

HOMOTOPY CLASSIFICATION
OF
FILTERED COMPLEXES

by

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P R E F A C E

It is well known that the homology functor from the category of free abelian chain complexes and homotopy classes of maps to that of graded abelian groups is full and dense and reflects isomorphisms. Thus such a complex is determined to within homotopy equivalence (although not a unique homotopy equivalence) by its homology. The homotopy classes of maps between two such complexes should therefore be expressible in terms of the homology groups, and such an expression is in fact provided by the Künneth formula for Hom , sometimes called 'the homotopy classification theorem'.

In [15] Kelly showed that the functor assigning to a short exact sequence of free abelian chain complexes its long exact homology sequence is again full and dense and reflects isomorphisms. Partial information about the kernel of this functor was found in [16]: but not enough to provide a homotopy classification theorem for this case.

Since a short exact sequence of free abelian chain complexes may be considered as a free abelian chain complex with a filtration of length 2 the question arises whether the above results admit appropriate generalizations for complexes with a filtration of finite length n .

The main purpose of this thesis is to exhibit for such filtered complexes a functor which is full and dense and reflects isomorphisms, and to provide a homotopy classification theorem for this case. We do not entirely restrict ourselves to free abelian complexes, but then we must content ourselves with an analogue of the Künneth spectral sequence instead of the short exact sequence.

Because of the degree of complication of the situations to be studied, conceptual methods are necessary. These are developed in Chapter 2, where we study the existence of such classifying functors, homotopy classification theorems, and analogues of the Künneth spectral sequence, in a very general context of triangulated categories. The necessary background material about graded categories, differential graded categories and triangulated categories is included in Chapter 1.

In Chapter 3 we apply the abstract theory to filtrations of length 1 and 2, going beyond the work of Kelly in the latter case by providing a homotopy classification theorem. The case of a general n is treated in Chapter 4, and constitutes the most technically difficult part of the thesis. The central problem here is to determine the projectives in the image category

(whose objects are in fact diagrams as considered by Wall in [19]), and this requires a long technical argument. Our reasons for including the case $n = 2$ (besides the trivial case $n = 1$) in the earlier chapter were in part a desire to make the essence of the argument clear in a case deep enough to be of interest but simple enough to be treated without the long technical considerations of Chapter 4; and in part that the determination of the projectives in the range category in the general case requires as a starting point the determination in this case.

The work done in this thesis is entirely original except for some of the sections of Chapter 1 where known but not easily available results are summarized in the form in which we need them.

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CHAPTER I. - PRELIMINARIES

§1. DG-categories and graded categories.

In this section we recall the definitions of DG-category, functor and natural transformation; graded categories and additive categories are dealt with as special cases. Functor categories are then considered.

A DG-category $\mathcal{A}$ consists of the following data:

- (i) a class whose elements are called objects of $\mathcal{A}$;
 - (ii) for each pair A, A' of objects of $\mathcal{A}$, a complex $[A, A'; \mathcal{A}] = [A, A']$ of abelian groups;
 - (iii) for each triple A, A', A'' of objects of $\mathcal{A}$, a chain map $[A', A''] \otimes [A, A'] \rightarrow [A, A'']$ called composition, the image of $g \otimes f$ being written gf for $f \in [A, A']_p, g \in [A', A'']_q$;
- such that the axioms:

DG1. composition is associative; and

DG2. for each object A of $\mathcal{A}$, there is an element $1_A \in [A, A]_0$ which behaves as an identity under composition;

are satisfied.

The differential of the complex $[A, A']$ will be denoted by D for all objects A, A' . The condition that the composition map as a graded abelian group map should be a chain map is equivalent to:

$$D_{p+q}(gf) = (D_q g)f + (-1)^q g(D_p f)$$

for all $f \in [A, A']_p, g \in [A', A'']_q$.

An element $f \in [A, A']_p$ will be called a protomorphism of degree p . If $D_p f = 0$ so that $f \in Z_p[A, A']$ then f will be called a morphism of degree p . Morphisms of degree zero will simply be called morphisms.

A morphism $f: A \rightarrow A'$ of degree p will be called an isomorphism of degree p (simply an isomorphism when $p = 0$) if there exists a protomorphism $g: A' \rightarrow A$ of degree $-p$ (necessarily a morphism of degree $-p$) with $gf = 1_A$ and $fg = 1_{A'}$.

Suppose $\mathcal{A}, \mathcal{B}$ are DG-categories. A DG-functor $T: \mathcal{A} \rightarrow \mathcal{B}$ consists of the following data:

- (i) a function which assigns to each object A of $\mathcal{A}$ an object TA of $\mathcal{B}$;
- (ii) for each pair A, A' of objects of $\mathcal{A}$ a chain map $T: [A, A'; \mathcal{A}] \rightarrow [TA, TA'; \mathcal{B}]$;

such that the axioms:

F1. for all $A \in \mathcal{A}$, $T_0(1_A) = 1_{TA}$;

F2. $T_{p+q}(gf) = T_q g \cdot T_p f$ for all protomorphisms f, g of degree p, q respectively;

are satisfied.

Suppose $T, S: \mathcal{A} \rightarrow \mathcal{B}$ are DG-functors. A DG-natural transformation $\alpha: T \rightarrow S$ is a family $(\alpha_A)_{A \in \mathcal{A}}$ of morphisms $\alpha_A: TA \rightarrow SA$ of $\mathcal{B}$ such that:

NT. for all protomorphisms $f: A \rightarrow A'$ of any degree p ,

$$\alpha_{A'} \cdot Tf = Sf \cdot \alpha_A.$$

When no confusion is possible we will write "functor" and "natural transformation" in place of "DG-functor" and "DG-natural transformation" respectively.

To save repetition we define graded categories as follows. A graded category $\mathcal{A}$ is a DG-category such that each complex $[A, A']$ has zero differential. A graded functor is just a DG-functor between graded categories, and a graded natural transformation is just a DG-natural transformation between graded functors.

An additive category $\mathcal{A}$ is a graded category such that $[A, A']_p = 0$ for $p \neq 0$ for all $A, A' \in \mathcal{A}$. An additive functor is a graded functor between additive categories, and an additive natural transformation is a graded natural transformation between additive functors.

The dual $\mathcal{A}^*$ of the DG-category $\mathcal{A}$ is the DG-category defined as follows:

- (i) the objects are the same as $\mathcal{A}$;
- (ii) $[A, A'; \mathcal{A}^*] = [A', A; \mathcal{A}]$;
- (iii) composition $[A', A''; \mathcal{A}^*] \otimes [A, A'; \mathcal{A}^*] \rightarrow [A, A''; \mathcal{A}^*]$ is given by $g \otimes f \mapsto (-1)^{pq} fg$, where

$$f \in [A', A; \mathcal{A}]_p, g \in [A'', A'; \mathcal{A}]_q.$$

The tensor product $\mathcal{A} \otimes \mathcal{B}$ of the DG-categories $\mathcal{A}, \mathcal{B}$ is the DG-category defined as follows:

- (i) the objects are pairs (A, B) where $A \in \mathcal{A}$ and $B \in \mathcal{B}$;
- (ii) $[(A, B), (A', B'); \mathcal{A} \otimes \mathcal{B}] = [A, A'; \mathcal{A}] \otimes [B, B'; \mathcal{B}]$;
- (iii) composition

$$\begin{aligned}
 &[(A', B'), (A'', B''); \mathcal{A} \otimes \mathcal{B}] \otimes [(A, B), (A', B'); \mathcal{A} \otimes \mathcal{B}] \\
 &= [A', A''; \mathcal{A}] \otimes [B', B''; \mathcal{B}] \otimes [A, A'; \mathcal{A}] \otimes [B, B'; \mathcal{B}] \\
 &\rightarrow [(A, B), (A'', B''); \mathcal{A} \otimes \mathcal{B}] = [A, A''; \mathcal{A}] \otimes [B, B''; \mathcal{B}]
 \end{aligned}$$

is given by

$$g \otimes k \otimes f \otimes h \mapsto (-1)^{ps} gf \otimes kh,$$

where $f \in [A, A'; \mathcal{A}]_p$, $g \in [A', A''; \mathcal{A}]_q$,
 $h \in [B, B'; \mathcal{B}]_r$, $k \in [B', B''; \mathcal{B}]_s$.

The DG-category $C\mathcal{D}$ of complexes over the additive category $\mathcal{D}$ is defined as follows:

(i) the objects are complexes A over $\mathcal{D}$ (the differential d^A of degree -1);

(ii) the complex $[A, B; C\mathcal{D}] = [A, B]$ is given by

$$[A, B]_p = \prod_{r \in \mathbb{Z}} (A_r, B_{r+p}; \mathcal{D}),$$

and if $f = (f_r)_{r \in \mathbb{Z}} \in [A, B]_p$, then

$$(D_p f)_r = d_{p+r}^B \cdot f_r - (-1)^p f_{r-1} \cdot d_r^A;$$

(iii) composition is given by

$$(g_r)(f_r) = (g_r f_r).$$

The graded category $G\mathcal{D}$, where $\mathcal{D}$ is an additive category, is the full sub-DG-category of $C\mathcal{D}$ with objects those complexes A over $\mathcal{D}$ with differential $d^A = 0$.

REMARK: DG-categories are categories over the closed category $C\mathcal{G}$, where $\mathcal{G}$ is the (additive) category of abelian groups and group homomorphisms, and so the general considerations of [3] apply. Functors and natural transformations are $C\mathcal{G}$ -functors and $C\mathcal{G}$ -natural transformations. Graded categories are $G\mathcal{G}$ -categories.

Let $\mathcal{A}$ be a DG-category. The DG-functor $\text{Hom}_{\mathcal{A}} : \mathcal{A}^* \otimes \mathcal{A} \rightarrow C\mathcal{G}$ is given as follows:

(i) $\text{Hom}_{\mathcal{A}}(A, A') = [A, A'; \mathcal{A}]$;

(ii) $\text{Hom}_{\mathcal{A}} : (A, B), (A', B'); \mathcal{A}^* \otimes \mathcal{A} = [A', A; \mathcal{A}] \otimes [B, B'; \mathcal{A}]$

$[[A, B; \mathcal{A}], [A', B'; \mathcal{A}]; \mathcal{C}_\mathcal{A}]$ is the chain map with

$$\text{Hom}_{\mathcal{C}_\mathcal{A}}(f, g)h = (-1)^{p(q+r)}ghf$$

where $f \in [A', A; \mathcal{A}]_p$, $g \in [B, B'; \mathcal{A}]_q$, and $h \in [A, B; \mathcal{A}]_r$.

From a DG-category $\mathcal{A}$ we obtain graded categories $Z\mathcal{A}$, $H\mathcal{A}$ with the same objects as $\mathcal{A}$, with $[A, A'; Z\mathcal{A}] = Z[A, A'; \mathcal{A}]$, $[A, A'; H\mathcal{A}] = H[A, A'; \mathcal{A}]$, and with compositions induced by the composition of $\mathcal{A}$.

Suppose $T, S: \mathcal{A} \rightarrow \mathcal{B}$ are DG-functors. A proto-natural transformation $\alpha: T \rightarrow S$ of degree n is a family $(\alpha_A)_{A \in \mathcal{A}}$ of protomorphisms $\alpha_A: TA \rightarrow SA$ of degree n in $\mathcal{B}$, such that:

PNT. for all objects A, A' of $\mathcal{A}$ the diagram

$$\begin{array}{ccc} [A, A'; \mathcal{A}] & \xrightarrow{T} & [TA, TA'; \mathcal{B}] \\ S \downarrow & & \downarrow [1, \alpha_{A'}; \mathcal{B}] \\ [SA, SA'; \mathcal{B}] & \xrightarrow{\quad} & [TA, SA'; \mathcal{B}] \\ & & [\alpha, 1; \mathcal{B}] \end{array}$$

commutes in $\mathcal{C}_\mathcal{B}$.

The diagram of PNT evaluated at $f \in [A, A'; \mathcal{A}]_p$ gives the equation: $\alpha_{A'} \cdot Tf = (-1)^{pn} Sf \cdot \alpha_A$.

If $\mathcal{A}, \mathcal{B}$ are DG-categories then we define the "DG-category" $[\mathcal{A}, \mathcal{B}]$ as follows:

- (i) the objects are DG-functors from $\mathcal{A}$ to $\mathcal{B}$;
- (ii) the complex $[T, S; [\mathcal{A}, \mathcal{B}]] = [T, S]$ is given by $[T, S]_n =$ the class of proto-natural transformations from T to S of degree n , and if $\alpha \in [T, S]_n$ then $D_n \alpha \in [T, S]_{n-1}$ is defined by $(D_n \alpha)_A = D_n(\alpha_A)$;

(iii) composition is given by $(\beta\alpha)_A = \beta_A\alpha_A$.

The following calculation shows that $D_n\alpha$ as given in (ii) is an element of $[T, S]_{n-1}$:

$$\begin{aligned}
 (D\alpha)_A \cdot Tf &= D(\alpha_A) \cdot Tf \\
 &= D(\alpha_A \cdot Tf) - (-1)^n \alpha_A \cdot DTf \\
 &= (-1)^{pn} D(Sf \cdot \alpha_A) - (-1)^n \alpha_A \cdot TDf \\
 &= (-1)^{pn} D(Sf \cdot \alpha_A) - (-1)^n (-1)^{n(p-1)} S Df \cdot \alpha_A \\
 &= (-1)^{pn} (D S f \cdot \alpha_A - D S f \cdot \alpha_A) \\
 &= (-1)^{pn} (-1)^{-p} S f \cdot D(\alpha_A) \\
 &= (-1)^{p(n-1)} S f \cdot (D\alpha)_A
 \end{aligned}$$

for $f \in [A, A']_p$. Elements of $Z_n[T, S]$ will be called natural transformations of degree n in accordance with our previous convention.

From this we observe that the "category" of DG-categories and DG-functors is closed. We have the natural isomorphism:

$$[A \otimes B, C] \cong [A, [B, C]].$$

It must be remarked that $[A, B]$ is only a legitimate DG-category when A is small, but we talk of it in any case.

For DG-categories A, B the evaluation functor $E_A: [A, B] \rightarrow B$ is defined for each $A \in A$ as follows:

- (i) $E_A T = TA$ for $T \in [A, B]$;
- (ii) $E_A: [T, S] \rightarrow [TA, SA]$ is the chain map given by $E_A \alpha = \alpha_A$.

That this is a DG-functor is a simple consequence of the definitions.

§2. Proto-split exact sequences.

For any DG-category $\mathcal{A}$ we examine sequences in $Z_0 \mathcal{A}$ which are direct sum situations at the proto-level; such sequences will be called proto-split exact sequences (pses). The deviation class of a pses is defined. The DG-categories with which we shall be dealing satisfy the "extension axiom" which ensures the existence of a pses with any given deviation. DG-functors are shown to preserve pses and their deviations. A theorem on maps of pses is proven.

Let $\mathcal{F}$ be the additive category defined as follows:

- (i) the objects are $-1, 0$ and 1 ;
- (ii) $[-1, 0] = [0, 1] = [-1, -1] = [0, 0] = [1, 1] = \mathbb{Z}$; and
 $[-1, 1] = [1, 0] = [0, -1] [1, -1] = 0$;
- (iii) all composites not involving identities are zero.

If $\mathcal{A}$ is any DG-category then the DG-category $[\mathcal{F}, \mathcal{A}]$ will be denoted by $\text{Seq } \mathcal{A}$. The objects are diagrams $A' \xrightarrow{i} A \xrightarrow{p} A''$ in $Z_0 \mathcal{A}$ with $pi = 0$; such objects will be called sequences in $\mathcal{A}$. A pair $(\tilde{p}, \tilde{i})$ of protomorphisms $\tilde{p} : A \rightarrow A', \tilde{i} : A'' \rightarrow A$ of degree zero such that

$$p\tilde{i} = 1_{A''}, \tilde{p}i = 1_{A'}, \text{ and } i\tilde{p} + \tilde{i}p = 1_A$$

will be called a splitting of the sequence $A' \xrightarrow{i} A \xrightarrow{p} A''$. A sequence in $\mathcal{A}$ which has a splitting will be called a proto-split exact sequence (pses) in $\mathcal{A}$; the full sub-DG-category of $\text{Seq } \mathcal{A}$ with objects the pses in $\mathcal{A}$ will be denoted by $\text{Pses } \mathcal{A}$.

If $(\tilde{p}, \tilde{i})$ is a splitting of the sequence $A' \xrightarrow{i} A \xrightarrow{p} A''$ then

$$\delta = \tilde{p} \cdot \tilde{D}i = -D\tilde{p} \cdot \tilde{i}$$

will be called the deviation of the splitting $(\tilde{p}, \tilde{i})$.

Notice that $0 = D(1_A) = i \cdot D\tilde{p} + D\tilde{i} \cdot p$, so

$$D\tilde{p} = -\delta \cdot p, \quad D\tilde{i} = i \cdot \delta \quad \text{and} \quad D\delta = 0.$$

So $\delta \in Z_{-1}[A'', A']$. Suppose (p', i') is another splitting of the sequence; then $p' = \tilde{p} + \gamma p$, $i' = \tilde{i} - i\gamma$ for some protomorphism $\gamma: A'' \rightarrow A'$ of degree zero. Conversely, $(\tilde{p} + \gamma p, \tilde{i} - i\gamma)$ is a splitting for all protomorphisms $\gamma: A'' \rightarrow A'$ of degree zero; the deviation of this splitting is

$$\delta' = (\tilde{p} + \gamma p)(D\tilde{i} - iD\gamma) = \delta - D\gamma. \quad \text{We shall call}$$

$\Delta = [\delta] \in H_{-1}[A'', A']$ the deviation class of the sequence;

it has the following properties readily deduced from the above:

- (a) Δ is independent of the splitting of the pses;
- (b) if $\delta' \in \Delta$ then there is a splitting of the pses with δ' as its deviation;
- (c) $\Delta = 0$ if and only if the sequence comes from a direct sum situation in $Z_0 A$.

If A is a graded category then $\Delta = 0$ so all pses in A come from direct sum situations.

A DG-category A will be said to have the extension axiom if the following condition is satisfied:

EA. for any morphism $\delta: A'' \rightarrow A'$ of degree -1 there exists a pses $A' \xrightarrow{i} A \xrightarrow{p} A''$ in A with deviation class $[\delta]$.

EXAMPLES

(a) If $\mathcal{Q}$ is an additive category, the pses in $C\mathcal{Q}$ are, to within isomorphism, sequences $A' \xrightarrow{i} A \xrightarrow{p} A''$ where $A_n = A'_n \oplus A''_n$, with differential of A given by the matrix $\begin{pmatrix} d^{A'} & \delta \\ 0 & d^{A''} \end{pmatrix}$ for some $\delta \in Z_{-1}[A'', A']$, and with $i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $p = (0, 1)$. Then $((1, 0), \begin{pmatrix} 0 \\ 1 \end{pmatrix})$ is a splitting with deviation δ ; and so the deviation class is $[\delta]$. If $\mathcal{Q}$ has finite direct sums then $C\mathcal{Q}$ has the extension axiom.

(b) The sequence $A'' \xrightarrow{p^*} A \xrightarrow{i^*} A'$ in $\mathcal{A}^*$ is a pses with deviation class Δ if and only if $A' \xrightarrow{i} A \xrightarrow{p} A''$ in $\mathcal{A}$ is a pses with deviation class $-\Delta$; for $\tilde{i}^*(D\tilde{p})^* = (D\tilde{p}) \cdot \tilde{i} = -\tilde{p} \cdot D\tilde{i}$. If $\mathcal{A}$ has the extension axiom then so does $\mathcal{A}^*$.

(c) The sequence $(A', B') \xrightarrow{(i, j)} (A, B) \xrightarrow{(p, q)} (A'', B'')$ in $\mathcal{A} \otimes \mathcal{B}$ is pses with deviation class (Δ, Γ) if and only if $A' \xrightarrow{i} A \xrightarrow{p} A''$, $B' \xrightarrow{j} B \xrightarrow{q} B''$ in $\mathcal{A}$, $\mathcal{B}$ respectively are pses with deviation classes Δ, Γ respectively. If $\mathcal{A}, \mathcal{B}$ have the extension axiom then so does $\mathcal{A} \otimes \mathcal{B}$.

(d) Suppose $\alpha: \underline{A} \xrightarrow{i} \underline{B} \xrightarrow{p} \underline{C}$ is a sequence in $\text{Pses } \mathcal{A}$ where $\underline{A}, \underline{B}, \underline{C}$ are the pses $A' \xrightarrow{j'} A \xrightarrow{q'} A''$, $B' \xrightarrow{j} B \xrightarrow{q} B''$, $C' \xrightarrow{j''} C \xrightarrow{q''} C''$ in $\mathcal{A}$, and $\underline{i} = (i', i, i'')$, $\underline{p} = (p', p, p'')$ are morphisms of $\text{Pses } \mathcal{A}$. Let $\underline{K}', \underline{K}, \underline{K}''$ denote the sequences $A' \xrightarrow{i'} B' \xrightarrow{p'} C'$, $A \xrightarrow{i} B \xrightarrow{p} C$, $A'' \xrightarrow{i''} B'' \xrightarrow{p''} C''$ in $\mathcal{A}$. Clearly if α is a pses in $\text{Pses } \mathcal{A}$ with deviation class $[\delta', \delta, \delta'']$ then $\underline{K}', \underline{K}, \underline{K}''$ are

are pses in $\mathcal{A}$ with deviation classes $[\delta'], [\delta], [\delta'']$ respectively. The following converse is also true.

Proposition 1. If α (as above) is a sequence in $\text{Pses } \mathcal{A}$ such that $\tilde{K}', \tilde{K}''$ are pses in $\mathcal{A}$ with splittings $(\tilde{p}', \tilde{i}'), (\tilde{p}'', \tilde{i}'')$ respectively, then there exists a splitting $(\tilde{p}, \tilde{i})$ of $\tilde{K}$ such that $((\tilde{p}', p, \tilde{p}''), (\tilde{i}', i, \tilde{i}''))$ is a splitting of α .

Proof. At the proto-level we may suppose $A = A' \oplus A''$, $B = B' \oplus B''$, $C = C' \oplus C''$; so $\begin{pmatrix} A \\ B \\ C \end{pmatrix}$ becomes $A' \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} A' \oplus A'' \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} A''$; similarly B, C . Then $i = \begin{pmatrix} i' & f \\ 0 & i'' \end{pmatrix}$, $p = \begin{pmatrix} p' & g \\ 0 & p'' \end{pmatrix}$ for some f, g such that $p'f + gi'' = 0$. Let $h = -\tilde{p}'f\tilde{p}''$, $k = -\tilde{i}'g\tilde{i}''$, and put $\tilde{p} = \begin{pmatrix} \tilde{p}' & h \\ 0 & \tilde{p}'' \end{pmatrix}$, $\tilde{i} = \begin{pmatrix} \tilde{i}' & k \\ 0 & \tilde{i}'' \end{pmatrix}$. Then we have $\tilde{p}'f + hi'' = 0$, $p'k + g\tilde{i}'' = 0$, $i'h + f\tilde{p}'' + \tilde{i}'g + kp'' = 0$, so $(\tilde{p}, \tilde{i})$ is a splitting of $\tilde{K}$. Moreover, $(\tilde{p}', \tilde{p}, \tilde{p}'')$, $(\tilde{i}', \tilde{i}, \tilde{i}'')$ are protomorphisms of degree zero in $\text{Pses } \mathcal{A}$, and so provide a splitting of α . //

Suppose $T: \mathcal{A} \rightarrow \mathcal{B}$ is a DG-functor and $A' \xrightarrow{i} A \xrightarrow{p} A''$ is a sequence in $\mathcal{A}$ with splitting $(\tilde{p}, \tilde{i})$ and deviation δ corresponding to this splitting. Then $T\tilde{p}.Ti = 1$, $Tp.T\tilde{i} = 1$, $T\tilde{i}.Tp + Ti.T\tilde{p} = 1$ and $T\tilde{p}.DT\tilde{i} = T\tilde{p}.TD\tilde{i} = T(\tilde{p}.D\tilde{i}) = T\delta$. So $(T\tilde{p}, T\tilde{i})$ is a splitting of the sequence $TA' \xrightarrow{Ti} TA \xrightarrow{Tp} TA''$ in $\mathcal{B}$ with deviation $T\delta$. In other words, DG-functors preserve pses, their splittings, and their deviation classes.

Theorem 2. Suppose $A: A' \xrightarrow{i} A \xrightarrow{p} A'' \in \text{Pses } \mathcal{A}$ with deviation class $[\delta]$ and $B: B' \xrightarrow{j} B \xrightarrow{q} B'' \in \text{Seq } \mathcal{A}$ such

that the sequence

$$0 \rightarrow [A'', B'] \xrightarrow{[1, j]} [A'', B] \xrightarrow{[1, q]} [A'', B''] \rightarrow 0$$

is exact with $\Gamma: H[A'', B''] \rightarrow H[A'', B']$ the connecting map of the exact homology triangle.

Then:

(a) the sequence

$$0 \rightarrow [A'', B'] \xrightarrow{u} [A, B] \xrightarrow{v} [A', B'] \oplus [A'', B''] \rightarrow 0$$

is exact, where $u(h) = (0, jhp, 0)$ and $v(f', f, f'') = (f', f'')$;

(b) if $f' \in Z_n[A', B']$, $f'' \in Z_n[A'', B'']$ then the existence of f such that $(f', f, f'') \in Z_n[A, B]$ is equivalent to the condition

$$\Gamma([f'']) = (-1)^n [f' \delta] .$$

Proof. Let $\underline{A}', \underline{A}''$ denote the sequences $A' \xrightarrow{1} A' \rightarrow 0$, $0 \rightarrow A'' \xrightarrow{1} A''$ respectively. Then $\underline{A}' \xrightarrow{(1, i, 0)} \underline{A} \xrightarrow{(0, p, 1)} \underline{A}''$ is a pses in Pses $\mathcal{A}$ and $[\underline{A}', \underline{B}] \cong [A', B']$, $[\underline{A}'', \underline{B}] \cong [A'', B']$.

But $[-, \underline{B}]$ preserves pses, so the sequence

$$0 \rightarrow [\underline{A}'', \underline{B}] \xrightarrow{u'} [\underline{A}, \underline{B}] \xrightarrow{v'} [\underline{A}', \underline{B}] \rightarrow 0$$

is exact, where $u'(k) = (0, kp, qk)$ and $v'(f', f, f'') = f'$.

It follows that the rows and columns of the commutative diagram:

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & [A'', B'] & \xrightarrow{[p, 1]} & [A, B'] & \xrightarrow{[i, 1]} & [A', B'] \rightarrow 0 \\ & & [1, j] \downarrow & & \downarrow u'' & & \downarrow 1 \\ 0 & \rightarrow & [A'', B] & \xrightarrow{u'} & [A, B] & \xrightarrow{v'} & [A', B'] \rightarrow 0 \\ & & [1, q] \downarrow & & \downarrow v'' & & \downarrow \\ 0 & \rightarrow & [A'', B''] & \xrightarrow{1} & [A'', B''] & \rightarrow & 0 \\ & & \downarrow 0 & & \downarrow 0 & & \downarrow 0 \end{array}$$

are exact, where $u''(k) = (ki, jk, 0)$ and $v''(f'f, f'') = f''$.
 Since $u = u'[1, j] = u''[p, 1]$ and $v = \begin{pmatrix} v' \\ v'' \end{pmatrix}$ the result
 (a) follows. The connecting map of the homology triangle
 of the first row of the last diagram is $[\delta, 1]_* :$

$H[A', B'] \rightarrow H[A'', B']$. Thus the connecting map of the
 homology triangle of the sequence of (a) is $\Delta = ([\delta, 1]_*, \Gamma)$;
 in evaluated form $\Delta([f'], [f'']) = \Gamma([f'']) + (-1)^{n+1} [f' \delta]$,

where $f' \in Z_n[A', B']$, $f'' \in Z_n[A'', B'']$. Let

$\eta: Z[A', B'] \oplus Z[A'', B''] \rightarrow H[A', B'] \oplus H[A'', B'']$ be the canonical
 epimorphism. The result (b) follows from the exact
 sequence:

$$Z[A, B] \xrightarrow{Zv} Z[A', B'] \oplus Z[A'', B''] \xrightarrow{\Delta\eta} H[A'', B']$$

(see [1] Ch. IV §3 p.59) . //

§3. Complete graded categories.

Weakly stable and stable DG-categories are defined in this section; most categories with which we deal in this work will be stable. Attention is then turned specifically to graded categories — although most of the ensuing results hold for DG-categories. Stable functors and stable natural transformations are defined and each graded category is shown to admit a canonical stable extension. The definitions of graded adjoint and tensored (cotensored) graded categories are recalled. Complete (cocomplete) graded categories are defined and connections with cotensored (tensored) graded categories are found. A functor category is shown to be cocomplete if its range category is cocomplete; in this case moreover, adjoints are found for the evaluation functors.

The DG-category $\mathcal{A}$ will be called weakly stable if there exist:

- L1. a "suspension" functor $L: \mathcal{A} \rightarrow \mathcal{A}$ with a left and right inverse to within isomorphism (that is, there exists L^{-1} such that $LL^{-1} \cong 1$, $L^{-1}L \cong 1$); and
- L2. a natural isomorphism $\ell: 1 \rightarrow L$ of degree 1.

Weakly stable DG-categories are those in which each object has an isomorph of any given degree. The pair (L, ℓ) is essentially unique; if $(\bar{L}, \bar{\ell})$ also satisfies L1, L2 then there exists a natural isomorphism $\theta: L \rightarrow \bar{L}$ (namely $\bar{\ell} \ell^{-1}$) such that $\bar{\ell} = \theta \ell$. The symbols L, ℓ will be used in all weakly stable categories.

The DG-category $\mathcal{A}$ will be called stable if it is weakly stable and if there is some choice of the pair (L, ℓ) such that L has an inverse (that is a functor L^{-1} with $LL^{-1} = 1$ and $L^{-1}L = 1$).

EXAMPLES.

(a) For any additive category $\mathcal{A}$, the DG-category $C\mathcal{A}$ is stable. The suspension functor $L: C\mathcal{A} \rightarrow C\mathcal{A}$ is defined by:

(i) LA is the complex with

$$(LA)_n = A_{n-1} \text{ and } d^{LA} = -d^A;$$

(ii) the chain map $L: [A, B] \rightarrow [LA, LB]$ evaluated at $f = (f_r) \in [A, B]_n$ is given by:

$$(Lf)_r = (-1)^n f_{r-1}.$$

The inverse L^{-1} is given on objects by:

$$(L^{-1}A)_n = A_{n+1} \text{ and } d^{L^{-1}A} = -d^A.$$

The natural isomorphism $\ell: 1 \rightarrow L$ of degree 1 is given by: $\ell_{Ar} \in [A_r, (LA)_{r+1}] = [A_r, A_r]$ is 1_{A_r} . If A has zero differential then so does LA ; it follows that $C\mathcal{A}$ is also stable.

(b) If $\mathcal{A}, \mathcal{B}$ are weakly stable {respectively stable} then so are $H\mathcal{A}, \mathcal{A}^*$, $\mathcal{A} \otimes \mathcal{B}$ and $[K, \mathcal{A}]$; the pairs (L, ℓ) are given (in obvious notation) by $(HL, H\ell)$, (L^*, ℓ^*) , $(L \otimes L, \ell \otimes \ell)$, $([1, L], [1, \ell])$ respectively.

Suppose $\mathcal{A}$ is a weakly stable DG-category. For any object A and integer m we write $L^m A$ for $LL \dots LA$ (m applications of L) when $m \geq 0$ and for $L^{-1}L^{-1} \dots L^{-1}A$ ($-m$ applications of L^{-1}) when $m < 0$. The isomorphisms $L^{-1}L \cong 1$, $LL^{-1} \cong 1$ and $\eta: 1 \rightarrow L$ give, for each integer m , a natural isomorphism $\eta^m: 1 \rightarrow L^m$ of degree m .

Thus there is a natural isomorphism:

$$[L^m A, L^n A'; \mathcal{A}] \cong L^{n-m}[A, A'; \mathcal{A}].$$

The remainder of this section deals only with graded categories; all script letters will be understood to denote graded categories.

If $\mathcal{A}, \mathcal{B}$ are stable a functor $T: \mathcal{A} \rightarrow \mathcal{B}$ will be called stable if $TL = LT$ and $T\eta = \eta T$. If $S, T: \mathcal{A} \rightarrow \mathcal{B}$ are stable functors, a natural transformation $\alpha: T \rightarrow S$ of degree n will be called stable if $\alpha L = L\alpha$. Let $[\mathcal{A}, \mathcal{B}]_L$ denote the sub-graded-"category" of $[\mathcal{A}, \mathcal{B}]$ with objects the stable functors and morphisms of degree n the natural transformations of degree n .

