

THE UNIVERSITY OF SYDNEY

CAPTURING STUDENTS' CONCEPTUAL CHANGE WHEN EXPLORING DECIMALS THROUGH DYNAMIC DIGITAL REPRESENTATIONS

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Statement of Originality

This is to certify that the content of this thesis is my own work. This thesis has not been submitted for any other degree or purpose.

I certify that the intellectual content of this thesis is the product of my own work, and that all assistance received in preparing this thesis and all sources have been acknowledged.

Name: Amelia Kate Gorman

Date: 10 December 2025

Abstract

Decimals hold significant importance within the primary mathematics curriculum because of their close connection to real-world applications, such as measurement, money, and data interpretation. Despite their importance, research has consistently shown that many students experience persistent difficulties in understanding decimal fractions, often carrying misconceptions into secondary schooling and beyond. Addressing these challenges requires pedagogical approaches and learning environments that not only make the structure of decimals explicit but also engage students in ways that foster conceptual change.

The purpose of this study was to examine the potential of dynamic digital representations to support the development of students' knowledge of decimal fractions. Dynamic representations, by enabling learners to manipulate and visualise mathematical ideas, offer unique affordances for making underlying concepts more accessible. This research investigated how primary students interact with such tools and whether their engagement can facilitate meaningful shifts in understanding.

To explore these questions, task-based interviews were conducted with six Year 4 (9-10 years old) students, each of whom worked with four different dynamic digital representations of decimals: *Wishball-hundredths, Decimal Strips, Zooming in on Place Value*, and *Zoomable Number Line.* Data were gathered through video- and audio-recordings, alongside screen-captures that traced the students' real-time interactions with the software. The analysis focused on detecting shifts in students' attention during their engagement with the representations. These shifts were examined using microgenetic methods, which enabled moment-by-moment tracking of the ways in which conceptual understanding evolved across the tasks.

The findings revealed that specific features of the digital representations acted as catalysts for productive cognitive conflict. In several cases, this confusion challenged students' initial assumptions about decimals, prompting them to reconstruct and refine their conceptualisations. The unique affordances of each tool were found to support different pathways of exploration: some encouraged students to reason about place value structure, others promoted the use of visual models to connect symbolic and spatial representations, and still others enabled iterative testing of hypotheses through immediate feedback.

This study contributes to the growing body of research on the role of digital technologies in mathematics education by highlighting how dynamic representations can support conceptual change in the domain of decimals. The results suggest that carefully

designed digital tools can provide students with opportunities to actively test, revise, and consolidate their thinking about decimal fractions. For teachers, these tools have the potential to enrich classroom practice by creating contexts where misconceptions surface and can be addressed through exploration and dialogue. Overall, the study underscores the importance of integrating dynamic digital representations into primary mathematics instruction as a means of deepening students' conceptual understanding and preparing them for future mathematical learning.

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As a co-author of the paper Changing structures of attention when learning about decimal fractions with digital tools. I confirm that Amelia Gorman's contribution to the paper is

consistent with her being named first author. In particular, the candidate's contribution to the following items should be noted:

- design of the study including concept of idea and research approach;
- critical reading and synthesis of topic academic texts;
- analysis of the data and writing of results and discussion.

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Artificial Intelligence Statement

During the preparation of this thesis, the author used GPT-4, paraphrase content for clarity, enhance sentence structure, and perform text enhancement. The author confirms that where text was modified by generative AI, the content was reviewed for possible errors, inaccuracies, and bias. The author takes full responsibility for the submitted thesis, confirms the work is their own, and has used generative AI in accordance with University guidelines and policies.

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CHAPTER 1: INTRODUCTION

1.1 Opening Statement

"Wait how can 0.25 be smaller than 0.3? Twenty-five is bigger than three!" This response from a Year 4 student captures a common misconception in children's early encounters with decimal fractions which is the belief that a longer numeral must represent a larger value. Such moments highlight the challenge of learning decimals, where whole-number thinking can conflict with the abstract structure of the decimal system. Despite the prevalence of decimals in everyday life including interpreting measurements, comparing prices and analysing data, many students continue to experience persistent difficulties in understanding them, often carrying misconceptions well into secondary schooling and beyond (Lamon, 2001).

Understanding why these misconceptions occur and how to address them remains a significant challenge for mathematics educators. Representations such as concrete materials, diagrams and models are widely used to help students visualise abstract mathematical ideas. However, there is still uncertainty about which representations are most effective for supporting conceptual understanding of decimals. In particular, static representations concrete models may be limited in making the invisible structure of decimals, such as the relationship between place value, magnitude, and density, meaningfully visible to learners.

Dynamic digital representations offer unique affordances for exploration and feedback. These digital tools allow students to actively manipulate mathematical ideas, make conjectures and receive immediate responses to their actions. By engaging students in visualising and testing relationships, dynamic representations can help make the unseeable aspects of mathematical structure visible and accessible. As such, they provide a potential aide for supporting conceptual change in mathematics particularly in the complex domain of decimal fractions.

1.2 Purpose of the Study

The aim of this research study was to explore what conceptual changes that emerged while children in middle-primary grades (9-10 years old) engaged with dynamic digital representations of decimal fractions. Decimals are of significance in the primary mathematics curriculum due to their application and use in everyday life. However, decimals and the related areas of ratio and proportion, are recognised as one of the most challenging and complex areas of mathematics for young children to learn (Lamon, 2001). Such complexities arise as students

are exposed to rational numbers in numerous symbolic representations, including ratios, fractions, decimals, and percentages, without developing deep understanding and knowledge of how to use each notation correctly (Bobis, 2011). Furthermore, decimal fractions present additional complications as part of the number relationship needed to make sense of the numeral, the denominator, is not visible. Steinle and Stacey (2004a) suggest that students find such a concept difficult to comprehend due to the abstractness of a 'hidden' number.

Children have difficulty in understanding the abstract nature of decimals, and a plethora of misconceptions arise from incomplete understandings (Irwin, 2001; Stacey & Steinle, 2003). Representations such as concrete materials, models and drawings are used in the mathematics classroom to assist students in understanding abstract concepts that are central to mathematics learning. An important feature of mathematics education is working with appropriate mathematical diagrammatic representations, but there is uncertainty around the best representations for decimals (Resnick et al., 2019; Steinle & Stacey, 2004). Dynamic representations are potentially an effective support for student learning within mathematics because of their ability to demonstrate relationships between quantities, highlight emerging patterns, and make otherwise abstract concepts more visible. Unlike static representations, students can actively interact with the mathematics content by making conjectures and testing them immediately to experience the concept from various perspectives and receive feedback on their thinking (Orrill & Polly, 2013). In particular, learner-centred digital tools that encourage interactive engagement with mathematics, allow students to 'see' notions that would otherwise be 'unseeable' (Orrill & Polly, 2013). Although there is an abundance of research into the use of digital learning objects in mathematics education (Moyer-Packenham, et al., 2016; Litster, et al., 2019), relatively little work has focused on the role of dynamic digital representations in teaching and learning decimals. The intention of this study was to establish a foundation for research on this topic by exploring how children's interactions with dynamic digital representations of decimal fractions may develop their conceptual understanding in this topic.

1.3 Research Questions

This study was guided by an overarching question designed to explore how conceptual changes emerge as children engage with dynamic digital representations of decimal fractions. The findings contribute to a broader understanding of how students develop decimal sense through interactive technologies and how this knowledge may inform more effective pedagogical practices in primary mathematics classrooms.

The overarching research question used to guide the study was:

How do children experience conceptual changes related to decimal fractions when they attend to features of dynamic digital representations?

The sub-research questions were: *How do students overcome misconceptions about decimal fractions when they attend to features of dynamic digital representations?*

As well as: *What features of Wishball-hundredths, Decimal Strips, Zooming in on Place Value, and Zoomable Number Line, do children attend to that prompt conceptual changes related to decimal fractions?*

1.4 Significance of the Study

This study contributes to the growing body of research on digital technologies in mathematics education by highlighting how dynamic representations can act as catalysts for conceptual change in the learning of decimals. The findings show how particular design features of digital tools can provoke productive cognitive conflict, prompting students to confront and reconstruct misconceptions. Each tool offered distinct pathways for exploration with some that highlighted decimal place value structure, while others emphasised decimal magnitude and density.

For teachers, this research underscores the pedagogical value of integrating dynamic representations into primary mathematics classrooms. By creating spaces where students can visualise, test and discuss mathematical ideas, these digital tools may support deeper conceptual understanding and encourage flexible thinking. The study's insights offer practical implications for designing and facilitating digital learning experiences that promote mathematical reasoning and address persistent misunderstandings about decimals.

1.5 Structure of the Thesis

This thesis is organised into nine chapters. Chapter 1 introduces the study, outlining the background, rationale, purpose, research questions and significance. Chapters 2 and 3 present a review of the literature related to decimal fractions and mathematical representations, with particular focus on the affordances of dynamic digital representations. These chapters establish the conceptual foundations of the study and situate it within existing research on how digital tools can support the development of students' understanding of decimals.

Chapter 4 outlines the theoretical frameworks that underpin the study, drawing on perspectives from constructivism, the role of representations in mathematics education, and the notion of shifts of attention. Together, these frameworks provide a productive lens for

examining how students interact with dynamic digital representations of decimal fractions. In particular, the theoretical framing supports an exploration of students' developing conceptual understanding within a learning environment structured around rich problem-solving tasks, by attending to the ways their attention shifts throughout the process of task engagement.

Chapter 5 describes the methodology employed in this study. A qualitative research design was adopted, using task-based interviews in conjunction with microgenetic methods for data collection and analysis. This chapter explains the participants, procedures, analytical approaches, and ethical considerations that informed the research design.

Chapters 6, 7, and 8 present the findings and discussion in response to the research questions. Chapter 6 addresses the first overarching research question: *How do children experience conceptual changes related to decimal fractions when they attend to features of dynamic digital representations?* Analysis of the data from students' interactions with the digital tools revealed distinct learning pathways and important changes in conceptual understanding, as identified through shifts in students' attention during the tasks.

Chapter 7 revisits the overarching research question and also focuses on the additional guiding question: *How do students overcome misconceptions about decimal fractions when they attend to features of dynamic digital representations?* This chapter highlights moments where students' prior misconceptions were challenged and restructured through engagement with the dynamic features of the digital tasks.

Chapter 8 addresses the sub-research question that examined the specific features of each dynamic digital representation:

What features of Wishball-hundredths, Decimal Strips, Zooming in on Place Value, and Zoomable Number Line, do children attend to that prompt conceptual changes related to decimal fractions?

This chapter provides a detailed analysis, highlighting areas of difference and similarity in conceptual understanding and strategy use across the four tasks involving dynamic digital representations of decimal fractions.

Finally, Chapter 9 concludes the thesis by summarising the key findings and reflecting on their implications for theory, research and classroom practice. It also considers the limitations of the study and proposes directions for future research in supporting conceptual change through dynamic digital representations in mathematics education.

CHAPTER 2: DECIMAL FRACTIONS

2.1 Literature Review

The literature review for the study is presented across three chapters. The three-part review of both seminal and current literature relates to the areas of *decimal fractions*, *mathematics representations* and relevant *theoretical frameworks* (shifts of attention theory, representations theory and constructivism). The review outlines existing research into the role of conceptual understanding in the primary mathematics classroom as well as the importance of decimal fractions in the primary mathematics curriculum. The difficulties students may encounter when learning about decimal fractions, including the misconceptions that often arise from incomplete understandings, and finally the three key concepts related to decimal fraction understanding; *decimal density, place value,* and *relative magnitude* are outlined. In the second chapter, the role of representations in mathematics learning including the importance of multiple representations related to a single concept, the affordances and constraints of dynamic representations and lastly the emerging field of dynamic digital representations are discussed. Finally, the theoretical framework that underpins the study is outlined, drawing on perspectives from constructivism, the role of representations in mathematics education, and the notion of 'shifts of attention'. Together, these perspectives provide a productive lens for examining how students interact with dynamic digital representations of decimal fractions. In particular, the framework supports an exploration of students' developing conceptual understanding within a learning environment structured around rich problem-solving tasks, by attending to the ways their attention shifts throughout the process of task engagement.

2.1.1 Conceptual Understanding of Decimal Fractions

Decimal fractions are derived from the base-ten number system. Martinie (2014) suggests when students lack an understanding of place value and the base-ten number system they experience increased frustration and difficulties with computing decimal fractions. Further difficulties arise when learners fail to recognise that decimal fractions are a special class of fractions with base-ten denominators (Martinie, 2014). Students need prior knowledge and experience with whole-numbers and common fractions to build conceptual understanding of decimal fractions as decimal representation is compounded in complexity by the merging of these rational number systems.

An understanding of decimal fractions is commonly referred to as decimal sense. This concept is connected to an overarching idea of number sense which refers to a person's general understanding of numbers and operations as well as the ability to handle daily-life situations that include numbers. This ability is used to develop useful, flexible and efficient strategies for handling numerical problems (Yang, 2005). Students with good number sense possess several important but elusive capabilities, including flexible mental computation, numerical estimation, and quantitative judgment (Greeno, 1991). They can move seamlessly between the real world of quantities and the mathematical world of numbers and numerical expressions. In addition, these learners can invent their own procedures for conducting numerical operations and represent the same number in multiple ways depending on the context and purpose of this representation (Case, 1998). The capabilities of these students illustrate why number sense is important in mathematics education. The relationship between rational number understanding and broader mathematical achievement is well established. Fraction and decimal reasoning involves connecting whole-number thinking to more complex mathematical ideas, and difficulties in this domain can inhibit students' further learning. Resnick et al. (2019) highlight that early rational number knowledge, particularly comparing and reasoning about magnitudes, predicts later mathematics achievement beyond general cognitive factors. More recent work by Resnick et al. (2023) suggests that reasoning flexibly across fractions and decimals supports structural understanding and relational reasoning which are both fundamental to later algebra and problem-solving. These findings underscore the importance of supporting deep conceptual understanding of decimals in the primary years.

Number Sense

Number sense is a complex process involving many different components of numbers, operations, and their relationships (Yang, et al., 2008). Based on a review of the number sense literature (McIntosh et. al, 1992; Markovits & Sowder, 1994; NCTM, 2000; Yang & Wu, 2010; Li & Yang, 2010), the basic characteristics of number sense are defined to include:

- Understanding the basic meaning of numbers: This implies an understanding of the base 10-number system including whole numbers, fractions, and decimals, as well as knowledge of place value, pattern of numbers and using multiple ways to represent numbers (McIntosh et. al, 1992).
- Recognising relative number size: This component determines understanding of relativity of numbers and their absolute magnitudes.

- Being able to use a benchmark appropriately: This requires the use of benchmarks to provide mental referents when thinking about numbers to solve problems appropriately (McIntosh et. al, 1992).
- Being able to judge the reasoning of a computational result: This component refers to using mental computation strategies to solve problems and to articulate reasoning of results.

Stacey et al. (2001) stated that students should develop number sense about whole numbers, fractions, decimals, and percent. Among these, decimal numbers are seen as a problematic area and have been recognised as a significant source of learning and teaching difficulties (Stacey et al., 2001).

McIntosh et al. (1992) extends the notion of number sense and provides an influential definition of 'mature' number sense. Number sense reflects an ability to use numbers and quantitative methods as a means of communicating, processing, and interpreting information. The theory of number sense holds an expectation that numbers are useful and that mathematics has a certain regularity (McIntosh et al., 1992). With this understanding, it is clear that the ability to reach a level of mature number sense requires mathematical knowledge and fluency of application of all rational numbers including decimal fractions. A balance of sound conceptual understanding around decimal fractions and efficient procedural knowledge of operations related to this mathematical idea is necessary to achieve well-rounded number sense where real-world knowledge transfer can occur.

Conceptual Understanding vs Procedural Understanding

In mathematics education, conceptual understanding is characterised as "an integrated and functional grasp of mathematical ideas" (National Research Council, 2001, p. 118). It is more than knowing facts and computational procedures but rather involves knowing why a mathematical idea is important and when it is useful. Conceptual understanding develops over time as students organise their knowledge into coherent, integrated mental structures (Caldwell, 2020). This understanding is most useful when the structure relates mathematical concepts to methods and computational procedures. Conceptual understanding includes being able to represent a variety of mathematical situations in different ways, to see the connections between those representations, and to know how to use each representation for different purposes (Caldwell, 2020).

Conceptual understanding begins with the real-life, self-constructed knowledge that may be correct or incorrect based on the student's understanding and experience (Hiebert,

1986). This knowledge can be drawn upon and used to apply to similar situations, as well as used to build knowledge of other new, related mathematical ideas (Rittle-Johnson et al., 2015). It is suggested that this informal knowledge may be unrelated to symbols and procedures (Mack, 1999). As a result, when students encounter formal fraction notation they often struggle to integrate this informal knowledge with symbolic representations. Students' understanding of operating with fractions consisting of rote procedures without connections to other mathematical concepts are most often incorrect. Researchers Gunderson and Gunderson (1957) and Leinhardt (1988) have shown that students can use informal knowledge to reason about joining and separating fractional quantities when real-life situations are presented.

In contrast to conceptual understanding, procedures are a series of steps or actions employed to achieve a task and reach a solution (Rittle-Johnson et al., 2017). Martin (2009) cautions that teaching procedures in an isolated pedagogy which employs rules without reason can often lead to incorrect solutions as well as the development of misconceptions when students attain the right answer for the wrong reason. Procedural knowledge is characterised by Rittle-Johnson et al. (2015) as the capacity to follow steps in sequence to solve mathematical problems or reach a mathematical goal. This requires a familiarity with and a knowledge of the system of symbols to construct algorithms. It can also pertain to a knowledge of procedural rules necessary to solve problems (Rittle-Johnson et al., 2015). Baroody et al. (2007) observed that procedures can often be interconnected or embedded within other procedures and disagree with teachers who may view procedural knowledge to be devoid of relationships.

The importance of conceptual understanding within mathematics education is evident in the real-world context of the 21st century. Learning how to perform computational algorithms is not sufficient preparation for life in outside the classroom in a technological world. People carry many devices that can perform most calculations, using mobile phones, tablets and computers. Students as 21st century learners require a deeper level of knowledge with an understanding of what mathematics they are doing and why they are doing it. They must select appropriate ways of solving problems and doing computations in a range of mathematical situations. Hiebert (2003) summarises the importance of conceptual understanding in saying "understanding is the key to remembering what is learned and being able to use it flexibly" (p.53).

Research suggests a focus on conceptual understanding for instruction will develop into a deeper understanding for procedural knowledge (Kieren, 1992). Moss and Case (1999) and Saxe et al. (2005) posit instruction emphasising procedural knowledge of rational number, as opposed to conceptual understanding, may discourage students from understanding rational

numbers in a meaningful way. Mack (1990) proposes the notation children devise themselves will lead to multiplicative reasoning instead of additive reasoning which is a critical element for adequate understanding of fractions and decimals. Teachers confront a challenge between balancing conceptual understanding and procedural knowledge of fractional numbers when instructing young learners in the classroom. The use of procedural knowledge and efficiency may be the teachers' end goal for students however the intermediate step to achieve this end must integrate conceptual understanding into everyday mathematics instruction (Lamon, 2006).

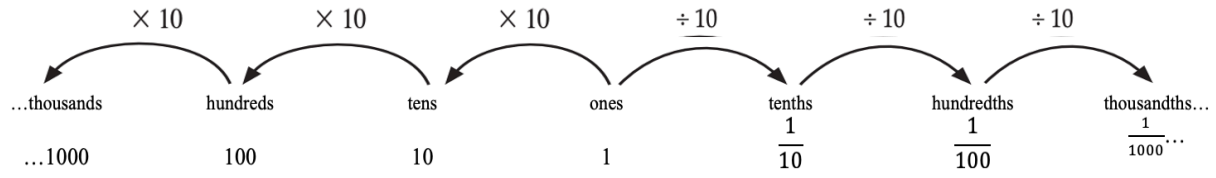
The growth of students' conceptual understanding is synonymous with mathematics learning. Conceptual understanding develops over time as students organise their knowledge into a coherent, integrated mental structure (Caldwell, 2020). The learning of mathematics occurs through shifts of students' attention which "constitutes changes in understanding" (Mason, 2002, p. 25). A student's understanding evolves not in a linear progression, but through the accumulation of more refined microqualities of attention, leading to a deeper grasp of a mathematical concept (Mason, 2008). Pirie and Kieren's (1991) theoretical work on acquiring mathematical understanding proposes a similar learning trajectory, highlighting that 'folding back' is crucial for the development of mathematical thinking. A growth in mathematical understanding is seen to emerge through the continual movement back and forth through levels of knowing, as students reflect on and reconstruct their current understandings (Pirie & Kieren, 1991). For example, a student grappling with flexible strategies for addition and subtraction of decimals may face a problem that is not immediately solvable. They will need to revisit earlier understandings of decimal concepts and use these to inform their new thinking at the higher learning level, leading to a deeper understanding of the concept. For example, performing a mental subtraction calculation requires a complex chain of reasoning and cognitive strategies including a "series of multi-step associations and procedures that involve facts, verbal chains, discriminations, concepts, and rules designed to bring about a response to a specified problem" (Kame'enui & Simmons, 1990, p. 294). Clements and Sarama (2004) developed a cognitive model of students' learning that described the processes involved in the construction of conceptual knowledge across several qualitatively distinct structural levels of increasing complexity. Importantly, these levels of structured understanding are not distinct classes of mathematical thinking that suddenly change, rather higher levels are elaborations and advancements of lower levels (Clements & Sarama, 2004).

2.1.2 Decimal Fractions and Decimal Sense

Currently, there is insufficient research regarding the optimal learning trajectory through which students develop a conceptual understanding of decimals (Tian & Siegler, 2018). However, it is agreed that educational researchers and teachers should have a strong understanding of the key concepts related to decimal fractions to investigate optimal teaching strategies (Moloney & Stacey, 1996). The most common representation of equivalent fractions in real life is decimals (Lamon, 2012). Decimals were created to reduce the difficulty of operating on fractions, especially with addition (Stacey & Steinle, 1998). They have become commonplace because of their application to the standard international system of measures, e.g., 1.8 metres, 1.450 tonnes (Moloney & Stacey, 1996). Decimals are constrained equivalent fractions in that the denominators are restricted to tenths, hundredths, thousandths, and so forth. Any fraction can be expressed in equivalent form to a desired degree of accuracy. The place value system is positional in that the total value of a digit is the multiplication of its face value by its place value position as illustrated in Figure 2.1. For example, the 7 in 23.479 represents 7 x $\frac{1}{100} = \frac{7}{100}$. The system is infinite in both left and right directions, with the connection between consecutive places being multiplication or division by ten (Siemon, et al., 2011).

Figure 2.1

Diagram of place names and multiplicative relationship between adjacent places.



The power of the place value system is in the simplicity of calculation. To add $\frac{3}{4} + \frac{2}{5}$ requires conversion of both fractions to equivalent forms with the same denominator, $\frac{3}{4} = \frac{15}{20}$, $\frac{2}{5} = \frac{8}{20}$, so $\frac{3}{4} + \frac{2}{5} = \frac{15}{20} + \frac{8}{20} = \frac{23}{20} = 1\frac{3}{20}$. If the decimals are known, the corresponding calculation is 0.75 + 0.4 = 1.15. The addition calculation is represented in Figure 2.2. Decimals can also be used to compare fractions by converting each fraction into its decimal form, which allows students to compare quantities using familiar place value ideas. For example, when $\frac{1}{3}$ is converted to 0.33 as a decimal, and $\frac{2}{5}$ converts exactly to 0.4. Once expressed as decimals, students can then directly compare 0.33 to 0.4 and deduce that $\frac{1}{3}$ is less than $\frac{2}{5}$.

Figure 2.2

Diagram of addition calculation 0.75 + 0.4 = 1.15 using decimal fractions.



Note. Sourced from (National Council of Teachers of Mathematics, 2014).

Three Key Decimal Fraction Concepts

Decimal fractions are a complex representation of rational numbers that many students struggle to understand when the curriculum shifts from whole numbers to rational number in the middle-primary years (Steinle & Stacey, 2004a). Developing conceptual understandings of decimal density, place value and relative magnitude of decimal fractions is considered key for consolidating number sense, since these properties unify all numbers (Siegler, et al., 2011) and rational number magnitude knowledge is related to future mathematics achievement (DeWolf, et al., 2015). Furthermore, decimal fractions need to be understood as fractions before they are introduced as an extension of the base-10 number system (Siemon et al., 2011).

A decimal fraction refers to a numeral written with digits to the right of a decimal point that represents a part of a whole, for example 0.75. Place value of decimal fractions involves symmetry around the ones place value with the first digit after the decimal represents the tenths place. The next digit after the decimal represents the hundredths place and following this the thousandths place value. The number 36.271 can be placed on a place value chart as shown in Figure 2.3. The place value columns facilitate reading the number in a manner consistent with how place value positions are used for whole numbers, such as “thirty-six and two-hundred and seventy-one thousandth”.

Figure 2.3

Diagram of place names with decimal point as a marker for the Ones column.

Tens	Ones with Decimal Point	Tenths	Hundredths	Thousandths
3	6 .	2	7	1

Saxe et al. (2005) contend that correct student understanding of *decimal place value* knowledge does not automatically mean decimal numbers can be completely understood. This is because decimals are a large content area within the primary mathematics curriculum, linked with other mathematics topics, and therefore require additional numerical knowledge. This

includes the use of appropriate fractional language, understanding and correctly representing relative size and knowing and acknowledging that there are an infinite number of decimal possibilities that exist (Moody, 2010; Zevenbergen et al., 2004). For example, the common pronunciation of "point four two" does little to convey meaning. Encouraging students to decode the symbol as four tenths and two hundredths may help reinforce the connection between the new symbol and the more familiar expression of that quantity (Moody, 2010). As Goldin and Schteingold (2001) suggest, developing clear links between quantity, vocabulary and symbol is critical for new understanding.

Researchers generally argue that children need a good understanding of *decimal place value* concepts before moving on to working with these numbers in operative problems (Irwin, 2001; Sowder, 1997; Wearne & Kouba, 2000). Children need to recognise the decimal unit a digit represents to form a complete understanding of place value in decimals, which according to Nuerk et al. (2014) is acquired in three learning stages. Firstly, place value identification refers to the correct locating and naming of digit positions in a multi-digit number (Nuerk et al., 2014). From here, place value activation is developed, which refers to the use of the numerical information of a decimal unit, often seen in number comparison tasks (Nuerk et al., 2014). Finally, place value computation describes the integration of place value information in arithmetic tasks (Nuerk et al., 2014). A pillar in the Herzog et al. (2019) developmental model of place value understanding is the relation of dependence and inclusion between the stages of conceptual acquisition. When applying the theoretical findings to changes in student thinking, it is clear that they require the knowledge of lower levels to develop successive levels. To progress beyond their current decimal thinking they also need to have previously developed the prior levels of concept knowledge and associated numeracy skills (Battista, 2012).

One of the key concepts required for understanding decimal fractions is *decimal density*. Decimal density refers to the continuity property of rational numbers, whereby between any two decimals there are an infinite number of other decimals (Widjaja et al., 2008). *Decimal magnitude* understanding refers to the ability to estimate and reason about the size of a decimal (Resnick et al., 2019). One way in which students can demonstrate their knowledge of *decimal magnitude* is by determining the location of a decimal on a number line, rationalising about the size of a decimal compared to the benchmark fraction (e.g. ½) and judging the validity of decimal and fraction equivalencies (Resnick et al., 2019). Students must be exposed to quality learning experiences that build decimal knowledge inclusive of *decimal place value*, *decimal density*, and *relative magnitude*.

2.1.3 Conceptual Approaches to Decimal Fraction Understanding

Numerous studies have investigated the teaching and learning of decimal fractions, offering valuable insights into the conceptual hurdles students face and effective strategies for instruction. Across diverse research efforts, three interconnected themes emerge as essential for building decimal sense: understanding decimal structure and place value, using models and representations, and applying real-world and contextualised experiences.

Decimal Structure and Place Value Understanding

A recurring challenge in decimal learning is students' misunderstanding of decimal place value and the additive structure of the number system. Even preservice teachers who had advanced mathematical exposure commonly misconstrued decimals by applying whole-number reasoning, as highlighted by D'Ambrosio and Kastberg (2012). Their study revealed that a core difficulty lies in recognising the hierarchical subdivision of units (e.g., tenths into hundredths, hundredths into thousandths) and in understanding that decimal numerals are additive in nature. For example, failing to interpret 0.606 as 6 tenths and 6 thousandths, rather than only 606 thousandths, demonstrates a critical gap in understanding. This difficulty is not isolated. Cramer et al. (2015) noted that students with strong decimal comprehension consistently demonstrated the ability to decompose and compose decimals based on place value understanding. This flexibility was crucial for comparison, estimation, and operations involving decimals. Research therefore stresses that a solid grasp of place value, understanding both the structure and the additive properties of decimal numbers, is fundamental to deeper mathematical reasoning.

Use of Representations and Models

Another central concept across studies of decimal understanding development is the importance of visual and concrete representations. D'Ambrosio and Kastberg (2012) pointed to decimal grids as powerful tools for illustrating decimal density and supporting visualisation of numerical relationships. However, their findings also cautioned against procedural strategies that might obscure true understanding including padding decimals with zeros for comparison, for example writing 0.5 as 0.50 to compare it with 0.47. Importantly, they highlighted that such tactics could reinforce whole number reasoning if not paired with meaningful representation. Cramer et al. (2015) identified five instructional indicators for decimal understanding, many of which relate to model use. Students who effectively used visual models, such as hundreds-grids and were able to connect these models to precise mathematical language and estimation strategies, displayed stronger conceptual understanding. These findings reinforce the idea that

models serve not only as visual aids but also as cognitive bridges between abstract symbols and tangible understanding.

A foundational connection exists between measurement and decimal fractions through their shared reliance on length as a representational form. When students engage with decimal fractions in measurement contexts such as number lines, rulers or area models, they are interpreting and partitioning continuous quantities, which supports a more tangible and visual understanding of decimal magnitude and place value (Gravemeijer et al., 2017; Wilkins & Norton, 2018). In particular, number lines feature prominently in research as effective tools for supporting decimal understanding. Denton (2016) used handmade metric tapes and Cuisenaire Rods in a measurement and mapping project that provided students with extensive hands-on experience. These materials, embedded in physical activity, acted as embodied number line models that enabled sustained exploration of decimal notation and equivalencies in meaningful and context-rich ways. This connection between measurement and decimals is particularly relevant in the context of the current study, where the selected dynamic digital tools relied on length-based representations. Although each tool employed this representation differently, they all foreground quantity as a measurable extension or subdivision of length. These dynamic and interactive digital environments offered students opportunities to manipulate and visualise decimal values, thereby supporting conceptual development through spatial reasoning and visual patterning (Hillmayr et al., 2020). Such representations make abstract concepts more accessible by grounding them in familiar experiences of measuring and comparing lengths.

Understanding decimal magnitude requires integration of the place value convention of recording digits in columns alongside a fractional understanding of denominators (Moody, 2010). Teachers should view existing student constructs as knowledge sites to anchor new meaning. The linkage of new symbols and systems to concrete referents is seen as an important first step in understanding new concepts (Goldin & Shteingold, 2001). The integrated theory of numerical development identifies a unifying feature of mathematics learning whereby all real numbers, including fractions and decimals, have magnitudes that can be ordered along the number line (Siegler & Lortie-Forgues, 2014; Siegler et al., 2011). As children expand their mathematics knowledge base, they concurrently increase their mental number lines to include larger whole numbers as well as non-whole rational numbers (Siegler & Lortie-Forgues, 2014; Siegler et al., 2011). To achieve complete conceptual understanding of decimal fraction magnitudes, students need to reorganise their conceptions of the number line to accommodate new numerical properties (Siegler & Lortie-Forgues, 2014; Siegler et al., 2011). Therefore, it

is worthwhile exploring the capabilities and limitations of a number line representation to assist with students activating numerical representations to express decimal magnitude.

Real-World and Contextualised Learning Experiences

Moving beyond abstract instruction, several researchers championed the use of real-world contexts to develop decimal concepts. Irwin (2001) and Bonotto (2004, 2009) argue that students lived experiences, especially with measurement, can be powerful entry points for learning decimals. Measurement inherently involves magnitude and comparison, providing a transparent connection between the symbolic and the concrete. This concept underpins the approach taken by Rafiepour and Karimianzade (2018), who designed a teaching experiment in which students measured lengths, calibrated units and discovered decimal representations through contextualised activities. Their work echoes the principles of Realistic Mathematics Education (RME), which emphasises the value of authentic, meaningful contexts in supporting student-led knowledge construction. Similarly, Lai and Tsang (2009) advocate for RME in decimal instruction, observing that traditional methods often fail to connect with students' real-life knowledge or encourage active learning. Denton's (2016) classroom mapping project provided students with a tangible purpose for using decimals, offering the necessary time and tools to develop understanding at the concrete level before moving toward abstraction.

When designing learning experiences to explore concepts tied to decimal fractions, teachers should consider the affordances and merit of using a real-world context in which to ground the learning. This is also applicable for researchers designing tasks that aim to reveal the cognitive conflict and conceptual understanding that may emerge from engaging with mathematical experiences connected to problems found in the real-world. For example, a task that requires students to compare and order a set of gymnastics competition scores that are recorded to two decimal places, allows them to tangibly grasp the mathematical concepts within a context that they may relate to. The findings from the studies outlined above were pivotal in informing the task design for this research study. The tasks spanned a range of decimal concepts inclusive of place value, decimal density and relative magnitude, as well as incorporating different representations such as number lines, a place value abacus, relative magnitude area models, and numerical notation. In addition, some of the activities included in the interviews were contextualised in a real-world scenario such as a sport score problem.

Together, these studies suggest that building decimal sense requires more than procedural fluency as it demands instructional strategies that foreground conceptual understanding. Understanding decimal fractions is a complex, multi-dimensional process.

Instructional design must account for students' developmental readiness, prior knowledge, and the cognitive demands of decimal concepts. When conceptual foundations are laid through carefully chosen models and real-world contexts, students can develop strong decimal sense.

2.1.4 Decimal Fraction Misconceptions

Decimal fraction sense is associated with building a conceptual understanding rather than a procedural understanding of fractions and forms a basis for developing appropriate strategies in mathematical tasks (Way, 2011). If students do not have a strong conceptual understanding of decimals, they are more susceptible to developing a set of misconceptions associated with the topic area. Furthermore, as students incorporate new concepts associated with decimal fractions into their existing knowledge of rational numbers, systematic and predictable errors of understanding arise (Martinie, 2014).

The term misconception is used by some researchers to describe situations in which a learner's understanding is in conflict with accepted meanings and understandings of mathematics (Barmby, et al., 2009). Other researchers object to this term because, while these ideas are technically incorrect, from the student's viewpoint, the ideas expressed are logical (Schnepper & McCoy, 2013). In this paper the term misconception is used to express the viewpoint that these ideas form a natural part of the development of mathematical understanding but are errors in thinking that are well-documented in the literature.

Misconceptions are predictable, systematic mistakes that arise from an incorrect interpretation; broadly they are a mistaken way of thinking and understanding about a mathematical concept (Steinle, 2004). Young students' misconceptions about the meaning of decimal number notation have been well documented in the works of Steinle and Stacey (Steinle & Stacey, 2004a; Steinle, 2004, Steinle & Stacey, 2001). Misconceptions often arise when students inappropriately apply rules to decimal fractions that result in the right answer for the wrong reason, reinforcing the fabricated rule (Hiebert & Wearne, 1985; Swan & Marshall, 2010). The application of this thinking then persists, and procedural flaws are not corrected (Hiebert and Wearne 1985; Steinle and Stacey 2004a). Instruction that favours rule-based knowledge produces correct answers in the beginning; however, students who only encounter procedural methods in their learning of decimal fractions tend to regress in performance over time (Woodward, et al., 1997; Steinle & Stacey, 2004a). Mack (1990) suggests that with time and meaningful instruction, students can separate whole-number from rational-number constructs and thus develop an accurate understanding of how decimal

fractions are represented symbolically. Understanding misconceptions is an essential step to improving instruction in decimal fraction lessons (Steinle & Stacey, 2004a)

Teachers need to be equipped with appropriate instructional tools that manage cognitive conflict in the context of student learning about decimal fractions. Whilst a plethora of studies have been conducted into the specific causes of student misconceptions, persistent procedural errors in decimal comparison tasks indicates a need to identify the factors that will assist the design of successful teaching interventions (Okazaki & Koyama, 2005; Moody, 2010). Therefore, future studies should aim to investigate what instructional tools can be implemented in a mathematics classroom to assist students' in developing correct decimal fraction sense and avoid common misconceptions from arising.

Many students encounter difficulties grasping concepts linked to decimal fractions when they fail to connect prior number knowledge to new number knowledge and consequently confuse decimals with whole numbers or with fractions (Irwin, 1996). Young learners find it very difficult to understand decimal fractions as an extension of place value as used for ones, tens, hundreds, and thousands and even more difficult to comprehend that decimals are always the result of a division (Irwin, 2001). This difficulty is perpetuated when teachers attempt to simplify the learning of decimal fractions by encouraging students to annex zeros and compare the numbers only to the right of the decimal point as if they were two whole numbers. The instructional practice fosters the view that decimal fractions are mathematically congruent with whole numbers. The teacher unknowingly prevented the students from creating meaning and limit their complete conceptual understanding (D'Ambrosio & Kastberg, 2012). Teaching of decimal fractions as an extension of the whole-number system frequently occurs with insufficient understanding of the place value structure (Fuson, 1990; Charalambous & Pitta-Pantazi, 2007). In consequence, a number of incorrect understandings begin to percolate and often persist without appropriate remediation from a more knowledgeable other (Martinie & Bay-Williams, 2003). For example, students often believe that there is symmetry about the decimal point resulting in a "oneths" place value that mirrors the ones digit to the left of the decimal point. Another well-documented example permeates when students think that a decimal fraction with more digits to the right of the decimal point is larger in the value of the number regardless of the place value of each digit.

Comparative decimal fractions strategies reveal insight into students' thinking as they compare pairs of decimal fractions. Seminal research conducted by Steinle and Stacey (2004b) established a set of persistent decimal misconception categories that emerged from major results of a large-scale longitudinal study of students' erroneous thinking about decimals. The

misconceptions were organised in three groups, as depicted in Table 1 below, according to how the student ordered the decimals when comparing a pair of decimal numerals to determine the larger number.

Table 2.1

Three significant decimal misconception types with decimal comparison examples.

Longer-is-Larger		**Shorter-is-Larger**		**Apparent Expert**	
Pattern 1:		Pattern 2:		Pattern 3:	
4.8	4.75	4.8	4.75	4.8	4.75
4.65	4.3	4.65	4.3	4.65	4.3
Decimal with *more* digits is chosen in each case		Decimal with *fewer* digits is chosen in each case		Correct choices in each case	

Longer-is-Larger Misconception

The first major misconception is labelled Longer-is-Larger Misconceptions. The thinking behind this misconception category is related to a student considering a longer decimal to be a larger number than a shorter decimal. The reasons this pattern of decimal misunderstanding emerges include some children having not made the decimal-fraction link and others having place value difficulties. For example, a student that holds such a misconception may assume that digits after the decimal point make another whole number rather than the digit representing part of a whole. Whole-number thinking also includes children who have not completely made the decimal-fraction link and think of two different types of whole numbers making up a decimal, perhaps a set of "whole numbers" and then more bits of unspecified size. These students are also likely to have difficulty coordinating the number of parts and the size of the parts in a fraction, because they do not understand the decimal-fraction link. Another reason students may possess Longer-is-Larger Misconceptions is because they have made the decimal-fraction link but have trouble with fundamentals of place value, this is labelled as Column Overflow Thinking (Ashlock, 2010). These students have typically learnt the correct column names for decimal numbers but attempt to write too many digits into a column. For example, they mistakenly believe 0.12 is 12 tenths (as there is no zero after the point) while 0.012 is 12 hundredths (as there is one zero after the point). In effect, they squeeze the number 12 into one column. This thinking pattern often leads to selecting the longer decimals as larger except when there are zeros in the first decimal places. The final sub-category of students who display Longer-is-Larger Misconceptions are Reverse Thinkers who don't understand that the place value of the columns decreases the further right the digits progress after the decimal point. A student with this thinking patterns reads a decimal

number such as 0.428 to mean 4 tens + 2 hundreds + 8 thousands, or possibly 4 ones + 2 tens + 8 hundreds if another ones-column has been inserted after the decimal point for symmetry. These students might select 0.35 as larger than 0.41 because 53 is larger than 14 or because 530 is larger than 140. Longer-is-larger misconceptions are most common in primary school, with about 40% of Year 5 students interpreting decimals this way, reducing in prevalence to about 5% by Year 10 (Steinle & Stacey, 2004b).

Shorter-is-Larger Misconception

The second category of misconceptions related to decimal fractions are called Shorter-is-Larger Misconceptions. These students generally pick shorter decimals to be larger numbers. One dominant reason for this type of thinking is the student's inability to coordinate the numerator and denominator of a fraction, and is referred to as Denominator Focussed Thinking. Such a student has knowledge of decimal place value names but simultaneously has difficulty coordinating the size of the numerator and denominator of a fraction. For example, they understand that 0.4 is equal to 4 tenths and 0.83 is equal to 83 hundredths. The child can articulate that tenths are larger than hundredths however the misconception occurs when they incorrectly conclude that 0.4 is greater than 0.83 because they think only about the size of the parts, the tenths or hundredths, and cannot simultaneously consider how many parts there are. So, the child has a strong vision of the decimal parts of the numbers as 4 tenths and 83 hundredths but does not also see the latter in the expanded form 8 tenths + 3 hundredths. Another reason for Shorter-is-Larger thinking is that children are trying to interpret decimal notation in terms of the more familiar common fraction notation. These students have misunderstood the decimal-fraction link and do not consider place value. Instead, they comprehend the decimal part as the denominator of a fraction, with larger denominators creating smaller fractions. For example, they think that 0.12 is equivalent to 1/12 and 0.3456 is equivalent to 1/3456. In consequence, these Reciprocal Thinkers act as if longer decimals give smaller numbers. Finally, Negative Thinkers are students who have forgotten about the decimal-fraction link and have replaced the concept with new knowledge about negative integers. Decimal place value names are symmetric around the ones-column, and this reminds Negative Thinkers of the way in which the positive and negative parts of the number line are symmetric about zero. As such, they are disposed to interpret decimals as negative numbers and will select, for example, 0.20 to be larger than 0.35 because thinking along a number line and considering decimal numbers to be equivalent to negative numbers means -20 is larger

than -35. Steinle and Stacey's (2004b) research data showed that at any one time, about 10% of students in all years from 5 to 10 have Shorter-is-Larger Misconceptions.

Apparent-Expert Behaviour

The third category of decimal fraction misconceptions are named Apparent-Expert Behaviour, referring to students who succeed in deciding which of two decimals is larger. Some of these students are True Experts, with a strong understanding of decimal notation. However, other students follow procedural comparison rules correctly but have minimal understanding of decimal place value and its fractional aspects. Money Thinkers are students who confidently deal with decimals in everyday life because they understand one and two decimal place numbers well and relate these decimals to money. For example, they cogitate 4.63 as 4 dollars and 63 cents. Beyond two decimal places, Money Thinkers are unable to order decimal numbers on the number line. For example, when asked to place numbers between 3.14 and 3.15 on a number line they omit 3.141, 3.142, 3.143 and 3.144 and have a lack of understanding about the general relationships between the place value columns. Furthermore, some Money Thinkers consider decimal numbers such as 4.45 and 4.4502 to be equal. These students believe that the extra digits on the end are "mis-hits" and don't hold number value, in effect their internal number system is discrete with integer numbers of cents. Steinle and Stacey (2004b) Australian research data shows that about a quarter of Year 5 students are 'experts' but this rises to only about two thirds of Year 10 students.

Focusing on errors concerning decimal fractions, researchers have found almost complete correspondence between student answers and their underlying schema (Nesher & Peled, 1986). This is evidence that these answers are not mistakes in terms of the students' perspective but indicative of their conceptual understanding. Cognitive conflict is a term used to describe the tension created when new evidence is recognised by the student as contradicting previous knowledge (Swan, 1983; Confrey, 1990; Moody, 2010). Resolution of that conflict may result in new learning, as points of both connection and difference are recognised and responded to (Harel & Tall, 1991). Cognitive conflict can be resolved through effective intervention methods including the use of appropriate mathematical representations.

Gorman and Way (2018) investigated how middle-primary students interacted with a dynamic zoomable digital number line to demonstrate knowledge and understandings of decimal fractions. Results from a task-based interview indicated that the zoomable number line can assist students in developing conceptual understandings of decimal density, magnitude and place value. The students' interactions with the zoomable number line, at times, generated

productive cognitive conflict, as the dynamic representation of decimal fractions challenged their existing understandings and misconceptions. Outside of the research-interview situation, these would have been ideal ‘teaching moments’. This idea of appropriate mathematical representations having the capability to build conceptual understanding of decimal fractions will be further explored in the next chapter.

Summary

This chapter has reviewed the literature on the role of conceptual understanding in the primary mathematics classroom with particular focus on decimal fractions that emerge as an important yet challenging component of the primary curriculum. Research consistently demonstrates that young students often encounter significant difficulties when learning about decimals, with misconceptions often emerging from incomplete understandings of number structure. These challenges are closely linked to three foundational concepts: decimal density, place value, and relative magnitude that each shape how students make sense of decimals and apply them in problem-solving contexts. Together, these insights underscore the need for instructional approaches that support conceptually robust mathematical learning. This review therefore provides a critical foundation for the present study, which investigated how Year 4 students experience conceptual change in these key areas when engaging with dynamic digital representations designed to support their understanding of decimal fractions.

CHAPTER 3: MATHEMATICAL REPRESENTATIONS

3.1 Situating the Chapter

In Chapter Three the critical role of representations in mathematics learning is explored, with a particular focus on how they support students' conceptual understanding. The chapter begins by outlining the foundational role of representations in mathematical thinking and communication, highlighting how different forms such as symbolic, visual, verbal, and physical, contribute to the development of meaning. Then, the importance of coordinating multiple representations of a single concept is examined, drawing on research that illustrates how students benefit from making connections across different forms. The chapter also addresses the affordances and constraints of dynamic representations, including those that allow for interaction and transformation over time. Finally, attention is given to the emerging field of dynamic digital representations, which increasingly feature in mathematics classrooms and offer unique opportunities for student engagement and conceptual change. Together, these discussions provide a comprehensive foundation for understanding how representations mediate mathematical learning and shape students' experiences with key concepts such as decimal fractions.

Mathematical representations are widely acknowledged as valuable tools that provide experiential learning through concrete and virtual manipulatives. The theories of Piaget (1952, 1970), Bruner (1966, 1986) and Skemp (1987) agree that concept development follows a progression from physical objects to representational forms and then abstract thought. Students' ability to create mental images and access abstract ideas relies on their experiences (Kennedy, 1986). Through interaction and manipulation of physical objects, students can create stronger mental images and can represent abstract ideas more completely (Kennedy, 1986).

Representations are used in the mathematics classroom to assist students in understanding abstract concepts that are central to mathematics learning. Specific to the field of mathematics, the creation of representation refers to "the act of externalising an internal, mental abstraction" (Pape & Tchoshanov, 2001, p. 119).

3.1.1 Theoretical Foundations of Mathematical Representations

Mathematical objects are indirectly accessible through representations and contribute to the paradoxical nature of mathematical knowledge whereby "the only way of gaining access to them is using signs, words or symbols, expressions or drawings. But at the same time,

mathematical objects must not be confused with the used semiotic representations" (Duval, 2000, p.60). Mathematical objects themselves remain abstract ideas, and the representations created such as numerals, diagrams, physical models, or digital visuals, are not the mathematical ideas but rather signify or point toward them. For instance, the concept of 'three' is an abstract quantity that can be shown as three dots, three blocks, the numeral '3', or the word 'three', each offering a different viewpoint of the same underlying idea. These representations are examples of 'threeness', which give learners access to an abstract concept that cannot be directly observed. Similarly, decimal fractions require students to navigate multiple representations that point to underlying quantitative relationships, reinforcing the importance of distinguishing between the abstract mathematical object and the signs used to express it. Key scholars in the field have emphasised the value of these semiotic systems which Duval (2000) refers to as registers.

Bosch and Chevallard (1999) articulate the distinction between ostensive and non-ostensive objects, where ostensive refers to the visible, tangible or symbolically expressed forms through which mathematical ideas are shown. These include written expressions, diagrams, formulas and graphical representations, and are the things students can point to, draw or manipulate, however are not the mathematical ideas themselves. In contrast, non-ostensive objects refer to the abstract mathematical concepts that remain intangible including number, function or limit. Bosch and Chevallard (1999) argue that mathematicians view their work as operating primarily with these non-ostensive objects and therefore ostensive objects play only an auxiliary role in their field. In contrast, Moreno-Armella (1999) claims that mathematical actions which emerge from cognitive activity are mediated by material or symbolic tools, suggesting that these ostensive forms play a fundamental role in shaping how mathematical thinking develops.

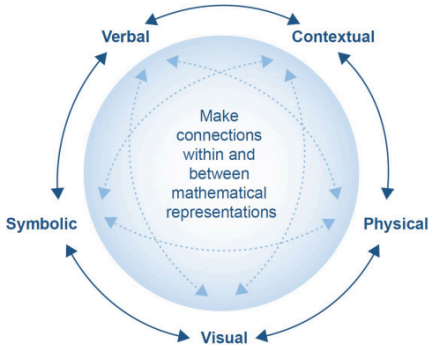
In addition, a different school of thought suggests that the use of 'objects with properties' facilitates the transition from the ostensive representation of the mathematical concept having been written on the board, drawn with the computer or made with concrete materials, to an ideal and non-ostensive representation (Font et al., 2007). The teachers' discourse in a classroom always features objectual metaphor as the mathematical ideas need to be presented as 'objects with properties' as a physical representation, either on the board or digital screen, with manipulatives, or with gestures, to be understood by the learners (Font et al., 2007).

More recent studies have claimed that mathematical representations are extremely important in developing fractional thinking (Siemon et al., 2011; Siemon et al., 2021). Students

who can efficiently partition area, set and line models, and name fractions in a variety of ways are more likely to have a deeper understanding of the link between fractions involving tenths and the base-10 number system (Siemon et al., 2011). Effective teaching of number engages students in making connections within and between the five types of representations as interconnectedly illustrated in Figure 3.1: physical, visual, symbolic, verbal and contextual (National Council of Teachers of Mathematics, 2014).

Figure 3.1

Diagram of five types of mathematical representations: physical, visual, symbolic, verbal and contextual.



Note. Sourced from (National Council of Teachers of Mathematics, 2014).

3.1.2 Multiple Representations and Conceptual Understanding

As mathematical concepts can only be accessed through representations, they are crucial for the construction processes of the learner's conceptual understanding (Duval, 2006; Goldin & Shteingold, 2001). Students employ multiple representations to support their understanding of mathematical concepts and facilitate problem-solving (Ainsworth, et al., 2002). Multiple representations have been generally defined as providing the same information in more than one form of external mathematical representation by Goldin and Shteingold (2001).

Seminal work by Janvier (1987) found when teachers enriched their mathematics classrooms with multiple representations, the students were more efficient in making connections between the meaning of algebraic concepts and the way of representing them, which results in conceptual understanding. When students are given opportunities to connect more than one representation of a mathematics concept, they can deduce the key properties as well as gradually generalise and abstract the meaning behind each representation.

Dynamic Digital Representations and Real-Time Connections

Dynamic digital representations are computational artifacts that exist within a digital medium and possess multiple co-occurring dynamic properties, namely interactivity, responsiveness, and manipulability (Ainsworth, 2022). Interactivity refers to the capacity for bidirectional exchange between the user and the representation, enabling direct engagement rather than passive observation. Responsiveness denotes the representation's ability to adapt or transform in real time as a function of user input, environmental data, or underlying computational processes. Manipulability describes the degree to which users can modify, reconfigure, or act upon the representational elements themselves by altering both the view and the structure of what is being represented. Together, these properties distinguish dynamic digital representations from both static digital displays, which lack interactivity and manipulability, and from non-digital dynamic representations, which lack the computational substrate that enables real-time responsiveness (Ainsworth, 2022).

One of the key affordances of dynamic digital representations in mathematics education is their capacity to simultaneously display multiple representations, including symbolic alongside numerical, and visual forms, on a single screen, allowing students to make connections between them in real time (Sinclair & Pimm, 2015). This affordance supports learners in coordinating different mathematical ideas and representations, which is central to developing conceptual understanding (Ainsworth, 2022; diSessa, 2017b).

Research has shown that when learners can interact with linked representations such as number lines, area models, and decimal symbols, they are better able to recognise structure, detect patterns and revise misconceptions (Torbeyns, 2020; McCulloch et al., 2021). The integration of multiple representations in a dynamic format also facilitates productive struggle, defined as "the effortful but manageable cognitive engagement that occurs when students work through challenging mathematical tasks without immediate teacher intervention" (Hiebert & Grouws, 2007), by making abstract ideas more tangible and open to manipulation (Reinholz & Shah, 2018).

3.1.3 Traditional Mathematical Representations

Mathematical Manipulatives: Evolution and Theory

The use of manipulatives is well-situated in the mainstream of mathematics instruction (Moyer, 2001). Manipulatives refer to the expansive category of mathematics tools that can be used to explicitly and concretely represent abstract mathematical ideas (Moyer, 2001). The

pedagogical basis for using manipulatives embeds the notion that firsthand interaction with manipulatives assists students' mathematical understanding (Moyer, 2001).

Manipulatives have been used in the primary mathematics classroom for over fifty years. The seminal work of Piaget (1952) outlines how young children between the ages of seven to ten years work in predominantly concrete forms and that the abstract notions of mathematics are primarily accessible to them through embodiment in practical resources. The development of concrete number representations such as the base apparatus by Zoltan Dienes (1969) and the Cuisenaire rods by Caleb Gattegno and Georges Cuisenaire (1954) was in direct response to the research and published findings of Piaget (Moyer, 2001). The design of these mathematics manipulatives drew attention to the capabilities of concrete representations in progress towards abstract work in symbolic forms.

Bruner (1966) provides an elaboration for enactive, iconic and symbolic modes of working. This was further supported by the emergence of a constructivist view of learning whereby students construct their own mathematical meanings relating concrete manipulatives to the abstract symbols to achieve cognitive clarity (von Glasersfeld, 1995).

Manipulatives have been categorised as 'structured' or 'unstructured' in the literature to assist with distinction (Drews & Hansen, 2007). Structured manipulatives refer to tangible objects intended to reflect a specific conceptual structure, whereas unstructured manipulatives are an open-ended and versatile tool without an intended specific conceptual structure. Since the early 2000s a debate has emerged in the field of mathematics education as to how concrete manipulatives facilitate learning as a stand-alone representation. Chao et al. (2000) proposed a name for this dualism, the 'mental tool view' versus the 'abstraction view'. The mental tool view proposes that young students process new learnings through the creation of powerful mental images made in response to interaction with physical materials. The theory proposes that as learners become adept with a particular representation, they form a mental image of that specific material which they can access to guide and constrain their mathematical thinking and problem-solving (Schwartz, 1994).

The consistent use of highly-structured representations are advocated by educational scholars, based on the notion that if a representation is well-selected, it will make salient the structure of the mathematical concept being taught (Chao, et al., 2000). However, an opposing school of thought is evident in the abstraction view, which proposes that the value of using physical materials lies in the ability of students to abstract the mathematical relations of interest from a variety of concrete representations (Dienes, 1964). This abstraction view agrees with the perceptual variability principle, which recommends as much variation as possible during

concept formation. “During long periods of exposure to different forms of materials, students would abstract the common core of the mathematical idea represented in various materials. The irrelevant features of specific materials would fade away” (Chao, et al., 2000, p. 286).

It is widely acknowledged in the literature that students’ thinking and reasoning are critical when engaging in learning with manipulatives including physical, virtual and pictorial objects (Fyfe, et al., 2014). The concrete manipulatives and the learning activities in which they are embedded, are only as valuable as the students' reflection on the mathematical concepts (Fyfe, et al., 2014). Manipulatives can be powerful tools to support sense making, mathematical thinking and reasoning when they are used as tools to support these thinking processes that build conceptual understanding (Manches & O'Malley, 2016).

Virtual manipulatives represent an extension of the manipulative tradition, carrying forward the foundational principle that mathematical understanding is deepened through active engagement with concrete, tangible representations. As technology has advanced, the capacity of these hands-on tools in digital form has been reimagined by preserving their pedagogical intent while offering new possibilities for interaction, accessibility and dynamic exploration. Virtual manipulatives are discussed further in depth in Section 3.1.5.

Number Lines in Decimal Instruction

An example of a common mathematical representation utilised in the classroom to teach fractional thinking is a number line. Number lines are a teaching and learning model used to support the positioning of fractions and decimals through a rational number counting sequence (Teppo & van den Heuvel-Panhuizen, 2014). Teppo and van den Heuvel-Panhuizen (2014) define a number line as a figural device that represents mathematical abstractions to assist the consideration of and operation with different types of numbers. Number lines are a powerful representation of decimal fractions as they show where a decimal ‘fits’ between two-whole numbers, a concept that is quite difficult for young children to understand (Thomson & Walker, 1996).

3.1.4 Digital and Dynamic Representations

Dynamic representations can be an effective support for student learning within mathematics because of their dynamic affordances. Students actively interact with the mathematical content by making conjectures and testing them immediately to ‘see’ the concept from various perspectives (Orrill & Polly, 2013). Learner-centred computer tools that

encourage interactive engagement with mathematics, make visible for students concepts that would otherwise remain hidden from view (Orrill & Polly, 2013).

As such, the role of representations when engaging with digital technologies is crucial to assisting students' mathematical conceptual understanding. Digital technologies encompass a broad range of tools, from static resources to more sophisticated environments that support active mathematical engagement. Within this broader category, a specific subset is referred to throughout this thesis as dynamic digital technologies and are considered "dynamic manipulable and interactive representational forms" that "mediate and are mediated by mathematical thinking and expression" (Hoyles & Noss, 2003, p.326). By contrast in this study, traditional representations refer to static, print-based mathematical representations, including symbolic notation, drawn diagrams, and physical number lines, that do not respond to or change with student interaction.

3.1.5 Virtual Manipulatives: Bridging Concrete and Abstract

Virtual manipulatives represent a specific category of dynamic digital representations that have gained prominence in decimal instruction. Virtual manipulatives are "an interactive... visual representation of a dynamic object that presents opportunities for constructing mathematical knowledge" (Moyer et al., 2002, p. 373). They support teachers in modelling abstract mathematical concepts for deeper student comprehension (Moyer et al., 2002).

Both virtual and tangible manipulatives are now commonly used in primary education to support children's acquisition of mathematical concepts. Online mathematics manipulatives offer advantages in the mathematics classroom in a similar manner to traditional manipulatives that have been used for decades by teachers. Moyer-Packenham et al. (2005) suggest virtual manipulatives are uniquely suited for teaching mathematics with children due to their practicality and tangibility.

Interactive Features and Capabilities

The potential for alteration is common among digital manipulatives as users are capable of changing aspects of the interface for example colour, size, scale and view-point. This interactivity allows children to be engrossed in the problem-solving process (Moyer-Packenham et al., 2005). Furthermore, students are exposed to links between symbolic and iconic notation as many virtual manipulatives save numerical information or provide mathematics notations that label the on-screen objects (Moyer-Packenham et al., 2005).

Virtual manipulatives enable students to interact with digital shapes on a screen, often resembling familiar physical manipulatives and being used in similar ways, such as virtual fraction circles that allow students to put same-size pieces together to represent fraction sums (Satsangi & Raines, 2023). This approach bridges the gap between concrete and abstract representations while maintaining the interactive elements which are thought to be crucial for conceptual development.

Virtual manipulatives are intended to both engage and enable children to develop their understandings of abstract mathematical concepts. The exact connection between manipulatives and cognitive development, however, is not entirely clear (Ahmed et al., 2004). It is also unclear how the use of virtual or physical materials influence the development of different strategies to help children solve instructional tasks (Manches & O'Malley, 2016). This is why further research is needed into what conceptual learning emerges when children engage with virtual manipulatives.

Digital technologies facilitate new representational systems with increased capabilities in manipulation and processing compared to their static counterparts (Laborde & Laborde, 2014). For example, the digital resource called Zoomable Number Line offers a dynamic representation of decimal fractions that students can manipulate (Maths is fun, 2016). The digital tool is capable of illustrating concepts that cannot be effectively represented on a traditional static number line. It is very difficult to display hundredths, and thousandths on a paper-based number line, however the sophisticated functions of the interactive web-app make this easily achievable. For example, children can stretch the number line and zoom-in to see how tenths 'fit' between two consecutive whole numbers, and zoom-in again to see how hundredths are located between a pair of tenths. A further feature is moving forward and backward along the number line, to see the continuous sequence of numbers and to locate specific whole-numbers and decimals.

A number line in a dynamic digital context is no longer a static diagram representing a snapshot of integers, but a manipulative tool capable of "representing invariant relationships among variable elements" (Laborde & Laborde, 2014, p.187). The zooming-in and scrolling affordance in dynamic number line environments demonstrate the transformation that technology can have on representations offered for mathematical activity and subsequently for the meaning of mathematical objects. The dynamic number line represents the relationship of decimal fractions existing between whole numbers, and smaller decimals existing between those decimal fractions, that is, the concept of decimal density.

