infty} V_x \left| m^{+\varepsilon}\left(x;\lambda_j\right) \right|^2 \mathrm{d}x = \nu\left(\lambda_j\right)\int_{-\infty}^{\infty} m^{-\varepsilon}\left(x;\lambda_j\right)\left\{m^{+\varepsilon}\left(x;\lambda_j\right)\right\}^* \mathrm{d}x$$

$$-\nu^*\left(\lambda_j\right)\int_{-\infty}^{\infty} \quad {}^{+\varepsilon}\left(x;\lambda_j\right)\left\{m^{-\varepsilon}\left(x;\lambda_j\right)\right\}^* \mathrm{d}x$$

$$\left\{\mu^*\left(\lambda_j\right) - \mu\left(\lambda_j\right)\right\}\left\| m^{+\varepsilon}\left(x;\lambda_j\right) \right\|^2, \qquad (4.3.77)$$

where $\|\bullet\|$ denotes the $L_2(\Re)$ norm, that is

$$\left\| m^{+\varepsilon}\left(x;\lambda_j\right) \right\| \underline{\underline{\mathrm{def}}} \left(\int_{-\infty}^{\infty} \left| m^{+\varepsilon}\left(x;\lambda_j\right) \right|^2 \mathrm{d}x\right)^{1/2}.$$

It is appropriate that we pause to compare and contrast (4.3.77) with the corresponding equation in the IST for the real-valued ILW equation. The relevant analogue to (4.3.77) is equation (D1) in Ref. 63; the reader who consults Ref. 63 will find that the pertinent equation is in differential form. Equation (4.3.77) and equation (D1) in Ref. 63 have several features in common, in particular the combination of eigenfunctions in the right hand side of (4.3.77) appears within equation (D1). The interesting point is that the solution of the relevant evolution equation is absent from equation (D1) of Ref. 63, whereas in (4.3.77) the solution of the MILW equation appears *explicitly*. The current appearance of V is attributed to the fact that the way we arrive at (4.3.77) is sensitive to the inequality $U^* \neq U$, whereas in Ref. 63 U denotes a real-valued solution of the ILW equation, and therefore no distinction between U and U^* is possible.

Kodama et al. [63] simplify their versions of the two integrals that appear in the right hand side of (4.3.77) by making adroit use of the formulae $\left(m^{\pm}\right)^* = -\left(m^*\right)^{\mp}$; the reader should be aware that these formulae are incorrectly printed in Ref. 63. We will now show that

$$\int_{-\infty}^{\infty} m^{\pm\varepsilon}\left(x;\lambda_j\right)\left\{m^{\mp\varepsilon}\left(x;\lambda_j\right)\right\}^* \mathrm{d}x = \left\| m\left(x;\lambda_j\right) \right\|^2 \qquad (4.3.78)$$

by using a procedure that is *not* contingent on the formulae $\left(m^{\pm}\right)^{*} = -\left(m^{*}\right)^{\mp}$. Our method relies on the contour integral

$$\oint_{C} m^{-\varepsilon}\left(z;\lambda_{j}\right)\left\{m^{+\varepsilon}\left(z^{*};\lambda_{j}\right)\right\}^{*}\mathrm{d}z, \tag{4.3.79}$$

where $z = x + iy$ denotes the complex extension of x, and C is the rectangular contour whose vertices are located at the points $\left(-R_{1},0\right)$, $\left(R_{2},0\right)$, $\left(-R_{1},-\varepsilon\delta\right)$ and $\left(R_{2},-\varepsilon\delta\right)$ in the complex z-plane; R_{1} and R_{2} denote two (large) positive real numbers. We note that the integrand of (4.3.79) is analytic in the horizontal strip between $\mathrm{Im}\left(z\right) = 0$ and $\mathrm{Im}\left(z\right) = -2\varepsilon\delta$ in the complex z
that

$$\oint_{C} m^{-\varepsilon}\left(z;\lambda_{j}\right)\left\{m^{+\varepsilon}\left(z^{*};\lambda_{j}\right)\right\}^{*}\mathrm{d}z = 0 \tag{4.3.80}$$

because (by construction) the integrand of (4.3.79) is analytic inside the region bounded by the contour C. If we traverse C in the anticlockwise direction, then

$$\int_{-R_{1}}^{R_{2}}\left\{m^{-\varepsilon}\left(x;\lambda_{j}\right)\left\{m^{+\varepsilon}\left(x;\lambda_{j}\right)\right\}^{*} - m^{-\varepsilon}\left(x - i\varepsilon\delta;\lambda_{j}\right)\left\{m^{+\varepsilon}\left(x + i\varepsilon\delta;\lambda_{j}\right)\right\}^{*}\right\}\mathrm{d}x \tag{4.3.81}$$

is the contribution to the integral (4.3.79) from the *horizontal* segments of C. In the situation $R_{1} \to \infty$ and $R_{2} \to \infty$ (where R_{1} is independent of R_{2}) the boundary conditions (4.3.65) and (4.3.71) allow us to ignore the contributions to (4.3.79) from the *vertical* segments of C. Concatenating the information we have extracted from (4.3.79), and then substituting the limiting form of (4.3.81) into (4.3.80) we obtain

$$\int_{-\infty}^{\infty} m^{-\varepsilon}\left(x;\lambda_{j}\right)\left\{m^{+\varepsilon}\left(x;\lambda_{j}\right)\right\}^{*}\mathrm{d}x = \int_{-\infty}^{\infty} m^{-\varepsilon}\left(x - i\varepsilon\delta;\lambda_{j}\right)\left\{m^{+\varepsilon}\left(x + i\varepsilon\delta;\lambda_{j}\right)\right\}^{*}\mathrm{d}x,$$

which, by virtue of (4.2.18$^{\pm}$), is equivalent to the equation

$$\int_{-\infty}^{\infty} m^{-\varepsilon}\left(x;\lambda_{j}\right)\left\{m^{+\varepsilon}\left(x;\lambda_{j}\right)\right\}^{*}\mathrm{d}x = \left\| m\left(x;\lambda_{j}\right)\right\|^{2}. \tag{4.3.82}$$

An application in (4.3.82) of the simple involution $-\varepsilon \to +\varepsilon$ will establish the validity of the remaining component of (4.3.78). The final result

$$i\int_{-\infty}^{\infty} V_x \left| m^{+\varepsilon}(x;\lambda_j) \right|^2 \mathrm{d}x = \left\{ \nu(\lambda_j) - \nu^*(\lambda_j) \right\} \left\| m(x;\lambda_j) \right\|^2$$

$$- \left\{ \mu^*(\lambda_j) - \mu(\lambda_j) \right\} \left\| m^{+\varepsilon}(x;\lambda_j) \right\|^2 \qquad (4.3.83)$$

results when we make suitable use of (4.3.78) in equation (4.3.77).

Now let us assume that each discrete eigenvalue is pure imaginary, that is $\lambda_j = i\varepsilon\hat{\lambda}_j$ (where $\hat{\lambda}_j$ denotes an appropriate positive real number). If we evaluate at $\lambda_j = i\varepsilon\hat{\lambda}_j$ the functions defined by equations (4.3.73) and (4.3.74), then we obtain

$$\mu(\lambda_j) = -\tfrac{1}{2}\hat{\lambda}_j \cot\left(\delta\hat{\lambda}_j\right) \qquad (4.3.84)$$

and

$$\nu(\lambda_j) = \tfrac{1}{2}\hat{\lambda}_j \operatorname{cosec}\left(\delta\hat{\lambda}_j\right), \qquad (4.3.85)$$

respectively. Equations (4.3.84) and (4.3.85) prove that $\mu(\lambda_j)$ and $\nu(\lambda_j)$ are real-valued whenever λ_j is pure imaginary. The inference we can procure from (4.3.84) and (4.3.85) is that the right hand side of (4.3.83) vanishes because of the equalities $\mu(\lambda_j) = \mu^*(\lambda_j)$ and $\nu(\lambda_j) = \nu^*(\lambda_j)$. We now have an incompatibility because (generically) the two functions V_x and $\left| m^{+\varepsilon}(x;\lambda_j) \right|^2$ are *not* orthogonal in the interval $-\infty < x < \infty$. In the IST for the real-valued ILW equation [63] the

(4.3.83) leads us to conclude that discrete eigenvalues with $\operatorname{Re}(\lambda_j) \neq 0$ are possible in the IST for the complex-valued ILW equation; the reader may recall that this phenomenon is also present in the IST for the complex-valued KdV equation.

