Chapter 7

A Study of Natural Convection

Natural convection flows are driven by the density differences that can arise in a fluid, with these density differences causing a buoyancy force (in the presence of acceleration or a gravitational field) which drives the flow. The density differences can be caused by temperature or concentration gradients in the fluid, although in this chapter we discuss only thermally driven convection. Thermal convection flows are common in nature, where in conjunction with the Coriolis forces they drive circulation in the atmosphere, ocean currents, and possibly the flows in the magma which drive continental drift. From an engineering perspective such flows are of interest in any situation where heat transfer is taking place through a fluid, since they increase the rate of heat transfer over that which would occur due to conduction alone.

In this chapter two natural convection flows are examined. The first, Rayleigh–Bénard convection, has long been studied and there is a wealth of literature available on the subject. The problem was modelled as a two dimensional flow in order to test a CFD code and to see how well the code modelled the physics of the flow for which there is an abundance of experimental data to compare it with.

The second flow considered is a three dimensional intrusion flow created by the impulsive heating of one wall of a cavity containing an initially isothermal and stationary fluid. This flow has previously been studied as a two dimensional flow, but a recent experiment revealed three dimensional structures in the flow, and the CFD code was used to help improve the understanding of the nature of these structures.

7.1 Rayleigh–Bénard Convection

Rayleigh–Bénard convection arises in a layer of fluid lying between two isothermal horizontal plates, where a temperature difference exists between the plates. If the upper plate is hotter than the lower then the fluid is stable and so no motion occurs. However, if the lower plate is the hotter of the two (as shown in Figure 7.1) then the fluid is unstable and a convective flow can be generated in the fluid, typically in the form of a series of counter-rotating parallel rolls (Figure 7.2). The original experimental study of this flow was by Bénard, with Lord Rayleigh making the first theoretical analysis[137]. Reviews of the literature include the papers by Pellew and Southwell[128] and Bodenschatz et al[13] and the books by Chandrasekhar[21] and Koschmieder[84].

From the linear analysis of Rayleigh, the critical parameter of the flow field is shown to be what is now known as the Rayleigh number,

\[ \text{Ra} = \frac{g \beta \Delta T d^3}{\nu \alpha}, \]  

(7.1)
where $g$ is the acceleration due to gravity, $\beta$ the thermal expansion coefficient, $\Delta T$ the temperature difference between the two plates, $d$ the plate separation, and $\nu$ and $\alpha$ the kinematic viscosity and the thermal diffusivity respectively. For low Rayleigh numbers the viscous forces in the fluid exceed the buoyancy force and so the fluid is stable and remains stationary. However at the critical value of the Rayleigh number, $Ra_c$, the buoyancy forces equal the viscous forces, and for Rayleigh numbers above this critical value the fluid layer becomes unstable and convection occurs.

The value of the limiting Rayleigh number is a function of the wavelength of the convecting rolls, a plot of the function being given in Figure 7.3. For Rayleigh numbers below the line the fluid is stable, but above it convection can occur. The critical Rayleigh number $Ra_c$ is the minimum of the stability limit, and occurs for a unique wavelength $\lambda_c$, this wavelength being non-dimensionalised by $d$. For supercritical conditions however the rolls can take on a range of wavelengths—there is a continuum of possible wavelengths in the flow.

For a fluid constrained between two rigid walls the critical Rayleigh number is $Ra_c = 1707.8$ whilst the critical wavelength is $\lambda_c = 2.016$. Other boundary conditions modelled with linear theory include flow between two free surfaces, and flow between a rigid wall and a free surface. The critical Rayleigh numbers and wavelengths for each set of boundary conditions are given in Table 7.1.

The flow field for four convection rolls at a marginally supercritical Rayleigh number of 1750 is shown in Figure 7.4. Contour plots are given for the streamfunction, the x and y components of velocity, the pressure and the temperature fields.

A series of runs were made to capture two features of the flow. The measurement of the critical Rayleigh number is discussed in Section 7.1.1, whilst attempts to measure the roll wavelength for moderately supercritical Rayleigh numbers is discussed in Section 7.1.2.
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Figure 7.3: The critical Rayleigh number as a function of wavelength (\( \lambda \)). Regions above the curve are unstable and convection occurs.

<table>
<thead>
<tr>
<th>Boundaries</th>
<th>( R_{ae} )</th>
<th>( \lambda_c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rigid-rigid</td>
<td>1707.8</td>
<td>2.016</td>
</tr>
<tr>
<td>Rigid-free</td>
<td>1100.7</td>
<td>2.342</td>
</tr>
<tr>
<td>Free-free</td>
<td>657.5</td>
<td>2.828</td>
</tr>
</tbody>
</table>

Table 7.1: Critical parameters at the onset of Rayleigh–Bénard convection.