3.1.6 Theoretical Perspectives on Digital Mathematical Representations

Arcavi (2005) outlines important elements that characterise the structure and development of students' competence in symbol sense. Particular students' mathematical behaviours and dispositions are identified to highlight the importance of students' development and appreciation for symbol sense to deal with mathematical situations or problems. The characteristics identified by Arcavi (2005) include familiarity with symbols; reading and manipulation of symbolic expressions; construction of symbolic expressions for a desired graph; selection of proper symbolic representation, checking for symbolic meaning and the recognition of distinct roles in using symbols in different contexts. This list can constrain mathematics educators when designing learning activities, as it limits attention to symbolic representations associated with a single mathematical concept. A holistic mathematics education that enhances mathematical thinking involves searching for meaning beyond symbol sense. Santos-Trigo (2006) suggests that integrating technology into the mathematics classroom can foster the dispositions, resources and problem-solving strategies needed for this expanded way of thinking.

Technology can generate a dynamic representation of a problem in which students can pose and pursue questions that might lead them to identify and examine connections and extensions of that problem (Santos-Trigio, 2006; Drijvers & Sinclair, 2024). In this process, it is argued that representing the problem dynamically might offer students the opportunity to be engaged in a line of thinking in which they constantly need to search for mathematical relationships and utilise the knowledge they possess to explore and support the pertinence and validity of those relationships (Roschelle et al., 2000). For example, when students interact with dynamic digital representations of decimal fractions, each user adjustment made on the digital tool prompts immediate visual or numerical feedback. This could then prompt students to consider how changes in place value position affect the overall magnitude of the decimal number and to test whether their reasoning is maintained across different situations. In doing so, students can actively search for relationships within the decimal system and use their existing knowledge to justify and refine their strategies thus mirroring the relationship-seeking processes described by Roschelle et al. (2000).

The integration of dynamic digital representations in mathematics education has emerged as a transformative approach to supporting student learning and conceptual understanding. These representations constitute a crucial component of mathematical infrastructure, providing systematic methods for presenting mathematical thoughts, creating communicative records across contexts, and supporting mathematical reasoning and

computation (Laborde & Laborde, 2014). Dynamic digital representations benefit mathematics learning through four key dimensions: epistemological considerations in design, cognitive transformations in student thinking, progressive learning through interaction modalities and pedagogical reimagining of mathematical tasks (Laborde & Laborde, 2014).

3.1.7 Pedagogical Authenticity and Design Principles

Central to the effective implementation of dynamic digital representations is the principle of epistemological fidelity, which ensures that digital mathematical objects maintain consistency with the abstract concepts they represent (Laborde & Laborde, 2014). This principle operates at two critical levels: the graphical representation of mathematical objects and their behavioural responses under direct manipulation. The significance of this fidelity becomes particularly pronounced in educational software designed for young learners, where contradictions between digital representations and abstract mathematical concepts can generate cognitive dissonance during foundational knowledge formation (Moreno & Mayer, 1999; Ruamba et al., 2025).

Direct Manipulation versus Basic Interactivity

The literature establishes a crucial distinction between basic interactivity and authentic direct manipulation in educational software design (Squires, 1995). Basic interactivity, characterised by users requesting software to execute predetermined commands through button presses or dialog completion, represents a limited and impoverished form of user interface. In contrast, authentic direct manipulation enables students to act directly upon mathematical objects, transcending the limitations of simple command-based interactions. Examples of direct manipulation techniques are outlined in the section below on Interaction Modalities.

Zazkis and Liljedahl (2004) argue that when representations facilitate sophisticated direct engagement, users inevitably perceive the digital representation and the abstract mathematical object as unified entities. This unification is fundamental to mathematical learning, as it allows students to engage with mathematical concepts through direct manipulation rather than through intermediary command structures. This assertion by Zazkis and Liljedahl (2004) aligns closely with the digital tools selected in the present study, which are designed to allow students to interact directly with decimal quantities rather than operate them through indirect commands or static notation. Tools that enable zooming, dragging and dynamically partitioning quantities can support a connection between the digital representation and the underlying abstract decimal idea. When students can manipulate dynamic

representations such as, stretching a number line to reveal increasing precision, or adjusting a value and immediately observing its effect, they begin to view the representation and the abstract notion of decimal magnitude as a single, coherent concept. This unification is essential when learning about decimals, where conceptual challenges often arise from viewing digits, place-value structure and magnitudes as disconnected ides. By selecting digital tools that encouraged direct interaction with decimal quantities, the present study intentionally fostered an integrated perception that enabled students to reason about decimal density, place value and relative magnitude through meaningful engagement with the mathematical concepts.

3.1.8 Cognitive Transformations in Student Thinking

Dynamic representations provide mathematics educators with unprecedented opportunities to observe and understand student thinking processes (Hoyles et al., 2006). The interactive nature of these tools fundamentally transforms traditional approaches to mathematical object manipulation, as students cannot apply familiar paper-and-pencil routines when engaging with dynamic digital environments. This constraint can be pedagogically valuable, as it compels students to make deliberate decisions about their mathematical actions (Engelbrecht & Borba, 2024).

The relationship between student perception, understanding and interaction with mathematical objects reveals important insights into mathematical thinking processes. Hoyles et al. (2006) demonstrate that students' conceptual understanding directly influences their interaction patterns with dynamic representations, thereby making their mathematical reasoning visible and accessible for educational analysis. Drawing on the reviewed literature, this study identifies a gap in research examining student interactions with dynamic representations of decimal fractions, which this study aims to address by investigating how students engage with these digital tools to construct and refine their understanding of decimal fraction concepts.

Progressive Learning Through Interaction Modalities

Research into student interaction patterns with dynamic representations has revealed systematic progressions in mathematical exploration behaviours. Arzarello et al. (1998) identified three distinct categories of organized experimentation that demonstrate increasing sophistication in mathematical thinking:

1. Wandering dragging: Characterised by exploratory manipulation aimed at discovering mathematical regularities and patterns.

2. "Lieu muet" dragging: Purposeful manipulation designed to preserve identified mathematical regularities.

3. Hypothesis testing dragging: Systematic manipulation employed to test presupposed mathematical regularities.

The progression from wandering exploration to systematic hypothesis testing represents more than a mere change in interaction technique; it signifies a fundamental cognitive shift in students' understanding of mathematical concepts represented within dynamic tools (Vos, 2018). This progression suggests that dynamic representations not only support existing mathematical understanding but may fundamentally alter students' perceptions of mathematics itself (Kaput, 2008).

3.1.9 Pedagogical Reimagining of Mathematical Tasks

The integration of dynamic representations in mathematics education necessitates comprehensive redesign of traditional mathematical tasks to capitalise on the unique affordances of digital manipulation tools. This didactic dimension requires educators to move beyond simple digitisation of existing activities toward creating learning experiences that leverage specific dynamic capabilities including dragging, zooming, and scrolling functionalities (Hoyles et al., 2006).

Dynamic representations introduce interaction modalities that create fundamentally new pathways for mathematical learning. These tools do not merely digitise traditional mathematical activities but enable entirely new forms of mathematical engagement that were previously impossible with static materials (Hoyles, 2014). The challenge for educators lies in recognising and harnessing these new pedagogical possibilities while maintaining mathematical rigor and conceptual clarity.

The literature demonstrates that dynamic digital representations benefit mathematics learning through multiple interconnected mechanisms, and a potentially transformative impact within the mathematics classroom (Moyer et al., 2002). At the epistemological level, properly designed representations maintain mathematical fidelity while enabling authentic direct manipulation. Cognitively, these tools make student thinking visible while transforming traditional approaches to mathematical reasoning (Hoyles, 2014). The progression of interaction modalities reveals increasingly sophisticated mathematical understanding, while the didactic dimension opens new possibilities for mathematical task design and pedagogy (Laborde & Laborde, 2014).

3.1.10 Impact on Mathematics Classroom Practice

The incorporation of dynamic digital technologies has afforded insights for mathematics education. Noss and Hoyles (2006) posit that the field of research on computationally-based technologies in mathematical learning is interrelated to the broader field of mathematics education research. Changes in the computational domain open up the potential for change in mathematics education however the effect cannot be radical without supporting research.

Tensions in dynamic digital technologies integration into mathematics classrooms exist as learners need to be able to understand the syntax and semantics of the digital tools. More specifically, they must deduce how it works, what it affords, and how it could potentially be employed (Noss & Hoyles, 2006). Some researchers claim that this tension is not a barrier to the mathematics education as knowledge about software is seen as being symbiotic with mathematics learning (diSessa, 2000; Wilensky & Stroup, 2001).

Another tension noted in the literature is that students may use the affordances of dynamic digital representations to avoid the cognitive load of mathematical thinking. Noss and Hoyles (2006) distinguish the labels of mathematical learners versus mathematical users to address this tension and explain the need of each group. Mathematical learners are understood to search for and appreciate generality and structure while mathematical users access the technology to complete a task or solve a problem. This is where the role of the teacher in selecting, implementing, and designing tasks that incorporate digital tools, becomes paramount.

Broader meta-analytic research supports the effectiveness of specific types of digital tools in mathematics education. Evidence indicates that use of intelligent tutoring systems or simulations such as dynamic mathematical tools was significantly more beneficial than hypermedia systems, with effect sizes being larger when digital tools were used in addition to other instruction methods and not as a substitute (Hillmayr et al., 2020). This finding underscores the importance of complementary rather than replacement approaches when implementing dynamic digital representations.

Research Findings on Virtual versus Physical Manipulatives

A study conducted by Burns and Hamm (2011) compared the use of concrete and virtual manipulatives on student review of fraction concepts in third grade students. The results implied that using virtual manipulatives was as effective as using concrete manipulatives for

the intended learning experience. This concurs with the findings of Ainsa (1999) and Steen et al. (2006).

Furthermore, several benefits of virtual manipulatives have been recorded in the literature that suggest the ease of managing student-use of digital representations. For example, Moyer et al. (2002) suggest that on-screen manipulatives may be preferred by teachers as they are easier to manage in terms of distribution and pack-up. From the perspective of older students, they consider virtual manipulatives to be more sophisticated than the concrete form and therefore display increased eagerness to engage with mathematical representations to support learning and problem solving (Moyer et al., 2002).

However, it is important to acknowledge the literature presents nuanced findings regarding the relative effectiveness of digital versus physical manipulatives in fraction and decimal learning. A significant study examining this comparison found that student achievement in learning fractions using concrete manipulatives was better than virtual manipulatives, with girls' progress in learning fractions being more significant than boys (Farra et al., 2024). This finding suggests that while digital representations offer certain advantages, the tactile experience of concrete manipulatives may still provide superior learning outcomes in some contexts.

Teacher Decision-Making and Feedback

Teachers possess an invaluable understanding of the mathematical learners in their classroom which they can use to design learning tasks at the point of need. Dynamic digital representations can address students' misconceptions and challenge students' thinking when presented with cognitive conflict to be resolved (GebreYohannes et al., 2016).

Bell's (1993) principles for the design of teaching note the importance of timely feedback. With respect to dynamic representations many feature constant feedback as an affordance within the digital tool itself. However, other dynamic representations such as the Zoomable Number line and the interactive decimal fraction wall do not provide feedback to the learner. In Bell's diagnostic teaching, cognitive conflict is resolved through students working collaboratively on tasks followed by teacher-led discussion to facilitate reflection and review of learning.

Strässer (2001) investigated the connection between students' use of drag functionality and the development of their understanding of geometric figures. The results indicated that the power of the drag mode in exploration is not spontaneously grasped by students and can only be mastered with explicit introduction from the teacher. A key finding suggested that students

were unable to successfully interact with the dynamic geometric representation to uncover the notion of geometric figures as a set of relationships between variable elements, when experimenting with the digital tool through 'wandering dragging'.

However, teacher input was transformative in assisting the students with recognising the mathematical concepts embedded in the dynamic geometric representation. A clear shift was observed from 'wandering dragging' to 'lieu muet' dragging reflecting a cognitive change. Strässer's (2001) research highlights the capabilities of dynamic representations in demonstrating students' conceptual understanding of mathematical notions, as teachers can view the cognitive progression of students' thinking through the shift in their physical dragging manipulation from exploratory to purposeful. While this offers valuable diagnostic potential, the feasibility of monitoring such behaviours across an entire class simultaneously warrants further consideration, particularly in resource-constrained classroom contexts.

Theoretical Perspectives on Technology-Modelling

In a dynamic mathematics context, current theoretical perspectives on technology-modelling domains of theory propose that human knowledge is embarked in the representations of theoretical objects. Laborde & Laborde (2014) suggest that representational objects reflect the theoretical model underlying the technology, independent of the pre-constructed thinking of the user. Furthermore, the mathematical objects share knowledge with the user through the "creation of artifacts offering a dynamic model of theoretical objects" (Laborde & Laborde, 2014, p.200). A Vygotskian lens on this concept proposes the application of human knowledge led to a transformation within technology design where artifacts embody theoretical knowledge at a higher degree than before. Simultaneously, the role and reach of these psychological tools on the mental activity of the user may be significantly extended.

3.1.11 Affordances and Constraints of Dynamic Representations

Affordances of Virtual Manipulatives

Integral to this research into dynamic representations is investigating the affordances and constraints of these digital learning tools. Established but evolving research has explored the potential benefits of touchscreen digital tools, and this body of research suggests that apps for virtual manipulatives offer various advantages that may impact learning, which children utilise in diverse ways (Tucker & Moyer-Packenham, 2014; Tucker, 2015).

The rapid advancement of technology in mathematics education presents new challenges for educators, as it involves "using new kinds of mathematical tasks and modifying

the nature of mathematical activities in classroom based on a set of pedagogical principles" (Shahmohammadi, 2019, p.1834). This critical perspective demonstrates the importance of closely examining how technology influences and enhances students' learning and conceptual understanding of mathematics, particularly with emerging developments, specifically the integration of artificial intelligence, augmented reality, and adaptive learning systems (Engelbrecht & Borba, 2024).

An affordance requires something that "relates attributes of something in the environment to an interactive activity by an agent who has some ability based on its own attributes, which are characteristics of the environment or agent" (Greeno, 1994, p.383). The qualification of affordances in the literature is broad and ranges from the artefact being objectively beneficial, neutral, or injurious (Gibson, 1986; Greeno, 1994). Chemero (2003) built upon the work of Greeno (1994) to suggest that affordances exist in direct relation to a corresponding ability and are paired in continuous systems. Two key theoretical views of affordances in technology have emerged with both having a shared interpretation of affordances being "action possibilities relative to the agent" (Burlamaqui & Dong, 2014, p.13).

Affordances of dynamic representations enacted through virtual manipulatives are strongly influenced by the user's ability to interact with the digital tool. Clements and Sarama (2009) posit an overarching but underemphasised reason for the positive effects of computer manipulatives in that computer manipulatives provide unique affordances for the development of integrated-concrete knowledge. Through a far-reaching synthesis of literature, they collated seven hypothesised, interrelated affordances, along with evidence for their empirical validity. These included bringing mathematical ideas and processes to conscious awareness; encouraging and facilitating complete, precise explanations; supporting mental 'actions on objects'; changing the very nature of the manipulative; symbolising mathematical concepts; linking the concrete and symbolic with feedback; and recording and replaying students' actions (Clements & Sarama, 2009).

Five Categories of Virtual Manipulatives Affordances

The effectiveness of virtual manipulatives is well documented in the literature, with a meta-analysis of studies comparing virtual manipulatives with other instructional treatments resulting in a moderate effect in favour of virtual manipulatives (Moyer-Packenham & Westenskow, 2013). In their meta-analysis, Moyer-Packenham and Westenskow (2013) identified five categories of affordances of virtual manipulatives that contribute to learning mathematics:

1. Focused constraint: Virtual manipulatives focus and constrain student attention on mathematical objects and processes.

2. Creative variation: Virtual manipulatives encourage creativity and increase the variety of students' solutions.

3. Simultaneous linking: Virtual manipulatives simultaneously link representations with each other and with students' actions.

4. Efficient precision: Virtual manipulatives contain precise representations that allow for accurate and efficient use.

5. Motivation: Virtual manipulatives motivate students to persist at mathematical tasks.

Interaction modality has been suggested as a factor influencing results, as long-term interaction with touchscreen virtual manipulatives may be more effective than long-term interaction with mouse-controlled virtual manipulatives (Paek, 2012). Furthermore, researchers found that children adjusted their external representations of mathematical content by refining their input gestures when interacting with touchscreen virtual manipulatives (Ladel & Kortenkamp, 2012).

Evolving research has focused on affordances of touchscreen virtual manipulatives. Tucker and Moyer-Packenham (2014) concluded that children's affordance access varied depending on their ability. For example, when interacting with Motion Math: Zoom, efficient precision was evident when children with greater technological proficiency chose specific places to use the app's features to zoom in within a given range. Other participants who zoomed inaccurately or inefficiently were neither as precise nor efficient as children who could effectively zoom. Access to affordances may vary by approach and degree, change over time and lead to different outcomes (Tucker, 2015). Together, this research suggests that virtual manipulative touch-screen tablet apps offer an array of affordances that may influence learning, and that children access these affordances in different ways. However, there remains limited evidence on how these affordances specifically shape the development of decimal fraction understanding, highlighting the need for studies such as this, that explore the intersection of tool design, student interaction and conceptual growth in this domain.

Constraints and Considerations

Dynamic representations constrain objects to have mathematical properties, relevant to the mathematical activity (Roschelle et al., 2000). Furthermore, Moyer's (2001) research attends to the notion that students may potentially utilise manipulatives in a rote-learned procedure without an understanding of how the apparatus reflects mathematical structures. It

is crucial that students examine how the manipulative reflects and embodies mathematical structure as this enables the meaning of the manipulative to become transparent to the user (Moyer, 2001).

A user's interaction with a dynamic representation is an individual experience strongly tied to choices (Baccaglini-Frank, 2021). Within any engagement with digital tools there are local considerations. These local considerations are often temporal in nature, in that they concern a particular moment in time or part of an interaction (Santos-Trigio, 2006). The technique of *stressing and ignoring* is a subconscious tool that users' access to process information presented to them. At any one moment in time there is so much available information for a person to attend to through all their senses, that they simply cannot attend to everything. Therefore, choices are made as to what their eyes turn to and in what direction, and also what they focus on.

Santos-Trigio (2006) suggest that issues around *stressing and ignoring* arise with the level of conscious awareness in the way that a person is doing so. This is due to certain visual opportunities being stressed and consequently others being ignored. The implication in a mathematics education context is that what a student learns is partly determined by those decisions as to what mathematics educators stress and what they risk overlooking. The constraint the action of *stressing and ignoring* puts on a dynamic representation in mathematics is that there is a risk that a critical aspect of the representation will be ignored in the filtering process.

3.2 Conclusion: Maximising the Potential of Dynamic Representations

Arguably, manipulatives can be used to provoke children to think and explore number concepts independently and deeply (Manches & O'Malley, 2016). For students to engage with the full potential of dynamic manipulatives, external sources of input must be present. These may include teacher scaffolding, engagement with learning tasks that embed or require manipulative use, feedback from external sources including teacher, peers, or the virtual manipulative itself, and self-reflection to surmise conceptual learnings (Baccaglini-Frank, 2021).

The convergence of these benefits suggests that dynamic representations represent more than technological enhancements to traditional mathematics education as they constitute a fundamental shift toward more interactive, visible, and authentic forms of mathematical learning (Drijvers & Sinclair, 2024). Future research should prioritise investigation of specific mathematical domains, particularly decimal fractions, to further understand how these

transformative tools can optimise mathematical conceptual development. Given the well-documented challenges students face in the mathematical domain of decimal fractions, this study seeks to address this gap by examining how such dynamic digital tools can optimise students' development of robust decimal fraction concepts.

CHAPTER 4: THEORETICAL FRAMEWORKS

4.1 Situating the Chapter

This chapter presents the theoretical framework that informs and guides the study, drawing on three key perspectives: constructivism, the role of representations in mathematics education, and the concept of 'shifts of attention'. These perspectives collectively offer a valuable lens for investigating how students engage with dynamic digital representations of decimal fractions, particularly in the context of rich, task-based learning environments. Constructivist principles underpin the view that learning occurs through active engagement and meaning making, while research on mathematical representations highlights the importance of multiple, coordinated representations in supporting conceptual understanding. The notion of shifts of attention adds a finer-grained lens for examining how students' focus changes over time in response to different visual and interactive features of digital tools. Together, these perspectives form a coherent framework for exploring how students develop their understanding of decimal fractions through dynamic interactions.

In addition to outlining the theoretical underpinnings of the study, this chapter introduces the research questions that guide the investigation. These questions were developed in response to a gap in the existing literature concerning how primary students' attention and understanding evolve during interactions with multiple digital tools designed to support decimal learning. While previous studies have explored the affordances of individual digital tools, fewer have examined how students' attention shifts across multiple tools and how this may contribute to conceptual change. The research questions presented here seek to address this gap and frame the study's inquiry into the learning processes that unfold within these digitally mediated environments.

4.1.1 Constructivist Theory

The theoretical framework adopted in this study is supported by the work of Piaget (1970), whose research was seminal in introducing task-based interviews as a method to uncover children's thinking about posed problems and who believed that "true understanding takes place when the student makes discoveries for themselves" (Assad, 2015, p.17). Constructivism within mathematics research has been extensively documented by several key researchers (Cobb, et al., 1992; von Glasersfeld, 1995; Katic, et al., 2009). A major principle of Piaget's theory of constructivism suggests that students construct their understanding and

knowledge through perceptions and experiences, which are themselves "mediated through previous knowledge" (Simon, 1995, p.115). Therefore, this study recognises students' existing knowledge of decimal fractions and anticipates that individual students will respond to the digital tools in different ways. Additionally, constructivism embodies the notion that "learning is a social process" (Jorgensen et al., 2010), indicating the need for active engagement in problem-solving (Zain et al., 2012) and opportunities for the sharing of ideas, explanations and opinions (Cobb et al., 1992).

Increasingly, students are active participants in the construction of mathematical knowledge and in this mode of learning are seen to be more extrinsically motivated and learn important skills such as critical thinking and problem solving (Zakaria & Iksan, 2007; Johnson et al., 2009; Froyd & Simpson, 2010). Constructivism is a theory based on the premise that all humans construct their own perspective of the world. A person adjusts their mental model to incorporate new experiences and make sense of this new information, and in consequence their schema is constantly readjusting. Learners are not passive recipients of information but are active agents engaging in constructing their own knowledge. Constructivism emphasises preparing learners to engage in problem-solving situations and to negotiate and generate solutions through the sharing and exchange of ideas (Zain et al., 2012). Constructivism presents the idea that the learner is much more actively involved in a collaboration with both their peers and the teacher when creating new meanings (Fosnot, 2005).

Constructivist learning theory posits that students construct knowledge for themselves, and learning is established when meaning is found. According to this theory, learning involves language that is used and directly influences knowledge acquisition (Zain et al., 2012). People talk to themselves and others as they learn, and as such language and learning are inextricably intertwined. Learning is also viewed as contextual where students do not learn through isolated facts and theories of abstract pieces of information, separated from the rest of their lives. Rather they learn in relation to what they already know and understand to be accurate knowledge. Unlike the Cognitivist theory, learning is an active process that requires students to utilise their sensory inputs and constructs meaning from it. It is also a social activity, encouraging learning through interactions with peers, teachers, or other people. Each of these key features of constructivist learning theory positions the student as the active agent of knowledge acquisition. In adopting a constructivist framework for this study, the researcher worked directly with the student to understand the learning processes they engaged with to acquire conceptual knowledge of decimal fractions.

Constructivist learning theory upholds the role of social interaction in the construction of thinking. Bermejo et al. (2021) affirm that the learning process becomes a communicative process in which students are listened to, thereby understanding their goals and assuming their logical reasoning. Often, the teacher acts as a guide by asking students to clarify and justify their ideas both orally and in writing. This allows the timely influence and deepening of mathematical ideas which the teacher gauges as most relevant. The teacher achieves this when they raise questions and provide tasks that provoke, compromise and challenge each student's mind. Teacher subject-knowledge should not be overlooked as "in order for the teacher to manage the discussion in class correctly and efficiently, he needs have not only extensive knowledge on the topic being discussed, but also, and above all, he needs to know how the child learns that specific topic, that is, the levels of development in understanding the topic or content, the difficulties and typical errors that usually arise" (Bermejo et al., 2020, p.40). When selecting data collection methods aligned with a constructivist approach, it is important to use strategies that actively engage students in articulating their thinking. In the present study, oral prompts and open dialogue were employed to facilitate communication and verbal reasoning, which uncovered students' mathematical thinking as they interacted with dynamic digital representations of decimal fractions. Such approaches not only provide rich data to address the research question but also as an active step in the learning process where students can articulate their current thinking, verbally work through instances of cognitive conflict and then justify their newfound understandings.

In constructivist learning approaches, the individual's pace is considered of importance as it is accepted that each student progresses at a different rate. Learning is recognised as an active dynamic process in which connections between different facts, ideas and processes are constantly changing and their structure is continually reformatted (Schumacher & Kennedy, 2008). Such connections are fostered through dialogue between teacher and students, students with their peers (Schumacher & Kennedy, 2008). Learning in a constructivist environment is achieved through talking, listening, reading, writing and reflecting all the content, ideas, issues that are presented to them, as well as building on the existing perceptions that they may already have to their learning (Schumacher & Kennedy, 2008). Furthermore, physical actions including hands-on experiences are also necessary for learning and can be especially beneficial for children as these activities engage the mind as well as the hands. The dynamic affordances of the digital tools used in this study aimed to link the physical engagement with mathematics to the cognitive processes of learning which would translate to conceptual understanding of decimal fractions.

According to Bermejo et al. (2021), an appropriate constructivist intervention allows for the student to interact with the knowledgeable other, always using different means to reason, relate, solve problems and communicate. It also allows the student to anticipate and make conjectures of solutions based on mathematical arguments that validate what is stated. Furthermore, it aids the learner to focus on solving problems that permit examples and counterexamples to be explored. Considering these aspects, constructivist learning theory concludes that the student can establish conjectures and solutions through reasoning, a process that combines their prior knowledge with the concepts that they engage with interactively by creating new knowledge structures (Bermejo et al., 2021). The present study employed the use of a dynamic digital representation to stimulate the learner's physical engagement with the mathematical concept of decimal fractions. Furthermore, the ongoing dialogue between the researcher and the student-participant prompted mathematical reasoning to test current thinking and actively engage in forming new ideas related to this specific area of mathematics.

4.1.2 Representation Theory

Representations are used in the mathematics classroom to assist students in understanding abstract concepts that are central to mathematics learning. Specific to the field of mathematics, representation refers to "the act of externalising an internal, mental abstraction" (Pape & Tchoshanov, 2001, p. 119). The improvement of mathematical understanding and representational thinking of students require flexible use of multiple representations and the interaction of external and internal representations (Pape & Tchoshanov, 2001). Making meaningful translations between representational modes plays a crucial role in the acquisition of mathematical concepts, and student engagement with multiple representations can facilitate this (Lesh et al., 1987).

The present study draws on a framework of Hiebert and Carpenter (1992) to describe the structural aspects of mathematical knowledge, learning processes and how learning from representations occurs. The framework of Hiebert and Carpenter draws on three key assumptions from research in the cognitive sciences. The first assumption is "knowledge is represented internally, and that these internal representations are structured" (p. 66). Hiebert and Carpenter metaphorically characterise students' internal knowledge representations as a network, where the nodes represent pieces of information and the connections between them reflect the relationships among that information. Hiebert and Carpenter's second assumption is learning results in the connection of internal representations of knowledge in ways beneficial for understanding. According to Hiebert and Carpenter, a mathematical concept is understood

if its internal representation is part of the internal network of knowledge. A key finding from their research maintains that the greater the number of connections in an internal network of knowledge and the stronger the connections within the network, the greater the degree of understanding. Therefore, students' internal representations of knowledge of mathematical concepts can be characterised as structured and organised networks of knowledge. Hiebert and Carpenter make a distinction between internal and external representations, where internal knowledge representations are the cognitive structures of knowledge in a learner's mind, and external representations often assume the forms of spoken language, pictures, written symbols, and manipulative models (Lesh et al., 1987). Hiebert and Carpenter's third assumption is external representations can influence students' internal representations of knowledge, where external representations of mathematical concepts can facilitate and support students' learning of those mathematical concepts.

Mathematics educators have long perceived external representations as an effective means of making mathematical ideas understandable for students (Hiebert & Carpenter, 1992). According to Hiebert and Carpenter (1992), the instructional use of external representations of mathematical concepts can facilitate students' construction of 22 mental models. Additionally, Goldin (2000, 2003) maintains that external representations should play a fundamental role in empirical investigations of students' reasoning and understanding of mathematical concepts, such as during task-based interviews. The present study incorporated four different dynamic digital representations of decimal fractions that students engaged with to consolidate and build their conceptual understanding within this area of mathematics.

4.1.3 Shift of Attention Theory

The theory of shifts of attention provides a scaffold for educators' understanding of teaching mathematics as well as proposing five observable attending stages that indicate what and how students are learning in the mathematics classroom. Several shifts of attention may occur in the process of thinking mathematically related to the object or focus of attention (what the learner attends to) and the form or structure of attention (how the learner attends) (Mason, 2008). Structure of attention refers to "holding wholes, discerning details, recognising relationships, perceiving properties, and reasoning on the basis of agreed properties" (Mason, 2008, p.35). As attention shifts from one concept to another, its structure alters and connections are made, relationships are perceived, and new ideas are appreciated. *Holding wholes* implies observing a mathematical object with or without focusing on mathematical characteristics. Such an action may only occur for a brief period before the learner begins *discerning details*

which involves marking distinctions and recognising important details. *Recognising relationships* occurs when the learner identifies particular relations between elements of a mathematical concept such as a model, calculation, diagram or equation. Shifting attention from *recognising relationships* to *perceiving properties* of specific elements is vital to growth of mathematical understanding. *Perceiving properties* requires the learner to hold an "awareness of a possible relationship and look for elements to fit it, thus leading to generalisation" (Mason & Johnston-Wilder, 2004, p.60). *Reasoning on the basis of agreed properties* is a form of attention that requires formal mathematical reasoning related to previously identified properties, "independent of particular objects, properties, as the only basis for further reasoning" (Mason & Johnston-Wilder, 2004; Mason, 2008, p.37). As learners gaze and detect apparent motion through saccadic eye movements, there is a shift that can cause them to suddenly notice specific details within a crowd of otherwise unperceived information (Wei et al., 2023).

The theory of shifts of attention has been applied to study progressions of mathematical problem solving (Voutsina et al., 2019). It can also be applied to account for changes related to shifts in the object and structure of learners' attention of any concept of mathematics including decimal fractions. Voutsina et al. (2019) investigated dynamic changes in how problem-solving strategies were communicated and conceptualised when students shifted the structure of their attention because of shifts in the mathematical object being attended to. The findings revealed microqualities that influence different learners' structure of attention during phases of mathematics learning which provide significant insights for understanding qualitative dynamics of change in learning (Voutsina et al., 2019).

4.2 The Relationship between the Theories

Mason (2010) views learning as the shifting of attention in form and focus which is what is being attended to and how it is being attended by the learner. A strong claim emerges from his research that argues engaging with a mathematical task is not sufficient for learning to occur (Mason, 2010). There is need for the integration of experience and sense-making which requires transformation of attention, whereby the learner's attention is shifted from complete immersion in the task to a broader perspective that internalises the new concept or approach (Mason, 2010). Dynamic representations of decimal fractions can offer direction to learners' shifts of attention, guiding them to focus on mathematically relevant features while minimising distractions from less pertinent details, thereby supporting the development of meaningful mathematical understanding (Santos-Trigio, 2006). The dynamic nature of digital

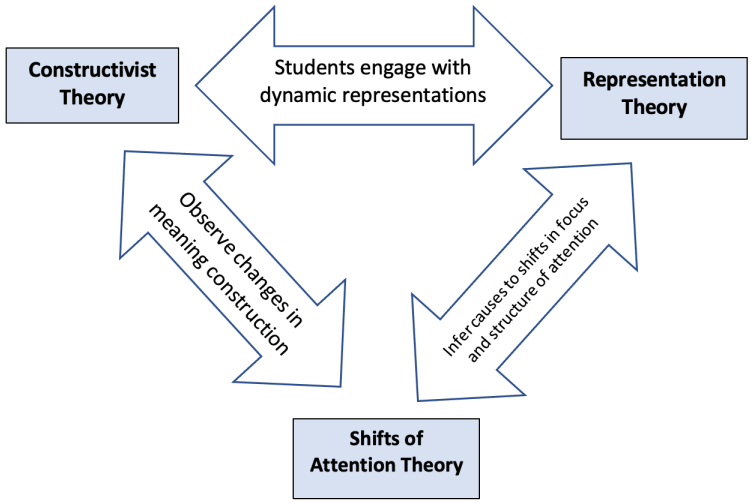
manipulatives forces shifts in the object of children's attention when they are actively interacting with the mathematics content as well as the structure of their attention in how they attend to features of the mathematics representation and the embedded mathematics concepts.

Constructivist approaches foster the idea that knowledge is not fixed or predetermined which has implications for the ways in which learning tasks are constructed and carried out in the classroom. If students are responsible for constructing knowledge, then it follows that teachers need to encourage and accept student autonomy and initiative (Fung, 2000). Learning tasks should be student-centered or provide opportunities for at least some degree of student choice. Learning should be "participatory, proactive, communal, collaborative and given over to the construction of meanings rather than receiving them" (Bruner, 1996, p.84). Consequently, teachers must source "inventive ways of engaging students in meaningful practices, of providing access to resources that enhance their participation, of opening their horizons so they can put themselves on learning trajectories they can identify with, and of involving them in actions, discussions, and reflections that make a difference to the communities that they value" (Wenger, 1998, p.10). Representations are a way for students to engage with the mathematical ideas being taught. They are an essential element in "supporting students' understanding of mathematical concepts and relationships, in communicating mathematical approaches, arguments, and understanding to oneself and to others, and in applying mathematics into realistic problems situations through modelling" (NCTM, 2000, p.67).

One of the tenets of a constructivist approach is that of helping students to adapt and build on their existing knowledge constructs to develop new levels of conceptual understanding (Fung, 2000). Representations act as a tool for manipulation and communication, and tools for conceptual understanding of mathematical ideas (Zazkis & Liljedahl, 2004). The interdependence between representations, conceptual understanding and shifting attentions is illustrated in Figure 4.1. Representations also play essential role in teaching and learning of mathematics because they help teachers and students to grasp the abstract notion of mathematics (Roubicek, 2006). This study was designed to stimulate engagement with multiple dynamic representations to develop understanding of decimal fractions. Acquiring new mathematical knowledge is a fundamental cognitive process, where there in ongoing association between the representations and the model itself (Duval, 2017). Goldin and Shteingold (2001) contend that teachers need to understand the effects of representation on students' learning to teach mathematics most effectively.

Figure 4.1

Diagram of relationship between three theoretical perspectives: constructivism, representations in mathematics, and 'shifts of attention'.



Gorman and Way (2018) observed students' shifts in attention when actively interacting with decimal fractions within the Zoomable number line web-app, by making conjectures and testing them immediately to 'see' the mathematics concept. When analysing the data, the researchers were able to explore shifts of children's explicit attention to decimal fraction concepts and to relationships between concepts as they verbalised their thinking. Understanding these shifts in what and how children are attending to features of decimal fractions is significant for advancing researchers' understanding of the dynamics of change in learning. As such, future research should be dedicated to exploring a potential theoretical account for inter- and intra-individual differences and changes in phases of attention to raise educators' understanding of individual learners' trajectories of conceptual learning related to decimal fractions.

4.3 Research Questions

The aim for the research is to explore what conceptual changes emerge while children in middle-primary grades engage with dynamic digital representations of decimal fractions. This new-found understanding of how students develop decimal sense through dynamic digital representations could translate into effective pedagogy within the primary mathematics classroom. The overarching research question sufficiently communicates the aim of the study. An additional research question is asked to guide exploration into the capabilities of these dynamic digital tools in addressing misconceptions about decimal fractions.

In addition, a sub-research question was proposed that addressed each separate dynamic digital representation of decimal fractions (*Wishball, Decimal Strips, Zooming in on Place Value*, and *Zoomable Number Line*). This allowed for specific data analysis of the features embedded within each digital tool that the students engaged with and interpretation of how these features prompted conceptual change in understanding of decimal fractions.

The overarching research question used to guide the study was:

How do children experience conceptual changes related to decimal fractions when they attend to features of dynamic digital representations?

The sub-research questions were: *How do students overcome misconceptions about decimal fractions when they attend to features of dynamic digital representations?*

As well as: *What features of Wishball-hundredths, Decimal Strips, Zooming in on Place Value, and Zoomable Number Line, do children attend to that prompt conceptual changes related to decimal fractions?*

In summary, this chapter has established the theoretical perspectives that inform the study and articulated the research questions that guide the investigation. Together, these provide the conceptual grounding for the methodological decisions that follow. The subsequent chapter outlines the research design and analysis approaches employed by detailing the selection of participants, the design of interview tasks, and the procedures used for data collection and analysis. In doing so, the chapter demonstrates how the methodological choices are consistent with the theoretical framework and are appropriately aligned with addressing the research questions.

CHAPTER 5: METHODOLOGY

5.1 Introduction

All decisions linked to the design of research and data collection were guided by theory as it provides the foundation and rationale for research. In addition, the theoretical perspective of the study provided the framework for interpreting research findings and for understanding social phenomena (Bryman, 2012). This research project sought to explore the cognitive and affective domains of young learners which meant the uniqueness of each participant would impact the collected data (Di Martino & Zan, 2011). The responses collected from the participants was within a qualitative methodology where "both numerical and non-numerical data are viewed in the same way; all data is a symbolic representation, which needs to be interpreted and thus its meaning is subjective and context dependent" (Twining et al., 2017, p. 2). As such, it was appropriate to adopt an interpretivist approach from a qualitative research perspective with the view of human experience being an interpretive activity.

The application and context of utilising qualitative methods in research is related to the theoretical stance of the ontology and epistemology of reality or knowledge (Twining et al., 2017). This form of methodology recognises multiple realities and the cultural construction of knowledge (Twining et al., 2017). Underneath the broad approach of qualitative methods stems interpretivism as a paradigm that endorses a view that the subject matter of the social sciences being people and their institutions is fundamentally different from that of the natural sciences (Bryman, 2012). Interpretivism accepts individual and collective understandings and social norms from a participant's point of view. As such, these personal interpretations, perceptions, meanings and understandings are each regarded as legitimate sources of primary data (Mason, 2002). In this study, the collection of self-reported data formed the totality of the data source as it accepted that each participants' responses and answers were subjective, social realities and worthy of analysis.

The interpretivist perspective views the social world as a creation of the minds of individuals and the social world is further reinforced as people interact with one another. Humans are naturally self-aware and people do respond differently when they are conscious of being studied. This principle equally applies to the participants and the researcher, who may sometimes be tempted to alter their behaviours because of such self-awareness (Denscombe, 2002). This understanding is accepted in the collection and analysis of responses and actions which form the data for this study.

This study employed a qualitative methodology, using task-based interviews in conjunction with microgenetic methods for data collection and analysis (Chinn & Sherin, 2014). Well designed and effectively implemented task-based interviews can reveal information about how students construct knowledge, their cognitive processes, and their interpretations of learning situations and tasks (Ginsburg, 1997). Microgenetic methods facilitate detailed analysis of students' reasoning, particularly in research designs incorporating task-based interviews (Chinn & Sherin, 2014; Siegler, 2006). The analysis approach of closely studying student approaches to knowledge acquisition was initially developed by Piaget (1970) as well as by Vygotsky and Luria (1994). In more recent times, the microgenetic method has been consolidated in the research of Robert Siegler and his colleagues (Siegler & Chen, 1998; Siegler & Crowley, 1991; Siegler & Stern, 1998). The method of investigating moment-to-moment processes of learning is important for this study as it connects 'how' the students are engaging with the dynamic digital representations with 'what' decimal concepts are being attended to by the students. In doing so, identification of specific affordances that prompt changes in conceptual understanding can occur.

5.2 Task-Based Interviews

The methodology considered most appropriate for the study is *structured, task-based interview*. The methodology complements the theoretical framework from a constructivist, learner-centred view. Constructivist learning theory has been linked with structured task-based interviews in the work of Gerald Goldin (2003) who found the research method to be an effective means of understanding conceptual knowledge, higher-level problem-solving processes, and children's internal constructions of mathematical meanings in relation to procedural and algorithmic learning. Task-based interview has the dual purpose of observing mathematical behaviour and "... drawing inferences from the observations about the problem solver's cognition and/or affect" (Goldin, 1993, p.303). Therefore, the tasks must be sufficiently open-ended to allow each student to respond in their own preferred way, and designed to maximise dialogue between the researcher and student (Hunting, 1997, p.153). The researcher used probing questions to encourage further explanation, for example "Can you tell me what you are thinking?", but remained neutral regarding 'correctness' of responses (Gorman & Way, 2018). The purpose of the interview was neither to teach nor to test knowledge, but rather to explore the student's ways of thinking when interacting with the mathematical representations (Gorman & Way, 2018).

Task-based interviews have their origin in clinical interviews that emerged with the theoretical work of Piaget, who is credited with pioneering the clinical interview. In the early 1960s, the clinical interview was used to gain a deeper understanding of children's cognitive development (Piaget 1965, 1975). Task-based interviews have been used by researchers in qualitative research in mathematics education to gain knowledge about an individual or group of students' existing and developing mathematical knowledge and problem-solving behaviours.

Gerald Goldin's (2003) research links constructivist theory with structured task-based interviews as a means of understanding conceptual knowledge, higher-level problem-solving processes, and children's internal constructions of mathematical meanings in relation to procedural and algorithmic learning. Task-based interviews align with qualitative research by studying mathematical learning and problem solving without the reliance on correct or incorrect answers associated with formal tests (Goldin, 2003). Task-based interviews are a valuable research tool as they allow researchers to gauge student conceptual knowledge as well as having the capacity to extend that understanding. Therefore, the data that was collected from task-based interviews has the potential to assist teachers' content knowledge and pedagogical approaches as they become aware of individual students' needs (Gorman & Way, 2018).

The most effective method for data collection in task-based interview is video, with audio, recording (Hunting, 1997; Maher & Sigley, 2014). Video-recording the interview was critical to collecting quality data as it was necessary to capture what is said and done by the child simultaneously, including gestures, expressions and emotional responses. In addition, a screen-recording tool from the application QuickTime Player was used to document the manipulative actions that the students made on all four digital web-apps, which were accessed on an iPad. Observational notes were documented using an original template designed for this study. The video recordings, screen recordings, observation notes, participant's work samples, and other related metadata, formed the data for analysis.

5.3 Task Design

Data collection for this study consisted of three key phases; an initial task-based interview, four proceeding task-based interviews each focussed on a different dynamic digital representation, and a final task-based interview (refer to Table 5.1 for timeline). The researcher worked individually with each participant in six 30-minute sessions spanning a six-week period. The task-based interviews began with a base-line task involving a conversation about decimal fractions where the student was asked to identify and explain the concept of a decimal

fraction. They then located a sequence of one and two decimal place numbers on a static paper-based number line. The initial task-based interview was used to collect data on students' "existing and developing decimal fraction knowledge, ways of reasoning and explaining how connections are built to other ideas as they extend their knowledge" (Maher & Sigley, 2014, p.580). In doing so, the researcher was able to compare the data collected from the task-based interviews with the results from the introductory activity and therefore form some conclusions in relation to the students' prior knowledge.

Following this, four separate task-based interviews were conducted, over a four-week period, where the students interacted with a different dynamic digital representation during each session. For two of the task-based interviews, a set of tasks were designed with increasing levels of challenge, to encourage the students to interact conceptually with decimal notation, number line positioning and area models. For the other two task-based interviews, the participants engaged with questions and tasks in-built to the digital tool. At the conclusion of each task-based interview, the participant was asked the reflective question, "Is there something that you learned in our last session that helped you with this task?" The purpose of which was to ascertain whether the student was making any connections between the four different dynamic digital representations of decimal fractions. The data collected from the task-based interviews depended on the "quality and incisiveness of the task" and accompanying questions (Hunting, 1997, p.148). Therefore, rehearsal interviews were conducted with a small group of voluntary adult participants to practice and refine giving task instructions and asking stimulating questions.

Table 5.1

Timeline of Data Collection and Analysis

Research Event	Details	Timeframe	Phase
Pre-Test and Initial Task-Based Interview	Base-line interview	Week 1	Phase 1
Preliminary analysis of pre-test results to deduce conceptual starting point at the first task-based interview.			
First Task-Based Interview	Wishball-hundredths	Week 3	Phase 2
Initial analysis to deduce new understandings gathered during the first task-based interview and infer participant's new conceptual starting point at the next interview.			
Second Task-Based Interview	Decimal Strips *Note: researcher-designed tasks*	Week 3	
Initial analysis to deduce new understandings gathered during the second task-based interview and infer participant's new conceptual starting point at the next interview.			
Third Task-Based Interview	Zooming in on Place Value	Week 4	
Initial analysis to deduce new understandings gathered during the third task-based interview and infer participant's new conceptual starting point at the next interview.			
Fourth Task-Based Interview	Zoomable Number Line (most abstract and unguided representation, transfer knowledge from prompted and animated Zooming in on Place Value manipulative) *Note: researcher-designed tasks*	Week 5	
Initial analysis to deduce new understandings gathered during the fourth task-based interview and infer participant's new conceptual starting point at the final interview.			
Final Task-Based Interview	Review interview	Week 6	Phase 3
Analysis – Interpretation of quotes and actions through analysis in relation to Structure and Microqualities of Attention (Mason, 2008) and corresponding Decimal Fraction Learning Progression conceptual ideas, as outlined in the Framework for Analysis table (Table 2).			
Secondary Analysis – Patterns of changes in understanding. The quotes, actions, and accompanying analysis were coded using key concepts that emerged from the data. The data was then categorised into the themes for example place value, relative size of decimals, decimal density, and abstractness of decimals (dynamic representation)			

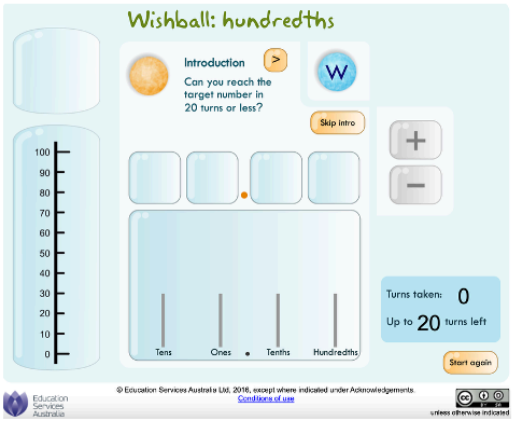
The dynamic digital representations incorporated in this study were selected due to their compatibility with the aim of the research questions. Four virtual manipulatives were chosen from a larger list after exploring and analysing the affordances and constraints of each potential digital tool, with each selected tool connected to one or more key conceptual demands for understanding decimal fractions identified in the literature, namely decimal density, place value and relative magnitude. The number-line based, place-value symbolic, and area-model dynamic digital representations of decimal fractions used were in the form of applets which were embedded into a webpage and displayed using a web-browser (Moyer et al., 2002). These virtual math manipulatives aimed to support students' ability to model abstract mathematical

concepts for deeper comprehension. Retaining the well-documented affordances of manipulatives that have been used for decades by teachers and students in classrooms, the four dynamic digital representations; *Wishball-hundredths*, *Decimal Strips*, *Zooming in on Place Value*, and *Zoomable Number Line*, offered additional advantages and affordances to their static counterparts.

The set of figures (Figure 5.1, Figure 5.2, Figure 5.3, Figure 5.4) presents an overview of the four digital tools and the related tasks used in this research study. Further details about the use of these digital tools in the task-based interviews is provided in Section 5.4 Participants and Interview Procedure under the subheading *Materials.*

Figure 5.1

Wishball-hundredths

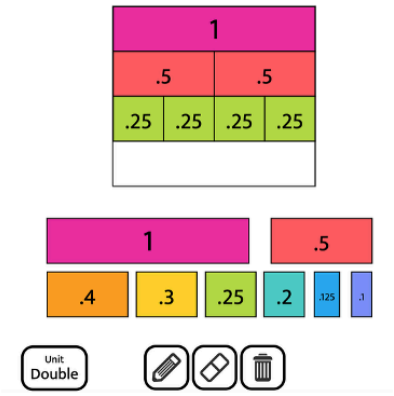


Note. Screenshot of the Wishball virtual manipulative and description of tasks (https://www.scootle.edu.au/ec/viewing/L869/index.html)

Wishball-hundredths is a learning tool with a specified in-built aim where students try to reach a target decimal fraction by adding or subtracting in under 20 moves. The spinner generates a random number, and users assign it to a place value to then either add or subtract to the base number. Text prompts guide understanding of decimal place value up to tenths and hundredths. A number line and counting frame show each move dynamically. To commence this task-based interview, students read the on-screen instructions, then engaged with the Wishball-hundredths app for five rounds, ensuring both familiarity and adequate interactive time without losing engagement.

Figure 5.2

Decimal Strips

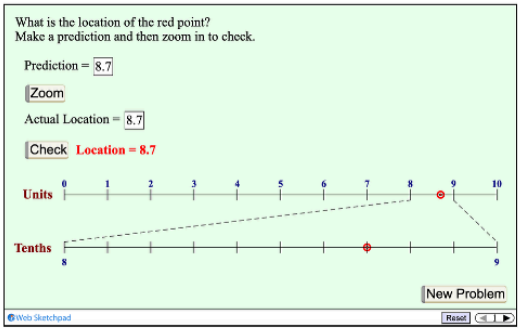


Note. Screenshot of the Decimal Strips virtual manipulative and description of tasks (https://toytheater.com/decimal-strips/)

The second task-based interview involved students using the dynamic digital representation Decimal Strips to complete a number of tasks posed by the researcher. Decimal Strips allow users to explore the relative magnitude of decimal fractions by clicking and dragging coloured tiles representing different decimal values (1-whole, 0.5, 0.4, 0.3, 0.25, 0.2, 0.125, 0.1) onto a decimal fraction wall. The tasks included ordering decimal fractions in ascending and descending order, comparing decimal numbers to identify the larger one, and matching decimal fractions with their common fraction representations.

Figure 5.3

Zooming in on Place Value

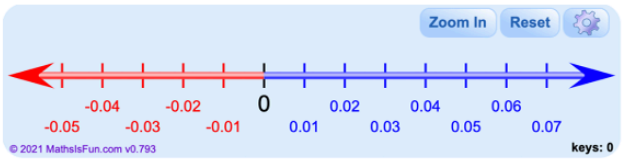


Note. Screenshot of the Zooming in on Place Value virtual manipulative and description of tasks (http://www.sineofthetimes.org/zooming-in-on-place-value/)

In the third task-based interview, students used the interactive digital tool Zooming in on Place Value to place decimal fractions on a number line. The in-built task is to determine the location of a decimal fraction and self-check the answer using an embedded Zoom function. After reading aloud the question and instructions, students interacted with the dynamic model showing a point on a number line. They estimated its location, then used the Zoom button to expand the interval between whole numbers, revealing a magnified view divided into tenths. Finally, students named the precise location of the point using the tenths integer marks. Each student engaged with the interactive Zooming in on Place Value app for ten rounds.

Figure 5.4

Zoomable Number Line



Note. Screenshot of the Zoomable Number Line virtual manipulative and description of task (https://www.mathsisfun.com/numbers/number-line-zoom.html)

In the fourth task-based interview, students engaged with the Zoomable Number Line virtual manipulative. This digital tool represents decimal fractions on a number line dynamically through a Zoom In function. The interview consisted of five tasks of increasing difficulty, designed to encourage conceptual interaction with decimal notation and number line positioning. Tasks included locating one and two decimal place numbers on the digital number line, comparing pairs of decimal numbers, identifying the next hundredths decimal after a given fraction, and solving a multi-step problem.

5.4 Participants

The participants in this study consisted of Year 4 students (aged 9-10 years old) in Australia, a grade level chosen because the New South Wales (NSW) state syllabus introduces decimal notation for tenths and hundredths at this point (NESA, 2012). Typically, students in this grade are still in the early stages of understanding of rational numbers in the context of decimal fractions which was true for the participants in this study. Individual participants held a different level of understanding that is influenced by their acquisition of the anticipated prerequisite knowledge related to decimals from Year 3. The NSW Mathematics Syllabus positions Year 3 students to consolidate understandings of fractions and their relationship to decimals, as well as explore decimal fractions through practical activities and contexts such as money or measurements. Furthermore, Year 4 students commonly have a limited understanding of how to represent decimals using number line representations, despite being capable and age-suited to learn from this form of representation (Moss & Case, 1999; Siegler et al., 2011).

Six participants were selected from a single Year 4 classroom based on their decimal comparison pre-test results. One criterion for participation was the students' ability to effectively articulate their mathematical thinking and reasoning verbally. The class teacher confirmed that the selected children would feel comfortable engaging in one-on-one discussions with the researcher. Although the participants were not intended to be representative of the entire class, the researcher chose six students encompassing a range of

mathematical achievement levels to account for potential differences in how low and high achievers interacted with the dynamic digital representations. Due to the time-consuming nature of data collection and analysis required in microgentic designs of research, generally a small number of participants are selected to provide the rich data sets (diSessa, 2017a). Interviewing six children was deemed sufficient to capture a variety of student responses.

The researcher contacted the principal of an independent primary school in the outer suburbs of the state's capital city, Sydney, by email. Upon first contact, the researcher introduced herself and briefly outlined the intention of the study using the Introductory Letter (see appendix A), as well as providing the Participant Information Sheet (see appendix B), and ethics approvals from HREC (see appendix C). The researcher anticipated a level of non-response or polite rejection from schools, but the initial principal agreed to host the data collection for this study. The researcher provided a template wording invitation email that was sent by the school office to the Year 4 teachers asking them to consider supporting the study. Once a Year 4 teacher had nominated their class, a three-way meeting was held between the researcher, principal and classroom teacher. The timeline for data collection was outlined and suitable days were negotiated between the stakeholders. In addition, a request was made for the Year 4 class to not be exposed to explicit mathematics teaching involving decimal fractions during the 6-week period of task-based interviews as it would impact the data being gathered.

As the four dynamic digital representations of decimal fractions were foreign virtual manipulatives to the students, it was important that the content and skills were connected to the students' prior knowledge so that they could successfully engage with the new representation. The tasks used in the interviews were designed appropriately for Year 4 students and were presented on palm-cards using a large plain font. The task-based interviews lasted approximately thirty minutes which proved to be sufficient time to collect quality data and simultaneously brief enough to maintain age-appropriate student concentration levels. The interviews were conducted one-on-one between the interviewer and a Year 4 student, in a quiet location organised by the teacher.

5.5 Research Design

Pre-Test and Initial Task-Based Interview

To commence the data collection phase, a pre-test was designed and administered to all students in one Year 4 class from which the participants were then selected. The pre-test required students to compare pairs of decimal fractions and determine the larger value, these decimal fraction pairs were taken from Stacey and Steinle's (2003) Decimal Comparison Test

which is used to classify students' thinking about decimal notation. Steinle and Stacey (2004a) found that approximately about half of 11-year-olds who completed the given Decimal Comparison Test, understood whole number place value well and many less understood decimals. Comparing the size of decimals is an efficient task to diagnose difficulties, because the instructions for students are simple and the test is quick to administer, whilst the different patterns amongst the answers are very revealing. The research-based test was accompanied by marking instructions and an analysis tool to allocate a code to each question that indicated the students' thinking with a classification of either Longer-is-larger, Shorter-is-larger, or Apparent-expert (see Appendix D). Steinle and Stacey (2004a) have attested to the reliability of the test and stated, "children's answers are consistent on the Decimal Comparison Tests, even across long periods of time" (p.547). In one test-retest experiment (Moloney & Stacey, 1996) found that only 6 of 50 junior-secondary students changed their individual-classification over one year. Furthermore, an observation that emerged from Steinle and Stacey's data analysis highlighted the number of students who consistently answered the separate item following one of the known misconception patterns. As such, they concluded that the data from the comparison test is sufficiently valid over time and therefore provides a useful guide for addressing students' specific learning needs. Steinle and Stacey (2004a) also acknowledge that occasionally students are not consistent which has been attributed to the structure of the comparison test prompting them to think more carefully about the task allowing the students to learn as they progress through the items. Alternatively, the inconsistency emerges because the student has several different ideas in response to different items posed. Steinle and Stacey (2004a) hypothesise that there are rare patterns of thinking about decimals that have not yet been discovered.

The pre-test results were analysed using Steinle and Stacey's Decimal Comparison Test marking instructions (see Appendix E). 21 students in a Year 4 class completed the pre-test and each students' answers were scored and coded with a primary classification as well as a finer classification. The raw scores spanned from the lowest 10/30 to the highest 29/30. The class results featured all three primary classifications of Longer-is-larger, Shorter-is-larger, and Apparent-expert, with a majority of students demonstrating a Longer-is-larger misconception with decimal fractions. Test results were entered onto a spreadsheet. Using the results from the Decimal Comparison Test the researcher nominated six participants from the class that spanned a range of results and appeared to hold different mathematical thinking within the area of decimal fractions. The pre-test results of the six participants are displayed below in Figure 5.5. While the students selected to participate in the task-based interviews are not representative of

the whole-class, they do show a variety of mathematical learning needs and between the six students all three primary classifications of decimal misconceptions are evident with a range of finer classifications are also held. In addition, the six students nominated were cross-checked with the supervisory team to validate selected participants demonstrating an ideal spread of prior knowledge.

Figure 5.5

Whole-class Decimal Comparison Test results

	A	B	C	D	E	F
1	**Student Name**	**Score /30**	**Primary Classification**	**Finer Classification**		**Six Participants**
2	Tehillah F	10	Longer-Larger	Whole Number and Reverse		Tehillah F
3	Isabella D	18	Longer-Larger	Whole Number		Yehali W
4	Sebastian S	19	Longer-Larger	Whole Number		Valerie C
5	Maddie L	19	Longer-Larger	Whole Number		Aarav K
6	Annelise D	24	Unclassified	Whole Number		Annelise D
7	Asher M	23	Longer-Larger	Column Overflow		Andrew L
8	Ashin J	23	Longer-Larger	Column Overflow		
9	Morgan D	19	Longer-Larger	Whole Number		
10	Krishika P	21	Longer-Larger	Whole Number and Column Overflow		
11	Hazelle G	19	Longer-Larger	Whole Number		
12	Aarav K	20	Shorter-Larger	Denominator Focussed		
13	Valerie C	18	Shorter-Larger	Denominator Focussed		
14	Kailynn S	19	Longer-Larger	Whole Number		
15	Evelyn L	21	Longer-Larger	Whole Number and Column Overflow		
16	Yehali W	17	Longer-Larger	Whole Number		
17	Elyssa K	17	Longer-Larger	Whole Number		
18	Malachi P	19	Longer-Larger	Whole Number		
19	Dominic M	22	Unclassified	Whole Number		
20	Theodore S	19	Longer-Larger	Whole Number		
21	Hunter V	19	Longer-Larger	Whole Number		
22	Andrew L	29	Apparent-Expert	Task Expert		
23						
24	**Class Analysis**					
25	Average	19.7619048				
26	Median	19				
27						
28	**Score**	**Number of Students**	**Primary Classification**	**Number of Students**		
29	10	1	Longer-Larger	16		
30	17	2	Shorter-Larger	2		
31	18	2	Apparent-Expert	1		
32	19	8	Unclassified	2		
33	20	1				
34	21	2	**Finer Classification**	**Number of Students**		
35	22	1	Whole Number	16		
36	23	2	Column Overflow	4		
37	24	1	Reverse	1		
38	29	1	Denominator Focussed	2		
39			Task Expert	1		

Note. Screenshot of Year 4 class decimal comparison pre-test data collated in an Excel spreadsheet.

An initial task-based interview was held for approximately 30 minutes. The purpose was to assess the participants' current knowledge of decimal fractions including their misconceptions, conceptual understanding of decimal fractions involving place value, magnitude and density, and their ability to locate decimal fractions on number lines. The tasks (See Table 5.2) included a series of questions that required the students to place and locate decimal fractions on a static number line to gauge their current knowledge and understandings

of linear representations of decimals. In doing so, the data collected from the static number line activity could be compared with the results from the dynamic digital task-based interviews and therefore be able to form some conclusions in relation to the students' prior knowledge.

Table 5.2

Outline of initial interview tasks and probing questions

Task (examples of probing questions in *italics*)	
1. Decimal fractions introductory questions: The participant was shown a card with the decimal fraction '4.2' on it. The interviewer asked the participant these questions:	*What is this number? 'four point two', 'four and two tenths'* *What is this?* (interviewer gestures to decimal point) *What does it mean? 'part of a whole number'*
These introductory questions helped establish a shared language for the remainder of the task-based interviews.	
2. Using the static, paper-based number line provided, locate the following decimals on the number line. The participant was given a set of cards each with a decimal fraction (0.7, 0.1, 7.7, and 1.89). The participant then attempted to place each card in its correct location on a paper-based static number line in front of them.	0.7 0.1 7.7 1.89

Task-Based Interviews

Four task-based interviews proceeded the initial task-based interview. During the four task-based interviews, the students engaged in tasks incorporating the four dynamic digital representations designed to elicit their mathematical thinking of decimal fractions. To gain information about the features of the digital tools that afforded opportunities for students' understanding of decimal fractions, students were frequently prompted for verbal explanation of how they used the virtual manipulates during the task-based interviews or whether their thinking had changed at any point whilst interacting with the digital interfaces. During the analysis phase of the study, students' responses to prompts were examined to determine if they provided information about affordances of the dynamic digital representations for supporting students' conceptual understanding of decimal fractions. It was important to reflect on and take detailed notes after each task-based interview to develop "impressions" of each participant's understanding of and reasoning regarding decimal fractions (Chi, 1997, p.281). Documented impressions were validated against key literature and the results were used to form initial broad themes during data analysis.