Each discrete eigenvalue is a zero of the function defined by equation (4.3.57). The *order* of the zero in $A(\lambda)$ at $\lambda = \lambda_j$ now requires investigation. Equation (4.3.72), but with λ_j replaced by λ, will once again be the location from which our investigation will commence. Differentiating the generalized version of (4.3.72) with respect to λ, and then evaluating the ensuing equation at $\lambda = \lambda_j$ provide us with the equation

$$m_{\lambda x}^{+\varepsilon}(x;\lambda_j) = i\varepsilon\left\{ U - \mu(\lambda_j) \right\} m_{\lambda}^{+\varepsilon}(x;\lambda_j) - i\varepsilon\dot{\mu}(\lambda_j) m^{+\varepsilon}(x;\lambda_j)$$

$$- i\varepsilon\dot{\nu}(\lambda_j) m^{-\varepsilon}(x;\lambda_j) - i\varepsilon\nu(\lambda_j) m_{\lambda}^{-\varepsilon}(x;\lambda_j), \qquad (4.3.86)$$

total derivative with respect to λ, for example

$$\dot{\mu}\left(\lambda_j\right) \underline{\underline{\text{def}}} \left.\frac{\mathrm{d}\mu(\lambda)}{\mathrm{d}\lambda}\right|_{\lambda=\lambda_j} .$$

Equation (4.3.86) can be used to derive the prolonged (and obstreperous) equation

$$\begin{aligned}\frac{\partial}{\partial x}\left\{m_\lambda^{+\,\varepsilon}\left(x;\lambda_j\right)\left\{m^{+\,\varepsilon}\left(x;\lambda_j\right)\right\}^*\right\} &= i\varepsilon\nu^*\left(\lambda_j\right)m_\lambda^{+\,\varepsilon}\left(x;\lambda_j\right)\left\{m^{-\,\varepsilon}\left(x;\lambda_j\right)\right\}^* \\ &\quad + i\varepsilon m_\lambda^{+\,\varepsilon}\left(x;\lambda_j\right)\left\{m^{+\,\varepsilon}\left(x;\lambda_j\right)\right\}^*\left(U-U^*\right) \\ &\quad + i\varepsilon m_\lambda^{+\,\varepsilon}\left(x;\lambda_j\right)\left\{m^{+\,\varepsilon}\left(x;\lambda_j\right)\right\}^*\left[\mu^*\left(\lambda_j\right)-\mu\left(\lambda_j\right)\right] \\ &\quad - i\varepsilon\dot{\mu}\left(\lambda_j\right)\left|m^{+\,\varepsilon}\left(x;\lambda_j\right)\right|^2 \\ &\quad - i\varepsilon\dot{\nu}\left(\lambda_j\right)m^{-\,\varepsilon}\left(x;\lambda_j\right)\left\{m^{+\,\varepsilon}\left(x;\lambda_j\right)\right\}^* \\ &\quad - i\varepsilon\nu\left(\lambda_j\right)m_\lambda^{-\,\varepsilon}\left(x;\lambda_j\right)\left\{m^{+\,\varepsilon}\left(x;\lambda_j\right)\right\}^* . \qquad (4.3.87)\end{aligned}$$

The reader who wishes to derive equation (4.3.87) is invited to complete the following procedure:

1) multiply both sides of (4.3.86) by $\left\{m^{+\,\varepsilon}\left(x;\lambda_j\right)\right\}^*$;

2) multiply both sides of (4.3.75) by $m_\lambda^{+\,\varepsilon}\left(x;\lambda_j\right)$;

3) combine the two (nonlinear) equations from Steps 1 and 2.

We will choose to make conclusions about the order of the zero in $A(\lambda)$ at $\lambda=\lambda_j$ based on information extracted from (4.3.87) in the case U is a *real-valued* solution of the ILW equation. The rationale that supports our decision is as follows:

1) the analysis of (4.3.87) whenever U is a real-valued solution of the ILW equation is easier than the analysis required whenever U is a complex-valued solution of the ILW equation, but the analysis for the real-valued case retains the essential features of the complex-valued case;

2) we wish to correct a certain error made in Appendix D of Ref. 63.

Let us reiterate that the discrete eigenvalues allied to the IST [63] for the real-valued ILW equation are pure imaginary. If U denotes a *real-valued* solution of the ILW equation, then (4.3.87) simplifies in some degree to the equation

$$\begin{aligned}\frac{\partial}{\partial x}\left\{m_{\lambda}^{+\varepsilon}\left(x;\lambda_j\right)\left\{m^{+\varepsilon}\left(x;\lambda_j\right)\right\}^*\right\} &= i\varepsilon\nu\left(\lambda_j\right)m_{\lambda}^{+\varepsilon}\left(x;\lambda_j\right)\left\{m^{-\varepsilon}\left(x;\lambda_j\right)\right\}^* \\ &\quad - i\varepsilon\nu\left(\lambda_j\right)m_{\lambda}^{-\varepsilon}\left(x;\lambda_j\right)\left\{m^{+\varepsilon}\left(x;\lambda_j\right)\right\}^* \\ &\quad - i\varepsilon\dot{\nu}\left(\lambda_j\right)m^{-\varepsilon}\left(x;\lambda_j\right)\left\{m^{+\varepsilon}\left(x;\lambda_j\right)\right\}^* \\ &\quad - i\varepsilon\dot{\mu}\left(\lambda_j\right)\left|m^{+\varepsilon}\left(x;\lambda_j\right)\right|^2\end{aligned}$$

because $U = U^*$ (by construction) and equations (4.3.84) and (4.3.85) inform us that $\mu\left(\lambda_j\right)$ and $\nu\left(\lambda_j\right)$ are real-valued quantities whenever λ_j is pure imaginary. Integrating both sides of the above equation over the interval $-\infty < x < \infty$, and then using (4.3.78) to evaluate the integral that multiplies $\dot{\nu}\left(\lambda_j\right)$ in the ensuing equation we obtain

$$\begin{aligned}\dot{A}\left(\lambda_j\right)B^*\left(\lambda_j\right)e^{-i\varepsilon\delta\hat{\lambda}_j} &= i\varepsilon\nu\left(\lambda_j\right)\int_{-\infty}^{\infty}\Theta\left(x;\lambda_j\right)\mathrm{d}x \\ &\quad - i\varepsilon\left\{\dot{\mu}\left(\lambda_j\right)\left\|m^{+\varepsilon}\left(x;\lambda_j\right)\right\|^2 + \dot{\nu}\left(\lambda_j\right)\left\|m\left(x;\lambda_j\right)\right\|^2\right\}. \qquad (4.3.88)\end{aligned}$$

Equation (4.3.88) requires supplementation with the following explanatory remarks:

- $\Theta\left(x;\lambda_j\right) \underline{\underline{\mathrm{def}}}\ m_{\lambda}^{+\varepsilon}\left(x;\lambda_j\right)\left\{m^{-\varepsilon}\left(x;\lambda_j\right)\right\}^* - m_{\lambda}^{-\varepsilon}\left(x;\lambda_j\right)\left\{m^{+\varepsilon}\left(x;\lambda_j\right)\right\}^*$;
- $\lambda_j = i\varepsilon\hat{\lambda}_j$, where [63] $0 < \delta\hat{\lambda}_j < \pi$;
- $\lim_{x\to\pm\infty} m_{\lambda}^{+\varepsilon}\left(x;\lambda_j\right)\left\{m^{+\varepsilon}\left(x;\lambda_j\right)\right\}^*$ have been determined from the asymptotic data (4.3.8) and (4.3.9), and the identities embodied in (4.3.69) and (4.3.70).