Let $\mathcal{A}^L$ denote the stable graded category defined as follows:

- (i) the class of objects of $\mathcal{A}^L$ is the disjoint union of copies of the class of objects of $\mathcal{A}$, one copy for each integer - let the object in the m -th copy corresponding to A be denoted by $L^m A$;
- (ii) $[L^m A, L^n A'; \mathcal{A}^L] = L^{n-m}[A, A'; \mathcal{A}]$;

(iii) the composite of $f \in [L^m A, L^n A']_q = [A, A']_{q+m-n}$ and $g \in [L^n A', L^p A'']_r = [A', A'']_{r+n-p}$ is just $gf \in [A, A'']_{q+r+m-p} = [L^m A, L^p A'']_{q+r}$.

$\mathcal{A}$ may be regarded as a full sub-graded-category of $\mathcal{A}^L$ by identifying A and $L^0 A$.

Now we shall show that $\mathcal{A}^L$ is stable. Define $L: \mathcal{A}^L \rightarrow \mathcal{A}^L$ on objects by $L^m A \rightarrow L^{m+1} A$. For

$f \in [L^m A, L^n A']_q = [A, A']_{q+m-n}$, $Lf \in [L^{m+1} A, L^{n+1} A']_q = [A, A']_{q+m-n}$ is defined to be the element $(-1)^q f$.

Clearly L is a functor with inverse L^{-1} given by $L^m A \mapsto L^{m-1} A$ on objects. Let $1_A \in [L^m A, L^{m+1} A]_1 = [A, A]_0$ be the identity of A . It is readily checked that $1 = (1_A): 1 \rightarrow L$ is a natural isomorphism of degree 1.

Theorem 3. If $\mathcal{B}$ is stable then restriction gives an isomorphism:

$$[\mathcal{A}^L, \mathcal{B}]_L \cong [\mathcal{A}, \mathcal{B}]$$

of graded categories (the restriction functor has an inverse).

Proof. For each functor $T: \mathcal{A} \rightarrow \mathcal{B}$ define $T^L: \mathcal{A}^L \rightarrow \mathcal{B}$ by:

$$(i) \quad T^L L^m A = L^m T A;$$

$$(ii) \quad \text{if } f \in [L^m A, L^n A']_q \text{ then } T^L f = 1^n \cdot T f \cdot 1^{-m} \in [L^m T A, L^n T A']_q.$$

Clearly T^L is a functor and $T^L L = L T^L$ on objects; but also, if f is as in (ii), we have:

$$\begin{aligned}
LT^L f &= L(\mathcal{L}^n \cdot \text{Tr} \cdot \mathcal{L}^{-m}) \\
&= (-1)^q \mathcal{L} \cdot \mathcal{L}^n \cdot \text{Tr} \cdot \mathcal{L}^{-m} \cdot \mathcal{L}^{-1} \quad (\text{by naturality of } \mathcal{L}) \\
&= (-1)^q \mathcal{L}^{n+1} \cdot \text{Tr} \cdot \mathcal{L}^{-(m+1)} \\
&= \mathcal{L}^{n+1} \cdot \text{TL}f \cdot \mathcal{L}^{-(m+1)} \\
&= T^L Lf.
\end{aligned}$$

So T^L is stable.

For each natural transformation $\alpha: T \rightarrow S$ of degree r define $\alpha_{L^m A}^L = (-1)^{mr} \mathcal{L}^m \cdot \alpha_A \cdot \mathcal{L}^{-m} \in [L^m T A, L^m S A]_r$.

Then, for f as in (ii), we have:

$$\begin{aligned}
\alpha_{L^n A}^L \cdot T^L f &= (-1)^{nr} \mathcal{L}^n \cdot \alpha_A \cdot \mathcal{L}^{-n} \cdot \mathcal{L}^n \cdot \text{Tr} \cdot \mathcal{L}^{-m} \\
&= (-1)^{nr} \mathcal{L}^n \cdot \alpha_A \cdot \text{Tr} \cdot \mathcal{L}^{-m} \\
&= (-1)^{r(q+m-n)} (-1)^{nr} \mathcal{L}^n \cdot \text{Sf} \cdot \alpha_A \cdot \mathcal{L}^{-m} \\
&\quad (\text{by the naturality of } \alpha) \\
&= (-1)^{rq} \mathcal{L}^n \cdot \text{Sf} \cdot \mathcal{L}^{-m} \cdot (-1)^{mr} \mathcal{L}^m \cdot \alpha_A \cdot \mathcal{L}^{-m} \\
&= (-1)^{rq} \text{S}^L f \cdot \alpha_{L^m A}^L.
\end{aligned}$$

Thus $\alpha^L = (\alpha_{L^m A}^L): T^L \rightarrow S^L$ is a natural transformation of

degree r . Moreover, we have:

$$\begin{aligned}
\alpha_{L^{m+1} A}^L &= \alpha_{L^{m+1} A}^L \\
&= (-1)^{(m+1)r} \mathcal{L}^{m+1} \cdot \alpha_A \cdot \mathcal{L}^{-(m+1)} \\
&= (-1)^r \mathcal{L} \cdot \alpha_{L^m A}^L \cdot \mathcal{L}^{-1} \\
&= L \alpha_{L^m A} \quad (\text{by the naturality of } \mathcal{L}).
\end{aligned}$$

So α^L is stable. If $T = S$, $\alpha = 1$ so that $r = 0$, then $\alpha^L = 1$; also $(\beta\alpha)^L = \beta^L \alpha^L$. Thus we have an "extension" functor $T \mapsto T^L$, $\alpha \mapsto \alpha^L$ which is clearly a left inverse to restriction; using the fact that γ is a natural transformation of degree 1 it is also readily checked that $F^L = F$, $\gamma^L = \gamma$ for stable F, γ . So extension and restriction are mutually inverse isomorphisms, //

The graded functor $S: \mathcal{B} \otimes \mathcal{C} \rightarrow \mathcal{A}$ will be called a graded adjoint of the graded functor $T: \mathcal{C}^* \otimes \mathcal{A} \rightarrow \mathcal{B}$ if there is a natural isomorphism:

$$[S(B, C), A; \mathcal{A}] = [B, T(C, A); \mathcal{B}];$$

we shall write $S \dashv T$. Graded adjoints are unique to within natural isomorphisms.

We do not distinguish between $\mathbb{Z}$ and the graded group with $\mathbb{Z}$ in degree zero and 0 in other degrees. Denote by $\mathcal{I}_0$ the category with one object 1 and $[1, 1; \mathcal{I}_0] = \mathbb{Z}$. For any graded category $\mathcal{A}$ we do not distinguish between $\mathcal{A}$ and $\mathcal{A} \otimes \mathcal{I}_0$.

Taking $\mathcal{C} = \mathcal{I}_0$ in the definition of graded adjoint gives an important special case.

If $\text{Hom}_{\mathcal{A}}: \mathcal{A}^* \otimes \mathcal{A} \rightarrow G\mathcal{I}$ has a graded adjoint then we shall say $\mathcal{A}$ is tensored; we shall write $\text{Ten}_{\mathcal{A}}: G\mathcal{I} \otimes \mathcal{A} \rightarrow \mathcal{A}$ for the graded adjoint and put $\text{Ten}_{\mathcal{A}}(X, A) = X \otimes A$, $\text{Ten}_{\mathcal{A}}(x, f) = x \otimes f$. We shall say $\mathcal{A}$ is cotensored if $\mathcal{A}^*$ is tensored.

For any graded category $\mathcal{A}$ we shall write $\mathcal{A}_0$ for the additive category with the same objects as $\mathcal{A}$ and morphisms the morphisms of $\mathcal{A}$ of degree zero.

If $T: \mathcal{K} \rightarrow \mathcal{A}$ is an additive functor then we shall denote the limit of T (if it exists) by a pair (M, μ) where $M \in \mathcal{A}$ and μ is a family $(\mu_K)_{K \in \mathcal{K}}$ of morphisms $\mu_K \in [M, TK; \mathcal{A}]$. A graded functor $T: \mathcal{K} \rightarrow \mathcal{A}$ is essentially an additive functor $T: \mathcal{K} \rightarrow \mathcal{A}_0$ if $\mathcal{K}$ is additive; by a limit of $T: \mathcal{K} \rightarrow \mathcal{A}$ will be meant a limit of $T: \mathcal{K} \rightarrow \mathcal{A}_0$; dually for colimits.

Theorem 4. Suppose we have the following:

- (a) a small additive category $\mathcal{K}$;
- (b) a weakly stable graded category $\mathcal{A}$;
- (c) a functor $T: \mathcal{K} \rightarrow \mathcal{A}$ with limit (M, μ) ;
- (d) an object A of $\mathcal{A}$.

Then $([A, M], [1, \mu])$ is the limit of $[A, T-]; \mathcal{K} \rightarrow \text{Gly}$.

Proof. Let $u_r: L^r A \rightarrow A$ be an isomorphism of degree $-r$ for each integer r . Suppose $N, \nu = (\nu_K)_{K \in \mathcal{K}}$ are such that all diagrams:

$$\begin{array}{ccc} N & \xrightarrow{\nu_K} & [A, TK] \\ & \searrow \nu_{K'} & \downarrow [1, Tk] \\ & & [A, TK'] \end{array}$$

commute in $(\text{Gly})_0$; clearly $[A, M], [1, \mu]$ satisfy this condition. Take $x \in N_r$, then $\nu_K(x): A \rightarrow TK$ is of degree r , and $L^r A \rightarrow TK$ given by $\nu_K(x)$. u_r is of degree zero. Since $\nu_{K'}(x) \cdot u_r = Tk \cdot \nu_K(x) \cdot u_r$ we obtain a unique map

$f(x): L^r A \rightarrow M$ such that $v_K(x) = \mu_K \cdot f(x) \cdot u_r^{-1}$. Define $g: N \rightarrow [A, M]$ by $g(x) = f(x) \cdot u_r^{-1}$. Then g is unique such that the diagrams:

$$\begin{array}{ccc} N & \xrightarrow{v_K} & [A, TK] \\ g \downarrow & \nearrow [1, \mu_K] & \\ [A, M] & & \end{array}$$

commute. The result follows. //

Theorem 5. A tensored graded category is weakly stable.

Proof. Suppose $\mathcal{A}$ is tensored. Put $L = L\mathbb{Z} \otimes -$, $L^{-1} = L^{-1}\mathbb{Z} \otimes -$; and take $1_A \in [A, L\mathbb{Z} \otimes A]_1 = [\mathbb{Z} \otimes A, L\mathbb{Z} \otimes A]_1$ to be $1_{\mathbb{Z}} \otimes 1_A$. Then $L^{-1}L = L^{-1}\mathbb{Z} \otimes (L\mathbb{Z} \otimes -) \cong (L^{-1}\mathbb{Z} \otimes L\mathbb{Z}) \otimes - \cong \mathbb{Z} \otimes - = 1$, and $1 = (1_A)$ is a natural isomorphism of degree 1. //

$\mathcal{A}$ will be called complete [finitely complete] if $\mathcal{A}_0$ is complete [finitely complete] and $\mathcal{A}$ is weakly stable. Dually define cocomplete (as we saw earlier $\mathcal{A}$ is weakly stable if and only if $\mathcal{A}^*$ is).

Theorem 6. A cocomplete graded category is tensored.

Proof. Suppose $\mathcal{A}$ is a cocomplete graded category. Then $\mathcal{A}_0$ is a cocomplete additive category and so tensored. For $X \in \text{Gly}$, $A \in \mathcal{A}$ define $X \otimes A = \sum_{r \in \mathbb{Z}} X_r \otimes L^r A$ where $\otimes$ and $\sum$ on the right pertain to $\mathcal{A}_0$. But then we have isomorphisms:

$$\begin{aligned}
[X, [A, A'; A]; \text{Gly}]_n &= \prod_r (X_r, [A, A'; A]_{r+n}; \text{ly}) \\
&\cong \prod_r (X_r, [L^r A, L^{-n} A'; A]_0; \text{ly}) \\
&= \prod_r (X_r, (L^r A, L^{-n} A'; A_0); \text{ly}) \\
&\cong \prod_r (X_r \otimes L^r A, L^{-n} A'; A_0) \\
&\cong (\sum_r X_r \otimes L^r A, L^{-n} A'; A_0) \\
&= [X \otimes A, L^{-n} A'; A]_0 \\
&\cong [X \otimes A, A'; A]_n,
\end{aligned}$$

giving an isomorphism:

$$[X, [A, A'; A]; \text{Gly}] \cong [X \otimes A, A'; A]$$

natural in A' . By the general theory of adjunction

- $\otimes$ - may be defined on morphisms of any degree uniquely in such a way that we obtain a functor $\text{Ten}_A: \text{Gly} \otimes A \rightarrow A$ with $\text{Ten}(X, A) = X \otimes A$ and the above isomorphism is natural in X, A, A' . So A is tensored. //

Theorem 7. If $\mathcal{H}$ is small and A is cocomplete
[finitely cocomplete] then $[\mathcal{H}, A]$ is cocomplete [finitely
cocomplete] and colimits are formed evaluationwise.

Proof. Suppose $\mathcal{K}$ is an additive category and has $\mathcal{K}$ -colimits and is weakly stable. We show that each functor $T: \mathcal{K} \rightarrow [\mathcal{H}, A]$ has a colimit. For each $H \in \mathcal{H}$ select a colimit $(MH, \mu_{-, H})$ of $E_H T: \mathcal{K} \rightarrow A$. For each $f \in [H, H'; \mathcal{H}]_n$ all the diagrams:

$$\begin{array}{ccccc}
 (\text{TK})_H & \xrightarrow{(\text{TK})f} & (\text{TK})_{H'} & \xrightarrow{\mu_{K,H'}} & \text{MH}' \\
 (\text{TK})_H \downarrow & & (\text{TK})_{H'} \downarrow & \nearrow \mu_{K',H'} & \\
 (\text{TK}')_H & \xrightarrow{(\text{TK}')f} & (\text{TK}')_{H'} & &
 \end{array}$$

commute. Since $(\text{MH}, \mu_{-,H})$ is a colimit the dual of Theorem 4 gives the existence of a unique $\text{Mf} \in [\text{MH}, \text{MH}'; \mathcal{A}]_n$ such that the diagrams:

$$\begin{array}{ccc}
 (\text{TK})_H & \xrightarrow{\mu_{K,H}} & \text{MH} \\
 (\text{TK})f \downarrow & & \downarrow \text{Mf} \\
 (\text{TK})_{H'} & \xrightarrow{\mu_{K,H'}} & \text{MH}'
 \end{array}$$

commute. From the uniqueness it follows that $M: \mathcal{H} \rightarrow \mathcal{A}$ is a functor and from the commutativity it follows that $\mu_{K,-}: \text{TK} \rightarrow M$ is a natural transformation. We show that (M, μ) , where $\mu = (\mu_{K,-})$, is the colimit of T . The commuting condition holds since it holds on evaluation at each H . Suppose $\{\text{TK} \xrightarrow{\nu_K} N \mid K \in \mathcal{K}\}$ satisfies the commuting condition. Evaluating at H we obtain a family satisfying the commuting condition for a colimit of E_H^T . So there exists a unique $\alpha_H \in [\text{MH}, \text{NH}]_0$ such that the triangles:

$$\begin{array}{ccc}
 (\text{TK})_H & \xrightarrow{\mu_{K,H}} & \text{MH} \\
 \searrow \nu_{K,H} & & \downarrow \alpha_H \\
 & & \text{NH}
 \end{array}$$

all commute. The proof is completed on showing that $\alpha = (\alpha_H): M \rightarrow N$ is a natural transformation. But for $f \in [H, H']_n$ we have:

$$\begin{aligned}
\text{Nf} \cdot \alpha_H \cdot \mu_{K,H} &= \text{Nf} \cdot \nu_{K,H} \quad (\text{definition of } \alpha_H) \\
&= \nu_{K,H'} \cdot (\text{TK})f \quad (\text{naturality of } \nu_K) \\
&= \alpha_{H'} \cdot \mu_{K,H'} \cdot (\text{TK})f \quad (\text{definition of } \alpha_{H'}) \\
&= \alpha_{H'} \cdot \text{Mf} \cdot \mu_{K,H} \quad (\text{definition of Mf}) .
\end{aligned}$$

Since $(\text{MH}, \mu_{-,H})$ is a colimit the dual of Theorem 4 gives $\text{Nf} \cdot \alpha_H = \alpha_{H'} \cdot \text{Mf}$; so α is natural.

We have already seen that $[\mathcal{H}, \mathcal{A}]$ is weakly stable if $\mathcal{A}$ is. //

Theorem 8. If $\mathcal{H}$ is small then there is a natural isomorphism:

$$[[H, -; \mathcal{H}], [A, F-; \mathcal{H}]; [\mathcal{H}, G_L]] \cong [A, FH; \mathcal{A}] .$$

Proof. Define $\Gamma: [[H, -], [A, F-]] \rightarrow [A, FH]$ by $\Gamma_n(\alpha) = \alpha_H(1_H)$. That Γ is natural is trivial. For $f \in [A, FH]_n$ define $\Omega_n(f)_{H'}: [H, H'] \rightarrow [A, FH']$ of degree n to be the composite:

$$[H, H'] \xrightarrow{F} [FH, FH'] \xrightarrow{[f, 1]} [A, FH'] .$$

Then $\Omega_n(f) = (\Omega_n(f)_{H'}) : [H, -] \rightarrow [A, F-]$ is a natural transformation of degree n . So we have

$\Omega: [A, FH] \rightarrow [[H, -], [A, F-]]$. Moreover

$\Gamma_n \Omega_n(f) = \Omega_n(f)_H(1_H) = [f, 1] \cdot F1 = f$ so $\Gamma \Omega = 1$; and for α as above and $h \in [H, H']_m$, the degree n naturality of α gives $\alpha_{H'}(h) = \alpha_H \cdot [1, h] \cdot (1_H)$

$$= (-1)^{mn} [1, Fh] \cdot \alpha_H(1_H)$$

$$= (-1)^{mn} [1, Fh] \cdot \Gamma_n(\alpha)$$

$$= [\Gamma_n(\alpha), 1] \cdot Fh$$

$$= \Omega_n(\Gamma_n(\alpha))_{H'}(h) ,$$

so $\Omega\Gamma = 1$. //

Theorem 9. If $\mathcal{H}$ is small and $\mathcal{A}$ is cocomplete then the evaluation functor $E_H: [\mathcal{H}, \mathcal{A}] \rightarrow \mathcal{A}$, where $H \in \mathcal{H}$, has an adjoint $J_H: \mathcal{A} \rightarrow [\mathcal{H}, \mathcal{A}]$ whose value on the object $A \in \mathcal{A}$ is given by:

$$(i) \quad (J_H A)H' = [H, H'] \otimes A \text{ for } H' \in \mathcal{H} ;$$

(ii) $J_H A: [H', H''] \rightarrow [[H, H'] \otimes A, [H, H''] \otimes A]$ is the map corresponding, under the tensor adjunction, to the map $([H', H''] \otimes [H, H']) \otimes A \rightarrow [H, H''] \otimes A$ induced by composition in $\mathcal{H}$.

Proof. By Theorem 6, $\mathcal{A}$ is tensored, so $J_H A$ as in the theorem is defined. Then the following isomorphism are natural in F :

$$[J_H A, F] = [[H, -] \otimes A, F]$$

$$\cong [[H, -], [A, F-]] \text{ (from the tensor adjunction)}$$

$$\cong [A, FH] \text{ (by Theorem 8). } //$$

A graded category $\mathcal{G}$ will be called a graded model if:

GM1. it is small;

GM2. for all objects X, Y , either $[X, Y] = 0$ or $[X, Y] = L^{n(X, Y)} \mathbb{Z}$ for some integer $n(X, Y)$;

GM3. for $[X, Y], [Y, Z], [X, Z] \neq 0$ composition

$[Y, Z] \otimes [X, Y] \rightarrow [X, Z]$ is zero unless $n(X, Z) =$

$n(X, Y) + n(Y, Z)$ in which case it is given by

$1 \otimes 1 \mapsto 1$ from $\mathbb{Z} \otimes \mathbb{Z}$ to $\mathbb{Z}$ in dimension $-n(X, Z)$.

In a graded model we shall write $X \rightarrow Y$ for the distinguished map $1 \in \mathbb{Z} = [X, Y]_{-n(X, Y)}$ if $[X, Y] \neq 0$; if $[X, Y] = 0$ then $X \rightarrow Y$ will denote the zero map. If $G: \mathcal{I} \rightarrow \mathcal{B}$ is a graded functor we shall write $GX \rightarrow GY$ for $G(X \rightarrow Y)$.

Theorem 10. If $\mathcal{I}$ is a graded model and $\mathcal{B}$ is stable then the evaluation functor $E_X: [\mathcal{I}, \mathcal{B}] \rightarrow \mathcal{B}$, where $X \in \mathcal{I}$, has an adjoint $J_X: \mathcal{B} \rightarrow [\mathcal{I}, \mathcal{B}]$ whose value on the object $B \in \mathcal{B}$ is given by:

$$(i) \quad (J_X B)Y = L^m B \text{ if } [X, Y]_m \neq 0, \\ = 0 \text{ otherwise;}$$

$$(ii) \quad (J_X B)Y \rightarrow (J_X B)Y' \text{ is the isomorphism } L^{m,n}; L^m B \rightarrow L^{m+n} B \text{ of degree } n \text{ if } [X, Y]_m, [Y, Y']_n, [X, Y']_{m+n} \neq 0, \text{ it is zero otherwise.}$$

Proof. Define $\Gamma: [J_X B, G] \rightarrow [B, GX]$ by $\Gamma_n(\alpha) = \alpha_X$.

Clearly Γ is natural in G . For $f \in [B, GX]_n$ define $(\Omega_n f)_Y$ to be zero if $[X, Y] = 0$ and $(-1)^{mn}$ times the composite $(J_X B)Y = L^m B \xrightarrow{\lambda^{-m}} B \xrightarrow{f} GX \rightarrow GY$ if $[X, Y]_m \neq 0$.

Then $\Omega_n f: J_X B \rightarrow G$ is natural of degree n . So we have $\Omega: [B, GX] \rightarrow [J_X B, G]$. It is simply checked that Γ, Ω are mutually inverse isomorphisms. //

§4. Abelian graded categories.

The purpose of this section is to define abelian graded categories and show that the familiar homological algebra may be done in them. Exactness, projectivity and graded groups $\text{Ext}^n[B, B']$ are defined for an abelian graded category. A graded functor category is shown to be abelian if the range category is; if the range category is moreover cocomplete and projectively perfect then the functor category is shown to be also, and the nature of the projectives is given. In particular we look at the functor category of triangles over a graded category.

A graded category $\mathcal{B}$ will be called abelian if it is finitely complete and cocomplete, every monomorphism is a kernel and every epimorphism is a cokernel. So $\mathcal{B}$ is abelian if and only if $\mathcal{B}_0$ is abelian and $\mathcal{B}$ is weakly stable.

Suppose $\mathcal{B}$ is abelian. A sequence $B' \xrightarrow{f} B \xrightarrow{g} B''$ in $\mathcal{B}$, where $\deg f = m$, $\deg g = n$, will be called exact if the sequence $L^m B' \xrightarrow{f \cdot \tau^m} B \xrightarrow{\tau^n g} L^{-n} B''$ is exact in $\mathcal{B}_0$. An object P will be called projective in $\mathcal{B}$ if it is projective in $\mathcal{B}_0$. From the isomorphisms $[P, L^m B'] \cong L^m [P, B']$, $[P, L^{-n} B''] \cong L^{-n} [P, B'']$ we see that $B' \xrightarrow{f} B \xrightarrow{g} B''$ exact in $\mathcal{B}$ and P projective in $\mathcal{B}$ imply that the sequence $[P, B'] \xrightarrow{[1, f]} [P, B] \xrightarrow{[1, g]} [P, B'']$ is exact in Ab . For $B, B' \in \mathcal{B}$, the graded abelian group $\text{Ext}^n[B, B'; \mathcal{B}]$ is defined by:

$$\text{Ext}^n[B, B'; \mathcal{B}]_m = \text{Ext}^n(B, L^{-m}B'; \mathcal{B}_0).$$

It is readily checked that exactness of the sequence
 $0 \rightarrow B' \xrightarrow{f} B \xrightarrow{g} B'' \rightarrow 0$ in $\mathcal{B}$ with $\deg f = m$, $\deg g = n$
implies exactness of a long sequence:

$$\begin{aligned} \dots \rightarrow \text{Ext}^r[B'', C] &\xrightarrow{\text{Ext}[g, 1]} \text{Ext}^r[B, C] \xrightarrow{\text{Ext}[f, 1]} \\ &\text{Ext}^r[B', C] \xrightarrow{\Delta} \text{Ext}^{r+1}[B'', C] \xrightarrow{\text{Ext}[g, 1]} \dots \end{aligned}$$

in $G\mathcal{Y}$, natural in such short exact sequences and in C ,
where $\deg \text{Ext}[g, 1] = +n$, $\deg \text{Ext}[f, 1] = +m$ and
 $\deg \Delta = -m-n$.

Theorem 11. If $\mathcal{D}$ is an abelian category then $G\mathcal{D}$
is an abelian graded category, $P \in G\mathcal{D}$ is projective if
and only if each P_r is projective in $\mathcal{D}$, and if each
 B_r has a projective resolution in $\mathcal{D}$ then:

$$\text{Ext}^n[B, B'; G\mathcal{D}]_m = \prod_{r \in \mathbb{Z}} \text{Ext}^n(B_r, B'_{r+m}; \mathcal{D}) \quad .//$$

Theorem 12. If $\mathcal{A}$ is a small graded category and $\mathcal{B}$
is an abelian graded category then $[\mathcal{A}, \mathcal{B}]$ is an abelian
graded category with the sequence $F' \rightarrow F \rightarrow F''$ exact if
and only if the sequences $F'H \rightarrow FH \rightarrow F''H$ are exact in $\mathcal{B}$
for all $H \in \mathcal{A}$.

Proof. By Theorem 7 and its dual $[\mathcal{A}, \mathcal{B}]$ is
finitely complete and cocomplete. Suppose $\alpha: F' \rightarrow F$ is a
monomorphism; then $\ker \alpha = 0$, so, by Theorem 7,
 $\ker \alpha_H = 0$, so α_H is a monomorphism. Let $\beta: F \rightarrow F''$ be

the cokernel of α in $[\mathcal{A}, \mathcal{B}]$; then $\alpha_H = \ker(\text{coker } \beta_H)$ since $\beta_H = \text{coker } \alpha_H$ by Theorem 5 and every monomorphism in $\mathcal{B}$ is the kernel of its cokernel. So by Theorem 5 again, $\alpha = \ker(\text{coker } \beta)$. So every monomorphism is co-normal, and dually every epimorphism is normal. The exactness clause follows from Theorem 5 and the definition of exactness in a graded abelian category. //

A graded abelian category $\mathcal{B}$ will be called projectively perfect if $\mathcal{B}_0$ is projectively perfect (= has enough projectives). A projective resolution of an object of $\mathcal{B}$ will mean a projective resolution of it in $\mathcal{B}_0$.

Theorem 13. If $\mathcal{A}$ is a small graded category, and $\mathcal{B}$ is a cocomplete projectively perfect abelian graded category then $[\mathcal{A}, \mathcal{B}]$ is a cocomplete projectively perfect abelian graded category. The projectives of $[\mathcal{A}, \mathcal{B}]$ are the objects of the form $\sum_{H \in \mathcal{A}} J_H P(H)$, where $P(H)$ is projective in $\mathcal{B}$, and their retracts.

Proof. We apply Theorem 3.1 of [5] Ch. II §3 p 17 to

$J_H \longrightarrow E_H: ([\mathcal{A}, \mathcal{B}]_0, \mathcal{B}_0), H \in \mathcal{A}$; the family $\{J_H\}, H \in \mathcal{A}$ being cointegrable by Theorem 7; the adjunctions existing by Theorem 9. By Theorem 12 the class $\mathcal{E}^\#$ is the class of exact sequences of $[\mathcal{A}, \mathcal{B}]_0$, and the statement that this is a projective class is exactly the statement that

$[A, B]$ is projectively perfect; moreover the projectives of $[A, B]$ are as stated, by the theorem of [5]. //

Theorem 14. If the graded model $\mathcal{I}$ has a finite class of objects and $\mathcal{B}$ is a projectively perfect stable abelian graded category then $[\mathcal{I}, \mathcal{B}]$ is projectively perfect.
The projectives of $[\mathcal{I}, \mathcal{B}]$ are the objects of the form $\sum_{X \in \mathcal{I}} P(X)$, where $P(X)$ is projective in $\mathcal{B}$, and their retracts.

Proof. Again we apply the theorem of [5] Ch. II §3 p 17 to $J_X \rightarrow E_X$. $\mathcal{I}$ has a finite class of objects so $\{J_X\}$ is cointegrable since $[\mathcal{I}, \mathcal{B}]$ is finitely cocomplete. The adjunctions exist this time by Theorem 10. //

Let $\mathcal{I}_3$ denote the graded model with three objects X', X, X'' , and $[X', X] = [X, X''] = L[X'', X'] = [X', X'] = [X, X] = [X'', X''] = \mathbb{Z}$, $[X', X''] = [X, X'] = [X'', X] = 0$.

If $\mathcal{B}$ is a graded category then we will denote the graded category $[\mathcal{I}_3, \mathcal{B}]$ by $\text{Tgl } \mathcal{B}$. An object of $\text{Tgl } \mathcal{B}$ is a triangle:

$$\beta : B' \xrightarrow{b''} B \xrightarrow{b'} B'' \xrightarrow{b} B'$$

in $\mathcal{B}$ where $\deg b = -1$, $\deg b' = \deg b'' = 0$ and $b''b = b'b'' = bb' = 0$. If $\gamma : C' \xrightarrow{c''} C \xrightarrow{c'} C \xrightarrow{c} C'$ is another triangle, then a morphism $\beta \rightarrow \gamma$ of degree n is a triple (f', f, f'') of morphisms $f' : B' \rightarrow C'$, $f : B \rightarrow C$, $f'' : B'' \rightarrow C''$ of degree n such that

$$c''f' = fb'', \quad c'f = f''b' \quad \text{and} \quad cf'' = (-1)^n f'b.$$

The evaluation functors $E' = E_{X'}$, $E = E_X$, $E'' = E_{X''}$ have adjoints $J' = J_{X'}$, $J = J_X$, $J'' = J_{X''}$ given by:

$$J'B : B \xrightarrow{1} B \rightarrow 0 \rightarrow B$$

$$JB : 0 \rightarrow B \xrightarrow{1} B \rightarrow 0$$

$$J''B : L^{-1}B \rightarrow 0 \rightarrow B \xrightarrow{L^{-1}} L^{-1}B$$

when $\mathcal{B}$ is stable.

If $\mathcal{B}$ is abelian a triangle $\beta: B' \xrightarrow{b''} B \xrightarrow{b'} B'' \xrightarrow{b} B'$ $\in \text{Tgl } \mathcal{B}$ will be called exact over $\mathcal{B}$ when the sequence $B' \xrightarrow{b''} B \xrightarrow{b'} B'' \xrightarrow{b} B' \xrightarrow{b''} B$ is exact in $\mathcal{B}$.

Theorem 15. If $\mathcal{B}$ is stable abelian and projectively perfect then so is $\text{Tgl } \mathcal{B}$. The projectives in $\text{Tgl } \mathcal{B}$ are those triangles $\chi: X' \xrightarrow{x''} X \xrightarrow{x'} X'' \xrightarrow{x} X'$ such that:

(a) χ is exact;

(b) X', X, X'' are projective; and

(c) one (and hence all) of $\ker x'$, $\ker(\chi)$, $\ker x''$ is projective.

If $\mathcal{B}$ has finite projective dimension then (c) may be relaxed.

Proof. By Theorem 14 the projectives are retracts of objects of the form $J'P \oplus JQ \oplus J''R$ with P, Q, R projective; but such objects are isomorphic to objects of this form. So

if χ is projective then (a),(b),(c) are satisfied.

