# *CHAPTER 1*

THE SIGNIFICANCE OF THE

MODIFIED INTERMEDIATE LONG WAVE

**AND**

MODIFIED BENJAMIN-ONO

**EQUATIONS TO SOLITON**

# Section 1.1: Introduction

In a remarkable paper that was published in an issue of *Physical Review Letters*, Zabusky and Kruskal [122] reported the recurrence of initial states for the nonlinear evolution equation

$$u_t + 6uu_x + u_{xxx} = 0. \qquad (1.1.1)$$

Notation of particular relevance to (1.1.1) is as follows: $t$ is the temporal variable, $x$ is the spatial variable and $u \equiv u(x,t)$. Throughout this thesis $x$ and $t$ will denote independent variables with the ranges $t \geq 0$ and $-\infty < x < \infty$. Subscripts that involve $x$ or $t$ will be used to denote partial derivatives, for example $u_t = \partial u / \partial t$ and $u_{xx} = \partial^2 u / \partial x^2$. Equations in this thesis will be numbered with the format (a.b.c), where **a** designates the chapter number, **b** refers to the section number, and **c** is the equation number within a given section.

Zabusky and Kruskal [122] discovered that distinct travelling wave solutions of (1.1.1) interact in an elastic manner, and emerge from their collision to regain (as $t \rightarrow \infty$) their initial form and speed. The realization that a mechanism far more complicated than linear superposition was operative at the scattering phase prompted Zabusky and Kruskal

are presented in the review article by Scott, Chu and McLaughlin [113].

Equation (1.1.1) has derived its name from the work of Korteweg and de Vries [65], but in fact Boussinesq [see C.R. Acad. Paris, vol. 73, pp. 256-360 (1871)] first derived (1.1.1) as a model for wave propagation in rectangular channels. The physical problem that motivated the work of Zabusky and Kruskal is documented in Refs 6 and 36. In this thesis

The exciting development reported by Zabusky and Kruskal stimulated an intense search for an analytical method to solve the initial value problem for equation (1.1.1). Gardner, Greene, Kruskal and Miura (GGKM) in a pioneering work [45] developed an exact solution of the Cauchy problem for (1.1.1). The method developed by GGKM employed the direct and inverse scattering problems for the one-dimensional time

$$\psi_{xx} = -(u + \kappa)\psi, \qquad (1.1.2)$$

where $\psi \equiv \psi(x,t)$ is the wave function, $u$ is the potential that induces a scattering effect and $\kappa$ is the spectral parameter. The ingenious observation made by GGKM was to identify the potential in (1.1.2) with the desired solution of the initial value problem for (1.1.1).

associated with (1.1.2) to express $u(x, t)$ in terms of the solution of a *linear* integral equation and quantities (scattering data) that could be determined from the known initial value $u(x, 0)$. Two factors [45] contributed to the success of the method employed by GGKM:

**1)** The eigenvalues of the spectral problem (1.1.2) are independent of *t*;

**2)** The *linear* equation

$$\psi_t = 2(2\kappa - u)\psi_x + (u_x + \omega)\psi \tag{1.1.3}$$

governs the temporal evolution of the scattering data that appear in the formula for $u(x, t)$. The constant $\omega$ in equation (1.1.3) is determined from any boundary condition imposed on the eigenfunction $\psi$.

The Modified Korteweg-de Vries (MKdV) equation,

$$v_t + 6v^2 v_x + v_{xxx} = 0, \tag{1.1.4}$$

was significant in the development of the procedure used by GGKM to solve the initial value problem for the KdV equation [46,87]. Miura [86] has shown that the explicit nonlinear transformation

$$u = v^2 + iv_x \tag{1.1.5}$$

maps solutions of the MKdV equation into solutions of the KdV equation. Ablowitz et al.

and Shadwick [99]. Historically, the BT (1.1.5) led GGKM [46] to consider the inverse

The method used by GGKM to solve the initial value problem for the KdV

solutions that can be constructed from an appropriate IST. For example, Wadati [119] has solved the Cauchy problem for (1.1.4) by using the IST. References 6, 11, 36, 39, 64 and 125 illustrate the manifold of applications for the IST.

The power, beauty and versatility of the IST method are exemplified in its application to nonlinear singular integro-differential equations. The prototype nonlinear singular integro-differential equation soluble by an IST [27,48,63] is the Intermediate Long Wave (ILW) equation

$$U_t + \frac{1}{\delta} U_x + 2UU_x + \mathbf{T}(U_{xx}) = 0, \tag{1.1.6}$$

where $U \equiv U(x, t; \delta)$, $\delta$ is a positive parameter and

$$(\mathbf{T}f)(x) \ \underline{\underline{\mathrm{def}}}\ \frac{1}{2\delta}(\mathrm{P})\int_{-\infty}^{\infty} \coth\left(\frac{\pi}{2\delta}(\xi - x)\right) f(\xi)\,\mathrm{d}\xi. \tag{1.1.7}$$

The symbol $(\mathrm{P})$ adjacent to an integral designates that the integral is to be evaluated in the Cauchy principal value sense. Equation (1.1.6) was derived by Joseph [56] and Kubota et al. [66] as a model for the one-dimensional propagation of weakly nonlinear long internal gravity waves in a stratified fluid of *finite* depth.

In an adroit implementation of the IST, Kodama et al. [63] demonstrated that the initial value problem for (1.1.6) can be solved by an IST that is conceptually similar to the IST used by GGKM to solve the KdV equation.

Despite superficial differences in the appearances of equations (1.1.1) and (1.1.6), the KdV and ILW equations are connected by virtue of the physical systems they describe. Several researchers [56,66,108] have observed that subject to certain mild conditions being satisfied, equation (1.1.6) contracts as $\delta \to 0^+$ to equation (1.1.1). The mathematical limit $\delta \to 0^+$ has the physical interpretation of being the shallow water limit. In the deep water limit ($\delta \to \infty$) it has been shown [56,66,108] that equation (1.1.6) bifurcates into the nonlinear singular integro-differential equation

$$q_t + 2qq_x + \mathbf{H}(q_{xx}) = 0, \tag{1.1.8}$$

where $q \equiv q(x, t)$ and the Hilbert transform $\mathbf{H}$ is

$$(\mathbf{H}f)(x) \ \underline{\underline{\mathrm{def}}}\ \frac{1}{\pi}(\mathrm{P})\int_{-\infty}^{\infty} \frac{f(\xi)}{\xi - x}\,\mathrm{d}\xi. \tag{1.1.9}$$

Equation (1.1.8) is the Benjamin-Ono (BO) equation, and was first derived by Benjamin [15] to model uni-directional internal waves of finite amplitude in stable heterogeneous fluid systems of *infinite* total depth. Fokas and Ablowitz [43] have solved the initial value problem for equation (1.1.8) using an implementation of the IST that is conceptually similar to the IST for nonlinear equations that involve three independent variables, namely two spatial variables and one temporal variable.

A natural extrapolation from equations (1.1.6) and (1.1.8) is to construct *modified* versions of the ILW and BO equations. The modified versions of the ILW and BO

appropriately generalizes the Miura transformation between the KdV and MKdV equations. The nomenclature that we will use to refer to the two modified-type nonlinear singular integro-differential equations is Modified Intermediate Long Wave (MILW) equation for the modified version of the ILW equation, and Modified Benjamin-Ono (MBO) equation for the modified version of the BO equation. Desirable attributes of the MILW and MBO equations are as follows:

**1)** MILW $\to$ MKdV as $\delta \to 0^+$, in a replication of the limit ILW $\to$ KdV as $\delta \to 0^+$;

**2)** existence of a BT between the ILW-MILW pair, and the $\delta \to 0^+$ limit of this particular BT should produce the Miura transformation between the KdV-MKdV pair;

**3)** MILW $\to$ MBO as $\delta \to \infty$, in a direct analogue of the limit ILW $\to$ BO as $\delta \to \infty$;

**4)** existence of a BT between the BO-MBO pair, and this particular BT should be retrievable from the $\delta \to \infty$ limit of the BT that connects the ILW-MILW pair;

**5)** the MILW and MBO equations should each support the algebraic structure characteristic of an integrable soliton equation;

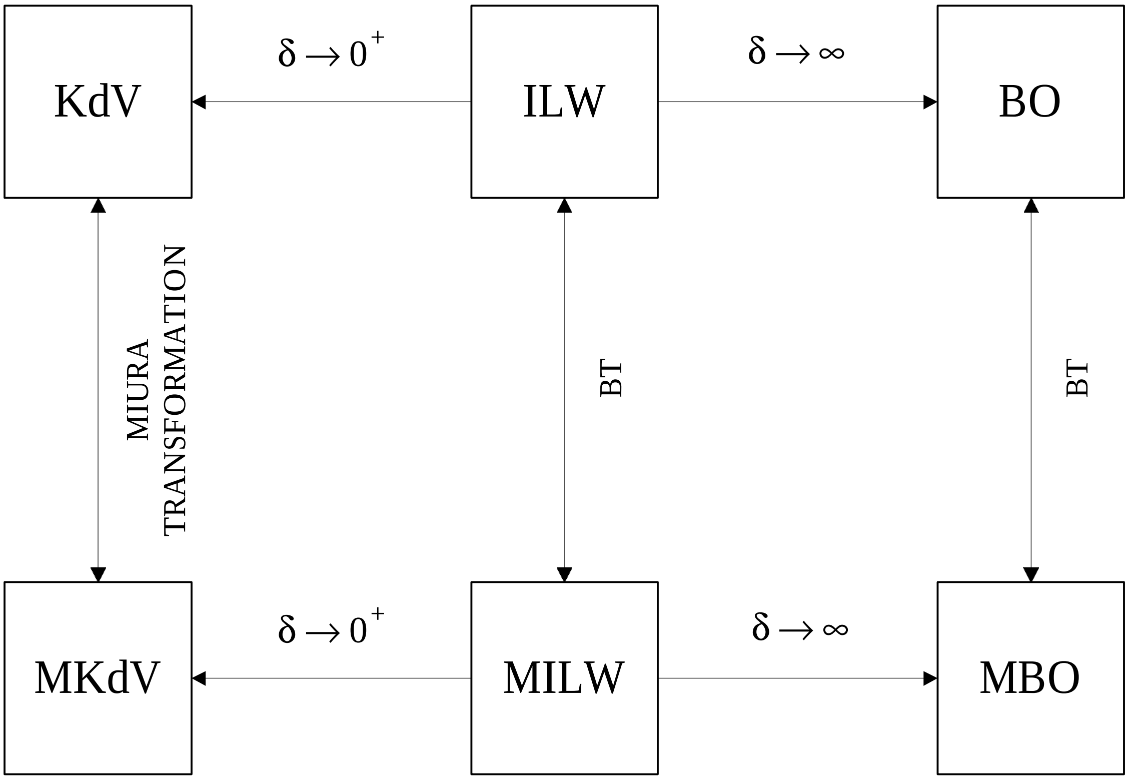
A schematic representation of the interrelationships between the standard and modified versions of the equations that have entered into our discussion is shown in the display on the next page of this thesis.

The two candidates to represent the MILW and MBO equations are the following nonlinear singular integro-differential equations [48,89,90,112]:

**MILW:**
$$V_t + \beta V_x\left(e^V - 1\right) + \frac{1}{\delta} V_x + V_x \mathbf{T}\left(V_x\right) + \mathbf{T}\left(V_{xx}\right) = 0; \qquad (1.1.10)$$

**MBO:**
$$Q_t + \alpha Q_x\left(e^Q - 1\right) + Q_x \mathbf{H}\left(Q_x\right) + \mathbf{H}\left(Q_{xx}\right) = 0, \qquad (1.1.11)$$

where $V \equiv V(x,t)$, $Q \equiv Q(x,t)$, $\mathbf{T}$ is defined by (1.1.7), $\mathbf{H}$ is defined by (1.1.9) and $\alpha$ and $\beta$ are parameters (independent of $x$ and $t$). Independent studies of the MILW and MBO equations by several researchers [89,90,112] have shown that these equations exhibit




*Schema of Interrelationships amongst Equations that are Significant to the Subject Matter in this Thesis*

the signature characteristic of an integrable soliton equation. For example, the MILW equation possesses an infinite number of conservation laws [112], a linear scattering problem [112], a BT [89,112] and multi-soliton solutions [89]. The reader inclined towards physical applications will have observed that the MILW and MBO equations have the same dispersion laws as their standard (unmodified) counterparts, and therefore the MILW and MBO equations could be relevant to the accurate description of physical systems modelled by the ILW and BO equations, respectively.

