CHAPTER 3

THE INVERSE SCATTERING TRANSFORM FOR THE MODIFIED KORTEWEG-de VRIES EQUATION: A NEW PERSPECTIVE

Section 3.1: Introduction

In Section 2.1 we verified that the Modified Korteweg-de Vries (MKdV) equation,

$$v_t + 6v^2 v_x + v_{xxx} = 0, \tag{3.1.1}$$

is the shallow water limit associated with the Modified Intermediate Long Wave (MILW) equation

$$V_t + \beta V_x\left(e^V - 1\right) + \frac{1}{\delta}V_x + V_x\mathbf{T}\left(V_x\right) + \mathbf{T}\left(V_{xx}\right) = 0.$$

Wadati [119] has solved the initial value problem for (3.1.1) by using the

that $v \to 0$ as $x \to \pm\infty$

value problem for the MKdV equation seems to have been overlooked by all practitioners of the IST. The cardinal point in the KdV based scheme to solve (3.1.1) is to use the Gardner-Greene-Kruskal-Miura (GGKM) IST for the KdV equation [45] to first solve the initial value problem for the *complexified* (dependent variable only) KdV equation, and thence use (2.1.17) to construct the solution of (3.1.1). In this chapter we will develop and elucidate the KdV based scheme to solve the MKdV equation.

The technique whereby one solves a nonlinear evolution equation by first using an

the Miura transformation connecting the auxiliary and original equations so as to determine

the initial value problem for the Gardner equation [86]

$$\varpi_t + 6a\varpi\varpi_x + 6b\varpi^2\varpi_x + \varpi_{xxx} = 0 \qquad (b > 0), \tag{3.1.2}$$

where a and b are constants and $\varpi \equiv \varpi(x,t)$. Wadati [120] used the following *matrix* scattering problem to solve the Cauchy problem for (3.1.2):

$$\Im\phi = \kappa^2\phi, \tag{3.1.3}$$

where

$$\Im = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\frac{\partial^2}{\partial x^2} + \begin{pmatrix} \Omega & 0 \\ 0 & \Omega^* \end{pmatrix},$$

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix},$$

$\kappa \in \Re$ and Ω^* denotes the complex conjugate of Ω. The complex-valued potential, $\Omega \equiv \Omega(x,t)$, in the eigenvalue problem (3.1.3) is related to the dependent variable in (3.1.2) by the equation [120]

$$\Omega = -a\varpi - b\varpi^2 - i\sqrt{b}\varpi_x.$$

Equations (3.1.1) and (3.1.2) are (gauge) equivalent, that is the transformations

$$\varpi = \frac{v}{\sqrt{b}} - \frac{a}{2b},\ x' = x + \frac{3a^2}{2b}t \text{ and } t' = t$$

converts (3.1.2) into (3.1.1). Hence, the work in Ref. 120 uses the Miura transformation (2.1.17) to solve the initial value problem for (3.1.1).

The method that we will present to solve the initial value problem for (3.1.1)

1) an *explicit* complexification of the KdV equation will be used in our method, whereas Wadati [120] makes no reference to the complexified KdV equation in his work;

2) the KdV scattering problem (a scalar scattering problem) will be used in our method to solve the complexified KdV equation, whereas Wadati [120] employs the *matrix* scattering problem (3.1.3) in his method.

solutions of the KdV and MKdV equations can be used to solve the initial value problem for the latter equation. Although the material in this chapter provides a new procedure from which to build and view solutions of (3.1.1), we consider that the primary significance of this chapter resides in its potential to establish a conceptual bridge to the material in Chapters 4 and 5 of this thesis.

The Miura transformation that connects solutions of the KdV equation,

$$u_t + 6uu_x + u_{xxx} = 0, \tag{3.1.4}$$

to solutions of equation (3.1.1) is pivotal to the development of our procedure to solve the initial value problem for (3.1.1). According to (2.1.17), the Miura transformation that connects solutions of the KdV and MKdV equations is

$$u = v^2 + iv_x, \tag{3.1.5}$$

where $v \equiv v(x,t)$ and $u \equiv u(x,t)$ satisfy (3.1.1) and (3.1.4), respectively.

Given a real-valued solution, $v \equiv v(x,t)$, of (3.1.1), the reader can verify that (3.1.5) produces a complex-valued solution of (3.1.4), namely a complex-valued function of the two real variables x and t that satisfies equation (3.1.4). If the *known* real-valued initial solution, $v_0 \equiv v(x,0)$, of (3.1.1) is used in (3.1.5), then

$$u_0 \equiv u(x,0) = v_0^2 + iv_{0,x} \tag{3.1.6}$$

is a complex-valued solution of (3.1.4) at $t = 0$. Here and henceforth, we will use the notation $v_{0,x}$ to denote $\partial v_0 / \partial x$. We note that if v_0 and $v_{0,x}$ decay to zero as $|x| \to \infty$, then according to (3.1.6) we have $u_0 \to 0$ as $|x| \to \infty$.

Suppose, now, that $u(x,t)$ is a complex-valued solution of (3.1.4), which has evolved from the *known* initial value (3.1.6). The source from which the initial value (3.1.6) originates permits us to conclude that

$$v_x(x,t) = \text{Im}\,\{u(x,t)\} \tag{3.1.7}$$

for all $x \in \Re$ and $t \geq 0$. According to (3.1.7), the solution of the initial value problem for (3.1.1) with initial value v_0 and boundary conditions $v_0 \to 0$ and $v_{0,x} \to 0$ as $|x| \to \infty$ can be constructed from the solution of the initial value problem for the KdV equation with *complex* initial value (3.1.6). The solution of the complexified initial value problem for the KdV equation can be achieved by an extension of the GGKM IST for the KdV equation that incorporates the situation in which the desired solution evolves from the complex initial value (3.1.6). Once the complexified initial value problem for $u(x,t)$ is solved, $v_x(x,t)$ is determined from (3.1.7), and thence $v(x,t)$ is constructed from a single quadrature with respect to x. The function of integration that remains in $v(x,t)$ is fixed by the boundary conditions $v(x,t) \to 0$ (uniformly in t) as $|x| \to \infty$.

The procedure that has been outlined to solve the Cauchy problem for (3.1.1) *cannot* be applied to solve the initial value problem for the defocusing form of the MKdV equation:

$$v_t - 6v^2 v_x + v_{xxx} = 0. \tag{3.1.8}$$

The source of the limitation associated with (3.1.8) resides in the Miura transformation [86],

$$u = -\left(v^2 + v_x\right), \tag{3.1.9}$$

that connects solutions of equations (3.1.4) and (3.1.8). Equation (3.1.9) maps *real-valued*

solutions of (3.1.8) into *real-valued* solutions of (3.1.4). The absence of a coupling between complex-valued solutions of (3.1.4) and real-valued solutions of (3.1.8) renders ineffective any complexification procedure designed to solve the latter equation. Huang, Chen and Chen [54] have recently used the IST to solve the Cauchy problem for (3.1.8) with a nonzero (and symmetrical) boundary value as $|x| \to \infty$.

At the present time, the Riemann-Hilbert (RH) problem provides the most effective mathematical framework in which to implement the IST. In order to achieve the desired complex extension of the IST for the KdV equation, we will adopt a RH based approach to the IST for the KdV equation [11,39,42,125], rather than the original IST used by GGKM [45,46] to solve the KdV equation.

The remainder of this chapter is divided into four sections. Section 3.2 is devoted to an analysis of the direct problem for the *x*-part of the Lax Pair associated with the complex KdV equation. Once suitable Jost functions are introduced into our work, the scattering data associated with the IST for the complex-valued KdV equation are calculated in Section 3.2. It is well known that the discrete eigenvalues in the IST for the *real* KdV equation [45,46] reside on the imaginary axis of the plane formed by an analytic continuation of the spectral parameter off its real axis. An interesting feature of the scattering data for the complexified KdV initial value problem is that the discrete eigenvalues are *not* restricted to lie on the imaginary axis of the plane formed from the complex extension of the spectral parameter. Section 3.3 is devoted to a formulation and solution of the inverse problem for the complexified KdV equation. A specific extension of the RH boundary value problem (1.3.64) will be used extensively in Section 3.3 to satisfy our objectives. The time evolution of the scattering data is computed in Section 3.4. The last section of this chapter connects the results of Sections 3.2, 3.3 and 3.4 with the solution of the initial value problem for equation (3.1.1). In Section 3.5 we also demonstrate how our method delivers the correct 1-soliton solution for the MKdV equation (3.1.1). We conclude this section by offering the following two remarks:

1) in the remainder of this chapter we have endeavoured to be consistent with the notation used in the well known works [11,42] in which the RH based approach to the IST for the real-valued KdV equation is presented, particularly Ref. 11;

2) our basic philosophy throughout this chapter is to work towards establishing a paradigm that will assist us to solve the initial value problems for the MILW and MBO equations.