7.1.1 Using CFD to Measure the Critical Rayleigh Number

For Rayleigh–Bénard convection between solid boundaries, the critical Rayleigh number, \( R_{ae} \), at which the fluid starts to convect, was calculated by Jeffreys[70] using a linear theory as 1707.8, with the critical wavelength \( \lambda_c \) being 2.016. These values for the onset of convection have been measured many times in experiment, and it was decided to see how well the CFD code modelled the problem.

To capture the critical Rayleigh number is a difficult task, since as the Rayleigh number of the flow tends to \( R_{ae} \), the settling time for the flow (ie: the time taken for the flow to reach steady state) tends to infinity. Therefore to measure \( R_{ae} \), a method following the experimental procedure of Schmidt and Milverton[150] was used. The flow problem is modelled for a range of Rayleigh numbers that spans \( R_{ae} \), and the heat transfer coefficient is evaluated for each case and plotted as a function of \( R_a \). Above and below \( R_{ae} \), the heat transfer coefficients are smooth and continuous. However at \( R_{ae} \) there is a discontinuity in the derivative of the heat transfer coefficient with respect to \( R_a \), and from this the value of \( R_{ae} \) can be approximated.

For the numerical model the Nusselt number was used as a measure of heat transfer rather than the heat transfer coefficient, the Nusselt number being the ratio of the measured heat transfer to the heat transfer that would occur if only conduction took place (ie: the heat transfer if there was no fluid motion). For the Rayleigh–Bénard problem, if \( q \) is the measured heat transfer, and \( q_{cond} = k \Delta T/d \) is the heat transfer when there is no convection, the Nusselt number is

\[
\text{Nu} = \frac{q}{q_{cond}} = \frac{q d}{k \Delta T}. \tag{7.2}
\]

For conditions where \( R_a < R_{ae} \) and there is no convection \( \text{Nu} = 1 \). When \( R_a > R_{ae} \) convection occurs augmenting the rate of heat transfer and \( \text{Nu} > 1 \).

The flow was modelled as a transient process using a two-dimensional version of the CFD code, with third order upwind differencing (QUICK–see Section 2.2.4) and fractional–step time integration (see Section 2.4.4). The fluid had the properties \( \nu = 1/400 \), \( Pr = 7 \) with the height of the domain, gravity and the temperature difference being fixed at 1, the Rayleigh number being modified by varying...
Figure 7.4: The flow field for marginally supercritical Rayleigh–Bénard convection, Ra = 1750. Plots of streamfunction, x component of velocity, y component of velocity, pressure and temperature.
the thermal expansion coefficient $\beta$, with $\beta = \text{Ra} \nu^2 / g \text{Pr}$. The domain was modelled as a two-dimensional cavity, with solid isothermal walls on the top and bottom boundaries, and with periodic end boundary conditions. The modelled cavity had an aspect ratio of either 9 or $20\lambda_c (= 18.144$ or $40.32$), with a mesh of $180 \times 22$ being used for the cases with an aspect ratio of 9, and meshes of $806 \times 22$ and $1613 \times 42$ being used where the cavity was modelled with an aspect ratio of $20\lambda_c$.

The initial flow field was a stationary fluid ($u = v = p = 0$), with the initial temperature field being that for steady state conduction (ie: a linear profile between the upper and lower boundaries). This temperature profile was perturbed by a random value of $\pm 0.005$ in order to start convection (if not perturbed the fluid would remain motionless). The code was integrated forward using a constant $\Delta t$, with the value of $\Delta t$ being chosen by trial and error such that the Courant number did not exceed $0.01^1$. The Nusselt number was calculated at each wall where $u = v = 0$ and the heat transfer is solely due to conduction and so it can be calculated from the ratio of the average temperature gradient at the wall to the average gradient across the cavity.

![Figure 7.5: The Nusselt number and heat flux for sub- and super-critical Rayleigh–Bénard convection. Experimental data of Koschmieder and Pallas[85] and Farhadieh and Tankin[39] compared with the numerical model.](image)

A plot of the Nusselt number obtained with the CFD model, plotted as a function of Rayleigh number, is shown in Figure 7.5, with the Rayleigh number being scaled to the theoretical value of $R_{ac} = 1707.8$. The numerical values are compared with the experimental data of Koschmieder and Pallas[85] and that of Farhadieh and Tankin[39]. The former data is evaluated from experiments performed using oil as a working fluid, whilst the experiments of Farhadieh and Tankin used water.