Final Task-Based Interview

For the concluding interview session of the study, a final task-based interview was held with each student. This task-based interview comprised of the same decimal fraction tasks used in the initial interview. The purpose of the subsequent task-based interview was to gather data that allowed conclusions to be drawn as to whether any changes occurred in each of the students' mathematical thinking and reasoning related to decimal fraction concepts. The structure and timing of the final task-based interview was concurrent with the initial task-based interview.

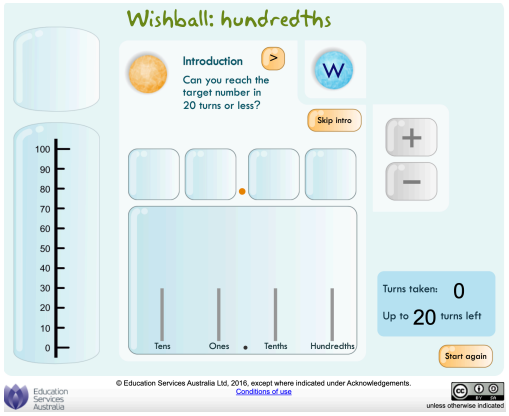
5.6 Interview Procedure

Digital Tools used in Task-Based Interviews

This section presents an overview of the four digital tools used in this research study during the task-based interviews.

First Task-Based Interview with Wishball-hundredths

Task-Based Interview One (Wishball-hundredths) (https://www.scootle.edu.au/ec/viewing/L869/index.html)



Note. Screenshot of the Wishball virtual manipulative where students try to reach a target decimal fraction number by adding or subtracting in fewer than 20 moves. The Spinner function generates a random number and the user must choose a place value to assign to the digit then either add or subtract it to the base number.

The first task-based interview focused exclusively on the interactive dynamic digital tool called Wishball. Students explore place value of decimal fractions in a game-like environment with opportunities for mental addition and subtraction. Wishball provides a structured learning environment with a specified aim for successful engagement. The digital tool features text-prompts to guide learning about the place value of decimal fractions up to tenths and hundredths. A key learning objective is that students identify the place value of each

digit in a two-digit decimal fraction and the relative value of a given digit in different places within a two-digit decimal fraction.

Steinle and Stacey (2004a) have documented the prominence of whole-number thinking that students' hold when learning about decimal fractions. Teaching of decimal fractions as an extension of the whole-number system frequently occurs with insufficient understanding of the place value structure (Fuson, 1990). This difficulty is perpetuated when teachers attempt to simplify the learning of decimal fractions by encouraging students to annex zeros and compare the numbers only to the right of the decimal point as if they were two whole numbers. The instructional practice fosters the view that decimal fractions are mathematically congruent with whole-numbers. As such, it is imperative that students be able to recognise the role of place value in communicating the relative size of decimal fractions. Wishball features a number line and counting frame that dynamically display the moves the user makes when adding and subtracting decimals with respect to place value. These dynamic affordances are designed to visually represent the increasing and decreasing value of the decimal fraction as ones, tenths and hundredths are added and subtracted during each move of play.

The user aims to try and reach a target number by adding or subtracting in fewer than 20 moves. Firstly, the Spinner randomly serves up a digit from which the user chooses a place value to assign to the digit then either adds or subtracts it to the base number. For example, the user could receive a starting number, such as 3.59, and then works toward turning it into a target number, such as 7.82. The Wishball option, which can be used only once, allows the user to choose the final digit required to finish the game. In the Wishball challenge subseries, only one function (addition or subtraction) is made available for students to use to reach the target number. This requires students to be more careful about selection of place-value so they don't 'overshoot' the target number. Importantly, a number line and counting frame dynamically display the moves the user makes.

To commence this task-based interview, the researcher asked each participant to read aloud the decimal fraction 3.84 and to also articulate what digit was in the tenths, hundredths, and ones. This helped establish a common language between the researcher and the participant as well as ascertaining the student's current understanding of decimal place value. The participant was then required to read the Wishball application instructions aloud as they appeared on the iPad screen and then given the opportunity to ask any clarifying questions. Each participant engaged with the interactive Wishball app for five rounds. This allowed for sufficient familiarisation with the dynamic digital representation and then adequate interactive time without disengagement occurring.

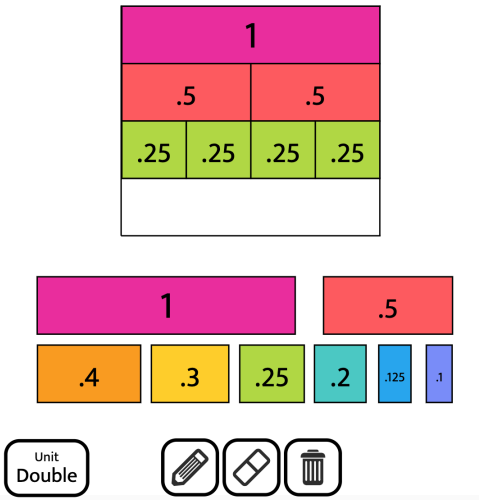
Table 5.3

Outline of Wishball-hundredths interview tasks and probing questions

Task (examples of probing questions in *italics*)	
Read this number. *3.84* *What digit is in the tenths?* *What digit is in the hundredths?* *What digit is in the ones?*	*Can you read the instructions aloud as they appear on the screen?* *Do you have any questions?*

Second Task-Based Interview with Decimal Strips

Task-Based Interview Two (Decimal Strips) (https://toytheater.com/decimal-strips/)



Note. Screenshot of the Decimal Strips virtual manipulative where students click and drag coloured tiles representing different decimal values onto the canvas.

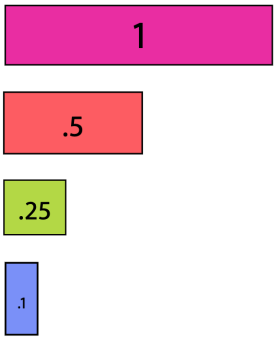
The second task-based interview involved the participants accessing dynamic digital area-representation of decimal fractions called Decimal Strips to complete several tasks posed by the researcher. This dynamic area-model representation of decimal fractions provides opportunities for the user to investigate the relative magnitude of decimal fractions through manipulating coloured tiles that are sized to represent different decimal values including 1-whole, 0.5, 0.4, 0.3, 0.25, 0.2, 0.125, 0.1. The representation of decimals up to thousandths encourages the conceptual understanding of relative size and magnitude of decimals as the user can dynamically shift two 0.125 tiles and directly compare the value with a single 0.25 tile to visually see the equal values. Many young students' think that the more digits a decimal fraction number has corresponds to it being a larger value. The affordance of being able to compare the three-digit decimal of 0.125 with the two-digit decimal 0.25 and seeing that it has a smaller value is a powerful conceptual idea for the students to uncover.

To begin the task-based interview, the student ordered a series of six decimal fractions in ascending order and then a different set of six decimals in descending order. After which,

they compared a series of two decimal numbers by identifying which of the pair was larger. Finally, the participants matched three decimal fractions with the common fraction symbolic representation.

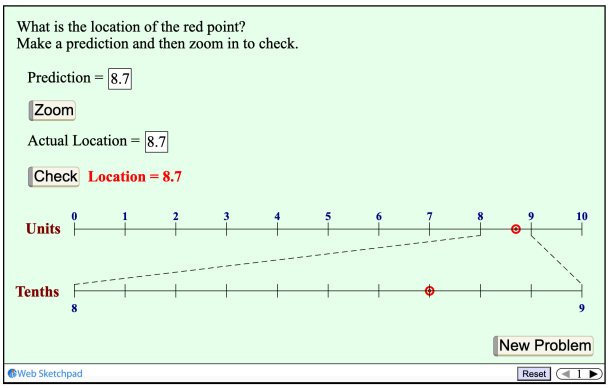
Table 5.4

Outline of Decimal Strips interview tasks and probing questions

Task (examples of probing questions in *italics*)	
1. Order a series of six decimal numbers (ascending). The participant was given six cards with a decimal fraction on each and asked to put the decimals in ascending order using the Decimal Strips. *What strategy did you use to order the decimals?* *How do you know you have ordered them correctly?*	0.3 0.8 0.2 0.6 0.4 0.1
2. Order a series of six decimal numbers (descending). The participant was given six cards with a decimal fraction on each and asked to put the decimals in descending order using the Decimal Strips. *What strategy did you use to order the decimals?* *How do you know you have ordered them correctly?*	0.6 0.4 0.7 0.1 1.0 0.9
3. Compare a series of two decimal numbers. The participant was given two cards with a decimal fraction on each and asked to identify which of the pair is LARGER, using the Decimal Strips. *What strategy did you use to predict which decimal was larger?* *How do you know it is larger?*	0.2 0.5 0.3 0.1 0.25 0.2 1.0 0.9 0.125 0.1
4. Tenths as decimal fractions. The participant was given a set of 9 cards to match up. Three cards were coloured decimal strips, three cards were decimal fractions, and three cards were common fractions. *If a pink decimal strip is 1-whole, then match the common fraction and decimal fraction that is represented by these decimal strips.*	

Third Task-Based Interview with Zooming in on Place Value

Task-Based Interview Three (Zooming in on Place Value) (http://www.sineofthetimes.org/zooming-in-on-place-value/)



Note. Screenshot of the Zooming in on Place Value virtual manipulative where students predict the location of a decimal fraction on a number line and then self-check their answer through the embedded Zoom function.

The third task-based interview required each student to place decimal fractions on a number line through the interactive digital tool called Zooming in on Place Value. Decimal fractions are derived from the base-ten number system. Martinie (2014) suggests when students lack an understanding of place value and the base-ten number system they experience increased frustration and difficulties with computing decimal fractions. Further difficulties arise when learners fail to recognise that decimal fractions are a special class of fractions with base-ten denominators (Martinie, 2014). As such, students require an understanding of place value of decimal fractions involves symmetry around the ones place value with the first digit after the decimal represents the tenths place. As Goldin and Schteingold (2001) suggest, developing clear links between quantity, vocabulary and symbol is critical for conceptual understanding and the visual affordances of the digital tool illustrate the symbolic notation. A key affordance of the Zooming in on Place Value virtual manipulative is its ability to dynamically displaying ten-tenths between two whole-numbers. Through a 'zooming in' animation the digital tool reveals ten-tenths between two whole-numbers, and ten-hundredths between two-tenths in a two-step question sequence. This gradual release of information allows the student to build knowledge of decimal density as to the continuity property of rational numbers, whereby, between any two decimals there are an infinite number of other decimals (Widjaja et al., 2008).

To successfully engage with the dynamic digital representation the participant is prompted to read aloud the question and instructions on the display screen. The interactive model displays a point sitting on a number line. The user is prompted to describe its location. After typing an estimate into the box, the user presses the 'Zoom' button, and the dynamic representation responds as the interval between two whole numbers expands, showing a

magnified view divided into tenths. The point still sits at the same location, but now it lies at a tick mark. The user names the precise location of the point using the tenths integer marks. A new decimal point on the number line is displayed by pressing 'New Problem' for the user to estimate. Each participant engaged with the interactive Zooming in on Place Value app for ten rounds.

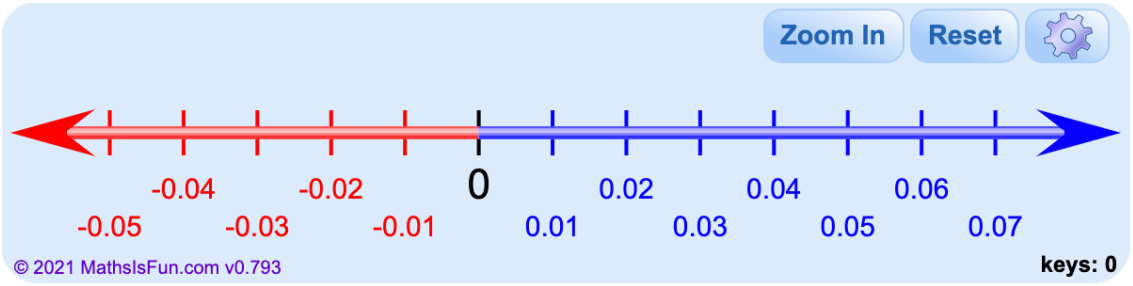
Table 5.5

Outline of Zooming in on Place Value interview tasks and probing questions

Task (examples of probing questions in *italics*)
Can you read the instructions aloud as they appear on the screen?
Do you have any questions?

Fourth Task-Based Interview with Zoomable Number Line

Task-Based Interview Four (Zoomable Number Line) (https://www.mathsisfun.com/numbers/number-line-zoom.html)



Note. Screenshot of the Zoomable Number Line virtual manipulative which dynamically represents decimal fractions on a number line through a Zoom In capability.

The fourth task-based interview involved a sequence of tasks that required the participants to engage with the Zoomable Number Line virtual manipulative for successful completion. Many young children have extensive difficulties in recognising the density of decimals, as it is void in their whole-number knowledge (Widjaja et al., 2008). The discreteness feature of whole numbers is incongruous for understanding the density of decimals (Widjaja et al., 2008). The Zoomable Number Line with its dynamic capabilities can assist students in understanding the continuity aspects of decimals as the user is able to continually zoom in to reveal decimals between each rational number. Understanding what a decimal fraction represents is a difficult concept, and therefore students should be encouraged to interact with quality representations such as the Zoomable Number Line to assist them in developing their decimal sense.

Furthermore, Steinle and Stacey (2004b) argue that the number line is one of the few models that is functional for discussing the density of decimals. However, Steinle and Stacey

(2004b) have commented on the limitations of a static number line, as they note that some students view it as a discrete representation. These students picture the number line as a series of isolated points, close together, not as a continuous line without any gaps (Steinle & Stacey, 2004a). The Zoomable Number Line has the potential to reverse this thinking as its zooming functions illustrate the density of decimals, which allows students to physically see the infinite number of parts that fit in between two whole numbers.

The fourth task-based interview incorporated a set of five tasks with increasing levels of challenge, to encourage the students to interact conceptually with decimal notation and number line positioning through a Zoomable Number line virtual manipulative. The task-based interview began with students locating a sequence of one and two decimal place numbers on the digital number line. They then compared a series of two decimal numbers by identifying which of the pair was larger. The participants also described what hundredths decimal came after a given decimal fraction. Finally, a multi-step problem solving question was given to the students to solve.

Table 5.6

Outline of Zoomable Number Line interview tasks and probing questions

Task (examples of probing questions in *italics*)	
1. The participant freely investigated the 'zoomable number line' app by manipulating the toggles to zoom in and out, and scroll along the number line. As the participant was experimenting with the 'app', the interviewer paused the student at several points to prompt them to explain and/or predict what the number line was displaying as they zoom in to reveal tenths, and hundredths of numbers.	*Where is the number line zooming in between?* *What do those lines represent? (tenths, hundredths)* *What does that decimal number mean?* *What do you notice about the 'zoomable number line' app?* *Are there any features of the app that are confusing or difficult to use?* *What are some interesting or useful features of the app?*
2. Locate a sequence of one and two decimal place numbers (and describe its location on the number line e.g. between __ and __). The participant was given a card with a decimal fraction on it and asked to locate the decimal using the 'zoomable number line' app. They described the decimal's location by stating "it's between ___ and ___". For example, the decimal 0.71 is between 0.70 and 0.72. *How did you locate the decimal fraction on the number line?* *How did you know what numbers the decimal was between?*	0.3 0.12 0.9 1.55 4.915* 9.819* **To be given if students locate previous decimals easily.*

Task (examples of probing questions in *italics*)		
3. Compare a series of two decimal numbers. The participant was given two cards with a decimal fraction on each and asked to identify which of the pair is LARGER, using the 'zoomable number line'. *What strategy did you use to predict which decimal was larger?* *How do you know it is larger?*	0.4 0.86 0.3 1.85 3.71 3.92* 4.08* **To be given if students compare previous decimals easily.*	4.5 1.3 0.4 1.84 3.76 3.481 4.8
4. Describe what hundredths decimal comes next. The participant as given a card with a decimal fraction on it and asked to articulate what hundredths decimal comes next using the 'zoomable number line' app.	2.34 6.57 8.23 23.54 10.2* 0.99* **To be given if students describe previous decimals easily.*	
5. Problem solving question.* The participant was given a problem solving question on a card to solve, using the 'zoomable number line'. *Five gymnasts are entered into a competition. Four of the gymnasts have performed their routines. Their scores, out of ten, were 9.8, 9.75, 9.79, and 9.76. What score must the last gymnast get in order to win the competition?*	*Did you use the number line to work out the answer? Why or why not?* *What strategy did you use to find the answer?*	
Involved ordering decimals from smallest to largest, and then determining how the highest compares with the winning score. **To be given if students answer previous tasks easily.*		

The researcher conducted rehearsal interviews using a group of voluntary adult participants to practice delivering the interview protocol. The exercise exposed some aspects of the interview design that needed to be refined including the pre-programed instructions accompanying the Wishball application. A number of participants expressed confusion when initially stepping through the instructional prompts both with the wording and sequence of information. As such, the researcher created a one-page user guide that featured simple word instructions with a corresponding screen-capture of the application that circled or pointed using an arrowhead the exact digital feature being described. Another point of interest that was observed in the Decimal Strips trial interview involved the lack of interaction with the digital application when answering the problems posed. As the rehearsal interviews were conducted with adults it likely that this was due to them not requiring a digital aide to help with problem solving, however to avoid this happening when conducting the interviews with children

participants the researcher added a follow-up question “Can you prove your answer and thinking on the Decimal Strips app?” at the end of each task, which was asked if the child had not engaged with the digital tool independently.

5.7 Microgenetic Data Analysis

The data collected from task-based interviews was analysed using a microgenetic approach. The correlation between one-on-one task-based interviews and the principles of the microgenetic method is evident when gathering intensive data from learners over a period of time to capture a record of moment-by-moment learning processes (Chinn, 2006). The aim of the approach is to detect learning changes by determining what specific shifts in the learner’s attention have occurred and how they have transpired (Voutsina et al., 2019). Attaining an understanding of the precise experiences that lead to conceptual change provides a powerful insight for researchers and teachers to recognise potential approaches when teaching key mathematical concepts (George & Voustina, 2021). Microgenetic methods of analysis involve closely examining what learners do and say over several task-based interviews, thus allowing researchers to draw conclusions about what prompts knowledge change, how learning occurs, and other aspects of the processes of acquiring conceptual understanding (Voutsina et al., 2019). The findings gathered from studying microqualities that underlie different learners’ structure of attention can provide new insights for understanding qualitative dynamics of change in learning (Voutsina et al., 2019).

A microgenetic approach was highly suited to capture these important details of student learning as it engages individuals in an intensive learning experience over a period of time which is likely to accelerate the process of change and therefore allows researchers to capture the change as it occurs (Kuhn, 1995). Highly-concentrated observation of learning behaviour over a series of task-based interviews requires intense data collection, followed by trial-by-trial analysis of observed sayings and doings (Siegler, 2006).

Microgenetic data analysis involved a combination of inductive and deductive approaches. The process began with repeated viewings of the video-audio recordings and screen-captures at the conclusion of each task-based interview where the researcher deduced new understandings that the participants had gathered. Particular attention was given to the manipulation of each digital tool, the mathematics concepts demonstrated, and the relationship between the two. These significant sayings and doings were accurately documented in chronological order. As qualitative microgenetic studies focus on tracing and documenting points of change in learning through analysis of each individual’s verbal and physical

behaviours, a high density of observations was required. This entailed multiple viewings of the data from task-based interviews to explore "development of individual students' knowledge, reasoning, problem solving and understanding of mathematics concepts through a qualitative lens" (Ginsburg, 2009, p.113). A framework for analysis was designed by the researcher as a reference tool when analysing the verbal reasoning and manipulative actions the participants made. The 'Shifting Attention with Decimal Concepts Framework' (see Table 5.7) correlated Mason's five components of structures for attending with key decimal fraction concepts that form a sound conceptual understanding as outlined in the ACARA National Numeracy Learning Progression. This analysis tool was used alongside a microgenetic approach to data analysis to discern observable shifts in how and what the participants were attending to as they engaged with the four dynamic digital representations of decimal fractions. To enhance reliability beyond expert review, a subset of transcripts was independently coded by a second researcher to ensure consistency in the application of the framework. Discrepancies were discussed and resolved collaboratively, leading to refinements in the coding scheme. Validity was further strengthened through triangulation across multiple data sources, including screen recordings, student verbalisations and interviewer observation notes. This allowed for a more comprehensive interpretation of the students' evolving conceptual understandings.

5.8 Operational Definition and Criteria for Conceptual Change

Successful detection of conceptual changes relied upon a strong distintion between conceptual change episodes and transient performance adjustments. Therefore, a three-criterion performance threshold was maintained throughout the microgenetic data analysis. An episode was identified as conceptual change only when: (1) the student spontaneously revised a prior stated belief or strategy in response to a representational affordance of the digital tool (not merely corrects an arithmetic error); (2) the revised understanding was sustained across at least one subsequent task or verbalisation turned within the same interview session; and (3) the student articulated, however partially, why their earlier conception was insufficient. Transient performance adjustments, by contrast, were labelled when a student produced a correct response but reverted to the prior conception or couldn't explain the shift.

Table 5.7

Shifting Attention with Decimal Concepts Framework

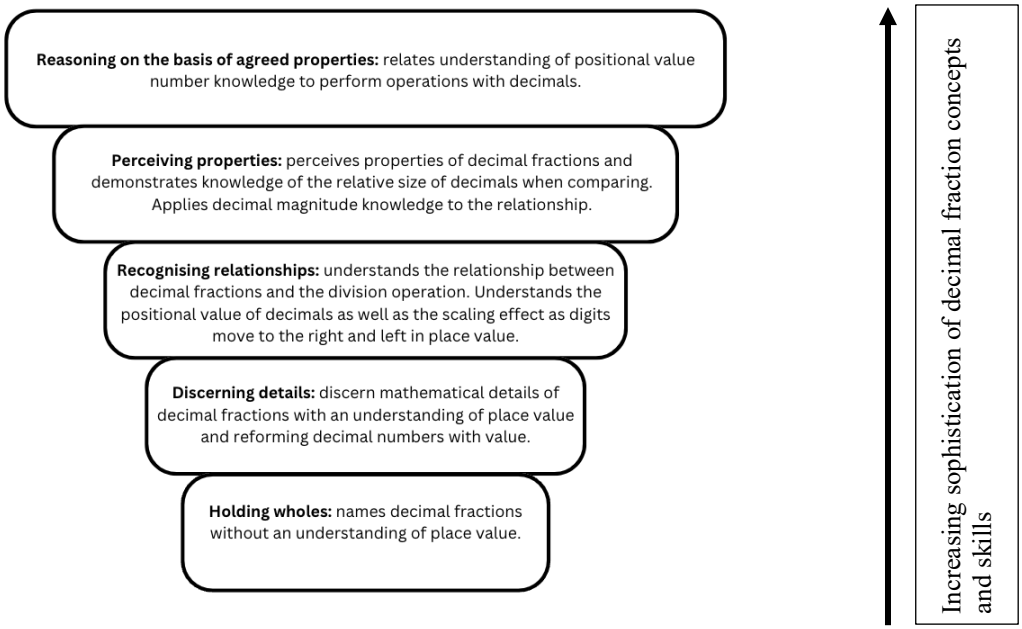
Structure and Microqualities of Attention (Mason, 2008)	Decimal Fraction Learning Progression (ACARA, 2020)	Connection of Theory of Structures of Attention and Decimal Fraction Learning Progression Points (Researcher's own reasoning)	Key Concept(s) of Decimal Fractions in which Conceptual Change has Occurred (Researcher's own reasoning)
Holding wholes	Naming decimal fractions.	The student verbally names decimal fractions without an understanding of place value.	Place value
Discerning details	State the place value of digits in decimal numbers of up to two decimal places. Use place value to partition decimals of up to two decimal places.	The student can discern mathematical details of decimal fractions with an understanding of place value and reforming decimal numbers with value.	Place value Decimal magnitude
Recognising relationships	Understand what a decimal fraction is and recognise that they arise out of division. Model, compare and represent decimals of up to two decimal places. Understanding and estimating relative size of decimals.	The student understands the relationship between decimal fractions and the division operation, as the decimal is parts of a unit resulting from dividing the whole into equal parts. Furthermore, the student understands the positional value of decimals as well as the scaling effect as digits move to the right and left in place value for example 3.509 has 35.09 tenths, 350.9 hundredths, 3509 thousandths.	Place value Decimal density
Perceiving properties	Place decimals of up to two decimal places on a number line. Apply knowledge of decimals to record measurements, for example 123 cm = 1.23 m.	The student perceives properties of decimal fractions as they demonstrate knowledge of the relative size of decimals when comparing and applies this knowledge to the relationship with measurement and position when plotting values on a number line.	Decimal magnitude
Reasoning on the basis of agreed properties	Flexible strategies for addition, subtraction, multiplication and division of decimals.	The student relates their understanding of positional value number knowledge to perform operations with decimals.	Place value Decimal density

Note. Alignment of Mason's (2008) five structures and Microqualities of attention with key decimal fraction concepts, and the researcher's reasoning to connect each structure of attention with the decimal fraction learning progression (ACARA, 2020).

In addition, a 'stepped' visual analysis tool (See Figure 5.6) was designed to help map the thinking progression that each student made when shifting structures of attention to build conceptual understanding of decimal fractions. The layered tiles are annotated with each structure of attention alongside the corresponding key decimal knowledge concepts and skills that contribute to decimal sense. Whilst employing a microgenetic analysis approach, links were made between the actions and sayings demonstrated by each student and a specific decimal concept with the corresponding structure of attention. The changes in decimal conceptual understanding were recorded along the side of the diagram and were sequentially numbered to represent each observable attention shift during the student's interaction with the dynamic digital representation. Annotated versions of this visual analysis tool are used in the findings and discussion section of this thesis.

Figure 5.6

'Stepped' Visual Analysis Tool Diagram



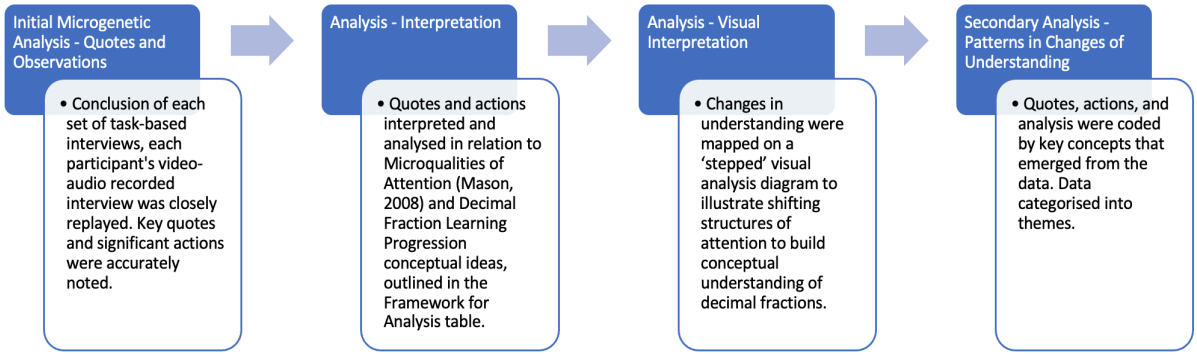
Note. Diagram showing structures of attention with linked decimal fraction concepts.

Following this, individual student's responses and actions were interpreted to determine shifts in what each student was attending to and consequently make strong inferences about how the change had occurred, see Figure 5.7. This interpretation was achieved through a secondary analysis, by searching for similarities and differences across the students' profiles and identifying patterns that emerged. The most prominent moments of conceptual change were identified from the learning progressions that students exhibited when shifting attention from one structure to another. This layer of analysis was completed for four key episodes of

pivotal learning for all six students and examined the capabilities of each dynamic digital tool. The analysis and interpretation were used to draw inferences into how students build knowledge of specific decimal concepts and what facilitates this conceptual learning. Four decimal principles that linked to the conceptual changes and emerged from the students' engagement with the dynamic representation are used as the key findings in the results section.

Figure 5.7

Analysis Process Diagram



Note. Diagram showing four key stages of data analysis.

5.9 Researcher Positionality

As the researcher in this study, I acknowledge that my background as a primary mathematics educator and my sustained interest in digital technologies in mathematics classrooms inevitably shaped the questions pursued, the interpretive lens applied, and the assumptions brought to the field. Consistent with an interpretivist paradigm, I understand knowledge as co-constructed and context-dependent, and recognise that my own perspectives are dimensions of the research process to be critically examined rather than eliminated (Finlay, 2002). Given my familiarity with the digital tools under investigation, I was alert to the risk of confirmation bias during analysis. To manage this, I maintained an open lens throughout data collection and analysis, conducted systematic negative analysis to ensure disconfirming evidence received equal attention, and employed cross-checking where appropriate. Where proximity to a teacher-researcher dynamic was present, I remained mindful of the power differential inherent in researcher-student relationships and the potential for participants to respond in ways shaped by researcher expectations. I acknowledge that such relational dynamics cannot be fully neutralised through procedural protocols alone (Pillow, 2003). This study proceeds from the position that transparency about researcher positionality strengthens rather than undermines the trustworthiness of qualitative inquiry.

5.10 Ethical Considerations

The tasks and interview questions were designed using appropriate instructional language and mathematical content for Year 4 participants. The research project was conducted with permission from the school's principal and with respect for the classroom teacher to minimise disruption to the classroom routine. Written consent was sought from the parent or legal guardian of the students, and was also accompanied with a detailed information statement. Additionally, consent was sought from the young students in the form of written consent with an age-appropriate information statement. The researcher maintained sensitivity to the students' comfort throughout the interview process and assured the students that they could stop at any time. The researcher had a valid Working with Children Check to legally work with the young students. Ethics approval was sought and granted from the University of Sydney's Human Research Ethics Committee prior to commencing data collection.

5.11 Conclusion

This chapter has outlined the methodological framework underpinning the study by detailing the research design, participants, data collection methods and analytical approaches used to investigate Year 4 students' conceptual understanding of decimal fractions through dynamic digital representations. Together, these decisions established a rigorous and purposeful design for exploring the research questions and ensuring the trustworthiness of the findings. The subsequent three chapters present the results of this study. Chapter 6 examines in detail students' conceptual changes across the four digital tools by examining *what* and *how* these conceptual changes occurred. Chapter 7 explores challenges in students' conceptual understandings linked with prior misconceptions and how these can be addressed with dynamic digital representation of decimals. Finally, Chapter 8 closely examines the specific dynamic affordances within each digital tool that students utilised when shifting conceptions about decimal fractions.

CHAPTER 6: CONCEPTUAL CHANGES WHEN LEARNING ABOUT DECIMAL FRACTIONS WITH DYNAMIC DIGITAL REPRESENTATIONS

6.1 Situating the Chapter

The focus of this chapter is on the conceptual changes experienced by the participants during their interactions with the dynamic digital representations across four separate task-based interviews. A microgenetic approach to data analysis occurred with the use of the researcher-designed 'Shifting Attention with Decimal Concepts Framework' to plot each participant's progression of understanding as they worked to achieve the decimal tasks. This focus of data analysis related to the first overarching research question: *How do children experience conceptual changes related to decimal fractions when they attend to features of dynamic digital representations?* Understanding the individual learning pathways that students took when completing the decimal fraction tasks and interacting with each dynamic digital representation, uncovered a number of important changes in conceptual understanding related to decimals as detected through shifts in their attention.

The chapter has been separated into two distinct sections. The first section is in the form of a published paper titled *Changing Structures of Attention When Learning About Decimal Fractions with Digital Tools*. The first half of the paper repeats information provided in the first five chapters of this thesis. The new material begins with the Results section on page 14 of the article. The paper has been embedded in its original, published format.

The second section outlines additional findings and discussion that looks more closely at the relationship between conceptual changes and patterns of shifts in attention by focusing on student responses to one of the digital tools, the *Zoomable Number Line*. Each of the six participants demonstrated changes in conceptual understanding related to decimal density whilst engaging with the sophisticated affordances embedded within this dynamic digital representation. Microgenetic analysis revealed the unique thinking pathways that students took when reacting to the symbolic, visual and dynamic representations of decimals in the *Zoomable Number Line* and then self-connecting these details with knowledge of decimal concepts to develop a deeper understanding.

The chapter concludes with a summary of significant findings and discussion related to the study's overarching research question: *How do children experience conceptual changes related to decimal fractions when they attend to features of dynamic digital representations?*

6.2 Changing Structures of Attention when Learning about Decimal Fractions with Digital Tools

education sciences

MDPI

Article

Changing Structures of Attention When Learning About Decimal Fractions with Digital Tools

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Abstract

Decimals are of great significance in the primary mathematics curriculum due to their application and use in everyday life. The purpose of this study was to investigate the effectiveness of certain dynamic digital representations in developing students' knowledge of decimal fractions. Task-based interviews were used with six Year 4 (9–10 years old) students that incorporated four different dynamic digital representations of decimals. Data collected via video–audio recordings were used to detect shifts in students' attention while using the digital representations. Attention shifts were analysed using microgenetic methods to determine conceptual changes over time. Findings uncovered specific features of the digital representations that generated productive cognitive confusion that prompted changes in students' understanding of decimal fractions. The unique affordances of each digital tool offered students opportunities to dynamically explore decimal concepts, and the digital tools could be used to enrich the teaching of decimals within the primary mathematics classroom.

Keywords: decimal fractions; dynamic digital representations; conceptual understanding; mathematics education

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1. Introduction

Children have difficulty in understanding the abstract nature of decimals, and a plethora of misconceptions arise from incomplete understandings (Steinle & Stacey, 2004). Such complexities arise as students are exposed to rational numbers in numerous symbolic representations, including ratios, fractions, decimals, and percentages, without deep understanding and knowledge of how to use each notation correctly (Bobis, 2011). Furthermore, decimal fractions present additional complications as part of the number-relationship needed to make sense of the numeral, the denominator, is not visible (Steinle & Stacey, 2004). Developing conceptual understanding of *decimal density*, *place value* and *relative magnitude* of decimal fractions is considered key for consolidating number sense, since these properties unify all numbers (Siegler et al., 2011) and rational number magnitude knowledge is related to future mathematics achievement (DeWolf et al., 2016). The current research sought to understand how the use of certain interactive digital tools influenced students' understanding and conceptualisation of decimal fractions. This study involved mapping how their understanding evolved through shifts in attention, what challenges they encountered with the dynamic affordances, and what new insights they gained through their engagement with these digital representations. The new-found understanding of how students develop decimal sense through dynamic digital representations could translate

into advice about effective pedagogy within the primary mathematics classroom. Two research questions communicate the aim of the study:

1. What conceptual changes occur when children attend to features of dynamic digital representations of decimal fractions?

2. How do shifts in attention mediate engagement with these representations?

In this section literature related to the areas of *decimal fractions* and *mathematics representations* is presented. The review outlines existing research into the role of conceptual understanding in the primary mathematics classroom and the importance of decimal fractions in the primary mathematics curriculum. In addition, a summary of the difficulties that students encounter when learning about decimal fractions, including misconceptions that often arise from incomplete understandings, is presented. The second part of the review discusses the role of representations in mathematics learning extending to the field of dynamic digital representations. A description of the affordances and constraints of dynamic representations is included.

1.1. Conceptual Understanding of Decimal Fractions

Decimal fractions stem from the base-ten numerical system. According to Martinie (2014), students encountering difficulties in comprehending place value with the base-ten system often struggle with decimal fraction computations. Additionally, learners may face challenges when failing to recognise decimal fractions as a distinct category of fractions with base-ten denominators (Martinie, 2014). Students need prior knowledge and experience with whole-numbers and common fractions to build conceptual understanding of decimal fractions as decimal notation is compounded in complexity by the merging of these two rational number systems. In the realm of mathematics education, conceptual understanding is defined as "an integrated and functional grasp of mathematical ideas" (National Research Council, 2001, p. 118). This understanding encompasses the familiarity with presenting various scenarios in diverse ways, discern connections between those representations, and employ each representation for distinct purposes (Caldwell, 2020). The development of students' conceptual understanding parallels their progress in mathematics learning.

The growth of students' conceptual understanding is synonymous with mathematics learning. Conceptual understanding develops over time as students organise their knowledge into a coherent, integrated mental structure (Caldwell, 2020). The learning of mathematics occurs through shifts of students' attention which "constitutes changes in understanding" (Mason, 2002, p. 25). A student's understanding evolves not in a linear progression, but through the accumulation of more refined 'microqualities' of attention, leading to a deeper grasp of a mathematical concept (Mason, 2008). Microqualities of attention refer to the specific and rapid ways in which students direct and regulate their focus when engaging with mathematical concepts. These subtle shifts in attention can significantly impact how learners develop a deep conceptual understanding of decimal fractions. The development of more a sophisticated and mature grasp of the concept of decimal numbers occurs through the hurried re-experience of shifts of attention whenever a learner is invited to respond to a probe about decimals (Mason, 2008). Pirie and Kieren's (1991) theoretical work on acquiring mathematical understanding proposes a similar learning trajectory, highlighting that 'folding back' is crucial for the development of mathematical thinking. A growth in mathematical understanding is seen to emerge through the continual movement back and forth through levels of knowing, as students reflect on and reconstruct their current understandings (Pirie & Kieren, 1991). For example, a student grappling with flexible strategies for addition and subtraction of decimals may face a problem that is not immediately solvable. They will need to revisit earlier understandings of decimal concepts and use these to inform their new thinking at the higher learning level, leading to

a deeper understanding of the concept. Clements and Sarama (2004) developed a cognitive model of students' learning that described the processes involved in the construction of conceptual knowledge across several qualitatively distinct structural levels of increasing complexity. Importantly, these levels of structured understanding are not distinct classes of mathematical thinking that suddenly change, rather higher levels are elaborations and advancements of lower levels (Clements & Sarama, 2004).

Currently, there is insufficient research regarding the optimal learning process through which students develop a conceptual understanding of decimals (Tian & Siegler, 2018). Decimals are constrained equivalent fractions, in that the denominators are limited to tenths, hundredths, thousandths, and so forth. *Place value* of decimal fractions involves symmetry around the ones place value. The decimal system extends infinitely in both directions of magnitude, with the relationship between consecutive places directed by multiplication or division by ten. *Decimal density* refers to the continuity property of rational numbers, whereby between any two decimals there are an infinite number of other decimals (Widjaja et al., 2008). *Decimal magnitude* understanding refers to familiarity with estimating and reasoning about the size of a decimal (Resnick et al., 2019). One way in which students can demonstrate their knowledge of decimal magnitude is by determining the location of a decimal on a number line, rationalising about the size of a decimal compared to the benchmark fraction (e.g., $\frac{1}{2}$), and judging the validity of decimal and fraction equivalencies (Resnick et al., 2019).

Understanding decimal fractions conceptually rather than only procedurally is essential, as it lays the groundwork for employing appropriate strategies in mathematical tasks (Way, 2011). Without a robust conceptual grasp of decimals, students are prone to developing misconceptions related to the topic (Steinle & Stacey, 2004). Misconceptions often arise when students inappropriately apply rules to decimal fractions that result in the right answer for the wrong reason, reinforcing the fabricated rule (Hiebert & Wearne, 1985). Four common misconception categories that emerge from the literature include; longer-is-larger thinking, shorter-is-larger thinking, zero-makes-small thinking, and money rule (Moloney & Stacey, 1997; Steinle & Stacey, 2004; Martinie, 2014). Students must be equipped with effective learning tools to address cognitive confusions in their thinking about decimal fractions. Therefore, it is necessary to consider what instructional tools should be implemented in mathematics classrooms to aid students in cultivating accurate decimal fraction sense and preventing the emergence of common misconceptions.

1.2. Mathematics Representations

Representations are widely recognised as valuable tools in mathematics, offering experiential learning opportunities through both concrete and virtual manipulatives. Piaget (1970), Bruner (1986), and Skemp (1987) concur that the development of concepts progresses from physical objects to representational forms and then to abstract thought. According to Kennedy (1986), students' capacity to form mental images and access abstract ideas is dependent on their experiences. By interacting with and manipulating physical objects, students can form stronger mental images and represent abstract concepts more comprehensively (Kennedy, 1986). This aligns with Pirie and Kieren's (1994) early phases of mathematical understanding, where students construct mental images through engaging with tangible representations before progressing to more abstract reasoning. Studies have emphasised the significance of mathematical representations in fostering fractional thinking (Siemon et al., 2011). Students proficient in partitioning areas, sets, and line models, as well as naming fractions in various ways, are more likely to grasp the connection between fractions involving tenths and the base-10 number system (Siemon et al., 2011).

Dynamic representations have proven to be effective aids in student learning within mathematics due to their dynamic affordances. These representations enable students to actively engage with mathematical content by formulating conjectures and promptly testing them, thus allowing them to perceive concepts from various perspectives (Orrill & Polly, 2013). Specifically, learner-centred computer tools that promote interactive engagement with mathematics empower students to 'see' notions that would otherwise be 'unseeable' (Orrill & Polly, 2013). Digital technologies are regarded as "dynamic, manipulable, and interactive representational forms" that both facilitate and are influenced by mathematical thinking and expression (Hoyles & Noss, 2003, p. 326).

When designing learning activities, mathematics educators are frequently restricted to using symbolic representations that isolate a single mathematical concept. A holistic mathematics education that enhances mathematical thinking involves searching for meaning beyond symbol sense. Santos-Trigo (2006) suggests that the integration of technology in a mathematics classroom has the potential to develop a disposition, resources and problem-solving strategies associated with this extended way of thinking. Technology can generate a dynamic representation of a problem in which students can pose and pursue questions that might lead them to identify and examine connections and extensions of that problem (Santos-Trigo, 2006). In this process, it is argued that representing the problem dynamically might offer students the opportunity to be engaged in a line of thinking in which they constantly need to search for mathematical relationships and utilise the knowledge they possess to explore and support the pertinence and validity of those relationships (Roschelle et al., 2000). The integration of dynamic digital representations in mathematics education has emerged as a transformative approach to supporting student learning and conceptual understanding. These representations constitute a crucial component of cultural infrastructure, providing systematic methods for presenting mathematical thoughts, creating communicative records across contexts, and supporting mathematical reasoning and computation (Laborde & Laborde, 2014). This examination of dynamic digital representations reveals their benefits to mathematics learning through four key dimensions: epistemological considerations in design, cognitive transformations in student thinking, progressive learning through interaction modalities and pedagogical reimagining of mathematical tasks (Ruthven, 2022).

Integral to this research into dynamic representations is investigating the affordances and constraints of these digital learning tools. Recent research has explored the potential benefits of touchscreen digital tools, and this body of research suggests that apps for virtual manipulatives offer various advantages that may impact learning, which children utilise in diverse ways (Moyer-Packenham & Westenskow, 2016; Tucker & Moyer-Packenham, 2014; Tucker et al., 2016). The rapid advancement of technology in mathematics education presents new challenges for educators, as it involves "using new kinds of mathematical tasks and modifying the nature of mathematical activities in classroom based on a set of pedagogical principles" (Shahmohammadi, 2019, p. 1834). This critical perspective demonstrates the importance of closely examining how technology influences and enhances students' learning and conceptual understanding of mathematics. Specifically, teachers must comprehend how different instructional tools can aid students in improving their understanding of mathematical concepts through various cognitive processes.

2. Theoretical Frameworks

The theoretical perspectives of constructivism, representations in mathematics and 'shifts of attention' yield a theoretical framework useful for investigating students' interactions with dynamic digital representations of decimal fractions. In particular, a learning environment incorporating rich problem-solving tasks is suited to exploring students' con-

ceptual understanding of decimal fractions through noticing their shifts in attention during the process of task completion (Voutsina et al., 2019).

2.1. Constructivist Theory

The theoretical framework employed in this study is underpinned by the work of Piaget (1970), who was seminal in using task-based interviews to elucidate children's thought processes regarding posed problems and advocated for the belief that "true understanding takes place when the student makes discoveries for themselves" (Assad, 2015, p. 17). Constructivism, a prominent theoretical perspective in mathematics research, has been extensively explored by several key scholars (Cobb et al., 1992; von Glasersfeld, 1995; Katic et al., 2009). A fundamental tenet of Piaget's constructivist theory posits that students construct their understanding and knowledge through perceptions and experiences, which are in turn "mediated through previous knowledge" (Simon, 1995, p. 115). Consequently, this study acknowledged students' pre-existing knowledge of decimal fractions and anticipates that individual students will interact with digital tools in varying ways. Constructivist mathematics teaching with digital tools emphasises the interplay between mathematical content knowledge, constructivist pedagogies, and technology to enhance conceptual understanding (Mishra & Koehler, 2006). This study positions digital tools to enhance constructivist mathematics learning by offering interactive, visual, and collaborative experiences.

2.2. Representation Theory

In the mathematics classroom, representations serve as tools to assist students in grasping abstract concepts that are fundamental to mathematical learning. Specifically, within the realm of mathematics, representation is defined as "the act of externalising an internal, mental abstraction" (Pape & Tchoshanov, 2001, p. 119). Hiebert and Carpenter (1992) make a distinction between internal and external representations, where internal knowledge representations are the cognitive structures of knowledge in a learner's mind, and external representations often assume the forms of spoken language, pictures, written symbols, and manipulative models (Lesh et al., 1987). Mathematics educators have long perceived external representations as an effective means of making mathematical ideas understandable for students (Hiebert & Carpenter, 1992). G. A. Goldin (2000, 2003) maintains that external representations should play a fundamental role in empirical investigations of students' reasoning and understanding of mathematical concepts, such as during task-based interviews. Enhancing students' mathematical understanding and their ability to think representationally necessitates the flexible use of multiple representations, along with the interaction between external and internal representations (Pape & Tchoshanov, 2001). Effectively translating concepts across different representational modes plays a critical role in the acquisition of mathematical ideas, and engaging with multiple representations can facilitate this process (Lesh et al., 1987; Duval, 2006).

2.3. Shifts of Attention Theory

The theory of shifts of attention serves as a framework for researchers to comprehend students' mathematical thinking, offering five observable ways of attending that indicate both the content and manner in which students learn in the mathematics classroom (Mason, 2008). Mason's (2008) conceptualisation of attention serves to explain students' mathematical learning processes. Throughout the development of mathematical thinking, various shifts of attention may occur, relating to the object or focus of attention (what the learner attends to) and the form or structure of attention (how the learner attends) (Mason, 2008). The structure of attention encompasses activities such as "holding wholes, discerning details, recognising relationships, perceiving properties, and reasoning based on agreed properties"

(Mason, 2008, p. 35). As attention shifts from one focus to another and/or from one form to another, its structure alters which enables the learner to establish connections, perceive relationships, and appreciate new ideas. *Holding wholes* involves observing a mathematical object with or without focusing on its mathematical characteristics, typically for a brief period before transitioning to *discerning details*, which involves marking distinctions and recognising significant features. *Recognising relationships* occurs when the learner identifies specific relations between elements of a mathematical concept, such as a model, calculation, diagram, or equation. Shifting attention from *recognising relationships* to *perceiving properties* of specific elements is crucial for the development of mathematical understanding. *Perceiving properties* necessitates the learner's "awareness of a possible relationship and look for elements to fit it, thus leading to generalisation" (Mason & Johnston-Wilder, 2004, p. 60). Lastly, *reasoning based on agreed properties* involves formal mathematical reasoning related to previously identified properties, "independent of particular objects, properties, as the only basis for further reasoning" (Mason & Johnston-Wilder, 2004; Mason, 2008, p. 37).

The theory of shifts of attention has been applied in studying progressions of mathematical problem-solving (Voutsina et al., 2019). It can also be used to explain changes related to shifts in the object and structure of learners' attention regarding any mathematical concept, including decimal fractions. As students engage with mathematical tasks and representations, there are continual changes in the focus and structure of their attention (Attard et al., 2016). For a learner, this shift occurs while detecting apparent motion from saccadic eye movement, which can suddenly switch to an awareness of details amidst previously undiscerned information (Wei et al., 2023). Mathematics learning evolves through these shifts in attention, coupled with students' increasing awareness of them. Voutsina et al. (2019) explored dynamic changes in how problem-solving strategies were articulated and conceptualised when students shifted the structure of their attention in response to shifts in the mathematical object being focused on. The findings revealed microqualities that influence the structure of attention among different students during various phases of mathematics learning, offering valuable insights into the qualitative dynamics of change in learning (Voutsina et al., 2019).

2.4. The Relationship Between the Theories

Mason (2010) perceives learning as the dynamic shifting of attention in both form and focus, encompassing what is being attended to and how it is being attended to by the learner. From his research emerges a compelling assertion that mere engagement with a mathematical task is insufficient for effective learning. Instead, there is a necessity for the integration of experience and sense-making, requiring a transformation of attention. This transformation entails shifting the learner's attention from complete immersion in the task to a broader perspective that internalises the new concept or approach (Mason, 2010). Dynamic representations of decimal fractions have the potential to guide learners' shifts of attention. The dynamic nature of digital manipulatives induces shifts in the object of students' attention as they actively interact with the mathematics content, as well as in the structure of their attention concerning how they perceive features of the mathematics representation and the embedded mathematical concepts (refer to Figure 1). Understanding these shifts in what and how learners attend to features of decimal fractions is crucial for advancing researchers' comprehension of the dynamics of change in learning.

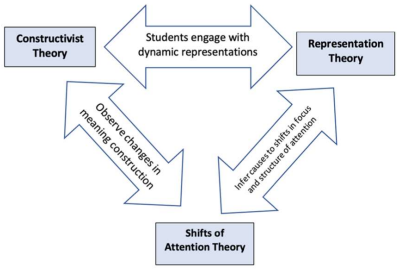


Figure 1. Diagram of relationship between three theoretical perspectives: constructivism, representations in mathematics, and 'shifts of attention'.

3. Materials and Methods

This study employed a qualitative methodology, using task-based interviews in conjunction with microgenetic methods for data collection and analysis (Chinn & Sherin, 2014). Task-based interviews, when well designed and effectively implemented, offer insights into how students construct knowledge, their cognitive processes, and their interpretations of learning situations and tasks (H. Ginsburg, 1997). Microgenetic methods facilitate detailed analysis of students' reasoning, particularly in research designs incorporating task-based interviews (Chinn & Sherin, 2014; Siegler, 2006). The method of investigating moment-to-moment processes of learning is important for this study as it connects 'how' the students are engaging with the dynamic digital representations with 'what' decimal concepts are being attended to by the students. In doing so, identification of specific affordances that prompt changes in conceptual understanding can occur.

3.1. Task-Based Interviews Overview

Task-based interviews serve the dual purpose of observing mathematical behaviour and drawing inferences about the problem solver's cognition and/or affect from the observations (G. Goldin, 1993). Consequently, tasks must be sufficiently open-ended to allow each student to respond in their preferred manner and designed to encourage dialogue between the researcher and student (Hunting, 1997). The primary aim of the interview was neither to teach nor to test knowledge, but rather to explore the student's thought processes and interactions with mathematical representations (Gorman & Way, 2018).

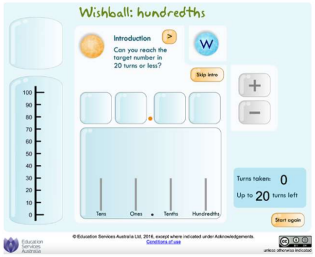
3.2. Task Design

The dynamic digital representations incorporated in this study were selected due to their compatibility with the aim of the research questions. Four virtual manipulatives were chosen from a larger list of seventeen web-based apps after exploring and analysing the affordances and constraints of each potential digital tool. The process of selecting the dynamic digital representations to be included in the study involved a systematic evaluation based on predefined criteria, including mathematical accuracy linked to decimal density, decimal magnitude and/or decimal place value, age-appropriate interactivity, pedagogical effectiveness, and alignment with the study objective. Each tool was assessed for its potential to provide meaningful visualisations, foster conceptual understanding, and support student engagement. The final selection prioritised tools that included a range of linear, area, and symbolic representations of decimal place value, density and magnitude.

The number-line-based, place-value symbolic, and area-model dynamic digital representations of decimal fractions used were in the form of applets, which were embedded into a webpage and displayed using a web browser (Moyer et al., 2002). Retaining the well-documented affordances of manipulatives that have been used for decades by teachers

and students in classrooms, the four dynamic digital representations; *Wishball-hundredths*, *Decimal Strips*, *Zooming in on Place Value*, and *Zoomable Number Line*, offered additional advantages and affordances to their static counterparts. An in-depth list of affordances and constraints for each digital tool is outlined below in Figures 2–5.

Wishball-hundredths



Wishball-hundredths is a learning tool with a specified in-built aim where students try to reach a target decimal fraction by adding or subtracting in under 20 moves. The spinner generates a random number, and users assign it to a place value to then either add or subtract to the base number. Text prompts guide understanding of decimal place value up to tenths and hundredths. A number line and counting frame show each move dynamically. To commence this task-based interview, students read the on-screen instructions, then engaged with the Wishball-hundredths app for five rounds, ensuring both familiarity and adequate interactive time without losing engagement.

Affordances

1. Multiple representations of decimals simultaneously displayed on one screen.
2. Focus on symbolic representation with on-screen visuals.
3. Interactive digital tool to understand place value of decimals.
4. Dynamic counting frame with moving abacus beads that responds to user input.
5. Dynamic vertical number line that responds to user input.
6. Structured learning environment with specific aim.
7. Prompts to guide learning of place value of tenths, hundredths and thousandths.
8. Feedback given with text if user does not reach the target number in less than 20 turns.
9. Represents decimals beyond 1-whole.

Constraints

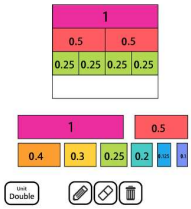
1. No feedback given by digital tool. Risk of consolidating or perpetuating misconceptions and inaccurate understanding of decimals without input from a more knowledgeable other (teacher input).
2. Only focus on concept of place value with decimals.
3. User could hold whole-number thinking and successfully achieve the digital tool's aim.

Figure 2. Screenshot of the Wishball virtual manipulative and description of tasks (https://www.scootle.edu.au/ec/viewing/L869/index.html) (accessed on 30 May 2023).

The task-based interviews began with a base-line task involving a conversation about decimal fractions where the student was asked to identify and explain the concept of a decimal fraction. They then located a sequence of one and two decimal place numbers on a static paper-based number line. In doing so, the researcher was able to compare the data collected from the task-based interviews with the results from the introductory activity and therefore form some conclusions in relation to the students' prior knowledge. Following this, four separate task-based interviews were conducted, over a four-week period, where the students interacted with a different dynamic digital representation during each session. For two of the task-based interviews, a set of tasks were designed with increasing levels of challenge, to encourage the students to interact conceptually with decimal notation, number line positioning and area models. Probing questions were used by the researcher to encourage further explanation, such as "Can you tell me what you are thinking?", while maintaining neutrality regarding the correctness of responses (Gorman & Way, 2018).

The set of figures (Figures 2–5) presents an overview of the four digital tools and the related tasks used in this research study.

Decimal Strips



Affordances

1. Visually represents decimal relationships and concept of magnitude.
2. Drag and drop bars to represent different decimal values onto the canvas.
3. Link symbolic, visual and dynamic representations by providing mathematical notations that label the on-screen decimal bars.
4. Exploratory digital tool to understand relative size.

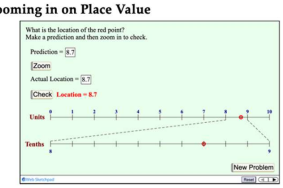
The second task-based interview involved students using the dynamic digital representation Decimal Strips to complete a number of tasks posed by the researcher. Decimal Strips allow users to explore the relative magnitude of decimal fractions by clicking and dragging coloured tiles representing different decimal values (1-whole, 0.5, 0.4, 0.3, 0.25, 0.2, 0.125, 0.1) onto a decimal fraction wall. The tasks included ordering decimal fractions in ascending and descending order, comparing decimal numbers to identify the larger one, and matching decimal fractions with their common fraction representations.

Constraints

1. No feedback given by digital tool. Risk of consolidating or perpetuating misconceptions and inaccurate understanding of decimals without input from a more knowledgeable other (teacher input).
2. Unable to represent decimal numbers beyond 1-whole.
3. No suggested learning activities or integrated tasks in digital tool. Teacher must facilitate purposeful interaction.

Figure 3. Screenshot of the Decimal Strips virtual manipulative and description of tasks (https://toytheater.com/decimal-strips/) (accessed on 6 June 2023).

Zooming in on Place Value



Affordances

1. Visually and physically represents density of decimals as it reveals tenths and hundredths but only with integer marks not numerical symbols.
2. Question prompts to guide learning of place value of tenths, hundredths and thousandths.
3. Illustrate concepts of place value, and decimal density so user can visualise in a fluid and dynamic way, that cannot effectively be represented with static tools.
4. Feedback given with chance to correct answer after 'zoom in' shows closer view of tenths between two whole-numbers.
5. Can manipulate numbers dynamically, receive immediate feedback, and explore concepts through trial and error, enhancing engagement and deeper learning.
6. Users are engaged in the problem-solving process.
7. Potential to address cognitive conflict related to decimal density and place value.

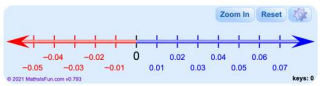
In the third task-based interview, students used the interactive digital tool Zooming in on Place Value to place decimal fractions on a number line. The in-built task is to determine the location of a decimal fraction and self-check the answer using an embedded Zoom function. After reading aloud the question and instructions, students interacted with the dynamic model showing a point on a number line. They estimated its location, then used the Zoom button to expand the interval between whole numbers, revealing a magnified view divided into tenths. Finally, students named the precise location of the point using the tenths integer marks. Each student engaged with the interactive Zooming in on Place Value app for ten rounds.

Constraints

1. Does not address concept of decimal magnitude as the tool only facilitates interaction with tenths, hundredths, or thousandths, one place value at a time.
2. Limited ability to generate cognitive conflict which could lead to overcoming misconceptions held by user.

Figure 4. Screenshot of the Zooming in on Place Value virtual manipulative and description of tasks (http://www.sineofthetimes.org/zooming-in-on-place-value/) (accessed on 13 June 2023).

Zoomable Number Line



In the fourth task-based interview, students engaged with the Zoomable Number Line virtual manipulative. This digital tool represents decimal fractions on a number line dynamically through a Zoom In function. The interview consisted of five tasks of increasing difficulty, designed to encourage conceptual interaction with decimal notation and number line positioning. Tasks included locating one and two decimal place numbers on the digital number line, comparing pairs of decimal numbers, identifying the next hundredths decimal after a given fraction, and solving a multi-step problem.

Affordances

1. Visually and physically represents density of decimals as it reveals tenths and hundredths.
2. Suggested learning activities to aide interaction with zoomable number line.
3. Consolidate understanding of place value.
4. Ordering and comparing decimals.
5. Illustrate concepts of place value, and decimal density so user can visualise in a fluid and dynamic way, that cannot effectively be represented with static tools.
6. Dynamic number line allows for zooming, precise adjustments, and an infinite number of representations without physical constraints of static number lines.
7. User can progress at their own pace, reset or adjust activities, and explore concepts in a way that suits their learning needs.
8. Potential to generate and address cognitive conflict related to decimal density and place value.

Constraints

1. No feedback given by digital tool. Risk of consolidating or perpetuating misconceptions and inaccurate understanding of decimals without input from a more knowledgeable other (teacher input).

Figure 5. Screenshot of the Zoomable Number Line virtual manipulative and description of task (https://www.mathsisfun.com/numbers/number-line-zoom.html) (accessed on 20 June 2023).

3.3. Participants and Interview Procedure

The participants in this study consisted of Year 4 students (9–10 years old), a grade level chosen because the New South Wales (NSW) state syllabus introduces decimal notation for tenths and hundredths at this point (NESA, 2012). Typically, students in this grade are still in the early stages of understanding of rational numbers in the context of decimal fractions which was true for the participants in this study. Individual participants held a different level of understanding that is influenced by their acquisition of the anticipated prerequisite knowledge related to decimals from Year 3. The NSW Mathematics Syllabus positions Year 3 students to consolidate conceptual knowledge of fractions and their relationship to decimals, as well as explore decimal fractions through practical activities and contexts such as money or measurements.

Six participants were selected from a single Year 4 classroom based on their decimal comparison pre-test results. One criterion for participation was the students' ability to articulate their mathematical thinking and reasoning verbally. The class teacher confirmed that the selected children would feel comfortable engaging in one-on-one discussions with the researcher. Although the participants were not intended to be representative of the entire class, the researcher chose six students encompassing a range of mathematical achievement levels to account for potential differences in how low and high achievers interacted with the dynamic digital representations. Due to the time-consuming nature of data collection and analysis required in microgentic designs of research, generally a small number of participants are selected to provide the rich data sets (diSessa, 2017). Choosing six children was deemed sufficient to capture a variety of student responses (Marton, 1986).

3.4. Procedure and Data Collection

Data collection for this study consisted of two key phases:

1. An initial decimal comparison pre-test;
2. Four individual task-based interviews each focussing on a different dynamic digital representation.