Section 3.2: The Direct Problem

The linear scattering problem for the KdV equation, in a form that is suitable for application to the MKdV equation, is [45]

$$\psi_{xx} = -\left(u + \kappa\right)\psi, \tag{3.2.1}$$

$$\psi_t = 2\left(2\kappa - u\right)\psi_x + \left(u_x + \omega\right)\psi. \tag{3.2.2}$$

In equations (3.2.1) and (3.2.2), $\psi \equiv \psi(x,t)$, κ and ω are arbitrary constants, and $u \equiv u(x,t)$ is a *complex-valued* solution of equation (3.1.4); the real and imaginary parts of u are specified by equation (3.1.5). It is an elementary task to verify that the compatibility condition $\psi_{txx} \equiv \psi_{xxt}$ results in the KdV equation,

$$u_t + 6uu_x + u_{xxx} = 0.$$

The direct problem maps the known initial value for u, which in our case is denoted as u_0 and specified by (3.1.6), into the scattering data. The direct problem is concerned exclusively with equation (3.2.1). In order to examine the direct problem for (3.2.1), we will introduce into our analysis a *real-valued* spectral parameter, denoted as k, through the boundary conditions

$$\psi \to e^{\pm i\varepsilon kx} \text{ as } |x| \to \infty. \tag{3.2.3}$$

$\varepsilon = \pm 1$, enters naturally into our analysis as a reflection of the intrinsic freedom available in certain normalizations that will appear in future calculations.

The boundary conditions (3.2.3) impose the requirement

$$\kappa = k^2$$

for (3.2.3) to be consistent with equations (3.2.1) and (3.2.2). Replacing κ with k^2 in equations (3.2.1) and (3.2.2) we obtain the following system of equations:

$$\psi_{xx} = -\left(u + k^2\right)\psi; \tag{3.2.4}$$

$$\psi_t = 2\left(2k^2 - u\right)\psi_x + \left(u_x + \omega\right)\psi. \tag{3.2.5}$$

Let $m(x;k)$ and $\bar{m}(x;k)$ denote *left* Jost functions, and let $n(x;k)$ and $\bar{n}(x;k)$ denote *right* Jost functions. The Jost functions satisfy equations (3.2.4) and (3.2.5), and are distinguished from each other by their asymptotic behaviour:

$$\left.\begin{aligned} m(x;k) &\to e^{-i\varepsilon kx} \\ \bar{m}(x;k) &\to e^{i\varepsilon kx} \end{aligned}\right\} \quad \text{as} \quad x \to -\infty \tag{3.2.6}$$

and

$$\left.\begin{aligned} n(x;k) &\to e^{i\varepsilon kx} \\ \bar{n}(x;k) &\to e^{-i\varepsilon kx} \end{aligned}\right\} \quad \text{as} \quad x \to \infty. \tag{3.2.7}$$

Two aspects of our notation require clarification:

1) Whenever there is no likelihood of confusion to the reader, we will *not* display the explicit t-dependence of any variable that is a function of t. For example, we will write $m(x;k)$ in lieu of $m(x,t;k)$;

2) $\bar{m}(x;k)$ is *not* the complex conjugate of $m(x;k)$. The notation $m^*(x;k)$ will be used to denote the complex conjugate of $m(x;k)$. The remark about $\bar{m}(x;k)$ also applies (suitably modified) to $\bar{n}(x;k)$.

Any implementation of the IST that uses the RH problem requires a knowledge of the analytical character of the Jost functions; the analyticity referred to here is with respect to the complex extension of the spectral parameter, k. In the remainder of this

k k_R and k_I denote two real numbers, then $k = k_R + ik_I$ constitutes the complex extension of the spectral parameter. The most efficient method by which we can ascertain the analytic character of the Jost functions is to use the transformation

$$W = \psi e^{i\varepsilon kx} \tag{3.2.8}$$

to modify the asymptotics of the Jost functions. The equations

$$W_{xx} = 2i\varepsilon k W_x - uW \tag{3.2.9}$$

and

$$W_t = 2\left(2k^2 - u\right)W_x + \left(u_x + 2i\varepsilon ku + \gamma\right)W, \tag{3.2.10}$$

where

$$\gamma = \omega - 4i\varepsilon k^3,$$

emerge when we use (3.2.8) to eliminate ψ from equations (3.2.4) and (3.2.5).

Equation (3.2.8) induces a change in the asymptotics of the Jost functions. The revised asymptotics of the Jost functions are as follows:

$$\left.\begin{aligned} M(x;k) &\to 1 \\ \bar{M}(x;k) &\to e^{2i\varepsilon kx} \end{aligned}\right\} \quad \text{as} \quad x \to -\infty \tag{3.2.11}$$

and

$$\left.\begin{aligned} N(x;k) &\to e^{2i\varepsilon kx} \\ \bar{N}(x;k) &\to 1 \end{aligned}\right\} \quad \text{as} \quad x \to \infty. \tag{3.2.12}$$

Each of the functions $M(x;k)$, $N(x;k)$, $\bar{M}(x;k)$ and $\bar{N}(x;k)$ satisfies equations (3.2.9) and (3.2.10). The relationships between the original and modified eigenfunctions are as follows:

$$(M,\, N) = (m,\, n)e^{i\varepsilon kx}$$

and

$$(\bar{M},\, \bar{N}) = (\bar{m},\, \bar{n})e^{i\varepsilon kx}.$$

We can determine the analytic character of the Jost functions M, N, $\bar{M}$ and $\bar{N}$ by converting (3.2.9) into an appropriate linear integral equation. The general solution of (3.2.9) in the form of a linear integral equation is

$$W(x;k) = W_0(x;k) + \int_{-\infty}^{\infty} G(x,\xi;k)\,u(\xi)\,W(\xi;k)\,\mathrm{d}\xi, \tag{3.2.13}$$

where $W_0(x;k)$ represents a solution of the homogeneous version ($u \equiv 0$) of (3.2.9), and $G(x,\xi;k)$ satisfies the equation

$$G_{xx}(x,\xi;k) - 2i\varepsilon k G_x(x,\xi;k) = -\delta(x-\xi), \tag{3.2.14}$$

where $\delta(x-\xi)$ denotes the Dirac delta function.

Equation (3.2.14) is autonomous, so it is natural to seek its solution in the form

$$G(x,\xi;k) = \frac{1}{2\pi}\int_C g(p;k)e^{i(x-\xi)p}\,\mathrm{d}p, \tag{3.2.15}$$

where C is a horizontal line (suitably chosen) of infinite extent in the complex p-plane, and $g(p;k)$ is an unknown function. Substituting (3.2.15) into (3.2.14), and then using a

suitable integral representation of the delta function [100],

$$\delta(x) = \frac{1}{2\pi}\int_C e^{ixp}\,\mathrm{d}p,$$

we obtain the concrete form of $g(p;k)$, namely

$$g(p;k) = \frac{1}{p(p-2\varepsilon k)}. \tag{3.2.16}$$

The function $g(p;k)$ has real simple poles at $p = 0$ and $p = 2\varepsilon k$. In order to avoid the poles on the axis $\mathrm{Im}(p) = 0$, the contour C must be either elevated into the half-plane $\mathrm{Im}(p) > 0$, or lowered into the half-plane $\mathrm{Im}(p) < 0$. The two possible outcomes that arise when (3.2.16) is used to evaluate (3.2.15) are as follows:

$$G_{-\varepsilon}(x,\xi;k) = \frac{i\varepsilon}{2k}\left(1 - e^{2i\varepsilon k(x-\xi)}\right)\theta(\xi - x); \tag{3.2.17}$$

$$G_{+\varepsilon}(x,\xi;k) = -\frac{i\varepsilon}{2k}\left(1 - e^{2i\varepsilon k(x-\xi)}\right)\theta(x - \xi). \tag{3.2.18}$$

Equation (3.2.17) corresponds to the selection $C = -\infty + i0^+$ to $\infty + i0^+$, whereas (3.2.18) is associated with contour $C = -\infty - i0^+$ to $\infty - i0^+$. Here and henceforth, $\theta(\bullet)$ will denote the Heaviside step function:

$$\theta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0. \end{cases}$$

The subscripts attached to $G(x,\xi;k)$ in equations (3.2.17) and (3.2.18) signify the analytic character of the function $G(x,\xi;k)$ in the complex k-plane: $G_{\pm\varepsilon}(x,\xi;k)$ admits analytic continuation into the half-plane $\mathrm{sgn}(\mathrm{Im}(k)) = \pm\varepsilon$, and we refer to such functions as $(\pm\varepsilon)$ functions in k

The jump in the sectionally analytic function $G(x,\xi;k)$ across the $\mathrm{Im}(k) = 0$ axis can now be quantified. According to equations (3.2.17) and (3.2.18), the jump in the function $G(x,\xi;k)$ across the $\mathrm{Im}(k) = 0$ axis is

$$G_{+\varepsilon}(x,\xi;k) - G_{-\varepsilon}(x,\xi;k) = \frac{\varepsilon}{2ik}\left(1 - e^{2i\varepsilon k(x-\xi)}\right). \tag{3.2.19}$$