The critical Rayleigh number for the flow can be found by fitting a straight line to the supercritical values of Nusselt number and finding the intersection with the line of $\text{Nu} = 1$. Using the numerical data for Rayleigh numbers in the range $1750 < \text{Ra} < 2200$ gave a critical Rayleigh number of 1703, which compares with 1707.8 from linear theory and $1682 \pm 56$ from the experiments of Koschmieder and Pallas.

Figure 7.5 shows a good agreement between the numerical data and the experiment of Farhadieh and Tankin. Koschmieder and Pallas’s data does not agree so well with the numerical model, with their data predicting a lower value of the critical Rayleigh number and a higher value of the Nusselt number. This may be due to uncertainties in their measurements of the material properties of their working fluid, a problem to which they allude to in their paper.

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1The Adams–Bashforth time stepping scheme uses a constant time step. Thus an automatic time step calculation is not possible, since this would require changing the time step in mid calculation and restarting the Adams–Bashforth scheme.
The growth of the velocity field is shown in Figure 7.6, where the maximum value of velocity is plotted as a function of time for a range of sub- and supercritical Rayleigh numbers. All cases show an initial increase in velocity up to $t = 5$ due to the perturbation of the initial temperature field, with this perturbation subsequently decaying. For Rayleigh numbers below $R_{a_c}$ (i.e., the runs made with $Ra = 1700$ and less) the perturbed velocity field decays at an exponential rate, with the rate of decay increasing as the Rayleigh number tends to zero. For supercritical flows the initial decay in the flow velocity is followed by a period of growth, with the velocity levelling off at its steady state value. The value of the steady state velocity increases with increasing Rayleigh number. The time taken to reach steady state also decreases with increasing Rayleigh number, so with the settling time $t_s$ being the time taken to reach steady state, then for both sub- and supercritical flows $t_s \to \infty$ as $Ra \to R_{a_c}$. This is illustrated in Figure 7.7 where the settling time for the flows is plotted as a function of Rayleigh number. The settling time has been calculated as the time for the maximum velocity to reach within $10^{-5}$ of its steady state value. Two curves of the form $1/(Ra - R_{a_c})$ have been fitted to the data points suggesting that this is the functional form of the relationship between the settling time and the Rayleigh number.

Figure 7.6: The maximum velocity as a function of time for impulsively heated sub- and supercritical Rayleigh number flows.

Figure 7.7: The settling time for impulsively started Rayleigh–Bénard convection. Time taken for the maximum velocity to reach maximum steady state velocity as a function of the Rayleigh number.
In Figure 7.8 the maximum velocity of the steady state flow is plotted as a function of the reduced normalised Rayleigh number $\varepsilon = (Ra - Ra_c)/Ra_c$. For the range of Rayleigh numbers studied the maximum velocity increases as $\sqrt{\varepsilon}$, which is in agreement with the experimental data of Dubois and Bergé[38].

Finally the critical wavelength for the flow was determined from a flow at the marginally supercritical Rayleigh number of 1750. Since the domain was modelled as a finite cavity a quantisation effect occurs in the wavelength selection, since the number of rolls present must be an integer (and thanks to the periodic boundaries it must be a multiple of 2). For a cavity of length $L$ containing $N$ rolls the wavelength is

$$\lambda = \frac{2L}{N}, \quad (7.3)$$

whilst the uncertainty in the wavelength is

$$\Delta \lambda = \frac{\lambda}{N}, \quad (7.4)$$

The critical wavelength was thus measured as $\lambda_c = 2.02 \pm 0.05$, compared to the theoretical value of 2.016.

![Figure 7.8: The maximum steady state velocity as a function of the reduced normalised Rayleigh number $\varepsilon = (Ra - Ra_c)/Ra_c$.](image)
7.1.2 Using CFD to Model Wavelength Selection

At the critical Rayleigh number the flow has a single wavelength $\lambda_c$, the critical wavelength. However for supercritical flow where $Ra > Ra_c$, linear theory predicts a continuum of wavelengths and there is no preferred wavelength to the flow.

However, whilst the linear theory allows for a range of wavelengths for values that are greater and less than $\lambda_c$, the non-linear theory of Schlüter et al. allows only wavelengths that are less than $\lambda_c$ on stability grounds, whilst still allowing for a continuum of wavelengths.

