A multi-point maximum principle to prove global parabolic Harnack inequalities

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Statement of Originality

I certify that the content of this thesis is my own work and all sources used to prepare this thesis have been acknowledged. In particular, no generative AI tools were used during the development of this thesis. This thesis has not been submitted previously for any other degree or purpose.

Jessica Rachel Slegers

Statement of Authorship Attribution

The content of Chapter 3, Section 3.1 is contained in the article [**7**], which was published in the journal *Calculus of Variations and Partial Differential Equations.*

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I am the primary writer of this article, however the authors are listed alphabetically by surname, which is a typical convention for mathematics articles.

Jessica Rachel Slegers, 28 February 2026

As supervisor for the candidature upon which this thesis is based, I can confirm that the authorship attribution statement above is correct.

Florica-Corina Cirstea, 28 February 2026

Abstract

In this work, we aim to introduce a novel methodology for proving global pointwise Harnack inequalities for parabolic partial differential equations on a Riemannian manifold (M, g). The main idea of our approach is to apply a multi-point maximum principle. We demonstrate our techniques by studying the Harnack inequalities satisfied by positive solutions of two different equations, namely the linear Schrödinger equation

$$\partial_t u = \Delta u - Vu \qquad \text{in } M \times (0, \infty)$$

for a potential function $V \in C^2(M)$, as well as the doubly nonlinear heat equation

$$\partial_t u = \Delta_p(u^m) \qquad \text{in } M \times (0, \infty)$$

where $m > 0$, $p \geq 2$, and

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$$

is the p-Laplace operator.

In Chapter 1, we recount the history of the study of parabolic Harnack inequalities, before reviewing the existence of solutions to our aforementioned equations of interest in Chapter 2. In Chapter 3, we present the first proofs of the Harnack inequality using our multi-point maximum principle approach, focusing on classical solutions of the equations given above. In Section 3.1, we analyse the Schrödinger equation, first in Euclidean space and then on a Riemannian manifold M with nonnegative Ricci curvature. This section also contains applications to Schrödinger equations with a gradient drift term, which includes the heat equation governed by the Ornstein-Uhlenbeck operator $\Delta - x \cdot \nabla$. In addition, we use our Harnack inequality to recover a differential Harnack inequality comparable to the famous result of Li and Yau [**104**]. The results of this first section of Chapter 3 appear in the author's article together with Andrews and Hauer [**7**], which was published in the journal *Calculus of Variations and Partial Differential Equations.*

In Section 3.2, we demonstrate how our techniques can be extended to prove the Harnack inequality for positive classical solutions of the doubly nonlinear heat equation. However, since solutions of this equation do not in general possess sufficient regularity to all be treated as classical solutions, we dedicate Chapter 4 to adapting our proof techniques to viscosity solutions of the linear Schrödinger equation and doubly nonlinear heat equation. After reviewing the basic notions associated with viscosity solutions, we develop a modified version of the parabolic theorem on sums by Crandall and Ishii [**41**], which is crucial in our methodology. Finally, we present a new proof of the Harnack inequality satisfied by positive viscosity solutions of the doubly nonlinear heat equation. These results have not yet been published.

We conclude in Chapter 5 by summarising the questions still remaining after the development of this thesis and we outline some potential directions for future research.

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Contents

CHAPTER 1

Introduction

When studying the regularity properties of partial differential equations, it is often important to know whether the solutions satisfy a Harnack inequality. By a Harnack inequality, we refer primarily to pointwise estimates such as the famous inequality

$$u(x,t) \geq u(y,s)\left(\frac{s}{t}\right)^{d/2} e^{-\frac{|x-y|^2}{4(t-s)}} \qquad \text{for all } x,y \in \mathbb{R}^d \text{ and } 0 < s < t \tag{1.1}$$

satisfied by every positive solution u of the heat equation

$$\partial_t u = \Delta u \qquad \text{on } \mathbb{R}^d \times (0,\infty). \tag{1.2}$$

Using the inequality (1.1), one can deduce that for every $r > 0$, there exists a constant $C = C(d, t_0, r)$ such that

$$\sup_{x\in B_r(x_0)} u(x, t_0 - r^2) \leq C \inf_{x\in B_r(x_0)} u(x, t_0)$$

for all $x_0 \in \mathbb{R}^d$ and $t_0 > r^2$, which is another common form of the parabolic Harnack inequality that one encounters in literature.

In this thesis, we aim to develop a new methodology, which utilises a multi-point maximum principle to prove global Harnack inequalities similar to (1.1), which are satisfied by positive solutions of parabolic equations on a Riemannian manifold (M, g) of dimension $d \geq 1$. The multi-point maximum principle is already a known technique in the study of differential equations. For instance, it is familiar from the analysis of viscosity solutions, where it is used to prove comparison principles and uniqueness results (see [**42**]). The particular approach employed in this thesis is most heavily inspired by Andrews' application of a multi-point maximum principle to study moduli of continuity for solutions of quasilinear heat equations [**5, 6**]. The proof techniques and results of Andrews have also been generalised by Li and Wang [**105, 106, 108**] to accommodate the notion of viscosity solutions.

In this work, we treat both classical and viscosity solutions of two specific parabolic equations. We first consider the *linear Schrödinger equation*

$$\partial_t u = \Delta u - Vu \qquad \text{in } M \times (0,\infty) \tag{1.3}$$

with potential $V \in C^2(M)$, which appears in physics as a model of diffusion in a system containing a sink or source. Our second equation of interest is the *doubly nonlinear heat equation*,

$$\partial_t u = \Delta_p(u^m) \qquad \text{in } M \times (0,\infty), \tag{1.4}$$

where $m > 0$, $p \geq 2$, and

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$$

is the *p-Laplace operator*. The equation (1.4) is also sometimes called Leibenson's equation after the physicist who proposed this equation as a model of the turbulent flow of a gas in a porous medium [**101**]. Setting $p = 2$ in equation (1.4) corresponds to the porous medium equation

$$\partial_t u = \Delta(u^m), \tag{1.5}$$

while choosing $m = 1$ yields the p-heat equation

$$\partial_t u = \Delta_p u. \tag{1.6}$$

Before presenting our new methodology, we will take a tour through the historical development of parabolic Harnack inequalities and recall the main proof techniques found in existing literature. While it would be impossible to present an exhaustive list of works concerning this ever-expanding area of research, we aim to highlight the most enduring ideas that continue to guide modern research on this topic.

1.1. A history of Harnack inequalities

The study of Harnack inequalities began with the work of the mathematician Carl Gustav Axel Harnack, who in his 1887 book [**78**] published the following intriguing fact about harmonic functions on a disk: For a given point $x_0 \in \mathbb{R}^2$ and $R > 0$, every positive solution $u : B_R(x_0) \subseteq \mathbb{R}^2 \to \mathbb{R}$ of $\Delta u = 0$ in $B_R(x_0)$ satisfies

$$\frac{R-r}{R+r} u(x_0) \leq u(x) \leq \frac{R+r}{R-r} u(x_0)$$

for all $x \in B_r(x_0)$, where $0 < r < R$. A standard proof of this result follows from the Poisson formula for harmonic functions. For positive harmonic functions $u : \Omega \to \mathbb{R}$ defined on a general domain $\Omega \subseteq \mathbb{R}^d$, $d \geq 1$, Harnack's inequality is often stated in the form

$$\gamma^{-1} \sup_{\Omega'} u \leq u(x) \leq \gamma \inf_{\Omega'} u, \tag{1.7}$$

where $\overline{\Omega'} \subseteq \Omega$, $x \in \Omega'$, and $\gamma > 0$ is a constant. What is most remarkable about the inequality (1.7) is that the constant γ is uniform across all positive harmonic functions u on Ω and depends only on the sets Ω and Ω'. This enables the Harnack inequality (1.7) to be a useful tool to prove several other properties of harmonic functions, for instance

(i) (Liouville's Theorem, [**62**]) Every nonnegative harmonic function on $\mathbb{R}^d$ is constant.
(ii) (Removable Singularity Theorem, [**14**]) Let $d \geq 3$. If u is a harmonic function defined on the punctured ball $B_r(0) \setminus \{0\}$ and $u(x) = o(|x|^{2-d})$ for $x \to 0$, then $u(0)$ can be defined such that $u : B_r(0) \to \mathbb{R}$ is harmonic.

(iii) (Harnack's Convergence Theorem, [**66**]) If a sequence $(u_n)_{n\geq 1}$ of harmonic functions on a connected open set $\Omega \subseteq \mathbb{R}^d$ is monotonically increasing and there exists $x_0 \in \Omega$ such that $(u_n(x_0))_{n\geq 1}$ is convergent, then $(u_n)_{n\geq 1}$ converges uniformly on every compact subset of Ω to a harmonic function u.

Linear heat equations

A natural question following Harnack's discovery was whether an analogous statement holds true for positive solutions of the heat equation. However, it was not immediately clear what form such an inequality should take, and thus it was not until almost seven decades later that a result was found. In 1954, Hadamard [**74**] and Pini [**126**] independently proved the following result in one spatial dimension: For given $r > 0$, and $t_0 > r^2$, there exists a constant $\gamma > 1$ such that every positive solution u of the heat equation (1.2) satisfies

$$\gamma^{-1} \sup_{x\in B_r(x_0)} u(x, t_0 - r^2) \leq u(x_0, t_0) \leq \gamma \inf_{x\in B_r(x_0)} u(x, t_0 + r^2) \tag{1.8}$$

for every $x_0 \in \mathbb{R}^d$. Their proof involved representing the solution u via the Green's function for the heat equation. This result was generalised for arbitrarily spatial dimensions the following year by Montaldo [**115**].

The most striking difference one should notice in the parabolic case is that we are no longer able to compare the supremum and infimum of u over the same ball at the same point in time. Instead, we see a *waiting time* appear in the inequality. In terms of the physical interpretation of the heat equation as a model for the distribution of heat in a medium, this means that if at some time t_0, the temperature u is known at a point x_0 in space, then after waiting some time, the temperature everywhere in a neighbourhood of x_0 will be at least $\frac{u(x_0,t_0)}{\gamma}$. This waiting time is in fact necessary, as the following counterexample shows [**117**].

Suppose for given $x_0 \in \mathbb{R}$ and $t_0, r > 0$, there is a constant γ such that

$$\sup_{x\in B_r(x_0)} u(x, t_0) \leq \gamma \inf_{x\in B_r(x_0)} u(x, t_0) \tag{1.9}$$

for all positive solutions u of the one-dimensional heat equation

$$u_t = u_{xx} \qquad \text{in } \mathbb{R} \times (0, \infty). \tag{1.10}$$

Now consider the particular solutions

$$u_\xi(x, t) := t^{-1/2} e^{-(x+\xi)^2/4t}$$

of (1.10) for each $\xi \in \mathbb{R}$. Applying (1.9) for $t_0 = 1$ and any $r > 0$ large enough so that $0 \in B_r(x_0)$, we have

$$\left(\frac{u_\xi(0, 1)}{u_\xi(x_0, 1)}\right)^{-1} \leq \gamma.$$

However

$$\lim_{\xi \to -\infty} \frac{u_\xi(0,1)}{u_\xi(x_0,1)} = \lim_{\xi \to -\infty} e^{x_0^2/4} e^{x_0 \xi/4} = 0.$$

and hence the left-hand side in the previous inequality becomes unbounded in the limit as $\xi \to \infty$. With this contradiction we conclude the inequality (1.9) is false.

The next breakthrough occurred in the 1960s when Moser introduced a new methodology to prove Harnack inequalities for positive solutions of elliptic and parabolic equations governed by divergence form operators

$$\mathcal{L}u := \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left(a_{ij}(x,t) \frac{\partial u}{\partial x_j} \right),$$

where the coefficients a_{ij} are assumed to be bounded, measurable functions satisfying $a_{ij} = a_{ji}$ as well as the uniform ellipticity condition

$$\frac{1}{\lambda}|\xi|^2 \leq \sum_{i,j=1}^{d} a_{ij}\xi_i\xi_j \leq \lambda|\xi|^2 \qquad \text{for all } \xi \in \mathbb{R}^d$$

for some $\lambda > 0$. Moser first established his famous iteration technique in the elliptic case [**116**], before adapting it to the parabolic case [**117, 118**]. In particular, he demonstrated that all positive weak solutions of the equation $\partial_t u = \mathcal{L}u$ on $\Omega \times (0,T)$ with $\Omega \subseteq \mathbb{R}^d$ satisfy an inequality of the form

$$\sup_{B_r(x_0)\times(t_1^-,t_2^-)} u \leq C \inf_{B_r(x_0)\times(t_1^+,t_2^+)} u \tag{1.11}$$

where $r > 0$ is such that $\overline{B}_r(x_0) \subseteq \Omega$, $0 < t_1^- < t_2^- < t_1^+ < t_2^+ < T$, and C is a constant depending only on $d, \lambda, r, t_1^-, t_2^-, t_1^+$, and t_2^+, and not on the solution u. Throughout his proof of the Harnack inequality (1.11), Moser uses the quantity

$$M(p,Q) := \left(\frac{1}{\mu(Q)} \iint_Q u^p \,\mathrm{d}x \,\mathrm{d}t \right)^{1/p},$$

where $p \in \mathbb{R} \setminus \{0\}$ and $Q \subseteq \Omega$ is a cylinder. His main strategy follows by proving estimates of the form

$$M(p,Q^+) \leq cM(p,Q^-), \tag{1.12}$$

where Q^+ and Q^- are disjoint cylinders in Ω and $c > 0$ is a constant. By iterating the inequality (1.12) for a particularly chosen sequence of cylinders, and leveraging the fact that

$$\lim_{p\to+\infty} M(p,Q) = \sup_Q u \quad \text{and} \quad \lim_{p\to-\infty} M(p,Q) = \inf_Q u$$

the Harnack inequality (1.11) is proven. The derivation of the inequality (1.12) is long and technical, and relies on three main tools: the Poincaré inequality, the Sobolev inequality, and a modified version of an inequality by John and Nirenberg for functions of bounded mean oscillation. The last of these inequalities was rather cumbersome and required particular care to establish. Moser even wrote that "it was desirable to avoid it entirely" in favour of another approach [**119**]. Using an argument by Bombieri and Giusti [**25**] in the elliptic case as inspiration, Moser was later able to simplify his proof of both the elliptic and parabolic Harnack inequalities, so that the John-Nirenberg-type lemma was no longer needed.

In addition, Moser could apply his Harnack inequality to contribute a new proof of the Hölder continuity of solutions in both the elliptic [**116**] and parabolic cases [**117**], although these results had previously been shown via other methods by De Giorgi [**46**] and Nash [**121**]. The main idea of Moser's proof is to use the Harnack inequality (1.11) to control the oscillation of the solution u on a nested sequence of cylinders to derive a Hölder estimate. These ideas of Moser are perhaps some of the most influential in the study of Harnack inequalities so far and his techniques remain as some of the most commonly used today.

Before proceeding, we briefly mention that Harnack inequalities for elliptic and parabolic equations associated to non-divergence form operators were found by Krylov and Safonov [**95, 131**]. Their work introduced a very different approach to Harnack inequalities than Moser's and relies on concepts from probability theory and stochastic differential equations. A concise explanation of their approach can be found in the survey article [**88**] of Kassmann.

Nonlinear heat equations

Alongside the progress being made in the linear scenario, the next important aim was to achieve Harnack inequalities for nonlinear equations. A main object of study was quasilinear parabolic equations of p-Laplace type, that is, equations of the form

$$\partial_t u - \operatorname{div} A(x,t,u,\nabla u) = 0 \qquad \text{on } \Omega \times (0,\infty), \tag{1.13}$$

where $A : \Omega \times (0,\infty) \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is a measurable function satisfying the structure conditions

$$\begin{cases} A(x,t,u,\nabla u) \cdot \nabla u \geq C_0 |\nabla u|^p \\ \quad |A(x,t,u,\nabla u)| \leq C_1 |\nabla u|^{p-1} \end{cases} \tag{1.14}$$

for some $1 < p < \infty$ and positive constants C_0, C_1. The prototype of the equation (1.13) is the p-heat equation

$$\partial_t u - \operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0.$$

For the equation (1.13), one defines the *modulus of ellipticity* as a function $\lambda = \lambda(x,t,u,\nabla u)$ such that

$$A \cdot \nabla u \geq \lambda |\nabla u|^2$$

on $\Omega \times (0,\infty) \times \mathbb{R} \times \mathbb{R}^d$. If A satisfies the structure condition (1.14), a suitable modulus of ellipticity is $\lambda = C_0 |\nabla u|^{p-2}$. Thus, one sees that the behaviour of solutions of the equation (1.13) is strongly influenced by the value of the parameter p in (1.14). If $p > 2$, we notice that λ vanishes whenever $\nabla u = 0$, and one calls the equation (1.13) *degenerate* in this case. If $1 < p < 2$, then $\lambda \to \infty$ as $\nabla u \to 0$ and one calls the equation (1.13) *singular*.

Following Moser's success in the 1960s in studying the Harnack inequality for linear equations, his techniques were swiftly adapted before the end of the decade by Serrin [**134**] and Trudinger [**145**] to prove both Harnack inequality and Hölder continuity results for the elliptic counterpart

$$\operatorname{div} A(x,t,u,\nabla u) = 0 \qquad \text{on } \Omega \tag{1.15}$$

of equation (1.13), where A satisfies the same structure conditions in (1.14). However, progress in the parabolic setting followed a very different story. The problem of establishing a Harnack inequality of the form (1.8) for the equation (1.13) was intertwined with parallel efforts to prove the Hölder continuity of weak solutions. Ladyžhenskaya and Ural'tseva [**99**] established the Hölder continuity of weak solutions of the elliptic equation (1.15) using the earlier methods of De Giorgi, which did not rely on a Harnack inequality. However, in the parabolic case, they only managed to prove similar results for $p = 2$ [**98**]. It was not until the latter half of the 1980s that the Hölder continuity of solutions of (1.13) with $p \neq 2$ was proven by DiBenedetto, first in the degenerate range $p > 2$ [**48**], and later in the singular range $1 < p < 2$ together with Chen [**35**]. Harnack inequality results eventually followed as well, however, the inequalities obtained were not exactly of the form (1.8). In fact, the inequality (1.8) is in general false for solutions of (1.13), as counterexamples show (see Chapter 4, Section 3, and Chapter 5, Section 1.3 in [**57**]).

In order to achieve these new results, DiBenedetto created a new technique, referred to as the *method of intrinsic scaling*, which is elaborated upon in great detail in [**57**] (see also [**147**]). His strategy was not entirely unlike Moser's, in that it still involved iterating estimates of the solution u over cylinders, but the key difference was that the height of the cylinders in the time variable was chosen to depend on the value of u in a certain manner, which was more accommodating of the structure of the equation (1.13). This in turn meant that the dimensions of the cylinders in the resulting Harnack inequalities also depended on u. More specifically, DiBenedetto's approach led to inequalities of the form

$$\gamma^{-1} \sup_{x\in B_r(x_0)} u(x, t_0 - \theta r^p) \le u(x_0,t_0) \le \gamma \inf_{x\in B_r(x_0)} u(x, t_0 + \theta r^p), \tag{1.16}$$

where

$$\theta = \left(\frac{c}{u(x_0,t_0)}\right)^{p-2},$$

the constants c and γ depend only on d, p, C_0, and C_1, and $(x_0,t_0) \in \Omega \times (0,T)$ is such that the cylinders

$$B_{4r}(x_0) \times (t_0, t_0 + \theta r^p] \quad \text{and} \quad B_{4r}(x_0) \times (t_0 - \theta r^p, t_0]$$

are contained in $\Omega \times (0,T)$. Using this approach, DiBenedetto was able to prove Harnack inequalities for the p-heat equation (1.6) in both the cases $p \ge 2$ [**49**], as well as for $\frac{2d}{d+1} < p < 2$ together with Kwong [**58**]. Interestingly, DiBenedetto and Kwong showed in these articles, that these techniques could also be applied to prove Harnack inequalities for positive solutions of the porous medium equation (1.5) for $m > \frac{(d-2)_+}{d}$. Results for the doubly nonlinear heat equation (1.4) were also achieved by Gianazza and Vespri [**65, 149**]. We note that for the equation (1.4) with $m = \frac{1}{p-1}$, sometimes called Trudinger's equation, the operator $\partial_t u - \Delta_p(u^{\frac{1}{p-1}})$ becomes homogeneous, and so one does not encounter the same difficulties as with general $m > 0$ and $p > 1$. This enabled the traditional approach of Moser to be applied more instantly to obtain Harnack inequalities without the techniques of DiBenedetto [**93, 97, 146**].

Harnack inequalities for equation (1.13) with the full quasilinear structure were found much later by DiBenedetto, Gianazza, and Vespri [**53, 55**], again subject to the condition that $p > p_* := \frac{2d}{d+1}$. The work of these three authors revealed several interesting phenomena, and we refer to their book [**57**] for a detailed, self-contained account of their findings. Firstly, when $p_* < p < 2$, the solutions of (1.13) satisfy additional Harnack inequalities, which are forward and backward in time, and elliptic. By this, we mean that an inequality of the form

$$\gamma^{-1} \sup_{x \in B_r(x_0)} u(x,t) \le u(x_0,t_0) \le \gamma \inf_{x \in B_r(x_0)} u(x,s) \tag{1.17}$$

holds, where $t, s > 0$ are allowed to be any points in time satisfying

$$t_0 - \delta[u(x_0,t_0)]^{2-p} r^p \le t, s \le t_0 + \delta[u(x_0,t_0)]^{2-p} r^p$$

and γ, δ are positive constants depending only on p and d. In particular, an elliptic Harnack inequality holds, in the sense that one may take $t = s = t_0$ in (1.17), meaning that the inequality (1.9), which we showed to be false for the heat equation ($p = 2$), now becomes true when $p_* < p < 2$. Most importantly, the value $p_* = \frac{2d}{d+1}$ was shown to be the critical threshold for the Harnack inequality to hold, that is, solutions of (1.13) satisfying structure condition (1.14) with $1 < p \le p_*$ do not in general satisfy the Harnack inequality (1.16) or (1.17). For example, solutions of the p-heat equation for $1 < p \le p_*$ become extinct in finite time [**50**], so the inequality (1.16) could not possibly hold. Hence, solutions of (1.13) are important examples of functions, which are Hölder continuous despite not satisfying the Harnack inequality, showing that the Harnack inequality is a stronger property. The authors later demonstrated that solutions of (1.13) for p in the subcritical range $1 < p \le p_*$ do, however, satisfy weaker forms of the Harnack inequality [**54, 56**].

There has been a great deal of further progress on Harnack inequalities for nonlinear heat equations, but we will revisit this topic after discussing the contributions of Li and Yau and their differential Harnack inequality.

Heat equations on manifolds

Following the early progress on parabolic Harnack inequalities, there was great curiosity to explore similar results for heat equations posed on Riemannian manifolds. However, early attempts to adapt Moser's iteration techniques were often met with limitations, primarily due to the lack of suitable Sobolev and Poincaré inequalities, and were often not sharp. Thus, the next major milestone came in 1986, when Li and Yau [**104**] presented a novel strategy to prove parabolic Harnack inequalities, which avoided Moser's approach entirely, signalling the introduction of a completely different perspective through which to view this area of research. In the article [**104**], they prove pointwise Harnack inequalities for positive solutions of the Schrödinger equation (1.3) on a Riemannian manifold

(M, g) with Ricci curvature bounded from below. In the particular case that the manifold has nonnegative Ricci curvature, it is shown that every positive solution u of (1.3) satisfies

$$u(x,t) \geq u(y,s) \left(\frac{s}{t}\right)^{d/2} e^{-C_2(\theta^{2/3}+C_1^{1/2})(t-s)-\omega(x,y;t,s)} \tag{1.18}$$

for all $x, y \in M$ and $0 < s < t$. Here, $V = V(x,t) \in C^{2,1}(M \times (0,\infty))$, the constant C_1 is such that $\Delta V \leq C_1$ on $M \times (0,\infty)$, C_2 is a constant depending on the dimension d of the manifold M, θ is a constant related to the growth of ∇V, and

$$\omega(x,y;t,s) := \inf_{\gamma \in \Gamma_{x,y}} \frac{1}{4(t-s)} \int_0^1 |\dot{\gamma}|^2 \,\mathrm{d}\tau + (t-s) \int_0^1 V(\gamma(\tau), (1-\tau)s + \tau t) \,\mathrm{d}\tau,$$

where $\Gamma_{x,y}$ denotes the set of curves $\gamma : [0,1] \to M$ with $\gamma(0) = y$ and $\gamma(1) = x$. The techniques used in [**104**] to prove this result were an adaptation of previous work by Yau and Cheng in the elliptic setting [**36, 157**]. Given the strong influence of Li and Yau's work on the development of our own methodology, we find it appropriate to explore their techniques in greater detail.

The essence of the proof of (1.18) lies in first establishing the local gradient estimate

$$\begin{aligned}\frac{|\nabla u|^2}{u^2} - \alpha\frac{\partial_t u}{u} - \alpha V \leq C_3 \frac{\alpha^2}{R^2}\left(1 + \frac{\alpha^2}{\alpha - 1}\right) + \alpha^2 \frac{d}{2t} \\ + \left(C_4 (\sigma(R)^4 (\alpha-1)^2 \alpha^4 \varepsilon^{-1})^{1/3} + \alpha^3 \frac{d}{2} C_1\right)^{1/2}\end{aligned} \tag{1.19}$$

on a geodesic ball of radius $R > 0$ for every $\alpha > 1$ and $0 < \varepsilon < 1$, where C_3 and C_4 are depend only on d and $\sigma(R)$ describes the growth of ∇V. This is achieved by applying a maximum principle to the quantity $F := t(|\nabla v|^2 - \alpha\partial_t v - \alpha V)$, where $v = \log u$. While we omit the details of the argument for brevity, we do mention that an important step in the computation involves proving the lemma

$$\begin{aligned}(\Delta - \partial_t)F \geq -2g(\nabla v, \nabla F) - \frac{F}{t} + \frac{2t}{d}(|\nabla v|^2 - \partial_t v - V)^2 \\ - \alpha t \Delta V - 2(\alpha-1) t g(\nabla v, \nabla V),\end{aligned}$$

where $v = \log u$. Observe here that this estimate involves the third derivatives in space and the second derivative in time of the solution u.

Next, under the condition $\lim_{R\to\infty} \frac{\sigma(R)}{R} \leq \theta$ for some constant θ, choosing $\alpha - 1 = R^{-2}\theta^{-1/2}$ and letting $R \to \infty$ in (1.19) produces the gradient estimate

$$-\Delta(\log u) = \frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u} - V \leq \frac{d}{2t} + C_2\theta^{2/3} + \left(\frac{d}{2}C_1\right)^{1/2}. \tag{1.20}$$

The estimate (1.20) is often referred to in literature as the *differential Harnack inequality* because as Li and Yau demonstrated in [**104**], this inequality can be integrated along curves in space-time to yield the pointwise global Harnack inequality (1.18). Indeed, for fixed points $x, y \in M$, let $\gamma : [0,1] \to M$ be any C^1

curve satisfying $\gamma(0) = y$ and $\gamma(1) = x$. Starting with the equivalent form

$$\partial_t(\log u) \geq |\nabla \log u|^2 - V - \frac{d}{2t} - C,$$

of (1.20), where $C := C_2\theta^{2/3} + \left(\frac{d}{2}C_1\right)^{1/2}$ and combining with the fact that

$$\begin{aligned}\frac{d}{d\tau}\log u(\gamma(\tau), (1-\tau)s + \tau t) &= (t-s)\partial_t \log u(\gamma(\tau), (1-\tau)s + \tau t)) \\ &\quad + g(\nabla \log u(\gamma(\tau), (1-\tau)s + \tau t)), \dot{\gamma}(\tau))\end{aligned}$$

yields

$$\begin{aligned}&\frac{d}{d\tau}\log u(\gamma(\tau), (1-\tau)s + \tau t) \\ &\quad \geq g(\nabla \log u(\gamma(\tau), (1-\tau)s + \tau t)), \dot{\gamma}(\tau)) \\ &\qquad + (t-s)|\nabla \log u|^2 - V(t-s) - \frac{d(t-s)}{2((1-\tau)s + \tau t)} - C(t-s).\end{aligned}$$

Integrating over $[0, 1]$, we have

$$\begin{aligned}&\log u(x,t) - \log u(y,s) + C(t-s) + \frac{d}{2}\log\left(\frac{t}{s}\right) \\ &\quad \geq \int_0^1 g(\nabla \log u, \dot{\gamma}) + (t-s)|\nabla \log u|^2 \, \mathrm{d}\tau - (t-s)\int_0^1 V(\gamma) \, \mathrm{d}\tau.\end{aligned}$$

By completing the square in the first integral, we see that

$$\begin{aligned}&\log u(x,t) - \log u(y,s) + C(t-s) + \frac{d}{2}\log\left(\frac{t}{s}\right) \\ &\quad \geq -\frac{1}{4(t-s)}\int_0^1 |\dot{\gamma}|^2 \, \mathrm{d}\tau - (t-s)\int_0^1 V(\gamma) \, \mathrm{d}\tau\end{aligned}$$

which implies the Harnack inequality (1.18) after taking the infimum over all $\gamma \in \Gamma_{x,y}$.

In the particular case that $V = 0$ and $M = \mathbb{R}^d$ corresponding to the heat equation, this derivation leads directly to the inequality (1.1) stated at this beginning of this manuscript. This inequality is considered sharp, since the fundamental solution Γ of the heat equation given by

$$\Gamma(x,t) = (4\pi t)^{-d/2} e^{\frac{-|x|^2}{4t}} \tag{1.21}$$

for all $x \in \mathbb{R}^d$ and $t > 0$ satisfies (1.1) with equality on the set of points $x, y \in \mathbb{R}^d$, $0 < s < t$ such that $sx = ty$. In addition, the differential Harnack inequality (1.20) is satisfied with equality on all of $\mathbb{R}^d \times (0, \infty)$. We mention that Moser previously obtained a result very close to (1.1), although the version stated in [**117**] did not include the explicit constants needed to gain this sharpness. The strategy of proving a differential Harnack inequality and then integrating along curves has also been adapted for nonlinear equations, as we will soon discuss.

We also note that the differential Harnack inequality (1.20) is of independent interest, since it implies several other properties of solutions to (1.3), including heat kernel estimates [**104**]. For a discussion of some further consequences, we refer the reader to the classical book by Davies [**45**, Section 5.4].

The results obtained in [**104**] have been improved by various authors using similar techniques, especially for the heat equation ($V \equiv 0$) (see, for instance, [**18, 20, 45, 76, 103**]), but also in the general potential case [**102, 130, 158, 159**]. Furthermore, Li and Yau's results were expanded upon by Hamilton [**76**], who introduced a new form of differential Harnack inequality, referred to as a *matrix Harnack inequality.* In 1993, Hamilton demonstrated that under slightly stronger curvature assumptions on the manifold M, if u is a positive solution of the heat equation on a compact manifold M and X is a vector field on M, then

$$\nabla_i \nabla_j f + \frac{1}{2t} f g_{ij} + g(\nabla_i f, X_j) + g(\nabla_j f, X_i) + f X_i X_j \geq 0 \tag{1.22}$$

holds in $M \times (0, \infty)$ for each $1 \leq i, j \leq d$. This result can be considered a generalisation of the Li-Yau differential Harnack inequality (1.20) for $V \equiv 0$, since one may recover (1.20) by choosing $X = -\frac{\nabla f}{f}$ in (1.22) and taking the "trace", that is, letting $i = j$ and summing over i. Several authors have studied such Hamilton-type inequalities in various settings, although we will not focus on this here. We refer the reader to the articles [**9, 30, 40, 107, 109, 123, 156**] and the references therein.

The achievements of Li and Yau spawned considerable interest in studying Harnack inequalities for other problems in the manifold setting, including for geometric flows. Several important contributions in this direction were made in the 1990s by Chow [**37, 38, 39**] and Hamilton, who proved Harnack inequalities for the Ricci flow [**75**] and the mean curvature flow [**77**]. The former of Hamilton's results has particular significance amongst the wider mathematical community as it informed a crucial element of Perelman's proof of the famous Poincaré conjecture. Another popular direction of research has been Harnack inequalities for heat equations on time-dependent manifolds, evolving under some particular geometric flow. Some examples of this can be found in the articles [**32, 79, 82, 110, 143, 160**].

Another interesting development was the findings of Grigor'yan [**69**] and Saloff-Coste [**132**], who discovered a certain equivalence between the parabolic Harnack inequality and the manifold M possessing the properties of volume doubling and a Poincaré inequality. The main idea behind this correspondence is that these two properties are sufficient to imply a Sobolev-type inequality strong enough for Moser iteration to proceed. We state their result more precisely in the case of the heat equation. A manifold M endowed with a measure μ is said to have the volume doubling property if for a given $0 < r_0 \leq \infty$, there is a $C > 0$ such that

$$\mu(B_{2r}(x)) \leq C\mu(B_r(x))$$

for all $0 < r < r_0$ and $x \in M$. Moreover, we say that M admits a (weak) Poincaré inequality if for a given $0 < r_0 \leq \infty$, there is a $C > 0$ such that

$$\int_{B_r(x)} |f - f_{x,r}|^2 \, \mathrm{d}\mu \leq Cr^2 \int_{B_{2r}(x)} |\nabla f|^2 \, \mathrm{d}\mu$$

for all $0 < r < r_0$, $x \in M$, and $f \in C^\infty(M)$. Then, the main result of [**69**] and [**132**] states that positive solutions of the heat equation on M satisfy a Harnack inequality of the form

$$\sup_{Q^-} u \leq C \inf_{Q^+} u$$

for particularly chosen sets $Q^-, Q^+ \subseteq M$ if and only if M has the volume doubling property and admits a weak Poincaré inequality. Importantly, manifolds with nonnegative Ricci curvature satisfy both these properties, and thus the results in the articles [**69**] and [**132**] are consistent with the work of Li and Yau. We note that this result also applies for more general divergence form operators. For more details on this matter, we refer the reader to [**133**].

Differential Harnack inequalities and nonlinear heat equations

Prior to the work of DiBenedetto on the p-heat and porous medium equations, differential Harnack inequalities for these equations had already been proven, although the connection to the Harnack inequality was not acknowledged at the time of their discovery. In 1979, Aronson and Bénilan proved that if $m > \frac{(d-2)_+}{d}$, then every positive solution of the porous medium equation (1.5) on $\mathbb{R}^d \times (0,\infty)$ satisfies

$$\Delta v \geq -\frac{K}{t}, \tag{1.23}$$

where $v = \frac{m}{m-1}u^{m-1}$ if $m \neq 1$, $v = \log u$ if $m = 1$, and $K = (m - 1 + \frac{2}{d})^{-1}$. This inequality should be compared with the differential Harnack inequality (1.20) for the linear heat equation. The derivation of the inequality (1.23) is performed using a comparison principle and we briefly explain this methodology in the simple case $m = 1$ in Appendix B. The corresponding estimate for the p-heat and doubly nonlinear heat equations was later proven by Esteban and Vázquez [**60, 61**]. This estimate takes the near-identical form

$$\Delta_p v \geq -\frac{K}{t}, \tag{1.24}$$

except one must now define

$$v := \begin{cases} \dfrac{m}{\lambda} u^\lambda & \text{if } \lambda \neq 0, \\[2ex] \dfrac{1}{p-1} \log u & \text{if } \lambda = 0, \end{cases}$$

where $\lambda = m - \frac{1}{p-1}$. Importantly, the estimate (1.24) is only valid for $m > 0$ and $p > 1$ such that $K = (m(p-1) - 1 + \frac{p}{d})^{-1} > 0$. We remark that in the case $m = 1$ corresponding to the p-heat equation, p must satisfy $p > p_* = \frac{2d}{d+1}$, which coincides with the range identified by DiBenedetto, Gianazza and Vespri [**57**] for the Harnack inequality to hold. Moreover, much like the differential Harnack inequality (1.20) for the heat equation on $\mathbb{R}^d$, each of the estimates (1.23) and (1.24) is sharp, since they are satisfied with equality by the corresponding Barenblatt solution (see Section 2.3 for the definition). However, instead of Harnack inequalities, the authors who proved (1.23) and (1.24) were rather interested in proving regularisation effects of the respective equations. The space-time integration of (1.23) and (1.24) to obtain pointwise Harnack inequalities was not explicitly performed until Auchmuty and Bao's 1994 article [**13**], which credits Li, Yau and Hamilton for the main idea of how to proceed with the integration step. An estimate analogous to (1.23) was also made available by Crandall and Pierre [**44**, Lemma 2] for the generalised porous medium equation

$$\partial_t u = \Delta(\varphi(u)), \tag{1.25}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is a non-decreasing function, although we are not aware of any works, which integrate this estimate to obtain a Harnack inequality. However, it is very common in literature that authors will prove a differential Harnack inequality in a form similar to (1.23) or (1.20) without integrating it, either because it is technically difficult to do so, or because the differential form of the Harnack inequality already supplies enough information.

More recently, researchers have been interested in studying differential Harnack inequalities for nonlinear heat equations on Riemannian manifolds. One of the first major results in this area appeared in Vázquez's treatise on the porous medium equation, in which he proved the analogue of (1.23) on manifolds with nonnegative Ricci curvature [**148**, Proposition 11.12]. Together with Lu, Ni and Villani [**111**], Vázquez also proved local versions of (1.23) on manifolds with Ricci curvature bounded from below. In the case $m > 1$, these results were later improved by Huang, Huang, and Li [**80**], who also obtained a Harnack inequality. Additional results for variations of the porous medium equation, especially the generalised porous medium equation (1.25), have been obtained in [**113, 127, 151, 158**], and also for the porous medium equation on time-dependent manifolds in [**17, 31, 112, 152**]. Some work has also been done on the p-heat [**94**] and doubly nonlinear heat equations on manifolds [**16, 34, 153, 154**], although there are fewer works available in this direction than for the porous medium equation.

Recent trends

Since its inception, the study of parabolic Harnack inequalities has spread in many other directions with a wide variety of applications beyond what is possible to include in this overview. Nonetheless, we would like still to briefly acknowledge some of the other significant advancements, which have emerged throughout the 21st century, including, but not limited to, Harnack inequalities for stochastic partial differential equations (see [**150**] and the references therein), on graphs [**47, 59, 68**], and applications to the theory of optimal transportation [**19**]. Most recently, a popular field of research has been Harnack inequalities for nonlocal diffusion equations, the prototypical example being the fractional heat equation

$$\partial_t u + (-\Delta)^s u = 0,$$

where $(-\Delta)^s$ denotes the fractional Laplace operator defined by

$$(-\Delta)^s u(x) := C_{d,s} \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^d \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x-y|^{d+2s}} \, \mathrm{d}y,$$

$0 < s < 1$ and $C_{d,s}$ is a normalisation constant. Several results have been attained by adapting the classical methods of Moser and De Giorgi (see, for instance, [**64, 89, 90, 135, 140**]). An alternative approach was also found by Weber and Zacher [**155**], who used heat kernel representations of solutions to first derive a Li-Yau-type differential Harnack inequality.

1.2. Overview of thesis

In this thesis, we wish to contribute another piece to this ever-growing story by presenting a new methodology to prove global pointwise parabolic Harnack inequalities, which is based on a multi-point maximum principle argument. Our approach allows us to directly prove Harnack inequalities, and then recover the differential form afterwards using space-time differentiation. As a consequence, we only require the solution to be $C^{2,1}$ on $M \times (0,\infty)$ instead of needing the $C^{3,2}$-regularity required by Li and Yau [**104**]. In the Euclidean setting, a version of our argument also applies to viscosity solutions, which only need to be continuous functions. While the regularity of solutions is typically not an obstacle in the linear case, since solutions of (1.3) are generally smooth, this relaxation of the regularity assumptions on the solutions is advantageous, when studying solutions of nonlinear equations, which often have weaker regularity properties.

This remainder of thesis is organised in the following way.

In **Chapter 2** we briefly review the known existence theory for our two main equations of interest, the linear Schrödinger equation

$$\partial_t u = \Delta u - Vu \qquad \text{in } M \times (0,\infty) \tag{1.3}$$

and the doubly nonlinear heat equation

$$\partial_t u = \Delta_p(u^m) \qquad \text{in } M \times (0,\infty). \tag{1.4}$$

Since we work on a Riemannian manifold M, we first explain the required functional-analytic setup by defining and giving basic properties of the Sobolev space $W^{1,p}(M)$. We then apply the theory of gradient systems to show the existence of solutions to the linear equation (1.3) for all potential functions $V \in C(M)$, which are bounded from below. Appealing to the concept of Mosco convergence, we then demonstrate that solutions of (1.3) on a complete Riemannian manifold M can be approximated locally uniformly by solutions of the homogeneous Neumann problem posed on an increasing sequence of compact submanifolds $M_n \subseteq M$, which cover M. This property will be crucial later to study the Harnack inequality on non-compact manifolds. The theory regarding the doubly nonlinear heat equation (1.4) is markedly more complex than for the linear equation (1.3) and therefore, instead of explaining this in detail, we find it more appropriate to provide references to literature on the topic.

In **Chapter 3**, we begin our presentation of our multi-point maximum principle approach to prove Harnack inequalities, first for positive classical solutions of our equations of interest. This chapter is divided into two sections, with Section 3.1 focused on the Schrödinger equation (1.3) and Section 3.2 on the doubly nonlinear heat equation (1.4). We note that the results of Section 3.1 appear in the author's article [**7**] with Andrews and Hauer, which was published recently. The results in the other sections of this thesis have not yet been published.

In Section 3.1, we first treat the most basic case, which is the Schrödinger equation (1.3) on the Euclidean space $M = \mathbb{R}^d$. Our main task in Section 3.1.1 is to establish a Harnack inequality for positive solutions of the homogeneous Neumann problem associated to (1.3) on smooth bounded subsets $\Omega \subseteq \mathbb{R}^d$, which is achieved by Theorem 3.2. By approximating solutions of (1.3) on the full space $\mathbb{R}^d$ by solutions of the Neumann problem, we extend the conclusion of Theorem 3.2 to solutions of the full-space problem (see Theorem 3.1). In each of the Theorems 3.1 and 3.2, we provide sufficient conditions in order for a Harnack inequality of the form

$$u(x,t) \geq u(y,s)\frac{\beta(s)}{\beta(t)}e^{-\omega(x,y;t,s)}$$

to hold, where ω and β are functions satisfying certain properties. In Section 3.1.2, we study good candidates for optimal choices of the functions ω and β, which we motivate using the work of Li and Yau [**104**]. In the case of the classical heat equation ($V \equiv 0$) and the Schrödinger equation with quadratic potential $V(x) = |x|^2$, we show that our choice of ω and β leads to sharp Harnack inequalities. Our result for the heat equation recovers the famous inequality (1.1), while the result for the quadratic potential is new.

Working with our optimal choices for ω and β, in Section 3.1.3, we prove the analogues of Theorem 3.1 and Theorem 3.2 on general Riemannian manifolds with nonnegative Ricci curvature (see Theorem 3.11 and 3.12). The proofs in the Riemannian case follow much of the same strategy as in the Euclidean case, although due to the limited regularity of the function ω, we make some purely technical modifications to our approach.

Next, in Section 3.1.4, we apply our results from the previous sections to obtain Harnack inequalities for positive solutions of the equation

$$\partial_t u = \Delta u - 2\nabla f \cdot \nabla u - Vu \qquad \text{in } M \times (0,\infty)$$

for a given function $f \in C^4(M)$. Our conclusions follow from the simple observation that the above equation can be reduced to the equation (1.3) via a change of variables. In particular, we apply our sharp results for quadratic potentials to obtain a Harnack inequality for the heat equation associated to the Ornstein-Uhlenbeck operator $\Delta - x \cdot \nabla$.

Finally, in Section 3.1.5, we use our Harnack inequality to obtain a differential Harnack inequality comparable to the result (1.20) of Li and Yau.

In Section 3.2, we repeat our analysis from the previous section to obtain Harnack inequalities for positive classical solutions of the doubly nonlinear heat equation (1.4) with $m > 0$ and $p \geq 2$. Section 3.2.1 is devoted to studying (1.4) on the Euclidean space $\mathbb{R}^d$, while Section 3.2.2 concerns the equation (1.4) on a Riemannian manifold with nonnegative Ricci curvature.

Given that solutions of the doubly nonlinear heat equation (1.4) are not classical in general, it is necessary to expand our proof techniques to weaker notions of solutions. In **Chapter 4**, we choose to work with viscosity solutions. In Section 4.1, we present a summary of the basic theory associated with viscosity solutions, including their characterisation via second-order semijets. In our proof of the Harnack inequality, the most important tools we utilise are the theorem on sums and maximum principle for semicontinuous functions. Therefore, we use Section 4.2 to firstly revise the proofs of these theorems as they appeared in the article of Crandall and Ishii [**41**], before stating and proving our own modified versions of these results in the parabolic case. Finally, in Section 4.3, we discuss a new proof of the Harnack inequality for positive viscosity solutions of the equations (1.3) and (1.4) in the Euclidean case $M = \mathbb{R}^d$. While there are some existing articles about Harnack inequalities for viscosity solutions of fully nonlinear equations (see, for instance, [**29, 73, 92, 96, 125**]), our proof offers a very different approach to this matter.

We conclude the thesis in **Chapter 5**, where we summarise the main questions remaining at the end of this work and suggest some potential directions for future research.

CHAPTER 2

Existence Theory

Before studying the Harnack inequalities satisfied by solutions of equations (1.3) and (1.4), we first give an exposition of the existence theory for these equations.

We study the linear Schrödinger equation

$$\partial_t u = \Delta u - Vu \qquad \text{in } M \times (0, \infty) \tag{1.3}$$

with a potential function $V \in C(M)$, which is bounded from below, on both a complete Riemannian manifold M, as well as the homogeneous Neumann problem on a compact manifold with boundary ∂M. We employ the theory of gradient systems to demonstrate that equation (1.3) has a unique solution for all initial data $u(\cdot, 0) := u_0 \in L^2(M)$. We will also explain the important property, that solutions of the equation (1.3) on a complete Riemannian manifold can be approximated by solutions of the homogeneous Neumann problem on compact submanifolds.

We also make some comments regarding the doubly nonlinear heat equation

$$\partial_t u = \Delta_p(u^m) \qquad \text{in } M \times (0, \infty), \tag{1.4}$$

however, the theory of this equation is significantly more complex. Therefore, in order to not detract from the focus of this thesis, we prefer to briefly summarise the most relevant findings from literature.

Before we begin, we introduce the function spaces needed throughout this chapter.

2.1. Sobolev spaces on Riemannian manifolds

We briefly review the essential definitions and properties concerning Sobolev spaces on Riemannian manifolds. Here, we follow the definitions and notations used by Grigor'yan [**67**]. For a reminder of the fundamentals of Riemannian geometry, one may consult Appendix A.

Let (M, g) be a Riemannian manifold. We first endow M with the structure of a measure space. We build a σ-algebra $\Lambda(M)$ on M by declaring a set $E \subseteq M$ to be measurable if $\varphi(E \cap U)$ is a (Lebesgue) measurable set in $\mathbb{R}^d$ for any chart (U, φ) of M. In particular, $\Lambda(M)$ contains all open (and closed) sets on M. We may then define a measure $\nu : \Lambda(M) \to [0, \infty]$ in a chart (U, φ) by setting

$$\mathrm{d}\nu := \sqrt{\det g}\ \mathrm{d}\mu,$$

where $\mathrm{d}\mu$ denotes the Lebesgue measure on $\mathbb{R}^d$. Where there is no ambiguity, we also write $\mathrm{d}x$ instead of $\mathrm{d}\nu$. We may thus consider the triple $(M, \Lambda(M), \mathrm{d}x)$ as a measure space. Then, we can define the Lebesgue spaces $L^p(M) := L^p(M, \nu)$ for $1 \leq p \leq \infty$ with the norm

$$\|u\|_p := \left(\int_M |u|^p \, \mathrm{d}x\right)^{1/p} \qquad \text{for } 1 \leq p < \infty,\ u \in L^p(M),$$
$$\|u\|_\infty := \operatorname*{ess\,sup}_{x \in M} |u(x)| \qquad \text{for all } u \in L^\infty(M)$$

via the usual construction. We also define the Lebesgue space $\vec{L}^p(M)$ of vector fields X on M by setting $X \in \vec{L}^p(M)$ if and only if $|X| := \sqrt{g(X,X)} \in L^p(M)$. We set

$$\|X\|_{\vec{L}^p(M)} := \||X|\|_p$$

for all $X \in \vec{L}^p(M)$. Where there is no confusion, we also write $\|\cdot\|_p$ instead of $\|\cdot\|_{\vec{L}^p(M)}$. Finally, we write $L^p_{\mathrm{loc}}(M)$ to denote the space of functions $u : M \to \mathbb{R}$ such that $u \in L^p(K)$ for all compact sets $K \subseteq M$.

We now recall some important properties of the space $L^p(M)$, which follow from the general theory of Lebesgue spaces [**28**].

Proposition 2.1. *The spaces* $(L^p(M), \|\cdot\|_p)$ *and* $(\vec{L}^p(M), \|\cdot\|_{\vec{L}^p(M)})$ *are*

(i) Banach spaces for all $1 \leq p \leq \infty$*;*
(ii) separable for $1 \leq p < \infty$*;*
(iii) reflexive for $1 < p < \infty$*.*

In order to define the Sobolev space $W^{1,p}(M)$, we must first understand the notion of the weak gradient. For this, we require the space of (Schwartz) distributions on M.

Definition 2.1. The space of test functions $\mathscr{D}(M)$ on M is defined as $C_c^\infty(M)$ equipped with the following notion of convergence. We say that $\xi_n \to \xi$ in $\mathscr{D}(M)$ if

(i) there exists a compact set $K \subseteq M$ such that $\mathrm{supp}(\xi_n) \subset K$ for every $n \geq 1$, and
(ii) for any chart U in M, $\lim_{n\to\infty} D^\alpha \xi_n = D^\alpha \xi$ uniformly in U, for each multi-index $\alpha \in \mathbb{N}_0^d$, where

$$D^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_d^{\alpha_d}}$$

and $|\alpha| := \sum_{i=1}^d \alpha_i$.

Definition 2.2. The space of continuous linear functionals on $\mathscr{D}(M)$ is denoted $\mathscr{D}'(M)$ and one calls the elements of $\mathscr{D}'(M)$ *distributions*. For a given $u \in \mathscr{D}'(M)$ we write $\langle u, \xi \rangle_{\mathscr{D}'(M)}$ to denote the value of u at $\xi \in \mathscr{D}'(M)$. In addition, we say that $u_n \to u$ in $\mathscr{D}'(M)$, or in the sense of distributions, if

$$\langle u_n, \xi \rangle_{\mathscr{D}'(M)} \to \langle u, \xi \rangle_{\mathscr{D}'(M)}$$

for all $\xi \in \mathscr{D}(M)$.

By replacing $C_c^\infty(M)$ in Definition 2.1 and Definition 2.2 with the space of compactly supported smooth vector fields on M, we define the spaces $\vec{\mathscr{D}}(M)$ and $\vec{\mathscr{D}}'(M)$ analogously.

Example 2.1. Every function $u \in L^1_{\mathrm{loc}}(M)$ is associated with a distribution by setting

$$\langle u, \xi \rangle_{\mathscr{D}'(M)} := \int_M u\xi \, \mathrm{d}x$$

for all $\xi \in \mathscr{D}(M)$.

Given a vector field $X := (X_1, \dots, X_d)$ on M, we define the divergence of X in a chart U with coordinates $x_1, \dots, x_d$ by

$$\operatorname{div} X := \frac{1}{\sqrt{\det g}} \sum_{i=1}^{d} \frac{\partial}{\partial x_i} \left(\sqrt{\det g} X_i \right).$$

Definition 2.3. If for a given distribution $u \in \mathscr{D}'(M)$, there is a vector field $X \in \vec{\mathscr{D}}'(M)$ such that

$$\langle X, \xi \rangle_{\vec{\mathscr{D}}'(M)} = -\langle u, \operatorname{div} \xi \rangle_{\mathscr{D}'(M)} \qquad \text{for all } \xi \in \vec{\mathscr{D}}(M),$$

we call $\nabla u := X$ the distributional gradient of u. In the case that $u \in L^1_{\mathrm{loc}}(M)$ and $\nabla u \in \vec{L}^1_{\mathrm{loc}}(M)$ can be realised as locally integrable functions on M, we call ∇u the weak gradient of u.

With this definition, we are finally ready to define the Sobolev space $W^{1,p}(M)$.

Definition 2.4. For $1 \leq p < \infty$, we define

$$W^{1,p}(M) := \{u \in L^p(M) \mid \nabla u \in \vec{L}^p(M) \text{ exists}\}.$$

We also write $H^1(M) := W^{1,2}(M)$.

We equip $W^{1,p}(M)$ with the norm $\|\cdot\|_{W^{1,p}(M)}$ defined via

$$\|u\|^p_{W^{1,p}(M)} := \|u\|^p_p + \|\nabla u\|^p_p$$

for all $u \in W^{1,p}(M)$.

As in the case for Sobolev spaces on $\mathbb{R}^d$, $W^{1,p}(M)$ inherits from $L^p(M)$ the same properties already described in Proposition 2.1. In addition, we note the following useful properties, which follow immediately from the previous definitions.

Proposition 2.2. *Let $(u_n)_{n\geq 1}$ and $u \in W^{1,p}(M)$.*

(i) If $u_n \to u$ in $\mathscr{D}'(M)$, then $\nabla u_n \to \nabla u$ in $\vec{\mathscr{D}}'(M)$.
(ii) $u_n \to u$ in $W^{1,p}(M)$ if and only if $u_n \to u$ in $L^p(M)$ and $\nabla u_n \to \nabla u$ in $\vec{L}^p(M)$.

2.2. The linear Schrödinger equation

In this section, we study the existence of solutions to the Schrödinger equation

$$\partial_t u = \Delta u - Vu \qquad \text{in } M \times (0,\infty), \tag{1.3}$$

where the potential function $V \in C(M)$ is bounded from below and (M, g) is a Riemannian manifold, which is assumed to either be compact with boundary ∂M, or complete. In the case that M is compact, we consider the equation (1.3) together with the homogeneous Neumann boundary condition

$$g(\nabla u, \nu) = 0 \qquad \text{on } \partial M \times (0,\infty), \tag{2.1}$$

where ν denotes the outward pointing unit normal vector to ∂M.

Remark 2.1. Without loss of generality, we may assume throughout that $V \geq 0$ because if V is merely bounded from below by $-\alpha$ for some $\alpha > 0$, applying the transformation $\hat{u} := ue^{-\alpha t}$ to the equation (1.3) yields

$$\partial_t \hat{u} = \Delta \hat{u} - (V_\alpha)\hat{u}$$

with $V_\alpha := V + \alpha \geq 0$. After obtaining a result for the new, nonnegative potential $V + \alpha$, one can reverse the transformation to transfer the result to the solution u of the original equation.

Traditionally, a linear equation such as (1.3) might be studied using the framework of sesquilinear forms and linear semigroup theory (see, for example, [**124**]). However, one might encounter some complications in the case that the manifold M is not compact and the potential V is unbounded. Therefore, we elect to instead use the theory of gradient systems. We will use the following definition.

Definition 2.5 (Subdifferential operator)**.** Let $E : H \to (-\infty, +\infty]$ be a proper, convex, lower semicontinuous function defined on a Hilbert space H. The mapping $\partial E \subseteq H \times H$ defined by

$$\partial E := \left\{ (u,v) \in H \times H \;\middle|\; \liminf_{t \to 0^+} \frac{E(u+th) - E(u)}{t} \geq \langle v, h \rangle \text{ for all } h \in H \right\}$$

is called the *subdifferential operator* of E. We write $v \in \partial E(u)$ if and only if $(u, v) \in \partial E$ and call $D(\partial E) := \{u \in H \mid \partial E(u) \neq \emptyset\}$ the *domain* of ∂E.

In general, the set $\partial E(u) \subseteq H$ may contain more than one element. However, if E is Gâteaux differentiable at $u \in D(\partial E)$, then ∂E is single-valued at u and $\partial E(u) = \{E'(u)\}$ [**136**, Proposition 7.6].

We aim to study the equation (1.3) as an abstract Cauchy problem

$$\begin{cases} \dfrac{du}{dt}(t) + Au(t) \ni 0 & \text{for } t \in [0,T], \\ \qquad\quad u(0) = u_0 \end{cases} \tag{2.2}$$

in a Hilbert space H, where $u_0 \in H$ and $A = \partial E$ is the subdifferential operator of a suitably chosen function E.

Definition 2.6. We call a function $u \in W^{1,1}((0,T],H) \cap C([0,T],H)$ a *(strong) solution* to the Cauchy problem (2.2) if

$$-\frac{du}{dt}(t) \in Au(t) \qquad \text{a.e. } t \in (0,T)$$

and $u(0) = u_0$. Here,

$$W^{1,1}((0,T],H) = \{u \in L^1(0,T;H) \mid u' \in L^1(\delta,T;H) \text{ for all } \delta \in (0,T)\}.$$

It is well-documented in the literature that the Cauchy problem (2.2) has a unique (strong) solution u for all $u_0 \in \overline{D(A)}$ (see, for example [**27**]). Some further properties of this solution are provided by the following theorem.

Theorem 2.3 ([**27**]). *Let u be the unique (strong) solution to* (2.2). *If $u_0 \in \overline{D(A)}$, then*

(i) $u \in W^{1,\infty}(0,T;H)$;
(ii) $u(t) \in D(A)$ for all $t \in (0,T)$;
(iii) $E(u) \in L^1(0,T)$.

Theorem 2.3 indicates that the operator $A = \partial E$ has a smoothing effect on the initial datum. Namely, one sees that even when $u_0 \in \overline{D(A)}$, one has that $u(t) \in D(A)$ for all $t \in (0,T)$.

Our task is now to identify an appropriate function $E : H \to (-\infty, +\infty]$ such that the problem (2.2) corresponds to the equation (1.3). Consider a function $u : M \to \mathbb{R}$ on a manifold M, which satisfies the equation (1.3) in the classical sense. Then for all functions $v \in C_c^\infty(M)$, one has

$$\int_M v\partial_t u \ \mathrm{d}x - \int_M v\Delta u \ \mathrm{d}x + \int_M Vuv \ \mathrm{d}x = 0.$$

Applying the integration by parts formula on M, which is a standard inclusion in many textbooks (see, for example, [**100**]), it follows that

$$\int_M v\partial_t u \ \mathrm{d}x - \int_{\partial M} g(\nabla u, \nu)v \ \mathrm{d}x + \int_M g(\nabla u, \nabla v) \ \mathrm{d}x + \int_M Vuv \ \mathrm{d}x = 0.$$

If M is compact and u satisfies (2.1), this reduces to

$$\int_M v\partial_t u \ \mathrm{d}x + \int_M g(\nabla u, \nabla v) \ \mathrm{d}x + \int_M Vuv \ \mathrm{d}x = 0. \tag{2.3}$$

If M is instead complete, this equation still holds, since $\partial M = \emptyset$. Now, let $H = L^2(M)$ and $A \subseteq H \times H$ be the operator defined by

$$\langle Au, v\rangle_{L^2(M)} := \int_M g(\nabla u, \nabla v) \ \mathrm{d}x + \int_M Vuv \ \mathrm{d}x$$

for all $u, v \in L^2(M)$. Using the operator A, we can rewrite (2.3) as

$$\langle \partial_t u, v\rangle_{L^2(M)} + \langle Au, v\rangle_{L^2(M)} = 0$$

for all $v \in C_c^\infty(M)$, which implies that the equation

$$\partial_t u + Au = 0$$

holds in $L^2(M)$. Moreover, we can identify the operator A as the subdifferential of the Gâteaux-differentiable functional $E : L^2(M) \to [0, +\infty]$ with domain

$$D(E) := \{u \in H^1(M) \mid Vu^2 \in L^1(M)\}$$

defined by

$$E(u) := \begin{cases} \frac{1}{2}\int_M |\nabla u|^2 \,\mathrm{d}x + \frac{1}{2}\int_M Vu^2 \,\mathrm{d}x & \text{if } u \in D(E) \\ +\infty & \text{otherwise} \end{cases} \tag{2.4}$$

for all $u \in L^2(M)$. It is clear to see that E is convex and proper, so our main task is to prove the following.