The boundary conditions (3.2.11) and (3.2.12) oblige us to associate $G_{+\varepsilon}(x,\xi;k)$ with the *left* Jost functions $M(x;k)$ and $\bar{M}(x;k)$, and to affiliate $G_{-\varepsilon}(x,\xi;k)$ with the *right* Jost functions $N(x;k)$ and $\bar{N}(x;k)$. A compendium of the linear integral equations that characterize the Jost functions is as follows:

$$\begin{pmatrix} M(x;k) \\ \bar{M}(x;k) \end{pmatrix} = \begin{pmatrix} 1 \\ e^{2i\varepsilon kx} \end{pmatrix} + \int_{-\infty}^{\infty} G_{+\varepsilon}(x,\xi;k)u(\xi)\begin{pmatrix} M(\xi;k) \\ \bar{M}(\xi;k) \end{pmatrix} d\xi; \tag{3.2.20}$$

$$\begin{pmatrix} \bar{N}(x;k) \\ N(x;k) \end{pmatrix} = \begin{pmatrix} 1 \\ e^{2i\varepsilon kx} \end{pmatrix} + \int_{-\infty}^{\infty} G_{-\varepsilon}(x,\xi;k)u(\xi)\begin{pmatrix} \bar{N}(\xi;k) \\ N(\xi;k) \end{pmatrix} d\xi. \tag{3.2.21}$$

Equations (3.2.20) and (3.2.21) are second-kind Volterra integral equations. Provided that all the appropriate regularity conditions are satisfied, we can deduce the following information from equations (3.2.20) and (3.2.21):

(a) $M(x;k)$ is a $(+\varepsilon)$ function in k;

(b) $\bar{N}(x;k)$ is a $(-\varepsilon)$ function in k;

(c) $\bar{M}(x;k)$ cannot be analytically continued off the real k-axis;

(d) $N(x;k)$ cannot be analytically continued off the real k-axis.

We are prohibited from analytically continuing the Jost functions $\bar{M}(x;k)$ and $N(x;k)$ off the real k-axis because of the exponential-type forcing term in the integral equations that characterize the Jost functions $\bar{M}(x;k)$ and $N(x;k)$.

The eigenfunctions $M(x;k)$, $N(x;k)$ and $\bar{N}(x;k)$ are connected by the completeness relation [42,125]

$$M(x;k) = a(k)\bar{N}(x;k) + b(k)N(x;k), \tag{3.2.22}$$

where $a(k)$ and $b(k)$, which are members of the *continuous spectrum*, are defined by the following functionals:

$$a(k) = 1 - \frac{i\varepsilon}{2k}\int_{-\infty}^{\infty} M(\xi;k)u(\xi)\,d\xi; \tag{3.2.23}$$

$$b(k) = \frac{i\varepsilon}{2k}\int_{-\infty}^{\infty} M(\xi;k)u(\xi)e^{-2i\varepsilon k\xi}d\xi. \tag{3.2.24}$$

The formula (3.2.23) reveals $a(k)$ to be a $(+\varepsilon)$ function in the complex k-plane because $a(k)$ inherits the analytic character of the eigenfunction $M(x;k)$. The exponential in the integrand of (3.2.24) prevents us from analytically continuing $b(k)$ off the real k-axis.

One method by which equations (3.2.23) and (3.2.24) can be derived relies on the behaviour of $M(x;k)$ as $x \to \infty$. According to equations (3.2.12) and (3.2.22), we have

$$M(x;k) \to a(k) + b(k)e^{2i\varepsilon kx} \text{ as } x \to \infty. \tag{3.2.25}$$

An alternative to equation (3.2.25) is

$$M(x;k) \to 1 - \frac{i\varepsilon}{2k}\int_{-\infty}^{\infty} M(\xi;k)u(\xi)\left\{1 - e^{2i\varepsilon k(x-\xi)}\right\}d\xi \text{ as } x \to \infty. \tag{3.2.26}$$

We can derive equation (3.2.26) by substituting (3.2.17) into (3.2.20), and then considering the limit $x \to \infty$ in the resultant equation. Equating the coefficients of $e^{2i\varepsilon kx}$ in equations (3.2.25) and (3.2.26), and then equating the residual terms from these equations we obtain the formulae (3.2.23) and (3.2.24). The reader who wishes to see a derivation of equations (3.2.23) and (3.2.24) that differs from the derivation we have presented can consult Ref. 42.

An important result that emerges from the formulae that characterize $a(k)$ and $M(x;k)$ is that each of these quantities possesses the canonical normalization, in other words the large-k asymptotics of $a(k)$ and $M(x;k)$ have unity as their leading term:

$$\left.\begin{aligned} M(x;k) &= 1 - \frac{i\varepsilon}{2k}\int_{-\infty}^{x} u(\xi)d\xi + O\left(\frac{1}{k^2}\right) \\ a(k) &= 1 - \frac{i\varepsilon}{2k}\int_{-\infty}^{\infty} u(\xi)d\xi + O\left(\frac{1}{k^2}\right) \end{aligned}\right\} \text{ as } k \to \infty. \tag{3.2.27}$$

Hitherto, our analysis of the direct problem for (3.2.4) has focused exclusively upon the *continuous spectrum* for this particular eigenvalue problem. Discrete (bound) states for the spectral problem (3.2.4) are also possible. The *discrete spectrum* for (3.2.4) consists of a finite number of discrete eigenvalues, which we will denote as k_j, and an associated finite set of exponentially localized eigenfunctions that are characterized by the boundary conditions

$$m(x;k_j) \to 0 \text{ as } x \to -\infty \tag{3.2.28}$$

and

$$n(x;k_j) \to 0 \text{ as } x \to \infty. \tag{3.2.29}$$

The discrete eigenvalues for (3.2.4) display the following attributes:

1) $j = 1, 2, \cdots, q$ (where q is a finite positive integer);

2) $\operatorname{sgn}\left(\operatorname{Im}\left(k_j\right)\right) = \varepsilon$;

3)
; (3.2.30)

4) (3.2.31)

, where $b_j \underline{\underline{\text{def}}}\ b\left(k_j\right)$.

In the case of the *real-valued* KdV equation, (3.2.4) is self-adjoint. Hence, any discrete eigenvalue of the spectral problem (3.2.4) is restricted to the imaginary axis in the complex k-plane [125]. Removal of the reality constraint on $u(x,t)$ results in discrete eigenvalues for which $\operatorname{Re}\left(k_j\right) \neq 0$ because (3.2.4) ceases to be self-adjoint. We can demonstrate the increase in mobility of the discrete eigenvalues that occurs whenever $u(x,t)$ is a complex-valued solution of the KdV equation by using the initial value

$$u(x,0) = 2\alpha\delta(x),\ \alpha = \alpha_R + i\alpha_I, \tag{3.2.32}$$

where $\operatorname{Re}(\alpha) > 0$, $\operatorname{Im}(\alpha) \neq 0$ and $\delta(x)$ is the Dirac delta function, to calculate the associated discrete eigenvalue(s). The reader may wish to note that the assumptions made about the complex constant α are consistent with (3.1.4). Substituting (3.2.32) into (3.2.20), and then using the filtering property of the delta function [100], namely the property

$$\int_{-\infty}^{\infty} f(x)\delta(x)\,\mathrm{d}x = f(0),$$

we obtain the formula

$$M(x,0;k) = 1 - \frac{i\varepsilon\alpha}{k}\left(1 - e^{2i\varepsilon kx}\right)M(0,0;k)\theta(x),$$

where $M(\bullet,0;k) \equiv M(\bullet,t;k)\big|_{t=0}$. It is appropriate that we make the assignment

$$M(0,0;k) = 1$$

because

$$\lim_{x \to 0^{\pm}} M(x,0;k) = 1.$$

Substituting (3.2.32) and the assigned value for $M(0,0;k)$ into equation (3.2.23) we obtain

$$a(k,0) = 1 - \frac{i\varepsilon\alpha}{k}.$$

According to (3.2.30), discrete eigenvalues occur at the zeroes of $a(k,0)$. Solving the equation $a(k,0) = 0$ we find that $k_1 = -\varepsilon\alpha_I + i\varepsilon\alpha_R$ is the solitary discrete eigenvalue.

The property $\mathrm{Re}\left(k_1\right) \neq 0$ reflects the increase in mobility of the discrete eigenvalues, namely whenever $u(x,t)$ is a complex-valued solution of the KdV equation the discrete eigenvalues are *not* restricted to the imaginary axis of the complex *k*-plane.