Despite the important generalizations that the MILW and MBO equations provide to the mathematical theory that buttresses the KdV-MKdV pairing, and the possible physical applications of the MILW and MBO equations, the initial value problems for these equations have not been solved. In the thesis we use the IST to derive the solutions of the initial value problems for the MILW and MBO equations. The only restrictions that we place on the initial values for the MILW and MBO equations are that they be real-valued, sufficiently smooth and decay to zero sufficiently fast as $|x| \to \infty$. A complete solution of the initial value problems for the MILW and MBO equations is presented in this thesis. For example, we use the results of our IST schemes to derive soliton solutions for equations (1.1.10) and (1.1.11).

The cardinal point in the solution schemes that we will develop to solve the MILW and MBO equations is to use an appropriate IST to solve the initial value problem for the *complex extension* of each unmodified equation, and then use the solution of the complexified initial value problem to solve for the modified equation by an inversion of the BT that connects the standard-modified pair. An outline of the procedure to solve, for example, the MILW equation will clarify matters for the reader. The explicit nonlinear transformation [89,112]

$$U(x,t) = \frac{1}{2}\left\{\mathbf{T}\left(V_x(x,t)\right) + \beta\left(e^{V(x,t)} - 1\right) + i\,V_x(x,t)\right\} \tag{1.1.12}$$

is the relevant BT between the ILW-MILW pair. Equation (1.1.12) maps a *real-valued* solution of equation (1.1.10) into a *complex-valued* solution of equation (1.1.6). In particular, the known initial value, $V_0$ say, for the MILW equation will be mapped by (1.1.12) into the complex-valued function $U_0 \equiv U(x,0)$

refers to a complex-valued function of the two real variables $x$ and $t$. Throughout this thesis, whenever the distinction between real-valued and complex-valued versions of a particular evolution equation is relevant we explicitly state such a distinction in our narrative. From equation (1.1.12) we deduce that

$$V_x(x,t) = 2\,\mathrm{Im}\,\{U(x,t)\}. \tag{1.1.13}$$

If we choose the function $U_0$ as our initial value for the ILW equation (1.1.6), then equation (1.1.13) is correct for all $t \geq 0$. The function $U(x,t)$ is the solution of the ILW equation (1.1.6) that has evolved from the *known* complex initial value $U_0$, where

$$U_0 \underline{\underline{\mathrm{def}}} \ \frac{1}{2}\left\{\mathbf{T}\left(V_x\right) + \beta\left(e^{V} - 1\right) + i\,V_x\right\}\Big|_{t=0}. \tag{1.1.14}$$

We solve for $U(x,t)$ by extending the IST for the real-valued ILW equation [63] to incorporate the complex initial value (1.1.14). Once $U(x,t)$ is known we then proceed to determine $V_x(x,t)$ from (1.1.13), and thence $V(x,t)$ by a quadrature with respect to $x$. A residual function of integration in the formula for $V(x,t)$ is determined from the boundary conditions $V(x,t) \to 0$ (uniformly in $t$) as $|x| \to \infty$.

We close this section by providing a chapter-by-chapter outline of how this thesis is arranged. The next section of this chapter is concerned with a review of integrable nonlinear singular integro-differential equations. Mathematical properties of the operators defined by equations (1.1.7) and (1.1.9) are collated and presented in the last section of Chapter 1; these mathematical properties will be referenced throughout this thesis.

Chapter 2 is the first chapter in this thesis that presents material specifically concerned with the MILW and MBO equations. The mathematical origins of the MILW and MBO equations are elucidated in Chapter 2. A direct method is used in the last section of Chapter 2 to derive the 1-soliton solution for the MILW equation.

Chapter 3 is critical to the development of the solution schemes for the MILW and MBO equations. At the centre of Chapter 3 is a solution of the initial value problem for equation (1.1.4) by means of the IST for the complex-valued KdV equation (1.1.1). Although the material in Chapter 3 provides a new perspective from which to view solutions of the MKdV equation, we consider that the primary significance of Chapter 3 resides in the paradigm we develop for the solutions of the MILW and MBO equations.

Chapter 4 is devoted to the solution of the initial value problem for the MILW equation (1.1.10). The procedure we have outlined in this section to solve the MILW equation is successfully implemented in Chapter 4. Soliton solutions for the real-valued ILW equation and the MILW equation are computed from the results derived in Chapter 4.

Chapter 5 is an expanded version of the recent paper by Scoufis and Cosgrove [114] on the solution of the MBO equation. Scoufis and Cosgrove did not (explicitly) derive in Ref. 114 the 1-soliton solution for the MBO equation, so we devote considerable effort in the last section of Chapter 5 to the derivation of the 1-soliton solution for the MBO equation.

A complete set of results for the limit $\delta \to 0^+$ of the complex-valued ILW

available for the $\delta \to 0^+$

correctness of our results for the solutions of the MKdV and MILW equations motivated the inclusion of Chapter 6 in this thesis.

Chapter 7 contains the $\delta \to \infty$ limit of the direct problem for the complex-valued

A conclusion in which we collate and identify two interesting (open) problems that merit attention is the means we have chosen to end this thesis.

## Section 1.2: A Survey of Integrable Nonlinear Singular Integro-differential Equations

Integral equations are now considered an indispensable mathematical tool for a rigorous and comprehensive discussion of many areas in modern Pure and Applied Mathematics. Difficult problems in Engineering and Mathematical Physics have either been solved in terms of some type of integral equation, or the theory of integral equations has expressed in mathematical language the practical problem of interest. For example, the

homogeneous Fredholm integral equation of the second-kind [117]. Tricomi [117] has published an excellent introduction to the mathematical theory of integral equations.

Until recently, integro-differential equations were considered as some exotic class of integral equations, and therefore did not receive separate consideration. The feature that characterizes an integro-differential equation is that the actions of differentiation *and* integration are applied to the unknown, or to combinations that involve the unknown. At first it may seem that the simultaneous presence of integration and differentiation in a single equation mitigates the complexity of issues that surrounds the separate study of differential and integral equations: unfortunately, such naive thoughts prove unjustified. Saaty [102, pp. 301-345] provides one of the few introductions available to the field of integro-differential equations.

A familiar problem sourced from differential calculus provides us with perhaps the classic example of an integro-differential equation. Suppose that we seek to maximize or minimize a function $I(\alpha)$ whose definition is

$$I(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)\,\mathrm{d}x,$$

where $\alpha$ is a parameter, the limits of integration are known and $f(x,\alpha)$ must be determined so that $I(\alpha)$ is stationary. Computing $\mathrm{d}I/\mathrm{d}\alpha$ by
equating this derivative to zero delivers to us the following integro-differential equation for $f(x,\alpha)$:

$$\int_{a(\alpha)}^{b(\alpha)} \frac{\partial f(x,\alpha)}{\partial \alpha}\,\mathrm{d}x + f(b(\alpha),\alpha)\frac{\mathrm{d}b}{\mathrm{d}\alpha} - f(a(\alpha),\alpha)\frac{\mathrm{d}a}{\mathrm{d}\alpha} = 0.$$

Singular integro-differential equations are characterized by at least one unbounded limit of integration in their nonlocal term(s), or the presence of at least one kernel in the nonlocal term(s) that has at least one singularity within its interval of integration. This section is a survey of the physical applications and mathematical theory of *integrable* nonlinear singular integro-differential equations. Santini [106] has collated recent important developments in the field of integrable nonlinear singular integro-differential equations.

by Santini (because it was published after his review) or of particular relevance to this thesis. In the context of our survey, a nonlinear singular integro-differential equation is integrable if it can be placed into at least one of the following categories:

Category 1: the equation can be linearized;

Category 2: the equation can be mapped into an equation (not necessarily linear) that admits exact solution;

Category 3: the equation is amenable to the Inverse Scattering Transform (IST).

The first class of integrable nonlinear singular integro-differential equations we consider in this review are those that can be placed into either Category 1 or 2 of our classification scheme. Satsuma [111] provides an example of a physically significant nonlinear singular integro-differential equation that can be linearized. The equation at the

$$u_t - d u_{xx} + \frac{k}{2\delta}\frac{\partial}{\partial x}\left\{(\mathrm{P})\int_{-\infty}^{\infty} \coth\left(\frac{\pi}{2\delta}(y-x)\right)u(y)\,\mathrm{d}y\right\} = 0, \qquad (1.2.1)$$

where $d$, $k$ and $\delta$ are positive constants, $u \equiv u(x,t)$ and throughout this section subscripts denote partial derivatives. Satsuma [111] has proposed (1.2.1) as a model for nonlinear diffusion, particularly because (1.2.1) contracts in the limit $\delta \to 0^+$

[20]. The linearization of (1.2.1) is achieved [111] with the assistance of two functions $U^{\pm}(x)$, defined by equation (4) in Ref. 111, which are analytic in the horizontal strips between $\mathrm{Im}(z) = 0$ and $\mathrm{Im}(z) = \pm 2\delta$; $z = x + iy$ defines the complex extension of $x$. Vertical periodicity [111] connects the two functions $U^{\pm}(x)$.

Constantin, Lax and Majda [31] have derived a physically significant nonlinear singular integro-differential equation that can be transformed into a nonlinear partial differential equation, the latter equation possessing explicit closed form solution.

derive the one-dimensional equation

$$\frac{\partial \omega}{\partial t} = \omega \mathbf{H}(\omega) \tag{1.2.2}$$

to model vorticity in three dimensions, where $\omega \equiv \omega(x, t)$ denotes the vorticity: $\omega$ is related $v$, through the equation $\omega = \text{curl}\,(v)$. Equation (1.2.2) is an example of a nonlinear singular integro-differential equation because **H** (the Hilbert transform) is

$$(\mathbf{H}f)(x) \ \underline{\underline{\text{def}}}\ \frac{1}{\pi}(\mathrm{P})\int_{-\infty}^{\infty} \frac{f(\xi)}{\xi - x}\,\mathrm{d}\xi. \tag{1.2.3}$$

authors of Ref. 31 assume that the initial value decays sufficiently fast as $|x| \to \infty$. Constantin, Lax and Majda [31] solve the initial value problem for equation (1.2.2) by transforming this equation into the nonlinear partial differential equation

$$Z_t(x, t) = \frac{1}{2} Z^2(x, t), \tag{1.2.4}$$

where the conformal mapping

$$Z(x, t) = (\mathbf{H}\omega)(x) + i\,\omega(x, t)$$

connects the vorticity to $Z(x, t)$. Clearly, (1.2.4) is a separable differential equation that admits an explicit (rational) solution.

Ablowitz, Fokas and Kruskal [10] have extended the work in Ref. 31 by showing that certain classes of nonlinear singular integro-differential equations can be mapped by means of explicit transformations into either ordinary differential equations or to linearizable partial differential equations. The work in Ref. 10 is significant because it demonstrates that many of the familiar (and ubiquitous) soliton-type equations can be generalized to integrable nonlocal equations.

