Graded Representation Theory of Quiver Hecke Algebras

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A thesis submitted to fulfil the requirements
for the degree of Doctor of Philosophy

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2026

Statement of Originality

This is to certify that the content of this thesis is my own work. This thesis has not been submitted for any other degree or purpose.

I certify that the intellectual content of this thesis is the product of my own work, and that all assistance received in preparing this thesis and all sources have been acknowledged.

Date: 2 March 2026

AI Statement

During the preparation of this thesis the author used **ChatGPT** to assist with (i) language polishing (e.g. grammar and typos), and (ii) drafting and debugging python code (e.g. suggesting implementations, identifying likely bugs).

The author confirms that no mathematical proof or theorem in this thesis relies on generative AI output, and all results and conclusions are the author's responsibility.

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CHAPTER 0

Introduction

Another possible title for this thesis is *Combinatorial Representation Theory of Quiver Hecke Algebras*. The central aim is to develop combinatorial frameworks motivated by representation-theoretic problems for Hecke algebras, and to understand the structures that these frameworks reveal.

This thesis is primarily about the representation theory of *(cyclotomic) quiver Hecke algebras*, also called *(cyclotomic) KLR algebras*; see Subsection 1.3.4. These algebras were introduced independently by Khovanov–Lauda [**KL09**] and Rouquier [**Rou08**] in order to categorify the negative half of the corresponding quantum group. A key link to classical objects is provided by Brundan–Kleshchev [**BK09a**], who constructed an isomorphism between cyclotomic KLR algebras and cyclotomic Hecke algebras (Ariki–Koike algebras) of type $A^{(1)}_{e-1}$. In particular, this provides cyclotomic Hecke algebras—including the level-one Iwahori–Hecke algebras—with a nontrivial grading. This makes it possible to consider graded representation-theoretic invariants.

Among the most important are the *graded decomposition numbers*, which record the multiplicity of D^μ in a graded composition series of S^λ; see Subsection 1.3.7. Computing these numbers is notoriously difficult; the failure of James' conjecture already illustrates how subtle decomposition behaviour can be, see [**Wil16**, **Spe26**]. Despite this, graded decomposition numbers satisfy striking structural constraints, and much of this thesis is concerned with providing new ones.

Cyclotomic quiver Hecke algebras are naturally organised by *level*. In level 1 (the Iwahori–Hecke algebra), graded decomposition numbers depend on the parameter $e \in \mathbb{Z}_{>1}$, the *quantum characteristic*. In higher level they also depend on a *charge* $\boldsymbol{\kappa}$, which determines the dominant weight; see Subsection 1.1.2 and Subsection 1.3.4.

Assume for the moment that the base field has characteristic 0. In this setting there are several *runner-removal theorems*—Theorem 3.3.1, Theorem 3.3.2, and Theorem 4.4.8—which can be phrased in terms of abacus combinatorics. They produce explicit maps on partitions $\lambda \mapsto \lambda^+$ with the property that the graded decomposition numbers $d^e_{\lambda,\mu}$ at quantum characteristic e coincide with $d^{e+1}_{\lambda^+,\mu^+}$ at quantum characteristic $e+1$. This phenomenon is closely related to *Jantzen's generic decomposition pattern* [**Lus80**]. While runner removal gives a concrete recipe for λ^+, it is less clear why such equalities should hold, or what mechanism lies behind them. One way to seek an explanation is via Kazhdan–Lusztig theory.

For any Coxeter group W, one can define the associated Hecke algebra together with a bar involution on this algebra, and hence the Kazhdan–Lusztig basis, characterised by bar-invariance and triangularity. The *Kazhdan–Lusztig polynomials* are the coefficients that arise when expressing the Kazhdan–Lusztig basis in terms of the standard basis; see Subsection 1.3.1. For a parabolic subgroup P, there are *parabolic Kazhdan–Lusztig polynomials* indexed by minimal-length representatives in W/P; see Subsection 1.3.2. In the affine Weyl group case, Goodman–Wenzl [**GW98**, Theorem 5.3] showed that these parabolic Kazhdan–Lusztig polynomials coincide with graded decomposition numbers; see Subsection 1.4.4 and Theorem 3.3.4.

Affine Weyl groups also control an arrangement of hyperplanes in $\mathbb{R}^n$; the connected components of the complement are the *alcoves*; see Section 1.4. There is a bijection between e-alcoves and the minimal-length representatives of W/W_0, where W_0 is the associated finite Weyl group. From this viewpoint, changing e to $e+1$ corresponds geometrically to dilating the relevant alcove, and the independence of parabolic Kazhdan–Lusztig polynomials from e becomes transparent. In Section 3.3 we make this explicit for the empty-runner-removal theorem Theorem 3.3.1. By contrast, the other runner-removal theorems Theorem 3.3.2 and Theorem 4.4.8 do not admit such clean descriptions in alcove combinatorics.

The alcove approach depends on a map Ω sending a partition with at most r parts to a dominant weight of $\mathfrak{sl}_r$; see Section 3.1. The set of partitions can be used to describe the blocks of Iwahori–Hecke algebras (and q-Schur algebras) in terms of e-cores and e-weights; see Subsection 1.2.4. The e-weight is a nonnegative integer measuring the complexity of the block. This leads to a natural question: if we fix an e-weight w and consider the set $\mathscr{P}_{r,e,w}$ of partitions with at most r parts and e-weight w, what is the image of $\mathscr{P}_{r,e,w}$ under Ω? Chapter 3 is devoted to this problem. See the beginning of Chapter 3 for a summary of the main results.

A recurring theme in modern representation theory is to lift algebraic structures to categorical ones, a process known as *categorification*. For example, KLR algebras categorify quantum groups [**KL09**, **KL10**, **Rou08**], and cyclotomic KLR algebras categorify irreducible highest-weight modules [**KK12**]. In the present context, one would like a categorical refinement of runner removal: a functor F between module categories (from R^{Λ}_{α}-modules to $R^{\Lambda'}_{\alpha'}$-modules) sending $S^{\lambda} \mapsto S^{\lambda^+}$ and $D^{\mu} \mapsto D^{\mu^+}$. If F is exact, then the runner-removal equalities follow formally.

Such a categorical approach was developed in [**CM10**] for Iwahori–Hecke algebras and q-Schur algebras, and has the advantage of not requiring characteristic 0. Our motivation in Chapter 4 is to obtain a categorical lift that applies to general cyclotomic KLR algebras. Maksimau [**Mak18**] constructed an isomorphism between KLR algebras of types $A^{(1)}_{e-1}$ and $A^{(1)}_{e}$; Mathas and Tubbenhauer [**MT23**] refer to this isomorphism as *subdivision* and extend it to the KLRW setting. Mathas further conjectured that subdivision should categorify runner removal theorems. In Chapter 4 we partially confirm this conjecture by showing that the combinatorics defining λ^+ agree with those arising from subdivision; see Section 4.4. We also show that subdivision relates Specht modules in a natural way; see Theorem 4.3.19. The exactness needed for a full categorification, however, is not addressed there and is left for future work.