Proposition 2.4. *The function E defined in* (2.4) *is lower semicontinuous.*

PROOF. We use a characterisation of lower semicontinuity and show that the sublevel set

$$E_\alpha := \{u \in L^2(M) \mid E(u) \leq \alpha\}$$

is closed in $L^2(M)$ for every $\alpha \in \mathbb{R}$. Let $\alpha \in \mathbb{R}$ and suppose that $(u_n)_{n\geq 1} \subseteq L^2(M)$ is such that $E(u_n) \leq \alpha$ for all $n \geq 1$ and $u_n \to u$ for some $u \in L^2(M)$. Then, in particular, one has that

$$\frac{1}{2}\int_M |\nabla u_n|^2 \,\mathrm{d}x \leq \alpha \quad \text{and} \quad \frac{1}{2}\int_M Vu_n^2 \,\mathrm{d}x \leq \alpha \tag{2.5}$$

for all $n \geq 1$. The first inequality in (2.5) implies that $(\nabla u_n)_{n\geq 1} \subseteq \vec{L}^2(M)$ is a bounded sequence. Since $\vec{L}^2(M)$ is reflexive, we can extract a subsequence of $(u_n)_{n\geq 1}$, which we also denote by $(u_n)_{n\geq 1}$, such that $\nabla u_n \rightharpoonup v$ for some $v \in \vec{L}^2(M)$. In particular, $\nabla u_n \to v$ in $\vec{\mathscr{D}}'(M)$, and so by Proposition 2.2, we can identify $v = \nabla u$ in $\vec{L}^2(M)$. Therefore, we have that $u \in H^1(M)$ and $\nabla u_n \rightharpoonup \nabla u$. Moreover, we may extract a further subsequence of $(u_n)_{n\geq 1}$ such that $u_n \to u$ pointwise a.e. in M (see [**28**, Theorem 4.9]). From this, it follows that

$$\liminf_{n\to\infty} \frac{1}{2}Vu_n^2 = \frac{1}{2}Vu^2 \qquad \text{a.e. in } M.$$

In particular, Fatou's lemma and the second inequality in (2.5) show that

$$\frac{1}{2}\int_M Vu^2 \,\mathrm{d}x = \frac{1}{2}\int_M \liminf_{k\to\infty} Vu_{n_k}^2 \,\mathrm{d}x \leq \liminf_{k\to\infty} \frac{1}{2}\int_M Vu_{n_k}^2 \,\mathrm{d}x \leq \alpha < \infty,$$

and so $Vu^2 \in L^1(M)$. This implies $u \in D(E)$. Finally, we use the weak convergence of $\nabla u_n \rightharpoonup \nabla u$ to see that

$$\begin{aligned} E(u) &= \frac{1}{2}\int_M |\nabla u|^2 \,\mathrm{d}x + \frac{1}{2}\int_M Vu^2 \,\mathrm{d}x \\ &\leq \liminf_{n\to\infty} \left(\frac{1}{2}\int_M |\nabla u_n|^2 \,\mathrm{d}x + \frac{1}{2}\int_M Vu_n^2 \,\mathrm{d}x\right) \\ &= \liminf_{n\to\infty} E(u_n) \\ &\leq \alpha \end{aligned}$$

Therefore, E_α is closed and E is lower semicontinuous. □

Remark 2.2. It is well-known that the solution u of the Cauchy problem (2.2) with E defined by (2.4) enjoys much higher regularity properties than already stated (see, for example, [**62**] or [**98**]). In particular, when we study (1.3) in Chapter 3, we always assume $V \in C^2(M)$, which is sufficient to ensure the solutions are themselves C^2, that is, the solutions are classical.

In Chapter 3, when we study the Harnack inequality satisfied by solutions of (1.3) on a complete, non-compact manifold, such as the full Euclidean space $\mathbb{R}^d$, we rely heavily on being able to approximate solutions of (1.3) on M by solutions of Neumann problems on compact subsets $M_n \subset M$. We use the remainder of this section to justify why this is possible and describe in which way the solutions of these problems converge. Throughout, we now assume M is a complete Riemannian manifold and that there exists an increasing sequence $(M_n)_{n\geq 1}$ of compact subsets $M_n \subseteq M_{n+1} \subseteq \ldots \subseteq M$ such that $\bigcup_{n\geq 1} M_n = M$.

Our first goal is to extend the Cauchy-Neumann problem (2.2) on M_n to a problem on the entire manifold M by extending the energy E defined in (2.4) from $L^2(M_n)$ to a new energy E_n on $L^2(M)$. Then, we demonstrate that E_n converges to E in the sense of Mosco convergence defined below, which has several powerful consequences.

Definition 2.7 (Mosco convergence)**.** Let H be a Hilbert space and suppose that $E_n, E : H \to (-\infty, +\infty]$ are proper, convex and lower semicontinuous. We say that $(E_n)_{n\geq 1}$ *Mosco converges* to E and write $\text{M-}\lim_{n\to\infty} E_n = E$ if

(i) for every sequence $(u_n)_{n\geq 1} \subseteq H$ and $u \in H$ such that $u_n \rightharpoonup u$ in H,
$$E(u) \leq \liminf_{n\to\infty} E_n(u_n),$$
and

(ii) for every $u \in H$, there exists $(u_n)_{n\geq 1} \subseteq H$ such that
$$\lim_{n\to\infty} u_n = u \quad \text{and} \quad \lim_{n\to\infty} E_n(u_n) = E(u).$$

For every $n \geq 1$, let
$$U_n := \{u \in H^1(M_n) \mid Vu^2|_{M_n} \in L^1(M_n)\}$$
and define $E_n : L^2(M) \to (-\infty, +\infty]$ with domain
$$D(E_n) := \{u \in L^2(M) \mid u|_{M_n} \in U_n \text{ and } u = 0 \text{ on } M \setminus M_n\}$$
by
$$E_n(u) := \begin{cases} E(u) & \text{if } u \in D(E_n) \\ +\infty & \text{otherwise} \end{cases} \tag{2.6}$$
for all $u \in L^2(M)$.

Before we proceed, it is imperative that the Cauchy problem (2.2) with E replaced by E_n still corresponds to the Neumann problem for the Schrödinger equation (1.3) on M_n. We do this by computing the subdifferential of E_n. By definition, $(u, v) \in \partial_{L^2(M)} E_n$ if and only if
$$\langle v, h\rangle_{L^2(M)} \leq \liminf_{t\to 0^+} \frac{E_n(u + th) - E_n(u)}{t}$$

for all $h \in L^2(M)$. However, if $h \notin D(E_n)$, then $E_n(u+th) = +\infty$ and the inequality is trivially satisfied. Therefore, it is equivalent to write

$$\begin{aligned}\langle v, h\rangle_{L^2(M)} &\le \liminf_{t\to 0^+} \frac{E_n(u+th) - E_n(u)}{t} \\ &= \liminf_{t\to 0^+} \frac{1}{2}\int_M \frac{|\nabla u + t\nabla h|^2 - |\nabla u|^2}{t}\,\mathrm{d}x + \frac{1}{2}\int_M \frac{V[(u+th)^2 - u^2]}{t}\,\mathrm{d}x\end{aligned}$$

for all $h \in D(E_n)$. However, since $\operatorname{supp}(u) \subseteq M_n$ for all $u \in D(E_n)$, this means

$$\langle v, h\rangle_{L^2(M_n)} \le \liminf_{t\to 0^+} \frac{1}{2}\int_{M_n} \frac{|\nabla u + t\nabla h|^2 - |\nabla u|^2}{t}\,\mathrm{d}x + \frac{1}{2}\int_{M_n} \frac{V[(u+th)^2 - u^2]}{t}\,\mathrm{d}x$$

for all $h \in U_n$. Since E is Gâteaux differentiable at all $u \in U_n$, we evaluate this limit to get

$$\langle v, h\rangle_{L^2(M_n)} \le \int_{M_n} \nabla u \cdot \nabla h + Vuh\,\mathrm{d}x$$

for all $h \in U_n$. Replacing h by $-h$ gives the reverse inequality. Therefore, we conclude that

$$\begin{aligned}\partial_{L^2(M)} E_n(u) &= \left\{ v \in L^2(M) \,\middle|\, \langle v, h\rangle_{L^2(M_n)} = \int_{M_n} \nabla u \cdot \nabla h + Vuh\,\mathrm{d}x \text{ for all } h \in U_n \right\} \\ &= \partial_{L^2(M_n)} E(u|_{M_n}).\end{aligned}$$

This has the important consequence, that the Cauchy problems (2.2) corresponding to the energy E defined on M_n and to the energy E_n defined on M are equivalent and (2.2) remains an appropriate weak formulation of the Schrödinger equation (1.3). Moreover, if we consider $u \in C^2(M_n)$, integrating by parts gives that

$$-\int_{M_n} h\Delta u\,\mathrm{d}x + \int_{\partial M_n} g(\nabla u, \nu)h\,\mathrm{d}x + \int_{M_n} Vuh\,\mathrm{d}x = \int_{M_n} vh\,\mathrm{d}x \tag{2.7}$$

for all $h \in U_n$. In particular, if $h \in C_c^\infty(M_n) \subseteq U_n$, then $u = 0$ on ∂M_n and so

$$-\int_{M_n} h\Delta u\,\mathrm{d}x + \int_{M_n} Vuh\,\mathrm{d}x = \int_{M_n} vh\,\mathrm{d}x.$$

However, since $C_c^\infty(M_n)$ is dense in $L^2(M_n)$, this implies

$$-\int_{M_n} h\Delta u\,\mathrm{d}x + \int_{M_n} Vuh\,\mathrm{d}x = \int_{M_n} vh\,\mathrm{d}x.$$

for all $h \in L^2(M_n)$. It follows from (2.7) that

$$\int_{\partial M_n} g(\nabla u, \nu)h\,\mathrm{d}x = 0$$

for all $h \in L^2(M_n)$, which is only possible if $g(\nabla u, \nu) = 0$ on ∂M_n. Thus, we have recovered the Neumann condition (2.1) from the operator $\partial_{L^2 M} E_n$ as well.

Proposition 2.5. *Let* $E_n, E : L^2(M) \to (-\infty, +\infty]$ *be defined as in* (2.4) *and* (2.6). *Then* $\text{M-lim}_{n\to\infty} E_n = E$.

PROOF. Firstly, observe that by the definition of E_n, one has $E(u) \leq E_n(u)$ for all $u \in L^2(M)$. Therefore, if $u_n \rightharpoonup u$ in $L^2(M)$, the lower semicontinuity of E implies

$$E(u) \leq \liminf_{n\to\infty} E(u_n) \leq \liminf_{n\to\infty} E_n(u_n), \tag{2.8}$$

which proves part (i) in Definition 2.7.

To prove part (ii), we first assume $u \in L^2(M) \setminus D(E)$ and suppose $u_n \to u$ in $L^2(M)$. Then $E(u) = +\infty$ and (2.8) implies

$$+\infty = E(u) \leq \liminf_{n\to\infty} E_n(u_n) \leq \limsup_{n\to\infty} E_n(u_n) \leq +\infty$$

and hence,

$$\lim_{n\to\infty} E_n(u_n) = +\infty = E(u).$$

Now suppose $u \in D(E)$ and consider the sequence $(u_n)_{n\geq 1} \subseteq L^2(M)$ given by $u_n := u\mathbb{1}_{M_n}$ for all $n \geq 1$. Since $u_n \to u$ pointwise on M and

$$|u_n|^2 \leq |u|^2 \in L^1(M),$$

Lebesgue's dominated convergence theorem gives that $u_n \to u$ in $L^2(M)$. Similarly, we show $E_n(u_n) \to E(u)$. Since $u \in D(E)$, we have that $u \in H^1(M)$ and $Vu^2 \in L^1(M)$. For any $n \geq 1$, it follows that $u_n|_{M_n} = u|_{M_n} \in H^1(M_n)$ and $Vu_n^2|_{M_n} \in L^1(M)$ and therefore $u_n \in D(E_n)$. This means $E_n(u_n) = E(u_n)$ for every $n \geq 1$, so we just need to show $E(u_n) \to E(u)$.

By the same argument as applied to the sequence $(u_n)_{n\geq 1}$, we have that $\nabla u_n \to \nabla u$ in $\vec{L}^2(M)$, that is,

$$\int_M |\nabla u_n|^2 \, \mathrm{d}x \to \int_M |\nabla u|^2 \, \mathrm{d}x.$$

Moreover, $|Vu_n^2| \to |Vu^2|$ pointwise on M and

$$|Vu_n^2| = |Vu^2\mathbb{1}_{M_n}| \leq |Vu^2| \in L^1(M),$$

and so again, by Lebesgue's dominated convergence theorem, we have

$$\int_M Vu_n^2 \, \mathrm{d}x \to \int_M Vu^2 \, \mathrm{d}x.$$

Together, these results give that $E(u_n) \to E(u)$. □

Some of the most important results concerning Mosco convergence come from the works of Attouch [**10, 11**]. In particular, he demonstrated that the Mosco convergence of functionals E_n to E implies the graph convergence of the subdifferentials ∂E_n to ∂E (see [**11**, Theorem 3.66]), which leads to the following strong approximation result [**11**, Theorem 3.74]. We will use the version of this result appearing in [**12**].

Theorem 2.6 ([**12**, Theorem 17.4.7])**.** *Let $E_n, E : H \to (-\infty, +\infty]$ be proper, convex and lower semicontinuous, $A_n = \partial E_n$, $A = \partial E$, $u_{0,n} \in \overline{D(A_n)}$, and $u_0 \in \overline{D(A)}$. Let u_n and u be the solutions of the Cauchy problems*

$$\begin{cases} \dfrac{du_n}{dt}(t) + A_n u_n(t) \ni 0 & \text{for } t \in [0,T] \\ \qquad\qquad u_n(0) = u_{0,n}, \end{cases} \tag{2.9}$$

$$\begin{cases} \dfrac{du}{dt}(t) + Au(t) \ni 0 & \text{for } t \in [0,T] \\ \qquad\qquad u(0) = u_0, \end{cases} \tag{2.10}$$

respectively. If

- *(i)* $\sup_{n\geq 1} E_n(u_{0,n}) < \infty$,
- *(ii)* $u_{0,n} \to u_0$ *in* H, *and*
- *(iii)* $\text{M-}\lim_{n\to\infty} E_n \to E$,

then $u_n \to u$ in $(C(0,T;H), \|\cdot\|_\infty)$ and $\frac{du_n}{dt} \rightharpoonup \frac{du}{dt}$ in $L^2(0,T;H)$. Moreover, if $E_n(u_{0,n}) \to E(u_0)$, then $\frac{du_n}{dt} \to \frac{du}{dt}$ strongly in $L^2(0,T;H)$.

Corollary 2.7. *Let u be a solution to*

$$\partial_t u = \Delta u - Vu \qquad \text{on } M \times (0,\infty).$$

Given an increasing sequence $(M_n)_{n\geq 1}$ of compact sets $M_n \subseteq M_{n+1} \subseteq \ldots \subseteq M$ such that $\bigcup_{n\geq 1} M_n = M$, there exists a sequence $(u_n)_{n\geq 1}$ of solutions to

$$\begin{cases} \partial_t u_n = \Delta u_n - Vu_n & \text{on } M_n \times (0,\infty) \\ g(\nabla u_n, \nu) = 0 & \text{on } \partial M_n \times (0,\infty), \end{cases} \tag{2.11}$$

such that $u_n \to u$ locally uniformly on M.

Proof. Let $u_0 := u(\cdot, 0)$ and set $u_{0,n} = u_0 \mathbb{1}_{M,n}$. Then the Cauchy problem (2.9) associated to the energy E_n has a unique solution, which we denote by u_n. We note that $u_n|_{M_n}$ is a solution of (2.11). Applying Theorem 2.6 yields that $u_n \to u$ in $(C(0,T;L^2(M)), \|\cdot\|_\infty)$, that is, $u_n \to u$ in $L^2(M)$ uniformly on $[0,T]$. In particular, there is a subsequence $(u_{k_n})_{n\geq 1} \subseteq (u_n)_{n\geq 1}$ such that $u_n(x,t) \to u(x,t)$ a.e. $x \in M$ and for all $t \in [0,T]$. Using interior regularity results, such as Schauder estimates (see, for example, [**63**, Theorem 5, Chapter 3]), the convergence in the spatial variable can be upgraded to local uniform convergence. □

Remark 2.3. We note that the Schauder estimates from [**63**] referred to in the above proof are stated for heat equations on Euclidean spaces governed by uniformly elliptic operators of the form

$$\mathcal{L}u := \sum_{i,j=1}^{d} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i \frac{\partial u}{\partial x_i} + cu.$$

These results are also applicable on a manifold M, since in each chart (U, φ) of M, we can understand a solution u of the equation

$$\partial_t u = \Delta u - Vu$$

via the map $\varphi^{-1} \circ u : \mathbb{R}^d \supseteq \varphi^{-1}(U) \to \mathbb{R}$, which still satisfies an equation of the form $\partial_t u = \mathcal{L}u$, where the coefficients a_{ij}, b_i, c in L depend on the metric g of M.

2.3. The doubly nonlinear heat equation

In comparison to the linear equation studied in the previous section, the existence of solutions to the doubly nonlinear heat equation

$$\partial_t u = \Delta_p(u^m) \qquad \text{in } M \times (0,\infty) \tag{1.4}$$

and the approximation of solutions on a non-compact manifold are far more delicate matters and in particular, the gradient-systems approach is no longer suitable. Therefore, giving a detailed presentation of such results extends beyond the scope of this thesis and we instead outline the most important results in existing literature.

For (1.4) on the full Euclidean space $M = \mathbb{R}^d$, a class of self-similar solutions was already identified in the 1950s by Barenblatt [**21**]. If $m > 0$ and $p > 1$ are such that

$$K := \frac{1}{m(p-1) - 1 + \frac{p}{d}} > 0,$$

then the function

$$\mathcal{B}(x,t) := \begin{cases} \dfrac{1}{t^K}\left(C - \dfrac{\lambda}{mp'}\left(\dfrac{K}{d}\right)^{p'-1}\left|\dfrac{x}{t^{K/d}}\right|^{p'}\right)_+^{1/\lambda}, & \lambda \neq 0 \\[2ex] \dfrac{C}{t^K}\exp\left(-\dfrac{1}{p'}\left(\dfrac{1}{p}\right)^{p'-1}\left|\dfrac{x}{t^{1/p}}\right|^{p'}\right), & \lambda = 0 \end{cases} \tag{2.12}$$

is a nonnegative solution to (1.4) on $\mathbb{R}^d \times (0,\infty)$, where $p' := \frac{p}{p-1}$, $\lambda := m - \frac{1}{p-1}$ and $C > 0$ is any positive constant. Importantly, the Barenblatt solution $\mathcal{B}$ plays the role of the fundamental solution in the theory of the doubly nonlinear heat equation and is consistent with the fundamental solution (1.21) of the heat equation when one takes the limits $m \to 1$ and $p \to 2$.

More generally, one often considers solutions of (1.4) in the weak sense.

Definition 2.8 (Nonnegative weak solution of (1.4))**.** Let M be a Riemannian manifold, $T > 0$, $m > 0$, and $p > 1$. We call a function $u : M \times [0,T] \to [0,\infty)$ a *nonnegative weak solution of* (1.4) if

$$u \in C([0,T]; L^{m+1}(M)), \quad u^m \in L^p(0,T; W^{1,p}(M)),$$

and

$$\int_0^T \int_M -u\partial_t\varphi + |\nabla u^m|^{p-2}\nabla u^m \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t = 0$$

for all nonnegative functions

$$\varphi \in W^{1,1+\frac{1}{m}}(0,T; L^{1+\frac{1}{m}}(M)) \cap L^p(0,T; W_0^{1,p}(M)).$$

Existence results for equation (1.4) in the Euclidean setting $M = \Omega \subseteq \mathbb{R}^d$ have been studied by several authors and are well-understood for the entire range $m > 0$, $p > 1$ (see, for example, [**3, 83, 84, 129, 142**]). In the case that M is a general Riemannian manifold, understanding the existence of solutions is still an active area of research. Several results have been achieved in the $p = 2$ case corresponding to the porous medium equation in both the slow-diffusion ($m > 1$) [**70, 71**] and fast-diffusion ($m < 1$) [**22, 26, 72**] ranges. For general $p > 1$ and $m > 0$, less is known about the existence of solutions, although significant progress in this direction has been made in recent years [**4, 114**]. Most recently, a result by Sürig [**144**] provided the existence of a weak solution for all nonnegative initial data $u_0 \in L^1(M) \cap L^\infty(M)$ when $mp \geq 1$ and M is complete.

An important consideration when studying solutions of (1.4) is that, even in the Euclidean setting, weak solutions are not in general classical solutions. For the p-Laplace heat equation ($m = 1$), a famous result by DiBenedetto and Friedman [**51, 52**] indicates that weak solutions u of this equation only have $C^{1,\alpha}$-regularity in general, that is, ∇u is Hölder continuous. More recent works suggest this property is also shared by solutions of the doubly nonlinear heat equation for $m \neq 1$ [**23, 141**]. In Chapter 4, we introduce another notion of solutions, called *viscosity solutions*, which have even lower regularity requirements than weak solutions and turn out to be more convenient to use in our proof of the Harnack inequality. Before this, in Chapter 3, we study the Harnack inequality satisfied by classical solutions of the equations (1.3) and (1.4) as motivation for the discussion which follows in Chapter 4.

CHAPTER 3

Harnack inequalities for classical solutions

We now turn our attention to the centrepiece of this work, which is the introduction of our new methodology to prove global Harnack inequalities in both the Euclidean and Riemannian settings. We begin in Section 3.1 by explaining our proof in the simplest possible case by first considering the linear Schrödinger equation. The work discussed in this section forms the main content of the author's article [**7**]. This is followed in Section 3.2 by a treatment of the doubly nonlinear heat equation. In both sections, we begin by considering solutions in Euclidean space, before studying solutions on a Riemannian manifold with nonnegative Ricci curvature. In this chapter, we limit our discussion to classical solutions of these equations, so that we are not distracted by the issues caused by a lack of regularity.

Throughout the remainder of this thesis, we will frequently use the following notations. For $\Omega \subseteq \mathbb{R}^d$ and a function $\omega : \Omega \times \Omega \to \mathbb{R}$, $(x, y) \mapsto \omega(x, y)$, we write

$$\Delta_x \omega := \sum_{i=1}^{d} \frac{\partial^2 \omega}{\partial x_i^2}, \quad \Delta_y \omega := \sum_{i=1}^{d} \frac{\partial^2 \omega}{\partial y_i^2}, \quad \text{and} \quad \Delta_{xy} \omega := \sum_{i=1}^{d} \frac{\partial^2 \omega}{\partial x_i \partial y_i}.$$

We also denote by S the set

$$S := \{(t, s) \mid 0 < s < t\}.$$

3.1. The Harnack inequality for the linear Schrödinger equation

We start by considering the equation

$$\partial_t u = \Delta u - V u \qquad \text{in } M \times (0, \infty), \tag{1.3}$$

where M is the Euclidean space $\mathbb{R}^d$ or a smooth bounded subset $\Omega \subseteq \mathbb{R}^d$, before moving onto the general manifold case.

3.1.1. The Schrödinger equation on Euclidean space

Our main result concerns positive solutions of the full-space problem on $\mathbb{R}^d$.

Theorem 3.1. *Let $V \in C^2(\mathbb{R}^d)$ be bounded from below. Suppose there exists a continuous function $\omega = \omega(x, y; t, s) : \mathbb{R}^d \times \mathbb{R}^d \times S \to \mathbb{R}$, which is twice differentiable in the first and second arguments and differentiable in the third and fourth arguments. Suppose that $\omega(x, y; t, s) \to \infty$ whenever $t \to s^+$ for $x \neq y$ and $\lim_{t\to s^+} \omega(x, x; t, s) \geq 0$ for all $x \in \mathbb{R}^d$. In addition, assume that ω satisfies*

$$\partial_t\omega + |\nabla_x\omega|^2 \geq V(x) \qquad \text{in } \mathbb{R}^d \times \mathbb{R}^d \times S, \tag{3.1}$$

$$\partial_s\omega - |\nabla_y\omega|^2 \geq -V(y) \qquad \text{in } \mathbb{R}^d \times \mathbb{R}^d \times S, \tag{3.2}$$

and that there exist strictly increasing, differentiable functions $\beta : [0, \infty) \to [0, \infty)$ satisfying $\beta(0) = 0$ and $A : [0, \infty) \to [0, \infty)$ such that

$$A(t)^2\Delta_x\omega + A(s)^2\Delta_y\omega + 2A(t)A(s)\Delta_{xy}\omega \leq A(t)^2\frac{\beta'(t)}{\beta(t)} - A(s)^2\frac{\beta'(s)}{\beta(s)} \tag{3.3}$$

holds in $\mathbb{R}^d \times \mathbb{R}^d \times S$. Finally, assume there exists an increasing sequence $(\Omega_n)_{n\geq1}$ of smooth, bounded domains $\Omega_n \subseteq \mathbb{R}^d$ such that $\bigcup_{n\geq1}\Omega_n = \mathbb{R}^d$ and

$$\begin{aligned} &\nabla_x\omega \cdot \nu_n(x) \geq 0, &&\text{for all } x \in \partial\Omega_n,\ y \in \overline{\Omega_n}, \text{ and } 0 < s < t,\\ &\nabla_y\omega \cdot \nu_n(y) \geq 0, &&\text{for all } x \in \overline{\Omega_n},\ y \in \partial\Omega_n, \text{ and } 0 < s < t. \end{aligned} \tag{3.4}$$

hold for all $n \geq 1$, where ν_n is the outward pointing unit normal vector to Ω_n. Then, every positive solution u of

$$\partial_t u = \Delta u - Vu \qquad \text{in } \mathbb{R}^d \times (0, \infty) \tag{3.5}$$

satisfies

$$u(x, t) \geq u(y, s)\frac{\beta(s)}{\beta(t)}e^{-\omega(x,y;t,s)} \tag{3.6}$$

for all $x, y \in \mathbb{R}^d$ and $0 < s < t$.

In order to prove this theorem, we approximate positive solutions u of (3.5) by a sequence of positive solutions u_n of the Neumann problem (3.8) defined below on an increasing sequence of bounded domains. In particular, we require the following result regarding positive solutions u of the Neumann problem.

Theorem 3.2. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with smooth boundary $\partial\Omega$ and let $V \in C^2(\Omega)$ be bounded from below. Let $\omega = \omega(x, y; t, s) : \overline{\Omega} \times \overline{\Omega} \times S \to \mathbb{R}$ be a continuous function, which is twice differentiable in the first and second arguments and differentiable in the third and fourth arguments. Suppose that $\omega(x, y; t, s) \to \infty$ whenever $t \to s^+$ for $x \neq y$ and $\lim_{t\to s^+} \omega(x, x; t, s) \geq 0$ for all $x \in \overline{\Omega}$. In addition, assume that ω satisfies* (3.1)–(3.3) *in $\overline{\Omega} \times \overline{\Omega} \times S$. Finally, assume that*

$$\begin{aligned} &\nabla_x\omega \cdot \nu(x) \geq 0, &&\text{for all } x \in \partial\Omega,\ y \in \overline{\Omega}, \text{ and } 0 < s < t,\\ &\nabla_y\omega \cdot \nu(y) \geq 0, &&\text{for all } x \in \overline{\Omega},\ y \in \partial\Omega, \text{ and } 0 < s < t. \end{aligned} \tag{3.7}$$

Then every positive solution u of

$$\begin{cases} \partial_t u = \Delta u - Vu & \text{in } \Omega \times (0,\infty), \\ \nabla u \cdot \nu = 0 & \text{on } \partial\Omega \times (0,\infty) \end{cases} \tag{3.8}$$

satisfies (3.6) *for all $x, y \in \overline{\Omega}$ and $0 < s < t$.*

Let us first outline the strategy of the proof of Theorem 3.2, which should be understood as a model for how a multi-point maximum principle can be employed to prove pointwise Harnack inequalities. Seeking to show that all positive solutions u of (3.8) satisfy (3.6), we first observe that after making the transformation $v = \log u$, proving (3.6) is equivalent to showing

$$Z(x,y;t,s) := v(x,t) - v(y,s) + \log\left(\frac{\beta(t)}{\beta(s)}\right) + \omega(x,y;t,s) + \frac{\varepsilon}{2}(t-s)^2 + \varepsilon > 0$$

for all $x, y \in \overline{\Omega}$, $0 < s < t$ and $\varepsilon > 0$. We demonstrate that $Z > 0$ "initially", that is, when t and s are both sufficiently close to zero. Then, we suppose there is a point $P_0 = (x_0, y_0; t_0, s_0) \in \overline{\Omega} \times \overline{\Omega} \times S$ such that the function Z touches zero for the "first time" at P_0 and use conditions on the derivatives of Z at P_0 to arrive at a contradiction, implying that Z must instead remain positive for all times $0 < s < t$. Of course, since the function Z depends on two time variables t and s and not just one, we must clarify what is meant by Z becoming zero for the "first time".

Definition 3.1 (First touching point). Let $\Omega \subseteq \mathbb{R}^d$, $Z : \Omega\times\Omega\times(0,\infty)\times(0,\infty) \to \mathbb{R}$, $c \in \mathbb{R}$ and $P_0 := (x_0, y_0; t_0, s_0) \in \Omega \times \Omega \times (0,\infty) \times (0,\infty)$. We say that Z *touches $c \in \mathbb{R}$ from above for the first time at P_0* if

$$Z(P_0) = c \quad \text{and} \quad Z(P) > c \tag{3.9}$$

for all $P = (x, y; t, s) \in \Omega \times \Omega \times (0,\infty) \times (0,\infty)$ such that

$$|(t,s)| < |(t_0, s_0)|,$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^2$.

By reversing the inequality in (3.9), we also understand what is meant by a function Z *touching $c \in \mathbb{R}$ from below for the first time.*

Remark 3.1. We essentially consider pairs of times $(t,s) \in (0,\infty) \times (0,\infty)$ as being partially ordered with respect to the Euclidean norm, that is, we consider (t,s) as occurring before (t_0,s_0) if and only if $|(t,s)| < |(t_0,s_0)|$. Through this idea, one may also generalise Definition 3.1 for functions of any finite number of spatial or time variables. However, Definition 3.1 only compares the function values of Z at (t_0,s_0) with those at times before (t_0,s_0). In particular, the definition makes no assumption about what occurs at other times (t,s) for which $|(t,s)| = |(t_0,s_0)|$. Therefore, if Z touches a value $c \in \mathbb{R}$ for the first time at a point $P_0 = (x_0, y_0; t_0, s_0)$, we do not exclude the possibility that there is another point $\hat{P}_0 = (\hat{x}_0, \hat{y}_0; \hat{t}_0, \hat{s}_0)$ with $|(\hat{t}_0, \hat{s}_0)| = |(t_0, s_0)|$, such that Z also touches c for the first time at P_0, but $(\hat{t}_0, \hat{s}_0) \neq (t_0, s_0)$.

Remark 3.2. We may deduce several facts about the behaviour of a function Z at a point, where it touches a value for the first time.

(i) If Z touches a value $c \in \mathbb{R}$ from above for the first time at $P_0 = (x_0, y_0; t_0, s_0)$, then the geometry of balls with respect to the Euclidean distance ensures that

$$Z(x_0, y_0; t_0 - \eta, s_0) < Z(x_0, y_0; t_0, s_0)$$

and

$$Z(x_0, y_0; t_0, s_0 - \eta) < Z(x_0, y_0; t_0, s_0)$$

for all $\eta > 0$ small enough. Therefore, Z is decreasing with respect to t at t_0 and with respect to s at s_0. Moreover, if Z is differentiable in t and s, this means

$$\partial_t Z(P_0) \leq 0 \quad \text{and} \quad \partial_s Z(P_0) \leq 0.$$

(ii) If Z is continuous, condition (3.9) implies that

$$Z(x, y; t_0, s_0) \geq Z(x_0, y_0; t_0, s_0)$$

for all $x, y \in \Omega$, that is, considered as a function of x and y only, $Z(x, y; t_0, s_0)$ has a local minimum at (x_0, y_0). If Z is twice differentiable in x and y, and x_0, y_0 are interior points of Ω, then

$$\nabla_{(x,y)} Z(P_0) = 0 \quad \text{and} \quad D^2_{(x,y)} Z(P_0) \geq 0.$$

Proof of Theorem 3.2. Let u be a positive solution of (3.8) and set $v = \log u$. Then v is a solution to

$$\begin{cases} \partial_t v = \Delta v + |\nabla v|^2 - V & \text{in } \Omega \times (0, \infty), \\ \nabla v \cdot \nu = 0 & \text{on } \partial\Omega \times (0, \infty). \end{cases}$$

Fix $\varepsilon > 0$ and define a function Z on $\overline{\Omega} \times \overline{\Omega} \times S$ by

$$Z(x, y; t, s) = v(x, t) - v(y, s) + \log\left(\frac{\beta(t)}{\beta(s)}\right) + \omega(x, y; t, s) + \frac{\varepsilon}{2}(t - s)^2 + \varepsilon.$$

We make some initial observations about Z for (t, s) near the boundary of S. Firstly, as $s \to 0^+$, the term $\log\left(\frac{\beta(t)}{\beta(s)}\right) \to \infty$ due to the assumption that $\beta(0) = 0$. Therefore, $Z > 0$ for s near 0. Next, consider the behaviour of Z as $t \to s^+$. If $x \neq y$, then the assumption that $\omega \to \infty$ as $t \to s^+$ guarantees that $Z > 0$ for t near s. Otherwise, if $x = y$, then the difference $v(x, t) - v(x, s) \to 0$, so we can assume $v(x, t) - v(x, s) + \varepsilon > 0$. By assumption, all other terms in the limit of Z are nonnegative, so once again, $Z > 0$ for t near s. In particular, we see that $Z > 0$ for all times (t, s) in a sufficiently small neighbourhood of $(0, 0)$ in S. Then, Z will either remain strictly positive for all times, or there exists a point $P_0 := (x_0, y_0; t_0, s_0) \in \overline{\Omega} \times \overline{\Omega} \times S$ such that Z touches 0 for the first time at P_0 in the sense of Definition 3.1, that is,

$$Z(x_0, y_0; t_0, s_0) = 0 \quad \text{and} \quad Z(x, y; t, s) > 0$$

for all $x, y \in \overline{\Omega}$ and for all $0 < s < t$ with $|(t, s)| < |(t_0, s_0)|$. Suppose the latter of these possibilities is true and such a point P_0 exists.

We first consider the case that $x_0, y_0 \in \Omega$, and more specifically, that they do not occur on the boundary $\partial\Omega$. Then, as explained in Remark 3.2, it follows that

$$\partial_t Z(P_0) \leq 0, \quad \partial_s Z(P_0) \leq 0, \quad \nabla_{(x,y)} Z(P_0) = 0, \quad D^2_{(x,y)} Z(P_0) \geq 0.$$

By direct calculation,

$$\begin{aligned} \nabla_x Z(x, y; t, s) &= \nabla v(x, t) + \nabla_x \omega(x, y; t, s), \\ \nabla_y Z(x, y; t, s) &= -\nabla v(y, s) + \nabla_y \omega(x, y; t, s), \end{aligned}$$

and thus $\nabla_{(x,y)} Z(P_0) = 0$ implies that

$$\nabla v(x_0, t_0) = -\nabla_x \omega(P_0), \quad \nabla v(y_0, s_0) = \nabla_y \omega(P_0). \tag{3.10}$$

Next, we calculate

$$\begin{aligned} \partial_t Z(x, y; t, s) &= \partial_t v(x, t) + \frac{\beta'(t)}{\beta(t)} + \partial_t \omega(x, y; t, s) + \varepsilon(t - s), \\ \partial_s Z(x, y; t, s) &= -\partial_s v(y, s) - \frac{\beta'(s)}{\beta(s)} + \partial_s \omega(x, y; t, s) - \varepsilon(t - s). \end{aligned}$$

Using that $\partial_t v = \Delta v + |\nabla v|^2 - V$ in Ω, $\partial_t Z(P_0) \leq 0$, and $\partial_s Z(P_0) \leq 0$ we have that

$$\begin{aligned} \Delta v(x_0, t_0) + |\nabla v(x_0, t_0)|^2 - V(x_0) + \frac{\beta'(t_0)}{\beta(t_0)} + \partial_t \omega(P_0) + \varepsilon(t_0 - s_0) \leq 0, \\ -\Delta v(y_0, s_0) - |\nabla v(y_0, s_0)|^2 + V(y_0) - \frac{\beta'(s_0)}{\beta(s_0)} + \partial_s \omega(P_0) - \varepsilon(t_0 - s_0) \leq 0. \end{aligned}$$

Then, using (3.10), we obtain that

$$\begin{aligned} \Delta v(x_0, t_0) &\leq -|\nabla_x \omega(P_0)|^2 - \partial_t \omega(P_0) + V(x_0) - \frac{\beta'(t_0)}{\beta(t_0)} - \varepsilon(t_0 - s_0), \\ -\Delta v(y_0, s_0) &\leq |\nabla_y \omega(P_0)|^2 - \partial_s \omega(P_0) - V(y_0) + \frac{\beta'(s_0)}{\beta(s_0)} + \varepsilon(t_0 - s_0). \end{aligned}$$

Due to the assumed inequalities (3.1) and (3.2), it follows that

$$\Delta v(x_0, t_0) \leq -\frac{\beta'(t_0)}{\beta(t_0)} - \varepsilon(t_0 - s_0), \tag{3.11}$$

$$-\Delta v(y_0, s_0) \leq \frac{\beta'(s_0)}{\beta(s_0)} + \varepsilon(t_0 - s_0). \tag{3.12}$$

Next, we make use of the inequality $D^2_{(x,y)} Z(P_0) \geq 0$. Let $x(\tau)$ and $y(\tau)$ be two paths in Ω such that

$$\begin{aligned} x'(\tau) &= A(t_0) e_i, \quad x(\tau_0) = x_0, \\ y'(\tau) &= A(s_0) e_i, \quad y(\tau_0) = y_0 \end{aligned}$$

for some τ_0, where e_i is the i^{th} standard basis vector. We have that

$$\begin{aligned}\frac{d}{d\tau}Z(x(\tau),y(\tau);t,s) &= A(t_0)D_iv(x,t) - A(s_0)D_iv(y,s)\\ &\quad + A(t_0)\omega_{x_i}(x,y;t,s) + A(s_0)\omega_{y_i}(x,y;t,s)\\ \frac{d^2}{d\tau^2}Z(x(\tau),y(\tau);t,s) &= A(t_0)^2D_{ii}v(x,t) - A(s_0)^2D_{ii}v(y,s) + A(t_0)^2\omega_{x_ix_i}(x,y;t,s)\\ &\quad + A(s_0)^2\omega_{y_iy_i}(x,y;t,s) + 2A(t_0)A(s_0)\omega_{x_iy_i}(x,y;t,s)\end{aligned}$$

Therefore, since $\frac{d^2}{d\tau^2}Z(P_0) \geq 0$, we have that

$$\begin{aligned}A(t_0)^2D_{ii}v(x_0,t_0) &- A(s_0)^2D_{ii}v(y_0,s_0) + A(t_0)^2\omega_{x_ix_i}(P_0)\\ &+ A(s_0)^2\omega_{y_iy_i}(P_0) + 2A(t_0)A(s_0)\omega_{x_iy_i}(P_0) \geq 0,\end{aligned}$$

so that summing over i yields

$$\begin{aligned}A(t_0)^2\Delta v(x_0,t_0) &- A(s_0)^2\Delta v(y_0,s_0)\\ &+ A(t_0)^2\Delta_x\omega(P_0) + A(s_0)^2\Delta_y\omega(P_0) + 2A(t_0)A(s_0)\Delta_{xy}\omega(P_0) \geq 0.\end{aligned}$$

By using the inequality (3.3), as well as (3.11) and (3.12), we obtain

$$0 > -\varepsilon(A(t_0)^2 - A(s_0)^2)(t_0 - s_0) \geq 0,$$

which is a contradiction. Therefore, such a point P_0 with $x_0, y_0 \in \Omega$ cannot exist. Thus, either one or both of x_0, y_0 must be on the boundary $\partial\Omega$.

Without loss of generality, assume $x_0 \in \partial\Omega$. Then

$$\frac{d}{d\tau}Z(x_0 - \tau\nu(x_0), y_0; t_0, s_0)\bigg|_{\tau=0} = -\nabla v(x_0,t_0)\cdot\nu(x_0) - \nabla_x\omega(P_0)\cdot\nu(x_0) \leq 0, \tag{3.13}$$

where we have used that $\nabla v(x_0,t_0)\cdot\nu(x_0) = 0$ by the Neumann condition on v as well as the assumption (3.7) on ω. One of two cases must occur: either the inequality in (3.13) is strict, or we have equality.

Firstly, if

$$\frac{d}{d\tau}Z(x_0 - \tau\nu(x_0), y_0; t_0, s_0)\bigg|_{\tau=0} < 0,$$

since Z is at least continuously differentiable in the first argument, there is some $\tau^* > 0$ such that $\frac{d}{d\tau}Z(x_0 - \tau\nu(x_0), y_0; t_0, s_0) < 0$ for all $\tau \in [0, \tau^*)$. Integrating over this interval, we have

$$Z(x_0 - \tau^*\nu(x_0), y_0; t_0, s_0) < Z(x_0, y_0; t_0, s_0),$$

which contradicts x_0 minimising $Z(x, y_0; t_0, s_0)$ over all $x \in \Omega$.

The other possibility is that

$$\frac{d}{d\tau}Z(x_0 - \tau\nu(x_0), y_0; t_0, s_0)\bigg|_{\tau=0} = 0. \tag{3.14}$$

Considering $Z(x) = Z(x, y_0; t_0; s_0)$ as a function only of x, then (3.14) is equivalent to $\nabla_x Z(x_0)\cdot\nu(x_0) = 0$. Since $Z(x)$ has a minimum at $x = x_0$, then all derivatives of $Z(x)$ at x_0 in directions tangential to $\partial\Omega$ at x_0 must also be 0. Therefore, we conclude that $\nabla_x Z(x_0) = \nabla_x Z(P_0) = 0$.

Now consider whether $y \in \partial\Omega$ or $y \in \Omega$. If $y_0 \in \partial\Omega$, we could repeat our previous arguments to obtain $\nabla_y Z(P_0) = 0$. If $y_0 \in \Omega$, then the function $Z(y) = Z(x_0, y; t_0, s_0)$ attains a minimum at $y = y_0$ and therefore $\nabla_y Z(P_0) = 0$. In either case, we have that $\nabla_x Z(P_0) = 0$ and $\nabla_y Z(P_0) = 0$, so $\nabla_{(x,y)} Z(P_0) = 0$. Since Z has a minimum at P_0, it follows that $D^2_{(x,y)} Z(P_0) \geq 0$. Combining these facts with $\partial_t Z(P_0) \leq 0$ and $\partial_s Z(P_0) \leq 0$, we may repeat the argument from when $x_0, y_0 \in \Omega$ to arrive at a contradiction.

Therefore, $Z > 0$ for all $x, y \in \overline{\Omega}$ and all $0 < s < t$. Finally, taking $\varepsilon \to 0$ shows that the Harnack inequality (3.6) holds. $\square$

We now explain our result for the full-space problem.

PROOF OF THEOREM 3.1. Suppose that $(\Omega_n)_{n\geq 1}$ is an increasing sequence of smooth bounded domains $\Omega_n \subseteq \mathbb{R}^d$ with $\bigcup_{n\geq 1} \Omega_n = \mathbb{R}^d$ such that (3.4) holds. Let u be a positive solution of (1.3) with initial value $u_0 := u(x, 0) \in L^2(\mathbb{R}^d)$ and define $u_{0,n} := u_0|_{\Omega_n}$ for all $n \geq 1$. Let u_n be the unique positive solution to the problem

$$\begin{cases} \partial_t u_n = \Delta u_n - V u_n & \text{in } \Omega_n \times (0, \infty), \\ \nabla u_n \cdot \nu_n = 0 & \text{on } \partial\Omega_n \times (0, \infty), \\ u_n(\cdot, 0) = u_{0,n} & \text{in } \Omega_n. \end{cases}$$

By Theorem 3.2, u_n satisfies the Harnack inequality

$$u_n(x, t) \geq u_n(y, s) \frac{\beta(s)}{\beta(t)} e^{-\omega(x,y;t,s)} \tag{3.15}$$

for all $x, y \in \overline{\Omega_n}$ and $0 < s < t$. By Corollary 2.7, there exists a subsequence $(u_{k_n})_{n\geq 1} \subseteq (u_n)_{n\geq 1}$ such that $u_{k_n} \to u$ locally uniformly on $\mathbb{R}^d \times (0, \infty)$. Therefore, passing to the limit $n \to \infty$ in (3.15) yields the claimed result. $\square$

3.1.2. Analysis of a certain energy functional

The function ω in the premises of Theorem 3.1 and Theorem 3.2 could potentially be any function satisfying the conditions of these theorems, especially the inequalities (3.1)–(3.4). However, a natural question is whether such a function ω with the required properties even exists, and if it does, whether there is an optimal way to choose this ω. Of course, if one such ω exists, there is no reason for it to be unique, as one can always perturb ω by adding a nonnegative function $h = h(s, t)$, which is continuously differentiable with $\partial_t h, \partial_s h \geq 0$ in S, as this leaves the inequalities (3.1)–(3.4) unharmed. We dedicate this section to the analysis of a strong candidate for ω. In anticipation of our discussion in the next section in non-Euclidean settings, we let (M, g) be a d-dimensional Riemannian manifold with nonnegative Ricci curvature.

For a given potential function $V \in C^2(M)$, which is bounded from below, we consider the energy functional defined by

$$E[\gamma; t, s] := \frac{1}{4(t-s)} \int_0^1 |\dot{\gamma}|^2 \, \mathrm{d}\tau + (t - s) \int_0^1 V(\gamma(\tau)) \, \mathrm{d}\tau \tag{3.16}$$

for every C^1 curve $\gamma : [0,1] \to M$. We then set

$$\omega(x, y; t, s) := \min_{\gamma \in \Gamma_{x,y}} E[\gamma; t, s], \tag{3.17}$$

for each $x, y \in M$ and $0 < s < t$, where $\Gamma_{x,y}$ is the set of curves $\gamma : [0,1] \to M$ with $\gamma(0) = y$ and $\gamma(1) = x$.

We note that the energy E defined in (3.16) has already been studied extensively. For instance, this energy E can be recognised from classical mechanics as the action functional for a particle moving in a potential $-V$. In physics literature, the function ω defined in (3.17) is sometimes named the *Agmon metric*, after the mathematician Shmuel Agmon, who used this quantity during his study of the decay of eigenfunctions of the Schrödinger operator $-\Delta + V$ at infinity [**1**]. We note that the Agmon metric is typically not defined via (3.16) and (3.17), but rather (up to a constant) by

$$\varrho(x, y) := \min_{\gamma \in \Gamma_{x,y}} \int_0^1 |\dot\gamma| \sqrt{V(\gamma)}\, \mathrm{d}\tau, \tag{3.18}$$

which can be interpreted as the geodesic distance with respect to the weighted metric $\sqrt{V} g$. It was later reported by Carmona and Simon [**33**], that the quantity in (3.18) is in fact equivalent to our function ω. The idea of the argument presented in [**139**] to demonstrate this equivalence goes as follows. Assuming that $V \geq \alpha > 0$, for $0 < s < t$ fixed, Young's inequality implies

$$|\dot\gamma| \sqrt{V(\gamma)} = \frac{|\dot\gamma|}{\sqrt{2(t-s)}} \sqrt{2(t-s)V(\gamma))} \leq \frac{1}{2} \left(\frac{|\dot\gamma|^2}{2(t-s)} + 2(t-s)V(\gamma) \right)$$

for all $\gamma \in \Gamma_{x,y}$ and it follows immediately, that $\varrho \leq \omega$. To see that $\omega \leq \varrho$, take a regular curve $\gamma \in \Gamma_{x,y}$ and reparametrise it so that $|\dot\gamma| = 2(t-s)V(\gamma)$. This choice intentionally corresponds to the equality case of Young's inequality in the line above, meaning that we now have

$$\omega \leq \frac{1}{4(t-s)} \int_0^1 |\dot\gamma|^2\, \mathrm{d}\tau + (t-s) \int_0^1 V(\gamma(\tau))\, \mathrm{d}\tau = \int_0^1 |\dot\gamma| \sqrt{V(\gamma)}\, \mathrm{d}\tau.$$

Taking the infimum leads to $\omega \leq \varrho$.

The function ω defined in (3.17) also plays an important role in the work of Li-Yau [**104**, Theorem 6.1], since as seen in the introduction, it occurs naturally while performing space-time integration to derive the Harnack inequality (1.18) from its differentiated form (1.20). This function ω is also closely related to the fundamental solution Γ of the equation (1.3). In particular, it was proven first in the Euclidean case $M = \mathbb{R}^d$ by Simon [**138**] and later on a complete manifold M by Li and Yau [**104**], that ω can be expressed via the limit

$$\omega(x, y; t, s) = -\lim_{\lambda \to \infty} \frac{\log \Gamma_\lambda(x - y, \frac{t-s}{\lambda})}{\lambda}, \tag{3.19}$$

where Γ_λ denotes the fundamental solution of the equation

$$\partial_t u = \Delta u - \lambda^2 V u \qquad \text{in } M \times (0, \infty).$$

In general, we do not expect ω defined by (3.17) to be explicitly computable. However, we briefly discuss some important potential functions V, for which it is possible to calculate ω and present the relevant Harnack inequality obtained in each case.

Example 3.1 (Heat equation)**.** In the case of the heat equation ($V \equiv 0$) on the Euclidean space $M = \mathbb{R}^d$, the optimal choice of ω is the function

$$\omega(x, y; t, s) := \frac{|x - y|^2}{4(t - s)},$$

which is the value attained by E along the straight line segment, i.e. geodesic, connecting x and y. In addition, ω satisfies (3.1)–(3.3) with equality if $A(\tau) = \tau$ and $\beta(\tau) = \tau^{d/2}$. With this choice, we recover the classical parabolic Harnack inequality

$$u(x, t) \geq u(y, s) \left(\frac{s}{t}\right)^{d/2} e^{-\frac{|x-y|^2}{4(t-s)}} \tag{1.1}$$

stated in the introduction. As noted in Section 1.1, we remark that (1.1) is considered sharp, in the sense that it is satisfied by the fundamental solution

$$\Gamma(x, t) = (4\pi t)^{-d/2} e^{\frac{-|x|^2}{4t}}$$

with equality on the set of points

$$\{(x, y, t, s) \in \mathbb{R}^d \times \mathbb{R}^d \times S \mid sx = ty\}.$$

Example 3.2 (Quadratic potential)**.** Another noteworthy case to consider is that of the quadratic potential

$$V(x) = C_1^2 |x - a|^2 + C_2$$

on the Euclidean space $M = \mathbb{R}^d$ for $a \in \mathbb{R}^d$ and constants $C_1 \neq 0$, $C_2 \in \mathbb{R}$. In this case, one can compute explicitly that for fixed $x, y \in \mathbb{R}^d$ and $0 < s < t$, the energy E given by (3.16) is minimised by the curve defined by

$$\gamma_0(\tau) = a + \frac{1}{\sinh(2C_1(t-s))}\Big(\sinh(2C_1\tau(t-s))(x-a) + \sinh(2C_1(1-\tau)(t-s))(y-a)\Big)$$

for all $\tau \in [0, 1]$. It then follows that

$$\omega(x, y; t, s) = \frac{C_1}{2}\left(\frac{|x-y|^2}{\sinh(2C_1(t-s))} + (|x-a|^2 + |y-a|^2)\tanh(C_1(t-s))\right) + C_2(t-s).$$

One can verify that this choice of ω satisfies (3.1)–(3.3) with equality if we set

$$A(t) = \sinh(2C_1 t) \quad \text{and} \quad \beta(t) = \sinh^{d/2}(2C_1 t).$$

Then, Theorem 3.1 applied with Ω_n chosen to be the ball of radius n centred at a yields that every positive solution u of

$$\partial_t u = \Delta u - (C_1^2|x-a|^2 + C_2)u \qquad \text{in } \mathbb{R}^d \times (0, \infty) \tag{3.20}$$

satisfies the Harnack inequality

$$u(x, t) \geq u(y, s) \left(\frac{\sinh(2C_1 s)}{\sinh(2C_1 t)}\right)^{d/2} e^{-\omega(x,y;t,s)} \tag{3.21}$$

for every $x, y \in \mathbb{R}^d$ and $0 < s < t$.

We note that inequality (3.21) is sharp in a similar sense to how the classical result for the heat equation without potential is sharp. For $a = 0$, $C_2 = 0$, the fundamental solution of (3.20) is known from [**128**] to have the explicit formula

$$\Gamma(x,t) = \left(\frac{2\pi}{C_1}\sinh(2C_1 t)\right)^{-d/2} e^{\frac{-C_1|x|^2}{2\tanh(2C_1 t)}}.$$

One can verify that this solution satisfies (3.21) with equality on the set of points

$$\{(x,y,t,s) \in \mathbb{R}^d \times \mathbb{R}^d \times S \mid \sinh(2C_1 s)x = \sinh(2C_1 t)y\}.$$

This is an improvement of the inequality (1.18) obtained by Li and Yau [**104**].

We now seek to understand the properties of the function ω defined by (3.17) via the energy E in (3.16) for a general potential function V, and in particular, when it fulfils the requirements of Theorem 3.1 and Theorem 3.2. Importantly, we emphasise that many of the computations that follow will be merely formal in nature, since in general, ω may not possess sufficient regularity to satisfy the assumptions of these theorems in a classical sense. In fact, the function ω given by (3.17) might be merely Lipschitz continuous in x and y (see [**104**]). Therefore, the following analysis should serve primarily as a motivation for how to choose ω, while the computations will be made rigorous in the next section when we study the proof of the Harnack inequality on manifolds.

We begin by verifying that the energy E defined in (3.16) does indeed attain a minimum, meaning that the function ω in (3.17) is well-defined.

Proposition 3.3. *Let (M, g) be a Riemannian manifold, which is either compact or complete. Then for every $x, y \in M$ and $0 < s < t$, there exists $\gamma_0 \in \Gamma_{x,y}$ such that*

$$E[\gamma_0; t, s] = \inf_{\gamma \in \Gamma_{x,y}} E[\gamma; t, s].$$

In the proof of Proposition 3.3 we require the manifold-valued Sobolev space $H^1(0,1;M)$, which we now define. Recall from Nash's embedding theorem [**120**], that any d-dimensional Riemannian manifold M can be isometrically embedded in $\mathbb{R}^\ell$ for some $\ell \geq d+1$. Using this embedding, we set

$$H^1(0,1;M) := \{\gamma \in H^1(0,1;\mathbb{R}^\ell) \mid \gamma(\tau) \in M \text{ a.e. } \tau \in [0,1]\}.$$

However, since the Sobolev space $H^1(0,1;\mathbb{R}^\ell)$ is embedded in $C(0,1;\mathbb{R}^\ell)$, we may assume that every $\gamma \in H^1(0,1;M)$ is a continuous function and it is therefore equivalent to write

$$H^1(0,1;M) := \{\gamma \in H^1(0,1;\mathbb{R}^\ell) \mid \gamma(\tau) \in M \text{ for all } \tau \in [0,1]\}.$$

Then for $x, y \in M$ and $0 < s < t$, we set

$$\Gamma_{x,y} := \{H^1(0,1;M) \mid \gamma(0) = y, \gamma(1) = x\}.$$

PROOF OF PROPOSITION 3.3. We assume without loss of generality that $V \geq 0$. Then we immediately notice that $E(\gamma) \geq 0$ for all $\gamma \in H^1(0,1;M)$ and therefore $\inf_{\gamma\in\Gamma_{x,y}} E(\gamma) > -\infty$.

Let $(\gamma_n)_{n\geq 1} \subseteq \Gamma_{x,y}$ be a minimising sequence for E, that is, we suppose $E(\gamma_n) \to \inf_{\gamma\in\Gamma_{x,y}} E(\gamma)$ as $n \to \infty$. Then $(E(\gamma_n))_{n\geq 1}$ is a bounded sequence. In particular, this implies

$$0 \leq \frac{1}{4(t-s)} \int_0^1 |\dot{\gamma}_n|^2 \, \mathrm{d}\tau \leq E(\gamma_n) \leq C$$

for some $C \geq 0$, and so $(\dot{\gamma}_n)_{n\geq 1}$ is a bounded sequence in $L^2(0,1;\mathbb{R}^\ell)$. Moreover, since

$$\begin{aligned}
\|\gamma_n(\tau)\|_{\mathbb{R}^\ell} &\leq \|\gamma_n(\tau) - \gamma_n(0)\|_{\mathbb{R}^\ell} + \|\gamma_n(0)\|_{\mathbb{R}^\ell} \\
&= \|y\|_{\mathbb{R}^\ell} + \int_0^\tau |\dot{\gamma}_n(\hat{\tau})| \, \mathrm{d}\hat{\tau} \\
&\leq \|y\|_{\mathbb{R}^\ell} + \int_0^1 |\dot{\gamma}_n(\tau)| \, \mathrm{d}\tau \\
&\leq \|y\|_{\mathbb{R}^\ell} + \|\dot{\gamma}_n\|_{L^\infty(0,1;\mathbb{R}^\ell)},
\end{aligned}$$

for all $\tau \in [0,1]$, we see that $(\gamma_n)_{n\geq 1}$ is a bounded sequence in $L^\infty(0,1;\mathbb{R}^\ell)$, and thus also in $L^2(0,1;\mathbb{R}^\ell)$. Therefore $(\gamma_n)_{n\geq 1}$ is bounded in $H^1(0,1;\mathbb{R}^\ell)$. Since $H^1(0,1;\mathbb{R}^\ell)$ is reflexive, there exists $\gamma_0 \in H^1(0,1;\mathbb{R}^\ell)$ and a subsequence $(\gamma_{k_n})_{n\geq 1} \subseteq (\gamma_n)_{n\geq 1}$ such that $\gamma_{k_n} \rightharpoonup \gamma_0$ as $n \to \infty$. However, $H^1(0,1;\mathbb{R}^\ell)$ is compactly embedded in $(C(0,1;\mathbb{R}^\ell), \|\cdot\|_\infty)$, and therefore $\gamma_{k_n} \to \gamma_0$ in $C(0,1;\mathbb{R}^\ell)$. Since $\gamma_{k_n}(\tau) \in M$ for every $\tau \in [0,1]$ and $n \geq 1$ and M is a closed set in $\mathbb{R}^\ell$, it follows that $\gamma_0(\tau) \in M$ for all $\tau \in [0,1]$, Hence $\gamma_0 \in \Gamma_{x,y}$.

Next we note that the weak convergence of $\gamma_{k_n} \rightharpoonup \gamma_0$ in $H^1(0,1;M)$ implies $\dot{\gamma}_{k_n} \rightharpoonup \dot{\gamma}_0$ in $L^2(0,1;\mathbb{R}^\ell)$, and therefore

$$\|\dot{\gamma}_0\|^2_{L^2(0,1;\mathbb{R}^\ell)} \leq \liminf_{n\to\infty} \|\dot{\gamma}_{k_n}\|^2_{L^2(0,1;\mathbb{R}^\ell)}.$$

Moreover, since V is continuous, the pointwise convergence of γ_{k_n} to γ_0 implies $V(\gamma_{k_n}(\tau)) \to V(\gamma_0(\tau))$ as $n \to \infty$ for all $\tau \in [0,1]$. By Fatou's lemma, it follows that

$$\int_0^1 V(\gamma_0) \, \mathrm{d}\tau = \int_0^1 \liminf_{n\to\infty} V(\gamma_{k_n}) \, \mathrm{d}\tau \leq \liminf_{n\to\infty} \int_0^1 V(\gamma_{k_n}) \, \mathrm{d}\tau.$$

Putting these facts together shows that

$$E(\gamma_0) \leq \liminf_{n\to\infty} E(\gamma_{k_n}) = \lim_{n\to\infty} E(\gamma_{k_n}) = \inf_{\gamma\in\Gamma_{x,y}} E(\gamma)$$

and therefore

$$\inf_{\gamma\in\Gamma_{x,y}} E(\gamma) = E(\gamma_0) = \min_{\gamma\in\Gamma_{x,y}} E(\gamma_0).$$

□

Throughout our analysis, it is useful to understand the derivatives of $E[\gamma; t, s]$ under variations of the curve γ.