The key to the interpretation of all wave phenomena is the dispersion relation between the frequency $\omega$ and the wave-number $k$, where $k$ is a real number for the case of uni-directional waves and a $n$-dimensional vector for the case of $n$-dimensional wave propagation. Unless otherwise stated, in this section we will consider uni-directional waves that propagate along the $x$-axis. Also, $t$ will denote our temporal variable. The ranges for the independent variables are $-\infty < x < \infty$ and $t \geq 0$. For uni-directional waves in which, say, the dependent variable is $u(x, t)$, we determine the frequency $\omega$ by substituting the trial solution

$$u(x,t) = e^{i(\omega t - kx)} \tag{1.2.5}$$

into the *linearized* version of the governing equation.

Conventional weakly nonlinear surface waves that propagate over shallow water bounded below by a flat impenetrable bed typically have dispersion relations of the form

$$\omega(k) = k\left(a_1 - a_2 k^2\right), \tag{1.2.6}$$

where $a_1$ is an arbitrary real constant and $a_2$ is a positive constant. The Korteweg-de Vries (KdV) equation [65],

$$u_t + 6uu_x + u_{xxx} = 0, \tag{1.2.7}$$

is a nonlinear wave equation that exhibits a dispersion relation of the type shown in (1.2.6). The dependent variable, $u \equiv u(x,t)$, that appears in (1.2.7) measures the finite amplitude of a uni-directional long wave that propagates on the surface of a shallow fluid [65]; the characteristic feature of a long wave is that $\lambda / h >> 1$, where $\lambda$ denotes the wavelength and $h$ is the depth of the fluid that *directly*

to the well established convention of writing the KdV equation in dimensionless units. The reader who wishes to see a version of the KdV equation embellished with coefficients that include appropriate environmental parameters is directed to Refs 17, 35, 65 and 93. Throughout this thesis all evolution equations will be written in dimensionless units.

We have characterized (1.2.6) as a conventional dispersion law because the phase velocity $c(k) = \omega(k)/k$ for (1.2.6) has a smooth maximum $a_1$ at $k = 0$. The limit $k \to 0$ corresponds to the long wave limit. The dispersion law

$$\omega(k) = k(a_1 - a_2 |k|), \tag{1.2.8}$$

where $a_1$ and $a_2$ are as in (1.2.6), is not conventional because the phase velocity

$$c(k) = \frac{\omega(k)}{k} = a_1 - a_2 |k| \tag{1.2.9}$$

is not differentiable at $k = 0$. The absence from (1.2.9) of differentiability in the region that corresponds to the long wave limit has a profound effect on the structure of any evolution equation associated with the dispersion law (1.2.8). Benjamin [15] has derived the singular integro-differential equation

$$u_t + 2uu_x + \mathbf{H}(u_{xx}) = 0, \tag{1.2.10}$$

where **H** is defined by (1.2.3), as an example of a nonlinear evolution equation that supports the dispersion law (1.2.8). The principal value integral

$$(\mathrm{P})\int_{-\infty}^{\infty} \frac{e^{-ik\xi}}{\xi - x}\mathrm{d}\xi = -\pi i\frac{|k|}{k}e^{-ikx}$$

is useful to the reader who wishes to derive the dispersion law for the BO equation by

amplitude in stable *heterogeneous* fluid systems. Four scenarios were considered by Benjamin, and we refer the reader to Fig. 1 in Ref. 15 for a precise enumeration of the four cases mentioned. In all four cases considered by Benjamin [15] the total depth of the fluid system is infinite, but in each case there exists a layer of finite depth characterized by a continuous density variation.

The equation derived by Benjamin [15] models *uni-directional* wave propagation in the pycnocline (region where the density varies) and is relevant to the dynamics associated with energy exchange between different layers in the ocean. Experimental data collated and analysed by Ostrovsky and Stepanyants [93] corroborate the existence of

equation that is outside its original hydrodynamical setting is supplied by Roberts [98].

Ono [92] presented a derivation of equation (1.2.10) using a multiple-scales expansion, and clarified certain issues relating to the stability of the algebraic solitary wave

$$u(x,t) = \frac{2\kappa}{1 + \kappa^2(x - \kappa t)^2} \tag{1.2.11}$$

first derived by Benjamin [15]; $\kappa$ is a positive parameter. Equation (1.2.10) is now referred

[32] conducted work contemporaneously with Benjamin to show that algebraic solitary waves such as (1.2.11) can exist in fluids of great depth. The reader is referred to the recent review article by Matsuno [80] in which the work found in Refs 15, 32 and 92 is collated and scrutinized.

It is interesting to note that the Modified Benjamin-Ono (MBO) equation,

$$Q_t + \alpha Q_x\left(e^Q - 1\right) + Q_x\mathbf{H}(Q_x) + \mathbf{H}(Q_{xx}) = 0, \tag{1.2.12}$$

$e^Q$

about the vacuum solution is $e^Q = 1 + Q + \mathrm{O}\left(Q^2\right)$. Coincidence of the dispersion laws means that (1.2.12) could be a relevant model for the physical systems modelled by the BO equation. At the present time, equation (1.2.12) is a synthetic equation because it has not been derived from the appropriate hydrodynamical equations of motion.

Both the KdV and BO equations are examples of a general equation for weakly nonlinear waves first proposed by Whitham [121]:

$$u_t + A_0 u u_x + \int_{-\infty}^{\infty} K(x-\eta) u_\eta(\eta, t)\,\mathrm{d}\eta = 0, \tag{1.2.13}$$

where $A_0$ is a nonzero constant and the kernel $K(x)$ is determined from the phase velocity $c(k)$ through the Fourier transform

$$K(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} c(k) e^{ikx}\,\mathrm{d}k. \tag{1.2.14}$$

Let us demonstrate how, for example, the BO equation (1.2.10) can be retrieved from

velocity. Substituting (1.2.9) into (1.2.14) we (formally) obtain the result

$$K(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (a_1 - a_2|k|) e^{ikx}\,\mathrm{d}k,$$

which can be expressed in the form

$$K(x) = a_1\delta(x) - \frac{a_2}{2\pi}\int_{-\infty}^{\infty} |k| e^{ikx}\,\mathrm{d}k \tag{1.2.15}$$

because the Dirac delta function $\delta(x)$ has the integral representation

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,\mathrm{d}k.$$

$F_X\{\psi(x)\}$ of the

function $\psi(x)$ by means of the equation

$$F_X\{\psi(x)\} \underline{\underline{\mathrm{def}}} \frac{1}{2\pi}\int_{-\infty}^{\infty} |k| e^{ikx}\,\mathrm{d}k \int_{-\infty}^{\infty} \psi(\eta) e^{-ik\eta}\,\mathrm{d}\eta. \tag{1.2.16}$$

Benjamin [see equation (A4) in Ref. 15] has derived the elegant identity

$$F_X\{\psi(x)\} = -\mathbf{H}\left(\frac{\mathrm{d}\psi}{\mathrm{d}x}\right) \tag{1.2.17}$$

for $F_X\{\psi(x)\}$, where $\mathbf{H}$ is the Hilbert transform defined by (1.2.3). Substituting (1.2.15) into (1.2.13), and then using (1.2.16) and (1.2.17) we obtain

$$u_t + (A_0 u + a_1) u_x + a_2 \mathbf{H}(u_{xx}) = 0\ . \tag{1.2.18}$$

Equation (1.2.10) is merely a specialized version of (1.2.18).

The KdV and BO equations are inappropriate models in the *intermediate* region: $h/\lambda = O(1)$, where $h$ is the total fluid depth and $\lambda$ denotes the wavelength. Joseph [56] and Kubota et al. [66] derived a physically significant nonlinear wave equation that contains the KdV and BO equations as extreme limits. The Intermediate Long Wave (ILW) equation derived by the authors of Refs 56 and 66 is

$$U_t + \frac{1}{\delta} U_x + 2UU_x + \mathbf{T}(U_{xx}) = 0, \tag{1.2.19}$$

where $U \equiv U(x, t; \delta)$, $\delta$ is a (positive) parameter that measures total fluid depth, and

$$(\mathbf{T}f)(x) \underline{\underline{\text{def}}} \frac{1}{2\delta}(\mathrm{P})\int_{-\infty}^{\infty} \coth\left(\frac{\pi}{2\delta}(\xi - x)\right) f(\xi)\mathrm{d}\xi. \tag{1.2.20}$$

Maslowe and Redekopp [70] have shown that (1.2.19) is an accurate model for waves in a stratified shear flow, provided that the shear-layer thickness is small in comparison to the characteristic wavelength. In the *deep water limit* ($\delta \to \infty$) equation (1.2.19) tends smoothly to the BO equation (1.2.10), and the *shallow water limit* ($\delta \to 0^+$) of (1.2.19) is the KdV equation (1.2.7). Considerable controversy condensed around the deep water limit of equation (1.2.19) when this particular equation was first derived. Chen and Lee [25] disputed the claim originally made by Joseph [56] that solutions of (1.2.19) tend in a smooth manner as $\delta \to \infty$ to solutions of (1.2.10). Joseph [58], Henyey [50], and Satsuma

solutions for (1.2.19) by showing that solutions of the BO equation are indeed retrievable from the $\delta \to \infty$ limit of solutions for the ILW equation.

The first mathematical studies of the BO and ILW equations were numerically based solution schemes [66,84] that used the Fast Fourier Transform. At the centre of the pseudospectral schemes lies the identity [see, for example, equation (C3) in Ref. 16]

$$F\{(\mathbf{H}f)(x)\} = i\,\mathrm{sgn}\,(k)F\{f(x)\},$$

where

$$\mathrm{sgn}\,(k) = \begin{cases} 1, & k > 0 \\ -1, & k < 0 \end{cases}$$

and $F\{f(x)\}$ is the Fourier transform of the function $f(x)$:

$$F\{f(x)\} \underline{\underline{\text{def}}} \int_{-\infty}^{\infty} f(x)e^{-ikx}\mathrm{d}x.$$

Aided by the identity for $F\{(\mathbf{H}f)(x)\}$ it is possible to transform the BO equation into an equation that does not contain a Hilbert transform term [see equation (11) in Ref. 85]; see Burggraf and Duck [21] for a comprehensive description of the pseudospectral method.

Meiss and Pereira [84] numerically integrated equation (1.2.10) using the Lorentzian profile

$$u(x,0) = \frac{2aN}{1 + a^2x^2} \tag{1.2.21}$$

as their initial value. Equation (1.2.21) contains two free parameters: $a$ and $N$. The physical properties (amplitude, width and velocity) of any solution that evolves from the initial value (1.2.21) will be functionally dependent on $a$. The parameter $N$ is a positive integer that Meiss and Pereira [84] conjectured would correspond to the number of solitons in the final state. Meiss and Pereira [84] verified their conjecture for the cases $N = 2$ and $N = 3$, for example they observed [see Fig. 2 in Ref. 94] that the initial value (1.2.21) with $N = 2$ evolves into a final state that consists of two solitons.

Numerical solutions of the ILW and BO equations [66,84] corroborated the conjecture that the ILW and BO equations admit soliton-type solutions. Exact integrability (in the KdV sense) of the ILW and BO equations required these equations to support an infinite set of conservation laws. The quantity $C(t)$ is a conservation law if $C'(t) \equiv 0$, where $t$. Kubota et al. [see equation (32) in Ref. 66] derived the four conservation laws

$$C_1 = \int_{-\infty}^{\infty} U(x,t)\,\mathrm{d}x, \tag{1.2.22a}$$

$$C_2 = \int_{-\infty}^{\infty} U^2(x,t)\,\mathrm{d}x, \tag{1.2.22b}$$

$$C_3 = \frac{4}{3}\int_{-\infty}^{\infty} \left\{U^3(x,t) + \frac{3}{2}U(x,t)\mathbf{T}\big(U_x(x,t)\big)\right\}\mathrm{d}x \tag{1.2.22c}$$

and

$$C_4 = \frac{\partial}{\partial t}\left\{\int_{-\infty}^{\infty} xU(x,t)\,\mathrm{d}x\right\} \tag{1.2.22d}$$

for the ILW equation (1.2.19). In fact, the authors of Ref. 66 succeeded in deriving only the first *three* conservation laws for the ILW equation because it is possible to show that $C_4$ is equivalent to $C_2$. Equations (1.2.22a) and (1.2.22b) express the conservation of mass and momentum, respectively. A striking feature of (1.2.22a-d) is their nonlocal character, which differs significantly from the local-type conservations laws associated with the KdV equation [87]. Ono [see equations (5.1), (5.2) and (5.3) in Ref. 92] derived the first three conservation laws for the BO equation. Meiss and Pereira [see equations (4a) and (4b) in Ref. 84] derived the fourth and fifth conservation laws for the BO equation. Shortly after the work of Meiss and Pereira, Chen and Kaup [28], and Case [24] reported algorithms for the construction of the infinite set of conservation laws associated with the BO equation. Exact integrability of the ILW equation was confirmed by Chen and Lee [25] who used

Contemporaneously with the work of Chen and Lee [25], Joseph and Egri [57] also derived multi-soliton solutions for the ILW equation.