Suppose χ satisfies (a),(b),(c). Let $P = \ker x'$,
 $Q = \ker(\chi)$, $R = \ker x''$. From the exact sequences:

$$0 \rightarrow L^{-1}R \rightarrow X' \rightarrow P \rightarrow 0$$

$$0 \rightarrow P \rightarrow X \rightarrow Q \rightarrow 0$$

$$0 \rightarrow Q \rightarrow X'' \rightarrow R \rightarrow 0$$

and (b) we obtain isomorphisms $X' \cong L^{-1}R \oplus P$,

$X \cong P \oplus Q$, $X'' \cong Q \oplus R$ under which the three exact sequences above become:

$$0 \rightarrow L^{-1}R \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} L^{-1}R \oplus P \xrightarrow{(0,1)} P \rightarrow 0, \text{ \&c.}$$

Thus x', x, x'' become the composites:

$$P \oplus Q \xrightarrow{(0,1)} Q \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} Q \oplus R, Q \oplus R \xrightarrow{(0,1)} R \xrightarrow{\chi^{-1}} L^{-1}R \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} L^{-1}R \oplus P,$$

$$L^{-1}R \oplus P \xrightarrow{(0,1)} P \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} P \oplus Q \text{ respectively. So}$$

$\chi \cong J'P \oplus JQ \oplus J''R$ and hence is projective. To prove the last sentence of the theorem, take χ, P, Q, R as before except for condition (c). Then the sequence:

$$0 \rightarrow R \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow LX' \rightarrow \dots \rightarrow L^n X' \rightarrow L^n X \rightarrow L^n X'' \\ \rightarrow L^{n+1} R \rightarrow 0$$

is exact. Choose n such that $3n$ is greater than the projective dimension of β_0 . By the theorem of [14] Ch.VII §6 p 181, the projective dimension of $L^{n+1}R$ being $\leq 3n$ and all the $L^r X', L^r X, L^r X''$ being projective implies R projective. //

Remark. The foregoing results all hold when we replace abelian categories by additive categories with finite limits and colimits, and a projective class of objects (see [5] Ch. I §2 p5).

§5. Triangulated graded categories.

In this section we define Verdier triangulated graded categories and list some of their properties. The protosplit exact sequences of a weakly stable DG-category $\mathcal{A}$ which has the extension axiom are shown to give a Verdier triangulation on $H\mathcal{A}$.

Suppose $\mathcal{A}$ is a weakly stable graded category. A class $\mathcal{J}$ of objects of $\text{Tgl}\mathcal{A}$ will be called a triangulation of $\mathcal{A}$ if:

T0. all isomorphisms in $\text{Tgl}\mathcal{A}$ of objects in $\mathcal{J}$ are also in $\mathcal{J}$.

The triangulation $\mathcal{J}$ will be called a Puppe triangulation if the following axioms are satisfied:

T1. if $A \in \mathcal{A}$ then $0 \rightarrow A \xrightarrow{1} A \rightarrow 0 \in \mathcal{J}$;

T2. if $A' \xrightarrow{a''} A \xrightarrow{a'} A'' \xrightarrow{a} A' \in \mathcal{J}$ then so does $A \xrightarrow{a'} A'' \xrightarrow{a} LA' \xrightarrow{a''\tau^{-1}} A$;

T3. if $a' \in [A, A''; \mathcal{A}]_0$ then there exists $A' \xrightarrow{a''} A \xrightarrow{a'} A'' \xrightarrow{a} A' \in \mathcal{J}$;

T4. if $A' \xrightarrow{a''} A \xrightarrow{a'} A'' \xrightarrow{a} A'$, $C' \xrightarrow{c''} C \xrightarrow{c'} C'' \xrightarrow{c} C' \in \mathcal{J}$ and $f \in [A, C; \mathcal{A}]_m$, $f'' \in [A'', C''; \mathcal{A}]_m$ are such that $f'' \cdot a' = c' \cdot f$ then there exists $f' \in [A', C'; \mathcal{A}]_m$ such that (f', f, f'') is a morphism of $\text{Tgl}\mathcal{A}$ of degree m .

Sometimes we shall regard a triangulation $\mathcal{J}$ as being a full sub-graded-category of $\text{Tgl}\mathcal{A}$.

Theorem 16 (Puppe). Suppose $\mathcal{J}$ is a Puppe triangulation of $\mathcal{A}$.

- (a) If $A' \xrightarrow{a''} A \xrightarrow{a'} A'' \xrightarrow{a} A' \in \mathcal{J}$ and $C \in \mathcal{A}$ then $[C, A'] \xrightarrow{[1, a'']} [C, A] \xrightarrow{[1, a']} [C, A''] \xrightarrow{[1, a]} [C, A']$ and $[A'', C] \xrightarrow{[a', 1]} [A, C] \xrightarrow{[a'', 1]} [A', C] \xrightarrow{[a, 1]} [A'', C]$ are exact triangles over $G\mathcal{A}$.
- (b) If (f', f, f'') is a map of $\mathcal{J}$ and any two of f', f, f'' are isomorphisms, then so is the third.
- (c) $\mathcal{J}$ is closed under finite direct sums in $\text{Tgl } \mathcal{A}$.
- (d) If $\mathcal{J}'$ is another Puppe triangulation of $\mathcal{A}$ with $\mathcal{J}' \subset \mathcal{J}$ then $\mathcal{J}' = \mathcal{J}$.
- (e) If $A' \xrightarrow{a''} A \xrightarrow{a'} A'' \xrightarrow{a} A' \in \mathcal{J}$ then $A' = 0$ is equivalent to a' being an isomorphism.
- (f) If $A' \xrightarrow{a''} A \xrightarrow{a'} A'' \xrightarrow{0} A' \in \mathcal{J}$ then there exist $\tilde{a}': A \rightarrow A', \tilde{a}'': A'' \rightarrow A$ such that $A' \xleftarrow{\tilde{a}'} A \xrightarrow{\tilde{a}''} A'' \xrightarrow{a'} A' \xleftarrow{\tilde{a}'} A \xrightarrow{\tilde{a}''} A''$ is a direct sum situation in $\mathcal{A}$.

Proof. (a) Suppose $C \xrightarrow{f} A'' \xrightarrow{a} A'$ is zero. By T1, $0 \rightarrow C \xrightarrow{1} C \rightarrow 0 \in \mathcal{J}$. By T4 there exists $g: C \rightarrow A$ such that $f \cdot 1 = a' \cdot g$. So the first triangle of (a) in $G\mathcal{A}$ is exact at $[C, A'']$. Using T3 we obtain exactness at $[C, A]$ and $[C, A']$. A similar argument gives exactness of the second triangle.

(b) By T3 it suffices to prove f', f'' isomorphisms implies f is. Taking $[C, -]$ of the diagram:

$$\begin{array}{ccccccc}
 L^{-1} & A'' & \xrightarrow{a} & A' & \xrightarrow{a''} & A & \xrightarrow{a'} & A'' & \xrightarrow{\lambda \cdot a} & LA' \\
 L^{-1}f'' \downarrow & & & f' \downarrow & & f \downarrow & & f'' \downarrow & & Lf' \\
 L^{-1}C'' & \xrightarrow{c} & C' & \xrightarrow{c''} & C & \xrightarrow{c'} & C'' & \xrightarrow{\lambda \cdot c} & LC' ,
 \end{array}$$

using (a), and applying the "five lemma", we deduce that $[1, f]: [C, A] \rightarrow [C, C]$ is an isomorphism of some degree and a fortiori surjective, so $fg = 1$ for some g . Taking $[-, A]$ we similarly deduce $hf = 1$ for some h . So f is an isomorphism.

(c) Suppose $A' \xrightarrow{a''} A \xrightarrow{a'} A'' \xrightarrow{a} A'$,
 $C' \xrightarrow{c''} C \xrightarrow{c'} C'' \xrightarrow{c} C'$ have a direct sum in $Tgl A$.
 Then we have $a' \oplus c': A \oplus C \rightarrow A'' \oplus C''$, and so by T3 there exists a triangle $B \rightarrow A \oplus C \xrightarrow{a' \oplus c'} A'' \oplus C'' \rightarrow B \in \mathcal{J}$. By T4 there exist f, g such that the following diagram commutes:

$$\begin{array}{ccccccc}
 A' & \xrightarrow{a''} & A & \xrightarrow{a'} & A'' & \xrightarrow{a} & A' \\
 f \downarrow & & \downarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} & & \downarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} & & \downarrow f \\
 B & \xrightarrow{\quad} & A \oplus C & \xrightarrow{a' \oplus c'} & A'' \oplus C'' & \xrightarrow{\quad} & B \\
 g \uparrow & & \uparrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} & & \uparrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} & & \uparrow g \\
 C' & \xrightarrow{c''} & C & \xrightarrow{c'} & C'' & \xrightarrow{c} & C'
 \end{array}$$

This means we have a commutative diagram:

$$\begin{array}{ccccccc}
 A' \oplus C' & \xrightarrow{a'' \oplus c''} & A \oplus C & \xrightarrow{a' \oplus c'} & A'' \oplus C'' & \xrightarrow{a \oplus c} & A' \oplus C' \\
 (f, g) \downarrow & & \downarrow 1 & & \downarrow 1 & & \downarrow (f, g) \\
 B & \xrightarrow{\quad} & A \oplus C & \xrightarrow{a' \oplus c'} & A'' \oplus C'' & \xrightarrow{\quad} & B
 \end{array}$$

Observe that in the proof of (b) we needed only that the

triangles should have the properties in the conclusion of (a). Thus (f, g) is an isomorphism. The result follows by T0.

(d) Suppose $A' \xrightarrow{a''} A \xrightarrow{a'} A'' \xrightarrow{a} A' \in \mathcal{J}$. By T3 for $\mathcal{J}'$ there exists

$C' \xrightarrow{c''} A \xrightarrow{a'} A'' \xrightarrow{c} C' \in \mathcal{J}' \subset \mathcal{J}$. By T4 for $\mathcal{J}$ there exists f' such that $(f', 1, 1)$ is a map from the first triangle to the second. Then, by (b), f' is an isomorphism. So by T0 for $\mathcal{J}'$, the triangle

$$A' \xrightarrow{a''} A \xrightarrow{a'} A'' \xrightarrow{a} A'$$

is in $\mathcal{J}'$.

(e) If $A' = 0$ then, by (a), $[C, a'], [a', C]$ are isomorphisms for all C ; so a' is an isomorphism. Conversely, suppose a' is an isomorphism then, by (a), $[C, A'] = [A', C] = 0$ for all C ; so $A' = 0$.

(f) By (a), for all C , the sequence:

$$0 \rightarrow [C, A'] \xrightarrow{[1, a'']} [C, A] \xrightarrow{[1, a']} [C, A''] \rightarrow 0$$

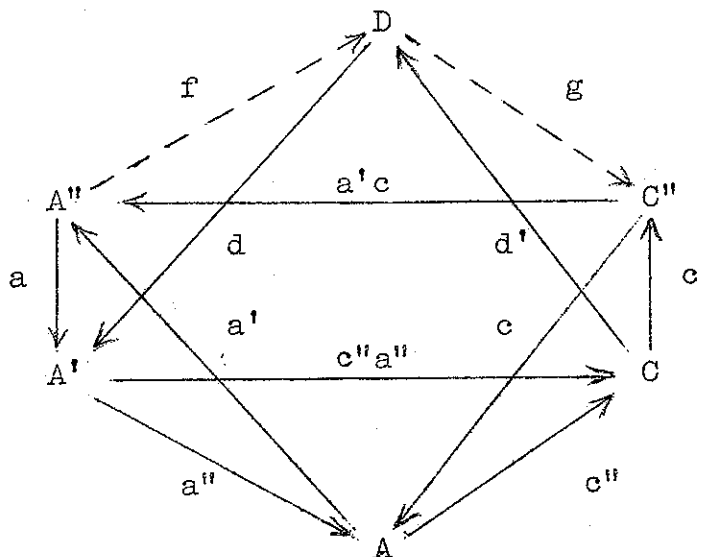
is exact. Taking $C = A''$ gives $[A'', a']$ surjective, and so there exists $\tilde{a}''$ such that $a' \tilde{a}'' = 1$. Then

$[A, a'](1 - \tilde{a}'' a') = 0$, and taking $C = A$ we find $\tilde{a}'$ such that $a'' \tilde{a}' = 1 - \tilde{a}'' a'$. Then $[A', a''](1 - \tilde{a}' a'') = (1 - a'' \tilde{a}') a'' = \tilde{a}'' a' a'' = \tilde{a}'' \cdot 0 = 0$. Taking $C = A'$ we thus have

$$1 - \tilde{a}' a'' = 0. //$$

A triangulation $\mathcal{T}$ of $\mathcal{A}$ will be called a Verdier triangulation if it is a Puppe triangulation satisfying the further "octohedral axiom":

T5. in the diagram:



excluding the dotted maps, if $A' \xrightarrow{a''} A \xrightarrow{a'} A'' \xrightarrow{a} A'$,
 $A \xrightarrow{c''} C \xrightarrow{c'} C'' \xrightarrow{c} A$, $A' \xrightarrow{c''a''} C \xrightarrow{d'} D \xrightarrow{d} A' \in \mathcal{T}$
 then there exist f, g as shown such that
 $A'' \xrightarrow{f} D \xrightarrow{g} C'' \xrightarrow{a'c} A'' \in \mathcal{T}$ and $d.f = a$, $gd' = c'$,
 $fa' = d'c''$, $cg = a''d$.

The octohedral axiom appearing in [6] Ch.I §1 p 21 does not require $fa' = d'c''$, $cg = a''d$.

Theorem 17. Suppose $\mathcal{A}$ is a DG-category which is weakly stable and has the extension axiom. Let $\mathcal{T}$ be the class of isomorphs in $\text{TglH}\mathcal{A}$ of triangles

$A' \xrightarrow{a''} A \xrightarrow{a'} A'' \xrightarrow{a} A'$ over $H\mathcal{A}$ such that there exists a pses $A' \xrightarrow{i} A \xrightarrow{p} A''$ in $\mathcal{A}$ with deviation class a and $[i] = a''$, $[p] = a'$. Then $\mathcal{J}$ is a Verdier triangulation of $H\mathcal{A}$.

Proof. Axiom T0 is satisfied. Each $0 \rightarrow A' \xrightarrow{1} A'$ is a pses in $\mathcal{A}$, so T1 is satisfied. We now prove T2. Suppose $A' \xrightarrow{i} A \xrightarrow{p} A''$ is pses in $\mathcal{A}$ with splitting $(\tilde{p}, \tilde{i})$ giving deviation δ . Then $i. \ell^{-1} \in Z_{-1}[LA', A]$ and so by the extension axiom there exists a pses $A \xrightarrow{j} B \xrightarrow{q} LA'$ with splitting $(\tilde{q}, \tilde{j})$ giving deviation $i. \ell^{-1}$. We seek an homotopy equivalence f such that the diagram:

$$\begin{array}{ccccccc}
 A' & \xrightarrow{i} & A & \xrightarrow{p} & A'' & \xrightarrow{i.\delta} & LA' & \xrightarrow{Li} & LA \\
 \downarrow 1 & \swarrow \tilde{p} & \downarrow 1 & \swarrow \tilde{i} & \downarrow f & \swarrow -1 & \downarrow -1 & \swarrow L\tilde{p} & \downarrow -1 \\
 A' & \xrightarrow{i} & A & \xrightarrow{j} & B & \xrightarrow{q} & LA' & \xrightarrow{Li} & LA \\
 & & & \tilde{q} & & \tilde{j} & & &
 \end{array}$$

(excluding the dotted maps) is commutative to within homotopy (i.e. induces a commutative diagram in $H\mathcal{A}$, with f inducing an isomorphism). First recall from §2 that

$$D\tilde{i} = i.\delta, \quad D\tilde{j} = j.i.\ell^{-1},$$

$$D\tilde{p} = -\delta.p, \quad D\tilde{q} = i.\ell^{-1}.q.$$

Let $f = j\tilde{i} - \tilde{j}\ell.\delta$. Then

$$\begin{aligned}
Df &= j.D\tilde{f} - D\tilde{j}.\ell\delta \\
&= j\iota\delta - j\iota\ell^{-1}\ell\delta \\
&= 0 ; \\
qf &= -\ell\delta ; \\
fp &= j\tilde{f}p - \tilde{j}\ell\delta p \\
&= j(1-i\ell^{-1}\ell\tilde{p}) + \tilde{j}\ell.D\tilde{p} \\
&= j + \tilde{j}\ell.D\tilde{p} - D\tilde{j}.\ell\tilde{p} \\
&= j - D(\tilde{j}\ell\tilde{p}) .
\end{aligned}$$

It remains to show that f is an homotopy equivalence.

Let $g = p\tilde{q}$. Then

$$\begin{aligned}
Dg &= p.D\tilde{q} \\
&= -p\iota\ell^{-1}q \\
&= 0 ; \\
gf &= p\tilde{q}(j\tilde{f} - \tilde{j}\ell\delta) \\
&= p\tilde{f} \\
&= 1 ; \\
fg &= f p \tilde{q} \\
&= (j - D(\tilde{j}\ell\tilde{p}))\tilde{q} \\
&= 1 - \tilde{j}q - D(\tilde{j}\ell\tilde{p}\tilde{q}) - \tilde{j}\ell\tilde{p}.D\tilde{q} \\
&= 1 - \tilde{j}q - D(\tilde{j}\ell\tilde{p}\tilde{q}) + \tilde{j}\ell\tilde{p}\iota\ell^{-1}q \\
&= 1 - D(\tilde{j}\ell\tilde{p}\tilde{q}) .
\end{aligned}$$

Thus $A \xrightarrow{[p]} A' \xrightarrow{[\ell\delta]} LA' \xrightarrow{[1].\ell^{-1}} A$, being isomorphic to a triangle of $\mathcal{J}$, is in $\mathcal{J}$. So T2 holds.

Suppose $a' \in [A, A''; H_0 \mathcal{A}]_0 = H_0[A, A'']$. Then $\ell^{-1} a' \in H_{-1}[A, L^{-1} A'']$; choose $\delta \in \ell^{-1} \cdot a'$, so $\delta \in Z_{-1}[A, L^{-1} A'']$. Since $\mathcal{A}$ has extensions, there exists a pses $L^{-1} A'' \xrightarrow{i} A' \xrightarrow{p} A$ in $\mathcal{A}$ with deviation δ for some splitting. So $L^{-1} A'' \xrightarrow{[i]} A' \xrightarrow{[p]} A \xrightarrow{\ell^{-1} a'} LA'' \in \mathcal{J}$. Now applying T2 we see that T3 holds. Theorem 2(b) and T2 imply T4.

It remains to prove T5, which we shall do in the presence of T2. Suppose $A \xrightarrow{i} A'' \xrightarrow{p} LA'$, $A \xrightarrow{j} C \xrightarrow{q} C''$, $C \xrightarrow{k} D \xrightarrow{r} LA'$ are pses with splittings $(\tilde{p}, \tilde{i}), (\tilde{q}, \tilde{j}), (\tilde{r}, \tilde{k})$ giving deviations $\delta_A, \delta_C, \delta_D$ respectively, with $\delta_D = j\delta_A + Ds$. The three triangles of T5 come from T2 and the notation $a' = [i]$, $a = \ell^{-1} \cdot [p]$, $a'' = [\delta_A]$, $c'' = [j]$, $c' = [q]$, $d' = [k]$, $d = \ell^{-1} [r]$. Put $\gamma = q\tilde{r} + qsr$, and $\phi = kj\tilde{p} + \tilde{k}p - ksp$. Then $D\gamma = 0$, $D\phi = 0$, $\gamma k = q$, $\phi i = kj$, $r\phi = p$ and $\gamma\phi = 0$. So from the commutative diagram:

$$\begin{array}{ccccc}
 A & \xrightarrow{j} & C & \xrightarrow{q} & C'' \\
 i \downarrow & & k \downarrow & & \parallel \\
 A'' & \xrightarrow{\phi} & D & \xrightarrow{\gamma} & C'' \\
 p \downarrow & & r \downarrow & & \downarrow \\
 LA' & \xlongequal{\quad} & LA' & \longrightarrow & 0
 \end{array}$$

and Proposition 1, it follows that $A'' \xrightarrow{\phi} D \xrightarrow{\gamma} C''$ is a pses in $\mathcal{A}$ with deviation class $[i\delta_C]$ by Theorem 2(b). Moreover,

$$\begin{aligned}
\delta_c \gamma &= \delta_c (q\tilde{r} + qsr) \\
&= D\tilde{q}.\tilde{r} + D\tilde{q}.sr \\
&= -\tilde{q}.D\tilde{r} - \tilde{q}.Ds.r \\
&= \tilde{q}.\delta_D.r - \tilde{q}.(\delta_D - j\delta_A).r \\
&= \tilde{q}.j\delta_A.r \\
&= \delta_A r .
\end{aligned}$$

Put $f = [\phi]$, $g = [\gamma]$. Axiom T5 follows. //

CHAPTER 2. - GENERALITIES

§1. A general classification theorem.

Suppose $\mathcal{A}$ is a graded category with a Puppe triangulation $\mathcal{J}$, $\mathcal{B}$ is a stable abelian graded category, and $T: \mathcal{A} \rightarrow \mathcal{B}$ is a graded functor. In this section we develop a type of homological algebra in the category $\mathcal{A}$ and derive a classification theorem for the functor T . If $\mathcal{A}$ in a sense has enough projectives then the projectives with which we deal are shown to be those objects A with $\dim_T A = 0$ as defined in [16].

An object $X \in \mathcal{A}$ will be called T-projective if TX is projective in $\mathcal{B}$ and the map

$$T: [X, \mathcal{A}; \mathcal{A}] \rightarrow [TX, \mathcal{B}; \mathcal{B}]$$

is an isomorphism for all $A \in \mathcal{A}$.

A triangle $\alpha: A' \xrightarrow{a''} A \xrightarrow{a'} A'' \xrightarrow{a} A'$ over will be called T-simple if $\alpha \in \mathcal{J}$ and the sequence

$$0 \rightarrow TA' \xrightarrow{Ta''} TA \xrightarrow{Ta'} TA'' \rightarrow 0$$

is exact in $\mathcal{B}$.

Theorem 18. If $\alpha: A' \xrightarrow{a''} X \xrightarrow{a'} A'' \xrightarrow{a} A' \in \mathcal{J}$ with X T-projective, and $\gamma: C' \xrightarrow{c''} C \xrightarrow{c'} C'' \xrightarrow{c} C'$ is T-simple, then any $f'' \in [A'', C'']_n$ extends to $(f', f, f'') \in [\alpha, \gamma]_n$. If also $(g', g, f'') \in [\alpha, \gamma]_n$ then there exist $s', s \in [X, C']_n$ such that

$$g' - f' = s'a'', \quad g - f = c''s.$$

Proof. Since TX is projective and Tc' is an epimorphism, $T(f''a'): TX \rightarrow TC''$ lifts to $h: TX \rightarrow TC$ with

$T(f''a') = Tc'.h$. Since $T:[X,C] \rightarrow [TX,TC]$ is an isomorphism, $h = Tf$ for some f , and $f''.a' = c'.f$. By T4 we obtain $(f',f,f'') \in [\alpha,\gamma]_n$ as required.

If $(f',f,0) \in [\alpha,\gamma]_n$ then $c'.f = 0$ and $f'a = 0$. By Theorem 16 (a) it follows that $f = c''s$ and $f' = s'a''$ for some s,s' . //

An object $A \in \mathcal{A}$ will be said to be T-developable if there exists a T-simple triangle $\alpha: A' \xrightarrow{a} X \xrightarrow{a'} A \xrightarrow{x} A'$ with X T-projective; α will be called a T-development of A .

Suppose $\alpha: A' \xrightarrow{a} X \xrightarrow{a'} A \xrightarrow{x} A'$ is a T-development of A . Define graded abelian groups $M_T[A,C], M_T^*[A,C]$ by the exact sequence:

$$0 \rightarrow M_T[A,C] \xrightarrow{[a,1]} [X,C] \xrightarrow{[a',1]} [A',C] \rightarrow M_T^*[A,C] \rightarrow 0.$$

Suppose $\bar{\alpha}: \bar{A}' \xrightarrow{\bar{a}} \bar{X} \xrightarrow{\bar{a}'} \bar{A} \xrightarrow{\bar{x}} \bar{A}'$ is a T-development of $\bar{A}$; define $M_T[\bar{A},C], M_T^*[\bar{A},C]$ correspondingly. For $f \in [A,\bar{A}]_m$ there exists $(g',g,f) \in [\alpha,\bar{\alpha}]_m$ by Theorem 18. Then the

square:

$$\begin{array}{ccc} [\bar{X},C] & \xrightarrow{[\bar{a},1]} & [\bar{A}',C] \\ [g,1] \downarrow & & \downarrow [g',1] \\ [X,C] & \xrightarrow{[a,1]} & [A',C] \end{array}$$

commutes. So maps

$$M_T[f,1]: M_T[\bar{A},C] \rightarrow M_T[A,C],$$

$$M_T^*[f,1]: M_T^*[\bar{A},C] \rightarrow M_T^*[A,C]$$

are induced. These maps are independent of the choice of

g', g . For suppose $(h', h, f) \in [\alpha, \bar{\alpha}]_m$. By Theorem 18 there exist s, s' such that $h' - g' = s'a$, $h - g = \bar{a}s$. If $u \in [\bar{X}, C]_n$ and $[\bar{a}, 1]u = u\bar{a} = 0$ then $[h, 1]u = (-1)^{mn}uh = (-1)^{mn}u(g + \bar{a}s) = (-1)^{mn}ug = [g, 1]u$; so $[M_T f, 1]$ is independent of the choice of g', g . If $v \in [\bar{A}', C]_n$ then $[h', 1]v = (-1)^{mn}vh' = (-1)^{mn}(vg' + vs'a) = [g', 1]v + [a, 1]t$ for some t ; so $M_T^*[f, 1]$ is independent of the choice of g', g .

It follows that $M_T[A, C]$, $M_T^*[A, C]$ are independent of the T-development of A to within isomorphism. Let A_T denote the full sub-graded-category of A with objects the T-developable objects of A . Then we have graded functors

$$M_T, M_T^* : A_T^* \otimes A \rightarrow Gg$$

determined uniquely up to natural isomorphism.

Theorem 19. The sequence:

$$0 \rightarrow M_T^* \xrightarrow{U} \text{Hom}_{A_T} \xrightarrow{V} M_T \rightarrow 0$$

is exact in $[A_T^* \otimes A, Gg]$ with $\deg U = -1$ and
 $\deg V = 0$ where, if $A \in A_T$, $C \in A$ and

$A' \xrightarrow{a} X \xrightarrow{a'} A \xrightarrow{x} A''$ is a T-development of A , then
 $U_{A,C}$ is induced by $[x, C]$ and $V_{A,C}$ by $[a', C]$.

Proof. In the notation of the theorem, Theorem 16(a) gives the exact sequence:

$$[X, C] \xrightarrow{[a, 1]} [A', C] \xrightarrow{[x, 1]} [A, C] \xrightarrow{[a', 1]} [X, C] \xrightarrow{[a, 1]} [A', C].$$

The result now follows from the definitions of $M_T[A, C]$ and $M_T^*[A, C]$. //

For any object $A \in \mathcal{A}$ and non-negative integer r we define inductively what will be meant by $T\text{-dim } A \leq r$. Firstly, $T\text{-dim } A = 0$ will mean A is T -projective. For $r > 0$, $T\text{-dim } A \leq r$ will mean there exists a T -development $A' \xrightarrow{a} X \xrightarrow{a'} A \xrightarrow{x} A'$ of A with $T\text{-dim } A' \leq r-1$. We write $T\text{-dim } A = r$ if $T\text{-dim } A \leq r$ but $T\text{-dim } A \not\leq r-1$.

Theorem 20. If $T\text{-dim } A \leq 1$, there are isomorphisms:

$$W_{A,C}: M_T[A,C] \cong [TA, TC; \mathcal{B}] ,$$

$$W_{A,C}^*: M_T^*[A,C] = \text{Ext}^1[TA, TC; \mathcal{B}] .$$

These isomorphisms are natural in A with $T\text{-dim } A \leq 1$ and in C . Moreover, $T = WV$.

Proof. Suppose $T\text{-dim } A \leq 1$. Then there exists a T -development $X' \xrightarrow{a} X \xrightarrow{x'} A \xrightarrow{x} X'$ with X' T -projective. The exact sequence $0 \rightarrow TX' \xrightarrow{Ta} TX \xrightarrow{Tx'} TA \rightarrow 0$ gives the exact lower row of the commutative diagram:

$$\begin{array}{ccccccc} 0 \rightarrow M_T[A,C] \rightarrow [X,C] & \xrightarrow{[a,1]} & [X',C] \rightarrow M_T^*[A,C] \rightarrow 0 \\ & T \downarrow & & T \downarrow & & & \\ 0 \rightarrow [TA, TC] & \xrightarrow{[Tx', 1]} & [TX, TC] & \xrightarrow{[Ta, 1]} & [TX', TC] \rightarrow \text{Ext}^1[TA, TA'] \rightarrow 0 \end{array}$$

Since X, X' are T -projectives, the columns are isomorphisms; so we obtain isomorphisms W, W^* as required. //

Corollary 21. If $T\text{-dim } A \leq 1$ there is a short exact sequence

$$0 \rightarrow \text{Ext}^1[\text{TA}, \text{TC}; \mathcal{B}] \xrightarrow{R} [\text{A}, \text{C}; \mathcal{A}] \xrightarrow{T} [\text{TA}, \text{TC}; \mathcal{B}] \rightarrow 0$$

of graded abelian groups, where $\deg R = -1$. The sequence is natural in $\mathcal{A}$ with $\text{T-dim } \mathcal{A} \leq 1$, and in $\mathcal{C}$. //

Corollary 22. If $f \in [\text{A}', \text{A}; \mathcal{A}]_p$, $g \in [\text{A}, \text{C}; \mathcal{A}]_q$ where $\text{T-dim } \mathcal{A}, \mathcal{A}' \leq 1$ and $\text{T}f = 0$, $\text{T}g = 0$, then $gf = 0$.

Proof. Since $\text{T}f = 0$ there exists $\phi \in \text{Ext}^1[\text{TA}', \text{TA}]_{p+1}$ with $R\phi = f$ by Corollary 21. By the naturality of R , the following square commutes up to a factor of $(-1)^q$.

$$\begin{array}{ccc} \text{Ext}^1[\text{TA}', \text{TA}] & \xrightarrow{R} & [\text{A}', \text{A}] \\ \text{Ext}[1, \text{T}g] \downarrow & & \downarrow [1, g] \\ \text{Ext}^1[\text{TA}', \text{TC}] & \xrightarrow{R} & [\text{A}', \text{C}] \end{array}$$

But $\text{T}g = 0$, so evaluating the square at ϕ we obtain the result. //

Now we investigate how our definitions compare with those of [16] §3 p 850. If $\mathcal{A} = \text{H}\mathcal{C}$ for some DG-category $\mathcal{C}$ with the triangulation of Theorem 17, then clearly $\text{T-dim } \mathcal{A} \leq r$ implies $\dim_{\text{T}} \mathcal{A} \leq r$ in the sense of Kelly. But even in our general situation we can prove a theorem corresponding to Theorem 3 of [16]; Corollary 22 is a particular case.

Theorem 23. In the diagram

$$\text{A}_0 \xrightarrow{f_0} \text{A}_1 \xrightarrow{f_1} \dots \xrightarrow{f_r} \text{A}_{r+1},$$

let $f_i \in [\text{A}_i, \text{A}_{i+1}; \mathcal{A}]_{p_i}$ be such that $\text{T}f_i = 0$.

If $T\text{-dim } A_0 \leq r$ then $f_r f_{r-1} \dots f_0 = 0$.

Proof. For $r = 0$, $T: [A_0, A_1] \rightarrow [TA_0, TA_1]$ is an isomorphism and a fortiori injective. We prove the theorem by induction. Suppose the theorem true for $r-1$ where $r > 0$. If $T\text{-dim } A_0 \leq r$ then there is a T-development $A' \xrightarrow{a} X \xrightarrow{a'} A_0 \xrightarrow{x} A'$ with $T\text{-dim } A' \leq r-1$. Now $Tf_0 = 0$ and X is T-projective, so the map $[X, f_0]: [X, A_0] \rightarrow [X, A_1]$ is zero. But the following diagram commutes:

$$\begin{array}{ccccc} [A', A_0] & \longrightarrow & [A_0, A_0] & \longrightarrow & [X, A_0] \\ (-1)^{p_0} [1, f_0] \downarrow & & [1, f_0] \downarrow & & 0 \downarrow \\ [A', A_1]_{[x, 1]} & \longrightarrow & [A_0, A_1] & \longrightarrow & [X, A_1] \end{array}$$

with exact rows. So there exists $g \in [A', A_1]_{p_0+1}$ with $[1, f_0](1) = [x, 1](g)$; that is, $f_0 = (-1)^{p_0+1} g x$.