Ablowitz and Clarkson [11] provide a detailed argument that determines the location (in a qualitative sense) of the discrete eigenvalues in the IST for the real-valued KdV equation. Suitable modifications of the Ablowitz and Clarkson proof will allow us to scrutinize the direct problem associated with (3.2.4) in the transition from a real-valued to a complex-valued solution of the KdV equation. Consider, then, the discrete eigenfunction $m\left(x;k_j\right)$; the large-*x* asymptotics of $m\left(x;k_j\right)$ are fixed by equations (3.2.28), (3.2.29) and (3.2.31). The eigenfunction $m\left(x;k_j\right)$ satisfies [see (3.2.4)] the differential equation

$$m_{xx}\left(x;k_j\right) + u(x)m\left(x;k_j\right) + k_j^2 m\left(x;k_j\right) = 0, \tag{3.2.33}$$

where

$$u(x) = v^2(x) + iv_x(x).$$

It is an opportune moment to remind the reader that subscripts involving *x* or *t* denote partial derivatives, for example $m_{xx} = \partial^2 m/\partial x^2$. The complex conjugate of (3.2.33) is

$$\left\{m^*\left(x;k_j\right)\right\}_{xx} + u^*(x)m^*\left(x;k_j\right) + \left(k_j^2\right)^* m^*\left(x;k_j\right) = 0, \tag{3.2.34}$$

where

$$u^*(x) = v^2(x) - iv_x(x).$$

Multiplying (3.2.33) by $m^*\left(x;k_j\right)$, multiplying (3.2.34) by $m\left(x;k_j\right)$, and then subtracting one of the resultant equations from the other enables us to derive the *balance equation*,

$$m^*\left(x;k_j\right)m_{xx}\left(x;k_j\right) - m\left(x;k_j\right)\left\{m^*\left(x;k_j\right)\right\}_{xx} + 2iv_x\left|m\left(x;k_j\right)\right|^2 + \left\{k_j^2 - \left(k_j^2\right)^*\right\}\left|m\left(x;k_j\right)\right|^2 = 0.$$

The simplified equation

$$\left\{k_j^2 - \left(k_j^2\right)^*\right\}\left\|m\left(x;k_j\right)\right\|^2 = -2i\int_{-\infty}^{\infty} v_x\left|m\left(x;k_j\right)\right|^2 \mathrm{d}x, \tag{3.2.35}$$

where

$$\left\|m\left(x;k_j\right)\right\| \underline{\underline{\mathrm{def}}} \left(\int_{-\infty}^{\infty} m\left(x;k_j\right)m^*\left(x;k_j\right)\mathrm{d}x\right)^{1/2},$$

is formed when we integrate the balance equation over the interval $-\infty < x < \infty$.

At this stage of our work it is prudent to compare equation (3.2.35) to the corresponding equation when $u(x,t)$ is a real-valued solution of the KdV equation. Ablowitz and Clarkson [11] show that the equation

$$\left\{k_j^2 - \left(k_j^2\right)^*\right\} \left\| m\left(x; k_j\right) \right\|^2 = 0$$

is the analogue of (3.2.35), whenever $u(x,t)$ is a real-valued solution of the KdV equation. The equation derived by Ablowitz and Clarkson forces k_j^2 to be *real*, and because the localized boundary conditions that characterize the discrete spectrum require $\mathrm{Im}\left(k_j\right) \neq 0$, each discrete eigenvalue in the IST for the real-valued KdV equation is pure imaginary.

Let us now return to the situation in which $u(x,t)$ satisfies the complex-valued KdV equation. Generically, we would expect v_x and $\left| m\left(x; k_j\right) \right|^2$ *not* to be orthogonal in the interval $-\infty < x < \infty$. Therefore, the presence in (3.2.35) of a (generically) nonzero right hand side liberates us from the restriction $k_j^2 \in \Re$. Freedom from the restriction $k_j^2 \in \Re$ means that whenever $u(x,t)$ is a complex-valued solution of the KdV equation, the spectral problem (3.2.4) can display discrete eigenvalues that possess a non-zero real part.

Extrapolating from the conclusions we reached as a result of calculating the discrete spectrum for the complex initial value (3.2.32) prompts us to suggest that $a(k)$ possesses a *simple* zero at $k = k_j$, where $j = 1, 2, \cdots, q$. The zeroes of $a(k)$ are indeed simple whenever $u(x,t)$ is a real-valued solution of the KdV equation [11,125]. We now examine the order of the zero in $a(k)$ at $k = k_j$ for the case in which $u(x,t)$ satisfies the complex-valued KdV equation. Consider the differential equation satisfied by the left Jost function $m(x;k)$:

$$m_{xx}(x;k) + \left\{u(x) + k^2\right\} m(x;k) = 0. \tag{3.2.36}$$

The equation

$$m_{kxx}(x;k) + 2km(x;k) + \left\{u(x) + k^2\right\} m_k(x;k) = 0 \tag{3.2.37}$$

is formed when we differentiate both sides of (3.2.36) with respect to k. We can eliminate $u(x)$ by multiplying (3.2.36) by $m_k(x;k)$ and (3.2.37) by $m(x;k)$, and then subtracting the new version of equation (3.2.37) from the new version of equation (3.2.36). The equation at the end of the process we have described is as follows:

$$m_{xx}(x;k)m_k(x;k) - m_{kxx}(x;k)m(x;k) = 2km^2(x;k).$$

The left hand side of the above equation, surprisingly, is a perfect derivative with respect to x, namely

$$\frac{\partial}{\partial x}\{m_x(x;k)m_k(x;k) - m_{kx}(x;k)m(x;k)\} = 2km^2(x;k). \tag{3.2.38}$$

Now let us suppose that $k = k_j$ is a discrete eigenvalue of the spectral problem (3.2.4). As a direct consequence of our assumption about $k = k_j$ we have $\operatorname{sgn}\left(\operatorname{Im}(k_j)\right) = \varepsilon$ and also that equations (3.2.30) and (3.2.31) are satisfied. Evaluating (3.2.38) at $k = k_j$, integrating over the interval $-\infty < x < \infty$, and then using (3.2.31) we obtain the equation

$$\left[m_x(x;k_j)m_k(x;k_j) - m_{kx}(x;k_j)m(x;k_j)\right]_{x=-\infty}^{x=\infty} = 2k_j b_j^2 \int_{-\infty}^{\infty} n^2(x;k_j)\,\mathrm{d}x. \tag{3.2.39}$$

The contribution to the *left hand side* of (3.2.39) as $x \to -\infty$ is

$$\lim_{x\to-\infty}\left\{m_x(x;k_j)m_k(x;k_j) - m_{kx}(x;k_j)m(x;k_j)\right\} = 0 \tag{3.2.40}$$

because from (3.2.28) we see that $m(x;k_j)$ is recessive as $x \to -\infty$. At the right extremity we have

$$\lim_{x\to\infty}\left\{m_x(x;k_j)m_k(x;k_j) - m_{kx}(x;k_j)m(x;k_j)\right\} = 2i\varepsilon k_j \dot{a}(k_j)b(k_j), \tag{3.2.41}$$

where

$$\dot{a}(k_j) \equiv \left.\frac{\partial a}{\partial k}\right|_{k=k_j}.$$

The derivation of equation (3.2.41) employs (3.2.7), the identity [see (3.2.8) and (3.2.22)]

$$m(x;k) = a(k)\bar{n}(x;k) + b(k)n(x;k),$$

(3.2.30) and (3.2.31) to compute

$$\lim_{x\to\infty}\left\{m_x(x;k_j)m_k(x;k_j) - m_{kx}(x;k_j)m(x;k_j)\right\}.$$

Equations (3.2.40) and (3.2.41) make it possible for us to simplify equation (3.2.39). Substituting equations (3.2.40) and (3.2.41) into equation (3.2.39), and then rearranging the resultant equation so as to make the nonlocal term the subject of the formula we arrive at the following equation:

$$\int_{-\infty}^{\infty} n^2\left(x;k_j\right)\mathrm{d}x = i\varepsilon\frac{\dot{a}\left(k_j\right)}{b\left(k_j\right)}. \tag{3.2.42}$$

Equation (3.2.42) is also valid when $u(x,t)$ is a real-valued solution of the KdV equation [11,125]. We have already mentioned that each discrete eigenvalue in the IST for the real-valued KdV equation has the form $k_j = i\varepsilon\kappa_j$, where $\kappa_j > 0$. The reader can verify by using (3.2.17) that $G_{-\varepsilon}\left(x,\xi;i\varepsilon\kappa_j\right)$ is *real*. If $u(x,t)$ and $G_{-\varepsilon}\left(x,\xi;i\varepsilon\kappa_j\right)$ are real-valued functions, then the kernel of the integral equation (3.2.21) is real, which in turn forces the function $n^2\left(x;k_j\right)$ to be real [60]. The function $n^2\left(x;k_j\right)$ ceases being real whenever $u(x,t)$ is a complex-valued solution of the KdV equation because in this situation the kernel of the integral equation (3.2.21) is complex at all points of the discrete spectrum. We close this section with the observation that the complex-valued character of $n^2\left(x;k_j\right)$ exposes the possibility that the left hand side of (3.2.42) can vanish, and therefore second-order zeroes in $a(k)$ at $k = k_j$ are permitted in the discrete spectrum for the complex-valued KdV equation. Generically, however, the integral in (3.2.42) will be non-zero, so it is plausible to assume, as we will, that the zeroes of $a(k)$ are all simple.