Numerous studies have been devoted to the mathematical properties of the ILW equation. As a result of considerable effort, it has been established that the ILW equation possesses: a linear scattering problem [27,48,108]; multi-soliton solutions [25,57]; a

Recent work [97,105] on the mathematical properties of the ILW equation has been aimed at the identification of the common cause responsible for the exact integrability of local (for example KdV) and nonlocal (for example ILW) nonlinear evolution equations.

Concerning the BO equation, it has been proved that this equation possesses: a linear scattering problem [14,18]; periodic multi-soliton solutions [109]; rational multi-soliton solutions [22,55,71,75]; an infinite number of conservation laws [18,88]; a

be built from the solutions of an integrable many body problem [26]; a Hamiltonian

linearizing transform [5,8].

Variants of the standard ILW and BO equations have been derived by several researchers in order to accurately model certain physical systems that lie outside the framework of the systems modelled by the ILW and BO equations. We provide the following three examples of extensions to the canonical ILW and BO equations:

**1)** The derivation of the ILW equation requires the amplitude $U(x)$ to decay to zero as $|x| \to \infty$, where $x$ is the spatial variable in the direction of propagation. Ablowitz et al. [7] have derived a *periodic* analogue of the ILW equation to cater for the description of long internal waves of moderate amplitude that satisfy the spatial periodicity condition

$U(x+2L) = U(x)$, where $L$ is a positive constant. The equation derived by the authors of Ref. 7 has the same appearance as (1.2.19), except that the integral operator requires amendment to

$$(\mathbf{T} f)(x) \underline{\underline{\mathrm{def}}} \frac{1}{2L} (\mathrm{P}) \int_{-L}^{L} \hat{\Upsilon}(x-\xi; \delta; L) f(\xi) d\xi,$$

where

$$\hat{\Upsilon}(x; \delta; L) \underline{\underline{\mathrm{def}}} -\frac{2K}{\pi} \left\{ Z\left(\frac{Kx}{L}\right) + dn\left(\frac{Kx}{L}\right) cs\left(\frac{Kx}{L}\right) \right\}, \tag{1.2.23}$$

$K$ is the complete elliptic integral of the first-kind (with modulus $k$) $Z$ function and, $cs(a)$ and $dn(a)$ are Jacobian elliptic functions. All of the elliptic functions involved in the definition of $\hat{\Upsilon}(x; \delta; L)$ have modulus $k$ determined by the condition

$$\frac{K'(k)}{K(k)} = \frac{\delta}{L},$$

where $K'(k)$ is the associated elliptic integral of the first-kind. The reader is referred to Abramowitz and Stegun [13] for the definitions of all the elliptic functions associated with (1.2.23). Ablowitz et al. [7] have shown that the periodic ILW equation admits a BT and an infinity of motion constants. Lebedev and Radul [67] have constructed a hierarchy of integrable equations for the periodic ILW equation. The solution of the initial value problem for the ILW equation with periodic boundary conditions remains an open problem;

**2)** Bogdanova-Ryzhova and Ryzhov [see equation (5) in Ref. 19] derived a *forced* BO equation to explore nonlinear disturbances produced in an incompressible boundary layer by roughness on a wall;

**3)** Matsuno [see equation (2.22) in Ref. 81] has also derived a forced BO equation (different from the one in Ref. 19) to model internal wave propagation in a two-layer fluid system that rests over uneven topography.

The first studies of the initial value problems for the ILW and BO equations were concerned with stability for the *linearized* versions of these equations. Chen and Kaup [29] provided the first linear stability analysis of the BO equation, and these authors were particularly interested in the temporal development of solutions of the BO equation linearized about its algebraic 1-soliton solution [see (1.2.11)]. Let $S(x,t)$ denote the function defined by (1.2.11), and write the solution of the BO equation as $u(x,t) = S(x,t) + \psi(x,t)$.

Substituting our equation for $u(x,t)$ into (1.2.10), and then linearizing (with respect to $\psi$) the resultant equation we obtain

$$\psi_t + 2(S\psi)_x + \mathbf{H}(\psi_{xx}) = 0. \tag{1.2.24}$$

The stability analysis of (1.2.24) by Chen and Kaup [29] involved normal-mode solutions of (1.2.24) in the form

$$\psi(x,t) = \varphi(x,t)e^{-i\omega t},$$

where $\omega$ is a constant. The solution of the original evolution equation [in our case the BO equation (1.2.10)] is stable if all bounded solutions (with respect to $x$) of (1.2.24) have $\mathrm{Im}(\omega) \leq 0$. Success of the linear stability analysis is predicated on the assumption that the normal-modes comprise a complete set within some function space that is appropriate for the description of solutions of (1.2.24). Chen and Kaup [see sections IV and V in Ref. 29] demonstrated that the eigenfunctions $\varphi(\omega)$ of the version of (1.2.24) in normal-mode variables do not comprise a complete set. Practical considerations lead Chen and Kaup [29] to append an additional function to the collection of normal modes in order to achieve

Kaup [29], and they also showed that the appended function leads to solutions of (1.2.24) that grow linearly in time. Bennet et al. [16] succeeded in explaining the significance of the secular instability found by Chen and Kaup. Linear stability analysis of the BO equation has received renewed attention recently [82,115].

Exact integrability (in the KdV sense) of the ILW and BO equations stimulated interest in the application of the IST to the initial value problems for the ILW and BO equations. Kodama et al. [63] were the first to use the IST to solve the initial value problem for the ILW equation. The solution of the initial value problem for the BO equation by means of the IST was first presented by Fokas and Ablowitz [43]. Rigorous aspects of the IST for the ILW and BO equations have been studied by Abdelouhab et al. [1], Coifman and Wickerhauser [30], and Naumkin [91]. The results of Coifman and Wickerhauser [30] are particularly relevant to this thesis. The most pertinent result (in the context of this thesis) from Ref. 30 is Theorem 7.1, which states that if the solution of the *complex-valued* BO equation decays sufficiently fast as $|x| \to \infty$, then the scattering problem associated with the *complex-valued* BO equation has no bound states in the neighbourhood of the origin (with respect to the spectral parameter). In this review we will not discuss the differences between the IST schemes for the ILW and BO equations because in the body of this thesis we explore these differences in considerable detail.

The reader who has arrived at this point may have formed the impression that the ILW and BO equations are the only two nonlinear singular integro-differential equations that are integrable by the IST. We now widen the scope of our review to consider equations other than the ILW and BO equations. The first equation that our expanded view captures is the *matrix* evolution equation [33,34,107]

$$Q_t = \sigma_3 \gamma(L) Q, \tag{1.2.25}$$

where:

- $\sigma_3 = \mathrm{diag}\,(1, -1)$ is the third Pauli spin matrix;

- $Q \equiv Q(x,t)$ is a $2 \times 2$ matrix whose main diagonal contains only zeroes, and whose non-zero terms correspond to the off-diagonal terms of a $2 \times 2$ matrix $U(x,t)$ that satisfies the scalar identities $\mathrm{tr}\{\sigma_3 U(x,t)\} = 0$ and $\mathrm{tr}\{U(x,t)\} + \det\{U(x,t)\} = 0$;

- $\gamma(y)$ is an arbitrary polynomial in $y$;

- $L$ is the operator

$$L\,F \;\underline{\underline{\mathrm{def}}}\; i\sigma_3 \left\{ \sqrt{1+Q^2}\,DF + \tfrac{1}{2} Q D^{-1}\left( [Q, F] / \sqrt{1+Q^2} \right) \right\},$$

where $F$ is an off-diagonal $2 \times 2$ matrix, $[a, b] \equiv ab - ba$ is the standard commutator bracket, and $\sqrt{1+Q^2} = \sqrt{1 + u_{12}u_{21}}$; $u_{12}$ and $u_{21}$ denote (respectively) the 12- and 21- components of the matrix $U(x,t)$. The symbols $D$ and $D^{-1}$ in the definition of $L$ denote the integral operator

$$(D\,f)(x) \;\underline{\underline{\mathrm{def}}}\; \tfrac{1}{\delta}(\mathrm{P})\int_{-\infty}^{\infty} \left\{ \sinh\left( \tfrac{\pi}{\delta}(y-x) \right) \right\}^{-1} f(y)\,\mathrm{d}y$$

and its inverse

$$\left(D^{-1} f\right)(x) \;\underline{\underline{\mathrm{def}}}\; -\tfrac{1}{\delta}(\mathrm{P})\int_{-\infty}^{\infty} \left\{ \coth\left( \tfrac{\pi}{\delta}(y-x) \right) \right\}^{-1} f(y)\,\mathrm{d}y.$$

The mathematical origin of (1.2.25) can be found in Ref. 34. The simplest evolution equation that belongs to the class (1.2.25) corresponds to the linear function $\gamma(y) = -iy$ and the $2 \times 2$ matrices

$$U(x,t) = \begin{bmatrix} u(x,t) - 1 & v(x,t) \\ v(x,t) & u(x,t) - 1 \end{bmatrix} \text{ and } Q(x,t) = \begin{bmatrix} 0 & v(x,t) \\ v(x,t) & 0 \end{bmatrix},$$

where $u$ and $v$ denote two sufficiently smooth functions. The scalar constraints on the matrix $U$ translate into the condition $u = \sqrt{1+v^2}$, and we find that the action of $L$ produces the nonlinear singular integro-differential equation

$$v_t = \sqrt{1+v^2}\,Dv. \tag{1.2.26}$$

Interestingly, in the shallow water region if we let $v = \delta v' + \mathrm{O}(\delta^2)$, $t = 2t'/\delta$ and $x = x'$, and make use of the limit [see equation (18) in Ref. 107]

$$D \to \frac{\delta}{2}\frac{\partial}{\partial x} + \mathrm{O}(\delta^2) \text{ as } \delta \to 0^+,$$

then we retrieve from (1.2.26) the simplest model for linear waves: $v_t = v_x$.

In the context of this thesis, the most interesting case of (1.2.25) arises from the selection $\gamma(y) = iy^3$. Santini [107] has shown that the cubic $\gamma(y) = iy^3$ and the matrix $U(x,t)$ with the off-diagonal elements $u_{12} = u_{21} = v$ leads to the equation

$$v_t = \sqrt{1+v^2}\,D\left\{\sqrt{1+v^2}\,D\left(\sqrt{1+v^2}\,Dv + vD^{-1}vDv\right)\right\}. \tag{1.2.27}$$

The shallow water limit of (1.2.27) is the MKdV equation, and therefore (1.2.27) is a candidate for the intermediate modified KdV equation. However, we note two significant differences between the Modified Intermediate Long Wave (MILW) equation (1.1.10) and (1.2.27): **1)** equation (1.1.10) contains derivatives and integrals of the dependent variable with respect to the spatial variable, whereas (1.2.27) is purely nonlocal with respect to the spatial variable; **2)** equation (1.2.27) originates from a $2 \times 2$ matrix spectral problem [see equations (1a) and (1b) in Ref. 34] whereas (1.1.10) originates from the BT for the ordinary ILW equation (1.2.19).