3.4.1. Pre-Test

To initiate the data collection phase, the lead researcher designed and administered a pre-test to all students in a Year 4 class. This pre-test tasked students with comparing pairs of decimal fractions and determining the larger decimal, such as 0.4 and 0.457. These pairs of decimal fractions were sourced from Stacey and Steinle's (2003) Decimal Comparison Test, a tool utilised for categorising students' understanding of decimal notation. The research-based test is accompanied by marking instructions and an analysis tool to allocate a code to each question that indicates the students' thinking with a classification of either *Longer-is-larger*, *Shorter-is-larger*, or *Apparent-expert*. Using the results from the Decimal Comparison Test six participants were nominated that spanned a range of results and appeared to hold different mathematical knowledge within the area of decimal fractions.

3.4.2. Task-Based Interviews

The four dynamic digital representations used were *Wishball-hundredths*, *Decimal Strips*, *Zooming in on Place Value*, and *Zoomable Number Line*. The researcher worked individually with each participant in four 30 min sessions which proved to be sufficient time to collect quality data and simultaneously brief enough to maintain age-appropriate student concentration levels. In the task-based interviews, the role of the interviewer was to facilitate the student's engagement with the mathematical tasks while maintaining a neutral stance to ensure authentic insight into their decimal thinking. The interviewer guided the tasks by prompting reflection and encouraging explanation, rather than providing evaluative feedback or hints toward correct answers. Neutrality was maintained through the use of open-ended questions including "Can you tell me more about your thinking?" and non-judgmental verbal and facial responses. This approach minimised interviewer influence and allowed the student's reasoning, misconceptions and conceptual changes to emerge naturally through interaction with the tasks.

3.5. Microgenetic Analysis

The data collected from the task-based interviews were analysed using a microgenetic approach. The correlation between one-on-one task-based interviews and the principles of the microgenetic method is evident when gathering intensive data from learners over a period of time to capture a record of moment-by-moment learning processes (Chinn, 2006). Microgenetic methods of analysis involve closely examining what learners do and say over several task-based interviews, thus allowing researchers to draw conclusions about what prompts knowledge change, how learning occurs, and other aspects of the processes of acquiring conceptual understanding (Voutsina et al., 2019).

Data analysis began with repeated viewings of the video–audio recordings at the conclusion of each task-based interview where the researcher deduced new understandings that the participants had gathered. Particular attention was given to the manipulation of each digital tool, the mathematics concepts demonstrated, and the relationship between the two. These significant sayings and doings were accurately documented in chronological order. As qualitative microgenetic studies focus on tracing and documenting points of change in learning through analysis of each individual's verbal and physical behaviours, a high density of observations was required. This entailed multiple viewings of the data

from task-based interviews to explore "development of individual students' knowledge, reasoning, problem solving and understanding of mathematics concepts through a qualitative lens" (H. P. Ginsburg, 2009, p. 113). A framework for analysis was designed by the researcher as a reference tool when analysing the verbal reasoning and manipulative actions the participants made. The 'Shifting Attention with Decimal Concepts Framework' (see Table 1) correlated Mason's five components of structures for attending with key decimal fraction concepts that form a sound conceptual understanding as outlined in the ACARA National Numeracy Learning Progression. This analysis tool was used alongside a microgenetic approach to data analysis to discern observable shifts in how and what the participants were attending to as they engaged with the four dynamic digital representations of decimal fractions. The coding process for the microgenetic analysis was guided by the 'Shifting Attention with Decimal Concepts Framework', which provided a structured lens for identifying moments of conceptual change as students engaged with each dynamic digital representation. Each video-recorded task-based interview was transcribed and segmented into fine-grained episodes, which were then coded to capture shifts in the students' focus, strategies and reasoning about decimal concepts. To enhance reliability beyond expert review, a subset of transcripts was independently coded by a second researcher to ensure consistency in the application of the framework. Discrepancies were discussed and resolved collaboratively, leading to refinements in the coding scheme. Validity was further strengthened through triangulation across multiple data sources, including screen recordings, student verbalisations and interviewer observation notes. This allowed for a more comprehensive interpretation of the students' evolving conceptual understandings.

Table 1. Shifting Attention with Decimal Concepts Framework.

Structure and Microqualities of Attention (Mason, 2008)	**Decimal Fraction Learning Progression (ACARA, 2020)**	**Connection of Theory of Structures of Attention and Decimal Fraction Learning Progression Points (Researcher's Own Reasoning)**	**Key Concept(s) of Decimal Fractions in which Conceptual Change Has Occurred (Researcher's Own Reasoning)**
Holding wholes	Naming decimal fractions.	The student verbally names decimal fractions without an understanding of place value.	Place value
Discerning details	State the place value of digits in decimal numbers of up to two decimal places. Use place value to partition decimals of up to two decimal places.	The student can discern mathematical details of decimal fractions with an understanding of place value and reforming decimal numbers with value.	Place value Decimal magnitude
Recognising relationships	Understand what a decimal fraction is and recognise that they arise out of division. Model, compare and represent decimals of up to two decimal places. Understanding and estimating relative size of decimals.	The student understands the relationship between decimal fractions and the division operation, as the decimal is parts of a unit resulting from dividing the whole into equal parts. Furthermore, the student understands the positional value of decimals as well as the scaling effect as digits move to the right and left in place value, for example, 3.509 has 35.09 tenths, 350.9 hundredths, 3509 thousandths.	Place value Decimal density

Table 1. *Cont.*

Structure and Microqualities of Attention (Mason, 2008)	Decimal Fraction Learning Progression (ACARA, 2020)	Connection of Theory of Structures of Attention and Decimal Fraction Learning Progression Points (Researcher's Own Reasoning)	Key Concept(s) of Decimal Fractions in which Conceptual Change Has Occurred (Researcher's Own Reasoning)
Perceiving properties	Place decimals of up to two decimal places on a number line. Apply knowledge of decimals to record measurements, for example, 123 cm = 1.23 m.	The student perceives properties of decimal fractions as they demonstrate knowledge of the relative size of decimals when comparing and applies this knowledge to the relationship with measurement and position when plotting values on a number line.	Decimal magnitude
Reasoning on the basis of agreed properties	Flexible strategies for addition, subtraction, multiplication and division of decimals.	The student relates their understanding of positional value number knowledge to perform operations with decimals.	Place value Decimal density

Alignment of Mason's (2008) five structures and microqualities of attention with key decimal fraction concepts, and the researcher's reasoning to connect each structure of attention with the decimal fraction learning progression (ACARA, 2020).

In addition, a 'stepped' visual analysis tool (See Figure 6) was designed to help map the thinking progression that each student made when shifting structures of attention to build conceptual understanding of decimal fractions. The layered tiles are annotated with each structure of attention alongside the corresponding key decimal knowledge concepts and skills that contribute to decimal sense. Whilst employing a microgenetic analysis approach, links were made between the actions and sayings demonstrated by each student and a specific decimal concept with the corresponding structure of attention. In addition, considerations were made into how the students' attention flittered between any particular stage of the diagram. For example, attention shifts that occurred between discerning details (what is being referred to), recognising relationships (between discerned details), and perceiving properties (as being instantiated). The changes in decimal conceptual understanding were recorded along the side of the diagram and were sequentially numbered to represent each observable attention shift during the student's interaction with the dynamic digital representation. Annotated versions of this visual analysis tool are used in the findings and discussion section of this paper.

Following this, individual student's responses and actions were interpreted to determine shifts in what each student was attending to and consequently make inferences about how the change had occurred. This interpretation was achieved through a secondary analysis as detailed in Figure 7, by searching for similarities and differences across the students' profiles and identifying patterns that emerged. The most prominent moments of conceptual change were identified from the learning progressions that students exhibited when shifting attention from one structure to another. This layer of analysis was completed for four key episodes of pivotal learning for all six students and were examined for evidence of the role played by each dynamic tool. The analysis and interpretation were used to draw inferences into how students build knowledge of specific decimal concepts and what facilitates this conceptual learning. Four decimal principles that linked to the conceptual changes and emerged from the students' engagement with the dynamic representation are used as the key findings in Section 4 Results.

To enhance validity, the primary researcher's analysis and interpretation of results were cross-referenced with two additional experts in mathematics education research. One expert served as the researcher's tertiary supervisor, while the other was both a published

mathematics education researcher and an advisor for the NSW Department of Education. By involving these external reviewers in assessing the audio–video recordings of task-based interviews and the researcher's written analysis, credibility was established in the interpretation of the qualitative data.

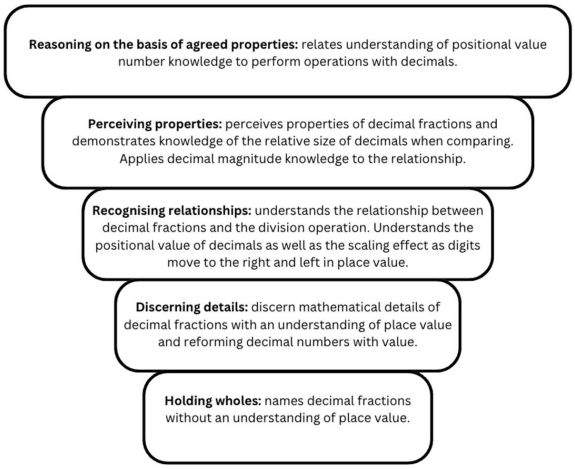


Figure 6. 'Stepped' Visual Analysis Tool Diagram.

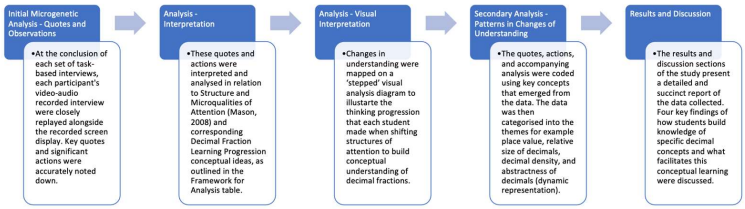


Figure 7. Diagram showing five key stages of data analysis.

4. Results

The findings outlined in this paper present a synthesis that emerged from detailed analysis during the concentrated interpretation of students' verbal reasoning and manipulative actions whilst engaging with four digital tools. An analysis of the specific affordances within each dynamic digital tool that enhanced students' conceptual understanding of decimal fractions is presented in Table 2. This table aims to present the mathematical concept or process that each digital tool supported. Discerning similar pathways that students followed when shifting attention from one structure to another allowed inferences to be drawn about how students build knowledge of specific decimal concepts and what facilitates this conceptual learning. Four key findings emerged from the data analysis that detail the formation of four distinct decimal principles and skills related to *decimal place value*, *relative magnitude* and *decimal density*. These findings are used as sub-headings in this section alongside student evidence and examples to illustrate each conceptual finding. The excerpts were chosen because these students' thinking patterns and verbal reasoning most clearly demonstrate the conceptual changes that occurred in their understanding of decimals. When an evidence snapshot from a task-based interview is described, it is interpretated using the 'Shifting Attention with Decimal Concepts Framework' to deduce the unique dynamic digital affordances that prompted changes in conceptual understanding related to decimals.

Table 2. Summary of findings highlighting key affordances used by students to prompt changes in decimal fraction conceptual understanding.

Dynamic Digital Representation	Affordances Utilised	Key Changes in Decimal Conceptual Understanding
Wishball-hundredths	Dynamic counting frame with moving abacus beads that responded to student input. Dynamic vertical number line that reflected student input. Multiple representations of decimals simultaneously displayed on one screen. Symbolic representation of tenths and hundredths decimal place value. On-screen prompts to guide learning of place value of tenths, and hundredths. Feedback given with text if student did not reach the target number in less than 20 turns.	The fluid movement between decimal place values on the dynamic counting frame modelled the reforming of decimal fractions which all students tuned into when they acknowledged that value could be shifted across place values to perform operations and reform the decimal number. The students utilised the vertical number line representation to assist with gauging the relative size of decimals, particularly the scaling effects as digits move to the right and left in place value. Students discerned how the dynamic counting frame featured a visible display of this addition operation principle, in-built constraint on column overflow.
Decimal Strips	Drag and drop feature helped dynamically represent decimal relationships and concept of magnitude. Links symbolic, visual and dynamic representations by providing mathematical notations with labelled on-screen decimal strips.	Exploratory digital tool assisted students with understanding relative size of decimals through interactive feature of combining and separating decimal place value strips on a multi-row decimal fraction wall.
Zooming in on Place Value	Visually and physically represents density of decimals as the number line zooms-in to reveals tenths and hundredths but only with integer marks not numerical symbols. Feedback given with chance to correct answer after 'zoom in' shows closer view of tenths between two whole-numbers. Manipulated numbers dynamically, received immediate feedback, and explored concepts through trial and error, enhancing engagement and deeper learning.	The virtual manipulative illustrates concepts of decimal place value and decimal density. The students came to understand these concepts in a fluid and dynamic way as the tool's affordances helped them visualise decimal fractions, that cannot effectively be represented with static tools. Potential to address students' cognitive conflict related to decimal density and place value.
Zoomable Number Line	Visually and physically represented density of decimals as it revealed tenths, hundredths, thousandths and beyond. Dynamic number line allowed for zooming, precise adjustments, and an infinite number of representations without physical constraints of static number lines. Scrolling back and forth along the dynamic number line assisted with comparing decimal place values.	Supports students in extending their understanding of decimal place value to tenths, hundredths, thousandths, and beyond. By utilising the zoom-in and zoom-out functions, all students visually explored how the decimal number system is structured around powers of ten. Unlike physical manipulatives, which may be limited to representing only a few decimal places, this digital number line provides a dynamic way for students to see that decimals can always be divided further. Additionally, continuous zooming reinforces the concept of decimal density which resulted in some students making connections to the idea of infinity by recognising that numbers can always be divided into smaller parts without limit, just as they can extend infinitely in the other direction. Potential to generate and address cognitive conflict related to decimal density and place value.

4.1. Finding One: Students Demonstrated a Change in Understanding Related to Decimal Density Through Noticing the Dynamic Display of Continuous Decimal Place Values on the Zoomable Number Line

The continuous property of numbers refers to the fact that between any two distinct numbers, there is an infinite quantity of other numbers. Understanding that between 0 and 1, there are infinitely many decimals, and the density of decimals is high, requires knowledge of the continuous property of numbers. This concept is highly abstract, and some learners experience cognitive conflict when they are first exposed to the idea as it opposes their whole-number knowledge.

During the Zoomable Number Line interview five out of the six students demonstrated a change in conceptual understanding related to decimal density directly linked to the research question *What conceptual changes occur when children attend to features of dynamic digital representations of decimal fractions?* These five students shifted attention from a starting place of either *holding wholes* or *discerning details* without knowledge of the relationship between decimals and the division operation, which is a precursor to understanding decimal density, to a more accurate understanding of this specific concept. They connected the details discerned in the dynamic zoomable number line with their decimal place value knowledge to uncover the continuous property of decimal numbers. The individual learning journeys that the students took to reach this new understanding had an interesting commonality whereby all students cycled through at least two-rounds of hypothesising when attempting to explain why the *Zoomable Number Line* continued to reveal an increasing number of decimal place values. This self-questioning was evidence of cognitive confusion as the students attempted to *recognise relationships* between the positional value of decimals and the continued division of place values that resulted in another decimal digit being displayed on the dynamic number line, a *discerned detail*. Annelise was one such student who completed two cycles of shifting attentions between *discerning details* and *recognising relationships* before she could articulate the continuous property of decimals and therefore describe decimal density. The evidence snapshot below details the interaction between Annelise and the interviewer that exposed a change in conceptual understanding related to decimal density. In addition, the 'structures of attention' diagram (see Figure 8) maps the observed learning progression Annelise moved through to gain familiarity and facility with the shifts of attention necessary in order to have *recognised relationships* readily available for further thinking.



Photo showing Annelise continually zooming-in to reveal more and more decimal place values on the zoomable number line.

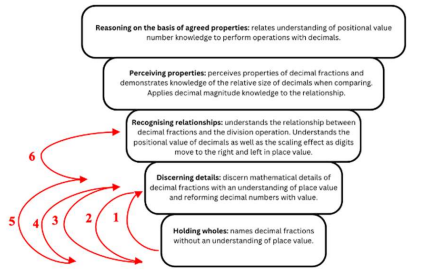


Diagram showing Annelise's thinking pathway back-and-forth between the structures of attention with linked decimal fraction concepts.

Figure 8. Annotated images of Annelise's interaction with the zoomable number line including a diagram showing Annelise's shifts of attention in sequential order.

Annelise was freely investigating the zoomable number line app by continuously zooming in which enabled her to recognise a relationship of never-ending betweenness

which is labelled 'density'. She was interested in discovering how many place values could be revealed in a decimal when she predicted a maximum and then tested her hypothesis. The accompanying verbal exchange is detailed below:

> Annelise: [Held down zoom-in function to continually reveal more and more decimal place values] Oh wow. . . there are so many numbers.
>
> Interviewer: How many numbers are after the decimal point?
>
> Annelise: Six.
>
> Interviewer: What do you think is the maximum number of decimals there are after a decimal point?
>
> Annelise: Maybe 10 [held down zoom-in function to continually reveal more and more decimal place values] I just counted 12 numbers after. . .maybe it stops at 20" [held down zoom-in function to continually reveal more and more decimal place values] "I don't think it will stop.
>
> Interviewer: Why don't you think it will stop.
>
> Annelise: Because there are infinite numbers.

At the commencement of the interview, Annelise displayed an incomplete understanding of the abstract concept of decimal density. In the example recounted above, Annelise was learning to extend the place value to tenths, hundredths, thousandths and beyond, supported by using the zoomable number line to recognise how the numbers increase and decrease. The zoom-in and zoom-out functions show how the decimal number system is based on powers of ten and helped build Annelise's knowledge of decimal density. Concrete manipulatives may only go as far as representing thousandths, however the digital zoomable number line with its dynamic affordance allowed Annelise to notice that decimals can always be divided into ten smaller pieces. Furthermore, Annelise appeared to connect what she witnessed in the continually zooming-in number line with her knowledge of the concept of infinity in that, the numbers continue to get smaller without ending, just as they can continue to become larger without ending. The analysis suggests that by exploring a powerful digital learning tool, Annelise could reason about her observations which ultimately helped her determine her own methods to explain the size and value of decimal numbers and therefore discover why the decimal place values continue forever.

One student's attention remained within the attention structure of *discerning details* when freely investigating the *Zoomable Number Line*. Yehali was questioned as to the maximum number of digits that come after the decimal point. Her response was "maybe seven. . ." before she held down the zoom-in function on digital tool which continually revealed more and more decimal place values. Yehali commented "I counted 9 [decimal places] . . .maybe it will go up to the 10" She again held down the zoom-in function and revealed additional decimal place values, her final explanation was "there's 14 now, it probably goes up and up." When asked to explain why it "goes up and up", Yehali was unable to produce a mathematical reason for the continuous property of decimals. At the commencement of this interview Yehali exhibited thinking which indicated that she was treating decimals as other whole numbers to the right of the decimal point. This was a result of her viewing the decimal point as separating two individual whole numbers. However, Yehali's attention shifted from *holding wholes* to *discerning details* when she witnessed the *Zoomable Number Line* dynamically expand to display ten-tenths between two whole numbers. A limitation with conceptual knowledge development is evident in Yehali's experience, as she could not independently relate what she uncovered on the *Zoomable Number Line* with the abstract concept of infinite decimals existing between any two whole-

numbers. This evidence excerpt highlights the importance of students understanding the continuous property of decimals in order to grasp the concept of decimal density.

4.2. Finding Two: Students' Knowledge of Decimal Magnitude and Decimal Place Value Determined Their Success with Decimal Comparison Tasks and How They Used Decimal Strips

All six students involved in this study demonstrated a shift in conceptual understanding linked with decimal place value which they used when comparing pairs of decimal numbers. Thus, a second form of conceptual change was found to answer the research question *What conceptual changes occur when children attend to features of dynamic digital representations of decimal fractions?* In order to successfully determine the larger decimal value when comparing a pair of decimal numbers, the students needed knowledge of the decimal unit that a digit represented. For example, when engaging in a decimal comparison task that used the *Decimal Strips* digital tool, an understanding of the magnitude of tenths and hundredths was needed to decide whether 0.2 or 0.25 had a larger value.

Those students who initially demonstrated a strong link between decimal understanding and the base-10 numeration system used the digital tool *Decimal Strips* differently to other students who displayed conceptual confusion based on their treatment of decimals as whole numbers. These same students operated with facility and familiarity when ordering decimals, and experienced shifts between forms of attention that enabled them to offer mathematical explanations which included the relationship between decimal place value and measurement concepts. For example, Annelise needed to compare a series of two decimal numbers to determine larger decimal value during Interview 2: Decimal Strips. Annelise acknowledged her insufficient decimal knowledge to answer the comparison question. The decimal task is detailed below:

> Interviewer: "Which decimal in this pair is larger?" [Presented card with 0.2 and 0.25 decimals].
>
> Annelise: "This is what I was looking at before. I'm not sure I'll have to check." [Dragged aqua 0.2 strip, and green 0.25 strip onto two blank fraction wall rows]. "They are very close but this one [pointed to green 0.25 strip] is the biggest."
>
> Interviewer: "Why do you think they are so close in value?"
>
> Annelise: "Maybe the zero-point-two has something else on it you just can't see it."
>
> Interviewer: "What would be on the end?"
>
> Annelise: "It could be twenty-something and then this one [pointed to yellow 0.3 strip at bottom of screen display] could be thirty-something."

Annelise was deeply aware of the affordances of the dynamic digital representation to display the relative size of decimal fractions as was needed in this decimal comparison challenge. She relied on the dynamic representation of *Decimal Strips* to accurately compare the relative sizes of 0.2 and 0.25 decimals and concluded the values were "very close but this one [pointed to green 0.25 strip] is the biggest." Her second attempt at explaining why 0.25 was slightly bigger than 0.2 was correct due to her use of the dynamic affordances within Decimal Strips to manipulate length-representations of the relative size of decimals when comparing pairs. This is an example of cognitive confusion being generated by the dynamic display of decimal fraction strips and their relative sizes. Annelise resolved her cognitive confusion through shifting attention structures as she engaged with the sophisticated functions of the *Decimal Strips* digital tool. She gave evidence of shifting between three separate structures of attention, resting eventually on recognising a relationship between number size and decimal-strip length, ultimately speaking as if she was calling upon decimal-size, with the on-screen decimal-strip lengths as her reference. In the beginning

Annelise operated within the microquality of *discerning details* to use place value knowledge to predict the larger decimal fraction. She then moved towards *recognising relationships* as she consolidated her understanding of the decimal-fraction link whereby decimals are the result of dividing the whole into equal parts. Finally, Annelise shifted her attention to *perceiving properties* by connecting her knowledge of the relative size of decimals when comparing and applying this concept to the relationship with measurement.

Annelise's learning progression was mirrored in all students' attempts to compare pairs of decimal numbers whereby knowledge of the decimal unit a digit represents determined their success with the task. This observation seeks to answer to the research question *How do shifts in attention meditate engagement with these representations?* Each student began with identifying the place values of the decimal by locating and naming the digit positions in a multi-digit number. Those students who correctly activated their place value knowledge used the numerical information of a decimal unit to complete number comparison tasks. Finally, the two students who demonstrated place value computation integrated their place value information in arithmetic tasks. The student evidence from this study highlighted the relation of dependence and inclusion between the stages of conceptual acquisition as a pillar in the learning developmental model. Students needed knowledge of lower levels to develop successive structures of attention and to progress beyond their current decimal thinking they had also developed the prior levels.

4.3. Finding Three: Relating Decimal Fractions with the Division Operation Was Prompted by Student Interaction with Zooming in on Place Value as Well as the Zoomable Number Line, and Positioned Students to Understand the Relative Magnitudes of Rational Numbers

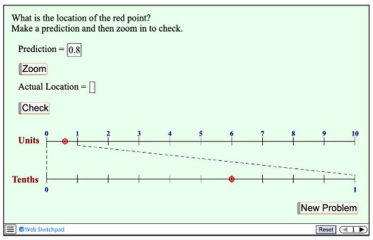
Understanding decimals is a multi-dimensional undertaking. Students need to coordinate place value concepts with aspects of whole number and fraction knowledge. Making the transition to understanding decimals relies on having a thorough understanding of previous concepts fully integrated with new information. Five of the students were identified, through a Decimal Comparison Test (Stacey & Steinle, 2003), as holding misconceptions that interfered with their knowledge of decimal concepts. Analysis of the test results indicated that three students' thinking was classified as *Longer-is-larger*, two students' thinking as *Shorter-is-larger*, and one student was an *Apparent-expert*. In the initial interview conducted to gather data on their current knowledge of decimal fractions including their misconceptions, conceptual understanding of decimal fractions involving *place value*, *magnitude* and *density*, the students were asked to explain what a decimal fraction was. No student responded with reasoning that included the idea of decimals representing parts of a unit resulting from dividing the whole into equal parts.

In this study, a number of the tasks required students to order sets of decimals correctly using knowledge of the relative sizes of decimals. A student's success with these tasks appeared to be determined by their knowledge of the relationship between decimal notation and the division operation. Two dynamic representations, *Zooming in on Place Value* and the *Zoomable Number Line*, seemed to be particularly effective in showing the relationship between decimal fractions and the division operation. The data revealed that all students developed a real sense of the size of decimal numbers through their engagement with these two dynamic representations. This was demonstrated in their consistent success with locating decimals on a number line as well as their verbal explanations that accompanied decimal comparison tasks when using the *Zoomable Number Line*.

Aarav was one such student who made observable growth in his understanding of decimal magnitude. The transcript below outlines the student's thinking in response to the interviewer's questioning during Interview 3: Zooming in on Place Value. Visual references of Aarav's engagement with the digital tool are shown in Figure 9.



Photo showing Aarav's initial prediction of 0.8 and 'zooming in' animation showing tenths between zero and one as integer dashes on number line.



Screenshot of the 'Zooming in on Place Value' web-app that was recorded using QuickTime Player whilst Aarav was manipulating it.

Figure 9. Annotated images of Aarav's interaction with Zooming in on Place Value.

> Aarav: I think the dot is three away from one. . . so zero point eight. [Typed the predicted location in and watched the 'zooming in' animation reveal the location between zero and one with tenths integers marked as dashes on the number line].
>
> Aarav: It's actually zero point six.
>
> Interviewer: How do you know that is the answer?
>
> Aarav: Because I counted along zero point one, zero point two, zero point three, zero point four, zero point five, and zero point six.
>
> Interviewer: What is the second number line showing?
>
> Aarav: Oh, I understand it's showing the number after the decimal point, the tenths.
>
> Interviewer: Did the zoomed-in number line help you answer the question?
>
> Aarav: Yes, seeing the closer numbers on the number line helped me, I could see exactly where the decimal was with it zoomed in and it was easier.

Aarav had difficulty initially predicting what tenths decimal was located on the number line when prompted to name the decimal that was marked on an unlabelled position. However, with assistance in the form of the 'zooming in' animation as the interval between two whole numbers expands showing a magnified view divided into tenths, he successfully solved later questions. Aarav engaged with the dynamic nature of the *Zooming in on Place Value* web-app to assist with locating tenths decimal fractions on a number line. The student used the animated zooming feature embedded in the learning tool to eliminate the abstractness of decimals. His manipulative interaction with the dynamic representation, enhanced his understanding of decimals, as he was able to 'see' notions that would otherwise be 'unseeable'.

The *Zoomable Number Line* representation also offered dynamic experiences for the students to see decimal concepts in motion. Each student overcame the misconceptions they initially held to successfully compare the relative size of decimal pairs and determine the larger value. The pathway of thinking progressed from *holding wholes* with an incorrect understanding of decimal place value to a more accurate internal representation of decimals by *discerning details* presented on the dynamic number line. A clear supporting answer to the research question *What conceptual changes occur when children attend to features of dynamic digital representations of decimal fractions?* In the task-based interview the students were asked to locate the decimal 0.3 on the zoomable number line and describe its location by stating "it's between _ and _". A majority of the students responded with incorrect initial answers due to the presence of whole-number thinking or incomplete decimal-fraction link thinking. However, as a result of their interactions with the *Zoomable Number Line*, each student *recognised relationships* between digits in the decimal as they understood the

positional value to the right and left of a decimal point and used this knowledge to deduce that tenths were a smaller value than wholes, and hundredths as an even smaller value. The diagram below, labelled Figure 10, outlines the three structures of attention that one student, Yehali, held during a moment of conceptual change about decimals. The verbal exchange between the interviewer and Yehali uncovered a change in understanding from her thinking that the location of 0.3 was between "Two and four. . ." to then articulating that the decimal was "smaller" than the predicted whole numbers and she needed to "scroll backwards to zero" to locate the decimal. As she transitioned back and forth between gazing, *discerning details* and *recognising relationships*, Yehali's decimal narrative became more succinct which is evidence of her increasing sophistication of decimal comprehension. Each microquality of attention is labelled alongside the corresponding decimal fraction concept that Yehali acknowledged as well as the specific affordances within the *Zoomable Number Line* representation that prompted each shift in attention.

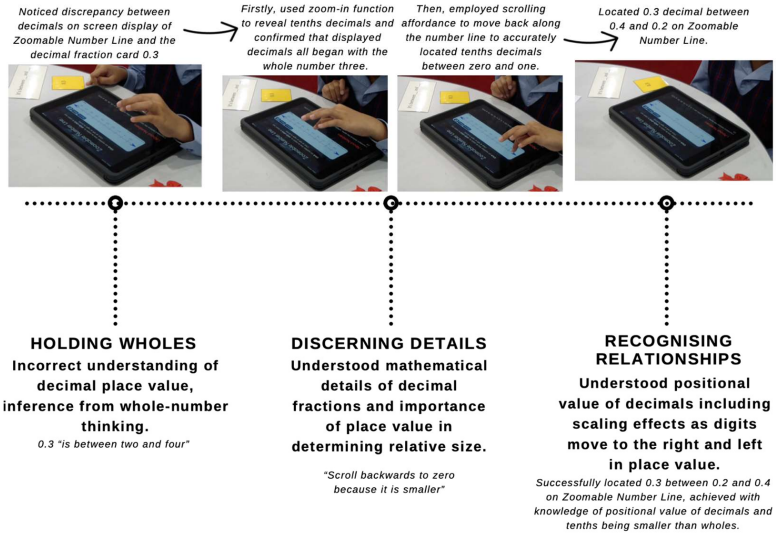


Figure 10. Diagram showing three structures of attention with linked decimal fraction concepts that Yehali showed evidence of are noted below the timeline. Yehali's actions and sayings associated with each shift in attention and consequent change in decimal conceptual understanding are annotated in italics above the timeline with a screenshot from interview video-recordings.

4.4. Finding Four: Discovering Decimal Magnitude with Tenths and Hundredths Required an Understanding of Reforming Decimal Numbers with Value, as Dynamically Displayed in Wishball-Hundredths

Students who developed a sound understanding of decimal magnitude through the feedback and dynamic features of *Wishball-hundredths* experienced shifts in attention related to *recognising relationships* between decimals and the division operation such as ten-tenths being equal to one-whole. These same students could apply this knowledge of reforming decimals when estimating the addition and subtracting of decimals. A finding directly related to the research question *How do shifts in attention mediate engagement with these representations?* Aarav was one-such student who experienced cognitive confusion during the *Wishball-hundredths* interview when he viewed the decimal place values as separate digits without any relation to each other. Aarav encountered the spinner number 9 and

current starting number 51.45 early in his interactions with the *Wishball-hundredths* game. He articulated his confusion:

> Aarav: I don't know what to do with this number because if I minus for any of them [place value digits] it will automatically be zero but if I plus then it is going to be too high. . . okay I am going to risk it and add it to one to make it ten and then I'll subtract on the next spin.
>
> [Screen displayed 60.45 as the new starting number whilst the dynamic counting frame simultaneously carried over 10-ones to the tens place value].
>
> Aarav: I understand it now. . . because I added nine-ones to make ten-ones in total so it had to jump over to the tens place value.
>
> [Screen displayed next spinner number was 8 alongside the current starting number 60.45].
>
> Aarav: I could add it here [gestured to the 4-tenths place value] which would give one to this number [gestured to ones place value] and then it would be closer to two [the target ones digit].
>
> Interviewer: Is the counting frame helping you?
>
> Aarav: Yes, I can see how many I need to add or minus to each number.

At this point of cognitive confusion, Aarav's attention had shifted back to *discerning details*. Without the ability to partition and reform place values across the decimal fraction it is difficult to successfully *recognise relationships* between the positional value of decimals and the scaling effects as decimals move right and left in place value. This is a key point of understanding between attending to *discerning details* and the higher-order mathematical thinking process of *recognising relationships*. A shift between these structures of attention can only be completed with conceptual knowledge of using place value to partition decimals which is a precursor of estimating relative size of decimals. As such, a change in understanding depends on students being able to fluently reform decimal numbers with value. Aarav shifted between *discerning details* and *recognising relationships* six observable times across the timeframe of one-round of *Wishball-hundredths* (See Figure 11). Three other students demonstrated the same pattern of thinking when they shifted back and forth between these two structures of attention. The time spent within and between *discerning details* and *recognising relationships* suggests that the process of acquiring mathematical knowledge involves many numerical and symbolic details being discerned, with each new mathematical detail being understood through a reasonable explanation that is an extension of current understandings. As such, the students who had limited prior-conceptual understanding of decimal place value experienced cognitive confusion when attempting to explain why their digital input during the *Wishball-hundredths* interactive activity resulted in different size movements on the vertical number line.

The feedback embedded within the interactive app prompted conceptual changes related to the students' decimal number knowledge. Multiple representations of the changing decimal fraction were displayed on a dynamic vertical number line, and a dynamic place value counting frame. The students utilised both of these representations to assist with gauging the relative size of decimals, particularly the scaling effects as digits move to the right and left in place value. This shift in attention from *discerning details* to *recognising relationships* demonstrated an expansion of conceptual understanding from the students initially stating decimal place values to then understanding the decimal magnitude that each place value represented.

Interestingly, two of these same students were then able to shift attention structures into *reasoning on the basis of agreed properties*. Both these students acknowledged that their

thinking had changed from their prior understanding of performing operations with decimal fractions. The conceptual change from *discerning details* of the place values within a decimal to then being able to *recognise relationships*, *perceive properties*, and *reason on the basis of agreed properties* by performing operations with decimals with an understanding of positional value number knowledge, was the result of new learnings that occurred during the *Wishball-hundredths* game. These were prompted by unique affordances embedded within the web-app including the dynamic counting frame. The students were no longer only attending to *discerning details* but were able to hold multiple structures of attention simultaneously whilst tuning into a number of intertwined conceptual principles to demonstrate in-depth knowledge of the mathematical concepts related to decimal fractions. The students who gained facility in *reasoning on the basis of agreed properties* experienced brief but dynamic back and forth shifts between forms of attention before engaging in similar thinking. The two students showed their ability to flexibly use strategies for subtraction of decimals which was linked to the students attending to *reasoning on the basis of agreed properties* when they applied their understanding of positional place value to perform operations with decimals. The students demonstrated consideration of the relative magnitude of each place value in the two-digit decimal fraction alongside the carry over addition and subtraction principle.

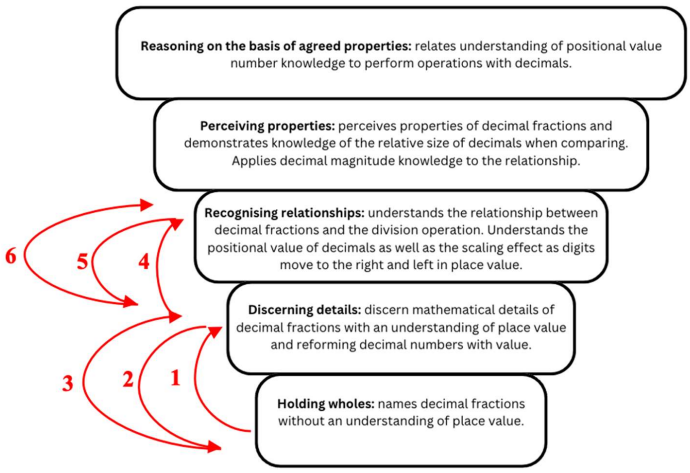


Figure 11. Diagram showing Aarav's thinking pathway back-and-forth between the structures of attention with linked decimal fraction concepts, shifts of attention are numbered in sequential order.

5. Discussion

The discussion connects the findings from this study to the research questions and central prior research in the field. Limitations of the study are outlined alongside suggestions for further research with consideration of implications for theory and practice. A newfound understanding of how students developed decimal sense through their engagement with four dynamic digital representations is discussed in this section. Students noticed the Zoomable Number Line's dynamic display of the continuous property of number which prompted a change in conceptual understanding related to decimal density, they also engaged with the Decimal Strips digital tool differently when comparing pairs of decimal fractions depending on levels of decimal magnitude and place value prior knowledge. Relating decimal fractions to the division operation was prompted by student interaction with Zooming in on Place Value and the Zoomable Number Line, positioning students to

understand the relative magnitudes of rational numbers. Students also better understood decimal magnitude when they acknowledged reforming decimal numbers with value, as dynamically displayed in Wishball-hundredths. The dynamic affordances embedded within the digital tools interrupted the students' flow of attention, to guide their conceptual learning journey of decimal fractions by generating cognitive conflict.

5.1. Structures of Attention and Changing Conceptual Understanding

It is acknowledged that students experience microqualities of attention in the learning journey and will switch back and forth between the multiple structures, often very rapidly (Mason, 2010). As such, it is important to notice shifts between these states of attention which can happen very quickly as these are vital in mathematical thinking. The structures are not a single progression pathway as learners will go back and forth as well as around in a cycle. In particular, noticing shifts from specific relationships to perceiving properties is pivotal, as this is the essence of mathematics. For a learner this shift unfolds while gazing to detect apparent motion produced from saccadic eye movement which can suddenly switch to awareness of details amongst a mass of other, undiscerned detail (Wei et al., 2023). As details are detected and discriminated, the mind automatically looks for relationships; differences and similarities. To do this requires something being invariant as a background against which to detect change (Mason, 2010). *Recognising relationships* focuses on particulars, however *perceiving properties* shifts to generalisable principles, and then to the particular as exemplary or paradigmatic. Formalising in mathematics is the overt action which accompanies a shift from perceiving properties to taking certain properties as definitive and so as the basis for reasoning. In each episode of evidence presented in the findings section, the student was no longer only attending to the entry-level structure of attention but was able to rapidly move between multiple structures whilst tuning into a number of conceptual understandings simultaneously to demonstrate in-depth knowledge of the mathematical concepts related to decimal fractions. This finding sought to answer to the study's first research question *What conceptual changes occur when children attend to features of dynamic digital representations of decimals fractions?* For four of the students, this sequence occurred six times before they formalised certain properties as definitive and so as the basis for reasoning. This is suggestive that learners require time to articulate the relationships they recognise, and then need to confirm the relationship by searching for other instances of it within an object, before prompting a move to articulating the relationship as a property. The theoretical work of Pirie and Kieren (1991) on the nature of acquiring mathematical understanding posits a similar learning trajectory where the process of "folding back" is necessary for dynamical growth of mathematical thinking. What was observed through microgentic analysis of the students' interactions with four dynamic digital tools, was a layering of increasingly sophisticated structures of attention as a compounded 'stacking' of mathematical details that were grouped in relationships to be extrapolated as properties before being generalised as principles to be reasoned on the basis of agreed properties.

A students' change in understanding is not a unidirectional progression through structures of attention as separate modes of learning but rather a layering of more sophisticated microqualities of attention that build a more robust understanding of a mathematical concept (Clements & Sarama, 2004). This was particularly evident in the students' engagement with the Wishball-hundredths interactive activity. They recognised the digital tool's ability to reform decimal numbers from the feedback of a new starting number. Two students extended this concept to understand the subtraction trading principle being shown through the movement of the beads on the counting frame which prompted a shift in their attention. Furthermore, these students were able to learn from the feedback and reform the decimal independently during a later decimal subtraction calculation. Being able to

estimate this mathematical move by performing both mental addition and subtraction calculations required a complex chain of reasoning and cognitive strategies including a "series of multi-step associations and procedures that involve facts, verbal chains, discriminations, concepts, and rules designed to bring about a response to a specified problem" (Kame'enui & Simmons, 1990, p. 294). The students linked understandings of decimal place value with the procedural knowledge and steps to add and subtract value from a decimal fraction. Within the short timeframe of this task-based interview, the two students formed an understanding of how subtracting decimals followed the same principles of renaming and trading as whole numbers and could exchange one unit for ten tenths and one tenth for ten hundredths (Martinie, 2014). Most importantly, they noticed this pattern was continuous and understood that when moving to the right of the decimal point, the numbers become ten times smaller (Martinie, 2014). The growth in the students' conceptual understanding of decimal fractions was facilitated by the dynamic affordances embedded within the interactive activity of *Wishball-hundredths* namely the counting frame and vertical number line.

5.2. Provoking Productive Cognitive Confusion

All students demonstrated conceptual change as the result of productive cognitive conflict. Complete resolution of their confusion was dependent on the ability to generalise a discovery from the dynamic digital representation by linking what they saw displayed with their prior knowledge of decimal concepts. Across the four interviews, all six students engaged with a task that prompted them to question their current understandings of decimal fractions and begin to develop mathematical thinking that consistently included the decimal-fraction link. The thinking pathway provided an answer to the second research question *How do shifts in attention mediate engagement with these representations?* This was particularly evident when they worked through the decimal comparison tasks alongside the *Decimal Strips* and *Zoomable Number Line* dynamic representations. Seminal work by Swan (1983) found that students who experienced 'cognitive conflict' when learning gained better outcomes overall than students who learnt through 'positive only teaching'. *Decimal Strips* generated cognitive conflict as it challenged the students' current understandings and present misconceptions about decimals. It was not until they used the functions of the *Decimal Strips* web-app and dragged the coloured tiles onto the blank fraction wall that they explicitly recognised the relative values of tenths and hundredths. Steinle and Stacey (2004) detailed how teachers simply telling a student that their thinking is incorrect does not eradicate that student's misconceptions. Rather, instructional techniques need to be implemented to assist students in overcoming or controlling the influence of misconceptions and systematic errors. Most of the students were able to self-correct their thinking by interacting with this powerful dynamic representation as it challenged their prior understanding of decimals and allowed them to examine their errors in thinking and misconceptions. For example, Annelise needed a mathematical reason as to why 0.3 was a longer decimal strip than 0.25, which was currently conflicting with her prior thinking that a longer decimal number was always larger in value. By dynamically comparing the relative sizes of these two decimals on the *Decimal Strips* web-app, she was able to connect this active learning with her understanding of decimal place value recently consolidated in the same interview, to determine that tenths are a larger value than hundredths and therefore despite 0.3 only having one decimal place it can be understood as '30-something' as it is in the tenths place value. As such, this interview moment is a display of Annelise extrapolating her *recognising relationships* between the dynamically visual representation and correcting her 'longer is larger' decimal thinking to a generalisable understanding of decimal place value after she *perceived properties* of scaling place values to the left of

a decimal point. Annelise's experience was not a linear progression through structures of attention as separate modes of learning but rather a layering of more sophisticated microqualities of attention that build a more robust understanding of a mathematical concept.

5.3. Abstractness of Decimal Fractions

Students relied on the dynamic representations to reduce the abstractness of decimal fractions by visually showing the relative positions of decimals on a number line. This enabled them to recognise the relationship between decimal notation and the division operation, which leads to *perceiving properties* with the relative magnitude of rational numbers. The students could then reason and problem-solve to produce solutions to decimal location questions. Such a finding provided a second answer to both research questions *What conceptual changes occur when children attend to features of dynamic digital representations of decimals fractions?* and *How do shifts in attention mediate engagement with these representations?* Engaging with the dynamic zooming and stretching of the *Zooming in on Place Value* number line, eliminated the abstractness of the decimals and clearly presented the sequence of scores for the students to see. As a result, they were able to visualise the location of each tenths decimal to correctly predict the answer to the question. The digital functions of the *Zooming in on Place Value* interactive activity allowed them to uncover decimal concepts that a static number line is not able to illustrate. The ability to stretch out the number line between two whole-numbers, enabled students to see the continuous sequence of decimals, which assisted them when operating with different types of numbers (Teppo & van den Heuvel-Panhuizen, 2014). G. Goldin and Shteingold (2001) suggest, developing clear links between quantity, vocabulary and symbol is critical for conceptual understanding and the visual affordances of the digital tool illustrate the symbolic notation. A key affordance of the *Zooming in on Place Value* virtual manipulative is its ability to dynamically displaying ten-tenths between two whole-numbers. Through a 'zooming in' animation the digital tool reveals ten-tenths between two whole-numbers, and ten-hundredths between two-tenths in a two-step question sequence. This gradual release of information allows the student to build knowledge of decimal density as to the continuity property of rational numbers, whereby, between any two decimals there are an infinite number of other decimals (Widjaja et al., 2008). Furthermore, Aarav's critique of the app in saying that he was able to see the number line with the decimals on it, indicated that the web-app was able to assist him in visualising the 'abstractness' of decimal fractions. Young students need support when developing their decimal sense and teachers therefore need to be sensitive to students' ways of thinking and learning (Way, 2011). Quality teaching that employs quality representations such as a dynamic number line can assist students in better understanding decimal fractions.

Students may follow erroneous rules linked to whole-number thinking when trying to assimilate new knowledge about decimals with what they already know about whole numbers, place value and fractions (Moloney & Stacey, 1997). To develop their understanding of the continuous nature of the base-10 system, students need to relate decimal fractions with the division operation which can be difficult to illustrate using static representations.

5.4. Practical Implications for Classroom Practice

This research demonstrates that dynamic digital representations may act as powerful catalysts for conceptual change in mathematics classrooms. When integrated thoughtfully, tools such as Wishball-hundredths, Decimal Strips, Zooming in on Place Value and Zoomable Number Line could help students visualise abstract relationships, test assumptions, and confront misconceptions about decimals. For teachers, this underscores the

potential importance of purposeful selection and timing—introducing digital tools at key moments may help draw attention to underlying mathematical structures that might otherwise remain implicit. Strategic modelling by the teacher, such as demonstrating how to use zooming functions to explore density or decimal placement, could help students focus on conceptual ideas rather than procedural mechanics. These findings suggest that digital tools may be most effective when used not merely as enrichment but as pedagogical interventions to support transitions in student thinking.

The findings also highlight the potential role of teacher noticing and mathematical discourse in supporting conceptual development. Teachers who are attuned to shifts in students' gestures, language, and interactions with representations may be better able to recognise pivotal moments of emerging understanding and intervene productively. Encouraging students to articulate and justify their reasoning during digital interactions could provide valuable diagnostic insight and foster deeper conceptual connections. Professional learning that develops teachers' capacity to interpret student attention and reasoning may therefore be particularly valuable. Embedding opportunities for small-group or one-to-one discussion, reflection, and guided exploration with dynamic representations could strengthen the link between procedural fluency and conceptual understanding, an integration that is often difficult to achieve through static representations alone.

5.5. Limitations and Implications for Further Research

Although six students provided sufficient variety in responses to satisfy the exploratory nature of this study, a larger number of participants from several different schools would allow more robust conclusions to be drawn about the effectiveness of the four dynamic digital tools; *Wishball-hundredths*, *Decimal Strips*, *Zooming in on Place Value*, and *Zoomable Number Line*, for supporting shifts in understanding of decimal fractions.

Applying the theory of shifts of attention (Mason, 2008) and using the researcher-designed framework as an analysis tool provided a practical scaffold for plotting each participant's progression of understanding as they worked to achieve the decimal tasks during each of the interviews. The framework proved to be useful in detecting and mapping the non-linear learning path students took when making the delicate shift to abstraction. Some students appeared to experience a shift in understanding as a direct result of interaction with the tasks and representations of decimals. Specifically, the dynamic affordances focused the students' attention on important decimal concepts that changed their current mathematical thinking by providing an extended awareness to move beyond their existing conceptual understanding. This analysis framework has the potential to help describe how students learn important decimal fraction concepts and may prove to be a useful tool in further research into decimal knowledge progression. The value of this extended theory of shifts of attention when used alongside a microgenetic analysis approach enabled the discernment of observable shifts in how and what the students were attending to as they developed key decimal concepts. The same process of analysis could be applied to other research situations focused on different mathematical concepts to understand how conceptual knowledge is learned by students.

An emerging data source in mathematics education research involves the use of eye-tracking technology to investigate various aspects of learning behaviours and processes. The inclusion of eye-tracking technology could add further objectivity and reliability to the data by precisely discerning the details that students are viewing when making observable changes to their current decimal fraction understandings.

6. Conclusions

This study explored what conceptual changes emerged from Year 4 (9–10 years old) students' engagement with dynamic digital representations of decimal fractions embedded within the interactive activities, *Wishball-hundredths*, *Decimal Strips*, *Zooming in on Place Value*, and *Zoomable Number Line*. Mapping the students' attention shifts revealed insights into how their understanding of decimal fractions evolved through their interaction with the dynamic digital representations. This provided evidence for evaluating the effectiveness of these interactive activities in promoting conceptual understanding and proficiency of decimal fractions among the group of Year 4 students. The participants demonstrated changes in their conceptual knowledge related to decimal fractions as they reacted to the dynamic affordances which prompted shifts in their attention and consequently new learnings were formed, within a short amount of time. The responses of the Year 4 students revealed that interacting with features such as the animated vertical number line and counting frame in *Wishball-hundredths*, the manipulative decimal fraction tiles in *Decimal Strips*, the zoom-in number line in *Zooming in on Place Value*, and the zoomable and scrollable number line in *Zoomable Number Line*, prompted new learnings about the key decimal concepts. In analysing the thinking pathways students took when completing the decimal tasks, it appeared that engagement with the four dynamic digital tools stimulated thinking about the relationship between whole numbers and decimal fractions, as well as place value, the relative size of decimals, and the density of numbers as an infinite system. The students' interactions with the dynamic representations of decimal fractions embedded within each digital tool, at times, generated productive cognitive conflict, as the representations challenged their existing understandings and misconceptions. Outside of the research-interview situation, these would have been ideal 'teaching moments'.

Mason (2010) views learning as the shifting of attention in form and focus, which is *what* is being attended to and *how* it is being attended by the learner. A strong claim emerges from his research that argues that engaging with a mathematical task is not sufficient for learning to occur rather there is need for the integration of experience and sense-making which requires transformation of attention (Mason, 2010). Dynamic representations of decimal fractions are capable of offering direction to learners' shifts of attention. The dynamic nature of digital manipulatives forces shifts in the object of children's attention when they are actively interacting with the mathematics content as well as the structure of their attention in how they attend to features of the mathematics representation and the embedded mathematics concepts. Conceptual development occurred when the learners experienced a shift between attending to their current knowledge, discerning new mathematical details exposed in the dynamic digital tools, and perceiving relationships as properties that might be applicable in other numerical situations. Being aware of these shifts occurring can alert teachers to the possibility that learners are not making appropriate shifts and therefore implement appropriate strategies and learning tools such as dynamic digital representations, to prompt shifts and consequently a change in conceptual understanding.

Examining the responses of a group of participants completing all four task-based interviews revealed the full scope of conceptual understanding of decimal fractions inclusive of *decimal place value*, *relative magnitude* and *decimal density*, which emerged from the students' interactions with four dynamic digital tools. Together this collection of dynamic digital representations of decimals offered powerful learning opportunities for students to develop decimal sense and implement this knowledge in a number of mathematical contexts. Important and difficult skills such as comparing decimals, locating decimals on a number line and adding and subtracting decimals were all studied between the four digital tools. While keeping in mind the inferential limitations of a small-scale study, the findings clearly indicate the potential of dynamic, digital representations to be used as

effective learning and teaching tools. Students experienced shifts of attention when interacting with the unique dynamic digital affordances that prompted changes in conceptual understanding related to decimals.

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6.3 Additional Findings and Discussion: Students demonstrated a change in understanding related to decimal density through noticing the dynamic display of continuous decimal place values on the Zoomable Number Line.

The concept of the continuous nature of numbers means that between any two distinct numbers, there exists an infinite number of other numbers. Understanding that there are infinitely many decimals between 0 and 1, with a high density of these decimals, relies on comprehending this continuous property. This idea is quite abstract, and some learners initially struggle with it as it contrasts with their knowledge of whole numbers. In the *Zoomable Number Line* interview, five out of six students showed a shift in their understanding of decimal density. These five students moved from a focus on whole numbers or noticing details without grasping the connection between decimals and division, to a better understanding of this concept. They used the dynamic *Zoomable Number Line* to link their decimal place value knowledge to the continuous nature of decimal numbers. The learning journeys of the students revealed a common pattern where each student went through at least two rounds of hypothesising to explain why the *Zoomable Number Line* kept showing more decimal place values. This self-questioning indicated cognitive confusion as they tried to understand the relationship between the positional value of decimals and the ongoing division of place values, which resulted in another decimal digit appearing on the dynamic number line. The findings below are presented student-by-student with a transcript of the student's sayings and doings and an interpretation of their actions and thinking. Finally, an overall discussion documents the significant commonalities and distinctive differences between the students' engagement with the *Zoomable Number Line* to uncover the concept of decimal density.

6.3.1 Annelise

Annelise was one of the students who went through two cycles of shifting her focus between noticing details and recognising relationships before she could explain the continuous nature of decimals and thus describe decimal density. The evidence snapshot below captures the interaction between Annelise, the interviewer, and the dynamic digital tool, highlighting her change in understanding of decimal density. Additionally, the 'structures of attention' diagram outlines the learning progression Annelise experienced as she attained this new level of knowledge. This evidence transcript is also presented in the article titled "Changing structures of attention when learning about decimal fractions with digital tools", however the version here has been added to with annotations of shifts in attention within the transcript and additional interpretation details.

Annelise was freely investigating the *Zoomable Number Line* app by continuously zooming in to discover the density of the dynamic representation (see Figure 6.1). She was interested in discovering how many place values could be revealed in a decimal when she predicted a maximum and then tested her hypothesis. The accompanying verbal exchange is detailed below:

Annelise: [Held down zoom-in function to continually reveal more and more decimal place values] Oh wow… there are so many numbers. *Shift in attention holding wholes to discerning details.*

Interviewer: How many numbers are after the decimal point?

A: Six. *Shift in attention discerning details back to holding wholes.*

I: What do you think is the maximum number of decimals there are after a decimal point?

A: Maybe 10 [held down zoom-in function to continually reveal more and more decimal place values] *shift in attention holding wholes to discerning details* I just counted 12 numbers after…maybe it stops at 20 *shift in attention discerning details back to holding wholes* [held down zoom-in function to continually reveal more and more decimal place values] I don't think it will stop. *Shift in attention holding wholes to discerning details.*

I: Why don't you think it will stop.

A: Because there are infinite numbers. *Shift in attention discerning details to recognising relationships.*

Figure 6.1

Annelise evidence snapshot with Zoomable Number Line



Photo showing Annelise continually zooming-in to reveal more and more decimal place values on the Zoomable Number Line.

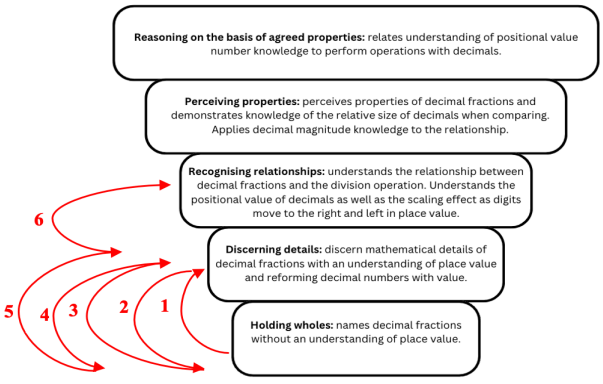


Diagram showing Annelise's thinking pathway back-and-forth between the structures of attention with linked decimal fraction concepts.

At the start of the interview, Annelise had an incomplete understanding of the abstract concept of decimal density. In the example mentioned above, she was learning to extend place value to tenths, hundredths, thousandths, and beyond. Using the *Zoomable Number Line* helped her recognise how numbers increase and decrease. The zoom-in and zoom-out functions illustrated that the decimal number system is based on powers of ten, enhancing Annelise's understanding of decimal density. While concrete manipulatives might only represent up to thousandths, the digital *Zoomable Number Line's* dynamic capability allowed Annelise to observe that decimals can always be divided into ten smaller parts. Additionally, Annelise seemed to connect her observations of the continuously zooming number line with her understanding of infinity, realising that numbers can infinitely get smaller just as they can infinitely get larger. This analysis suggests that by using a powerful digital learning tool Annelise could reason through her observations which ultimately helped her develop her own methods to explain the size and value of decimal numbers and understand why decimal place values extend indefinitely.

6.3.2 Yehali

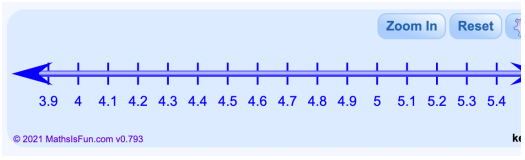
One student's attention remained within the attention structure of discerning details when freely investigating the *Zoomable Number Line*. The transcript details Yehali's initial engagement with the zooming-in function of the *Zoomable Number Line*.

> Yehali: [Zoomed into number line slowly, stopped when integer markings were displayed between whole numbers 4 and 5 without decimal notation]. *Shift in attention holding wholes to discerning details.*
>
> Interviewer: What do you think these little markings are on the number line?
>
> Y: How to count between two numbers… *shift in attention discerning details back to holding wholes.*
>
> I: What numbers do you think will be shown when you continue zooming in?
>
> Y: … maybe one [continued to zoom-in to reveal tenths decimal notation on screen display, see Figure 6.2] it's showing the decimals now *shift in attention holding wholes to discerning details* [held down zoom-in function to show increasing number of place values after decimal point].
>
> I: What is the maximum number of digits that can come after the decimal point?
>
> Y: Maybe seven… *shift in attention discerning details back to holding wholes* [held down zoom-in function to continually reveal more and more decimal place values] I counted nine …maybe it will go up to the 10 [held down zoom-in function to continually reveal more

and more decimal place values] there's 14 now, it probably goes up and up. *Shift in attention holding wholes to discerning details.*

Figure 6.2

Yehali evidence snapshot with Zoomable Number Line



Screenshot of the Zoomable Number Line web-app that was recorded using QuickTime Player whilst Yehali was manipulating it.

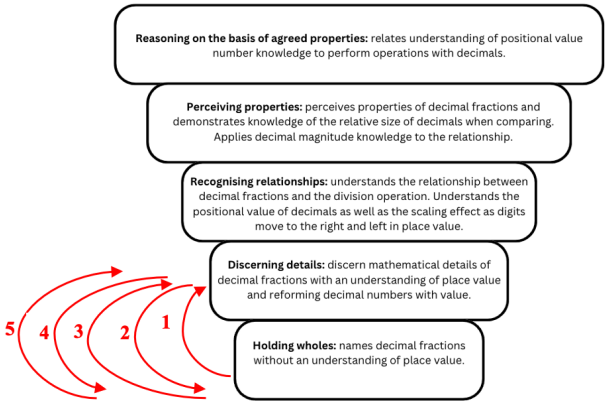


Diagram showing Yehali's thinking pathway back-and-forth between the structures of attention with linked decimal fraction concepts.

Yehali was unable to provide a mathematical explanation for the continuous nature of decimals. At the beginning of the interview, her thinking suggested that she treated decimals as whole numbers to the right of the decimal point. This was due to her perception of the decimal point as merely separating two whole numbers. However, Yehali's focus shifted from considering whole numbers to noticing details when she observed the *Zoomable Number Line* dynamically expanding to show ten-tenths between two whole numbers. A limitation in her conceptual knowledge development became evident, as she struggled to independently connect her observations on the *Zoomable Number Line* with the abstract idea of infinite decimals existing between any two whole numbers. This excerpt underscores the importance of students understanding the continuous nature of decimals to fully grasp the concept of decimal density.

6.3.3 Aarav

Aarav uncovered how between any two distinct numbers there are an infinite number of decimal fractions as he continually zooming in around the number four. He was interested in discovering how many place values could be revealed in a decimal. This was observed when he was freely investigating the *Zoomable Number Line* app by continuously zooming in to discover the density of the dynamic representation. He described his actions aloud, "it's zooming in... *shift in attention holding wholes to discerning details* [continued to zoom-in to reveal

tenths decimal notation on screen display] *shift in attention discerning details to recognising relationships* and showing the decimals, the tenths... [continued to zoom-in to reveal hundredths decimal notation on screen display] *shift in attention recognising relationships back to discerning details* and now it's showing the hundredths... *shift in attention discerning details to recognising relationships* [continued to zoom-in to reveal thousandths decimal notation on screen display] *shift in attention recognising relationships back to discerning details* it's going into the thousandths... *shift in attention discerning details to recognising relationships* I wonder if it will stop?... [continued to zoom-in to reveal increasing decimal digits on screen display, see Figure 6.3] *shift in attention recognising relationships back to discerning details* I think it will continue forever because the decimal can go on forever" *shift in attention discerning details to recognising relationships.*

Figure 6.3

Aarav evidence snapshot with Zoomable Number Line



Photo showing student tapping the zoom-in function to reveal an increasing number of decimal place values around the whole-number four.