Section 3.3: The Inverse Problem

All the preliminary results required to proceed to a solution of the *inverse problem* were derived in Section 3.2. The solution of the inverse problem expresses $u(x)$ in terms of the scattering data,

$$a(k),\ b(k),\ \text{and}\ k_j,\ \text{where}\ j = 1, 2, \cdots, q.$$

Generating the solution of the inverse problem for the complex-valued KdV equation necessitates that we interpret (3.2.22) as an appropriate Riemann-Hilbert Boundary Value Problem (RHBVP) in the complex k-plane. We will initiate the process of viewing (3.2.22) as a RHBVP by collecting the analytic and asymptotic properties of all terms that appear in equation (3.2.22). The relevant properties, designated P1, P2 and so forth, are as follows:

P1: $M(x;k)$ and $a(k)$ are $(+\varepsilon)$ functions in k;

P2: $\bar{N}(x;k)$ is a $(-\varepsilon)$ function in k;

P3: $N(x;k)$ and $b(k)$ *cannot* be analytically continued off the real k-axis;

P4: $a(k)$ has a *simple* zero at the discrete eigenvalue k_j, where $j = 1, 2, \cdots, q$ and $\operatorname{sgn}\left(\operatorname{Im}(k_j)\right) = \varepsilon$;

P5: The large-k asymptotics of $M(x;k)$, $\bar{N}(x;k)$ and $a(k)$ are

$$\begin{pmatrix} M(x;k) \\ \bar{N}(x;k) \\ a(k) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{i\varepsilon}{2k} \int_{-\infty}^{\infty} \begin{pmatrix} \theta(x-\xi) \\ -\theta(\xi - x) \\ 1 \end{pmatrix} d\xi + O\left\{ \frac{1}{k^2} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} \text{ as } k \to \infty.$$

Section 3.2 of this thesis contains derivations of P1-P5.

In order to interpret (3.2.22) as a RHBVP in the complex k-plane, we must supplement the information contained in P1-P5 with the connection formula

$$\bar{N}(x;k) = N(x;-k)e^{2i\varepsilon kx}. \tag{3.3.1}$$

Equation (3.3.1) can be derived as follows:

1) Perform the transition $k \to -k$ in equation (3.2.17), and then verify the correctness of the identity

$$G_{-\varepsilon}(x, \xi; -k) = G_{-\varepsilon}(x, \xi; k) e^{-2i\varepsilon k(x-\xi)}; \tag{3.3.2}$$

2) Use (3.2.21) and (3.3.2) to derive the homogeneous Volterra integral equation

$$\vartheta(x; k) = \int_{-\infty}^{\infty} G_{-\varepsilon}(x, \xi; k) u(\xi) \vartheta(\xi; k) d\xi,$$

where

$$\vartheta(x; k) \underline{\text{def}} \; \bar{N}(x; k) - N(x; -k) e^{2i\varepsilon kx};$$

3) Assume that the homogeneous version of (3.2.13) possesses only the *trivial* solution. The assumption that the homogeneous version of (3.2.13) possesses only the trivial solution leads immediately to the conclusion $\vartheta(x; k) \equiv 0$, and hence we arrive at (3.3.1).

The function $M(x; k) / a(k)$, by virtue of P1 and P4, is a $(+\varepsilon)$ function in k, except for a finite number of simple poles at $k = k_j$, where $j = 1, 2, \cdots, q$. The meromorphic function $M(x; k) / a(k)$ admits the representation

$$\frac{M(x; k)}{a(k)} = 1 + \sum_{j=1}^{q} \frac{\hat{A}_j(x)}{k - k_j} + \mu_{+\varepsilon}(x; k), \tag{3.3.3}$$

where $\mu_{+\varepsilon}(x; k)$ is a $(+\varepsilon)$ function in k, and $\hat{A}_j(x)$ denotes the residue of $M(x; k) / a(k)$ at $k = k_j$. The function $\mu_{+\varepsilon}(x; k)$, as a result of P5, satisfies the boundary condition $\mu_{+\varepsilon}(x; k) \to 0$ as $k \to \infty$ (along the real k-axis).

Substituting (3.3.1) and (3.3.3) into (3.2.22) we obtain

$$\sum_{j=1}^{q} \frac{\hat{A}_j(x)}{k - k_j} + \mu_{+\varepsilon}(x; k) = \bar{N}(x; k) - 1 + r(k) \bar{N}(x; -k) e^{2i\varepsilon kx}, \tag{3.3.4}$$

where

$$r(k) \underline{\text{def}} \; \frac{b(k)}{a(k)}. \tag{3.3.5}$$

As a result of the nomenclature that has developed around the IST, practitioners of the IST frequently refer to $r(k)$

The residue, $\hat{A}_j(x)$, is fixed when we expand both sides of equation (3.3.4) into a Laurent series about the isolated simple pole at $k = k_j$, and then match the coefficients of $1/(k - k_j)$ either side of this equation. We find that

$$\hat{A}_j(x) = \frac{b(k_j)}{\dot{a}(k_j)} \bar{N}(x; -k_j) e^{2i\varepsilon k_j x}. \tag{3.3.6}$$

A compact way to express (3.3.6) is achieved when we introduce into our work the *normalizing coefficient*, which we will denote as c_j, defined by the equation

$$c_j \underline{\underline{\text{def}}} \frac{b(k_j)}{i\dot{a}(k_j)}; \; j = 1, 2, \cdots, q. \tag{3.3.7a}$$

$$\varepsilon c_j \int_{-\infty}^{\infty} n^2(x; k_j)\, dx = 1, \tag{3.3.7b}$$

which can be confirmed through a comparison of equation (3.2.42) to equation (3.3.7a). Also, a comparison of equation (3.3.6) to equation (3.3.7a) reveals that the former equation can be written in the form

$$\hat{A}_j(x) = i c_j \bar{N}(x; -k_j) e^{2i\varepsilon k_j x},$$

and hence equation (3.3.4) now reads

$$i \sum_{j=1}^{q} c_j \frac{\bar{N}(x; -k_j)}{k - k_j} e^{2i\varepsilon k_j x} + \mu_{+\varepsilon}(x; k) = \bar{N}(x; k) - 1 + r(k) \bar{N}(x; -k) e^{2i\varepsilon k x}. \tag{3.3.8}$$

We pause (briefly) to remark that the normalizing coefficient is real whenever $u(x, t)$ satisfies the real-valued KdV equation [11]. As mentioned in Section 3.2 of this thesis, the discrete eigenfunction $n(x; k_j)$ is complex whenever $u(x, t)$ is a complex-valued solution of the KdV equation. The normalizing coefficient defined by (3.3.7a) is also complex because c_j inherits through (3.3.7b) the complex-valued character of $n(x; k_j)$.

The function

$$r(k) \bar{N}(x; -k) e^{2i\varepsilon k x}$$

cannot be analytically continued off the real k-axis. Nevertheless, we can employ the Plemelj formulae [11,100] to decompose the function

$$r(k)\bar{N}(x;-k)e^{2i\varepsilon kx}$$

into its $(\pm\varepsilon)$-parts. An application of the Plemelj formulae produces the partitioning

$$r(k)\bar{N}(x;-k)e^{2i\varepsilon kx} = \varepsilon\{\hat{u}_{+\varepsilon}(x;k) - \hat{u}_{-\varepsilon}(x;k)\}, \tag{3.3.9}$$

where $\hat{u}_{\pm\varepsilon}(x;k)$, which denote the $(\pm\varepsilon)$-parts, are determined through the functionals [11,100]

$$\hat{u}_{\pm\varepsilon}(x;k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{r(\zeta)\bar{N}(x;-\zeta)}{\zeta-(k\pm i\varepsilon 0^{+})}e^{2i\varepsilon\zeta x}\mathrm{d}\zeta. \tag{3.3.10}$$

The reader who recalls the derivation of equations (1.3.60) and (1.3.61) may prefer to express (3.3.10) in the form

$$\hat{u}_{\pm\varepsilon}(x;k) = \frac{1}{2\pi i}(\mathrm{P})\int_{-\infty}^{\infty}\frac{r(\zeta)\bar{N}(x;-\zeta)}{\zeta-k}e^{2i\varepsilon\zeta x}\mathrm{d}\zeta \pm \frac{\varepsilon}{2}r(k)\bar{N}(x;-k)e^{2i\varepsilon kx}.$$

Equation (3.3.8) can be expressed in the form

$$i\sum_{j=1}^{q}c_j\frac{\bar{N}(x;-k_j)}{k-k_j}e^{2i\varepsilon k_j x} + \mu_{+\varepsilon}(x;k) = \bar{N}(x;k) - 1 + \varepsilon\{\hat{u}_{+\varepsilon}(x;k) - \hat{u}_{-\varepsilon}(x;k)\}, \tag{3.3.11}$$

as soon as (3.3.9) is used in (3.3.8). Equation (3.3.11) contains two equations: the first is derived when we equate the $(+\varepsilon)$ terms either side of (3.3.11); the second equation (and the one that we will use) results when we equate the $(-\varepsilon)$ terms either side of (3.3.11). The equation formed when we equate the $(-\varepsilon)$-parts either side of (3.3.11) is

$$\bar{N}(x;k) = \mathfrak{G} + i\sum_{j=1}^{q}c_j\frac{\bar{N}(x;-k_j)}{k-k_j}e^{2i\varepsilon k_j x} + \varepsilon\hat{u}_{-\varepsilon}(x;k),$$

where the constant $\mathfrak{G}$ is determined from (see P5, page 69) the boundary condition $\bar{N}(x;k)\to 1$ as $k\to\infty$. Consistency of our equation for $\bar{N}(x;k)$ with P5 necessitates that $\mathfrak{G} = 1$. Therefore, $\bar{N}(x;k)$ satisfies the following singular integral equation:

$$\bar{N}(x;k) = 1 + i\sum_{j=1}^{q} c_j \frac{\bar{N}(x;-k_j)}{k-k_j} e^{2i\varepsilon k_j x} + \frac{\varepsilon}{2\pi i}\int_{-\infty}^{\infty} \frac{r(\zeta)\bar{N}(x;-\zeta)}{\zeta - (k - i\varepsilon 0^{+})} e^{2i\varepsilon\zeta x}\mathrm{d}\zeta. \qquad (3.3.12)$$