Before we leave the class of evolution equations associated with (1.2.25), we wish to revisit equation (1.2.26). Introduce into (1.2.26) the transformation $v(x,t) = i\sin(\theta(x,t))$. The reader will find that the equation for $\theta(x,t)$ is

$$\theta_t = D\sin(\theta),$$

and because $D^{-1}$ is the inverse of $D$ the equation for $\theta(x,t)$ can be written in the form

$$D^{-1}\theta_t = \sin(\theta). \tag{1.2.28}$$

The Hilbert transform $\mathbf{H}$ defined by (1.2.3) appears in the limit $D^{-1} \to -\mathbf{H}$ as $\delta \to \infty$, and therefore the deep water analogue of (1.2.28) is the sine-Hilbert equation:

$$\mathbf{H}(\theta_t) = -\sin(\theta). \tag{1.2.29}$$

The dispersion law for the sine-Hilbert equation is particularly simple. Recalling that $\sin(\theta) = \theta + O(\theta^3)$ we can easily see that the linearized version of equation (1.2.29) is

$$\mathbf{H}(\theta_t) = -\theta.$$

Substituting (1.2.5) into the linearized sine-Hilbert equation we obtain the dispersion law $\omega(k) = -\operatorname{sgn}(k)$. Santini et al. [104] have used the IST to solve the initial value problem for the sine-Hilbert equation.

In our discussion of the BO equation we noted that this particular equation has a dispersion law that produces a phase velocity of the type shown in (1.2.9). Matsuno [77] has observed and commented on a certain anomaly that is associated with the phase velocity for the BO equation. The author of Ref. 77 has noted that for high wave-numbers ($k \to \infty$) the phase velocity for the BO equation becomes negative, in stark violation of the assumption of uni-directional wave propagation made by Benjamin [15] in his original derivation of the BO equation. Matsuno [78] has proposed the nonlinear singular integro-differential equation

$$u_t + u_x - 2uu_t + 2u_x \int_x^\infty u_t(x', t)\,\mathrm{d}x - \mathbf{H}(u_{tx}) = 0 \tag{1.2.30}$$

to resolve the anomalies inherent in the BO equation. The derivation of (1.2.30) presented by Matsuno [78] is purely heuristic, being arrived at by a consideration of the terms required to achieve certain balances. The dispersion law for equation (1.2.30) is $\omega = k/(1 + |k|)$, and the corresponding phase velocity $c(k) = 1/(1 + |k|)$ is always positive. In the long wave limit $k \to 0$

into $\omega = k\left\{1 - |k| + O(k^2)\right\}$

Therefore, (1.2.30) can model the same physical systems described by the BO equation. Matsuno [79] has shown that (1.2.30) is a completely integrable soliton-type equation. For example, (1.2.30) has a known Lax pair [see equations (2.31a-d) in Ref. 79]. At the present time the solution of the Cauchy problem for (1.2.30) by the IST is an open problem.

A physically significant integrable nonlinear singular integro-differential equation that has been derived from an asymptotic multi-scale technique has recently been supplied by Pelinovsky [94]. The equation derived by Pelinovsky is as follows:

$$iA_t = A_{xx} + iA\left(|A|^2\right)_x + A\mathbf{T}\left(\left\{|A|^2\right\}_x\right), \tag{1.2.31}$$

where $A \equiv A(x,t)$ is a *complex-valued* function of the two real variables $x$ and $t$. Assuming that $|A|^2 \sim O(\delta)$ as $\delta \to 0^+$ we find that in the shallow water region (1.2.31) transforms

$$iA_t = A_{xx} - \frac{1}{\delta}A|A|^2,$$

and therefore (1.2.31) represents an intermediate version of the NLS equation. Pelinovsky and Grimshaw [95] have used the IST to solve the initial value problem for (1.2.31) with the nonzero boundary conditions $|A| \to \rho$ as $|x| \to \infty$, where $\rho$ is a positive real constant.

We close this review with the remark that nonlinear singular integro-differential equations that involve two spatial variables and one temporal variable have been derived [4,83,101].

# Section 1.3: Properties of the Operators $\mathbf{T}$ and $\mathbf{H}$

The singular integral operators

$$\left(\mathbf{T}f\right)(x) \underline{\underline{\mathrm{def}}} \frac{1}{2\delta}\left(\mathrm{P}\right)\int_{-\infty}^{\infty} \coth\left(\frac{\pi}{2\delta}(\xi - x)\right) f\left(\xi\right)\mathrm{d}\xi \tag{1.3.1}$$

and

$$\left(\mathbf{H}f\right)(x) \underline{\underline{\mathrm{def}}} \frac{1}{\pi}\left(\mathrm{P}\right)\int_{-\infty}^{\infty} \frac{f\left(\xi\right)}{\xi - x}\mathrm{d}\xi, \tag{1.3.2}$$

where $\left(\mathrm{P}\right)$ indicates that an integral is in the *principal value* sense, will manifest themselves continually in this thesis. The $\mathbf{T}$ operator, for example, appears in the MILW equation

$$V_t + \beta V_x\left(e^V - 1\right) + \frac{1}{\delta}V_x + V_x\mathbf{T}\left(V_x\right) + \mathbf{T}\left(V_{xx}\right) = 0,$$

and the $\mathbf{H}$ operator, the *Hilbert transform*, is present in the MBO equation

$$Q_t + \alpha Q_x\left(e^Q - 1\right) + Q_x\mathbf{H}\left(Q_x\right) + \mathbf{H}\left(Q_{xx}\right) = 0.$$

The operators $\mathbf{T}$ and $\mathbf{H}$ possess several interesting and useful attributes. In this section we will collate several of the properties associated with the operators defined by equations (1.3.1) and (1.3.2). The results in this section provide the infrastructure for certain derivations that appear elsewhere in this thesis. We will present derivations of all the results that appear in this section; our sources [42,62,110,112] either provide an outline of a particular proof or are content to merely quote the properties of the operators $\mathbf{T}$ and $\mathbf{H}$ that are used in their works. Throughout this section, we will assume that $f\left(x\right)$, the generic argument for $\mathbf{T}$ and $\mathbf{H}$, satisfies all the conditions necessary to ensure the validity of our computations. For example, we will assume that $f\left(x\right) \in L_1\left(\Re\right) \cap H_{\mathrm{O}}\left(\Re\right)$, where $H_{\mathrm{O}}\left(\Re\right)$

$f\left(x\right) \in L_1\left(\Re\right)$ is sufficient to guarantee the convergence of the integrals that define (1.3.1) and (1.3.2), whereas the condition $f\left(x\right) \in H_{\mathrm{O}}\left(\Re\right)$ renders integrable the singularity at $\xi = x$ in the integrands of (1.3.1) and (1.3.2).

In physical applications of integrable nonlinear singular integro-differential equations [56], the parameter $\delta$ measures the total depth of the media in which an internal wave is propagating. Two extremes of $\delta$ are possible: shallow water limit ($\delta \to 0^+$); deep water limit ($\delta \to \infty$). The first results that we will present in this section are the deep and shallow water limits of the $\mathbf{T}$ operator. Consider, first, the *deep water* region: $\delta >> 1$. The kernel of (1.3.1) admits the expansion

$$\frac{1}{2\delta}\coth\left(\frac{\pi}{2\delta}(\xi - x)\right) = \frac{1}{\pi\left(\xi - x\right)} + \frac{\pi\left(\xi - x\right)}{12\delta^2} + O\left(\frac{1}{\delta^4}\right) \tag{1.3.3}$$

in the deep water region. Substituting (1.3.3) into (1.3.1) we immediately obtain the equation

$$\lim_{\delta\to\infty}\left(\mathbf{T}f\right)(x) = \left(\mathbf{H}f\right)(x). \tag{1.3.4}$$

The calculation that elucidates the behaviour of $\mathbf{T}$ in the *shallow water* region ($0 < \delta << 1$) is more difficult than the preceding calculation. The author is unaware of any reference that presents a detailed exposition of the $\mathbf{T}$
$0 < \delta << 1$, so we will devote considerable effort to rectify this situation. Motivated by the work of Satsuma et al. [112], we will introduce the integral operator

$$\left(\hat{\mathbf{T}}f\right)(x) \underline{\underline{\text{def}}} \frac{1}{2\delta}\left(\mathrm{P}\right)\int_{-\infty}^{\infty}\left\{\coth\left(\frac{\pi}{2\delta}(\xi - x)\right) - \operatorname{sgn}\left(\xi - x\right)\right\}f\left(\xi\right)\mathrm{d}\xi \tag{1.3.5}$$

to assist us in determining the behaviour of $\left(\mathbf{T}f\right)(x)$ in the region $0 < \delta << 1$. In equation (1.3.5), and henceforth, $\operatorname{sgn}\left(x\right) = \pm 1$ if $\pm x > 0$. The connection between the operators $\mathbf{T}$ and $\hat{\mathbf{T}}$ is expressed by the equation

$$\left(\mathbf{T}f_x\right)(x) = \left(\hat{\mathbf{T}}f_x\right)(x) - \frac{1}{\delta}f\left(x\right), \tag{1.3.6}$$

where $f_x \equiv \mathrm{d}f/\mathrm{d}x$.

Next, we shall introduce the change of variable

$$y = \frac{\xi - x}{\delta}$$

into the integrand of (1.3.5). In terms of the new dummy variable, $y$, the $\hat{\mathbf{T}}$ operator reads

$$\left(\hat{\mathbf{T}}f\right)(x) = \frac{1}{2}\left(\mathrm{P}\right)\int_{-\infty}^{\infty}\left\{\coth\left(\frac{\pi y}{2}\right) - \operatorname{sgn}\left(y\right)\right\}f\left(x + \delta y\right)\mathrm{d}y. \tag{1.3.7}$$

The reader is politely requested to note that on account of the restriction $\delta > 0$ we have

$$\operatorname{sgn}\left(\delta y\right) = \operatorname{sgn}\left(y\right) \ .$$

The Taylor series for the function $f\left(x+\delta y\right)$ when expressed in *ascending* powers of $\delta y$ is

$$f\left(x+\delta y\right)=\sum_{k=0}^{\infty}\frac{\left(\delta y\right)^{k}f^{(k)}\left(x\right)}{k!}, \tag{1.3.8}$$

where $f^{(k)}\left(x\right)\equiv \mathrm{d}^{k}f/\mathrm{d}x^{k}$ whenever $k\geq 1,$ and $f^{(0)}\left(x\right)\equiv f\left(x\right).$ Substituting (1.3.8) into (1.3.7), and then interchanging the order of summation and integration will generate for us the infinite series

$$\left(\hat{\mathbf{T}}f\right)(x)=\frac{1}{2}\sum_{k=0}^{\infty}\frac{\delta^{k}f^{(k)}(x)}{k!}\left(\mathrm{P}\right)\int_{-\infty}^{\infty}\left\{\coth\left(\frac{\pi y}{2}\right)-\operatorname{sgn}\left(y\right)\right\}y^{k}\mathrm{d}y. \tag{1.3.9}$$

The procedure that we will use to evaluate the integral

$$\left(\mathrm{P}\right)\int_{-\infty}^{\infty}\left\{\coth\left(\frac{\pi y}{2}\right)-\operatorname{sgn}\left(y\right)\right\}y^{k}\mathrm{d}y$$

is sensitive to whether $k=0$ or $k\geq 1$. Consider, first, the case $k=0$:

$$\left(\mathrm{P}\right)\int_{-\infty}^{\infty}\left\{\coth\left(\frac{\pi y}{2}\right)-\operatorname{sgn}\left(y\right)\right\}\mathrm{d}y.$$