Proposition 3.4. *Fix a curve* $\gamma_0 : [0,1] \to M$ *and let* $\gamma(\tau, r) : [0,1] \times (-\delta, \delta) \to M$ *be a smooth family of curves such that* $\gamma(\tau, 0) = \gamma_0(\tau)$. *Then we have the following formulas:*

$$\frac{d}{dr} E[\gamma(\tau,r); t, s] = \frac{1}{2(t-s)} \left(g(\gamma_r, \dot\gamma)|_0^1 - \int_0^1 g(\nabla_\tau \dot\gamma - 2(t-s)^2 \nabla V(\gamma), \gamma_r) \, \mathrm{d}\tau \right) \tag{3.22}$$

$$\begin{aligned} \frac{d^2}{dr^2} E[\gamma(\tau,r); t, s] &= \frac{1}{2(t-s)} \int_0^1 R(\gamma_r, \dot\gamma, \gamma_r, \dot\gamma) + g(\nabla_\tau \nabla_r \gamma_r, \dot\gamma) + |\nabla_r \dot\gamma|^2 \, \mathrm{d}\tau \\ &\quad + (t-s) \int_0^1 g(D^2 V(\gamma)\gamma_r, \gamma_r) + g(\nabla V(\gamma), \gamma_{rr}) \, \mathrm{d}\tau \end{aligned} \tag{3.23}$$

Proof. Firstly, we have

$$\begin{aligned} \frac{d}{dr} E[\gamma(\tau,r); t, s] &= \frac{1}{4(t-s)} \int_0^1 \frac{d}{dr} g(\dot\gamma, \dot\gamma) \, \mathrm{d}\tau + (t-s) \int_0^1 \frac{d}{dr} V(\gamma) \, \mathrm{d}\tau \\ &= \frac{1}{2(t-s)} \int_0^1 g(\nabla_r \dot\gamma, \dot\gamma) \, \mathrm{d}\tau + (t-s) \int_0^1 g(\nabla V(\gamma), \gamma_r) \, \mathrm{d}\tau \\ &= \frac{1}{2(t-s)} \int_0^1 g(\nabla_\tau \gamma_r, \dot\gamma) \, \mathrm{d}\tau + (t-s) \int_0^1 g(\nabla V(\gamma), \gamma_r) \, \mathrm{d}\tau \\ &= \frac{1}{2(t-s)} \left(g(\gamma_r, \dot\gamma)|_0^1 - \int_0^1 g(\gamma_r, \nabla_\tau \dot\gamma) \, \mathrm{d}\tau \right) \\ &\qquad + (t-s) \int_0^1 g(\nabla V(\gamma), \gamma_r) \, \mathrm{d}\tau \\ &= \frac{1}{2(t-s)} \left(g(\gamma_r, \dot\gamma)|_0^1 - \int_0^1 g(\nabla_\tau \dot\gamma - 2(t-s)^2 \nabla V(\gamma), \gamma_r) \, \mathrm{d}\tau \right) \end{aligned}$$

Then, differentiating again,

$$
\begin{aligned}
&\frac{d^2}{dr^2}E[\gamma(\tau,r);t,s] \\
&\quad = \frac{d}{dr}\left(\frac{1}{2(t-s)}\int_0^1 g(\nabla_r\dot\gamma,\dot\gamma)\ \mathrm{d}\tau + (t-s)\int_0^1 g(\nabla V(\gamma),\gamma_r)\ \mathrm{d}\tau\right) \\
&\quad = \frac{1}{2(t-s)}\int_0^1 g(\nabla_r\nabla_r\dot\gamma,\dot\gamma) + |\nabla_r\dot\gamma|^2\ \mathrm{d}\tau \\
&\qquad\qquad + (t-s)\int_0^1 g(D^2V(\gamma)\gamma_r,\gamma_r) + g(\nabla V(\gamma),\gamma_{rr})\ \mathrm{d}\tau \\
&\quad = \frac{1}{2(t-s)}\int_0^1 g(\nabla_r\nabla_\tau\gamma_r,\dot\gamma) + |\nabla_r\dot\gamma|^2\ \mathrm{d}\tau \\
&\qquad\qquad + (t-s)\int_0^1 g(D^2V(\gamma)\gamma_r,\gamma_r) + g(\nabla V(\gamma),\gamma_{rr})\ \mathrm{d}\tau \\
&\quad = \frac{1}{2(t-s)}\int_0^1 R(\gamma_r,\dot\gamma,\gamma_r,\dot\gamma) + g(\nabla_\tau\nabla_r\gamma_r,\dot\gamma) + |\nabla_r\dot\gamma|^2\ \mathrm{d}\tau \\
&\qquad\qquad + (t-s)\int_0^1 g(D^2V(\gamma)\gamma_r,\gamma_r) + g(\nabla V(\gamma),\gamma_{rr})\ \mathrm{d}\tau
\end{aligned}
$$

□

Using the formula (3.22), we see that the Euler-Lagrange equation associated to the minimisation problem used in (3.17) to define ω is

$$\nabla_\tau\dot\gamma_0 = 2(t-s)^2\nabla V(\gamma_0). \tag{3.24}$$

Definition 3.2. For given $x, y \in M$ and $0 < s < t$, we call a curve $\gamma_0 : [0,1] \to M$ a *V-geodesic* from y to x if γ_0 is a solution of

$$\begin{cases} \nabla_\tau\dot\gamma_0 = 2(t-s)^2\nabla V(\gamma_0) & \text{on } [0,1], \\ \gamma_0(0) = y,\ \gamma_0(1) = x. \end{cases}$$

Now, we would like to conclude that the minimiser of E found in the proof of Proposition 3.3 is a V-geodesic, however, this is not always guaranteed to be the case. For example, in the case that M is compact and the minimising curve lies partly on ∂M, we do not necessarily expect the Euler-Lagrange equation (3.24) to be satisfied. Therefore, we restrict the class of manifolds in which we work by introducing a certain notion of convexity adapted to the potential V.

Definition 3.3. We call a compact manifold M *V-convex* if for all $x, y \in M$ and $0 < s < t$, there exists a V-geodesic γ_0 such that

$$E[\gamma_0; t, s] = \min_{\gamma\in\Gamma_{x,y}} E[\gamma; t, s]$$

and

$$h(\xi,\xi) + g(\nabla V(x), \nu(x)) > 0 \tag{3.25}$$

holds for all $x \in \partial M$ and $\xi \in T_x\partial M \setminus \{0\}$, where h denotes the second fundamental form on ∂M and $\nu(x)$ is the outward pointing unit normal vector to the boundary at $x \in \partial M$.

Moreover, a complete manifold M is called *V-approximable* if there exists an increasing sequence $(M_n)_{n\geq 1}$ of V-convex regions $M_n \subseteq M$ with $\bigcup_{n\geq 1} M_n = M$.

Remark 3.3.

(i) By rescaling ξ, it is easy to show that the condition (3.25) is equivalent to

$$h(\xi, \xi) + \alpha g(\nabla V(x), \nu(x)) > 0 \tag{3.26}$$

for any $\alpha > 0$.

(ii) By taking $\alpha \to 0$ in (3.26), we see that the condition (3.25) implies the nonnegativity of the second fundamental form and therefore every V-convex domain is necessarily convex. In addition, dividing (3.26) by $\alpha > 0$ and taking $\alpha \to \infty$ implies that $\nabla V \cdot \nu \geq 0$ on ∂M, which is a condition previously assumed by Yau [**159**].

(iii) Recall that for all curves $\gamma_0 : [0,1] \to M$, the second fundamental form is related to the normal curvature of γ_0 through the formula

$$h(\dot{\gamma}_0, \dot{\gamma}_0) = -g(\nabla_\tau \dot{\gamma}_0, \nu(\gamma_0)).$$

Inserting this into (3.26) with $\alpha = 2(t-s)^2$ implies that if $\gamma_0(\tau) \in \partial M$ for some $\tau \in [0,1]$ and $\dot{\gamma}_0(\tau) \neq 0$, then

$$g(-\nabla_\tau \dot{\gamma}_0(\tau) + 2(t-s)^2 \nabla V(\gamma_0(\tau)), \nu(\gamma_0(\tau)) > 0.$$

However, if γ_0 satisfies the Euler-Lagrange equation (3.24), then the expression in the left-hand side of the above inequality is equal to 0, producing a contradiction. Therefore, a V-geodesic γ_0 may never touch the boundary of a V-convex manifold, unless the velocity of γ_0 vanishes at the point of contact.

Example 3.3 (Geodesically convex potentials)**.** Suppose that the potential function $V : M \to \mathbb{R}$ is geodesically convex, that is, $V \circ \gamma : [0,1] \to \mathbb{R}$ is convex for every geodesic $\gamma : [0,1] \to M$ in M. Then $(V \circ \gamma)' = g(\nabla V, \dot{\gamma})$ is monotone and so

$$g(\nabla V, \dot{\gamma})(1) \geq g(\nabla V, \dot{\gamma})(0).$$

In addition, suppose that V attains a local minimum at a point $x_0 \in M$, so that $\nabla V(x_0) = 0$, and let B be a geodesic ball in M centred at x_0. If we choose γ such that $\gamma(0) = x_0$ and $\gamma(1) = x$ lies on the boundary of B, then

$$g(\nabla V(x), \nu(x)) = g(\nabla V, \dot{\gamma})(1) \geq 0.$$

for all $x \in \partial B$. Once we combine this with the fact that the second fundamental form of ∂B is strictly positive, we have proven that balls centred at a minimiser of a geodesically convex potential V satisfy the condition (3.25).

With the important assumption of V-convexity, we are ready to continue investigating whether ω satisfies the remaining assumptions of Theorem 3.1 and Theorem 3.2. Next, we observe that $\omega(x, y; t, s)$ has the correct behaviour in the limit as $t \to s^+$.

Proposition 3.5. *Let ω be defined as in* (3.17)*. Then* $\lim_{t\to s^+}\omega(x,y;t,s)\to\infty$ *for $x\neq y$ and* $\lim_{t\to s^+}\omega(x,x;t,s)\geq 0$ *for all $x\in M$.*

PROOF. Suppose γ_0 minimises $E[\gamma;t,s]$. Assuming $V\geq-\alpha$ in M for some $\alpha>0$, it follows that

$$\begin{aligned}\omega(x,y;t,s)=E[\gamma_0,t,s]&=\frac{1}{4(t-s)}\int_0^1|\dot\gamma_0|^2\,\mathrm{d}\tau+(t-s)\int_0^1V(\gamma_0(\tau))\,\mathrm{d}\tau\\&\geq\frac{1}{4(t-s)}\int_0^1|\dot\gamma_0|^2\,\mathrm{d}\tau-\alpha(t-s).\end{aligned}$$

Now, the Cauchy-Schwarz inequality implies that

$$L(\gamma_0)^2=\left(\int_0^1|\dot\gamma_0|\,\mathrm{d}\tau\right)^2\leq\int_0^1|\dot\gamma_0|^2\,\mathrm{d}\tau.$$

Therefore,

$$\omega(x,y;t,s)\geq\frac{L(\gamma_0)^2}{4(t-s)}-\alpha(t-s)\geq\frac{d^2(x,y)}{4(t-s)}-\alpha(t-s).$$

If $d(x,y)>0$, this inequality implies $\omega\to\infty$ as $t\to s^+$ as desired. Otherwise, if we assume $x=y$, we still obtain

$$\lim_{t\to s^+}\omega(x,x;t,s)\geq 0.$$

□

The most important properties to verify are that ω satisfies the differential inequalities (3.1)–(3.4). In their paper, Li and Yau [**104**] already demonstrated that ω satisfies (3.1) and (3.2) with equality. Through the next two propositions, we review their proof of this fact.

Proposition 3.6 (Time derivatives of E)**.** *Let γ_0 be a V-geodesic connecting $x,y\in M$. Then*

$$\frac{\partial}{\partial t}E[\gamma_0;t,s]=-\frac{|\dot\gamma_0|^2(1)}{4(t-s)^2}+V(x),\tag{3.27}$$

$$\frac{\partial}{\partial s}E[\gamma_0;t,s]=\frac{|\dot\gamma_0|^2(0)}{4(t-s)^2}-V(y).\tag{3.28}$$

PROOF. We have

$$\begin{aligned}
&\frac{\partial}{\partial t}E[\gamma_0;t,s]\\
&\quad=\frac{\partial}{\partial t}\left(\frac{1}{4(t-s)}\int_0^1 g(\dot{\gamma_0},\dot{\gamma_0})\ \mathrm{d}\tau+(t-s)\int_0^1 V(\gamma_0)\ \mathrm{d}\tau\right)\\
&\quad=-\frac{1}{4(t-s)^2}\int_0^1 g(\dot{\gamma_0},\dot{\gamma_0})\ \mathrm{d}\tau+\int_0^1 V(\gamma_0)\ \mathrm{d}\tau\\
&\qquad+\frac{1}{2(t-s)}\int_0^1 g\left(\frac{\partial}{\partial t}\dot{\gamma_0},\dot{\gamma_0}\right)+2(t-s)^2 g\left(\nabla V(\gamma_0),\frac{\partial}{\partial t}\gamma_0\right)\ \mathrm{d}\tau\\
&\quad=-\frac{1}{4(t-s)^2}\int_0^1 g(\dot{\gamma_0},\dot{\gamma_0})\ \mathrm{d}\tau+\int_0^1 V(\gamma_0)\ \mathrm{d}\tau\\
&\qquad-\frac{1}{2(t-s)}\int_0^1 g\left(\nabla_\tau\dot{\gamma_0},\frac{\partial}{\partial t}\gamma_0\right)-2(t-s)^2 g\left(\nabla V(\gamma_0),\frac{\partial}{\partial t}\gamma_0\right)\ \mathrm{d}\tau
\end{aligned}$$

When integrating by parts to obtain the last line, we have used that $\gamma_0(1)\equiv x$ and $\gamma_0(0)\equiv y$ independent of t. Therefore, $\frac{\partial}{\partial t}\gamma_0(1)=\frac{\partial}{\partial t}\gamma_0(0)=0$. Recalling that γ_0 satisfies the Euler-Lagrange equation

$$\nabla_\tau\dot{\gamma_0}=2(t-s)^2\nabla V(\gamma_0), \tag{3.24}$$

we have that

$$\frac{\partial}{\partial t}E(\gamma_0)=-\frac{1}{4(t-s)^2}\int_0^1 g(\dot{\gamma_0},\dot{\gamma_0})-4(t-s)^2V(\gamma_0)\ \mathrm{d}\tau.$$

However, multiplying the Euler-Lagrange equation (3.24) through by $2\dot{\gamma_0}$, we have

$$2g(\nabla_\tau\dot{\gamma_0},\dot{\gamma_0})=2(t-s)^2 g(\nabla V(\gamma_0),\dot{\gamma_0}),$$

which implies

$$\frac{d}{d\tau}g(\dot{\gamma_0},\dot{\gamma_0})=4(t-s)^2\frac{d}{d\tau}V(\gamma_0).$$

Integrating from 0 to τ, we find that

$$g(\dot{\gamma_0}(\tau),\dot{\gamma_0}(\tau))-4(t-s)^2V(\gamma_0(\tau))=g(\dot{\gamma_0}(0),\dot{\gamma_0}(0))-4(t-s)^2V(\gamma_0(0))$$

for all $\tau\in[0,1]$. In other words, $g(\dot{\gamma_0(\tau)},\dot{\gamma_0}(\tau))-4(t-s)^2V(\gamma_0(\tau))$ is constant in τ. Applying this to our formula for $\frac{\partial}{\partial t}E(\gamma_0)$, we see that

$$\frac{\partial}{\partial t}E(\gamma_0)=-\frac{|\dot{\gamma_0}|^2(1)}{4(t-s)^2}+V(x).$$

By a similar process, we can also prove (3.28). ☐

Proposition 3.7 (First-order equations satisfied by ω)**.** *Let γ_0 be a V-geodesic connecting $x,y\in M$ and set $\omega(x,y;t,s)=E[\gamma_0;t,s]$. Then ω satisfies*

$$\partial_t\omega+|\nabla_x\omega|^2=V(x) \tag{3.29}$$

$$\partial_s\omega-|\nabla_y\omega|^2=-V(y) \tag{3.30}$$

in $M\times M\times S$.

PROOF. Since γ_0 satisfies the Euler-Lagrange equation, (3.22) simplifies to become

$$\frac{d}{dr}E[\gamma(\tau,r);t,s]\bigg|_{r=0} = \frac{1}{2(t-s)}(g(\gamma_r,\dot{\gamma_0})(1) - g(\gamma_r,\dot{\gamma_0})(0)).$$

If we consider a variation $\gamma(\tau,r)$ of γ_0 with $\gamma(0,r)=y$ for all r, then $\gamma_r(0,r)=0$ and therefore

$$g(\nabla_x\omega,\gamma_r(1,r)) = \frac{d}{dr}E[\gamma(\tau,r);t,s]\bigg|_{r=0} = \frac{g(\gamma_r(1,r),\dot{\gamma_0}(1))}{2(t-s)}$$

for all choices of $\gamma_r(1,r)$. Therefore,

$$\nabla_x\omega = \frac{1}{2(t-s)}\dot{\gamma_0}(1).$$

In light of (3.27), we conclude that

$$\partial_t\omega + |\nabla_x\omega|^2 = V(x).$$

By a similar argument, instead considering variations $\gamma(\tau,r)$ such that $\gamma(1,r)=x$ for all r, we find that

$$\nabla_y\omega = -\frac{1}{2(t-s)}\dot{\gamma_0}(0)$$

and hence,

$$\partial_s\omega - |\nabla_y\omega|^2 = -V(y)$$

as claimed. □

Using the expressions for the gradient of ω found during the proof of Proposition 3.7, we can also understand when ω satisfies the condition,

$$\begin{aligned} &g(\nabla_x\omega,\nu(x)) \geq 0, \qquad \text{for all } x\in\partial M,\ y\in M, \text{ and } 0<s<t,\\ &g(\nabla_y\omega,\nu(y)) \geq 0, \qquad \text{for all } x\in M,\ y\in\partial M, \text{ and } 0<s<t. \end{aligned} \tag{3.4}$$

on a compact manifold with boundary. These inequalities are equivalent to

$$\nabla_x\omega\cdot\nu = \frac{g(\dot{\gamma_0}(1),\nu)}{2(t-s)} \geq 0 \quad \text{and} \quad \nabla_y\omega\cdot\nu = -\frac{g(\dot{\gamma_0}(0),\nu)}{2(t-s)} \geq 0$$

for all V-geodesics with an endpoint, either $\gamma_0(0)$ or $\gamma_0(1)$ or both, occurring on the boundary ∂M. However, as commented in Remark 3.3 (ii), once we assume that M is V-convex, M will also be convex and both the above inequalities will be satisfied.

Our final task is to find sufficient conditions under which ω satisfies an inequality of the form

$$A(t)^2\Delta_x\omega + A(s)^2\Delta_y\omega + 2A(t)A(s)\Delta_{xy}\omega \leq A(t)^2\frac{\beta'(t)}{\beta(t)} - A(s)^2\frac{\beta'(s)}{\beta(s)} \tag{3.3}$$

The following comparison result can aid in answering this question.

Theorem 3.8. *Let M be a Riemannian manifold with nonnegative Ricci curvature, and let $V_1, V_2 \in C^2(M)$ be two potential functions, which satisfy $\Delta V_1 \leq \Delta V_2$ in M. Assume that there exist strictly increasing, differentiable functions $\beta : [0,\infty) \to [0,\infty)$ satisfying $\beta(0) = 0$ and $A : [0,\infty) \to [0,\infty)$ such that*

$$(t-s)\int_0^1 \frac{d}{2}A'(\tau t + (1-\tau)s)^2 + A(\tau t + (1-\tau)s)^2 \Delta V_2(\gamma_0)\,\mathrm{d}\tau \leq A(t)^2\frac{\beta'(t)}{\beta(t)} - A(s)^2\frac{\beta'(s)}{\beta(s)}$$

for all V_2-geodesics γ_0 and $0 < s < t$. Then the function ω_1 defined by (3.17) for the potential $V = V_1$ (formally) satisfies

$$A(t)^2\Delta_x\omega_1 + A(s)^2\Delta_y\omega_1 + 2A(t)A(s)\Delta_{xy}\omega_1 \leq A(t)^2\frac{\beta'(t)}{\beta(t)} - A(s)^2\frac{\beta'(s)}{\beta(s)}$$

in $M \times M \times S$. Moreover, if M is compact and V_1-convex, then every positive solution u of

$$\begin{cases} \partial_t u = \Delta u - V_1 u & \text{in } M \times (0,\infty), \\ g(\nabla u, \nu) = 0 & \text{on } \partial M \times (0,\infty) \end{cases}$$

satisfies the Harnack inequality

$$u(x,t) \geq u(y,s)\frac{\beta(s)}{\beta(t)}e^{-\omega_1(x,y;t,s)} \tag{3.31}$$

for all $x, y \in M$ and $0 < s < t$. In addition, if M is complete and V_1-approximable, then every positive solution u of

$$\partial_t u = \Delta u - V_1 u \qquad \text{in } M \times (0,\infty)$$

satisfies (3.31).

This result follows immediately from a short lemma.

Lemma 3.9. *Let M be a Riemannian manifold with nonnegative Ricci curvature, $V \in C^2(M)$ be bounded from below and $A : [0,\infty) \to [0,\infty)$ be differentiable. Then ω as defined in* (3.17) *satisfies*

$$\begin{aligned} A(t)^2\Delta_x\omega &+ A(s)^2\Delta_y\omega + 2A(t)A(s)\Delta_{xy}\omega \\ &\leq \frac{d}{2}(t-s)\int_0^1 A'(\tau t + (1-\tau)s)^2\,\mathrm{d}\tau \\ &\quad + (t-s)\int_0^1 A(\tau t + (1-\tau)s)^2\Delta V(\gamma_0)\,\mathrm{d}\tau \end{aligned} \tag{3.32}$$

for all V-geodesics γ_0 in M and $0 < s < t$.

Proof. Fix $0 < s < t$ and let $\gamma_0 : [0,1] \to M$ be a V-geodesic such that $\gamma_0(0) = y$ and $\gamma_0(1) = x$ for $x, y \in M$ fixed. Choose an orthonormal basis $(e_i(0))_{i=1}^d$ of T_yM and parallel transport it along γ_0 to obtain an orthonormal basis $e_i := e_i(\tau)$ at each point on the curve. We then define a particular variation of γ_0 by setting

$$\gamma(\tau, r) = \exp_{\gamma_0(\tau)}(A(\tau t + (1-\tau)s)re_i).$$

We note that $\gamma_r(1,r) = A(t)e_i$, $\gamma_r(0,r) = A(s)e_i$, and $\gamma_{rr}(\tau,r) = 0$ for all $r \in (-\delta,\delta)$.

In addition, the variation of the endpoints of the curves is described by

$$\begin{aligned} x(r) &:= \gamma(1,r) = \exp_x(A(t)re_i), \\ y(r) &:= \gamma(0,r) = \exp_y(A(s)re_i). \end{aligned}$$

Now, for any r,

$$\omega(x(r),y(r);t,s) = \min_{\gamma\in\Gamma_{x(r),y(r)}} E[\gamma;t,s] \le E[\gamma(\tau,r);t,s],$$

where equality is attained for $r=0$. Therefore, it follows that as a function of r, $\omega(x(r),y(r);t,s) - E[\gamma(\tau,r);t,s]$ is negative and attains a maximum value of 0 at $r=0$. Therefore,

$$\frac{d^2}{dr^2}\omega(x(r),y(r);t,s)\bigg|_{r=0} \le \frac{d^2}{dr^2}E[\gamma;t,s]\bigg|_{r=0}. \tag{3.33}$$

Explicit computation of the left-hand side yields

$$A(t)^2\frac{\partial^2\omega}{\partial x_i^2} + A(s)^2\frac{\partial^2\omega}{\partial y_i^2} + 2A(t)A(s)\frac{\partial^2\omega}{\partial x_i\partial y_i},$$

while the right-hand side can be computed using (3.23):

$$\begin{aligned} &\frac{\partial^2}{\partial r^2}E[\gamma(\tau,r);t,s]\bigg|_{r=0} \\ &= \frac{1}{2(t-s)}\bigg(\int_0^1 |(t-s)A'(\tau t+(1-\tau)s)e_i|^2 \\ &\qquad + A(\tau t+(1-\tau)s)^2R(e_i,\dot\gamma_0,e_i,\dot\gamma_0)\ \mathrm{d}\tau\bigg) \\ &\qquad + (t-s)\int_0^1 A(\tau t+(1-\tau)s)^2\nabla_i\nabla_i V(\gamma_0)\ \mathrm{d}\tau \\ &= \frac{1}{2}(t-s)\int_0^1 A'(\tau t+(1-\tau)s)^2\ \mathrm{d}\tau \\ &\qquad - \frac{1}{2(t-s)}\int_0^1 A(\tau t+(1-\tau)s)^2R(e_i,\dot\gamma_0,\dot\gamma_0,e_i)\ \mathrm{d}\tau \\ &\qquad + (t-s)\int_0^1 A(\tau t+(1-\tau)s)^2\nabla_i\nabla_i V(\gamma_0)\ \mathrm{d}\tau. \end{aligned}$$

Inserting these expressions back into (3.33) and summing over i gives

$$\begin{aligned} &A(t)^2\Delta_x\omega + A(s)^2\Delta_y\omega + 2A(t)A(s)\Delta_{xy}\omega \\ &\quad\le \frac{d}{2}(t-s)\int_0^1 A'(\tau t+(1-\tau)s)^2\ \mathrm{d}\tau \\ &\qquad\quad - \frac{1}{2(t-s)}\int_0^1 A(\tau t+(1-\tau)s)^2\,\mathrm{Ric}(\dot\gamma_0,\dot\gamma_0)\ \mathrm{d}\tau \\ &\qquad\quad + (t-s)\int_0^1 A(\tau t+(1-\tau)s)^2\Delta V(\gamma_0)\ \mathrm{d}\tau. \end{aligned}$$

Since the manifold M is assumed to have nonnegative Ricci curvature, this estimate reduces to

$$\begin{aligned}
&A(t)^2\Delta_x\omega + A(s)^2\Delta_y\omega + 2A(t)A(s)\Delta_{xy}\omega \\
&\quad\le \frac{d}{2}(t-s)\int_0^1 A'(\tau t+(1-\tau)s)^2\,\mathrm{d}\tau \\
&\qquad + (t-s)\int_0^1 A(\tau t+(1-\tau)s)^2\Delta V(\gamma_0)\,\mathrm{d}\tau,
\end{aligned}$$

which is the claimed inequality (3.32). □

Proof of Theorem 3.8. Applying Lemma 3.9 to ω_1, we have

$$\begin{aligned}
&A(t)^2\Delta_x\omega_1 + A(s)^2\Delta_y\omega_1 + 2A(t)A(s)\Delta_{xy}\omega_1 \\
&\quad\le \frac{d}{2}(t-s)\int_0^1 A'(\tau t+(1-\tau)s)^2\,\mathrm{d}\tau \\
&\qquad + (t-s)\int_0^1 A(\tau t+(1-\tau)s)^2\Delta V_1(\gamma_0)\,\mathrm{d}\tau \\
&\quad\le \frac{d}{2}(t-s)\int_0^1 A'(\tau t+(1-\tau)s)^2\,\mathrm{d}\tau \\
&\qquad + (t-s)\int_0^1 A(\tau t+(1-\tau)s)^2\Delta V_2(\gamma_0)\,\mathrm{d}\tau \\
&\quad\le A(t)^2\frac{\beta'(t)}{\beta(t)} - A(s)^2\frac{\beta'(s)}{\beta(s)}.
\end{aligned}$$

Hence the function ω_1 corresponding to V_1 satisfies (3.3) with the functions A and β borrowed from V_2. In the case that $M = \Omega \subseteq \mathbb{R}^d$, all other necessary assumptions from Theorem 3.1 (if $\Omega = \mathbb{R}^d$) and Theorem 3.2 (if $\Omega \neq \mathbb{R}^d$) hold and the Harnack inequality (3.31) follows as a consequence. If M is a general Riemannian manifold, the same conclusion follows by applying Theorem 3.11 or Theorem 3.12 from the next section. □

Applying Theorem 3.8 with the quadratic potential $V_2 = C^2|x|^2$, we find that a Harnack inequality holds for the class of potentials V with ΔV bounded from above.

Corollary 3.10. *Let $\Omega \subseteq \mathbb{R}^d$ and $V \in C^2(\Omega)$ be bounded from below, such that either Ω is V-convex ($\Omega \neq \mathbb{R}^d$) or $\Omega = \mathbb{R}^d$ is V-approximable. Suppose there is a constant $C \neq 0$ such that $\Delta V \le 2dC^2$. Then, every positive solution u of* (3.8) *satisfies*

$$u(x,t) \ge u(y,s)\left(\frac{\sinh(2Cs)}{\sinh(2Ct)}\right)^{d/2} e^{-\omega(x,y;t,s)} \tag{3.34}$$

for every $x, y \in \overline{\Omega}$ and $0 < s < t$, where ω is defined by (3.37).

3.1.3. The manifold case

Now that we have identified an appropriate candidate for the function ω, which possesses suitable properties, even in the manifold setting, we are ready to present our main result in this case.

Theorem 3.11. *Let (M, g) be a complete Riemannian manifold without boundary and nonnegative Ricci curvature. Let $V \in C^2(M)$ be bounded from below and such that M is V-approximable. Suppose there exist strictly increasing, differentiable functions $\beta : [0, \infty) \to [0, \infty)$ satisfying $\beta(0) = 0$ and $A : [0, \infty) \to [0, \infty)$ such that*

$$(t-s)\int_0^1 \frac{d}{2}A'(\tau t + (1-\tau)s)^2 + A(\tau t + (1-\tau)s)^2 \Delta V(\gamma_0)\ \mathrm{d}\tau \\ \leq A(t)^2 \frac{\beta'(t)}{\beta(t)} - A(s)^2 \frac{\beta'(s)}{\beta(s)} \tag{3.35}$$

holds for all V-geodesics γ_0 and $0 < s < t$. Then every positive solution u of

$$\partial_t u = \Delta u - Vu \qquad \text{in } M \times (0, \infty) \tag{1.3}$$

satisfies

$$u(x,t) \geq u(y,s)\frac{\beta(s)}{\beta(t)} e^{-\omega(x,y;t,s)} \tag{3.36}$$

for all $x, y \in M$ and $0 < s < t$, where $\omega : M \times M \times S$ is given by

$$\omega(x,y;t,s) := \min_{\gamma \in \Gamma_{x,y}} \frac{1}{4(t-s)} \int_0^1 |\dot\gamma|^2\ \mathrm{d}\tau + (t-s)\int_0^1 V(\gamma(\tau))\ \mathrm{d}\tau. \tag{3.37}$$

As in the Euclidean case, Theorem 3.11 follows by an approximation argument after obtaining the following result for the Neumann problem.

Theorem 3.12. *Let (M, g) be a compact Riemannian manifold with nonnegative Ricci curvature and (possibly empty) boundary ∂M. Let $V \in C^2(M)$ be bounded from below and such that ∂M is V-convex. In addition, suppose that* (3.35) *holds for all V-geodesics γ_0. Then every positive solution u of*

$$\begin{cases} \partial_t u = \Delta u - Vu & \text{in } M \times (0,\infty), \\ g(\nabla u, \nu) = 0 & \text{on } \partial M \times (0, \infty) \end{cases} \tag{3.38}$$

satisfies the Harnack inequality (3.36) *for all $x, y \in M$ and $0 < s < t$.*

The core strategy of the proof of Theorem 3.12 is identical to how we proceeded to prove Theorem 3.2, however, we must overcome some additional technical complications. In Theorem 3.1 and Theorem 3.2, we assumed that the function ω admitted enough regularity to satisfy the inequalities (3.1)—(3.4) pointwise. This assumption was not unreasonable to make, since as outlined in Section 3.1.2, it is possible to find such a sufficiently smooth function ω for many important potentials V. However, for general Riemannian manifolds M, these regularity properties of ω might no longer be true. To bypass this issue in our proofs, we work directly with the energy E and analyse the function Z in our proof under smooth variations of curves connecting points x and y in M.

As we did in the Euclidean case, we start by considering the case, where M is compact.

PROOF OF THEOREM 3.12. Let u be a positive solution of (3.38) and set $v = \log u$. Then v satisfies

$$\begin{cases} \partial_t v = \Delta v + |\nabla v|^2 - V & \text{in } M \times (0,\infty), \\ g(\nabla v, \nu) = 0 & \text{on } \partial M \times (0,\infty). \end{cases}$$

Fix $\varepsilon > 0$ and define Z by

$$Z(x,y;t,s) := v(x,t) - v(y,s) + \log\left(\frac{\beta(t)}{\beta(s)}\right) + \omega(x,y;t,s) + \frac{\varepsilon}{2}(t-s)^2 + \varepsilon.$$

As in the proof of Theorem 3.2, we note that $Z > 0$ as $s \to 0^+$, and in light of Proposition 3.5, we can conclude that $Z > 0$ for t near s, and in particular, for (t,s) sufficiently close to $(0,0)$.

We again proceed as in the proof of Theorem 3.2, by supposing Z touches 0 from above for the first time at a point $P_0 = (x_0, y_0; t_0, s_0)$. For now, we assume that $x_0, y_0 \notin \partial M$.

We define a new function $\tilde{Z} : \mathcal{C}^\infty([0,1], M) \times S \to \mathbb{R}$ by setting

$$\tilde{Z}(\gamma;t,s) = v(\gamma(1),t) - v(\gamma(0),s) + \log\left(\frac{\beta(t)}{\beta(s)}\right) + E[\gamma;t,s] + \frac{\varepsilon}{2}(t-s)^2 + \varepsilon,$$

which we analyse under smooth variations of γ. Clearly, Z and $\tilde{Z}$ are related by the inequality

$$\tilde{Z}(\gamma;t,s) \geq Z(\gamma(1),\gamma(0);t,s),$$

and equality holds if γ minimises $E[\gamma;t,s]$ over the set of all paths connecting $\gamma(0)$ and $\gamma(1)$ fixed. Thus, we can conclude that if Z has a minimum at $(x_0,y_0;t_0,s_0)$, then $\tilde{Z}$ has a minimum at (γ_0,t_0,s_0), where γ_0 is such that $E[\gamma_0;t_0,s_0] \leq E[\gamma;t_0,s_0]$ for all $\gamma \in \Gamma_{x,y}$. From this, we conclude that

$$\partial_t \tilde{Z}(\gamma_0;t_0,s_0) \leq 0 \tag{3.39a}$$

$$\partial_s \tilde{Z}(\gamma_0;t_0,s_0) \leq 0 \tag{3.39b}$$

$$\left.\frac{\partial}{\partial r}\tilde{Z}(\gamma(\tau,r);t_0,s_0)\right|_{r=0} = 0 \tag{3.39c}$$

$$\left.\frac{\partial^2}{\partial r^2}\tilde{Z}(\gamma(\tau,r);t_0,s_0)\right|_{r=0} \geq 0 \tag{3.39d}$$

for all smooth variations $\gamma = \gamma(\tau,r)$ of γ_0, with $r \in (-\delta,\delta)$ for some $\delta > 0$ and $\gamma(\tau,0) = \gamma_0(\tau)$.

Using (3.39c) and the formula (3.22) for the first variation of E, we obtain that

$$\begin{aligned} g(\nabla v(x_0,t_0),&\gamma_r(1)) - g(\nabla v(y_0,s_0),\gamma_r(0)) \\ &= \frac{-1}{2(t_0-s_0)}(g(\gamma_r(1),\dot{\gamma}_0(1)) - g(\gamma_r(0),\dot{\gamma}_0(0))) \end{aligned}$$

for all smooth variations $\gamma(\tau,r)$ of γ_0. By considering variations with $\gamma_r(1)=0$, and then with $\gamma_r(0)=0$, it follows that

$$\nabla v(x_0,t_0)=\frac{-1}{2(t_0-s_0)}\dot\gamma_0(1) \quad \text{and} \quad \nabla v(y_0,s_0)=\frac{-1}{2(t_0-s_0)}\dot\gamma_0(0). \tag{3.40}$$

Next, (3.39a) implies

$$\begin{aligned}0&\geq \partial_t v(x_0,t_0)+\frac{\beta'(t_0)}{\beta(t_0)}+\frac{\partial}{\partial t}E[\gamma_0;t_0,s_0]+\varepsilon(t_0-s_0)\\ &=\Delta v(x_0,t_0)+|\nabla v(x_0,t_0)|^2-V(x_0)+\frac{\beta'(t_0)}{\beta(t_0)}+\frac{\partial}{\partial t}E[\gamma_0;t_0,s_0]+\varepsilon(t_0-s_0)\end{aligned}$$

However, using (3.40) and the expression for $\frac{\partial}{\partial t}E[\gamma_0;t_0,s_0]$ in Proposition 3.6, we see that

$$\frac{\partial}{\partial t}E[\gamma_0;t_0,s_0]=-\frac{|\dot\gamma_0(1)|^2}{4(t_0-s_0)^2}+V(x_0)=-|\nabla v(x_0,t_0)|^2+V(x_0).$$

Therefore,

$$\Delta v(x_0,t_0)\leq -\frac{\beta'(t_0)}{\beta(t_0)}-\varepsilon(t_0-s_0). \tag{3.41}$$

By a similar argument, (3.39b) implies

$$-\Delta v(y_0,s_0)\leq \frac{\beta'(s_0)}{\beta(s_0)}+\varepsilon(t_0-s_0). \tag{3.42}$$

We now wish to make use of our last condition (3.39d). Choose an orthonormal basis $(e_i(0))_{i=1}^d$ of $T_{y_0}M$ and parallel transport it along γ_0 to obtain an orthonormal basis $e_i:=e_i(\tau)$ at each point on the curve. We then define a particular variation of γ_0 by setting

$$\gamma(\tau,r)=\exp_{\gamma_0(\tau)}(A(\tau t_0+(1-\tau)s_0)re_i).$$

We note that $\gamma_r(1,r)=A(t_0)e_i$, $\gamma_r(0,r)=A(s_0)e_i$, and $\gamma_{rr}(\tau,r)=0$ for all $r\in(-\delta,\delta)$. Then, by computing $\frac{\partial^2}{\partial r^2}\tilde Z(\gamma(\tau,r);t_0,s_0)\Big|_{r=0}$ explicitly, we find that

$$0\leq A(t_0)^2\nabla_i\nabla_i v(x_0,t_0)-A(s_0)^2\nabla_i\nabla_i v(y_0,s_0)+\frac{\partial^2}{\partial r^2}E[\gamma(\tau,r);t_0,s_0]\Big|_{r=0}. \tag{3.43}$$

Proceeding identically as in the proof of Lemma 3.9 to compute and estimate $\frac{\partial^2}{\partial r^2}E[\gamma(\tau,r);t_0,s_0]\Big|_{r=0}$, we obtain

$$\begin{aligned}0\leq{}& A(t_0)^2\Delta v(x_0,t_0)-A(s_0)^2\Delta v(y_0,s_0)\\ &+\frac{d}{2}(t_0-s_0)\int_0^1 A'(\tau t_0+(1-\tau)s_0)^2\,\mathrm{d}\tau\\ &+(t_0-s_0)\int_0^1 A(\tau t_0+(1-\tau)s_0)^2\Delta V(\gamma_0)\,\mathrm{d}\tau.\end{aligned} \tag{3.44}$$

Inserting the inequalities (3.41) (3.42), and (3.35) into (3.44) yields

$$0\leq -\varepsilon(t_0-s_0)(A(t_0)^2-A(s_0)^2)<0.$$

This is a contradiction since A is assumed to be strictly increasing. If $\partial M = \emptyset$, then this concludes the proof. Otherwise, at least one of the points x_0, y_0 needs to lie on the boundary ∂M.

Suppose $y_0 \in \partial M$. Let $e(0) := -\nu(y_0) \in T_{y_0}M$ and parallel transport $e(0)$ along γ_0, so that $e(\tau) \in T_{\gamma_0(\tau)}M$. We choose a variation $\gamma : [0,1] \times [0,\delta) \to M$ of γ_0 defined by

$$\gamma(\tau, r) = \exp_{\gamma_0(\tau)}(r(1-\tau)e(\tau)). \tag{3.45}$$

Then $\gamma_r(\tau, r) = (1-\tau)e(\tau)$ for all $r \in [0,\delta)$. In particular, $\gamma_r(0,r) = -\nu(y_0)$ and $\gamma_r(1,r) = 0$. With the help of (3.22), we see that

$$\begin{aligned}\frac{\partial}{\partial r}\tilde{Z}(\gamma(\tau,r);t_0,s_0)\bigg|_{r=0} &= g(\nabla v(x_0,t_0),\gamma_r(1,0)) - g(\nabla v(y_0,s_0),\gamma_r(0,0)) \\ &\quad + \frac{1}{2(t_0-s_0)}(g(\gamma_r(1,0),\dot{\gamma}_0(1)) - g(\gamma_r(0,0),\dot{\gamma}_0(0))) \\ &= g(\nabla v(y_0,s_0),\nu(y_0)) + \frac{g(\nu(y_0),\dot{\gamma}_0(0)}{2(t_0-s_0)} \\ &= \frac{g(\nu(y_0),\dot{\gamma}_0(0))}{2(t_0-s_0)},\end{aligned}$$

where the last equality follows as a consequence of the Neumann condition on v. The assumption that M is V-convex and in particular convex (see Remark 3.3 (ii)) implies $g(\nu(y_0),\dot{\gamma}_0(0)) \le 0$ and so

$$\frac{\partial}{\partial r}\tilde{Z}(\gamma(\tau,r));t_0,s_0)\bigg|_{r=0} \le 0.$$

If this inequality is strict, we may apply a similar argument as in the proof of Theorem 3.2 to reach a contradiction. Therefore, we must have equality, which can occur only if $g(\nu(y_0),\dot{\gamma}_0(0)) = 0$, that is, if $\dot{\gamma}_0(0) \in T_{y_0}\partial M$. Since the manifold was assumed to be V-convex, Remark 3.3 (iii) implies this is only possible if $\dot{\gamma}_0(0) = 0$. Therefore, for any variation of γ_0, not necessarily the one defined by (3.45), we have

$$\begin{aligned}\frac{\partial}{\partial r}\tilde{Z}(\gamma(\tau,r);t_0,s_0)\bigg|_{r=0} &= g(\nabla v(x_0,t_0),\gamma_r(1,0)) - g(\nabla v(y_0,s_0),\gamma_r(0,0)) \\ &\quad + \frac{1}{2(t_0-s_0)}g(\gamma_r(1,0),\dot{\gamma}_0(1)).\end{aligned}$$

In particular, for variations $\gamma(\tau,r) : [0,1] \times (-\delta,\delta) \to M$ with $\gamma_r(1,0) = 0$ we have

$$\frac{\partial}{\partial r}\tilde{Z}(\gamma(\tau,r);t_0,s_0)\bigg|_{r=0} = -g(\nabla v(y_0,s_0),\gamma_r(0,0)).$$

If we instead consider choosing $\gamma_r(0,0) \in T_{y_0}\partial M$, then we have

$$0 = \frac{\partial}{\partial r}\tilde{Z}(\gamma(\tau,r);t_0,s_0)\bigg|_{r=0} = -g(\nabla v(y_0,s_0),\gamma_r(0,0)).$$

Since the Neumann boundary condition implies $g(\nabla v(y_0,s_0),\gamma_r(0,0)) = 0$ for all $\gamma_r(0,0) \in (T_{y_0}\partial M)^\perp$, we conclude that

$$g(\nabla v(y_0,s_0),\gamma_r(0,0)) = 0$$

for all choices of $\gamma_r(0,0) \in T_{y_0}M$ and therefore

$$\nabla v(y_0, s_0) = 0 = \frac{-1}{2(t_0 - s_0)}\dot{\gamma}_0(0)$$

holds as before. It follows that

$$\frac{\partial}{\partial r}\tilde{Z}(\gamma(\tau,r);t_0,s_0)\bigg|_{r=0} = g(\nabla v(x_0,t_0), \gamma_r(1,0)) + \frac{1}{2(t_0-s_0)}g(\gamma_r(1,0), \dot{\gamma}_0(1))$$

for any variation of γ_0.

If $x_0 \notin \partial M$, then we may consider variations $\gamma(\tau, r) : [0,1] \times (-\delta, \delta) \to M$ with any choice of $\gamma_r(1,0) \in T_{x_0}M$. We then have

$$0 = \frac{\partial}{\partial r}\tilde{Z}(\gamma(\tau,r);t_0,s_0)\bigg|_{r=0} = g(\nabla v(x_0,t_0), \gamma_r(1,0)) + \frac{1}{2(t_0-s_0)}g(\gamma_r(1,0), \dot{\gamma}_0(1))$$

and therefore

$$\nabla v(x_0, t_0) = \frac{-1}{2(t_0 - s_0)}\dot{\gamma}_0(1)$$

holds as before. If $x_0 \in \partial M$, by repeating the argument for y_0, it must be that $\dot{\gamma}_0(1) = 0$ and $\nabla v(x_0, t_0) = 0$ and again we recover the above identity. From this point on, we can continue to follow the argument as in the case that $x_0, y_0 \notin \partial M$ to again reach a contradiction.

□

Proof of Theorem 3.11. Since M is assumed to be V-approximable, we may assume there exists a sequence $(M_n)_{n\geq 1}$ of V-convex domains approximating M. From here, the argument follows precisely as in the proof of Theorem 3.1 with $\mathbb{R}^d$ replaced by M and Ω_n replaced by M_n. □

3.1.4. The Schrödinger equation with gradient drift

Let M be a complete Riemannian manifold and $f \in C^4(M)$. We consider positive solutions of the equation

$$\partial_t u = \Delta u - 2\nabla f \cdot \nabla u - Vu \qquad \text{in } M \times (0, \infty). \tag{3.46}$$

Under the change of variables $v := e^{-f}u$, (3.46) is transformed into the Schrödinger equation

$$\partial_t v = \Delta v - \tilde{V}v \qquad \text{in } M \times (0, \infty) \tag{3.47}$$

for the potential $\tilde{V} := |\nabla f|^2 - \Delta f + V$. Therefore, if the conditions of Theorem 3.11 are satisfied for the potential $\tilde{V}$, we may conclude that every positive solution v of (3.47) satisfies the Harnack inequality

$$v(x,t) \geq v(y,s)\frac{\beta(s)}{\beta(t)}e^{-\tilde{\omega}(x,y;t,s)},$$

where

$$\tilde{\omega}(x,y;t,s) := \min_{\Gamma_{x,y}} \frac{1}{4(t-s)}\int_0^1 |\dot{\gamma}|^2 \, \mathrm{d}\tau + (t-s)\int_0^1 (|\nabla f|^2 - \Delta f + V)(\gamma(\tau)) \, \mathrm{d}\tau.$$

Reversing the transformation, this leads to the Harnack inequality

$$u(x,t) \geq u(y,s)\frac{\beta(s)}{\beta(t)}e^{f(x)-f(y)-\tilde{\omega}(x,y;t,s)}$$

satisfied by every positive solution u of the original equation (3.46). We summarise this finding in the following corollary.

Corollary 3.13. *Let (M,g) be a complete Riemannian manifold without boundary and nonnegative Ricci curvature. Let $f \in C^4(M)$, $V \in C^2(M)$, and let u be a positive solution of*

$$\partial_t u = \Delta u - 2\nabla f \cdot \nabla u - Vu \qquad \text{in } M \times (0,\infty). \tag{3.46}$$

If the conditions of Theorem 3.11 hold for the potential $\tilde{V} := |\nabla f|^2 - \Delta f + V$, then u satisfies

$$u(x,t) \geq u(y,s)\frac{\beta(s)}{\beta(t)}e^{f(x)-f(y)-\tilde{\omega}(x,y;t,s)} \tag{3.48}$$

for all $x, y \in M$ and $0 < s < t$, where

$$\tilde{\omega}(x,y;t,s) := \min_{\Gamma_{x,y}} \frac{1}{4(t-s)}\int_0^1 |\dot{\gamma}|^2 \, \mathrm{d}\tau + (t-s)\int_0^1 \tilde{V}(\gamma(\tau)) \, \mathrm{d}\tau.$$

We obtain another similar corollary for the Neumann problem on compact manifolds by applying Theorem 3.12.

Corollary 3.14. *Let (M,g) be a compact Riemannian manifold with nonnegative Ricci curvature and (possibly empty) boundary ∂M. Let $f \in C^4(M)$, $V \in C^2(M)$, and let u be a positive solution of*

$$\begin{cases} \partial_t u = \Delta u - 2\nabla f \cdot \nabla u - Vu & \text{in } M \times (0,\infty), \\ \nabla u \cdot \nu = (\nabla f \cdot \nu)u & \text{on } \partial M \times (0,\infty). \end{cases} \tag{3.49}$$

If the conditions of Theorem 3.12 hold for the potential $\tilde{V} := |\nabla f|^2 - \Delta f + V$, then u satisfies (3.48) *for all $x, y \in M$ and $0 < s < t$.*

Example 3.4 (Ornstein-Uhlenbeck with quadratic potential)**.** For $M = \mathbb{R}^d$, we consider when $V = C_1^2|x|^2$ and $f = -\frac{C_2}{2}|x|^2$ for $C_1, C_2 \in \mathbb{R}$. These choices correspond to the equation

$$\partial_t u = \Delta u + 2C_2 x \cdot \nabla u - C_1^2|x|^2 u,$$

which under the transformation $v = e^{-\frac{C_1}{4}|x|^2}$ becomes

$$\partial_t v = \Delta v - (C^2|x|^2 + dC_2)v$$

for C such that $C^2 = C_1^2 + C_2^2$. From Example 3.2, it follows that a positive solution v of this equation satisfies

$$v(x,t) \geq v(y,s)\left(\frac{\sinh(2Cs)}{\sinh(2Ct)}\right)^{d/2} e^{-\tilde{\omega}(x,y;t,s)}$$

for all $x, y \in \mathbb{R}^d$ and $0 < s < t$ with

$$\tilde{\omega}(x,y;t,s) = \frac{C}{2}\left(\frac{|x-y|^2}{\sinh(2C(t-s))} + (|x|^2+|y|^2)\tanh(C(t-s))\right) + dC_2(t-s).$$

Reversing the transformation, we have that u satisfies

$$u(x,t) \geq u(y,s) \left(\frac{\sinh(2Cs)}{\sinh(2Ct)}\right)^{d/2} e^{\frac{C_2}{2}(|x|^2-|y|^2)-\tilde{\omega}(x,y;t,s)}.$$

One may compare this inequality with the results obtained in [**122**].

3.1.5. Differential Harnack inequalities

Suppose $u : M \to [0,\infty)$ satisfies the Harnack inequality

$$u(x,t) \geq u(y,s)\frac{\beta(s)}{\beta(t)}e^{-\omega(x,y;t,s)}$$

for a given function $\beta : [0,\infty) \to [0,\infty)$ and ω defined as in Theorem 3.11. We note that after reparametrising the curve γ, the energy E equivalent to

$$E[\gamma;t,s] = \frac{1}{4}\int_s^t |\dot{\gamma}|^2 \, \mathrm{d}\tau + \int_s^t V(\gamma(\tau)) \, \mathrm{d}\tau.$$

Then, after making the change of variables $v = \log u$, we have that

$$v(x,t) - v(y,s) + \log(\beta(t)) - \log(\beta(s)) + \frac{1}{4}\int_s^t |\dot{\gamma}|^2 \, \mathrm{d}\tau + \int_s^t V(\gamma(\tau)) \, \mathrm{d}\tau \geq 0$$

for all $x, y \in M$, $0 < s < t$, where γ is a V-geodesic connecting x and y in M. For $h > 0$ small and $\vec{e} \in T_yM$ to be determined later, set $x = \exp_y(h\vec{e})$ and $t = s+h$. We write $\gamma_h : [s, s+h] \to M$ to represent the V-geodesic with $\gamma_h(s) = y$ and $\gamma_h(s+h) = x = \exp_y(h\vec{e})$. Then

$$\frac{v(\gamma_h(s+h), s+h) - v(y,s)}{h} + \frac{\log(\beta(s+h)) - \log(\beta(s))}{h} + \frac{1}{4h}\int_s^{s+h} |\dot{\gamma}_h|^2 \, \mathrm{d}\tau + \frac{1}{h}\int_s^{s+h} V(\gamma_h(\tau)) \, \mathrm{d}\tau \geq 0.$$

Passing to the limit as $h \to 0^+$ yields

$$\nabla v(y,s) \cdot \vec{e} + \partial_s v(y,s) + \frac{\beta'(s)}{\beta(s)} + \frac{1}{4}|\vec{e}|^2 + V(y) \geq 0.$$

Writing $\frac{1}{4}|\vec{e}|^2 + \nabla v(y,s) \cdot \vec{e} = |\nabla v(y,s) + \frac{1}{2}\vec{e}|^2 - |\nabla v(y,s)|^2$ and then choosing $\vec{e} = -2\nabla v(y,s)$, we have that

$$\partial_s v(y,s) + \frac{\beta'(s)}{\beta(s)} - |\nabla v(y,s)|^2 + V(y) \geq 0,$$

or equivalently if u solves (1.3),

$$\Delta(\log u) \geq -\frac{\beta'(s)}{\beta(s)}$$

in $M \times (0,\infty)$. Alternatively, if $u = e^{-f}\hat{u}$, where $\hat{u}$ solves (3.46), then

$$\Delta(\log \hat{u}) \geq \Delta f - \frac{\beta'(s)}{\beta(s)}$$

in $M \times (0,\infty)$.

3.2. The Harnack inequality for the doubly nonlinear heat equation

In this section, we are interested in studying the Harnack inequality satisfied by positive solutions of the doubly nonlinear heat equation

$$\partial_t u = \Delta_p(u^m) \qquad \text{in } M \times (0,\infty) \tag{1.4}$$

on a Riemannian manifold M with nonnegative Ricci curvature, where $m > 0$ and $p \geq 2$. Using the results in this section, one may deduce a Harnack inequality satisfied by positive solutions of the porous medium equation ($p = 2$) and p-heat equation ($m = 1$) in the degenerate range $p > 2$, as well as recover the results obtained in Section 3.1 for the heat equation ($m = 1$, $p = 2$).

3.2.1. Classical solutions in Euclidean space

We begin by discussing classical solutions of the doubly nonlinear heat equation

$$\partial_t u = \Delta_p(u^m) \qquad \text{in } \mathbb{R}^d \times (0,\infty) \tag{3.50}$$

in the Euclidean setting with $M = \mathbb{R}^d$. The main goal of this section is to prove the following theorem.

Theorem 3.15. *Let $m > 0$, $p \geq 2$ and set $\lambda := m - \frac{1}{p-1}$. Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with smooth boundary $\partial\Omega$. Let $\omega = \omega(x,y;t,s) : \overline{\Omega} \times \overline{\Omega} \times S \to \mathbb{R}$ be a continuous function, which is twice differentiable in the first and second arguments and differentiable in the third and fourth arguments. Suppose that $\omega(x,y;t,s) \to \infty$ whenever $t \to s^+$ for $x \neq y$ and $\lim_{t\to s^+} \omega(x,x;t,s) \geq 0$ for all $x \in \overline{\Omega}$. In addition, suppose there exists a strictly increasing, differentiable function $\beta : [0,\infty) \to [0,\infty)$ with $\beta(0) = 0$ and a function $A : [0,\infty) \to [0,\infty)$ such that $A^2\beta^\lambda$ is increasing and ω satisfies*

$$\partial_t \omega + \beta^{-\lambda(p-1)}(t)|\nabla_x \omega|^p \geq 0 \tag{3.51}$$

$$\partial_s \omega - \beta^{-\lambda(p-1)}(s)|\nabla_y \omega|^p \geq 0 \tag{3.52}$$

$$\begin{aligned} A^2(t)\beta^{-\lambda(p-2)}(t)\Delta_{p,x}\omega &+ A^2(s)\beta^{-\lambda(p-2)}(s)\Delta_{p,y}\omega \\ &+2A(t)A(s)\beta^{-\frac{\lambda(p-2)}{2}}(t)\beta^{-\frac{\lambda(p-2)}{2}}(s)\Delta_{p,xy}\omega \\ &\qquad \leq A^2(t)\beta^{\lambda-1}(t)\beta'(t) - A^2(s)\beta^{\lambda-1}(s)\beta'(s) \end{aligned} \tag{3.53}$$

in $\overline{\Omega} \times \overline{\Omega} \times S$, where

$$\Delta_{p,xy}\omega := |\nabla_x\omega|^{\frac{p-4}{2}}|\nabla_y\omega|^{\frac{p-4}{2}} \times \left(|\nabla_x\omega||\nabla_y\omega| \sum_{i=1}^d \frac{\partial^2\omega}{\partial x_i \partial y_i} - (p-2)\sum_{i,j=1}^d \frac{\partial\omega}{\partial x_i}\frac{\partial^2\omega}{\partial x_i \partial y_j}\frac{\partial\omega}{\partial y_j} \right).$$

Finally, assume that

$$\begin{aligned} \nabla_x\omega \cdot \nu(x) \geq 0, \qquad &\text{for all } x \in \partial\Omega,\ y \in \overline{\Omega}, \text{ and } 0 < s < t, \\ \nabla_y\omega \cdot \nu(y) \geq 0, \qquad &\text{for all } x \in \overline{\Omega},\ y \in \partial\Omega, \text{ and } 0 < s < t. \end{aligned} \tag{3.54}$$

Then every positive solution u of

$$\begin{cases} \partial_t u = \Delta_p(u^m) & \text{in } \Omega \times (0,\infty) \\ |\nabla(u^m)|^{p-2}\nabla(u^m)\cdot\nu = 0 & \text{on } \partial\Omega \times (0,\infty). \end{cases} \tag{3.55}$$

satisfies

$$\psi(\beta(t)u(x,t)) \geq \psi(\beta(s)u(y,s)) - \omega(x,y;t,s) \tag{3.56}$$

for all $x, y \in \overline{\Omega}$ *and* $0 < s < t$, *where*

$$\psi(u) := \begin{cases} \dfrac{m}{\lambda}u^\lambda & \text{if } \lambda \neq 0, \\ \dfrac{1}{p-1}\log u & \text{if } \lambda = 0. \end{cases} \tag{3.57}$$

Assuming that one can approximate positive solutions u of the doubly nonlinear heat equation (3.50) by solutions of the corresponding Neumann problem on bounded domains $\Omega \subseteq \mathbb{R}^d$, one could apply the idea of the proof of Theorem 3.1 to obtain the following result.

Theorem 3.16. *Let* $m > 0$, $p \geq 2$ *and set* $\lambda := m - \frac{1}{p-1}$. *Suppose there exists a continuous function* $\omega = \omega(x,y;t,s) : \mathbb{R}^d \times \mathbb{R}^d \times S \to \mathbb{R}$, *which is twice differentiable in the first and second arguments and differentiable in the third and fourth arguments. Suppose* $\omega(x,y;t,s) \to \infty$ *whenever* $t \to s^+$ *for* $x \neq y$ *and* $\lim_{t\to s}\omega(x,y;t,s) \geq 0$ *for all* $x \in \mathbb{R}^d$. *In addition, assume that* ω *satisfies* (3.51)–(3.53) *in* $\mathbb{R}^d \times \mathbb{R}^d \times S$. *Finally, assume there exists an increasing sequence* $(\Omega_n)_{n\geq 1}$ *of smooth, bounded domains* $\Omega_n \subset \mathbb{R}^d$ *such that* $\bigcup_{n\geq 1}\Omega_n = \mathbb{R}^d$ *and*

$$\begin{aligned} \nabla_x\omega\cdot\nu(x) \geq 0, \qquad & \text{for all } x \in \partial\Omega_n,\ y \in \overline{\Omega_n}, \text{ and } 0 < s < t, \\ \nabla_y\omega\cdot\nu(y) \geq 0, \qquad & \text{for all } x \in \overline{\Omega_n},\ y \in \partial\Omega_n \text{ and } 0 < s < t \end{aligned} \tag{3.58}$$

hold for all $n \geq 1$. *Then every positive* $C^{2,1}$*-solution u of*

$$\partial_t u = \Delta_p(u^m) \qquad \text{in } \mathbb{R}^d \times (0,\infty) \tag{3.50}$$

satisfies the Harnack inequality (3.56) *for all* $x, y \in \mathbb{R}^d$ *and* $0 < s < t$.