The (explicit) nonlocal term in equation (4.3.88) can be evaluated by recourse to a procedure that is fundamentally similar to the method employed to derive (4.3.78). Motivated by the form of (4.3.79), we now endeavour to evaluate

$$\int_{-\infty}^{\infty}\left\{ m_{\lambda}^{+\varepsilon}\left(x;\lambda_j\right)\left\{m^{-\varepsilon}\left(x;\lambda_j\right)\right\}^{*} - m_{\lambda}^{-\varepsilon}\left(x;\lambda_j\right)\left\{m^{+\varepsilon}\left(x;\lambda_j\right)\right\}^{*}\right\}\mathrm{d}x$$

by using the contour integral

$$\oint_{\Gamma} m_{\lambda}^{+\varepsilon}\left(z;\lambda_j\right)\left\{m^{-\varepsilon}\left(z^{*};\lambda_j\right)\right\}^{*}\mathrm{d}z,$$

where Γ denotes the rectangular contour whose vertices are located at the points $\left(-R_1, 0\right)$, $\left(R_2, 0\right)$, $\left(R_2, \varepsilon\delta\right)$ and $\left(-R_1, \varepsilon\delta\right)$ in the complex z-plane; R_1 and R_2 denote two large

$$\int_{-\infty}^{\infty}\Theta\left(x;\lambda_j\right)\mathrm{d}x = -2i\varepsilon\delta\dot{A}\left(\lambda_j\right)B^{*}\left(\lambda_j\right). \tag{4.3.89}$$

Substituting (4.3.89) into (4.3.88), and then performing a few rearrangements in the ensuing equation we find that

$$\left\{2\delta\nu\left(\lambda_j\right) - e^{-i\varepsilon\delta\hat{\lambda}_j}\right\}\dot{A}\left(\lambda_j\right)B^{*}\left(\lambda_j\right) = i\varepsilon\dot{\mu}\left(\lambda_j\right)\left\| m^{+\varepsilon}\left(x;\lambda_j\right)\right\|^{2} + i\varepsilon\dot{\nu}\left(\lambda_j\right)\left\| m\left(x;\lambda_j\right)\right\|^{2}. \tag{4.3.90}$$

The most convenient form (for our intended use) of equation (4.3.90) is

$$\left\{2\delta\nu\left(\lambda_j\right) - e^{-i\varepsilon\delta\hat{\lambda}_j}\right\}\frac{\dot{A}\left(\lambda_j\right)}{B\left(\lambda_j\right)} = i\varepsilon\dot{\mu}\left(\lambda_j\right)\left\| n^{+\varepsilon}\left(x;\lambda_j\right)\right\|^{2} + i\varepsilon\dot{\nu}\left(\lambda_j\right)\left\| n\left(x;\lambda_j\right)\right\|^{2}, \tag{4.3.91}$$

which is attained by use of the identity (4.3.70) to eliminate the discrete eigenfunction $m^{+\varepsilon}\left(x;\lambda_j\right)$ from the right hand side of (4.3.90).

It remains for us to demonstrate that the right hand side of equation (4.3.91) is nonzero in the interval $0 < \delta\hat{\lambda}_j < \pi$. The reader who uses equations (4.3.6) and (4.3.7) to compute $\dot{\mu}\left(\lambda_j\right)$ and $\dot{\nu}\left(\lambda_j\right)$, where $\lambda_j = i\varepsilon\hat{\lambda}_j$ and $0 < \delta\hat{\lambda}_j < \pi$, will find that

$$i\varepsilon\dot{\mu}\left(\lambda_j\right) = \tfrac{1}{2}\delta\hat{\lambda}_j \operatorname{cosec}^2\left(\delta\hat{\lambda}_j\right) - \tfrac{1}{2}\cot\left(\delta\hat{\lambda}_j\right) \tag{4.3.92a}$$

and

$$i\varepsilon\dot{\nu}\left(\lambda_j\right) = \tfrac{1}{2}\operatorname{cosec}\left(\delta\hat{\lambda}_j\right) - \tfrac{1}{2}\delta\hat{\lambda}_j \cot\left(\delta\hat{\lambda}_j\right)\operatorname{cosec}\left(\delta\hat{\lambda}_j\right). \tag{4.3.92b}$$

Equations (4.3.92a) and (4.3.92b), when viewed simultaneously, disclose that the two quantities $i\varepsilon\dot{\mu}\left(\lambda_j\right)$ and $i\varepsilon\dot{\nu}\left(\lambda_j\right)$ are real-valued, so the entire right hand side of (4.3.91) is real-valued. Given that the right hand side of (4.3.91) is real-valued, an efficient means by which we can show that the right hand side of (4.3.91) is nonzero in the interval $0 < \delta\hat{\lambda}_j < \pi$ is to execute the MACSYMA [68] commands

and

.

The graphical output from the two MACSYMA commands is sufficient to demonstrate

ε *positive definite* in the

interval $0 < \delta\hat{\lambda}_j < \pi$. Therefore, we agree with the authors of Ref. 63 who conclude from their work that $A(\lambda)$ has a simple zero at $\lambda = \lambda_j$. It is now plausible for us to assume that when we revert to the complex-valued ILW equation $A(\lambda)$ will exhibit a simple zero at $\lambda = \lambda_j$, where all quantities now refer to scattering data allied to the complex-valued ILW equation and $j = 1, 2, \ldots, q$. In what remains of this chapter we will indeed assume that each discrete eigenvalue in the IST for the complex-valued ILW equation is a simple zero of $A(\lambda)$.

Define C_j

$$C_j \;\underline{\underline{\mathrm{def}}}\; \frac{B\left(\lambda_j\right)}{i\dot{A}\left(\lambda_j\right)}. \tag{4.3.93}$$

We emphasize that the definition of C_j is independent of whether the potential in the eigenvalue problem (4.3.12) is real-valued or complex-valued. The authors of Ref. 63 claim, as the result of some calculations in Appendix D of Ref. 63, that C_j is a positive real number (for a fixed value of t). Our calculations contradict the claim made (and proof provided) in Ref. 63 about the reality of C_j. We believe that the normalizing coefficient in the IST for the real-valued ILW equation is *complex-valued* (for all t) with nonzero real and imaginary parts. Pursuant with (4.3.91), a real-valued C_j requires

$$\mathrm{Re}\left\{2\delta\nu\left(\hat{\lambda}_j\right) - e^{-i\varepsilon\delta\hat{\lambda}_j}\right\} \equiv \delta\hat{\lambda}_j \operatorname{cosec}\left(\delta\hat{\lambda}_j\right) - \cos\left(\delta\hat{\lambda}_j\right) = 0,$$

which is impossible: the function $x\operatorname{cosec}(x) - \cos(x)$ is an increasing function in $(0, \pi)$. Elsewhere in this thesis we will explore the implications of a complex-valued normalizing coefficient for the IST of the real-valued ILW equation. At the present time it will suffice for us to assert that the normalizing coefficient, C_j, in the IST for the complex-valued ILW equation is complex-valued (for all t). The author of this thesis has derived an analogue of (4.3.91) for the case of the complex-valued ILW equation and verified that C_j *can* be pure imaginary in such a situation; this feature of the normalizing coefficient (as mentioned) is not observed in the IST for the real-valued ILW equation.

We close this section by offering the following remark: the correction made to the work in Ref. 63 should *not* be interpreted as a criticism of the authors responsible for Ref. 63. The decision to present the correction was motivated by the concern to consolidate the theory developed to solve the initial value problem for the MILW equation.

Section 4.4: The Inverse Problem and its Solution

In this section we will formulate and solve the inverse problem associated with the IST for the complex-valued ILW equation. An extension of the Riemann-Hilbert Boundary Value Problem (RHBVP) (4.2.14) will facilitate the formulation and solution of the inverse problem. Our eventual target is to use the solution of the inverse problem to express the solution of the complex-valued ILW equation in terms of the scattering data

$$A(\lambda),\ B(\lambda) \text{ and } \left\{\lambda_j,\ C_j\right\}_{j=1}^{j=q},$$

where the notation $\{\bullet\}_{j=1}^{j=q}$ indicates a finite set in which the discrete index j assumes all integer values in the range 1 to q.

Equation (4.3.53) is the main equation in this section. Considerable effort is required to interpret (4.3.53) as a suitable RHBVP in the complex $\zeta_+(\lambda)$-plane. The complex $\zeta_+(\lambda)$-plane is formed when λ is allowed to assume a continuum of complex values and $\zeta_+(\lambda)$ is partitioned into real and imaginary parts, the map into the $\zeta_+(\lambda)$-plane being specified by (4.3.14). The interpretation of (4.3.53) as a RHBVP hinges on the connection formula

$$\bar{N}^{+\varepsilon}_{-\varepsilon}(x;-\lambda) = N^{+\varepsilon}(x;\lambda)\,e^{-i\varepsilon\lambda(x-i\varepsilon\delta)}, \tag{4.4.1}$$

which in turn is inextricably connected to the discrete symmetry relationship

$$G^{+\varepsilon}_{-\varepsilon}(x,\eta;-\lambda) = G^{+\varepsilon}_{-\varepsilon}(x,\eta;\lambda)\,e^{-i\varepsilon\lambda(x-\eta)}. \tag{4.4.2}$$