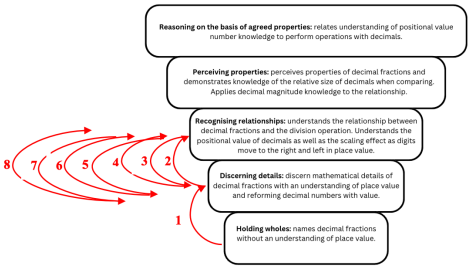


Diagram showing Aarav's thinking pathway back-and-forth between the structures of attention with linked decimal fraction concepts.

Aarav explored the concept of decimal fractions between whole numbers and the smaller decimals within those fractions. He extended this understanding by commenting on the continuity property of decimals, observing that an infinite number of decimals exist between any two decimals. His insights into decimal density were driven by the zoom function of the digital number line. As he continually zoomed in around the whole-number four, Aarav discovered that there are infinite decimal fractions between any two distinct numbers. He was intrigued by how many place values could be revealed in a decimal, which fascinated him as he freely investigated the *Zoomable Number Line* web-app, continually zooming in to uncover the density of the dynamic representation.

6.3.4 Valerie

Valerie commenced the *Zoomable Number Line* interview with an understanding that decimal fractions are located between whole numbers, however she held incomplete knowledge about naming specific decimals could be found between the whole numbers one and two. The transcript below shows how her thinking changed based on the feedback she received from the *Zoomable Number Line* when she used the zoom-in function to reveal decimals between whole numbers (see Figure 6.4).

Valerie: [Zoomed into number line slowly, stopped when tenths integer markings were displayed between whole numbers 1 and 2 without decimal notation].

Interviewer: What is the number line doing?

V: It's making the number spread further apart.

I: What are these markings showing on the number line?

V: Where the numbers in between are. *Shift in attention holding wholes to discerning details.*

I: And what number will be displayed here? [pointed to midpoint integer marking between whole numbers 1 and 2].

V: Maybe zero-point-five. *Shift in attention discerning details back to holding wholes.*

I: Continue exploring the zoomable number line.

V: [Scrolled to the right to display tenths integer marking between whole numbers 2 and 3].

I: What do you notice about the line markings on the number line here?

V: They are showing how many numbers go in between. *Shift in attention holding wholes to discerning details.*

I: How many numbers do go in between?

V: [Counted 10-tenths integer markings between whole numbers 2 and 3] ten.

I: What numbers do you think will be displayed when you continue to zoom-in?

V: Zero-point-one, zero-point-two, zero-point-three, zero-point-four, zero-point-five, zero-point-six, zero-point-seven, zero-point-eight, zero-point-nine, and zero-point … no actually three. *Shift in attention discerning details back to holding wholes.*

I: Well done, keep going.

V: [Held down zoom-in function to reveal tenths decimal fractions between whole numbers 2 and 3].

I: What numbers can you see displayed?

V: Two-point-one, two-point-two, two-point-three, two-point-four… *Shift in attention holding wholes to discerning details.*

I: Is that a different set of numbers to what you thought would be between 2 and 3?

V: Yes, I didn't know it could be a ones and then a decimal *shift in attention discerning details to recognising relationships* [held down zoom-in function to reveal hundredths integer markings between 2.3 and 2.4 without decimal notation].

I: What are these line markings showing on the number line?

V: Maybe… maybe… [long pause] maybe point-one *shift in attention recognising relationships back to discerning details* [held down zoom-in function to reveal hundredths decimal fractions between 2.3 and 2.4].

I: Tell me what numbers you can see displayed now.

V: Two-point-thirty-one, two-point-thirty-two, two-point-thirty-three…

I: Is that what you expected?

V: No, it is different because it has a double-digit at the end and I haven't seen that before *shift in attention discerning details to recognising relationships* [held down zoom-in function to continually reveal more and more decimal place values].

I: What are you noticing?

V: Every time the lines appear between the numbers there are always 10 and they are evenly spaced out to scale. *Shift in attention recognising relationships to perceiving properties.*

I: What do you notice about the number of digits after the decimal point?

V: There are three.

I: What do you think the maximum number of decimal digits could be on the number line?

V: Four *shift in attention perceiving properties back to holding wholes* [held down zoom-in function to continually reveal more and more decimal place values].

I: Has it shown more than four?

V: Yes, maybe it's actually eight for the maximum [held down zoom-in function to continually reveal more and more decimal place values] that's more than eight now… *Shift in attention holding wholes to discerning details.*

I: Would you like to change your answer?

V: I think it might actually go on forever because numbers aren't limited. *Shift in attention discerning details to perceiving properties.*

Figure 6.4

Valerie evidence snapshot with Zoomable Number Line

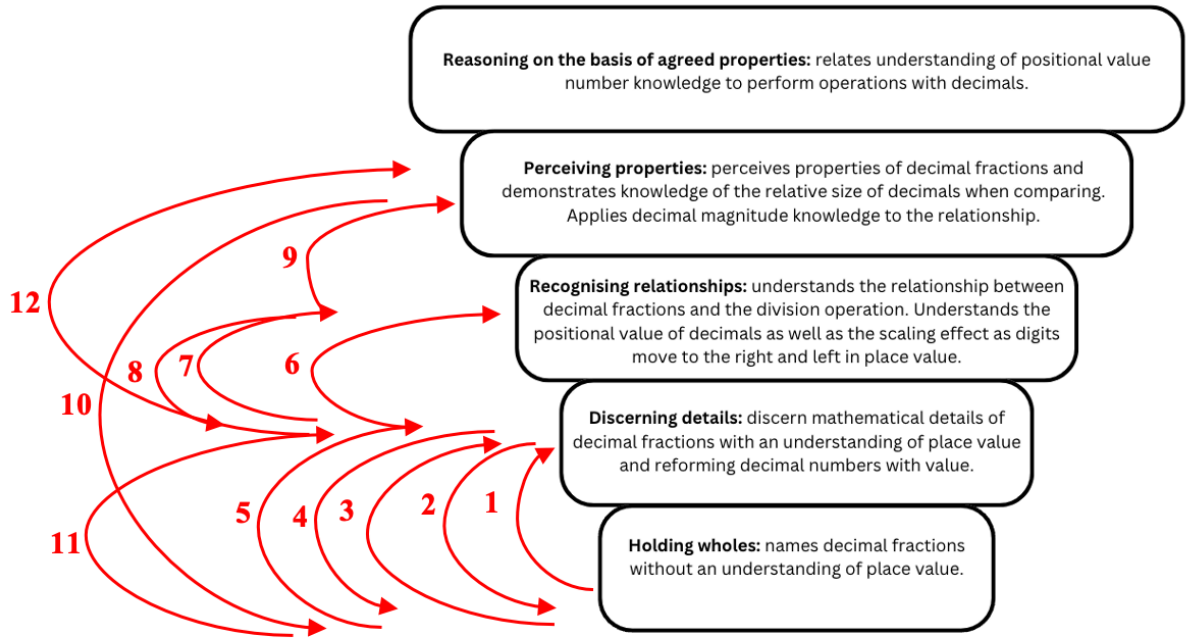


Diagram showing Valerie's thinking pathway back-and-forth between the structures of attention with linked decimal fraction concepts.

When beginning to explore the capabilities of the *Zoomable Number Line*, Valerie was discerning details about decimal numbers with tenths however was doing so with an inaccurate and incomplete understanding. At this point she understood that there were ten-tenths between two whole numbers, however she demonstrated confusion on two occasions when she was asked to locate a decimal between one and two and also when she was asked to count the tenths between two and three. In both of these instances, Valerie only named decimals starting "zero-point" rather than correctly saying the whole number and then the decimal value. Through the zoom-in function and the dynamic number line, she recognised her erroneous thinking that all decimals were less than one-whole, when she saw the sequential display of tenths between two and three. Valerie stated, "I didn't know it could be a ones and then a decimal".

Although Valerie had prior knowledge of decimal fractions, this only extended to the tenths place value. Her understanding of hundredths was developed through the dynamic digital manipulative. Valerie's attention shifted when she observed to the dynamic display of hundredths between the decimals 2.3 and 2.4. She explained her change in understanding, "…it is different because it has a double-digit at the end and I haven't seen that before". Valerie's actions and reasoning indicated that her understanding of hundredths place value in decimals

was developed through her exploration of the *Zoomable Number Line*. Initially, the student did not understand the concept of place value in decimals as suggested by her incorrect answer that "point-one" would be revealed between 2.3 and 2.4. Using the 'zoom in' function and the scrolling capability of the *Zoomable Number Line* helped Valerie recognise the place value of hundredths in decimal fractions. Valerie extended her thinking beyond hundredths place value to recognise the continuity property of decimal when she witnessed the *Zoomable Number Line* display an increasing number of decimal place values. The dynamic display prompted a shift in Valerie's attention to perceiving properties as evidenced in her reasoning, "I think it might actually go on forever because numbers aren't limited". She recognised the infinite density of the decimal numeration system, which resulted directly from the zoom function of the digital number line.

6.3.5 Andrew

Andrew was a capable mathematics student with sound prior understanding of decimal fractions and knowledge of the concept of decimal density. However, the capabilities of the *Zoomable Number Line* broadened Andrew's awareness of decimals beyond tenths and hundredths, and revealed the concept of infinite decimals which was new knowledge for Andrew (see Figure 6.5).

Interviewer: Tell me what you notice about this number line?

Andrew: [Scrolled to the right to dynamically display whole number markings from -3 to 10] that you can scroll to show more numbers along the number line… and it shows negative one and negative two.

I: What else can the number line do and show?

A: [Held down zoom-in function to reveal tenths integer markings without decimal notation, between whole numbers 8, 9, 10 and 11].

I: What are those line markings on the number line showing?

A: The decimals along like point-one, point-two and point-three, it's showing all the decimals with the tenths column. *Shift in attention holding wholes to discerning details.*

I: Good thinking… you can continue exploring.

A: [Held down zoom-in function to reveal tenths decimal notation between whole numbers 10 and 11, paused when hundredths integer markings without decimal notation was displayed] Now it has split it into hundredths between ten-point-seven and ten-point-eight. *Shift in attention discerning details to recognising relationships.*

I: Can you name the decimals that being represented by the line markings?

A: Here [gestured to first hundredths integer marking between 10.7 and 10.8] would be ten-point-seven-one, and then ten-point-seven-two and so on until you reach ten-point-seven-nine, and then ten-point-eight.

I: If you were to continue zooming in, how many decimal place values could be displayed, what is the maximum number of decimal digits a number can have?

A: Maybe up to 100 *shift in attention recognising relationships back to discerning details* [held down zoom-in function to continually reveal more and more decimal place values] it's splitting it up a lot on the number line. *Shift in attention discerning details to recognising relationships.*

I: Do you think it will stop eventually or will it continue zooming?

A: I don't think it will stop because every number after the ones can be divided over and over again so it's shows many different numbers. *Shift in attention recognising relationships to perceiving properties.*

Figure 6.5

Andrew evidence snapshot with Zoomable Number Line

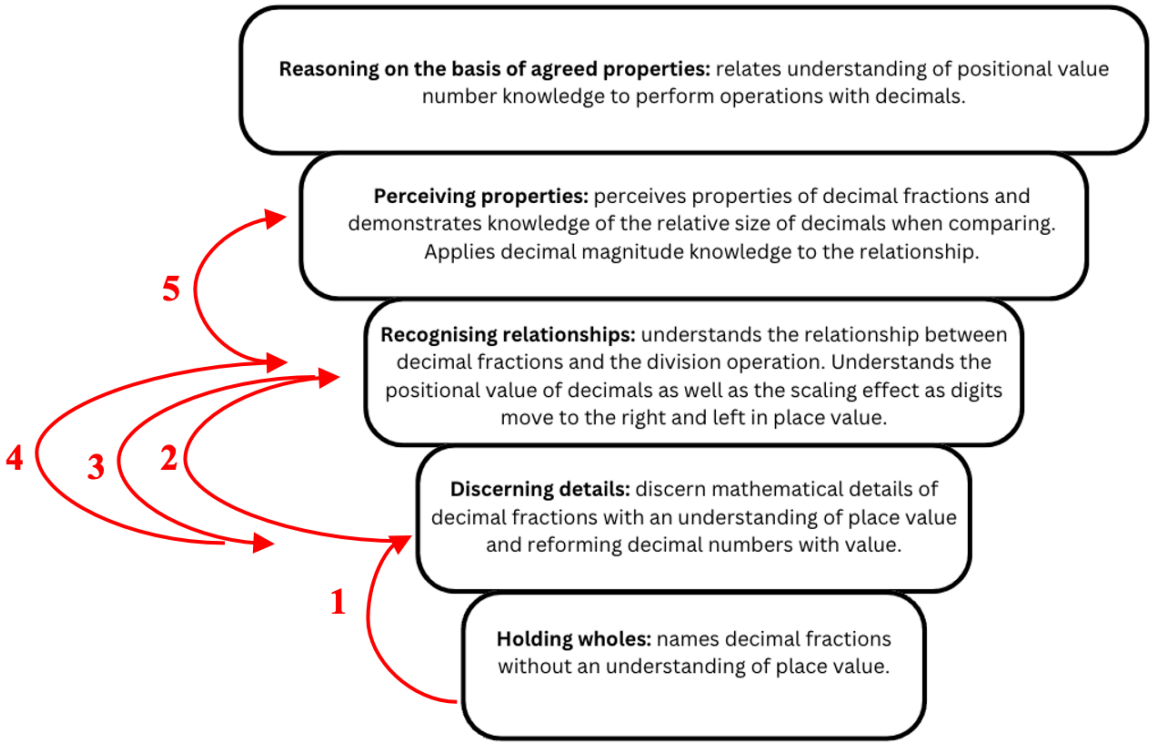


Diagram showing Andrew's thinking pathway back-and-forth between the structures of attention with linked decimal fraction concepts.

Early in the *Zoomable Number Line* interview, Andrew attended to the idea of decimal fractions existing between whole numbers, and smaller decimals existing between those

decimal fractions, the concept of decimal density. He used the zoom-in function of the digital tool to reveal the relationship between whole numbers and decimal fractions, and could mathematically explain the idea of density of rational numbers. Andrew also uncovered how between any two distinct numbers there are an infinite number of decimal fractions as he continually zoomed in between the numbers 10.7 and 10.8. He was interested in discovering how many place values could be revealed in a decimal. The function of zooming-in to reveal increasing place values fascinated him and provoked a shift of attention to perceiving properties when he correctly reasoned that decimal place values can be "divided over and over again". The *Zoomable Number Line* assisted Andrew in verbally explaining his developed understanding of decimal fractions.

6.3.6 Tehillah

Tehillah displayed significant conceptual confusion around decimal fractions across a number of tasks in the *Zoomable Number Line* interview. She presented initial confusion during the exploration time due to the presence of whole-number thinking. However, as a result of her interaction with the zoomable number line, particularly tuning into the zoom-in function, the student as able to question her current thinking and begin to develop more appropriate decimal-fraction link thinking (see Figure 6.6).

Interviewer: Tell me what you notice as you explore this zoomable number line.

Tehillah: [Held down zoom-in function to reveal tenths decimal notation, between whole numbers seven and eight].

I: What is the number line showing?

T: It is become smaller and is showing numbers close to each other.

I: What are the numbers on the line that you can see?

T: The seven-points, and eight-points *shift in attention holding wholes to discerning details* [held down zoom-in function to continually reveal more and more decimal place values between whole numbers seven and eight].

I: What numbers can you see displayed now?

T: It's showing seven-point, with high numbers. *Shift in attention discerning details back to holding wholes.*

I: How many numbers are after the decimal point?

T: Six. *Shift in attention holding wholes to discerning details.*

I: How many do you think the number line may show?

T: I don't know *shift in attention discerning details back to holding wholes* [held down zoom-in function to continually reveal more and more decimal place values] just lots. *Shift in attention holding wholes to discerning details.*

Figure 6.6

Tehillah evidence snapshot with Zoomable Number Line

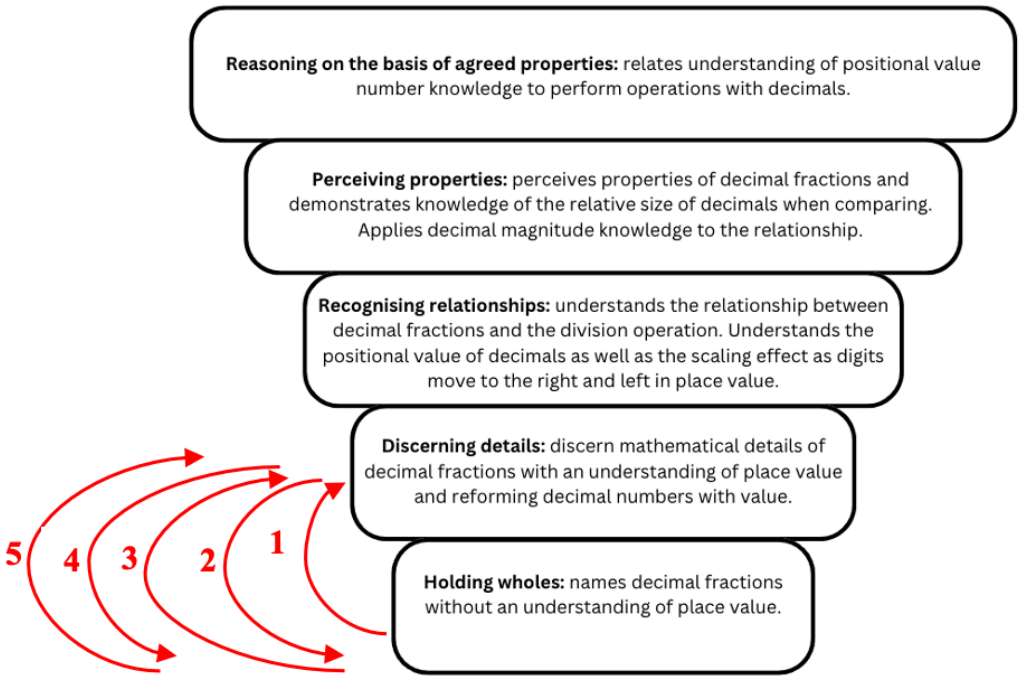


Diagram showing Tehillah's thinking pathway back-and-forth between the structures of attention with linked decimal fraction concepts.

There was limited decimal understanding communicated in Tehillah's mathematical reasoning, "the seven-points, and eight-points", however she was tuned into the capabilities on the *Zoomable Number Line* and used these to aide her answers throughout the interview. The digital functions of the *Zoomable Number Line* were capable of dynamically illustrating decimal concepts that would otherwise be difficult to show on a static number line. This appeared to be particularly useful for a student such as Tehillah who relied on the visual, symbol and dynamic elements working synergistically to show decimals located between whole numbers. Number lines are a powerful representation of decimal fractions and the abstract concept of decimal density, as they show where a decimal 'fits' between two-whole numbers, an idea that was quite difficult for Tehillah to understand. In this evidence transcript, Tehillah's attention shifted between *holding wholes* and *discerning details* to eventually reason that "there are smaller numbers in between the ones". Her manipulative interaction with the

dynamic representation encouraged a more enhanced understanding of decimals, as she was able to 'see' notions that would otherwise be 'unseeable'.

6.4 Discussion of Additional Findings

Mapping the learning trajectories that students took whilst engaging with the *Zoomable Number Line* to build conceptual understanding of decimal density. Tracing and documenting points of change in learning through analysis of each individual's verbal and physical behaviours, allowed inferences to be made as to which specific features of the digital tool prompted shifts in the students' attention. This provided evidence for evaluating the effectiveness of the virtual manipulative in promoting conceptual understanding and proficiency in decimal fractions among Year 4 students. All participants showed changes in their conceptual knowledge of decimal density as they responded to dynamic affordances embedded within the *Zoomable Number Line* that prompted shifts in their attention, leading to new learnings within a short period. The responses of the Year 4 students indicated that interacting with features such as the zoomable and scrollable number line in *Zoomable Number Line*, led to new learnings about key decimal concepts. Analysing the thinking pathways students took when freely investigating the number line showed that engagement with this dynamic digital tool stimulated thinking about the relationship between whole numbers and decimal fractions, as well as the density of numbers as an infinite system. Students' interactions with the dynamic representations of decimal fractions within the *Zoomable Number Line* generated productive cognitive conflict, challenging their existing understandings and misconceptions. Outside the research-interview setting, these moments would have been ideal teaching opportunities.

Mason (2010) views learning as a shift in attention in form and focus, referring to what is being attended to and how it is attended by the learner. His research strongly argues that engaging with a mathematical task alone is insufficient for learning and instead requires the integration of experience and sense-making, which involves transforming attention. Dynamic representations of decimal fractions can guide learners' shifts in attention. The dynamic nature of digital manipulatives forces changes in the object and structure of children's attention as they interact with mathematical content and representation features, leading to the development of embedded mathematical concepts. Conceptual development occurs when learners shift between their current knowledge, discerning new mathematical details revealed by dynamic digital tools, and perceiving relationships as properties applicable in other numerical situations. Awareness of these shifts can alert teachers to the possibility that learners are not making

appropriate shifts, prompting the use of strategies and learning tools like dynamic digital representations to effect changes in conceptual understanding.

It is acknowledged that students experience microqualities of attention in the learning journey and will switch back and forth between the multiple structures, often very rapidly (Mason, 2010). As such, it is important to notice shifts between these states of attention which can happen very quickly as these are vital in mathematical thinking. The structures are not a single progression pathway as learners will go back and forth as well as around in a cycle. In particular, noticing shifts from specific relationships to perceiving properties is pivotal, as this is the essence of mathematics. For a learner this shift unfolds while gazing to detect apparent motion produced from saccadic eye movement which can suddenly switch to awareness of details amongst a mass of other, undiscerned detail (Wei et al., 2023). As details are detected and discriminated, the mind automatically looks for relationships including differences and similarities. To do this requires something being invariant as a background against which to detect change (Mason, 2010). Recognising relationships focuses on particulars, however perceiving properties shifts to generalisable principles, and then to the particular as exemplary or paradigmatic. Formalising in mathematics is the overt action which accompanies a shift from perceiving properties to taking certain properties as definitive and so as the basis for reasoning. In each episode of evidence presented in the findings section, the student was no longer only attending to the entry-level structure of attention but was able to rapidly move between multiple structures whilst tuning into both visual and symbolic representations simultaneously to develop knowledge of concepts related to decimal fractions. For four of the students, this sequence occurred upwards of six times before they formalised certain properties as definitive and so as the basis for reasoning. This is suggestive that learners require time to articulate the relationships they recognise, and then need to confirm the relationship by searching for other instances of it within an object, before prompting a move to articulating the relationship as a property. The theoretical work of Pirie and Kieren (1991) on the nature of acquiring mathematical understanding posits a similar learning trajectory where the process of "folding back" is necessary for dynamical growth of mathematical thinking. What was observed through microgentic analysis of the students' interactions with the dynamic digital tool, was a layering of increasingly sophisticated structures of attention as a compounded 'stacking' of mathematical details that were grouped in relationships to be extrapolated as properties before being generalised as principles.

Decimal density refers to the continuity property of rational numbers whereby between any two decimals there are an infinite number of other decimals (Widjaja, Stacey & Steinle,

2008). Many young children have extensive difficulties in recognising the density of decimals, as it is void in their whole-number knowledge (Widjaja et al., 2008). The discreteness feature of whole numbers is incongruous for understanding the density of decimals (Widjaja et al., 2008). The *Zoomable Number Line* with its dynamic capabilities, assisted the students in this study in understanding the continuity aspects of decimals as they were able to continually zoom in to reveal decimals between each rational number. As Aarav stated, "it's zooming in... and showing the decimals, the tenths... and now it's showing the hundredths... it's going into the thousandths..." Understanding what a decimal fraction represents is a difficult concept therefore students should be encouraged to interact with quality representations such as the *Zoomable Number Line* to assist them in developing their decimal sense. Steinle and Stacey (2004b) argue that the number line is one of the few models that is functional for discussing the density of decimals. However, Steinle and Stacey (2004b) have commented on the limitations of a static number line, as they note that some students view it as a discrete representation. These students picture the number line as a series of isolated points, close together, not as a continuous line without any gaps (Steinle & Stacey, 2004b). The *Zoomable Number Line* seems to have the potential to reverse this thinking as its zooming functions illustrate the density of decimals, which allows students to physically see the infinite number of parts that fit in between two whole numbers. Both Andrew and Annelise had sophisticated conceptual understandings of place value in decimal fractions and were fascinated by the number lines' zooming feature to reveal the density of decimals. This attitude suggests that the *Zoomable Number Line* has sophisticated functions that have the potential to challenge and extend all students in developing their current conceptual understandings. Steinle and Stacey's (2004b) work concurs with this observation as they argue that students who understand that decimals are not confined to just tenths or just hundredths rather they can be continually divided into ten equal parts have a more robust conceptual understanding.

6.5 Concluding Remarks on the Overall Chapter

This chapter was divided into two distinct sections to present findings and discussion related to the study's overarching research question: *How do children experience conceptual changes related to decimal fractions when they attend to features of dynamic digital representations?*

The first section, titled "Changing Structures of Attention When Learning About Decimal Fractions with Digital Tools" explored the significant conceptual shifts resulting from

students' interactions with four dynamic digital representations. This part was presented in the format of a published journal article, which was included in its original form.

The second section delved into additional findings and discussions, focusing on how conceptual changes are related to shifts in attention as prompted by affordances within dynamic digital representations of decimal fractions. It specifically examined student responses to one of the digital tools, the *Zoomable Number Line*. Through engagement with this sophisticated tool, each of the six participants demonstrated changes in their understanding of *decimal density*. A microgenetic analysis of their interactions revealed the unique cognitive pathways students followed as they responded to the symbolic, visual and dynamic representations of decimals. This process allowed them to connect these representations with their existing knowledge and achieve a deeper conceptual understanding.

This chapter outlined the individual learning pathways that students took when completing the decimal fraction tasks and interacting with each dynamic digital representation, and therefore has uncovered a number of important changes in conceptual understanding related to decimals as detected through shifts in their attention. The next findings and discussion chapter seeks to demonstrate the powerful role that these four dynamic digital technologies can have in both revealing and addressing decimal misconceptions, ultimately fostering deeper decimal sense in young learners.

CHAPTER 7: STUDENTS CHANGING THINKING RELATED TO MISCONCEPTIONS ABOUT DECIMAL FRACTIONS

7.1 Situating the Chapter

The results and associated discussion in this chapter, examines how the six Year 4 students from this study overcame misconceptions about decimal fractions through engagement with the digital tools Wishball-hundredths, Decimal Strips, Zooming in on Place Value, and Zoomable Number Line. The detailed analysis highlights the misconceptions about decimals that each of the six students demonstrated in the pre-test and initial interview and then examines areas of conceptual change and the overcoming of misconceptions across the task-based interviews about each dynamic digital representation. The research question used to focus this results and discussion chapter is: *How do students overcome misconceptions about decimal fractions when they attend to features of dynamic digital representations?*

7.2 Identifying Misconceptions - Pre-test and Initial Interview

The Year 4 class (age 9 to 10 year olds) from which this study's group of participants were selected, completed a pre-test that asked them to compare pairs of decimal fractions and determine the larger value (See Methodology Chapter 5, and Appendix F). The pre-test results of the six participants are displayed in Table 7.1.

Table 7.1

Pre-test results and initial interview observations related to misconception thinking for six students selected to participate in task-based interviews

Student Name	Score /30	Primary Classification	Finer Classification	Initial Interview Observations
Aarav	20	Shorter-larger	Denominator Focussed	Understood decimals to be used as the answer to as division problem that doesn't result in a whole number. Incorrect placement of 0.1 and 0.7 between -1 and 0 whole numbers on static number line.
Andrew	29	Apparent Expert	Task Expert	Incomplete explanation of what a decimal fraction is. Student response focused on counting tenths in sequence without reference to the relationship between decimals and the division of the whole into equal parts. Justification of placing 0.7 between 0 and 1, "zero point seventy is closer to 1 because I always count decimals as going up by tenths, so I know whether it's closer to the next [whole] number".
Annelise	24	Unclassified	Whole Number	Demonstrated conceptual confusion around the part-whole relationship of decimal fractions due to the interference of their dominant whole-number knowledge. Incorrectly placed 0.1 in line with -1, and 0.7 in line with 7-wholes. Placement of 1.89 at the furthest possible location to the right along the number line. The decimal was seen as being larger than 7.7 and placed between 7 and 8 as the furthest point to the right of zero on the paper-based number line.

Tehillah	10	Longer-larger	Whole Number and Reverse	No acknowledgement of decimal 4.2 having, 4-wholes and two-tenths. Read the decimal as "forty-two". Displayed thinking of decimals as having the same numeral properties as whole numbers. Demonstrated conceptual confusion around the part-whole relationship of decimal fractions due to the interference of their dominant whole-number knowledge. Incorrectly placed 0.1 in line with -1, and 0.7 in line with 7-wholes. Placement of 1.89 was at the furthest possible location to the right along the number line. The decimal was seen as being larger than 7.7 and placed between 7 and 8 as the furthest point to the right of zero on the paper-based number line.
Valerie	18	Shorter-larger	Denominator Focussed	Attempted static number line task with incorrect placement of 0.1 in line with -2, and 0.7 in line with 0, 1.89 in line with 1-whole and 7.7 in line with 7-wholes. Reasoning to support placement of 0.7, 1.89, and 7.7 on static number line reveals that she is focused on the whole-number when estimating the position. "0.1 is here [gestured to -2 integer mark on static number line] because it looks like the smallest number, so I put it with the smallest number on the line." "0.7 goes here [gestured to 0 integer mark on static number line] because it is below 1 and has a zero." Suggests she lacks understanding of decimal density when plotting values on a number line.
Yehali	17	Longer-larger	Whole Number	No acknowledgement of decimal 4.2 having, 4-wholes and two-tenths. Read the decimal as "forty-two". Displayed thinking of decimals as having the same numeral properties as whole numbers.

	Demonstrated conceptual confusion around the part-whole relationship of decimal fractions due to the interference of their dominant whole-number knowledge. Incorrectly placed 0.1 in line with -1, and 0.7 in line with 7-wholes. Placement of 1.89 at the furthest possible location to the right along the number line. The decimal was seen as being larger than 7.7 and placed between 7 and 8 as the furthest point to the right of zero on the paper-based number line.

The initial interview assessed the participants' current knowledge of decimal fractions including their misconceptions, conceptual understanding of decimal fractions involving place value, magnitude and density, and their ability to locate decimal fractions on number lines (See Methodology Chapter 5 for details of tasks). The 'static number line' task asked students to order and place four decimal fractions on a number line and gave the opportunity to gather further information about students' misunderstandings and misconceptions.

Five of the participants, excluding Andrew, demonstrated conceptual confusion around the part-whole relationship of decimal fractions due to the interference of their dominant whole-number knowledge. Such confusion was most evident in the task that required students to place four separate decimal fractions on a static number line marked between -2 and 8. For example, Annelise, Tehillah, and Yehali, incorrectly placed 0.1 in line with -1, and 0.7 in line with 7-wholes. The mathematical reasoning that the students shared suggested they were initially focused on the digits of each decimal without considering the place values that each digit represented.

Another common observed response of the students who held longer-is-larger misconception thinking was the placement of 1.89 at the furthest possible location to the right along the number line. The decimal was seen as being larger than 7.7 and placed between 7 and 8 as the furthest point to the right of zero on the paper-based number line. This suggests that these students were ignoring the decimal point and reading each decimal as a whole number so 1.89 was being treated mathematically as 189 and therefore the largest value number in the set of decimals on the number line.

Analysis of the initial interview data revealed a range of misunderstandings related to decimal fractions, specifically in decimal place value, relative magnitude, and decimal density. These findings prompted the question of how dynamic digital tools might support students in addressing their existing misconceptions about decimals.

7.3 Changing Misconceptions – Engaging with Dynamic Digital Representations

During the data analysis phase of the task-based interviews, decimal misconceptions held by different students became evident and erroneous thinking patterns became obvious. Four key incidents of the students' learning linked to overcoming misconception thinking, and are presented as four main findings. Each finding is explored by detailing how the students changed their conceptions of decimal fractions that were unproductive patterns of thinking and overcame misconception thinking to return to a productive pathway of thinking about decimals. The interview snapshots were interpreted using the 'Shifting Attention with Decimal Concepts

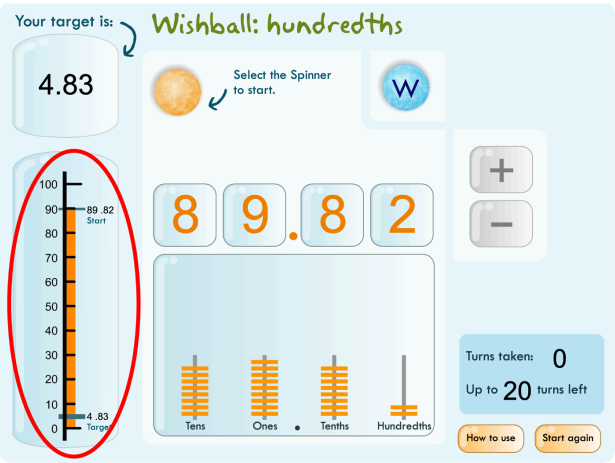
Framework' for analysis alongside a microgenetic approach to data analysis, as presented in the Methodology chapter. The data analysis revealed what specific features of each dynamic digital representation prompted attention shifts and how these translated to changes in misconception thinking related to decimal fractions. The findings will then be examined in greater depth through a detailed illustration of one student's misconceptions and how these changed over the sequence of task-based interviews. The chapter will conclude with a discussion of the broader findings, highlighting how the affordances of the four digital tools supported shifts in students' misconception thinking.

7.3.1 Finding One: Students demonstrated a shift from dominant whole-number thinking to a conceptual understanding of the decimal-fraction link through active engagement with a dynamic number line

The reliance on whole number thinking when learning about decimal fractions is well-documented in the literature (Steinle & Stacey, 2004a; Moloney & Stacey, 1996). The teaching of decimal fractions as an extension of the whole-number system often lacks a sufficient focus on the structure of place value (Fuson, 1990). This issue is compounded when educators simplify the concept by prompting students to add zeros and compare only the digits to the right of the decimal point as though they were whole numbers. Such practices reinforce a misconception that decimal fractions behave mathematically congruent with whole numbers. Therefore, it is crucial for students to understand how place value determines the relative size of decimal fractions.

Figure 7.1

Wishball-hundredths interactive activity



Note. Screenshot of the Wishball interactive activity showing the dynamic vertical number line as indicated by red circle.

All students involved in this study developed an accurate understanding of decimal fractions with regards to place value throughout their engagement with the Wishball-hundredths digital tool. Wishball-hundredths incorporates a number line and a counting frame to dynamically illustrate the effects of adding and subtracting decimals in relation to place value. Each of the students recognised the affordance of these interactive features in visually demonstrating the changing value of decimal fractions as ones, tenths, and hundredths are added or subtracted during each move of play.

The students who initially had dominant whole-number thinking, appeared to rely on the relative-sized movements of the vertical number line to help understand the decimal-fraction link whereby ten-hundredths is equal to one-tenth, and ten-tenths is equal to one-whole. This was particularly helpful for Yehali, who's change in decimal place value understanding was prompted by her engagement with the dynamic vertical number line. She observed the hundredths subtraction making only a very small movement on the number line and compared this with the movement of the ones-position subtraction to correctly reason that the hundredths must be smaller than the ones place value. Initially, she believed that hundredths would be a large value similar to whole-number value hundreds. However, Yehali's thinking changed from whole-number focused to a more appropriate understanding of decimal numeration with tenths and hundredths, whilst engaging with the Wishball-hundredths interactive activity.

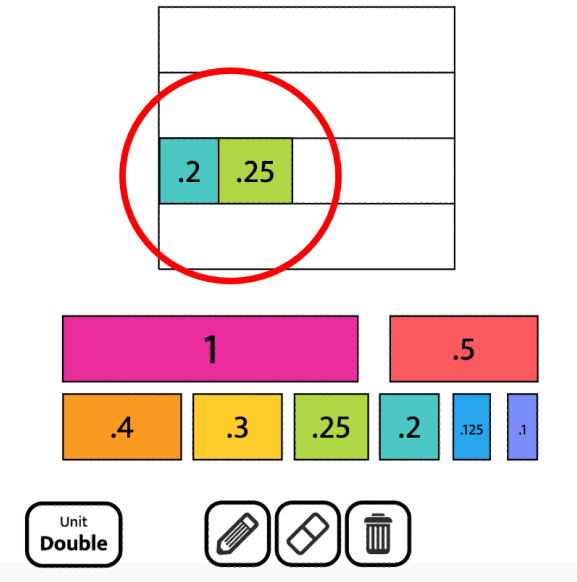
7.3.2 Finding Two: Students shifted from the Shorter-is-larger misconception to accurately recognizing relative decimal magnitude through interaction with a dynamic length representation

Two of the students, Valerie and Aarav, demonstrated dominant shorter-is-larger thinking based on their Decimal Comparison pre-test results. Steinle & Stacey (2004a) maintain that despite the smaller cohort of students with this misconception classification, it remains a challenging misconception to overcome as difficulty coordinating the size of the numerator and denominator of a fraction and also think only about the size of the parts (the tenths or hundredths) without simultaneously considering how many parts there are (Steinle & Stacey, 2004a). The misconception interfered with the two students' abilities to correctly compare pairs of decimal fractions as asked in the Decimal Strips task-based interview. Both students incorrectly selected decimals with tenths-only as always being the larger decimal value despite the alternate decimal option having hundredths and being a larger decimal

fraction. For example, 0.2 was selected as being larger in value compared with 0.25 (see Figure 7.2). A full transcript and interpretation of this evidence example is presented in the published conference proceeding paper, *Student Engagement with Dynamic Digital Representations of Decimal Fractions to Prompt Conceptual Change* (Gorman, 2024a), which is included in Appendix G.

Figure 7.2

Decimal Strips virtual manipulative



Note. Screenshot of Decimal Strips virtual manipulative with display of 0.2 and 0.25 coloured strips as indicated by red circle.

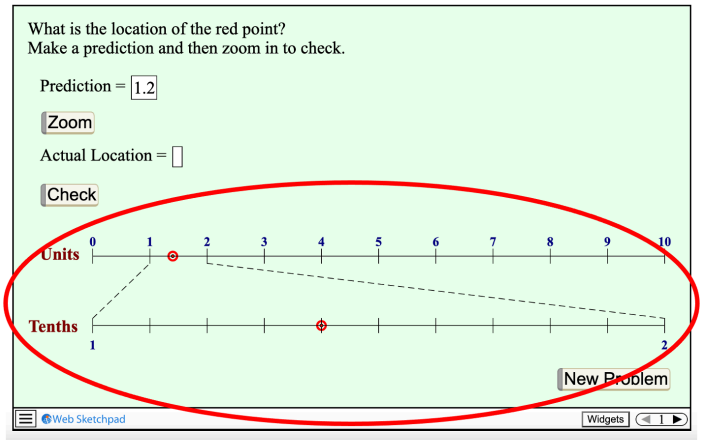
The dynamic digital tool, Decimal Strips, generated cognitive conflict as it challenged the students' current misconception thinking about decimals. It was not until they used the functions of the Decimal Strips web-app and dragged the coloured decimal tiles onto the blank fraction wall that they recognized the value of hundredths in a decimal fraction. Both students were able to self-correct their thinking by interacting with this dynamic representation as it challenged their prior understanding of decimals and allowed them to examine their errors in thinking and misconceptions. A detailed example of this misconception thinking in conflict with a decimal comparison question during the task-based interview is provided in the section *Aarav's shorter-is-larger thinking in Decimal Strips*.

7.3.3 Finding Three: Conceptions were altered from incomplete part-whole knowledge to an accurate understanding of relative decimal magnitude through engaging with a dynamic linear model

The dynamic digital representation Zooming in on Place Value helped each student participant to 'see' notions that would otherwise be 'unseeable' through its unique 'zooming in' animation between two whole numbers to show a magnified view divided into tenths (see Figure 7.3). Five of the students initially had difficulty predicting what tenths decimal was located on the number line as they held insufficient decimal magnitude knowledge to ascertain the relative placement of decimal fraction on a number line. However, by engaging with the dynamic nature of the Zooming in on Place Value web-app, they successfully solved later questions by making mathematical decisions using decimal magnitude thinking. This showed that within the timeframe of the 30-minute interview, the students were comfortable and confident enough to use the learning tool to eliminate the abstractness of decimals.

Figure 7.3

Zooming in on Place Value dynamic linear model



Note. Screenshot of the Zooming in on Place Value web-app that shows 'zoomed-in' animation of ten-tenths between one and two as indicated by red circle.

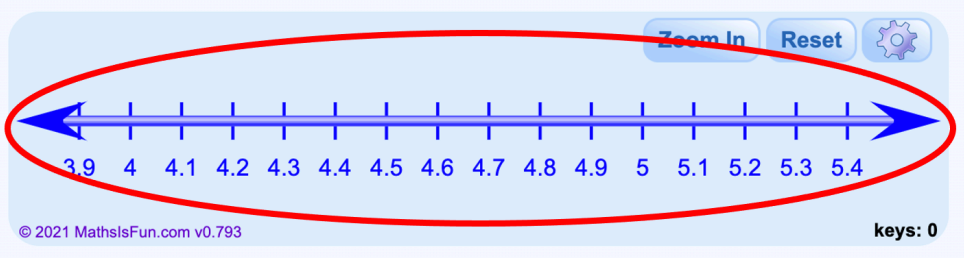
7.3.4 Finding Four: Students shifted from relying on the strategy of annexing zeros to developing a deeper understanding of decimal place value beyond whole numbers through engagement with a dynamic zoomable number line

Several of the students', Annelise, Yehali, and Aarav's, actions suggested that their knowledge of place value in decimals, particularly tenths and hundredths, was enhanced through their interaction with the features and functions of the dynamic Zoomable Number Line (see Figure 7.4). Initially, these students did not fully understand the concept of place

value in decimals as indicated by their consideration of tenths decimal place values being the same as hundredths place value. For example, the decimal 0.1 was considered to be equal with 0.01. Using the zooming and scrolling functions of the Zoomable Number Line the same students were able to correctly distinguish and describe the relative sizes of the decimal fractions they encountered in tasks. Ultimately, their mathematical mindset of whole-number thinking shifted to recognise decimal place value. The adaptability and interactive features of dynamic digital tools, which go beyond the limitations of traditional static representations, played a crucial role in helping the students successfully complete decimal fraction tasks, such as the decimal location activity in the Zoomable Number Line interview.

Figure 7.4

Zoomable Number Line dynamic digital tool



Note. Screenshot of the 'Zoomable Number Line' web-app that shows 'zoomed-in' number line with tenths between 3.9 and 5.4 as indicated by red circle.

7.4 Illustrative Case of Aarav

The four key findings presented in the previous section are now explored in greater detail using evidence episodes from one student, Aarav, as his initial understandings around decimal changed significantly whilst engaging with the dynamic digital representations in each task-based interview. Aarav was selected because, despite his decimal thinking initially being classified as shorter-is-larger, he revealed more complicated decimal thinking patterns throughout the initial interview and proceeding task-based interviews which changed in response to his engagement with the dynamic digital tools. Furthermore, his learning journey is reflective of the other students' experiences as each participant demonstrated a change in conceptual understanding to address prior misconceptions of decimals.

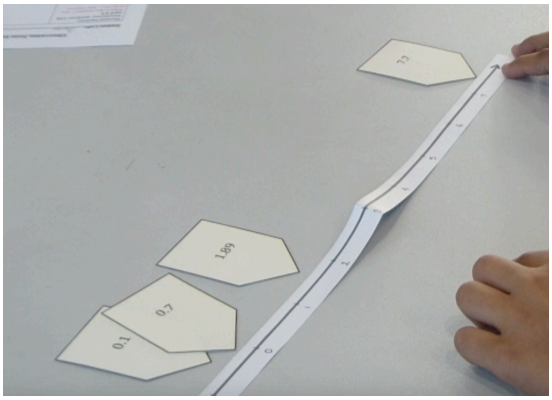
7.4.1 Aarav's pre-test and initial interview

Aarav's pre-test results showed that he held misconceptions around shorter decimal fractions being larger as detected in the Decimal Comparison Test analysis with a primary

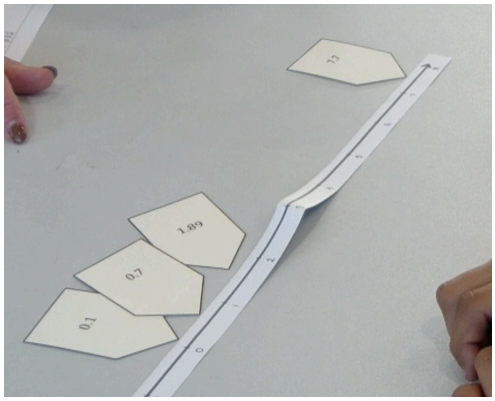
classification of shorter-is-larger and a finer classification of denominator focussed. When asked to read the decimal 4.2 aloud, Aarav confidently pronounced “four point two” and continued to explain that the decimal point means “when you are dividing something and you can’t divide it equally then you have to use a decimal to show remainders”. Aarav’s decimal fraction knowledge was still being consolidated in a number of conceptual areas. Whilst he correctly described the need for a decimal fraction as the answer to as division problem that doesn’t result in a whole number, Aarav did not demonstrate an understanding of the decimal representing parts of a unit resulting from dividing the whole into equal parts. When asked to place four decimal fractions on the static number line, Aarav’s first attempt highlighted some cognitive confusion with the decimal place value of numbers between 0 and 1-whole with the examples 0.1 and 0.7 used in the task. Initially he placed both of these numbers between -1 and 0 on the static number line before changing the position of both decimals to correctly sit between 0 and 1 (refer to Figure 7.5). Aarav was able verbalise his thinking behind this shift of understanding by rationalising that “0.1 is really close to 0 but on this side, and 0.7 is less than 1 but 6 more than 0.1”. Furthermore, his self-correction showed the beginning of understanding relative size of decimal fractions through the description of 0.7 being 6 more than 0.1, although there was no reference to the value of 6 being tenths and as such no inference can be made as to whether he understands the positional value of decimals.

Figure 7.5

Aarav’s first and second attempt static number line task



Note. Photo showing Aarav’s first attempt of static number line task with incorrect placement of 0.1 and 0.7 between -1 and 0 whole numbers.



Note. Photo showing Aarav’s second attempt of static number line task with corrected placement of 0.1 and 0.7 between 0 and 1 whole numbers.

7.4.2 Aarav's whole-number thinking in Wishball-hundredths

Aarav encountered the spinner number 7 and current starting number 98.04 early in his interactions with the Wishball-hundredths game when targeting the decimal number 63.68. He experienced cognitive confusion related to decimal place value as he viewed the decimal place values as separate digits without any relation to each other. He articulated his confusion:

> Aarav: [Tapped 7 above hundredths place value column, tapped add function, screen displayed 98.11 as starting number and dynamic counting frame carried over 10-hundredths to the tenths place value].
>
> Interviewer: Why did you pick that number to add to?
>
> A: Because the decimal is very low and it needs to go higher.
>
> I: And what happened on the screen?
>
> A: It added more than 10 here [gestured to hundredths place value] so it had to move across.

The student was clearly viewing the decimal fraction as two different whole numbers separated by a decimal point. He justified adding 7-hundredths to the starting number despite it being 30 over the target number, by stating that the decimal part of the whole number is "very low and needs to go higher". An inference can be drawn that Aarav was experiencing persistent misconception thinking related to his dominant whole-number knowledge and was treating the digits to the right of the decimal point as a separate whole number from which to add or subtract value. This inhibited his ability to partition and reform the decimal fraction with values across the decimal point. At this stage the student was holding wholes as his attention was not computing the mathematical details of a decimal fraction being parts of a unit resulting from dividing the whole into equal parts. Further evidence of Aarav's whole-number thinking being challenged during Interview 1: Wishball-hundredths is presented in the published conference proceeding paper titled *Capturing conceptual changes with dynamic digital representations* (Gorman, 2024b) and is included in Appendix H.

Interpretation: For Aarav, the feedback embedded within the interactive app prompted conceptual changes related to his decimal number knowledge. Multiple representations of the changing decimal fraction were displayed on a dynamic vertical number line, and a dynamic place value counting frame. The student utilised both of these representations to assist with gauging the relative size of decimals, particularly the scaling effects as digits move to the right and left in place value. This shift in attention from *discerning details* to *recognising relationships* demonstrated a growth in conceptual understanding, starting from Aarav only

stating decimal place values to then understanding the decimal magnitude that each place value represented.

What Aarav observed on the dynamic counting frame was a visible display of this addition operation principle, in-built constraint on 'column overflow' (Ashlock, 2010). If there are ten or more pieces of any one size place value, they will not fit on the appropriate rod of the counting frame and so ten of them must be exchanged for a single component of the next highest place value. The fluid movement between decimal place values on the counting frame modelled the reforming of decimal fractions and the student acknowledged that value can be shifted across place values to perform operations and reform the decimal number. The proceeding spinner number that appeared allowed Aarav to consolidate his understanding and also evidenced his new-found ability to reform tenths within a decimal fraction. He became highly tuned into the dynamic affordances of the Wishball-hundredths interactive activity and used these to assist with reforming decimals as knowledge of place value changed from whole-number focused to understanding part-whole value of decimals.

7.4.3 Aarav's shorter-is-larger thinking in Decimal Strips

The challenge-task required students to compare a series of pairs of decimal numbers to determine larger decimal value.

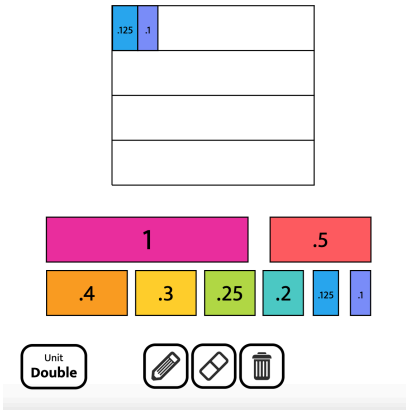
> Interviewer: Which decimal in this pair is larger? [Presented card with 0.125 and 0.1 decimals].
>
> Aarav: This one [pointed to 0.1 decimal card] but I am going to check. [Dragged turquoise 0.125 strip onto empty row of fraction wall before dragging pink 1.0 strip onto empty row below. Quickly removed pink 1.0 strip from the fraction wall. Replaced with correct blue 0.1 strip]. Actually no, this one [pointed to 0.125 decimal card] is bigger.
>
> I: Why did you change your answer?
>
> A: Because I can see that this one [gestured to turquoise 0.125 strip, see Figure 7.6] is a tiny bit longer than the other.

Figure 7.6

Aarav and Decimal Strips



Note. Photo showing student gesturing along the length of 0.1 and 0.125 decimal strips to explain why 0.125 is a larger decimal value than 0.1 decimal fraction.



Note. Screenshot of the 'Decimal Strips' digital tool that was recorded using QuickTime Player whilst Aarav was manipulating it.

Interpretation: During the decimal comparison task, Aarav incorrectly predicted that 0.1 was larger than 0.125. Upon dragging the respective decimal strips onto the fraction wall, he realized his thinking was incorrect and adjusted his conceptual understanding to include the value of hundredths adding magnitude to a decimal fraction. The dynamic ability to compare decimals on this digital tool caused a shift in attention when Aarav uncovered the value of hundredths as being an important decimal place value that changes the magnitude of decimals. Aarav was able to reconcile his "shorter-is-larger" thinking through self-correcting. When asked to explain his change in thinking the student used gesture as he referred to the visual decimal fraction area-representation of two different coloured decimal strips on the screen to assist his succinct and accurate mathematical explanation of why 0.125 is larger in value compared to 0.1. The dynamic capabilities of the decimal strips provoked a shift in attention when Aarav questioned his "shorter-is-larger" thinking that was conflicting with the decimal-fraction link represented by the moveable decimal strips. Aarav changed his conceptual thinking related to decimal magnitude by considering the relative size of decimals.

Aarav initially demonstrated limited understanding of the place values of hundredths and thousands when he first predicted that 0.1 was larger than 0.125. However, through his ability to manipulate the decimal strips and correctly link the length of each strip to the respective value of the decimal fractions, he could shift his thinking to recognizing relationships between the position of each decimal digit and its place value. A powerful change

in understanding was experienced by Aarav as he self-corrected the misconception and consequently correctly compared the two decimal fractions.

7.4.4 Aarav's consolidation of relative magnitude of decimals with Zooming in on Place Value

Aarav demonstrated observable progress in his understanding of decimal magnitude. The transcript below details the student's thinking in response to the researcher's questioning during Interview 3: Zooming in on Place Value.

Interviewer: Where is the red point located?

Aarav: Zero-decimal-one. [Typed predicted location in as 0.1 and watched the 'zooming-in' animation reveal the location between zero and one with tenths integers marked as dashes on the number line, See Figure 7.7].

I: What has it zoomed-in to show now?

A: The dot is over here [pointed to red point located at the three-tenths place on the number line between 0 and 1].

I: What do you think the actual location is?

A: Zero-decimal-three.

I: What makes you think that?

A: Because it started with the same zero here [pointed to zero integer marking on tenths number line] and then it moved to three [pointed to three-tenths integer marking on tenths number line]. [Typed second predicted location in as 0.3 and then saw the actual location revealed as 0.3].

I: You were correct. What helped you get the correct answer?

A: I realised that the number line below is lined up with the numbers up the top so I can see what the number after the decimal is.

I: And what are the numbers after the decimal?

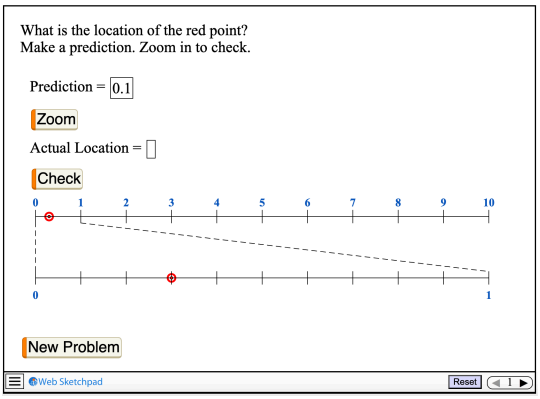
A: I think they are the tenths in between the zero and one.

I: How many tenths are between zero and one?

A: [Counted each tenths integer mark along the number line] ten… that makes sense because there have to be ten-tenths in total.

Figure 7.7

Zooming in on Place Value web-app



Note. Screenshot of the 'Zooming in on Place Value' web-app showing student's initial prediction of 0.1 and 'zooming in' animation showing tenths between zero and one as integer dashes on number line.

Interpretation: Aarav had difficulty initially predicting what tenths decimal was located on the number line due to his limited understanding of decimal magnitude and persistent whole-number thinking that currently interfered with a developing a complete understanding of decimal fraction concepts. During the second round of the Zooming in on Place Value interview, Aarav was able to extract more of the mathematical meaning from what he observed on the dynamically moving number line when it stretched out to show tenths between zero and one as his prediction correctly noted. He was able to link the animated number line with his strong prior knowledge of whole numbers to appreciate the relationship between decimals and fractions. Aarav noted that the zero remained in the same place on the second number line and therefore the red point must be between the whole numbers zero and one, "I realised that the number line below is lined up with the numbers up the top so I can see what the number after the decimal is." He expanded on this concept when reasoning that the place value after the decimal point was tenths and between two whole numbers there are ten-tenths. His more accurate second prediction of 0.3 was informed by the shift in his thinking to recognise relationships between decimals and division of the unit into ten-tenths shown as equally spaced integer marks on the dynamic number line. The student used the learning tool to eliminate the abstractness of decimals. His manipulative interaction with the dynamic representation, enhanced his understanding of decimals, as he was able to 'see' notions that would otherwise be 'unseeable'.

7.4.5 Aarav's building of decimal place value knowledge and Zoomable Number Line

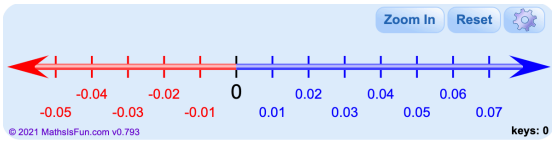
Aarav displayed significant conceptual confusion around decimal fractions across a number of tasks in the Zoomable Number Line interview. He was asked to locate the decimal 0.9 on the zoomable number line and describe its location by stating "it's between _ and _". Aarav immediately, zoomed-in to show hundredths on the number line and scrolled left from 0.12, he paused when 0.09 was shown on screen display exclaiming, "here is the number." The interviewer questioned, "is that the same number as shown on the card?" The student considered and then answered, "no it's even littler" before he continued to scroll left from 0.09 and stopped when 0 was displayed on zoomable number line. The interviewer asked, "what are you thinking?" to which Aarav replied, "I zoomed in too much because the numbers are hundredths but this one [pointed to 0.9 decimal card] is tenths". He tapped the 'zoom out' button and adjusted the number line to show tenths only (See Figure 7.8), he then scrolled back to the right along the number line and stopped upon 0.9 being shown, "I found it… zero-point-nine is between zero-point-eight and one-whole."

Figure 7.8

Aarav and Zoomable Number Line



Note. Photo showing student tapping the zoom-in function to reveal an increasing number of decimal place values around the whole-number zero.



Note. Screenshot of the 'Zoomable Number Line' web-app that was recorded using QuickTime Player whilst Aarav was manipulating it.

Interpretation: Aarav's actions indicated that his understanding of tenths and hundredths place value in decimals was developed through his exploration of the Zoomable Number Line. Initially, the student did not understand the concept of place value in decimals as suggested by his consideration of 0.9 to be the same value as 0.09. With prompting, Aarav re-examined the 0.9 decimal card and compared it with the 0.09 decimal fraction displayed on

the number line, before realising that he had scrolled to “hundredths but this one [pointed to 0.9 decimal card] is tenths”. Using the ‘zoom out’ function of the Zoomable Number Line helped Aarav correctly locate 0.9 as being “between zero-point-eight and one-whole.” Aarav’s attention shifted across this decimal comparison task. Initially, he was operating with a mathematical mindset of whole number thinking and thus an incomplete ability to discern details related to decimal place value. However, through his interactions with the features of the zoomable number line, Aarav could state the place value of digits in decimal numbers of up to two digits as well as understanding the positional value of decimals with the scaling effects as digits move to the right and left in place value. Specifically, the scrolling function prompted Aarav to change his incorrect thinking to now understand digits after the decimal point to be a fraction not a whole number. The zoom-in and zoom-out affordances facilitated a shift in attention when Aarav distinguished between tenths and hundredths with an understanding of the different place value’s relative size, an important concept related to the attention structure of *recognising relationships*.

7.5 Discussion

Students often face persistent challenges when shifting from understanding numbers solely as natural numbers to a conception of the broader number system that includes rational and real numbers. This process of conceptual change frequently gives rise to misconceptions about decimal magnitude, such as believing that longer decimals are larger or relying solely on place value column names for comparison. The six participants in this study exhibited thinking consistent with misconceptions commonly identified in the literature (e.g., Resnick et al., 2019; Stacey et al., 2001). These include both rule-based errors and more subtle misunderstandings among students who appeared to reason accurately but lacked deeper conceptual grasp of the decimal-fraction link. Addressing such misconceptions is critical, especially as mathematical content becomes more complex. This discussion explores findings across all participants highlighting shared patterns in their decimal reasoning. There is also detailed focus on one participant, Aarav, whose words and actions during the task-based interviews serve as illustrative examples throughout this discussion.

7.5.1 Whole-number thinking

Many students struggle with decimal fractions due to a tendency to apply whole-number reasoning, a challenge consistently documented in research (Irwin, 2001; Resnick et al., 2019). Decimal fractions are often introduced as an extension of whole numbers, which can

obscure the conceptual complexities of place value and part-whole relationships (Charalambous & Pitta-Pantazi, 2007). In this study, five of the six students demonstrated conceptual confusion stemming from dominant whole-number thinking. These participants showed difficulty in differentiating the additive structure of whole numbers from the multiplicative and fractional relationships inherent in decimals.

Aarav's thinking exemplified one common form of this difficulty that Steinle & Stacey (2004b) term "column overflow thinking." In his initial attempts at decimal addition, he treated each digit position as unrelated to the others. When summing values that exceeded nine in a particular column, Aarav placed the result directly in that column without regrouping. This reflected a misunderstanding of the base-ten structure in decimal notation. Several other students in the study showed similar reasoning, highlighting the need for targeted interventions that address misconceptions about place value and decimal magnitude.

Research shows that understanding decimal numeration requires coordination between students' prior knowledge of whole-number place value and their developing understanding of fractions (McIntosh et al., 1992). For instance, when comparing 0.3 and 0.34, students must not only recognise the numerators as 3 and 34 but also coordinate the implicit denominators as tenths and hundredths. This form of reasoning was initially absent in many students' responses but began to emerge as they engaged with the Wishball-hundredths digital tool. For Aarav, the feedback from the dynamic counting frame was particularly influential. He paid close attention to how values changed across place columns in response to input, helping him visualise the connections between digits in adjacent positions. This interaction supported his developing ability to coordinate fraction and place value knowledge, a shift observed in other students as well. The visual and interactive nature of Wishball-hundredths proved valuable for challenging and restructuring the whole-number-based reasoning that had previously dominated students' thinking.

7.5.2 Shorter-is-larger thinking

On the Decimal Comparison pre-test, Valerie and Aarav exhibited a prevalent form of misunderstanding known as the shorter-is-larger misconception. Although only a small proportion of students displayed this type of thinking, this aligns with findings from Steinle and Stacey's (2004a) large-scale study, which also reported a relatively low frequency of this misconception. However, Steinle and Stacey (2004a) argue that despite its lower occurrence this misconception is particularly persistent and challenging to remediate. Students often struggle to coordinate the relationship between the size of the parts (such as tenths or

hundredths) and the number of those parts, leading to flawed reasoning about the size of decimal fractions (Steinle & Stacey, 2004a).