At $k=0$ (in accordance with the definition of a principal value integral) we are required to compute

$$\lim_{\varepsilon\to 0^{+}}\left(\int_{-\infty}^{-\varepsilon}+\int_{+\varepsilon}^{\infty}\right)\left\{\coth\left(\frac{\pi y}{2}\right)-\operatorname{sgn}\left(y\right)\right\}\mathrm{d}y,$$

and this particular limit is equivalent to

$$\lim_{\varepsilon\to 0^{+}}\left(\int_{-\infty}^{-\varepsilon}\left\{\coth\left(\frac{\pi y}{2}\right)+1\right\}\mathrm{d}y+\int_{+\varepsilon}^{\infty}\left\{\coth\left(\frac{\pi y}{2}\right)-1\right\}\mathrm{d}y\right). \tag{1.3.10}$$

If we use the change of variable $z=-\,y$ in the *first* of the two integrals within (1.3.10), then we obtain the equation

$$\int_{-\infty}^{-\varepsilon}\left\{\coth\left(\frac{\pi y}{2}\right)+1\right\}\mathrm{d}y=-\int_{+\varepsilon}^{\infty}\left\{\coth\left(\frac{\pi z}{2}\right)-1\right\}\mathrm{d}z. \tag{1.3.11}$$

Equations (1.3.10) and (1.3.11) make possible the following conclusion:

$$(\mathrm{P})\int_{-\infty}^{\infty}\left\{\coth\left(\frac{\pi y}{2}\right)-\operatorname{sgn}(y)\right\}\mathrm{d}y=0. \tag{1.3.12}$$

Now consider the generic case: $k \geq 1$. In the generic case, each nonlocal term in the series (1.3.9) has an integrand that is bounded at $y = 0$. The boundedness arises because in the neighbourhood of $y = 0$ it can be shown that

$$\lim_{y \to 0}\coth\left(\frac{\pi y}{2}\right)y^k=\begin{cases}\frac{2}{\pi}, & k=1\\ 0, & k\geq 2.\end{cases}$$

Therefore, whenever $k \geq 1$ we can discard the principal value notation in (1.3.9), and proceed to evaluate each nonlocal term in the series (1.3.9) as a standard Riemann integral,

$$\int_{-\infty}^{\infty}\left\{\coth\left(\frac{\pi y}{2}\right)-\operatorname{sgn}(y)\right\}y^k\mathrm{d}y. \tag{1.3.13}$$

The integrand of (1.3.13) is either an odd or even function, whose character alternates as the discrete parameter $k$ moves over the positive integers. The possible values for (1.3.13), given the situation $k \geq 1$, are

$$\int_{-\infty}^{\infty}\left\{\coth\left(\frac{\pi y}{2}\right)-\operatorname{sgn}(y)\right\}y^k\mathrm{d}y=2\int_{0}^{\infty}\left\{\coth\left(\frac{\pi y}{2}\right)-1\right\}y^k\mathrm{d}y \tag{1.3.14}$$

in the case of odd-$k$ and zero for even-$k$.

Equations (1.3.12) and (1.3.14) restrict the series (1.3.9) to a sum over odd-$k$ values only, that is

$$\left(\hat{\mathbf{T}}f\right)(x)=\sum_{k=1}^{\infty}\frac{\delta^{2k-1}f^{(2k-1)}(x)}{(2k-1)!}I_k, \tag{1.3.15}$$

where

$$I_k \underline{\underline{\mathrm{def}}} \int_0^{\infty}\left\{\coth\left(\frac{\pi y}{2}\right)-1\right\}y^{2k-1}\mathrm{d}y.$$

The integral that defines $I_k$ (see the previous page) is equivalent to

$$I_k = 2\int_0^{\infty} \frac{y^{2k-1}}{e^{\pi y} - 1}\,\mathrm{d}y \tag{1.3.16}$$

because

$$\coth\left(\frac{\pi y}{2}\right) = \frac{e^{\pi y} + 1}{e^{\pi y} - 1}.$$

At this stage of our calculation it is appropriate to reveal the strategy that we will use to evaluate (1.3.16). Ideally, we would like to express (1.3.16) in terms of a germane tabulated (special) function. The *Riemann Zeta function*, $\zeta(s)$, will prove beneficial to our work. The standard definition of $\zeta(s)$ is [13, p. 807]

$$\zeta(s) = \sum_{l=1}^{\infty} \frac{1}{l^s}, \tag{1.3.17}$$

where Re $(s) > 1$. An alternative to (1.3.17) is [13, p. 807]

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^{\infty} \frac{z^{s-1}}{e^z - 1}\,\mathrm{d}z, \tag{1.3.18}$$

where $\Gamma(s)$, which denotes the *Gamma function*, is defined by the integral formula

$$\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x}\,\mathrm{d}x.$$

The connection between (1.3.16) and the Riemann Zeta function will become transparent once we introduce the transformation $z = \pi y$ into (1.3.16); the result of this transformation is

$$I_k = \frac{2}{\pi^{2k}}\int_0^{\infty} \frac{z^{2k-1}}{e^z - 1}\,\mathrm{d}z. \tag{1.3.19}$$

A comparison of equations (1.3.18) and (1.3.19) leads us to the conclusion

$$I_k = \frac{2\Gamma(2k)\zeta(2k)}{\pi^{2k}}, \tag{1.3.20}$$

and because

$$\Gamma(2k) = (2k - 1)!$$

we can express (1.3.20) in the form

$$I_k = \frac{2\zeta(2k)(2k-1)!}{\pi^{2k}}. \tag{1.3.21}$$

The identity [13, p. 807]

$$\zeta(2k) = \frac{|B_{2k}|(2\pi)^{2k}}{2(2k)!} \tag{1.3.22}$$

will facilitate a version of (1.3.21) that is more convenient than the current version of equation (1.3.21). The expression $B_{2k}$ in (1.3.22) denotes a *Bernoulli number*, which is defined by the following equation [13, p. 804]

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{x^n}{n!} B_n.$$

A sample from the set of Bernoulli numbers is as follows [13, p. 804]:

$$B_0 = 1;\ B_1 = -\tfrac{1}{2};\ B_{2k-1} = 0\ \ (k \ge 2);\ B_2 = \tfrac{1}{6};\ B_4 = -\tfrac{1}{30}.$$

Replacing the $\zeta(2k)$ term in (1.3.21) with the identity (1.3.22) we obtain the equation

$$I_k = \frac{2^{2k}}{2k}|B_{2k}|. \tag{1.3.23}$$

Substituting (1.3.23) into (1.3.15) will produce for us the series

$$\left(\hat{\mathbf{T}} f\right)(x) = \sum_{k=1}^{\infty} \frac{2^{2k}|B_{2k}|\delta^{2k-1}}{(2k)!} f^{(2k-1)}(x),$$

which is the shallow water expansion of $\hat{\mathbf{T}}$, expressed in terms of known quantities. The explicit form of the above series when truncated after the *second* term is

$$\left(\hat{\mathbf{T}} f\right)(x) = \frac{\delta}{3} f^{(1)}(x) + \frac{\delta^3}{45} f^{(3)}(x) + \mathrm{O}\left(\delta^5\right). \tag{1.3.24}$$

If we (formally) replace $f$ with $f_x$ in equation (1.3.24), then we obtain the series

$$\left(\hat{\mathbf{T}}\ {}_x\right)(x) = \frac{\delta}{3} f^{(2)}(x) + \frac{\delta^3}{45} f^{(4)}(x) + \mathrm{O}\left(\delta^5\right) \tag{1.3.25}$$

Finally, the shallow water expansion

$$\left(\mathbf{T} f_x\right)(x) = -\frac{1}{\delta} f(x) + \frac{\delta}{3} f^{(2)}(x) + \frac{\delta^3}{45} f^{(4)}(x) + \mathrm{O}\left(\delta^5\right) \tag{1.3.26}$$

is the result of our substituting (1.3.25) into (1.3.6).

The deep and shallow water limits that have occupied our attention in this section

**T** operator. Our knowledge about the limiting values of the **T** operator is enriched by the result [106]

$$\left(\mathbf{T}f\right)(x) \to \mp \frac{1}{2\delta}\int_{-\infty}^{\infty} f\left(\xi\right)\mathrm{d}\xi \;\text{ as }\; x \to \pm\,\infty, \tag{1.3.27}$$

which can be derived if we substitute the limit

$$\coth\left(\frac{\pi}{2\delta}(\xi - x)\right) \to \mp\, 1 \;\text{ as }\; x \to \pm\,\infty$$

into equation (1.3.1). An important corollary to (1.3.27) is that if $f \to 0$ and $f_x \to 0$ as $\left|x\right| \to \infty$, then

$$\left(\mathbf{T}f_x\right)(x) \to 0 \;\text{ as }\; \left|x\right| \to \infty. \tag{1.3.28}$$

The question as to whether the two linear operators **T** and $\partial/\partial x$ *commute* merits a response because the MILW equation contains the terms $\mathbf{T}\left(V_x\right)$ and $\mathbf{T}\left(V_{xx}\right)$. Therefore, let us consider (1.3.1), but expressed in the form

$$\left(\mathbf{T}f\right)(x) = \lim_{\varepsilon \to 0^+}\left(\mathbf{I}_- + \mathbf{I}_+\right), \tag{1.3.29}$$

where

$$\mathbf{I}_{\pm} \;\underline{\underline{\mathrm{def}}}\; \pm\frac{1}{2\delta}\int_{x\,\pm\,\varepsilon}^{\pm\,\infty}\coth\left(\frac{\pi}{2\delta}(\xi - x)\right) f\left(\xi\right)\mathrm{d}\xi.$$

Differentiating both sides of (1.3.29) with respect to $x$ we obtain

$$\frac{\partial}{\partial x}\left\{\left(\mathbf{T}f\right)(x)\right\} = \lim_{\varepsilon \to 0^+}\left(\frac{\partial \mathbf{I}_-}{\partial x} + \frac{\partial \mathbf{I}_+}{\partial x}\right). \tag{1.3.30}$$

The procedure to simplify $\partial\mathbf{I}_{\pm}/\partial x$ employs to differentiate under an integral sign and a single integration by parts. As an example of how to simplify the derivatives in the right hand side of (1.3.30), consider $\partial\mathbf{I}_-/\partial x$:

$$\frac{\partial \mathbf{I}_-}{\partial x} = \frac{\partial}{\partial x}\left\{\frac{1}{2\delta}\int_{-\infty}^{x-\varepsilon}\coth\left(\frac{\pi}{2\delta}(\xi - x)\right) f\left(\xi\right)\mathrm{d}\xi\right\}. \tag{1.3.31}$$

(1.3.31) we obtain the following equation:

$$\frac{\partial \mathbf{I}_-}{\partial x} = -\frac{1}{2\delta} f\left(x-\varepsilon\right)\coth\left(\frac{\varepsilon\pi}{2\delta}\right) + \frac{1}{2\delta}\int_{-\infty}^{x-\varepsilon} f\left(\xi\right)\frac{\partial}{\partial x}\left\{\coth\left(\frac{\pi}{2\delta}(\xi - x)\right)\right\}\mathrm{d}\xi. \qquad (1.3.32)$$

The operators $\partial/\partial x$ and $\partial/\partial$   form a skew-symmetric pair when applied to the function

$$\coth\left(\frac{\pi}{2}(\xi - x)\right),$$

that is

$$\frac{\partial}{\partial x}\left\{\coth\left(\frac{\pi}{2\delta}(\xi - x)\right)\right\} = -\frac{\partial}{\partial \xi}\left\{\coth\left(\frac{\pi}{2\delta}(\xi - x)\right)\right\}. \qquad (1.3.33)$$

Replacing the derivative in the integrand of (1.3.32) with the *right hand side* of (1.3.33) we obtain

$$\frac{\partial \mathbf{I}_-}{\partial x} = -\frac{1}{2\delta} f\left(x-\varepsilon\right)\coth\left(\frac{\varepsilon\pi}{2\delta}\right) - \frac{1}{2\delta}\int_{-\infty}^{x-\varepsilon} f\left(\xi\right)\frac{\partial}{\partial \xi}\left\{\coth\left(\frac{\pi}{2\delta}(\xi - x)\right)\right\}\mathrm{d}\xi. \qquad (1.3.34)$$

An *integration by parts* applied to the integral in (1.3.34) produces the simplified equation

$$\frac{\partial \mathbf{I}_-}{\partial x} = \frac{1}{2\delta}\int_{-\infty}^{x-\varepsilon} \coth\left(\frac{\pi}{2\delta}(\xi - x)\right) f_\xi\left(\xi\right)\mathrm{d}\xi. \qquad (1.3.35)$$

The reader is now in a position to verify that the equation

$$\frac{\partial \mathbf{I}_+}{\partial x} = \frac{1}{2\delta}\int_{x+\varepsilon}^{\infty} \coth\left(\frac{\pi}{2\delta}(\xi - x)\right) f_\xi\left(\xi\right)\mathrm{d}\xi \qquad (1.3.36)$$

is formed when we repeat the procedure used to derive (1.3.35), but applied to the integral that determines $\partial \mathbf{I}_+ / \partial x$.