Finally, our analysis of Specht modules relies crucially on the highest-weight presentation of Kleshchev–Mathas–Ram [**KMR12**]. In that presentation, the Specht module S^λ is realised as a quotient of a permutation module M^λ by Garnir relations; see Subsection 1.3.8. Although these permutation modules differ substantially from the classical constructions for symmetric groups or Iwahori–Hecke algebras (cf. [**Mat99**, Page 30]), it is still natural to ask whether the classical theorem persists—namely, whether every permutation module admits a Specht filtration; see [**Mat99**, Corollary 4.10]. In Chapter 2 we study this question in several cases and obtain a negative answer; this leads instead to the notion of a *generalized Specht filtration*, see Theorem 2.3.2.

The thesis is organized as follows. Chapter 1 provides the necessary background and introduces the definitions used throughout. In Chapter 2, we construct explicit (generalized) Specht filtrations of permutation modules. In Chapter 3, we study the pattern of dominant weights arising from certain families of partitions. Finally, in Chapter 4, we introduce the subdivision map in its algebraic, combinatorial, and categorical forms, and explain its connection with runner-removal theorems.

CHAPTER 1

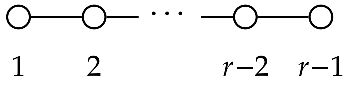
Preliminaries

This chapter collects notation and results from Lie theory, partition and alcove combinatorics, and Hecke algebras. Our goal is not to present the full theory; rather, we define only what we require and record key results without proof (with references to the literature for details). Throughout, we include examples to clarify the notation.

1.1. Lie theory

1.1.1. Finite type A_{r-1}. A standard reference for Lie algebras of finite type is [**Hum72**], and one can consult [**Car05**, Section 8.1] for type A_{r-1}.

Fix $r \geq 2$ an integer. Consider the finite type A_{r-1} quiver:



and set

$$E = \left\{ (x_1, \ldots, x_r) \in \mathbb{R}^r \;\middle|\; \sum_{i=1}^{r} x_i = 0 \right\},$$

equipped with the restriction of the standard inner product $(\cdot, \cdot)$ on $\mathbb{R}^r$. Let ε_i be the ith standard basis vector of $\mathbb{R}^r$.

The *root system* of type A_{r-1} is

$$R = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i \neq j \leq r\} \subset E.$$

We take the set of *simple roots* to be

$$\Delta = \{\alpha_1, \ldots, \alpha_{r-1}\}, \qquad \alpha_i = \varepsilon_i - \varepsilon_{i+1}.$$

This determines the set of *positive roots*

$$R^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq r\}.$$

The *highest root* is

$$\theta = \varepsilon_1 - \varepsilon_r = \alpha_1 + \alpha_2 + \cdots + \alpha_{r-1}. \tag{1.1.1}$$

For any root $\alpha \in R$, the corresponding *coroot* is

$$\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}.$$

We identify E with its dual E^* via $(\cdot,\cdot)$, so that the evaluation pairing $\langle\cdot,\cdot\rangle : E \times E^* \to \mathbb{R}$ agrees with the inner product. Hence, for any weight $x \in E$ and coroot $\alpha^\vee \in E$,

$$\langle x, \alpha^\vee\rangle = (x, \alpha^\vee) = \frac{2(x,\alpha)}{(\alpha,\alpha)}.$$

In type A, one has $(\alpha,\alpha) = 2$ for all $\alpha \in R$, so $\alpha^\vee = \alpha$ and $\langle x, \alpha^\vee\rangle = (x,\alpha)$.

The *root lattice* is

$$Q = \bigoplus_{i=1}^{r-1} \mathbb{Z}\alpha_i.$$

The *fundamental weights* $\Lambda_1, \ldots, \Lambda_{r-1} \in E$ are characterized by

$$\langle \Lambda_j, \alpha_i^\vee\rangle = \delta_{ij} \qquad (1 \le i, j \le r-1).$$

The *weight lattice* is

$$P = \bigoplus_{i=1}^{r-1} \mathbb{Z}\Lambda_i,$$

The set of *dominant weights* is

$$P^+ = \{\lambda \in P \mid \langle \lambda, \alpha_i^\vee\rangle \ge 0 \text{ for all } 1 \le i \le r-1\} = \bigoplus_{i=1}^{r-1} \mathbb{Z}_{\ge 0}\Lambda_i.$$

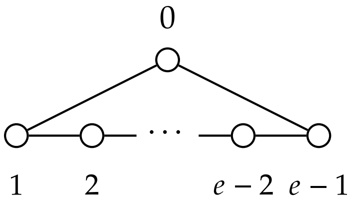
The *Weyl vector* is

$$\rho = \sum_{i=1}^{r-1} \Lambda_i,$$

equivalently, ρ is the half-sum of the positive roots; see [**Hum72**, Section 13.3].

1.1.2. Affine type $A_{e-1}^{(1)}$. In this section, we briefly introduce the Lie-theoretic data in type $A_{e-1}^{(1)}$ that will be used later in this thesis. For the general theory of Kac–Moody algebras, we refer to [**Kac83**, Chapter 1-6] and [**Car05**, Chapter 14-18].

Let $e > 2$ be an integer. In this thesis, let Γ be the quiver of type $A_{e-1}^{(1)}$:



It has vertex set $I = \{0, 1, \ldots, e-1\}$, which we identify with $\mathbb{Z}/e\mathbb{Z}$. In particular, we identify e with 0. Throughout this thesis, we fix the cyclic orientation $i \to i+1$ for each $i \in I$.

The (affine) Cartan matrix $(a_{ij})_{i,j\in I}$ of type $A^{(1)}_{e-1}$ is the following $e \times e$ matrix:

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & -1 & 2 & -1 & 0 \\ 0 & \cdots & 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & \cdots & 0 & 0 & -1 & 2 \end{pmatrix}$$

The simple roots are $\{\alpha_i \mid i \in I\}$, and $Q^+ := \bigoplus_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i$ is the *positive cone of root lattice*. For $\alpha \in Q^+$, let $\mathrm{ht}(\alpha)$ denote the *height of* α; that is, if $\alpha = \sum\limits_{i\in I} x_i\alpha_i$ with $x_i \in \mathbb{Z}_{\geq 0}$, then $\mathrm{ht}(\alpha) = \sum\limits_{i\in I} x_i$.

Let $\mathfrak{S}_n$ be the *symmetric group* on n letters and let $\sigma_r = (r, r+1)$, for $1 \leq r < n$, be the *simple transpositions* of $\mathfrak{S}_n$. Then $\mathfrak{S}_n$ acts on the set I^n from the left by place permutations.