We now present a proof of (4.4.2), which will be followed by a proof of (4.4.1). The derivation of (4.4.2) begins with equation (4.3.50$^-$), expressed in the form

$$G^{+\varepsilon}_{-\varepsilon}(x,\eta;\lambda) = \frac{1}{2\pi}\int_{C_-}\frac{e^{i(x-\eta)r}}{r-\varepsilon\left[\zeta_+(\lambda)+(2\delta)^{-1}\right]\left(1-e^{-2\varepsilon\delta r}\right)}\,\mathrm{d}r, \tag{4.4.3}$$

where C_- denotes the straight line from $-\infty + i0^+$ to $\infty + i0^+$ in the complex r-plane. The reader can demonstrate that (4.4.3) is equivalent to (4.3.50$^-$) by using [see (4.3.14)]

$$\zeta_+(x) = \frac{x}{2} + \frac{x}{2}\coth(\delta x) - \frac{1}{2\delta}$$

to eliminate $\zeta_+(\bullet)$ from the integrands of (4.3.50$^-$) and (4.4.3). Introducing into (4.4.3) the involution $\lambda \to -\lambda$, and then using the identities (4.3.17b) and (4.3.17c) in the subsequent equation we obtain

$$G^{+\,\varepsilon}_{-\,\varepsilon}(x,\eta;-\lambda) = \frac{1}{2\pi}\int_{C_-} \frac{e^{i(x-\eta)r}}{r - \varepsilon\left[\zeta_+(\lambda) + (2\delta)^{-1}\right]\left(e^{-2\delta\lambda} - e^{-2\varepsilon\delta(r+\varepsilon\lambda)}\right)}\mathrm{d}r.$$

The change of variable $R = r + \varepsilon\lambda$ allows us to convert the equation for $G^{+\,\varepsilon}_{-\,\varepsilon}(x,\eta;-\lambda)$ into the equation

$$G^{+\,\varepsilon}_{-\,\varepsilon}(x,\eta;-\lambda) = \frac{1}{2\pi}\int_{C_-} \frac{e^{i(x-\eta)(R-\varepsilon\lambda)}}{R - \varepsilon\lambda - \varepsilon\left[\zeta_+(\lambda) + (2\delta)^{-1}\right]\left(e^{-2\delta\lambda} - e^{-2\varepsilon\delta R}\right)}\mathrm{d}R. \qquad (4.4.4)$$

The identities (4.3.17a) and (4.3.17c) can now be employed by the reader to show that (4.4.4) is equivalent to (4.4.2).

We are now in a position to verify (4.4.1). According to (4.3.52), $\bar{N}^{+\,\varepsilon}_{-\,\varepsilon}(x;-\lambda)$ satisfies the integral equation

$$\bar{N}^{+\,\varepsilon}_{-\,\varepsilon}(x;-\lambda) = 1 + \varepsilon\int_{-\infty}^{\infty} G^{+\,\varepsilon}_{-\,\varepsilon}(x,\eta;-\lambda)\,U(\eta)\,\bar{N}^{+\,\varepsilon}_{-\,\varepsilon}(\eta;-\lambda)\,\mathrm{d}\eta,$$

and this equation, because of (4.4.2), can be expressed in the form

$$\bar{N}^{+\,\varepsilon}_{-\,\varepsilon}(x;-\lambda)e^{i\varepsilon\lambda x} = e^{i\varepsilon\lambda x} + \varepsilon\int_{-\infty}^{\infty} G^{+\,\varepsilon}_{-\,\varepsilon}(x,\eta;\lambda)\,U(\eta)\,\bar{N}^{+\,\varepsilon}_{-\,\varepsilon}(\eta;-\lambda)e^{i\varepsilon\lambda\eta}\,\mathrm{d}\eta. \qquad (4.4.5)$$

An inspection of (4.3.52) will also show the reader that the equation for $N^{+\,\varepsilon}(x;\lambda)$ can be adjusted to read

$$N^{+\,\varepsilon}(x;\lambda)e^{-\lambda\delta} = e^{i\varepsilon\lambda x} + \varepsilon\int_{-\infty}^{\infty} G^{+\,\varepsilon}_{-\,\varepsilon}(x,\eta;\lambda)\,U(\eta)\,N^{+\,\varepsilon}(\eta;\lambda)e^{-\lambda\delta}\,\mathrm{d}\eta. \qquad (4.4.6)$$

Equations (4.4.5) and (4.4.6) imply a *homogeneous* version of (4.3.19) in which

$$W^{+\,\varepsilon}(x;\lambda) \equiv \left\{\bar{N}^{+\,\varepsilon}_{-\,\varepsilon}(x;-\lambda) - N^{+\,\varepsilon}(x;\lambda)e^{-i\varepsilon\lambda(x-i\varepsilon\delta)}\right\}e^{i\varepsilon\lambda x}.$$

Homogeneous Volterra integral equations of the second-kind display only the trivial solution [60], therefore we conclude that $W^{+\,\varepsilon}(x;\lambda) \equiv 0$, and it is now a short step for the reader to arrive at the identity (4.4.1).

Define [11] $M^{+\,\varepsilon}_{+\,\varepsilon}(x;\zeta_+)$ through the equation

$$M^{+\,\varepsilon}_{+\,\varepsilon}(x;\zeta_+) \equiv M^{+\,\varepsilon}_{+\,\varepsilon}(x;\,\zeta_+(\lambda)) = M^{+\,\varepsilon}_{+\,\varepsilon}(x;\lambda),$$

and continue the process for $\bar{N}^{+\,\varepsilon}_{-\,\varepsilon}(x;\zeta_+)$, $N^{+\,\varepsilon}(x;\zeta_+)$, $A(\zeta_+)$ and $B(\zeta_+)$. The terminology

ζ_+ ζ_+,

defined as $\zeta_+ = \zeta_+^{(\mathrm{R})} + i\zeta_+^{(\mathrm{I})}$ for which $\zeta_+^{(\mathrm{R})}$ and $\zeta_+^{(\mathrm{I})}$ denote two real numbers. In terms of the notation defined, a summary of the results contained in Section 4.3 of this thesis is as follows:

- $M^{+\,\varepsilon}_{+\,\varepsilon}(x;\zeta_+)$ and $A(\zeta_+)$ are $(+\,\varepsilon)$ functions in the complex ζ_+-plane;
- $\bar{N}^{+\,\varepsilon}_{-\,\varepsilon}(x;\zeta_+)$ is a $(-\,\varepsilon)$ function in the complex ζ_+-plane;
- $N^{+\,\varepsilon}(x;\zeta_+)$ and $B(\zeta_+)$ cannot be analytically continued off the $\zeta_+^{(\mathrm{R})}$-axis;
- $A(\zeta_+)$ has a simple zero at ζ_{+_j}, where $\zeta_{+_j} \equiv \zeta_+(\lambda_j)$ and $j = 1, 2, \ldots, q$;
- $M^{+\,\varepsilon}_{+\,\varepsilon}(x;\zeta_+) \to 1$ and $A(\zeta_+) \to 1$ as $\zeta_+ \to \infty$ (along the $\zeta_+^{(\mathrm{R})}$-axis).

The ratio $M^{+\,\varepsilon}_{+\,\varepsilon}(x;\zeta_+)/A(\zeta_+)$ defines a meromorphic function, which has simple poles at ζ_{+_j} and displays the canonical normalization as $\zeta_+ \to \infty$. An equation that encapsulates all the qualities of the ratio $M^{+\,\varepsilon}_{+\,\varepsilon}(x;\zeta_+)/A(\zeta_+)$ is

$$\frac{M^{+\,\varepsilon}_{+\,\varepsilon}(x;\zeta_+)}{A(\zeta_+)} = 1 + \sum_{j=1}^{q} \frac{D_j(x)}{\zeta_+ - \zeta_{+_j}} + \phi_{+\,\varepsilon}(x;\zeta_+),$$

where

$$D_j(x) = \lim_{\zeta_+ \to \zeta_{+_j}} \left(\zeta_+ - \zeta_{+_j}\right) \frac{M^{+\,\varepsilon}_{+\,\varepsilon}(x;\zeta_+)}{A(\zeta_+)}$$

and $\phi_{+\,\varepsilon}(x;\zeta_+)$ denotes a suitable $(+\,\varepsilon)$ function in ζ_+. It is an opportune moment to remind the reader that in Section 3.3 of this thesis we evaluated a limit that is defined in a similar fashion to the limit associated with $D_j(x)$; the relevant equation from Section 3.3 is

find that

$$\frac{M^{+\,\varepsilon}_{+\,\varepsilon}(x;\zeta_+)}{A(\zeta_+)} = 1 + i\sum_{j=1}^{q} C_j \frac{N^{+\,\varepsilon}\left(x;\zeta_{+_j}\right)}{\zeta_+ - \zeta_{+_j}} + \phi_{+\,\varepsilon}(x;\zeta_+), \tag{4.4.7}$$

where C_j is defined by (4.3.93). The equation

$$1 + i\sum_{j=1}^{q} C_j \frac{N^{+\varepsilon}\left(x;\zeta_{+_j}\right)}{\zeta_+ - \zeta_{+_j}} + \phi_{+\varepsilon}(x;\zeta_+) = \bar{N}^{+\varepsilon}_{-\varepsilon}(x;\zeta_+) + R(\zeta_+)N^{+\varepsilon}(x;\zeta_+), \tag{4.4.8a}$$

where

$$R(\zeta_+) \;\underline{\underline{\text{def}}}\; \frac{B(\zeta_+)}{A(\zeta_+)}, \tag{4.4.8b}$$

is formed when we substitute (4.4.7) into (4.3.53).