Let $f'_1 = (-1)^{p_0+1} f_1 g \in [A', A_2]_{p_0+p_2+1}$. Then $Tf'_1 = 0$

since $Tf_1 = 0$. But $T\text{-dim } A' \leq r-1$ implies

$f_r f_{r-1} \dots f_2 f'_1 = 0$. So

$$\begin{aligned} f_r f_{r-1} \dots f_2 f_1 f_0 &= f_r f_{r-1} \dots f_2 f'_1 x \\ &= 0 \cdot x \\ &= 0 \quad // \end{aligned}$$

Lemma 24. If $\alpha: A' \xrightarrow{a''} A \xrightarrow{a'} A'' \xrightarrow{a} A'$ is T-simple and $T: [A'', A'] \rightarrow [TA'', TA']$ is injective, then $\alpha \cong J'A' \oplus JA''$.

Proof. The following diagram commutes:

$$\begin{array}{ccc}
 [A'', A'] & \xrightarrow{T_{A''A'}} & [TA'', TA'] \\
 [a', 1] \downarrow & & \downarrow [Ta', 1] \\
 [A, A'] & \xrightarrow{T_{AA'}} & [TA, TA']
 \end{array}$$

Now Ta' is an epimorphism so $[Ta', 1]$ is injective; also $T_{A''A'}$ is injective. So $[a', A']$ is injective. But $[a', A']a = aa' = 0$, so $a = 0$. The result now follows from Theorem 16(f). //

Lemma 25. If $X \oplus A$ is T -projective then so is X .

Proof. Since T is a graded functor $T(X \oplus A) = TX \oplus TA$ and $T: [X \oplus A, C] \rightarrow [T(X \oplus A), C]$ is

$T \oplus T: [X, C] \oplus [A, C] \rightarrow [TX, TC] \oplus [TA, TC]$. So $X \oplus A$ T -projective implies TX projective and $T: [X, C] \rightarrow [TX, TC]$ an isomorphism. //

The meaning given to $\dim_T X = 0$ in [16] is that $T: [X, A] \rightarrow [TX, TA]$ should be injective for all A .

Combining Lemmas 24 and 25 we have the following:

Theorem 26. If X is T -developable then it is T -projective if and only if $T: [X, A] \rightarrow [TX, TA]$ is injective for all $A \in \mathcal{A}$. //

The object $A = A_0 \in \mathcal{A}$ will be called T -resoluble if, for each integer $n \geq 0$, there exists a T -development

$$A_{n+1} \xrightarrow{a_n} X_n \xrightarrow{a'_n} A_n \xrightarrow{x_n} A_{n+1} \text{ of } A_n.$$

Note that $T\text{-dim } A \leq r$ for some integer $r \geq 0$ certainly implies that A is T -resoluble. If A is T -resoluble and $T\text{-dim } A$ is not less than any positive integer r then we write $T\text{-dim } A = \infty$.

Remark. Suppose $A = A_0$ is T -resoluble, and choose T -developments $A_{n+1} \xrightarrow{a_n} X_n \xrightarrow{a'_n} A_n \xrightarrow{x_n} A_{n+1}$ for each $n \geq 0$. For $C \in \mathcal{A}$ this gives a Massey exact couple:

$$\begin{array}{ccc}
 [A_{n+1}, C] & \xrightarrow{[x, 1]} & [A_n, C] \\
 & \nwarrow [a, 1] \quad \nearrow [a', 1] & \\
 & [X_n, C] &
 \end{array}$$

We shall examine the spectral sequence of this exact couple for a special case later.

§2. Homological functors.

The notion of an homological functor is defined here. An analogue of the dimension theorem of homological algebra (see [1] Ch. VI §2 p.111, or [18] Ch. VII §6 p.181) is proved. Suppose the triangulation of $\mathcal{A}$ comes from a DG-category $\mathcal{C}$ with $\mathcal{A} = H\mathcal{C}$ (see Theorem 17). Then $T: \mathcal{A} \rightarrow \mathcal{B}$ gives a functor $T^1: \text{HPses } \mathcal{C} \rightarrow \text{Tgl } \mathcal{B}$. In this situation T^1 -projectivity, T^1 -developability and T^1 -dimension are examined.

Suppose $\mathcal{A}, \mathcal{I}, \mathcal{B}$ are as in §1. A graded functor $T: \mathcal{A} \rightarrow \mathcal{B}$ will be called homological if

HF1. T is stable;

HF2. if $A' \xrightarrow{a''} A \xrightarrow{a'} A'' \xrightarrow{a} A' \in \mathcal{I}$ then the triangle

$$\begin{array}{ccccccc} & & Ta'' & & Ta' & & Ta \\ TA' & \longrightarrow & TA & \longrightarrow & TA'' & \longrightarrow & TA' \end{array}$$

is exact over $\mathcal{B}$.

For $\alpha: A' \xrightarrow{a''} A \xrightarrow{a'} A'' \xrightarrow{a} A' \in \mathcal{I}$ we shall write $T\alpha$ for the triangle $TA' \xrightarrow{Ta''} TA \xrightarrow{Ta'} TA'' \xrightarrow{Ta} TA'$ over $\mathcal{B}$.

It will be tacitly assumed in this section that $T: \mathcal{A} \rightarrow \mathcal{B}$ is an homological functor and $\mathcal{I}$ is a Verdier triangulation.

A morphism f of $\mathcal{A}$ will be called a T -epimorphism when Tf is an epimorphism in $\mathcal{B}$.

Lemma 27 (a). Suppose $\alpha: A' \xrightarrow{a''} A \xrightarrow{a'} A'' \xrightarrow{a} A' \in \mathcal{I}$. Then α is T -simple $\iff a'$ is a T -epimorphism
 $\iff Ta = 0$.

(b) A is T -developable $\iff$ there exists a T -epimorphism

$\epsilon: X \rightarrow A$ with X T-projective.

Proof. (a) is trivial and (b) follows from (a) and T3. //

Lemma 28. If $A' \xrightarrow{a} X \xrightarrow{a'} A \xrightarrow{x} A'$,
 $C' \xrightarrow{c} Y \xrightarrow{c'} A \xrightarrow{y} C'$ are T-developments of A then

$$X \oplus C' \cong Y \oplus A'.$$

Proof. Since X is T-projective,
 $T(ya') = Ty.Ta' = 0.Ta' = 0$ implies $ya' = 0$; similarly
 $xc' = 0$. Applying T5 to the three triangles

$$\begin{array}{c} L^{-1}X \xrightarrow{L^{-1}a'} L^{-1}A \xrightarrow{x\ell} A' \xrightarrow{\ell^{-1}a} L^{-1}X, \\ L^{-1}A \xrightarrow{y\ell} C' \xrightarrow{-c} Y \xrightarrow{\ell^{-1}c'} L^{-1}A, \\ L^{-1}X \xrightarrow{0} C' \xrightarrow{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} X \oplus C' \xrightarrow{(\ell^{-1}, 0)} L^{-1}X \end{array}$$

of $\mathcal{J}$ we obtain a triangle $A' \rightarrow X \oplus C' \rightarrow Y \xrightarrow{xc'} A'$ in $\mathcal{J}$.
 But $xc' = 0$, so the result follows from Theorem 16(f). //

Lemma 29. Suppose X is T-projective and A is T-resoluble. Then $T\text{-dim } A = T\text{-dim } X \oplus A$.

Proof. For $r = 0$ this follows from Lemma 25.
 Suppose $r > 0$. Since A is T-resoluble there is a
 T-development $A' \xrightarrow{a} Y \xrightarrow{a'} A \xrightarrow{y} A'$ of A with A'
 T-resoluble. By T1 and Theorem 16(c),

$$A' \xrightarrow{\begin{pmatrix} 0 \\ a \end{pmatrix}} X \oplus Y \xrightarrow{1 \oplus a'} X \oplus A \xrightarrow{(0y)} A' \text{ is then a T-development}$$

of $X \oplus A$. So $X \oplus A$ is T -resoluble.

If $T\text{-dim } A \leq r$ then we may suppose $T\text{-dim } A' \leq r-1$, and so by the above, $T\text{-dim } X \oplus A \leq r$.

We prove that $T\text{-dim } X \oplus A \leq r$ implies $T\text{-dim } A \leq r$ by induction. If $T\text{-dim } X \oplus A \leq r$ then there exists a T -development $C \rightarrow Z \rightarrow X \oplus A \rightarrow C$ of $X \oplus A$ with $T\text{-dim } C \leq r-1$ and so C is T -resoluble. By Lemma 28 we have $X \oplus Y \oplus C \cong Z \oplus A'$. By the above $T\text{-dim } X \oplus Y \oplus C \leq r-1$, so $T\text{-dim } Z \oplus A' \leq r-1$. But Z is T -projective and A' is T -resoluble, so by induction $T\text{-dim } A' \leq r-1$. Thus $T\text{-dim } A \leq r$. //

Theorem 30 (Dimension), Suppose $A' \rightarrow X \rightarrow A \rightarrow A'$ is a T -development of A with A' T -resoluble. If A is T -projective then so is A' ; otherwise

$$(T\text{-dim } A') = (T\text{-dim } A) - 1.$$

Proof. If A is T -projective then $X \cong A' \oplus A$ by Lemma 24; but X is T -projective so A' is T -projective. Suppose $T\text{-dim } A = r > 0$. Then there exists a T -development $C' \rightarrow Y \rightarrow A \rightarrow C'$ of A with $T\text{-dim } C' = r-1$. By Lemma 28, $X \oplus C' \cong Y \oplus A'$. By Lemma 29 $T\text{-dim } X \oplus C' = r-1$ and so $T\text{-dim } Y \oplus A' = r-1$. Again by Lemma 29, this implies $T\text{-dim } A' = r-1$. //

Theorem 31. Suppose $X: X' \xrightarrow{x''} X \xrightarrow{x'} X'' \xrightarrow{x} X'$ is in $\mathcal{J}$ and either of the following conditions is satisfied.

(a) $\text{im } Tx'' \cong TC$ for some $C \in \mathcal{A}$; or

(b) $\mathcal{B}$ has finite projective dimension.

Then X', X, X'' T-projective imply TX is projective in
 $\text{Tgl } \mathcal{B}$.

Proof. Under hypothesis (b) the result follows from
Theorem 15.

Suppose (a) holds. The sequence

$$[X, C] \xrightarrow{[x'', 1]} [X', C] \xrightarrow{[x, 1]} [X'', C]$$

is exact by Theorem 16(a), and X', X, X'' are T-projective;
so the sequence

$$[TX, TC] \xrightarrow{[Tx'', 1]} [TX', TC] \xrightarrow{[Tx, 1]} [TX'', TC]$$

is exact. But Tx'' factors $TX' \xrightarrow{e} TC \xrightarrow{m} TX$ with e
epimorphic, m monomorphic. Then

$$m.e.Tx = Tx''.Tx = T(x''x) = 0,$$

so $e.Tx = 0$. That is $e \in \ker[Tx, 1] = \text{im}[Tx'', 1]$, so
 $e = e'$. $Tx'' = e'.m.e$ for some e' . Thus $e'm = 1$ and
 TC is a retract of TX . So $\text{im } Tx'' = \ker Tx'$ is projective
in $\mathcal{B}$. The result again follows by Theorem 9. //

Theorem 32. If $\pi: P' \xrightarrow{p''} P \xrightarrow{p'} P'' \xrightarrow{p} P'$,
 $\beta: B' \xrightarrow{b''} B \xrightarrow{b'} B'' \xrightarrow{b} B' \in \text{Tgl } \mathcal{B}$ with π projective in
 $\text{Tgl } \mathcal{B}$ and β exact over $\mathcal{B}$, then the triangle

$$[P'', B'] \xrightarrow{u} [\pi, \beta] \xrightarrow{v} [P', B'] \oplus [P'', B''] \xrightarrow{w} [P'', B']$$

is exact over Gly , where $u(g) = (0, b''gp', 0)$,

$$v(f', f, f'') = (f', f'') \quad \text{and} \quad w(f', f'') = [l, b]f'' - [p, l]f'.$$

Proof. If the triangle is exact with π replaced by ρ, σ then it is exact for π replaced by $\rho \oplus \sigma$. So it suffices to prove the result for π replaced by $J'P, JP, J''P$ with P projective.

For $J'P$ the triangle becomes $0 \rightarrow [P, B'] \xrightarrow{1} [P, B'] \rightarrow 0$, which is exact.

For JP the triangle becomes

$$[P, B'] \xrightarrow{[1, b'']} [P, B] \xrightarrow{[1, b']} [P, B''] \xrightarrow{[1, b]} [P, B'] \quad \text{which is exact}$$

since β is exact and P is projective.

For $J''P$ the triangle becomes

$$[P, B'] \xrightarrow{0} [P, B''] \xrightarrow{(1)} [L^{-1}P, B'] \oplus [P, B''] \xrightarrow{(\ell^{-1}, 0)} [P, B'],$$

which is exact. //

Suppose $\mathcal{C}$ is a stable DG-category with the extension axiom, $A = H\mathcal{C}$ and $T: A \rightarrow B$ is as before. Then $\text{Pses } \mathcal{C}$ is stable with the extension axiom. Let $A^1 = H\text{Pses } \mathcal{C}$ with Verdier triangulation $\mathcal{T}^1$ obtained from the DG-structure of $\text{Pses } \mathcal{C}$ (see Theorem 17). Also let $B^1 = Tgl B$. If f is a morphism of degree m in $\mathcal{C}$ we shall write Tf for $T[f]$.

Define $T^1: A^1 \rightarrow B^1$ as follows:

- (i) if $A: A' \xrightarrow{i} A \xrightarrow{p} A''$ is pses in $\mathcal{C}$ with deviation class Δ , then $T^1 A$ is the triangle
- $$TA' \xrightarrow{Ti} TA \xrightarrow{Tp} TA'' \xrightarrow{T\Delta} TA';$$

(ii) if (f', f, f'') is a morphism of some degree in $\text{Pses } \mathcal{C}$, then $T^1(f', f, f'') = (Tf', Tf, Tf'')$ (recall Theorem 2(b)).

Then T^1 is an homological functor (recall Proposition 1).

Theorem 33. Suppose $\underline{X}: \underline{X}' \xrightarrow{m} \underline{X} \xrightarrow{e} \underline{X}'' \in \mathcal{A}^1$.

(a) If $\underline{X}$ is T^1 -projective then $\underline{X}', \underline{X}, \underline{X}''$ are T -projective;

(b) If $\underline{X}', \underline{X}''$ are T -projective and $T^1 \underline{X}$ is projective in $\mathcal{B}^1$ (see Theorem 31), then $\underline{X}$ is T^1 -projective.

Proof. Let $\underline{x}$ be the deviation class of $\underline{X}$, let $\underline{x}'' = [m]$, $\underline{x}' = [e]$, $\underline{x}: \underline{X}' \xrightarrow{\underline{x}''} \underline{X} \xrightarrow{\underline{x}'} \underline{X}'' \xrightarrow{\underline{x}} \underline{X}'$. For $C \in \mathcal{A}$, $JC \in \mathcal{J}$ by T1, and clearly $TJC = JTC$. Hence we have a commuting square:

$$\begin{array}{ccc} [\underline{X}, JC] & \xrightarrow{T^1} & [T^1 \underline{X}, JTC] \\ \downarrow & & \downarrow \\ [\underline{X}'', C] & \xrightarrow{T} & [T\underline{X}'', TC] \end{array}$$

in which the vertical maps are isomorphisms and we have identified JC and the pses giving JC as its triangle in $\mathcal{J}$.

(a) If $\underline{X}$ is T^1 -projective then T^1 in the square is an isomorphism, so $T: [\underline{X}'', C] \rightarrow [T\underline{X}'', TC]$ is an isomorphism; also $T^1 \underline{X}$ is projective, so by Theorem 15, $T\underline{X}''$ is projective. Hence $\underline{X}''$ is T -projective. By T2 it now follows

that X', X are also T -projective.

(b) By Theorem 2(a), taking the homology triangle, we have the exact triangle:

$$H[X'', A'] \rightarrow H[\underline{X}, \underline{A}] \rightarrow H[X', A'] \oplus H[X'', A''] \rightarrow H[X'', A']$$

for $\underline{A}: A' \rightarrow A \rightarrow A'' \in \text{Pses } \mathcal{C}$. If $T^1 \underline{X}$ is projective, Theorem 32 gives the exact triangle:

$$[TX'', TA'] \rightarrow [T^1 \underline{X}, T^1 \underline{A}] \rightarrow [TX', TA'] \oplus [TX'', TA''] \rightarrow [TX'', TA'] .$$

Also $(T_{X''A'}, T_{XA}^1, T_{X'A'} \oplus T_{X''A''})$ is a map from the first triangle to the second. If X', X'' are T -projective then $T_{X''A'}, T_{X'A'} \oplus T_{X''A''}$ are isomorphisms. By the 'five lemma' T_{XA}^1 is an isomorphism. //

Theorem 34. The pses $\underline{A}: A' \xrightarrow{i} A \xrightarrow{p} A''$ in $\mathcal{C}$ is T^1 -develorable if and only if A', A, A'' are T -develorable.

Proof. If $\underline{A}$ is T^1 -develorable there exists a T^1 -projective $\underline{X} (\Rightarrow X', X, X'' \text{ } T\text{-projective by Theorem 33})$ and $\underline{\epsilon} = (\epsilon', \epsilon, \epsilon'') \in Z_0[\underline{X}, \underline{A}]$ with $[\underline{\epsilon}]$ a T^1 -epimorphism ($\Leftrightarrow [\epsilon'], [\epsilon], [\epsilon'']$ T -epimorphisms). So A', A, A'' are T -develorable (see Lemma 27(b)).

Suppose A', A, A'' are T -develorable. Choose T -epimorphisms $[\eta'] \in H[Y', A'], [\eta] \in H[Y, A], [\eta''] \in H[Y'', A'']$ with Y', Y, Y'' T -projective. Let $\underline{Y}', \underline{Y}, \underline{Y}''$ be the pses $\underline{Y}' \xrightarrow{1} Y' \rightarrow 0, 0 \rightarrow Y \xrightarrow{1} Y, L^{-1}Y'' \rightarrow \bar{Y} \rightarrow Y''$ where the latter has deviation class ℓ^{-1} . We have $(\eta', i\eta', 0)$ $(0, \eta, p\eta), (\delta\eta'' \ell, \bar{\eta}, \eta'')$ in $Z_0[\underline{Y}', \underline{A}], Z_0[\underline{Y}, \underline{A}], Z_0[\underline{Y}'', \underline{A}]$

respectively where δ is the deviation of $\underline{A}$ for some splitting. Let $\underline{X}$ be the pses $\underline{Y}' \oplus \underline{Y} \oplus \underline{Y}''$. Clearly $\underline{Y}', \underline{Y}$ are T^1 -projective. Since $H[\underline{Y}'', \underline{C}] \cong H[\underline{Y}'', \underline{C}']$ for all $\underline{C}$ it follows that $\underline{Y}''$ is T^1 -projective. Hence $\underline{X}$ is T^1 -projective. Let

$$\epsilon' = (\eta', \delta\eta'' \ell): \underline{Y}' \oplus L^{-1}\underline{Y}'' \rightarrow \underline{A}',$$

$$\epsilon = (i\eta', \eta, \bar{\eta}): \underline{Y}' \oplus \underline{Y} \oplus \bar{\underline{Y}} \rightarrow \underline{A}, \quad \epsilon'' = (p\eta, \eta''): \underline{Y} \oplus \underline{Y}'' \rightarrow \underline{A}''.$$

Since $T\eta', T\eta, T\eta''$ are epimorphisms it follows that $T\epsilon', T\epsilon, T\epsilon''$ are epimorphisms. So $[\epsilon', \epsilon, \epsilon'']$ is a T^1 -epimorphism. By Lemma 27(b), $\underline{A}$ is T^1 -developable. (This proof is modelled on the last two paragraphs of [16]). //

Corollary 35. If each of the objects of $\underline{A}$ is T -developable then each of the objects of $\underline{A}^1$ is T^1 -developable. //

All the conditions so far placed on $T: \underline{A} \rightarrow \underline{B}$ also then hold for $T^1: \underline{A}^1 \rightarrow \underline{B}^1$; not so condition (b) of the followed theorem.

Theorem 36. Suppose

(a) each object of $\underline{A}$ is T -developable (and hence T -resoluble); and

(b) $\underline{X}: \underline{X}' \rightarrow \underline{X} \rightarrow \underline{X}'' \in \underline{A}^1$ is T^1 -projective if and only if $\underline{X}', \underline{X}, \underline{X}''$ are T -projective.

If $\underline{A}: \underline{A}' \rightarrow \underline{A} \rightarrow \underline{A}''$ is a pses in $\underline{C}$ then

$$T^1\text{-dim } \underline{A} = \max(T\text{-dim } \underline{A}', T\text{-dim } \underline{A}, T\text{-dim } \underline{A}'').$$

Proof. It is clear that $T^1\text{-dim } \underline{A} \leq r$ implies $T\text{-dim } A', A, A'' \leq r$. We prove the converse by induction. For $r = 0$ this is just condition (b). Suppose $r > 0$ and true for $r-1$. Suppose $T\text{-dim } A', A, A'' \leq r$. Then A', A, A'' are T -developable, so by Theorem 34, $\underline{A}$ is T^1 -developable. So there exists a T^1 -development $\underline{C} \rightarrow \underline{X} \rightarrow \underline{A} \rightarrow \underline{C}$ of $\underline{A}$. This gives T -developments $C' \rightarrow X' \rightarrow A' \rightarrow C'$ etc., of A', A, A'' . Using (a) we may apply the Dimension Theorem 30 to obtain $T\text{-dim } C', C, C'' \leq r-1$. By induction this means $T^1\text{-dim } \underline{C} \leq r-1$, and so $T^1\text{-dim } \underline{A} \leq r$. //

§3. A spectral sequence.

Suppose $\mathcal{C}$ is a weakly stable DG-category with the extension axiom, $\mathcal{A} = H\mathcal{C}$ is the induced triangulated graded category (see Theorem 17), $\mathcal{B}$ is an abelian graded category, and $T: \mathcal{A} \rightarrow \mathcal{B}$ is any functor. Under these conditions we find a spectral sequence generalising the characterization theorem (Corollary 21) of §1. First we recall some facts on spectral sequences (see [4]).

A filtration of an abelian group complex Γ is a sequence $\Phi = (\Phi_p)_{p \in \mathbb{Z}}$ of subcomplexes Φ_p of Γ such that $\Phi_{p-1} \leq \Phi_p$ for each p . Let m denote the family of monomorphisms $\Phi_p \rightarrow \Gamma$, and e the family of canonical epimorphisms $\Gamma \rightarrow \Gamma/\Phi_p$; we also have monomorphisms $\Phi_{p-1} \rightarrow \Phi_p$ and epimorphisms $\Gamma/\Phi_{p-1} \rightarrow \Gamma/\Phi_p$. The filtration Φ of Γ will be called complete if (Γ, m) is the direct limit of the directed system

$$\dots \rightarrow \Phi_{p-1} \rightarrow \Phi_p \rightarrow \Phi_{p+1} \rightarrow \dots$$

and (Γ, e) is the inverse limit of the directed system

$$\dots \rightarrow \Gamma/\Phi_{p-1} \rightarrow \Gamma/\Phi_p \rightarrow \Gamma/\Phi_{p+1} \rightarrow \dots$$

A filtration Φ of Γ gives rise to a spectral sequence (E_{Φ}^r, d^r) . We shall be interested only in the second term $E_{\Phi}^2 = E_{\Phi}^2$ which is the bigraded group given by:

$$E_{pq}^2 = \ker d_{pq}^1 / \text{im } d_{p+1,q}^1,$$

where $d_{pq}^1: H_{p+q}(\Phi_p/\Phi_{p-1}) \rightarrow H_{p+q-1}(\Phi_{p-1}/\Phi_{p-2})$ is the map

induced by the differential in Γ .

Let ϕ, ϕ' be filtrations of Γ, Γ' . A map of filtrations is a chain map $\phi: \Gamma \rightarrow \Gamma'$ such that

$$\phi\phi_p \leq \phi'_p$$

for all p . Such a map functorially induces a bigraded group map $E^2\phi: E^2_\phi \rightarrow E^2_{\phi'}$. The following proposition is proven in [4].

Proposition 37. Let ϕ, ϕ' be complete filtrations of the complexes Γ, Γ' and $\phi: \Gamma \rightarrow \Gamma'$ a map of filtrations. If $E^2\phi: E^2_\phi \rightarrow E^2_{\phi'}$ is an isomorphism then so is $H\phi: H\Gamma \rightarrow H\Gamma'$. //

Theorem 38. If $A, C \in \mathcal{A}$ and $T\text{-dim } A \leq N$ for some positive integer N , then there exists a complex $\Gamma_{A,C}$ of abelian groups with a complete filtration $\phi_{A,C}$ such that:

- (a) $H\Gamma_{A,C} \cong H[A, C; \mathcal{A}]$; and
- (b) $E^2_{\phi_{A,C}} \cong \text{Ext}^{-p}[TA, TC; \mathcal{B}]_q$.

Moreover, if also $A', C' \in \mathcal{A}$, with $T\text{-dim } A' \leq N'$ for some positive integer N' , and $f: A' \rightarrow A, g: C \rightarrow C'$ are morphisms of $\mathcal{A}$ then there exists a map of filtrations $\phi: \Gamma_{A,C} \rightarrow \Gamma_{A',C'}$ such that:

- (c) the map $H\phi$ corresponds to the map $H[f, g]$ under the isomorphisms of (a); and
- (d) the map $E^2\phi$ corresponds to the map $\text{Ext}[Tf, Tg]$ under the isomorphisms of (b).

Proof. Choose pses $A_{n+1} \xrightarrow{u_n} X_n \xrightarrow{v_n} A_n$ with X_n T-projective and the sequence

$$0 \rightarrow TA_{n+1} \xrightarrow{Tu_n} TX_n \xrightarrow{Tv_n} TA_n \rightarrow 0$$

exact for each n such that $0 \leq n \leq N$ where $A_0 = A$ and $A_N = X_N$. This is possible since $T\text{-dim } A \leq N$. Set $X_n = 0$ for $n < 0$ and $n > N$.

Define $\Gamma = \Gamma_{A,C}$ by :

$$\Gamma_n = \prod_{r \geq 0} [X_r, C]_{n+r}; \text{ and}$$

$$d^\Gamma = D + [u, v, 1].$$

Let Θ be the filtration of Γ given by:

$$\Theta_{pn} = \prod_{r \leq p-n} [X_r, C]_{n+r};$$

it is complete since $X_r = 0$ for $r < 0$. Now Θ_p / Θ_{p-1} is the complex with n -th component $[X_{p-n}, C]_p$ and differential $[u, v, 1]$. From the short exact sequences

$$0 \rightarrow [A_{-q}, C] \xrightarrow{[v, 1]} [X_{-q}, C] \xrightarrow{[u, 1]} [A_{-q+1}, C] \rightarrow 0$$

we obtain the long exact sequence

$$0 \rightarrow [A, C] \xrightarrow{[v_0, 1]} [X_0, C] \xrightarrow{[uv, 1]} [X_1, C] \xrightarrow{[uv, 1]} [X_2, C] \rightarrow \dots$$

of complexes. So $[v_0, 1]$ induces an isomorphism

$$\begin{aligned} H_n(\Theta_p / \Theta_{p-1}) &\cong [A, C]_p \text{ for } n = p, \\ &= 0 \text{ otherwise.} \end{aligned}$$

So $[v_0, 1]$ induces an isomorphism

$$\begin{aligned} E_{pq}^2 &\cong H_p[A, C] \quad \text{for } q = 0, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Let Ω be the filtration of the complex $[A, C]$ by degrees; that is

$$\begin{aligned} \Omega_{pn} &= [A, C]_n \quad \text{for } n \leq p, \\ &= 0 \quad \text{for } n > p; \end{aligned}$$

it is clearly complete. Then $[v_0, 1]: [A, C] \rightarrow \Gamma$ is a map of filtrations; by the above $E^2[v_0, 1]$ is an isomorphism. So by Proposition 37 we have (a).

Let Φ be the filtration of Γ given by:

$$\Phi_{pn} = \prod_{r \geq -p} [X_r, C]_{n+r};$$

it is complete since $X_r = 0$ for $r > N$. Then d_{pq}^1 is the map $H_q[X_{-p}, C] \rightarrow H_q[X_{-p+1}, C]$ induced on homologies by the chain map $[uv, 1]$. But X_{-p}, X_{-p+1} are T -projective; so under the appropriate isomorphisms T , d_{pq}^1 becomes $[T(uv), 1]: [TX_{-p}, TC]_q \rightarrow [TX_{-p+1}, TC]_q$. But the sequence

$$\dots \rightarrow TX_2 \xrightarrow{T(uv)} TX_1 \xrightarrow{T(uv)} TX_0 \xrightarrow{Tv} TA \rightarrow 0$$

is a projective resolution of TA . So we have the isomorphism of (b).

We now prove the mapping part of the theorem. Let objects and maps pertaining to A', C' be denoted by the same symbols as the corresponding ones for A, C only

dashed. Define f_r, k_r for $r \geq 0$ by induction as follows. Firstly $f_0 = f$. Suppose f_r is defined. The triangles $A'_{r+1} \rightarrow X'_r \rightarrow A'_r \rightarrow A'_{r+1}$, $A_{r+1} \rightarrow X_r \rightarrow A_r \rightarrow A_{r+1}$ are T-simple, and X'_r is T-projective. So by Theorem 18, $[f_r]: A'_r \rightarrow A_r$ can be extended to a map $([f_{r+1}], h, [f_r])$ of triangles. Hence the square

$$\begin{array}{ccc} A'_r & \longrightarrow & A'_{r+1} \\ [f_r] \downarrow & & \downarrow [f_{r+1}] \\ A_r & \longrightarrow & A_{r+1} \end{array}$$

commutes in $\mathcal{A}$. By Theorem 2 there exists $k_r \in Z_0[X'_r, X_r]$ such that the diagram

$$\begin{array}{ccccc} A'_{r+1} & \xrightarrow{u'_r} & X'_r & \xrightarrow{v'_r} & A'_r \\ f_{r+1} \downarrow & & k_r \downarrow & & f_r \downarrow \\ A_{r+1} & \xrightarrow{u_r} & X_r & \xrightarrow{v_r} & A_r \end{array}$$

commutes. Let $[k, g]: F \rightarrow F'$ be the chain map given by $[k, g]_n = \prod_{r \geq 0} [k_r, g]_{n+r}$. That $\phi = [k, g]$ is a map of filtrations satisfying (c), (d) is now readily checked. //

Corollary 39. If $f: C \rightarrow C'$ is a morphism of $\mathcal{C}$
such that $Tf: TC \rightarrow TC'$ is an isomorphism, then
 $H[1, f]: H[A, C] \rightarrow H[A, C']$ is an isomorphism for all A with
finite T-dimension. //

Corollary 40. If each object of $\mathcal{A}$ has finite
T-dimension then $T: \mathcal{A} \rightarrow \mathcal{B}$ reflects isomorphisms. //

Compare these results with [1] Ch.XVII §4 p.371
and [2]§3 p.286.

REMARK. In Theorem 38 the condition $T\text{-dim } A \leq N$ was
used to ensure that $\Phi_{A,C}$ is a complete filtration of
 $\Gamma_{A,C}$. If A is T -resoluble and the X_r can be chosen
with $[X_r, C]_{n+r} = 0$ for r large enough then $\Phi_{A,C}$ is
a complete filtration of $\Gamma_{A,C}$; in particular cases this
can be ensured by conditions alternative to $X_r = 0$ for
 r large enough.

S4. Right inverses; splittings.

The purpose of this section is to find conditions under which the characterization sequence of Corollary 21 splits.

In the situation of the last section suppose T is homological, $B^\#$ is the full sub-graded-category of B with objects those of B with projective dimension ≤ 1 , and $A^\#$ is the full sub-graded-category of A with objects those A of A with $TA \in B^\#$. Then T restricts to give $T^\# : A^\# \rightarrow B^\#$.

Theorem 41. Suppose B has a full sub-graded-category R with the following properties:

BT1. each $P \in R$ is a projective object of B ;

BT2. if B is a projective object of B then there exists $P \in R$ with $P \cong B$ in B ; and

BT3. there is a graded functor $N: R \rightarrow ZC$ such that

$TN: R \rightarrow B$ is the inclusion.