If $\mathbf{i} = (\mathbf{i}_1, \ldots, \mathbf{i}_n) \in I^n$, we define the associated positive root $\alpha(\mathbf{i})$ by:

$$\alpha(\mathbf{i}) := \alpha_{i_1} + \cdots + \alpha_{i_n} \in Q^+ \tag{1.1.2}$$

The $\mathfrak{S}_n$-orbits on I^n are the sets

$$I^\alpha := \{\mathbf{i} \in I^n \mid \alpha = \alpha(\mathbf{i})\},$$

which are parametrized by $\alpha \in Q^+$ of height n.

Let $\Lambda_0, \ldots, \Lambda_{e-1}$ be the fundamental weights. The weight lattice is

$$P := \bigoplus_{i\in I} \mathbb{Z}\Lambda_i,$$

and the *dominant weight lattice* is defined as

$$P^+ := \bigoplus_{i\in I} \mathbb{Z}_{\geq 0}\Lambda_i.$$

Any element $\Lambda = \sum_{i\in I} a_i\Lambda_i \in P^+$ is a *dominant weight*, and the sum of the coefficients $\sum_{i\in I} a_i$ is the *level* of Λ.

There is a natural embedding of the root lattice to the weight lattice $Q \to P$, see [Car05, Chapter 17] for details.

We denote the set of simple coroots by $\{\alpha_i^\vee \mid i \in I\}$. The *coroot lattice* is defined as

$$Q^\vee := \bigoplus_{i\in I} \mathbb{Z}\alpha_i^\vee.$$

There is a canonical bilinear pairing $\langle\cdot,\cdot\rangle : P \times Q^\vee \to \mathbb{C}$ satisfying

$$\langle \alpha_j, \alpha_i^\vee\rangle = a_{ji} \quad \text{and} \quad \langle \Lambda_j, \alpha_i^\vee\rangle = \delta_{ji}$$

for all $i, j \in I$.Since the Cartan matrix (a_{ij}) is symmetric, we can identify α_i with $\alpha_i^\vee$ to define a symmetric bilinear form $(\cdot|\cdot)$ on $\mathfrak{h}^*$. This form is determined by the values:

$$(\alpha_i|\alpha_j) = a_{ij} \quad \text{and} \quad (\Lambda_i|\alpha_j) = \delta_{ij}$$

for all $i, j \in I$.

1.2. Partition combinatorics

One appealing feature of the representation theory of Hecke algebras is its elegant description via natural combinatorial tools. In this section, we introduce the basic notions we will use.

1.2.1. Young diagrams and Abaci. A *partition* λ is a finite weakly decreasing sequence of positive integers $(\lambda_1, \ldots, \lambda_r)$. It is often convenient to regard λ as a longer (or even infinite) sequence $(\lambda_1, \cdots, \lambda_{r+1}, \cdots)$ by setting $\lambda_i = 0$ for all $i > r$. Each λ_i is called a *part* of λ.

The *length* $\ell(\lambda)$ is the number of nonzero parts, and the *size* is $|\lambda| = \sum_i \lambda_i$. Let $\mathscr{P}_n$ be the set of partitions of n, and set $\mathscr{P} := \bigsqcup_{n \geq 0} \mathscr{P}_n^\Lambda$.

Similarly, a *composition* μ is a finite sequence of positive integers $(\mu_1, \ldots, \mu_r)$. Its length and size are defined analogously. Every partition is a composition, but not conversely.

A *Young diagram* is a finite collection of square boxes arranged in left-justified rows. Given a composition $\mu = (\mu_1, \ldots, \mu_r)$, its Young diagram $[\mu]$ is obtained by drawing μ_i boxes in the ith row. Conversely, counting the number of boxes in each row from top to bottom recovers the composition. Thus, we may identify a composition μ with its Young diagram $[\mu]$. Let $[\mu]_j$ be the jth row of $[\mu]$.

The *conjugate* of a partition λ, denoted λ', is obtained by reflecting the Young diagram $[\lambda]$ across its main diagonal. Clearly, $|\lambda'| = |\lambda|$.

A *node* is a square box of the Young diagram $[\lambda]$, specified by its coordinates (r, c), where r and c denote the row and column in which the node lies. Its *content* is defined to be $c - r$. If there are two nodes $A = (r, c)$ and $B = (r', c')$ in one Young diagram, we say A is *above* B, or B is *below* A if $r < r'$.

Example 1.2.1. *Take $\lambda = (4, 3, 2, 2)$. This partition has length $\ell(\lambda) = 4$ and size $|\lambda| = 11$. The Young diagram $[\lambda]$ and its conjugate $[\lambda']$ are*

$$\begin{array}{cccc} 0 & 1 & 2 & 3 \\ -1 & 0 & 1 & \\ -2 & -1 & & \\ -3 & -2 & & \end{array} \qquad \begin{array}{cccc} 0 & 1 & 2 & 3 \\ -1 & 0 & 1 & 2 \\ -2 & -1 & & \\ -3 & & & \end{array}$$

where each node is filled with its content. ◇

Fix $e \in \mathbb{Z}_{\geq 2}$. An *$e$-abacus* has e *runners*, arranged from left to right and labeled $0, 1, \ldots, e-1$. The positions are indexed by $\mathbb{Z}_{\geq 0}$ and arranged on the runners as follows: if $x \in \mathbb{Z}_{\geq 0}$ is written uniquely as $x = ae + b$ with $a \in \mathbb{Z}_{\geq 0}$ and $b \in \{0, 1, \ldots, e-1\}$, then position x lies on the b-runner in row a. Rows increase downward.

An *e-abacus with M beads* is a subset $B \subset \mathbb{Z}_{\geq 0}$ of cardinality $|B| = M$. Elements of B are called *bead positions*, and all other positions are *gaps*. We depict the M-abacus by placing a bead at each position in B and leaving all other positions empty.

Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition. Fix an integer $M \geq \ell(\lambda)$. The *M-beta numbers* of λ are

$$\beta_i^M(\lambda) = \lambda_i - i + M \qquad (1 \leq i \leq M),$$

and the corresponding *M-beta set* is

$$B(\lambda; M) = \{\beta_i^M(\lambda) \mid 1 \leq i \leq M\} \subset \mathbb{Z}_{\geq 0} \tag{1.2.2}$$

The *e-abacus of λ with M beads*, denoted $\mathrm{Ab}_e^M(\lambda)$, is the M-abacus having beads precisely at the positions in $B(\lambda; M)$.

If $\beta_1 > \beta_2 > \cdots > \beta_M$ are the elements of $B(\lambda; M)$ in decreasing order, then

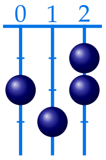
$$\lambda_i = \beta_i - M + i \qquad (1 \leq i \leq M),$$

so $B(\lambda; M)$ determines λ uniquely (discard trailing zeros).