The issue concerning the possible commutativity of the linear operators $\mathbf{T}$ and $\partial/\partial x$ is resolved when we substitute (1.3.35) and (1.3.36) into (1.3.30), as follows:

$$\frac{\partial}{\partial x}\left\{\left(\mathbf{T} f\right)(x)\right\} = \lim_{\varepsilon \to 0^+}\frac{1}{2\delta}\left(\int_{-\infty}^{x-\varepsilon}\mathrm{d}\xi + \int_{x+\varepsilon}^{\infty}\mathrm{d}\xi\right)\coth\left(\frac{\pi}{2\delta}(\xi - x)\right) f_\xi\left(\xi\right)\mathrm{d}\xi$$

$$\equiv \frac{1}{2\delta}\left(\mathrm{P}\right)\int_{-\infty}^{\infty}\coth\left(\frac{\pi}{2\delta}(\xi - x)\right) f_\xi\left(\xi\right)\mathrm{d}\xi$$

$$\frac{\partial}{\partial x}\left\{\left(\mathbf{T} f\right)(x)\right\} = \left(\mathbf{T} f_x\right)(x), \qquad (1.3.37)$$

and from (1.3.37) we conclude that the linear operators $\mathbf{T}$ and $\partial/\partial x$ commute.

The deep water ($\delta \to \infty$) analogue of (1.3.37) is

$$\frac{\partial}{\partial x}\left\{\left(\mathbf{H}f\right)(x)\right\} = \left(\mathbf{H}f_x\right)(x) \tag{1.3.38}$$

because of the limit (1.3.4). The proof of (1.3.38) is similar to the proof of (1.3.37). Hence, we will omit the proof designed to show that the two linear operators $\mathbf{H}$ and $\partial/\partial x$ commute.

Two equations that combine the $\mathbf{T}$ operator with an integration over the real line are [112]

$$\int_{-\infty}^{\infty}\left\{f(x)\left(\mathbf{T}g\right)(x) + g(x)\left(\mathbf{T}f\right)(x)\right\}\mathrm{d}x = 0 \tag{1.3.39}$$

and the following result attributed to Chen and Lee [25]:

$$\left(\mathbf{T}\frac{\partial}{\partial x}\left\{\ln\, f_+ / f_-\right\}\right)(x) = -\,i\frac{\partial}{\partial x}\left(\ln\, f_+\, f_-\right). \tag{1.3.40}$$

In equation (1.3.40), and henceforth, the following conventions will apply:

- ;
- ;
- .

We now proceed to the derivation of (1.3.39), after which we will prove (1.3.40). In respect of equation (1.3.39), we will assume that $f(x)$ and $g(x)$ are sufficiently well behaved to guarantee that

$$\left|\int_{-\infty}^{\infty} f(x)\left(\mathbf{T}g\right)(x)\mathrm{d}x\right| < \infty$$

and

$$\left|\int_{-\infty}^{\infty} g(x)\left(\mathbf{T}f\right)(x)\mathrm{d}x\right| < \infty.$$

The expression $f(x)\left(\mathbf{T}g\right)(x)$ when written in its entirety is

$$f(x)\big(\mathbf{T}g\big)(x) = \frac{1}{2\delta}(\mathrm{P})\int_{-\infty}^{\infty} f(x)\coth\left(\frac{\pi}{2\delta}(\xi - x)\right)g(\xi)\,\mathrm{d}\xi. \tag{1.3.41}$$

Integrating both sides of (1.3.41) over the interval $-\infty < x < \infty$ we obtain the equation

$$\int_{-\infty}^{\infty} f(x)(\mathbf{T}g)(x)\,\mathrm{d}x = \frac{1}{2\delta}\int_{-\infty}^{\infty} \mathrm{d}x\,(\mathrm{P})\int_{-\infty}^{\infty} f(x)\coth\left(\frac{\pi}{2\delta}(\xi - x)\right)g(\xi)\,\mathrm{d}\xi. \tag{1.3.42}$$

A reversal of the order of integration in the right hand side of (1.3.42) induces the following sequence of results:

$$\int_{-\infty}^{\infty} f(x)(\mathbf{T}g)(x)\,\mathrm{d}x = \frac{1}{2\delta}\int_{-\infty}^{\infty} \mathrm{d}\xi\,(\mathrm{P})\int_{-\infty}^{\infty} f(x)\coth\left(\frac{\pi}{2\delta}(\xi - x)\right)g(\xi)\,\mathrm{d}x$$

$$= -\frac{1}{2\delta}\int_{-\infty}^{\infty} g(\xi)\,\mathrm{d}\xi\,(\mathrm{P})\int_{-\infty}^{\infty} \coth\left(\frac{\pi}{2\delta}(x - \xi)\right)f(x)\,\mathrm{d}x$$

$$= -\int_{-\infty}^{\infty} g(\xi)(\mathbf{T}f)(\xi)\,\mathrm{d}\xi.$$

Therefore,

$$\int_{-\infty}^{\infty} f(x)(\mathbf{T}g)(x)\,\mathrm{d}x = -\int_{-\infty}^{\infty} g(x)(\mathbf{T}f)(x)\,\mathrm{d}x.$$

A rearrangement of the most recent equation will furnish us with (1.3.39).

If $f(x) \equiv g(x)$, then an important corollary to (1.3.39) is

$$\int_{-\infty}^{\infty} f(x)(\mathbf{T}f)(x)\,\mathrm{d}x = 0. \tag{1.3.43}$$

The derivation of (1.3.40) requires a procedure that is more sophisticated than the procedure used to derive (1.3.39). Motivated by the work of Matsuno [76] and, Satsuma and Ablowitz [110] we will begin the derivation of (1.3.40) by introducing into our work the contour integral

$$\frac{1}{2\delta}\oint_C \coth\left(\frac{\pi}{2\delta}(\omega - x)\right)\frac{\partial}{\partial\omega}\left\{\ln f_+(\omega)\right\}\mathrm{d}\omega,$$

where $\omega = \xi + i\eta$ is the complex extension of the real variable $\xi$, and $C$ is the rectangular contour shown in Fig. 1, Appendix A. The following restrictions will be imposed on the function $f_+(\omega)$:

$$\left.\begin{array}{ll}\textbf{1)} & f_+(\omega) \text{ is } \textit{analytic} \text{ inside the contour } C; \\ \textbf{2)} & f_+(\omega) \text{ is } \textit{nonzero} \text{ in the region } -2\delta \leq \operatorname{Im}(\omega) \leq 0; \\ \textbf{3)} & \left|\operatorname{cosech}^2(\omega)\cdot \ln\, f_+(\omega)\right| \to 0 \text{ as } \xi \to \pm\infty \text{ along } \textit{any} \text{ ray in the} \\ & \text{region } -2\delta \leq \operatorname{Im}(\omega) \leq 0. \end{array}\right\} \qquad (1.3.44)$$

theorem can be invoked, because of the first two conditions enumerated in (1.3.44), to provide us with the equation

$$\frac{1}{2\delta}\oint_C \coth\left(\frac{\pi}{2\delta}(\omega - x)\right)\frac{\partial}{\partial\omega}\left\{\ln\, f_+(\omega)\right\}d\omega = 0,$$

which is equivalent to the equation

$$\frac{1}{2\delta}\sum_{j=1}^{4}\int_{C_j} \coth\left(\frac{\pi}{2\delta}(\omega - x)\right)\frac{\partial}{\partial\omega}\left\{\ln\, f_+(\omega)\right\}d\omega = 0 \qquad (1.3.45)$$

because of (see Fig. 1, Appendix A) the decomposition $C = C_1 \oplus C_2 \oplus C_3 \oplus C_4$. Each term in the series (1.3.45) is as follows:

$$\boldsymbol{j = 1:} \quad \frac{1}{2\delta}(\mathrm{P})\int_{-R_1}^{R_2} \coth\left(\frac{\pi}{2\delta}(\xi - x)\right)\frac{\partial}{\partial\xi}\left\{\ln\, f_+(\xi)\right\}d\xi + i\frac{\partial}{\partial x}\left\{\ln\, f_+(x)\right\}; \qquad (1.3.46a)$$

$$\boldsymbol{j = 2:} \quad \frac{1}{2\delta}\int_{0}^{-2\delta} \coth\left(\frac{\pi}{2\delta}(R_2 - x + i\eta)\right)\frac{\partial}{\partial\eta}\left\{\ln\, f_+(R_2 + i\eta)\right\}d\eta; \qquad (1.3.46b)$$

$$\boldsymbol{j = 3:} \quad \frac{1}{2\delta}(\mathrm{P})\int_{R_2}^{-R_1} \coth\left(\frac{\pi}{2\delta}(\xi - x)\right)\frac{\partial}{\partial\xi}\left\{\ln\, f_-(\xi)\right\}d\xi + i\frac{\partial}{\partial x}\left\{\ln\, f_-(x)\right\}; \qquad (1.3.46c)$$

$$\boldsymbol{j = 4:} \quad \frac{1}{2\delta}\int_{-2\delta}^{0} \coth\left(\frac{\pi}{2\delta}(-R_1 - x + i\eta)\right)\frac{\partial}{\partial\eta}\left\{\ln\, f_+(-R_1 + i\eta)\right\}d\eta. \qquad (1.3.46d)$$

The limiting values of (1.3.46a) and (1.3.46c) as $R_k \to \infty$ $(k = 1, 2)$ are

$$\lim_{R_k \to \infty}\left\{(1.3.46a)\right\} = \left(\mathbf{T}\frac{\partial}{\partial x}\left\{\ln\, f_+\right\}\right)(x) + i\frac{\partial}{\partial x}\left\{\ln\, f_+(x)\right\} \qquad (1.3.47)$$

and

$$\lim_{R_k \to \infty} \Big\{(1.3.46c)\Big\} = -\left(\mathbf{T}\frac{\partial}{\partial x}\Big\{\ln\, f_-\Big\}\right)(x) + i\frac{\partial}{\partial x}\Big\{\ln\, f_-(x)\Big\}, \tag{1.3.48}$$

respectively. The expression (1.3.46b) vanishes in the limit $R_2 \to \infty$, whereas (1.3.46d) vanishes as $R_1 \to \infty$. We will now justify our assertion that (1.3.46b) vanishes in the limit $R_2 \to \infty$. A single integration by parts of the nonlocal term that determines (1.3.46b) yields

$$\frac{1}{2\delta}\int_0^{-2\delta} \coth\left(\frac{\pi}{2\delta}(R_2 - x + i\eta)\right)\frac{\partial}{\partial\eta}\Big\{\ln\, f_+\big(R_2 + i\eta\big)\Big\}\mathrm{d}\eta = \Lambda_1 + \Lambda_2, \tag{1.3.49}$$

where:

$$\Lambda_1 \underline{\underline{\mathrm{def}}}\ \frac{i\pi}{4\delta^2}\int_0^{-2\delta} \ln\Big(f_+(R_2 + i\eta)\Big)\,\mathrm{cosech}^2\left(\frac{\pi}{2\delta}(R_2 - x + i\eta)\right)\mathrm{d}\eta;$$

$$\Lambda_2 \underline{\underline{\mathrm{def}}}\ \frac{1}{2\delta}\coth\left(\frac{\pi}{2\delta}(R_2 - x)\right)\Big\{\ln\Big(f_+(R_2 - 2i\delta)\Big) - \ln\Big(f_+(R_2)\Big)\Big\}.$$