Significant results can be acquired from equation (3.3.12) when we examine the large-k asymptotics of this particular equation. Expanding the right hand side of (3.3.12) in ascending powers of $1/k$ we obtain the infinite series

$$\bar{N}(x;k) = 1 + \frac{\Xi(x)}{k} + \mathrm{O}\left(\frac{1}{k^2}\right), \qquad (3.3.13)$$

where

$$\Xi(x) \underline{\underline{\text{def}}}\; i\sum_{j=1}^{q} c_j \bar{N}(x;-k_j) e^{2i\varepsilon k_j x} - \frac{\varepsilon}{2\pi i}\int_{-\infty}^{\infty} r(\zeta)\bar{N}(x;-\zeta)e^{2i\varepsilon\zeta x}\mathrm{d}\zeta.$$

The equation

$$\bar{N}(x;k) = 1 + \frac{i\varepsilon}{2k}\int_{x}^{\infty} u(\xi)\mathrm{d}\xi + \mathrm{O}\left(\frac{1}{k^2}\right), \qquad (3.3.14)$$

which portrays the behaviour of (3.2.21) as $k \to \infty$ $(k \in \Re)$, provides an alternative to equation (3.3.13).

An *explicit* formula for $u(x)$ in terms of appropriate scattering data is now feasible. Equating the coefficients of the $\mathrm{O}(1/k)$ terms in equations (3.3.13) and (3.3.14), and then using the Fundamental Theorem of Calculus we obtain the reconstruction formula:

$$u(x) = -\frac{\partial}{\partial x}\left\{2\varepsilon\sum_{j=1}^{q} c_j N(x;k_j) + \frac{1}{\pi}\int_{-\infty}^{\infty} r(\zeta)N(x;\zeta)\mathrm{d}\zeta\right\}, \qquad (3.3.15)$$

provided we use (3.3.1) to eliminate from (3.3.15) any reference to the eigenfunction $\bar{N}(x;k)$.

problem associated with the linear eigenvalue problem (3.2.1). We still require a closed system of equations that relates the $q + 1$ unknowns $N(x;k)$ and $N(x;k_j)$ to the scattering data; the scattering data in question being

$$a(k),\ b(k),\ c_j \text{ and } k_j,$$

where $j = 1, 2, \cdots, q$. The equations that must supplement (3.3.15) are as follows:

$$N(x;k) = \left\{ 1 - i\sum_{j=1}^{q} \frac{c_j N(x;k_j)}{k + k_j} + \frac{\varepsilon}{2\pi i}\int_{-\infty}^{\infty} \frac{r(\zeta)N(x;\zeta)}{\zeta + k + i\varepsilon 0^{+}} \mathrm{d}\zeta \right\} e^{2i\varepsilon kx}, \tag{3.3.16}$$

where $\mathrm{Im}(k) = 0$, and the system of equations

$$N(x;k_p) = \left\{ 1 - i\sum_{j=1}^{q} \frac{c_j N(x;k_j)}{k_p + k_j} + \frac{\varepsilon}{2\pi i}\int_{-\infty}^{\infty} \frac{r(\zeta)N(x;\zeta)}{\zeta + k_p} \mathrm{d}\zeta \right\} e^{2i\varepsilon k_p x}. \tag{3.3.17}$$

The index p in equation (3.3.17) takes each of the values $p = 1, 2, \cdots, q$. A prerequisite for the derivation of (3.3.16) and (3.3.17) is the equation

$$N(x;-k) = \left\{ 1 + i\sum_{j=1}^{q} \frac{c_j N(x;k_j)}{k - k_j} + \frac{\varepsilon}{2\pi i}\int_{-\infty}^{\infty} \frac{r(\zeta)N(x;\zeta)}{\zeta - (k - i\varepsilon 0^{+})} \mathrm{d}\zeta \right\} e^{-2i\varepsilon kx}, \tag{3.3.18}$$

and the reader can derive (3.3.18) by using (3.3.1) to eliminate $\bar{N}(x;\bullet)$ from (3.3.12). The simple involution $k \to -k$ in (3.3.18) produces (3.3.16), whereas (3.3.17) is the limit of (3.3.18) as $k \to -k_p$. We remind the reader that

$$r(k) \underline{\underline{\mathrm{def}}} \frac{b(k)}{a(k)}.$$

Equations (3.3.16) and (3.3.17) relate the $q + 1$ unknowns $N(x;k)$ and $N(x;k_j)$ to the scattering data

$$a(k),\ b(k) \text{ and } \{c_j, k_j\}_{j=1}^{j=q},$$

where the notation $\{\bullet\}_{j=1}^{j=q}$ indicates that the *discrete* index j assumes all integers in the range 1 to q. Therefore, equation (3.3.15) expresses $u(x)$ in terms of quantities that have a known relationship to the scattering data.

We will close this section with the observation that the system of equations (3.3.15)-(3.3.17) reduces to a system of linear *algebraic* equations whenever $r(k) \equiv 0$.

Section 3.4: The Time Evolution of the Scattering Data

Hitherto, time has entered parametrically into our analysis of the direct and inverse problems for equation (3.2.9). The eigenfunctions $M(x;k)$, $\bar{M}(x;k)$, $N(x;k)$, $\bar{N}(x;k)$ and the scattering data

$$a(k),\ b(k) \text{ and } \left\{c_j, k_j\right\}_{j=1}^{j=q}$$

are all (theoretically) functions of t. In order to emphasize the time-dependence of the scattering data and eigenfunctions for (3.2.9), henceforth we will write $M(x,t;k)$ and so on for the remaining eigenfunctions and

$$a(k,t),\ b(k,t) \text{ and } \left\{c_j(t), k_j(t)\right\}_{j=1}^{j=q}$$

for the scattering data.

Our immediate objective is to complement the system of equations (3.3.15-17) with equations that describe the (time) evolution of the scattering data. Throughout this section we will assume that $u(x,t)$ satisfies the following boundary conditions:

$$u(x,t) \to 0 \text{ and } u_x(x,t) \to 0 \text{ (uniformly in } t) \text{ as } |x| \to \infty.$$

The evolution of the scattering data is ascertained from an appropriate specialization of the linear equation (3.2.10). For example, the eigenfunction $M(x,t;k)$ evolves according to the equation

$$M_t(x,t;k) = 2\left(2k^2 - u\right)M_x(x,t;k) + \left(u_x + 2i\varepsilon ku + \gamma\right)M(x,t;k), \tag{3.4.1}$$

where $u \equiv u(x,t)$ and γ is a constant that is determined by any boundary condition imposed upon $M(x,t;k)$. According to (3.2.11),

$$M(x,t;k) \to 1 \text{ as } x \to -\infty.$$

The boundary condition for the eigenfunction $M(x,t;k)$ will be satisfied if $\gamma = 0$, which means that (3.4.1) can now take its precise form,

$$M_t(x,t;k) = 2\left(2k^2 - u\right)M_x(x,t;k) + \left(u_x + 2i\varepsilon ku\right)M(x,t;k). \tag{3.4.2}$$

The independence of the scattering data from the spatial variable, x, allows us the freedom to select the value of x for use in equation (3.4.2). A convenient choice is $x \to \infty$ because (3.4.2) assumes a form that is independent of the potential:

$$M_t = 4k^2 M_x, \tag{3.4.3}$$

where M now denotes the asymptotic behaviour of $M\left(x,t;k\right)$ as $x \to \infty$. Substituting (3.2.25) into (3.4.3) we obtain the equation

$$a_t\left(k,t\right) + b_t\left(k,t\right)e^{2i\varepsilon kx} = 8i\varepsilon k^3 b\left(k,t\right)e^{2i\varepsilon kx},$$

from which we infer that

$$a_t\left(k,t\right) = 0 \tag{3.4.4}$$

and

$$b_t\left(k,t\right) = 8i\varepsilon k^3 b\left(k,t\right). \tag{3.4.5}$$

Equations (3.4.4) and (3.4.5) are readily integrated to produce the (general) solutions

$$a\left(k,t\right) = a\left(k,0\right) \tag{3.4.6}$$

and

$$b\left(k,t\right) = b\left(k,0\right)e^{8i\varepsilon k^3 t}. \tag{3.4.7}$$

We determine the quantities $a\left(k,0\right)$ and $b\left(k,0\right)$ by solving the relevant equations, using the known initial value (3.1.6) in the place of $u\left(x,t\right)$. Notwithstanding the technical difficulties involved in the determination of $a\left(k,0\right)$ and $b\left(k,0\right)$, we proceed as if these quantities are known.