Example 1.2.3. *Continue with Example 1.2.1, choose $M = \ell(\lambda) = 4$ and form the M–beta numbers $\beta_i^M(\lambda)$:*

$$7, 5, 3, 2$$

and the e–abacus of λ with M beads is



◇

1.2.2. Multipartitions, charges and residues.

The Young diagram has a natural filling of the contents defined in the last section, in this section, we introduce a more general notion of content, residue, which depends on the type $A_{e-1}^{(1)}$.

Fix type $A_{e-1}^{(1)}$ and a dominant weight $\Lambda = \Lambda_i$. Let λ be a partition and let $[\lambda]$ be its Young diagram. We fill each node of $[\lambda]$ with its *residue*, defined by

$$\mathrm{res}_\Lambda(r, c) \equiv c - r + i \pmod{e}.$$

A node with residue j is called an *j-node*. The Young diagram filled with residues is denoted $[\lambda]_\Lambda$. When Λ is clear from the context, we simply write $[\lambda]$ and res in place of $[\lambda]_\Lambda$ and res_Λ, respectively.

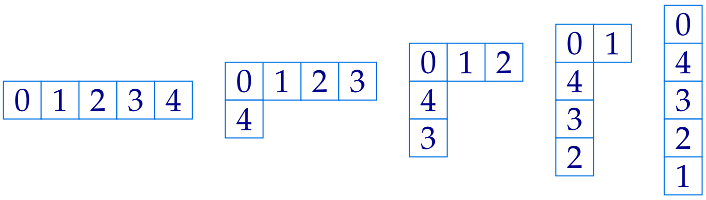
Following the convention from [**BKW11**], we define the *residue content* of a partition λ to be the positive root $\alpha_\lambda \in Q^+$ given by

$$\alpha_\lambda := \sum_{A\in[\lambda]} \alpha_{\operatorname{res} A} \in Q^+.$$

Let $\mathscr{P}^\Lambda$ be the set of partitions equipped with this residue labelling. For $\alpha \in Q^+$, we also set

$$\mathscr{P}^\Lambda_\alpha := \{\lambda \in \mathscr{P}^\Lambda \mid \alpha_\lambda = \alpha\}.$$

Example 1.2.4. *Take $\Lambda = \Lambda_0$ and $e = 5$, and consider the positive root $\alpha = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$. Then $\mathscr{P}^\Lambda_\alpha$ consists of the partitions with the following Young diagrams:*



In contrast, if we take $\beta = \alpha_1$, then $\mathscr{P}^\Lambda_\beta = \emptyset$. ◇

An *ℓ-partition* of n is an ℓ-tuple $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(\ell)})$ of partitions such that

$$\sum_{m=1}^{\ell} |\lambda^{(m)}| = n.$$

For each $1 \le m \le \ell$, let $[\lambda^{(m)}]$ be the Young diagram of $\lambda^{(m)}$, and let $\lambda^{(m)}_r$ be the rth part of $\lambda^{(m)}$. The Young diagram of $\boldsymbol{\lambda}$ is the disjoint union

$$[\boldsymbol{\lambda}] \;:=\; \bigsqcup_{m=1}^{\ell} \{m\} \times [\lambda^{(m)}] \;=\; \{(m,r,c) \in \mathbb{Z}^3_{>0} \mid 1 \le m \le \ell,\; 1 \le c \le \lambda^{(m)}_r\}.$$

As in the partition case, a *node* of $[\boldsymbol{\lambda}]$ is an element $A = (m,r,c) \in [\boldsymbol{\lambda}]$, where A is the square box in row r and column c of the mth component $[\lambda^{(m)}]$.

A *multipartition* is an ℓ-partition for any $\ell \ge 1$. The notions of ℓ-compositions and their Young diagrams, residues, etc. are defined analogously.

Fix a dominant weight Λ of level ℓ. To define residues of nodes, we use the notion of a charge. A *charge of* Λ is an ℓ-tuple $\boldsymbol{\kappa} = (\kappa_1, \dots, \kappa_\ell) \in I^\ell$ such that $\Lambda = \sum\limits_{m=1}^{\ell} \Lambda_{\kappa_m}$. Given such a charge and a node $A = (m,r,c) \in [\boldsymbol{\lambda}]$, we define its residue by

$$\operatorname{res}_{\boldsymbol{\kappa}}(A) \equiv \kappa_m + c - r \pmod{e}.$$

Equivalently, the residue function on the mth component $[\lambda^{(m)}]$ is $\operatorname{res}_{\Lambda_{\kappa_m}}$. The residue content of an ℓ-partition is the sum of the residue contents of its components, namely $\alpha_{\boldsymbol{\lambda}} = \sum_{1\le m\le \ell} \alpha_{\lambda^{(m)}}$.

Let $\mathscr{P}_n^{\boldsymbol{\kappa}}$ be the set of ℓ-partitions of n, with residues defined with respect to the charge $\boldsymbol{\kappa}$, and set $\mathscr{P}^{\boldsymbol{\kappa}} = \bigsqcup_{n\geq 0} \mathscr{P}_n^{\boldsymbol{\kappa}}$. For $\alpha \in Q^+$, let $\mathscr{P}_\alpha^{\boldsymbol{\kappa}}$ denote the set of ℓ-partitions of residue content α. We remark that if $\ell = 1$ and $\kappa \in I$, then $\mathscr{P}_\alpha^{\Lambda_\kappa}$ coincides with $\mathscr{P}_\alpha^{\kappa}$ for any $\alpha \in Q^+$.

Example 1.2.5. *Fix type $A_2^{(1)}$, $\Lambda = \Lambda_2 + \Lambda_1$. Consider the 2-partition $\boldsymbol{\lambda} = ((3,2,1),(5,1,1))$. Take a charge of Λ to be $\boldsymbol{\kappa} = (1,2)$, then $[\boldsymbol{\lambda}]$ is the following:*

$$\left(\begin{array}{|c|c|c|}\hline 1&2&0\\\hline 0&1&\\\cline{1-2} 2&&\\\cline{1-1}\end{array}\;\middle|\;\begin{array}{|c|c|c|c|c|}\hline 2&0&1&2&0\\\hline 1&&&&\\\cline{1-1} 0&&&&\\\cline{1-1}\end{array}\right)$$

The residue content of $\boldsymbol{\lambda}$ is $\beta := 5\alpha_0 + 4\alpha_1 + 4\alpha_2$ and hence $\boldsymbol{\lambda} \in \mathscr{P}_\beta^{\boldsymbol{\kappa}}$. Take another charge of Λ to be $\boldsymbol{\kappa}' = (2,1)$, then $[\boldsymbol{\lambda}]$ is the following:

$$\left(\begin{array}{|c|c|c|}\hline 2&0&1\\\hline 1&2&\\\cline{1-2} 0&&\\\cline{1-1}\end{array}\;\middle|\;\begin{array}{|c|c|c|c|c|}\hline 1&2&0&1&2\\\hline 0&&&&\\\cline{1-1} 2&&&&\\\cline{1-1}\end{array}\right)$$

The residue content of $\boldsymbol{\lambda}$ is $\gamma := 4\alpha_0 + 4\alpha_1 + 5\alpha_2$ and hence $\boldsymbol{\lambda} \in \mathscr{P}_\gamma^{\boldsymbol{\kappa}'}$. ◇

Let $\boldsymbol{\kappa}$ be a charge of Λ. It is sometimes convenient to choose an *integral charge* $\tilde{\boldsymbol{\kappa}} = (\tilde{\kappa}_1, \ldots, \tilde{\kappa}_\ell) \in \mathbb{Z}^\ell$, meaning an integral lift of $\boldsymbol{\kappa}$ such that $\tilde{\kappa}_m \equiv \kappa_m \pmod{e}$ for $1 \leq m \leq \ell$.