The *third* component of (1.3.44) compels us to make the conclusion

$$\lim_{R_2 \to \infty} \Lambda_1 = 0.$$

After a brief inspection of the equation that defines $\Lambda_2$ we discover that

$$\lim_{R_2 \to \infty} \Lambda_2 = 0.$$

The vanishing of $\Lambda_1$ and $\Lambda_2$ as $R_2 \to \infty$ interact with (1.3.49) to produce the desired result:

$$\lim_{R_2 \to \infty} \frac{1}{2\delta}\int_0^{-2\delta} \coth\left(\frac{\pi}{2\delta}(R_2 - x + i\eta)\right)\frac{\partial}{\partial\eta}\Big\{\ln\Big(f_+(R_2 + i\eta)\Big)\Big\}\mathrm{d}\eta = 0. \tag{1.3.50}$$

A repetition (with suitable modifications) of the procedure used to derive (1.3.50) will yield the equation

$$\lim_{R_1 \to \infty} \frac{1}{2\delta}\int_{-2\delta}^{0} \coth\left(\frac{\pi}{2\delta}(-R_1 - x + i\eta)\right)\frac{\partial}{\partial\eta}\Big\{\ln\Big(f_+(-R_1 + i\eta)\Big)\Big\}\mathrm{d}\eta = 0. \tag{1.3.51}$$

We have derived the limits (1.3.47), (1.3.48), (1.3.50) and (1.3.51) for the purpose of verifying the correctness of equation (1.3.40). Replacing each term in the series (1.3.45) with an appropriate limiting value will lead us to (1.3.40), once we rearrange the terms that remain in (1.3.45).

Expressions of the form

$$\left(\mathbf{T}\frac{\partial^2}{\partial x^2}\left\{\ln\, f_+ / f_-\right\}\right)(x)$$

can be accommodated by us differentiating both sides of (1.3.40) with respect to $x$, and then using (1.3.37) to interchange the $\mathbf{T}$ and $\partial/\partial x$ operators. The equation

$$\left(\mathbf{T}\frac{\partial^2}{\partial x^2}\left\{\ln\, f_+ / f_-\right\}\right)(x) = -\,i\frac{\partial^2}{\partial x^2}\left\{\ln\, f_+(x)\, f_-(x)\right\} \tag{1.3.52}$$

is produced when we complete the process outlined. The procedure used to derive (1.3.52) can also be extended to expressions of the form

$$\left(\mathbf{T}\frac{\partial^n}{\partial x^n}\left\{\ln\, f_+ / f_-\right\}\right)(x),$$

where $n$ denotes a positive integer in the range $n \geq 3$.

The significance of the operators $\mathbf{T}$ and $\mathbf{H}$ can be extended beyond the MILW and MBO equations by the observation that $\mathbf{T}$ and $\mathbf{H}$ appear in the solution of certain types of Riemann-Hilbert (RH) boundary value problems [11,42]. Ablowitz and Fokas [12] provide a detailed and lucid treatment of the RH boundary value problem, its various generalizations, for example the DBAR problem, and numerous applications of the RH problem to soliton theory. The most elementary version of the RH problem seeks to decompose a *known* function, which we will denote as $\varphi(x)$, into the form

$$\varphi(x) = \psi^+(x) - \psi^-(x), \tag{1.3.53}$$

where $\psi^\pm(x)$ denote the boundary values of functions that are analytic between the horizontal strips $\mathrm{Im}\,(z) = 0$ and $\mathrm{Im}\,(z) = \pm\,2\delta$ in the complex $z$-plane; $z = x + iy$ being the complex extension of the real variable $x$. Outside of the regions bounded by the lines $\mathrm{Im}\,(z) = 0$ and $\mathrm{Im}\,(z) = \pm\,2\delta$, the boundary values that appear in equation (1.3.53) are connected by the vertical periodicity condition

$$\psi^-(x) = \psi^+(x + 2i\delta), \tag{1.3.54}$$

where

$$\psi^+(x + 2i\delta) \equiv \lim_{y \to 2\delta^-} \psi(x + iy).$$

Santini [106] has shown that the solution of (1.3.53) is generated by the integral formula

$$\psi(z) = \frac{1}{4i\delta}\int_{-\infty}^{\infty} \coth\left(\frac{\pi}{2\delta}(\xi - z)\right)\varphi(\xi)\,\mathrm{d}\xi + \tilde{\alpha}, \tag{1.3.55}$$

where the constant $\tilde{\alpha}$ is determined from an asymptotic condition, for example in the case of the *canonical normalization* we impose the condition $\psi(z) \to 1$ as $z \to \infty$. The boundary values, $\psi^{\pm}(x)$, of the sectionally analytic function $\psi(z)$ are

$$\psi^{\pm}(x) = \lim_{y \to 0^{\pm}} \psi(x + iy),$$

which in view of equation (1.3.55) are determined by

$$\psi^{\pm}(x) = \lim_{y \to 0^{\pm}} \frac{1}{4i\delta}\int_{-\infty}^{\infty} \coth\left(\frac{\pi}{2\delta}\left(\xi - (x + iy)\right)\right)\varphi(\xi)\,\mathrm{d}\xi + \tilde{\alpha}. \tag{1.3.56$^{\pm}$}$$

The connection between $\psi^{\pm}(x)$ and **T** will become transparent as soon as the limits in (1.3.56$^{\pm}$) are computed. Roos [100] provides a lucid exposition of the technique to calculate boundary values of sectionally analytic functions that have an integral representation. For didactic purposes, we will determine $\psi^{+}(x)$

$\gamma$,

shown in Fig. 2 of Appendix A, with the property

$$\lim_{\rho \to 0^{+}} \frac{1}{4i\delta}\int_{\gamma} \Phi(\omega; x; y)\,\mathrm{d}\omega = \frac{1}{4i\delta}\int_{-\infty}^{\infty} \Phi(\xi; x; y)\,\mathrm{d}\xi, \tag{1.3.57}$$

where

$$\Phi(\omega; x; y) \underline{\underline{\mathrm{def}}} \coth\left(\frac{\pi}{2\delta}\left(\omega - (x + iy)\right)\right)\varphi(\omega),$$

$\omega = \xi + i\eta$ is the complex extension of the real variable $\xi$, and $\rho$ denotes the radius of the semicircular indentation in $\gamma$ (see Fig. 2, Appendix A). The equation

$$\psi^{+}(x) = \lim_{y \to 0^{+}}\left\{ \lim_{\rho \to 0^{+}} \frac{1}{4i\delta}\int_{\gamma} \Phi(\omega; x; y)\,\mathrm{d}\omega \right\} + \tilde{\alpha} \tag{1.3.58}$$

is produced when we substitute (1.3.57) into (1.3.56$^{+}$). The point $\omega = x$ *does not* lie on the contour $\gamma$. We can benefit from our judicious construction of the contour $\gamma$ by interchanging the order of the limits in (1.3.58), and then computing the *inner* limit by evaluating the integrand at $y = 0$; the result of this procedure is

$$\psi^{+}(x) = \lim_{\rho \to 0^{+}} \frac{1}{4i\delta} \int_{\gamma} \coth\left(\frac{\pi}{2\delta}(\omega - x)\right)\varphi(\omega)\,d\omega + \tilde{\alpha}, \tag{1.3.59}$$

where we have used the equality

$$\Phi(\omega; x; 0) = \coth\left(\frac{\pi}{2\delta}(\omega - x)\right)\varphi(\omega)$$

to remove $\Phi$ form the integrand of (1.3.59).

The simple pole at $\omega = x$ in the integrand of (1.3.59) necessitates that we evaluate (1.3.59) as the aggregate of a principal value integral over the real line and the *half-residue* of the function

$$\frac{1}{4i\delta}\coth\left(\frac{\pi}{2\delta}(\omega - x)\right)\varphi(\omega)$$

at the simple pole $\omega = x$. Therefore, the precise form of $\psi^{+}(x)$ is

$$\psi^{+}(x) = \frac{1}{2}(\mathbf{I} - i\mathbf{T})\varphi(x) + \tilde{\alpha}, \tag{1.3.60}$$

where $\mathbf{I}$ denotes the identity operator. The condition [11] $\varphi(x) \in L_1(\Re) \cap H_0(\Re)$ is usually imposed on $\varphi(x)$ to guarantee the existence of $\psi^{+}(x)$. Modifying the contour $\gamma$ in Fig. 2 (Appendix A) so as to position the semicircular indentation in the *upper half* of the complex $\omega$-plane, and then repeating the procedure used to derive (1.3.60) we obtain

$$\psi^{-}(x) = -\frac{1}{2}(\mathbf{I} + i\mathbf{T})\varphi(x) + \tilde{\alpha}. \tag{1.3.61}$$

Kodama et al. [62] have employed the convenient redefinition

$$\psi^{\pm}(x) = \tilde{\psi}(x \mp i\delta) \tag{1.3.62$^{\pm}$}$$

$\psi(z)$ appearing in equation (1.3.55). An important consequence of (1.3.62$^{\pm}$) is that the Taylor series

$$\psi^{\pm}(x) = \tilde{\psi} \mp i\delta\tilde{\psi}_x - \frac{\delta^2}{2!}\tilde{\psi}_{xx} \pm i\frac{\delta^3}{3!}\tilde{\psi}_{xxx} + \cdots, \tag{1.3.63}$$

where $\psi \equiv \tilde{\psi}(x)$ and $\tilde{\psi} \in C^{\infty}(\Re)$, determine $\psi^{\pm}(x)$ in the shallow water region $(0 < \delta << 1)$.

The analogue of equation (1.3.53) in the deep water region ($\delta >> 1$) is

$$\Phi(x) = \Psi^{+}(x) - \Psi^{-}(x), \tag{1.3.64}$$

where

$$\lim_{\delta \to \infty} \begin{pmatrix} \varphi(x) \\ \\ \psi^{\pm}(x) \end{pmatrix} = \begin{pmatrix} \Phi(x) \\ \\ \Psi^{\pm}(x) \end{pmatrix}.$$

In the limit $\delta \to \infty$, the regions between $\operatorname{Im}(z) = 0$ and $\operatorname{Im}(z) = \pm 2\delta$ expand to occupy the half-planes $\pm \operatorname{Im}(z) > 0$. Hence, $\Psi^{\pm}(x)$ denote the boundary values of functions that are analytic in the half-planes $\pm \operatorname{Im}(z) > 0$.

The solution of the RH problem (1.3.64) is [11,100]

$$\Psi^{\pm}(x) = \frac{1}{2}\left(\pm \mathbf{I} - i\mathbf{H}\right)\Phi(x) + \tilde{\beta}, \tag{1.3.65$^{\pm}$}$$

obtained through the replacement of **T** with **H** and $\varphi(x)$ with $\Phi(x)$ in equations (1.3.60) and (1.3.61). Unfortunately, equations (1.3.54) and (1.3.62$^{\pm}$) have no analogues in the deep water region. Equations (1.3.65$^{\pm}$) complete the collation of those properties associated with the operators **T** and **H** that will be used in this thesis. The reader who wishes to see properties of the Hilbert transform not mentioned in this section is encouraged to consult Titchmarsh [116]. The preparatory results required to understand the material in this thesis have now been assembled, and therefore in the next chapter we consider issues that relate directly to the MILW and MBO equations.