Although the function $R(\zeta_+)N^{+\varepsilon}(x;\zeta_+)$ *cannot* be analytically continued off the $\operatorname{Im}(\zeta_+) = 0$

$$R(\zeta_+)N^{+\varepsilon}(x;\zeta_+) = \varepsilon\left\{\hat{U}_{+\varepsilon}(x;\zeta_+) - \hat{U}_{-\varepsilon}(x;\zeta_+)\right\}, \tag{4.4.9}$$

where:

- $-1/(2\delta) < \zeta_+ < \infty$;
- $\hat{U}_{\pm\varepsilon}(x;\zeta_+)$ denote the $(\pm\varepsilon)$-parts (with respect to ζ_+) of $R(\zeta_+)N^{+\varepsilon}(x;\zeta_+)$;
- $\hat{U}_{\pm\varepsilon}(x;\zeta_+)$ are defined by the functionals [103]

$$\hat{U}_{\pm\varepsilon}(x;\zeta_+) \;\underline{\underline{\text{def}}}\; \frac{1}{2\pi i}\int_{-1/(2\delta)}^{\infty} \frac{R(\lambda_+)N^{+\varepsilon}(x;\lambda_+)}{\lambda_+ - \left(\zeta_+ \pm i\varepsilon 0^+\right)}\mathrm{d}\lambda_+. \tag{4.4.10}$$

Equation (4.4.9) has its origin in (4.2.14). Substituting (4.4.9) into (4.4.8a), and then equating $(-\varepsilon)$-parts either side of the resultant equation we are able to conclude that

$$\bar{N}^{+\varepsilon}_{-\varepsilon}(x;\zeta_+) = 1 + i\sum_{j=1}^{q} C_j \frac{N^{+\varepsilon}\left(x;\zeta_{+_j}\right)}{\zeta_+ - \zeta_{+_j}} + \frac{\varepsilon}{2\pi i}\int_{-1/(2\delta)}^{\infty} \frac{R(\lambda_+)N^{+\varepsilon}(x;\lambda_+)}{\lambda_+ - \left(\zeta_+ - i\varepsilon 0^+\right)}\mathrm{d}\lambda_+. \tag{4.4.11}$$

In the process of deriving (4.4.11) we have carefully sought to preserve the boundary condition $\bar{N}^{+\varepsilon}_{-\varepsilon}(x;\zeta_+) \to 1$ as $\zeta_+ \to \infty$. A standard asymptotic analysis of (4.4.11) as $\zeta_+ \to \infty$ reveals that

$$\bar{N}^{+\varepsilon}_{-\varepsilon}(x;\zeta_+) = 1 + \frac{\Xi\left(x;\zeta_{+_j}\right)}{\zeta_+} + \mathrm{O}\left\{\frac{1}{(\zeta_+)^2}\right\}, \tag{4.4.12}$$

where $\Xi\left(x;\zeta_{+_j}\right)$ is defined as follows:

$$\Xi\left(x;\zeta_{+_j}\right)\ \underline{\underline{\mathrm{def}}}\ i\sum_{j=1}^{q} C_j N^{+\varepsilon}\left(x;\zeta_{+_j}\right)-\frac{\varepsilon}{2\pi i}\int_{-1/(2\delta)}^{\infty} R(\lambda_+)N^{+\varepsilon}(x;\lambda_+)\mathrm{d}\lambda_+.$$

In addition to (4.4.11), equation (4.3.52) also has the potential to furnish us with the asymptotic expansion of $\bar{N}^{+\varepsilon}_{-\varepsilon}(x;\zeta_+)$ as $\zeta_+\to\infty$. First, we will use (4.3.52) to compute the asymptotic expansion of $\bar{N}^{+\varepsilon}_{-\varepsilon}(x;\zeta_+)$ as $\zeta_+\to\infty$, after which we will match this expansion with (4.4.12): our eventual target being a formula that expresses $U(x)$ in terms of the scattering data

$$A(\lambda),\ B(\lambda)\ \text{and}\ \left\{\lambda_j,\ C_j\right\}_{j=1}^{j=q}.$$

$G^{+\varepsilon}_{-\varepsilon}(x,\eta;\zeta_+)$ behaves when $\zeta_+ >> 1$ is indispensable; the reader who wishes to refresh their memory of $G^{+\varepsilon}_{-\varepsilon}(x,\eta;\zeta_+)$ can inspect (4.4.3). Equations (4.2.15$^+$) and (4.2.16$^\pm$) facilitate the expression of $G^{+\varepsilon}_{-\varepsilon}(x,\eta;\zeta_+)$ in the form

$$G^{+\varepsilon}_{-\varepsilon}(x,\eta;\zeta_+)=\frac{1}{4i\delta}\int_{-\infty}^{\infty}\coth\left(\frac{\pi}{2\delta}\left[\xi-\left(x+i\varepsilon 0^+\right)\right]\right)\Delta G(\xi,\eta;\zeta_+)\mathrm{d}\xi, \tag{4.4.13}$$

where

$$\Delta G(\xi,\eta;\zeta_+)\ \underline{\underline{\mathrm{def}}}\ \varepsilon\left\{G^{+\varepsilon}_{-\varepsilon}(\xi,\eta;\zeta_+)-G^{-\varepsilon}_{-\varepsilon}(\xi,\eta;\zeta_+)\right\}.$$

Incidentally, $\Delta G(\xi,\eta;\zeta_+)$ represents the jump in $G^{+\varepsilon}(\xi,\eta;\zeta_+)$ across the $\zeta_+^{(\mathrm{R})}$-axis. The quantity $\Delta G(\xi,\eta;\zeta_+)$ admits the representation

$$\Delta G(\xi,\eta;\zeta_+)=\frac{\varepsilon}{2\pi}\int_{C_-}\frac{\left(1-e^{-2\varepsilon\delta r}\right)e^{i(\xi-\eta)r}}{r-\varepsilon\left[\zeta_+ +(2\delta)^{-1}\right]\left(1-e^{-2\varepsilon\delta r}\right)}\mathrm{d}r, \tag{4.4.14}$$

which can be verified by suitable use of equations (4.2.17) and (4.4.3). Despite the intricacy of the integral present in (4.4.14), it is possible to prove that (4.4.14) admits the simple asymptotic expansion

$$\Delta G(\xi,\eta;\zeta_+)=-\frac{1}{\zeta_+}\delta(\xi-\eta)+\mathrm{O}\left\{\frac{1}{(\zeta_+)^2}\right\}\ \text{as}\ \zeta_+\to\infty, \tag{4.4.15}$$

where $\delta(\bullet)$ denotes the Dirac delta function. The information we require, namely the behaviour of $G^{+\varepsilon}_{-\varepsilon}(x,\eta;\zeta_+)$ as $\zeta_+\to\infty$, can clearly be seen from (4.4.13) and (4.4.15) to be

$$G^{+\varepsilon}_{-\varepsilon}(x,\eta;\zeta_+) = -\frac{1}{4i\delta\zeta_+}\coth\left(\frac{\pi}{2\delta}\left[\eta-\left(x+i\varepsilon0^+\right)\right]\right)+\mathrm{O}\left\{\frac{1}{(\zeta_+)^2}\right\} \tag{4.4.16}$$

as $\zeta_+ \to \infty$. It is now an elementary matter for the reader to use (4.4.16) in (4.3.52), and thereby derive the equation

$$\bar{N}^{+\varepsilon}_{-\varepsilon}(x;\zeta_+) = 1-\frac{\varepsilon}{\zeta_+}U^{+\varepsilon}(x)+\mathrm{O}\left\{\frac{1}{(\zeta_+)^2}\right\} \text{ as } \zeta_+\to\infty, \tag{4.4.17}$$

where

$$U^{+\varepsilon}(x) \;\underline{\underline{\mathrm{def}}}\; \frac{1}{4i\delta}\int_{-\infty}^{\infty}\coth\left(\frac{\pi}{2\delta}\left[\eta-\left(x+i\varepsilon0^+\right)\right]\right)U(\eta)\,\mathrm{d}\eta. \tag{4.4.18}$$