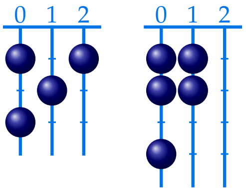
We now define the abacus configuration of an ℓ-partition with a given integral charge. Fix $\Lambda \in P^+$, a charge $\boldsymbol{\kappa} \in I^\ell$ of Λ, and $\alpha \in Q^+$. For any $\boldsymbol{\lambda} \in \mathscr{P}_\alpha^{\boldsymbol{\kappa}}$, choose an integral charge $\tilde{\boldsymbol{\kappa}} \in \mathbb{Z}^\ell$ such that $\tilde{\kappa}_m \equiv \kappa_m \pmod{e}$ and $\tilde{\kappa}_m \geq \ell\big(\lambda^{(m)}\big)$ for each $1 \leq m \leq \ell$. For each $1 \leq m \leq \ell$, define the $\tilde{\kappa}_m$-beta numbers of $\lambda^{(m)}$ by

$$\beta_i^m := \tilde{\kappa}_m + \lambda_i^{(m)} - i \qquad (1 \leq i \leq \tilde{\kappa}_m).$$

Form the e-abacus corresponding to the beta numbers $\{\beta_i^m \mid 1 \leq i \leq \tilde{\kappa}_m\}$, and write it as $\mathrm{Ab}_e^{\tilde{\kappa}_m}\big(\lambda^{(m)}\big)$. We then define the e-abacus of the ℓ-partition $\boldsymbol{\lambda}$ with integral charge $\tilde{\boldsymbol{\kappa}}$ to be the ℓ-tuple

$$\mathrm{Ab}_e^{\tilde{\boldsymbol{\kappa}}}(\boldsymbol{\lambda}) := \Big(\mathrm{Ab}_e^{\tilde{\kappa}_1}\big(\lambda^{(1)}\big), \ldots, \mathrm{Ab}_e^{\tilde{\kappa}_\ell}\big(\lambda^{(\ell)}\big)\Big).$$

Example 1.2.6. *Continue with Example 1.2.5. For the charge $\boldsymbol{\kappa} = (1,2)$ and partition $\boldsymbol{\lambda}$, we take the integral charge $\tilde{\boldsymbol{\kappa}} = (4,5)$, the corresponding e-abacus of $\boldsymbol{\lambda}$ is as follows:*



◇

We end this section with a definition that will be used in Chapter 3.

Definition 1.2.7. *Fix a composition $\mu = (\mu_1, \ldots, \mu_j)$ of length j, a j-partition $\boldsymbol{\lambda}$ is said to be* of type μ *if it satisfies the length constraint:*

$$\ell(\lambda^{(i)}) \leq \mu_i \quad \textit{for all } 1 \leq i \leq j.$$

For an integer $w \geq 0$, we define $A(\mu; w)$ to be the number of j-partitions of type μ with total size w:

$$A(\mu; w) \;:=\; \#\Big\{ (\lambda^{(1)}, \ldots, \lambda^{(j)}) : \; |\lambda^{(1)}| + \cdots + |\lambda^{(j)}| = w,\; \ell(\lambda^{(i)}) \leq \mu_i \;\forall i \Big\}.$$

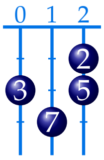
Example 1.2.8. *Let $\mu = (2,1)$ and $w = 2$. We calculate $A\big((2,1); 2\big)$. We look for pairs $(\lambda^{(1)}, \lambda^{(2)})$ such that $|\lambda^{(1)}| + |\lambda^{(2)}| = 2$, $\ell(\lambda^{(1)}) \leq 2$, and $\ell(\lambda^{(2)}) \leq 1$.*

- *Case $|\lambda^{(1)}| = 2, |\lambda^{(2)}| = 0$: $\lambda^{(1)} \in \{(2), (1,1)\}$, $\lambda^{(2)} = \emptyset$.*
- *Case $|\lambda^{(1)}| = 1, |\lambda^{(2)}| = 1$: $\lambda^{(1)} = (1)$, $\lambda^{(2)} = (1)$.*
- *Case $|\lambda^{(1)}| = 0, |\lambda^{(2)}| = 2$: $\lambda^{(1)} = \emptyset$. For $\lambda^{(2)}$, the partition (2) has length 1 which is valid, but $(1,1)$ has length 2 which is invalid since $\mu_2 = 1 < \ell(1,1)$.*

Thus, $A\big((2,1); 2\big) = 2 + 1 + 1 = 4$. ◇

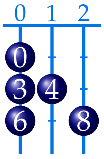
1.2.3. Moves in abacus. In an abacus configuration, a runner is called *flush* if each bead on that runner is placed as high as possible. Formally, the b–runner is flush if whenever there is a bead at position $ae + b$, there is also a bead at $(a-1)e + b$ (for $a \geq 1$).

Example 1.2.9. *Consider the partition $\lambda = (4,3,2,2)$. The 4–beta numbers are $(7,5,3,2)$, and the abacus $\mathrm{Ab}_3^4(\lambda)$ is*



Here the 2–runner is flush, whereas the 0– and 1–runners are not.

If we instead consider the 5-beta numbers, which are $(8,6,4,3,0)$, then the abacus $\mathrm{Ab}_3^5(\lambda)$ is



In this abacus the 0–runner is flush, while the 1– and 2–runners are not. ◇

We define elementary operations on abacus configurations by moving a single bead to an adjacent position. Let a bead be located at position $x = ae + b$ (row a, runner b). A move is valid only if the target position is empty (a *gap*). The atomic moves are:

- **Vertical moves (Sliding):**
  - *Slide down:* $x \mapsto x + e$. This moves the bead to row $a + 1$ on the same runner ($ae + b \mapsto (a+1)e + b$).