Using the fine mesh solutions for supercritical Rayleigh–Bénard flows described in the previous section, the wavelengths for supercritical flows were calculated by counting rolls in the solutions and using Equations (7.3) and (7.4). Care was made to ensure that the flow was calculated for a time longer than the horizontal relaxation time for the cavity, $t_{relax} = L^2/\nu$. The calculated wavelength is plotted as a function of Rayleigh number in Figure 7.9, together with the experimental data of Farhadieh and Tankin and Willis et al. Both sets of experimental data were made using water as a working fluid, so the Prandtl number of all flows is similar (a Prandtl number of 7 being used in the numerical model).

![Figure 7.9: The variation of wavelength with Rayleigh number for supercritical Rayleigh–Bénard convection. Experimental data of Farhadieh and Tankin and Willis et al. compared with the numerical model.](image)

As can be seen, for the numerical solutions the wavelength decreases with increasing Rayleigh number, a trend that agrees with the theory of Schlüter but which is the opposite to that of the experiments, which for both sets of data exhibits an increase in wavelength with increasing Rayleigh number.

To test the sensitivity of the solutions to initial conditions some supercritical runs were made where instead of using an initially motionless velocity field, the steady state solution for a marginally supercritical flow was used as the initial condition. As with other runs the initial temperature field was perturbed in order to trigger any instabilities in the flow. Unlike the previous solutions made with a stationary initial field, the steady state wavelengths of these runs remained the same as the initial wavelength, which was the critical wavelength of the flow $\lambda_c$, regardless of the Rayleigh number of the flow.

Finally to test the effect of the rate of heating the flow, some supercritical runs were made with the temperature difference across the cavity being increased linearly with time instead of being impulsively applied as with previous runs. As with the restart runs made using marginally supercritical flow fields as the initial condition, the wavelength of the steady state solution did not vary from the initial
wavelength of the flow, which was the critical wavelength, $\lambda_c$.

Whilst the trend for wavelengths to decrease or remain constant contradicts the experimental findings of Farhadieh and those of Willis, these trends agrees with those found by Carrière et al[20] in modelling two-dimensional Rayleigh–Bénard convection between conducting boundaries.

A subset of the flow field for a supercritical flow is shown in Figure 7.10, four rolls of the flow being shown for a flow with a Rayleigh number of 3500 ($Ra/Ra_c = 2.05$). In comparison with the flow field for the marginally supercritical flow (shown in Figure 7.4 above) the isotherms in the temperature field are strongly distorted by the convective motion, and the roll wavelength is slightly shortened.

Figure 7.10: The flow field for supercritical Rayleigh–Bénard convection, $Ra = 3500$. Plots of streamfunction, x component of velocity, y component of velocity, pressure and temperature. Comparison should be made with the flow in Figure 7.4.
7.1.3 Summary

The CFD code was used to model transient natural convection for two-dimensional Rayleigh–Bénard flow. The code accurately predicted the values of the critical Rayleigh number and the critical wavelength, and gave good agreement with experiment for the increase in Nusselt number for supercritical flows.

However, for supercritical flow the code predicted no unique dependence of wavelength on Rayleigh number. Impulsively started flows showed a decrease in the wavelength of the rolls with increasing Rayleigh number, whilst flows which were gradually heated to their steady state showed no variation in the wavelength from $\lambda_c$. The trend in the impulsively started flows agrees with non-linear theories in the literature and with results from similar CFD models, but differs from experimental results where a trend of increasing wavelength with increasing Rayleigh number is observed.
7.2 Three Dimensional Transient Convective Flow

The fluid mechanics problem of natural convection in a cavity heated from the side has an only slightly less illustrious history than that of the Rayleigh–Bénard problem. The earliest analysis of the problem is that of Batchelor[9] who drew upon the little experimental data available at that time. This was followed by the first numerical computation of the flow field by Poots[132], who used a spectral method and two female assistants as a computation engine. The earliest models to run on an electronic computer were those by Wilkes and Churchill[180] and de Vahl Davis[29], both modelling flow in a two-dimensional square cavity, the former modelling the problem with a transient code, whilst the latter author modelled the flow as a steady state process. The flow was later modelled by de Vahl Davis and Mallinson in a three-dimensional cavity[108], whilst the two-dimensional steady state problem has been compared with experiment[71] and used as a benchmark for a number of CFD codes[32, 31, 30, 73].

All these early works concentrated on the problem as a steady state process, with the transient process of imposing a temperature gradient on an initially isothermal and stationary flow being ignored or used as a numerical device to achieve the steady state solution. This was rectified by a numerical and scaling study of the transient flow by Patterson and Imberger[127], followed by a series of numerical and experimental studies of the two-dimensional transient flow by Patterson, Armfield and Schöpf[126, 3, 152]. Whilst these studies provided an understanding of the structure of the fundamental two-dimensional flow in the cavity, a recent experiment by Schöpf and Stiller[153] has revealed the interesting three-dimensional structure of the initial thermal intrusion, and it is this flow which forms the topic of this section.