The evolution of the *reflection coefficient*, $r\left(k,t\right)$, is determined as follows:

$$r\left(k,t\right) = \frac{b\left(k,t\right)}{a\left(k,t\right)} \Rightarrow r\left(k,t\right) = \frac{b\left(k,0\right)e^{8i\varepsilon k^3 t}}{a\left(k,0\right)},$$

from which we conclude that

$$r\left(k,t\right) = r\left(k,0\right)e^{8i\varepsilon k^3 t}. \tag{3.4.8}$$

Equations (3.4.6), (3.4.7) and (3.4.8) completely determine the (time) evolution of all quantities in the *continuous spectrum*.

The quantities $k_j(t)$ and $c_j(t)$ are members of the *discrete spectrum.* In order to ascertain the evolutions of $k_j(t)$ and $c_j(t)$, it will prove convenient to work with the exponentially localized eigenfunction n_j, where $n_j \equiv n(x,t;k_j)$. Equation (3.2.5) informs us that the evolution of n_j is governed by the equation

$$n_{j,t} = 2\left\{2k_j^2(t) - u(x,t)\right\}n_{j,x} + \left\{u_x(x,t) + \omega\right\}n_j, \tag{3.4.9}$$

where

$$n_{j,t} \equiv \frac{\partial n_j}{\partial t},\ n_{j,x} \equiv \frac{\partial n_j}{\partial x}$$

and multiple order partial derivatives will be depicted by appropriate subscripts. By virtue of (3.2.7), the boundary condition

$$n_j \to e^{i\varepsilon k_j x} \text{ as } x \to \infty$$

is associated with equation (3.4.9). The boundary condition for n_j imposes the requirement

$$\omega = -4i\varepsilon k_j^3(t),$$

and hence equation (3.4.9) now becomes

$$n_{j,t} = 2\left\{2k_j^2(t) - u(x,t)\right\}n_{j,x} + \left\{u_x(x,t) - 4i\varepsilon k_j^3(t)\right\}n_j. \tag{3.4.10}$$

In addition to satisfying equation (3.4.10), the discrete eigenfunction n_j also satisfies equation (3.2.4):

$$n_{j,xx} = -\left\{u(x,t) + k_j^2(t)\right\}n_j. \tag{3.4.11}$$

Equations (3.4.10) and (3.4.11) form an *isospectral flow*, provided that $u(x,t)$ satisfies the KdV equation. The reader may verify the constancy of the discrete eigenvalues by checking the compatibility condition

$$n_{j,txx} \equiv n_{j,xxt}.$$

The equation

$$k_j(t) = k_j(0) \tag{3.4.12}$$

expresses the immutability of the discrete eigenvalues. In the remainder of this chapter we will adopt the convention

$$k_j \equiv k_j(0).$$

The procedure that quantifies the evolution of the *normalizing coefficient*, $c_j(t)$, is more sophisticated than the methods used to determine the evolutions of $a(k,t)$, $b(k,t)$ and $k_j(t)$. The orthogonality condition

$$\int_{-\infty}^{\infty} u_x(x,t) n_j^2 \, dx = 0 \tag{3.4.13}$$

is central to the method we will employ to compute the evolution of $c_j(t)$. Equation (3.4.13) can be derived through the following procedure:

1. Differentiate both sides of (3.4.11) with respect to x, and hence obtain the equation

$$n_{j,xxx} = -u_x(x,t) n_j - \left\{ u(x,t) + k_j^2(t) \right\} n_{j,x}; \tag{3.4.14}$$

2. Multiply both sides of (3.4.14) by n_j to form the nonlinear equation

$$n_j n_{j,xxx} = -u_x(x,t) n_j^2 - \left\{ u(x,t) + k_j^2(t) \right\} n_j n_{j,x},$$

which, because of equation (3.4.11), simplifies to

$$n_j n_{j,xxx} = -u_x(x,t) n_j^2 + n_{j,x} n_{j,xx}; \tag{3.4.15}$$

3. Integrate (3.4.15) over the interval $-\infty < x < \infty$, and then rearrange the integrated equation to form the identity

$$\int_{-\infty}^{\infty} u_x(x,t) n_j^2 \, dx = \int_{-\infty}^{\infty} \left(n_{j,x} n_{j,xx} - n_j n_{j,xxx} \right) dx. \tag{3.4.16}$$

4. Use the boundary conditions

$$n_j \to 0 \text{ and } n_{j,x} \to 0 \text{ as } |x| \to \infty \tag{3.4.17}$$

and a single integration by parts to evaluate the *right hand side* of (3.4.16), as follows:

$$\int_{-\infty}^{\infty}\left(n_{j,x}n_{j,xx}-n_j n_{j,xxx}\right)\mathrm{d}x=-\int_{-\infty}^{\infty}n_j^2\frac{\partial}{\partial x}\left\{\frac{n_{j,xx}}{n_j}\right\}\mathrm{d}x$$

$$=-\left[n_j n_{j,xx}\right]_{-\infty}^{\infty}+\int_{-\infty}^{\infty}\frac{n_{j,xx}}{n_j}\frac{\partial}{\partial x}\left\{n_j^2\right\}\mathrm{d}x$$

$$=2\int_{-\infty}^{\infty}n_{j,x}n_{j,xx}\mathrm{d}x$$

$$\int_{-\infty}^{\infty}\left(n_{j,x}n_{j,xx}-n_j n_{j,xxx}\right)\mathrm{d}x=\int_{-\infty}^{\infty}\frac{\partial}{\partial x}\left\{\left(n_{j,x}\right)^2\right\}\mathrm{d}x,$$

from which, with the aid of (3.4.17), we conclude that

$$\int_{-\infty}^{\infty}\left(n_{j,x}n_{j,xx}-n_j n_{j,xxx}\right)\mathrm{d}x=0.$$

Hence, we arrive at the desired result:

$$\int_{-\infty}^{\infty}u_x(x,t)n_j^2\mathrm{d}x=0.$$

We will now initiate the task of determining the evolution of $c_j(t)$. A minor rearrangement of equation (3.3.7b) will show that an *explicit* formula for the normalizing coefficient is

$$\left\{c_j(t)\right\}^{-1}=\varepsilon\int_{-\infty}^{\infty}n_j^2\mathrm{d}x. \tag{3.4.18}$$

Differentiating both sides of (3.4.18) with respect to *t* we obtain the equation

$$\frac{\mathrm{d}c_j(t)}{\mathrm{d}t}=-2\varepsilon\left\{c_j(t)\right\}^2\int_{-\infty}^{\infty}n_j n_{j,t}\mathrm{d}x. \tag{3.4.19}$$

The $n_{j,t}$ term that appears in the integrand of (3.4.19) can be eliminated. Substituting (3.4.10) into the integrand of (3.4.19) we obtain

$$\frac{\mathrm{d}c_j(t)}{\mathrm{d}t}=-2\varepsilon\left\{c_j(t)\right\}^2\int_{-\infty}^{\infty}\left(2\left\{2k_j^2-u(x,t)\right\}n_j n_{j,x}+\left\{u_x(x,t)-4i\varepsilon k_j^3\right\}n_j^2\right)\mathrm{d}x. \tag{3.4.20}$$

The term $u_x(x,t)n_j^2$ in the integrand of (3.4.20) does not contribute because of the orthogonality condition (3.4.13). Therefore, equation (3.4.20) simplifies to

$$\frac{\mathrm{d}c_j(t)}{\mathrm{d}t} = -2\varepsilon\left\{c_j(t)\right\}^2\int_{-\infty}^{\infty}\left(2\left\{2k_j^2 - u(x,t)\right\}n_j n_{j,x} - 4i\varepsilon k_j^3 n_j^2\right)\mathrm{d}x.$$

Simplifications of the above equation can be achieved through a single integration by parts and an appropriate use of the information contained in equations (3.4.13) and (3.4.17) to evaluate the term

$$\int_{-\infty}^{\infty}\left\{2k_j^2 - u(x,t)\right\}n_j n_{j,x}\mathrm{d}x.$$

The reader who pursues the strategy advocated will find that the differential equation for $c_j(t)$ reduces to

$$\frac{\mathrm{d}c_j(t)}{\mathrm{d}t} = 8ik_j^3\left\{c_j(t)\right\}^2\int_{-\infty}^{\infty} n_j^2\mathrm{d}x,$$

which, with the assistance of (3.4.18), simplifies to

$$\frac{\mathrm{d}c_j(t)}{\mathrm{d}t} = 8i\varepsilon k_j^3 c_j(t). \tag{3.4.21}$$

Equation (3.4.21) admits the general solution

$$c_j(t) = c_j(0)e^{8i\varepsilon k_j^3 t}, \tag{3.4.22}$$

where $c_j(0)$ denotes the initial value of the normalizing coefficient.