  - *Slide up:* $x \mapsto x - e$. This moves the bead to row $a-1$ on the same runner ($ae + b \mapsto (a-1)e + b$), provided $a \geq 1$.
- **Horizontal moves (Shifting):**
  - *Shift right:* $x \mapsto x + 1$. Generally, this moves the bead to the adjacent runner on the right ($b \mapsto b + 1$) within the same row. However, if the bead is on the last runner ($b = e - 1$), it moves to the first runner of the next row:

$$ae + (e-1) \longmapsto (a+1)e + 0.$$

  - *Shift left:* $x \mapsto x - 1$. Generally, this moves the bead to the adjacent runner on the left ($b \mapsto b - 1$) within the same row. However, if the bead is on the first runner ($b = 0$), it moves to the last runner of the previous row:

$$ae + 0 \longmapsto (a-1)e + (e-1).$$

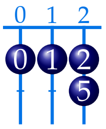
General bead movements are obtained by iterating these atomic steps, provided the target position at each step is empty.

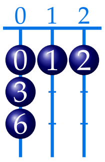
1.2.4. Core, weight, and quotient. A partition λ is called an *e–core* if there exists some $r \geq \ell(\lambda)$ such that every runner of $\mathrm{Ab}_e^r(\lambda)$ is flush. For any partition λ, choose $r \geq \ell(\lambda)$, form the e–abacus with r beads, and slide each bead upwards along its runner as far as possible. The resulting abacus corresponds to an e–core partition, denoted $\mathrm{core}_e(\lambda)$. We call $\mathrm{core}_e(\lambda)$ the *e–core* of λ, and define the *e–weight* of λ to be the total number of upward moves needed to obtain $\mathrm{core}_e(\lambda)$ from $\mathrm{Ab}_e^r(\lambda)$, denoted by $w_e(\lambda)$. Equivalently,

$$w_e(\lambda) = \frac{|\lambda| - |\mathrm{core}_e(\lambda)|}{e} \in \mathbb{Z}_{\geq 0},$$

The e–core and e–weight of a partition are independent of the choice of r; see [**Mat99**, Section 5.3].

Example 1.2.10. *In Example 1.2.9, for $r = 4$ or $r = 5$, there are runners that are not flush, hence $(4, 3, 2, 2)$ is not a 3-core. The corresponding 3-cores correspond to the following abaci in the two cases, respectively:*





Both abaci yield the same 3-core partition (2), and it is easy to verify that the 3-weight is 3. ◇

The *e–quotient* of λ captures the relative positions of the beads on each runner. Fix an e-abacus of λ with r beads. For each runner $j \in \{0, \ldots, e-1\}$, let r_j be the number of beads on that runner. Let the row indices of these beads be $y_{j,1} > y_{j,2} > \cdots > y_{j,r_j} \geq 0$. We view this sequence as the set of r_j-beta numbers for a new partition $\lambda^{(j)}$. Explicitly, the parts of $\lambda^{(j)}$ are given by:

$$\lambda_k^{(j)} = y_{j,k} - (r_j - k) \qquad (1 \leq k \leq r_j).$$

The *e–quotient* of λ is the e-tuple of partitions

$$\operatorname{quot}_e(\lambda) \;=\; (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(e-1)}). \tag{1.2.11}$$

A fundamental property relating the core, quotient, and weight is:

$$|\lambda| \;=\; |\operatorname{core}_e(\lambda)| + e\sum_{j=0}^{e-1} |\lambda^{(j)}|.$$

Comparing this with the definition of the e-weight, we see that $w_e(\lambda) = \sum_{j=0}^{e-1} |\lambda^{(j)}|$. Thus, the partition λ is uniquely determined by its e-core and its e-quotient.

Example 1.2.12. *In Example 1.2.9, $\lambda = (4,3,2,2)$ and $e = 3$. The 4-beta numbers are $(7,5,3,2)$. 0-runner has a single bead at position 3 (row 1), which corresponds to the partition $\lambda^{(0)} = (1)$. 1-runner has a bead at position 7 (row 2), yielding $\lambda^{(1)} = (2)$. 2-runner contains beads at positions $5,2$ (rows $1,0$), which correspond to the empty partition $\lambda^{(2)} = \emptyset$. The 3-quotient is $\big((1),(2),\emptyset\big)$. The total size is $1+2+0 = 3$, which matches the 3-weight of $w_3(4,3,2,2) = 3$.* ◇

1.2.5. Tableaux. Fix the quiver of type $A^{(1)}_{e-1}$ and let I be the vertex set. Fix $\ell \in \mathbb{Z}_{>0}$ and a charge $\boldsymbol{\kappa} = (\kappa_1, \dots, \kappa_\ell) \in I^\ell$. Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(\ell)}) \in \mathscr{P}^{\boldsymbol{\kappa}}_n$ be an ℓ-partition of n.

A $\boldsymbol{\lambda}$-*tableau* is a bijection

$$T : [\boldsymbol{\lambda}] \xrightarrow{\sim} \{1, 2, \dots, n\}.$$

The *shape* of a tableau T, written Shape(T), is the ℓ-partition $\boldsymbol{\lambda}$. A $\boldsymbol{\lambda}$-tableau T can be viewed as a filling of the nodes of the Young diagram $[\boldsymbol{\lambda}]$ with the numbers in $\{1, 2, \dots, n\}$ without repetitions.

Let $r, s \in \{1, 2, \cdots, n\}$. If $T^{-1}(r)$ and $T^{-1}(s)$ are horizontally adjacent or vertically adjacent in $[\boldsymbol{\lambda}]$, we write $r \rightarrow_T s$ or $r \downarrow_T s$, respectively.

For $1 \le m \le n$, write $T \downarrow m$ for the tableau obtained from T by deleting all nodes occupied by entries $> m$. In other words, $T \downarrow m$ is the restriction of T to $T^{-1}(\{1, 2, \dots, m\})$.

For $r \in \{1, \dots, n\}$, let $A_r := T^{-1}(r) \in [\boldsymbol{\lambda}]$ and define the residue of r in T by

$$\operatorname{res}_r(T) \;:=\; \operatorname{res}_{\boldsymbol{\kappa}}(A_r) \in I.$$

The *residue sequence* of T is

$$\mathbf{i}^T \;=\; \operatorname{res}(T) \;:=\; \big(\operatorname{res}_1(T), \dots, \operatorname{res}_n(T)\big) \in I^n.$$

A tableau T of shape $\boldsymbol{\lambda}$ is *row-standard* if, for each component $\boldsymbol{\lambda}^{(m)}$, its entries increase strictly from left to right along each row of $\boldsymbol{\lambda}^{(m)}$. It is *column-standard* if, for each component $\boldsymbol{\lambda}^{(m)}$, its entries increase strictly from top to bottom along each column of $\boldsymbol{\lambda}^{(m)}$. It is *standard* if it is both row-standard and column-standard. Equivalently, T is standard if and only if, for each $d \in \{1, \dots, n\}$, the Shape($T \downarrow d$) is an ℓ-partition. Let

$\operatorname{RStd}(\boldsymbol{\lambda})$ and $\operatorname{Std}(\boldsymbol{\lambda})$ be the sets of row-standard $\boldsymbol{\lambda}$-tableaux and standard $\boldsymbol{\lambda}$-tableaux, respectively.