Equations (4.4.12) and (4.4.17) are two distinct asymptotic expansions of $\bar{N}^{+\varepsilon}_{-\varepsilon}(x;\zeta_+)$ for $\zeta_+ >> 1$, and when we match these expansions we find that

$$U^{+\varepsilon}(x) = \frac{1}{2\pi i}\int_{-1/(2\delta)}^{\infty}R(\lambda_+)N^{+\varepsilon}(x;\lambda_+)\,\mathrm{d}\lambda_+ - i\varepsilon\sum_{j=1}^{q}C_jN^{+\varepsilon}\left(x;\zeta_{+_j}\right). \tag{4.4.19}$$

Equation (4.4.19) requires supplementation with: an equation that associates $N^{+\varepsilon}(x;\zeta_+)$ with the scattering data; a system of equations that relates $N^{+\varepsilon}\left(x;\zeta_{+_j}\right)$ to the scattering data. The requisite equations are as follows:

$$\begin{aligned} N^{+\varepsilon}(x;\zeta_+) = \Bigg\{ & 1 - i\sum_{j=1}^{q}C_j\frac{N^{+\varepsilon}\left(x;\zeta_{+_j}\right)}{\zeta_-+\zeta_{+_j}} \\ & + \frac{\varepsilon}{2\pi i}\int_{-1/(2\delta)}^{\infty}\frac{R(\lambda_+)N^{+\varepsilon}(x;\lambda_+)}{\lambda_++\zeta_-+i\varepsilon0^+}\,\mathrm{d}\lambda_+\Bigg\}e^{i\varepsilon\lambda(x-i\varepsilon\delta)}; \end{aligned} \tag{4.4.20}$$

$$\begin{aligned} N^{+\varepsilon}\left(x;\zeta_{+_p}\right) = \Bigg\{ & 1 - i\sum_{j=1}^{q}C_j\frac{N^{+\varepsilon}\left(x;\zeta_{+_j}\right)}{\zeta_{-_p}+\zeta_{+_j}} \\ & + \frac{\varepsilon}{2\pi i}\int_{-1/(2\delta)}^{\infty}\frac{R(\lambda_+)N^{+\varepsilon}(x;\lambda_+)}{\lambda_++\zeta_{-_p}}\,\mathrm{d}\lambda_+\Bigg\}e^{i\varepsilon\lambda_p(x-i\varepsilon\delta)}, \end{aligned} \tag{4.4.21}$$

where $\zeta_{-_p} \equiv \zeta_-(\lambda_p)$ and the discrete index p assumes each of the values $p = 1, 2, \ldots, q$.

The crucial step towards the derivation of (4.4.20) and (4.4.21) is to use the identity (4.4.1), but expressed in the form

$$\bar{N}^{+\varepsilon}_{-\varepsilon}(x;\zeta_+) = N^{+\varepsilon}\big(x;\zeta_+(-\lambda)\big)e^{i\varepsilon\lambda(x-i\varepsilon\delta)},$$

to derive from (4.4.11) the following equation:

$$N^{+\varepsilon}\big(x;\zeta_+(-\lambda)\big)e^{i\varepsilon\lambda(x-i\varepsilon\delta)} = 1 + i\sum_{j=1}^{q} C_j \frac{N^{+\varepsilon}\big(x;\zeta_{+_j}\big)}{\zeta_+ - \zeta_{+_j}}$$

$$+\frac{\varepsilon}{2\pi i}\int_{-1/(2\delta)}^{\infty} \frac{R(\lambda_+)N^{+\varepsilon}(x;\lambda_+)}{\lambda_+ - \left\{\zeta_+(\lambda) - i\varepsilon 0^+\right\}}\,\mathrm{d}\lambda_+ . \qquad (4.4.22)$$

The involution $\zeta_+ \to -\zeta_-$ when introduced into (4.4.22) produces (4.4.20), whereas passage to the limit $\zeta_+ \to -\zeta_{-_p}$ is sufficient to procure (4.4.21) from (4.4.22).

Equations (4.4.19), (4.4.20) and (4.4.21) make it possible for us to construct the complex-valued solution of the ILW equation at a fixed *t*. Our next objective is to ascertain how the scattering data

$$A(\lambda),\ B(\lambda) \text{ and } \left\{\lambda_j,\ C_j\right\}_{j=1}^{j=q}$$

evolve when viewed as functions of *t*. We will address the time evolution of the scattering data in the next section of this thesis.

Section 4.5: The Evolution of the Scattering Data

fixed t

complete solution of the initial value problem for the complex-valued ILW equation is achieved when we supplement the system of equations that solves the inverse problem with equations for the time evolution of

$$R(\lambda, t)\ \underline{\underline{\text{def}}}\ \frac{B(\lambda, t)}{A(\lambda, t)} \text{ and } \left\{\lambda_j(t),\, C_j(t)\right\}_{j=1}^{j=q}.$$

The equations that specify the time evolution of the scattering data are as follows:

$$\lambda_j(t) = \lambda_j(0); \tag{4.5.1}$$

$$R(\lambda, t) = R(\lambda, 0)\, e^{i\varepsilon\vartheta(\lambda, t)}; \tag{4.5.2}$$

$$C_j(t) = C_j(0)\, e^{i\varepsilon\vartheta(\lambda_j, t)}; \tag{4.5.3}$$

$$\vartheta(\lambda, t)\ \underline{\underline{\text{def}}}\ \lambda\left\{\lambda\coth(\delta\lambda) - \frac{1}{\delta}\right\}t, \tag{4.5.4}$$

where $R(\lambda, 0)$, $\lambda_j(0)$ and $C_j(0)$ are considered *known* because these quantities can be determined from the solution of the direct problem for (4.3.12) in which (4.1.5) is the input potential.

We will omit the derivation of equations (4.5.1), (4.5.2) and (4.5.3) because these equations can be derived by elementary modifications of those methods used to ascertain the time evolution of the scattering data in the IST for the complex-valued KdV equation; the interested reader is referred to Section 3.4 of this thesis for the preparatory material necessary to derive equations (4.5.1), (4.5.2) and (4.5.3). It will suffice for us to record that after using (4.5.1), (4.5.2) and (4.5.3) in an appropriate manner within each of the equations (4.4.19), (4.4.20) and (4.4.21) we obtain the following three equations:

$$U^{+\varepsilon}(x, t) = \frac{1}{2\pi i}\int_{-1/(2\delta)}^{\infty} R(\lambda_+, 0)\, N^{+\varepsilon}(x, t; \lambda_+)\, e^{i\varepsilon\vartheta(\lambda_+, t)}\, \mathrm{d}\lambda_+ - i\varepsilon\sum_{j=1}^{q} C_j(0)\, N^{+\varepsilon}\left(x, t; \zeta_{+_j}\right) e^{i\varepsilon\vartheta(\lambda_j, t)}; \tag{4.5.5}$$

$$N^{+\varepsilon}(x,t;\zeta_+) = \left\{ 1 - i\sum_{j=1}^{q} C_j(0)\,\frac{N^{+\varepsilon}\left(x,t;\zeta_{+_j}\right)}{\zeta_- + \zeta_{+_j}}\,e^{i\varepsilon\vartheta(\lambda_j,t)} \right.$$

$$\left. + \frac{\varepsilon}{2\pi i}\int_{-1/(2\delta)}^{\infty} \frac{R(\lambda_+,0)\,N^{+\varepsilon}(x,t;\lambda_+)}{\lambda_+ + \zeta_- + i\varepsilon 0^+}\,e^{i\varepsilon\vartheta(\lambda_+,t)}\mathrm{d}\lambda_+ \right\} e^{i\varepsilon\lambda(x - i\varepsilon\delta)}; \qquad (4.5.6)$$

$$N^{+\varepsilon}\left(x,t;\zeta_{+_p}\right) = \left\{ 1 - i\sum_{j=1}^{q} C_j(0)\,\frac{N^{+\varepsilon}\left(x,t;\zeta_{+_j}\right)}{\zeta_{-_p} + \zeta_{+_j}}\,e^{i\varepsilon\vartheta(\lambda_j,t)} \right.$$

$$\left. + \frac{\varepsilon}{2\pi i}\int_{-1/(2\delta)}^{\infty} \frac{R(\lambda_+,0)\,N^{+\varepsilon}(x,t;\lambda_+)}{\lambda_+ + \zeta_{-_p}}\,e^{i\varepsilon\vartheta(\lambda_+,t)}\mathrm{d}\lambda_+ \right\} e^{i\varepsilon\lambda_p(x - i\varepsilon\delta)}. \qquad (4.5.7)$$

In writing equations (4.5.5), (4.5.6) and (4.5.7) we have adopted the convention $\lambda_j \equiv \lambda_j(0)$.