Then $T^\# : A^\# \rightarrow B^\#$ has a right inverse V up to isomorphism. Moreover, if $\mathcal{L}$ is a class of objects of containing all the NP , and $\mathcal{L}$ is closed under extensions, then $\mathcal{L}$ contains all the $VB, B \in B^\#$.

Proof. For any $B \in B^\#$ we may choose (by BT2) a short exact sequence

$$0 \rightarrow Y \xrightarrow{\kappa} X \xrightarrow{\epsilon} B \rightarrow 0$$

in B_0 with $X, Y \in R$, such that $Y = 0$ if B is projective, and $X = B, \epsilon = 1_B$ if $B \in R$. Then

$N\kappa \in Z_0[NY, NX]$; so we may choose a pses

$$NX \xrightarrow{i} VB \xrightarrow{p} LNY$$

in $\mathcal{L}$, and a splitting $(\tilde{p}, \tilde{i})$, such that the pses has deviation $N\kappa \cdot \mathcal{L}^{-1}$ with this splitting; if $B \in \mathcal{R}$ choose $i = i'_{NB}$, $VB = NX$. For any new $B \in \mathcal{B}^\#$ which comes into consideration, fix such choices for all time. Take $B' \in \mathcal{B}^\#$ and let the choices pertaining to B' be denoted as for B only dashed. Suppose $g \in [B, B']_n$; then there exist $h \in [X, X']_n$, $k \in [Y, Y']_n$ such that $h\kappa = \kappa'k$ and $g\epsilon = \epsilon'h$. Let $f \in Z_n[VB, VB']_n$ be the unique map such that $i'.Nh = f.i$, $p'.f = LNK.p$ and $\tilde{p}'.f.\tilde{i} = 0$. Suppose that $h'\kappa = \kappa'k'$, $g\epsilon = \epsilon'h'$; then $h' - h = \kappa'u$, $k' - k = u\kappa$ for some $u \in [X, Y']_n$. Let f' be the unique map obtained from h', k' as f was from h, k . Let $s = \tilde{i}'. \mathcal{L}. Nu. \tilde{p} \in [VB, VB']_{n+1}$; then

$$\begin{aligned} Ds &= D\tilde{i}'. \mathcal{L}. Nu. \tilde{p} + (-1)^{n+1} \tilde{i}'. \mathcal{L}. Nu. D\tilde{p} \\ &= i'. N\kappa'. Nu. \tilde{p} + (-1)^{n+2} \tilde{i}'. \mathcal{L}. Nu. N\kappa. \mathcal{L}^{-1}. p \\ &= i'. N(\kappa'u). \tilde{p} + \tilde{i}'. LN(u\kappa). p \\ &= i'. N(h' - h). \tilde{p} + \tilde{i}'. LN(k' - k). p. \tilde{i}. p \\ &= (f' - f). i. \tilde{p} + \tilde{i}'. p'. (f' - f). \tilde{i}. p \\ &= (f' - f). i. \tilde{p} + (\tilde{i}'p' + i'\tilde{p}'). (f' - f). \tilde{i}. p \\ &= (f' - f). (i\tilde{p} + \tilde{i}p) \\ &= f' - f ; \end{aligned}$$

so $[f'] = [f]$. Define $Vg = [f] \in H_n[VB, VB']$. Thus we have an extension $V: \mathcal{B}^\# \rightarrow \mathcal{A}$ of $\mathcal{R} \xrightarrow{N} Z\mathcal{L} \rightarrow H\mathcal{L} = \mathcal{A}$

which is readily seen to be a functor.

For $B \in \mathcal{B}^\#$ we have the pses $NX \xrightarrow{i} NB \xrightarrow{p} LNY$ with deviation class $[N\kappa, \ell^{-1}]$. But T is homological, so the sequence

$$\begin{array}{ccccccc} & TN\kappa & Ti & Tp & & LTN\kappa & \\ TNY & \longrightarrow & TNX & \longrightarrow & TVB & \longrightarrow & LTNY & \longrightarrow & LTNX \end{array}$$

is exact. By BT3, $TN\kappa = \kappa$. But κ is a monomorphism, so the sequence

$$0 \rightarrow Y \xrightarrow{\kappa} X \xrightarrow{Ti} TVB \rightarrow 0$$

is exact. Thus there is an isomorphism $\lambda_B: TVB \cong B$ unique with the property $\lambda_B \cdot Ti = \epsilon$. The calculation

$$\begin{aligned} \lambda_{B'} \cdot TVg \cdot Ti &= \lambda_{B'} \cdot Tf \cdot Ti \\ &= \lambda_{B'} \cdot Ti' \cdot h \\ &= \epsilon' \cdot h \\ &= g \cdot \epsilon \\ &= g \cdot \lambda_B \cdot Ti \end{aligned}$$

shows that $\lambda_{B'} \cdot TVg = g \cdot \lambda_B$ since Ti is an epimorphism. Thus λ is a natural isomorphism $TV \cong 1$. //

Corollary 42. Suppose the conditions of the theorem hold, and further

BT4. for all $A \in \mathcal{A}$, TA has projective dimension ≤ 1 . Then the short exact sequence of Corollary 21 splits giving an isomorphism:

$$H[A, C; \mathcal{C}] \cong [TA, TC; \mathcal{B}] \oplus \text{Ext}^1[TA, TC; \mathcal{B}]$$

when $T\text{-dim } A \leq 1$.

Proof. For $P \in \mathcal{R}$, NP is T -projective, and for

$B \in \mathcal{B}^\#$, $T\text{-dim VB} \leq 1$. So $T\text{-dim VTC} \leq 1$. By
 Corollary 21, $T:H[A, VTA] \rightarrow [TA, TVTA]$,
 $T:H[VTC, C] \rightarrow [TVTC, TC]$ are surjective, so there exist
 $\sigma \in H_0[A, VTA]$, $\tau \in H_0[VTC, C]$ with $T\sigma = \lambda_{TA}^{-1}$, $T\tau = \lambda_{TC}$.
 Define $\tilde{R}: [TA, TC] \rightarrow H[A, C]$ by $\tilde{R}f = \tau.Vf.\sigma$; then

$$\begin{aligned}
 T\tilde{R}f &= T\tau.TVf.T\sigma \\
 &= \lambda_{TC}.TVf.\lambda_{TA}^{-1} \\
 &= f \text{ by the naturality of } \lambda . //
 \end{aligned}$$

CHAPTER 3. - THE EXACT HOMOLOGY TRIANGLE

§1. The homology functor.

Suppose $\mathcal{Q}$ is an abelian category with enough projectives, $\mathcal{C}$ is the DG-category $C\mathcal{Q}$, $\mathcal{A}$ is the Verdier triangulated graded category $H\mathcal{C}$ (see Theorem 17), $\mathcal{B}$ is the stable abelian graded category $G\mathcal{Q}$, and $T: \mathcal{A} \rightarrow \mathcal{B}$ is the homology functor. The general theory of the last chapter is shown to give familiar results (see [1] Ch XVII) in this case.

Let $\mathcal{C}^P$ denote the full sub-DG-category of $\mathcal{C}$ with objects the projective complexes over $\mathcal{Q}$. Then $\mathcal{C}^P$ is stable and satisfies the extension axiom, so $\mathcal{A}^P = H\mathcal{C}^P$ is Verdier triangulated (see Theorem 17). Let $T^P: \mathcal{A}^P \rightarrow \mathcal{B}$ be the restriction of T to $\mathcal{A}^P$. It is well-known that T^P is homological.

An object X of $\mathcal{A}$ will be called CE-projective when each $X_n, B_n X, Z_n X, H_n X$ is projective in $\mathcal{Q}$ (such complexes are used in [1] Ch. XVII). From [5] Ch. IV pp 28-35 we deduce that X is CE-projective if and only if it is chain isomorphic to a complex $C \oplus P$ (the direct sum as complexes) where C is a contractible projective complex and P is a projective complex with zero differential.

Theorem 43. A complex is T-projective if and only if it is chain isomorphic to a complex $C \oplus P$ (direct sum as complexes) where C is contractible and P is projective

with zero differential. For projective complexes
 T^P -projectivity, T -projectivity and CE -projectivity are
Equivalent conditions.

Proof. Suppose X is T -projective. Let $P = HX$.
 Since $0 \rightarrow BX \xrightarrow{j} ZX \xrightarrow{\xi} P \rightarrow 0$ is exact, $ZX \cong BX \oplus P$.
 We can suppose $ZX = BX \oplus P$, that the sequence
 $0 \rightarrow ZX \xrightarrow{i} X \xrightarrow{\eta} BX \rightarrow 0$ is $0 \rightarrow BX \oplus P \xrightarrow{(i', i'')} X \xrightarrow{\eta} BX \rightarrow 0$,
 and that X has differential $(i', i'') \begin{pmatrix} 1 \\ 0 \end{pmatrix} \eta = i' \eta$. But
 $T: H[X, P] \rightarrow [HX, HP]$ is an isomorphism, and is induced by
 $[i'', 1]$, so there exists a chain map $p: X \rightarrow P$ with
 $pi'' = 1$; since p is a chain map $pi' \eta = 0$, so $pi' = 0$,
 η being epimorphic. So i'' is a retract, and we may put
 $X = C \oplus P$, $i'' = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $p = (0, 1)$. Then $pi' = 0$ implies
 i' has the form $\begin{pmatrix} i' & 0 \\ 0 & 0 \end{pmatrix}: BX \rightarrow C \oplus P$; and $\eta(i', i'') = 0$
 implies η is of the form $(\eta_0, 0): C \oplus P \rightarrow BX$. So
 $X = C \oplus P$ then has differential $\begin{pmatrix} i' & 0 \\ 0 & 0 \end{pmatrix} (\eta_0, 0) = \begin{pmatrix} i' \eta_0 & 0 \\ 0 & 0 \end{pmatrix}$.
 So X is the direct sum of the complex C , with differential
 $i' \eta_0$, and P with zero differential. So
 $0 \rightarrow BX \oplus P \xrightarrow{i' \oplus 1} C \oplus P \xrightarrow{(\eta_0, 0)} BX \rightarrow 0$ is exact; so
 $0 \rightarrow BX \xrightarrow{i'} C \xrightarrow{\eta_0} BX \rightarrow 0$ is exact. Thus $HC = 0$. But
 C is T -projective, so $T: H[C, C] \rightarrow [HC, HC]$ is an iso-
 morphism. Thus $H[C, C] = 0$ and C is contractible.

If X in the above is projective then C is projective

and only T^P -projectivity is needed in the argument. Then $C \oplus P$ is CE-projective. For projective complexes we thus have: T -projectivity $\Rightarrow T^P$ -projectivity $\Rightarrow$ CE-projectivity.

Suppose $X = C \oplus P$ with C, P as in the theorem. Then $HC = 0$ and $H[C, A] = 0$ for all complexes A , so C is T -projective. Also, $P = HP$ is projective, so to prove X is T -projective it remains to prove

$T: H[P, A] \rightarrow [P, HA]$ is an isomorphism for all A . The sequences $0 \rightarrow ZA \xrightarrow{i} A \xrightarrow{\eta} BA \rightarrow 0$,

$$0 \rightarrow BA \xrightarrow{j} ZA \xrightarrow{\zeta} HA \rightarrow 0 \text{ are exact. Take}$$

$f: P \rightarrow HA$; since P is projective and ζ is epimorphic,

$f = \zeta \cdot f'$ for some f' . Let $g = i \cdot f'$; then

$dg = ij\eta g = ij\eta if' = 0$, and $Tg = f$. So T_{PA} is sur-

jective. Suppose $Tg = 0$ for some $g \in Z[P, A]$. Then

$\zeta \cdot Zg = 0$, so $Zg = j \cdot k$ for some $k: P \rightarrow BA$. But P is projective and η is epimorphic, so $k = \eta \cdot h$ for some

$h: P \rightarrow A$. Then $g = i \cdot Zg = ijk = ij\eta k = dh$, so $g = 0$.

Hence T_{PA} is injective, and so an isomorphism.

Taking X projective in the last paragraph (i.e. C projective) we see that CE-projectivity $\Rightarrow T$ -projectivity //

A sequence $A' \rightarrow A \rightarrow A''$ in $Z_0 \mathcal{C}$ will be called CE-exact if each of the sequences $A'_n \rightarrow A_n \rightarrow A''_n$,

$Z_n A' \rightarrow Z_n A \rightarrow Z_n A''$ is exact in $\mathcal{D}$. If $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$

is CE-exact then two applications of the "three-by-three diagram lemma" (see [17] Ch. II §5 p 49) yield that each of

the sequences $0 \rightarrow B_n A' \rightarrow B_n A \rightarrow B_n A'' \rightarrow 0$,

$$0 \rightarrow H_n A' \rightarrow H_n A \rightarrow H_n A'' \rightarrow 0 \text{ is exact.}$$

A chain map $f \in Z_0[A, A']$ will be called a CE-epimorphism if each $B_n f, H_n f$ is an epimorphism of $\mathcal{Q}$. If f is a CE-epimorphism, by the "short five lemma", each $f_n, Z_n f$ is an epimorphism of $\mathcal{Q}$.

From [5] Ch. IV §3 p 34 we find that CE-projective complexes and CE-exact sequences form a projective class in $Z_0 \mathcal{C}$. We write $\text{CE-dim } A \leq r$ for $A \in \mathcal{C}$ when there exists a CE-exact sequence

$$0 \rightarrow X_r \rightarrow X_{r-1} \rightarrow \dots \rightarrow X_1 \rightarrow X_0 \rightarrow A \rightarrow 0$$

with each X_i CE-projective. The obvious meanings are given to $\text{CE-dim } A = r$, $\text{CE-dim } A = \infty$; the latter is true for all A .

Theorem 44. Each object of $\mathcal{A}^P$ is T^P -developable.

Proof. If A is a projective complex then there exists a CE-exact sequence $0 \rightarrow A' \rightarrow X \rightarrow A \rightarrow 0$ of projective complexes with X CE-projective; the sequence being pses since A is projective. Thus we have a 1-simple triangle $A' \rightarrow X \rightarrow A \rightarrow A'$; and X is T^1 -projective by Theorem 43. //

Now combine Theorems 28 and 44.

Corollary 45. The projective complex X is CE-projective if and only if $H: H[X, A] \rightarrow [HX, HA]$ is injective for all projective complexes A . //

Theorem 46. If A is a projective complex then

$$T\text{-dim } A \leq T^P\text{-dim } A = \text{CE-dim } A .$$

Proof. From Theorem 43 and the fact that a T^P -simple triangle is T-simple the inequality follows. We prove that $T^P\text{-dim } A \leq r \iff \text{CE-dim } A \leq r$ by induction on r . Theorem 43 gives the result for $r = 0$. Suppose $r > 0$ and the result true for $r - 1$. Let $0 \rightarrow A' \rightarrow X \rightarrow A \rightarrow 0$ be CE-exact with X CE-projective. This gives a T^P -development $A' \rightarrow X \rightarrow A \rightarrow A'$ of A . If $T^P\text{-dim } A \leq r$ then Theorem 30 gives $T^P\text{-dim } A' \leq r-1$ since A' is T^P -resolvable by Theorem 44; so $\text{CE-dim } A' \leq r-1$ by induction; so $\text{CE-dim } A \leq r$. If $\text{CE-dim } A \leq r$ then $\text{CE-dim } A' \leq r-1$, so $T^P\text{-dim } A' \leq r-1$ by induction; so $T^P\text{-dim } A \leq r$. //

Theorem 47.

If the projective dimension of $\mathcal{Q}$ is 1 then there is a short exact sequence

$$0 \rightarrow \text{Ext}^1[HA, HC; G\mathcal{Q}] \xrightarrow{R} H[A, C; C\mathcal{Q}] \xrightarrow{H} [HA, HC; G\mathcal{Q}] \rightarrow 0$$

of graded abelian groups where $\deg R = -1$, natural in complexes A, C with A projective. Moreover, the sequence splits.

Proof. If A is projective, $\text{CE-dim } A \leq 1$ since the projective dimension of $\mathcal{Q}$ is 1, and so $T\text{-dim } A \leq 1$ by Theorem 46. Corollary 21 now gives the short exact sequence. In Theorem 41 take $\mathcal{R}$ to be the class of

projective objects of $\mathcal{B}$, and N to be the inclusion

$\mathcal{B} = \mathcal{G}\mathcal{D} \rightarrow \mathcal{ZC}\mathcal{D}$ restricted to $\mathcal{R}$. Then BT1,2,3,4 are satisfied so Corollary 42 applies. //

All the theory of the last chapter can now be interpreted in this context.

§2. The exact homology triangle functor.

Let $\mathcal{P}$ be the graded category $\text{HPses } \mathcal{C}^P$ in the notation of the last section. The objects of $\mathcal{P}$ are just short exact sequences of projective complexes over $\mathcal{Q}$. Let $\mathcal{E}$ be the full sub-graded-category of $\text{Tgl } \mathcal{B}$ with objects the exact triangles over $\mathcal{B}$. Then $T^1: \mathcal{A}^1 \rightarrow \text{Tgl } \mathcal{B}$ induces $K: \mathcal{P} \rightarrow \mathcal{E}$. We shall not distinguish between K and $\mathcal{P} \rightarrow \mathcal{E} \rightarrow \text{Tgl } \mathcal{B}$. In this section we show how the theory of the last chapter may be applied to K . Indeed, more information is found concerning this functor than is given by Theorem 5 of [15] §5 p 746; we find its kernel and show that it has a right inverse (up to isomorphism).

Theorem 48. The exact sequence $\underline{X}$:

$0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0$ of projective complexes is K -projective if and only if X', X, X'' are CE-projective.

Proof. By Theorem 34, X', X, X'' are CE-projective if and only if they are T^P -projective. Let χ be the triangle of $\mathcal{J}$ obtained from $\underline{X}$; then $T\chi = K\underline{X}$. Each $B \in \mathcal{B}$ is a complex over $\mathcal{Q}$ with zero differential and so $B \in \mathcal{A}$, and $TB = B$. So (a) of Theorem 31 is satisfied. Suppose X', X, X'' are T^P -projective; then they are T -projective, and so Theorem 31 gives $T\chi$ projective. So $K\underline{X}$ is projective. Also $K = T^{Pl}$. Applying Theorem 33(b) we thus have $\underline{X}$ K -projective.

The converse follows from Theorem 33(a). //

Combining Corollary 35 and Theorem 44, we have:

Theorem 49. Each object of $\mathcal{P}$ is K-developable. //

Combining Theorem 36 and 46, we have:

Theorem 50. If $A: 0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$ is an object of $\mathcal{P}$ then

$K\text{-dim } A = \max(\text{CE-dim } A', \text{CE-dim } A, \text{CE-dim } A''). //$

Lemma 51. If the projective dimension of $\mathcal{D}$ is k then the objects of $\mathcal{E}$ are the objects of $\text{Tgl } \mathcal{B}$ with projective dimension $\leq k$.

Proof. An object $\beta \in \text{Tgl } \mathcal{B}$ is exact if and only if the object

$$\dots \rightarrow L^{n-1}B'' \rightarrow L^n B' \rightarrow L^n B \rightarrow L^n B'' \rightarrow L^{n+1}B' \rightarrow \dots$$

of $C\mathcal{B}_0$ has zero homology (where β is $B' \rightarrow B \rightarrow B'' \rightarrow B'$).

From the exact homology triangle it follows that, if

$0 \rightarrow \gamma \rightarrow X \rightarrow \beta \rightarrow 0$ is a short exact sequence in

$\text{Tgl } \mathcal{B}$ and X is exact then β is exact if and only if

γ is exact. Take $\beta \in \text{Tgl } \mathcal{B}$ and choose an exact sequence

$$0 \rightarrow \gamma \rightarrow X_{k-1} \rightarrow \dots \rightarrow X_1 \rightarrow X_0 \rightarrow \beta \rightarrow 0$$

in $\text{Tgl } \mathcal{B}$ with each X_i projective. The projective

dimension of $\mathcal{B}$ is k , so the objects C', C, C'' in γ

are projective. But by the above, since $X_0, X_1, \dots, X_{k-1}$

are exact (see Theorem 15), β is exact if and only if γ

is exact; that is, (again by Theorem 15) if and only if γ

is projective; that is, if and only if β has projective dimension $\leq k$. //

Theorem 52. Suppose the projective dimension of $\mathcal{D}$ is 1. The functor $K: \mathcal{P} \rightarrow \mathcal{E}$ has a right inverse up to isomorphism. If $A, C \in \mathcal{P}$ then there is a natural short exact sequence:

$$0 \rightarrow \text{Ext}^1[K_A, K_C; \text{Tgl } \mathcal{B}] \xrightarrow{R} H[A, C; \text{Seq } \mathcal{C}] \xrightarrow{K} [K_A, K_C; \text{Tgl } \mathcal{B}] \rightarrow 0$$

with $\deg R = -1$, which splits.

Proof. In the notation of Ch. 2 §4 we have, by Lemma 51, $\mathcal{B}^{1\#} = \mathcal{E}$; also $\mathcal{P} = \mathcal{A}^{\text{Pl}} = \mathcal{A}^{\text{Pl}\#}$ since $T^{\mathcal{P}}$ is homological, and $T^{\text{Pl}\#} = K$.

Let $\mathcal{R}$ be the class of triangles χ over $\mathcal{B}$ of the form $J'P \oplus JQ \oplus J''R$, with P, Q, R projective. By Theorem 15, BT1 and 2 of Theorem 41 are satisfied. Define $N: \mathcal{R} \rightarrow \text{ZPses } \mathcal{C}^{\mathcal{P}}$ as follows.

(i) $N\chi$ is the short exact sequence

$$0 \rightarrow X' \xrightarrow{u} X \xrightarrow{v} X'' \rightarrow 0,$$

where $X' = L^{-1}R \oplus P$, $X'' = Q \oplus R$ with zero differentials, and $X = L^{-1}R \oplus P \oplus Q \oplus R$ with differential given by the 4×4 matrix with 1^{-1} in the top right corner and zeros elsewhere; u is the coprojection into, and v the projection from the direct sum $X = X' \oplus X''$; they are chain maps.

(ii) Suppose also $\psi \in \mathcal{P}$. A morphism $w: \chi \rightarrow \psi$ of degree n is a triple

$$\left(\begin{pmatrix} L^{-1}h & k' \\ 0 & f \end{pmatrix}, \begin{pmatrix} f & k \\ 0 & g \end{pmatrix}, \begin{pmatrix} g & k'' \\ 0 & h \end{pmatrix} \right).$$

Define Nw to be the morphism

$$\left(\begin{pmatrix} L^{-1}h & k' \\ 0 & f \end{pmatrix}, \begin{pmatrix} L^{-1}h & k' & 0 & 0 \\ 0 & f & k & 0 \\ 0 & 0 & g & k'' \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} g & k'' \\ 0 & h \end{pmatrix} \right)$$

of sequences, of degree n .

A simple calculation shows that N is a functor, and KN is the identity of $\mathcal{E}$, so BT3 is satisfied. But $\mathcal{P} = \mathcal{A}^{Pl} = \mathcal{A}^{Pl\#}$ so BT4 of Corollary 42 is satisfied. Theorem 41 and Corollary 42 now give the result. //

The short exact sequence of Theorem 52 holds when $\underline{C}$ is replaced by any exact sequence of complexes over $\underline{Q}$. The adjustment of the argument needed to give this result is left to the reader.

CHAPTER 4. - FILTERED COMPLEXES

Throughout this chapter $\mathcal{A}$ will denote a projectively perfect abelian category, $\mathcal{C}$ will denote the DG-category $C\mathcal{A}$, and $\mathcal{B}$ will denote the graded category $G\mathcal{A}$.

§1. The graded functor $K_n: \mathcal{P}_n \rightarrow \mathcal{E}_n$.

The aim of this section is to define a functor on complexes over $\mathcal{A}$ with filtrations of length $n-1$. The value of this functor on a complex A with filtration F is to be the collection of homology triangles of the short exact sequences

$$0 \rightarrow F_q A / F_p A \rightarrow F_r A / F_p A \rightarrow F_r A / F_q A \rightarrow 0$$

($p < q < r$). Before this definition can be made precise it is necessary to define a category which will receive such a functor.

Let $\text{Flt}_n \mathcal{C}$ denote the sub-DG-category of $\bigotimes_{n-1} \mathcal{C}$ (= the tensor product of $n-1$ copies of $\mathcal{C}$) with objects those $(n-1)$ -tuples $A = (A_p)_{0 < p < n}$ with A_{p-1} a sub-complex of A_p for $1 < p < n$, and if $A' = (A'_p)_{0 < p < n}$

is another such object then $[A, A'; \text{Flt}_n \mathcal{C}]$ is the sub-complex of $[A, A'; \bigotimes_{n-1} \mathcal{C}]$ with elements $f = (f_p)_{0 < p < n}$

in dimension m such that f_p restricted to A_{p-1} coincides with f_{p-1} . As a notational convenience we put

$$A_0 = 0 \text{ for } A \in \text{Flt}_n \mathcal{C}.$$

There is a DG-functor $\text{Flt}_n \mathcal{C} \rightarrow \mathcal{C}$ given by $A \mapsto A_{n-1}$, $f \mapsto f_{n-1}$. It is easily seen that $\text{Flt}_n \mathcal{C}$ may be given suspension and hence made stable uniquely in such a way that this DG-functor is stable. Moreover, $\text{Flt}_n \mathcal{C}$ has the extension axiom. Let $\mathcal{P}_n$ denote the Verdier triangulated (see Theorem 17) graded category $\text{HFlt}_n \mathcal{C}$.

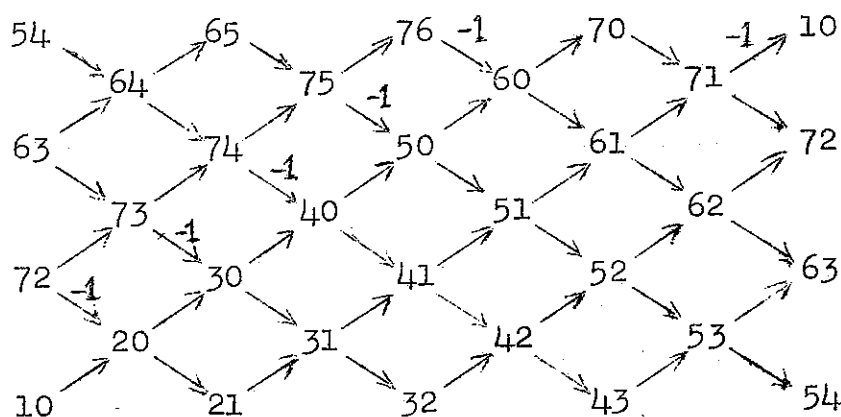
Let $\mathcal{J}_n$ ($n \geq 2$) be the graded model (see Ch.1 §3) defined as follows:

(i) the objects are pairs (u, v) of integers satisfying the condition $0 \leq v < u < n$;

(ii) if $(u, v), (u', v')$ are two such pairs, then

$$\begin{aligned} [(u, v), (u', v')] &= \mathbb{Z} \text{ if } v \leq v' < u \leq u', \\ &= L^{-1} \mathbb{Z} \text{ if } v' < v \leq u' < u, \\ &= 0 \text{ otherwise.} \end{aligned}$$

Notice that for $n = 3$ we obtain $\mathcal{J}_3$ as in Ch.1 §4. Each $\mathcal{J}_n$ may be drawn on a Möbius band; for each example we give $\mathcal{J}_8$:



where we have put $uv \rightarrow u'v'$ if $[(u,v), (u',v')] = \mathbb{Z}$
 and $uv \xrightarrow{-1} uv$ if $[(u,v), (u',v')] = L^{-1}\mathbb{Z}$; the other
 maps are obtained by composition, with the rule that
 composites $uv \rightarrow wv \rightarrow wu$, $wv \xrightarrow{-1} wu \xrightarrow{-1} uv$,
 $wu \xrightarrow{-1} uv \rightarrow wv$ are zero.

Consider the following data:

- (a) objects $G(u,v)$ of $\mathcal{B}$ for $0 \leq v < u < n$;
- (b) morphisms $G(u,v) \rightarrow G(u+1,v)$ of degree 0 in $\mathcal{B}$
 for $0 \leq v < u < n-1$;
- (c) morphisms $G(u,v) \rightarrow G(u,v+1)$ of degree 0 in $\mathcal{B}$
 for $0 < v+1 < u < n$;
- (d) morphisms $G(n-1,v) \rightarrow G(v,0)$ of degree -1 in $\mathcal{B}$
 for $0 < v < n-1$;

subject to the axioms:

- (e) the composites

$$\begin{aligned} G(v,v-1) &\rightarrow G(v+1,v-1) \rightarrow G(v+1,v) \quad \text{for } 0 < v < n-1, \\ G(n-1,n-2) &\rightarrow G(n-2,0) \rightarrow G(n-1,0), \\ G(n-1,0) &\rightarrow G(n-1,1) \rightarrow G(1,0), \end{aligned}$$

are all zero;

- (f) the squares

$$\begin{array}{ccc} G(u,v) & \rightarrow & G(u+1,v) \\ \downarrow & & \downarrow \\ G(u,v+1) & \rightarrow & G(u+1,v+1) \end{array} \quad \text{for } 0 < v+1 < u < n-1,$$

$$\begin{array}{ccc} G(n-1,v) & \rightarrow & G(v,0) \\ \downarrow & & \downarrow \\ G(n-1,v+1) & \rightarrow & G(v+1,0) \end{array} \quad \text{for } 0 < v < n-2$$

all commute.

If G is an $\mathcal{J}_n$ -diagram in $\mathcal{B}$ then we have
 (a), ..., (f) as above. Conversely, with such data we
 obtain an $\mathcal{J}_n$ -diagram G in $\mathcal{B}$ by defining

$$G(u, v) \rightarrow G(u', v') \quad \text{for } 0 \leq v < u < n, \quad 0 \leq v' < u' < n$$

to be:

(g) the composite

$$G(u, v) \rightarrow G(u+1, v) \rightarrow \dots \rightarrow G(u', v) \rightarrow G(u', v+1) \rightarrow \dots \rightarrow G(u', v')$$

$$\text{for } 0 \leq v \leq v' < u \leq u' < n;$$

(h) the composite

$$G(u, v) \rightarrow G(n-1, v) \rightarrow G(v, 0) \rightarrow G(v, v') \rightarrow G(u', v')$$

$$\text{for } 0 \leq v' < v \leq u' < u < n.$$

(i) zero otherwise.

We introduce the notation $G(u, v) = 0$ for $0 \leq u \leq v < n$.

Then (e), (f) together are equivalent to:

(f') the first square of (f) commutes for

$$0 < v+1 \leq u < n-1,$$

and the second square of (f) commutes for $0 \leq v \leq n-2$.

From (f') we deduce the commutativity of the diagram:

$$\begin{array}{ccccc}
 G(u, v) & \longrightarrow & G(u+1, v) & \rightarrow \dots \rightarrow & G(u', v) \\
 \downarrow & & \downarrow & & \downarrow \\
 G(u, v+1) & \longrightarrow & G(u+1, v+1) & \rightarrow \dots \rightarrow & G(u', v+1) \\
 \downarrow & & \downarrow & & \downarrow \\
 \vdots & & \vdots & & \vdots \\
 \downarrow & & \downarrow & & \downarrow \\
 G(u, v') & \longrightarrow & G(u+1, v') & \rightarrow \dots \rightarrow & G(u', v')
 \end{array}$$

for $0 \leq v \leq v' < n$, $0 \leq u \leq u' < n$, and of the diagram:

$$\begin{array}{ccccccc}
 G(u,v) & \longrightarrow & G(n-1,v) & \longrightarrow & G(v,0) & \longrightarrow & G(v,v') \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 G(u,u') & \longrightarrow & G(n-1,u') & \longrightarrow & G(u',0) & \longrightarrow & G(u',v')
 \end{array}$$

for $0 < v \leq u' < n-1$, $0 \leq u, v' < n$. So we have alternative decompositions of the maps of (g), (h).

We must show that the composite $G(r,s) \rightarrow G(t,s) \rightarrow G(t,u) \rightarrow G(v,w)$ is $G(r,s) \rightarrow G(v,w)$. Four separate cases must be distinguished, and in each case the result follows from a commutative diagram.