Our main task is therefore to prove Theorem 3.15. The proof will be broken into several cases depending on the sign of λ. Our strategy in each case closely mirrors the approach utilised in Section 3.1 for the heat equation, which ultimately amounts to demonstrating that a suitably chosen function Z remains nonnegative on $\overline{\Omega}\times\overline{\Omega}\times S$. We again employ a change of variables $v = \psi(u)$, with the function ψ in (3.57) being chosen as in the work of Esteban and Vázquez [**60, 61**]. While the core of our argument to prove Theorem 3.15 is largely unchanged when compared with the proof of Theorem 3.2 in the linear case, the nonlinear structure of the p-Laplace operator poses some potential technical difficulties, which we overcome by working with an appropriate basis of $\mathbb{R}^d$.

Proof of Theorem 3.15. We start with the case $\lambda > 0$.

Let u be a positive $C^{2,1}$-solution of (3.55) and set $v = \frac{m}{\lambda}u^\lambda$. Then v is a solution to

$$\begin{cases} \partial_t v = \lambda v \Delta_p v + |\nabla v|^p & \text{in } \Omega \times (0,\infty) \\ |\nabla v|^{p-2}\nabla v \cdot \nu = 0 & \text{on } \partial\Omega \times (0,\infty). \end{cases} \tag{3.59}$$

Fix $\varepsilon > 0$ and define a function Z on $\overline{\Omega} \times \overline{\Omega} \times S$ by

$$Z(x,y;t,s) = \beta^\lambda(t)v(x,t) - \beta^\lambda(s)v(y,s) + \omega(x,y;t,s) + \frac{\varepsilon}{2}(t-s)^2 + \varepsilon.$$

Firstly, observe that as $s \to 0^+$, the term including $\beta^\lambda(s)$ vanishes due to the assumption that $\beta(0) = 0$. Since the other terms in Z are nonnegative, we clearly have $Z > 0$ for s near 0. Next, if $t \to s^+$ and $x \neq y$, the assumption that $\omega \to \infty$ as $t \to s^+$ guarantees that $Z > 0$ for t near s. Otherwise, if $x = y$, then the difference $\beta^\lambda(t)v(x,t) - \beta^\lambda(s)v(x,s) \to 0$, so we can assume $\beta^\lambda(t)v(x,t) - \beta^\lambda(s)v(x,s) + \varepsilon > 0$. Again, the other terms in Z are nonnegative, so $Z > 0$ for t near s. We see that $Z > 0$ for all times (t,s) sufficiently close to $(0,0)$. Then, Z either remains strictly positive for all times, or there exists a point $P_0 := (x_0, y_0; t_0, s_0) \in \overline{\Omega} \times \overline{\Omega} \times S$ such that Z touches 0 for the first time at P_0 in the sense of Definition 3.1. Suppose the latter is true and such a point P_0 exists. First, assume that $x_0, y_0 \in \Omega$. Then, it follows that

$$\partial_t Z(P_0) \leq 0, \quad \partial_s Z(P_0) \leq 0, \quad \nabla_{(x,y)} Z(P_0) = 0, \quad D^2_{(x,y)} Z(P_0) \geq 0.$$

Firstly, the condition $\nabla_{(x,y)} Z(P_0) = 0$ implies

$$\begin{aligned} 0 &= \nabla_x Z(P_0) = \beta^\lambda(t_0)\nabla v(x_0) + \nabla_x \omega(P_0) \\ 0 &= \nabla_y Z(P_0) = -\beta^\lambda(s_0)\nabla v(x_0) + \nabla_y \omega(P_0) \end{aligned}$$

and therefore

$$\nabla v(x_0, t_0) = -\beta^{-\lambda}(t_0)\nabla_x \omega(P_0), \tag{3.60}$$

$$\nabla v(y_0, s_0) = \beta^{-\lambda}(s_0)\nabla_y \omega(P_0). \tag{3.61}$$

Next, the condition $\partial_t Z(P_0) \leq 0$ implies

$$0 \geq \lambda\beta^{\lambda-1}(t_0)\beta'(t_0)v(x_0,t_0) + \beta^\lambda(t_0)\partial_t v(x_0,t_0) + \partial_t \omega(P_0) + \varepsilon(t_0 - s_0).$$

Using the equation (3.59) as well as (3.60), it follows that

$$\begin{aligned} 0 &\geq \lambda\beta^{\lambda-1}(t_0)\beta'(t_0)v(x_0,t_0) + \lambda\beta^\lambda(t_0)v(x_0,t_0)\Delta_p v(x_0,t_0) \\ &\quad + \beta^\lambda(t_0)|\nabla v(x_0,t_0)|^p + \partial_t \omega(P_0) + \varepsilon(t_0 - s_0) \\ &= \lambda\beta^{\lambda-1}(t_0)\beta'(t_0)v(x_0,t_0) + \lambda\beta^\lambda(t_0)v(x_0,t_0)\Delta_p v(x_0,t_0) \\ &\quad + \beta^{-\lambda(p-1)}(t_0)|\nabla_x \omega(P_0)|^p + \partial_t \omega(P_0) + \varepsilon(t_0 - s_0). \end{aligned}$$

However, due to the assumption (3.51) on ω, we are left with

$$0 \geq \lambda\beta^{\lambda-1}(t_0)\beta'(t_0)v(x_0,t_0) + \lambda\beta^\lambda(t_0)v(x_0,t_0)\Delta_p v(x_0,t_0) + \varepsilon(t_0 - s_0),$$

which we rearrange to

$$\beta^\lambda(t_0)\Delta_p v(x_0,t_0) \leq -\beta^{\lambda-1}(t_0)\beta'(t_0) - \frac{\varepsilon}{\lambda}\frac{t_0 - s_0}{v(x_0,t_0)}. \tag{3.62}$$

By a similar argument using $\partial_s Z(P_0) \le 0$, (3.61) and (3.52), we derive that

$$-\beta^{\lambda}(s_0)\Delta_p v(y_0, s_0) \le \beta^{\lambda-1}(s_0)\beta'(s_0) + \frac{\varepsilon}{\lambda}\frac{t_0 - s_0}{v(y_0, s_0)}. \tag{3.63}$$

For the next step of the proof, it is most convenient to work with respect to a particular basis of $\mathbb{R}^d$. If $p = 2$, it is sufficient to work using the standard basis. However, if $p > 2$, the p-Laplace operator contains mixed derivatives, which we would like to avoid in our calculation. We distinguish between the cases, where either or both of $\nabla v(x_0, t_0)$ and $\nabla v(y_0, s_0)$ are zero. We can easily rule out the case $\nabla v(x_0, t_0) = 0$, since when $p > 2$, this would imply $\Delta_p v(x_0, t_0) = 0$. However, since β is assumed to be a strictly increasing nonnegative function and so $\beta'(t_0) > 0$, inequality (3.62) implies $\Delta_p v(x_0, t_0) < 0$, which is a contradiction. However, the case that $\nabla v(y_0, s_0) \ne 0$ is not so immediately ruled out. Therefore, we make two cases.

For the moment, we proceed assuming that both $\nabla v(x_0, t_0)$ and $\nabla v(y_0, s_0)$ are nonzero. Then $\nabla_x \omega(P_0)$ and $\nabla_y \omega(P_0)$ are also nonzero because of (3.60) and (3.61). This permits us to define the following two orthonormal bases of $\mathbb{R}^d$, firstly $B^{(x)} = (e_1^{(x)}, e_2^{(x)}, \ldots, e_d^{(x)})$ with $e_1^{(x)} = \frac{1}{|\nabla_x \omega(P_0)|}\nabla_x \omega(P_0)$ as well as $(e_1^{(y)}, e_2^{(y)}, \ldots, e_d^{(y)})$ with $B^{(y)} = e_1^{(y)} = -\frac{1}{|\nabla_y \omega(P_0)|}\nabla_y \omega(P_0)$. In the steps that follow, we will use $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial y_i}$ to denote derivatives in the $e_i^{(x)}$ and $e_i^{(y)}$ directions respectively.

Suppose $x(\tau)$ and $y(\tau)$ are two paths in Ω such that

$$\begin{aligned}
x'(\tau) &= A(t_0)\beta^{-\frac{\lambda(p-2)}{2}}(t_0)|\nabla_x \omega(P_0)|^{\frac{p-2}{2}} e_i^{(x)}, \quad x(\tau_0) = x_0, \\
y'(\tau) &= A(s_0)\beta^{-\frac{\lambda(p-2)}{2}}(s_0)|\nabla_y \omega(P_0)|^{\frac{p-2}{2}} e_i^{(y)}, \quad y(\tau_0) = y_0
\end{aligned}$$

for $1 \le i \le d$ and for some τ_0. Then

$$\begin{aligned}
&\frac{d}{d\tau} Z(x(\tau), y(\tau); t, s) \\
&\quad = A(t_0)\beta^{-\frac{\lambda(p-2)}{2}}(t_0)|\nabla_x \omega(P_0)|^{\frac{p-2}{2}} \beta^{\lambda}(t)\frac{\partial v}{\partial x_i}(x, t) \\
&\qquad - A(s_0)\beta^{-\frac{\lambda(p-2)}{2}}(s_0)|\nabla_y \omega(P_0)|^{\frac{p-2}{2}} \beta^{\lambda}(s)\frac{\partial v}{\partial y_i}(y, s) \\
&\qquad + A(t_0)\beta^{-\frac{\lambda(p-2)}{2}}(t_0)|\nabla_x \omega(P_0)|^{\frac{p-2}{2}} \frac{\partial \omega}{\partial x_i}(x, y; t, s) \\
&\qquad + A(s_0)\beta^{-\frac{\lambda(p-2)}{2}}(s_0)|\nabla_y \omega(P_0)|^{\frac{p-2}{2}} \frac{\partial \omega}{\partial y_i}(x, y; t, s)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{d^2}{d\tau^2} Z(x(\tau), y(\tau); t, s) \\
&\quad = A^2(t_0)\beta^{-\lambda(p-2)}(t_0)|\nabla_x \omega(P_0)|^{p-2}\beta^{\lambda}(t)\frac{\partial^2 v}{\partial x_i^2}(x,t) \\
&\qquad - A^2(s_0)\beta^{-\lambda(p-2)}(s_0)|\nabla_y \omega(P_0)|^{p-2}\beta^{\lambda}(s)\frac{\partial^2 v}{\partial y_i^2}(y,s) \\
&\qquad + A^2(t_0)\beta^{-\lambda(p-2)}(t_0)|\nabla_x \omega(P_0)|^{p-2}\frac{\partial^2 \omega}{\partial x_i^2}(x,y;t,s) \\
&\qquad + A^2(s_0)\beta^{-\lambda(p-2)}(s_0)|\nabla_y \omega(P_0)|^{p-2}\frac{\partial^2 \omega}{\partial y_i^2}(x,y;t,s) \\
&\qquad + 2A(t_0)A(s_0)\beta^{-\frac{\lambda(p-2)}{2}}(t_0)\beta^{-\frac{\lambda(p-2)}{2}}(s_0)\times \\
&\qquad\qquad |\nabla_x \omega(P_0)|^{\frac{p-2}{2}}|\nabla_y \omega(P_0)|^{\frac{p-2}{2}}\frac{\partial^2 \omega}{\partial x_i \partial y_i}(x,y;t,s).
\end{aligned}$$

Evaluating at $\tau = \tau_0$ and using that $D^2_{(x,y)}Z(P_0) \geq 0$ implies

$$\frac{d^2}{d\tau^2} Z(x(\tau), y(\tau); t, s)\bigg|_{\tau=\tau_0} \geq 0,$$

and so we have

$$\begin{aligned}
0 \leq\ & A^2(t_0)\beta^{-\lambda(p-3)}(t_0)|\nabla_x \omega(P_0)|^{p-2}\frac{\partial^2 v}{\partial x_i^2}(x_0,t_0) \\
&- A^2(s_0)\beta^{-\lambda(p-3)}(s_0)|\nabla_y \omega(P_0)|^{p-2}\frac{\partial^2 v}{\partial y_i^2}(y_0,s_0) \\
&+ A^2(t_0)\beta^{-\lambda(p-2)}(t_0)|\nabla_x \omega(P_0)|^{p-2}\frac{\partial^2 \omega}{\partial x_i^2}(P_0) \\
&+ A^2(s_0)\beta^{-\lambda(p-2)}(s_0)|\nabla_y \omega(P_0)|^{p-2}\frac{\partial^2 \omega}{\partial y_i^2}(P_0) \\
&+ 2A(t_0)A(s_0)\beta^{-\frac{\lambda(p-2)}{2}}(t_0)\beta^{-\frac{\lambda(p-2)}{2}}(s_0)\times \\
&\qquad |\nabla_x \omega(P_0)|^{\frac{p-2}{2}}|\nabla_y \omega(P_0)|^{\frac{p-2}{2}}\frac{\partial^2 \omega}{\partial x_i \partial y_i}(P_0).
\end{aligned}$$

Summing over i and then adding a further $(p-2)$ multiples of the inequality for $i=1$ yields

$$\begin{aligned}0 \leq\, & A^2(t_0)\beta^{-\lambda(p-3)}(t_0)|\nabla_x\omega(P_0)|^{p-2}\left(\Delta v(x_0,t_0)+(p-2)\frac{\partial^2 v}{\partial x_1^2}(x_0,t_0)\right)\\ & - A^2(s_0)\beta^{-\lambda(p-3)}(s_0)|\nabla_y\omega(P_0)|^{p-2}\left(\Delta v(y_0,s_0)+(p-2)\frac{\partial^2 v}{\partial y_1^2}(y_0,s_0)\right)\\ & + A^2(t_0)\beta^{-\lambda(p-2)}(t_0)|\nabla_x\omega(P_0)|^{p-2}\left(\Delta_x\omega(P_0)+(p-2)\frac{\partial^2 \omega}{\partial x_1^2}(P_0)\right)\\ & + A^2(s_0)\beta^{-\lambda(p-2)}(s_0)|\nabla_y\omega(P_0)|^{p-2}\left(\Delta_y\omega(P_0)+(p-2)\frac{\partial^2 \omega}{\partial y_1^2}(P_0)\right)\\ & + 2A(t_0)A(s_0)\beta^{-\frac{\lambda(p-2)}{2}}(t_0)\beta^{-\frac{\lambda(p-2)}{2}}(s_0)|\nabla_x\omega(P_0)|^{\frac{p-2}{2}}|\nabla_y\omega(P_0)|^{\frac{p-2}{2}}\\ & \times\left((p-2)\frac{\partial^2\omega}{\partial x_1\partial y_1}(P_0)+\sum_{i=1}^{d}\frac{\partial^2\omega}{\partial x_i\partial y_i}(P_0)\right).\end{aligned} \tag{3.64}$$

Note that when $p\neq 2$, (3.60) and (3.61) imply

$$\begin{aligned}\nabla v(x_0,t_0) &= -\beta^{-\lambda}(t_0)|\nabla_x\omega(P_0)|e_1^{(x)}\\ \nabla v(y_0,s_0) &= \beta^{-\lambda}(s_0)|\nabla_y\omega(P_0)|e_1^{(y)}\end{aligned}$$

and recall that in non-divergence form, the p-Laplace operator Δ_p may be expressed as

$$\Delta_p v = |\nabla v|^{p-4}\left(|\nabla v|^2\Delta v+(p-2)\nabla v^\top D^2 v\nabla v\right).$$

It follows that

$$\begin{aligned}\Delta_p v(x_0,t_0) &= \beta^{-\lambda(p-2)}(t_0)|\nabla_x\omega(P_0)|^{p-2}\left(\Delta v(x_0,t_0)+(p-2)\frac{\partial^2 v}{\partial x_1^2}(x_0,t_0)\right),\\ \Delta_p v(y_0,s_0) &= \beta^{-\lambda(p-2)}(s_0)|\nabla_y\omega(P_0)|^{p-2}\left(\Delta v(y_0,s_0)+(p-2)\frac{\partial^2 v}{\partial y_1^2}(y_0,s_0)\right),\end{aligned}$$

Therefore, the inequality (3.64) simplifies to

$$\begin{aligned}0 \leq\, & A^2(t_0)\beta^{\lambda}(t_0)\Delta_p v(x_0,t_0) - A^2(s_0)\beta^{\lambda}(s_0)\Delta_p v(y_0,s_0)\\ & + A^2(t_0)\beta^{-\lambda(p-2)}(t_0)\Delta_{p,x}\omega(P_0) + A^2(s_0)\beta^{-\lambda(p-2)}(s_0)\Delta_{p,y}\omega(P_0)\\ & + 2A(t_0)A(s_0)\beta^{-\frac{\lambda(p-2)}{2}}(t_0)\beta^{-\frac{\lambda(p-2)}{2}}(s_0)|\nabla_x\omega(P_0)|^{\frac{p-2}{2}}|\nabla_y\omega(P_0)|^{\frac{p-2}{2}}\\ & \times\left((p-2)\frac{\partial^2\omega}{\partial x_1\partial y_1}(P_0)+\sum_{i=1}^{d}\frac{\partial^2\omega}{\partial x_i\partial y_i}(P_0)\right).\end{aligned}$$

This inequality remains true in the case $p = 2$. We may now insert the inequalities (3.62) and (3.63), which produces

$$\begin{aligned}
0 \leq & \frac{\varepsilon}{\lambda}(t_0 - s_0)\left(\frac{A^2(s_0)}{v(y_0, s_0)} - \frac{A^2(t_0)}{v(x_0, t_0)}\right) \\
& + A^2(s_0)\beta^{\lambda-1}(s_0)\beta'(s_0) - A^2(t_0)\beta^{\lambda-1}(t_0)\beta'(t_0) \\
& + A^2(t_0)\beta^{-\lambda(p-2)}(t_0)\Delta_{p,x}\omega(P_0) + A^2(s_0)\beta^{-\lambda(p-2)}(s_0)\Delta_{p,y}\omega(P_0) \\
& + 2A(t_0)A(s_0)\beta^{-\frac{\lambda(p-2)}{2}}(t_0)\beta^{-\frac{\lambda(p-2)}{2}}(s_0)|\nabla_x\omega(P_0)|^{\frac{p-2}{2}}|\nabla_y\omega(P_0)|^{\frac{p-2}{2}} \\
& \quad \times \left((p-2)\frac{\partial^2\omega}{\partial x_1 \partial y_1}(P_0) + \sum_{i=1}^{d}\frac{\partial^2\omega}{\partial x_i \partial y_i}(P_0)\right).
\end{aligned}$$

However, after rewriting the derivatives in the above inequality with respect to the standard basis, the assumed inequality (3.53) implies

$$0 \leq \frac{\varepsilon}{\lambda}(t_0 - s_0)\left(\frac{A^2(s_0)}{v(y_0, s_0)} - \frac{A^2(t_0)}{v(x_0, t_0)}\right).$$

We claim that the expression of the right-hand side is in fact strictly negative, and we have thus reached a contradiction. Since $Z(P_0) = 0$, it follows that

$$\beta^\lambda(t_0)v(x_0, t_0) - \beta^\lambda(s_0)v(y_0, s_0) = -\omega(P_0) - \frac{\varepsilon}{2}(t_0 - s_0) - \varepsilon < 0.$$

Therefore

$$\frac{1}{\beta^\lambda(s_0)v(y_0, s_0)} - \frac{1}{\beta^\lambda(t_0)v(x_0, t_0)} < 0.$$

Since $A^2\beta^\lambda$ is assumed to be increasing, $A^2(s_0)\beta^\lambda(s_0) < A^2(t_0)\beta^\lambda(t_0)$ and so

$$\frac{A^2(s_0)}{v(y_0, s_0)} - \frac{A^2(t_0)}{v(x_0, t_0)} = \frac{A^2(s_0)\beta^\lambda(s_0)}{\beta^\lambda(s_0)v(y_0, s_0)} - \frac{A^2(t_0)\beta^\lambda(t_0)}{\beta^\lambda(t_0)v(x_0, t_0)} < 0,$$

which leads to a contradiction.

Returning to the case that $p > 2$ and $\nabla v(y_0, s_0) = 0$, because of (3.61), this would imply that $\nabla_y\omega(P_0)$, $\Delta_{p,y}\omega(P_0)$, and $\Delta_{p,xy}\omega(P_0)$ all vanish as well. This means that (3.53) reduces to

$$\begin{aligned}
& A^2(t_0)\beta^{-\lambda(p-2)}(t_0)\Delta_{p,x}\omega(P_0) \\
& \qquad \leq A^2(t_0)\beta^{\lambda-1}(t_0)\beta'(t_0) - A^2(s_0)\beta^{\lambda-1}(s_0)\beta'(s_0).
\end{aligned} \tag{3.65}$$

We may repeat the arguments from when $\nabla v(y_0, s_0) \neq 0$, except we only use the basis $B^{(x)}$ and analyse $\frac{d^2}{d\tau^2}Z(x(\tau), y_0; t_0, s_0)|_{\tau=\tau_0}$. This leads to

$$0 \leq A^2(t_0)\beta^\lambda(t_0)\Delta_p v(x_0, t_0) + A^2(t_0)\beta^{-\lambda(p-2)}(t_0)\Delta_{p,x}\omega(P_0).$$

Inserting (3.65) and (3.62) yields

$$0 \leq -A^2(s_0)\beta^{\lambda-1}(s_0)\beta'(s_0) - \frac{\varepsilon}{\lambda}\frac{t_0 - s_0}{v(x_0, t_0)} < 0,$$

so again we have a contradiction. Therefore, the assumption that $x_0, y_0 \in \Omega$ is false and at least one of these points must occur on the boundary $\partial\Omega$. The argument to finish the proof from here follows identically as in the linear case.

We now consider the case $\lambda = m - \frac{1}{p-1} < 0$. We again define $v := \frac{m}{\lambda} u^\lambda$, bearing in mind that v is now a negative function. Fixing $\varepsilon > 0$, we define

$$\begin{aligned} Z(x, y; t, s) &:= \beta^\lambda(t) v(x,t) - \beta^\lambda(s) v(y,s) + \omega(x,y;t,s) \\ &\quad + \frac{\varepsilon}{2}\left(\int_s^t \frac{1}{A^2(\tau)\beta^\lambda(\tau)}\,\mathrm{d}\tau\right)^{-2} + \varepsilon. \end{aligned}$$

We first note that as $s \to 0^+$ one has $-\beta^\lambda(s)v(y,s) \to \infty$ since $\beta(0) = 0$, $\lambda < 0$ and $-v(y,s) > 0$. Therefore $Z > 0$ for $s > 0$ small. In addition, when $t \to s^+$, the integral term in Z becomes unbounded, so $Z > 0$ for t sufficiently close to s. In particular, Z will be strictly positive for $0 < s < t$ in a small enough neighbourhood of $(0, 0)$.

As in the case $\lambda > 0$, we suppose Z vanishes for the first time at some point $P_0 := (x_0, y_0; t_0, s_0)$. Proceeding as in our earlier proof and first assuming x_0, y_0 are both in Ω, we find that

$$\nabla v(x_0, t_0) = -\beta^{-\lambda}(t_0)\nabla_x \omega(P_0) \tag{3.60}$$

$$\nabla v(y_0, s_0) = \beta^{-\lambda}(s_0)\nabla_y \omega(P_0) \tag{3.61}$$

remain true. However, due to the change in the definition of Z, inequalities (3.62) and (3.63) are replaced by

$$\begin{aligned} \beta^\lambda(t_0)\Delta_p v(x_0, t_0) &\le -\beta^{\lambda-1}(t_0)\beta'(t_0) \\ &\quad + \varepsilon\left(\int_s^t \frac{1}{A^2(\tau)\beta^\lambda(\tau)}\,\mathrm{d}\tau\right)^{-3} \frac{1}{\lambda A^2(t_0)\beta^\lambda(t_0)v(x_0,t_0)} \end{aligned}$$

and

$$\begin{aligned} -\beta^\lambda(s_0)\Delta_p v(y_0, s_0) &\le \beta^{\lambda-1}(s_0)\beta'(s_0) \\ &\quad - \varepsilon\left(\int_s^t \frac{1}{A^2(\tau)\beta^\lambda(\tau)}\,\mathrm{d}\tau\right)^{-3} \frac{1}{\lambda A^2(s_0)\beta^\lambda(s_0)v(y_0,s_0)} \end{aligned}$$

respectively. Following the approach of the proof when $\lambda > 0$, we reach

$$0 \le \varepsilon\left(\int_s^t \frac{1}{A^2(\tau)\beta^\lambda(\tau)}\,\mathrm{d}\tau\right)^{-3}\left(\frac{1}{\lambda\beta^\lambda(t_0)v(x_0,t_0)} - \frac{1}{\lambda\beta^\lambda(s_0)v(y_0,s_0)}\right). \tag{3.66}$$

As before, since $Z(P_0) = 0$, one has

$$\beta^\lambda(t_0)v(x_0,t_0) - \beta^\lambda(s_0)v(y_0,s_0) = -\omega(P_0) - \frac{\varepsilon}{2}\left(\int_s^t \frac{1}{A^2(\tau)\beta^\lambda(\tau)}\,\mathrm{d}\tau\right)^{-2} - \varepsilon < 0.$$

Therefore, since $\lambda < 0$, this implies

$$\lambda\beta^\lambda(t_0)v(x_0,t_0) > \lambda\beta^\lambda(s_0)v(y_0,s_0).$$

Since the quantities on both sides of the inequality are positive, this yields

$$\frac{1}{\lambda\beta^\lambda(t_0)v(x_0,t_0)} < \frac{1}{\lambda\beta^\lambda(s_0)v(y_0,s_0)}.$$

As the integral in (3.66) is positive, we have reached

$$0 \le \varepsilon\left(\int_s^t \frac{1}{A^2(\tau)\beta^\lambda(\tau)}\,\mathrm{d}\tau\right)^{-3}\left(\frac{1}{\lambda\beta^\lambda(t_0)v(x_0,t_0)} - \frac{1}{\lambda\beta^\lambda(s_0)v(y_0,s_0)}\right) < 0,$$

which is a contradiction. By the same argument as earlier, we can rule out the possibility that either x_0 or y_0 occurs on the boundary $\partial\Omega$. Therefore, the proof in this case is complete.

The final case to be treated is when $\lambda = 0$, that is $m = \frac{1}{p-1}$. The case $p = 2$ corresponds to the heat equation, which was already covered in the previous section. Therefore, we assume $p > 2$.

Let u be a $C^{2,1}$-solution of

$$\begin{cases} \partial_t u = \Delta_p(u^{\frac{1}{p-1}}) & \text{in } \Omega \times (0,\infty) \\ |\nabla(u^{\frac{1}{p-1}})|^{p-2}\nabla(u^{\frac{1}{p-1}}) \cdot \nu = 0 & \text{on } \partial\Omega \times (0,\infty). \end{cases} \tag{3.67}$$

on a smooth domain $\Omega \subseteq \mathbb{R}^d$ and set $v = \frac{1}{p-1}\log u$. Then v solves

$$\begin{cases} \partial_t v = \frac{1}{p-1}\Delta_p v + |\nabla v|^p & \text{in } \Omega \times (0,\infty) \\ |\nabla v|^{p-2}\nabla v \cdot \nu = 0 & \text{on } \partial\Omega \times (0,\infty). \end{cases} \tag{3.68}$$

Fix $\varepsilon > 0$ and define a function Z on $\overline{\Omega} \times \overline{\Omega} \times S$ by

$$Z(x,y;t,s) = v(x,t) - v(y,s) + \frac{1}{p-1}\log\left(\frac{\beta(t)}{\beta(s)}\right) + \omega(x,y;t,s) + \frac{\varepsilon}{2}(t-s)^2 + \varepsilon.$$

Firstly, observe that as $s \to 0^+$, the term $\frac{1}{p-1}\log\left(\frac{\beta(t)}{\beta(s)}\right) \to \infty$ due to the assumption that $\beta(0) = 0$. Therefore, $Z > 0$ for s near 0. Next, if $t \to s^+$ and $x \neq y$, the assumption that $\omega \to \infty$ as $t \to s^+$ guarantees that $Z > 0$ for t near s. Otherwise, if $x = y$, then the difference $v(x,t) - v(x,s) \to 0$, so we can assume $v(x,t) - v(x,s) + \varepsilon > 0$. Again, the other terms in Z are nonnegative, so $Z > 0$ for t near s.

Suppose Z touches 0 for the first time at a point $P_0 := (x_0, y_0; t_0, s_0) \in \overline{\Omega} \times \overline{\Omega} \times S$ and assume for the moment that $x_0, y_0 \in \Omega$. Then, it follows that

$$\partial_t Z(P_0) \leq 0, \quad \partial_s Z(P_0) \leq 0, \quad \nabla_{(x,y)} Z(P_0) = 0, \quad D^2_{(x,y)} Z(P_0) \geq 0.$$

In the analysis that follows, it is useful to note that when $\lambda = m - \frac{1}{p-1} = 0$, the three inequalities (3.51)–(3.53) satisfied by ω simplify to

$$\partial_t \omega + |\nabla_x \omega|^p \geq 0, \tag{3.69}$$

$$\partial_s \omega - |\nabla_y \omega|^p \geq 0, \tag{3.70}$$

$$A^2(t)\Delta_{p,x}\omega + A^2(s)\Delta_{p,y}\omega + 2A(t)A(s)\Delta_{p,xy}\omega \leq A^2(t)\frac{\beta'(t)}{\beta(t)} - A^2(s)\frac{\beta'(s)}{\beta(s)}. \tag{3.71}$$

Firstly, the condition $\nabla_{(x,y)} Z(P_0) = 0$ implies

$$0 = \nabla_x Z(P_0) = \nabla v(x_0) + \nabla_x \omega(P_0)$$
$$0 = \nabla_y Z(P_0) = -\nabla v(x_0) + \nabla_y \omega(P_0)$$

and therefore

$$\nabla v(x_0, t_0) = -\nabla_x \omega(P_0) \tag{3.72}$$

$$\nabla v(y_0, s_0) = \nabla_y \omega(P_0). \tag{3.73}$$

Next, the condition $\partial_t Z(P_0) \leq 0$ implies

$$0 \geq \partial_t v(x_0, t_0) + \frac{1}{p-1}\frac{\beta'(t_0)}{\beta(t_0)} + \partial_t \omega(P_0) + \varepsilon(t_0 - s_0).$$

Using the equation (3.68) as well as (3.72), it follows that

$$\begin{aligned} 0 &\geq \frac{1}{p-1}\Delta_p v(x_0, t_0) + |\nabla v(x_0, t_0)|^p + \frac{1}{p-1}\frac{\beta'(t_0)}{\beta(t_0)} + \partial_t \omega(P_0) + \varepsilon(t_0 - s_0) \\ &= \frac{1}{p-1}\Delta_p v(x_0, t_0) + |\nabla_x \omega(P_0)|^p + \frac{1}{p-1}\frac{\beta'(t_0)}{\beta(t_0)} + \partial_t \omega(P_0) + \varepsilon(t_0 - s_0). \end{aligned}$$

However, due to the assumption (3.69) on ω, this reduces to

$$0 \geq \frac{1}{p-1}\Delta_p v(x_0, t_0) + |\nabla_x \omega(P_0)|^p + \frac{1}{p-1}\frac{\beta'(t_0)}{\beta(t_0)} + \partial_t \omega(P_0) + \varepsilon(t_0 - s_0),$$

which we rearrange to

$$\Delta_p v(x_0, t_0) \leq -\frac{\beta'(t_0)}{\beta(t_0)} - \varepsilon(p-1)(t_0 - s_0). \tag{3.74}$$

By a similar argument using $\partial_s Z(P_0) \leq 0$, (3.72) and (3.70), we derive that

$$-\Delta_p v(y_0, s_0) \leq \frac{\beta'(s_0)}{\beta(s_0)} + \varepsilon(p-1)(t_0 - s_0). \tag{3.75}$$

As in the $\lambda > 0$ case, we can rule out the possibility that $\nabla v(x_0, t_0) = 0$. The case $\nabla v(y_0, s_0) = 0$ can also be excluded by a similar argument. Therefore, we assume that both $\nabla v(x_0, t_0)$ and $\nabla v(y_0, s_0)$ are nonzero. Then $\nabla_x \omega(P_0)$ and $\nabla_y \omega(P_0)$ are also nonzero because of (3.72) and (3.73). This permits us to define two orthonormal bases $B^{(x)} = (e_1^{(x)}, e_2^{(x)}, \ldots, e_d^{(x)})$ with $e_1^{(x)} = \frac{1}{|\nabla_x \omega(P_0)|}\nabla_x \omega(P_0)$ and $(e_1^{(y)}, e_2^{(y)}, \ldots, e_d^{(y)})$ with $B^{(y)} = e_1^{(y)} = -\frac{1}{|\nabla_y \omega(P_0)|}\nabla_y \omega(P_0)$ of $\mathbb{R}^d$.

Suppose $x(\tau)$ and $y(\tau)$ are two paths in Ω such that

$$\begin{aligned} x'(\tau) &= A(t_0)|\nabla_x \omega(P_0)|^{\frac{p-2}{2}} e_i^{(x)}, \quad x(\tau_0) = x_0, \\ y'(\tau) &= A(s_0)|\nabla_y \omega(P_0)|^{\frac{p-2}{2}} e_i^{(y)}, \quad y(\tau_0) = y_0 \end{aligned}$$

for $1 \leq i \leq d$ and for some τ_0. Then

$$\begin{aligned} &\frac{d}{d\tau} Z(x(\tau), y(\tau); t, s) \\ &= A(t_0)|\nabla_x \omega(P_0)|^{\frac{p-2}{2}} \frac{\partial v}{\partial x_i}(x, t) - A(s_0)|\nabla_y \omega(P_0)|^{\frac{p-2}{2}} \frac{\partial v}{\partial y_i}(y, s) \\ &\quad + A(t_0)|\nabla_x \omega(P_0)|^{\frac{p-2}{2}} \frac{\partial \omega}{\partial x_i}(x, y; t, s) + A(s_0)|\nabla_y \omega(P_0)|^{\frac{p-2}{2}} \frac{\partial \omega}{\partial y_i}(x, y; t, s) \end{aligned}$$

and

$$\begin{aligned}
&\frac{d^2}{d\tau^2}Z(x(\tau),y(\tau);t,s)\\
&\quad=A^2(t_0)|\nabla_x\omega(P_0)|^{p-2}\frac{\partial^2 v}{\partial x_i^2}(x,t)-A^2(s_0)|\nabla_y\omega(P_0)|^{p-2}\frac{\partial^2 v}{\partial y_i^2}(y,s)\\
&\qquad+A^2(t_0)|\nabla_x\omega(P_0)|^{p-2}\frac{\partial^2\omega}{\partial x_i^2}(x,y;t,s)+A^2(s_0)|\nabla_y\omega(P_0)|^{p-2}\frac{\partial^2\omega}{\partial y_i^2}(x,y;t,s)\\
&\qquad+2A(t_0)A(s_0)|\nabla_x\omega(P_0)|^{\frac{p-2}{2}}|\nabla_y\omega(P_0)|^{\frac{p-2}{2}}\frac{\partial^2\omega}{\partial x_i\partial y_i}(x,y;t,s).
\end{aligned}$$

Evaluating at $\tau=\tau_0$ and using that $D^2_{(x,y)}Z(P_0)\geq 0$ implies

$$\frac{d^2}{d\tau^2}Z(x(\tau),y(\tau);t,s)\bigg|_{\tau=\tau_0}\geq 0,$$

and we have

$$\begin{aligned}
0\leq{}& A^2(t_0)|\nabla_x\omega(P_0)|^{p-2}\frac{\partial^2 v}{\partial x_i^2}(x_0,t_0)-A^2(s_0)|\nabla_y\omega(P_0)|^{p-2}\frac{\partial^2 v}{\partial y_i^2}(y_0,s_0)\\
&+A^2(t_0)|\nabla_x\omega(P_0)|^{p-2}\frac{\partial^2\omega}{\partial x_i^2}(P_0)+A^2(s_0)|\nabla_y\omega(P_0)|^{p-2}\frac{\partial^2\omega}{\partial y_i^2}(P_0)\\
&+2A(t_0)A(s_0)|\nabla_x\omega(P_0)|^{\frac{p-2}{2}}|\nabla_y\omega(P_0)|^{\frac{p-2}{2}}\frac{\partial^2\omega}{\partial x_i\partial y_i}(P_0).
\end{aligned}$$

Summing over i and then adding a further $(p-2)$ multiples of the inequality for $i=1$ yields

$$\begin{aligned}
0\leq{}& A^2(t_0)|\nabla_x\omega(P_0)|^{p-2}\left(\Delta v(x_0,t_0)+(p-2)\frac{\partial^2 v}{\partial x_1^2}(x_0,t_0)\right)\\
&-A^2(s_0)|\nabla_y\omega(P_0)|^{p-2}\left(\Delta v(y_0,s_0)+(p-2)\frac{\partial^2 v}{\partial y_1^2}(y_0,s_0)\right)\\
&+A^2(t_0)|\nabla_x\omega(P_0)|^{p-2}\left(\Delta_x\omega(P_0)+(p-2)\frac{\partial^2\omega}{\partial x_1^2}(P_0)\right)\\
&+A^2(s_0)|\nabla_y\omega(P_0)|^{p-2}\left(\Delta_y\omega(P_0)+(p-2)\frac{\partial^2\omega}{\partial y_1^2}(P_0)\right)\\
&+2A(t_0)A(s_0)|\nabla_x\omega(P_0)|^{\frac{p-2}{2}}|\nabla_y\omega(P_0)|^{\frac{p-2}{2}}\\
&\times\left((p-2)\frac{\partial^2\omega}{\partial x_1\partial y_1}(P_0)+\sum_{i=1}^{d}\frac{\partial^2\omega}{\partial x_i\partial y_i}(P_0)\right).
\end{aligned}\tag{3.76}$$

Note that (3.72) and (3.73) imply

$$\begin{aligned}
\nabla v(x_0,t_0)&=-|\nabla_x\omega(P_0)|e_1^{(x)},\\
\nabla v(y_0,s_0)&=|\nabla_y\omega(P_0)|e_1^{(y)}.
\end{aligned}$$

As for when $\lambda > 0$, one has

$$\Delta_p v(x_0,t_0) = |\nabla_x \omega(P_0)|^{p-2}\left(\Delta v(x_0,t_0) + (p-2)\frac{\partial^2 v}{\partial x_1^2}(x_0,t_0)\right),$$
$$\Delta_p v(y_0,s_0) = |\nabla_y \omega(P_0)|^{p-2}\left(\Delta v(y_0,s_0) + (p-2)\frac{\partial^2 v}{\partial y_1^2}(y_0,s_0)\right),$$

and hence, the inequality (3.76) simplifies to

$$\begin{aligned}0 \le\ & A^2(t_0)\Delta_p v(x_0,t_0) - A^2(s_0)\Delta_p v(y_0,s_0)\\ &+ A^2(t_0)\Delta_{p,x}\omega(P_0) + A^2(s_0)\Delta_{p,y}\omega(P_0)\\ &+ 2A(t_0)A(s_0)|\nabla_x\omega(P_0)|^{\frac{p-2}{2}}|\nabla_y\omega(P_0)|^{\frac{p-2}{2}}\times\\ &\qquad\left((p-2)\frac{\partial^2\omega}{\partial x_1\partial y_1}(P_0) + \sum_{i=1}^d \frac{\partial^2\omega}{\partial x_i\partial y_i}(P_0)\right).\end{aligned}$$

We may now insert the inequalities (3.74) and (3.75), which produces

$$\begin{aligned}0 \le\ & \varepsilon(p-1)(t_0-s_0)\left(A^2(s_0)-A^2(t_0)\right)\\ &+ A^2(s_0)\frac{\beta'(s_0)}{\beta(s_0)} - A^2(t_0)\frac{\beta'(t_0)}{\beta(t_0)}\\ &+ A^2(t_0)\Delta_{p,x}\omega(P_0) + A^2(s_0)\Delta_{p,y}\omega(P_0)\\ &+ 2A(t_0)A(s_0)|\nabla_x\omega(P_0)|^{\frac{p-2}{2}}|\nabla_y\omega(P_0)|^{\frac{p-2}{2}}\times\\ &\qquad\left((p-2)\frac{\partial^2\omega}{\partial x_1\partial y_1}(P_0) + \sum_{i=1}^d \frac{\partial^2\omega}{\partial x_i\partial y_i}(P_0)\right).\end{aligned}$$

However, the assumed inequality (3.71) implies

$$0 \le \varepsilon(p-1)(t_0-s_0)\left(A^2(s_0)-A^2(t_0)\right) < 0,$$

which is a contradiction. Therefore, the assumption that $x_0, y_0 \in \Omega$ is false and at least one of these points must occur on the boundary $\partial\Omega$. However, we can also rule this out using the argument from the linear case. □

The condition $D^2_{(x,y)}Z(P_0) \ge 0$ can be utilised in a different manner than we have just shown. The following alternative calculation makes greater use of properties from linear algebra to obtain a contradiction. We provide this computation below in the case $\lambda > 0$ to demonstrate the analogue between the proofs in the cases of classical and viscosity solutions, which we discuss later.

PROOF 2 OF THEOREM 3.17. We return to the estimates (3.62) and (3.63). Taking a linear combination of these inequalities with the positive coefficients $A^2(t_0)$ and $A^2(s_0)$ yields

$$\begin{aligned}0 \le\ & A^2(s_0)\beta^{\lambda-1}(s_0)\beta'(s_0) - A^2(t_0)\beta^{\lambda-1}(t_0)\beta'(t_0)\\ &+ \frac{\varepsilon}{\lambda}(t_0-s_0)\left(\frac{A^2(s_0)}{v(y_0,s_0)} - \frac{A^2(t_0)}{v(x_0,t_0)}\right)\\ &- \left(A^2(t_0)\beta^\lambda(t_0)\Delta_p v(x_0,t_0) - A^2(s_0)\beta^\lambda(s_0)\Delta_p v(y_0,s_0)\right).\end{aligned}\tag{3.77}$$

We aim to obtain a lower bound on

$$Q := A^2(t_0)\beta^\lambda(t_0)\Delta_p v(x_0,t_0) - A^2(s_0)\beta^\lambda(s_0)\Delta_p v(y_0,s_0).$$

Making use of the non-divergence form of the operator Δ_p, this quantity is equal to

$$\begin{aligned} Q = A^2(t_0)\beta^\lambda(t_0)|\nabla v(x_0,t_0)|^{p-2}\operatorname{tr} D^2 v(x_0,t_0) \\ - A^2(s_0)\beta^\lambda(s_0)|\nabla v(y_0,s_0)|^{p-2}\operatorname{tr} D^2 v(y_0,s_0) \\ + (p-2)A^2(t_0)\beta^\lambda(t_0)|\nabla v(x_0,t_0)|^{p-4}\times \\ \nabla v(x_0,t_0)^\top D^2 v(x_0,t_0)\nabla v(x_0,t_0) \\ - (p-2)A^2(s_0)\beta^\lambda(s_0)|\nabla v(y_0,s_0)|^{p-4}\times \\ \nabla v(y_0,s_0)^\top D^2 v(y_0,s_0)\nabla v(y_0,s_0). \end{aligned}$$

Inserting the relationships (3.60) and (3.61) gives

$$\begin{aligned} Q = A^2(t_0)\beta^{-\lambda(p-2)}(t_0)|\nabla_x\omega(P_0)|^{p-2}\operatorname{tr}(\beta^\lambda(t_0)D^2 v(x_0,t_0)) \\ + A^2(s_0)\beta^{-\lambda(p-2)}(s_0)|\nabla_y\omega(P_0)|^{p-2}\operatorname{tr}(-\beta^\lambda(s_0)D^2 v(x_0,t_0)) \\ + (p-2)A^2(t_0)\beta^{-\lambda(p-2)}(t_0)|\nabla_x\omega(P_0)|^{p-4}\times \\ \nabla_x\omega(P_0)^\top(\beta^\lambda(t_0)D^2 v(x_0,t_0))\nabla_x\omega(P_0) \\ - (p-2)A^2(s_0)\beta^{-\lambda(p-2)}(s_0)|\nabla_y\omega(P_0)|^{p-4}\times \\ \nabla_y\omega(P_0)^\top(\beta^\lambda(s_0)D^2 v(y_0,s_0))\nabla_x\omega(P_0). \end{aligned}$$

Now, let

$$F := \begin{pmatrix} A^2(t_0)\beta^{-\lambda(p-2)}(t_0)|\nabla_x\omega(P_0)|^{p-2}I_d & 0 \\ 0 & A^2(s_0)\beta^{-\lambda(p-2)}(s_0)|\nabla_y\omega(P_0)|^{p-2}I_d \end{pmatrix},$$

$$G := \begin{pmatrix} \beta^\lambda(t_0)D^2 v(x_0,t_0) & 0 \\ 0 & -\beta^\lambda(s_0)D^2 v(y_0,s_0) \end{pmatrix},$$

$$\zeta := \begin{pmatrix} A(t_0)\beta^{-\frac{\lambda(p-2)}{2}}(t_0)|\nabla_x\omega(P_0)|^{\frac{p-4}{2}}\nabla_x\omega(P_0) \\ A(s_0)\beta^{-\frac{\lambda(p-2)}{2}}(s_0)|\nabla_y\omega(P_0)|^{\frac{p-4}{2}}\nabla_y\omega(P_0) \end{pmatrix},$$

so that

$$Q = \operatorname{tr}(FG) + (p-2)\zeta^\top G\zeta.$$

By the properties of the trace, this formula remains true if we replace F by the matrix

$$\tilde F := \begin{pmatrix} F_{11} & F_{12} \\ F_{12} & F_{22} \end{pmatrix},$$

where

$$\begin{aligned} F_{11} &:= A^2(t_0)\beta^{-\lambda(p-2)}(t_0)|\nabla_x\omega(P_0)|^{p-2}I_d \\ F_{12} &:= A(t_0)A(s_0)\beta^{-\frac{\lambda(p-2)}{2}}(t_0)\beta^{-\frac{\lambda(p-2)}{2}}(s_0)|\nabla_x\omega(x_0,t_0)|^{\frac{p-2}{2}}|\nabla_y\omega(y_0,s_0)|^{\frac{p-2}{2}}I_d \\ F_{22} &:= A^2(s_0)\beta^{-\lambda(p-2)}(s_0)|\nabla_y\omega(P_0)|^{p-2}I_d. \end{aligned}$$

One may verify that this choice of $\tilde{F}$ remains a positive semi-definite matrix. We also recall that $D^2_{(x,y)} Z(P_0) \geq 0$. Applying the definition of the function Z, this implies

$$G = \begin{pmatrix} \beta^\lambda(t_0) D^2 v(x_0, t_0) & 0 \\ 0 & -\beta^\lambda(s_0) D^2 v(y_0, s_0) \end{pmatrix} \geq -D^2_{(x,y)} \omega(P_0).$$

It follows that

$$\begin{aligned} Q &= \operatorname{tr}(\tilde{F} G) + (p-2) \zeta^\top G \zeta \\ &\geq -\operatorname{tr}(\tilde{F} D^2_{xy} \omega(P_0)) - (p-2) \zeta^\top D^2_{(x,y)} \omega(P_0) \zeta. \end{aligned}$$

Expanding the products, this inequality is equivalent to

$$\begin{aligned} Q \geq &-A^2(t_0) \beta^{-\lambda(p-2)}(t_0) \Delta_{p,x} \omega(P_0) - A^2(s_0) \beta^{-\lambda(p-2)}(s_0) \Delta_{p,y} \omega(P_0) \\ &- 2A(t_0) A(s_0) \beta^{-\frac{\lambda(p-2)}{2}}(t_0) \beta^{-\frac{\lambda(p-2)}{2}}(s_0) \Delta_{p,xy} \omega(P_0) \end{aligned}$$

and so combining this with (3.77) produces

$$\begin{aligned} 0 \leq\ & A^2(s_0) \beta^{\lambda-1}(s_0) \beta'(s_0) - A^2(t_0) \beta^{\lambda-1}(t_0) \beta'(t_0) \\ &+ \frac{\varepsilon}{\lambda}(t_0 - s_0) \left(\frac{A^2(s_0)}{v(y_0, s_0)} - \frac{A^2(t_0)}{v(x_0, t_0)} \right) \\ &+ A^2(t_0) \beta^{-\lambda(p-2)}(t_0) \Delta_{p,x} \omega(P_0) + A^2(s_0) \beta^{-\lambda(p-2)}(s_0) \Delta_{p,y} \omega(P_0) \\ &+ 2A(t_0) A(s_0) \beta^{-\frac{\lambda(p-2)}{2}}(t_0) \beta^{-\frac{\lambda(p-2)}{2}}(s_0) \Delta_{p,xy} \omega(P_0). \end{aligned}$$

Then, the assumption (3.53) once again implies

$$0 \leq \frac{\varepsilon}{\lambda}(t_0 - s_0) \left(\frac{A^2(s_0)}{v(y_0, s_0)} - \frac{A^2(t_0)}{v(x_0, t_0)} \right)$$

and the remainder of the proof follows identically.

□

The optimal ω

As in the linear case, we identify an optimal candidate for the function ω. For a given increasing function $\beta : [0, \infty) \to [0, \infty)$, we define an energy

$$E[\gamma; t, s] = \frac{C_p}{\left((B(t) - B(s)\right)^{p'-1}} \int_0^1 |\dot{\gamma}|^{p'} \, \mathrm{d}\tau \tag{3.78}$$

for curves $\gamma : [0,1] \to M$, where B is such that $B'(t) = \beta^{-\lambda(p-1)}(t)$, $p' := \frac{p}{p-1}$, and $C_p := (p-1)p^{-p'}$. For fixed $x, y \in M$, the energy E is minimised over the set of curves γ joining x and y by the geodesic passing between these points and therefore we choose

$$\omega(x, y; t, s) := \min_{\gamma \in \Gamma_{x,y}} E[\gamma; t, s] = C_p \frac{d(x,y)^{p'}}{(B(t) - B(s))^{p'-1}}. \tag{3.79}$$

Following the ideas in Section 3.1.2, one sees that the function ω defined in (3.79) satisfies (3.51) and (3.52) with equality. However, it is again not immediately clear if there exists an increasing function A such that (3.53) holds.

Looking first in the Euclidean case $M = \mathbb{R}^d$, we now aim to determine for which choices of β, there exists a function A, such that inequality (3.53) is satisfied. By an explicit computation, we find that

$$\begin{aligned}\nabla_x\omega(x, y; t, s) &= \frac{p'C_p}{(B(t) - B(s))^{p'-1}}|x - y|^{p'-2}(x - y)\\ \nabla_y\omega(x, y; t, s) &= -\frac{p'C_p}{(B(t) - B(s))^{p'-1}}|x - y|^{p'-2}(x - y) = -\nabla_x\omega(x, y; t, s).\end{aligned}$$

Then, since

$$\begin{aligned}|\nabla_x\omega|^{p-2}\nabla_x\omega &= \left(\frac{p'C_p}{(B(t) - B(s))^{p'-1}}\right)^{p-1}|x - y|^{(p'-2)(p-1)+p-2}(x - y)\\ &= \frac{1}{p}\frac{x - y}{B(t) - B(s)},\end{aligned}$$

it follows that

$$\Delta_{p,x}\omega = \operatorname{div}_x(|\nabla_x\omega|^{p-2}\nabla_x\omega) = \frac{d}{p}\frac{1}{B(t) - B(s)}.$$

Similarly, one has

$$\Delta_{p,y}\omega = \operatorname{div}_y(|\nabla_y\omega|^{p-2}\nabla_y\omega) = \frac{d}{p}\frac{1}{B(t) - B(s)}.$$

In addition, the relationship $\nabla_x\omega = -\nabla_y\omega$ means that the formula for the mixed p-Laplacian $\Delta_{p,xy}\omega$ simplifies greatly and

$$\begin{aligned}\Delta_{p,xy}\omega = |\nabla_x\omega|^{\frac{p-4}{2}}|\nabla_y\omega|^{\frac{p-4}{2}}\Big(&|\nabla_x\omega||\nabla_y\omega|\operatorname{div}_x(\nabla_y\omega)\\ &- (p - 2)\nabla_x\omega^\top D^2_{xy}\omega\nabla_y\omega\Big)\\ = -|\nabla_x\omega|^{p-4}(&|\nabla_x\omega|^2\Delta_x\omega + \nabla_x\omega^\top D^2_x\omega\nabla_x\omega) = \Delta_{p,x}\omega\\ = -\frac{d}{p}&\frac{1}{B(t) - B(s)}.\end{aligned}$$

Therefore, (3.53) is equivalent to

$$\begin{aligned}\frac{d}{p}\frac{\left(A(t)\beta^{-\frac{\lambda(p-2)}{2}}(t) - A(s)\beta^{-\frac{\lambda(p-2)}{2}}(s)\right)^2}{B(t) - B(s)}&\\ \leq A^2(t)\beta^{\lambda-1}(t)\beta'(t) - A^2(s)\beta^{\lambda-1}(s)\beta(s).&\end{aligned}$$

One simple way in which this condition can be satisfied is if

$$B(t) = A(t)\beta^{-\frac{\lambda(p-2)}{2}}(t), \tag{3.80}$$

$$\frac{d}{p}A(t)\beta^{-\frac{\lambda(p-2)}{2}}(t) = A^2(t)\beta^{\lambda-1}(t)\beta'(t) \tag{3.81}$$

hold for all $t > 0$. Rearranging (3.80) for $A(t)$ and inserting into (3.81) leads to

$$\frac{d}{p}B(t) = B^2(t)\beta^{\lambda(p-1)-1}(t)\beta'(t) \tag{3.82}$$

and therefore

$$B(t) = \frac{\lambda(p-1)d}{p}\left(\frac{d}{dt}(\beta^{\lambda(p-1)}(t))\right)^{-1}.$$

Recalling that $B'(t) = \beta^{-\lambda(p-1)}(t)$, this implies

$$B(t) = -\frac{\lambda(p-1)d}{p}\frac{(B'(t))^2}{B''(t)},$$

which, when rearranged, gives

$$\frac{B''(t)}{B'(t)} = -\frac{\lambda(p-1)d}{p}\frac{B'(t)}{B(t)}.$$

Therefore

$$\log(B'(t)) = -\frac{\lambda(p-1)d}{p}\log B(t) + C$$

for some constant of integration C, which we do not track in the computations that follow. Given that B may be any antiderivative of $\beta^{-\lambda(p-1)(t)}$, we may assume $B(0) = 0$. Solving for $B(t)$ then produces

$$B(t) = Ct^{\frac{1}{1+\frac{\lambda(p-1)d}{p}}} = Ct^{\frac{p}{d}K},$$

where

$$K = \frac{1}{m(p-1) - 1 + \frac{p}{d}}.$$

Finally, by differentiating B and rearranging, it follows that

$$\beta(t) = \left(\frac{CKp}{d}\right)^{-1/\lambda(p-1)} t^K.$$

We note that we have only determined β up to a positive multiplicative constant, however, the structure of the Harnack inequality (3.56) implies the resulting inequality is equivalent for all choices of this constant. Therefore, for convenience, we select $C = \frac{d}{Kp}$, so that

$$\beta(t) = t^K, \qquad B(t) = \frac{d}{Kp}t^{Kp/d}, \qquad A(t) = \frac{d}{Kp}t^{(\frac{p}{d}+\frac{\lambda(p-2)}{2})K}.$$

With these choices, we consider

$$\omega(x, y; t, s) = \frac{1}{p'}\left(\frac{K}{d}\right)^{p'-1}\frac{|x-y|^{p'}}{(t^{Kp/d} - s^{Kp/d})^{p'-1}} \tag{3.83}$$

to be the optimal choice of ω in the Euclidean setting. Our choice is supported by the fact that, analogous to the situation for the heat equation (see Example 3.1), the Barenblatt solution $\mathcal{B}$ for the doubly nonlinear heat equation given by (2.12) satisfies the resulting Harnack inequality (3.56) with equality for all $x, y \in \mathbb{R}^d$ and $0 < s < t$ in the support of $\mathcal{B}$ satisfying

$$s^{Kp/d}x = t^{Kp/d}y.$$

Remark 3.4. Theorem 3.16 requires that the function β is strictly increasing, which if $\beta = t^K$, only occurs if $K = \frac{1}{m(p-1)-1+\frac{p}{d}} > 0$. This introduces the restriction

$$p > \frac{d(m+1)}{1+dm},$$

which must be satisfied in order to deduce a Harnack inequality using Theorem 3.16. In the case $m = 1$ corresponding to the p-heat equation, this restriction reads as

$$p > p_* := \frac{2d}{d+1},$$

which matches the range for which both the Harnack inequality (see, for instance, [**57**]) and its differentiated form (1.24) due to Esteban and Vázquez [**61**] are known to hold. Moreover, with ω chosen as in (3.83), we recover the Harnack inequality

$$\frac{1}{\lambda}(t^K u(x,t))^\lambda - \frac{1}{\lambda}(s^K u(y,s))^\lambda + \frac{1}{p'}\left(\frac{K}{d}\right)^{p'-1} \frac{|x-y|^{p'}}{(t^{Kp/d} - s^{Kp/d})^{p'-1}} \geq 0$$

satisfied by every positive solution of the p-heat equation (1.6) for every $x, y \in \mathbb{R}^d$, $0 < s < t$, which coincides with the result in [**13**].

Similarly, for $p = 2$, one has the restriction

$$m > 1 - \frac{2}{d},$$

which already appeared in the work of Aronson and Bénilan [**8**] on the porous medium equation. In addition, we also obtain the Harnack inequality

$$\frac{m}{m-1}(t^K u(x,t))^{m-1} - \frac{m}{m-1}(s^K u(y,s))^{m-1} + \frac{K}{2d}\frac{|x-y|^2}{t^{2K/d} - s^{2K/d}} \geq 0$$

satisfied by every positive solution of the porous medium equation (1.5) for every $x, y \in \mathbb{R}^d$, $0 < s < t$ [**13**].

3.2.2. Classical solutions on Riemannian manifolds

The arguments from the previous section can also be adapted to solutions of the equation (1.4) on Riemannian manifolds. We have the following results.

Theorem 3.17. *Let $m > 0$, $p \geq 2$ be such that $K = \frac{1}{m(p-1)-1+p/d} > 0$. Let (M, g) be a compact Riemannian manifold with (possibly empty) convex boundary and nonnegative Ricci curvature. Then every positive $C^{2,1}$-solution u of*

$$\begin{cases} \partial_t u = \Delta_p(u^m) & \text{in } M \times (0,\infty) \\ g(|\nabla(u^m)|^{p-2}\nabla(u^m), \nu) = 0 & \text{on } \partial M \times (0,\infty) \end{cases} \tag{3.84}$$

satisfies the Harnack inequality

$$\psi(\beta(t)u(x,t)) \geq \psi(\beta(s)u(y,s)) - \omega(x,y;t,s) \tag{3.85}$$

for all $x, y \in M$ *and* $0 < s < t$*, where* ψ *is defined as in Theorem 3.16,* $\beta(t) := t^K$*,* $\lambda := m - \frac{1}{p-1}$*,* B *is an antiderivative of* $\beta^{-\lambda(p-1)}$*, and*

$$\omega(x, y; t, s) = \frac{C_p\, d(x, y)^{p'}}{(B(t) - B(s))^{p'-1}}.$$

Assuming one can approximate solutions of (1.4) on a complete manifold M by solutions of the homogeneous Neumann problem (3.84) on convex submanifolds, the following theorem holds as well.

