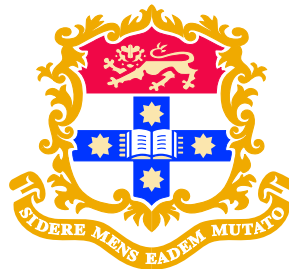


# Bifurcation problems in chaotically stirred reaction-diffusion systems

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# Abstract

A detailed theoretical and numerical investigation of the behaviour of reactive systems under the influence of chaotic stirring is presented. These systems exhibit stationary solutions arising from the balance between chaotic advection and diffusion. Excessive stirring of such systems results in the termination of the reaction via a saddle-node bifurcation. The solution behaviour of these systems is analytically described using a recently developed nonperturbative, non-asymptotic variational method. This method involves fitting appropriate parameterised test functions to the solution, and also allows us to describe the bifurcations of these systems. This method is tested against numerical results obtained using a reduced one-dimensional reaction-advection-diffusion model. Four one- and two-component reactive systems with multiple homogeneous steady-states are analysed, namely autocatalytic, bistable, excitable and combustion systems. In addition to the generic stirring-induced saddle-node bifurcation, a rich and complex bifurcation scenario is observed in the excitable system. This includes a previously unreported region of bistability characterised by a hysteresis loop, a supercritical Hopf bifurcation and a saddle-node bifurcation arising from propagation failure. Results obtained with the nonperturbative method provide a good description of the bifurcations and solution behaviour in the various regimes of these chaotically stirred reaction-diffusion systems.

# Declaration of originality

The work presented in this thesis is original except where stated otherwise. No part of this thesis has been submitted for the award of any other degree or diploma at this or any other university.

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# Contents

<b>Abstract</b>	<b>i</b>
<b>Declaration of originality</b>	<b>ii</b>
<b>Acknowledgements</b>	<b>iii</b>
<b>1 Introduction</b>	<b>1</b>
<b>2 The Lagrangian filament model</b>	<b>7</b>
2.1 Description of the model . . . . .	7
2.2 Generic solution behaviour . . . . .	8
2.3 Asymptotic solution behaviour . . . . .	9
2.3.1 Asymptotic solution for large $Da$ . . . . .	9
2.3.2 Asymptotic solution for small $Da$ . . . . .	10
<b>3 A variational method to approximate solution profiles</b>	<b>12</b>
3.1 Introduction . . . . .	12
3.2 Description of method . . . . .	13
3.3 Generic test functions . . . . .	15
<b>4 One component systems: Autocatalytic and bistable systems</b>	<b>17</b>
4.1 Introduction . . . . .	17
4.2 Description of the models . . . . .	18
4.2.1 The autocatalytic system . . . . .	18
4.2.2 The bistable system . . . . .	19
4.3 The solution near the critical value . . . . .	20
4.3.1 The autocatalytic system . . . . .	21
4.3.2 The bistable system . . . . .	25
4.4 The solution far from the critical value . . . . .	29
4.4.1 The autocatalytic system . . . . .	30
4.4.2 The bistable system . . . . .	33

4.5	Summary . . . . .	36
<b>5</b>	<b>Two component systems: Excitable media</b>	<b>38</b>
5.1	Introduction . . . . .	38
5.2	Description of the model . . . . .	39
5.3	Numerical simulations of the 1-D filament model . . . . .	41
5.3.1	Solution behaviour for large Damköhler numbers . . . . .	46
5.3.2	Saddle-node at small Damköhler numbers . . . . .	48
5.3.3	The hysteresis loop . . . . .	51
5.3.4	The Hopf bifurcation . . . . .	53
5.3.5	The bifurcation scenario near the high codimension points . . . . .	56
5.4	Analysis using the nonperturbative variational method . . . . .	59
5.5	Results obtained with the nonperturbative variational method . . . . .	63
5.5.1	Solutions far from the saddle-node . . . . .	63
5.5.2	Solutions near the large Damköhler number saddle-node . . . . .	66
5.5.3	Solutions near the low Damköhler number saddle-node . . . . .	69
5.5.4	Solution behaviour in the hysteresis loop . . . . .	74
5.5.5	Solution behaviour near the Hopf bifurcation . . . . .	78
5.6	Summary . . . . .	82
<b>6</b>	<b>Two component systems: Flame filaments in a combustion system</b>	<b>83</b>
6.1	Introduction . . . . .	83
6.2	Description of the model . . . . .	85
6.3	Analysis using the nonperturbative variational method . . . . .	87
6.4	Solution behaviour near the saddle-node . . . . .	91
6.4.1	Small Lewis numbers . . . . .	91
6.4.2	Equal diffusion: $Le = 1$ . . . . .	97
6.4.3	Large Lewis numbers . . . . .	101
6.4.4	The solution at the saddle-node . . . . .	108
6.5	Comparison of results far from the saddle-node . . . . .	109
6.6	Summary . . . . .	117
<b>7</b>	<b>Summary</b>	<b>119</b>
<b>A</b>	<b>Description of numerical methods</b>	<b>122</b>
A.1	Solving the partial differential equations . . . . .	122
A.2	Solving the ordinary differential equations . . . . .	125

<b>B</b>	<b>Derivation of the expressions for the free parameters of a Gaussian test function in an excitable system</b>	<b>127</b>
B.1	Derivation of the amplitude and inverse pulse width . . . . .	127
B.2	Limiting cases . . . . .	130
B.2.1	The limit $a \rightarrow 0$ . . . . .	130
B.2.2	The limit $\epsilon \rightarrow 0$ . . . . .	130
B.2.3	The limit $Da \rightarrow \infty$ . . . . .	130
	<b>Bibliography</b>	<b>133</b>