(i) $0 \leq s \leq u < r \leq t < n$, $0 \leq u \leq w < t \leq v < n$:

$$\begin{array}{ccccc}
 G(r,s) & \longrightarrow & G(t,s) & \longrightarrow & G(t,u) \\
 \downarrow & & \searrow & & \downarrow \\
 & & & & G(t,w) \\
 \downarrow & \swarrow & & \searrow & \downarrow \\
 G(v,s) & \longrightarrow & & \longrightarrow & G(v,w)
 \end{array}$$

(ii) $0 \leq s \leq u < r \leq t < n$, $0 \leq w < u \leq v < t < n$:

$$\begin{array}{ccccc}
 G(r,s) & \longrightarrow & G(t,s) & \longrightarrow & G(t,u) \\
 \downarrow & & \searrow & & \downarrow \\
 G(r,v) & \longrightarrow & & \longrightarrow & G(t,v) \\
 \downarrow & & & & \downarrow \\
 G(n-1,v) & \xlongequal{\hspace{2cm}} & & & G(n-1,v) \\
 & \searrow & & & \downarrow \\
 & & & & G(v,w)
 \end{array}$$

(iii) $0 \leq u < s \leq t < r < n$, $0 \leq u \leq w < t \leq v < n$:

$$\begin{array}{ccccccc}
 G(r,s) & \longrightarrow & G(s,0) & \longrightarrow & G(s,u) & \longrightarrow & G(t,u) \\
 & \searrow & & & \downarrow & & \downarrow \\
 & & & & G(s,w) & \longrightarrow & G(t,w) \\
 & & & & & & \downarrow \\
 & & & & & & G(v,w) .
 \end{array}$$

(iv) $0 \leq u < s \leq t < r < n$, $0 \leq w < u \leq v < t < n$:

$$\begin{array}{ccccccc}
 G(r,s) & \longrightarrow & G(s,0) & \longrightarrow & G(s,u) & \longrightarrow & G(t,u) \\
 & \searrow & & & \downarrow & & \downarrow \\
 & & & & G(s,v) & \longrightarrow & G(t,v) \\
 & & & & \downarrow & & \downarrow \\
 & & & & G(0,0) & \longrightarrow & G(v,0) \\
 & & & & & & \downarrow \\
 & & & & & & G(v,w) .
 \end{array}$$

Let $\text{Tgl}_n \mathcal{B}$ denote the graded category $[\mathcal{I}_n, \mathcal{B}]$.
 Let $\mathcal{E}_n$ denote the full sub-graded-category of $\text{Tgl}_n \mathcal{B}$ which has objects those functors $G: \mathcal{I}_n \rightarrow \mathcal{B}$ with each of the triangles

$G(v,u) \rightarrow G(w,u) \rightarrow G(w,v) \rightarrow G(v,u)$ for $0 \leq u < v < w < n$, exact over $\mathcal{B}$ (see Ch.1 §4). Let $J_{(u,v)}$ denote the adjoint of the evaluation functor $E_{(u,v)}: \text{Tgl}_n \mathcal{B} \rightarrow \mathcal{B}$ (see Theorem 10).

We now come to the definition of the graded functor $K_n: \mathcal{P}_n \rightarrow \mathcal{E}_n$. Take any object A of $\mathcal{P}_n$; then $K_n A = G: \mathcal{I}_n \rightarrow \mathcal{B}$ is the functor determined by the following

definitions:

(a) $G(u, v) = H(A_u/A_v)$ for $0 \leq v < u < n$;

(b) $G(u, v) \rightarrow G(u+1, v)$ is the morphism of degree 0 induced on homologies by the chain map

$A_u/A_v \rightarrow A_{u+1}/A_v$ coming from the inclusion $A_u \leq A_{u+1}$, for $0 \leq v < u < n-1$;

(c) $G(u, v) \rightarrow G(u, v+1)$ is the morphism of degree 0 induced on homologies by the chain map

$A_u/A_v \rightarrow A_u/A_{v+1}$ coming from the inclusion $A_v \leq A_{v+1}$, for $0 < v+1 < u < n$;

(d) $G(n-1, v) \rightarrow G(v, 0)$ is the morphism of degree -1 in the homology triangle of the short exact sequence $0 \rightarrow A_v \rightarrow A_{n-1} \rightarrow A_{n-1}/A_v \rightarrow 0$, for $0 < v < n-1$.

That the above definitions do determine a functor G follows from the above since:

(e) each of the composites

$$A_v/A_{v-1} \rightarrow A_{v+1}/A_{v-1} \rightarrow A_{v+1}/A_v \quad \text{for } 0 < v < n-1$$

is zero, and the diagram

$$\begin{array}{ccccccc}
 0 & \rightarrow & A_{n-2} & \rightarrow & A_{n-1} & \rightarrow & A_{n-1}/A_{n-2} \rightarrow 0 \\
 & & \downarrow & & \parallel & & \downarrow \\
 0 & \rightarrow & A_{n-1} & \xrightarrow{=} & A_{n-1} & \rightarrow & 0 \\
 & & \downarrow & & \parallel & & \downarrow \\
 0 & \rightarrow & A_1 & \rightarrow & A_{n-1} & \rightarrow & A_{n-1}/A_1 \rightarrow 0
 \end{array}$$

commutes with exact rows;

(f) each of the squares

$$\begin{array}{ccc} A_u/A_v & \longrightarrow & A_{u+1}/A_v \\ \downarrow & & \downarrow \\ A_u/A_{v+1} & \longrightarrow & A_{u+1}/A_{v+1} \end{array} \text{ for } 0 < v+1 < u < n-1$$

commutes, and each of the diagrams

$$\begin{array}{ccccccc} 0 & \rightarrow & A_v & \longrightarrow & A_{n-1} & \longrightarrow & A_{n-1}/A_v \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & A_{v+1} & \longrightarrow & A_{n-1} & \longrightarrow & A_{n-1}/A_{v+1} \rightarrow 0 \end{array} \text{ for } 0 < v < n-2$$

commutes with exact rows.

G is defined on morphisms of degree $0, -1$ as in (g), (h), (i).

For $0 \leq u < v < w < n$ we have a commutative diagram:

$$\begin{array}{ccccccc} 0 & \rightarrow & A_v & \longrightarrow & A_{n-1} & \longrightarrow & A_{n-1}/A_v \rightarrow 0 \\ & & \parallel & & \uparrow & & \uparrow \\ 0 & \rightarrow & A_v & \longrightarrow & A_w & \longrightarrow & A_w/A_v \rightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \rightarrow & A_v/A_u & \longrightarrow & A_w/A_u & \longrightarrow & A_w/A_v \rightarrow 0 \end{array}$$

with exact rows. From (g), (h) it now follows that the homology triangle of the bottom row of this diagram is:

$$G(v, u) \rightarrow G(w, u) \rightarrow G(w, v) \rightarrow G(v, u).$$

So $G = K_n A \in \mathcal{E}_n$.

Take any morphism $f: A \rightarrow A'$ of degree p of $\text{Flt}_n^{\mathcal{C}}$, and put $G = K_n A$, $G' = K_n A'$. For $0 \leq v < u < n$ let

$f_{u,v}$ be the morphism of degree p defined by the commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A_v & \longrightarrow & A_u & \longrightarrow & A_u/A_v \longrightarrow 0 \\ & & \downarrow f_v & & \downarrow f_u & & \downarrow f_{u,v} \\ 0 & \longrightarrow & A'_v & \longrightarrow & A'_u & \longrightarrow & A'_u/A'_v \longrightarrow 0 . \end{array}$$

We prove that $\alpha = K_n[f]; G \rightarrow G'$ given by:

$$\alpha(u,v) = Hf_{u,v} ,$$

is a natural transformation of degree p ; it is clear that $f \in B_p[A, A']$ implies $f_{u,v} \in B_p[A_u/A_v, A'_u/A'_v]$ and hence $\alpha_{(u,v)} = 0$. By (g), (h) it suffices to show that α commutes with morphisms of the type (b), (c), and commutes with morphisms of degree -1 of the type (d) when p is even and anticommutes when p is odd. The first part is clear and the second part follows from the commutative diagrams:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A_v & \longrightarrow & A_{n-1} & \longrightarrow & A_{n-1}/A_v \longrightarrow 0 \\ & & \downarrow f_v & & \downarrow f_{n-1} & & \downarrow f_{n-1,v} \\ 0 & \longrightarrow & A'_v & \longrightarrow & A'_{n-1} & \longrightarrow & A'_{n-1}/A'_v \longrightarrow 0 \text{ for } 0 < v < n-1, \end{array}$$

with exact rows and the equation:

$$d^{A'} \cdot f_{n-1} = (-1)^p f_{n-1} \cdot d^A .$$

Hence we have defined $K_n: \mathcal{P}_n \rightarrow \mathcal{E}_n$; it is clearly a functor since H is. We shall not distinguish between

K_n and the composite $\mathcal{P}_n \xrightarrow{K_n} \mathcal{E}_n \rightarrow \text{Tgl}_n \mathcal{B}$ unless confusion is likely.

Let $\text{Flt}_n^{\mathcal{P}} \mathcal{C}$ denote the full sub-DG-category of $\text{Flt}_n \mathcal{C}$ with objects those $A = (A_p)_{0 \leq p \leq n}$ with A_p/A_{p-1} projective for $0 < p \leq n$. Clearly $\text{Flt}_n^{\mathcal{P}} \mathcal{C}$ is stable and has the extension axiom. Put $\mathcal{P}_n^{\mathcal{P}} = \text{HFlt}_n^{\mathcal{P}} \mathcal{C}$; by Theorem 17 this is Verdier triangulated.

If $A' \rightarrow A \rightarrow A''$ is a pses in $\text{Flt}_n \mathcal{C}$ with deviation $\delta = (\delta_p)$ for some splitting, then each $A'_u/A'_v \rightarrow A_u/A_v \rightarrow A''_u/A''_v$ is a pses in $\mathcal{C}$ with deviation $\delta_{u,v}$ (see definition of K_n on morphisms) for the induced splitting. It follows that $K_n: \mathcal{P}_n \rightarrow \mathcal{E}_n$ is homological.

Let $K_n^{\mathcal{P}}$ be the restriction of K_n to $\mathcal{P}_n^{\mathcal{P}}$; then $K_n^{\mathcal{P}}$ is also homological.

Note that, for $A \in \mathcal{P}_n^{\mathcal{P}}$, each A_u/A_v (for $0 \leq v < u \leq n$) is projective.

§2. K_n -dimension.

In order to apply the theory of Ch.2 to the functor K_n we investigate which filtered complexes A have $K_n\text{-dim } A = r$ for a given $r \geq 0$. To do this it is necessary to have an intrinsic characterisation of the projectives in the graded category $\text{Tgl}_n \mathcal{B}$. The proof of the lemma giving this characterisation is postponed until a later section.

We now prove a result corresponding to Theorem 2.

Lemma 53. If $A \in \mathcal{P}_n^P$, $C \in \mathcal{P}_n$ and $A' \in \mathcal{P}_{n-1}^P$, $C'' \in \mathcal{P}_{n-1}$ are given by:

$$A'_r = A_{r+1}/A, \quad C''_r = C_r \quad \text{for } 0 < r < n-1;$$

then the sequence

$$0 \rightarrow [A', C''; \text{Flt}_{n-1} \mathcal{C}] \xrightarrow{u} [A, C; \text{Flt}_n \mathcal{C}] \xrightarrow{v} \sum_{0 < r < n} [A_r/A_{r-1}, C_r/C_{r-1}; \mathcal{C}] \rightarrow 0$$

is exact, where $u(g)_{r+1}$ is the composite

$$A_{r+1} \rightarrow A_{r+1}/A_1 \xrightarrow{g_r} C_r \xrightarrow{j_r} C_{r+1} \quad \text{and} \quad v(f)_r = f_{r,r-2}.$$

The connecting map Δ of the homology triangle of this short exact sequence is given by:

$$\Delta(h)_r = \Delta''_r(h_{r+1}t'_r) - (-1)^P[t''_r h_1 \delta'_r]$$

for $h \in \sum_{0 < r < n} \mathbb{Z}_P[A_r/A_{r-1}, C_r/C_{r-1}]$, where δ'_r is the deviation of the pses $A_1 \rightarrow A_{r+1} \rightarrow A_{r+1}/A_1$ for some splitting, Δ''_r is the connecting map of the homology

triangle of the short exact sequence:

$$0 \rightarrow [A'_r, C_r] \rightarrow [A'_r, C_{r+1}] \xrightarrow{w_r} [A'_r, C_{r+1}/C_r] \rightarrow 0 ,$$

t'_r is the epimorphism $A_{r+1}/A_1 \rightarrow A_{r+1}/A_r$, and t''_r is the monomorphism $C_1 \rightarrow C_r$.

Proof. Since $A_{r+1} \rightarrow A_{r+1}/A_1$ is epimorphic and $C_r \rightarrow C_{r+1}$ is monomorphic, u is injective. Also $C_r \rightarrow C_{r+1}$ induces the zero map $C_r/C_{r-1} \rightarrow C_{r+1}/C_r$, so $vu = 0$. Suppose $f \in [A, C]$ with each $f_{r+1, r} = 0$. For $0 < r < n-1$, f_{r+1} then induces $g'_r: A_{r+1} \rightarrow C_r$,

whence f_r is the composite $A_r \rightarrow A_{r+1} \xrightarrow[g'_r]{g'_r} C_r$. But $f_1 = 0$, so the composite $A_1 \rightarrow A_{r+1} \xrightarrow[g'_r]{g'_r} C_r$ is zero, and g'_r induces $g_r: A_{r+1}/A_1 \rightarrow C_r$. This gives g with $u(g) = f$.

We now prove that v is surjective. Suppose we have $h_r \in [A_r/A_{r-1}, C_r/C_{r-1}]$ for $0 < r < n$. Let $f_1 = h_1$. Given f_{r-1} (with $r > 1$), there exists f_r such that the diagram:

$$\begin{array}{ccccc} A_{r-1} & \longrightarrow & A_r & \longrightarrow & A_r/A_{r-1} \\ \downarrow f_{r-1} & & \downarrow f_r & & \downarrow h_r \\ C_{r-1} & \longrightarrow & C_r & \longrightarrow & C_r/C_{r-1} \end{array}$$

commutes, since A_r/A_{r-1} is projective. Choose such a sequence $f = (f_r)$ inductively. Then $h_r = v(f)_r$.

It remains to prove the formula for Δ . Take $h_r \in Z_p[A_r/A_{r-1}, C_r/C_{r-1}]$ for $0 < r < n$. Choose f such that $v(f)_r = h_r$. Let $(\tilde{p}_r, \tilde{i}_r)$ be a splitting of

$A_1 \xrightarrow{i_r} A_{r+1} \xrightarrow{p_r} A_{r+1}/A_1$ with deviation δ'_r . Now

$h_{r+1} t'_r \in Z_p[A'_r, C_{r+1}/C_r]$, and from the commutative diagram:

$$\begin{array}{ccccc} A_{r+1}/A_1 & \xrightarrow{i_r} & A_{r+1} & \xrightarrow{f_{r+1}} & C_{r+1} \\ & \searrow t'_r & \downarrow & & \downarrow \\ & & A_{r+1}/A_r & \xrightarrow{h_{r+1}} & C_{r+1}/C_r \end{array}$$

we have $w_r(f_{r+1} \tilde{i}_r) = h_{r+1} t'_r$. Thus we may choose

$k_r \in \Delta''_r(h_{r+1} t'_r)$ with $j_r k_r = D(f_{r+1} i)$. Then:

$$\begin{aligned} (Df)_{r+1} &= D(f_{r+1}(\tilde{i}_r p_r + i_r \tilde{p}_r)) \\ &= j_r k_r p_r + D(t''_{r+1} h_1 \tilde{p}_r) \\ &= j_r k_r p_r + (-1)^{p_{t''_{r+1} h_1}} D(\tilde{p}_r) \\ &= j_r (k_r - (-1)^{p_{t''_{r+1} h_1}} \delta'_r) p_r \end{aligned}$$

since $D(\tilde{p}_r) = -\delta'_r p_r$ (see Ch.1 §2). Thus

$$k_r - (-1)^{p_{t''_{r+1} h_1}} \delta'_r \in \Delta(h)_r$$

and the result follows. //

The proof of the following lemma will be left until a later section.

Lemma 54. The following conditions on an object G of $\text{Tgl}_n \mathcal{B}$ are equivalent:

(a) G is projective in $\text{Tgl}_n \mathcal{B}$;

(b) $G \cong \sum_{0 \leq v < u < n} J_{(u,v)} P(u,v)$ where each $P(u,v)$ is projective in $\mathcal{B}$;

(c) each triangle

$$G(v,u) \rightarrow G(w,u) \rightarrow G(w,v) \rightarrow G(v,u)$$

is projective in $\text{Tgl} \mathcal{B}$ for $0 \leq u < v < w < n$.

Lemma 55. If F is a projective object of $\text{Tgl}_n \mathcal{B}$ and G is an object of $\mathcal{E}_n$ then the triangle

$$[F', G''; \mathcal{E}_{n-1}] \xrightarrow{U} [F, G; \mathcal{E}_n] \xrightarrow{V} \sum_{0 < r < n} [F(r, r-1), G(r, r-1); \mathcal{B}] \xrightarrow{W} [F', G''; \mathcal{E}_{n-1}]$$

is exact over G^l , where $F'(x, y) = F(x+1, y+1)$,

$G''(x, y) = G(x, y)$ for $0 \leq y < x < n-1$ and the maps of

F', G'' come from those of F, G , $u(\alpha)_{(x,y)}$ is the composite

$F(x, y) \rightarrow F(x+1, y+1) \xrightarrow{\alpha(x,y)} G(x, y)$, $v(\alpha)_r = \alpha(r, r-1)$, and

$w(\gamma)_{(x,y)}$ is the sum of the map

$$F(x+1, y+1) \rightarrow F(x+1, x) \xrightarrow{\gamma_x} G(x+1, x) \rightarrow G(x, y)$$

and $(-1)^p$ of the map

$$F(x+1, y+1) \rightarrow F(y+1, y) \xrightarrow{\gamma_y} G(y+1, y) \rightarrow G(x, y) .$$

Proof. It is readily checked that u, v, w as given above are well-defined. If the triangle is exact with F replaced by U, V then it is exact with F replaced by

$U \oplus V$. So, by Lemma 54, it suffices to prove the result for $F = J_{(\mathbb{K}, y)} P$, where P is projective in $\mathcal{B}$. Note that

$$\begin{aligned} F(r, s) &= P \quad \text{for } y \leq s < x \leq r, \\ &= L^{-1}P \quad \text{for } s < y \leq r < x, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

(a) Suppose $y > 0$. In this case the triangle becomes:

$$[P, G(x-1, y-1)] \rightarrow [P, G(x, y)] \rightarrow [P, G(x, x-1)] \oplus [P, LG(y, y-1)] \rightarrow [P, G(x-1, y-1)].$$

But P is projective and the triangle:

$$G(x-1, y-1) \rightarrow G(x, y) \rightarrow G(x, x-1) \oplus LG(y, y-1) \rightarrow G(x-1, y-1)$$

is exact since the rows of the commutative diagram:

$$\begin{array}{ccccccc} G(x-1, y) & \rightarrow & G(x, y) & \rightarrow & G(x, x-1) & \rightarrow & G(x-1, y) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ G(x-1, y) & \rightarrow & LG(y, y-1) & \rightarrow & LG(x-1, y-1) & \rightarrow & G(x-1, y) \end{array}$$

are exact triangles (see [19]), so we have the result.

(b) Suppose $y = 0$. In this case the triangle becomes:

$$[P, G(n-1, x-1)] \rightarrow [P, G(x, 0)] \rightarrow [P, G(x, x-1)] \oplus [P, G(n-1, 0)] \rightarrow [P, G(n-1, x-1)].$$

As in (a) we have P projective and a Mayer-Vietoris exact triangle:

$$G(n-1, x-1) \rightarrow G(x, 0) \rightarrow G(x, x-1) \oplus G(n-1, 0) \rightarrow G(n-1, x-1),$$

so again the result follows. //

For $0 \leq v < u < n$ we now define a DG-functor

$$\Gamma_{(u, v)}: \mathcal{L} \rightarrow \text{Flt}_n \mathcal{L} \quad \text{which induces the adjoints } \mathcal{P}_2 \rightarrow \mathcal{P}_n$$

to the evaluation functor given by $A \mapsto A_u/A_v$ and $f \mapsto f_{u,v}$.

For $C \in \mathcal{C}$, let $\gamma(C) \in \mathcal{C}$ be the complex determined by the graded object $L^{-1}C \oplus C$ and the differential

$$\begin{pmatrix} L^{-1}d & 1^{-1} \\ 0 & d \end{pmatrix}. \text{ Let } \gamma(f) \in [\gamma(C), \gamma(C')]_p \text{ denote } \begin{pmatrix} L^{-1}f & 0 \\ 0 & f \end{pmatrix}$$

for $f \in [C, C']_p$. Then $\gamma: \mathcal{C} \rightarrow \mathcal{C}$ is a DG-functor.

For $0 \leq v < u < n$ and $C \in \mathcal{C}$ we define

$\Gamma_{(u,v)}C \in \text{Flt}_n \mathcal{C}$. If $v = 0$ then:

$$\begin{aligned} (\Gamma_{(u,0)}C)_r &= 0 \text{ for } 1 \leq r < u, \\ &= C \text{ for } u \leq r < n, \end{aligned}$$

If $v > 0$ then:

$$\begin{aligned} (\Gamma_{(u,v)}C)_r &= 0 \text{ for } 1 \leq r < v, \\ &= L^{-1}C \text{ for } v \leq r < u, \\ &= \gamma(C) \text{ for } u \leq r < n. \end{aligned}$$

Each $\Gamma_{(u,v)}$ becomes a DG-functor if we define $\Gamma_{(u,v)}^f$ by replacing C by f in the above, for any protomorphism f of any degree.

Notice that $K_n \Gamma_{(u,v)} = J_{(u,v)}^H$.

Let $\theta_{u,v}: [\Gamma_{(u,v)}C, A; \text{Flt}_n \mathcal{C}] \rightarrow [C, A_u/A_v; \mathcal{C}]$ be the DG-natural transformation given by $\theta_{u,v}(f) = f_{u,v}$ for $f \in [\Gamma_{(u,v)}C, A]_p$ (noting that $C = (\Gamma_{(u,v)}C)_u / (\Gamma_{(u,v)}C)_v$).

Lemma 56. For $0 \leq v < u < n$ and $A \in \text{Flt}_n^P \mathcal{C}$ the chain map $\theta = \theta_{u,v}: [\Gamma_{(u,v)}C, A] \rightarrow [C, A_u/A_v]$ has a right

chain inverse which is a left homotopy inverse.

Proof. If $v = 0$ then θ is clearly an isomorphism, so we suppose $v > 0$. Let $(\tilde{p}, \tilde{i})$ be a splitting of the pses $A_v \xrightarrow{i} A_u \xrightarrow{p} A_u/A_v$ with deviation δ . Define $\omega: [C, A_u/A_v] \rightarrow [\Gamma_{(u,v)} C, A]$ as follows. For $f \in [C, A_u/A_v]_m$ let $g' = (-1)^m \delta f \tilde{i} \in [L^{-1}C, A_v]_m$ and $g = (ig', \tilde{i}f) \in [\gamma(C), A_u]_m$; then:

$$\begin{aligned} \omega(f)_r &= 0 \quad \text{for } 1 \leq r < v, \\ &= L^{-1}C \xrightarrow{g'} A_v \rightarrow A_r \quad \text{for } v \leq r < u, \\ &= \gamma(C) \xrightarrow{g} A_u \rightarrow A_r \quad \text{for } u \leq r < n. \end{aligned}$$

From the calculation:

$$\begin{aligned} D_g &= d(ig', \tilde{i}f) - (-1)^m (ig', \tilde{i}f) \begin{pmatrix} L^{-1}d & \tilde{i}^{-1} \\ 0 & d \end{pmatrix} \\ &= (dig' - (-1)^m ig' L^{-1}d, D(\tilde{i}f) - (-1)^m ig' \tilde{i}^{-1}) \\ &= ((-1)^m i\delta.Df.\tilde{i}, D\tilde{i}.f + \tilde{i}.Df - i\delta f) \\ &= ((-1)^m i\delta.Df.\tilde{i}, \tilde{i}.Df), \end{aligned}$$

it follows that ω is a chain map. From the commutative diagram:

$$\begin{array}{ccccc} L^{-1}C & \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} & \gamma(C) & \xrightarrow{(0,1)} & C \\ g' \downarrow & & g \downarrow & & \downarrow f \\ A_v & \xrightarrow{i} & A_u & \xrightarrow{p} & A_u/A_v, \end{array}$$

it follows that $\theta\omega = 1$.

Define $\sigma: [\Gamma_{(u,v)} C, A] \rightarrow [\Gamma_{(u,v)} C, A]$ of degree 1 as follows. If $h \in [\Gamma_{(u,v)} C, A]_m$ then $h_u = (ih_v, k)$ for

some $k \in [C, A_u]_m$ with $pk = h_{u,v}$. Let $s' = (-1)^m \tilde{p}k \wr \in [L^{-1}C, A_v]_{m+1}$ and $s = (is', 0) \in [\gamma(C), A]_{m+1}$. Then:

$$\begin{aligned} \sigma(h) &= 0 \text{ for } 1 \leq r < v, \\ &= L^{-1}C \xrightarrow{s'} A_v \rightarrow A_r \text{ for } v \leq r < u, \\ &= \gamma(C) \xrightarrow{s} A_u \rightarrow A_r \text{ for } u \leq r < n. \end{aligned}$$

Let $f = pk$, $g' = (-1)^m \delta f$, $g = (ig', \tilde{f}f)$. Then:

$$\begin{aligned} D\sigma(h)_u &= Ds \\ &= D((-1)^m i\tilde{p}k \wr, 0) \\ &= ((-1)^m i.D(\tilde{p}k).\wr, i\tilde{p}k) \\ &= ((-1)^m i(-\delta pk + \tilde{p}.Dk)\wr, k - \tilde{f}f) \\ &= (-g' + (-1)^m i\tilde{p}.Dk.\wr, k - \tilde{f}f), \text{ and} \end{aligned}$$

$$\begin{aligned} \sigma(Dh)_u &= \sigma(i.Dh_v, Dk - (-1)^m ih_v \wr^{-1})_u \\ &= ((-1)^{m+1} i\tilde{p}(Dk - (-1)^m ih_v \wr^{-1})\wr, 0) \\ &= (ih_v - (-1)^m i\tilde{p}.Dk.\wr, 0); \text{ so} \end{aligned}$$

$$\begin{aligned} D\sigma(h)_u + \sigma(Dh)_u &= (ih_v, k) - (g', \tilde{f}f) \\ &= h_u - g. \end{aligned}$$

It follows that $\sigma: 1 \simeq \omega\theta$. //

For $0 < u < n$ let $\theta_{u,0}^*: [A, \Gamma_{(u,0)} C; \text{Flt}_n \mathcal{L}] \rightarrow [A_{n-1}/A_{u-1}, C; \mathcal{L}]$ be the DG-natural isomorphism given by $\theta_{u,0}^*(f) = f_{n-1,u-1}$. For $0 < v < u < n$ let $\theta_{u,v}^*: [A, \Gamma_{(u,v)} C; \text{Flt}_n \mathcal{L}] \rightarrow [A_{u-1}/A_{v-1}, L^{-1}C; \mathcal{L}]$ be the

DG-natural transformation given by $\theta^*_{u,v}(f) = f_{u-1,v-1}$.

The following lemma can be proven similarly to

Lemma 56.

Lemma 57. For $0 < v < u < n$ and $\Lambda \in \text{Flt}_n^P \mathcal{C}$ the chain map $\theta^*_{u,v}: [A, \Gamma_{(u,v)} C] \rightarrow [A_{u-1}/A_{v-1}, L^{-1}C]$ induces an isomorphism on homologies. //

Theorem 58. For $X \in \mathcal{P}_n^P$ the following conditions are equivalent:

- (a) X is K_n -projective;
- (b) X is K_n^P -projective;
- (c) each X_u/X_v is CE-projective for $0 \leq v < u < n$.

Proof. (a) $\Rightarrow$ (b) is trivial.

(b) $\Rightarrow$ (c). Suppose X is K_n^P -projective and $0 \leq v < u < n$. Then $K_n^P X$ is projective in $\text{Tgl}_n(\mathcal{B})$, and so by Lemma 54 $(K_n^P X)(u,v) = H(X_u/X_v)$ is projective. For any $C \in \mathcal{C}^P$ set $\Gamma C = \Gamma_{(v+1,0)} C$ and $JHC = J_{(v+1,0)} HC$ if $u = n-1$, $\Gamma C = \Gamma_{(u+1,v+1)} LC$ and $JHC = J_{(u+1,v+1)} LHC$ if $u < n-1$. Then $\phi^*: [K_n X, JHC] \rightarrow [H(X_u/X_v), HC]$ is an isomorphism, where $\phi^*(\alpha) = \alpha_{(u,v)}$; and the diagram:

$$\begin{array}{ccc} H X, \Gamma C & \xrightarrow{K_n} & [K_n X, JHC] \\ H\theta^* \downarrow & & \downarrow \phi^* \\ H[X_u/X_v, C] & \xrightarrow{H} & [H(X_u/X_v), HC] \end{array}$$

commutes. By Lemma 57 $H\theta^*$ is an isomorphism, and K_n

is an isomorphism since X is K_n^P -projective. So $H: H[X_u/X_v, C] \rightarrow [H(X_u/X_v), HC]$ is an isomorphism. Thus X_u/X_v is H^P -projective. So by Theorem 43, X_u/X_v is CE-projective.

(c) $\Rightarrow$ (a). For $n = 2$ the result follows from Theorem 43. We prove the result by induction on n . Suppose $n > 2$ and that (c) $\Rightarrow$ (a) for $n-1$. Take $X \in \mathcal{P}_n^P$ satisfying (c). For $0 \leq u < v < w < n$, $0 \rightarrow X_v/X_u \rightarrow X_w/X_u \rightarrow X_w/X_v \rightarrow 0$ is a short exact sequence of CE-projective complexes. Thus, by Theorem 43, the triangles:

$$H(X_v/X_u) \rightarrow H(X_w/X_u) \rightarrow H(X_w/X_v) \rightarrow H(X_v/X_u),$$

are projective. So, by Lemma 54, $K_n X$ is projective in $\text{Tgl}_n \mathcal{B}$. Suppose $A \in \mathcal{P}_n$. By Lemma 53 we have a short exact sequence:

$$0 \rightarrow [X', A''] \rightarrow [X, A] \rightarrow \sum_{0 < r < n} [X_r/X_{r-1}, A_r/A_{r-1}] \rightarrow 0,$$

giving an exact triangle:

$$H[X', A''] \rightarrow H[X, A] \rightarrow \sum_{0 < r < n} H[X_r/X_{r-1}, A_r/A_{r-1}] \rightarrow H[X', A''].$$

By Lemma 55 we have an exact triangle:

$$[K_{n-1} X', K_{n-1} A''] \rightarrow [K_n X, K_n A] \rightarrow \sum_{0 < r < n} [H(X_r/X_{r-1}), H(A_r/A_{r-1})] \rightarrow [K_{n-1} X', K_{n-1} A''].$$

From the descriptions of the maps in Lemmas 53 and 55 we see that $(K_{n-1}, K_n, \Sigma H)$ gives a morphism of the first triangle into the second. By induction X' is K_{n-1} -projective since

it satisfies (c). So this K_{n-1} and ΣH are isomorphisms. By the "five-lemma" it now follows that $K_n: H[X, A] \rightarrow [K_n X, K_n A]$ is an isomorphism. So X is K_n -projective. //

Theorem 59. Each $A \in \mathcal{P}_n^P$ is K_n^P -developable (and hence K_n^P -resoluble).

Proof. For each A_u/A_v ($0 \leq v < u < n$) choose a CE-projective $Y_{u,v}$ and a CE-epimorphism $\eta_{u,v}: Y_{u,v} \rightarrow A_u/A_v$. By Lemma 56 there exists $\bar{\eta}_{u,v}: \Gamma(u,v)Y_{u,v} \rightarrow A$ with $\theta_{u,v}(\bar{\eta}_{u,v}) = \eta_{u,v}$. Put $X = \sum_{0 \leq v < u < n} \Gamma(u,v)Y_{u,v}$, and let $\epsilon: X \rightarrow A$ be the unique map determined by the $\bar{\eta}_{u,v}$. Then each $\epsilon_{u,v}: X_u/X_v \rightarrow A_u/A_v$ is a CE-epimorphism and X_u/X_v is CE-projective. By Theorem 58, X is K_n^P -projective. Also ϵ is a K_n -epimorphism. So A is K_n^P -developable by Lemma 27(b). //

Theorem 60. For $A \in \mathcal{P}_n^P$, $K_n\text{-dim } A \leq K_n^P\text{-dim } A = \max\{\text{CE-dim } A_u/A_v \mid 0 \leq v < u < n\}$.