Theorem 3.18. *Let* $m > 0$*,* $p \geq 2$ *be such that* $K = \frac{1}{m(p-1)-1+p/d} > 0$*. Let* (M, g) *be a complete Riemannian manifold without boundary and nonnegative Ricci curvature. Then every positive* $C^{2,1}$*-solution* u *of*

$$\partial_t u = \Delta_p(u^m) \qquad \text{in } M \times (0, \infty) \tag{1.4}$$

satisfies the Harnack inequality (3.85) *for all* $x, y \in M$ *and* $0 < s < t$*.*

For the same reasons as in the linear case, we work directly with the energy

$$\begin{aligned} E[\gamma; t, s] &= \frac{C_p}{\big((B(t) - B(s)\big)^{p'-1}} \int_0^1 |\dot\gamma|^{p'}\, \mathrm{d}\tau \\ &= \frac{C_p}{\big((B(t) - B(s)\big)^{p'-1}} \int_0^1 g(\dot\gamma, \dot\gamma)^{\frac{p'}{2}}\, \mathrm{d}\tau \end{aligned}$$

defined in (3.78) for all C^1 curves γ. To assist us in the proof of Theorem 3.18, we briefly analyse some properties of the energy E.

For $x, y \in M$ fixed, we first consider the problem of minimising E over $\Gamma_{x,y}$. Since E is Gâteaux differentiable with

$$\langle E'[\gamma; t, s], h\rangle = \frac{p' C_p}{\big((B(t) - B(s)\big)^{p'-1}} \int_0^1 |\dot\gamma|^{p'-2} g(\dot\gamma, \dot h)\, \mathrm{d}\tau$$

the Euler-Lagrange equation associated to this problem is

$$\nabla_\tau \Big(|\dot\gamma|^{p'-2} \dot\gamma \Big) = 0, \tag{3.86}$$

which is satisfied by the geodesic γ_0 connecting x and y. Thus,

$$\omega(x, y; t, s) := \min_{\gamma \in \Gamma_{x,y}} E[\gamma; t, s] = E[\gamma_0; t, s] = \frac{C_p\, d(x, y)^{p'}}{\big((B(t) - B(s)\big)^{p'-1}}.$$

Next, we consider the derivatives of $E[\gamma; t, s]$ under smooth variations of the curve γ.

Proposition 3.19. *Fix a curve $\gamma_0 : [0,1] \to M$ and let $\gamma(\tau, r) : [0,1] \times (-\delta, \delta)$ be a smooth family of curves such that $\gamma(\tau, 0) = \gamma_0(\tau)$. Then we have the formulas*

$$\begin{aligned}\frac{d}{dr}&E[\gamma(\tau,r); t, s]\\ &= \left(\frac{1}{p\big(B(t)-B(s)\big)}\right)^{p'-1} |\dot\gamma|^{p'-2} g(\gamma_r, \dot\gamma)\Bigg|_0^1 \\ &\quad - \left(\frac{1}{p\big(B(t)-B(s)\big)}\right)^{p'-1} \int_0^1 g\Big(\gamma_r, \nabla_\tau\big(|\dot\gamma|^{p'-2}\dot\gamma\big)\Big)\,\mathrm{d}\tau\end{aligned} \tag{3.87}$$

and

$$\begin{aligned}\frac{d^2}{dr^2}&E[\gamma(\tau,r); t, s]\\ &= \left(\frac{1}{p\big(B(t)-B(s)\big)}\right)^{p'-1} \int_0^1 (p'-2)|\dot\gamma|^{p'-4} g(\nabla_r\dot\gamma, \dot\gamma)^2\,\mathrm{d}\tau \\ &\quad + \left(\frac{1}{p\big(B(t)-B(s)\big)}\right)^{p'-1} \int_0^1 |\dot\gamma|^{p'-2}\Big(R(\gamma_r, \dot\gamma, \gamma_r, \dot\gamma) \\ &\qquad + g(\nabla_\tau\nabla_r\gamma_r, \dot\gamma) + |\nabla_r\dot\gamma|^2\Big)\,\mathrm{d}\tau.\end{aligned} \tag{3.88}$$

PROOF. By direct calculation, one has

$$\begin{aligned}\frac{d}{dr}&E[\gamma(\tau,r); t, s]\\ &= \frac{C_p}{\big((B(t)-B(s)\big)^{p'-1}} \int_0^1 \frac{d}{dr} g(\dot\gamma, \dot\gamma)^{\frac{p'}{2}}\,\mathrm{d}\tau \\ &= \frac{p'C_p}{\big((B(t)-B(s)\big)^{p'-1}} \int_0^1 g(\dot\gamma, \dot\gamma)^{\frac{p'}{2}-1} g(\nabla_r\dot\gamma, \dot\gamma)\,\mathrm{d}\tau \\ &= \left(\frac{1}{p\big(B(t)-B(s)\big)}\right)^{p'-1} \int_0^1 g(\dot\gamma, \dot\gamma)^{\frac{p'}{2}-1} g(\nabla_\tau\gamma_r, \dot\gamma)\,\mathrm{d}\tau \\ &= \left(\frac{1}{p\big(B(t)-B(s)\big)}\right)^{p'-1} g(\dot\gamma, \dot\gamma)^{\frac{p'}{2}-1} g(\gamma_r, \dot\gamma)\Bigg|_0^1 \\ &\quad - \left(\frac{1}{p\big(B(t)-B(s)\big)}\right)^{p'-1} \int_0^1 g\Big(\gamma_r, \nabla_\tau\big(g(\dot\gamma, \dot\gamma)^{\frac{p'}{2}-1}\dot\gamma\big)\Big)\,\mathrm{d}\tau\end{aligned}$$

and

$$
\begin{aligned}
&\frac{d^2}{dr^2}E[\gamma(\tau,r);t,s] \\
&\quad= \left(\frac{1}{p\big(B(t)-B(s)\big)}\right)^{p'-1} \frac{d}{dr}\left(\int_0^1 g(\dot\gamma,\dot\gamma)^{\frac{p'}{2}-1} g(\nabla_r\dot\gamma,\dot\gamma)\ \mathrm{d}\tau\right) \\
&\quad= \left(\frac{1}{p\big(B(t)-B(s)\big)}\right)^{p'-1} \int_0^1 (p'-2) g(\dot\gamma,\dot\gamma)^{\frac{p'}{2}-2} g(\nabla_r\dot\gamma,\dot\gamma)^2 \\
&\qquad+ \left(\frac{1}{p\big(B(t)-B(s)\big)}\right)^{p'-1} \int_0^1 g(\dot\gamma,\dot\gamma)^{\frac{p'}{2}-1}\Big(g(\nabla_r\nabla_r\dot\gamma,\dot\gamma)+|\nabla_r\dot\gamma|^2\Big)\ \mathrm{d}\tau \\
&\quad= \left(\frac{1}{p\big(B(t)-B(s)\big)}\right)^{p'-1} \int_0^1 (p'-2) g(\dot\gamma,\dot\gamma)^{\frac{p'}{2}-2} g(\nabla_r\dot\gamma,\dot\gamma)^2 \\
&\qquad+ \left(\frac{1}{p\big(B(t)-B(s)\big)}\right)^{p'-1} \int_0^1 g(\dot\gamma,\dot\gamma)^{\frac{p'}{2}-1}\Big(R(\gamma_r,\dot\gamma,\gamma_r,\dot\gamma) \\
&\qquad\qquad\qquad+ g(\nabla_\tau\nabla_r\gamma_r,\dot\gamma)+|\nabla_r\dot\gamma|^2\Big)\ \mathrm{d}\tau
\end{aligned}
$$

□

We also have the analogue of Proposition 3.7.

Proposition 3.20. *Let γ_0 be a geodesic connecting $x, y \in M$ and set $\omega(x,y;t,s) = E[\gamma_0;t,s]$. Then ω satisfies*

$$\partial_t\omega + \beta^{-\lambda(p-1)}(t)|\nabla_x\omega|^p = 0 \tag{3.89}$$

$$\partial_t\omega - \beta^{-\lambda(p-1)}(s)|\nabla_y\omega|^p = 0 \tag{3.90}$$

in $M \times M \times S$.

Proof. When $\gamma_0(\tau) := \gamma(\tau,0)$ is a geodesic, the formula (3.87) condenses to

$$\frac{d}{dr}E[\gamma(\tau,r);t,s]\bigg|_{r=0} = \left(\frac{1}{p\big(B(t)-B(s)\big)}\right)^{p'-1} |\dot\gamma_0(\tau)|^{p'-2} g(\gamma_r(\tau,0),\dot\gamma_0(\tau))\bigg|_0^1.$$

If we take the family $\gamma(\tau,r)$ such that $\gamma(0,r) = y$ for all $r \in (-\delta,\delta)$, then $\gamma_r(0,r) = 0$ and so

$$g(\nabla_x\omega,\gamma_r(1)) = \left(\frac{1}{p\big(B(t)-B(s)\big)}\right)^{p'-1} |\dot\gamma_0(1)|^{p'-2} g(\gamma_r(1),\dot\gamma_0(1))$$

is true for all choices of $\gamma_r(1)$. Therefore,

$$\nabla_x\omega = \left(\frac{1}{p\big(B(t)-B(s)\big)}\right)^{p'-1} |\dot\gamma_0(1)|^{p'-2}\dot\gamma_0(1)$$

and it follows that

$$|\nabla_x \omega| = \left(\frac{d(x,y)}{p\big(B(t)-B(s)\big)}\right)^{p'-1}.$$

In addition, we note that

$$\begin{aligned}\partial_t \omega &= \frac{\partial}{\partial t}\frac{C_p\, d(x,y)^{p'}}{\big((B(t)-B(s)\big)^{p'-1}} \\ &= C_p(1-p')\beta^{-\lambda(p-1)}(t)\left(\frac{d(x,y)}{B(t)-B(s)}\right)^{p'} \\ &= -\beta^{-\lambda(p-1)}(t)\left(\frac{d(x,y)}{p\big(B(t)-B(s)\big)}\right)^{p'},\end{aligned}$$

and it is now easy to see that ω satisfies

$$\partial_t \omega + \beta^{-\lambda(p-1)}(t)|\nabla_x \omega|^p = 0.$$

Using a similar derivation, one may demonstrate that ω satisfies

$$\partial_t \omega - \beta^{-\lambda(p-1)}(s)|\nabla_y \omega|^p = 0$$

as well. □

PROOF OF THEOREM 3.17. To avoid repetition, we give the proof only in the case $\lambda = m - \frac{1}{p-1} > 0$ and the other cases can be handled by making similar adjustments as in the Euclidean case.

Let u be a positive $C^{2,1}$ solution of (3.84) and set $v = \frac{m}{\lambda}u^\lambda$ so that v satisfies

$$\begin{cases}\partial_t v = \lambda v \Delta_p v + |\nabla v|^p & \text{in } M\times(0,\infty),\\ g(|\nabla v|^{p-2}\nabla v, \nu) = 0 & \text{on } \partial M \times (0,\infty).\end{cases} \tag{3.91}$$

For a fixed $\varepsilon > 0$, we let

$$Z(x,y;t,s) := \beta^\lambda(t)v(x,t) - \beta^\lambda(s)v(y,s) + \frac{C_p\, d(x,y)^{p'}}{(B(t)-B(s))^{p'-1}} + \frac{\varepsilon}{2}(t-s)^2 + \varepsilon.$$

As in the proof of Theorem 3.15, we see that $Z > 0$ for s sufficiently small and for t near s. In particular, $Z > 0$ for (t,s) near $(0,0)$ with $0 < s < t$. Suppose there exists a point $P_0 := (x_0, y_0; t_0, s_0)$ such that Z touches 0 for the first time at P_0. We assume for the moment that $x_0, y_0 \notin \partial M$.

If we now define a new function $\tilde{Z} : C^\infty([0,1], M) \times S \to \mathbb{R}$ by

$$\begin{aligned}\tilde{Z}(\gamma;t,s) &:= \beta^\lambda(t)v(\gamma(1),t) - \beta^\lambda(s)v(\gamma(0),s) \\ &\qquad + \frac{C_p}{(B(t)-B(s))^{p'-1}}\int_0^1 |\dot\gamma|^{p'}\, \mathrm{d}\tau + \frac{\varepsilon}{2}(t-s)^2 + \varepsilon,\end{aligned}$$

and set γ_0 to be the geodesic such that $\gamma_0(0) = y_0$ and $\gamma_0(1) = x_0$, then $\tilde{Z}(\gamma_0; t_0, s_0) = 0$ and $\tilde{Z} > 0$ whenever $t^2 + s^2 < t_0^2 + s_0^2$. Therefore

$$\partial_t \tilde{Z}(\gamma_0; t_0, s_0) \leq 0 \tag{3.92a}$$

$$\partial_s \tilde{Z}(\gamma_0; t_0, s_0) \leq 0 \tag{3.92b}$$

$$\frac{\partial}{\partial r} \tilde{Z}(\gamma(\tau, r); t_0, s_0)\bigg|_{r=0} = 0 \tag{3.92c}$$

$$\frac{\partial^2}{\partial r^2} \tilde{Z}(\gamma(\tau, r); t_0, s_0)\bigg|_{r=0} \geq 0 \tag{3.92d}$$

for all smooth variations $\gamma = \gamma(\tau, r)$ of γ_0, with $r \in (-\delta, \delta)$ for some $\delta > 0$ and $\gamma(\tau, 0) = \gamma_0(\tau)$. The condition (3.92c) implies

$$\begin{aligned} &g\left(\beta^\lambda(t_0)\nabla v(x_0, t_0), \gamma_r(1)\right) - g\left(\beta^\lambda(s_0)\nabla v(y_0, s_0), \gamma_r(0)\right) \\ &\quad = -\left(\frac{1}{p\big(B(t) - B(s)\big)}\right)^{p'-1} \times \\ &\qquad\qquad \left(g\left(|\dot{\gamma_0}(1)|^{p'-2}\dot{\gamma_0}(1), \gamma_r(1)\right) - g\left(|\dot{\gamma_0}(0)|^{p'-2}\dot{\gamma_0}(0), \gamma_r(0)\right)\right). \end{aligned}$$

By considering the cases when $\gamma_r(1) = 0$ and $\gamma_r(0) = 0$, we reach the conclusion that

$$\begin{aligned} \nabla v(x_0, t_0) &= -\beta^{-\lambda}(t_0)\left(\frac{1}{p\big(B(t) - B(s)\big)}\right)^{p'-1} |\dot{\gamma_0}(1)|^{p'-2}\dot{\gamma_0}(1), \\ \nabla v(y_0, s_0) &= -\beta^{-\lambda}(s_0)\left(\frac{1}{p\big(B(t) - B(s)\big)}\right)^{p'-1} |\dot{\gamma_0}(0)|^{p'-2}\dot{\gamma_0}(0). \end{aligned} \tag{3.93}$$

Moreover, since γ_0 is a geodesic, it has constant speed equal to $d(x_0, y_0)$. Therefore,

$$\begin{aligned} |\nabla v(x_0, t_0)| &= \beta^{-\lambda}(t_0)\left(\frac{d(x_0, y_0)}{p\big(B(t_0) - B(s_0)\big)}\right)^{p'-1}, \\ |\nabla v(y_0, s_0)| &= \beta^{-\lambda}(s_0)\left(\frac{d(x_0, y_0)}{p\big(B(t_0) - B(s_0)\big)}\right)^{p'-1}. \end{aligned} \tag{3.94}$$

Next, using the condition (3.92a) and equation (3.91), it follows that

$$\begin{aligned} 0 &\geq \lambda\beta^{\lambda-1}(t_0)\beta'(t_0)v(x_0, t_0) + \beta^\lambda(t_0)\partial_t v(x_0, t_0) \\ &\quad + \frac{\partial}{\partial t}E[\gamma_0; t_0, s_0] + \varepsilon(t_0 - s_0) \\ &= \lambda\beta^{\lambda-1}(t_0)\beta'(t_0)v(x_0, t_0) + \lambda\beta^\lambda(t_0)v(x_0, t_0)\Delta_p v(x_0, t_0) \\ &\quad + \beta^\lambda(t_0)|\nabla v(x_0, t_0)|^p + \frac{\partial}{\partial t}E[\gamma_0; t_0, s_0] + \varepsilon(t_0 - s_0). \end{aligned}$$

However, as in the proof of Proposition 3.20, we find that

$$\beta^\lambda(t_0)|\nabla v(x_0,t_0)|^p + \frac{\partial}{\partial t}E[\gamma_0; t_0, s_0] = 0,$$

and therefore

$$0 \geq \lambda\beta^{\lambda-1}(t_0)\beta'(t_0)v(x_0,t_0) + \lambda\beta^\lambda(t_0)v(x_0,t_0)\Delta_p v(x_0,t_0) + \varepsilon(t_0 - s_0),$$

or equivalently,

$$\beta^\lambda(t_0)\Delta_p v(x_0,t_0) \leq -\beta^{\lambda-1}(t_0)\beta'(t_0) - \frac{\varepsilon}{\lambda}\frac{t_0 - s_0}{v(x_0,t_0)}. \tag{3.95}$$

By a similar argument beginning with (3.92b), we obtain

$$-\beta^\lambda(s_0)\Delta_p v(y_0,s_0) \leq \beta^{\lambda-1}(s_0)\beta'(s_0) + \frac{\varepsilon}{\lambda}\frac{t_0 - s_0}{v(y_0,s_0)}. \tag{3.96}$$

As in the Euclidean case, the next step of the proof proceeds after making a certain choice of basis. If $p = 2$, we choose $(e_i(0))_{i=1}^d$ to be any orthonormal basis of $T_{y_0}M$, which we parallel transport along γ_0 to obtain an orthonormal basis $e_i := e_i(\tau)$ at each point on the curve. If $p \neq 2$, we choose a particular basis of $T_{y_0}M$ such that $e_1(0) = \frac{\dot\gamma(0)}{|\dot\gamma(0)|}$. When parallel transporting $(e_i(0))_{i=1}^d$ along γ_0, which is a geodesic, we have $e_1(\tau) = \frac{\dot\gamma(\tau)}{|\dot\gamma(\tau)|}$ for all $\tau \in [0,1]$. We note that we can rule out the possibility that $\dot\gamma$ is ever 0 in a similar fashion as in the Euclidean case. Indeed, if $\dot\gamma = 0$, then (3.93) implies $\nabla v(x_0,t_0) = \nabla v(y_0,s_0) = 0$, and thus $\Delta_p v(x_0,t_0) = \Delta_p v(y_0,s_0) = 0$ as well. However, this would create a contradiction with (3.95). Therefore $\dot\gamma \neq 0$.

We choose a particular variation of γ_0 by

$$\gamma(\tau,r) := \exp_{\gamma_0(\tau)}\left((B(t_0)\tau + (1-\tau)B(s_0))\left(\frac{d(x_0,y_0)}{p(B(t_0)-B(s_0))}\right)^{\frac{(p'-1)(p-2)}{2}} re_i(\tau)\right)$$

so that

$$\gamma_r(0,r) = B(s_0)\left(\frac{d(x_0,y_0)}{p(B(t_0)-B(s_0))}\right)^{\frac{(p'-1)(p-2)}{2}} e_i(0)$$

$$\gamma_r(1,r) = B(t_0)\left(\frac{d(x_0,y_0)}{p(B(t_0)-B(s_0))}\right)^{\frac{(p'-1)(p-2)}{2}} e_i(1)$$

and $\nabla_r\gamma_r = 0$. Then condition (3.92d) implies

$$\begin{aligned}0 \leq\ & \beta^\lambda(t_0)B(t_0)^2\left(\tfrac{d(x_0,y_0)}{p(B(t_0)-B(s_0))}\right)^{(p'-1)(p-2)}\nabla_i\nabla_i v(x_0,t_0)\\ &- \beta^\lambda(s_0)B(s_0)^2\left(\tfrac{d(x_0,y_0)}{p(B(t_0)-B(s_0))}\right)^{(p'-1)(p-2)}\nabla_i\nabla_i v(y_0,s_0)\\ &+ \frac{\partial^2}{\partial r^2}E(\gamma(\tau,r);t_0,s_0)\bigg|_{r=0}.\end{aligned}$$

Using the formula (3.88), one may compute that

$$\begin{aligned}
&\frac{\partial^2}{\partial r^2}E(\gamma(\tau,r);t_0,s_0)\bigg|_{r=0}\\
&\quad=\frac{p'-2}{p}(B(t_0)-B(s_0))\delta_{i1}\\
&\qquad+\tfrac{1}{p(B(t_0)-B(s_0))}\int_0^1 d(x_0,y_0)^{(p'-1)(p-2)+p'-2}\times\\
&\qquad\qquad\Big((B(t_0)\tau+(1-\tau)B(s_0))^2R(e_i,\dot\gamma,e_i,\dot\gamma)\\
&\qquad\qquad\qquad+(B(t_0)-B(s_0))^2\Big)\,\mathrm{d}\tau
\end{aligned}$$

and therefore

$$\begin{aligned}
0\le{}&\beta^\lambda(t_0)B(t_0)^2\left(\tfrac{d(x_0,y_0)}{p(B(t_0)-B(s_0))}\right)^{(p'-1)(p-2)}\nabla_i\nabla_i v(x_0,t_0)\\
&-\beta^\lambda(s_0)B(s_0)^2\left(\tfrac{d(x_0,y_0)}{p(B(t_0)-B(s_0))}\right)^{(p'-1)(p-2)}\nabla_i\nabla_i v(y_0,s_0)\\
&+\frac{p'-2}{p}(B(t_0)-B(s_0))\delta_{i1}\\
&+\tfrac{1}{p(B(t_0)-B(s_0))}\int_0^1 d(x_0,y_0)^{(p'-1)(p-2)+p'-2}\times\\
&\qquad\Big((B(t_0)\tau+(1-\tau)B(s_0))^2R(e_i,\dot\gamma,e_i,\dot\gamma)\\
&\qquad\qquad+(B(t_0)-B(s_0))^2\Big)\,\mathrm{d}\tau.
\end{aligned}$$

Summing over i and adding a further $(p-2)$ multiples of the inequality for $i=1$ produces

$$\begin{aligned}
0\le{}&\beta^\lambda(t_0)B(t_0)^2\left(\tfrac{d(x_0,y_0)}{p(B(t_0)-B(s_0))}\right)^{(p'-1)(p-2)}\Big(\Delta v(x_0,t_0)+(p-2)\nabla_1\nabla_1 v(x_0,t_0)\Big)\\
&-\beta^\lambda(s_0)B(s_0)^2\left(\tfrac{d(x_0,y_0)}{p(B(t_0)-B(s_0))}\right)^{(p'-1)(p-2)}\Big(\Delta v(y_0,s_0)+(p-2)\nabla_1\nabla_1 v(y_0,s_0)\Big)\\
&+\frac{(p-1)(p'-2)}{p}(B(t_0)-B(s_0))+\frac{d+p-2}{p}(B(t_0)-B(s_0))\\
&-\tfrac{1}{p(B(t_0)-B(s_0))}\int_0^1 d(x_0,y_0)^{(p'-1)(p-2)+p'-2}\times\\
&\qquad\Big((B(t_0)\tau+(1-\tau)B(s_0))^2(\mathrm{Ric}(\dot\gamma,\dot\gamma)\\
&\qquad\qquad+(p-2)R(e_1,\dot\gamma,\dot\gamma,e_1))\Big)\,\mathrm{d}\tau.
\end{aligned}$$

Since for $p>2$, the vector $e_1(\tau)$ is defined to be proportional to $\dot\gamma(\tau)$ for each $\tau\in[0,1]$, the coefficient $R(e_1,\dot\gamma,\dot\gamma,e_1)$ vanishes. Moreover, since we assume the manifold M has nonnegative Ricci curvature, the inequality simplifies to

$$
\begin{aligned}
0 \le \beta^{\lambda}(t_0)B(t_0)^2 &\left(\tfrac{d(x_0,y_0)}{p(B(t_0)-B(s_0))}\right)^{(p'-1)(p-2)} \Big(\Delta v(x_0,t_0) + (p-2)\nabla_1\nabla_1 v(x_0,t_0)\Big) \\
&- \beta^{\lambda}(s_0)B(s_0)^2 \left(\tfrac{d(x_0,y_0)}{p(B(t_0)-B(s_0))}\right)^{(p'-1)(p-2)} \Big(\Delta v(y_0,s_0) + (p-2)\nabla_1\nabla_1 v(y_0,s_0)\Big) \\
&+ \frac{d}{p}(B(t_0) - B(s_0)).
\end{aligned}
$$

Next, we recognise using (3.94) that

$$
\Delta_p v(x_0,t_0) = \beta^{-\lambda(p-2)}(t_0) \left(\tfrac{d(x_0,y_0)}{p(B(t_0)-B(s_0))}\right)^{(p'-1)(p-2)} \Big(\Delta v(x_0,t_0) + (p-2)\nabla_1\nabla_1 v(x_0,t_0)\Big)
$$

$$
\Delta_p v(y_0,s_0) = \beta^{-\lambda(p-2)}(s_0) \left(\tfrac{d(x_0,y_0)}{p(B(t_0)-B(s_0))}\right)^{(p'-1)(p-2)} \Big(\Delta v(y_0,s_0) + (p-2)\nabla_1\nabla_1 v(y_0,s_0)\Big)
$$

and so

$$
\begin{aligned}
0 \le \beta^{\lambda(p-1)}(t_0)B(t_0)^2\Delta_p v(x_0,t_0) &- \beta^{\lambda(p-1)}(s_0)B(s_0)^2\Delta_p v(y_0,s_0) \\
&+ \frac{d}{p}(B(t_0) - B(s_0)).
\end{aligned}
$$

Inserting inequalities (3.95) and (3.96), we have

$$
\begin{aligned}
0 \le \frac{\varepsilon}{\lambda}(t_0 - s_0) &\left(\frac{\beta^{\lambda(p-2)}B(s_0)^2}{v(y_0,s_0)} - \frac{\beta^{\lambda(p-2)}B(t_0)^2}{v(x_0,t_0)}\right) \\
&+ B(s_0)^2\beta^{\lambda(p-1)-1}(s_0)\beta'(s_0) - B(t_0)^2\beta^{\lambda(p-1)-1}(t_0)\beta'(t_0) + \frac{d}{p}(B(t_0) - B(s_0)).
\end{aligned}
$$

However, according to (3.82), this becomes

$$
\begin{aligned}
0 &\le \frac{\varepsilon}{\lambda}(t_0 - s_0)\left(\frac{\beta^{\lambda(p-2)}B(s_0)^2}{v(y_0,s_0)} - \frac{\beta^{\lambda(p-2)}B(t_0)^2}{v(x_0,t_0)}\right) \\
&= \frac{\varepsilon}{\lambda}(t_0 - s_0)\left(\frac{A^2(s_0)}{v(y_0,s_0)} - \frac{A^2(t_0)}{v(x_0,t_0)}\right) < 0,
\end{aligned}
$$

which is a contradiction as in the Euclidean case. Therefore, $x_0, y_0 \in \partial M$. However, we can rule out this possibility as well by the same arguments as in the linear case. Therefore the point P_0 cannot exist and the proof is complete. □

CHAPTER 4

Harnack inequalities for viscosity solutions

As discussed in Section 2.3, solutions of the doubly nonlinear heat equation do not possess enough regularity in general to apply the theorems presented in Chapter 3. Therefore, the goal of this chapter is to introduce a weaker notion of solution, namely viscosity solutions, and prove results analogous to those found in Chapter 3 for such solutions. The main appeal of working with viscosity solutions is that they require very low regularity assumptions, needing only to be continuous and not necessarily differentiable. Yet, one can still prove several meaningful results about their existence, uniqueness, and other properties, such as comparison principles.

The notion of viscosity solutions was first proposed in 1983 by Michael G. Crandall and Pierre-Louis Lions [**43**] in the context of Hamilton-Jacobi equations

$$H(x, u, \nabla u) = 0$$

and their time-dependent counterparts. The name "viscosity solution" was chosen as a reference to the already well-established "vanishing viscosity" method for proving existence of solutions to Hamilton-Jacobi equations. Described concisely, the vanishing viscosity method involves regularising the equation $H = 0$ by introducing a viscosity term $-\varepsilon\Delta u_\varepsilon$ and solving

$$-\varepsilon\Delta u_\varepsilon + H_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) = 0$$

for $\varepsilon > 0$ and a suitable approximation H_ε of H. Then, one shows that the solution u_ε of the regularised problem converges in the limit as $\varepsilon \to 0$ to a solution u of the original problem. It turned out that this method is also applicable for demonstrating the existence of viscosity solutions to such problems, providing a motivation for the name.

Since their introduction, the concept of viscosity solutions has been generalised to fully nonlinear second-order equations in both Euclidean space and Riemannian manifolds (see, for instance [**15**]). However, our presentation of this material, as well as our discussion of parabolic Harnack inequalities, will be restricted to the Euclidean setting.

We begin by precisely defining what is meant by a viscosity solution of a fully nonlinear elliptic equation and providing a characterisation of these solutions via the concept of second-order semijets. We also outline the changes needed to adapt this theory for parabolic equations. Next, we give a detailed exploration of the theorem on sums and maximum principle for semicontinuous functions by Crandall and Ishii [**41**], which are essential tools in our analysis that replace fundamental results from classical calculus when working with functions with no assumed differentiability properties. In particular, we develop a new, slightly modified version of the existing parabolic theorem on sums, before applying it to study the Harnack inequalities satisfied by positive viscosity solutions of both the linear Schrödinger equation and doubly nonlinear heat equation.

4.1. The notion of viscosity solutions

We devote this section to introducing and discussing the notion of *viscosity solutions* of elliptic and parabolic equations. Unless stated otherwise, all definitions used in this section can be found in the survey article of Crandall, Ishii and Lions [**42**], which is a standard reference on the topic.

Definition 4.1 (Upper and lower semicontinuous functions)**.** Let $\Omega \subseteq \mathbb{R}^d$. Recall that a function

(i) $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ is *upper semicontinuous* if for all sequences $x_n \to x$ converging in Ω, one has

$$u(x) \geq \limsup_{n\to\infty} u(x_n);$$

(ii) $u : \Omega \to \mathbb{R} \cup \{+\infty\}$ is *lower semicontinuous* if for all sequences $x_n \to x$ converging in Ω, one has

$$u(x) \leq \liminf_{n\to\infty} u(x_n).$$

We use the notations

$$\begin{aligned} \mathrm{USC}(\Omega) &:= \{u : \Omega \to \mathbb{R} \cup \{-\infty\} \mid u \text{ is upper semicontinuous}\}, \text{ and} \\ \mathrm{LSC}(\Omega) &:= \{u : \Omega \to \mathbb{R} \cup \{+\infty\} \mid u \text{ is lower semicontinuous}\}. \end{aligned}$$

Throughout this chapter, we use $\langle \cdot, \cdot \rangle$ to denote the inner product on $\mathbb{R}^d$ and Sym_d to refer to the set of $d \times d$ real symmetric matrices. We equip Sym_d with the *operator norm* $\|\cdot\|$ defined by

$$\|A\| := \sup_{|x|\leq 1} |Ax|.$$

We mention that on the space Sym_d, the operator norm coincides with the *spectral norm* $\|\cdot\|_2$, which is defined by

$$\|A\|_2 := \sqrt{\max\left\{\lambda \in \mathbb{R} \mid \lambda \text{ is an eigenvalue of } A^\top A\right\}}.$$

We also make use of the usual partial order on Sym_d by writing $X \leq Y$ if and only if $X - Y$ is negative semi-definite.

4.1.1. Fully nonlinear degenerate elliptic equations

Definition 4.2 (Fully nonlinear degenerate elliptic equation)**.** By a *fully nonlinear degenerate elliptic equation*, we refer to a partial differential equation of the form

$$F(x, u, \nabla u, D^2 u) = 0 \qquad \text{in } \Omega, \tag{4.1}$$

where $u : \Omega \to \mathbb{R}$ is defined on a domain $\Omega \subseteq \mathbb{R}^d$ and $F : \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathrm{Sym}_d \to \mathbb{R}$ is a continuous function satisfying the *(degenerate) ellipticity condition*

$$F(x, r, p, X) \leq F(x, r, p, Y) \qquad \text{if } X \geq Y \tag{4.2}$$

for all $x \in \Omega$, $r \in \mathbb{R}$, $p \in \mathbb{R}^d$ and $X, Y \in \mathrm{Sym}_d$.

Definition 4.3 (Viscosity solution of (4.1))**.** We call a function

(i) $u \in \mathrm{USC}(\Omega)$ a *viscosity subsolution of* (4.1) if for any $\varphi \in C^2(\Omega)$ and $x_0 \in \Omega$ such that $u - \varphi$ has a local maximum at $x_0 \in \Omega$, one has

$$F(x_0, u(x_0), \nabla\varphi(x_0), D^2\varphi(x_0)) \leq 0;$$

(ii) $u \in \mathrm{LSC}(\Omega)$ a *viscosity supersolution of* (4.1) if for any $\varphi \in C^2(\Omega)$ and $x_0 \in \Omega$ such that $u - \varphi$ has a local minimum at $x_0 \in \Omega$, one has

$$F(x_0, u(x_0), \nabla\varphi(x_0), D^2\varphi(x_0)) \geq 0;$$

(iii) $u \in C(\Omega)$ a *viscosity solution of* (4.1) if u is both a viscosity subsolution and a viscosity supersolution of (4.1).

Let us briefly motivate why Definition 4.3 describes a sensible notion of solution for the equation $F = 0$. Suppose $u \in C^2(\Omega)$ is a classical solution of (4.1), $\varphi \in C^2(\Omega)$, and that $u - \varphi$ has a local maximum at $x_0 \in \Omega$. Then $\nabla(u - \varphi)(x_0) = 0$ and so $\nabla u(x_0) = \nabla\varphi(x_0)$. In addition, $D^2(u - \varphi)(x_0) \leq 0$ and therefore $D^2 u(x_0) \leq D^2\varphi(x_0)$. From the ellipticity condition (4.2) on F, it follows that

$$F(x_0, u(x_0), \nabla\varphi(x_0), D^2\varphi(x_0)) \leq F(x_0, u(x_0), \nabla u(x_0), D^2 u(x_0)) = 0. \tag{4.3}$$

Similarly, if $u - \varphi$ has a local minimum at x_0, then we can conclude

$$F(x_0, u(x_0), \nabla\varphi(x_0), D^2\varphi(x_0)) \geq 0. \tag{4.4}$$

Of course, if (4.3) and (4.4) hold for any $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a local maximum or minimum respectively, we can choose $\varphi = u \in C^2(\Omega)$ since $u - \varphi \equiv 0$ has a local maximum and minimum in every point $x_0 \in \Omega$. From this we can conclude that u satisfies (4.1) in the classical sense.

What we have shown here is that for functions $u \in C^2(\Omega)$, the conditions in Definition 4.3 characterise what it means for u to be a classical solution of (4.1). In other words, the notions of classical and viscosity solutions coincide when $u \in C^2(\Omega)$.

Next, we seek to provide a characterisation of viscosity solutions, which is more useful in practice. For this, we first need to introduce the concept of semijets.

Definition 4.4 (Second-order semijets of semicontinuous functions)**.** Let $\Omega \subseteq \mathbb{R}^d$.

(i) For a function $u \in \mathrm{USC}(\Omega)$ and $x_0 \in \Omega$, we define the *second-order superjet* $J^{2,+}_\Omega u(x_0)$ as the set of $(p, X) \in \mathbb{R}^d \times \mathrm{Sym}_d$ such that

$$u(x) \leq u(x_0) + \langle p, x - x_0 \rangle + \frac{1}{2}\langle X(x - x_0), x - x_0 \rangle + o(|x - x_0|^2) \text{ as } x \to x_0.$$

(ii) For a function $u \in \mathrm{LSC}(\Omega)$ and $x_0 \in \Omega$, we define the *second-order subjet* $J^{2,-}_\Omega u(x_0)$ as the set of $(p, X) \in \mathbb{R}^d \times \mathrm{Sym}_d$ such that

$$u(x) \geq u(x_0) + \langle p, x - x_0 \rangle + \frac{1}{2}\langle X(x - x_0), x - x_0 \rangle + o(|x - x_0|^2) \text{ as } x \to x_0.$$

Remark 4.1.

(i) If x_0 is an interior point of Ω, then $J^{2,+}_\Omega u(x_0) = J^{2,+}_{\Omega'} u(x_0)$ for any other set $\Omega' \subseteq \mathbb{R}^d$ for which x_0 is an interior point. Therefore, we often omit the domain from the notation and simply write $J^{2,+}u(x_0)$. A similar statement is true about $J^{2,-}_\Omega u(x_0)$.

(ii) It is clear from the definitions that $J^{2,-}u(x_0) = -J^{2,+}(-u)(x_0)$. This relationship can be used to reformulate any properties and statements about $J^{2,+}u(x_0)$ in terms of $J^{2,-}u(x_0)$.

(iii) If $\varphi \in C^2(\Omega)$ and there is $(p, X) \in J^2\varphi(x_0) := J^{2,+}u(x_0) \cap J^{2,-}\varphi(x_0)$, then it follows immediately from the definitions that $\nabla\varphi(x_0) = p$ and $D^2\varphi(x_0) = X$.

(iv) If $\varphi \in C^2(\Omega)$, then it follows quickly from the definition of $J^{2,+}\varphi(x_0)$ and the second-order Taylor expansion of φ that

$$J^{2,+}\varphi(x_0) = \{(\nabla\varphi(x_0), X) \mid D^2\varphi(x_0) \leq X\}.$$

Geometrically, one may understand a pair $(p, X) \in J^{2,+}u(x_0)$ as the gradient and Hessian of a local quadratic approximation of u near x_0, which sits above the graph of u. This idea is made more precise by the following characterisation.

Proposition 4.1. *Let $u \in \mathrm{USC}(\Omega)$ and $x_0 \in \Omega$. Then*

$$J^{2,+}u(x_0) = \left\{ (\nabla\varphi(x_0), D^2\varphi(x_0)) \;\middle|\; \begin{array}{l} \varphi \in C^2(\Omega) \text{ and } u - \varphi \text{ has} \\ \text{a local maximum at } x_0 \end{array} \right\}.$$

PROOF. If $\varphi \in C^2(\Omega)$ is such that $u - \varphi$ has a local maximum at x_0, then Taylor's theorem implies

$$\begin{aligned} u(x) &\leq u(x_0) + \varphi(x) - \varphi(x_0) \\ &= u(x_0) + \langle \nabla\varphi(x_0), x - x_0 \rangle + \frac{1}{2}\langle D^2\varphi(x_0)(x - x_0), x - x_0 \rangle + o(|x - x_0|^2) \end{aligned}$$

for $x \to x_0$. Therefore $(\nabla\varphi(x_0), D^2\varphi(x_0)) \in J^{2,+}u(x_0)$.

To prove the reverse inclusion, we explicitly construct a function $\varphi \in C^2(\Omega)$ with the necessary properties. Suppose $(p, X) \in J^{2,+}u(x_0)$. By definition,

$$u(x) \leq u(x_0) + \langle p, x - x_0\rangle + \frac{1}{2}\langle X(x - x_0), x - x_0\rangle + o(|x - x_0|^2) \tag{4.5}$$

as $x \to x_0$ in Ω.

Define a function by

$$R(r) := \sup_{|x-x_0|\leq r} \frac{\big(u(x) - u(x_0) - \langle p, x - x_0\rangle - \frac{1}{2}\langle X(x - x_0), x - x_0\rangle\big)^+}{|x - x_0|^2}$$

for $r \geq 0$ so that

$$u(x) \leq u(x_0) + \langle p, x - x_0\rangle + \frac{1}{2}\langle X(x - x_0), x - x_0\rangle + R(|x - x_0|)|x - x_0|^2.$$

It is clear that $R : [0, \infty) \to [0, \infty)$ is non-decreasing and because of (4.5), R satisfies $\lim_{r\to 0} R(r) = 0$. We may also replace R by a continuous, non-decreasing majorant, which we also call R, such that $R(0) = 0$. Then we can define a new function $\tilde{R}$ by

$$\tilde{R}(t) := \int_0^{3t}\int_0^r R(s)\ \mathrm{d}s\ \mathrm{d}r$$

with $\tilde{R} \in C^2([0, \infty))$ and $\tilde{R}(0) = 0$. In addition, using the positivity and monotonicity of R to manipulate the integration bounds, we have

$$\begin{aligned}\tilde{R}(t) &\geq \int_{2t}^{3t}\int_0^r R(s)\ \mathrm{d}s\ \mathrm{d}r\\ &\geq \int_{2t}^{3t}\int_t^{2t} R(s)\ \mathrm{d}s\ \mathrm{d}r\\ &\geq \int_{2t}^{3t}\int_t^{2t} R(t)\ \mathrm{d}s\ \mathrm{d}r = t^2R(t).\end{aligned}$$

In particular,

$$u(x) \leq u(x_0) + \langle p, x - x_0\rangle + \frac{1}{2}\langle X(x - x_0), x - x_0\rangle + \tilde{R}(|x - x_0|).$$

If we define the function $\varphi : \Omega \to \mathbb{R}$ in the statement of Proposition 4.1 by

$$\varphi(x) = u(x_0) + \langle p, x - x_0\rangle + \frac{1}{2}\langle X(x - x_0), (x - x_0)\rangle + \tilde{R}(|x - x_0|),$$

then $u(x) - \varphi(x) \leq 0$ holds with equality at $x = x_0$. In other words, $u - \varphi$ has a maximum at $x = x_0$.

We claim $\varphi \in C^2(\Omega)$ with $\nabla\varphi(x_0) = p$, $D^2\varphi(x_0) = X$. It is clear that φ is C^2 at points $x \in \Omega \setminus \{x_0\}$, however, we must justify why this property extends also to the point $x = x_0$. In particular, we would like to prove that the gradient and Hessian of $\psi(x) := \tilde{R}(|x - x_0|)$ exist, are continuous, and vanish at $x = x_0$.

Since R is non-decreasing, we make the estimates

$$\tilde{R}(t) = \int_0^{3t}\int_0^r R(s)\ \mathrm{d}s\ \mathrm{d}r \leq \int_0^{3t}\int_0^r R(3t)\ \mathrm{d}s\ \mathrm{d}r = \frac{9}{2}t^2R(3t) \tag{4.6}$$

and

$$\tilde{R}'(t) = 3\int_0^{3t} R(r)\,\mathrm{d}r \le 3\int_0^{3t} R(3t)\,\mathrm{d}r = 9tR(3t). \tag{4.7}$$

We know that ψ is clearly differentiable on $\Omega \setminus \{0\}$ and for $x = x_0$ one has by (4.6) that

$$0 \le \frac{|\psi(x_0+h) - \psi(x_0))|}{|h|} = \frac{|\tilde{R}(|h|)|}{|h|} \le \frac{9}{2}|h|R(3|h|) \to 0.$$

Therefore

$$\nabla\psi(x) = \begin{cases} \tilde{R}'(|x-x_0|)\frac{x-x_0}{|x-x_0|} & x \ne x_0, \\ 0 & x = x_0. \end{cases}$$

Moreover, $\nabla\psi$ is continuous on Ω. In particular, $\nabla\psi$ is continuous at 0 since, again using (4.6),

$$\begin{aligned} |\nabla\psi(x)| &= \left|\tilde{R}'(|x-x_0|)\frac{x-x_0}{|x-x_0|}\right| \\ &= \left|3\int_0^{3|x-x_0|} R(s)\,\mathrm{d}s\right| \\ &\le 9|x-x_0|R(|x-x_0|) \to 0 \end{aligned}$$

as $x \to x_0$. Next, we see that ψ is twice differentiable on Ω, and in particular at $x = x_0$ since by (4.7), one has

$$0 \le \frac{|\nabla\psi(x_0+h) - \nabla\psi(x_0)|}{|h|} = \frac{\tilde{R}'(|h|)}{|h|} \le 9R(3|h|) \to 0$$

as $h \to 0$. Therefore

$$D^2\psi(x) = \begin{cases} \tilde{R}''(|x-x_0|)\dfrac{(x-x_0)(x-x_0)^\top}{|x-x_0|^2} & \\ \qquad + \tilde{R}'(|x-x_0|)\left(\dfrac{1}{|x-x_0|}I_d - \dfrac{(x-x_0)(x-x_0)^\top}{|x-x_0|^3}\right) & x \ne x_0, \\ 0 & x = x_0. \end{cases}$$

In addition, $D^2\psi$ is continuous on Ω. To see that $D^2\psi$ is continuous at $x = x_0$, we use (4.7) to obtain that

$$\begin{aligned} \|D^2\psi(x)\| &= \Bigg\|\tilde{R}''(|x-x_0|)\frac{(x-x_0)(x-x_0)^\top}{|x-x_0|^2} \\ &\qquad + \tilde{R}'(|x-x_0|)\left(\frac{1}{|x-x_0|}I_d - \frac{(x-x_0)(x-x_0)^\top}{|x-x_0|^3}\right)\Bigg\| \\ &\le 9R(3|x-x_0|) + 2\frac{\tilde{R}'(|x-x_0|)}{|x-x_0|} \\ &\le 9R(3|x-x_0|) + 18|x-x_0|\frac{R(3|x-x_0|)}{|x-x_0|} = 27R(3|x-x_0|) \to 0 \end{aligned}$$

as $x \to x_0$. Hence, $\psi \in C^2(\Omega)$, and as a result, $\varphi \in C^2(\Omega)$ as well. Then, it is clear that $\nabla\varphi(x_0) = p$ and $D^2\varphi(x_0) = X$. □

Proposition 4.1 justifies the following characterisation of viscosity solutions.

Theorem 4.2. *Let $u \in \mathrm{USC}(\Omega)$. Then u is a viscosity subsolution of* (4.1) *if and only if*

$$F(x_0, u(x_0), p, X) \leq 0$$

for all $(p, X) \in J^{2,+}u(x_0)$ and $x_0 \in \Omega$.

A similar characterisation can also be formulated for viscosity supersolutions in the obvious way.

One may also define the closures of the semijets in the following way.

Definition 4.5.

(i) For a given function $u \in \mathrm{USC}(\Omega)$, we define $\overline{J}^{2,+}u(x_0)$ to be the set of all pairs $(p, X) \in \mathbb{R}^d \times \mathrm{Sym}_d$, for which there exist sequences

$$(p_n, X_n)_{n\geq 1} \subseteq \mathbb{R}^d \times \mathrm{Sym}_d \quad \text{and} \quad (x_n)_{n\geq 1} \subseteq \Omega$$

such that $(p_n, X_n) \in J^{2,+}u(x_n)$ and

$$(p_n, X_n) \to (p, X), \quad x_n \to x_0, \quad u(x_n) \to u(x_0)$$

as $n \to \infty$.

(ii) For a given function $u \in \mathrm{LSC}(\Omega)$, we define $\overline{J}^{2,-}u(x_0)$ to be the set of all pairs $(p, X) \in \mathbb{R}^d \times \mathrm{Sym}_d$, for which there exist sequences

$$(p_n, X_n)_{n\geq 1} \subseteq \mathbb{R}^d \times \mathrm{Sym}_d \quad \text{and} \quad (x_n)_{n\geq 1} \subseteq \Omega$$

such that $(p_n, X_n) \in J^{2,-}u(x_n)$ and

$$(p_n, X_n) \to (p, X), \quad x_n \to x_0, \quad u(x_n) \to u(x_0)$$

as $n \to \infty$.

Proposition 4.3. *Let $u \in \mathrm{USC}(\Omega)$. Then u is a viscosity subsolution of* (4.1) *if and only if*

$$F(x_0, u(x_0), p, X) \leq 0$$

for all $(p, X) \in \overline{J}^{2,+}u(x_0)$.

Proof. By the definition of $\overline{J}^{2,+}u(x_0)$, there exist sequences

$$(p_n, X_n)_{n\geq 1} \subseteq \mathbb{R}^d \times \mathrm{Sym}_d \quad \text{and} \quad (x_n)_{n\geq 1} \subseteq \Omega$$

such that $(p_n, X_n) \in J^{2,+}u(x_n)$ and

$$(p_n, X_n) \to (p, X), \quad x_n \to x_0, \quad u(x_n) \to u(x_0)$$

as $n \to \infty$. Then the characterisation of viscosity subsolutions in Theorem 4.2 implies

$$F(x_n, u(x_n), p_n, X_n) \leq 0$$

for all $n \geq 1$. Taking $n \to 0$ yields

$$F(x_0, u(x_0), p, X) \leq 0$$

since F is continuous by assumption. □

4.1.2. Viscosity solutions of parabolic equations

In this work, we are interested in studying solutions u of parabolic equations of the form

$$\partial_t u + F(x, u(x), \nabla u, D^2 u) = 0 \qquad \text{in } \Omega \times (0,T) \tag{4.8}$$

where $\Omega \subseteq \mathbb{R}^d$ is a domain, $T > 0$, and F is again continuous and satisfies the ellipticity condition (4.2). Therefore, we must extend the notion of viscosity solutions to such equations. In the literature, there are generally two approaches to do this. One option is to continue using the definitions introduced in the previous section while interpreting the variable x as a pair (x,t). This means the gradient ∇u and Hessian matrix $D^2 u$ should now be understood as $\nabla_{(x,t)} u$ and $D^2_{(x,t)} u$ respectively. However, this is not entirely natural to do, since one must then account for the second-order time derivative $\partial_{tt} u$, which is not usually a consideration in the parabolic setting. Therefore, many authors, including this one, prefer to utilise a second approach, which in a way disregards the derivative $\partial_{tt} u$. This is achieved by adjusting the definitions of the semijets.

Definition 4.6 (Parabolic semijets of semicontinuous functions)**.** Let $\Omega \subseteq \mathbb{R}^d$, $I_i \subset \mathbb{R}$ be an interval for $i = 1, \dots, k$, and $S := I_1 \times \dots \times I_k \subseteq \mathbb{R}^k$.

(i) For a function $u \in \mathrm{USC}(\Omega \times S)$, $x_0 \in \Omega$, $t_0 \in S$, we define the *parabolic superjet* $\mathcal{P}^{2,+}_{\Omega \times S} u(x_0, t_0)$ as the set of $(a, p, X) \in \mathbb{R}^k \times \mathbb{R}^d \times \mathrm{Sym}_d$ such that

$$\begin{aligned} u(x,t) \le\ & u(x_0,t_0) + \langle a, t - t_0 \rangle + \langle p, x - x_0 \rangle \\ & + \frac{1}{2} \langle X(x - x_0), (x - x_0) \rangle + o(|t_0 - t|_1 + |x - x_0|^2) \end{aligned}$$

as $x \to x_0$ in Ω and $t_i \to t_{0i}^-$ for $i = 1, \dots, k$.

(ii) For a function $u \in \mathrm{LSC}(\Omega \times S)$, $x_0 \in \Omega$, $t_0 \in S$, we define the *parabolic subjet* $\mathcal{P}^{2,-}_{\Omega \times S} u(x_0, t_0)$ as the set of $(a, p, X) \in \mathbb{R}^k \times \mathbb{R}^d \times \mathrm{Sym}_d$ such that

$$\begin{aligned} u(x,t) \ge\ & u(x_0,t_0) + \langle a, t - t_0 \rangle + \langle p, x - x_0 \rangle \\ & + \frac{1}{2} \langle X(x - x_0), (x - x_0) \rangle + o(|t_0 - t|_1 + |x - x_0|^2) \end{aligned}$$

as $x \to x_0$ in Ω and $t_i \to t_{0i}^-$ for $i = 1, \dots, k$.

Remark 4.2. We immediately mention that our definition of $\mathcal{P}^{2,\pm} u(x_0, t_0)$ does not coincide with the standard definition encountered in literature, which may be found, for example, in [**42**].

(i) We allow for the possibility that the function u may depend on multiple time variables, as is the case for the function ω utilised in our proofs of the Harnack inequality.

(ii) In the typical definition in literature (for $k = 1$), the inequalities are understood in the two-sided limit $t \to t_0$. However, as commented by Juutinen [**86**], one often proceeds in the parabolic setting with the intuition that the behaviour of a solution to a parabolic equation at time t_0 is determined only by information about the solution at times before t_0 and not after. Therefore, in our definition, we only consider $t_i < t_{0i}$ in the one-sided limits $t_i \to t_{0i}^-$.

(iii) In Definition 4.6, $|\cdot|_1$ denotes the 1-norm and

$$|t_0 - t|_1 = \sum_{i=1}^{k} |t_{0i} - t_i| = \sum_{i=1}^{k} (t_{0i} - t_i),$$

since we always take $t_i < t_{0i}$.

(iv) As is the case for $J_\Omega^{2,\pm} u(x_0)$, the domain of the function $u : \Omega \times S \to \mathbb{R}$ is often not relevant when considering $\mathcal{P}^{2,\pm}_{\Omega\times S} u(x_0, t_0)$, since the definition is equivalent when one replaces $\Omega \times S$ with any other domain for which (x_0, t_0) is an interior point. In addition, if $I_i = (a, b]$ for some i and we consider the parabolic jets of u at a point (x_0, t_0) such that $t_{0i} = b$, then since we are only concerned with the behaviour of u to the left of $t_{0i} = b$, we have that $\mathcal{P}^{2,\pm}_{\Omega\times S} u(x_0, t_0) = \mathcal{P}^{2,\pm}_{\Omega\times S'} u(x_0, t_0)$, where S' is constructed by replacing the interval I_i in the definition of S by another interval I_i' such that $I_i \subset I_i'$ and b is an interior point of I_i'.

As in the elliptic case, we also define the closures of the parabolic semijets.

Definition 4.7.

(i) For given a function $u \in \mathrm{USC}(\Omega \times S)$, $x_0 \in \Omega$, $t_0 \in S$, we define $\overline{\mathcal{P}}^{2,+} u(x_0, t_0)$ to be the set of all triples $(a, p, X) \in \mathbb{R}^k \times \mathbb{R}^d \times \mathrm{Sym}_d$, for which there exist sequences

$$(a_n, p_n, X_n)_{n\geq 1} \subseteq \mathbb{R}^k \times \mathbb{R}^d \times \mathrm{Sym}_d \quad \text{and}$$
$$(x_n, t_n)_{n\geq 1} \subseteq \Omega \times S$$

such that $(a_n, p_n, X_n) \in \mathcal{P}^{2,+} u(x_n, t_n)$ and

$$(a_n, p_n, X_n) \to (a, p, X), \quad x_n \to x_0, \quad t_{ni} \to t_{0i}^-, \quad \text{and}$$
$$u(x_n, t_n) \to u(x_0, t_0)$$

as $n \to \infty$.

(ii) For given a function $u \in \mathrm{LSC}(\Omega \times S)$, $x_0 \in \Omega$, $t_0 \in S$, we define $\overline{\mathcal{P}}^{2,-} u(x_0, t_0)$ to be the set of all triples $(a, p, X) \in \mathbb{R}^k \times \mathbb{R}^d \times \mathrm{Sym}_d$, for which there exist sequences

$$(a_n, p_n, X_n)_{n\geq 1} \subseteq \mathbb{R}^k \times \mathbb{R}^d \times \mathrm{Sym}_d \quad \text{and}$$
$$(x_n, t_n)_{n\geq 1} \subseteq \Omega \times S$$

such that $(a_n, p_n, X_n) \in \mathcal{P}^{2,-} u(x_n, t_n)$ and

$$(a_n, p_n, X_n) \to (a, p, X), \quad x_n \to x_0, \quad t_{ni} \to t_{0i}^-, \quad \text{and}$$
$$u(x_n, t_n) \to u(x_0, t_0)$$

as $n \to \infty$.

In view of Theorem 4.2, we define viscosity solutions of (4.8) in the following manner.

Definition 4.8 (Viscosity solution of (4.8))**.** We call a function

(i) $u \in \mathrm{USC}(\Omega \times (0,T))$ a *viscosity subsolution of* (4.8) if

$$a + F(x_0, u(x_0), p, X) \leq 0$$

for all $(a, p, X) \in \mathcal{P}^{2,+}u(x_0, t_0)$ and $(x_0, t_0) \in \Omega \times (0,T)$;

(ii) $u \in \mathrm{LSC}(\Omega \times (0,T))$ a *viscosity supersolution of* (4.8) if

$$a + F(x_0, u(x_0), p, X) \geq 0$$

for all $(a, p, X) \in \mathcal{P}^{2,-}u(x_0, t_0)$ and $(x_0, t_0) \in \Omega \times (0,T)$;

(iii) $u \in C(\Omega \times (0,T))$ a *viscosity solution of* (4.8) if u is both a viscosity subsolution and viscosity supersolution of (4.8).

4.2. The maximum principle for semicontinuous functions

So far to prove Harnack inequalities, we have been analysing functions of the form

$$Z(x, y; t, s) = u_1(x,t) + u_2(y,s) - \varphi(x, y; t, s),$$

where the functions u_1, u_2, and φ are at least C^2 in the spatial variables and C^1 in the time variables. Therefore, if such a function Z attains a maximum at some point $P_0 := (x_0, y_0; t_0, s_0)$, we are permitted to apply classical results in calculus to conclude $\nabla_{(x,y)} Z(P_0) = 0$ and $D^2_{(x,y)} Z(P_0) \leq 0$. In particular, this implies

$$\nabla u_1(x_0, t_0) = \nabla_x \varphi(P_0), \quad \nabla u_2(y_0, s_0) = \nabla_y \varphi(P_0), \quad \text{and}$$
$$\begin{pmatrix} D^2 u_1(x_0, t_0) & 0 \\ 0 & D^2 u_2(y_0, s_0) \end{pmatrix} \leq D^2_{(x,y)} \varphi(P_0).$$

However, if the functions u_1 and u_2 have lower regularity, for instance, if they are viscosity solutions of some parabolic equation, which may not even be differentiable, then this analysis is no longer valid.

The goal of the theorems discussed in this section is to provide analogues of these statements, which are true even when the functions u_1 and u_2 are merely semicontinuous. The main theorem we study is the *theorem on sums*, which can be found in the work of Crandall and Ishii [**41**].

Theorem 4.4 (Theorem on sums, [**41**, Theorem 1])**.** *Let $u_i \in \mathrm{USC}(\mathbb{R}^{d_i})$ for $i = 1, \ldots, k$ and set $d := d_1 + \ldots + d_k$. Consider the function $u : \mathbb{R}^d \to \mathbb{R} \cup \{-\infty\}$ defined by*

$$u(x) := u_1(x_1) + \ldots + u_k(x_k),$$

for all $x := (x_1, \ldots, x_k)^\top \in \mathbb{R}^d$. Suppose that for $p := (p_1, \ldots, p_k)^\top \in \mathbb{R}^d$ and $X \in \mathrm{Sym}_d$, one has $(p, X) \in J^{2,+}u(x)$. Then for every $\kappa > 0$, there exists

$X_i \in \mathrm{Sym}_{d_i}$ *such that*

$$(p_i, X_i) \in \overline{J}^{2,+} u_i(x_i) \qquad \textit{for } i = 1, \ldots, k$$

and

$$-\left(\frac{1}{\kappa} + \|X\|\right) I_d \leq \begin{pmatrix} X_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X_k \end{pmatrix} \leq X + \kappa X^2, \tag{4.9}$$

where $\|X\|$ denotes the spectral norm of X.

Before we prove this result, we briefly introduce some concepts that will be encountered in the proof.

Definition 4.9 (Semiconvex function)**.** A function $\varphi : \mathbb{R}^d \to \mathbb{R}$ is called *semiconvex* if there is a $\lambda \in \mathbb{R}$ such that the shifted function $\varphi_\lambda(x) := \varphi(x) + \frac{\lambda}{2}|x|^2$ is convex. If φ_λ is convex for a particular $\lambda \in \mathbb{R}$, we say that φ is λ-*semiconvex.*

A basic property of semiconvex functions is as follows.

Proposition 4.5. *Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be λ-semiconvex. If φ is twice differentiable at $x \in \mathbb{R}^d$, then $D^2\varphi(x) \geq -\lambda I$.*

Proof. The statement follows immediately from the classical result, that the Hessian matrix of a convex function is positive semi-definite. ☐

In addition to Proposition 4.5, we require two more advanced results concerning semiconvex functions. Proofs of the following propositions can be found in the appendix of [**42**].