Equip $[\boldsymbol{\lambda}]$ with the *row-reading order* $\prec$ defined by

$$(m,a,b) \prec (m',a',b') \quad \leftrightarrow \quad \big(m < m'\big) \text{ or } \big(m = m' \text{ and } (a < a' \text{ or } (a = a' \text{ and } b < b'))\big).$$

Informally, we read the nodes of $[\boldsymbol{\lambda}]$ from the first component to the last component, and within each component we read from left to right along each row, taking rows from top to bottom.

Define the *initial tableau* $T^{\boldsymbol{\lambda}}$ to be the unique $\boldsymbol{\lambda}$-tableau whose entries increase strictly along $\prec$. Set

$$\mathbf{i}^{\boldsymbol{\lambda}} := \mathbf{i}^{T^{\boldsymbol{\lambda}}} \in I^n. \tag{1.2.13}$$

The symmetric group $\mathfrak{S}_n$ acts on the set of $\boldsymbol{\lambda}$-tableaux from the left by permuting entries:

$$(\sigma \cdot T)(A) := \sigma\big(T(A)\big) \qquad (\sigma \in \mathfrak{S}_n,\ A \in [\boldsymbol{\lambda}]).$$

For each $\boldsymbol{\lambda}$-tableau T, define $w^T \in \mathfrak{S}_n$ by the condition

$$w^T \cdot T^{\boldsymbol{\lambda}} = T. \tag{1.2.14}$$

Example 1.2.15. *Continue with Example 1.2.5, the initial $\boldsymbol{\lambda}$-tableau $T^{\boldsymbol{\lambda}}$ is the following:*

$$\left(\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & \\ 6 & & \end{array} \;\middle|\; \begin{array}{ccccc} 7 & 8 & 9 & 10 & 11 \\ 12 & & & & \\ 13 & & & & \end{array}\right)$$

If the charge is $\boldsymbol{\kappa} = (1,2)$, then the residue sequence $\mathbf{i}^{\boldsymbol{\lambda}}$ of $T^{\boldsymbol{\lambda}}$ is (1200122012010). *Take T to be the following tableau:*

$$\left(\begin{array}{ccc} 1 & 2 & 7 \\ 4 & 12 & \\ 9 & & \end{array} \;\middle|\; \begin{array}{ccccc} 3 & 6 & 8 & 10 & 11 \\ 5 & & & & \\ 13 & & & & \end{array}\right)$$

the residue sequence $\mathbf{i}^T$ is (1220100122010) *and*

$$w^T = \sigma_5\sigma_6\sigma_7\sigma_8\sigma_9\sigma_{10}\sigma_{11}\sigma_{10}\sigma_9\sigma_7\sigma_8\sigma_5\sigma_6\sigma_7\sigma_3\sigma_4\sigma_5\sigma_6\sigma_5\sigma_3$$

where σ_i is the simple transposition $(i,i+1) \in \mathfrak{S}_{13}$. ◇

1.2.6. Dominance order and Bruhat order on tableaux. For ℓ-compositions $\boldsymbol{\lambda}, \boldsymbol{\mu}$ of n, write $\boldsymbol{\lambda} \trianglerighteq \boldsymbol{\mu}$ if for every $1 \le t \le \ell$ and every $k \ge 1$ one has

$$\sum_{m=1}^{t-1} |\lambda^{(m)}| + \sum_{i=1}^{k} \lambda_i^{(t)} \ \ge\ \sum_{m=1}^{t-1} |\mu^{(m)}| + \sum_{i=1}^{k} \mu_i^{(t)}.$$

Write $\boldsymbol{\lambda} \triangleright \boldsymbol{\mu}$ if $\boldsymbol{\lambda} \trianglerighteq \boldsymbol{\mu}$ and $\boldsymbol{\lambda} \ne \boldsymbol{\mu}$.

Fix charge $\boldsymbol{\kappa}$ and $\alpha \in Q^+$ with $\mathrm{ht}(\alpha) = n$. Let $\boldsymbol{\nu} \in \mathscr{P}^{\boldsymbol{\kappa}}_{\alpha}$, and let S, T be row-standard $\boldsymbol{\nu}$-tableaux. Write $S \trianglerighteq T$ if

$$\mathrm{Shape}(S \downarrow m) \trianglerighteq \mathrm{Shape}(T \downarrow m) \qquad \text{for all } m = 1, 2, \ldots, n,$$

and write $S \triangleright T$ if $S \trianglerighteq T$ and $S \neq T$.

Let $\leq$ denote the (strong) Bruhat order on $\mathfrak{S}_n$, with respect to the Coxeter generators $\sigma_1, \ldots, \sigma_{n-1}$. Thus, for $u, w \in \mathfrak{S}_n$, one has $u \leq w$ if and only if there exists a reduced expression $w = \sigma_{r_1} \cdots \sigma_{r_m}$ and indices $1 \leq a_1 < \cdots < a_b \leq m$ such that

$$u = \sigma_{r_{a_1}} \cdots \sigma_{r_{a_b}}.$$

See, for example, [**Hum90**, Section 5.10]. In particular, $1 \leq w$ for all $w \in \mathfrak{S}_n$.

Lemma 1.2.16. *Let S, T be row-standard $\boldsymbol{\nu}$-tableaux. Then*

$$S \trianglerighteq T \quad \Longleftrightarrow \quad w^S \leq w^T.$$

Proof. This is the Ehresmann–James theorem; see, for example, [**Mat99**, Theorem 3.8]. □

1.2.7. Degree of tableaux. Following [**BKW11**], we recall the degree on standard tableaux.

Fix the quiver $A^{(1)}_{e-1}$ and a charge $\boldsymbol{\kappa} \in I^{\ell}$. Let $\boldsymbol{\lambda} \in \mathscr{P}^{\boldsymbol{\kappa}}$ be an ℓ-partition of n. A node $A \in [\boldsymbol{\lambda}]$ is *removable* if $[\boldsymbol{\lambda}] \setminus \{A\}$ is the Young diagram of an ℓ-partition, and a node $B \notin [\boldsymbol{\lambda}]$ is *addable* if $[\boldsymbol{\lambda}] \cup \{B\}$ is the Young diagram of an ℓ-partition. If $\mathrm{res}(A) = i$ and A is removable, then A is called a removable i-node of $[\boldsymbol{\lambda}]$. Similarly, if $\mathrm{res}(B) = i$ (in $[\boldsymbol{\lambda}] \cup \{B\}$) and B is addable, then B is called an addable i-node of $[\boldsymbol{\lambda}]$.