Proof. Using Theorem 58 we see that a K_n^P -development of A is a K_n -development, and so $K_n\text{-dim } A \leq K_n^P\text{-dim } A$. We now prove inductively that $K_n^P\text{-dim } A \leq r \iff \text{CE-dim } A_u/A_v \leq r$ for $0 \leq v < u < n$. For $r = 0$ we have the result by Theorem 58. Suppose the result true for $r-1$ where $r > 0$, and take $A \in \mathcal{P}_n^P$. By

Theorem 59 there is a K_n^P -development $C \rightarrow X \rightarrow A \rightarrow C$ of A . For $0 \leq v < u < n$ this gives an H^P -development $C_u/C_v \rightarrow X_u/X_v \rightarrow A_u/A_v \rightarrow C_u/C_v$ of A_u/A_v . By Theorem 30, $K_n^P\text{-dim } A \leq r \iff K_n^P\text{-dim } C \leq r-1$ (using Theorem 59). By induction $K_n^P\text{-dim } C \leq r-1 \iff \text{CE-dim } C_u/C_v \leq r-1$ for $0 \leq v < u < n$. By Theorem 30, $\text{CE-dim } C_u/C_v \leq r-1 \iff \text{CE-dim } A_u/A_v \leq r$ (using Theorems 44 and 46). //

Main Theorem. If the projective dimension of $\mathcal{D}$ is 1 then $K_n^P: \mathcal{P}_n^P \rightarrow \mathcal{E}_n$ is full and dense; the kernel $\mathcal{K}$ of K_n^P is given by a natural isomorphism:

$$L[A, C; \mathcal{K}] \cong \text{Ext}^1[K_n A, K_n C; \text{Tgl}_n \mathcal{B}].$$

Proof. Each object of $\mathcal{E}$ has CE-dimension ≤ 1 and so each object of $\mathcal{P}_n^P$ has K_n^P -dimension ≤ 1 by Theorem 60. So K_n^P is full with $\mathcal{K}$ as above by Corollary 21. It remains to prove that K_n^P is dense.

Suppose $G \in \mathcal{E}_n$. $\text{Tgl}_n \mathcal{B}$ has enough projectives (Theorem 15) so there exists a short exact sequence

$$0 \rightarrow F' \xrightarrow{\mathcal{K}} F \rightarrow G \rightarrow 0 \text{ with } F \text{ projective. By Lemma 54}$$

all the triangles $F(v, u) \rightarrow F(w, u) \rightarrow F(w, v) \rightarrow F(v, u)$

$(0 \leq u < v < w < n)$ are projective. But the corresponding

triangles of G are exact. By Theorem 51 this implies

that the triangles of F' are projective. So F' is

projective by Lemma 54. By Lemma 54 we may suppose

$F' = \sum J_{(u,v)} P'(u,v)$, $F = \sum J_{(u,v)} P(u,v)$ with $P'(u,v)$, $P(u,v)$ projectives in $\mathcal{B}$, with the summations over $0 \leq v < u < n$. $P'(u,v), P(u,v)$ may be regarded as complexes over $\mathcal{Q}$ with zero differentials, and so are CE-projectives.

Now put

$$Q' = \sum \Gamma_{(u,v)} P'(u,v), \quad Q = \sum \Gamma_{(u,v)} P(u,v).$$

Then Q', Q are K_n^P -projective by Theorems 43 and 58; moreover, from the equations $J_{(u,v)}^H = K_n \Gamma_{(u,v)}$, we have

$F' = K_n Q'$, $F = K_n Q$. So $K_n: H[Q', Q] \rightarrow [F', F]$ is an isomorphism. Let $\Delta = K_n^{-1} \kappa$. But $\text{Flt}_n^P \mathcal{C}$ has the extension axiom so there exists a pses $Q \rightarrow A \rightarrow LQ'$ with deviation $\Delta \cdot \mathcal{I}^{-1}$. Now K_n is homological and κ is a monomorphism so the sequence $0 \rightarrow F' \xrightarrow{\kappa} F \rightarrow K_n A \rightarrow 0$ is exact; hence $G \cong K_n A$. //

§3. Three-diagrams in p dimensions.

The remainder of this chapter is aimed at proving Lemma 54. The results of this section will be needed in the proof.

Recall the definition of the additive category (Ch.1 §2). A functor $F: \mathcal{F} \rightarrow \mathcal{D}$ is a short sequence in $\mathcal{D}$. We will say F is exact when the sequence $0 \rightarrow F(-1) \rightarrow F(0) \rightarrow F(1) \rightarrow 0$ is exact.

Proposition 61. If $F: \mathcal{F} \rightarrow \mathcal{D}$ is exact with $F(-1), F(1)$ projective then F is projective in $[\mathcal{F}, \mathcal{D}]$.

Proof. If P, Q are projectives in $\mathcal{D}$ then the short sequences $P \xrightarrow{1} P \rightarrow 0, 0 \rightarrow Q \xrightarrow{1} Q$ are easily checked to be projectives in $[\mathcal{F}, \mathcal{D}]$. But F is isomorphic to the direct sum of such sequences, and so also is projective. //

Let $\mathcal{F}_p$ denote the tensor product of p (≥ 0) copies of $\mathcal{F}$. Those $\underline{x} = (x_1, \dots, x_p) \in \mathcal{F}_p$ with no $x_i = 0$ will be called p -corners, for $p > 0$; the object of $\mathcal{F}_0$ is the 0-corner.

Let $t_i^-, t_i, t_i^+: \mathcal{F}_{p-1} \rightarrow \mathcal{F}_p$ be the functors given by:

$$t_i^-(x_1, \dots, x_{p-1}) = (x_1, \dots, x_{i-1}, -1, x_i, \dots, x_{p-1}),$$

$$t_i(x_1, \dots, x_{p-1}) = (x_1, \dots, x_{i-1}, 0, x_i, \dots, x_{p-1}),$$

$$t_i^+(x_1, \dots, x_{p-1}) = (x_1, \dots, x_{i-1}, 1, x_i, \dots, x_{p-1}),$$

for $1 \leq i \leq p$. The maps $-1 \rightarrow 0, 0 \rightarrow 1$ give natural transformations $t_i^- \rightarrow t_i, t_i \rightarrow t_i^+$. Then

$F \mapsto (Ft_i^- \rightarrow Ft_i \rightarrow Ft_i^+)$ gives an isomorphism:

$$(1) \quad [f_p, \mathcal{Q}] \cong [f, [f_{p-1}, \mathcal{Q}]] .$$

A functor $F: f_p \rightarrow \mathcal{Q}$ will be called a p-dimensional three-diagram in $\mathcal{Q}$.

The p-dimensional three-diagram F will be called exact if for each i with $1 \leq i \leq p$ the sequence:

$$(2) \quad 0 \rightarrow Ft_i^- \rightarrow Ft_i \rightarrow Ft_i^+ \rightarrow 0 ,$$

is exact in $[f_{p-1}, \mathcal{Q}]$.

Proposition 62. (a) If F is an exact p-dimensional three-diagram in $\mathcal{Q}$ with Fx projective for all p-corners x , then F is projective in $[f_p, \mathcal{Q}]$.

(b) If $f: F \rightarrow F'$ is a map of $[f_p, \mathcal{Q}]$, where F, F' are exact and $f_x: Fx \rightarrow F'x$ is an isomorphism (respectively monomorphism; epimorphism) for all p-corners x , then $f: F \rightarrow F'$ is an isomorphism (respectively monomorphism; epimorphism).

Proof. (a) follows by a simple induction using Proposition 61 and (1); (b) follows by a simple induction using the "short-five lemma" and (1). //

For $y \in f_p$ let

$$(3) \quad S(y) = \{x | x \text{ is a p-corner, } x_i = y_i \text{ for } y_i \neq 0\} .$$

In particular, $S(0)$ is the set of p-corners.

We prove now, for $y \leq y'$, $x \in S(y)$, $x' \in S(y')$

and $\underline{x}' \leq \underline{x}$, that:

$$(4) \quad \underline{x} \in S(\underline{y}') \iff \underline{x}' \in S(\underline{y}) .$$

Suppose $\underline{x} \in S(\underline{y}')$ and $y_i \neq 0$. If $y_i' \neq 0$ then $x_i' = y_i' = x_i = y_i$; if $y_i' = 0$ then $\underline{y} \leq \underline{y}'$ implies $y_i = -1$, and $\underline{x}' \leq \underline{x}$ implies $-1 \leq x_i' = x_i = y_i = -1$, and so $x_i' = y_i$. This proves $\Rightarrow$. Suppose $\underline{x}' \in S(\underline{y})$ and $y_i' \neq 0$. If $y_i \neq 0$ then $x_i = y_i = x_i' = y_i'$; if $y_i = 0$ then $\underline{y} \leq \underline{y}'$ implies $y_i' = 1$, and $\underline{x}' \leq \underline{x}$ implies $1 = y_i' = x_i' \leq x_i \leq 1$, and so $x_i = y_i'$. This proves $\Leftarrow$.

Suppose for each p-corner $\underline{x}$ an object $D(\underline{x})$ of is given; for example, D could be a p-dimensional three-diagram. Define $\Sigma D: \mathcal{F}_p \rightarrow \mathcal{D}$ by:

$$(5) \quad (\Sigma D)\underline{y} = \sum_{\underline{x} \in S(\underline{y})} D(\underline{x}) ;$$

(6) if $\underline{y} \leq \underline{y}'$ then $(\Sigma D)\underline{y} \rightarrow (\Sigma D)\underline{y}'$ is the matrix with typical element $D(\underline{x}) \rightarrow D(\underline{x}')$ the identity if $\underline{x} = \underline{x}'$, zero otherwise.

We shall prove that the sequences:

$$0 \rightarrow (\Sigma D)t_i^- \rightarrow (\Sigma D)t_i \rightarrow (\Sigma D)t_i^+ \rightarrow 0 ,$$

are all split exact, and so:

(7) ΣD is an exact p-dimensional three-diagram.

This follows from:

(8) $S(t_j \underline{y})$ is the disjoint union of $S(t_j^- \underline{y})$ and $S(t_j^+ \underline{y})$ for $\underline{y} \in \mathcal{F}_{p-1}$.

To prove (8) we observe that $\underline{x} \in S(t_{j\underline{y}}^-)$ implies $x_j = -1$, and $\underline{x} \in S(t_{j\underline{y}}^+)$ implies $x_j = +1$, so $S(t_{j\underline{y}}^-)$ and $S(t_{j\underline{y}}^+)$ are disjoint. Also, if $\underline{x} \in S(t_{j\underline{y}})$ and $\underline{x} \notin S(t_{j\underline{y}}^-)$ then $\underline{x}$ is a corner and $x_j \neq -1$, so $x_j = 1$ and $\underline{x} \in S(t_{j\underline{y}}^+)$; this proves $S(t_{j\underline{y}})$ is the union of $S(t_{j\underline{y}}^-)$ and $S(t_{j\underline{y}}^+)$.

Thus we have

$$(9) \quad (\Sigma D)t_i = (\Sigma D)t_i^- \oplus (\Sigma D)t_i^+.$$

If F is a p -dimensional three-diagram then:

$$(10) \quad \Sigma(Ft_i^-) = (\Sigma F)t_i^-, \quad \Sigma(Ft_i^+) = (\Sigma F)t_i^+,$$

$$\text{and } \Sigma(Ft_i) = (\Sigma F)t_i.$$

A map $f: F \rightarrow \Sigma F$ which is the identity at the p -corners will be called a splitting of F ; if such an f exists we will say F splits.

Proposition 63. An exact p -dimension three-diagram which takes projective values at the p -corners splits.

Proof. Suppose $0 \rightarrow A \xrightarrow{i} B \xrightarrow{p} C \rightarrow 0$ is exact, A, C are projective and $f: A \rightarrow A'$, $g: C \rightarrow C'$ are maps in any abelian category. Let p be a left inverse of i . Then the following diagram commutes:

$$\begin{array}{ccccc} A & \xrightarrow{i} & B & \xrightarrow{p} & C \\ f \downarrow & & \begin{pmatrix} fp \\ gp \end{pmatrix} \downarrow & & \downarrow g \\ A' & \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} & A' \oplus C' & \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} & C' \end{array}.$$

We now prove the proposition by induction on p . For $p = 1$ the result follows from the above by taking $A' = A$, $C' = C$, $f = 1$ and $g = 1$. Suppose $p > 1$ and the result true for $p-1$. If F is an exact p -dim. three-diagram taking projective values at the corners then Ft_1^- , Ft_1^+ are exact and take projective values at the $(p-1)$ -corners, so by Proposition 62 are projective in $[J_{p-1}, \mathcal{Q}]$; also, in this category, the sequence (2) is exact with $i = 1$. By induction there exist maps $f: Ft_1^- \rightarrow \Sigma(Ft_1^-)$, $g: Ft_1^+ \rightarrow \Sigma(Ft_1^+)$ which are the identities at the $(p-1)$ -corners. By the above there exists a commutative diagram:

$$\begin{array}{ccccc} Ft_1^- & \longrightarrow & Ft_1 & \longrightarrow & Ft_1^+ \\ f \downarrow & & h \downarrow & & g \downarrow \\ (\Sigma F)t_1^- & \longrightarrow & (\Sigma F)t_1 & \longrightarrow & (\Sigma F)t_1^+ \end{array}$$

(using (10)). All the p -corners are images of $(p-1)$ -corners under t_1^- and t_1^+ . From (1) with $i = 1$ it follows that f, h, g determine a splitting of F . //

Proposition 64. Suppose $D(\underline{x})$, $D'(\underline{x})$ are objects of $\mathcal{Q}$ for each p -corner $\underline{x}$. Then $(\Sigma D, \Sigma D', [J_p, \mathcal{Q}])$ is isomorphic to the additive group of matrices

$$\sum_{\underline{x} \in S(\underline{Q})} D(\underline{x}) \longrightarrow \sum_{\underline{x} \in S(\underline{Q})} D'(\underline{x}) \text{ with elements } D(\underline{x}) \rightarrow D'(\underline{x}')$$

zero whenever $\underline{x}' \not\leq \underline{x}$; the isomorphism is given by

$$f = (f_{\underline{y}}) \mapsto f_{\underline{Q}}.$$

Proof. Suppose $f: \Sigma D \rightarrow \Sigma D'$; we show that f_0 is such a matrix. If $\underline{x}' \not\leq \underline{x}$, where $\underline{x}, \underline{x}'$ are corners, then there exists a j such that $x_j = -1, x'_j = 1$; so $\underline{x} \in S(t_j^- 0)$ and $\underline{x}' \in S(t_j^+ 0)$. Thus $D(\underline{x}) \rightarrow D'(\underline{x}')$ induced by f_0 factors as:

$$D(\underline{x}) \rightarrow (\Sigma D)t_j^- 0 \rightarrow (\Sigma D)_0 \xrightarrow{f_0} (\Sigma D')_0 \rightarrow (\Sigma D')t_j^+ 0 \rightarrow D'(\underline{x}')$$

but this is zero since the diagram:

$$\begin{array}{ccccc} (\Sigma D)t_j^- 0 & \longrightarrow & (\Sigma D)_0 & \longrightarrow & (\Sigma D)t_j^+ 0 \\ f_{t_j^- 0} \downarrow & & f_0 \downarrow & & \downarrow f_{t_j^+ 0} \\ (\Sigma D')t_j^- 0 & \longrightarrow & (\Sigma D')_0 & \longrightarrow & (\Sigma D')t_j^+ 0 \end{array}$$

commutes.

Suppose $g(\underline{x}, \underline{x}') : D(\underline{x}) \rightarrow D'(\underline{x}')$ is a map of $\mathcal{Q}$ for pairs $\underline{x}, \underline{x}'$ of p -corners with $\underline{x}' \leq \underline{x}$. For any $\underline{y} \in \mathcal{P}_p$ let $f_{\underline{y}} : (\Sigma D)\underline{y} \rightarrow (\Sigma D')\underline{y}$ be the matrix $\sum_{\underline{x} \in S(\underline{y})} D(\underline{x}) \rightarrow \sum_{\underline{x} \in S(\underline{y})} D'(\underline{x})$ with typical element $D(\underline{x}) \rightarrow D'(\underline{x}')$ equal to $g(\underline{x}, \underline{x}')$ if $\underline{x}' \leq \underline{x}$, and zero otherwise.

From (4) it follows that, for $\underline{y} \leq \underline{y}'$, the following square commutes:

$$\begin{array}{ccc} \sum_{\underline{x} \in S(\underline{y})} D(\underline{x}) & \longrightarrow & \sum_{\underline{x} \in S(\underline{y}')} D(\underline{x}) \\ f_{\underline{y}} \downarrow & & \downarrow f_{\underline{y}'} \\ \sum_{\underline{x} \in S(\underline{y})} D'(\underline{x}) & \longrightarrow & \sum_{\underline{x} \in S(\underline{y}')} D'(\underline{x}) \end{array}$$

where the rows are given by (6). Thus we have defined a map $f: \Sigma D \rightarrow \Sigma D'$; and clearly $f_0 = (g(\underline{x}, \underline{x}'))$.

Thus $f \mapsto f_0$ is surjective.

Suppose $f = (f_{\underline{y}}): \Sigma D \rightarrow \Sigma D'$ is a map with $f_0 = 0$. We show that $f_{\underline{y}} = 0$ for each $\underline{y}$ and hence prove $f \mapsto f_0$ is injective, and so an isomorphism. Take $\underline{y} \in \mathcal{G}_P$ and define $\underline{y}'$ by:

$$\begin{aligned} y'_i &= y_i \quad \text{if } y_i \neq -1, \\ &= 0 \quad \text{if } y_i = -1. \end{aligned}$$

Then $S(\underline{y}) \leq S(\underline{y}') \leq S(0)$. The following diagrams commute:

$$\begin{array}{ccc} (\Sigma D)_0 & \longrightarrow & (\Sigma D)_{\underline{y}'} \\ 0 \downarrow & & \downarrow f_{\underline{y}'} \\ (\Sigma D')_0 & \longrightarrow & (\Sigma D')_{\underline{y}'} \end{array} \quad \begin{array}{ccc} (\Sigma D)_{\underline{y}} & \longrightarrow & (\Sigma D)_{\underline{y}'} \\ f_{\underline{y}} \downarrow & & \downarrow f_{\underline{y}'} \\ (\Sigma D')_{\underline{y}} & \longrightarrow & (\Sigma D')_{\underline{y}'} \end{array}.$$

By (6) the horizontal maps of the first diagram are projections (and so epimorphisms) and those of the second are coprojections (and so monomorphisms). Thus $f_{\underline{y}'} = 0$, and then $f_{\underline{y}} = 0$ as required. //

Theorem 65. If the following are given:

- (a) two functors $F, F': \mathcal{G}_P \rightarrow \mathcal{Q}$ which are exact and projective at the corners;
- (b) for each non-zero $\underline{x} \in \mathcal{G}_P$, a map $f_{\underline{x}}: F\underline{x} \rightarrow F'\underline{x}$ such that, whenever $0 \neq \underline{x} \leq \underline{x}' \neq 0$, the following square commutes

$$\begin{array}{ccc} F\underline{x} & \xrightarrow{f_{\underline{x}}} & F'\underline{x} \\ \downarrow & & \downarrow \\ F\underline{x}' & \xrightarrow{f_{\underline{x}'}} & F'\underline{x}' \end{array};$$

then there exists $f_0: F_0 \rightarrow F'_0$ such that $f = (f_{\underline{x}}): F \rightarrow F'$ is a map of $[f_p, \mathcal{Q}]$.

Proof. By Proposition 63 and the nature of the result we may suppose $F = \Sigma D$, $F' = \Sigma D'$. Suppose $\underline{x}, \underline{x}'$ are p-corners with $\underline{x}' \leq \underline{x}$ but not both $\underline{x}' = (-1, -1, \dots, -1)$ and $\underline{x} = (1, 1, \dots, 1)$. If $\underline{x}' \neq (-1, -1, \dots, -1)$ then $x'_j = 1$ for some j , so $1 = x'_j \leq x_j \leq 1$ and $x_j = 1$. Similarly if $\underline{x} \neq (1, 1, \dots, 1)$ there exists j such that $x_j = x'_j = -1$. Thus there exists j such that $x_j = x'_j$. Let $s_j: f_{p-1} \rightarrow f_p$ be defined by:

$$s_j(y_1, \dots, y_{p-1}) = (y_1, \dots, y_{j-1}, x_j, y_j, \dots, y_{p-1}).$$

Then $(f_{s_j \underline{y}}): F s_j \rightarrow F' s_j$ is a map of $[f_{p-1}, \mathcal{Q}]$ and so

by Proposition 64 is completely determined by

$$f_{s_j \underline{Q}}: \begin{matrix} \Sigma D(\underline{z}) \\ \underline{z} \in S(s_j \underline{Q}) \end{matrix} \longrightarrow \begin{matrix} \Sigma D'(\underline{z}) \\ \underline{z} \in S(s_j \underline{Q}) \end{matrix}.$$

But $\underline{x}, \underline{x}' \in S(s_j \underline{Q})$ and $\underline{x}' \leq \underline{x}$, so we have a map $D(\underline{x}) \rightarrow D'(\underline{x}')$. If also $x_k = x'_k$ for some $k \neq j$ we show that the resulting map $D(\underline{x}) \rightarrow D'(\underline{x}')$ induced by $f_{s_k \underline{Q}}$ is the same as the one induced by $f_{s_j \underline{Q}}$. Suppose $j < k$ and let $s_{jk}: f_{p-2} \rightarrow f_p$ be defined by:

$$s_{jk}(y_1, \dots, y_{p-2}) = (\bar{y}_1, \dots, y_{j-1}, x_j, y_j, \dots, y_{k-2}, x_k, y_{k-1}, \dots, y_{p-2}).$$

From (b) it follows that $f_{s_j \underline{Q}}, f_{s_k \underline{Q}}$ both induce $f_{s_{jk} \underline{Q}}: F(s_{jk} \underline{Q}) \rightarrow F'(s_{jk} \underline{Q})$, and this in turn induces

$D(\underline{x}) \rightarrow D'(\underline{x}')$, unique by Proposition 64.

Hence the $f_{\underline{y}}$ determine a map $D(\underline{x}) \rightarrow D'(\underline{x}')$ whenever $\underline{x}, \underline{x}'$ are corners with $\underline{x}' \leq \underline{x}$ but not both $\underline{x}' = (-1, -1, \dots, -1)$ and $\underline{x} = (1, 1, \dots, 1)$. Let $D(1, 1, \dots, 1) \rightarrow D'(-1, -1, \dots, -1)$ be any such map (for example, the zero map). Let $f_{\underline{0}}: \sum_{\underline{x} \in S(\underline{0})} D(\underline{x}) \rightarrow \sum_{\underline{x} \in S(\underline{0})} D'(\underline{x})$ be the matrix with typical element $D(\underline{x}) \rightarrow D'(\underline{x}')$ as above when $\underline{x}, \underline{x}'$ are corners with $\underline{x}' \leq \underline{x}$, and zero otherwise.

Let $h = (h_{\underline{y}}): F \rightarrow F'$ be the unique map (Proposition 64) with $h_{\underline{0}} = f_{\underline{0}}$. By definition of $f_{\underline{0}}$ the following diagram commutes:

$$\begin{array}{ccc} F_{\underline{0}} & \xrightarrow{\quad} & F t_{\underline{0}} \\ f_{\underline{0}} \downarrow & & \downarrow f_{t_{\underline{0}}} \\ F'_{\underline{0}} & \xrightarrow{\quad} & F' t_{\underline{0}}, \end{array}$$

whenever $t = t_1^+$ or t_1^- for any i . But the corresponding diagram with h replacing f also commutes and $f_{\underline{0}} = h_{\underline{0}}$ and $F_{\underline{0}} \rightarrow F t_{\underline{0}}$ is an epimorphism. So $f_{t_{\underline{0}}} = h_{t_{\underline{0}}}$. All $\underline{y} \in \mathcal{G}_p$ with $\underline{y} \neq \underline{0}$ have the form $t\underline{x}$ for some t and some $\underline{x}$. So $h_{\underline{y}} = f_{\underline{y}}$ for $\underline{y} \neq \underline{0}$. Thus $h = f: F \rightarrow F'$ is a map as required. //

§4. Proof of Lemma 54.

The plan of proof is: $(b) \Rightarrow (a) \Rightarrow (c) \Rightarrow (b)$.
 From Theorem 14, $(b) \Rightarrow (a)$ is clear. The proof of
 $(a) \Rightarrow (c)$ is routine, but that of $(c) \Rightarrow (b)$ is lengthy
 and requires the preceding theory.

Suppose $G = J_{(x,y)} B$ with $0 \leq y < x < n$, and
 $0 \leq u < v < w < n$. We examine the triangle:

$$(1) \quad G(v,u) \rightarrow G(w,u) \rightarrow G(w,v) \rightarrow G(v,u).$$

From the definition of J_n (see §1) and Theorem 10 we have:

$$\begin{aligned} G(v,u) &= B \quad \text{if } y \leq u < x \leq v, \\ &= L^{-1}B \quad \text{if } u < y \leq v < x, \\ &= 0 \quad \text{otherwise;} \end{aligned}$$

$$\begin{aligned} G(w,u) &= B \quad \text{if } y \leq u < x \leq w, \\ &= L^{-1}B \quad \text{if } u < y \leq w < x, \\ &= 0 \quad \text{otherwise;} \end{aligned}$$

$$\begin{aligned} G(w,v) &= B \quad \text{if } y \leq v < x \leq w, \\ &= L^{-1}B \quad \text{if } v < y \leq w < x, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

If $0 \leq y \leq u < x \leq v < w < n$ then (1) becomes
 $B \xrightarrow{1} B \rightarrow 0 \rightarrow B.$

If $0 \leq y \leq u < v < x \leq w < n$ then (1) becomes
 $0 \rightarrow B \xrightarrow{1} B \rightarrow 0.$

If $u < v$ and $0 \leq u \leq y \leq v < x \leq w < n$ then (1) becomes
 $L^{-1}B \rightarrow 0 \rightarrow B \xrightarrow{L^{-1}} L^{-1}B.$

If $0 \leq u < y \leq v < w < x < n$ then (1) becomes

$$L^{-1}B \xrightarrow{1} L^{-1}B \rightarrow 0 \rightarrow L^{-1}B.$$

If $0 \leq u < v < y \leq w < x < n$ then (1) becomes

$$0 \rightarrow L^{-1}B \xrightarrow{1} L^{-1}B \rightarrow 0.$$

Otherwise (1) is the zero triangle (this is left for the reader to check).

It follows that (1) is projective in $\text{Tgl } \mathcal{B}$ if B is projective in $\mathcal{B}$.

If $G = G' \oplus G''$ then the triangle (1) for G is the direct sum of the triangle (1) for G' and the triangle (1) for G'' . Thus, if $G = \sum_{0 \leq y < x < n} J(x,y)P(x,y)$ with the $P(x,y)$ projective in $\mathcal{B}$, then G has the property (c).

But a retract of a G with the property (c) also has the property (c). So, from Theorem 14 we have (a) $\Rightarrow$ (c).

We identify $\mathcal{J}_n^L$. Let $S_n = \{(u,v) \mid v < u < v+n\}$,
 $S_n^{2r} = \{(u+rn, v+rn) \mid 0 \leq v < u < n\}$, $S_n^{2r-1} = \{(v+rn, u+(r-1)n) \mid 0 \leq v < u < n\}$.
 Then S_n is the disjoint union of the S_n^r and each S_n^r

has $\frac{1}{2}n(n-1)$ elements. For $(u,v) \in \mathcal{J}_n$, put

$$L^{2r}(u,v) = (u+rn, v+rn) \in S_n^{2r}$$

$$L^{2r-1}(u,v) = (v+rn, u+(r-1)n) \in S_n^{2r-1}.$$

Thus we have isomorphism $L^r: S_n^0 \rightarrow S_n^r$ for each r . Recall the definition of $\mathcal{J}_n^L$ given in Ch.1 § 3. We see that $\mathcal{J}_n^L$ is the stable graded model given as follows:

(i) the objects are pairs (u, v) of integers satisfying the condition $v < u < v+n$;

(ii) if $(u, v), (u', v')$ are two such pairs, then

$$\begin{aligned} [(u, v), (u', v')] &= L^{2r} \mathbb{Z} \text{ if } u' - n < v + rn \leq v' < u + rn \leq u', \\ &= L^{2r-1} \mathbb{Z} \text{ if } v' < v + rn \leq u' < u + rn \leq v' + n, \\ &= 0 \text{ otherwise.} \end{aligned}$$

(iii) composition is that of a graded model (Ch.1 §3);

(iv) $L(u, v) = (v+n, u)$ and $\ell_{(u, v)} \in [(u, v), (v+n, u)]_1 = \mathbb{Z}$ is $1 \in \mathbb{Z}$.

By Theorem 3 we may identify $\text{Tgl}_n \mathcal{B}$ and $[\mathcal{J}_n^L, \mathcal{B}]_L$.

For $y < x < y+n$ and $B \in \mathcal{B}$, $J_{(x, y)}^B \in [\mathcal{J}_n^L, \mathcal{B}]_L$ is given as follows:

(i) for $v < u < v+n$,

$(J_{(x, y)}^B)(u, v) = L^p B$ if there is a non-zero morphism $(x, y) \rightarrow (u, v)$ of degree p in $\mathcal{J}_n^L$,
 $= 0$ otherwise.

(ii) for $v < u < v+n, s < r < s+n$, $(J_{(x, y)}^B)(u, v) \rightarrow (J_{(x, y)}^B)(r, s)$ is zero unless the composite $(x, y) \rightarrow (u, v) \rightarrow (r, s)$ is non-zero in which case it is $\ell^q: L^p B \rightarrow L^{p+q} B$, where p is the degree of $(x, y) \rightarrow (u, v)$ and q is the degree of $(u, v) \rightarrow (r, s)$.

Suppose $P(x, y)$ is an object of $\mathcal{B}$ for $0 \leq y < x < n$. Define $P(u, v)$ for $v < u < v+n$ by:

$$PL^p(x, y) = L^p P(x, y).$$

(2) Let $\bar{G} = \sum_{0 \leq y < x < n} J(x,y) P(x,y)$. Then from the above we see that:

(i) for $v < u < v + n$,

$$\bar{G}(u,v) = \sum_{u-n < y \leq v < x \leq u} P(x,y) ;$$

(ii) for $v < u < v+n$, $s < r < s+n$ and $(u,v) \rightarrow (r,s)$ of degree q , $\bar{G}(u,v) \rightarrow \bar{G}(r,s)$ is the matrix with typical element $P(x,y) \rightarrow P(x',y')$ equal to $\int_{P(x,y)}^q$ if $(x',y') = L^q(x,y)$, and zero otherwise.

(3) With $\bar{G}$ as above and $v < u < v+n$, $P(u,v)$ is the image of the map $\bar{G}(u,v) \rightarrow \bar{G}(n+v-1, u-1)$.

Proof. $\bar{G}(u,v)$ is given in (2)(i), and

$$\bar{G}(n+v-1, u-1) = \sum_{v-1 < y \leq u-1 < x \leq n+v-1} P(x,y) . \text{ But } u-n < y \leq v < x \leq u \text{ and } v-1 < y \leq u-1 < x \leq n+v-1$$

imply $x = u$, $y = v$. So from

(2)(ii), $\bar{G}(u,v) \rightarrow \bar{G}(n+v-1, u-1)$ is the matrix with all elements zero except $P(u,v) \rightarrow P(u,v)$, which is the identity //

Suppose $G \in \text{Tgl}_n \mathcal{B}$ satisfies the condition (c).

From the definition of $\mathcal{I}_n^L$ it follows that:

(4) each triangle

$$G(v,u) \rightarrow G(w,u) \rightarrow G(w,v) \rightarrow G(v,u)$$

is projective in $\text{Tgl} \mathcal{B}$ for $u < v < w < u + n$.

Let $I(u,v;t,w)$ denote the image of $G(u,v) \rightarrow G(t,w)$ if $v \leq w < u \leq t < v + n$, and 0 otherwise. In particular we set $P(u,v) = I(u,v;n+v-1, u-1)$, and note (since G is

stable), $LP(u,v) = PL(u,v)$. With this choice of $P(u,v)$, let $\bar{G}$ be given by (2).