Proposition 4.6 (Alexandrov's theorem, [**2**])**.** *Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be semiconvex. Then φ is twice continuously differentiable almost everywhere in $\mathbb{R}^d$.*

Proposition 4.7 (Jensen's lemma, [**85**])**.** *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be continuous and λ-semiconvex for $\lambda > 0$, and suppose φ has a strict local maximum at x_0. For $p \in \mathbb{R}^d$, set $\varphi_p(x) := \varphi(x) + \langle p, x - x_0\rangle$. Then for all $r, \delta > 0$, the set*

$$K_{\delta,r} := \left\{ x \in \overline{B}_r(x_0) \,\middle|\, \exists p \in B_\delta(0) \textit{ such that } \varphi_p \textit{ has a local maximum at } x \right\}$$

has positive measure. Moreover, one has the estimate

$$\mu(K_{\delta,r}) \geq \left(\frac{\delta}{\lambda}\right)^d V_d,$$

where V_d denotes the volume of the unit ball in $\mathbb{R}^d$.

Idea of proof. We present the main idea of the proof in the simpler case that $\varphi \in C^2(\Omega)$. If f is merely continuous, the proof may be completed by first mollifying the function f. The details of this argument can be found in [**42**] or [**91**].

We first show that if φ_p attains a maximum in $\overline{B}_r(x_0)$ for some $r > 0$, then the maximiser x occurs away from the boundary of this ball.

Suppose $r > 0$ is small enough so that $\varphi(x_0) > \varphi(x)$ for all $x \in \overline{B}_r(x_0)$. Then

$$\rho_r := \varphi(x_0) - \max_{\overline{B}_r(x_0)\setminus B_{r/2}(x_0)} \varphi > 0.$$

Let $p \in \mathbb{R}^d$ with $|p| = \delta > 0$. For any $x \in \overline{B}_r(x_0) \setminus B_{r/2}(x_0)$,

$$\begin{aligned}\max_{\overline{B}_r(x_0)} \varphi_p - \varphi_p(x) &\geq \varphi(x_0) - \max_{\overline{B}_r(x_0)\setminus B_{r/2}(x_0)} \varphi_p \\ &\geq \varphi(x_0) - \max_{\overline{B}_r(x_0)\setminus B_{r/2}(x_0)} \varphi - \delta r \\ &= \rho_r - \delta r.\end{aligned}$$

By taking δ small enough so that $\rho_r - \delta r > 0$ and x to be the maximiser of φ_p over $\overline{B}_r(x_0) \setminus B_{r/2}(x_0)$, it follows that

$$\max_{\overline{B}_r(x_0)} \varphi_p > \max_{\overline{B}_r(x_0)\setminus B_{r/2}(x_0)} \varphi_p.$$

Therefore, we may assume φ_p attains its maximum on $\overline{B}_r(x_0)$ at an interior point $x \in B_r(x_0)$.

Then, at a maximum point $x \in K_{\delta,r}$ of $\varphi_p(x) = \varphi(x) + \langle p, x - x_0\rangle$ with $|p| < \delta$, one has $p = -\nabla\varphi(x)$. It follows that for δ small enough, $B_\delta(0) \subseteq \nabla\varphi(K_{\delta,r})$. Moreover, one has

$$D^2\varphi(x) = D^2\varphi_p(x) \leq 0.$$

Since φ is λ-semiconvex, by Proposition 4.5, we have

$$-\lambda I \leq D^2\varphi(x) \leq 0.$$

This implies that all eigenvalues of $D^2\varphi(x)$ are negative and lie between $-\lambda$ and 0. Therefore

$$|\det D^2\varphi(x)| \leq \lambda^d$$

for each $x \in K_{\delta,r}$. Integrating over $K_{\delta,r}$, and using that $B_\delta(0) \subseteq \nabla\varphi(K_{\delta,r})$, we have

$$\lambda^d \mu(K_{\delta,r}) \geq \int_{K_{\delta,r}} |\det D^2\varphi(x)| \,\mathrm{d}x \geq \mu(\nabla\varphi(K_{\delta,r})) \geq \mu(B_\delta(0)).$$

We note that if $\nabla\varphi$ is a diffeomorphism on $K_{\delta,r}$, then the middle inequality holds with equality as a consequence of the change-of-variables formula. However, in general, we may only have an inequality (see [**91**]).

From this, we deduce the claimed estimate

$$|K_{\delta,r}| \geq \left(\frac{\delta}{\lambda}\right)^d V_d.$$

□

Definition 4.10 (Sup-convolution)**.** Let $\Omega \subseteq \mathbb{R}^d$ and suppose $u \in \mathrm{USC}(\Omega)$ is bounded from above. For a fixed $\lambda > 0$, the *sup-convolution* $\hat{u} : \mathbb{R}^d \to \mathbb{R}$ of u is defined by

$$\hat{u}(\xi) := \sup_{x\in\Omega} \left(u(x) - \frac{\lambda}{2}|x - \xi|^2 \right)$$

for every $\xi \in \mathbb{R}^d$.

Proposition 4.8. *Let $\Omega \subseteq \mathbb{R}^d$ and suppose $u \in \mathrm{USC}(\Omega)$ is bounded from above. Then the sup-convolution $\hat{u}$ is λ-semiconvex.*

PROOF. The claim follows by noticing that

$$\hat{u}(\xi) + \frac{\lambda}{2}|\xi|^2 = \sup_{x \in \Omega} \left(u(x) - \frac{\lambda}{2}|x|^2 + \lambda\langle x, \xi\rangle \right)$$

is convex as the supremum of convex functions. □

We now begin to outline the proof of Theorem 4.4. Our first aim is to reduce the premises of Theorem 4.4 to a simpler case. First, applying the definition of $(p, X) \in J^{2,+}u(x)$, we know that

$$u(x+y) \le u(x) + \langle p, y\rangle + \frac{1}{2}\langle Xy, y\rangle + o(|y|^2),$$

which, because $u = u_1 + \ldots + u_k$, is equivalent to

$$\begin{aligned} u_1(x_1+y_1) + \ldots + u_k(x_k+y_k) &\le u_1(x_1) + \ldots + u_k(x_k) \\ &\quad + \langle p_1, y_1\rangle + \ldots + \langle p_k, y_k\rangle + \frac{1}{2}\langle Xy, y\rangle + o(|y|^2) \end{aligned}$$

for $y \to 0$. Therefore, if we define $\tilde{u}_i(y_i) := u_i(x_i + y_i) - u_i(x_i) - \langle p_i, y_i\rangle$, then we have

$$\tilde{u}_1(y_1) + \ldots + \tilde{u}_k(y_k) \le \frac{1}{2}\langle Xy, y\rangle + o(|y|^2).$$

This implies that for all $\eta > 0$, there is $\delta > 0$ such that

$$\tilde{u}_1(y_1) + \ldots + \tilde{u}_k(y_k) \le \frac{1}{2}\langle Xy, y\rangle + \frac{\eta}{2}\sum_{i=1}^{k} |y_i|^2$$

for all $y_i \in \mathbb{R}^{d_i}$ such that $0 < |y_i| \le \delta$ and $i = 1, \ldots, k$. In addition, we note that by the definition of $\tilde{u}_i(y_i)$, we have $\tilde{u}_i(0) = 0$ for all $i = 1, \ldots, k$. Hence, if we define

$$v_i(y_i) := \begin{cases} 0 & y_i = 0, \\ \tilde{u}_i(y_i) - \frac{\eta}{2}|y_i|^2 & |y_i| \le \delta, \\ -\infty & |y_i| > \delta \end{cases} \tag{4.10}$$

for each $i = 1, \ldots, k$, it follows that

$$v_1(y_1) + \ldots + v_k(y_k) \le \frac{1}{2}\langle Xy, y\rangle$$

is true for all $x \in \mathbb{R}^d$. Thus, we focus on proving the following result, which is a special case of Theorem 4.4.

Theorem 4.9. *Let* $v_i \in \mathrm{USC}(\mathbb{R}^{d_i})$ *for all* $i = 1, \ldots, k$, $d = d_1 + \ldots + d_k$, *and* $X \in \mathrm{Sym}_d$ *be such that* $v_i(0) = 0$ *for all* $i = 1, \ldots, k$ *and*

$$v_1(y_1) + \ldots + v_k(y_k) \leq \frac{1}{2}\langle Xy, y\rangle \tag{4.11}$$

holds for all $y \in \mathbb{R}^d$. *Then for every* $\kappa > 0$, *there exists* $X_i \in \mathrm{Sym}_{d_i}$ *such that*

$$(0, X_i) \in \overline{J}^{2,+} v_i(0) \qquad \textit{for } i = 1, \ldots, k$$

and (4.9) *holds.*

Before proving Theorem 4.9, we will explain why it implies Theorem 4.4. Suppose the functions v_i are defined as in (4.10). Then Theorem 4.9 implies that for every $\eta > 0$ and $i = 1, \ldots, k$, there is $X_{i,\eta} \in \mathrm{Sym}_{d_i}$ such that one has $(0, X_{i,\eta}) \in \overline{J}^{2,+} v_i(0)$. By definition, there are sequences $(x_{i,\eta}^{(n)})_{n \geq 1} \subseteq \mathbb{R}^{d_i}$ and $(p_{i,\eta}^{(n)}, X_{i,\eta}^{(n)})_{n\geq 1} \subseteq \mathbb{R}^{d_i} \times \mathrm{Sym}_{d_i}$ such that $(p_{i,\eta}^{(n)}, X_{i,\eta}^{(n)}) \in J^{2,+} v_i(x_{i,\eta}^{(n)})$ and

$$x_{i,\eta}^{(n)} \to 0, \quad v_i(x_{i,\eta}^{(n)}) \to 0, \quad \text{and} \quad (p_{i,\eta}^{(n)}, X_{i,\eta}^{(n)}) \to (0, X_{i,\eta})$$

as $n \to \infty$. Then, for each $n \geq 1$, one has

$$v_i(x_{i,\eta}^{(n)} + y_i) \leq v_i(x_{i,\eta}^{(n)}) + \langle p_{i,\eta}^{(n)}, y_i\rangle + \frac{1}{2}\langle X_{i,\eta}^{(n)} y_i, y_i\rangle + o(|y_i|^2)$$

for $y_i \to 0$. Expressing this in terms of the original functions u_i yields

$$\begin{aligned} u_i(x_i + x_{i,\eta}^{(n)} + y_i) \leq u_i(x_i + x_{i,\eta}^{(n)}) &+ \langle p_i + p_{i,\eta}^{(n)} + \eta x_{i,\eta}^{(n)}, y_i\rangle \\ &+ \frac{1}{2}\langle (X_{i,\eta}^{(n)} + \eta I_{d_i}) y_i, y_i\rangle + o(|y_i|^2), \end{aligned}$$

which implies

$$(p_i + p_{i,\eta}^{(n)} + \eta x_{i,\eta}^{(n)}, X_{i,\eta}^{(n)} + \eta I_{d_i}) \in J^{2,+} u(x_i + x_{i,\eta}^{(n)})$$

for all $n \geq 1$, $\eta > 0$, and $i = 1, \ldots, k$. Since $v_i(x_{i,\eta}^{(n)}) \to 0$ and $x_{i,\eta}^{(n)} \to 0$ imply $u(x_i + x_{i,\eta}^{(n)}) \to u(x_i)$, and we also have that $p_{i,\eta}^{(n)} \to 0$, $x_{i,\eta}^{(n)} \to 0$ and $X_{i,\eta}^{(n)} \to X_{i,\eta}$, we may conclude $(p_i, X_{i,\eta} + \eta I_{d_i}) \in \overline{J}^{2,+} u(x_i)$ for all $\eta > 0$ and $i = 1, \ldots, k$.

Now, applying (4.9), we have

$$-\left(\frac{1}{\kappa} + \|X\|\right) I_d \leq \begin{pmatrix} X_{1,\eta} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X_{k,\eta} \end{pmatrix} \leq X + \kappa X^2,$$

which implies, in particular, that

$$-\left(\frac{1}{\kappa} + \|X\|\right) I_{d_i} \leq X_{i,\eta} \leq (X + \kappa X^2)_i$$

holds for all $\eta > 0$, where $(X + \kappa X^2)_i$ denotes a particular principal minor of $X + \kappa X^2$. Hence, the matrices $X_{i,\eta}$ lie within a compact set. Moreover, there exists a sequence $\eta_n \to 0$ for which $X_{i,\eta}$ converges to some limit, which we call X_i. It is clear that X_i satisfies (4.9) and $(p_i, X_i) \in \overline{J}^{2,+} u(x_i)$ as desired.

We now focus our attention on proving Theorem 4.9. First, we observe that for any $d \times d$ symmetric matrix X, we can estimate $\langle Xy, y\rangle$ by

$$\langle Xy, y\rangle \leq \left(\frac{1}{\kappa} + \|X\|\right)|y-\xi|^2 + \langle (X+\kappa X^2)\xi, \xi\rangle \tag{4.12}$$

for any $\kappa > 0$ and $\xi \in \mathbb{R}^d$. Indeed, with the help of Young's inequality, we see that

$$\begin{aligned}\langle Xy, y\rangle &= \langle X(y-\xi+\xi), y-\xi+\xi\rangle \\ &= \langle X(y-\xi), y-\xi\rangle + 2\langle X\xi, y-\xi\rangle + \langle X\xi, \xi\rangle \\ &\leq \|X\||y-\xi|^2 + \frac{1}{\kappa}|y-\xi|^2 + \kappa|X\xi|^2 + \langle X\xi, \xi\rangle \\ &= \left(\frac{1}{\kappa} + \|X\|\right)|y-\xi|^2 + \langle (X+\kappa X^2)\xi, \xi\rangle.\end{aligned}$$

Throughout the remainder of the proof, we will write $\lambda := \left(\frac{1}{\kappa} + \|X\|\right)$. Combining the inequalities (4.11) and (4.12) therefore leads to

$$v_1(y_1) + \ldots + v_k(y_k) - \frac{\lambda}{2}|y_i - \xi_i|^2 \leq \frac{1}{2}\langle (X+\kappa X^2)\xi, \xi\rangle.$$

Therefore, the sup-convolutions $\hat{v}_i(\xi_i) = \sup_{y_i \in \mathbb{R}^{d_i}} \left(v_i(y_i) - \frac{\lambda}{2}|y_i - \xi_i|^2\right)$ satisfy

$$\hat{v}_1(\xi_1) + \ldots + \hat{v}_k(\xi_k) \leq \frac{1}{2}\langle (X+\kappa X^2)\xi, \xi\rangle$$

for every $\xi \in \mathbb{R}^d$. In particular,

$$\hat{v}_1(0) + \ldots + \hat{v}_k(0) \leq 0.$$

However, $\hat{v}_i(0) = \sup_{y_i \in \mathbb{R}^{d_i}} \left(v_i(y_i) - \frac{\lambda}{2}|y_i|^2\right) \geq 0$, since $v_i(y_i) - \frac{\lambda}{2}|y_i|^2 = 0$ for $y_i = 0$. Therefore, $\hat{v}_i(0) = 0$ for all $i = 1, \ldots, k$. In other words,

$$\hat{v}_1(\xi_1) + \ldots + \hat{v}_k(\xi_k) - \frac{1}{2}\langle (X+\kappa X^2)\xi, \xi\rangle \leq 0$$

for every $\xi \in \mathbb{R}^d$ and equality is attained at $\xi = 0$.

The next step of the proof will be achieved through the following lemma.

Lemma 4.10. *If $f : \mathbb{R}^d \to \mathbb{R}$ is a continuous and λ-semiconvex function, and $B \in \mathrm{Sym}_d$ is such that*

$$\max_{\xi \in \mathbb{R}^d} \left(f(\xi) - \frac{1}{2}\langle B\xi, \xi\rangle\right) = f(0),$$

then there is a matrix $A \in \mathrm{Sym}_d$ such that

$$(0, A) \in \overline{J}^2 f(0) := \overline{J}^{2,+} f(0) \cap \overline{J}^{2,-} f(0)$$

and $-\lambda I \leq A \leq B$.

Proof. By assumption, the function $f_B(\xi) := f(\xi) - \frac{1}{2}\langle B\xi, \xi\rangle$ has a maximum at $\xi = 0$. Then for all $\delta > 0$, the function $f_B(\xi) - \delta|\xi|^2$ has a strict maximum at $\xi = 0$. Since the function $f_B - \delta|\cdot|^2$ is also semiconvex, we may apply Jensen's lemma (Proposition 4.7) to deduce that for all $\delta > 0$, there is a point $q_\delta \in \mathbb{R}^d$ such that $|q_\delta| \le \delta$ and

$$f(\xi) + \langle q_\delta, \xi\rangle - \frac{1}{2}\langle B\xi, \xi\rangle - \delta|\xi|^2$$

has a local maximum at some point $\xi_\delta \in \mathbb{R}^d$ with $|\xi_\delta| \le \delta$. In light of Alexandrov's theorem (Proposition 4.6), we may assume ξ_δ is such that f is C^2 at ξ_δ. Therefore $\nabla f(\xi_\delta)$ and $D^2 f(\xi_\delta)$ exist and satisfy

$$\nabla f(\xi_\delta) = 2\delta\xi_\delta + B\xi_\delta - q_\delta = O(\delta) \quad \text{and} \quad -\lambda I \le D^2 f(\xi_\delta) \le B + 2\delta I,$$

where the lower bound on $D^2 f(\xi_\delta)$ is due to the semiconvexity of f and Proposition 4.5. Moreover, $(\nabla f(\xi_\delta), D^2 f(\xi_\delta)) \in J^2 f(\xi_\delta)$. In the limit as $\delta \to 0$, we have $\nabla f(\xi_\delta) \to 0$ and, possibly after passing to a subsequence, $D^2 f(\xi_\delta)$ will converge to some limit, which we call A. Then $(0, A) \in \overline{J}^2 f(0)$ and $-\lambda I \le A \le B$. □

We continue with the proof of Theorem 4.9. Letting

$$v(y) = v_1(y_1) + \ldots + v_k(y_k),$$

we apply Lemma 4.10 with $B = X + \kappa X^2$ to the function

$$\hat{v}(\xi) = \sup_{y \in \mathbb{R}^d} \left(v(y) - \frac{\lambda}{2}|y - \xi|^2 \right),$$

which is λ-semiconvex because of Proposition 4.8. Therefore, there is a matrix $A \in \mathrm{Sym}_d$ such that $(0, A) \in \overline{J}^2 \hat{v}(0)$ and $-\lambda I \le A \le X + \kappa X^2$. We may also assume A takes the form

$$A = \begin{pmatrix} X_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X_k \end{pmatrix}, \tag{4.13}$$

where the matrices $X_i \in \mathrm{Sym}_{d_i}$ are such that $(0, X_i) \in \overline{J}^2 \hat{v}_i(0)$. Indeed, by the definition of $(0, A) \in \overline{J}^2 \hat{v}(0) = \overline{J}^{2,+} \hat{v}(0) \cap \overline{J}^{2,-} \hat{v}(0)$, more specifically, the definition of $\overline{J}^{2,+} \hat{v}(0)$, there exists $(x^{(n)}, p^{(n)}, X^{(n)})_{n \ge 1} \subseteq \mathbb{R}^d \times \mathbb{R}^d \times \mathrm{Sym}_d$ such that $(p^{(n)}, X^{(n)}) \in J^{2,+}\hat{v}(x^{(n)})$ for all $n \ge 1$ and

$$(x^{(n)}, \hat{v}(x^{(n)}), p^{(n)}, X^{(n)}) \to (0, \hat{v}(0), 0, A)$$

as $n \to \infty$. This means

$$\begin{aligned} &\hat{v}_1(x_1^{(n)} + y_1) + \ldots + \hat{v}_k(x_k^{(n)} + y_k) \\ &\qquad \le \hat{v}_1(x_1^{(n)}) + \ldots + \hat{v}_k(x_k^{(n)}) + \langle p_1^{(n)}, y_1\rangle + \ldots + \langle p_k^{(n)}, y_k\rangle \\ &\qquad\quad + \frac{1}{2}\langle X^{(n)} y, y\rangle + o(|y|^2) \end{aligned}$$

for all $y \in \mathbb{R}^d$ with $|y|^2 > 0$ small enough. If for a particular $i \in \{1, \ldots, k\}$, we choose $y_j = 0$ for all $j \ne i$, then

$$\hat{v}_i(x_i^{(n)} + y_i) \le \hat{v}_i(x_i^{(n)}) + \langle p_i^{(n)}, y_i\rangle + \frac{1}{2}\langle X_i^{(n)} y_i, y_i\rangle + o(|y_i|^2) \tag{4.14}$$

as $|y_i|^2 \to 0$, which implies $(0, X_i) \in \overline{J}^{2,+}\hat{v}_i(0)$. Moreover, summing (4.14) over i shows that $(0, A) \in \overline{J}^{2,+}\hat{v}(0)$, where A is the matrix in (4.13). By a similar argument, we obtain $(0, A) \in \overline{J}^{2,-}\hat{v}(0)$ as well, and so $(0, A) \in \overline{J}^{2}\hat{v}(0)$.

Now that we have identified an element $(0, X_i) \in \overline{J}^{2,+}\hat{v}_i(0)$, the only step remaining in the proof of Theorem 4.9 is to justify why $(0, X_i) \in \overline{J}^{2,+}v_i(0)$. This will be handled by the next lemma, but otherwise the proof is complete.

Lemma 4.11. *Let $\lambda > 0$, $v \in \mathrm{USC}(\mathbb{R}^d)$ be bounded from above. If $q, r \in \mathbb{R}^d$, $Y \in \mathrm{Sym}_d$, and $(q, Y) \in J^{2,+}\hat{v}(r)$, then $(q, Y) \in J^{2,+}v(r + \frac{q}{\lambda})$ and*

$$\hat{v}(r) + \frac{|q|^2}{2\lambda} = v(r + \frac{q}{\lambda}).$$

Moreover, if $(0, X) \in \overline{J}^{2,+}\hat{v}(0)$, then $(0, X) \in \overline{J}^{2,+}v(0)$.

Proof. First, we note that since $v(x) - \frac{\lambda}{2}|x - r|^2$ is upper semicontinuous on $\mathbb{R}^d$ and $\lim_{|x|\to\infty}\big(v(x) - \frac{\lambda}{2}|x - r|^2\big) = -\infty$, standard optimisation theorems imply there exists $y \in \mathbb{R}^d$ such that

$$\hat{v}(r) = \sup_{x\in\mathbb{R}^d}\left(v(x) - \frac{\lambda}{2}|x - r|^2\right) = v(y) - \frac{\lambda}{2}|y - r|^2.$$

Suppose $(q, Y) \in J^{2,+}\hat{v}(r)$. Then for all $x, \xi \in \mathbb{R}^d$,

$$\begin{aligned} v(x) - \frac{\lambda}{2}|x - \xi|^2 \le \hat{v}(\xi) &\le \hat{v}(r) + \langle q, (\xi - r)\rangle \\ &\qquad + \frac{1}{2}\langle Y(\xi - r), (\xi - r)\rangle + o(|\xi - r|^2) \\ &= v(y) - \frac{\lambda}{2}|y - r|^2 + \langle q, \xi - r\rangle \\ &\qquad + \frac{1}{2}\langle Y(\xi - r), (\xi - r)\rangle + o(|\xi - r|^2). \end{aligned}$$

On one hand, if we choose $\xi = x - y + r$, we have

$$v(x) \le v(y) + \langle q, (x - y)\rangle + \frac{1}{2}\langle Y(x - y), x - y\rangle + o(|x - y|^2),$$

which tells us $(q, Y) \in J^{2,+}v(y)$. On the other hand, if we set $x = y$ and $\xi = r + \alpha(\lambda(r - y) + q)$ for $\alpha \in \mathbb{R}$ arbitrary, then

$$\begin{aligned} 0 &\le \frac{\lambda}{2}|y - \xi|^2 - \frac{\lambda}{2}|y - r|^2 + \langle q, \xi - r\rangle + O(\alpha^2) \\ &= \frac{\lambda}{2}(|\xi|^2 - |r|^2) + \langle q - \lambda y, \xi - r\rangle + O(\alpha^2) \\ &= \frac{\lambda}{2}(|r + \alpha(\lambda(r - y) + q)|^2 - |r|^2) + \alpha\langle q - \lambda y, \lambda(r - y) + q\rangle + O(\alpha^2) \\ &= \alpha\lambda r(\lambda(r - y) + q) + \alpha\langle q - \lambda y, \lambda(r - y) + q\rangle + O(\alpha^2) \\ &= \alpha|\lambda(r - y) + q|^2 + O(\alpha^2). \end{aligned}$$

If we consider $\alpha < 0$, this implies $|\lambda(r-y)+q|^2 = 0$ and so $y = r + \frac{q}{\lambda}$. Therefore $(q, Y) \in J^{2,+}v(r + \frac{q}{\lambda})$ and

$$\hat{v}(r) = v(y) - \frac{\lambda}{2}|y-r|^2 = v\left(r + \frac{q}{\lambda}\right) - \frac{|q|^2}{2\lambda}$$

as claimed.

To prove the final statement of the lemma, we suppose $(0, X) \in \overline{J}^{2,+}\hat{v}(0)$. Then there is $(q_n, X_n) \in J^{2,+}\hat{v}(\xi_n)$ with

$$\xi_n \to 0, \quad \hat{v}(\xi_n) \to \hat{v}(0), \quad \text{and} \quad (q_n, X_n) \to (0, X)$$

as $n \to \infty$. By the first part of the lemma, we have that $(q_n, X_n) \in J^{2,+}v\left(\xi_n + \frac{q_n}{\lambda}\right)$ for all $n \geq 1$ and $v\left(\xi_n + \frac{q_n}{\lambda}\right) = \hat{v}(\xi_n) + \frac{|q_n|^2}{2\lambda}$. Since $x_n \to 0$, $q_n \to 0$, and $X_n \to X$, if we can show $v\left(\xi_n + \frac{q_n}{\lambda}\right) \to v(0)$, we will have proven the claim that $(0, X) \in \overline{J}^{2,+}v(0)$. Since $\xi_n + \frac{q_n}{\lambda} \to 0$ and v is upper semicontinuous,

$$v(0) \geq \limsup_{n\to\infty} v\left(\xi_n + \frac{q_n}{\lambda}\right) = \limsup_{n\to\infty}\left(\hat{v}(\xi_n) + \frac{1}{2\lambda}|q_n|^2\right) = \hat{v}(0) \geq v(0).$$

Therefore, $\limsup_{n\to\infty} v\left(\xi_n + \frac{q_n}{\lambda}\right) = v(0)$ and after passing to a subsequence, we have $\lim_{n\to\infty} v\left(\xi_n + \frac{q_n}{\lambda}\right) = v(0)$ as desired. □

In practice, the following corollary of Theorem 4.4 is quite useful.

Corollary 4.12 (Maximum principle for semicontinuous functions, [**41**, Example 1])**.** *Let $\Omega \subseteq \mathbb{R}^d$ be locally compact, $u_i \in \mathrm{USC}(\Omega)$ for $i = 1, \ldots, k$, and $\varphi \in C^2(\Omega^k)$. Consider the function*

$$u(x) := u_1(x_1) + \ldots + u_k(x_k) - \varphi(x)$$

defined for all $x = (x_1, ..., x_k)^\top \in \Omega^k$. If $Z := u - \varphi$ has a maximum at $x_0 \in \Omega^k$, then for all $\kappa > 0$, there exist $X_i \in \mathrm{Sym}_d$ such that $(\nabla_{x_i}\varphi(x_0), X_i) \in \overline{J}^{2,+}u_i(x_{0i})$ for $i = 1, \ldots k$ and

$$-\left(\frac{1}{\kappa} + \|X\|\right) I_d \leq \begin{pmatrix} X_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X_k \end{pmatrix} \leq X + \kappa X^2, \tag{4.15}$$

for $X = D^2\varphi(x_0)$.

Proof. Let K_i be a compact neighbourhood of x_{0i} in Ω for $i = 1, \ldots, k$. We define functions $\tilde{u}_i \in \mathrm{USC}(\mathbb{R}^d)$ by

$$\tilde{u}_i(x_i) = \begin{cases} u_i(x_i) & \text{if } x_i \in K_i, \\ -\infty & \text{if } x_i \in \mathbb{R}^d \setminus K_i. \end{cases}$$

If we set $\tilde{u}(x) = \tilde{u}_1(x_1) + \ldots + \tilde{u}_k(x_k)$, then $\tilde{Z} := \tilde{u} - \varphi$ has a maximum at x_0. Therefore, since φ is C^2, Taylor's theorem implies

$$\begin{aligned}\tilde{u}(x) &\leq \tilde{u}(x_0) + \varphi(x) - \varphi(x_0) \\ &= \tilde{u}(x_0) + \nabla\varphi(x_0)(x - x_0) + \frac{1}{2}\langle D^2\varphi(x_0)(x-x_0), x - x_0\rangle + o(|x-x_0|^2)\end{aligned}$$

for $x \to x_0$. Thus $(\nabla\varphi(x_0), D^2\varphi(x_0)) \in J^{2,+}\tilde{u}(x_0)$. Using Theorem 4.4, for every $\kappa > 0$, we find a matrix $X_i \in \mathrm{Sym}_d$ such that the conclusion of Corollary 4.12 holds with u_i replaced by $\tilde{u}_i$. In particular, we have $(\nabla_{x_i}\varphi(x_0), X_i) \in \overline{J}^{2,+}\tilde{u}_i(x_{0i})$. However, any sequence $(x_n)_{n\geq 1} \subseteq \Omega$ from the definition of $\overline{J}^{2,+}\tilde{u}_i(x_{0i})$ such that $x_n \to x_{0i}$ and $\tilde{u}_i(x_n) \to \tilde{u}_i(x_{0i})$ must eventually lie within K_i, since $\tilde{u}_i$ takes the value $-\infty$ outside of K_i and $\tilde{u}_i(x_n)$ should approach the finite value $\tilde{u}_i(x_{0i})$. Since u_i and $\tilde{u}_i$ coincide on K_i, it follows that $(\nabla_{x_i}\varphi(x_0), X_i) \in \overline{J}^{2,+}u_i(x_{0i})$. $\square$

Remark 4.3. The statements of the theorem on sums (Theorem 4.4) and maximum principle for semicontinuous functions (Corollary 4.12) are no longer true if one replaces the closure $\overline{J}^{2,+}u_i(x_i)$ in the conclusion by $J^{2,+}u_i(x_i)$. We justify this by means of a counterexample, which was presented in the article by Crandall and Ishii [**41**]. If one defines functions $u, v : \mathbb{R} \to \mathbb{R}$ by

$$u(x) := \begin{cases} x^2 & x \geq 0, \\ -2x^2 & x < 0 \end{cases}$$

and $v(x) := u(-x)$, then by checking each case for the signs of x and y, we find that

$$u(x) + v(y) \leq 2(x-y)^2$$

for all $x, y \in \mathbb{R}$, with equality attained at $x = y = 0$. If Corollary 4.12 were true without the closure in the conclusion, then for every $\kappa > 0$, we would be able to find numbers $X, Y \in \mathbb{R}$ such that $(0, X) \in J^{2,+}u(0)$, $(0, Y) \in J^{2,+}v(0)$, and

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq \kappa \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

In particular, this would imply $X \leq \kappa$. However, one may compute that

$$J^{2,+}u(0) = \{(0, X) \mid X \geq 2\},$$

and therefore, for $\kappa > 0$ small enough, it is no longer possible to find such an X.

For our purposes, we would like to apply the idea of Corollary 4.12 to situations, where the functions u_i are viscosity solutions of a parabolic equation. In particular, we will require analogues of Theorem 4.4 and Corollary 4.12, where the functions u_i depend also on a time variable t_i. In the case that each u_i depends on the same time variable t, results of this kind are readily available (see, for instance, Section 3 in [**41**]). However, these will not be suitable in our context, since we always work with functions depending on different time variables. Thus, we will now prove a new parabolic version of Theorem 4.4 and Corollary 4.12.

Theorem 4.13 (Parabolic theorem on sums)**.** *For $i = 1, \ldots, k$, let $\Omega_i \subseteq \mathbb{R}^{d_i}$ be locally compact and set $d := d_1 + \ldots + d_k$. Let $u_i \in \mathrm{USC}(\overline{\Omega_i} \times [S_i, T_i])$ and consider the function*

$$u(x,t) := u_1(x_1, t_1) + \ldots + u_k(x_k, t_k)$$

defined for all $x := (x_1, \ldots, x_k)^\top$, $t := (t_1, \ldots, t_k)^\top$ with $x_i \in \Omega_i$ and $t_i \in (S_i, T_i]$. Suppose that for $a := (a_1, \ldots, a_k)^\top \in \mathbb{R}^k$, $p := (p_1, \ldots, p_k)^\top \in \mathbb{R}^d$, $X \in \mathrm{Sym}_d$, *and* $(x,t) \in \Omega_1 \times \ldots \times \Omega_k \times (S_1,T_1] \times \ldots \times (S_k,T_k]$, *one has that*

$$(a,p,X) \in \mathcal{P}^{2,+}u(x,t).$$

Then for all $\kappa > 0$, *there are matrices* $X_i \in \mathrm{Sym}_{d_i}$ *such that for all* $i = 1,\ldots,k$,

$$(a_i,p_i,X_i) \in \overline{\mathcal{P}}^{2,+}u_i(x_i,t_i)$$

and

$$-\left(\frac{1}{\kappa} + \|X\|\right) I_d \le \begin{pmatrix} X_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X_k \end{pmatrix} \le X + \kappa X^2.$$

Proof. By the definition of $(a,p,X) \in \mathcal{P}^{2,+}u(x,t)$, one has

$$\begin{aligned} u(y,s) &\le u(x,t) + \langle a, s-t\rangle + \langle p, y-x\rangle \\ &\quad + \frac{1}{2}\langle X(y-x), y-x\rangle + o(t_1-s_1) + o\left(|y-x|^2 + \sum_{i=1}^k (t_i - s_i)\right) \end{aligned}$$

as $(y,s) \to (x,t)$ such that $s_i < t_i$ for all $i = 1,\ldots,k$. More precisely, for every $\eta > 0$, one can find a sufficiently small compact neighbourhood K_i of (x_i,t_i) such that $s_i \le t_i$ whenever $(y_i,s_i) \in K_i$ and

$$\begin{aligned} &u_1(y_1,s_1) + \ldots + u_k(y_k,s_k) \\ &\quad \le u_1(x_1,t_1) + \ldots + u_k(x_k,t_k) \\ &\qquad + \langle a, s-t\rangle + \langle p, y-x\rangle + \frac{1}{2}\langle X(y-x), y-x\rangle \\ &\qquad + \frac{\eta}{2}(t_1-s_1) + \ldots + \frac{\eta}{2}(t_k-s_k) + \frac{\eta}{2}|y-x|^2 \end{aligned}$$

for all $(y,s) \in U_1 \times \ldots \times U_k$.

If we define functions $\tilde{u}_i : \mathbb{R}^{d_i+1} \to \mathbb{R} \cup \{-\infty\}$ by

$$\tilde{u}_i(y_i,s_i) := \begin{cases} u(y_i,s_i) - u(x_i,t_i) & \\ \quad - a_i(s_i-t_i) - \langle p_i, y_i - x_i\rangle & \text{if } (y_i,s_i) \in K_i, \\ \quad - \frac{\eta}{2}(t_i-s_i) - \frac{\eta}{2}|y_i-x_i|^2 & \\ -\infty & \text{if } (y_i,s_i) \in \mathbb{R}^{d_i+1} \setminus K_i, \end{cases}$$

then

$$\tilde{u}_1(y_1,s_1) + \ldots + \tilde{u}_k(y_k,s_k) \le \frac{1}{2}\langle X(y-x), y-x\rangle$$

holds on $\mathbb{R}^{d+k}$. In particular, since $\tilde{u}_i(x_i,t_i) = 0$ for each $i = 1,\ldots,k$, we see that equality is attained at the point $(y,s) = (x,t)$. Therefore, the function

$$\tilde{u}_1 + \ldots + \tilde{u}_k - \frac{1}{2}\langle X(\cdot - x), \cdot - x\rangle$$

attains a maximum at the point (x,t). We note that the upper semicontinuity of u and the compactness of K_i will imply $\tilde{u}_i \in \mathrm{USC}(\mathbb{R}^{d_i+1})$ for all $i = 1,\ldots,k$, and thus we may apply the maximum principle for semicontinuous functions

(Theorem 4.4) to conclude that for every $\kappa > 0$, there exist matrices $\tilde{X}_{i,\eta} \in \mathrm{Sym}_d$ such that

$$(0, \tilde{X}_{i,\eta}) \in \overline{J}^{2,+}\tilde{u}_i(x_i, t_i)$$

and

$$-\left(\frac{1}{\kappa} + \|A\|\right) I_{d+k} \leq \begin{pmatrix} \tilde{X}_{1,\eta} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{X}_{k,\eta} \end{pmatrix} \leq A + \kappa A^2,$$

where

$$A = \begin{pmatrix} X_{11} & 0 & \cdots & X_{1k} & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ X_{k1} & 0 & \cdots & X_{kk} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

is the Hessian of $\frac{1}{2}\langle X(\cdot - x), \cdot - x\rangle$ in all variables $x_1, t_1, \ldots, x_k, t_k$. Here X_{ij} denotes the ij^{th} block of the matrix $X \in \mathrm{Sym}_d$ when partitioned in the natural way according to the decomposition $d = d_1 + \ldots + d_k$. By considering the appropriate principle minor of the above matrices, this condition implies

$$-\left(\frac{1}{\kappa} + \|A\|\right) I_d \leq \begin{pmatrix} X_{1,\eta} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X_{k,\eta} \end{pmatrix} \leq A + \kappa A^2,$$

where we now use $X_{i,\eta}$ to denote the truncation of the matrix $\tilde{X}_{i,\eta}$, which removes the last row and column. In addition, $(0, \tilde{X}_{i,\eta}) \in \overline{J}^{2,+}\tilde{u}_i(x_i, t_i)$ implies that $(0, 0, X_{i,\eta}) \in \overline{\mathcal{P}}^{2,+}\tilde{u}_i(x_i, t_i)$. Indeed, by the definition of $(0, \tilde{X}_{i,\eta}) \in \overline{J}^{2,+}\tilde{u}_i(x_i, t_i)$, there are sequences

$$(\tilde{p}_{i,\eta}^{(n)}, \tilde{X}_{i,\eta}^{(n)})_{n\geq 1} \subseteq \mathbb{R}^{d_i+1} \times \mathrm{Sym}_{d_i+1} \quad \text{and} \quad (x_{i,\eta}^{(n)})_{n\geq 1} \subseteq \mathbb{R}^{d_i+1}$$

such that

$$\begin{aligned} &(\tilde{p}_{i,\eta}^{(n)}, \tilde{X}_{i,\eta}^{(n)}) \in J^{2,+}\tilde{u}_i(x_{i,\eta}^{(n)}, t_{i,\eta}^{(n)}), \quad (\tilde{p}_{i,\eta}^{(n)}, \tilde{X}_{i,\eta}^{(n)}) \to (0, \tilde{X}_{i,\eta}) \\ &(x_{i,\eta}^{(n)}, t_{i,\eta}^{(n)}) \to (x_i, t_i), \quad \text{and} \quad \tilde{u}_i(x_{i,\eta}^{(n)}, t_{i,\eta}^{(n)}) \to \tilde{u}_i(x_i, t_i) \end{aligned}$$

as $n \to \infty$. Therefore

$$\begin{aligned} \tilde{u}_i(y_i, s_i) &\leq \tilde{u}_i(x_{i,\eta}^{(n)}, t_{i,\eta}^{(n)}) + \langle \tilde{p}_{i,\eta}^{(n)}, (y_i, s_i) - (x_{i,\eta}^{(n)}, t_{i,\eta}^{(n)})\rangle \\ &\quad + \frac{1}{2}\langle \tilde{X}_{i,\eta}^{(n)}((y_i, s_i) - (x_{i,\eta}^{(n)}, t_{i,\eta}^{(n)})), (y_i, s_i) - (x_{i,\eta}^{(n)}, t_{i,\eta}^{(n)})\rangle \\ &\quad + o(|y_i - x_{i,\eta}^{(n)}|^2 + (t_{i,\eta}^{(n)} - s_i)^2) \\ &= \tilde{u}_i(x_{i,\eta}^{(n)}, t_{i,\eta}^{(n)}) + a_{i,\eta}^{(n)}(s_i - t_{i,\eta}^{(n)}) + \langle p_{i,\eta}^{(n)}, y_i - x_{i,\eta}^{(n)}\rangle \\ &\quad + \frac{1}{2}\langle X_{i,\eta}^{(n)}(y_i - x_{i,\eta}^{(n)}), y_i - x_{i,\eta}^{(n)}\rangle + o(|y_i - x_{i,\eta}^{(n)}|^2 + |t_{i,\eta}^{(n)} - s_i|) \end{aligned}$$

as $(y_i, s_i) \to (x_{i,\eta}^{(n)}, t_i^{(n)})$ in $\mathbb{R}^{d_i+1}$, where $\tilde{p}_{i,\eta}^{(n)} = (p_{i,\eta}^{(n)}, a_{i,\eta}^{(n)})$ and $X_{i,\eta}^{(n)}$ is the truncation of $\tilde{X}_{i,\eta}^{(n)}$, which removes the final row and column. In particular, for $s_i \leq t_{i,\eta}^{(n)}$,

$$\begin{aligned}\tilde{u}_i(y_i, s_i) \leq \tilde{u}_i(x_{i,\eta}^{(n)}, t_{i,\eta}^{(n)}) &+ a_{i,\eta}^{(n)}(s_i - t_{i,\eta}^{(n)}) + \langle p_{i,\eta}^{(n)}, y_i - x_{i,\eta}^{(n)} \rangle \\ &+ \frac{1}{2}\langle X_{i,\eta}^{(n)}(y_i - x_{i,\eta}^{(n)}), y_i - x_{i,\eta}^{(n)} \rangle + o(|y_i - x_{i,\eta}^{(n)}|^2 + t_{i,\eta}^{(n)} - s_i)\end{aligned}$$

as $y_i \to x_{i,\eta}^{(n)}$ in $\mathbb{R}^{d_i}$ and $s_i \to t_i^{(n)}$ in $\mathbb{R}$ from below. We note that since $\tilde{u}_i(x_i, t_i)$ is finite and $\tilde{u}_i$ was set to be $-\infty$ outside of K_i, there must be a subsequence of $(x_{i,\eta}^{(n)}, t_{i,\eta}^{(n)})_{n\geq 1}$, which lies in K_i. In particular, we have $t_{i,\eta}^{(n)} \leq t_i$. We work with this subsequence, again denoting it by $(x_{i,\eta}^{(n)}, t_{i,\eta}^{(n)})_{n\geq 1}$. Then, it follows that $(a_{i,\eta}^{(n)}, p_{i,\eta}^{(n)}, X_{i,\eta}^{(n)}) \in \mathcal{P}^{2,+}\tilde{u}(x_{i,\eta}^{(n)}, t_{i,\eta}^{(n)})$ for each $n \geq 1$, and so we have $(0, 0, X_{i,\eta}) \in \overline{\mathcal{P}}^{2,+}\tilde{u}_i(x_i, t_i)$.

Since the points $(x_{i,\eta}^{(n)}, t_{i,\eta}^{(n)})$ and (y_i, s_i) can be taken to lie in K_i, we use the definition of $\tilde{u}_i$ on K_i, to obtain

$$\begin{aligned}u_i(y_i, s_i) \leq u_i(x_{i,\eta}^{(n)}, t_{i,\eta}^{(n)}) &+ \left(a_{i,\eta}^{(n)} + a_i - \frac{\eta}{2}\right)(s_i - t_{i,\eta}^{(n)}) + \langle p_i + p_{i,\eta}^{(n)}, y_i - x_{i,\eta}^{(n)} \rangle \\ &+ \frac{1}{2}\langle X_{i,\eta}^{(n)}(y_i - x_{i,\eta}^{(n)}), y_i - x_{i,\eta}^{(n)} \rangle + \frac{\eta}{2}|y_i - x_i|^2 - \frac{\eta}{2}|x_{i,\eta}^{(n)} - x_i|^2 \\ &+ o(|y_i - x_{i,\eta}^{(n)}|^2 + t_{i,\eta}^{(n)} - s_i).\end{aligned}$$

Rewriting $\frac{\eta}{2}|y_i - x_i|^2 - \frac{\eta}{2}|x_{i,\eta}^{(n)} - x_i|^2$ as

$$\begin{aligned}&\frac{\eta}{2}\left(|y_i - x_{i,\eta}^{(n)} + x_{i,\eta}^{(n)} - x_i|^2 - |x_{i,\eta}^{(n)} - x_i|^2\right) \\ &\quad = \frac{\eta}{2}|y_i - x_{i,\eta}^{(n)}|^2 + \eta\langle x_{i,\eta}^{(n)} - x_i, y_i - x_{i,\eta}^{(n)} \rangle\end{aligned}$$

leads to

$$\begin{aligned}u_i(y_i, s_i) \leq u_i(x_{i,\eta}^{(n)}, t_{i,\eta}^{(n)}) &+ \left(a_{i,\eta}^{(n)} + a_i - \frac{\eta}{2}\right)(s_i - t_{i,\eta}^{(n)}) \\ &+ \langle p_i + p_{i,\eta}^{(n)} + \eta(x_{i,\eta}^{(n)} - x_i), y_i - x_{i,\eta}^{(n)} \rangle \\ &+ \frac{1}{2}\langle (X_{i,\eta}^{(n)} + \eta I_{d_i})(y_i - x_{i,\eta}^{(n)}), y_i - x_{i,\eta}^{(n)} \rangle \\ &+ o(|y_i - x_{i,\eta}^{(n)}|^2 + t_{i,\eta}^{(n)} - s_i).\end{aligned}$$

This expresses that

$$\left(a_{i,\eta}^{(n)} + a_i - \frac{\eta}{2}, p_i + p_{i,\eta}^{(n)} + \eta(x_{i,\eta}^{(n)} - x_i), X_{i,\eta}^{(n)} + \eta I_{d_i}\right) \in \mathcal{P}^{2,+}u_i(x_{i,\eta}^{(n)}, t_{i,\eta}^{(n)}).$$

Considering the behaviour as $n \to \infty$, we find that

$$\left(a_i - \frac{\eta}{2}, p_i, X_{i,\eta} + \eta I_{d_i}\right) \in \overline{\mathcal{P}}^{2,+}u_i(x_i, t_i).$$

Taking $\eta \to 0$ as in the proof of Crandall and Ishii, we find that there is a matrix $X_i \in \mathrm{Sym}_{d_i}$ such that $X_{i,\eta} \to X_i$ and so we have that $(a_i, p_i, X_i) \in \overline{\mathcal{P}}^{2,+}u_i(x_i, t_i)$. □

We may use the parabolic theorem of sums to deduce a similar statement to Corollary 4.12.

Corollary 4.14 (Parabolic maximum principle for semicontinuous functions I)**.** *For $i = 1, \dots, k$, let $\Omega_i \subseteq \mathbb{R}^{d_i}$ be locally compact, and set $\Omega := \Omega_1 \times \ldots \times \Omega_k$ and $S := (S_1, T_1] \times \ldots \times (S_k, T_k]$. Let $u_i \in \mathrm{USC}(\overline{\Omega_i} \times [S_i, T_i])$ and let $\varphi : \Omega \times S \to \mathbb{R}$, $\varphi := \varphi(x_1, \dots, x_k, t_1, \dots, t_k)$ be twice continuously differentiable in each of the variables x_i and continuously differentiable in the variables t_i for $i = 1, \dots, k$. Consider the function*

$$Z(x,t) := u_1(x_1,t_1) + \ldots + u_k(x_k,t_k) - \varphi(x,t)$$

defined for all $x := (x_1, \dots, x_k)^\top \in \Omega$ and $t := (t_1, \dots, t_k)^\top \in S$. If there exists a point $(x_0, t_0) \in \Omega \times S$ such that Z touches the value $Z(x_0, t_0)$ from below for the first time at (x_0, t_0), then for all $\kappa > 0$, there are matrices $X_i \in \mathrm{Sym}_{d_i}$ such that for all $i = 1, \dots, k$,

$$(\partial_{t_i}\varphi(x_0,t_0), \nabla_{x_i}\varphi(x_0,t_0), X_i) \in \overline{\mathcal{P}}^{2,+} u_i(x_{0i}, t_{0i})$$

and

$$-\left(\frac{1}{\kappa} + \|X\|\right) I_d \le \begin{pmatrix} X_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X_k \end{pmatrix} \le X + \kappa X^2,$$

where $X = D_x^2\varphi(x_0,t_0)$ and $d := d_1 + \ldots + d_k$.

Proof. The premise of Corollary 4.14 is described by the inequality

$$u(x,t) \le u(x_0,t_0) + \varphi(x,t) - \varphi(x_0,t_0)$$

holding for $x \in \Omega$ and t with $t_i \le t_{0,i}$. Since $\varphi \in C^2(Q)$, we may apply the Taylor expansion of φ to obtain

$$\begin{aligned} u(x,t) \le u(x_0,t_0) &+ \langle \nabla_t\varphi(x_0,t_0), t - t_0\rangle + \langle \nabla_x\varphi(x_0,t_0), x - x_0\rangle \\ &+ \frac{1}{2}\langle D_x^2\varphi(x_0,t_0)(x-x_0), x - x_0\rangle + o\left(|t_0 - t|_1 + |x - x_0|^2\right) \end{aligned}$$

as $x \to x_0$ in Ω and $t_i \to t_{0i}^-$, which exactly expresses that

$$(\nabla_t\varphi(x_0,t_0), \nabla_x\varphi(x_0,t_0), D_x^2\varphi(x_0,t_0)) \in \mathcal{P}^{2,+}u(x_0,t_0).$$

The conclusion now follows immediately from Theorem 4.13. □

We also state the following variation of Corollary 4.14, which will be more readily applicable to the proofs of Harnack inequalities, in which we typically consider a function attaining a minimum rather than a maximum. The corollary that follows can be obtained using the relation $\mathcal{P}^{2,-}u(x_0,t_0) = -\mathcal{P}^{2,+}(-u)(x_0,t_0)$.

Corollary 4.15 (Parabolic maximum principle for semicontinuous functions II)**.** *For $i = 1, \dots, k$, let $\Omega_i \subseteq \mathbb{R}^{d_i}$ be locally compact, and set $\Omega := \Omega_1 \times \ldots \times \Omega_k$ and $S := (S_1, T_1] \times \ldots \times (S_k, T_k]$. Let $u_i \in \mathrm{LSC}(\overline{\Omega_i} \times [S_i, T_i])$ and let $\varphi : \Omega \times S \to \mathbb{R}$, $\varphi := \varphi(x_1, \dots, x_k, t_1, \dots, t_k)$ be twice continuously differentiable in each of the variables x_i and continuously differentiable in the variables t_i for $i = 1, \dots, k$. Consider the function*

$$Z(x,t) := u_1(x_1,t_1) + \ldots + u_k(x_k,t_k) - \varphi(x,t)$$

defined for all $x := (x_1, \dots, x_k)^\top \in \Omega$ *and* $t := (t_1, \dots, t_k)^\top \in S$. *If there exists a point* $(x_0, t_0) \in \Omega \times S$ *such that* Z *touches the value* $Z(x_0, t_0)$ *from above for the first time at* (x_0, t_0), *then for all* $\kappa > 0$, *there are matrices* $X_i \in \mathrm{Sym}_{d_i}$ *such that for all* $i = 1, \dots, k$,

$$(\partial_{t_i}\varphi(x_0, t_0), \nabla_{x_i}\varphi(x_0, t_0), X_i) \in \overline{\mathcal{P}}^{2,-} u_i(x_{0i}, t_{0i})$$

and

$$\left(\frac{1}{\kappa} + \|X\|\right) I_d \geq \begin{pmatrix} X_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X_k \end{pmatrix} \geq X - \kappa X^2,$$

where $X = D_x^2\varphi(x_0, t_0)$.

4.3. Harnack inequalities for viscosity solutions

In this section, we demonstrate how to employ our multi-point maximum principle approach to prove Harnack inequalities satisfied by viscosity solutions of parabolic equations. Following the structure of Chapter 3, we begin by discussing viscosity solutions of the linear Schrödinger equation to first illuminate the core ideas of the proof before we re-introduce the computational complexities encountered when dealing with the doubly nonlinear heat equation, which we will discuss in the second part of this section. We assume throughout that $\Omega \subseteq \mathbb{R}^d$ is a bounded domain.

4.3.1. The linear Schrödinger equation

Here, we are concerned with viscosity solutions of the Neumann problem

$$\begin{cases} \partial_t u = \Delta u - Vu & \text{in } \Omega \times (0,T), \\ \nabla u \cdot \nu = 0 & \text{on } \partial\Omega \times (0,T) \end{cases} \tag{4.16}$$

for the Schrödinger equation for a given C^2-potential function V.

Definition 4.11. We call a function

(i) $u \in \mathrm{USC}(\overline{\Omega} \times (0,T))$ a *viscosity subsolution of* (4.16) if

$$a - \operatorname{tr} X + V(x_0)u(x_0, t_0) \leq 0$$

for all $(a, p, X) \in \overline{\mathcal{P}}^{2,+}_{\overline{\Omega}\times(0,T)} u(x_0, t_0)$ and $(x_0, t_0) \in \Omega \times (0,T)$, and

$$\min\{a - \operatorname{tr} X + V(x_0)u(x_0, t_0),\ p \cdot \nu(x_0)\} \leq 0$$

for all $(a, p, X) \in \overline{\mathcal{P}}^{2,+}_{\overline{\Omega}\times(0,T)} u(x_0, t_0)$ and $(x_0, t_0) \in \partial\Omega \times (0,T)$;

(ii) $u \in \mathrm{LSC}(\overline{\Omega} \times (0,T))$ a *viscosity supersolution of* (4.16) if

$$a - \operatorname{tr} X + V(x_0)u(x_0, t_0) \geq 0$$

for all $(a, p, X) \in \overline{\mathcal{P}}^{2,-}_{\overline{\Omega}\times(0,T)} u(x_0, t_0)$ and $(x_0, t_0) \in \Omega \times (0, T)$, and

$$\max\{a - \operatorname{tr} X + V(x_0)u(x_0, t_0),\ p \cdot \nu(x_0)\} \geq 0$$

for all $(a, p, X) \in \overline{\mathcal{P}}^{2,-}_{\overline{\Omega}\times(0,T)} u(x_0, t_0)$ and $(x_0, t_0) \in \partial\Omega \times (0, T)$;

(iii) $u \in C(\overline{\Omega} \times (0, T))$ a *viscosity solution of* (4.16) if u is both a viscosity subsolution and viscosity supersolution of (4.16).

As in the proof of Theorem 3.2 for classical solutions, we wish to make the transformation $v = \log u$. Therefore, we must also understand the transformed problem

$$\begin{cases} \partial_t v = \Delta v + |\nabla v|^2 - V & \text{in } \Omega \times (0, T), \\ \nabla v \cdot \nu = 0 & \text{on } \partial\Omega \times (0, T) \end{cases} \tag{4.17}$$

in the viscosity sense.

Definition 4.12. We call a function

(i) $v \in \mathrm{USC}(\overline{\Omega} \times (0, T))$ a *viscosity subsolution of* (4.17) if

$$a - \operatorname{tr} X - |p|^2 + V(x_0) \leq 0$$

for all $(a, p, X) \in \overline{\mathcal{P}}^{2,+}_{\overline{\Omega}\times(0,T)} v(x_0, t_0)$ and $(x_0, t_0) \in \Omega \times (0, T)$, and

$$\min\{a - \operatorname{tr} X - |p|^2 + V(x_0),\ p \cdot \nu(x_0)\} \leq 0$$

for all $(a, p, X) \in \overline{\mathcal{P}}^{2,+}_{\overline{\Omega}\times(0,T)} v(x_0, t_0)$ and $(x_0, t_0) \in \partial\Omega \times (0, T)$;

(ii) $v \in \mathrm{LSC}(\overline{\Omega} \times (0, T))$ a *viscosity supersolution of* (4.17) if

$$a - \operatorname{tr} X - |p|^2 + V(x_0) \geq 0$$

for all $(a, p, X) \in \overline{\mathcal{P}}^{2,-}_{\overline{\Omega}\times(0,T)} v(x_0, t_0)$ and $(x_0, t_0) \in \Omega \times (0, T)$, and

$$\max\{a - \operatorname{tr} X - |p|^2 + V(x_0),\ p \cdot \nu(x_0)\} \geq 0$$

for all $(a, p, X) \in \overline{\mathcal{P}}^{2,-}_{\overline{\Omega}\times(0,T)} v(x_0, t_0)$ and $(x_0, t_0) \in \partial\Omega \times (0, T)$;

(iii) $v \in C(\overline{\Omega} \times (0, T))$ a *viscosity solution of* (4.17) if v is both a viscosity subsolution and viscosity supersolution of (4.17).

Proposition 4.16. *A positive function $u \in C(\overline{\Omega} \times (0, T))$ is a viscosity solution of* (4.16) *if and only if $v := \log u$ is a viscosity solution of* (4.17).

Proof. We will prove that if u is a subsolution of (4.16), then $v = \log u$ is a subsolution of (4.17). The reverse direction follows by a similar argument, as does the correspondence between supersolutions of the two equations.

Let $(a, p, X) \in \overline{\mathcal{P}}^{2,+}_{\overline{\Omega}\times(0,T)} v(x_0, t_0)$ for some $(x_0, t_0) \in \overline{\Omega} \times (0, T)$. Then, there exist sequences $(a_n, p_n, X_n)_{n\geq 1} \subseteq \mathbb{R} \times \mathbb{R}^d \times \mathrm{Sym}_d$ and $(x_n, t_n)_{n\geq 1}$ with $x_n \in \overline{\Omega}$ and $t_n \leq t_0$, such that $(a_n, p_n, X_n) \in \overline{\mathcal{P}}^{2,+}_{\overline{\Omega}\times(0,T)} v(x_n, t_n)$ for each $n \geq 1$, $(a_n, p_n, X_n) \to (a, p, X)$, $x_n \to x_0$ in $\overline{\Omega}$, $t_n \to t_0^-$ and $v(x_n, t_n) \to v(x_0, t_0)$ as $n \to \infty$. By definition, this means

$$\log u(x, t) \leq \log u(x_n, t_n) + a_n(t - t_n) + \langle p_n, x - x_n\rangle \\ + \frac{1}{2}\langle X_n(x - x_n), x - x_n\rangle + o(t_n - t + |x - x_n|^2)$$

as $x \to x_n$ in $\overline{\Omega}$ and $t \to t_n^-$. Using the monotonicity of the exponential function, as well as the Taylor expansion at $\log u(x_n, t_n)$, this implies

$$\begin{aligned} u(x,t) &\le \exp\Bigg(\log u(x_n,t_n) + a_n(t-t_n) + \langle p_n, x-x_n\rangle \\ &\qquad + \frac{1}{2}\langle X_n(x-x_n), x-x_n\rangle + o(t_n - t + |x-x_n|^2)\Bigg) \\ &= u(x_n,t_n) + u(x_n,t_n)h + \frac{1}{2}u(x_n,t_n)h^2 + o\left(|h|^2\right) \\ &= u(x_n,t_n) + u(x_n,t_n)a_n(t-t_n) + u(x_n,t_n)\langle p_n, x-x_n\rangle \\ &\qquad + \frac{1}{2}\langle u(x_n,t_n)(X_n + p_n p_n^\top)(x-x_n), x-x_n\rangle + o(t_n - t + |x-x_n|^2), \end{aligned}$$

where we used

$$\begin{aligned} h = a_n(t-t_n) + \langle p_n, x-x_n\rangle + \frac{1}{2}\langle X_n(x-x_n), x-x_n\rangle \\ + o(t_n - t + |x-x_n|^2). \end{aligned}$$

Therefore

$$\Big(u(x_n,t_n)a_n, u(x_n,t_n)p_n, u(x_n,t_n)(X_n + p_n p_n^\top)\Big) \in \mathcal{P}^{2,+}u(x_n,t_n).$$

Since u is a subsolution of (4.16), we may apply part (i) of Definition 4.11. If $x_0 \in \Omega$, we may assume that since $x_n \to x_0$, the sequence $(x_n)_{n\ge1}$ is eventually contained in Ω. Therefore, after possibly passing to a subsequence, one has

$$u(x_n,t_n)a_n - \operatorname{tr}(u(x_n,t_n)(X_n + p_n p_n^\top)) + V(x_n)u(x_n,t_n) \le 0$$

for each $n \ge 1$. Using the properties of the trace and the positivity of $u(x_n,t_n)$, this is equivalent to

$$a_n - \operatorname{tr} X_n - |p_n|^2 + V(x_n) \le 0. \tag{4.18}$$

Taking $n \to \infty$ gives

$$a - \operatorname{tr} X - |p|^2 + V(x) \le 0.$$

Therefore, the first inequality in Definition 4.12 is satisfied for all triples $(a,p,X) \in \overline{\mathcal{P}}^{2,+}_{\overline{\Omega}\times(0,T)}v(x_0,t_0)$ when $(x_0,t_0) \in \Omega \times (0,T)$.