The basic problem consists of an initially isothermal and motionless fluid contained in a cavity of square cross section. The left wall is impulsively heated creating a thermal boundary layer that rises up the heated wall and which then intrudes into the cavity along the roof of the cavity, whilst the right wall is cooled leading to a descending boundary layer and an intrusion on the cavity floor. A schematic of the experiment is shown in Figure 7.11.

The flow is characterised by two dimensionless groups, the Rayleigh number,

$$Ra = \frac{g \beta \Delta T l^3}{\nu \alpha}, \quad (7.5)$$

and the Prandtl number,

$$Pr = \frac{\nu}{\alpha}, \quad (7.6)$$

where $g$ is the acceleration due to gravity, $\beta$ the thermal expansion coefficient, $\Delta T$ the temperature difference across the cavity, $l$ the length scale of the cavity (the width or height of the cavity), and $\nu$ and $\alpha$ the kinematic viscosity and the thermal diffusivity respectively.

For the following studies the time scale has been non-dimensionalised with respect to the viscosity and the cavity length scale using a viscosity based Fourier number scaling

$$\tau = \frac{\nu t}{l^2}, \quad (7.7)$$

whilst the velocity has been similarly treated,

$$U = \frac{ul}{\nu}, \quad (7.8)$$

with $U$ being the Fourier velocity.

The experiments of Patterson and that of Schöpf and Stiller all utilised a cavity with a width and height of 0.24m, and a length in the $z$ axis of 0.5m. The hot and cold side walls were constructed of 0.001m copper plates, whilst the remaining adiabatic boundaries were constructed with 0.019m
Perspex sheets, which allowed the visualisation of the flow within the cavity, with water used as the working fluid. Numerical results will be compared to the Schöpf experiments for which the ambient temperature of the surroundings and the initial temperature of the cavity and fluid was approximately 21°C. The left wall of the cavity was impulsively heated by a temperature increase in the range of 1 to 7°C, whilst the right wall was maintained at the ambient temperature. For the case of a temperature increase of \( \Delta T = 4.8^\circ C \) this results in a Rayleigh number for the cavity of \( \text{Ra} = 10^6 \), and a Prandtl number of \( \text{Pr} = 6.8 \).

The flow was modelled using the same code that was used to model the Rayleigh–Bénard convection described in the first half of this chapter. The momentum and temperature equations were discretised using the QUICK third order upwind differencing scheme, with the fractional step time integration being used. The problem was modelled on three progressively finer meshes, using \( 30 \times 30, 58 \times 58 \) and \( 88 \times 88 \) meshes in two dimensions, and \( 30 \times 30 \times 50, 58 \times 58 \times 100 \) and \( 88 \times 88 \times 148 \) meshes for the three dimensional problem. In the \( x \) and \( y \) axis a log distribution of mesh points was used to concentrate points at the boundaries, whilst in the \( z \) axis a regular distribution of points was used. The time step was varied from mesh to mesh to ensure that the maximum Courant number at no time exceeded 0.1.

The initial flow field was set to be motionless and isothermal, with \( u = v = w = T = 0 \), with the high and low \( x \) boundaries (ie: the left and right walls to the cavity) being modelled as isothermal walls with a no-slip boundary condition for velocity. At time \( t = 0 \) the low \( x \) boundary (the left side of the cavity) was set at \( T = 1 \), whilst the opposite boundary remained at the same temperature as the cavity interior (\( T = 0 \)). Initially the \( y \) boundaries (the upper and lower surfaces) were modelled as adiabatic solid walls, with a no-slip boundary condition for velocity and a zero temperature gradient normal to the wall.

For the three dimensional problem cyclic (or periodic) boundary conditions were used in the \( z \) axis to ensure any disturbances in this axis were due to instabilities in the flow and not due to the presence of end walls. To trigger three dimensional structures in the flow a random fluctuation of 0.1T was added to the left wall for the initial time step only.