We will close this section by noting that the combination of equations (4.5.6) and (4.5.7) defines $N^{+\varepsilon}(x,t;\zeta_+)$ and $N^{+\varepsilon}\left(x,t;\zeta_{+_j}\right)$ in terms of $R(\lambda_+,0)$, λ_j and $C_j(0)$. Therefore, (4.5.5) defines $U^{+\varepsilon}(x,t)$ in terms of $R(\lambda_+,0)$, λ_j and $C_j(0)$, where $j = 1, 2, \ldots, q$.

Section 4.6: Soliton Solutions for the ILW and MILW Equations

Pure soliton solutions for the complex-valued ILW equation arise when we make the substitution $R(\bullet, 0) \equiv 0$ in equations (4.5.5), (4.5.6) and (4.5.7). The particular system of equations that governs the production of *pure soliton* solutions for the complex-valued ILW equation is as follows:

$$U^{+\varepsilon}(x,t) = -i\varepsilon \sum_{j=1}^{q} C_j(0;\varepsilon) N^{+\varepsilon}\left(x,t;\zeta_{+_j}\right) e^{i\varepsilon\vartheta(\lambda_j,t)}; \tag{4.6.1}$$

$$N^{+\varepsilon}\left(x,t;\zeta_{+_p}\right) = \left\{1 - i\sum_{j=1}^{q} C_j(0;\varepsilon) \frac{N^{+\varepsilon}\left(x,t;\zeta_{+_j}\right)}{\zeta_{+_j} + \zeta_{-_p}} e^{i\varepsilon\vartheta(\lambda_j,t)}\right\} e^{i\varepsilon\lambda_p(x-i\varepsilon\delta)}; \tag{4.6.2}$$

$$\vartheta(\lambda,t) \underline{\underline{\mathrm{def}}} \lambda\left\{\lambda\coth(\delta\lambda) - \frac{1}{\delta}\right\}t,$$

where $p = 1, 2, \ldots, q$ and we have decided to show the parametric dependence of the normalizing coefficient on ε by writing $C_j(0;\varepsilon)$. Equations (4.6.1) and (4.6.2) are also valid for the real-valued ILW equation [103]. However, in the case of the real-valued ILW equation the associated scattering data assume values that are different to those particular values for the scattering data allied to the complex-valued ILW equation.

At the conclusion of Section 4.3 we suggested that at a suitable time we would explore the implications of our discovery that the normalizing coefficient in the IST for the real-valued ILW equation is complex-valued for all t, and not real-valued as asserted by Kodama et al. [63]. An opportune moment has now arisen for us to examine the consequences of the correction we have made to the work in Ref. 63. Let us temporarily assume that $U(x,t) \equiv \{U(x,t)\}^*$. The reader who is familiar with the work in Ref. 63 may be lead to conclude that the correction we have alluded to is incompatible with the requirement of a real-valued solution of the ILW equation. We now proceed to show that the correction we have made to the work in Ref. 63 is in fact consistent with the reality constraint on the solution, and in fact our correction clears the path towards a solution that is more general than the solution contained in Ref. 63. The construction of the 1-soliton solution for the real-valued ILW equation will serve as an adequate illustration.

Kodama et al. [63] have determined that the 1-soliton solution for the real-valued ILW equation requires $p = q = 1$ and $\lambda_1 = i\hat{\lambda}_1$, where $0 < \delta\hat{\lambda}_1 < \pi$; for the sake of convenience we will let $\varepsilon = 1$ in equations (4.6.1) and (4.6.2). Substituting into (4.6.1) and (4.6.2) our particular choices of p, q, ε and λ_1 we obtain

$$U^{+}(x,t) = -\,iC_1(0)N^{+}\left(x,t;\zeta_{+_1}\right)e^{\hat{\lambda}_1\phi t}; \tag{4.6.3}$$

$$N^{+}\left(x,t;\zeta_{+_1}\right)e^{\hat{\lambda}_1(x-i\delta)} = 1 - \frac{C_1(0)}{\hat{\lambda}_1}N^{+}\left(x,t;\zeta_{+_1}\right)e^{\hat{\lambda}_1\phi t}, \tag{4.6.4}$$

where $C_1(0) \equiv C_1(0;\varepsilon = 1)$ and

$$\phi \;\underline{\underline{\mathrm{def}}}\; \frac{1}{\delta} - \hat{\lambda}_1\cot\left(\delta\hat{\lambda}_1\right). \tag{4.6.5}$$

Linear algebraic equations such as (4.6.3) and (4.6.4) can be solved by application of standard techniques, and we find that the solution for $U^{+}(x,t)$ is

$$U^{+}(x,t) = -\,\frac{i\hat{\lambda}_1 R_1\,e^{i\hat{\lambda}_1(\delta+\theta_1)}}{\hat{\lambda}_1 e^{\hat{\lambda}_1(x-\phi t)} + R_1\,e^{i\hat{\lambda}_1(\delta+\theta_1)}}, \tag{4.6.6}$$

where $R_1 > 0$ and $\theta_1 \neq 0$ are two real parameters (independent of x and t) such that

$$C_1(0) = R_1\,e^{i\theta_1\hat{\lambda}_1}.$$

It is important for us to record that $\theta_1 = 0$ in the work of Kodama et al. [63] because the

the work in Ref. 63 necessitates that $\theta_1 \neq 0$, always.

According to equations ($4.2.16^{+}$) and (4.4.18), we determine the (dependent) physical variable in the real-valued ILW equation through the equation

$$U(x,t) = U^{+}(x,t) + \left\{U^{+}(x,t)\right\}^{*}. \tag{4.6.7}$$

Substituting (4.6.6) into (4.6.7), and then performing some algebra on the expression formed, we find that our amended IST for the real-valued ILW equation delivers the 1-soliton solution

$$U(x,t) = \frac{\hat{\lambda}_1\sin\left(\hat{\lambda}_1\left[\delta+\theta_1\right]\right)}{\cos\left(\hat{\lambda}_1\left[\delta+\theta_1\right]\right) + \cosh\left(\hat{\lambda}_1\left[x-\phi t+\hat{x}_0\right]\right)}, \tag{4.6.8}$$

where

$$\hat{x}_0 \;\underline{\underline{\mathrm{def}}}\; \frac{\ln\left(\hat{\lambda}_1 / R_1\right)}{\hat{\lambda}_1}.$$

Equation (4.6.8) *is* indeed real-valued, despite concerns the reader may have held about the reality of the final solution. The correction we have made to the work in Ref. 63 manifests itself through the appearance of an essential parameter, which we denote as θ_1, that is absent from the solution derived by Kodama et al. [63]. The 1-soliton solution for the real-valued ILW equation presented by Kodama et al. [63] is retrieved from (4.6.8) in the limit $\theta_1 \to 0$. It is interesting to note that the additional parameter we have discovered in the 1-soliton solution for the real-valued ILW equation is also absent from the work of Satsuma et al. [108]. We anticipate that the extension of (4.6.8) to multi-soliton solutions and solutions with a non-soliton part (radiation component) will also produce solutions that are enriched by the presence of parameters that are absent from the corresponding complex-valued normalizing coefficients.

Let us now return to the situation in which $U(x,t) \neq \left\{U(x,t)\right\}^*$, and demonstrate how the IST we have developed for the complex-valued ILW equation delivers the correct 1-soliton solution for the MILW equation. In the case $p = q = 1$, we can solve equations (4.6.1) and (4.6.2) to show that

$$U^{+\,\varepsilon}(x,t) = -\frac{i\varepsilon\lambda_1 C_1(0;\varepsilon)e^{\delta\lambda_1}}{\lambda_1 e^{-i\varepsilon\lambda_1 x - i\varepsilon\vartheta(\lambda_1,t)} + iC_1(0;\varepsilon)e^{\delta\lambda_1}}, \tag{4.6.9}$$

where λ_1 denotes the solitary eigenvalue in the discrete spectrum of (4.2.12). The eigenvalue λ_1 depends parametrically on the indicator ε. We will use the notation λ_1^+ to refer to λ_1 when $\varepsilon = 1$, and the notation λ_1^- to refer to λ_1 when $\varepsilon = -1$. In Section 4.3 of this thesis we showed that $\lambda_1^\pm$ are complex numbers for which

$$\pm\,\mathrm{Im}\left(\lambda_1^\pm\right) > 0. \tag{4.6.10}$$

The map from $U^{+\,\varepsilon}(x,t)$ into the physical variable that solves the complex-valued ILW equation is provided by (4.2.10). Using equation (4.6.9) in (4.2.10) we find that the *preliminary* form of the 1-soliton solution for the complex-valued ILW equation is

$$U(x,t) = -\frac{i\lambda_1^+ C_1^+(0)e^{\delta\lambda_1^+}}{\lambda_1^+ e^{-i\lambda_1^+ x - i\vartheta(\lambda_1^+,t)} + iC_1^+(0)e^{\delta\lambda_1^+}} - \frac{i\lambda_1^- C_1^-(0)e^{\delta\lambda_1^-}}{\lambda_1^- e^{i\lambda_1^- x + i\vartheta(\lambda_1^-,t)} + iC_1^-(0)e^{\delta\lambda_1^-}}, \qquad (4.6.11)$$

where $C_1^\pm(0) \equiv C_1(0; \varepsilon = \pm 1)$ and $\vartheta(\lambda, t)$ is defined by equation (4.5.4).