If $x_0 \in \partial\Omega$, we can no longer make any assumptions about whether or not the points x_n lie in Ω or $\partial\Omega$. If $x_n \in \Omega$, we may use the same argument as before to conclude that (4.18) holds. However, if $x_n \in \partial\Omega$ for some $n \ge 1$, then Definition 4.11 (i) tells us only that

$$\min\{u(x_n,t_n)a_n - \operatorname{tr}(u(x_n,t_n)(X_n + p_n p_n^\top)) + V(x_n)u(x_n,t_n),\ u(x_n,t_n)p_n \cdot \nu(x_n)\} \le 0.$$

If

$$u(x_n,t_n)a_n - \operatorname{tr}(u(x_n,t_n)(X_n + p_n p_n^\top)) + V(x_n)u(x_n,t_n) \le 0,$$

then (4.18) holds once again. If $u(x_n,t_n)p_n \cdot \nu(x_n) \le 0$, then $p_n \cdot \nu(x_n) \le 0$. From this, we conclude

$$\min\{a_n - \operatorname{tr} X_n - |p_n|^2 + V(x_n),\ p_n \cdot \nu(x_n)\} \le 0$$

regardless of whether $x_n \in \Omega$ or $x_n \in \partial\Omega$. Therefore, we may take $n \to \infty$ to conclude that

$$\min\{a - \operatorname{tr} X - |p|^2 + V(x),\ p \cdot \nu(x)\} \leq 0$$

holds for all $(a, p, X) \in \overline{\mathcal{P}}^{2,+}_{\overline{\Omega}\times(0,T)} v(x_0, t_0)$ with $(x_0, t_0) \in \partial\Omega \times (0, T)$. Thus, we conclude that v is a subsolution of (4.17) in $\Omega \times (0, T)$. □

Theorem 4.17. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded, convex domain with smooth boundary $\partial\Omega$ and let $V \in C^2(\Omega)$ be bounded from below. Let $\omega : \overline{\Omega} \times \overline{\Omega} \times S \to \mathbb{R}$ be a continuous function, which is twice differentiable in the first and second arguments and differentiable in the third and fourth arguments. Suppose that $\omega(x, y; t, s) \to \infty$ whenever $t \to s^+$ for $x \neq y$ and $\lim_{t\to s^+} \omega(x, x; t, s) \geq 0$ for all $x \in \overline{\Omega}$. In addition, assume that ω satisfies*

$$\partial_t\omega + |\nabla_x\omega|^2 \geq V(x) \qquad \text{in } \overline{\Omega} \times \overline{\Omega} \times S, \tag{4.19}$$

$$\partial_s\omega - |\nabla_y\omega|^2 \geq -V(y) \qquad \text{in } \overline{\Omega} \times \overline{\Omega} \times S, \tag{4.20}$$

and there exist strictly increasing, differentiable functions $\beta : [0, \infty) \to [0, \infty)$ satisfying $\beta(0) = 0$ and $A : [0, \infty) \to [0, \infty)$ such that

$$A(t)^2\Delta_x\omega + A(s)^2\Delta_y\omega + 2A(t)A(s)\Delta_{xy}\omega \leq A(t)^2\frac{\beta'(t)}{\beta(t)} - A(s)^2\frac{\beta'(s)}{\beta(s)} \tag{4.21}$$

holds in $\overline{\Omega} \times \overline{\Omega} \times S$. Finally, assume that

$$\begin{aligned} \nabla_x\omega \cdot \nu(x) &\geq 0, && \text{for all } x \in \partial\Omega,\ y \in \overline{\Omega}, \text{ and } 0 < s < t, \\ \nabla_y\omega \cdot \nu(y) &\geq 0, && \text{for all } x \in \overline{\Omega},\ y \in \partial\Omega, \text{ and } 0 < s < t. \end{aligned} \tag{4.22}$$

Then, every positive viscosity solution u of

$$\begin{cases} \partial_t u = \Delta u - Vu & \text{in } \Omega \times (0, T), \\ \nabla u \cdot \nu = 0 & \text{on } \partial\Omega \times (0, T) \end{cases} \tag{4.16}$$

satisfies

$$u(x, t) \geq u(y, s)\frac{\beta(s)}{\beta(t)}e^{-\omega(x,y;t,s)} \tag{4.23}$$

for all $x, y \in \Omega$ and $0 < s < t$.

Proof. Let u be a positive viscosity solution of (4.16) and set $v = \log u$. Then, by Proposition 4.16, v is a viscosity solution of (4.17).

Fix $\varepsilon > 0$ and define a function Z on $\overline{\Omega} \times \overline{\Omega} \times S$ by

$$Z(x, y; t, s) = v(x, t) - v(y, s) + \log\left(\frac{\beta(t)}{\beta(s)}\right) + \omega(x, y; t, s) + \frac{\varepsilon}{2}(t - s)^2 + \varepsilon.$$

Identically as in the case for classical solutions, we know that $Z > 0$ whenever s is sufficiently small, or s is sufficiently close to t. In particular, $Z > 0$ in a small enough neighbourhood of $(0, 0)$. Suppose there is a point $P_0 := (x_0, y_0; t_0, s_0)$ such that $Z = 0$ for the first time at P_0 in the sense of Definition 3.1. We first assume that $x_0, y_0 \in \Omega$.

Applying Corollary 4.15 with $u_1(x, t) = v(x, t)$, $u_2(y, s) = -v(y, s)$, and

$$\varphi(x, y; t, s) = -\log\left(\frac{\beta(t)}{\beta(s)}\right) - \omega(x, y; t, s) - \frac{\varepsilon}{2}(t - s)^2 - \varepsilon$$

for $(x, y, t, s) \in \Omega \times \Omega \times (s_0, t_0] \times (0, s_0]$, we deduce that for every $\kappa > 0$, there exist matrices $X, Y \in \mathrm{Sym}_d$ such that

$$(\partial_t \varphi(P_0), \nabla_x \varphi(P_0), X) \in \overline{\mathcal{P}}^{2,-} v(x_0, t_0) \tag{4.24a}$$

$$(\partial_s \varphi(P_0), \nabla_y \varphi(P_0), -Y) \in \overline{\mathcal{P}}^{2,-} (-v)(y_0, s_0) \tag{4.24b}$$

$$\left(\frac{1}{\kappa} + \|H\|\right) I_{2d} \geq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \geq H - \kappa H^2, \tag{4.24c}$$

where $H = D^2_{(x,y)} \varphi(P_0) = -D^2_{(x,y)} \omega(P_0)$. Computing the first-order derivatives of φ explicitly, the condition (4.24a) implies

$$\left(-\frac{\beta'(t_0)}{\beta(t_0)} - \partial_t \omega(P_0) - \varepsilon(t_0 - s_0), -\nabla_x \omega(P_0), X\right) \in \overline{\mathcal{P}}^{2,-} v(x_0, t_0).$$

Using that v is a supersolution of (4.17), this implies

$$-\frac{\beta'(t_0)}{\beta(t_0)} - \partial_t \omega(P_0) - \varepsilon(t_0 - s_0) - \operatorname{tr} X - |\nabla_x \omega(P_0)|^2 + V(x_0) \geq 0.$$

However, by (4.19), this reduces to

$$\frac{\beta'(t_0)}{\beta(t_0)} + \varepsilon(t_0 - s_0) + \operatorname{tr} X \leq 0. \tag{4.25}$$

In a similar way, we understand (4.24b) to mean that

$$\left(\frac{\beta'(s_0)}{\beta(s_0)} - \partial_s \omega(P_0) + \varepsilon(t_0 - s_0), -\nabla_y \omega(P_0), -Y\right) \in \overline{\mathcal{P}}^{2,-} (-v)(y_0, s_0),$$

which is equivalent to

$$\left(-\frac{\beta'(s_0)}{\beta(s_0)} + \partial_s \omega(P_0) - \varepsilon(t_0 - s_0), \nabla_y \omega(P_0), Y\right) \in \overline{\mathcal{P}}^{2,+} v(y_0, s_0).$$

Since v is also a subsolution of (4.17), we have

$$-\frac{\beta'(s_0)}{\beta(s_0)} + \partial_s \omega(P_0) - \varepsilon(t_0 - s_0) - \operatorname{tr} Y - |\nabla_y \omega(P_0)|^2 - V(y_0) \leq 0,$$

which, because of (4.20) becomes

$$-\frac{\beta'(s_0)}{\beta(s_0)} - \varepsilon(t_0 - s_0) - \operatorname{tr} Y \leq 0. \tag{4.26}$$

Taking a linear combination of the inequalities (4.25) and (4.26) with the coefficients $A(t_0)^2$ and $A(s_0)^2$ yields

$$\begin{aligned} 0 \geq A(t_0)^2 \frac{\beta'(t_0)}{\beta(t_0)} - A(s_0)^2 \frac{\beta'(s_0)}{\beta(s_0)} + \varepsilon(t_0 - s_0)(A(t_0)^2 - A(s_0)^2) \\ + A(t_0)^2 \operatorname{tr} X - A(s_0)^2 \operatorname{tr} Y. \end{aligned}$$

Analysing first the expression $A(t_0)^2 \operatorname{tr} X - A(s_0)^2 \operatorname{tr} Y$ using inequality (4.24c), we note that

$$\begin{aligned} A(t_0)^2 \operatorname{tr} X - A(s_0)^2 \operatorname{tr} Y &= \operatorname{tr}(A(t_0)^2 X - A(s_0)^2 Y) \\ &= \operatorname{tr} \begin{pmatrix} A(t_0)^2 I_d & A(t_0)A(s_0) I_d \\ A(t_0)A(s_0) I_d & A(s_0)^2 I_d \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \\ &\geq \operatorname{tr} \begin{pmatrix} A(t_0)^2 I_d & A(t_0)A(s_0) I_d \\ A(t_0)A(s_0) I_d & A(s_0)^2 I_d \end{pmatrix} (H - \kappa H^2). \end{aligned}$$

Therefore

$$\begin{aligned} 0 \geq A(t_0)^2 \frac{\beta'(t_0)}{\beta(t_0)} - A(s_0)^2 \frac{\beta'(s_0)}{\beta(s_0)} + \varepsilon(t_0 - s_0)(A(t_0)^2 - A(s_0)^2) \\ + \operatorname{tr} \begin{pmatrix} A(t_0)^2 I_d & A(t_0)A(s_0) I_d \\ A(t_0)A(s_0) I_d & A(s_0)^2 I_d \end{pmatrix} (H - \kappa H^2) \end{aligned}$$

for all $\kappa > 0$. Taking $\kappa \to 0$ and applying (4.21) leads to

$$\begin{aligned} 0 &\geq A(t_0)^2 \frac{\beta'(t_0)}{\beta(t_0)} - A(s_0)^2 \frac{\beta'(s_0)}{\beta(s_0)} + \varepsilon(t_0 - s_0)(A(t_0)^2 - A(s_0)^2) \\ &\quad + \operatorname{tr} \begin{pmatrix} A(t_0)^2 I_d & A(t_0)A(s_0) I_d \\ A(t_0)A(s_0) I_d & A(s_0)^2 I_d \end{pmatrix} \begin{pmatrix} -D_x^2 \omega(P_0) & -D_{xy}^2 \omega(P_0) \\ -D_{xy}^2 \omega(P_0) & -D_y^2 \omega(P_0) \end{pmatrix} \\ &= A(t_0)^2 \frac{\beta'(t_0)}{\beta(t_0)} - A(s_0)^2 \frac{\beta'(s_0)}{\beta(s_0)} + \varepsilon(t_0 - s_0)(A(t_0)^2 - A(s_0)^2) \\ &\quad - A(t_0)^2 \Delta_x \omega(P_0) - A(s_0)^2 \Delta_y \omega(P_0) - 2A(t_0)A(s_0)\Delta_{xy}\omega(P_0) \\ &\geq \varepsilon(t_0 - s_0)(A(t_0)^2 - A(s_0)^2) > 0, \end{aligned}$$

which is a contradiction. Therefore, if such a point P_0 exists, it must occur with at least one of x_0 or y_0 on the boundary $\partial\Omega$.

By definition of the point $P_0 = (x_0, y_0; t_0, s_0)$, there exists $0 < \eta < t_0 - s_0$ such that

$$Z(x_0, y_0; t_0, s_0) \leq Z(x, y; t, s)$$

for all $(x, y; t, s) \in K := \overline{\Omega} \times \overline{\Omega} \times [t_0 - \eta] \times [s_0 - \eta, s_0]$, that is, Z attains its minimum on K at P_0. We can ensure that P_0 is the unique minimiser of Z on K by modifying this function. We define

$$\tilde{Z}(x, y; t, s) := Z(x, y; t, s) + |x - x_0|^4 + |y - y_0|^4 + |t - t_0|^4 + |s - s_0|^4$$

for all $(x, y; t, s) \in K$.

Before we can begin the main step of the argument, we first perturb the solution v, and in doing so, we perturb the equation satisfied by v as well. Fix a point $z_0 \in \Omega$ so that $\delta := d(z_0, \partial\Omega) > 0$ and define

$$h(x) := \frac{1}{2}|x - z_0|^2 \qquad \text{for all } x \in \overline{\Omega}.$$

Then for $\theta > 0$ small, we define

$$v^\theta(x, t) := v(x, t) + \theta h(x) \quad \text{and} \quad v_\theta(x, t) := v(x, t) - \theta h(x)$$

for all $(x,t) \in \overline{\Omega} \times (0,T)$. A simple calculation shows that v^θ satisfies

$$\begin{cases} \partial_t v^\theta - \Delta(v^\theta - \theta h) - |\nabla(v^\theta - \theta h)|^2 + V = 0 & \text{in } \Omega \times (0,T), \\ \nabla v^\theta \cdot \nu - \theta \nabla h \cdot \nu = 0 & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{4.27}$$

and v_θ satisfies

$$\begin{cases} \partial_t v_\theta - \Delta(v_\theta + \theta h) - |\nabla(v_\theta + \theta h)|^2 + V = 0 & \text{in } \Omega \times (0,T), \\ \nabla v_\theta \cdot \nu + \theta \nabla h \cdot \nu = 0 & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{4.28}$$

both in the sense of viscosity solutions. Using the functions v^θ and v_θ, we then define an approximation of $\tilde{Z}$ by setting

$$\begin{aligned} \tilde{Z}_\theta(x,y;t,s) := v^\theta(x,t) - v_\theta(y,s) + \log\left(\frac{\beta(t)}{\beta(s)}\right) + \omega(x,y;t,s) + \frac{\varepsilon}{2}(t-s)^2 + \varepsilon \\ + |x-x_0|^4 + |y-y_0|^4 + |t-t_0|^4 + |s-s_0|^4 \end{aligned}$$

for all $(x,y;t,s) \in K$. Since for each $\theta > 0$, $\tilde{Z}_\theta$ is a continuous function on the compact set K, there exists $P_\theta := (x_\theta, y_\theta; t_\theta, s_\theta) \in K$ such that

$$\tilde{Z}_\theta(P_\theta) = \min_{P \in K} Z(P).$$

We claim that since $\tilde{Z}_\theta \to \tilde{Z}$ uniformly on K as $\theta \to 0$ and because P_0 is the unique minimiser of $\tilde{Z}$ in K, there is a sequence $(\theta_n)_{n\geq 1}$ of positive numbers such that $P_{\theta_n} \to P_0$ as $n \to \infty$. Since $(P_\theta)_{\theta>0}$ is contained in the bounded set K, there is a sequence $(\theta_n)_{n\geq 1}$ and a point $\hat{P} \in K$ such that $P_{\theta_n} \to \hat{P}$ in K. Then $\tilde{Z}_{\theta_n}(P_{\theta_n}) \to \tilde{Z}(\hat{P})$ as $n \to \infty$ since

$$\begin{aligned} |\tilde{Z}_{\theta_n}(P_{\theta_n}) - \tilde{Z}(\hat{P})| &\leq |\tilde{Z}_{\theta_n} - \tilde{Z}(P_{\theta_n})| + |\tilde{Z}(P_{\theta_n}) - \tilde{Z}(\hat{P})| \\ &\leq \|\tilde{Z}_{\theta_n} - \tilde{Z}\|_\infty + |\tilde{Z}(P_{\theta_n}) - \tilde{Z}(\hat{P})| \to 0 \end{aligned}$$

because $\tilde{Z}_{\theta_n} \to \tilde{Z}$ uniformly as $n \to \infty$ and $\tilde{Z}$ is continuous. Now, by the definition of P_{θ_n}, one has that

$$\tilde{Z}_{\theta_n}(P_{\theta_n}) \leq \tilde{Z}_{\theta_n}(P)$$

for all points $P \in K$. Taking $n \to \infty$ gives

$$\tilde{Z}(\hat{P}) \leq \tilde{Z}(P)$$

for all $P \in K$, which expresses that $\hat{P}$ minimises Z over K. However, $\tilde{Z}$ was defined in such a way that P_0 is the unique minimiser of $\tilde{Z}$ over K, and $\hat{P} = P_0$ and $P_{\theta_n} \to P_0$ as claimed. Moreover, since $(t_{\theta_n}, s_{\theta_n}) \to (t_0, s_0)$, we may assume the point P_{θ_n} does not occur with $t_{\theta_n} = t_0 - \eta$ or $s_{\theta_n} = s_0 - \eta$. Therefore, $\tilde{Z}_{\theta_n}$ attains a minimum at $P_{\theta_n} \in \tilde{K} := \overline{\Omega} \times \overline{\Omega} \times (t_0 - \eta] \times (s_0 - \eta, s_0]$. Thus, we may apply the parabolic maximum principle for semicontinuous functions (Corollary 4.15) to the function

$$\tilde{Z}_{\theta_n}(x,y;t,s) = v^{\theta_n}(x,t) - v_{\theta_n}(y,s) - \varphi(x,y;t,s)$$

at P_{θ_n}, where

$$\varphi(x,y;t,s) = -\log\left(\frac{\beta(t)}{\beta(s)}\right) - \omega(x,y;t,s) - \frac{\varepsilon}{2}(t-s)^2 - \varepsilon \\ - |x-x_0|^4 - |y-y_0|^4 - |t-t_0|^4 - |s-s_0|^4.$$

Then for every $\kappa > 0$, there are matrices X_n, Y_n such that

$$(\partial_t\varphi(P_{\theta_n}), \nabla_x\varphi(P_{\theta_n}), X_n) \in \overline{\mathcal{P}}^{2,-}_{\tilde{K}} v^{\theta_n}(x_{\theta_n}, t_{\theta_n}), \tag{4.29}$$

$$(-\partial_s\varphi(P_{\theta_n}), -\nabla_y\varphi(P_{\theta_n}), Y_n) \in \overline{\mathcal{P}}^{2,+}_{\tilde{K}} v_{\theta_n}(y_{\theta_n}, s_{\theta_n}), \tag{4.30}$$

$$\left(\frac{1}{\kappa} + \|H_n\|\right) I_{2d} \geq \begin{pmatrix} X_n & 0 \\ 0 & -Y_n \end{pmatrix} \geq H_n - \kappa H_n^2, \tag{4.31}$$

where $H_n = D^2_{(x,y)}\varphi(P_{\theta_n})$. We now use that v^{θ_n} is a viscosity supersolution of (4.27). If $x_{\theta_n} \in \Omega$, we have

$$\begin{aligned} Q_n := &-\frac{\beta'(t_{\theta_n})}{\beta(t_{\theta_n})} - \partial_t\omega(P_{\theta_n}) - \varepsilon(t_{\theta_n} - s_{\theta_n}) - 4|t_{\theta_n} - t_0|^2(t_{\theta_n} - t_0) - \operatorname{tr} X_n \\ &- |-\nabla_x\omega(P_{\theta_n}) - 4|x_{\theta_n} - x_0|^2(x_{\theta_n} - x_0) - \theta_n\nabla h(x_{\theta_n})|^2 \\ &+ V(x_{\theta_n}) + \theta_n\Delta h(x_{\theta_n}) \geq 0, \end{aligned} \tag{4.32}$$

whereas if $x_{\theta_n} \in \partial\Omega$, we have

$$\max\left\{\begin{matrix} Q_n, -\nabla_x\omega(P_{\theta_n})\cdot\nu(x_{\theta_n}) - 4|x_{\theta_n} - x_0|^2(x_{\theta_n} - x_0)\cdot\nu(x_{\theta_n}) \\ -\theta_n\nabla h(x_{\theta_n})\cdot\nu(x_{\theta_n}) \end{matrix}\right\} \geq 0. \tag{4.33}$$

However, $\nabla_x\omega(P_{\theta_n})\cdot\nu(x_{\theta_n}) \geq 0$ by assumption, $(x_{\theta_n} - x_0)\cdot\nu(x_{\theta_n}) \geq 0$ since Ω is convex, and finally

$$\nabla h(x_{\theta_n})\cdot\nu(x_{\theta_n}) = (x_{\theta_n} - z_0)\cdot\nu(x_{\theta_n}) \geq d(z_0, \partial\Omega) = \delta.$$

Therefore

$$-\nabla_x\omega(P_{\theta_n})\cdot\nu(x_{\theta_n}) - 4|x_{\theta_n} - x_0|^2(x_{\theta_n} - x_0)\cdot\nu(x_{\theta_n}) - \theta_n\nabla h(x_{\theta_n})\cdot\nu(x_{\theta_n}) \leq -\theta_n\delta < 0.$$

Hence, in order for (4.33) to be satisfied, we must have that $Q_n \geq 0$, and so (4.32) holds regardless of whether $x_{\theta_n} \in \Omega$ or $x_{\theta_n} \in \partial\Omega$. Applying a similar argument using that v_{θ_n} is a viscosity subsolution of (4.28), we find that

$$\begin{aligned} &\frac{\beta'(s_{\theta_n})}{\beta(s_{\theta_n})} - \partial_s\omega(P_{\theta_n}) + \varepsilon(t_{\theta_n} - s_{\theta_n}) - 4|s_{\theta_n} - s_0|^2(s_{\theta_n} - s_0) + \operatorname{tr} Y_n \\ &\quad + |-\nabla_y\omega(P_{\theta_n}) - 4|y_{\theta_n} - y_0|^2(y_{\theta_n} - y_0) + \theta_n\nabla h(y_{\theta_n})|^2 \\ &\quad + V(y_{\theta_n}) - \theta_n\Delta h(y_{\theta_n}) \geq 0. \end{aligned} \tag{4.34}$$

Seeking to take $n \to \infty$ in (4.31), (4.32), and (4.34), we note that

$$\begin{aligned} H_n &= D^2_{(x,y)}\varphi(P_{\theta_n}) \\ &= -D^2_{(x,y)}\omega(P_{\theta_n}) - 4\begin{pmatrix} 2(x_{\theta_n} - x_0)(x_{\theta_n} - x_0)^\top & 0 \\ 0 & 2(y_{\theta_n} - y_0)(y_{\theta_n} - y_0)^\top \end{pmatrix} \\ &\quad - 4\begin{pmatrix} |x_{\theta_n} - x_0|^2 I_d & 0 \\ 0 & |y_{\theta_n} - y_0|^2 I_d \end{pmatrix} \end{aligned}$$

converges to $D^2_{(x,y)}\omega(P_0)$ as $n \to \infty$, the sequences $(X_n)_{n\geq 1}$ and $(Y_n)_{n\geq 1}$ converge, possibly after taking a subsequence, because of (4.31), to some matrices $X, Y \in \mathrm{Sym}_d$ respectively. It follows that

$$-\frac{\beta'(t_0)}{\beta(t_0)} - \partial_t\omega(P_0) - \varepsilon(t_0 - s_0) - \operatorname{tr} X - |\nabla_x\omega(P_0)|^2 + V(x_0) \geq 0$$

$$\frac{\beta'(s_0)}{\beta(s_0)} - \partial_s\omega(P_0) + \varepsilon(t_0 - s_0) + \operatorname{tr} Y + |\nabla_y\omega(P_0)|^2 + V(y_0) \geq 0$$

$$\left(\frac{1}{\kappa} + \|H\|\right) I_{2d} \geq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \geq H - \kappa H^2$$

hold with $H = -D^2_{(x,y)}\omega(P_0)$ as before. Therefore, we may continue using the same argument as when $x_0, y_0 \notin \partial\Omega$ to reach a contradiction once again. Thus, no such point P_0 exists and the proof is complete.

□

4.3.2. The doubly nonlinear heat equation

We now return to the Neumann problem

$$\begin{cases} \partial_t u = \Delta_p(u^m) & \text{in } \Omega \times (0,T) \\ |\nabla(u^m)|^{p-2}\nabla(u^m)\cdot\nu = 0 & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{4.35}$$

for the doubly nonlinear heat equation with $m > 0$ and $p \geq 2$. To avoid repetition, we present our ideas only in the case that $\lambda = m - \frac{1}{p-1} > 0$, although the same approach yields results in the cases $\lambda < 0$ and $\lambda = 0$ as well once one modifies the proofs as in the classical case discussed in Section 3.2.

Given that when expanded, the doubly nonlinear operator takes the form

$$\Delta_p(u^m) = m^{p-1}u^{(m-1)(p-1)}|\nabla u|^{p-4}\times \Big((p-1)(m-1)u^{-1}|\nabla u|^4 + |\nabla u|^2\Delta u + (p-2)\nabla u^\top D^2 u \nabla u\Big),$$

we define viscosity solutions of (4.35) in the following manner.

Definition 4.13. We call a function

(i) $u \in \mathrm{USC}(\overline{\Omega} \times (0,T))$ a *viscosity subsolution of* (4.35) if

$$a - m^{p-1}u(x_0,t_0)^{(m-1)(p-1)}|\xi|^{p-4} \Big((p-1)(m-1)u(x_0,t_0)^{-1}|\xi|^4 + |\xi|^2\operatorname{tr} X + (p-2)\xi^\top X\xi\Big) \leq 0$$

for all $(a,\xi,X) \in \overline{\mathcal{P}}^{2,+}_{\overline{\Omega}\times(0,T)}u(x_0,t_0)$ and $(x_0,t_0) \in \Omega \times (0,T)$, and

$$\min\left\{\begin{matrix} a - m^{p-1}u(x_0,t_0)^{(m-1)(p-1)}|\xi|^{p-4} \\ \Big((p-1)(m-1)u(x_0,t_0)^{-1}|\xi|^4 + |\xi|^2\operatorname{tr} X + (p-2)\xi^\top X\xi\Big), \\ m^{p-1}u(x_0,t_0)^{(m-1)(p-1)}|\xi|^{p-2}\xi\cdot\nu(x_0) \end{matrix}\right\} \leq 0$$

for all $(a,\xi,X) \in \overline{\mathcal{P}}^{2,+}_{\overline{\Omega}\times(0,T)}u(x_0,t_0)$ and $(x_0,t_0) \in \partial\Omega \times (0,T)$;

(ii) $u \in \mathrm{LSC}(\overline{\Omega}\times(0,T))$ a *viscosity supersolution of* (4.35) if

$$a - m^{p-1}u(x_0,t_0)^{(m-1)(p-1)}|\xi|^{p-4} \Big((p-1)(m-1)u(x_0,t_0)^{-1}|\xi|^4 + |\xi|^2 \operatorname{tr} X + (p-2)\xi^\top X\xi\Big) \geq 0$$

for all $(a,\xi,X) \in \overline{\mathcal{P}}^{2,-}_{\overline{\Omega}\times(0,T)}u(x_0,t_0)$ and $(x_0,t_0) \in \Omega \times (0,T)$, and

$$\max\begin{Bmatrix} a - m^{p-1}u(x_0,t_0)^{(m-1)(p-1)}|\xi|^{p-4} \\ \quad\Big((p-1)(m-1)u(x_0,t_0)^{-1}|\xi|^4 + |\xi|^2 \operatorname{tr} X + (p-2)\xi^\top X\xi\Big), \\ m^{p-1}u(x_0,t_0)^{(m-1)(p-1)}|\xi|^{p-2}\xi\cdot\nu(x_0) \end{Bmatrix} \geq 0$$

for all $(a,\xi,X) \in \overline{\mathcal{P}}^{2,-}_{\overline{\Omega}\times(0,T)}u(x_0,t_0)$ and $(x_0,t_0) \in \partial\Omega \times (0,T)$;

(iii) $u \in C(\overline{\Omega}\times(0,T))$ a *viscosity solution of* (4.35) if u is both a viscosity subsolution and viscosity supersolution of (4.35).

We also study the problem

$$\begin{cases} \partial_t v = \lambda v \Delta_p v + |\nabla v|^p & \text{in } \Omega\times(0,T) \\ |\nabla v|^{p-2}\nabla v\cdot\nu = 0 & \text{on } \partial\Omega\times(0,T) \end{cases} \tag{4.36}$$

obtained by applying the transformation $v = \frac{m}{\lambda}u^\lambda$ to (4.35).

Definition 4.14. We call a function

(i) $v \in \mathrm{USC}(\overline{\Omega}\times(0,T))$ a *viscosity subsolution of* (4.36) if

$$a - \lambda v(x_0,t_0)|\xi|^{p-4}(|\xi|^2 \operatorname{tr} X + (p-2)\xi^\top X\xi) - |\xi|^p \leq 0$$

for all $(a,\xi,X) \in \overline{\mathcal{P}}^{2,+}_{\overline{\Omega}\times(0,T)}v(x_0,t_0)$ and $(x_0,t_0) \in \Omega \times (0,T)$ and

$$\min\begin{Bmatrix} a - \lambda v(x_0,t_0)|\xi|^{p-4}(|\xi|^2 \operatorname{tr} X + (p-2)\xi^\top X\xi) - |\xi|^p, \\ |\xi|^{p-2}\xi\cdot\nu(x_0) \end{Bmatrix} \leq 0$$

for all $(a,\xi,X) \in \overline{\mathcal{P}}^{2,+}_{\overline{\Omega}\times(0,T)}u(x_0,t_0)$ and $(x_0,t_0) \in \partial\Omega \times (0,T)$;

(ii) $v \in \mathrm{LSC}(\overline{\Omega}\times(0,T))$ a *viscosity supersolution of* (4.36) if

$$a - \lambda v(x_0,t_0)|\xi|^{p-4}(|\xi|^2 \operatorname{tr} X + (p-2)\xi^\top X\xi) - |\xi|^p \geq 0$$

for all $(a,\xi,X) \in \overline{\mathcal{P}}^{2,-}_{\overline{\Omega}\times(0,T)}v(x_0,t_0)$ and $(x_0,t_0) \in \Omega \times (0,T)$ and

$$\max\begin{Bmatrix} a - \lambda v(x_0,t_0)|\xi|^{p-4}(|\xi|^2 \operatorname{tr} X + (p-2)\xi^\top X\xi) - |\xi|^p, \\ |\xi|^{p-2}\xi\cdot\nu(x_0) \end{Bmatrix} \geq 0$$

for all $(a,\xi,X) \in \overline{\mathcal{P}}^{2,-}_{\overline{\Omega}\times(0,T)}u(x_0,t_0)$ and $(x_0,t_0) \in \partial\Omega \times (0,T)$;

(iii) $v \in C(\overline{\Omega}\times(0,T))$ a *viscosity solution of* (4.36) if v is both a viscosity subsolution and viscosity supersolution of (4.35).

Proposition 4.18. *A positive function $u \in C(\Omega \times (0,T))$ is a viscosity solution of* (4.35) *if and only if $v := \frac{m}{\lambda} u^\lambda$ is a viscosity solution of* (4.36).

PROOF. It is sufficient to prove that if u is a subsolution of (4.35), then $v = \frac{m}{\lambda} u^\lambda$ is a subsolution of (4.36).

Let $(a, \xi, X) \in \overline{\mathcal{P}}^{2,+}_{\overline{\Omega}\times(0,T)} v(x_0, t_0)$ for some $(x_0, t_0) \in \overline{\Omega} \times (0,T)$. Then, we can find sequences $(a_n, p_n, X_n)_{n\geq 1} \subseteq \mathbb{R} \times \mathbb{R}^d \times \mathrm{Sym}_d$ and $(x_n, t_n)_{n\geq 1}$ with $x_n \in \overline{\Omega}$ and $t_n \leq t_0$, such that $(a_n, p_n, X_n) \in \overline{\mathcal{P}}^{2,+}_{\overline{\Omega}\times(0,T)} v(x_n, t_n)$ for each $n \geq 1$, $(a_n, \xi_n, X_n) \to (a, \xi, X)$, $x_n \to x_0$ in $\overline{\Omega}$, $t_n \to t_0^-$ and $v(x_n, t_n) \to v(x_0, t_0)$ as $n \to \infty$. By definition, this means

$$\frac{m}{\lambda} u(x,t)^\lambda \leq \frac{m}{\lambda} u(x_n, t_n)^\lambda + a_n(t - t_n) + \langle \xi_n, x - x_n \rangle \\ + \frac{1}{2} \langle X_n(x - x_n), x - x_n \rangle + o(t_n - t + |x - x_n|^2)$$

as $x \to x_n$ in $\overline{\Omega}$ and $t \to t_n^-$. Using the monotonicity of the transformation, as well as the Taylor expansion of $\tau \mapsto \tau^{1/\lambda}$ at $\tau = u(x_n, t_n)^\lambda$, this implies

$$\begin{aligned} u(x,t) \leq \Bigg(& u(x_n, t_n)^\lambda + \frac{\lambda}{m} a_n(t - t_n) + \frac{\lambda}{m} \langle \xi_n, x - x_n \rangle \\ & + \frac{1}{2} \left\langle \frac{\lambda}{m} X_n(x - x_n), x - x_n \right\rangle + o(t_n - t + |x - x_n|^2) \Bigg)^{1/\lambda} \\ = {} & u(x_n, t_n) + \frac{1}{m} u(x_n, t_n)^{1/\lambda} h + \frac{1}{2\lambda} \left(\frac{1}{\lambda} - 1 \right) u(x_n, t_n)^{1-2\lambda} h^2 + o(|h|^2) \\ = {} & u(x_n, t_n) + \frac{1}{m} u(x_n, t_n)^{1-\lambda} a_n(t - t_n) + \frac{1}{m} u(x_n, t_n)^{1-\lambda} \langle \xi_n, x - x_n \rangle \\ & + \frac{1}{2} \left\langle \left(\frac{1}{m} u(x_n, t_n)^{1-\lambda} X_n + \frac{1-\lambda}{m^2} u(x_n, t_n)^{1-2\lambda} \xi_n \xi_n^\top \right) (x - x_n), x - x_n \right\rangle \\ & + o(t_n - t + |x - x_n|^2), \end{aligned}$$

where we used

$$h := \frac{\lambda}{m} a_n(t - t_n) + \frac{\lambda}{m} \langle \xi_n, x - x_n \rangle + \frac{1}{2} \left\langle \frac{\lambda}{m} X_n(x - x_n), x - x_n \right\rangle \\ + o(t_n - t + |x - x_n|^2).$$

This implies

$$\Bigg(\frac{1}{m} u(x_n, t_n)^{1-\lambda} a_n, \frac{1}{m} u(x_n, t_n)^{1-\lambda} \xi_n, \\ \frac{1}{m} u(x_n, t_n)^{1-\lambda} X_n + \frac{1-\lambda}{m^2} u(x_n, t_n)^{1-2\lambda} \xi_n \xi_n^\top \Bigg) \in \mathcal{P}^{2,+} u(x_n, t_n).$$

Since u is a subsolution of (4.35), we may apply Definition 4.13 (i). If $x_0 \in \Omega$, then the sequence $(x_n)_{n\geq 1}$ eventually lies in Ω, and so we may use the first inequality from this definition to obtain the following inequality:

$$
\begin{aligned}
0 \geq\ & \frac{1}{m}u^{1-\lambda}a_n - m^{p-1}u^{(m-1)(p-1)}\left(\frac{1}{m}u^{1-\lambda}\right)^{p-4}|\xi_n|^{p-4}\times \\
&\Bigg((m-1)(p-1)u^{-1}\left(\frac{1}{m}u^{1-\lambda}\right)^{4}|\xi_n|^4 \\
&\quad + \left(\frac{1}{m}u^{1-\lambda}\right)^{2}\left(\frac{1}{m}u^{1-\lambda}\operatorname{tr}X_n + \frac{1-\lambda}{m^2}u^{1-2\lambda}|\xi_n|^2\right)|\xi_n|^2 \\
&\quad + (p-2)\left(\frac{1}{m}u^{1-\lambda}\right)^{2}\xi_n^\top\left(\frac{1}{m}u^{1-\lambda}X_n + \frac{1-\lambda}{m^2}u^{1-2\lambda}\xi_n\xi_n^\top\right)\xi_n\Bigg) \\
=\ & \frac{1}{m}u^{1-\lambda}a_n - m^{p-1}u^{(m-1)(p-1)}\left(\frac{1}{m}u^{1-\lambda}\right)^{p-1}|\xi_n|^{p-4}(|\xi_n|^2\operatorname{tr}X_n + (p-2)\xi_n^\top X_n\xi_n) \\
&- \Bigg((p-1)(1-\lambda)m^{p-3}u^{(m-1)(p-1)+1-2\lambda}\left(\frac{1}{m}u^{1-\lambda}\right)^{p-2} \\
&\quad + m^{p-1}(m-1)(p-1)u^{(m-1)(p-1)-1}\left(\frac{1}{m}u^{1-\lambda}\right)^{p}\Bigg)|\xi_n|^p,
\end{aligned}
$$

where we have omitted (x_n, t_n) from the notation for brevity. Dividing through by $\frac{1}{m}u^{1-\lambda}$ gives

$$
\begin{aligned}
0 \geq\ & a_n - mu^{(m-1)(p-1)+(1-\lambda)(p-2)}|\xi_n|^{p-4}(|\xi_n|^2\operatorname{tr}X_n + (p-2)\xi_n^\top X_n\xi_n) \\
&- \Bigg((p-1)(1-\lambda)u^{(m-1)(p-1)+1-2\lambda+(1-\lambda)(p-3)} \\
&\quad + (m-1)(p-1)u^{(m-1)(p-1)-1+(1-\lambda)(p-1)}\Bigg)|\xi_n|^p.
\end{aligned}
$$

In light of the facts that

$$
\begin{aligned}
&(m-1)(p-1) + (1-\lambda)(p-2) = \lambda, \\
&(m-1)(p-1) + 1 - 2\lambda + (1-\lambda)(p-3) = 0, \\
&(m-1)(p-1) - 1 + (1-\lambda)(p-1) = 0, \\
&(m-1)(p-1) + (p-1)(1-\lambda) = 1,
\end{aligned}
$$

it follows that

$$
a_n - \lambda v(x_n, t_n)|\xi_n|^{p-4}(|\xi_n|^2\operatorname{tr}X_n + (p-2)\xi_n^\top X_n\xi_n) - |\xi_n|^p \leq 0. \tag{4.37}
$$

Taking $n \to \infty$, we have

$$
a - \lambda v(x_0, t_0)|\xi|^{p-4}(|\xi|^2\operatorname{tr}X + (p-2)\xi^\top X\xi) - |\xi|^p \leq 0.
$$

Therefore, the first part of Definition 4.14, that is for when $x_0 \in \Omega$, is satisfied. If $x_0 \in \partial\Omega$, then we again consider both possibilities, that $x_n \in \Omega$ or $x_n \in \partial\Omega$.

If $x_n \in \Omega$, we again reach (4.37), but if $x_0 \in \partial\Omega$, Definition 4.13 implies

$$\min\left\{\begin{aligned} &\frac{1}{m}u^{1-\lambda}a_n - m^{p-1}u^{(m-1)(p-1)}\left(\frac{1}{m}u^{1-\lambda}\right)^{p-4}|\xi_n|^{p-4}\times \\ &\qquad\Bigg((m-1)(p-1)u^{-1}\left(\frac{1}{m}u^{1-\lambda}\right)^4|\xi_n|^4 \\ &\qquad\quad + \left(\frac{1}{m}u^{1-\lambda}\right)^2\left(\frac{1}{m}u^{1-\lambda}\operatorname{tr}X_n + \frac{1-\lambda}{m^2}u^{1-2\lambda}|\xi_n|^2\right)|\xi_n|^2 \\ &\qquad\quad + (p-2)\left(\frac{1}{m}u^{1-\lambda}\right)^2\xi_n^\top\left(\frac{1}{m}u^{1-\lambda}X_n + \frac{1-\lambda}{m^2}u^{1-2\lambda}\xi_n\xi_n^\top\right)\xi_n\Bigg), \\ &u(x_n,t_n)^{(m-1)(p-1)+(p-1)(1-\lambda)}|\xi_n|^{p-2}\xi_n\cdot\nu \end{aligned}\right\} \le 0.$$

If the first element inside the minimum above is non-positive, then we are again led to (4.37). If

$$u(x_n,t_n)^{(m-1)(p-1)+(p-1)(1-\lambda)}|\xi_n|^{p-2}\xi_n\cdot\nu \le 0,$$

then

$$|\xi_n|^{p-2}\xi_n\cdot\nu(x_n) \le 0,$$

and so we conclude that

$$\min\left\{\begin{aligned}&a_n - \lambda v(x_n,t_n)|\xi_n|^{p-4}(|\xi_n|^2\operatorname{tr}X_n + (p-2)\xi_n^\top X_n\xi_n) - |\xi_n|^p, \\ &|\xi_n|^{p-2}\xi_n\cdot\nu(x_n)\end{aligned}\right\} \le 0$$

regardless of whether $x_0 \in \Omega$ or $x_0 \in \partial\Omega$. Therefore, taking $n \to \infty$, we have

$$\min\left\{\begin{aligned}&a - \lambda v(x_0,t_0)|\xi|^{p-4}(|\xi|^2\operatorname{tr}X + (p-2)\xi^\top X\xi) - |\xi|^p, \\ &|\xi|^{p-2}\xi\cdot\nu(x_0)\end{aligned}\right\} \le 0$$

for all $(a,\xi,X) \in \overline{P}^{2,+}_{\overline{\Omega}\times(0,T)}v(x_0,t_0)$ with $(x_0,t_0) \in \partial\Omega\times(0,T)$, and so v is a viscosity subsolution of (4.14). ☐

Theorem 4.19. *Let $m > 0$, $p \ge 2$ and set $\lambda := m - \frac{1}{p-1}$. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary $\partial\Omega$. Suppose $\omega : \overline{\Omega}\times\overline{\Omega}\times S \to \mathbb{R}$ is a continuous function, which is twice differentiable in the first and second arguments and differentiable in the third and fourth arguments. Suppose that $\omega(x,y;t,s) \to \infty$ whenever $t \to s^+$ for $x \ne y$ and $\lim_{t\to s^+}\omega(x,x;t,s) \ge 0$ for all $x \in \overline{\Omega}$. In addition, suppose there exists a strictly increasing, differentiable function $\beta : [0,\infty) \to [0,\infty)$ with $\beta(0) = 0$ and a function $A : [0,\infty) \to [0,\infty)$ such that $A^2\beta^\lambda$ is increasing and ω satisfies*

$$\partial_t\omega + \beta^{-\lambda(p-1)}(t)|\nabla_x\omega|^p \ge 0 \qquad \text{in } \overline{\Omega}\times\overline{\Omega}\times S, \tag{4.38}$$

$$\partial_s\omega - \beta^{-\lambda(p-1)}(s)|\nabla_y\omega|^p \ge 0 \qquad \text{in } \overline{\Omega}\times\overline{\Omega}\times S, \tag{4.39}$$

$$\begin{aligned} &A^2(t)\beta^{-\lambda(p-2)}(t)\Delta_{p,x}\omega + A^2(s)\beta^{-\lambda(p-2)}(s)\Delta_{p,y}\omega \\ &\quad +2A(t)A(s)\beta^{-\frac{\lambda(p-2)}{2}}(t)\beta^{-\frac{\lambda(p-2)}{2}}(s)\Delta_{p,xy}\omega \\ &\qquad\qquad \le A^2(t)\beta^{\lambda-1}(t)\beta'(t) - A^2(s)\beta^{\lambda-1}(s)\beta'(s) \end{aligned} \tag{4.40}$$

in $\overline{\Omega} \times \overline{\Omega} \times S$. *Finally, assume that*

$$\begin{aligned} \nabla_x \omega \cdot \nu(x) \geq 0, & \qquad \textit{for all } x \in \partial\Omega,\ y \in \overline{\Omega}, \textit{ and } 0 < s < t, \\ \nabla_y \omega \cdot \nu(y) \geq 0, & \qquad \textit{for all } x \in \overline{\Omega},\ y \in \partial\Omega, \textit{ and } 0 < s < t. \end{aligned} \tag{4.41}$$

Then every positive viscosity solution u *of*

$$\begin{cases} \partial_t u = \Delta_p(u^m) & \textit{in } \Omega \times (0,\infty) \\ |\nabla(u^m)|^{p-2}\nabla(u^m) \cdot \nu = 0 & \textit{on } \partial\Omega \times (0,\infty) \end{cases} \tag{4.42}$$

satisfies the Harnack inequality

$$\psi(\beta(t)u(x,t)) \geq \psi(\beta(s)u(y,s)) - \omega(x,y;t,s) \tag{4.43}$$

for all $x, y \in \overline{\Omega}$ *and* $0 < s < t$, *where*

$$\psi(u) := \begin{cases} \dfrac{m}{\lambda} u^\lambda & \textit{if } \lambda \neq 0, \\ \dfrac{1}{p-1} \log u & \textit{if } \lambda = 0. \end{cases}$$

Our proof of Theorem 4.19 proceeds in a similar fashion to the analogous Theorem 3.15 for classical solutions, in that for any positive (viscosity) solution u of (4.35) we again define a function

$$Z(x,y;t,s) := \beta^\lambda(t)v(x,t) - \beta^\lambda(s)v(y,s) + \omega(x,y;t,s) + \frac{\varepsilon}{2}(t-s)^2 + \varepsilon,$$

where $v = \frac{m}{\lambda}u^\lambda$. However, our application of the parabolic maximum principle for semicontinuous functions (Corollary 4.15) only initially provides elements of $\overline{\mathcal{P}}^{2,\pm}(\beta^\lambda v)$ instead of $\overline{\mathcal{P}}^{2,\pm} v$. Therefore, we must first understand the following "product rule" type results.

Lemma 4.20 (Product rule for parabolic semijets). *Let* $u \in \mathrm{USC}(\Omega \times (S_i, T_i))$ *or* $u \in \mathrm{LSC}(\Omega \times (S_i, T_i))$, *and* $\mu : [0,\infty) \to [0,\infty)$.

(i) If μ *is differentiable at* $t_0 \in (S_i, T_i)$ *and* $(a, \xi, X) \in \mathcal{P}^{2,\pm}u(x_0,t_0)$ *for* $(x_0,t_0) \in \Omega \times (S_i,T_i)$, *then*

$$(a\mu(t_0) + \mu'(t_0)u(x_0,t_0), \mu(t_0)\xi, \mu(t_0)X) \in \mathcal{P}^{2,\pm}(\mu(t_0)u(x_0,t_0)).$$

(ii) If μ *is continuously differentiable and* $(a,\xi,X) \in \overline{\mathcal{P}}^{2,\pm}u(x_0,t_0)$ *for some* $(x_0,t_0) \in \Omega \times (0,T)$, *then*

$$(a\mu(t_0) + \mu'(t_0)u(x_0,t_0), \mu(t_0)\xi, \mu(t_0)X) \in \overline{\mathcal{P}}^{2,\pm}(\mu(t_0)u(x_0,t_0)).$$

(iii) If μ *is continuously differentiable,* $(a,\xi,X) \in \overline{\mathcal{P}}^{2,\pm}(\mu(x_0,t_0)u(x_0,t_0))$ *and* $\mu(t_0) \neq 0$, *then*

$$\left(\frac{a}{\mu(t_0)} - u(x_0,t_0)\frac{\mu'(t_0)}{\mu(t_0)}, \frac{1}{\mu(t_0)}\xi, \frac{1}{\mu(t_0)}X\right) \in \overline{\mathcal{P}}^{2,\pm}u(x_0,t_0).$$

PROOF.

(i) We give the details of the proof for $\mathcal{P}^{2+}u$ and the claim about $\mathcal{P}^{2,-}u$ follows similarly.

Since μ is differentiable at t_0,

$$\mu(t) = \mu(t_0) + \mu'(t_0)(t - t_0) + o(|t - t_0|)$$

as $t \to t_0$, and since $(a, \xi, X) \in \mathcal{P}^{2,+}u(x_0, t_0)$,

$$u(x, t) \le u(x_0, t_0) + a(t - t_0) + \langle \xi, x - x_0\rangle + \frac{1}{2}\langle X(x - x_0), x - x_0\rangle + o(t_0 - t + |x - x_0|^2)$$

as $x \to x_0$ and $t \to t_0^-$. Since $\mu \ge 0$, it follows that

$$\begin{aligned}\mu(t)u(x,t) \le{}& (\mu(t_0) + \mu'(t_0)(t - t_0) + o(t_0 - t))\times \\ &\Big(u(x_0, t_0) + a(t - t_0) + \langle \xi, x - x_0\rangle \\ &+ \frac{1}{2}\langle X(x - x_0), x - x_0\rangle + o(t_0 - t + |x - x_0|^2)\Big) \\ ={}& \mu(t_0)u(x_0, t_0) + (a\mu(t_0) + \mu'(t_0)u(x_0, t_0))(t - t_0) \\ &+ \mu(t_0)\langle \xi, x - x_0\rangle + \frac{1}{2}\langle \mu(t_0)X(x - x_0), (x - x_0)\rangle \\ &+ o(|t_0 - t + |x - x_0|^2)\end{aligned}$$

as $x \to x_0$ and $t \to t_0^-$ and so the claim is true.

(ii) By the definition of $(a, \xi, X) \in \overline{\mathcal{P}}^{2,\pm}u(x_0, t_0)$, one can find sequences $(a_n, \xi_n, X_n)_{n\ge1} \subseteq \mathbb{R} \times \mathbb{R}^d \times \mathrm{Sym}_d$ and $(x_n, t_n)_{n\ge1} \subseteq \Omega \times (0, T)$ such that $t_n \le t_0$ for $n \ge 1$, $(a_n, \xi_n, X_n) \in \mathcal{P}^{2,\pm}u(x_n, t_n)$, $(a_n, \xi_n, X_n) \to (a, \xi, X)$, $(x_n, t_n) \to (x, t)$ and $u(x_n, t_n) \to u(x, t)$ as $n \to \infty$. By part (i),

$$(a_n\mu(t_n) + \mu'(t_n)u(x_n, t_n), \mu(t_n)\xi_n, \mu(t_n)X_n) \in \mathcal{P}^{2,\pm}(\mu(t_n)u(x_n, t_n)).$$

In addition to the convergences we have already assumed, since μ is continuously differentiable, we have $\mu(t_n) \to \mu(t_0)$ and $\mu'(t_n) \to \mu'(t_0)$ as $n \to \infty$. The claim now follows immediately.

(iii) The statement follows immediately by writing

$$u(x_0, t_0) = \frac{1}{\mu(t_0)}(\mu(t_0)u(x_0, t_0))$$

and applying part (ii).

□

PROOF OF THEOREM 4.19. Let u be a positive viscosity solution of (4.35) and set $v = \frac{m}{\lambda}u^\lambda$. Then, as already shown, v is a viscosity solution of (4.36).

Fix $\varepsilon > 0$ and define a function Z on $\overline{\Omega} \times \overline{\Omega} \times S$ by

$$Z(x, y; t, s) = \beta^\lambda(t)v(x, t) - \beta^\lambda(s)v(y, s) + \omega(x, y; t, s) + \frac{\varepsilon}{2}(t - s)^2 + \varepsilon.$$

Identically as in the case for classical solutions, we know that $Z > 0$ whenever s is sufficiently small, or s is sufficiently close to t. In particular, $Z > 0$ in a small enough neighbourhood of $(0,0)$. Suppose there is a point $P_0 := (x_0, y_0; t_0, s_0)$ such that $Z = 0$ for the first time at P_0 in the sense of Definition 3.1. We first assume that $x_0, y_0 \in \Omega$.

Applying Corollary 4.15 with $u_1(x,t) = \beta^\lambda(t)v(x,t)$, $u_2(y,s) = -\beta^\lambda(s)v(y,s)$, and $\varphi = -\omega(x,y;t,s) - \frac{\varepsilon}{2}(t-s)^2 - \varepsilon$ for $(x,y,t,s) \in \Omega \times \Omega \times (s_0, t_0] \times (0, s_0]$, we deduce that for every $\kappa > 0$, there exist matrices $X, Y \in \mathrm{Sym}_d$ such that

$$(\partial_t \varphi(P_0), \nabla_x \varphi(P_0), X) \in \overline{\mathcal{P}}^{2,-}(\beta^\lambda v)(x_0, t_0), \tag{4.44}$$

$$(\partial_s \varphi(P_0), \nabla_y \varphi(P_0), -Y) \in \overline{\mathcal{P}}^{2,-}(-\beta^\lambda v)(y_0, s_0), \tag{4.45}$$

and

$$\left(\frac{1}{\kappa} + \|A\|\right) I_{2d} \geq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \geq H - \kappa H^2, \tag{4.46}$$

where $H = D^2_{(x,y)}\varphi(P_0) = -D^2_{(x,y)}\omega(P_0)$. Computing the first-order derivatives of φ explicitly and applying the product rule in Lemma 4.20 (iii), the condition (4.44) implies

$$\begin{aligned}\Bigg(\frac{-\partial_t \omega(P_0) - \varepsilon(t_0 - s_0)}{\beta^\lambda(t_0)} - \lambda v(x_0, t_0)\frac{\beta'(t_0)}{\beta(t_0)}&,\\ -\frac{1}{\beta^\lambda(t_0)}\nabla_x \omega(P_0), \frac{1}{\beta^\lambda(t_0)} X\Bigg) &\in \overline{\mathcal{P}}^{2,-} v(x_0, t_0).\end{aligned}$$

Using that v is a supersolution of (4.36), this implies

$$\begin{aligned}0 \leq\ & \frac{-\partial_t \omega(P_0) - \varepsilon(t_0 - s_0)}{\beta^\lambda(t_0)} - \lambda v(x_0, t_0)\frac{\beta'(t_0)}{\beta(t_0)} \\ & - \lambda v(x_0, t_0)\beta^{-\lambda(p-1)}(t_0)|\nabla_x \omega(P_0)|^{p-4} \times \\ & \qquad (|\nabla_x \omega(P_0)|^2 \operatorname{tr} X + (p-2)\nabla_x \omega(P_0)^\top X \nabla_x \omega(P_0)) \\ & - \beta^{-\lambda p}(t_0)|\nabla_x \omega(P_0)|^p.\end{aligned}$$

However, since $\beta^\lambda(t_0) > 0$, we may apply (4.38) to obtain

$$\begin{aligned}0 \leq\ & -\frac{\varepsilon(t_0 - s_0)}{\lambda v(x_0, t_0)} - \beta^{\lambda - 1}(t_0)\beta'(t_0) \\ & - \beta^{-\lambda(p-2)}(t_0)|\nabla_x \omega(P_0)|^{p-4} \times \\ & \quad (|\nabla_x \omega(P_0)|^2 \operatorname{tr} X + (p-2)\nabla_x \omega(P_0)^\top X \nabla_x \omega(P_0)).\end{aligned} \tag{4.47}$$

In a similar way, one may derive from (4.45) using the product rule, that

$$\begin{aligned}\Bigg(\frac{\partial_s \omega(P_0) - \varepsilon(t_0 - s_0)}{\beta^\lambda(s_0)} - \lambda v(y_0, s_0)\frac{\beta'(s_0)}{\beta(s_0)}&,\\ \frac{1}{\beta^\lambda(s_0)}\nabla_y \omega(P_0), \frac{1}{\beta^\lambda(s_0)} Y\Bigg) &\in \overline{\mathcal{P}}^{2,+} v(y_0, s_0).\end{aligned}$$

Since v is also a subsolution of (4.36), we have

$$\begin{aligned}0 \geq \frac{\partial_s \omega(P_0) - \varepsilon(t_0 - s_0)}{\beta^{\lambda}(s_0)} &- \lambda v(y_0, s_0)\frac{\beta'(s_0)}{\beta(s_0)} \\ &- \lambda v(y_0, s_0)\beta^{-\lambda(p-1)}(s_0)|\nabla_y \omega(P_0)|^{p-4}\times \\ &\qquad (|\nabla_y \omega(P_0)|^2 \operatorname{tr} Y + (p-2)\nabla_y \omega(P_0)^\top Y \nabla_y \omega(P_0)) \\ &- \beta^{-\lambda p}(s_0)|\nabla_y \omega(P_0)|^p,\end{aligned}$$

which because of (4.39) becomes

$$\begin{aligned}0 \geq -\frac{\varepsilon(t_0 - s_0)}{\lambda v(y_0, s_0)} &- \beta^{\lambda-1}(s_0)\beta'(s_0) \\ &- \beta^{-\lambda(p-2)}(s_0)|\nabla_y \omega(P_0)|^{p-4}\times \\ &\qquad (|\nabla_y \omega(P_0)|^2 \operatorname{tr} Y + (p-2)\nabla_y \omega(P_0)^\top Y \nabla_y \omega(P_0)).\end{aligned} \tag{4.48}$$

Taking a linear combination of the inequalities (4.47) and (4.48) with the coefficients $A(t_0)^2$ and $A(s_0)^2$ yields

$$\begin{aligned}0 \geq A(t_0)^2\beta^{\lambda}(t_0)\beta'(t_0) &- A(s_0)^2\beta^{\lambda}(s_0)\beta'(s_0) \\ &+ \frac{\varepsilon}{\lambda}(t_0 - s_0)\left(\frac{A(t_0)^2}{v(x_0, t_0)} - \frac{A(s_0)^2}{v(y_0, s_0)}\right) \\ &+ A(t_0)^2\beta^{-\lambda(p-2)}(t_0)|\nabla_x \omega(P_0)|^{p-4}\times \\ &\qquad (|\nabla_x \omega(P_0)|^2 \operatorname{tr} X + (p-2)\nabla_x \omega(P_0)^\top X \nabla_x \omega(P_0)) \\ &- A(s_0)^2\beta^{-\lambda(p-2)}(s_0)|\nabla_y \omega(P_0)|^{p-4}\times \\ &\qquad (|\nabla_y \omega(P_0)|^2 \operatorname{tr} Y + (p-2)\nabla_y \omega(P_0)^\top Y \nabla_y \omega(P_0)).\end{aligned} \tag{4.49}$$

We first work to estimate the quantity

$$\begin{aligned}Q := A(t_0)^2&\beta^{-\lambda(p-2)}(t_0)|\nabla_x \omega(P_0)|^{p-4}\times \\ &(|\nabla_x \omega(P_0)|^2 \operatorname{tr} X + (p-2)\nabla_x \omega(P_0)^\top X \nabla_x \omega(P_0)) \\ &- A(s_0)^2\beta^{-\lambda(p-2)}(s_0)|\nabla_y \omega(P_0)|^{p-4}\times \\ &\qquad (|\nabla_y \omega(P_0)|^2 \operatorname{tr} Y + (p-2)\nabla_y \omega(P_0)^\top Y \nabla_y \omega(P_0)) \\ = \operatorname{tr}&\Big(A(t_0)^2\beta^{-\lambda(p-2)}(t_0)|\nabla_x \omega(P_0)|^{p-2}X - A(s_0)^2\beta^{-\lambda(p-2)}(s_0)|\nabla_y \omega(P_0)|^{p-2}Y\Big) \\ &+ (p-2)\Big(A(t_0)^2\beta^{-\lambda(p-2)}(t_0)|\nabla_x \omega(P_0)|^{p-4}\nabla_x \omega(P_0)^\top X \nabla_x \omega(P_0) \\ &\qquad - A(s_0)^2\beta^{-\lambda(p-2)}(s_0)|\nabla_y \omega(P_0)|^{p-4}\nabla_y \omega(P_0)^\top Y \nabla_y \omega(P_0))\Big)\end{aligned}$$

Setting

$$F := \begin{pmatrix} A(t_0)^2\beta^{-\lambda(p-2)}(t_0)|\nabla_x\omega(P_0)|^{p-2}I_d & 0 \\ 0 & A(s_0)^2\beta^{-\lambda(p-2)}(s_0)|\nabla_y\omega(P_0)|^{p-2}I_d \end{pmatrix},$$

$$G := \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix},$$

$$\zeta := \begin{pmatrix} A(t_0)\beta^{-\frac{\lambda(p-2)}{2}}(t_0)|\nabla_x\omega(P_0)|^{\frac{p-4}{2}}\nabla_x\omega(P_0) \\ A(s_0)\beta^{-\frac{\lambda(p-2)}{2}}(s_0)|\nabla_y\omega(P_0)|^{\frac{p-4}{2}}\nabla_y\omega(P_0) \end{pmatrix},$$

we may express Q by

$$Q = \operatorname{tr}(FG) + (p-2)\zeta^\top G\zeta.$$

If we set

$$\tilde{F} := \begin{pmatrix} F_{11} & F_{12} \\ F_{12} & F_{22} \end{pmatrix},$$

where

$$\begin{aligned} F_{11} &:= A^2(t_0)\beta^{-\lambda(p-2)}(t_0)|\nabla_x\omega(P_0)|^{p-2}I_d, \\ F_{12} &:= A(t_0)A(s_0)\beta^{-\frac{\lambda(p-2)}{2}}(t_0)\beta^{-\frac{\lambda(p-2)}{2}}(s_0)|\nabla_x\omega(x_0,t_0)|^{\frac{p-2}{2}}|\nabla_y\omega(y_0,s_0)|^{\frac{p-2}{2}}I_d, \\ F_{22} &:= A^2(s_0)\beta^{-\lambda(p-2)}(s_0)|\nabla_y\omega(P_0)|^{p-2}I_d, \end{aligned}$$

then it is equivalent to write

$$Q = \operatorname{tr}(\tilde{F}G) + (p-2)\zeta^\top G\zeta.$$

According to (4.46), we have $G \geq -D^2_{(x,y)}\omega(P_0) - \kappa(D^2_{(x,y)}\omega(P_0))^2$, and therefore

$$\begin{aligned} Q \geq &-\operatorname{tr}(\tilde{F}(D^2_{(x,y)}\omega(P_0) + \kappa(D^2_{(x,y)}\omega(P_0))^2)) \\ &-(p-2)\zeta^\top(D^2_{(x,y)}\omega(P_0) + \kappa(D^2_{(x,y)}\omega(P_0))^2)\zeta \end{aligned}$$

We may now take $\kappa \to 0$ to obtain

$$Q \geq -\operatorname{tr}(\tilde{F}(D^2_{(x,y)}\omega(P_0)) - (p-2)\zeta^\top D^2_{(x,y)}\omega(P_0)\zeta.$$

Inserting this in (4.49) with the products written out and using (4.40), we have

$$\begin{aligned} 0 \geq\ & A(t_0)^2\beta^\lambda(t_0)\beta'(t_0) - A(s_0)^2\beta^\lambda(s_0)\beta'(s_0) \\ &+ \frac{\varepsilon}{\lambda}(t_0 - s_0)\left(\frac{A(t_0)^2}{v(x_0,t_0)} - \frac{A(s_0)^2}{v(y_0,s_0)}\right) \\ &- A^2(t_0)\beta^{-\lambda(p-2)}(t_0)\Delta_{p,x}\omega(P_0) - A^2(s_0)\beta^{-\lambda(p-2)}(s_0)\Delta_{p,y}\omega(P_0) \\ &- 2A(t_0)A(s_0)\beta^{-\frac{\lambda(p-2)}{2}}(t_0)\beta^{-\frac{\lambda(p-2)}{2}}(s_0)\Delta_{p,xy}\omega(P_0) \\ \geq\ & \frac{\varepsilon}{\lambda}(t_0 - s_0)\left(\frac{A(t_0)^2}{v(x_0,t_0)} - \frac{A(s_0)^2}{v(y_0,s_0)}\right) > 0, \end{aligned}$$

which is a contradiction, as before. Therefore, if such a point P_0 exists, it must occur with at least one of x_0 or y_0 on the boundary $\partial\Omega$.