Plots of streamlines and isotherms for the two-dimensional flow are shown in Figure 7.12 for a fluid impulsively heated at the left boundary, plots being given at non-dimensional times \( \tau = 0.51 \times 10^{-3} \), \( 0.86 \times 10^{-3} \) and \( 1.2 \times 10^{-3} \). At \( \tau = 0.51 \times 10^{-3} \) a thermal boundary layer has formed on the left side of the cavity, which has generated an intrusion that is starting to cross the cavity below the upper boundary. For the middle plots, at a non-dimensional time of \( \tau = 0.86 \times 10^{-3} \), the intrusion has crossed two-thirds of the cavity, whilst for the final plots, at a non-dimensional time of \( \tau = 1.2 \times 10^{-3} \), the intrusion has reached the far wall and a body of heated fluid lies across the full width of the cavity, below the upper boundary of the cavity.
Figure 7.12: Transient natural convection in a side heated cavity. Numerical two-dimensional model of the intrusion flow, with streamlines on the left, isotherms on the right, for a flow with $Ra = 10^9$, $Pr = 6.7$ at $\tau = 0.51 \times 10^{-3}$, $0.86 \times 10^{-3}$ and $1.2 \times 10^{-3}$. The flow is modelled as having adiabatic upper and lower boundaries.
Figure 7.13: Transient natural convection in a side heated cavity. Experimental data from Schöpf and Stiller[153]. Looking down at the cavity for $t = 30\ s$ (top) and $78\ s$ (bottom), for a flow with $Ra = 10^9$. 
Shadowgraphs of an equivalent experimental flow are shown in Figure 7.13, taken from above looking down and through the cavity. These are from the paper of Schöpf and Stiller[153], and were taken at 30 s and 78 s after the heating of the hot wall, corresponding to non-dimensional times of \( \tau = 0.51 \times 10^{-3} \) and \( \tau = 1.33 \times 10^{-3} \) respectively. In the first shadowgraph, at \( \tau = 0.51 \times 10^{-3} \), the thermal intrusion has only just started to move across the upper boundary of the cavity. The leading edge of this thermal intrusion can be seen as a vertical white line in the left side of the image. The second shadowgraph image shows the flow as the thermal intrusion reaches the far side of the cavity, at a non-dimensional time of \( \tau = 1.3 \times 10^{-3} \). A pattern of light and dark regions aligned with the flow direction have appeared in the intrusion.

Figure 7.14: Rayleigh–Bénard convection rolls aligned perpendicular to the mean flow direction, after Kelly[76].

A close examination of the isotherms for the modelled flow in Figure 7.12 at \( \tau = 1.2 \times 10^{-3} \) reveals that an unstable thermal gradient exists between the core of the thermal intrusion and the top of the cavity. It is thought that the structures seen in the experimental shadowgraph arise from Rayleigh–Bénard cells forming with their axes aligned in the flow direction, a sketch of such a flow being shown in Figure 7.14.

For Rayleigh–Bénard convection between a rigid and a free boundary, which corresponds to the boundary conditions for a cell generated in the upper layer of the intrusion, the critical Rayleigh number was given in Table 7.1 as \( \text{Ra}_c = 1100.7 \). If the structures seen in the experimental images are a result of a Rayleigh–Bénard instability, the intrusion Rayleigh number, based on the temperature difference and the vertical extent of the region over which the intrusion is unstable, must be greater than the critical Rayleigh number. However, a calculation of the Rayleigh number for the numerical results gave a number much lower than this value. It was decided to recalculate the problem using a more realistic boundary condition for the upper and lower boundaries. Instead of imposing an adiabatic boundary condition on the temperature equation, the upper and lower surfaces were instead modelled as conducting slabs of Perspex, having a thickness that was \( \frac{10}{9} \) of the cavity width, with the slabs having an adiabatic boundary condition on their outer surfaces. The thermal diffusivity of Perspex is \( 1.35 \times 10^{-4} \text{ m}^2/\text{s} \), which is approximately equal to that of water which is \( 1.47 \times 10^{-4} \text{ m}^2/\text{s} \).
Figure 7.15: Numerical two-dimensional model of the intrusion flow, with streamlines on the left, isotherms on the right, for a flow with $Ra = 10^6$, $Pr = 6.7$ at $\tau = 0.51 \times 10^{-3}, 0.86 \times 10^{-3}$ and $1.2 \times 10^{-3}$. The flow is modelled as having a conducting slab at the upper and lower boundaries, with adiabatic boundaries on the outside of this slab.
Figure 7.16: Numerical two-dimensional model of the intrusion flow. Isotherms for a flow with \( \text{Ra} = 10^9 \), \( \text{Pr} = 6.7 \) at \( \tau = 0.51 \times 10^{-3}, 0.86 \times 10^{-3} \) and \( 1.2 \times 10^{-3} \), are shown with an expanded vertical scale, with only the top 0.18 of the cavity being shown. The flow on the left has an adiabatic upper boundary, whilst that on the right has a conducting solid boundary.