Aarav's reasoning prior to instruction was heavily influenced by denominator-focused thinking. He correctly interpreted 0.1 as one-tenth and 0.125 as 125-hundredths, but concluded that 0.1 was greater, because "tenths are bigger than hundredths," failing to account for the number of parts involved. This reasoning reflects a disconnect between understanding the size of the fractional parts and the quantity of those parts, a pattern also present in Valerie's explanations. The use of the Decimal Strips tool helped address this misconception by creating cognitive conflict. As Aarav dragged coloured tiles onto a blank fraction wall, he could visually compare the lengths of 0.1 and 0.125. Seeing that 0.125 extended further prompted him to question his earlier reasoning and begin developing more appropriate decimal-fraction link thinking. Several students similarly revised their thinking after engaging with the tool.

This finding aligns with Swan and Marshall (2010), who argue that productive learning often occurs through experiences that challenge students' current understandings. Aarav's self-initiated exploration using the dynamic features of Decimal Strips, without teacher prompting, demonstrates the potential for dynamic digital tools to prompt conceptual change. His interaction typified a shift seen in other participants who began to use these affordances to verify or revise their decimal magnitude thinking.

7.5.3 Abstractness of decimals

Understanding decimals is a multi-dimensional undertaking. Students need to coordinate place value concepts with aspects of whole number and fraction knowledge. Aarav's responses in the initial interview did not feature reasoning that included the idea of decimals representing parts of a unit resulting from dividing the whole into equal parts. However, his mathematical thinking in this task indicated that he is building an understanding of decimal density with tenths being ten equal parts of one-whole. The dynamic stretching of the number line to zoom-in between two whole numbers was particularly helpful for Aarav when he needed to accurately describe the position of the 0.3 decimal fraction. The student was able to use the dynamic representation to eliminate the abstractness of the decimal fractions by visually presenting their relative positions on the number line. This allowed Aarav to clearly reason and problem-solve in order to produce a solution to the decimal location question. The student engaged with the dynamic zooming and stretching of the number line, which eliminated the abstractness of the decimals and clearly presented the sequence of scores for the student to see. As a result, Aarav was able to visualise the location of each tenths decimal to correctly predict

the answer to the question. The digital functions of the Zooming in on Place Value interactive activity allowed Aarav to uncover decimal concepts that a static number line is not able to illustrate.

A key affordance of the Zooming in on Place Value virtual manipulative is its ability to dynamically displaying ten-tenths between two whole-numbers. Through a 'zooming in' animation the digital tool reveals ten-tenths between two whole-numbers, and ten-hundredths between two-tenths in a two-step question sequence. The ability to stretch out the number line between two whole-numbers, enabled the student to see the continuous sequence of decimals, which assisted him when operating with different types of numbers (Teppo & van den Heuvel-Panhuizen, 2014). Young students need support when developing their decimal sense and teachers therefore need to be sensitive to students' ways of thinking and learning (Way, 2011). Quality teaching that employs quality representations such as Zooming in on Place Value can assist students in better understanding decimal fractions. This digital tool offers dynamic and interactive visualisations that enable learners to explore the base-ten structure underlying decimal notation. These representations make abstract mathematical ideas more accessible by allowing students to "zoom in" on the place value system, thereby illustrating the link between whole numbers and decimal fractions. Research emphasises the role of digital technologies in enhancing mathematical understanding, particularly through their capacity to represent mathematical ideas in multiple, interconnected ways (Sinclair & Pimm, 2015). Furthermore, Siemon et al. (2011) argue that purposeful and pedagogically sound representations are critical for fostering deep mathematical learning and supporting differentiated instruction. Thus, integrating digital representations into teaching practices can enhance students' engagement whilst avoiding the emergence of misconceptions in the domain of decimal fractions.

Representations are used in the mathematics classroom to assist students in understanding abstract concepts that are central to mathematics learning. Specific to the field of mathematics, representation refers to "the act of externalising an internal, mental abstraction" (Pape & Tchoshanov, 2001, p. 119). The affordances of the Zoomable Number Line were used to develop conceptual understanding of decimal place value, magnitude and density. The facility of the digital tool to adapt, change, and provide various interactive features and capabilities that traditional static number lines may not offer was pivotal in enabling Aarav to successfully complete this decimal location task in the Zoomable Number Line interview. By actively "scrolling backwards" Aarav could mathematically reason that the decimal 0.09 must be smaller in value than 0.9 as it was located closer to the whole number 0 on the number line. The affordances of the dynamic zoomable number line, particularly its ability to scroll

backwards and forwards infinitely along the number line sequence, stimulated learning about decimal density and how to plot values on a number line. The zoomable number line has sophisticated functions that have the potential to challenge and extend students in developing their current conceptual understandings. Steinle and Stacey's (2004a) work concurs with this observation as they argue that students who understand that decimals are not confined to just tenths or just hundredths, rather they can be continually divided into ten equal parts have a more robust conceptual understanding. The literature underscores the value of such representations. Sinclair and Pimm (2015) highlight the capacity of digital technologies to dynamically externalise abstract ideas, making them more accessible to learners. Aarav's progress, alongside similar patterns in other students, reinforces the importance of integrating pedagogically sound dynamic digital tools into mathematics instruction to support decimal sense and address misconceptions.

7.6 Concluding Statement

Examining the responses of the participants across the four task-based interviews revealed the breadth and depth of their conceptual understanding of decimal fractions encompassing decimal *place value*, *magnitude*, and *density*, which emerged through their interactions with four dynamic digital tools. Collectively, these tools provided rich opportunities for students to construct and refine their understanding of decimals by engaging with representations that were interactive, scalable and visually grounded in the base-ten system. Each tool also directly assisted the students in addressing misconceptions commonly held such as overgeneralising whole number principles, misinterpreting decimal place value, and misunderstanding the density of the number line. For example, *Wishball-hundredths* supported place value recognition and challenged misconceptions related to the position and value of digits; *Zooming in on Place Value* reinforced the recursive structure of the number system, helping to change the conception that there are no numbers between tenths and hundredths; *Decimal Strips* facilitated the comparison of decimal magnitudes, addressing the misconception that shorter decimals are always larger; and the *Zoomable Number Line* made visible the density of decimal numbers, confronting the idea that decimals are discrete or countable in the same way as whole numbers. Through these dynamic representations, students engaged with complex skills such as comparing, locating, adding and subtracting decimals, where misconceptions often surface. The interactive nature of the digital tools allowed students to test, reflect on and revise their thinking, thereby supporting meaningful conceptual change. These findings demonstrate the powerful role that dynamic digital technologies can play in

both revealing and addressing decimal misconceptions, ultimately fostering deeper decimal sense in young learners.

In particular, Aarav relied on multiple representations to support his understanding of decimal fraction concepts and facilitate problem-solving. Multiple representations have been generally defined as providing the same information in more than one form of external mathematical representation by Goldin and Shteingold (2001). Seminal work by Janvier (1987) found when teachers enriched their mathematics classrooms with multiple representations, the students were more efficient in making connections between the meaning of algebraic concepts and the way of representing them, which results in conceptual understanding. Aarav was given opportunities to connect more than one representation of a decimal place value which resulted in him deducing key properties as well as gradually generalising and abstracting the meaning behind each representation. By engaging with four different dynamic digital representations of decimal fractions Aarav developed his conceptual understanding of place value, specifically the positional value of decimals as digits move to the right and left in place value. Aarav also applied new structures of attention and levels of conceptual understanding in later interview sessions, which suggested that the multiple representations as well as the multiple opportunities to apply new learnings were important in his decimal conceptual understanding development.

A key affordance of dynamic digital representations is their ability to display multiple representations on a single screen. They can provide various types of representations including numerals, symbols, linear number lines, dynamic counting frames, relative area-models, and numbers in words, simultaneously on one screen-display. Goldin and Shteingold (2001) suggest, developing clear links between quantity, vocabulary and symbol is critical for conceptual understanding and the visual affordances of the digital tool illustrate the symbolic notation. Dynamic representations improve mathematical understanding and representational thinking of students as they can flexibly use multiple representations and are exposed to the interaction of external and internal representations (Pape & Tchoshanov, 2001). Making meaningful translations in representational modes plays a crucial role in acquisition of mathematical concepts and student engagement with multiple representations such as those found in *Wishball-hundredths, Decimal Strips, Zooming in on Place Value*, and *Zoomable Number Line* can facilitate this (Lesh et al., 1987). All students tuned into the multiple representations of the changing decimal fraction whilst interacting with *Wishball-hundredths*. They utilised the decimal numeral representation alongside the dynamic number line, and the dynamic counting frame to assist with gauging the relative size of decimals, particularly the scaling effects as digits move to the right and left in place value. This shift in attention from

discerning details to recognising relationships demonstrated a growth in conceptual understanding from only stating decimal place values to holding an understanding of the decimal magnitude that each place value represented. The next findings and discussion chapter seeks to closely examine the four digital tools and analyse how the affordances within each representation provoked conceptual changes related to decimal fractions.

CHAPTER 8: HOW FEATURES OF *WISHBALL-HUNDREDTHS, DECIMAL STRIPS, ZOOMING IN ON PLACE VALUE* AND *ZOOMABLE NUMBER LINE* PROMPT CONCEPTUAL CHANGES RELATED TO DECIMAL FRACTIONS

8.1 Situating the Chapter

The focus of the third results and associated discussion chapter is on identifying the areas of difference and similarity in conceptual understanding and strategy use prompted by the students' interactions with each of the four digital dynamic representations of decimal fractions. Microgenetic analysis methods were used to detect conceptual changes that could be attributed to the specific affordances of each digital tool. This focus of data analysis related to the research sub-question:

What features of Wishball-hundredths, Decimal Strips, Zooming in on Place Value and Zoomable Number Line, do children attend to that prompt conceptual changes related to decimal fractions?

The findings for each of the four digital dynamic representations are presented separately and detail the dynamic capabilities that appeared to force shifts in the object and structure of the students' attention when they were actively interacting with the mathematics content. The chapter concludes with a discussion of several major themes related to the unique thinking pathways that students took when reacting to the symbolic, visual and dynamic representations of decimals in each digital tool and then self-connecting these details with knowledge of decimal concepts to develop a deeper understanding.

8.2 What features of *Wishball-hundredths* do children attend to that prompt conceptual changes related to decimal fractions?

During the first task-based interview, students used the dynamic digital representation Wishball-hundredths, with their interactions prompted through a game-like structure. The findings regarding this digital tool are contained within the published article titled *Learning about decimal fractions with dynamic digital representations*. The paper has been embedded in its original, published format. The first half of the paper contains information already

presented in this thesis, with the new material beginning with the Results and Discussion section of page 13.

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Learning about decimal fractions with dynamic digital representations

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ABSTRACT

The abstract nature of decimal notation often causes difficulty for young students who struggle to grasp the meaning of the decimal point, place-value structure, and fractional magnitude. To address these challenges, teachers need to employ strategies such as visual representations, manipulatives, and interactive activities to make decimals more tangible and understandable. The study examined six students (ages 9–10) from an Australian primary school engaging with the interactive activity Wishball-hundredths. Analysis of oral responses revealed moments of conceptual change beyond knowledge of a correct procedure and strategic competence within the interactive activity. Findings suggested that students' understanding of decimal place value and magnitude shifted through engagement with the digital dynamic representations, namely a vertical number line and an animated place value counting frame. These results suggest that the dynamic representation Wishball-hundredths can be a powerful learning tool for young children's comprehension of decimal fractions.

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1. Introduction

The aim of this study was to investigate how middle-primary students interact with dynamic digital representations to demonstrate knowledge and understandings of decimal fractions. Decimals are considered to be of great significance in the primary mathematics curriculum due to their application and use in everyday life. However, decimals and the related areas of ratio and proportion, are recognised as one of the most challenging and complex areas of mathematics for young children to learn (Lamon, 2001). Students encounter complexities when exposed to rational numbers in various symbolic forms, such as ratios, fractions, decimals, and percentages, without fully understanding how to correctly use each notation (Bobis, 2011). Furthermore, decimal fractions present additional complications as part of the number-relationship needed to make sense of the number, the denominator, is not directly visible. Steinle and Stacey

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(2004) suggest that students find such a concept difficult to comprehend due to the abstractness of a "hidden" number.

Children have difficulty in understanding the abstract nature of decimals, and a plethora of misconceptions arise from incomplete understandings (Irwin, 2001; Stacey & Steinle, 2003). Representations such as concrete materials, models and drawings are used in the mathematics classroom to assist students in understanding abstract concepts that are central to mathematics learning. An important feature of mathematics education is working with appropriate mathematical diagrammatic representations, but there is uncertainty around the best representations for decimals. Dynamic representations are potentially an effective support for student learning within mathematics because of their dynamic affordances. Unlike with static representations, students are able to actively interact with the mathematics content by making conjectures and testing them immediately to experience the concept from various perspectives and receive feedback on their thinking (Orrill & Polly, 2013). In particular, learner-centred computer tools that encourage interactive engagement with mathematics, allow students to "see" notions that would otherwise be "unseeable" (Orrill & Polly, 2013). Although there is an abundance of research into the use of digital learning objects in mathematics education (Litster et al., 2019; Moyer-Packenham et al., 2016), relatively little work has focused on the role of dynamic digital representations in teaching and learning decimals. The intention of this study was to progress research in this topic by exploring how children's interactions with dynamic digital representations of decimal fractions may develop their conceptual understanding.

2. Literature review

2.1. Conceptual understanding of decimal fractions

Decimal fractions are derived from the base-ten number system. Martinie (2014) suggests when students lack an understanding of place value and the base-ten number system they experience increased frustration and difficulties with computing decimal fractions. Further difficulties arise when learners fail to recognise that decimal fractions are a special class of fractions with base-ten denominators (Martinie, 2014). Students need prior knowledge and experience with whole-numbers and common fractions to build conceptual understanding of decimal fractions as decimal representation is compounded in complexity by the merging of these rational number systems.

In mathematics education, conceptual understanding is characterised as "an integrated and functional grasp of mathematical ideas" (National Research Council, 2001, p. 118). Conceptual understanding includes being able to represent a variety of situations in different ways, to see the connections between those representations, and to know how to use each representation for different purposes (Caldwell, 2020). The growth of students' conceptual understanding is synonymous with mathematics learning. Mason's (2008) theorisation of the notion of attention has been employed to account for students' learning of mathematics. The theory is underpinned by the idea that students have to experience shifts of attention for learning to occur. As students interact with mathematical tasks and representations there are ongoing shifts in their focus and structure of attention (Attard et al., 2016). The learning of mathematics occurs through these shifts

of attention and students' developing awareness of them. Mason suggests that shifts in attention "constitute change in understanding" (Mason, 2002, p. 25).

At present, insufficient research has been conducted on how students best build a conceptual understanding of decimals. Decimal fractions are a complex representation of rational numbers that many students struggle to understand when the curriculum shifts from whole numbers to rational number in the middle-primary years. Developing conceptual understandings of decimal density, place value and relative magnitude of decimal fractions is considered key for consolidating number sense, since these properties unify all numbers (Siegler et al., 2011) and rational number magnitude knowledge is related to future mathematics achievement (DeWolf et al., 2016). Furthermore, decimal fractions need to be understood as fractions before they are introduced as an extension of the base-10 number system (Siemon et al., 2011).

2.2. Decimal fraction misconceptions

Decimal fraction sense is associated with building a conceptual understanding rather than a procedural understanding of fractions, and forms a basis for developing appropriate strategies in mathematical tasks (Way, 2011). If students do not have a strong conceptual understanding of decimals they are more susceptible to developing a set of misconceptions associated with the topic area. Misconceptions in interpreting and renaming decimal numbers often stem from reliance on inappropriate rules and procedures, leading to "longer-is-larger" and "shorter-is-larger" thinking (Steinle & Stacey, 2004). These dominant behaviours arise from clusters of misconceptions, such as whole-number thinking, column overflow, zero-makes-small thinking, and reverse thinking, which contribute to longer-is-larger errors. Similarly, shorter-is-larger misconceptions are linked to denominator-focused thinking, reciprocal thinking, and negative thinking, as explored by researchers such as Resnick et al. (1989), Stacey and Steinle (1998), and others.

Teachers need to be equipped with appropriate instructional tools that manage cognitive conflict in the context of student learning about decimal fractions. Whilst a plethora of studies have been conducted into the specific causes of student misconceptions, a persistent lack of student achievement indicates a need to identify the factors that will assist the design of successful teaching interventions (Moody, 2010; Okazaki & Koyama, 2005). Therefore, future studies should aim to investigate what instructional tools can be implemented in a mathematics classroom to assist students' in developing correct decimal fraction sense and avoid common misconceptions from arising.

2.3. Mathematics representations

Representations are widely acknowledged as a valuable mathematical tool that provide experiential learning through concrete and virtual manipulatives. The theories of Piaget (1970), Bruner (1986) and Skemp (1987) agree that concept development follows a progression from physical objects to representational forms and then abstract thought. Students' ability to create mental images and access abstract ideas relies on their experiences (Kennedy, 1986). Through interaction and manipulation of physical objects, students are able to create stronger mental images and can represent abstract ideas more

completely (Kennedy, 1986). More recent studies have claimed that mathematical representations are extremely important in developing fractional thinking (Siemon et al., 2011). Students who can efficiently partition area, set and line models, and name fractions in a variety of ways are more likely to have a deeper understanding of the link between fractions involving tenths and the base-10 number system (Siemon et al., 2011).

2.4. Dynamic representations

Dynamic representations can be an effective support for student learning within mathematics because of their dynamic affordances. Students are able to actively interact with the maths content by making conjectures and testing them immediately to "see" the concept from various perspectives (Orrill & Polly, 2013). In particular, learner-centred computer tools that promote interactive engagement with mathematics help students visualise concepts that would otherwise remain abstract (Orrill & Polly, 2013). The role of representations when engaging with digital technologies is crucial to assisting students' mathematical conceptual understanding. Digital technologies are considered "dynamic manipulable and interactive representational forms" that "mediate and are mediated by mathematical thinking and expression" (Hoyles & Noss, 2003, p. 326).

Manipulatives refer to the expansive category of mathematics tools that can be used to explicitly and concretely represent abstract mathematical ideas (Moyer, 2001). The pedagogical basis for using manipulatives embeds the notion that firsthand interaction with manipulatives assists students' mathematical understanding (Moyer, 2001). Manipulatives can be powerful tools to support sense making, mathematical thinking and reasoning when they are used as tools to support these thinking processes that build conceptual understanding (Manches & O'Malley, 2016). Virtual manipulatives are "an interactive … visual representation of a dynamic object that presents opportunities for constructing mathematical knowledge" (Moyer et al., 2002, p. 373). They support teachers in modelling abstract mathematical concepts for deeper student comprehension (Moyer et al., 2002). Both virtual and tangible manipulatives are now commonly used in primary education to support children's acquisition of mathematical concepts. Furthermore, digital manipulatives are often gamified, offering challenges or rewards to maintain interest (Russo et al., 2021).

Emergent research has focused on affordances of touchscreen virtual manipulatives. Digital manipulatives offer additional affordances when compared to their static counterparts including increased interactivity which can allow instant modification and exploration, the inclusion of immediate and specific feedback such as highlighting errors, giving hints, or confirming correct answers (Moyer-Packenham & Westenskow, 2016). Furthermore, dynamic digital representations offer a multimodal learning experience by providing various types of representations including numerals, symbols, linear number lines, dynamic counting frames, relative area-models, and numbers in words, simultaneously on one screen-display (Suh & Moyer-Packenham, 2016). Together, the research suggests that virtual manipulative touch-screen tablet apps offer an array of affordances that may influence learning, and that children access these affordances in different ways (Tucker, 2015; Tucker & Moyer-Packenham, 2014). This critical perspective demonstrates the importance of close analysis as to how technology affects and strengthens students' learning and conceptual understanding of mathematics.

3. Theoretical frameworks

The theoretical perspectives of constructivism, representations in mathematics and "shifts of attention" yield a theoretical framework useful for investigating students' interactions with dynamic digital representations of decimal fractions. In particular, a learning environment incorporating rich problem-solving tasks is suited to exploring students' conceptual understanding of decimal fractions through noticing their shifts in attention during the process of task completion.

3.1. Constructivist theory

The theoretical framework adopted in this study is supported by the work of Piaget (1970), who was seminal in introducing task-based interviews as a method to uncover children's thinking about posed problems and believed that "true understanding takes place when the student makes discoveries for themselves" (Assad, 2015, p. 17). Constructivism within mathematics research has been extensively documented by a number of key researchers (Cobb et al., 1992; Katic et al., 2009; von Glasersfeld, 1991). A major principle of Piaget's theory of constructivism suggests that students construct their understanding and knowledge through perceptions and experiences, which are themselves "mediated through previous knowledge" (Simon, 1995, p. 115). Therefore, this study recognises students' existing knowledge of decimal fractions and anticipates that individual students will respond to the digital tools in different ways. Additionally, constructivism embodies the notion that "learning is a social process" (Jorgensen et al., 2010), indicating the need for active engagement in problem-solving (Zain et al., 2012) and opportunities for the sharing of ideas, explanations and opinions (Cobb et al., 1992).

3.2. Representation theory

Representations are used in the mathematics classroom to assist students in understanding abstract concepts that are central to mathematics learning. Specific to the field of mathematics, representation refers to "the act of externalising an internal, mental abstraction" (Pape & Tchoshanov, 2001, p. 119). The improvement of mathematical understanding and representational thinking of students require flexible use of multiple representations and the interaction of external and internal representations (Pape & Tchoshanov, 2001). Making meaningful translations in representational modes plays a crucial role in acquisition of mathematical concepts and student engagement with multiple representations can facilitate this (Lesh et al., 1987).

3.3. Shift of attention theory

The theory of shifts of attention provides a scaffold for educators' understanding of teaching mathematics as well as proposing five observable attending stages that indicate what and how students are learning in the mathematics classroom. Several shifts of attention may occur in the process of thinking mathematically related to the object or focus of attention (what the learner attends to) and the form or structure of attention (how the learner attends) (Mason, 2008). Structure of attention refers to "holding wholes,

discerning details, recognising relationships, perceiving properties, and reasoning on the basis of agreed properties" (Mason, 2008, p. 35). As attention shifts from one concept to another, its structure alters and connections are made, relationships are perceived, and new ideas are appreciated. Holding wholes implies observing a mathematical object with or without focusing on mathematical characteristics. Such an action may only occur for a brief period before the learner begins discerning details which involves marking distinctions and recognising important details. Recognising relationships occurs when the learner identifies particular relations between elements of a mathematical concept such as a model, calculation, diagram or equation. Shifting attention from recognising relationships to perceiving properties of specific elements is vital to growth of mathematical understanding. Perceiving properties requires the learner to hold an "awareness of a possible relationship and look for elements to fit it, thus leading to generalisation" (Mason & Johnston-Wilder, 2004, p. 60). Reasoning on the basis of agreed properties is a form of attention that requires formal mathematical reasoning related to previously identified properties, "independent of particular objects, properties, as the only basis for further reasoning" (Mason & Johnston-Wilder, 2004; Mason, 2008, p. 37).

The theory of shifts of attention has been applied to study progressions of mathematical problem solving (Voutsina et al., 2019). It can also be applied to account for changes related to shifts in the object and structure of learners' attention of any concept of mathematics including decimal fractions. Voutsina et al. (2019) investigated dynamic changes in how problem solving strategies were communicated and conceptualised when students' shifted the structure of their attention as a result of shifts in the mathematical object being attended to. The findings revealed microqualities that influence different learners' structure of attention during phases of mathematics learning which provide significant insights for understanding qualitative dynamics of change in learning (Voutsina et al., 2019).

3.4. The relationship between the theories

Mason (2010) views learning as the shifting of attention in form and focus which is what is being attended to and how it is being attended by the learner. A strong claim emerges from his research that engaging with a mathematical task is not sufficient for learning to occur (Mason, 2010). There is need for the integration of experience and sense-making which requires transformation of attention, whereby the learner's attention is shifted from complete immersion in the task to a broader perspective that internalises the new concept or approach (Mason, 2010). Dynamic representations of decimal fractions are capable of offering direction to learners' shifts of attention. The dynamic nature of digital manipulatives forces shifts in the object of children's attention when they are actively interacting with the mathematics content as well as the structure of their attention in how they attend to features of the mathematics representation and the embedded mathematics concepts (refer to Figure 1). Understanding these shifts in what and how children are attending to features of decimal fractions is significant for advancing researchers' understanding of the dynamics of change in learning. Further detail of these specific shifts of attention relevant to decimal conceptions are explored in Section 6.5.

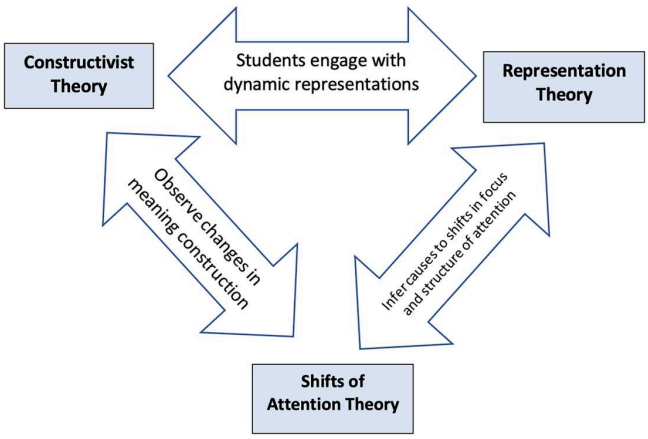


Figure 1. Diagram of relationship between three theoretical perspectives: constructivism, representations in mathematics, and "shifts of attention".

4. Study context

The aim for the research was to explore what conceptual changes emerge while children in middle-primary grades engage with dynamic digital representations of decimal fractions. Findings discussed in this paper are gathered from a broader research project exploring students' conceptual changes related to decimal fractions as prompted by interactions with dynamic digital representations. This new-found understanding of how students develop decimal sense through dynamic digital representations could translate into effective pedagogy within the primary mathematics classroom. The overarching research question sufficiently communicated the aim of the study. In addition, a sub-research question was proposed that addresses the specific dynamic digital representation of decimal fractions in focus, Wishball-hundredths. This allowed for specific data analysis of the features embedded within the digital tool that the students engaged with and interpretation of how these features prompted conceptual change in understanding of decimal fractions.

The overarching research question used to guide the study asks:

> What do observed conceptual changes suggest about 'what' children attend to and 'how' they attend to features (affordances and constraints) of the dynamic digital representations of decimal fractions?

The sub-question probes:

> What features of Wishball do children attend to that prompt conceptual changes related to decimal fractions?

5. Methodology

This study was conducted using a qualitative methodology, in which the researcher used task-based interviews combined with microgenetic methods to collect and analyse data

(Chinn & Sherin, 2014). Well designed and effectively implemented task-based interviews are able to reveal information about how students construct knowledge, their cognitive processes, and their interpretations of learning situations and tasks (Ginsburg, 1997). Microgenetic methods allow for the detailed analysis of students' reasoning, particularly for research designs incorporating task-based interviews (Chinn & Sherin, 2014; Siegler, 2006). Task-based interview has the dual purpose of observing mathematical behaviour and "… drawing inferences from the observations about the problem solver's cognition and/or affect" (Goldin, 1993, p. 303). Therefore, the tasks must be sufficiently open-ended to allow each student to respond in their own preferred way, and designed to maximise dialogue between the researcher and student (Hunting, 1997, p. 153). The researcher used probing questions to encourage further explanation, for example "Can you tell me what you are thinking?", but remained neutral regarding "correctness" of responses (Gorman & Way, 2018). The purpose of the interview was neither to teach nor to test knowledge, but rather to explore the student's ways of thinking and interacting with the mathematical representations (Gorman & Way, 2018)

The most effective method for data collection in task-based interview is video, with audio, recording (Hunting, 1997; Maher & Sigley, 2014). Video-recording the interviews was critical to collecting quality data as it was necessary to capture what was said and done by the child simultaneously, including gestures, expressions and emotional responses. In addition, a screen-recording tool from the application QuickTime Player was used to document the manipulative actions that the students' made on the digital web-app, which was accessed on an iPad. The video recordings, screen recordings, observation notes, participant's work samples, and other related metadata, formed the data for analysis.

6. Research design

Data collection for this study consisted of one task-based interview focussing on the dynamic digital representation Wishball-hundredths. The researcher worked individually with each of the six participants from a Year 4 (nine-ten years old) class in a New South Wales primary school, in a 30-min session. This provided sufficient time to collect quality data whilst being brief enough to maintain student concentration levels. The data collected from the task-based interviews depended on the "quality and incisiveness of the task" and accompanying questions (Hunting, 1997, p. 148). Therefore, a trial was conducted with a small group of voluntary adult participants to practice and refine giving task instructions and asking stimulating questions.

6.1. Participants and interview procedure

The participants of this study were Year 4 (nine-ten years old) students. This school level was selected because, according to the New South Wales state syllabus, students are introduced to decimal notation for tenths and hundredths in this year (NESA, 2012). Students have generally attained only a basic knowledge of rational numbers in the representation of decimal fractions. The syllabus stipulates these students are also expected to state and partition the place value of digits in decimal numbers of up to two decimal places (NESA, 2012). In addition, students learn to model, compare and represent decimals of up to two

decimal places using concrete materials, diagrams and number lines as well as being required to locate and compare decimals on number lines (NESA, 2012). The mathematical concepts and skills were therefore still in the early stages of development at the time of the study.

Six participants in total from one Year 4 classroom were invited to participate in this study. One criterion for participation was that the student must be able to effectively express their mathematical thinking and reasoning orally. The class teacher was asked to verify that the children nominated based on their pre-test results would be comfortable in talking one-to-one with the researcher. Although the children were not intended to be representative of the class, the researcher selected six participants that spanned a range of general mathematical achievement levels, in case there were differences in the ways in which low and high achievers interacted with the dynamic digital representation. Six children were considered sufficient to reveal a variety of student responses.

6.2. Pre-test

To commence the data collection phase, the researcher designed and administered a pre-test to all students in one Year 4 class from which the participants were selected. This pre-test required students to compare pairs of decimal fractions and determine the larger decimal, for example 0.4 and 0.457. These decimal fraction pairs were taken from Stacey and Steinle's (2003) Decimal Comparison Test which is used to classify students' thinking about decimal notation. Using the results, the researcher nominated six participants from the class that spanned a range of results and appeared to hold different mathematical thinking within the area of decimal fractions.

6.3. Materials

This section presents an overview of the digital tool Wishball-hundredths (refer to Figure 2) that the researcher used in this research study during the task-based interview.

The task-based interview focused exclusively on the interactive dynamic digital tool called Wishball. Students explored place value of decimal fractions in a game-like environment with opportunities for mental addition and subtraction. Wishball provides a structured learning environment with a specified aim for successful engagement. The digital tool features text-prompts to guide learning about the place value of decimal fractions up to tenths and hundredths. A key learning objective is that students identify the place value of each digit in a two-digit decimal fraction and the relative value of a given digit in different places within a two-digit decimal fraction. Wishball features a number line and counting frame that dynamically display the moves the user makes when adding and subtracting decimals with respect to place value. These dynamic affordances are designed to visually represent the increasing and decreasing value of the decimal fraction as tens, ones, tenths and hundredths are added and subtracted during each move of play.

The user aims to try and reach a target number by adding or subtracting in fewer than 20 moves. Firstly, the Spinner randomly serves up a digit from which the user chooses a place value to assign to the digit then either adds or subtracts it to the base number. For example, the user could receive a starting number, such as the example in Figure 2 above 31.93 and then works toward turning it into a target number, such as 1.03. The Wishball

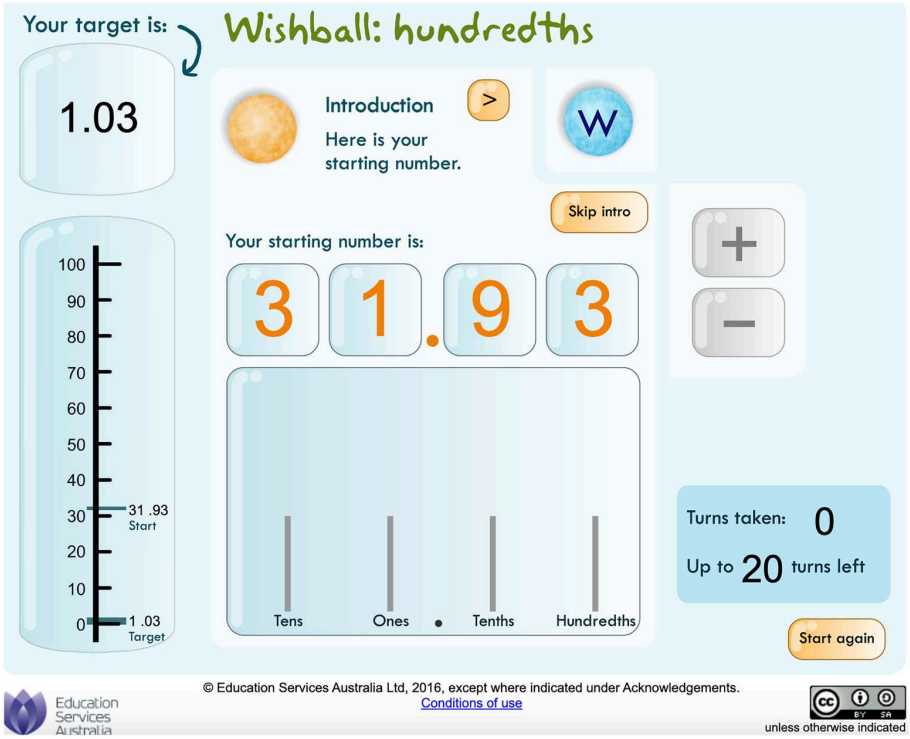


Figure 2. Screenshot of the Wishball interactive activity. Sourced from (Education Services Australia, 2014).

option, which can be used only once, allows the user to choose the final digit required to finish the game. In the Wishball challenge subseries, only one function (addition or subtraction) is made available for students to use to reach the target number. This requires students to be more careful about selection of place value so they don't "overshoot" the target number. Each participant engaged with the interactive Wishball app for between two to three rounds. This allowed for sufficient familiarisation with the dynamic digital representation and then adequate interactive time without disengagement occurring.

6.4. Data collection and analysis

The data collected from task-based interviews was analysed using a microgenetic approach. The correlation between one-on-one task-based interviews and the principles of the microgenetic method is evident when gathering intensive data from learners over a period of time to capture a record of moment-by-moment learning processes (Chinn, 2006). The aim of the approach is to detect learning changes by determining what specific shifts in the learner's attention have occurred and how they have transpired (Voutsina et al., 2019). Attaining an understanding of the precise experiences that lead to conceptual change provides a powerful insight for researchers and teachers to recognise potential approaches when teaching key mathematical concepts. Microgenetic

methods of analysis involve closely examining what learners do and say over task-based interviews, thus allowing researchers to draw conclusions about what prompts knowledge change, how learning occurs, and other aspects of the processes of acquiring conceptual understanding (Voutsina et al., 2019). The findings gathered from studying microqualities that underlie different learners' structure of attention can provide new insights for understanding qualitative dynamics of change in learning (Voutsina et al., 2019). Due to the time-consuming nature of data collection and analysis required in microgentic designs of research, generally a small number of participants are selected to provide the rich data sets (diSessa, 2017).

Microgenetic data analysis involved a combination of inductive and deductive approaches to repeated viewings of the video-audio recordings and screen-captures. This occurred at the conclusion of the task-based interview and formed an initial analysis where the researcher deduced new understandings that the participant had gathered. Firstly the key features of each child's responses was documented, with particular attention to manipulation of the digital tool, the mathematics concepts demonstrated, and the relationship between the two. These significant sayings and doings were accurately documented in chronological order. As qualitative microgenetic studies focus on tracing and documenting points of change in learning through analysis of each individual's verbal and physical behaviours, a high density of observations was required. This entailed multiple viewings of the data from the task-based interview to explore "development of individual students' knowledge, reasoning, problem solving and understanding of mathematics concepts through a qualitative lens" (Ginsburg, 2009, p. 113). Subsequently, the researcher interpreted and analysed each individual student's responses and actions to determine shifts in what the student was attending to and consequently made strong inferences about how the change had occurred. This was achieved by searching for similarities and differences across the students' profiles and identifying any patterns that emerged.

The primary researcher's analysis and interpretation of results were cross-referenced with two additional experts in the field of mathematics education research to increase validity. One expert was the researcher's tertiary supervisor and the other was a published mathematics education researcher as well as an advisor of mathematics for the NSW Department of Education. By engaging two external checking sources to view the audio-video recordings of the task-based interviews and the researcher's written analysis, the researcher established credibility in the interpretation of the qualitative data gathered.

6.5. Framework for analysis

A framework for analysis was designed by the researcher as a reference tool when analysing the verbal reasoning and manipulative actions each participant made. Table 1 presents a detailed outline of the specific decimal concepts students should to attend to when developing their conceptual understanding of decimal place value, magnitude and density. The framework correlates Mason's (2008) five components of structure for attending with key decimal fraction concepts that form a sound conceptual understanding. The value of this extended theory of shifts of attention enabled the researcher to understand what each participant experienced in the learning process and identify what elements of the dynamic digital representations prompted shifts of attention and

Table 1. Outline of specific decimal concepts required when developing students' conceptual understanding of decimal place value, magnitude and density by aligning Mason's (2008) five structures and microqualities of attention with key decimal fraction concepts, and the researcher's reasoning to connect each structure of attention with the decimal fraction learning progression (ACARA, 2020).

Structure and Microqualities of Attention (Mason, 2008)	Decimal Fraction Learning Progression (ACARA, 2020)	Connection of Theory of Structures of Attention and Decimal Fraction Learning Progression Points (Researcher's own reasoning)	Key Concept(s) of Decimal Fractions in which Conceptual Change has Occurred (Researcher's own reasoning)
Holding wholes	Naming decimal fractions.	The student verbally names decimal fractions without an understanding of place value.	Place Value
Discerning details	State the place value of digits in decimal numbers of up to two decimal places. Use place value to partition decimals of up to two decimal places.	The student can discern mathematical details of decimal fractions with an understanding of place value and reforming decimal numbers with value.	Place Value Decimal magnitude
Recognising relationships	Understand what a decimal fraction is and recognise that they arise out of division. Model, compare and represent decimals of up to two decimal places. Understanding and estimating relative size of decimals.	The student understands the relationship between decimal fractions and the division operation, as the decimal is parts of a unit resulting from dividing the whole into equal parts. Furthermore, the student understands the positional value of decimals as well as the scaling effect as digits move to the right and left in place value for example 3.509 has 35.09 tenths, 350.9 hundredths, 3509 thousandths.	Place Value Decimal density
Perceiving properties	Place decimals of up to two decimal places on a number line. Apply knowledge of decimals to record measurements, for example 123 cm = 1.23 m.	The student perceives properties of decimal fractions as they demonstrate knowledge of the relative size of decimals when comparing and applies this knowledge to the relationship with measurement and position when plotting values on a number line.	Decimal Magnitude
Reasoning on the basis of agreed properties	Flexible strategies for addition, subtraction, multiplication and division of decimals.	The student relates their understanding of positional value number knowledge to perform operations with decimals.	Place Value Decimal density

changes in understanding. As such, the researcher connected the separate structures for attending with the key decimal fraction learning progression points as outlined in the National Numeracy Learning Progression document published by the Australian Curriculum, Assessment and Reporting Authority (ACARA, 2020) by linking the specific decimal fraction concepts with each stage of Mason's (2008) theory. The framework was used alongside a microgenetic approach to analysis while the researcher discerned

observable shifts in how and what the participant was attending to as they engaged with the dynamic digital representation of decimal fractions.

Framework for identifying changes in conceptual understanding of decimal fractions.

7. Results and discussion

7.1. Introduction

The results and associated discussion, reports on six Year 4 students (nine-ten years old) from this small-scale study examining engagement with the interactive activity Wishball-hundredths. The results, interpretation and discussion sections have been combined to present a more detailed and succinct report of the data collected. The most prevalent observed changes in decimal fraction knowledge unfolded as three themes. These three key themes are used as sub-headings in this section. Each theme is explored by outlining the central finding, presenting the participant evidence, providing an interpretation using the framework for analysis as presented in the methodology section, and finally unpacking the theme in a balanced discussion. The analysis highlights the areas of difference and similarity in conceptual understanding and strategy use across the challenge of Wishball. It also interprets the participants' sayings and doings with reference to the theory of shifts of attention (Mason, 2008) which proposes five observable attending stages that indicate what and how students are learning mathematics. Students who demonstrated a strong link between decimal understanding and the base-10 numeration system engaged with the interactive activity differently to other students who displayed conceptual confusion based on their treatment of decimals as whole numbers. The examples students selected to support each theme demonstrate a variety of mathematical thinking within the area of decimal fractions and highlight the different ways learners interacted with the dynamic digital representation.

7.2. Theme one: conceptual changes from holding wholes to perceiving properties

Several shifts of attention may occur in the process of thinking mathematically related to the object or focus of attention (what the learner attends to) and the form or structure of attention (how the learner attends) (Mason, 2008). All students experienced a shift in attention and change in thinking related to decimal fractions during the timeframe of the task-based interview. Some students demonstrated this shift in conceptual understanding more rapidly than others. These same students also demonstrated more sophisticated levels of attending including *perceiving properties*. The evidence and accompanying interpretation and discussion below, outlines one such participant example showing how this shift in attention occurred as they engaged with Wishball-hundredths.

7.2.1. Theme one finding

Students experienced cognitive conflict when interacting with the student-directed representation, which assisted them in overcoming or controlling the influence of misconceptions and systematic errors.

7.2.2. Theme one evidence

Andrew was progressing through the Wishball-hundredths round when the spinner displayed 9 and the current starting number was 64.59. He tapped 9 above tenths place value column, tapped subtract function, and then the screen displayed 63.69 as the new starting number. In addition, the dynamic counting frame carried over 10-tenths to the tenths place value and then 4-tenths were erased.

The interviewer asked a follow-up question, "did you know in your mind that it would carry over?" Andrew considered the question and responded, "yes I knew that one of the ones would need to be traded to the tenths to subtract from, because I had seen it [the counting frame] move before."



Photo showing the dynamic movement of counting frame when 1-ones traded to tenths place value as indicated by red circle.

7.2.3. Theme one interpretation

In this observed shift of student understanding, Andrew acknowledged that his thinking had changed from his prior understanding of performing operations with decimal fractions. The conceptual change from *discerning details* of the place values within a decimal to then being able to *recognise relationships*, *perceive properties*, and *reason on the basis of agreed properties* by performing operations with decimals with an understanding of positional value number knowledge, was the result of new learnings that occurred during the Wishball-hundredths game. These were prompted by unique affordances embedded within the web-app including the dynamic counting frame. The student was no longer only attending to *discerning details* but was able to hold multiple structures of attention simultaneously whilst tuning into a number of intertwined conceptual principles to demonstrate in-depth knowledge of the mathematical concepts related to decimal fractions. Andrew showed his ability to flexibly use strategies for subtraction of decimals which was linked to the student attending to *reasoning on the basis of agreed properties* when he applied his understanding of positional place value to perform operations with decimals. The student knew that in order to reach the target number of 61.57 for this round he had to subtract small integers to not overshoot the number, this required consideration of the relative magnitude of each place value in the two-digit decimal fraction. In addition, Andrew knew that subtracting 9-tenths from the current 5-tenths in the

starting number would mean that value would be subtracted from the ones place value aligned with the carry over subtraction principle. Being able to predict this mathematical move by performing a mental subtraction calculation required a complex chain of reasoning and cognitive strategies including a "series of multi-step associations and procedures that involve facts, verbal chains, discriminations, concepts, and rules designed to bring about a response to a specified problem" (Kame'enui & Simmons, 1990, p. 294). In this example, Andrew linked understandings of decimal place value with the procedural knowledge and steps to subtract value from a decimal fraction.

7.2.4. Theme one discussion

A students' change in understanding is not a linear progression through structures of attention as separate modes of learning but rather a layering of more sophisticated micro-qualities of attention that build a more robust understanding of a mathematical concept (Sarama & Clements, 2009). The multiple opportunities to practice performing operations with decimals enabled Andrew to become more confident and mathematically secure with this skill. Earlier in the same interview with Andrew, the researcher has observed his confusion when faced with another subtraction move as he explained that he was considering subtracting 7 from the hundredths place value but was hesitant as he didn't know what going below zero in a place value column would do to the overall number. At that stage he demonstrated cognitive confusion with reforming place values across the decimal fraction. Andrew's attention was held by *discerning details* and stating the decimal place values without an understanding of relationship between positional values and being able to partition and reform using these decimal values. The leap between structures of attention can only be completed with conceptual knowledge of using place value to partition decimals as this is a precursor of estimating relative size of decimals. The levels of the structures of attention framework are not distinct classes of decimal fraction understanding that suddenly change. Rather, higher levels are elaborations and advancements of lower levels (Sarama & Clements, 2009). In the case of this young student, he recognised the digital tool's ability to reform decimal numbers from the feedback of new starting number and the subtraction trading principle being shown through the dynamic movement of counting frame which prompted a shift in his attention. Within the short timeframe of this task-based interview, Andrew formed an understanding of how subtracting decimals follows the same principles of renaming and trading as whole numbers and could exchange one unit for ten tenths and one tenth for ten hundredths (Martinie, 2014). Most importantly, he noticed this pattern is continuous and understood that when moving to the right of the decimal point, the numbers become ten times smaller (Martinie, 2014). The growth in this student's conceptual understanding of decimal fractions was facilitated by the dynamic affordances embedded within the interactive activity of Wishball-hundredths namely the counting frame and vertical number line.

7.3. Theme two: whole-number thinking

Five out of the six students interviewed in this study displayed conceptual confusion based on their treatment of decimals as whole numbers. These students also demonstrated difficulty coordinating the number of parts and the size of the parts, as they

had not developed the decimal-fraction link. The decimal-fraction link refers to the idea that decimals, like fractions, symbolise parts of a unit quantity (Steinle & Stacey, 2001). This was most prevalent in the reading of decimals as whole numbers for example, "one thousand, one hundred and seventy-one" when saying the decimal 11.71. The evidence detailed below clearly highlights the capabilities of the interactive activity Wishball-hundredths in addressing such errors in thinking that result from misconceptions or systematic errors.

7.3.1. Theme two finding

The students' conceptual understanding of decimals shifted from whole-number knowledge to recognising and applying the decimal-fraction link through their engagement with the interactive activity Wishball-hundredths.

7.3.2. Theme two evidence

Yehali displayed significant conceptual confusion around decimal fractions at various moments throughout both rounds of Wishball-hundredths. The student was faced with a spinner number of 7 to add or subtract from the starting number 30.36. Yehali nominated to subtract 7 from the ones place value column, and then the screen displayed 23.36 as next starting number. She noticed the movement of the dynamic vertical number line and commented "that moved a bit more that time" when comparing the magnitude of the ones place value to the previous hundredths place value subtraction. The interviewer asked a probing question "what does that tell you about that number compared to the one you chose before?" to which she replied, "that it is bigger so the one before must be smaller than ones but I'm not sure how."



Photo showing Yehali selecting 7-ones to be subtracted from the starting number 30.36.

7.3.3. Theme two interpretation

The student engaged with the dynamic vertical number line and linked what she saw being displayed with her selections of each place value within the decimal. In doing so, she attended to the mathematical idea of decimal magnitude as she began to bring

together various mathematical elements such as the positional value of decimals as well as recognising the relationship between decimal fractions and the division operation, as it is part of one-whole resulting from dividing the whole into equal parts. The student demonstrated cognitive confusion around the notion of hundredths being a smaller place value than ones as she reasoned "the one before [hundredths] must be smaller than ones but I'm not sure how". The differing sized movements on the dynamic vertical number line were in direct opposition to what she expected to happen because the student believed that hundredths would be a large value similar to whole-number value hundreds. However, when she observed the hundredths subtraction making only a very small movement on the number line and compared this with the movement of the ones subtraction she was able to correctly reason that the hundredths must be smaller than the ones place value. This was evidence of the student beginning to *recognise relationships* between decimal fractions and the division operation, as decimals are the result of dividing one-whole into equal parts. As such, Wishball-hundredths has great potential for developing understanding of decimal numeration because it incorporates numbers of different place values from hundredths up to tens.

7.3.4. Theme two discussion

Martinie (2014) suggests when students lack an understanding of place value and the base-ten number system they experience increased frustration and difficulties with computing decimal fractions. Yehali demonstrated limited knowledge of the connection between fractions and decimal notation. Initially, she was unable to recall the multiplicative nature of the base-ten place value system in her work with whole numbers as she could not divide each place value by 10 and extend this knowledge to tenths, and then hundredths (Siemon et al., 2011). As such, the student had persistent whole-number thinking that interfered with developing the decimal-fraction link. However, the dynamic affordance observed by Yehali prompted a shift in thinking as she questioned her prior understanding of digits in a decimal fraction having uniform value when she recognised evidence in conflict with her mathematical thinking. Yehali deepened her understanding of decimal fractions when she acknowledged the concept of place value as she verbalised that hundredths must be a smaller value compared to the ones place value. Children need to recognise the decimal unit a digit represents in order to form a complete understanding of place value in decimals (Nuerk et al., 2014). Yehali could identify place values and compare the differing magnitudes, which is an important first stage in decimal conceptual acquisition and essential for further knowledge development (Nuerk et al., 2014).

7.4. Theme three: place value and relative size of decimals

Three students exhibited thinking which indicated that they were treating decimals as another whole number to the right of the decimal point. The students' persistent whole-number thinking led them to believe, in the context of adding and subtracting decimals, that each place value represented the same magnitude. The dynamic place value counting frame enabled the students to correctly compare the relative sizes of each digit in the two-decimal place number, which assisted them in discovering the concept of place value in decimals.

7.4.1. Theme three finding

Students were able to understand the relative size of decimals by challenging incorrect understandings and simultaneously developing strategies for dealing with decimals as they reacted to the dynamic displays and feedback of Wishball-hundredths.

7.4.2. Theme three evidence

Tehillah recognised the spinner displayed 6 for the current starting number 67.98. She explained, "I need to pick a high number and minus it because if we only use a small number and minus it there won't be much change." The researcher probed for further reasoning, "what do you mean by small number?" Tehillah answered, "these two numbers [gestured to 6-tens and 7-ones]." After some thinking time, she decided "I'm going to pick the nine [gestured to 9-tenths] and minus the six." Before following through with this action she reconsidered her mathematical thinking and said, "Actually I think I'll minus 60 because it must be bigger than this [gestured to 0.6] which has a zero at the start". As such, Tehillah tapped 6 above tens place value column, tapped the subtract function, and the screen displayed the new starting number 7.98. The interviewer asked a follow up question, "did you notice anything on the screen change then?" The student evaluated, "yes the start number is now next to the target here [gestured to dynamic vertical number line]."



Photo showing the dynamic movement of vertical number line when 6-tens were subtracted from the tens place value as indicated by red circle

7.4.3. Theme three interpretation

Tehillah referred to 6-tens and 7-ones as being small numbers that she wouldn't subtract from. She reasoned that when compared with subtracting value from 9-tenths and 8-hundredths, it wouldn't make enough of a difference to the number in making it smaller and closer to the target number of 4.49. With this thinking, the student had transitioned back to the structure of attention *holding wholes* as she was viewing each digit with the same ones place value rather than their relative positional value of tens, ones, tenths and hundredths. A feature of the Wishball-hundredths interactive activity is the numerical display of the relative sizes of the spinner number for each place value option. For example, when the spinner shows 6 it is displayed as 60 for tens, 6 for

ones, 0.6 for tenths, and 0.06 for hundredths. The student tuned into this detail and was able to discern mathematical details by *recognising relationships* between the digit and which place value it held within the number. She uncovered this concept for herself as she reconsidered her prior understanding of place value and attempted to comprehend why the digit was shown as a different numeral depending on which place value column it was displayed on top of. As these two conflicting ideas were reconciled, she powerfully uncovered that the positional value changed with the scaling effect as digits moved to the right and left in place value. Tehillah's new conceptual understanding of decimal place value was reaffirmed from the feedback of the dynamic vertical number line moving the orange linear display lower towards the target number as she predicted it would (*see photo of student gesturing to new starting number displayed on the dynamic vertical number line*).

7.4.4. Theme three discussion

Many children have trouble understanding decimals particularly those who confuse them with whole numbers or with fractions (Irwin, 1996). Young learners find it very difficult to understand decimal fractions as an extension of place value as used for ones, tens, hundreds, and thousands and even more difficult to comprehend that decimals are always the result of a division (Irwin, 2001). A major shift in attention needs to occur for a student to understand decimals as a multiplicative structure where each place value is ten times smaller than the column to the immediate left of it. Whole number place value only requires students to apply additive thinking to successfully engage with the mathematics, whereas in order to truly understand decimals they must grasp this more complicated multiplicative understanding (Irwin, 2001). Wishball-hundredths facilitated a more complete understanding of decimals as it involved facing ideas that conflicted with what Tehillah already knew about whole numbers. She could appreciate that the further right the number sat away from the decimal point the smaller its relative value was as a result of equal division into 10 or 100 parts depending on whether it was the tenths or hundredths place value. The literature argues that children need a good understanding of decimal place value concepts before moving on to working with these numbers in operative problems (Irwin, 2001; Sowder, 1997; Wearne & Kouba, 2000). Thus highlighting the power of a learning tool such as Wishball-hundredths which is capable of prompting this important preliminary knowledge development.

8. Conclusions

This study explored how Year 4 (9–10 years old) students engaged with a dynamic, digital representation of decimal fractions embedded within the interactive activity, Wishball-hundredths. The participants demonstrated conceptual changes related to decimal fractions as they reacted to the dynamic affordances which prompted shifts in their attention and consequently new learnings were formed, within a short amount of time. The responses of the Year 4 students revealed that interacting with the animated vertical number line and counting frame, stimulated thinking about the relationship between whole numbers and decimal fractions, as well as place value and the relative size of decimals. The students' interactions with the dynamic representations of decimal fractions embedded within Wishball-hundredths, at times, generated productive cognitive

conflict, as the representations challenged their existing understandings and misconceptions. Outside of the research-interview situation, these would have been ideal "teaching moments".

Applying the theory of shifts of attention (Mason, 2008) and using the researcher-designed framework as an analysis tool provided a practical scaffold for plotting each participant's progression of understanding as they worked to achieve the aim of the Wishball-hundredths game. The framework proved to be useful in detecting and mapping the non-linear learning path students took when making the delicate shift to abstraction. Some students appeared to experience a shift in understanding as a direct result of interaction with the Wishball tasks and representations of decimals. Specifically, the dynamic affordances focused the students' attention on important decimal concepts that changed their current mathematical thinking by providing an extended awareness to move beyond their current conceptual understanding. This analysis framework has the potential to help describe how students learn important decimal fraction concepts and may prove to be a useful tool in further research into decimal knowledge progression.

Although six students provided sufficient variety in responses to satisfy the exploratory nature of this study, a larger number of participants from several different schools would allow more robust conclusions to be drawn about the effectiveness of the Wishball Hundredths digital game for supporting shifts in understanding of decimal fractions. The educational design of Wishball-hundredths targeted place value understanding and the associated relative magnitude of digits, and the changes in students' thinking detected in this study were directly related to these concepts. However, there are other important concepts for understanding decimals, such as number density, that are not addressed by Wishball, and were not detected in the responses of the participants.

While keeping in mind the inferential limitations of a small-scale study, the findings clearly indicate the potential of dynamic, digital representations to be used as effective learning and teaching tools to develop conceptual understanding of decimal fractions, and to address previously established misconceptions.

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Previous publications

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Ethical approval

The ethical aspects of this study have been approved by the Human Research Ethics Committee (HREC) of The University of Sydney 2022/893 according to the National Statement on Ethical

Conduct in Human Research (2007). Written consent was granted from the parent or legal guardian of the participants and written consent was also given by the young students involved in the study.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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8.3 What features of *Decimal Strips* do children attend to that prompt conceptual changes related to decimal fractions?

During the second task-based interview, students used the dynamic digital representation *Decimal Strips* to complete a number of researcher-designed tasks. Decimal Strips allowed the students to explore the relative magnitude of decimal fractions by clicking and dragging coloured tiles representing different decimal values (1-whole, 0.5, 0.4, 0.3, 0.25, 0.2, 0.125, 0.1) onto a decimal fraction wall. The tasks included ordering decimal fractions in ascending and descending order, comparing decimal numbers to identify the larger one, and matching decimal fractions with their common fraction representations.

The dynamic area-model representation of decimal fractions up to thousandths appeared to encourage conceptual understanding of the relative magnitude of decimals as the students dynamically shifted decimal tiles to compare their relative sizes. This was particularly useful when directly comparing a pair of decimal fractions to determine the larger decimal value and also when comparing equivalent decimals such as using two 0.125 tiles and directly comparing the value with a single 0.25 tile to visually see the equal values. A number of the young students' initially thought that the more digits a decimal fraction number has corresponded to it being a larger value. The dynamic affordance of the students being able to compare the three-digit decimal of 0.125 with the two-digit decimal 0.25 and seeing that it had a smaller value was a powerful conceptual learning for the students to uncover.

The key features of *Decimal Strips* that stimulated changes in conceptual thinking are explored in greater detail within the two evidence snapshots below.

Evidence Snapshot: Yehali and Decimal Strips

Yehali was asked to compare a series of two decimal numbers to determine the larger decimal value whilst completing the *Decimal Strips* task-based interview. One comparison challenge required the student to decide whether 0.9 or 1.0 was a larger decimal. Yehali initially answered the comparison question with insufficient decimal knowledge and dominant whole-number thinking. The transcript below highlights the change in thinking that Yehali experienced during the *Decimal Strips* interview.

Interviewer: Which is larger? [Placed card with 0.9 and 1.0 in front of student].

Yehali: The nine.

I: Can you prove your answer on the Decimal Strips screen?

Y: [Dragged red 0.5 strip and 0.4 strip to represent 0.9 decimal fraction] This is the decimal nine." [Dragged blue 0.1 decimal strip onto the same row] No I don't think that can be right. [Removed blue 0.1 decimal strip from fraction wall].

I: Why did you take that decimal strip off?

Y: Because it's not decimal one [0.1] it's a one and zero [1.0] so I need a different colour strip. [Dragged a pink 1-whole strip onto the empty row below, See Figure 8.1].

I: So which decimal number is larger?

Y: The one.

I: What made you change your thinking?

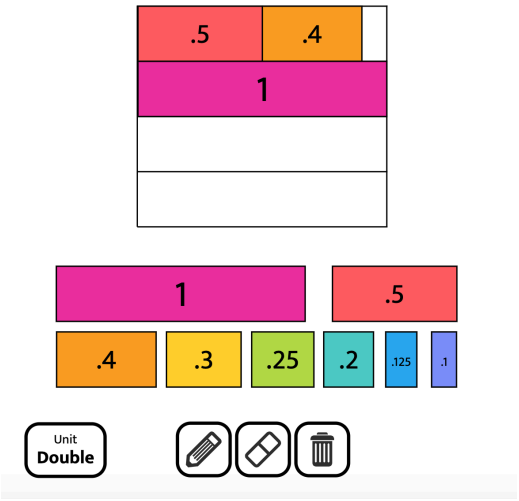
Y: I realised that the numbers were the other way around when I moved the strip and that meant the one was actually bigger than the decimal nine [0.9].

Figure 8.1

Photo and Screenshot of Decimal Strips display used by Yehali



Note. Photo showing Yehali using the area-representations of 0.5 and 0.4 to represent 0.9, and 1.0 decimal strips to explain why 1.0 is a larger decimal value than 0.9 decimal fraction.



Note. Screenshot of Decimal Strips virtual manipulative with display of 0.5 and 0.4 to represent 0.9, and 1.0 coloured strips as placed by Yehali.

The area representation seems to be a powerful model for comparing decimal fractions. It also appeared to encourage conceptual learning related to decimal place value. This was evident when Yehali attempted to represent 1.0 using a 0.1 decimal strip without acknowledging the order of the digits and simply focusing on a one being smaller than a nine as she was comparing the decimal 0.9 to 1.0. However, her attention shifted when she noticed that the 0.1 decimal strip she used did not have the digits in the correct order and instead she

needed to use a different strip to accurately represent 1-whole rather than one-tenth. In her reflection on the challenge posed she stated, “I realised that the numbers were the other way around when I moved the strip and that meant the one was actually bigger than the decimal nine [0.9].” A clear shift in attention was observed when Yehali moved from *holding wholes* and only reading the digits within each decimal using whole-number knowledge to *discerning details* as she acknowledged the place value order before and after the decimal point is important when comparing the relative values of decimals. Yehali stated that her change in understanding was materialised through the manipulation of the decimal strips as she considered the size of the strip being moved and also the symbolic decimal notation on each strip. Yehali’s existing misconception thinking “whole-number thinking” (Steinle & Stacey, 2004a) was being challenged through her engagement with the virtual manipulative as she dynamically compared the decimal lengths in this area-model representation. A significant first step to overcoming her erroneous thinking.

Furthermore, Yehali noticed that capabilities of the dynamic digital representation in comparing decimal fractions and as such independently nominated to use the dynamic affordances to assist her in checking her prediction to later comparison questions. The observation of her autonomously eliciting *Decimal Strips* to assist with completing decimal tasks attests to the dynamic representation being an accessible digital learning tool that students feel confident using to develop decimal fraction knowledge related to decimal magnitude.