The function defined by (4.6.11) must satisfy the boundary conditions [see (4.1.6)]

$$U(x,t) \to 0 \text{ (uniformly in } t\text{) as } x \to \pm\infty.$$

Equation (4.6.11) can be used to show that

$$U(x,t) \to \begin{cases} 0, \text{ as } x \to \infty \\ -\left(\lambda_1^+ + \lambda_1^-\right), \text{ as } x \to -\infty. \end{cases}$$

Therefore, compatibility of (4.6.11) with the localized boundary conditions for $U(x,t)$ necessitates that $\lambda_1^- = -\lambda_1^+$. In passing we should mention that the relationship between λ_1^+ and λ_1^- that we have discovered is consistent with (4.6.10).

Pure soliton solutions are examples of travelling wave solutions whose speed (usually) depends in some way on the discrete scattering data. Travelling wave solutions for the ILW and MILW equations require Re $\left(\lambda_1^+\right) = 0$. We can now conclude that the discrete eigenvalue associated with the 1-soliton solution for the complex-valued ILW equation has the form $\lambda_1^\pm = \pm i\hat{\lambda}$, where $\hat{\lambda}$ denotes an appropriate positive real number.

It will prove expedient for us to express $C_1^\pm(0)$ in the form

$$C_1^\pm(0) = R_\pm e^{i\hat{\lambda}\theta_\pm}, \qquad (4.6.12)$$

where $R_\pm > 0$ and $0 < \left|\hat{\lambda}\theta_\pm\right| < 2\pi$; we ask the reader to note the strict inequality of the upper and lower bounds for $\left|\hat{\lambda}\theta_\pm\right|$. Substituting $\lambda_1^\pm = \pm i\hat{\lambda}$ and (4.6.12) into (4.6.11) we find that

$$U(x,t) = \frac{i\hat{\lambda}R_{-}e^{-i\hat{\lambda}(\delta-\theta_{-})}}{R_{-}e^{-i\hat{\lambda}(\delta-\theta_{-})} - \hat{\lambda}e^{\hat{\lambda}(x-\phi t)}} - \frac{i\hat{\lambda}R_{+}e^{i\hat{\lambda}(\delta+\theta_{+})}}{R_{+}e^{i\hat{\lambda}(\delta+\theta_{+})} + \hat{\lambda}e^{\hat{\lambda}(x-\phi t)}}, \qquad (4.6.13)$$

where ϕ is defined by (4.6.5), but with $\hat{\lambda}_1$ replaced by $\hat{\lambda}$.

We are now in a position to apply equation (4.1.7), that is to transfer our attention from the complex-valued ILW equation to the (real-valued) MILW equation. Substituting (4.6.13) into (4.1.7), computing $\operatorname{Im}\{U(x,t)\}$, and then integrating both sides of the ensuing equation with respect to x we obtain

$$V(x,t) = \ln\left|\frac{\hat{\lambda}^2 e^{\hat{\lambda}(x-\phi t)} + R_{+}^2 e^{-\hat{\lambda}(x-\phi t)} + 2\hat{\lambda}R_{+}\cos\left(\hat{\lambda}[\delta+\theta_{+}]\right)}{\hat{\lambda}^2 e^{\hat{\lambda}(x-\phi t)} + R_{-}^2 e^{-\hat{\lambda}(x-\phi t)} - 2\hat{\lambda}R_{-}\cos\left(\hat{\lambda}[\delta-\theta_{-}]\right)}\right| + \hat{V}(t), \qquad (4.6.14)$$

where $\hat{V}(t)$ denotes the arbitrary function produced from the integration with respect to x. Equation (4.6.14) is the *primitive form* of the 1-soliton solution for the MILW equation because (4.6.14) requires adjustments to satisfy the boundary conditions $V(x,t) \to 0$ as $x \to \pm\infty$. Using (4.6.14) to compute the large-x behaviour of $V(x,t)$ we find that

$$V(x,t) \to \begin{cases} \hat{V}(t), \text{ as } x \to \infty \\ \\ \ln\left(\left[R_{+}/R_{-}\right]^2\right) + \hat{V}(t), \text{ as } x \to -\infty, \end{cases}$$

from which we infer that $\hat{V}(t) \equiv 0$ and $R_{+} = R_{-}$. Substituting $\hat{V}(t) \equiv 0$ and $\hat{R} = R_{+}$ into (4.6.14) we obtain

$$V(x,t) = \ln\left|\frac{\hat{\lambda}^2 e^{\hat{\lambda}(x-\phi t)} + \hat{R}^2 e^{-\hat{\lambda}(x-\phi t)} + 2\hat{\lambda}\hat{R}\cos\left(\hat{\lambda}[\delta+\theta_{+}]\right)}{\hat{\lambda}^2 e^{\hat{\lambda}(x-\phi t)} + \hat{R}^2 e^{-\hat{\lambda}(x-\phi t)} - 2\hat{\lambda}\hat{R}\cos\left(\hat{\lambda}[\delta-\theta_{-}]\right)}\right|. \qquad (4.6.15)$$

Define the parameter x_0 through the equation

$$x_0 \,\underline{\underline{\text{def}}}\, \frac{\ln\left(\hat{\lambda}/\hat{R}\right)}{\hat{\lambda}}. \qquad (4.6.16)$$

Through suitable use of (4.6.16) we can express (4.6.15) in the compact and elegant form

$$V(x,t) = \ln\left(\frac{\cosh\left[\hat{\lambda}\left(x - \phi t + x_0\right)\right] + \cos\left[\hat{\lambda}\left(\delta + \theta_+\right)\right]}{\cosh\left[\hat{\lambda}\left(x - \phi t + x_0\right)\right] - \cos\left[\hat{\lambda}\left(\delta - \theta_-\right)\right]}\right), \tag{4.6.17}$$

where by virtue of (4.6.5) we have

$$\phi \underline{\underline{\mathrm{def}}}\ \frac{1}{\delta} - \hat{\lambda}\cot\left(\delta\hat{\lambda}\right).$$

The reader may remember that Section 4.2 of this thesis contains an outline of the scheme to solve the initial value problem for the MILW equation. All but one stage of the solution scheme for the MILW equation has been completed: it remains for us to connect $\hat{\lambda}$ to the (fundamental) parameter β in the MILW equation (4.1.1). Stage VII (the last and incomplete stage) of the solution scheme entails use of (4.2.20) to achieve the desired relationship between $\hat{\lambda}$ and β. Fortunately, the effort we invested in collating the material within Section 2.2 now pays a substantial dividend. We can circumvent (4.2.20) by a *direct* comparison of (4.6.17) to (2.2.19), the latter equation being the 1-soliton solution for the

complete details). Comparing (4.6.17) to (2.2.19) we find that (4.6.17) is *identical* to (2.2.19) whenever the following two conditions are satisfied:

1) $\hat{\lambda}$ is connected to β through equation (2.2.17);

2) $\theta_{\pm} = \pm\dfrac{\pi}{2\hat{\lambda}}.$

We have now completed the solution of the initial value problem for the MILW equation. We close this chapter by noting that that the IST for the complex-valued ILW equation is conceptually similar to the IST for the complex-valued KdV equation. For example, if we use $\theta_{\pm} = \pm\pi/\left(2\hat{\lambda}\right)$ in (4.6.12), then we find that the normalizing coefficients $C_1^{\pm}(0)$ allied to the 1-soliton solution for the MILW equation are pure imaginary. The reader may recall that in Section 3.5 of this thesis we found that the normalizing coefficient in the 1-soliton solution for the MKdV equation is also pure imaginary.