Figure 7.17: Numerical two-dimensional model of the intrusion flow. The thickness of the intrusion is shown on the left, and the intrusion Rayleigh number, based on the temperature difference and thickness of the intrusion, is shown on the right. The critical Rayleigh number is approximately \( \text{Ra}_c = 1100 \). Plots are given for both adiabatic and conducting upper boundaries, at \( \tau = 0.51 \times 10^{-3}, 0.86 \times 10^{-3} \) and \( 1.2 \times 10^{-3} \).
A two dimensional flow calculated with the modified conducting boundary conditions is shown in Figure 7.15. For the bulk of the domain, the difference between this flow, and the flow shown previously in Figure 7.12 is minimal. However, the presence of a conducting upper boundary increases the temperature gradient in the intrusion adjacent to the boundary.

This is shown more clearly in Figure 7.16, where the vertical scale has been expanded to reveal the differences between the two flows. From inspection the temperature gradient for the conducting boundary flow is much higher midway along the intrusion, although both are similar at the intrusion head. This is borne out in Figure 7.17 which shows the depth of the intrusion (measured as the distance from the top wall to the location of maximum temperature of the intrusion) and the intrusion Rayleigh number (based on the intrusion depth described above, and the temperature difference between the upper boundary and the centre of the intrusion). The thickness of the intrusion is similar for both boundary conditions, as is the Rayleigh number of the head of the intrusion. However, in the remainder of the intrusion the conducting boundary case has a higher Rayleigh number than the adiabatic boundary case, with the intrusion Rayleigh number exceeding the critical Rayleigh number $Ra_c = 1100.7$ for most of the extent of the intrusion. For the adiabatic upper boundary the intrusion Rayleigh number exceeds the critical Rayleigh number only in the head region.

To reveal the three dimensional structure of the flow, the $z$ component of velocity for the flow is shown in Figures 7.18 and 7.19 for the adiabatic and conducting boundary cases respectively. Since the flow is three dimensional streamlines cannot be plotted for cross-sections of the intrusion. Instead the average magnitude of the $z$ axis component of the velocity is plotted, with plots being presented for the velocity magnitude averaged in the $x$ and $y$ axes, the contour levels being different for each plot to improve resolution.

The $z$ component of velocity clearly shows the three dimensional structure of the flow, however it is noted that the three-dimensionality is confined to the head region of the intrusion for the case with the adiabatic upper boundary, shown in Figure 7.18. In contrast, the conducting boundary condition results, shown in Figure 7.19, have rolls that run along nearly the full length of the intrusion. These figures show that for both cases rolls are formed, however for the conducting boundary condition the rolls are stronger and more extensive. The conducting boundary flow has a higher intrusion Rayleigh number over the full extent of the intrusion which correlates well with the stronger rolls and provides support for the hypothesis that the rolls are the result of a Rayleigh–Bénard instability.