Evidence Snapshot: Aarav and Decimal Strips

Aarav also encountered cognitive confusion with the same challenge-task during Interview 2: Decimal Strips, which required students to compare a series of two decimal numbers to determine the larger decimal value. The transcript below details the mathematical conversation between the interviewer and Aarav.

Aarav: The 0.2 is larger than 0.25.

Interviewer: Can you prove it?

A: [Dragged blue 0.2 strip, and green 0.25 strip onto blank fraction wall row]. Oh, that one [pointed to green 0.25 strip] is actually bigger because it has a five at the end.

I: What made you change your thinking then?

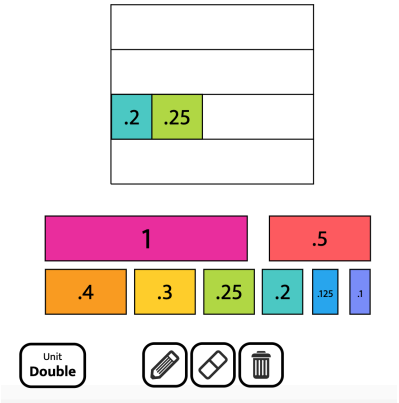
A: I saw the size of this one [gestures to green 0.25 strip, See Figure 8.2] is slightly bigger.

Figure 8.2

Photo and Screenshot of Decimal Strips display used by Aarav



Note. Photo showing Aarav gesturing along the length of 0.2 and 0.25 decimal strips to explain why 0.25 is a larger decimal value than 0.2 decimal fraction.



Note. Screenshot of Decimal Strips virtual manipulative with display of 0.2 and 0.25 coloured strips as placed by Aarav.

During the decimal comparison task, Aarav incorrectly predicted that 0.2 was larger than 0.25. Upon dragging the respective decimal strips onto the fraction wall, he realised his thinking was incorrect and adjusted his conceptual understanding to include the value of hundredths adding magnitude to a decimal fraction. The dynamic ability to compare decimals on this digital tool caused a shift in attention when Aarav powerfully uncovered the value of hundredths being an important decimal place value that changes the magnitude of decimals. Aarav was able to reconcile his "shorter-is-larger" thinking through self-correcting. When asked to explain his change in thinking the student used gesture and referred to the visual decimal fraction area-representation of two coloured decimal strips on the screen to assist his succinct and accurate mathematical explanation of why 0.25 was larger in value than 0.2. The dynamic capabilities of the decimal strips provoked a shift in attention when Aarav questioned his "shorter-is-larger" thinking that was conflicting with the decimal-fraction link represented by the moveable decimal strips. A powerful change in understanding was experienced by Aarav as he self-corrected the misconception preventing him from correctly comparing two decimal fractions.

8.4 What features of *Zooming in on Place Value* do children attend to that prompt conceptual changes related to decimal fractions?

Students used the interactive digital tool *Zooming in on Place Value* to place decimal fractions on a number line during the third task-based interview. The in-built task required students to determine the location of a decimal fraction and self-check the answer using an

embedded Zoom function. Students interacted with the unique affordance of the dynamic model as they estimated a given decimal point's location, then used the Zoom button to expand the interval between whole numbers, revealing a magnified view divided into tenths. Through a 'zooming in' animation the digital tool reveals ten-tenths between two whole-numbers, and ten-hundredths between two-tenths in a two-step question sequence. This gradual release of information allowed each student to build knowledge of decimal density as the continuity property of rational numbers, whereby, between any two decimals there are an infinite number of other decimals (Widjaja et al., 2008). The dynamic representation displayed clear links between quantity, vocabulary and symbol which was critical for the students' conceptual understanding as the visual affordances of the digital tool illustrated the symbolic notation. The unique affordances of *Zooming in on Place Value* that students engaged with to develop decimal knowledge are explored in two detailed evidence snapshots below.

Evidence Snapshot: Annelise and Zooming in on Place Value

Interview 3: Zooming in on Place Value, featured a dynamic number line that Annelise engaged with to learn about decimal density. Annelise began thinking aloud by saying the red dot was positioned at "one and a quarter" and was unable to convert the mixed fraction to a decimal. She typed the predicted location in as 1.2 and watched the 'zooming in' animation reveal the location between one and two with tenths integers marked as dashes on the number line (See Figure 8.3). The interviewer asked a series of prompting questions that revealed changes in Annelise's understanding of decimal density.

Interviewer: What is each of the integer dashes showing?

Annelise: They are showing the numbers between one and two.

I: What numbers are between one and two?

A: I can see one-point-one, one-point-two, one-point-three, one-point-four, one-point-five, one-point-six, one-point-seven, one-point-eight, one-point-nine.

I: What were you counting by then?

A: I was going up by tenths.

I: Are you able to tell me how many tenths are between one and two?

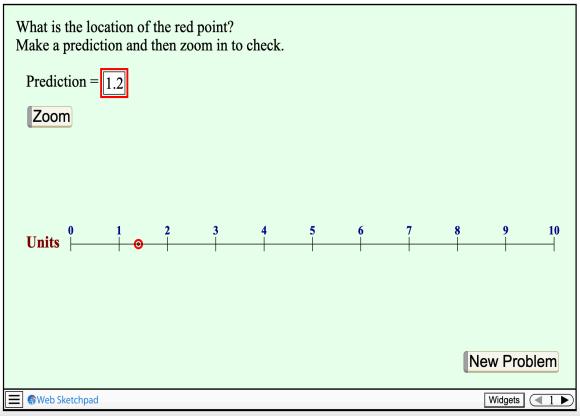
A: [Pointed to each tenth integer marking while counting in her head] There are ten-tenths on the number line… oh that makes so much sense… ten-tenths is the same as one-whole.

I: Can you tell me the actual location of the red point using your new thinking?

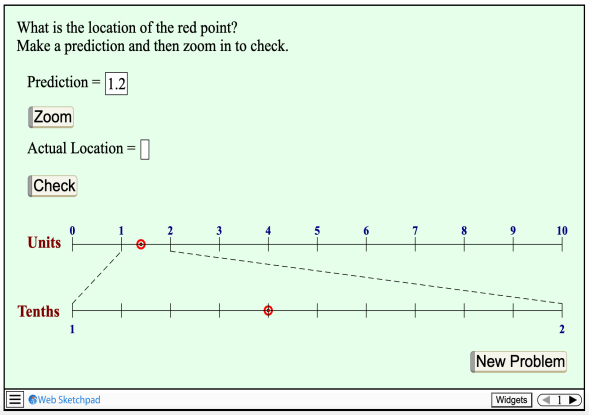
A: It's at zero-point-four.

Figure 8.3

Screenshots of Zooming in on Place Value display used by Annelise



Note. Screenshot of the 'Zooming in on Place Value' web-app that shows Annelise's initial prediction of 1.2 as red point location on number line.



Note. Screenshot of the 'Zooming in on Place Value' web-app that shows 'zoomed-in' animation of ten-tenths between one and two.

Annelise had difficulty initially predicting what tenths decimal was located on the number line, however with assistance in the form of the 'zooming in' animation as the interval between two whole numbers expands showing a magnified view divided into tenths, she successfully solved later questions. The dynamic animation of stretching the number line to reveal tenths between two whole-numbers prompted the student to consider the concept of decimal density and shifted her attention to understand that ten-tenths are located between any two whole-numbers. Annelise was *perceiving properties* integral to a complete conceptual understanding of decimals, including decimal density. She then instantly employed the help the dynamic zooming in on decimals to aid her in correcting her prediction. This showed that within the timeframe of the 30-minute interview the student was comfortable and confident enough to engage with the dynamic features of this app when solving a series of decimal fraction location questions. Annelise engaged with the dynamic nature of the *Zooming in on Place Value* web-app to assist with locating tenths decimal fractions on a number line. The student used the highly effective learning tool to eliminate the abstractness of decimals. Her manipulative interaction with the dynamic representation, enhanced her understanding of decimals, as she was able to 'see' notions that would otherwise be 'unseeable'.

Evidence Snapshot: Yehali and Zooming in on Place Value

Yehali's default numerical thinking was focused on whole numbers only. She did not hold an understanding of decimal density and therefore initially couldn't predict decimals located between whole numbers in the *Zooming in on Place Value* interview. The interactive

activity displayed the decimal 0.9 with the question, “what is the location of the red point? Can you make a prediction?” Yehali’s first guess was “one”, so she typed the predicted location in as 1.0 and watched the ‘zooming in’ animation reveal the location between zero and one with tenths integers marked as dashes on the number line. The researcher probed, “what did you see happen just then?” Yehali replied, “the numbers stretched out.” The screen displayed a second question, “what is the actual location of the red point?” The student answered “nine” and typed her second predicted location in as 9.0 and then saw the actual location revealed as 0.9. The researcher checked for understanding, “what is the actual location written as?” Yehali read, “zero-and-nine.” A follow-up question was posed, “why is that the actual answer for the location of the red point on the number line?” Yehali reasoned, “maybe because there is no number here it’s just zero [pointed to zero integer marking on number line] and it hasn’t gone to one yet [pointed to one integer marking on number line] so it’s in the middle somewhere.”

Yehali’s current thinking involved holding wholes as her understanding was constrained by the whole number details presented in the first-step of the interactive activity. When Yehali observed the animated number line dynamically zoom-in to show ten-tenths between zero and one, she used her whole-number thinking to change her prediction to “nine” as she was viewing the integer markings having ones value rather than tenths. The feedback Yehali received from the interactive activity prompted her to reconsider her dominant whole-number thinking. When she noticed that her second prediction of “nine” was incorrect she needed to produce a mathematical reason why the actual location was zero-point-nine. By closely viewing the dynamic number line she grasped a more accurate understanding that included some decimal knowledge, “maybe because there is no number here it’s just zero and it hasn’t gone to one yet so it’s in the middle somewhere.” Yehali demonstrated a shift in attention to recognising relationships when she understood that between two whole numbers, decimal values are located, the concept of decimal density.

8.5 What features of *Zoomable Number Line* do children attend to that prompt conceptual changes related to decimal fractions?

In the fourth task-based interview, students engaged with the *Zoomable Number Line* virtual manipulative. The digital tool represents decimal fractions on a number line dynamically through a zoom-in function. The interview consisted of five tasks of increasing difficulty, designed to encourage conceptual interaction with decimal notation and number line positioning. Tasks included locating one and two decimal place numbers on the digital number line, comparing pairs of decimal numbers, identifying the next hundredths decimal after a given

fraction, and solving a multi-step problem. The *Zoomable Number Line* with its dynamic capabilities was found to be capable of assisting students in understanding the continuity aspects of decimal fractions as well as the key concept of decimal density. The students were able to continually zoom-in on the dynamic number line to reveal decimals between each rational number. The number line is one of the few models that is functional for discussing the density of decimals, however a static number line holds limitations with some students viewing it as a discrete representation (Steinle & Stacey, 2004b). The *Zoomable Number Line* seemed to override this erroneous thinking that some students held, as its zooming functions illustrated the density of decimals, which allowed the students to physically see the infinite number of parts that fit in between two whole numbers. The two evidence episodes below highlight the key features of the *Zoomable Number Line* that prompted changes in decimal conceptual thinking.

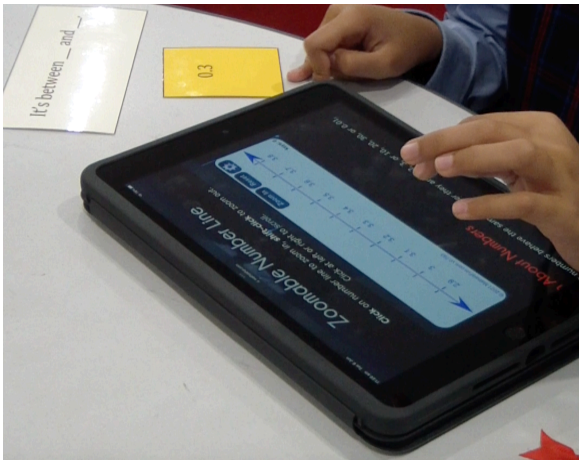
Evidence Snapshot: Yehali and Zoomable Number Line

During the Zoomable Number Line interview students were asked to locate a series of decimal fractions and describe the position on the number line by stating "it's between _ and _". Yehali was a student who demonstrated significant conceptual confusion around decimal fractions across a number of tasks in the *Zoomable Number Line* interview. The transcript details the verbal exchange between the researcher and Yehali, during this task in Interview 4: Zoomable Number Line.

> Interviewer: Can you describe the location of this decimal [pointed to 0.3 decimal card] on the number line by stating it's between __and __?
>
> Yehali: Two and four… [pointed to whole number 3 on number line].
>
> I: Does that number look like the one on the decimal card?
>
> Y: No.
>
> I: Can you use the zoomable number line to help you find the decimal location?
>
> Y: [Zoomed in around the whole number 3 to reveal tenths].
>
> I: Can you see the decimal?
>
> Y: No all of these numbers start with three-decimal-something.
>
> I: What do you need to do to locate the correct decimal?
>
> Y: Scroll backwards to zero because it's smaller. (See Figure 8.4)

Figure 8.4

Photo and Screenshot of Zoomable Number Line display used by Yehali



Note. Photo showing Yehali attempting to locate 0.3 by zooming-in to reveal tenths around the whole-number 3.



Note. Photo showing Yehali's successful location of 0.3 after "scrolling backwards to zero".

In this episode snapshot, the student was asked to locate the decimal 0.3 on the *Zoomable Number Line* and describe its location by stating "it's between _ and _". Yehali instantly responded, it's between "two and four". When prompted by the interviewer she zoomed into show tenths and was then able to locate 0.3 by "scrolling backwards to zero because it's smaller". The next question asked the student to locate 0.12 and describe its location on the number line. For this task, Yehali instantly zoomed into reveal hundredths around the 0 and was then able to scroll along accurately locating 0.12. Yehali presented initial confusion around decimals due to the presence of whole-number thinking. However, as a result of her interaction with the zoomable number line, particularly working through the decimal location task, she was able to question her current understandings and begin to develop more appropriate decimal-fraction link thinking. Consequently, Yehali's structure of attention had shifted from *holding wholes* with an incorrect understanding of decimal place value to a more accurate internal representation of decimals by *discerning details* of place value to help locate decimals on a number line. This was most evident when Yehali reasoned that she needed to "scroll backwards to zero because it's smaller" as she now understood that 0.3 was a smaller decimal than 3.0 by comparing the wholes and tenths place values. Furthermore, Yehali *recognised relationships* between digits in the decimal as she understood the positional value as they moved to the right and left and deduced that three-tenths was a smaller value than three-wholes.

Evidence Snapshot: Annelise and Zoomable Number Line

Annelise used the *Zoomable Number Line* to consolidate her understanding of decimal place value. She needed this knowledge of the importance of place value when comparing decimals during a task in Interview 4: Zoomable Number Line. The transcript highlights the student's engagement with the zoom-in function of the dynamic number line to overcome longer-is-larger thinking and understand the link between decimal notation and fractional numbers.

> Interviewer: Which is larger 0.86 or 1.3?
>
> Annelise: Maybe 1.3 [zoomed into locate 1.3 on number line, zoomed in further to display hundredths before scrolling backwards to locate 0.86].
>
> I: How do you know that 1.3 is larger from using the zoomable number line?
>
> A: Because I had to scroll backwards to the smaller number zero-point-eighty-nine, and also I know that you can get heaps of decimals here [pointed to place values after decimal point] but it's the first number that matters [pointed to 1 on 1.3 decimal card].
>
> I: What helped you understand that the number of digits in a decimal doesn't tell you whether it's larger or not?
>
> A: I learned it from this number line because I saw when I was zooming in before that the numbers all started with one even though they were getting longer and longer.

During the interview that focused on the *Zoomable Number Line* representation, Annelise was asked to deduce which decimal in a pair was larger. She predicted with uncertainty that 1.3 was larger than 0.89. The hesitancy in her tone suggested that she was having difficulty visualising the decimals and their relationship to each other in the comparison problem. Annelise independently engaged with the dynamic representation to check her thinking and afterwards demonstrated strong reasoning for her mathematically correct prediction. Furthermore, she had a strong understanding of a number line representation and was able to reason that 1.3 was the larger number as she needed to "scroll backwards to the smaller number zero-point-eighty-nine". The interviewer asked a follow-up question to which Annelise reflected that she was made aware of the mathematical concept that place value is the determiner of relative size of decimals. She also uncovered that decimals represent parts of the whole as her whole-number thinking was challenged through her exploration of the *Zoomable Number Line.*

8.6 Discussion of findings about the features of four dynamic digital tools and how students engaged with affordances which translated to decimal learning

8.6.1 Summary of Findings

For each of the digital tools a key change in conceptual understanding occurred as students responded to the dynamic affordances within each representation.

a) The *Zoomable Number Line's* dynamic presentation of number continuity encouraged shifts in students' conceptual understanding of decimal density.

b) Students utilised the *Decimal Strips* tool differently when comparing decimal fractions depending on their prior knowledge of decimal magnitude and place value.

c) Interaction with the tools *Zooming in on Place Value* and the *Zoomable Number Line* facilitated connections between decimal fractions and division, helping students grasp the relative sizes of rational numbers.

d) Engaging with *Wishball-hundredths* allowed students to deepen their comprehension of decimal magnitude by reforming decimal numbers as dynamically represented.

These digital tools leveraged dynamic features to redirect students' focus, creating cognitive conflict that supported their conceptual growth in understanding decimal fractions. The discussion explores newfound understandings of how students develop decimal sense through interactions with dynamic digital tools and respond to the affordances of the tools. Reflection on the findings in relation to previous research revealed four themes which are used to organise the discussion.

8.6.2 Productive Cognitive Conflict

All students involved in this study experienced moments of cognitive confusion generated by the dynamic affordances embedded within the digital tools which interrupted the students' flow of attention. Outside of the research-interview situation, these would have been ideal 'teaching moments'. Seminal work by Swan (1983) found that students who experienced 'cognitive conflict' when learning gained better outcomes overall than students who learnt through 'positive only teacher'. For example, *Decimal Strips* generated cognitive conflict as it challenged Aarav's current misconception thinking about decimals. It was not until he used the functions of the *Decimal Strips* web-app and dragged the 0.2 and 0.25 coloured tiles onto the blank fraction wall that he recognised the value of five-hundredths. Aarav was able to self-correct his thinking by interacting with this dynamic representation as it challenged his prior understanding of decimals and allowed him to examine his errors in thinking and misconceptions. Also, during Interview 2: Decimal Strips, Yehali generalised her discovery

during the decimal comparison task when she linked what she saw displayed by the dynamic decimal strips to her knowledge of decimal place value. Yehali needed a mathematical reason as to why 1.0 was a longer decimal strip than 0.9, which was currently conflicting with her prior thinking focused on whole-numbers. By dynamically comparing the relative sizes of these two decimals on the web-app, she was able to connect this active learning with her emerging understanding of decimal place value, to determine that tenths are a smaller value than ones and therefore despite 0.9 having a nine it can be understood as nine parts of one-whole. As such, this interview moment is a display of Yehali extrapolating her *recognising relationships* between the dynamically visual representation and connecting details with her understanding of decimal place value. Yehali's misconception with "whole-number thinking" (Steinle & Stacey, 2004a) was being challenged through her interaction with the Decimal Strips virtual manipulative as she dynamically compared the decimal lengths in this area-model representation, an important initial piece of knowledge to overcoming her erroneous thinking.

8.6.3 Dynamic digital representations can illustrate abstract concepts of decimal fractions that can otherwise be difficult to show

Representations are used in the mathematics classroom to assist students in understanding abstract concepts that are central to mathematics learning. Specific to the field of mathematics, representation refers to "the act of externalising an internal, mental abstraction" (Pape & Tchoshanov, 2001, p. 119). Number lines are a teaching and learning model used to support the positioning of fractions and decimals through a rational number counting sequence (Teppo & van den Heuvel-Panhuizen, 2014). Yehali uncovered how number lines can show where a decimal 'fits' between two-whole numbers, a concept that is quite difficult for young children to understand (Thomson & Walker, 1996). The affordances of the dynamic *Zoomable Number Line*, particularly its ability to scroll backwards and forwards infinitely along the number line sequence, stimulated learning about decimal density and how to plot values on a number line. The ability of digital tools to adapt, change, and provide various interactive features and capabilities that traditional educational materials may not offer were pivotal in enabling Yehali to successfully complete this decimal location task. By actively "scrolling backwards" Yehali could mathematically reason that the decimal 0.3 must be smaller in value than 3.0 as it was located closer to the whole number 0 on the number line.

Aarav presented initial confusion around decimals due to the presence of shorter-is-larger thinking. Prior to beginning the task-based interview Aarav was unable to coordinate the numerator and denominator of a fraction, and he held denominator focused thinking (Stacey &

Steinle, 2003). In this comparison question, Aarav understood that 0.2 was equal to 2-tenths and 0.25 was equal to 25 hundredths. However, the misconception occurred when he incorrectly concluded that 0.4 is greater than 0.83 because tenths are larger than hundredths, without any consideration of how many parts there are. However, as a result of his interaction with the *Decimal Strips*, particularly working through the decimal comparison task, he able to question his current understandings and begin to develop more appropriate decimal-fraction link thinking. *Decimal Strips* generated cognitive conflict as it challenged Aarav's current understandings and present misconceptions about decimals. It was not until he used the functions of the *Decimal Strips* web-app and dragged the 0.2 and 0.25 coloured tiles onto the blank fraction wall that he recognised the value of five-hundredths. Steinle and Stacey (2004a) detail how teachers simply telling a student that their thinking is incorrect does not eradicate that student's misconceptions. Rather, instructional techniques need to be implemented to assist students in overcoming or controlling the influence of misconceptions and systematic errors. Aarav was able to self-correct his thinking by interacting with this powerful dynamic representation as it challenged his prior understanding of decimals and allowed him to examine his errors in thinking and misconceptions.

8.6.4 Students' manipulative interaction with dynamic digital representations allows them to 'see' notions that would otherwise be 'unseeable'

Students were able to use the dynamic representations to eliminate the abstractness of the decimal fractions by visually showing the relative positions of decimals on a number line. For Annelise, she could then clearly reason and problem-solve to produce a solution to the decimal location question in the *Zooming in on Place Value* interview. Annelise engaged with the dynamic zooming and stretching of the number line, which eliminated the abstractness of the decimals and clearly presented the sequence of scores for her to see. As a result, Annelise was able to visualise the location of each tenths decimal to correctly predict the answer to the question. The digital functions of the *Zooming in on Place Value* interactive activity allowed Annelise to uncover decimal concepts that a static number line is not able to illustrate. The ability to stretch out the number line between two whole-numbers, enables students to see the continuous sequence of decimals, which assists them when operating with different types of numbers (Teppo & van den Heuvel-Panhuizen, 2014). Goldin and Schteingold (2001) suggest, developing clear links between quantity, vocabulary and symbol is critical for conceptual understanding and the visual affordances of the digital tool illustrate the symbolic notation. A key affordance of the *Zooming in on Place Value* virtual manipulative is its ability to

dynamically displaying ten-tenths between two whole-numbers. Through a 'zooming in' animation the digital tool reveals ten-tenths between two whole-numbers, and ten-hundredths between two-tenths in a two-step question sequence. This gradual release of information allows the student to build knowledge of decimal density as to the continuity property of rational numbers, whereby, between any two decimals there are an infinite number of other decimals (Widjaja et al., 2008). Furthermore, Annelise's critique of the app in saying that she was able to see the number line with the decimals on it, indicates that the web-app is able to assist students in visualising the 'abstractness' of decimal fractions. Young students need support when developing their decimal sense and teachers therefore need to be sensitive to students' ways of thinking and learning (Way, 2011). Quality teaching that employs quality representations such as the zoomable number line can assist students in better understanding decimal fractions.

8.6.5 Dynamic representations of decimals can challenge dominant whole-number thinking through shifting attention structures towards the decimal-fraction link

Dynamic representations of decimal fractions are capable of offering direction to learners' shifts of attention. The dynamic nature of digital manipulatives force shifts in the object of children's attention when they are actively interacting with the mathematics content as well as the structure of their attention in how they attend to features of the mathematics representation and the embedded mathematics concepts. Conceptual development occurred when the learners experienced a shift between attending to relationships within and between elements of current experience and perceiving relationships as properties that might be applicable in other situations (Mason, 2008). For example, Annelise transitioned between three separate structures of attention to demonstrate a more sophisticated conceptual understanding of decimals during Interview 4: Zoomable Number Line. In the beginning Annelise operated within the microquality of *discerning details* to use place value knowledge to predict the larger decimal fraction. She then moved towards *recognising relationships* as she consolidated her understanding of the decimal-fraction link whereby decimals are the result of dividing the whole into equal parts. Finally, Annelise shifted her attention to *perceiving properties* by connecting her knowledge of the relative size of decimals when comparing and applying this concept to the relationship with positioning values on a number line. These changes in students' decimal understanding were prompted by unique affordances embedded within the web-app, in particular the zooming-in and scrolling features, which provide layered perceptual access to relative magnitude and enabled shifts of attention between different scales of representation

(Tucker, 2015). By being aware of shifts occurring, teachers can be more alert to the possibility that learners are not making appropriate shifts and therefore implement teaching strategies and learning tools such as dynamic digital representations, to prompt these conceptual changes.

8.7 Concluding Statement

Teachers need to use various strategies, including visual representations, manipulatives, real-world examples, and interactive activities to make decimals more tangible and understandable. Examining a group of students completing four task-based interviews revealed the full scope of conceptual understanding of decimal fractions inclusive of decimal place value, relative magnitude and decimal density, which emerged from the students' interactions with four dynamic digital tools. This chapter incorporated a published journal article titled 'Learning about decimal fractions with dynamic digital representations' which examined how students' conceptual understanding of decimal place value evolved as they interacted with a dynamic counting frame and vertical number line. A number of the dynamic digital tools challenged the students' dominant whole-number thinking through shifting attention structures towards the decimal-fraction link as they engaged with the embedded features. At times, the students' experienced productive cognitive conflict, as the dynamic representation of decimal fractions challenged their existing understandings and misconceptions. Outside of the research-interview situation, these instances would have served as ideal teaching moments, where a teacher could have intervened to guide reflection, pose probing questions or provide scaffolding to support students in resolving the conflict in mathematically productive ways. The evidence snapshots included in this chapter highlight the capabilities of these four dynamic digital representations to illustrate abstract concepts of decimal fractions that can otherwise be difficult to show and allowed the students to 'see' notions that would otherwise be 'unseeable'. Digital representations have the potential to be used as effective learning and teaching tools to develop conceptual understanding of decimal fractions, and to address previously established misconceptions. The next chapter synthesises the findings of the three results and discussion chapters to discuss their implications in relation to existing literature and mathematics education practice.

CHAPTER 9: CONCLUSION

9.1 Situating the Chapter

The aim of this research study was to explore the conceptual changes that emerged while children in middle-primary grades engaged with dynamic digital representations of decimal fractions. The study employed video-recorded, individual task-based interviews to detect moments of students' conceptual change. A microgenetic analysis approach was used to identify specific shifts in students' attention that marked instances of conceptual change. The findings revealed which features of the dynamic digital tools prompted these shifts and how they contributed to addressing students' misconceptions about decimals.

The concluding chapter provides a synthesis of findings and reflection on the processes and outcomes of the overall research project. The first section discusses what the study revealed about nature and cause of conceptual change in understanding decimals experienced by the students, and how their misconceptions were challenged and addressed. The second section highlights the significant features of the dynamic representations that prompted the conceptual changes. The third section explores the contributions made by the study outcomes to the fields of teaching and research, within the limitations of the study's context and design.

9.2 The keys to conceptual change in understanding decimals

This section addresses the research questions:

How do children experience conceptual changes related to decimal fractions when they attend to features of dynamic digital representations?

How do students overcome misconceptions about decimal fractions when they attend to features of dynamic digital representations?

Three unifying ideas arose through a synthesis of the results as outlined in earlier chapters: the role of cognitive conflict in mathematical conceptual change, mediating the abstractness of decimals, and addressing decimal misconceptions. These three themes were identified as the most critical because they consistently surfaced across multiple student responses and interactions with the dynamic digital tools, highlighting their centrality to the learning process.

This research investigated the conceptual shifts that occurred among Year 4 students (ages 9-10) over a period of six-weeks as they engaged with dynamic digital representations of decimal fractions embedded within the interactive activities *Wishball-hundredths, Decimal*

Strips, Zooming in on Place Value, and *Zoomable Number Line*. By mapping the shifts in students' attention, the study provided insights into how their understanding of decimal fractions developed through these interactions. The findings offered evidence to assess the effectiveness of these activities in enhancing the students' conceptual grasp and proficiency with decimal fractions. The participants displayed noticeable changes in their understanding of decimals, influenced by the dynamic elements that guided their attention and lead to the formation of new knowledge in a relatively short period. The students' responses highlighted that features like the animated vertical number line and counting frame in *Wishball-hundredths*, the manipulable decimal tiles in *Decimal Strips*, the zooming number line in *Zooming in on Place Value*, and the scrollable number line in *Zoomable Number Line* were instrumental in fostering new insights into key decimal concepts. Analysis of the cognitive processes students employed while completing the decimal tasks revealed that these dynamic digital tools encouraged thinking about the relationships between whole numbers and decimals, place value, the relative size of decimals, and the infinite nature of number systems. At times, the students' interactions with these representations of decimal fractions generated productive cognitive conflict by challenging their pre-existing understandings and misconceptions, which outside of a research setting could serve as valuable teaching moments.

According to Mason (2010), learning involves a shift in both the form and focus of attention, influencing what is being attended to and how it is perceived by the learner. His research strongly suggests that simply engaging with a mathematical task is insufficient for learning, instead it requires the integration of experience and sense-making, which necessitates a transformation of attention. The five structures of attention as proposed by Mason involve distinctive forms of attention with particular learner behaviours and thought patterns as outlined below:

1. Holding wholes (gazing)
2. Discerning details (features and attributes)
3. Recognising relationships (part-part, part-whole)
4. Perceiving properties (leading to generalisation)
5. Deducing from definitions (axioms and definitions stated independently of particular objects)

Initially, attention comprises awareness of wholeness such as gazing to identify an object, which may be a part of some other object. After which, a student's focus shifts to discerning details comprised within that object, and so creating objects and sub-objects. This enables awareness of relationships or similarities between features comprising sub-objects at

any level of detail and then focus on properties as attributes that objects may possess. Finally, the learner holds awareness of reasoning solely on the basis of properties which requires specified properties to be seen as definitions, not just as attributes. Mason links learning with the student becoming aware of sufficient variation within an object and labels this as a shift in the structure of attention. Discernment may then progress to *recognising relationships*, *perceiving properties* and *reasoning on the basis of properties*, but often students do not independently follow the trajectory in this way. As such, learning also involves discerning the types of relationships worth attending to for mathematical purposes, the types of properties worth perceiving, and making the relevant but subtle shifts of attention which these discernments enable to develop the habit of mind to reason based on agreed properties.

Dynamic representations of decimal fractions can effectively direct learners' shifts of attention. The digital manipulatives' dynamic nature prompts changes not only in the focus of students' attention when they engage with mathematical content but also in how they structure their attention to features of the mathematical representation and the embedded concepts. Conceptual development takes place when learners experience a shift from relying on their current knowledge to discerning new mathematical details presented by dynamic digital tools and perceiving relationships that could be applied to other numerical contexts. Recognising these shifts can alert teachers to the possibility that learners might not be making the appropriate learning progression, allowing them to employ suitable strategies and tools such as dynamic digital representations to facilitate shifts and thus enhance conceptual understanding.

Understanding decimal fractions is a complex process that requires students to integrate their knowledge of place value with their understanding of whole numbers and fractions. Successfully grasping decimal concepts depends on how well students connect new ideas to previously established mathematical understandings. In this study, several tasks challenged students to accurately order decimal numbers based on their relative magnitudes. Their ability to perform these tasks appeared closely linked to their understanding of how decimal notation corresponds to division operations. The findings showed that all participants developed a stronger sense of decimal size through their engagement with the dynamic tools *Zooming in on Place Value* and the *Zoomable Number Line*. This was reflected in their consistent success at locating decimals accurately on number lines, as well as in the clarity and reasoning they demonstrated in verbal explanations during decimal comparison activities using the *Zoomable Number Line*. The findings identified three key factors in decimal conceptual change, including the role of cognitive conflict, the challenge of mediating decimal abstractness, and the need to address persistent misconceptions. Cognitive conflict was instrumental in prompting students

to question and revise their existing conceptions, marking it as a key mechanism for conceptual change. The abstract nature of decimals presented a recurring challenge for students, indicating the importance of providing accessible and meaningful representations to bridge the gap between symbolic notation and conceptual understanding. Finally, the persistence and variety of decimal misconceptions observed across the participant group reinforced the need for targeted instructional strategies to confront and resolve these misunderstandings. Together, these findings offer a comprehensive account of the key influences on students' developing decimal understanding.

9.2.1 The role of cognitive conflict

All students involved in this study experienced moments of cognitive confusion generated by the dynamic affordances embedded within the digital tools which interrupted the students' flow of attention, to guide their conceptual learning journey of decimal fractions. When engaging with the dynamic digital representations, students attempted to harmonise their prior-knowledge of decimal fractions with the symbolic, visual, and dynamic representations of decimals presented in the interactive tools. As the student restructured their mathematical schema to understand the new decimal concepts encountered in the dynamic digital representations, cognitive conflict occurred.

The findings of this research project reflect the viewpoint held by Moody (2010), that students will naturally encounter moments of confusion in the development of mathematical concepts. Teaching instruction which utilises effective representations to identify, expose and confront these difficult mathematical ideas, enable students to restructure their own thinking. Students actively construct their own understanding of mathematics and to be effective mathematicians, they need to think deeply about mathematics, connect ideas and be challenged (Harel & Tall, 1991). The data revealed how students responded to new decimal information gathered from the digital tools was either assimilated within the students' existing conceptual understanding or accommodated by changing their cognitive structures. Conceptual change was actualised when students could successfully modify an existing alternative conception which conflicted with new information, and then either modify their conceptual understanding or reject the original mathematical idea as they understood it to be no longer useful (Moody, 2010).

Several decimal fraction tasks prompted students to consider alternative conceptions which led them to reconceptualising their mathematical ideas. This was particularly evident when they worked through the decimal comparison tasks alongside the *Decimal Strips* and

Zoomable Number Line dynamic representations. Many students were able to independently revise their thinking through interaction with the dynamic representations, which prompted them to confront and reconsider their prior understanding about decimals. These digital tools not only challenged students' existing beliefs but also supported meaningful error analysis and conceptual refinement. For instance, Decimal strips generated cognitive conflict as it challenged Annelise's current misconception thinking about decimals. She initially struggled to reconcile why the decimal strip representing 0.3 appeared longer than the one for 0.25, which conflicted with her assumption that a number with more digits after the decimal point must be greater. However, through real-time manipulation and comparison of the decimal values using the Decimal Strips web-app, she actively engaged with the visual evidence. This experience, combined with her recently reinforced understanding of decimal place value during the same interview, enabled her to conclude that tenths represent a greater value than hundredths. She ultimately reasoned that although 0.3 has fewer decimal places than 0.25 it can be thought of as "30-something" in the tenths column making it the larger number.

Valerie's early engagement with the *Zoomable Number Line* revealed an initial attempt to discern key features of decimal numbers involving tenths, though her understanding at the time was limited. While she recognised that ten-tenths exist between two whole numbers, she encountered notable confusion when asked to identify a decimal between one and two, and again between two and three. On both occasions, she consistently referred to values beginning with "zero-point," revealing a persistent error in thinking that all decimals must fall between zero and one. This inaccurate understanding led to a productive cognitive conflict with a clash between her prior knowledge and the new information presented visually through the dynamic zoom-in function. As she observed the sequential unfolding of tenths between two and three on the number line, her misconception was directly challenged. This prompted a realisation as expressed in her words, "I didn't know it could be a ones and then a decimal." The dynamic representation enabled Valerie to revise her thinking by making visible what had previously been unclear. Although she had entered the activity with a partial understanding of decimal fractions, limited to tenths, her conceptual grasp expanded to include hundredths as well, facilitated by the digital tool's capacity to visually partition space and highlight decimal structure. This moment illustrates how this dynamic digital representation instigated cognitive conflict leading the student to question, confront and reconstruct her understanding of mathematical ideas.

The study highlighted the potential of dynamic digital representations, not merely as instructional aids but as pedagogically powerful tools. When embedded within tasks such as

positioning and estimating decimals on a number line to provoke cognitive conflict with whole-number thinking, zooming in to explore place value to expose the structure of the decimal system, comparing decimal magnitudes to challenge and refine existing connections between representations, and engaging with target-directed tasks such as *Wishball-hundredths* to create tension around misconceptions of decimal size and value, these tools are capable of disrupting established misunderstandings and enabling students to construct more coherent and accurate conceptualisations of decimal number systems.

9.2.2 Mediating the abstractness of decimals

Mathematical representations play a crucial role in helping students make sense of abstract concepts that are foundational to mathematics learning (Pape & Tchoshanov, 2001). In this study, every student engaged with four dynamic digital tools, which supported them in navigating a variety of cognitively demanding decimal tasks. These interactive representations effectively reduced the perceived abstractness of decimal numbers by providing dynamic visualisations of otherwise difficult concepts. One such concept is the idea of partitioning decimal numbers into fractional components which is widely acknowledged as particularly challenging for learners (Siegler & Lortie-Forgues, 2014). Students often find it difficult to grasp that each digit following the decimal point corresponds to a distinct fractional part of a whole (Siegler & Lortie-Forgues, 2014). To work confidently with decimals, a strong foundation in place value and number sense is essential (Way, 2011). Unlike whole numbers which students can often visualise easily, decimals lack inherent physical representations making it harder for learners to form mental models and contributing to frequent misunderstandings (Janvier, 1987). The four digital tools used in this research provided students with dynamic and visually supportive representations that addressed these challenges directly, particularly by strengthening their understanding of decimal place value. Students consistently drew on the features of *Wishball-hundredths* when working on addition and subtraction tasks involving decimals. The tool's dynamic number line and animated counting frame clearly illustrated students' moves and provided immediate visual feedback aligned with place value, enabling them to monitor and adjust their strategies in real time.

Students may follow erroneous rules linked to whole-number thinking when trying to assimilate new knowledge about decimals with what they already know about whole numbers, place value and fractions (Moloney & Stacey, 1996). To develop their understanding of the continuous nature of the base-10 system, students need to relate decimal fractions with the division operation which can be difficult to illustrate using static representations. Two dynamic

representations, *Zooming in on Place Value* and the *Zoomable Number Line*, appeared to be particularly effective in showing the relationship between decimal fractions and the division operation. The features of both these digital tools allowed the students see how decimal 'fit' between two whole-numbers namely through the zooming animation in *Zooming in on Place Value* as well as the *Zoomable Number Line's* zoom-in function. For example, the interactive digital tool *Zooming in on Place Value* allowed Yehali to visualise concepts that are often difficult to grasp, using a unique animation that 'zooms in' between two whole numbers to reveal divisions into tenths. Although she initially struggled to predict the correct tenths position on the number line, her engagement with the tool's dynamic features enabled her to answer later questions successfully. This demonstrated that, within the 30-minute interview, she became comfortable and confident in using this effective digital learning tool to clarify an abstract concept of decimals.

9.2.3 Addressing decimal misconceptions

Five out of the six students were identified through a Decimal Comparison Test (Steinle & Stacey, 2004a) as having misconceptions that affected their understanding of decimal concepts. The analysis of test results showed that three students, Annelise, Tehillah and Yehali, exhibited the Longer-is-larger misconception, two students, Aarav and Valerie, the Shorter-is-larger misconception, and one student, Andrew, was classified as an Apparent-expert. During an initial interview aimed at assessing their current understanding of decimal fractions including any misconceptions, students were asked to define a decimal fraction. None of the students provided an explanation that referenced the concept of decimals representing parts of a unit created by dividing a whole into equal parts. Examining the responses of a group of students who completed four different task-based interviews involving dynamic digital representations, provided a comprehensive view of their conceptual understanding of decimal fractions, including decimal place value, relative magnitude, and decimal density, which emerged from their interactions with the four dynamic digital tools. Collectively, these digital representations of decimals offered students powerful learning opportunities to develop decimal sense, overcome misconception thinking, and apply this knowledge across various mathematical contexts. The findings of the present study illustrate how engagement with dynamic digital representations can prompt substantial shifts in students' thinking over only a six-week period. These rapid conceptual changes highlight the potential of dynamic tools to disrupt established misconceptions. However, as the present study was not designed as a

longitudinal investigation, its scope does not allow for conclusions about the long-term retention or stability of the students' newly developed decimal understandings.

Addressing misconceptions is central to effective mathematics teaching. Carefully designed tasks, when paired with purposeful representations, can help surface and challenge students' incorrect assumptions. When tasks explicitly target specific misunderstandings and encourage students to articulate their reasoning, they create opportunities for students to transition from intuitive thinking to more formal mathematical reasoning. In this study, digital tools such as *Wishball-hundredths* and *Zooming in on Place Value* supported students in identifying and reconstructing their misconceptions about decimal fractions. For example, Wishball-hundredths revealed key place value relationships through its animated vertical number line and counting frame. These features played a pivotal role in helping three students, Yehali, Tehillah and Valerie, correct persistent misunderstandings about decimal place value and the relative size of decimal numbers.

Research supports the value of presenting mathematical ideas through multiple representations, as doing so allows students to make connections that deepen their understanding (Goldin & Shteingold, 2001). For students to build accurate mental models of decimal concepts, they must engage with varied symbolic, visual, and dynamic representations that reveal the structure and meaning behind decimal notation. In the four task-based interviews conducted for this study, students frequently overcame decimal misconceptions such as misinterpreting decimal magnitude or misaligning place value, by reasoning across these multiple forms. The use of tools like *Wishball-hundredths* provided simultaneous access to symbolic decimal notation, a linear representation via the number line and a visual abacus model. This multimodal approach helped to clarify the abstract nature of decimal fractions and supported students in understanding the relationships between decimal digits, their position and their relative value.

9.3 Significant features of the dynamic representations that prompt conceptual change

This section addresses the research question:

What features of Wishball-hundredths, Decimal Strips, Zooming in on Place Value, and Zoomable Number Line, do children attend to that prompt conceptual changes related to decimal fractions?

This study examined how Year 4 students (ages 9–10) interacted with four dynamic digital tools *Wishball-hundredths, Decimal Strips, Zooming in on Place Value*, and *Zoomable Number Line*, designed to support the representation and understanding of decimal fractions. Through their engagement with these tools, students demonstrated shifts in their conceptual understanding of decimals. These shifts appeared to be prompted by the dynamic and interactive features of the tools, which directed students' attention and supported the development of new insights over a relatively short period. A summation of the impactful features embedded within the four dynamic digital representations *Wishball-hundredths, Decimal Strips, Zooming in on Place Value,* and *Zoomable Number Line* are explored and connected with changes in students' thinking related to the key decimal fraction concepts of place value, relative magnitude and decimal density. The research aimed to identify the specific affordances of each digital tool that contributed to students' evolving understandings and to illuminate the role of dynamic digital representations in supporting conceptual change in mathematics education. Affordances in dynamic digital representations foster opportunities for learners to notice mathematical details, test conjectures and receive immediate visual feedback as they manipulate mathematical objects. Such opportunities can prompt shifts in attention, challenge existing misconceptions and encourage students to revise their decimal understandings in response to what they observe. As such, the affordances embedded within each digital tool acted as catalysts for conceptual change by enabling students to actively construct meaning through dynamic engagement with the decimal concepts. The following section highlights the dominant benefits afforded by each of the dynamic digital tools.

9.3.1 Attending to decimal place value in Wishball-hundredths

The responses of the Year 4 students revealed that interacting with features such as the animated vertical number line and dynamic counting frame, stimulated thinking about the relationship between whole numbers and decimal fractions, as well as place value and the relative size of decimals. The students' interactions with the dynamic representations of decimal fractions embedded within Wishball-hundredths, at times, generated productive cognitive conflict, as the representations challenged their existing understandings and misconceptions. The students acknowledged that their thinking had changed from their limited prior understanding of performing operations with decimal fractions. Two students, Andrew and Annelise, experienced shifts from discerning details of the place values within a decimal to then being able to recognise relationships, perceive properties, and reason on the basis of agreed properties by performing operations with decimals with an understanding of positional

value number knowledge. Conceptual development occurred when the learners experienced a shift between attending to relationships within and between elements of current experience and perceiving relationships as properties that might be applicable in other situations. Dynamic representations of decimal fractions can offer direction to learners' shifts of attention. The immediate feedback loop in *Wishball-hundredths* allowed students to see the result of each move in real time, enabling them to adjust their strategies, test hypotheses and reinforce their understanding of decimal place value and operations through continuous and responsive interaction. The dynamic nature of *Wishball-hundredths* forced shifts in the object of students' attention when they were actively interacting with the mathematics content as well as the structure of their attention in how they attended to features of the dynamic representation and the embedded decimal concepts.

The repeated opportunities to practice decimal operations allowed students to build confidence and a stronger mathematical foundation in this area. Progression through different stages of attention was made possible by understanding place value, which is essential for estimating relative size of decimals. According to Clements & Sarama (2004), the stages in the structures of attention framework aren't distinct classes of decimal comprehension, instead higher stages build on and expand upon the lower levels. This was clearly demonstrated during the students' interactions with the *Wishball-hundredths* digital tool. They observed how the interactive activity reformed decimal numbers from the feedback of new starting number. Two students, Andrew and Annelise, further applied this concept to grasp the principle of subtraction trading which was prompted by the counting frame's dynamic movements and consequently shifted their focus. Additionally, these students were able to independently apply this learning in a later decimal subtraction task. Within the limited timeframe of the task-based interview, the students gained an understanding that subtracting decimals follows the same renaming and trading principles as whole numbers, such as exchanging one unit for ten-tenths or one-tenth for ten-hundredths (Martinie, 2014). They recognised this pattern's continuity and understood that numbers become ten times smaller to the right of the decimal (Martinie, 2014). The development of their conceptual understanding of decimal fractions was supported by the dynamic affordances in Wishball-hundredths, particularly the counting frame and vertical number line.

9.3.2 Attending to decimal magnitude in Decimal Strips

Steinle and Stacey (2004a) have shown several widespread conceptions that are a barrier to understanding decimal fraction concepts. Because of the abstract nature of the

decimals, many of these concepts cannot be taught by showing an example and therefore building knowledge of these concepts is cognitively difficult. Students need to understand these key decimal ideas before they begin to solve problems involving decimal fractions. Across the four interviews, all six students engaged with a task that prompted them to question their current understandings of decimal fractions and begin to develop more appropriate decimal-fraction link thinking. This was particularly evident when they worked through the decimal comparison tasks alongside the *Decimal Strips* dynamic representations. *Decimal Strips* pieces can be arranged to represent a decimal number in an end-to-end representation. When placed end-to-end the pieces form a linear representation of decimals, which facilitates direct comparison of decimal fractions. Schwartz (1994) advises that "in the case of education reform, there need to be new curricular artifacts that allow the users to mark the newness of their undertaking" (p. 4). *Decimal Strips* was a digital tool the students had not seen before, and therefore served as the marker of this new conceptual knowledge. *Decimal Strips* generated challenge of the students' current understandings and simultaneously supported their learning of decimal concepts. For example, it was not until they used the functions of the *Decimal Strips* web-app and dragged the coloured tiles onto the blank fraction wall that they recognised the relative values of tenths and hundredths.

Earlier research into provoking cognitive conflict (Bell, 1993) suggests there is a more significant benefit to long-term learning when students encounter misconceptions through their own means, rather than teachers drawing attention to potential errors in thinking and misconceptions when providing instruction. As such, instructional techniques need to be implemented to assist students in overcoming or controlling the influence of misconceptions and systematic errors. Confrey (1990) claims "… learning … involves progressive consideration of alternative perspectives and the resolution of anomalies" (p. 32). This presents a need for 'conflict' in mathematics teaching and learning. Research shows that teaching is more effective when it assesses and uses prior knowledge to adapt to the needs of students (Battista, 2012). Therefore, successful decimal teaching needs to provide opportunities for students to modify their conceptions through exposure and rejection. The majority of the students in this study were able to self-correct their thinking by interacting with this powerful dynamic representation as it challenged their prior understanding of decimals and allowed them to examine their errors in thinking and misconceptions. For example, during a decimal comparison task Aarav initially predicted that 0.2 was greater than 0.25, revealing a common misconception that a shorter decimal is larger. However, when he used the *Decimal Strips* tool to drag both values onto the fraction wall, the visual comparison prompted him to reconsider.

Seeing the relative lengths of the coloured strips helped him recognise that hundredths contribute to the overall value of a decimal leading to the realisation that 0.25 is greater than 0.2. The dynamic representation shifted his attention to the importance of place value beyond the tenths, challenging his prior assumption. Aarav self-corrected his thinking and reconciled his misunderstanding through this visual feedback. When asked to explain his reasoning, he confidently used gestures and referenced the coloured area representations on the screen, offering a clear and mathematically sound explanation of why 0.25 is larger.

The interactivity of the tool provoked a productive moment of cognitive conflict that ultimately led to a powerful shift in his conceptual understanding of decimal magnitude. As such, this interview moment is a display of Aarav extrapolating his recognising relationships between the dynamically visual representation and correcting his 'shorter is larger' decimal thinking to a generalisable understanding of decimal place value after he perceived properties of scaling place values to the left of a decimal point. Aarav's experience was not a linear progression through structures of attention as separate modes of learning but rather a layering of more sophisticated microqualities of attention that build a more robust understanding of a mathematical concept.

9.3.3 Attending to decimal sequencing in Zooming in on Place Value

Students used dynamic representations to conceptualise decimal fractions more clearly by visually depicting their relative positions on a number line. This helped them understand the connection between decimal notation and division, ultimately enhancing their comprehension of the relative magnitude of rational numbers. Through this they were able to reason and solve decimal location problems effectively. The interactive *Zooming in on Place Value* tool, with its zooming and stretching features, made the concept of decimals less abstract and clearly presented the sequence of decimal numbers for the students to see. As a result, the students could accurately visualise the placement of each decimal which enabled them to correctly predict answers. The digital functions of the *Zooming in on Place Value* interactivity allowed them to uncover decimal concepts that a static number line is not able to illustrate. By stretching the number line between two whole numbers, students could see the continuous sequence of decimals, aiding them in working with various types of numbers (Teppo & van den Heuvel-Panhuizen, 2014).

Goldin and Schteingold (2001) emphasise that forming clear links between quantity, vocabulary, and symbol is essential for conceptual understanding, and the digital tool's visual features supported this by highlighting symbolic notation. A key advantage of the *Zooming in*

on Place Value virtual manipulative is its ability to dynamically show the division of tenths between whole numbers. Through a 'zooming-in' animation, the tool revealed tenths between whole numbers, and hundredths between tenths in a step-by-step process. This gradual release of information allowed students to develop an understanding of decimal density and the continuous nature of rational numbers, where infinite decimals exist between any two given decimals (Widjaja et al., 2008). Additionally, Aarav's feedback that the web-app helped him visualise decimals on the number line highlighted how the tool made the abstract nature of decimal fractions more tangible. Since young students need support when developing decimal sense, teachers should be attentive to their learning processes and ways of thinking (Way, 2011). Effective teaching that incorporates dynamic representations, like a dynamic digital number line, can greatly aid students in better grasping decimal fractions.

9.3.4 Attending to relationships and reasoning in Zoomable Number Line

A considerable body of research exists on students' understanding of decimals and the prevalence and persistence of misconceptions (Stacey & Steinle, 2003). Much of this knowledge has been inferred from responses to pencil and paper decimal comparison tests. When students are asked to choose the larger of two decimals, they are commonly categorised as using one of three erroneous rules; longer-is-larger, shorter-is-larger, and zero rule. Researchers (e.g., Skemp, 1987; Hiebert & Wearne, 1985) argues for teaching which focuses on conceptual understanding to a far greater extent that procedural understanding to overcome these misconceptions related to decimal fractions. The responses of the students revealed that interacting with the zoomable features of the digital number line, stimulated thinking about decimal density, the relationship between whole numbers and decimal fractions, place value and the relative size of decimals. Students used the digital number line to explore, check predictions, solve problems, and self-correct erroneous thinking.

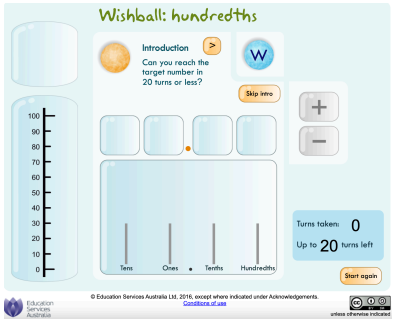
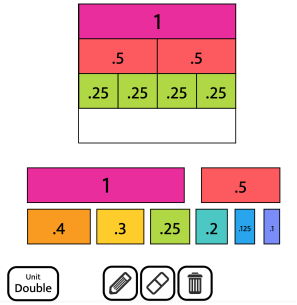
The *Zoomable Number Line* representation offered dynamic experiences for the students to see decimal concepts in motion. Each student demonstrated changes in their initial misconception to successfully compare the relative size of decimal pairs and determine the larger value. The pathway of thinking progressed from *holding wholes* with an incorrect understanding of decimal place value to a more accurate internal representation of decimals by *discerning details* presented on the dynamic number line. For example, in the task-based interview the students were asked to locate the decimal 0.3 on the zoomable number line and describe its location by stating "it's between _ and _". Five students, all participants excluding Andrew, responded with incorrect initial answers due to the presence of whole-number

thinking or incomplete decimal-fraction link thinking. However, as a result of their interactions with the *Zoomable Number Line*, particularly working through the decimal location tasks, they were able to question their current understandings and begin to develop more appropriate decimal-fraction link thinking. Furthermore, each student *recognised relationships* between digits in the decimal as they understood the positional value to the right and left of a decimal point and used this knowledge to deduce that tenths were a smaller value than wholes, and hundredths as an even smaller value. The findings indicated the potential of digital number lines being used as effective learning and teaching tools to develop conceptual understanding of decimal fractions, and to address previously established misconceptions.

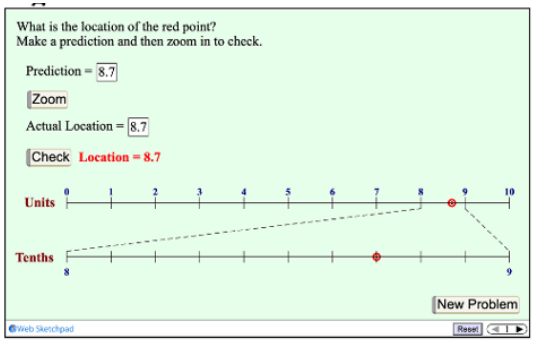
The visual summary in Table 9.1 highlights the key affordances and features of each digital tool that students drew on to support their conceptual change. It also details the significant changes in decimal understanding that were most apparent across the six students and links these with specific dynamic features for each digital tool. The summary table provides a clear and concise synthesis of the study's findings and outlines the distinctive affordances of each tool, the particular dynamic features that facilitated learning and the corresponding conceptual changes in students' decimal understanding.

Table 9.1

Summary of conceptual changes linked to affordances of digital tools

Digital Tool	Affordances Used	Changes in Decimal Understanding
Wishball-hundredths  Focus on place value. Involves adding or subtracting to reach a target decimal fraction number. (https://www.scootle.edu.au/ec/viewing/L8456/index.html)	Animated vertical number line – visually illustrates decimal increments between whole numbers. Dynamic counting frame – allows for interactive trading, for example ten hundredths for one tenth, supporting renaming. Immediate feedback loop – resets to new starting numbers, reinforcing decimal operations and place value understanding. Interactive gameplay – encourages repeated practice with decimal addition and subtraction. Forces shifts in attention – students must adjust focus based on changes in place value and magnitude. Supports conceptual understanding of decimal operations – particularly subtraction and regrouping.	Shifted attention to place value and positional value of digits in decimals. Prompted recognition of relationships between whole numbers and decimal fractions. Challenged existing misconceptions about decimal operations, creating productive cognitive conflict. Supported understanding of renaming and trading principles in decimal addition and subtraction. Enabled progression through structures of attention, deepening conceptual understanding. Helped students realise that decimal places decrease by a factor of ten for example, tenths and hundredths.
Decimal Strips  Focus on number magnitude. Involves dragging coloured tiles representing different decimal values. (https://toytheater.com/decimal-strips/)	Coloured decimal tiles (tenths, hundredths) – enable end-to-end linear representations of decimal values. Drag-and-drop interaction – supports physical manipulation and construction of visual decimal comparisons. Fraction wall display – helps compare the size of decimals side by side. Visual scaling and alignment – shows relative value clearly, for example 0.25 > 0.2 Promotes cognitive conflict – highlights inconsistencies in students' existing reasoning (shorter-is-larger misconception). Engages students in self-correction – encourages reflection and reasoning through visual evidence.	Facilitated visual comparison of decimals using end-to-end strip representations. Created moments of cognitive conflict, leading to self-correction of place value misconceptions (shorter-is-larger). Helped students understand the relative size of tenths vs hundredths. Encouraged development of decimal-fraction link thinking through dynamic interaction with representations. Enabled students to generalise conceptual knowledge beyond individual tasks.

Zooming in on Place Value

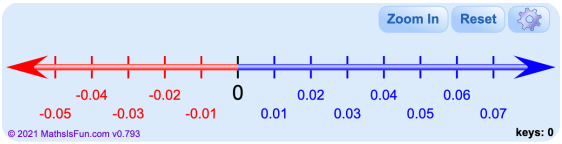


Focus on number magnitude.
Involves predicting the location of a decimal fraction on a number line, then self-checking with the embedded Zoom function.
(https://www.sineofthetimes.org/zooming-in-on-place-value/)

Zooming feature – allows deep exploration between whole numbers to tenths, then hundredths.
Dynamic number line – stretches and reveals hidden values, illustrating decimal density.
Step-by-step division of intervals – gradually reveals the continuous nature of rational numbers.
Connects symbols, quantity, and position – enhances understanding of the meaning behind decimal notation.
Supports prediction and estimation tasks – students test and refine their ideas about placement on the number line.

Improved understanding of decimal sequencing and relative positioning on a number line.
Revealed the density of decimals, showing that more values exist between any two decimals.
Supported conceptual links between symbolic notation and quantity.
Helped students predict decimal placement with greater accuracy.
Made abstract ideas more tangible through dynamic zooming and stretching.

Zoomable Number Line



Focus on number density.
Involves using Zoom-in/out capability to respond to teacher-designed tasks.
(https://www.mathsisfun.com/numbers/number-line-zoom.html)

Zoomable interface – enables detailed inspection of intervals between any two decimals.
Continuous, scalable number line – shows infinite decimals between any two values.
Interactive problem-solving tasks – such as locating, comparing, and predicting decimal values.
Visual emphasis on place value – helps distinguish tenths from hundredths and beyond.
Promotes relational reasoning – supports understanding of relative size and structure of decimals.
Dynamic corrections – allows students to test and revise their thinking in real time.

Addressed common misconceptions like whole-number thinking, longer-is-larger, and zero rule.
Supported development of accurate internal representations of decimals.
Helped students self-correct initial errors in locating and comparing decimals.
Encouraged reasoning based on positional value and decimal relationships.
Fostered a shift from procedural to conceptual understanding through interactive exploration.

9.4 Contributions

This section outlines the key contributions of the research, highlighting the new insights that have emerged and their significance within the field of mathematics education research. It discusses how the findings enhance current researcher and educator understandings of how students experience conceptual change when engaging with dynamic digital representations of decimal fractions. In addition to articulating these contributions, the section also acknowledges the limitations of the study thus providing a critical lens through which to interpret the results. Finally, recommendations for further research are proposed, identifying areas where future investigations could build upon the findings and address gaps that remain.

9.4.1 Implications for teaching

Across all four digital tools, the research addressed common yet critical conceptual challenges, including comparing decimal values, accurately placing decimals on number lines, and performing operations with decimal numbers. As students interacted with the dynamic affordances of each tool, their attention shifted toward previously overlooked mathematical features, resulting in meaningful changes in understanding. These findings underscore the potential of dynamic digital representations, not merely as instructional aids but as pedagogically powerful tools which, when used alongside carefully designed tasks, can disrupt established misconceptions and enabling students to construct more coherent and accurate conceptualisations of decimal number systems.

This study also highlighted several implications for classroom practice. Teachers who are attuned to shifts in student attention can more effectively identify when learners are not making necessary conceptual transitions. By recognising these moments, teachers can intervene strategically through the introduction of well-selected digital representations or by adjusting the structure and focus of instruction to support emerging understandings. For example, when addressing properties of mathematical concepts, teachers might pause to make relationships explicit, prompt students to articulate patterns or generalisations, or present alternative examples to reinforce critical ideas. As Mason (2008) notes, effective teaching depends on knowing both what to attend to and how to direct student attention productively. Supporting students' transitions from noticing individual relationships to perceiving general properties is central to mathematical thinking and learning. While this research sought to prompt changes in decimal thinking through student-led engagement with dynamic digital representations, teachers could also introduce and model the capabilities of each tool during purposeful teaching moments. Strategic use and modelling by the teacher may highlight not

just what the tool can do, but how and when it can be used to students' support thinking such as zooming in to test assumptions or using *Wishball-hundredths* to check decimal placement.