We now make no assumption regarding whether $x_0, y_0 \in \Omega$ or $x_0, y_0 \in \partial\Omega$. Proceeding in the same manner as in the proof of Theorem 4.17, we choose a point $z_0 \in \Omega$ with $\delta := d(z_0, \partial\Omega) > 0$ and define

$$h(x) := \frac{1}{2}|x - z_0|^2$$

for all $x \in \overline{\Omega}$. Letting

$$v^\theta(x,t) := v(x,t) + \theta h(x), \quad \text{and} \quad v_\theta(x,t) := v(x,t) - \theta h(x)$$

for $(x,t) \in \overline{\Omega} \times (0,T)$ and $\theta > 0$ small, we have that v^θ satisfies

$$\begin{cases} \partial_t v^\theta - \lambda(v^\theta - \theta h)\Delta_p(v^\theta - \theta h) - |\nabla(v^\theta - \theta h)|^p = 0 & \text{in } \Omega \times (0,T), \\ \qquad |\nabla v^\theta - \theta_n \nabla h|^{p-2}(\nabla v^\theta - \theta_n \nabla h) \cdot \nu(x_{\theta_n}) = 0 & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{4.50}$$

and v_θ satisfies

$$\begin{cases} \partial_t v^\theta - \lambda(v^\theta + \theta h)\Delta_p(v^\theta + \theta h) - |\nabla(v^\theta + \theta h)|^p = 0 & \text{in } \Omega \times (0,T), \\ \qquad |\nabla v^\theta + \theta_n \nabla h|^{p-2}(\nabla v^\theta + \theta_n \nabla h) \cdot \nu(x_{\theta_n}) = 0 & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{4.51}$$

both in the sense of viscosity solutions.

We again modify the function Z by defining

$$\tilde{Z}_\theta(x,y;t,s) := \beta^\lambda(t)v^\theta(x,t) - \beta^\lambda(s)v_\theta(y,s) - \varphi(x,y;t,s),$$

where

$$\begin{aligned} \varphi(x,y,;t,s) := -\omega(x,y;t,s) - \frac{\varepsilon}{2}(t-s)^2 - \varepsilon \\ - |x - x_0|^4 - |y - y_0|^4 - |t - t_0|^4 - |s - s_0|^4 \end{aligned}$$

for all $(x,y;t,s) \in \tilde{K} := \overline{\Omega} \times \overline{\Omega} \times (t_0 - \eta, t_0] \times (s_0 - \eta, s_0]$ and $\theta > 0$, where $0 < \eta < t_0 - s_0$. By the same argument as before, we can find a sequence of points $(P_{\theta_n})_{n \geq 1} \subseteq \tilde{K}$, such that P_{θ_n} minimises $\tilde{Z_{\theta_n}}$ over $\tilde{K}$ for each $n \geq 1$ and $P_{\theta_n} \to P_0$ in $\tilde{\tilde{K}}$. Applying Corollary 4.15 to $\tilde{Z}_{\theta_n}$ at P_{θ_n}, we find that for all $\kappa > 0$, there exist $X_n, Y_n \in \mathrm{Sym}_d$ such that

$$\begin{aligned} (\partial_t \varphi(P_{\theta_n}), \nabla_x \varphi(P_{\theta_n}), X_n) \in \overline{\mathcal{P}}^{2,-}_{\tilde{K}}(\beta^\lambda v^{\theta_n})(x_{\theta_n}, t_{\theta_n}), \\ (-\partial_s \varphi(P_{\theta_n}), -\nabla_y \varphi(P_{\theta_n}), Y_n) \in \overline{\mathcal{P}}^{2,+}_{\tilde{K}}(\beta^\lambda v_{\theta_n})(y_{\theta_n}, s_{\theta_n}), \end{aligned}$$

and also

$$\left(\frac{1}{\kappa} + \|H_n\|\right) I_{2d} \geq \begin{pmatrix} X_n & 0 \\ 0 & -Y_n \end{pmatrix} \geq H_n - \kappa H_n^2, \tag{4.52}$$

where $H_n := D^2\varphi(P_{\theta_n})$. Using the product rule and evaluating the derivatives of φ, this implies

$$\begin{aligned}(a^n, \xi^n, \Upsilon^n) := \Bigg(&\frac{-\partial_t\omega(P_{\theta_n}) - 4|t_{\theta_n} - t_0|^2(t_{\theta_n} - t_0) - \varepsilon(t_{\theta_n} - s_{\theta_n})}{\beta^\lambda(t_{\theta_n})} \\ &- \lambda v^{\theta_n}(x_{\theta_n}, t_{\theta_n})\frac{\beta'(t_{\theta_n})}{\beta(t_{\theta_n})}, \\ &- \frac{1}{\beta^\lambda(t_{\theta_n})}(\nabla_x\omega(P_{\theta_n}) + 4|x_{\theta_n} - x_0|^2(x_{\theta_n} - x_0)), \frac{1}{\beta^\lambda(t_{\theta_n})}X_n\Bigg) \\ &\qquad\qquad\in \overline{\mathcal{P}}^{2,-}_{\tilde K}(v^{\theta_n})(x_{\theta_n}, t_{\theta_n}), \\ (a_n, \xi_n, \Upsilon_n) := \Bigg(&\frac{\partial_s\omega(P_{\theta_n}) + 4|s_{\theta_n} - s_0|^2(s_{\theta_n} - s_0) - \varepsilon(t_{\theta_n} - s_{\theta_n})}{\beta^\lambda(s_{\theta_n})} \\ &- \lambda v_{\theta_n}(y_{\theta_n}, s_{\theta_n})\frac{\beta'(s_{\theta_n})}{\beta(s_{\theta_n})}, \\ &\frac{1}{\beta^\lambda(s_{\theta_n})}(\nabla_y\omega(P_{\theta_n}) + 4|y_{\theta_n} - y_0|^2(y_{\theta_n} - y_0)), \frac{1}{\beta^\lambda(s_{\theta_n})}Y_n\Bigg) \\ &\qquad\qquad\in \overline{\mathcal{P}}^{2,+}_{\tilde K}(v_{\theta_n})(y_{\theta_n}, s_{\theta_n}).\end{aligned}$$

Since v^{θ_n} is a viscosity subsolution of (4.50), we have that

$$\begin{aligned}Q_n := a^n &- \lambda(v^{\theta_n}(x_{\theta_n}, t_{\theta_n}) - \theta_n\nabla h(x_{\theta_n}))|\xi^n - \theta_n\nabla h(x_{\theta_n})|^{p-4}\times \\ &\Big(|\xi^n - \theta_n\nabla h(x_{\theta_n})|^2\operatorname{tr}(\Upsilon^n - \theta_n D^2h(x_{\theta_n})) \\ &\quad + (p-2)(\xi^n - \theta_n\nabla h(x_{\theta_n}))^\top(\Upsilon^n - \theta_n D^2h(x_{\theta_n}))(\xi^n - \theta_n\nabla h(x_{\theta_n}))\Big) \\ &- |\xi^n - \theta_n\nabla h(x_{\theta_n})|^p \geq 0\end{aligned} \tag{4.53}$$

if $x_{\theta_n} \in \Omega$ and

$$\max\{Q_n, |\xi^n - \theta_n\nabla h(x_{\theta_n})|^{p-2}(\xi^n - \theta_n\nabla h(x_{\theta_n}))\cdot\nu(x_{\theta_n})\} \geq 0$$

if $x_{\theta_n} \in \partial\Omega$. By the same argument as in the proof of Theorem 4.17, we conclude that

$$\begin{aligned}(\xi^n &- \theta_n\nabla h(x_{\theta_n}))\cdot\nu(x_{\theta_n}) \\ &= \left(-\frac{1}{\beta^\lambda(t_{\theta_n})}(\nabla_x\omega(P_{\theta_n}) + 4|x_{\theta_n} - x_0|^2(x_{\theta_n} - x_0)) - \theta_n\nabla h(x_{\theta_n})\right)\cdot\nu(x_{\theta_n}) \\ &< -\theta_n\delta < 0,\end{aligned}$$

and in particular $\xi^n - \theta_n\nabla h(x_{\theta_n}) \neq 0$. Therefore, we have

$$|\xi^n - \theta_n\nabla h(x_{\theta_n})|^{p-2}(\xi^n - \theta_n\nabla h(x_{\theta_n}))\cdot\nu(x_{\theta_n}) < 0,$$

which implies (4.53) holds whether $x_{\theta_n} \in \Omega$ or $x_{\theta_n} \in \partial\Omega$. By the same argument, we can also show that

$$\begin{aligned}
a_n &- \lambda(v_{\theta_n}(y_{\theta_n}, s_{\theta_n}) - \theta_n \nabla h(y_{\theta_n}))|\xi_n - \theta_n \nabla h(y_{\theta_n})|^{p-4} \times \\
&\Big(|\xi_n - \theta_n \nabla h(y_{\theta_n})|^2 \operatorname{tr}(\Upsilon_n - \theta_n D^2 h(y_{\theta_n})) \\
&\quad + (p-2)(\xi_n - \theta_n \nabla h(y_{\theta_n}))^\top (\Upsilon_n - \theta_n D^2 h(y_{\theta_n}))(\xi_n - \theta_n \nabla h(y_{\theta_n}))\Big) \\
&- |\xi_n - \theta_n \nabla h(y_{\theta_n})|^p \geq 0.
\end{aligned} \tag{4.54}$$

Taking $n \to \infty$ in (4.52), (4.53) and (4.54) yields that

$$\begin{aligned}
&\frac{-\partial_t \omega(P_0) - \varepsilon(t_0 - s_0)}{\beta^\lambda(t_0)} - \lambda v(x_0, t_0) \frac{\beta'(t_0)}{\beta(t_0)} \\
&\quad - \lambda v(x_0, t_0) \beta^{-\lambda(p-1)}(t_0) |\nabla_x \omega(P_0)|^{p-4} \times \\
&\qquad (|\nabla_x \omega(P_0)|^2 \operatorname{tr} X + (p-2) \nabla_x \omega(P_0)^\top X \nabla_x \omega(P_0)) \\
&\quad - \beta^{-\lambda p}(t_0) |\nabla_x \omega(P_0)|^p \geq 0,
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial_s \omega(P_0) - \varepsilon(t_0 - s_0)}{\beta^\lambda(s_0)} - \lambda v(y_0, s_0) \frac{\beta'(s_0)}{\beta(s_0)} \\
&\quad - \lambda v(y_0, s_0) \beta^{-\lambda(p-1)}(s_0) |\nabla_y \omega(P_0)|^{p-4} \times \\
&\qquad (|\nabla_y \omega(P_0)|^2 \operatorname{tr} Y + (p-2) \nabla_y \omega(P_0)^\top Y \nabla_y \omega(P_0)) \\
&\quad - \beta^{-\lambda p}(s_0) |\nabla_y \omega(P_0)|^p \geq 0,
\end{aligned}$$

$$\left(\frac{1}{\kappa} + \|H\|\right) I_{2d} \geq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \geq H - \kappa H^2$$

hold as before and we arrive to a contradiction from here. □

Remark 4.4. It was demonstrated in [**24**] (see also [**87, 137**] for the case $m = 1$), that all bounded weak solutions of the doubly nonlinear heat equation are viscosity solutions. Therefore, our results for viscosity solutions are also true for weak solutions.

CHAPTER 5

Final Remarks and Open Problems

The goal of this thesis has been to introduce a new technique to prove global pointwise parabolic Harnack inequalities on Riemannian manifolds using a multi-point maximum principle. While we have laid the groundwork for this new methodology, there are several directions in which our results could be expanded. We summarise the most pressing questions, which still remain.

The doubly nonlinear heat equation for $1 < p < 2$

In Chapters 3 and 4, we chose to restrict our attention to the doubly nonlinear heat equation with $p \geq 2$, so that the p-Laplace operator would be in its non-singular range. We expect that our techniques can be modified to treat the singular range $1 < p < 2$ as well. This would require a careful redefinition of viscosity solutions of the equation (1.4).

Viscosity solutions on manifolds

We would also like to be able to treat viscosity solutions of the doubly nonlinear heat equation on general Riemannian manifolds (M, g). This would require an extension of both the definition of viscosity solutions and the parabolic maximum principle for semicontinuous functions into the manifold setting, although these ideas have already been considered previously (see, for example [**15, 81**]).

Weakened regularity assumptions on ω

In Theorem 4.19, we assume that the function ω appearing in the Harnack inequality satisfies the collection of differential inequalities (4.38)–(4.41) in the classical sense. However, it is natural to consider whether Theorem 4.19 remains true if ω only satisfies these inequalities in the viscosity sense. This would have the advantage of reducing the regularity assumptions on ω, as ω would only need to be a continuous function. This would be especially helpful when carrying out proofs in the manifold setting, where our optimal choice of ω lacks in general the sufficient regularity to satisfy the necessary inequalities classically.

Weakening the condition Ric ≥ 0

For simplicity, we have assumed that all manifolds discussed in this thesis have nonnegative Ricci curvature. However, a great volume of the existing literature on parabolic Harnack inequalities, including the pioneering paper [**104**] by Li and Yau, deals with manifolds satisfying the more general bound Ric $\geq -K$ on the Ricci curvature for some $K \geq 0$. In order to treat such manifolds, we would need to better understand how the curvature bounds influence the structure of the Harnack inequality, and in turn, the required properties of the function ω.

Inclusion of lower order terms

It would be preferable to treat equations, which contain first and zero-order terms. In the linear setting, we already considered the heat equation with both a potential and gradient drift term in Section 3.1.4, however, we would like to expand the methodology to second-order linear equations with the full general structure

$$\partial_t u = \sum_{i,j=1}^{d} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i \frac{\partial u}{\partial x_i} + cu.$$

In the nonlinear setting, we would like to consider the doubly nonlinear heat equation with a potential term

$$\partial_t u = \Delta_p(u^m) + V|u^m|^{p-2}u^m, \tag{5.1}$$

and look for the analogues of the results in this thesis for the linear Schrödinger equation.

In addition, it would be interesting to consider generalisations of the doubly nonlinear operator $\Delta_p(u^m)$. For example, for $p = 2$, one may consider replacing the porous medium operator $\Delta(u^m)$ by the generalised porous medium operator $\Delta(\varphi(u))$, where φ is a non-decreasing function. Similarly, one could replace the p-Laplace operator $\Delta_p u$ by

$$\Delta_\varphi u := \operatorname{div}(\varphi(|\nabla u|)\nabla u),$$

The role of the fundamental solution

Throughout the development of this thesis, it has become increasingly clear that the optimal functional ω is inextricably connected to the fundamental solution of the given differential equation. This relationship is made clearest in the linear scenario by the limit formula (3.19) from Theorem 6.1 of [**104**]. After understanding the doubly nonlinear heat equation (5.1) with potential, we would be interested to investigate whether such a formula holds in the nonlinear setting as well.

CHAPTER A

Riemannian Geometry

We recall the most essential definitions from Riemannian geometry used throughout this thesis.

Definition A.1. Let M be a second-countable Hausdorff topological space. We say that M is a *d-dimensional differentiable manifold* if there is an open covering $(U_i)_{i\in I}$ of M, open sets $(V_i)_{i\in I}$ in $\mathbb{R}^d$, and homeomorphisms $\varphi : U_i \to V_i$ such that for all $i, j \in I$ with $U_i \cap U_j \neq \emptyset$, the map $\varphi_j \circ \varphi_i^{-1}$ is a diffeomorphism. We call each pair (U_i, φ_i) a *chart* on M and often identify the chart (U_i, φ_i) with the set U_i.

Definition A.2. For a function $f : M \to \mathbb{R}$, we say that $f \in C^k(M)$ for some $k \in \mathbb{N} \cup \{\infty\}$ if for every chart (U, φ), $\varphi : U \to V \subseteq \mathbb{R}^d$ on M, one has $f|_U \circ \varphi^{-1} \in C^k(V)$. In addition, we set $C_c^\infty(M)$ to be the subset of $C^\infty(M)$ containing functions with compact support in M.

Definition A.3. We call a function $\xi : C^\infty(M) \to \mathbb{R}$ a *derivation* at a point $x \in M$ if ξ is linear and satisfies the following "product rule":

$$\xi(fg) = \xi(f)g(x) + \xi(g)f(x_0)$$

for all $f, g \in C^\infty(M)$. The set of all derivations at x forms a d-dimensional vector space, which we call the tangent space T_xM at x.

Definition A.4. A vector field X on M is a map $x \mapsto X(x)$, which assigns to every $x \in M$ a unique vector $X(x) \in T_xM$. A vector field X is called smooth if $X(x)$ depends smoothly on x.

Definition A.5. Given two vector fields X, Y on M, we call the vector field given by

$$[X, Y] := XY - YX$$

the Lie bracket of X and Y.

Throughout this thesis, all manifolds we discuss have the structure of a Riemannian manifold.

Definition A.6 (Riemannian manifold)**.** Let M be a d-dimensional differentiable manifold. A *Riemannian metric* on M is a family $g = (g(x))_{x\in M}$, such that $g(x)$ is a symmetric, positive definite bilinear form on T_xM for every $x \in M$ and $g(x)$ depends smoothly on x. We call the pair (M, g) a *Riemannian manifold.*

We would also like to have a notion of the curvature of a Riemannian manifold (M, g). For this, we must first introduce the idea of a connection on M.

Definition A.7 (Connection, Levi-Civita connection)**.**

(i) A connection ∇ on a manifold M is an $\mathbb{R}$-bilinear map, which assigns to each pair X, Y of vector fields on M another vector field, denoted $\nabla_X Y$, such that
$$\nabla_{fX} Y = f \nabla_X Y$$
and
$$\nabla_X (fY) = X(f) Y + f \nabla_X Y$$
for all $f \in C^\infty(M)$.

(ii) The Levi-Civita connection on a Riemannian manifold (M, g) is the unique connection on M that is compatible with g in the sense that
$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$
for all vector fields X, Y, Z on M, and torsion-free, meaning that
$$\nabla_X Y - \nabla_Y X = [X, Y]$$
for all vector fields X, Y on M.

We always assume that the connection on the Riemannian manifolds we discuss is the Levi-Civita connection.

Definition A.8.

(i) We define the Riemann curvature tensor R by setting
$$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$
where X, Y, Z are vector fields on M. In addition, we write
$$R(X, Y, Z, W) := g(R(X, Y)Z, W),$$
where W is another vector field on M.

(ii) Given a basis $(e_i(x))_{1 \le i \le d}$ of $T_x M$ at each point $x \in M$, we define the Ricci curvature of M to be
$$\mathrm{Ric}(X, Y) := \sum_{i=1}^{d} R(e_i, X, Y, e_i).$$

Very importantly, all d-dimensional Riemannian manifolds can be isometrically embedded into $\mathbb{R}^\ell$ for some $\ell \ge d+1$, which is a theorem due to Nash [**120**]. Once we understand the manifold as existing in an ambient space, it makes sense to consider normal vectors to M.

Definition A.9. Let (M, g) be a Riemannian manifold embedded in $\mathbb{R}^\ell$ for some $\ell \ge 1$. We call a vector $N \in \mathbb{R}^\ell$ a normal vector to M at a point $x \in M$ if $g(X, N) = 0$ for all $X \in T_x M$, that is, N belongs to the orthogonal complement of $T_x M$ in $\mathbb{R}^\ell$. Then a normal vector field on M is a map $x \mapsto N(x)$, which assigns to each $x \in M$ a normal vector $N(x)$ at x.

In the special case that M is a d-dimensional hypersurface embedded in $\mathbb{R}^{d+1}$, the set of possible normal vectors at $x \in M$ forms a one-dimensional subspace of $\mathbb{R}^{d+1}$. Thus, after normalisation, there are only two choices of normal vector. If M is a closed manifold, that is, M is compact and without boundary, these choices correspond to the inward and outward pointing directions. For example, in this thesis, we often work on a compact manifold M with boundary ∂M and treat ∂M as a closed submanifold of M. In this situation, we typically choose to work with outward pointing unit normal vector at each point $x \in \partial M$.

We now define the second-fundamental form of M in the following way.

Definition A.10. Let X, Y be vector fields on M, which have been locally extended to vector fields in $\mathbb{R}^{d+1}$, and let N be an outward pointing normal vector field on M. We set

$$h(X, Y) := -g(\nabla_X Y, N),$$

where $\nabla_X Y$ denotes the covariant derivative in $\mathbb{R}^{d+1}$.

CHAPTER B

A classical proof of Li-Yau's differential Harnack inequality

In this brief appendix, we review the main idea of a standard proof of the Li-Yau inequality for the heat equation on the Euclidean space $\mathbb{R}^d$. The approach presented here most closely resembles the one used by Aronson and Bénilan [**8**] to obtain the analogous estimate for the porous medium equation, and later by Esteban and Vázquez [**61**] for the p-heat equation and doubly nonlinear heat equation.

Let u be a positive solution of the heat equation

$$\partial_t u = \Delta u \qquad \text{in } \mathbb{R}^d \times (0, \infty)$$

and set $v := \log u$. Then v is a solution of

$$\partial_t v = \Delta v + |\nabla v|^2. \tag{B.1}$$

Now set $p := \Delta v$. Using the equation (B.1) and the elementary inequality

$$\sum_{i,j=1}^{d} a_{ij}^2 \geq \frac{1}{d} \left(\sum_{i=1}^{d} a_{ii} \right)^2$$

applied to $(a_{ij}) = D^2 v$, it follows that

$$\begin{aligned} \partial_t p &= \Delta(v_t) = \Delta(\Delta v + |\nabla v|^2) \\ &= \Delta p + 2 \sum_{i,j=1}^{d} \left(\frac{\partial^2 v}{\partial x_i \partial x_j} \right)^2 + 2\nabla v \cdot \nabla(\Delta v) \\ &\geq \Delta p + \frac{2}{d} \left(\sum_{i=1}^{d} \frac{\partial^2 v}{\partial x_i^2} \right)^2 + 2\nabla v \cdot \nabla(\Delta v) \\ &= \Delta p + 2\nabla v \cdot \nabla p + \frac{2}{d} p^2. \end{aligned}$$

In other words, one has $\mathcal{L}p \geq 0$, where $\mathcal{L}$ is the operator defined by

$$\mathcal{L}p := \partial_t p - \Delta p - 2\nabla v \cdot \nabla p - \frac{2}{d} p^2.$$

Observe that for $\tilde{p} := -\frac{d}{2t}$, one has

$$\mathcal{L}\tilde{p} = \frac{d}{2t^2} - \frac{2}{d} \left(\frac{d^2}{4t^2} \right) = 0.$$

Therefore $\mathcal{L}p \geq \mathcal{L}\tilde{p}$ and a comparison principle implies $p \geq \tilde{p}$. Thus

$$\Delta v \geq -\frac{d}{2t}.$$

Using the equation (B.1) and reversing the transformation, this is equivalent to

$$\frac{\partial_t u}{u} + \frac{d}{2t} \geq \frac{|\nabla u|^2}{u^2},$$

which is the celebrated Li-Yau estimate.

Bibliography

[1] S. Agmon. *Lectures on exponential decay of solutions of second-order elliptic equations: bounds on eigenfunctions of N-body Schrödinger operators*, volume 29 of *Mathematical Notes*. Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1982.

[2] A. D. Alexandroff. Almost everywhere existence of the second differential of a convex function and some properties of convex surfaces connected with it. *Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser.*, 6:3–35, 1939.

[3] H. W. Alt and S. Luckhaus. Quasilinear elliptic-parabolic differential equations. *Math. Z.*, 183(3):311–341, 1983. doi:10.1007/BF01176474.

[4] D. Andreucci and A. F. Tedeev. Optimal decay rate for degenerate parabolic equations on noncompact manifolds. *Methods Appl. Anal.*, 22(4):359–376, 2015. doi:10.4310/MAA.2015.v22.n4.a2.

[5] B. Andrews. Multi-point maximum principles and eigenvalue estimates. In *2018 MATRIX annals*, volume 3 of *MATRIX Book Ser.*, pages 185–213. Springer, Cham, [2020] ©2020.

[6] B. Andrews and J. Clutterbuck. Time-interior gradient estimates for quasilinear parabolic equations. *Indiana Univ. Math. J.*, 58(1):351–380, 2009. doi:10.1512/iumj.2009.58.3756.

[7] B. Andrews, D. Hauer, and J. Slegers. A multi-point maximum principle to prove global Harnack inequalities for Schrödinger operators. *Calc. Var. Partial Differential Equations*, 65(4):Paper No. 112, 23, 2026. doi:10.1007/s00526-026-03277-6.

[8] D. G. Aronson and P. Bénilan. Régularité des solutions de l'équation des milieux poreux dans $\mathbf{R}^N$. *C. R. Acad. Sci. Paris Sér. A-B*, 288(2):A103–A105, 1979.

[9] G. Ascione, D. Castorina, G. Catino, and C. Mantegazza. A matrix Harnack inequality for semilinear heat equations. *Math. Eng.*, 5(1):Paper No. 003, 15, 2023. doi:10.3934/mine.2023003.

[10] H. Attouch. Convergence de fonctions convexes, des sous-différentiels et semi-groupes associés. *C. R. Acad. Sci. Paris Sér. A-B*, 284(10):A539–A542, 1977.

[11] H. Attouch. *Variational convergence for functions and operators*. Applicable Mathematics Series. Pitman (Advanced Publishing Program), Boston, MA, 1984.

[12] H. Attouch, G. Buttazzo, and G. Michaille. *Variational analysis in Sobolev and BV spaces*, volume 6 of *MPS/SIAM Series on Optimization*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA; Mathematical Programming Society (MPS), Philadelphia, PA, 2006. Applications to PDEs and optimization.

[13] G. Auchmuty and D. Bao. Harnack-type inequalities for evolution equations. *Proc. Amer. Math. Soc.*, 122(1):117–129, 1994. doi:10.2307/2160850.

[14] S. Axler, P. Bourdon, and W. Ramey. *Harmonic function theory*, volume 137 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1992. doi:10.1007/b97238.

[15] D. Azagra, J. Ferrera, and B. Sanz. Viscosity solutions to second order partial differential equations on Riemannian manifolds. *J. Differential Equations*, 245(2):307–336, 2008. doi:10.1016/j.jde.2008.03.030.

[16] S. Azami. Harnack inequality for porous medium equations with a blow-up term along the Finsler-geometric flow. *Filomat*, 39(1):155–171, 2025.

[17] M. Bailesteanu, X. Cao, and A. Pulemotov. Gradient estimates for the heat equation under the Ricci flow. *J. Funct. Anal.*, 258(10):3517–3542, 2010. doi:10.1016/j.jfa.2009.12.003.

[18] D. Bakry, F. c. Bolley, and I. Gentil. The Li-Yau inequality and applications under a curvature-dimension condition. *Ann. Inst. Fourier (Grenoble)*, 67(1):397–421, 2017. doi:10.5802/aif.3086.

[19] D. Bakry, I. Gentil, and M. Ledoux. On Harnack inequalities and optimal transportation. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 14(3):705–727, 2015.

[20] D. Bakry and Z. M. Qian. Harnack inequalities on a manifold with positive or negative Ricci curvature. *Rev. Mat. Iberoamericana*, 15(1):143–179, 1999. doi:10.4171/RMI/253.

[21] G. I. Barenblatt. On self-similar motions of a compressible fluid in a porous medium. *Akad. Nauk SSSR. Prikl. Mat. Meh.*, 16:679–698, 1952.

[22] D. Bianchi and A. G. Setti. Laplacian cut-offs, porous and fast diffusion on manifolds and other applications. *Calc. Var. Partial Differential Equations*, 57(1):Paper No. 4, 33, 2018. doi:10.1007/s00526-017-1267-9.

[23] V. Bögelein, F. Duzaar, U. Gianazza, N. Liao, and C. Scheven. Hölder continuity of the gradient of solutions to doubly non-linear parabolic equations, 2023. URL: https://arxiv.org/abs/2305.08539, arXiv:2305.08539.

[24] V. Bögelein and M. Strunk. A comparison principle for doubly nonlinear parabolic partial differential equations. *Ann. Mat. Pura Appl. (4)*, 203(2):779–804, 2024. doi:10.1007/s10231-023-01381-4.

[25] E. Bombieri and E. Giusti. Harnack's inequality for elliptic differential equations on minimal surfaces. *Invent. Math.*, 15:24–46, 1972. doi:10.1007/BF01418640.

[26] M. Bonforte, G. Grillo, and J. L. Vazquez. Fast diffusion flow on manifolds of nonpositive curvature. *J. Evol. Equ.*, 8(1):99–128, 2008. doi:10.1007/s00028-007-0345-4.

[27] H. Brezis. Propriétés régularisantes de certains semi-groupes non linéaires. *Israel J. Math.*, 9:513–534, 1971. doi:10.1007/BF02771467.

[28] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations.* Universitext. Springer, New York, 2011.

[29] L. A. Caffarelli. Interior a priori estimates for solutions of fully non-linear equations. *Ann. of Math. (2)*, 130(1):189–213, 1989. doi:10.2307/1971480.

[30] H.-D. Cao and L. Ni. Matrix Li-Yau-Hamilton estimates for the heat equation on Kähler manifolds. *Math. Ann.*, 331(4):795–807, 2005. doi:10.1007/s00208-004-0605-3.

[31] H.-D. Cao and M. Zhu. Aronson-Bénilan estimates for the porous medium equation under the Ricci flow. *J. Math. Pures Appl. (9)*, 104(4):729–748, 2015. doi:10.1016/j.matpur.2015.05.001.

[32] X. Cao and R. S. Hamilton. Differential Harnack estimates for time-dependent heat equations with potentials. *Geom. Funct. Anal.*, 19(4):989–1000, 2009. doi:10.1007/s00039-009-0024-4.

[33] R. Carmona and B. Simon. Pointwise bounds on eigenfunctions and wave packets in N-body quantum systems. V. Lower bounds and path integrals. *Comm. Math. Phys.*, 80(1):59–98, 1981. URL: http://projecteuclid.org/euclid.cmp/1103919806.

[34] D. Chen and C. Xiong. Gradient estimates for doubly nonlinear diffusion equations. *Nonlinear Anal.*, 112:156–164, 2015. doi:10.1016/j.na.2014.08.017.

[35] Y. Z. Chen and E. DiBenedetto. On the local behavior of solutions of singular parabolic equations. *Arch. Rational Mech. Anal.*, 103(4):319–345, 1988. doi:10.1007/BF00251444.

[36] S. Y. Cheng and S. T. Yau. Differential equations on Riemannian manifolds and their geometric applications. *Comm. Pure Appl. Math.*, 28(3):333–354, 1975. doi:10.1002/cpa.3160280303.

[37] B. Chow. On Harnack's inequality and entropy for the Gaussian curvature flow. *Comm. Pure Appl. Math.*, 44(4):469–483, 1991. doi:10.1002/cpa.3160440405.

[38] B. Chow. The Yamabe flow on locally conformally flat manifolds with positive Ricci curvature. *Comm. Pure Appl. Math.*, 45(8):1003–1014, 1992. doi:10.1002/cpa.3160450805.

[39] B. Chow. On an alternate proof of Hamilton's matrix Harnack inequality of Li-Yau type for the Ricci flow. *Indiana Univ. Math. J.*, 52(4):863–873, 2003. doi:10.1512/iumj.2003.52.2318.

[40] B. Chow and R. S. Hamilton. Constrained and linear Harnack inequalities for parabolic equations. *Invent. Math.*, 129(2):213–238, 1997. doi:10.1007/s002220050162.

[41] M. G. Crandall and H. Ishii. The maximum principle for semicontinuous functions. *Differential Integral Equations*, 3(6):1001–1014, 1990.

[42] M. G. Crandall, H. Ishii, and P.-L. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc. (N.S.)*, 27(1):1–67, 1992. doi:10.1090/S0273-0979-1992-00266-5.

[43] M. G. Crandall and P.-L. Lions. Viscosity solutions of Hamilton-Jacobi equations. *Trans. Amer. Math. Soc.*, 277(1):1–42, 1983. doi:10.2307/1999343.

[44] M. G. Crandall and M. Pierre. Regularizing effects for $u_t = \Delta\varphi(u)$. *Trans. Amer. Math. Soc.*, 274(1):159–168, 1982. doi:10.2307/1999502.

[45] E. B. Davies. *Heat kernels and spectral theory*, volume 92 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1989. doi:10.1017/CBO9780511566158.

[46] E. De Giorgi. Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari. *Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3)*, 3:25–43, 1957.

[47] T. Delmotte. Parabolic Harnack inequality and estimates of Markov chains on graphs. *Rev. Mat. Iberoamericana*, 15(1):181–232, 1999. doi:10.4171/RMI/254.

[48] E. DiBenedetto. On the local behaviour of solutions of degenerate parabolic equations with measurable coefficients. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 13(3):487–535, 1986. URL: http://www.numdam.org/item?id=ASNSP_1986_4_13_3_487_0.

[49] E. DiBenedetto. Intrinsic Harnack type inequalities for solutions of certain degenerate parabolic equations. *Arch. Rational Mech. Anal.*, 100(2):129–147, 1988. doi:10.1007/BF00282201.

[50] E. DiBenedetto. *Degenerate parabolic equations*. Universitext. Springer-Verlag, New York, 1993. doi:10.1007/978-1-4612-0895-2.

[51] E. DiBenedetto and A. Friedman. Addendum to: "Hölder estimates for nonlinear degenerate parabolic systems". *J. Reine Angew. Math.*, 363:217–220, 1985. doi:10.1515/crll.1985.363.217.

[52] E. DiBenedetto and A. Friedman. Hölder estimates for nonlinear degenerate parabolic systems. *J. Reine Angew. Math.*, 357:1–22, 1985. doi:10.1515/crll.1985.357.1.

[53] E. DiBenedetto, U. Gianazza, and V. Vespri. Harnack estimates for quasi-linear degenerate parabolic differential equations. *Acta Math.*, 200(2):181–209, 2008. doi:10.1007/s11511-008-0026-3.

[54] E. DiBenedetto, U. Gianazza, and V. Vespri. Alternative forms of the Harnack inequality for non-negative solutions to certain degenerate and singular parabolic equations. *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.*, 20(4):369–377, 2009. doi:10.4171/RLM/552.

[55] E. DiBenedetto, U. Gianazza, and V. Vespri. Forward, backward and elliptic Harnack inequalities for non-negative solutions to certain singular parabolic partial differential equations. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 9(2):385–422, 2010.

[56] E. DiBenedetto, U. Gianazza, and V. Vespri. Harnack type estimates and Hölder continuity for non-negative solutions to certain sub-critically singular parabolic partial differential equations. *Manuscripta Math.*, 131(1-2):231–245, 2010. doi:10.1007/s00229-009-0317-9.

[57] E. DiBenedetto, U. Gianazza, and V. Vespri. *Harnack's inequality for degenerate and singular parabolic equations*. Springer Monographs in Mathematics. Springer, New York, 2012. doi:10.1007/978-1-4614-1584-8.

[58] E. DiBenedetto and Y. C. Kwong. Harnack estimates and extinction profile for weak solutions of certain singular parabolic equations. *Trans. Amer. Math. Soc.*, 330(2):783–811, 1992. doi:10.2307/2153935.

[59] D. Dier, M. Kassmann, and R. Zacher. Discrete versions of the Li-Yau gradient estimate. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 22(2):691–744, 2021.

[60] J. R. Esteban and J. L. Vázquez. Homogeneous diffusion in $\mathbf{R}$ with power-like nonlinear diffusivity. *Arch. Rational Mech. Anal.*, 103(1):39–80, 1988. doi:10.1007/BF00292920.

[61] J. R. Esteban and J. L. Vázquez. Régularité des solutions positives de l'équation parabolique p-laplacienne. *C. R. Acad. Sci. Paris Sér. I Math.*, 310(3):105–110, 1990.

[62] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010. doi:10.1090/gsm/019.

[63] A. Friedman. *Partial differential equations of parabolic type*. Prentice-Hall, Inc., Englewood Cliffs, NJ, 1964.

[64] P. Garain and J. Kinnunen. Weak Harnack inequality for a mixed local and nonlocal parabolic equation. *J. Differential Equations*, 360:373–406, 2023. doi:10.1016/j.jde.2023.02.049.

[65] U. Gianazza and V. Vespri. A Harnack inequality for solutions of doubly nonlinear parabolic equations. *J. Appl. Funct. Anal.*, 1(3):271–284, 2006.

[66] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[67] A. Grigor'yan. *Heat kernel and analysis on manifolds*, volume 47 of *AMS/IP Studies in Advanced Mathematics*. American Mathematical Society, Providence, RI; International Press, Boston, MA, 2009. doi:10.1090/amsip/047.

[68] A. Grigor'yan and A. Telcs. Harnack inequalities and sub-Gaussian estimates for random walks. *Math. Ann.*, 324(3):521–556, 2002. doi:10.1007/s00208-002-0351-3.

[69] A. A. Grigor'yan. The heat equation on noncompact Riemannian manifolds. *Mat. Sb.*, 182(1):55–87, 1991.

[70] G. Grillo, D. D. Monticelli, and F. Punzo. The porous medium equation on noncompact manifolds with nonnegative Ricci curvature: a Green function approach. *J. Differential Equations*, 430:Paper No. 113191, 42, 2025. doi:10.1016/j.jde.2025.02.062.

[71] G. Grillo, M. Muratori, and F. Punzo. The porous medium equation with measure data on negatively curved Riemannian manifolds. *J. Eur. Math. Soc. (JEMS)*, 20(11):2769–2812, 2018. doi:10.4171/JEMS/824.

[72] G. Grillo, M. Muratori, and F. Punzo. Fast diffusion on noncompact manifolds: well-posedness theory and connections with semilinear elliptic equations. *Trans. Amer. Math. Soc.*, 374(9):6367–6396, 2021. doi:10.1090/tran/8431.

[73] G. Gripenberg. Harnack's inequality for viscosity solutions to degenerate elliptic equations in two variables. *Proc. Amer. Math. Soc.*, 137(11):3703–3708, 2009. doi:10.1090/S0002-9939-09-10030-8.

[74] J. Hadamard. Extension à l'équation de la chaleur d'un théorème de A. Harnack. *Rend. Circ. Mat. Palermo (2)*, 3:337–346, 1954. doi:10.1007/BF02849264.

[75] R. S. Hamilton. The Harnack estimate for the Ricci flow. *J. Differential Geom.*, 37(1):225–243, 1993. URL: http://projecteuclid.org/euclid.jdg/1214453430.

[76] R. S. Hamilton. A matrix Harnack estimate for the heat equation. *Comm. Anal. Geom.*, 1(1):113–126, 1993. doi:10.4310/CAG.1993.v1.n1.a6.

[77] R. S. Hamilton. Harnack estimate for the mean curvature flow. *J. Differential Geom.*, 41(1):215–226, 1995. URL: http://projecteuclid.org/euclid.jdg/1214456010.

[78] C.-G. A. Harnack. *Die Grundlagen der Theorie des logaritmischen Potentiales und der eindeutigen Potentialfunktion in der Ebene*. Teubner, Leipzig, 1887.

[79] S.-Y. Hsu. Gradient estimates for a nonlinear parabolic equation under Ricci flow. *Differential Integral Equations*, 24(7-8):645–652, 2011.

[80] G. Huang, Z. Huang, and H. Li. Gradient estimates for the porous medium equations on Riemannian manifolds. *J. Geom. Anal.*, 23(4):1851–1875, 2013. doi:10.1007/s12220-012-9310-8.

[81] T. Ilmanen. Generalized flow of sets by mean curvature on a manifold. *Indiana Univ. Math. J.*, 41(3):671–705, 1992. doi:10.1512/iumj.1992.41.41036.

[82] M. Ishida. Geometric flows and differential Harnack estimates for heat equations with potentials. *Ann. Global Anal. Geom.*, 45(4):287–302, 2014. doi:10.1007/s10455-013-9401-1.

[83] K. Ishige. On the existence of solutions of the Cauchy problem for a doubly nonlinear parabolic equation. *SIAM J. Math. Anal.*, 27(5):1235–1260, 1996. doi:10.1137/S0036141094270370.

[84] A. V. Ivanov. Regularity for doubly nonlinear parabolic equations. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 209:37–59, 261, 1994. doi:10.1007/BF02398459.

[85] R. Jensen. The maximum principle for viscosity solutions of fully nonlinear second order partial differential equations. *Arch. Rational Mech. Anal.*, 101(1):1–27, 1988. doi:10.1007/BF00281780.

[86] P. Juutinen. On the definition of viscosity solutions for parabolic equations. *Proc. Amer. Math. Soc.*, 129(10):2907–2911, 2001. doi:10.1090/S0002-9939-01-05889-0.

[87] P. Juutinen, P. Lindqvist, and J. J. Manfredi. On the equivalence of viscosity solutions and weak solutions for a quasi-linear equation. *SIAM J. Math. Anal.*, 33(3):699–717, 2001. doi:10.1137/S0036141000372179.

[88] M. Kassmann. Harnack inequalities: an introduction. *Bound. Value Probl.*, pages Art. ID 81415, 21, 2007. doi:10.1155/2007/81415.

[89] M. Kassmann and M. Weidner. The Harnack inequality fails for nonlocal kinetic equations. *Adv. Math.*, 459:Paper No. 110030, 14, 2024. doi:10.1016/j.aim.2024.110030.

[90] M. Kassmann and M. Weidner. The parabolic Harnack inequality for nonlocal equations. *Duke Math. J.*, 173(17):3413–3451, 2024. doi:10.1215/00127094-2024-0008.

[91] N. Katzourakis. *An introduction to viscosity solutions for fully nonlinear PDE with applications to calculus of variations in L^∞*. SpringerBriefs in Mathematics. Springer, Cham, 2015. doi:10.1007/978-3-319-12829-0.

[92] S. Kim and K.-A. Lee. Parabolic Harnack inequality of viscosity solutions on Riemannian manifolds. *J. Funct. Anal.*, 267(7):2152–2198, 2014. doi:10.1016/j.jfa.2014.07.023.

[93] J. Kinnunen and T. Kuusi. Local behaviour of solutions to doubly nonlinear parabolic equations. *Math. Ann.*, 337(3):705–728, 2007. doi:10.1007/s00208-006-0053-3.

[94] B. Kotschwar and L. Ni. Local gradient estimates of p-harmonic functions, $1/H$-flow, and an entropy formula. *Ann. Sci. Éc. Norm. Supér. (4)*, 42(1):1–36, 2009. doi:10.24033/asens.2089.

[95] N. V. Krylov and M. V. Safonov. A certain property of solutions of parabolic equations with measurable coefficients. *Mathematics of the USSR. Izvestiya*, 16(1):151–164, 1981. doi:10.1070/IM1981v016n01ABEH001283.

[96] T. Kurkinen and J. Siltakoski. Intrinsic Harnack's inequality for a general nonlinear parabolic equation in non-divergence form. *Potential Anal.*, 62(2):393–437, 2025. doi:10.1007/s11118-024-10141-9.

[97] T. Kuusi. Moser's method for a nonlinear parabolic equation. WorkingPaper A477, TKK, 2004.

[98] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural'ceva. *Linear and quasilinear equations of parabolic type*, volume Vol. 23 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1968. Translated from the Russian by S. Smith.

[99] O. A. Ladyzhenskaya and N. N. Ural'tseva. *Linear and quasilinear elliptic equations*. Academic Press, New York-London, 1968. Translated from the Russian by Scripta Technica, Inc, Translation editor: Leon Ehrenpreis.

[100] J. M. Lee. *Introduction to smooth manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2013.

[101] L. S. Leibenson. General problem of the movement of a compressible fluid in a porous medium. *Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR]*, 9:7–10, 1945.

[102] J. Li. Gradient estimates and Harnack inequalities for nonlinear parabolic and nonlinear elliptic equations on Riemannian manifolds. *J. Funct. Anal.*, 100(2):233–256, 1991. doi:10.1016/0022-1236(91)90110-Q.

[103] J. Li and X. Xu. Differential Harnack inequalities on Riemannian manifolds I: linear heat equation. *Adv. Math.*, 226(5):4456–4491, 2011. doi:10.1016/j.aim.2010.12.009.

[104] P. Li and S.-T. Yau. On the parabolic kernel of the Schrödinger operator. *Acta Math.*, 156(3-4):153–201, 1986. doi:10.1007/BF02399203.

[105] X. Li. Moduli of continuity for viscosity solutions. *Proc. Amer. Math. Soc.*, 144(4):1717–1724, 2016. doi:10.1090/proc/12850.

[106] X. Li. Modulus of continuity estimates for fully nonlinear parabolic equations. *Calc. Var. Partial Differential Equations*, 60(5):Paper No. 182, 23, 2021. doi:10.1007/s00526-021-02056-9.

[107] X. Li, H.-Y. Liu, and X.-A. Ren. Matrix Li-Yau-Hamilton estimates under Kähler-Ricci flow. *J. Geom. Anal.*, 35(4):Paper No. 113, 19, 2025. doi:10.1007/s12220-025-01948-y.

[108] X. Li and K. Wang. Moduli of continuity for viscosity solutions on manifolds. *J. Geom. Anal.*, 27(1):557–576, 2017. doi:10.1007/s12220-016-9690-2.

[109] X. Li and Q. S. Zhang. Matrix Li-Yau-Hamilton estimates under Ricci flow and parabolic frequency. *Calc. Var. Partial Differential Equations*, 63(3):Paper No. 63, 38, 2024. doi:10.1007/s00526-024-02668-x.

[110] Y. Li and X. Zhu. Li-Yau Harnack estimates for a heat-type equation under the geometric flow. *Potential Anal.*, 52(3):469–496, 2020. doi:10.1007/s11118-018-9739-x.

[111] P. Lu, L. Ni, J.-L. Vázquez, and C. Villani. Local Aronson-Bénilan estimates and entropy formulae for porous medium and fast diffusion equations on manifolds. *J. Math. Pures Appl. (9)*, 91(1):1–19, 2009. doi:10.1016/j.matpur.2008.09.001.

[112] B. Ma and J. Li. Gradient estimates of porous medium equations under the Ricci flow. *J. Geom.*, 105(2):313–325, 2014. doi:10.1007/s00022-013-0209-8.

[113] L. Ma, L. Zhao, and X. Song. Gradient estimate for the degenerate parabolic equation $u_t = \Delta F(u) + H(u)$ on manifolds. *J. Differential Equations*, 244(5):1157–1177, 2008. doi:10.1016/j.jde.2007.08.014.

[114] G. Meglioli, F. Oliva, and F. Petitta. Global existence for a Leibenson type equation with reaction on Riemannian manifolds. *Nonlinear Anal.*, 263:Paper No. 113967, 25, 2026. doi:10.1016/j.na.2025.113967.

[115] O. Montaldo. Sul primo problema di valori al contorno per l'equazione del calore. *Rend. Sem. Fac. Sci. Univ. Cagliari*, 25:1–14, 1955.

[116] J. Moser. On Harnack's theorem for elliptic differential equations. *Comm. Pure Appl. Math.*, 14:577–591, 1961. doi:10.1002/cpa.3160140329.

[117] J. Moser. A Harnack inequality for parabolic differential equations. *Comm. Pure Appl. Math.*, 17:101–134, 1964. doi:10.1002/cpa.3160170106.

[118] J. Moser. Correction to: "A Harnack inequality for parabolic differential equations". *Comm. Pure Appl. Math.*, 20:231–236, 1967. doi:10.1002/cpa.3160200107.

[119] J. Moser. On a pointwise estimate for parabolic differential equations. *Comm. Pure Appl. Math.*, 24:727–740, 1971. doi:10.1002/cpa.3160240507.

[120] J. Nash. The imbedding problem for Riemannian manifolds. *Ann. of Math. (2)*, 63:20–63, 1956. doi:10.2307/1969989.

[121] J. Nash. Continuity of solutions of parabolic and elliptic equations. *Amer. J. Math.*, 80:931–954, 1958. doi:10.2307/2372841.

[122] L. Negro. Harnack inequality for Ornstein-Uhlenbeck type operators. *Arch. Math. (Basel)*, 114(3):331–341, 2020. doi:10.1007/s00013-019-01429-z.

[123] L. Ni. Monotonicity and Li-Yau-Hamilton inequalities. In *Surveys in differential geometry. Vol. XII. Geometric flows*, volume 12 of *Surv. Differ. Geom.*, pages 251–301. Int. Press, Somerville, MA, 2008. doi:10.4310/SDG.2007.v12.n1.a7.

[124] E. M. Ouhabaz. *Analysis of heat equations on domains*, volume 31 of *London Mathematical Society Monographs Series*. Princeton University Press, Princeton, NJ, 2005.

[125] M. Parviainen and J. L. Vázquez. Equivalence between radial solutions of different parabolic gradient-diffusion equations and applications. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 21:303–359, 2020.

[126] B. Pini. Sulla soluzione generalizzata di Wiener per il primo problema di valori al contorno nel caso parabolico. *Rend. Sem. Mat. Univ. Padova*, 23:422–434, 1954. URL: http://www.numdam.org/item?id=RSMUP_1954__23__422_0.

[127] H. Qiu. Gradient estimates of a general porous medium equation for the V-Laplacian. *Kodai Math. J.*, 43(1):16–41, 2020. doi:10.2996/kmj/1584345686.

[128] A. Raich and M. Tinker. Schrödinger operators with A_∞ potentials. *Potential Anal.*, 45(2):387–402, 2016. doi:10.1007/s11118-016-9556-z.

[129] P. A. Raviart. Sur la résolution de certaines équations paraboliques non linéaires. *J. Functional Analysis*, 5:299–328, 1970. doi:10.1016/0022-1236(70)90031-5.

[130] Q.-h. Ruan. Gradient estimate for Schrödinger operators on manifolds. *J. Geom. Phys.*, 58(8):962–966, 2008. doi:10.1016/j.geomphys.2008.03.001.

[131] M. V. Safonov. Harnack's inequality for elliptic equations and Hölder property of their solutions. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 96:272–287, 312, 1980.

[132] L. Saloff-Coste. A note on Poincaré, Sobolev, and Harnack inequalities. *Internat. Math. Res. Notices*, 2:27–38, 1992. doi:10.1155/S1073792892000047.

[133] L. Saloff-Coste. Parabolic Harnack inequality for divergence-form second-order differential operators. *Potential Anal.*, 4(4):429–467, 1995. Potential theory and degenerate partial differential operators (Parma). doi:10.1007/BF01053457.

[134] J. Serrin. Local behavior of solutions of quasi-linear equations. *Acta Math.*, 111:247–302, 1964. doi:10.1007/BF02391014.

[135] B. Shang and C. Zhang. Harnack inequality for mixed local and nonlocal parabolic p-Laplace equations. *J. Geom. Anal.*, 33(4):Paper No. 124, 24, 2023. doi:10.1007/s12220-022-01173-x.

[136] R. E. Showalter. *Monotone operators in Banach space and nonlinear partial differential equations*, volume 49 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997. doi:10.1090/surv/049.

[137] J. Siltakoski. Equivalence of viscosity and weak solutions for a p-parabolic equation. *J. Evol. Equ.*, 21(2):2047–2080, 2021. doi:10.1007/s00028-020-00666-y.

[138] B. Simon. Instantons, double wells and large deviations. *Bull. Amer. Math. Soc. (N.S.)*, 8(2):323–326, 1983. doi:10.1090/S0273-0979-1983-15104-2.

[139] B. Simon. Semiclassical analysis of low lying eigenvalues. II. Tunneling. *Ann. of Math. (2)*, 120(1):89–118, 1984. doi:10.2307/2007072.

[140] M. Strömqvist. Harnack's inequality for parabolic nonlocal equations. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 36(6):1709–1745, 2019. doi:10.1016/j.anihpc.2019.03.003.

[141] M. Strunk. Gradient regularity for a class of doubly nonlinear parabolic partial differential equations. *NoDEA Nonlinear Differential Equations Appl.*, 32(4):Paper No. 66, 65, 2025. doi:10.1007/s00030-025-01076-0.

[142] S. Sturm. Existence of weak solutions of doubly nonlinear parabolic equations. *J. Math. Anal. Appl.*, 455(1):842–863, 2017. doi:10.1016/j.jmaa.2017.06.024.

[143] J. Sun. Gradient estimates for positive solutions of the heat equation under geometric flow. *Pacific J. Math.*, 253(2):489–510, 2011. doi:10.2140/pjm.2011.253.489.

[144] P. Sürig. Existence results for Leibenson's equation on Riemannian manifolds, 2026. URL: https://arxiv.org/abs/2601.20640, arXiv:2601.20640.

[145] N. S. Trudinger. On Harnack type inequalities and their application to quasilinear elliptic equations. *Comm. Pure Appl. Math.*, 20:721–747, 1967. doi:10.1002/cpa.3160200406.

[146] N. S. Trudinger. Pointwise estimates and quasilinear parabolic equations. *Comm. Pure Appl. Math.*, 21:205–226, 1968. doi:10.1002/cpa.3160210302.

[147] J. M. Urbano. *The method of intrinsic scaling*, volume 1930 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2008. A systematic approach to regularity for degenerate and singular PDEs. doi:10.1007/978-3-540-75932-4.

[148] J. L. Vázquez. *The porous medium equation*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2007. Mathematical theory.

[149] V. Vespri. Harnack type inequalities for solutions of certain doubly nonlinear parabolic equations. *J. Math. Anal. Appl.*, 181(1):104–131, 1994. doi:10.1006/jmaa.1994.1008.

[150] F.-Y. Wang. *Harnack inequalities for stochastic partial differential equations*. SpringerBriefs in Mathematics. Springer, New York, 2013. doi:10.1007/978-1-4614-7934-5.

[151] W. Wang. Harnack differential inequalities for the parabolic equation $u_t = \mathcal{L}F(u)$ on Riemannian manifolds and applications. *Acta Math. Sin. (Engl. Ser.)*, 33(5):620–634, 2017. doi:10.1007/s10114-016-6260-2.

[152] W. Wang, D. Xie, and H. Zhou. Local Aronson-Bénilan gradient estimates and Harnack inequality for the porous medium equation along Ricci flow. *Commun. Pure Appl. Anal.*, 17(5):1957–1974, 2018. doi:10.3934/cpaa.2018093.

[153] Y. Wang and W. Chen. Gradient estimates and entropy monotonicity formula for doubly nonlinear diffusion equations on Riemannian manifolds. *Math. Methods Appl. Sci.*, 37(17):2772–2781, 2014. doi:10.1002/mma.3016.

[154] Y.-Z. Wang and X. Wang. Gradient estimates for nonlinear reaction-diffusion equations on Riemannian manifolds. *Results Math.*, 77(1):Paper No. 34, 13, 2022. doi:10.1007/s00025-021-01518-y.

[155] F. Weber and R. Zacher. Li-Yau inequalities for general non-local diffusion equations via reduction to the heat kernel. *Math. Ann.*, 385(1-2):393–419, 2023. doi:10.1007/s00208-021-02350-z.

[156] X. Xu. Gradient estimates for $u_t = \Delta F(u)$ on manifolds and some Liouville-type theorems. *J. Differential Equations*, 252(2):1403–1420, 2012. doi:10.1016/j.jde.2011.08.004.

[157] S.-T. Yau. Harmonic functions on complete Riemannian manifolds. *Comm. Pure Appl. Math.*, 28:201–228, 1975. doi:10.1002/cpa.3160280203.

[158] S.-T. Yau. On the Harnack inequalities of partial differential equations. *Comm. Anal. Geom.*, 2(3):431–450, 1994. doi:10.4310/CAG.1994.v2.n3.a3.

[159] S.-T. Yau. Harnack inequality for non-self-adjoint evolution equations. *Math. Res. Lett.*, 2(4):387–399, 1995. doi:10.4310/MRL.1995.v2.n4.a2.

[160] G. Zhao. Gradient estimates and Harnack inequalities of a parabolic equation under geometric flow. *J. Math. Anal. Appl.*, 483(2):123631, 24, 2020. doi:10.1016/j.jmaa.2019.123631.