Isolines of the $z$ velocity field taken through the intrusion are shown in Figure 7.20 at the $x = 0.214, 0.352, 0.616, 0.821$ and $0.920$ sections, along with plots of the $x$ velocity profile and temperature gradient at each section. The double row of velocity contours shown along the upper boundary has a similar structure to that of the two dimensional Rayleigh–Bénard flow shown in Figure 7.4, the $z$ axis velocities in the three dimensional intrusion corresponding to the $x$ axis velocities in the two dimensional flow of Figure 7.4. The rolls are a secondary feature of the flow, with the $z$ component of velocity in the intrusion having a much lower value than the $x$ component. For the $Ra = 10^9$, $Pr = 6.7$ flow shown in Figure 7.20 the $z$ component of velocity is in the range $\pm 0.015$ whilst the $x$ component of velocity reaches a maximum of $6.6$. 
Figure 7.18: Isolines of the average magnitude of the $w$ velocity, the velocity being averaged along the $z$ axis (left) and $y$ (vertical) axis (right), for a flow with $Ra = 10^9$ and $Pr = 6.7$ at $\tau = 0.51 \times 10^{-3}$, $0.86 \times 10^{-3}$ and $1.2 \times 10^{-3}$, with an adiabatic upper boundary condition.
Figure 7.19: Isolines of the average magnitude of the $w$ velocity, the velocity being averaged along the $z$ axis (left) and $y$ (vertical) axis (right), for a flow with $Ra = 10^9$ and $Pr = 6.7$ at $\tau = 0.51 \times 10^{-3}$, $0.86 \times 10^{-3}$ and $1.2 \times 10^{-3}$, with a conducting upper boundary condition.
Figure 7.20: Sections normal to the $x$ axis taken through the cavity for an adiabatic upper boundary (left) and a conducting upper boundary (right). Sections are taken at $x = 0.214, 0.352, 0.616, 0.821$ and 0.920, with only the top 0.25 of the cavity being shown. For each graph the contours of $w$ are shown, with the contours being at 0.0005 intervals in the range ±0.01025, along with profiles of the $u$ velocity (the crossed line) and the temperature (the solid line). Profiles are for flows with $Ra = 10^9$ and $Pr = 6.7$ at $\tau = 1.2 \times 10^{-3}$. 
The intrusion depth and Rayleigh number of the adiabatic and conducting boundary flows are shown in Figures 7.21 and 7.22, for cavity Rayleigh numbers in the range \(Ra = 5 \times 10^7\) to \(1.2 \times 10^9\). The data is presented for the time at which the intrusion has just reached the right boundary of the cavity and stretches across the full width of the domain. For each cavity Rayleigh number there is only a moderate difference between the intrusion depths of the adiabatic and conducting boundary flows at the head of the intrusion. However, towards the midsection of the cavity the conducting boundary flow has a noticeably deeper intrusion. The intrusion Rayleigh number shows a much larger variation between the two types of flow, with the conducting boundary flow having a greater intrusion Rayleigh number across the full width of the cavity. Table 7.1 gives the critical Rayleigh number for a flow between one rigid and one free boundary as \(Ra_c = 1100.7\). Aside from the region at the very head of the intrusion, the adiabatic boundary flow reaches this value only over \(10\%\) of the width of the cavity, and only for cavity Rayleigh numbers greater than \(Ra = 2 \times 10^8\). For the conducting boundary flow the critical Rayleigh number is exceeded in the left side of the cavity for all flows modelled, and in the mid section of the cavity it is exceeded for cavity Rayleigh numbers greater than \(Ra = 1 \times 10^8\). Moreover, the region over which the critical Rayleigh number is exceeded extends across approximately \(70\%\) of the cavity width at the higher cavity Rayleigh numbers modelled, while for the adiabatic boundary flow the critical Rayleigh number is exceeded over only \(20\%\) of the cavity width.
Figure 7.23: Wavelength of the intrusion rolls for the numerical model and the experiment of Schöpf and Stiller[153].

The experiments of Schöpf and Stiller[153] revealed a relationship between the wavelengths of the rolls that formed in the thermal intrusion, and the Rayleigh number of the cavity, the roll wavelength decreasing with increasing cavity Rayleigh number. To model this effect, runs were made with both the adiabatic and the conducting upper boundaries for Rayleigh numbers in the range of $\text{Ra} = 5 \times 10^7$ to $\text{Ra} = 1.2 \times 10^9$. The wavelengths of the modelled flows are shown in Figure 7.23 along with the experimental data of Schöpf and Stiller. Whilst both the numerical and experimental wavelengths decrease with increasing Rayleigh number, they differ in magnitude by up to 50%.

### 7.2.1 Summary

A CFD code has been used to model the three dimensional transient natural convection flow due to the impulsive heating of the wall of a rectangular cavity. A flow pattern seen experimentally, where Rayleigh–Bénard cells form in the initial intrusion flow, is captured with the numerical model, and shown to depend upon the cavity roof having a finite thermal mass rather than being adiabatic. The trend in the wavelength of the rolls with respect to Rayleigh number is correctly modelled, although the wavelength of the rolls is over-predicted.
7.3 Conclusions

A Cartesian CFD code has been used to model two transient natural convection flows, two-dimensional Rayleigh–Bénard convection and a three-dimensional transient intrusion flow, with the results being compared with experiment.

The code captured the critical Rayleigh number and critical wavelength of the Rayleigh–Bénard flow, but the trend in the wavelength for supercritical flows was opposite that seen in experiment. However, the calculated wavelength was shown to depend on the initial conditions and the rate of increase in the temperature across the cavity, and agrees with an analytic theory. The relationship between the wavelength and the Rayleigh number of the flow therefore remains unclear.

The second problem modelled transient intrusion flow in a three-dimensional side heated cavity. The CFD code was used to recreate the experiments of Schöpf and Stiller[153] and it successfully captured the Rayleigh–Bénard rolls that form in the intrusion. The wavelength of the calculated rolls were in error when compared to the experiment, but displayed the same trend with increasing Rayleigh number. The generation of the rolls was shown to depend on the upper boundary not being truly adiabatic in the experiment, with the thermal mass of the boundary increasing the temperature gradient across the intrusion initiating stronger and more extensive rolls.