Teachers can purposefully integrate dynamic digital tools to support conceptual transitions in mathematics. This research demonstrates that dynamic digital representations can function as catalysts for conceptual change. When introduced at carefully considered points in a learning sequence, such tools can highlight structural relationships that may otherwise remain implicit. Their capacity to visualise and manipulate mathematical concepts in real time enables students to notice patterns, test hypotheses, and revise their understandings. For teachers, this points to the importance of deliberate selection and timing in the use of digital tools, not only as enrichment but also as tools for disrupting misconceptions and advancing conceptual understanding. For example, some students may benefit from seeing how the digital tool works first to reduce cognitive load so they can focus on the mathematical ideas rather than the mechanics of the virtual manipulative. The visual summary in Table 9.2 provides a practical reference point for educators and researchers seeking to understand which aspects of digital tools most effectively support conceptual change and how to potentially integrate these four digital tools in an educational context.

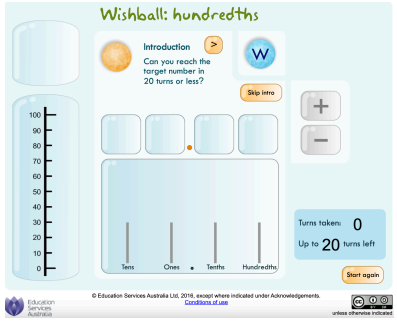
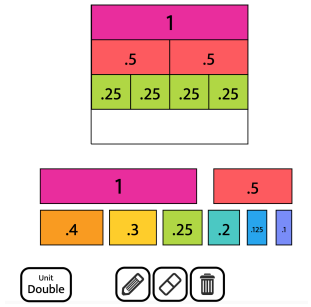
The findings illustrate the centrality of teacher noticing, particularly the capacity to interpret students' gestures, language, and interactions with representations, as a basis for pedagogical decision-making. The moment-to-moment analysis of student interactions revealed how small shifts in attention often preceded larger conceptual shifts. Teachers who are sensitive to these moments are better positioned to intervene productively. This underscores the importance of professional development that cultivates teachers' diagnostic expertise and equips them to use dynamic representations responsively and purposefully. Throughout the study, student articulation of thinking was valued as a lens into their conceptual understanding. The findings showed that critical insights into students' thinking emerged during verbal exchanges, particularly as students explained their reasoning or interpreted feedback from the digital tools. These moments provided valuable diagnostic information and often coincided with emerging conceptual breakthroughs. As such, the findings support instructional approaches that foreground mathematical discourse including opportunities for students to verbalise, justify and reflect on their thinking when engaging with representations. One-to-one or small-group settings were especially conducive to these conceptual changes, suggesting a need to embed such opportunities within regular classroom routines.

The dynamic nature of the tools enabled students to make connections between symbolic procedures and the underlying mathematical concepts. For example, zooming

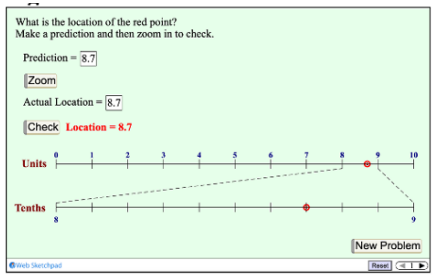
features helped students reconcile the density of the number system with the relative size of decimal values, while interactive number lines foregrounded the proportional relationships between tenths and hundredths. These experiences appeared to strengthen the link between procedural fluency and conceptual understanding, reinforcing the importance of designing learning experiences where both dimensions are mutually supportive. For educators, this highlights the potential of digital tools to mediate this integration in ways that are difficult to achieve through static representations alone.

Table 9.2

Summary of teaching advice linked to digital tools

Digital Tool	Teaching Advice
Wishball-hundredths  Focus on place value. Involves adding or subtracting to reach a target decimal fraction number. (https://www.scootle.edu.au/ec/viewing/L8456/index.html)	**Strengthening place-value understanding -** highlight how tenths create larger changes than hundredths and guide students to choose the place value that best refines their next guess. **Leveraging productive cognitive conflict -** use unexpected feedback (overshooting or undershooting) to challenge students' assumptions and prompt reconsideration of their strategies. **Supporting shifts in relative magnitude thinking -** prompt students to explain whether their guess is too high or too low and justify the size of the adjustment needed. **Making the affordances of Wishball-hundredths explicit -** draw attention to how the immediate higher/lower feedback supports strategic refinement and links actions to decimal reasoning. **Encouraging metacognitive noticing -** ask students to reflect briefly on what changed in their thinking after each round to surface emerging conceptual shifts.
Decimal Strips  Focus on number magnitude. Involves dragging coloured tiles representing different decimal values. (https://toytheater.com/decimal-strips/)	**Supporting shifts in relative magnitude thinking -** encourage students to compare strip lengths directly to judge which decimal is larger or smaller and by how much. **Strengthening place-value understanding** - draw attention to how each marked segment on the strip represents tenths and hundredths, reinforcing the structural meaning of each place value. **Leveraging productive cognitive conflict -** use moments when a visually longer or shorter strip contradicts a student's expectation to prompt reconsideration of their reasoning. **Making the affordances of Decimal Strips explicit** - highlight how aligning and comparing strips visually supports reasoning about magnitude, equivalence, and precision. **Encouraging metacognitive noticing -** prompt students to reflect on how visual comparisons helped them refine their understanding of decimal size.

Zooming in on Place Value



Focus on number magnitude.

Involves predicting the location of a decimal fraction on a number line, then self-checking with the embedded Zoom function.

(https://www.sineofthetimes.org/zooming-in-on-place-value/)

Supporting shifts in relative magnitude thinking - encourage students to observe how zooming reveals increasingly precise intervals, helping them judge how close one decimal is to another.

Strengthening place-value understanding - draw attention to how each zoom level exposes the structure of tenths and hundredths, making the role of each place value explicit.

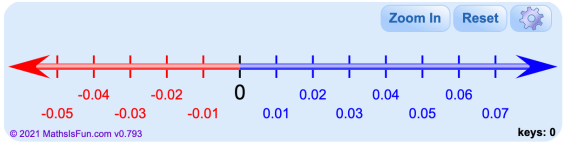
Developing noticing of decimal density - use the zoom feature to show that ten smaller interval numbers always appear between any two decimals, reinforcing density.

Leveraging productive cognitive conflict - highlight moments when zooming contradicts students' assumptions about spacing, prompting reassessment.

Making the affordances of Zooming in on Place Value explicit - emphasise how the zoom tool visually connects symbolic digits to proportional space on the number line.

Encouraging metacognitive noticing - ask students to reflect on how zooming helped them see the structure of decimals more clearly.

Zoomable Number Line



Focus on number density.

Involves using Zoom-in/out capability to respond to teacher-designed tasks.

(https://www.mathsisfun.com/numbers/number-line-zoom.html)

Supporting shifts in relative magnitude thinking - encourage students to use zooming to judge how far one decimal lies from another and refine their estimation of "how much bigger or smaller."

Strengthening place-value understanding - highlight how each zoom level reveals tenths, hundredths, and thousandths as increasingly finer partitions of the number line.

Developing noticing of decimal density - use the zoom function to show that new numbers always appear between any two decimals, reinforcing the idea of infinite density.

Leveraging productive cognitive conflict - draw attention to moments when spacing between decimals differs from students' expectations, prompting them to revise their reasoning.

Making the affordances of the Zoomable Number Line explicit - emphasise how zooming and scrolling allow students to coordinate symbolic decimal notation with proportional distance on the number line.

Encouraging metacognitive noticing - prompt students to reflect on how zooming clarified their understanding of decimal structure and relative size.

9.4.2 Methodological and theoretical contributions

This study makes a significant methodological contribution through its integration of Mason's (2008) theory of shifts of attention with a microgenetic approach to analysis. The combined use of this theoretical lens and methodological strategy offered a powerful means of investigating the complex and often non-linear processes underpinning students' conceptual change in mathematics. The application of shifts of attention theory, supported by a researcher-developed analytical framework (See Table 9.3), provided a valuable structure for tracing students' evolving understandings as they engaged with the decimal fraction tasks. This framework facilitated detailed tracking of the subtle, iterative ways in which students shifted their attention between visual, numerical, and structural aspects of the representations, and how such shifts contributed to more abstract reasoning about decimal fractions. In many cases, students' conceptual change appeared to be directly linked to their interaction with the dynamic features of the digital tools. These affordances successfully directed learners' attention to critical mathematical ideas, challenging and extending their existing understandings. The analytical framework thus proved effective in capturing the nuanced, situated processes through which students re-oriented their focus and reorganised their thinking in response to both tool-based prompts and internal cognitive shifts. Researchers could construct similar analytical frameworks to trace shifts of attention in relation to other mathematical concepts, which would be beneficial as it would offer a structured way to identify how specific representations or pedagogical features support conceptual change across varied content areas.

When paired with a microgenetic analysis approach, characterised by dense, moment-to-moment scrutiny of students' verbal and non-verbal behaviours over time, the shifts of attention model offered rich insights into how learners adjusted what they noticed, prioritised and reasoned about during decimal fraction tasks. This approach enabled the identification of specific moments of change in students' reasoning and highlighted how understandings were constructed, destabilised and re-formed across successive learning interactions. In contrast to broader pre/post methodologies, the microgenetic approach captured the process of learning in situ, offering explanatory power in relation to how and why conceptual change occurred. The combination of the shifts of attention framework and a microgenetic approach also made it possible to uncover the layered and recursive nature of students' conceptual development with decimal fractions. In each episode of evidence presented in the findings, students were observed to shift beyond discerning details as a structure of attention and instead demonstrated an ability to shift rapidly between multiple levels of structures involving recognising relationships and perceiving properties. This involved simultaneously tuning into a range of

conceptual understandings such as place value, relative magnitude and decimal density, to construct increasingly in-depth knowledge of decimal fractions. For four of the students, this pattern of shifting attention and layering conceptual insights occurred repeatedly up to six times before particular relationships were stabilised and formalised as mathematical properties that could serve as the basis for reasoning. This recursive sequence suggests that learners may require extended opportunities not only to notice mathematical relationships, but also to articulate and reconfirm them by locating analogous patterns across different representations and contexts.

The microgenetic method was critical in revealing the temporal dimension of learning where students' movement toward generalisation was not linear, but iterative and reflexive. The findings align with the theoretical work of Pirie and Kieren (1991), who proposed that the development of mathematical understanding involves cyclical movement, including the process of "folding back" to earlier understandings in order to consolidate and refine them. What was captured in this study through detailed, moment-by-moment analysis was a kind of conceptual 'stacking' as a compounding of increasingly sophisticated structures of attention that were grouped into meaningful relationships. These relationships were subsequently extrapolated as properties and eventually generalised into principles, forming the foundation for principled mathematical reasoning. Such insights would have been difficult to detect using more traditional methodological approaches, underscoring the value of combining the shifts of attention theory with a microgenetic lens in studies of conceptual change.

This dual-methodological approach holds considerable potential for future research investigating students' conceptual development in other mathematically challenging domains. Replicating this approach would enable researchers to document not only whether students' understandings shift but also how those shifts emerge through interaction with tools, tasks and teacher guidance. Importantly, the researcher-developed analytical framework also offers productive theoretical language for describing how learners allocate cognitive resources during mathematical activities and provides an alternative to more linear or stage-based models of conceptual change. Therefore, this study demonstrates the value of integrating shifts of attention theory with a microgenetic method to illuminate the intricate processes through which mathematical understanding develops. Future studies seeking to investigate conceptual change in mathematics may benefit from adopting a similar detailed and theoretically informed approach capable of capturing the dynamic nature of mathematical learning.

Table 9.3

Shifting Attention with Decimal Concepts Framework

Structure and Microqualities of Attention (Mason, 2008)	**Decimal Fraction Learning Progression (ACARA, 2020)**	**Connection of Theory of Structures of Attention and Decimal Fraction Learning Progression Points (Researcher's own reasoning)**	**Key Concept(s) of Decimal Fractions in which Conceptual Change has Occurred (Researcher's own reasoning)**
Holding wholes	Naming decimal fractions.	The student verbally names decimal fractions without an understanding of place value.	Place value
Discerning details	State the place value of digits in decimal numbers of up to two decimal places. Use place value to partition decimals of up to two decimal places.	The student can discern mathematical details of decimal fractions with an understanding of place value and reforming decimal numbers with value.	Place value Decimal magnitude
Recognising relationships	Understand what a decimal fraction is and recognise that they arise out of division. Model, compare and represent decimals of up to two decimal places. Understanding and estimating relative size of decimals.	The student understands the relationship between decimal fractions and the division operation, as the decimal is parts of a unit resulting from dividing the whole into equal parts. Furthermore, the student understands the positional value of decimals as well as the scaling effect as digits move to the right and left in place value for example 3.509 has 35.09 tenths, 350.9 hundredths, 3509 thousandths.	Place value Decimal density
Perceiving properties	Place decimals of up to two decimal places on a number line. Apply knowledge of decimals to record measurements, for example 123 cm = 1.23 m.	The student perceives properties of decimal fractions as they demonstrate knowledge of the relative size of decimals when comparing and applies this knowledge to the relationship with measurement and position when plotting values on a number line.	Decimal magnitude
Reasoning on the basis of agreed properties	Flexible strategies for addition, subtraction, multiplication and division of decimals.	The student relates their understanding of positional value number knowledge to perform operations with decimals.	Place value Decimal density

The theoretical insights that emerged from the study highlighted how dynamic digital representations can mediate learning in ways consistent with constructivist perspectives. In line with Bermejo et al. (2021), who emphasise the importance of interaction with a "knowledgeable other" in supporting learners' construction of mathematical meaning, the findings suggested that the design of these digital tools can partially assume this role. The built-in feedback, dynamically responsive animations and constraint-based prompts embedded in the representations provided students with challenge at moments when their thinking was uncertain. Thus dynamic digital representations were positioned not merely as visual aids but as active contributors to conceptual change.

Furthermore, the findings highlight the relevance of Santos-Trigio's (2006) theory of stressing and ignoring details in understanding shifts in attention. Productive learning with dynamic representations requires students to make decisions about what aspects of the display to attend to and what to temporarily disregard. Selective attention supports conceptual understanding, particularly when learners are guided by the digital tool's design or by teacher scaffolding, toward noticing critical features while ignoring less relevant information. Both stressing and ignoring as separate learning actions, enabled more efficient and focused engagement with the mathematical content.

9.4.3 Limitations and implications for future research into classroom practice

Although the responses of the six participating students provided rich and valuable insights appropriate for the exploratory aims of this study, the limited sample size presents constraints in terms of broader applicability. Future research would benefit from engaging a larger and more diverse group of students to enhance the generalisability of findings. Including participants from different schools, geographic regions, and educational contexts could yield a more representative understanding of how students interact with dynamic digital tools. Such an approach would also allow for the examination of contextual factors, such as teacher facilitation styles and task types, school technological infrastructure, and curriculum alignment that may influence the effectiveness of digital representations in supporting conceptual change related to decimal fractions. Expanding the sample would also make it possible to explore subgroup analyses, such as differences in outcomes between students with varying levels of prior knowledge, confidence in mathematics, or familiarity with digital technologies. These nuances could inform more targeted instructional strategies when using tools like *Wishball-hundredths, Decimal Strips, Zooming in on Place Value,* and *Zoomable Number Line*.

Another important consideration within the research design is the cumulative effect of students' repeated engagement with the four dynamic digital tools. By the time students participated in the fourth task-based interview, which involved the *Zoomable Number Line*, they had already encountered and interacted with a variety of representations that targeted key concepts related to decimal fractions. This sustained exposure likely contributed to a deeper and more connected understanding of decimal place value, magnitude and density. Because all participants experienced the digital tools in the same fixed sequence, improvements observed in later weeks cannot be attributed solely to that digital tool's affordances. Practice effects, conceptual priming from earlier tool use, and cumulative exposure are plausible confounding influences may have contributed to the students' stronger conceptual engagement observed in Interviews 3 and 4. The sequencing of these tools may have played a role in supporting conceptual progression, suggesting that the order in which students engage with different digital representations can influence their learning outcomes. Future research could investigate how varying the combination or sequence of digital tools might optimise conceptual change, and whether different students benefit from different learning pathways depending on their prior knowledge and misconceptions.

It is important to acknowledge that purposive sampling for think-aloud suitability, while methodologically appropriate and consistent with established protocol conventions (Ericsson & Simon, 1993), produced a sample that was not representative of the full range of Year 4 students. Students selected on the basis of metacognitive expressiveness may demonstrate richer and more visible conceptual engagement with digital representations than students who process understanding internally or who lack the verbal fluency to narrate their reasoning in real time. Whilst engagement with dynamic digital representations may not require students' verbal articulation to be effective, the methods of data collection and analysis were unable to capture non-verbal evidence of understanding and therefore the transferability of findings to students with lower verbalisation skills is indeterminate. Future research incorporating complementary methodologies such as clinical interviews, gesture analysis, or pre/post-test designs could extend these findings to broader cohorts without relying on concurrent verbalisation.

A promising direction for future research involves the integration of eye-tracking technology. As an emerging methodology in mathematics education research, eye-tracking can offer objective, real-time insights into students' attentional focus and cognitive engagement. By capturing where and how long students fixate on different elements of the screen, researchers can gain a more nuanced understanding of the specific visual and conceptual

features that either support or hinder learning. This type of data could help validate or challenge researcher interpretations of student thinking and provide empirical evidence of conceptual shifts during key learning moments. Eye-tracking is particularly well-suited for studying complex, dynamic representations like those employed in this study, where multiple visual elements compete for student attention and meaning-making is fluid and ongoing.

This study also holds practical implications for future research into mathematics teaching and classroom practice. The insights gained from students' interactions with the four digital tools can inform the development of evidence-based teaching resources and professional learning materials. For example, detailed guidance on how and when to introduce specific representations, how to scaffold student interpretation of visual features, and how to prompt productive student dialogue could be published as part of a broader effort to support teachers in leveraging digital technologies to enhance conceptual understanding of decimal fractions. Providing educators with pedagogically sound and classroom-ready strategies for using freely available tools would help bridge the gap between research and practice, ultimately improving mathematics learning experiences for students. While this study lays an important foundation for understanding how digital tools can support conceptual change, further research and practical translation efforts are needed to realise the full potential of these dynamic digital representations in diverse educational settings.

9.5 Final Statement

To support students in developing a robust understanding of decimal fractions, teachers must employ a range of instructional strategies, including visual models, manipulatives, real-life contexts, and interactive learning tasks. This study demonstrated how students' conceptual understanding of decimal fractions evolved through their engagement with the dynamic features embedded in four digital tools. Evidence of these changes emerged in students' verbal justifications and mathematical reasoning during the tasks. Analysis of their oral explanations revealed clear instances of conceptual development that extended beyond procedural knowledge and strategic competence. Notably, shifts in understanding related to decimal place value, magnitude, and density were closely linked to students' interactions with dynamic visual, linear, area-based, and symbolic representations. These novel findings indicate that tools such as *Wishball-hundredths, Decimal Strips, Zooming in on Place Value*, and *Zoomable Number Line* hold significant potential to enhance young learners' comprehension of key decimal concepts.

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Appendix A: Letter of Introduction/Email Content for Initial Contact with Principal

Dear Sir/Madam,

My name is Amelia Gorman and I am a Doctor of Philosophy student at The University of Sydney. As part of the program I am conducting a research project titled 'Learning Decimal Fractions with Apps' that will investigate how Year Four students interact with a series of dynamic digital representations to demonstrate knowledge and understandings of decimal fractions. The study will take place under the supervision of Dr Jennifer Way, Associate Professor, The University of Sydney. As such, I am approaching you to ask whether (pending ethics approval) your school might be willing to participate in this study?

Attached is an information sheet. Please do not hesitate in contacting me for further information regarding the nature of this study.

Kind regards,

Amelia Gorman.

Appendix B: Participant Information Statement (Principal/School Copy)

Participant Information Statement

Research Study: Learning Decimal Fractions with Apps

Associate Professor Jennifer Way (Head Researcher)
Sydney School of Education and Social Work
Phone: +61 2 9351 6272 | Email: jennifer.way@sydney.edu.au
Miss Amelia Gorman (Doctor of Philosophy student) | Email: agor3358@uni.sydney.edu.au

1. What is this study about?

We are conducting a research study about the ways in which children use digital tools to represent their mathematical thinking about decimal fractions. The purpose is to better understand how digital tools can be more effectively used to assist children's knowledge and understanding of decimal fractions. Taking part in this study is voluntary.

Please read this sheet carefully and ask questions about anything that you don't understand or want to know more about.

2. Who is running the study?

The study is being carried out by the following researchers:

- Amelia Gorman is conducting this study as the basis for the degree of Doctor of Philosophy at The University of Sydney. This will take place under the supervision of Dr Jennifer Way, Associate Professor, The University of Sydney.

3. Who can take part in the study?

We are seeking the cooperation of a Year 4 class from which we will select six students based on results from a pre-test. All students in the class are being invited to participate as the class teacher has volunteered to be involved in this study. Any child of a similar age would be eligible to participate in this study. Your school has been approached because of the interviewer's familiarity with the school community. Year 4 has been selected because of the sequence of content in the syllabus. As decimal fractions are first introduced in the syllabus in Year 4, the students will have varying degrees of knowledge of decimals which allows the researcher to investigate what understandings may develop from interaction with the digital tools. Six children are considered sufficient to give a variety of indicative responses.

4. What will the study involve for me?

If you decide to take part in this study, you will be asked to nominate a Year 4 teacher to facilitate task-based interviews with six children. The organisation required by the teacher is: administering a short paper-based pre-test to the class, confirming the six children selected would be comfortable in the interview situation; sending home information

statements and consent forms (prepared by the researchers) to the parents; and, arranging appropriate times for the interviews to be conducted.

The 'task-based interviews' involve the researcher (Amelia Gorman) asking an individual child to work on four or five 'locating and comparing decimal fractions' tasks, and encouraging him/her to talk about his/her strategies for solving the problem and using a digital tool. This interview process will be repeated over six weeks to investigate four different digital tools. Each interview will be approximately 30-minutes in length and each interview session timing will be negotiated with the classroom teacher to minimise disruption to regular class work.

The sessions will be video-audio recorded. This is necessary because the child will use many actions and gestures while they work and explain, and, as these are all forms of representations, they are very important in understanding what thinking is taking place. In addition, screen-captures and work samples will be collected as part of the data set.

5. Can I withdraw once I've started?

Being in this study is completely voluntary and you do not have to take part. The Principal can choose to withdraw permission at any time without affecting the school's relationship with the researchers or anyone else at the University of Sydney, now or in the future. Similarly, the co-operating teaching can withdraw from the role of facilitator at any time. Parents and individual children are also free to withdraw from the study at any time, with no consequences.

If you decide to take part in the study and then change your mind you can withdraw by simply telling Amelia Gorman that you no longer wish to participate (Email: agor3358@uni.sydney.edu.au), or if you prefer, contact Amelia's supervisor, Jennifer Way (Phone: 9351 6272, Email: jennifer.way@sydney.edu.au).

If the child takes part in an interview, they may refuse to answer any questions that they do not wish to answer.

If you or the child chooses to withdraw, we will not collect any more information. Please let us know at the time of withdrawal what you would like us to do with information we have collected about the child up to that point. If you wish, their information will be removed from our study records and will not be included in publications, up to the point that we have analysed and published the results.

6. Are there any risks or costs?

Aside from giving up your time, we do not expect that there will be any risks or costs associated with taking part in this study.

7. Are there any benefits?

You will not receive any direct benefits from being in the study.

8. What will happen to information that is collected?

Any information provided to us will be stored securely and we will only disclose identifiable information with your permission, unless we are required by law to release information. We are planning for the study findings to be published. The child will not be individually identifiable in these publications.

The audio-recorded conversations will be transcribed and annotated with notes about the associated actions. The findings will be written up in a report and possibly presented at a conference or published as a journal article. Quotations and video-extracts or stills may be published, but the child will not be identified in any way.

The child's information will be stored securely and their identity/information will be kept strictly confidential, except as required by law. Study findings may be published, but the child will not be individually identifiable in these publications.

9. Will I be told the results of the study?

You have a right to receive feedback about the overall results of this study. This feedback will be in the form of a short summary (1-2 pages). You are also welcome to discuss the results with the researcher. You will receive this feedback after the study is finished.

10. What if I would like further information?

When you have read this information, the following researcher/s will be available to discuss it with you further and answer any questions you may have:

- Amelia Gorman, (Email: agor3358@uni.sydney.edu.au) or Jennifer Way (Sydney School of Education & Social Work, University of Sydney, Phone: 9351 6272, Email: Jennifer.way@sydney.edu.au).

11. What if I have a complaint or any concerns?

The ethical aspects of this study have been approved by the Human Research Ethics Committee (HREC) of The University of Sydney 2022/893 according to the *National Statement on Ethical Conduct in Human Research (2007).*

If you are concerned about the way this study is being conducted or you wish to make a complaint to someone independent from the study, please contact the University:

Human Ethics Manager
human.ethics@sydney.edu.au
+61 2 8627 8176

This information sheet is for you to keep

Appendix C: Human Research Ethics Approval Letter

THE UNIVERSITY OF SYDNEY

Research Integrity & Ethics Administration
HUMAN RESEARCH ETHICS COMMITTEE

Wednesday, 1 February 2023

Assoc Prof Jennifer Way
Education; Faculty of Arts and Social Sciences
Email: jennifer.way@sydney.edu.au

Dear Jennifer,

The University of Sydney Human Research Ethics Committee (HREC) has considered your application.

I am pleased to inform you that after consideration of your response, your project has been approved.

Details of the approval are as follows:

Project No.:	**2022/893**
Project Title:	**Learning Decimal Fractions with Apps**
Authorised Personnel:	**Way Jennifer; Bobis Janette; Gorman Amelia;**
Approval Period:	**25/01/2023 to 25/01/2027**
First Annual Report Due:	**25/01/2024**

Documents Approved:

Date Uploaded	Version Number	Document Name
23/01/2023		Teacher Invitation Email
23/01/2023	Version 2	Participant Information Statement Principal Clean Copy
23/01/2023	Version 2	Participant Information Statement Parent Clean Copy
23/01/2023	Version 2	Participant Information Statement Child Clean Copy
23/01/2023	Version 2	Participant Consent Form Child Clean Copy
23/01/2023	Version 2	Participant Consent Form Parent Clean Copy
12/11/2022	Version 1	Confirmation Statement Selected Parent
12/11/2022	Version 1	Confirmation Statement Non-Selected Parent
12/11/2022		Appendices Interview Protocol
12/11/2022		Initial Contact Principal
12/11/2022		Interview Tasks Resources

Condition/s of Approval

- Research must be conducted according to the approved proposal.
- An annual progress report must be submitted to the Ethics Office on or before the anniversary of approval and on completion of the project.
- You must report as soon as practicable anything that might warrant review of ethical approval of the project including:
  - Serious or unexpected adverse events (which should be reported within 72 hours).
  - Unforeseen events that might affect continued ethical acceptability of the project.
- Any changes to the proposal must be approved prior to their implementation (except where an amendment is undertaken to eliminate *immediate* risk to participants).
- Personnel working on this project must be sufficiently qualified by education, training and experience for their role, or adequately supervised. Changes to personnel must be reported and approved.

Research Integrity & Ethics Administration
Research Portfolio
Level 3, F23 Administration Building
The University of Sydney
NSW 2006 Australia

T +61 2 9036 9161
E human.ethics@sydney.edu.au
W sydney.edu.au/ethics

ABN 15 211 513 464
CRICOS 00026A

- Personnel must disclose any actual or potential conflicts of interest, including any financial or other interest or affiliation, as relevant to this project.
- Data and primary materials must be retained and stored in accordance with the relevant legislation and University guidelines.
- Ethics approval is dependent upon ongoing compliance of the research with the *National Statement on Ethical Conduct in Human Research*, the *Australian Code for the Responsible Conduct of Research*, applicable legal requirements, and with University policies, procedures and governance requirements.
- The Ethics Office may conduct audits on approved projects.
- The Chief Investigator has ultimate responsibility for the conduct of the research and is responsible for ensuring all others involved will conduct the research in accordance with the above.

This letter constitutes ethical approval only.

Please contact the Ethics Office should you require further information or clarification.

Sincerely,

Associate Professor Stephen Fuller
Chair
Human Research Ethics Committee (HREC 2)

The University of Sydney of Sydney HRECs are constituted and operate in accordance with the National Health and Medical Research Council's (NHMRC) National Statement on Ethical Conduct in Human Research (2018) and the NHMRC's Australian Code for the Responsible Conduct of Research (2018)

Appendix D: Decimal Comparison Test Marking Instructions

Marking Instructions for the 30 item Decimal Comparison Test

Look at the accompanying version of the test, where the different groups of items have been colour coded. First mark Red and Blue groups (both out of 5) and assign a Primary Classification (Longer-is-larger, Shorter-is-larger, Apparent-expert).. Results other than those listed (scores of 2 or 3 or the Low/Low combination) are labelled as Unclassified. Many students will be able to be further classified using their responses to the remaining four groups of coloured items. (Note that Reciprocal/Negative thinkers cannot be separated with this test.)

Red Group	Blue Group	*Primary Classification*	Yellow Group	Azure Group	Green Group	Pink Group	*Finer Classification*
Low	High	***Longer-Larger***	Low	High	High	High	***whole number***
			High	High	High	High	***column overflow***
			Low	High	High	Low	***reverse***
High	Low	***Shorter-Larger***	High	Low	High	High	***denominator focussed***
			High	Low	Low	Low	***reciprocal/negative***
High	High	***Apparent-Expert***	High	Low	High	High	***money***
			High	High	High	High	***taskexpert***
Low = 0 or 1 High = 4 or 5			Low = 0 or 1 High = 3 or 4		Low = 0 or 1 High = 2 or 3		

Appendix E: Decimal Comparison Test Marking Analysis Tool

Name:_____ Class:_____ Date:_____

For each pair of decimal numbers circle the one which is LARGER.

0.4	0.457	1.06	1.053	4.4502	4.45
0.86	1.3	4.08	4.7	17.353	17.35
0.3	0.4	3.72	3.073	8.245	8.24563
1.85	1.84	2.621	2.0687986	3.2618	3.26
3.71	3.76	8.052573	8.514	3.741	3.746
4.8	4.63	5.62	5.736	0.35	0.42
0.5	0.36	0.5	0.75	2.186	2.954
0.75	0.8	0.426	0.3	0.872	0.813
0.37	0.216	2.516	2.8325	0.038	0.04
3.92	3.4813	7.942	7.63	0.006	0.53

Appendix F: Decimal Comparison Test Student Template

Name _____ Teacher _____

Class _____ Date _____

Compare the Pair

For each pair of decimal fractions, circle the one which is LARGER.

0.4	0.457	1.06	1.053	4.4502	4.45
0.86	1.3	4.08	4.7	17.353	17.35
0.3	0.4	3.72	3.073	8.245	8.24563
1.85	1.84	2.621	2.0687986	3.2618	3.26
3.71	3.76	8.052573	8.514	3.741	3.746
4.8	4.63	5.62	5.736	0.35	0.42
0.5	0.36	0.5	0.75	2.186	2.954
0.75	0.8	0.426	0.3	0.872	0.813
0.37	0.216	2.516	2.8325	0.038	0.04
3.92	3.4813	7.942	7.63	0.006	0.53

Appendix G: Paper published in MERGA (Mathematics Education Research Group of Australasia) Conference Proceedings for 2024.

Student Engagement with Dynamic Digital Representations of Decimal Fractions to Prompt Conceptual Change

Amelia Gorman
The University of Sydney
agor3358@uni.sydney.edu.au

Decimals and the related areas of ratio and proportion are recognised as one of the most challenging and complex areas of mathematics for young children to learn. In this study, four dynamic digital representations were used by Year 4 (9–10 years old) students in a set of video-recorded, individual task-based interviews to explore key concepts of decimals. Microgenetic analysis methods were used to detect conceptual changes through specific shifts in the learner's attention. The data collected revealed what features of each dynamic digital representation prompted attention shifts and how these translated to changes in conceptual understanding of decimal fractions.

Children have difficulty in understanding the abstract nature of decimals, and a plethora of misconceptions arise from incomplete understandings (Steinle & Stacey, 2004). Decimal fractions present complications as part of the number-relationship needed to make sense of the number, the denominator, is not visible. Developing conceptual understandings of decimal density, place value and relative magnitude of decimal fractions are considered key for consolidating number sense. The properties of density and place value unify all numbers and rational number magnitude knowledge is related to future mathematics achievement (Siegler et al., 2011). Minimal research has been conducted into what effective representations should be employed by mathematics educators to assist students in developing 'decimal sense' and managing 'cognitive conflict' in the context of decimal fractions (Swan, 1983). This study investigated a collection of dynamic digital representations that featured the three key concepts of decimals.

Representations are used in the mathematics classroom to assist students in understanding abstract concepts that are central to mathematics learning (Pape & Tchoshanov, 2001). An important feature of mathematics education is working with appropriate mathematical diagrammatic representations, but there is uncertainty around the best representations for decimals. Dynamic representations are potentially an effective support for student learning within mathematics because of their dynamic affordances. Unlike with static representations, students are able to actively interact with the mathematics content by making conjectures and testing them immediately to experience the concept from various perspectives and receive feedback on their thinking (Orrill & Polly, 2013). In particular, learner-centred computer tools that encourage interactive engagement with mathematics, allow students to 'see' notions that would otherwise be 'unseeable' (Orrill & Polly, 2013). Relatively little work has focused on the role of dynamic digital representations in teaching and learning decimals, although there is an abundance of research into the use of digital learning objects in mathematics education (Litster et al., 2018). The intention of this study was to establish a foundation for research in this topic by exploring how children's interactions with dynamic digital representations of decimal fractions may develop their conceptual understanding in this numerical area.

Theoretical Framework

In mathematics education, conceptual understanding involves being able to represent a variety of mathematical ideas in different ways, to see the connections between those representations, and to know how to use each representation for different purposes (Caldwell, 2020). The growth of students' conceptual understanding is synonymous with mathematics learning. Mason's (2008) theorisation of the notion of attention has been employed to account

(2024). In J. Višňovská, E. Ross, & S. Getenet (Eds.), *Surfing the waves of mathematics education. Proceedings of the 46th annual conference of the Mathematics Education Research Group of Australasia* (pp. 255–262). Gold Coast: MERGA.

for students' learning of mathematics. The theory is underpinned by the idea that students must experience shifts of attention for learning to occur. As students interact with mathematical tasks and representations there are ongoing shifts in their focus and structure of attention. The learning of mathematics occurs through these attention shifts and students' developing awareness of them (Mason, 2008). At present, there is little conclusive research on how students are best supported to build a conceptual understanding of decimals. The theory of shifts of attention provides a scaffold for educators' understanding of teaching mathematics as well as proposing five observable attending stages that indicate what and how students are learning in mathematics lessons. Mason (2008) described the 'focus of attention' (what the learner attends to) and the form or 'structure of attention' (how the learner attends) whereby several shifts of attention occur in the process of thinking mathematically related to an object. Structure of attention refers to "holding wholes, discerning details, recognising relationships, perceiving properties, and reasoning on the basis of agreed properties" (Mason, 2008, p. 35). See Table 1 for examples. As attention shifts from one concept to another, its structure alters and connections are made, relationships are perceived, and new ideas are appreciated. Applying the theory of shifts of attention may provide a scaffold to plot a progression of understanding that students may take when learning about key concepts related to decimal fractions.

The focus of this paper is to explore how children in middle-primary grades engaged with dynamic digital representations of decimal fractions to develop their conceptual understanding. The overarching research question communicates the aim of the study:

- What features of dynamic digital representation do children attend to that prompt conceptual changes related to decimal fractions?

Method

The methodology considered most appropriate for the study was structured, task-based interview because it complemented the theoretical framework from a constructivist, learner-centred view. Constructivist learning theory has been linked with structured task-based interviews in the work of Goldin (2003) who found the research method to be an effective means of understanding conceptual knowledge. In this study, task-based interviews were used to prompt students' interactions with mathematical tasks and representations and record ongoing shifts in their focus of attention and the structure of attention.

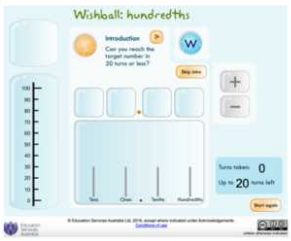
Data collection for this study consisted of four task-based interviews each focussing on a different dynamic digital representation: *Wishball-hundredths*, *Decimal Strips*, *Zooming in on Decimals*, and *Zoomable Number Line* (Figure 1). Four digital representations were selected from resources readily available to teachers on the internet, following an analysis of the affordances and constraints of each potential digital tool, and the focus on at least one of the three key decimal concepts. The number-line based, place-value symbolic, and area-model dynamic digital representations of decimals used were in the form of applets which are embedded into a webpage and displayed using a web-browser.

Four participants were from one Year 4 (9–10 years old) class situated in a school from the outer suburbs of Sydney, Australia. Although the students were not intended to be representative of the whole class, the researcher selected four that spanned a range of general mathematical achievement levels, based on results from a decimal comparison pre-test. This was in case there were differences in the ways in which students of varying abilities interacted with the dynamic digital representations. This grade level was selected because, according to the NSW syllabus (Board of Studies NSW, 2012), students are introduced to decimal notation for tenths and hundredths in Year 4. The decimal number concepts and skills were therefore still in the early stages of development at the time of the study.

Figure 1

Screenshots and Basic Information About the Digital Tools

Wishball

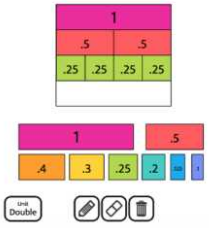


Focus on place value.

Involves adding or subtracting to reach a target decimal fraction number.

(https://www.scootle.edu.au/ec/viewing/L8456/index.html)

Decimal Strips

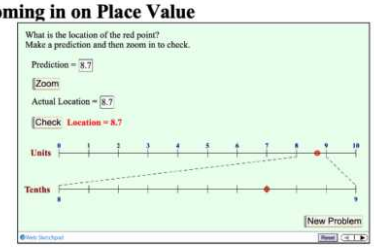


Focus on number magnitude.

Involves dragging coloured tiles representing different decimal values.

(https://toytheater.com/decimal-strips/)

Zooming in on Place Value

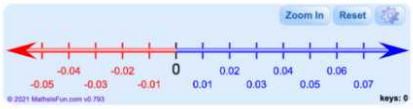


Focus on number magnitude.

Involves predicting the location of a decimal fraction on a number line, then self-checking with the embedded Zoom function.

(http://www.sineofthetimes.org/zooming-in-on-place-value/)

Zoomable Number Line



Focus on number density.

Involves using Zoom-in/out capability to respond to teacher-designed tasks.

(https://www.mathsisfun.com/numbers/number-line-zoom.html)

The researcher worked individually with each participant in four 30-minute sessions, over a period of four weeks. During the interviews, each digital resource (Figure 1) was introduced by inviting the student to freely investigate the tool for 5-minutes and them asking any clarifying questions. The tasks were then initiated by the researcher and completed by the student. In addition, the researcher, who was a qualified teacher, regularly asked probing questions to encourage further explanation by the student, for example 'Can you tell me what you are thinking?' but remained neutral regarding 'correctness' of responses. Video-recording the interviews was critical to collecting quality data as it was necessary to capture what was said and done by the child simultaneously, including gestures, expressions and emotional responses.

The data collected from the task-based interviews were analysed using a microgenetic approach that involved closely examining intensive data of what learners do and say over several task-based interviews to capture a record of moment-by-moment learning processes. This close examination allows researchers to draw conclusions about what prompts knowledge change, how learning occurs, and other aspects of the processes of acquiring conceptual understanding (Voutsina et al., 2019). Data analysis began with repeated viewings of the video-audio recordings of each task-based interview to deduce what new understandings the participants had gathered during the task-based interview. The initial analysis period occurred

over several weeks as the researcher transcribed and documented all observed changes in decimal understanding. Four evidence excerpts were selected for this paper from the larger data-set. These examples showed key tasks that provoked the most change in conceptual thinking as a result of the student's interaction with a dynamic digital representation of decimals. A framework to support analysis was designed by the researcher as a reference tool when analysing the verbal reasoning and manipulative actions made by the participants (Table 1). The 'Shifting Attention with Decimal Concepts Framework' correlated Mason's five components of structures for attending with the developmental progression of decimal fraction concepts outlined in the National Numeracy Learning Progression (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2020). The framework was developed prior to data collection by anticipating the specific decimal concepts students should attend to when developing their conceptual understanding of decimal place value, magnitude and density. This was then modified after the interviews based on observations of student understandings. The value of this extended theory of shifts of attention enabled the researcher to understand what each participant experienced in the learning process and identify what elements of the dynamic digital representations prompted shifts of attention and changes in understanding. This analysis tool was used alongside a microgenetic approach to data analysis when the researcher discerned observable shifts in how and what the participants were attending to as they engaged with the four dynamic digital representations of decimal fractions.

Table 1

Shifting Attention with Decimal Concepts Framework

Structure and microqualities of attention (Mason, 2008)	**Connecting Mason's structures with decimal fraction learning progression (ACARA, 2020)**	**Key concepts of decimal fractions**
Holding wholes	Names decimal fractions without an understanding of place value	Place value
Discerning details	Discern mathematical details of decimal fractions with an understanding of place value and reforming decimal numbers with value	Place value Decimal magnitude
Recognising relationships	Understands the relationship between decimal fractions and the division operation, as the decimal is parts of a unit resulting from dividing the whole into equal parts. Understands the positional value of decimals as well as the scaling effect as digits move to the right and left in place value	Place value Decimal density
Perceiving properties	Perceives properties of decimal fractions as they demonstrate knowledge of the relative size of decimals when comparing and applies this knowledge to the relationship with measurement and position when plotting values on a number line	Decimal magnitude
Reasoning on the basis of agreed properties	Relates their understanding of positional value number knowledge to perform operations with decimals.	Place value Decimal density

Findings

Four scenarios were selected to illustrate how each of the students interacted with one of the four dynamic representations. The evidence and accompanying interpretation reveal the specific dynamic affordances that facilitated shifts in attention which indicated changes in conceptual understanding of decimals. Pseudonyms are used in replacement of students' names throughout.

Example One: Annelise Wishball-Hundredths

In the interview examining the affordances of Wishball-hundredths, Annelise experienced shifts in attention where she was able to generalise mathematical principles pertinent to adding and subtracting decimals, including the concept of place value when reforming decimals. During the task-based interview, Annelise needed to reach the target number by adding or subtracting in fewer than 20 moves, the current starting number was 29.67 and the spinner displayed 9. Annelise attempted to calculate the subtraction $29.67 - 0.09$ before the screen displayed the answer, to decide whether it was an ideal move within the game. This was a key moment of conceptual learning as she exclaimed "which will actually make it closer to the target number!" Through her engagement with the dynamic affordances embedded in Wishball-hundredths, namely the dynamic counting frame, as well as the feedback she received when the starting number changed based on her own carefully considered mathematical moves, the student's thinking progressed to *reasoning on the basis of agreed properties* (Table 1) as she related her understanding of positional place value knowledge to perform a subtraction operation with decimals. However, her approximated answer suggested that she was consolidating her knowledge across multiple structures of attention. Annelise acknowledged her change in conceptual understanding was the result of new learnings that occurred during Wishball-hundredths when she explained, "I knew that one of the tenths would need to be traded to the hundredths to subtract from, because I had seen the counting frame move before." The powerful partnering of observed dynamic affordances and Annelise's knowledge of theoretical concepts related to decimals allowed a transition of attention to generalising principles and therefore being able to *reason on the basis of agreed properties* when calculating a subtraction problem with decimal numbers.

Example Two: Valerie Decimal Strips

Valerie was asked to compare a series of two decimal numbers to determine the larger decimal value whilst completing the Decimal Strips task-based interview. One comparison challenge required the student to decide whether 0.2 or 0.25 was a larger decimal. Valerie acknowledged her insufficient decimal knowledge to answer the comparison question. Without hesitation she stated, "I'm not sure I'll have to check." Valerie was aware of the affordances of the dynamic digital representation to display the relative size of decimal fractions as was needed in this decimal comparison challenge. She dragged the 0.2 and 0.25 coloured tiles onto the blank fraction wall. The researcher used a probe question, "why do you think they are so close in value?" to elicit further student thinking. Valerie considered the question before responding, "maybe the zero-point-two has something else on it you just can't see it." A second probe question was asked, "what would be on the end?" Valerie replied, "it could be twenty-something and then this one [pointed to yellow 0.3 strip] could be thirty-something."

The experience of Valerie unpacking this particular decimal comparison question is an example of how the abstract nature of decimals present significant computation difficulties for students. Connecting her pre-existing decimal knowledge with how she manipulated the decimal strips to compare relative sizes, Valerie posited that 0.2 and 0.3 were actually "twenty-something" and "thirty-something". She independently uncovered the concept of decimal place value and accurately described 0.2 and 0.3 as having a 'hidden' value of tenths. Valerie's existing misconception thinking "longer-is-larger" (Steinle & Stacey, 2004) was being challenged through her engagement with the virtual manipulative as she dynamically compared the decimal lengths in this area-model representation. A significant first step to overcoming her erroneous thinking.

Example Three: Yehali Zooming in on Place Value

Yehali's default numerical thinking was focused on whole numbers only. She did not hold an understanding of decimal density and therefore initially couldn't predict decimals located between whole numbers in the Zooming in on Place Value interview. The interactive activity displayed the decimal 0.9 with the question, "what is the location of the red point? Can you make a prediction?" Yehali's first guess was "one", so she typed the predicted location in as 1.0 and watched the 'zooming in' animation reveal the location between zero and one with tenths integers marked as dashes on the number line. The researcher probed, "what did you see happen just then?" Yehali replied, "the numbers stretched out." The screen displayed a second question, "what is the actual location of the red point?" The student answered "nine" and typed her second predicted location in as 9.0 and then saw the actual location revealed as 0.9. The researcher checked for understanding, "what is the actual location written as?" Yehali read, "zero-and-nine." A follow-up question was posed, "why is that the actual answer for the location of the red point on the number line?" Yehali reasoned, "maybe because there is no number here it's just zero [pointed to zero integer marking on number line] and it hasn't gone to one yet [pointed to one integer marking on number line] so it's in the middle somewhere."

Yehali's current thinking involved *holding wholes* as her understanding was constrained by the whole number details presented in the first-step of the interactive activity. When Yehali observed the animated number line dynamically zoom-in to show ten-tenths between zero and one, she used her whole-number thinking to change her prediction to "nine" as she was viewing the integer markings having ones value rather than tenths. The feedback Yehali received from the interactive activity prompted her to reconsider her dominant whole-number thinking. When she noticed that her second prediction of "nine" was incorrect she needed to produce a mathematical reason why the actual location was zero-point-nine. By closely viewing the dynamic number line she grasped a more accurate understanding that included some decimal knowledge, "maybe because there is no number here it's just zero and it hasn't gone to one yet so it's in the middle somewhere." Yehali demonstrated a shift in attention to *recognising relationships* when she understood that between two whole numbers, decimal values are located, the concept of decimal density.

Example Four: Aarav Zoomable Number Line

Aarav displayed significant conceptual confusion around decimal fractions across a number of tasks in the Zoomable Number Line interview. He was asked to locate the decimal 0.9 on the zoomable number line and describe its location. Aarav immediately, zoomed-in to show hundredths on the number line and scrolled left from 0.12, he paused when 0.09 was shown on screen display exclaiming, "here is the number." The researcher questioned, "is that the same number as shown on the card?" The student considered and then answered, "no it's even littler" before he continued to scroll left from 0.09 and stopped when 0 was displayed. The researcher asked, "what are you thinking?" Aarav replied, "I zoomed in too much because the numbers are hundredths but this one [pointed to 0.9 decimal card] is tenths". He tapped the 'zoom out' button and adjusted the number line to show tenths only, he then scrolled back to the right along the number line and stopped upon 0.9 being shown, "I found it … zero-point-nine is between zero-point-eight and one-whole." Aarav's actions indicated that his understanding of tenths and hundredths place value in decimals was developed through his exploration of the zoomable number line. Initially, the student did not understand the concept of place value in decimals as suggested by his consideration of 0.9 to be the same value as 0.09. Using the 'zoom out' function of the zoomable number line helped Aarav correctly locate 0.9. Aarav was originally operating with a mathematical mindset of whole-number thinking and thus an incomplete ability to *discern details* related to decimal place value. However, through his interactions with the features of the zoomable number line, Aarav's attention shifted to *recognising relationships*

by stating the place value of digits in the decimal number with an understanding of positional value.

Discussion

Representations are used in the mathematics classroom to assist students in understanding abstract concepts that are central to mathematics learning (Pape & Tchoshanov, 2001). The affordances of four dynamic digital representations were employed by the students in this study to develop conceptual understanding of decimal place value, magnitude and density. The facility of digital tools to adapt, change, and provide various interactive features and capabilities that traditional static representations may not offer were pivotal in enabling the students to successfully complete decimal fractions tasks such as the decimal location task in the Zoomable Number Line interview. By actively "scrolling backwards" Aarav could mathematically reason that the decimal 0.09 must be smaller in value than 0.9 as it was located closer to the whole number 0 on the number line. In addition, the dynamic digital representation Zooming in on Place Value helped Yehali 'see' notions that would otherwise be 'unseeable' through its unique 'zooming in' animation between two whole numbers to show a magnified view divided into tenths (Orrill & Polly, 2013). She initially had difficulty predicting what tenths decimal was located on the number line, however by engaging with the dynamic nature of the Zooming in on Place Value web-app, she successfully solved later questions. This showed that within the timeframe of the 30-minute interview, the student was comfortable and confident enough to use the highly effective learning tool to eliminate the abstractness of decimals.

All students involved in this study experienced moments of cognitive confusion generated by the dynamic affordances embedded within the digital tools which interrupted the students' flow of attention, to guide their conceptual learning journey of decimal fractions. Seminal work by Swan (1983) found that students who experienced 'cognitive conflict' when learning gained better outcomes overall than students who learnt through 'positive only teacher'. Decimal strips generated cognitive conflict as it challenged Valerie's current misconception thinking about decimals. It was not until she used the functions of the Decimal Strips web-app and dragged the 0.2 and 0.25 coloured tiles onto the blank fraction wall that she recognised the value of five-hundredths. Valerie was able to self-correct her thinking by interacting with this dynamic representation as it challenged her prior understanding of decimals and allowed her to examine her errors in thinking and misconceptions.

Dynamic representations of decimal fractions are capable of offering direction to learners' shifts of attention. The dynamic nature of digital manipulatives force shifts in the object of children's attention when they are actively interacting with the mathematics content as well as the structure of their attention in how they attend to features of the mathematics representation and the embedded mathematics concepts. Conceptual development occurred when the learners experienced a shift between attending to relationships within and between elements of current experience and perceiving relationships as properties that might be applicable in other situations (Mason, 2008). For example, Annelise acknowledged that her thinking had changed from her limited prior understanding of performing operations with decimal fractions. The shift from *discerning details* of the place values within a decimal to then being able to *recognise relationships, perceive properties*, and *reason on the basis of agreed properties* by performing operations with decimals with an understanding of positional value number knowledge, was the result of new learnings that occurred during the Wishball-hundredths game. These changes in decimal understanding were prompted by unique affordances embedded within the web-app including the dynamic counting frame. By being aware of shifts occurring, teachers can be more alert to the possibility that learners are not making appropriate shifts and therefore implement teaching strategies and learning tools such as dynamic digital representations, to prompt these conceptual changes.

Conclusion

Teachers need to use various strategies, including visual representations, manipulatives, real-world examples, and interactive activities to make decimals more tangible and understandable. Examining a group of students completing four task-based interviews revealed the full scope of conceptual understanding of decimal fractions inclusive of decimal place value, relative magnitude and decimal density, which emerged from the students' interactions with four dynamic digital tools. At times, the students' experienced productive cognitive conflict, as the dynamic representation of decimal fractions challenged their existing understandings and misconceptions. Outside of the research-interview situation, these would have been ideal 'teaching moments'. While keeping in mind the inferential limitations of a small-scale study, the findings clearly indicate the potential of dynamic, digital representations to be used as effective learning and teaching tools to develop conceptual understanding of decimal fractions, and to address previously established misconceptions.

Acknowledgements

Ethics approval 2022/893 was granted by Human Research Ethics Committee (HREC) of The University of Sydney and parents or legal guardians of the participants as well as the young students involved in the study gave informed consent.

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Appendix H: Paper published in ICME (International Congress on Mathematical Education) Conference Proceedings for 2024.

CAPTURING CONCEPTUAL CHANGES WITH DYNAMIC DIGITAL REPRESENTATIONS

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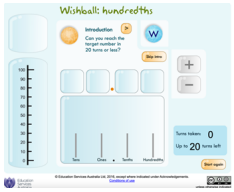
This paper reports one student case from a study that explored what conceptual changes emerged while children in middle-primary grades engaged with dynamic digital representations of decimal fractions. Two task-based interviews were conducted, each using a different digital tool that featured one or more of the three key concepts of decimals; place value, decimal density and relative magnitude. Video-audio recordings of the interviews and screen-captures were interpreted using microgenetic methods to detect specific shifts in the learner's attention and the resulting conceptual changes.

BACKGROUND

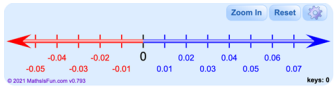
Children have difficulty in understanding the abstract nature of decimals, and a plethora of misconceptions arise from incomplete understandings (Steinle & Stacey, 2004). Developing conceptual understandings of *decimal density, place value and relative magnitude* of decimal fractions is considered key for consolidating number sense. Representations such as concrete materials, models, drawings and diagrams are used to assist students in understanding abstract concepts. Dynamic representations are potentially an effective support for student learning within mathematics because unlike static representations, students are able to actively interact with the mathematics content by making conjectures and testing them immediately to experience the concept from various perspectives and receive feedback on their thinking (Orrill & Polly, 2013). Learner-centred computer tools that encourage interactive engagement with mathematics, allow students to 'see' notions that would otherwise be 'unseeable' (Orrill & Polly, 2013). Although there is an abundance of research into the use of digital learning objects in mathematics education (e.g., Moyer-Packenham, et al., 2016), relatively little work has focused on the role of dynamic digital representations in teaching and learning decimals.

METHOD

This paper explores one Year 4 student's conceptual changes related to decimal fractions as prompted by their interactions with two dynamic digital representations (Figure 1). Task-based interviews were used to prompt the student's interaction with mathematical tasks and representations and record ongoing shifts in their 'focus of attention' and the 'structure of attention' (how the learner attends) (Mason, 2008). The figure below presents an overview of the two digital tools used during the task-based interviews, which each lasted for 30-minutes. The interviews were video recorded, and screen-captures were collected.



Screenshot of the Wishball virtual manipulative where students try to reach a target decimal number by adding or subtracting in fewer than 20 moves. The Spinner function generates a random number and the user must choose a place value to assign to the digit then



Screenshot of the Zoomable Number Line virtual manipulative which dynamically represents decimal fractions on a number line through a Zoom In capability. (https://www.mathsisfun.com/numbers/number-line-

Figure 1: *Wishball* and *Zoomable Number line* screenshots and description

The data collected from the task-based interviews was analysed using a microgenetic approach to capture moment-by-moment learning processes (Chinn, 2006), utilising Mason's (2008) 'five structures and microqualities of attention' (see Table 1). A framework for analysis was designed by the researcher by correlating Mason's five components with key decimal fraction concepts that form a sound conceptual understanding (Table 1).

Structure and Microqualities of Attention (Mason, 2008)	Connection of Theory of Structures of Attention and Decimal Fraction Learning Progression Points (Researcher's own reasoning)	Key Concept(s) of Decimal Fractions in which Conceptual Change has Occurred (Researcher's own reasoning)
Holding wholes	The student verbally names decimal fractions without an understanding of place value.	Place Value
Discerning details	The student can discern mathematical details of decimal fractions with an understanding of place value and reforming decimal numbers with value.	Place Value Decimal magnitude
Recognising relationships	The student understands the relationship between decimal fractions and the division operation, as the decimal is parts of a unit resulting from dividing the whole into equal parts. Furthermore, the student understands the positional value of decimals as well as the scaling effect as digits move to the right and left in place value for example 3.509 has 35.09 tenths, 350.9 hundredths, 3509 thousandths.	Place Value Decimal density
Perceiving properties	The student perceives properties of decimal fractions as they demonstrate knowledge of the relative size of decimals when comparing and applies this knowledge to the relationship with measurement and position when plotting values on a number line.	Decimal Magnitude
Reasoning on the basis of agreed properties	The student relates their understanding of positional value number knowledge to perform operations with decimals.	Place Value Decimal density

Table 1: Alignment of Mason's (2008) five structures and microqualities of attention with key decimal fraction concepts, and the researcher's reasoning to connect each structure of attention.

RESULTS

One extract from each of the two interviews are presented as examples of the relationship between the affordances of the dynamic digital representations of decimal fractions, the student's attentional shifts and changes in the student's understanding as the interview progressed.

Extract One: Wishball-hundredths

Representation One Evidence: Aarav encountered the spinner number 9 and current starting number 51.45 early in his interactions with the Wishball-hundredths game. He articulated internal cognitive conflict, "I don't know what to do with this number because if I minus for any of them [place value digits] it will automatically

be zero but if I plus then it is going to be too high." After a period of wait-time he reasoned, "okay I am going to risk it and add it to one to make it ten and then I'll subtract on the next spin." He tapped 1 above ones place value column, tapped the add function, and the screen displayed 60.45 as new starting number whilst the dynamic counting frame simultaneously carried over 10-ones to the tens place value. The interviewer probed "what happened there?" Aarav replied, "I understand it now… because I added nine-ones to make ten-ones in total so it had to jump over to the tens place value." The interviewer probed, "is the counting frame helping you?" and Aarav replied, "yes I can see how many I need to add or minus to each number."

Extract One Interpretation: Aarav observed the dynamic counting frame displaying the addition operation principle, in-built constraint on column overflow (Steinle & Stacey, 2004). He originally predicted that the addition of 9-ones and 1-ones would be recorded as 10-ones but all in the ones column. However, the fluid movement between decimal place values on the animated counting frame modelled the reforming of decimal fractions and the student acknowledged that value can be transferred across place values to perform operations and reform the decimal number. He demonstrated a shift in attention from discerning details to recognising relationships when he utilised the dynamic affordances to reform decimals with his knowledge of place value changing from whole-number focused to understanding part-whole value of decimals.

Extract Two: Zoomable Number Line

Representation Two Evidence: Aarav uncovered how between any two distinct numbers there are an infinite number of decimal fractions as he continually zooming in around the number four. He was interested in discovering how many place values could be revealed in a decimal. This was observed when he was freely investigating the zoomable number line app by continuously zooming in to discover the density of the dynamic representation. He described his actions aloud "it's zooming in... and showing the decimals, the tenths... and now it's showing the hundredths... it's going into the thousandths… I wonder if it will stop?... I think it will continue forever because the decimal can go on forever."

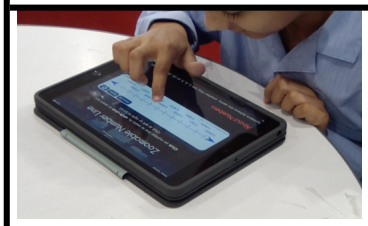
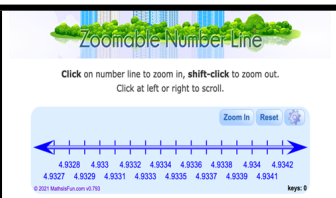


Photo showing student tapping the zoom-in function to reveal an increasing number of decimal place values around the whole-number four.



Screenshot of the 'Zoomable Number Line' web-app that was recorded using QuickTime Player whilst Aarav was manipulating it.

Extract Two Interpretation: During the interview Aarav attended to the idea of decimal fractions existing between whole numbers, the concept of decimal density. Aarav extended this idea as he commented on the continuity property of decimals, whereby between any two decimals there are an infinite number of other decimals. Aarav's conceptual understanding expanded to perceive properties of decimal fractions as he recognised the infinite density of

the decimal numeration system, which resulted directly from the zoom function of the digital number line.

DISCUSSION AND CONCLUSION

Aarav's incomplete decimal understanding was related to his whole-number thinking also described as "column overflow thinking" (Steinle & Stacey, 2004). In this misconception, each place is treated as having no relation to the place values on either side of it (Steinle & Stacey, 2004). The student in this study was able to coordinate place value knowledge with understandings of fractions, through the feedback delivered in Wishball-hundredths. In particular, Aarav attended to the dynamic movements of the counting frame as this demonstrated the relationship between relative place values in response to the place value addition or subtraction he inputted.

Many young children have extensive difficulties in recognising the density of decimals, as it is void in their whole-number knowledge. Steinle and Stacey (2004) claim the number line is one of the few models that is functional for discussing the density of decimals. However, they have commented on the limitations of a static number line with some students viewing it as a discrete representation. The zoomable number line with its dynamic capabilities, assisted Aarav in understanding the continuity aspects of decimals as he was able to continually zoom in to reveal decimals between each rational number.

Examining the case of one participant's interaction with two dynamic digital tools revealed a robust conceptual understanding of decimal fractions inclusive of decimal place value, density and relative magnitude. Together these dynamic digital representations offered powerful learning opportunities for the student to develop decimal sense and implement this knowledge in a number of mathematical contexts.

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