(5) If $y \leq v \leq x \leq u \leq y+n$ and $v \leq w \leq u \leq t \leq v+n$ then $(x,y) \rightarrow (u,v)$ induces a monomorphism

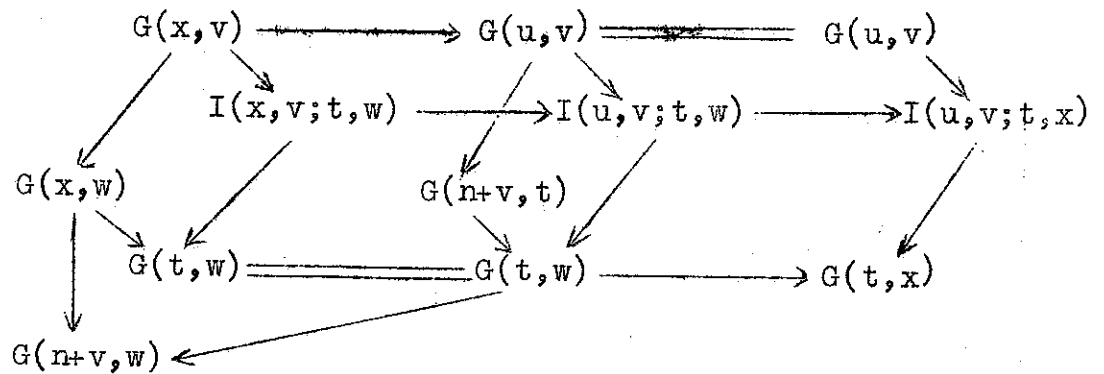
$I(x,y;t,w) \rightarrow I(u,v;t,w)$, and $(u,v) \rightarrow (t,w)$ induces an epimorphism $I(x,y;u,v) \rightarrow I(x,y;t,w)$.

(6) if $v \leq w \leq x \leq u \leq t \leq v+n$ and $v \neq x \neq t$, $w \neq u$ then the sequence:

$$0 \rightarrow I(x,v;t,w) \rightarrow I(u,v;t,w) \rightarrow I(u,v;t,x) \rightarrow 0,$$

is exact.

Proof. The following diagram commutes.



(where we replace $G(n+v,w)$ by 0 if $v = w$). A diagram chase using the exactness of the sequences:

$$G(x,w) \longrightarrow G(t,w) \longrightarrow G(t,x)$$

$$G(x,v) \longrightarrow G(x,w) \longrightarrow G(n+v,w)$$

$$G(n+v,t) \rightarrow G(t,w) \longrightarrow G(n+v,w)$$

yields the result. //

From (6), using suspension, we have the following result.

(7) If $y \leq v \leq w \leq u \leq t \leq n+y$ and $u \neq y+n, t \neq v+n, y \neq w$ then
the sequence

$$0 \rightarrow I(u, y; t, w) \rightarrow I(u, v; t, w) \rightarrow I(u, v; n+y, w) \rightarrow 0$$

is exact. //

A morphism $(u, v) \rightarrow (t, w)$ of degree zero in $\mathcal{J}_n^L$ with $u \leq t, v \leq w$ will be said to have length $t-u+w-v$. The morphisms of length zero are identities. All morphisms of length $\geq n-1$ are zero; for if $(u, v) \rightarrow (t, w)$ is of degree zero and is non-zero then $v \leq w \leq u \leq t \leq v+n$; but if it has length $\geq n-1$ then $t < v+n \leq t-u+w+1 \leq t$, a contradiction. Morphisms of length 1 have the form $(u, v) \rightarrow (u+1, v)$ or $(u, v) \rightarrow (u, v+1)$. A map of length $\lambda < n-1$ is a composite of λ maps of length 1; if $v \leq w \leq u \leq t < v+n$ then $(u, v) \rightarrow (t, w)$ is the composite $(u, v) \rightarrow (u+1, v) \rightarrow \dots \rightarrow (t, v) \rightarrow (t, v+1) \rightarrow \dots \rightarrow (t, w)$. Under suspension the morphism $(u, v) \rightarrow (u, v+1)$ gives a morphism of type $(u', v') \rightarrow (u'+1, v')$. If $(u, v) \rightarrow (t, w)$ is non-zero, of degree zero and length $n-2$, then $t = n+v-1$ and $w = u-1$; for $t < v+n = t-u+w+2$, so $-2 < w-u < 0$, and $w = u-1$; then $n-2 = t-u+w-v$ gives $t = n+v-1$. In this case then $I(u, v; t, w) = P(u, v)$.

(8) Suppose $v \leq w \leq u \leq t < v+n$. Put $\lambda = t-u+w-v$, $p = n-\lambda-2$ and $q = u-w-1$. Then $0 \leq q \leq p$.

(9) In the situation of (8) suppose $p > 0$ and $\underline{x} = (x_1, \dots, x_p) \in \mathcal{J}_p$ (see §3). Put:

$$r = \max\{i \mid 0 < i \leq q, x_i = -1\} \quad \text{if this set is non-empty,} \\ = 0 \quad \text{otherwise;}$$

$$r' = \min\{i \mid 0 < i \leq q, x_i = 1\} \quad \text{if this set is non-empty,} \\ = q+1 \quad \text{otherwise;}$$

$$s = \max\{i \mid q < i \leq p, x_i = -1\} \quad \text{if this set is non-empty,} \\ = q \quad \text{otherwise;}$$

$$s' = \min\{i \mid q < i \leq p, x_i = 1\} \quad \text{if this set is non-empty,} \\ = p+1 \quad \text{otherwise.}$$

$$\text{Then } 0 \leq r \leq q, 1 \leq r' \leq q+1, q \leq s \leq p, q+1 \leq s' \leq p+1.$$

$$\text{Now put } u' = u - r, v' = v + q - s, t' = t + p - s' + 1 \quad \text{and}$$

$$w' = w + q - r' + 1. \quad \text{Notice that:}$$

$$v' = v + q - s = v + u - w - 1 - s \leq u - 1 - s \leq u - r = u',$$

$$u' = u - r \leq u \leq t < t - s + p + 1 = t - \lambda - 1 + n - s = v + q - s + n = v' + n,$$

$$w' = w + q - r' + 1 \leq w + q - s' + p + 1 = u - 1 + t' - t < t',$$

$$t' < t' + 1 + w - v = u + p + \lambda + 2 - s' = u + n - s' = w' + n + r' - s' \leq w' + n.$$

$$\text{So } (u', v'), (t', w') \in \mathcal{J}_n^L. \quad \text{Also notice that:}$$

$$t' - u' = (p + 1 - s') + (t - u) + r \geq 0, \quad \text{and}$$

$$w' - v' = (w - v) + s - r' + 1 \geq 0,$$

so the length λ' of $(u', v') \rightarrow (t', w')$ is defined;

moreover:

$$\begin{aligned} \lambda' &= t' - u' + w' - v' \\ &= p + 2 + \lambda - s' + r - r' + s \\ &= n - s' + r - r' + s \\ &\geq n - p - 1 - q - 1 + q \\ &= \lambda \end{aligned}$$

with equality if and only if $\underline{x} = \underline{0}$.

(10) If $\underline{\bar{x}} = (\bar{x}_1, \dots, \bar{x}_p) \in \mathcal{J}_p$ obtain $\bar{r}, \bar{r}', \bar{s}, \bar{s}', \bar{u}', \bar{v}', \bar{t}', \bar{w}'$ as $r, r', s, s', u', v', t', w'$ were obtained from $\underline{x}$ in (9); we will use similar notation for $\underline{\hat{x}}, \underline{\tilde{x}}$ etc. Clearly $r \leq \bar{s}, s \leq \bar{r}+p, s' \leq \bar{r}'+p, r' \leq \bar{s}'$.

(11) If $\underline{x} \leq \underline{\bar{x}}$ then, from the definition in (9), we have $\bar{r} \leq r, \bar{r}' \leq r', \bar{s} \leq s, \bar{s}' \leq s'$. These inequalities together with those of (10) give the following inequalities:

$$v' \leq \bar{v}' < u' \leq \bar{u}' < v' + n, w' \leq \bar{w}' < t' \leq \bar{t}' < w' + n.$$

(12) In the situation of (8) we define a functor

$F = F_{u,v;t,w}: \mathcal{J}_p \rightarrow \mathcal{B}_0$. For $p = 0$, F is given by the object $P(u,v)$ of $\mathcal{B}_0$. For $p > 0$, F is given as follows:

(i) $F\underline{x} = I(u', v'; t', w')$ in the notation of (9);

(ii) if $\underline{x} \leq \underline{\bar{x}}$ then $F\underline{x} \rightarrow F\underline{\bar{x}}$ is the map induced on the images of the horizontal maps by the vertical maps of the diagram:

$$\begin{array}{ccc} G(u', v') & \longrightarrow & G(t', w') \\ \downarrow & & \downarrow \\ G(\bar{u}', \bar{v}') & \longrightarrow & G(\bar{t}', \bar{w}'), \end{array}$$

(recall (11)).

Since G is a functor, in order to check that F is a functor we need only verify that: $\underline{x} \leq \underline{\bar{x}}$ with

$(\underline{x}, \bar{x}; f_p) = 0$ implies $F\underline{x} \rightarrow F\bar{x}$ is zero. But in this case there exists an i with $x_i = -1$, $\bar{x}_i = 1$. If $i \leq q$ then $\bar{r}' \leq i \leq r$ (recall (9)), so $u' \leq \bar{w}'$ and the square

$$\begin{array}{ccc} (u', v') & \longrightarrow & (t', w') \\ \downarrow & & \downarrow \\ (\bar{u}', \bar{v}') & \longrightarrow & (\bar{t}', \bar{w}') \end{array}$$

has both legs zero, so $F\underline{x} \rightarrow F\bar{x}$ is zero. If $q < i$ then $\bar{s}' \leq i \leq s$ so $v' + n \leq \bar{t}'$ and again the map is zero.

(13) If $F\underline{x} \neq 0$ then $r < r'$ and $s < s'$. For $r' - r = u' - w'$ and $s' - s = n + v' - t'$; and $F\underline{x} \neq 0$ implies $(u', v') \rightarrow (t', w')$ non-zero, that is, $v' \leq w' < u' \leq t' < n + v'$.

(14) Suppose $(u', v') \rightarrow (u, v), (t, w) \rightarrow (t', w')$ are of degree zero in f_n^L , in the situation of (8), such that the composite $(u', v') \rightarrow (u, v) \rightarrow (t, w) \rightarrow (t', w')$ is non-zero. Put $r = u - u'$, $s = q + v - v'$, $s' = p + t - t' + 1$, $r' = q + w - w' + 1$, $\lambda' = t' - u' + w' - v'$, $p' = n - \lambda' - 2$; note that $r' - r = u' - w' > 0$, and $s' - s = n + v' - t' > 0$. Let $\underline{x} = (x_1, \dots, x_p)$ be given by:

$$\begin{aligned} x_i &= -1 \quad \text{for } 0 < i \leq r \quad \text{or } q < i \leq s, \\ &= 0 \quad \text{for } r < i < r' \quad \text{or } s < i < s', \\ &= 1 \quad \text{for } r' - 1 < i \leq q \quad \text{or } s' - 1 < i \leq p. \end{aligned}$$

Then $F\underline{x} = I(u', v'; t', w')$. So we have found exactly the non-zero values taken by F (see (13)).

(15) In the situation of (14), the number of x_i which are zero is $r'-r-l+s'-s-l = q+w-w'-u+u'+p+t-t'-q-v+v' = p'$, which is independent of u, v, t, w . Define

$F': \mathcal{G}_{p'} \rightarrow \mathcal{B}_0$ by:

(i) for $\underline{y} = (y_1, \dots, y_{p'})$, $F'\underline{y} = F\underline{z}$ where $\underline{z} = (z_1, \dots, z_p)$ is given by:

$$\begin{aligned} z_i &= -1 && \text{for } 0 < i \leq r \text{ or } q < i \leq s, \\ &= y_{i-r} && \text{for } r < i < r', \\ &= y_{p'-s'+l+i} && \text{for } s < i < s', \\ &= 1 && \text{for } r'-1 < i \leq q \text{ or } s'-1 < i \leq p; \end{aligned}$$

(ii) if $\underline{y} \leq \underline{y}'$ then $\underline{z} \leq \underline{z}'$, where $\underline{z}'$ is obtained from $\underline{y}'$ as $\underline{z}$ was obtained from $\underline{y}$ in (i), and $F'\underline{y} \rightarrow F'\underline{y}'$ is the map $F\underline{z} \rightarrow F\underline{z}'$.

From the definitions of $F_{u,v;t,w}$ and $F_{u',v';t',w'}$ it follows that $F' = F_{u',v';t',w'}$.

(16) In the notation of (8), (9), (10) suppose $\underline{x}, \bar{\underline{x}} \in \mathcal{G}_p$ are such that $r = \bar{r}$, $r' = \bar{r}'$, $s = \bar{s}$, $s' = \bar{s}'$. Then $F\underline{x} = F\bar{\underline{x}}$. We shall show that $F\underline{x}$ and $F\bar{\underline{x}}$ are actually linked in F by identity maps; more precisely, we shall show that there exist $\underline{y} = \underline{y}^0, \underline{y}^1, \dots, \underline{y}^m = \bar{\underline{x}}$ such that $F\underline{y}^i = F\underline{x}$ for $0 < i \leq m$ and either $\underline{y}^{i-1} \leq \underline{y}^i$ with $F\underline{y}^{i-1} \rightarrow F\underline{y}^i$ the identity map, or $\underline{y}^i \leq \underline{y}^{i-1}$ with

$F\tilde{y}^i \rightarrow F\tilde{y}^{i-1}$ the identity map. We may as well suppose $F\tilde{x} \neq 0$ and that $\tilde{x}$ has the form of the $\tilde{x}$ in (14). Since $r = \bar{r}$, $s = \bar{s}$, $r' = \bar{r}'$, $s' = \bar{s}'$, $\bar{x}$ satisfies the conditions:

$$\begin{aligned}\bar{x}_i &= 0 \text{ or } -1 \text{ for } 0 < i < r \text{ or } q < i < s, \\ &= 0 \text{ for } r < i < r' \text{ or } s < i < s', \\ &= 0 \text{ or } 1 \text{ for } r' < i \leq q \text{ or } s' < i \leq p; \\ \bar{x}_r &= -1 \text{ provided } r \neq 0; \quad \bar{x}_{r'} = 1 \text{ provided } r' \neq q+1, \\ \bar{x}_s &= -1 \text{ provided } s \neq q, \quad \bar{x}_{s'} = 1 \text{ provided } s' \neq p+1.\end{aligned}$$

Let $\tilde{y} = (y_1, \dots, y_p)$ be given by:

$$\begin{aligned}y_i &= -1 \text{ for } 0 < i \leq r \text{ or } q < i \leq s, \\ &= \bar{x}_i \text{ otherwise.}\end{aligned}$$

Then $\tilde{y} \leq \bar{x}$, $\tilde{y} \leq \tilde{x}$ and the maps $F\tilde{y} \rightarrow F\bar{x}$, $F\tilde{y} \rightarrow F\tilde{x}$ are the identities (see (12)). //

(17) Suppose $\tilde{x}$ is a p-corner and $\tilde{F}$ is as in (12).

If $q = 0$ then $r' = 1 = r+1$; if $q > 0$ then $x_i = 1$ for $r < i \leq q$, so $r' \leq r+1$. If $q = p$ then $s' = p+1 = s+1$; if $q < p$ then $x_i = 1$ for $s < i \leq p$, so $s' \leq s+1$. So

$$r' \leq r+1 \text{ and } s' \leq s+1.$$

If $r' = r+1$ and $s' = s+1$ then $t' = n+v'-1$ and $w' = u'-1$ so $F\tilde{x} = P(u', v')$. Otherwise the length of the map $(u', v') \rightarrow (t', w')$ is

$$t+p-s'+1-u+r+w+q-r'+1-v-q+s = p+\lambda+(r+1-r')+(s+1-s') > n-2.$$

So we have:

if $\underline{x}$ is a p-corner then $F_{\underline{x}} = P(u', v')$ if $r' = r+1$
and $s' = s+1$, $= 0$ otherwise.

The corners $\underline{x}$ with $r' = r+1$ and $s' = s+1$ are those

$\underline{x}$ with: $x_i = -1$ for $0 < i \leq r$ or $q < i \leq s$,
 $= 1$ for $r < i \leq q$ or $s < i \leq p$;

there are $(q+1)(p-q+1)$ such corners. It follows now

from (14) that the non-zero $F_{\underline{x}}$ with $\underline{x}$ a corner are

exactly the $P(u', v')$ where the map $(u', v') \rightarrow (u, v)$ of
degree zero is such that the composite

$(u', v') \rightarrow (u, v) \rightarrow (t, w)$ is non-zero; there are $(q+1)(p-q+1)$
such (u', v') 's.

(18) If $v \leq y < u \leq x < v+n$ then $I(u, v; xy)$ is
projective in $\mathcal{B}$.

Proof. If $y = v$ then the triangle:

$$G(u, y) \rightarrow G(x, y) \rightarrow G(x, u) \rightarrow G(u, y),$$

is projective (see (4)). So (see Theorem 15) $I(u, y; x, y)$

is projective. If $v < y$ then we have an exact sequence:

$$0 \rightarrow I(u, v; x, y) \rightarrow I(u, y; x, y) \rightarrow I(u, y; n+v, y) \rightarrow 0$$

by (7). But from the first part $I(u, y; x, y)$ and

$I(u, y; n+v, y)$ are projective, so $I(u, v; x, y)$ is projective. //

(19) $F = F_{u, v; t, w}$ as defined in (12) is exact and takes
projective values.

Proof. F takes projective values by (18). For λ, p, q as in (8). Suppose $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_p \in \mathcal{J}$

and put $\underline{x} = (x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_p)$,

$$\underline{x}^- = (x_1, \dots, x_{i-1}, -1, x_{i+1}, \dots, x_p),$$

$$\underline{x}^+ = (x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_p).$$

Obtain $r, r', s, s', u', v', t', w'$ from $\underline{x}$ as in (9). Then

$F\underline{x} = I(u', v'; t', w')$. Er shall show that the sequence:

$$(20) \quad 0 \rightarrow F\underline{x}^- \rightarrow F\underline{x} \rightarrow F\underline{x}^+ \rightarrow 0$$

is exact. Eight exhaustive cases must be distinguished.

(i) $r' \leq i \leq r$. Here $F\underline{x}^- = F\underline{x} = F\underline{x}^+$ and

$u' \leq w'$ so the map $(u', v') \rightarrow (t', w')$ is zero; hence the sequence (20) is zero.

(ii) $0 < i \leq r, i \leq r'$. Here $F\underline{x}^- = F\underline{x}$ and $F\underline{x}^+ = I(u', v'; t', w+q-i+1) = 0$ since $u' = u-r \leq w+q-i+1$; so (20) becomes $0 \rightarrow F\underline{x} \xrightarrow{1} F\underline{x} \rightarrow 0 \rightarrow 0$.

(iii) $r < i \leq q, r' \leq i$. Here $F\underline{x} = F\underline{x}^+$ and $F\underline{x}^- = I(u-i, v'; t', w') = 0$ since $u-i \leq w+q-r'+1 = w'$; so (20) becomes $0 \rightarrow 0 \rightarrow F\underline{x} \xrightarrow{1} F\underline{x} \rightarrow 0$.

(iv) $r < i \leq r'-1$. In this case (20) becomes:

$$0 \rightarrow I(u'', v'; t', w') \rightarrow I(u', v'; t', w') \rightarrow I(u', v'; t', u'') \rightarrow 0$$

where $u'' = u-i = w+q-i+1$, and this is exact by (6).

(v) $s' \leq i \leq s$. In this case $F_{\underline{x}}^- = F_{\underline{x}} = F_{\underline{x}}^+ = 0$ since $n+v' \leq t'$.

(vi) $q < i \leq s, i \leq s'$. Here $F_{\underline{x}}^- = F_{\underline{x}}$ and $F_{\underline{x}}^+ = I(u', v'; t+p-i+1, w') = 0$ since $v'+n \leq t+p-i+1$. So (20) is as in (ii).

(vii) $s < i \leq p, s' \leq i$. Here $F_{\underline{x}} = F_{\underline{x}}^+$ and $F_{\underline{x}}^- = I(u', v+q-i; t', w') = 0$ since $n+v+q-i \leq t'$. So (20) is as in (iii).

(viii) $s < i \leq s'-1$. Here (20) becomes:

$$0 \rightarrow I(u', v''; t', w') \rightarrow I(u', v'; t', w') \rightarrow I(u', v'; n+v'', w') \rightarrow 0$$

where $v'' = v+q-i = t+p-i+1-n$, and this is exact by (7). //

(21) Let $\bar{I}(u, v; t, w)$, $\bar{F} = \bar{F}_{u, v; t, w}$ be obtained from

$\bar{G}$ as $I(u, v; t, w)$, $F = F_{u, v; t, w}$ were from G . If

$v \leq w < u \leq t < v+n$ then:

$$\bar{I}(u, v; t, w) = \sum_{\substack{w < x < u \\ t-n < \bar{y} \leq v}} P(x, y).$$

If also $\bar{v} \leq \bar{w} < \bar{u} \leq \bar{t} < \bar{v}+n$, $v \leq \bar{v} < u \leq \bar{u} < v+n$,

$w \leq \bar{w} < t \leq \bar{t} < w+n$ then the map $\bar{I}(u, v; t, w) \rightarrow \bar{I}(\bar{u}, \bar{v}; \bar{t}, \bar{w})$

induced by the commuting square:

$$\begin{array}{ccc} \bar{G}(u, v) & \longrightarrow & \bar{G}(t, w) \\ \downarrow & & \downarrow \\ \bar{G}(\bar{u}, \bar{v}) & \longrightarrow & \bar{G}(\bar{t}, \bar{w}) \end{array}$$

is the matrix with typical element $P(x,y) \rightarrow P(\bar{x},\bar{y})$ the identity if $(x,y) = (\bar{x},\bar{y})$ and zero otherwise (see (2)).

$$(22) \quad \bar{F}_{u,v;t,w} = \sum F_{u,v;t,w}.$$

Proof. Take $\underline{x} = (x_1, \dots, x_p)$ in $\mathcal{J}_p$. In the notation of (9), if $r' < r$ then $\bar{F}\underline{x} = 0$; moreover, if $\underline{x}$ is a p-corner with $\bar{x}_i = x_i$ for $x_i \neq 0$ (i.e. $\bar{x} \in S(\underline{x})$) then $\bar{r}' < \bar{r}$, so $\bar{F}\bar{x} = 0$ (see (17)); thus $(\Sigma F)\underline{x} = 0$. Similarly both sides are zero at $\underline{x}$ if $s' < s$. So we may suppose $r < r'$, $s < s'$ (equality is impossible!).

Then:

$$\bar{F}\underline{x} = I(u',v';t',w') = \sum_{\substack{w' < a < u' \\ t' - n < b < v'}} P(a,b).$$

Suppose $\bar{x} \in S(\underline{x})$; then $r \leq \bar{r}$, $\bar{r}' \leq r'$, $s \leq \bar{s}$, $\bar{s}' \leq s'$.

Also $\bar{F}\bar{x} = 0$ unless $\bar{r}' = \bar{r} + 1$ and $\bar{s}' = \bar{s} + 1$. So:

$$(\Sigma F)\underline{x} = \sum_{\substack{r < \bar{r} < r' \\ s < \bar{s} < s'}} P(u - \bar{r}, v + q - \bar{s}) = \bar{F}\underline{x}.$$

That the maps of $\bar{F}$ and ΣF agree is clear from (12)(ii), (21) and §3 (6). //

(23) For $0 \leq \bar{p} \leq n-2$ let $\Pi_{\bar{p}}$ denote the following proposition:

there is a function f which assigns to each quadruplet $(u,v;t,w)$ of integers such that $v \leq w < u \leq t < v+n$ and $n-t+u-w+v-2 \leq \bar{p}$ (such quadruplets we shall temporarily call $\bar{p}$ -suitable) a map

$$f(u,v;t,w) : I(u,v;t,w) \rightarrow \bar{I}(u,v;t,w)$$

of $\mathcal{B}_0$, and if $(u,v;t,w)$ is $\bar{p}$ -suitable then the following conditions are satisfied:

- $\pi 1.$ if $t-u+w-v = n-2$ then $f(u,v;t,w)$ is the identity of $P(u,v)$;
- $\pi 2.$ $Lf(u,v;t,w) = f(v+n,u;w+n,t)$;
- $\pi 3.$ the family $g = (g_{\underline{x}})$ of maps $g_{\underline{x}} : F_{u,v;t,w}^{\underline{x}} \rightarrow \bar{F}_{u,v;t,w}^{\underline{x}}$ given by $g_{\underline{x}} = f(u',v';t',w')$, where u',v',t',w' are obtained from $\underline{x}$ as in (9), when the domain and range are non-zero, is a natural transformation $g : F_{u,v;t,w} \rightarrow \bar{F}_{u,v;t,w}$.

(24) Π_p is true for $0 \leq p \leq n-2$.

Proof. For $p = 0$ take f to be the function which assigns to each non-zero map $(u,v) \rightarrow (t,w)$ of degree zero and length $n-2$ the identity map of $P(u,v)$; $\pi 1, \pi 2, \pi 3$ are then satisfied. So Π_0 is true.

The proof now proceeds inductively on p . Suppose Π_{p-1} ($0 < p \leq n-2$) is true; then Π_p is true for $0 \leq p' \leq p-1$. There exists f defined on $(p-1)$ -suitable quadruplets satisfying $\pi 1, \pi 2, \pi 3$. We must define f now on those p -suitable quadruplets $(u,v;t,w)$ with $p = n-t+u-w+v-2$. Suppose $\underline{x} = (x_1, \dots, x_p) \neq \underline{0}$, and

obtain u', v', t', w' as in (9). Then

$p' = n - t' + u' - w' + v' - 2 \leq p - 1$ (see (9)), so we have

$f(u', v'; t', w') : F_{\underline{x}} \rightarrow \bar{F}_{\underline{x}}$ whenever the domain and range are non-zero, where we omit provisionally the $u, v; t, w$ subscripts of $F, \bar{F}$; if either the domain or range is zero we have the zero map. Suppose $0 \neq \underline{x} \leq \bar{\underline{x}} \neq 0$; we show that the diagram:

$$\begin{array}{ccc} F_{\underline{x}} & \xrightarrow{f'} & \bar{F}_{\underline{x}} \\ \downarrow & & \downarrow \\ F_{\bar{\underline{x}}} & \xrightarrow{\bar{f}'} & \bar{F}_{\bar{\underline{x}}} \end{array},$$

commutes, where $f' = f(u', v'; t', w')$, $\bar{f}' = \bar{f}(\bar{u}', \bar{v}'; \bar{t}', \bar{w}')$ in the notation of (10). If $F_{\underline{x}} = 0$ or $\bar{F}_{\bar{\underline{x}}} = 0$ this is clear; so we suppose $F_{\underline{x}} \neq 0$ and $\bar{F}_{\bar{\underline{x}}} \neq 0$. Then $(u', v') \rightarrow (t', w')$, $(\bar{u}', \bar{v}') \rightarrow (\bar{t}', \bar{w}')$ are non-zero, and so by (9), $(u', v'; t', w')$, $(\bar{u}', \bar{v}'; \bar{t}', \bar{w}')$ are $(p-1)$ -suitable. We may suppose $\underline{x}$ is an in (14) and $\bar{\underline{x}}$ is given similarly with r, r', s, s' replaced by $\bar{r}, \bar{r}', \bar{s}, \bar{s}'$ since this $\underline{x}, \bar{\underline{x}}$ give the same square by (16). Let $\hat{\underline{x}}$ be given by:

$$\begin{aligned} \hat{x}_i &= -1 \text{ for } 0 < i \leq r \text{ or } q < i \leq s, \\ &= 0 \text{ for } r < i < \bar{r}' \text{ or } s < i < \bar{s}', \\ &= 1 \text{ for } \bar{r}' - 1 < i \leq q \text{ or } \bar{s}' - 1 < i \leq p. \end{aligned}$$

Then $\hat{\underline{x}} \neq 0$ and $\underline{x} \leq \hat{\underline{x}} \leq \bar{\underline{x}}$. If $F_{\hat{\underline{x}}} = \bar{F}_{\hat{\underline{x}}} = 0$ then both legs of the square are zero, and so it commutes. So we may suppose either $F_{\hat{\underline{x}}} \neq 0$ or $\bar{F}_{\hat{\underline{x}}} \neq 0$. Then

$(\hat{u}', \hat{v}') \quad (\hat{t}', \hat{w}')$ is non-zero; so $(\hat{u}', \hat{v}'; \hat{t}', \hat{w}') = (u', \bar{v}'; t', \bar{w}')$ is $(p-1)$ -suitable. Thus our square splits up into the following two squares:

$$\begin{array}{ccc} F_{\underline{x}} & \xrightarrow{f'} & \bar{F}_{\underline{x}} \\ \downarrow & & \downarrow \\ F_{\hat{x}} & \xrightarrow{\hat{f}'} & \bar{F}_{\hat{x}} \end{array} , \quad \begin{array}{ccc} F_{\hat{x}} & \xrightarrow{\hat{f}'} & \bar{F}_{\hat{x}} \\ \downarrow & & \downarrow \\ F_{\underline{x}} & \xrightarrow{f'} & \bar{F}_{\underline{x}} \end{array} ,$$

where $\hat{f}' = f(u', \bar{v}'; t', \bar{w}')$. In (14) we observed that the diagrams $F, \bar{F}$ restricted to those $\underline{z}$ with $z_1 = x_1$ for $x_1 \neq 0$ are just the diagrams $F_{u', v'; t', w'}, \bar{F}_{u', v'; t', w'}$. Moreover, $\hat{x}$ is such a $\underline{z}$, so $F_{\underline{x}} \rightarrow F_{\hat{x}}, \bar{F}_{\underline{x}} \rightarrow \bar{F}_{\hat{x}}$ appear in these respective diagrams. Since $p' \leq p-1$, $\Pi_{p'}$ is true with the "f" taken to be the restriction of the "f" of Π_{p-1} . So by $\pi 3$, the first square commutes. Similarly, since $\hat{x}_1 = \bar{x}_1$ for $\bar{x}_1 \neq 0$ and $\bar{p}' \leq p-1$, the second square commutes. Thus the original square commutes.

This shows that we are in the situation of Theorem 65. For each $(u, v; t, w)$ with $n-t+u-w+v-2 = p$, $v \leq w < u \leq t < v+n$ and $0 \leq v < u < n$ choose a map $g: F_{u, v; t, w} \rightarrow \bar{F}_{u, v; t, w}$ such that $g_{\underline{x}} = f(u', v'; t', w')$ when $\underline{x} \neq 0$ and $F_{u, v; t, w}^{\underline{x}} \neq 0$, and $g_{\underline{x}} = 0$ otherwise: this choice is possible by (22) and Theorem 65.

Put $f(u, v; t, w) = g_0$.

Using the equation $Lf(u,v;t,w) = f(v+n,u;w+n,t)$ we thus obtain a map $f(u,v;t,w)$ for all p -suitable $(u,v;t,w)$. With this choice π_1, π_2, π_3 are satisfied. So Π_p is true. //

(25) Let f be as in Π_{n-2} . Then $h = (h_{(u,v)})$:
 $G \rightarrow \bar{G}$ given by $h_{(u,v)} = f(u,v;u,v): G(u,v) \rightarrow \bar{G}(u,v)$ is
a stable graded natural isomorphism.

Proof. h is stable by π_2 . By π_3 the diagrams:

$$\begin{array}{ccccc}
 G(u,v) & \longrightarrow & I(u,v;t,w) & \longrightarrow & G(t,w) \\
 \downarrow f(u,v;u,v) & & \downarrow f(u,v;t,w) & & \downarrow f(t,w;t,w) \\
 \bar{G}(u,v) & \longrightarrow & \bar{I}(u,v;t,w) & \longrightarrow & \bar{G}(t,w) ,
 \end{array}$$

commute for $(u,v) \rightarrow (t,w)$ of degree zero; using stability this implies h is natural. By π_1, π_3 f gives $g: F_{u,v;u,v} \rightarrow \bar{F}_{u,v;u,v}$ with $g_0 = h_{(u,v)}$ and $g_x = 1$ if x is a corner. So $h_{(u,v)}$ is an isomorphism by Proposition 62(b). So h is an isomorphism. //

(26) In Lemma 54, (c) $\Rightarrow$ (b) . //

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