Equations (3.4.6), (3.4.7), (3.4.12) and (3.4.22) completely determine the time evolution of the scattering data in terms of quantities that can be calculated from the known initial value (3.1.6). The reader who is familiar with the IST for the *real-valued* KdV equation will notice that (3.4.6), (3.4.7), (3.4.12) and (3.4.22) have a similar appearance to the corresponding equations that govern the evolution of the scattering data for the real-valued KdV equation [11,42,125]. The similarity of the equations we have derived with the equations that appear in the IST for the real-valued KdV equation reinforces the connection between the KdV and MKdV equations.

Now that we have derived equations for the evolution of the scattering data, we are able to provide the following system of equations for the solution of the inverse problem at $t > 0$:

$$u(x,t) = -\frac{\partial}{\partial x}\left\{ 2\varepsilon \sum_{j=1}^{q} c_j(0)\hat{N}(x,t;k_j) + \frac{1}{\pi}\int_{-\infty}^{\infty} r(\zeta,0)\hat{N}(x,t;\zeta)\,d\zeta \right\}; \tag{3.4.23}$$

$$N(x,t;k) = \left\{ 1 - i\sum_{j=1}^{q} c_j(0)\frac{\hat{N}(x,t;k_j)}{k+k_j} + \frac{\varepsilon}{2\pi i}\int_{-\infty}^{\infty} \frac{r(\zeta,0)\hat{N}(x,t;\zeta)}{\zeta+k+i\varepsilon 0^+}d\zeta \right\} e^{2i\varepsilon kx}; \tag{3.4.24}$$

$$N(x,t;k_p) = \left\{ 1 - i\sum_{j=1}^{q} c_j(0)\frac{\hat{N}(x,t;k_j)}{k_p+k_j} + \frac{\varepsilon}{2\pi i}\int_{-\infty}^{\infty} \frac{r(\zeta,0)\hat{N}(x,t;\zeta)}{\zeta+k_p}d\zeta \right\} e^{2i\varepsilon k_p x} \tag{3.4.25}$$

where

$$\hat{N}(x,t;\zeta) \underline{\underline{\text{def}}}\ N(x,t;\zeta)e^{8i\varepsilon\zeta^3 t},$$

$p = 1, 2, \ldots, q$ and (3.4.24) is valid for Im $(k) = 0$. The derivation of equations (3.4.23-25) is accomplished by a simple substitution of equations (3.4.8) and (3.4.22) into equations (3.3.15-17). A specialized version of equations (3.4.23-25) will be used in the next section to construct the 1-soliton solution for the MKdV equation.

Section 3.5: 1-Soliton Solution for the MKdV Equation

Pure soliton solutions for the MKdV and complex-valued KdV equations are generated from (3.4.23-25) when $r(\bullet, 0) \equiv 0$. The system of *algebraic* equations

$$u(x,t) = -2\varepsilon \frac{\partial}{\partial x}\left\{ \sum_{j=1}^{q} c_j(0) N(x,t;k_j) e^{8i\varepsilon k_j^3 t} \right\}, \tag{3.5.1}$$

$$N(x,t;k_p) = e^{2i\varepsilon k_p x} - i \sum_{j=1}^{q} c_j(0) \frac{N(x,t;k_j)}{k_p + k_j} e^{2i\varepsilon(k_p x + 4k_j^3 t)} \tag{3.5.2}$$

governs the production of *pure soliton* solutions for the MKdV and complex-valued KdV equations.

We now illustrate how the IST we have developed for the complex-valued KdV equation can be deployed to compute the 1-soliton solution for the MKdV equation,

$$v_t + 6v^2 v_x + v_{xxx} = 0.$$

For sake of convenience, we will make the selection $\varepsilon = 1$ in the remainder of this section; our decision forces Im $(k_j) > 0$. In the case where $q = 1$, we find from equation (3.5.2) that

$$N(x,t;k_1) = \frac{2k_1 e^{2ik_1 x}}{2k_1 + ic_1(0) e^{2ik_1(x + 4k_1^2 t)}} \tag{3.5.3}$$

is the 1-soliton eigenfunction. Substituting (3.5.3) into (3.5.1) we obtain

$$u(x,t) = -4 \frac{\partial}{\partial x}\left\{ \frac{k_1 c_1(0)}{ic_1(0) + 2k_1 e^{-2ik_1(x + 4k_1^2 t)}} \right\}, \tag{3.5.4}$$

which is the 1-soliton solution for the complex-valued KdV equation.

The author has used MACSYMA [68] to check that (3.5.4) satisfies the KdV equation,

$$u_t + 6uu_x + u_{xxx} = 0,$$

for arbitrary choices of the complex parameters k_1 and $c_1(0)$. Let us consider the situation in which $k_1 = i\hat{k}$, where $\hat{k} > 0$; our selection is motivated by the observation that (3.5.4) has the appearance of a travelling wave whenever Re $(k_1) = 0$. Using $k_1 = i\hat{k}$ in (3.5.4) we find that

$$u(x,t) = -\frac{\partial \Delta}{\partial x},$$

where

$$\Delta \underline{\underline{\text{def}}} \frac{4\hat{k}c_1(0)}{c_1(0) + 2\hat{k}e^{2\hat{k}(x-4\hat{k}^2 t)}}.$$

Let $c_1(0) = \alpha_1 + i\beta_1$, where α_1 and β_1 are arbitrary real numbers, except for the restriction $\beta_1 \neq 0$; the condition $\beta_1 \neq 0$ is imposed to avoid a real-valued $u(x,t)$. The separation of Δ into its real (Δ_R) and imaginary (Δ_I) parts yields

$$u(x,t) = -\frac{\partial \Delta_R}{\partial x} - i\frac{\partial \Delta_I}{\partial x}, \tag{3.5.5}$$

where

$$\Delta_R = \frac{4\beta_1^2\hat{k} + 4\alpha_1\hat{k}\left(\alpha_1 + 2\hat{k}e^{2\hat{k}(x-4\hat{k}^2 t)}\right)}{\beta_1^2 + \left(\alpha_1 + 2\hat{k}e^{2\hat{k}(x-4\hat{k}^2 t)}\right)^2} \tag{3.5.6}$$

and

$$\Delta_I = \frac{8\beta_1\hat{k}^2 e^{2\hat{k}(x-4\hat{k}^2 t)}}{\beta_1^2 + \left(\alpha_1 + 2\hat{k}e^{2\hat{k}(x-4\hat{k}^2 t)}\right)^2}. \tag{3.5.7}$$

The connection between the physical variable, *u*, that solves the complex-valued KdV equation, and the physical variable, *v*, that solves the MKdV equation is contained in (3.1.5). Equating (3.1.5) and (3.5.5) we find the pair of equations

$$v^2 = -\frac{\partial \Delta_R}{\partial x} \tag{3.5.8}$$

and

$$v_x = -\frac{\partial \Delta_I}{\partial x}, \tag{3.5.9}$$

where $v \equiv v(x,t)$. A quadrature (with respect to *x*) when applied to (3.5.9) will yield

$$v = -\Delta_I + \hat{v}(t), \tag{3.5.10}$$

where $\hat{v}(t)$ denotes an arbitrary function of integration. From (3.5.7) and (3.5.10) we deduce that

$$\lim_{x \to \pm\infty} v = \hat{v}(t).$$

Therefore, to satisfy the zero boundary conditions (imposed on *v*) requires $\hat{v}(t) \equiv 0$. A lengthy calculation will show the reader that (3.5.8) is satisfied only when $\alpha_1 = 0$. Substituting $\hat{v}(t) \equiv 0$, $\alpha_1 = 0$ and (3.5.7) into equation (3.5.10) we obtain

$$v(x,t) = -\frac{8\beta_1 \hat{k}^2 e^{2\hat{k}(x-4\hat{k}^2 t)}}{\beta_1^2 + 4\hat{k}^2 e^{4\hat{k}(x-4\hat{k}^2 t)}}. \tag{3.5.11}$$

Without loss of generality we can assume that $\beta_1 > 0$ because the MKdV equation is invariant under the action $v(x,t) \to -v(x,t)$. Assuming that $\beta_1 > 0$, we now define the parameter $\tilde{x}_0$ by the equation

$$\tilde{x}_0 = \frac{\ln\left(2\hat{k}/\beta_1\right)}{2\hat{k}}. \tag{3.5.12}$$

Some algebra will show that when (3.5.12) is employed in (3.5.11), the latter equation assumes the form

$$v(x,t) = -2\hat{k}\,\mathrm{sech}\left(2\hat{k}\left[x - 4\hat{k}^2 t + \tilde{x}_0\right]\right). \tag{3.5.13}$$

It is important to remark that equation (3.5.13) is identically equal to (2.2.7), the latter equation being the 1-soliton solution for the MKdV equation obtained through the use of the Bilinear Transformation Method.

The method used to derive (3.5.13) can be extended to produce multi-soliton solutions for the MKdV equation. Nevertheless, we will not pursue the calculation of multi-soliton solutions for the MKdV equation because we have now accomplished our objective in this chapter, namely the development of a paradigm for the solution of the initial value problems associated with the MILW and MBO equations.