We order nodes by declaring $B = (m', r', c')$ to be *below* $A = (m, r, c)$ if either $m' > m$, or $m' = m$ and $r' > r$. For a removable i-node A of $\boldsymbol{\lambda}$, set

(1.2.17)

$$d_A(\boldsymbol{\lambda}) := \#\{\text{addable } i\text{-nodes of } [\boldsymbol{\lambda}] \text{ below } A\} - \#\{\text{removable } i\text{-nodes of } [\boldsymbol{\lambda}] \text{ below } A\}.$$

Now let $T \in \mathrm{Std}(\boldsymbol{\lambda})$. Define $\deg(T)$ inductively on n by $\deg(\varnothing) := 0$, and, for $n > 0$,

$$\deg(T) := d_A(\boldsymbol{\lambda}) + \deg\big(T \downarrow (n-1)\big),$$

where A is the node occupied by n in T, and $T \downarrow (n-1)$ is the tableau obtained by deleting the entry n. This definition is valid because, by definition of a standard tableau, the shape of $T \downarrow (n-1)$ is an ℓ-partition, and $T \downarrow (n-1)$ is again standard.

Example 1.2.18. *Continue with Example 1.2.15, the degrees are* $\deg T^{\boldsymbol{\lambda}} = 4$ *and* $\deg T = 6$. ◇

1.2.8. Garnir tableaux. In this section, following [**KMR12**, Section 5], we introduce the Garnir combinatorics needed later. Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \cdots, \lambda^{(\ell)})$ be an ℓ-partition, and let $[\boldsymbol{\lambda}]$ be its Young diagram.

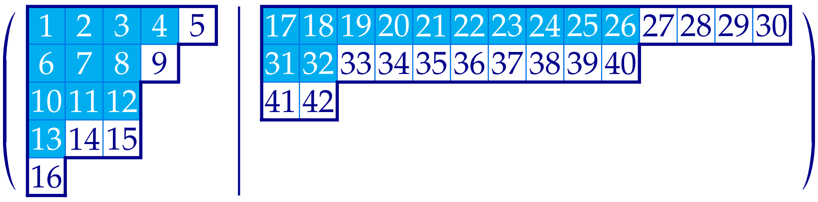
DEFINITION 1.2.19. *A node $A = (m, r, c) \in [\boldsymbol{\lambda}]$ is a* ***Garnir node*** *of $\boldsymbol{\lambda}$ if $(m, r+1, c) \in [\boldsymbol{\lambda}]$. The* ***Garnir belt*** *of A is the set $\mathcal{B}^A$ of nodes of $[\boldsymbol{\lambda}]$ consisting of A and all nodes directly to the right of A, together with the node directly below A and all nodes directly to the left of that node in the same component. Explicitly,*

$$\mathcal{B}^A = \{(m, r, z) \in [\boldsymbol{\lambda}] \mid c \le z \le \lambda_r^{(m)}\} \cup \{(m, r+1, z) \in [\boldsymbol{\lambda}] \mid 1 \le z \le c\}.$$

DEFINITION 1.2.20. *Let $A \in [\boldsymbol{\lambda}]$ be a Garnir node. The* ***Garnir tableau*** *G^A is the unique row-standard tableau satisfying:*

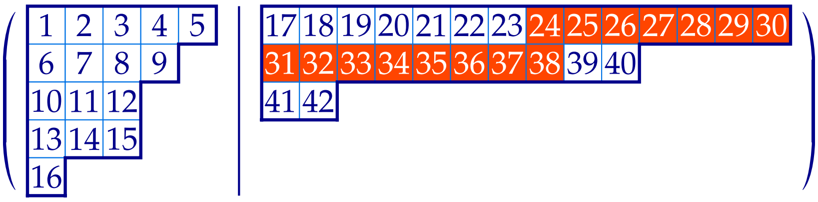
- *it agrees with $T^{\boldsymbol{\lambda}}$ on all nodes outside the Garnir belt $\mathcal{B}^A$,*
- *its entries in $\mathcal{B}^A$ increase from the bottom-left to the top-right.*

EXAMPLE 1.2.21. *Consider the 2-partition $\boldsymbol{\lambda} = (5, 4, 3, 3, 1 \mid 14, 10, 2)$. We color the Garnir nodes by cyan:*



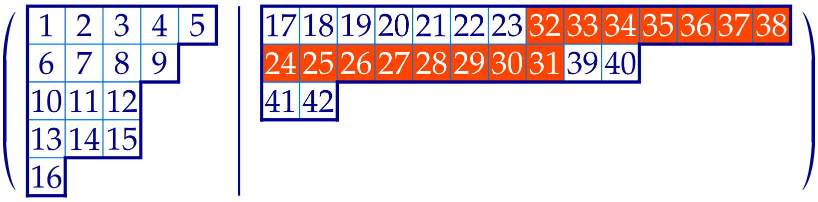
$$\left(\begin{array}{ccccc}1&2&3&4&5\\6&7&8&9&\\10&11&12&&\\13&14&15&&\\16&&&&\end{array}\;\middle|\;\begin{array}{cccccccccccccc}17&18&19&20&21&22&23&24&25&26&27&28&29&30\\31&32&33&34&35&36&37&38&39&40&&&&\\41&42&&&&&&&&&&&&\end{array}\right)$$

Let $A = (2, 1, 8)$, then $T(A) = 24$ and the Garnir belt $\mathcal{B}^A$ consists of the following orange nodes:



$$\left(\begin{array}{ccccc}1&2&3&4&5\\6&7&8&9&\\10&11&12&&\\13&14&15&&\\16&&&&\end{array}\;\middle|\;\begin{array}{cccccccccccccc}17&18&19&20&21&22&23&24&25&26&27&28&29&30\\31&32&33&34&35&36&37&38&39&40&&&&\\41&42&&&&&&&&&&&&\end{array}\right)$$

and the Garnir tableau G^A is the following:



$$\left(\begin{array}{ccccc}1&2&3&4&5\\6&7&8&9&\\10&11&12&&\\13&14&15&&\\16&&&&\end{array}\;\middle|\;\begin{array}{cccccccccccccc}17&18&19&20&21&22&23&32&33&34&35&36&37&38\\24&25&26&27&28&29&30&31&39&40&&&&\\41&42&&&&&&&&&&&&\end{array}\right)$$

$\diamond$

Fix the quiver of type $A_{e-1}^{(1)}$ and let I be its vertex set. Let $\Lambda \in P^+$, and let $\boldsymbol{\kappa} = (\kappa_1, \cdots, \kappa_\ell) \in I^\ell$ be a charge of Λ. Take $\boldsymbol{\lambda} \in \mathscr{P}^{\boldsymbol{\kappa}}$. In particular, we can assign a residue to each node of $[\boldsymbol{\lambda}]$, as discussed in Subsection 1.2.5.

Fix a Garnir node $A = (m, r, c) \in [\boldsymbol{\lambda}]$ and write $\mathcal{B}^A$ for its Garnir belt.

DEFINITION 1.2.22. *A* ***(row) A-brick*** *is a subset $B \subseteq \mathcal{B}^A$ of the form*

$$B = \{(m, x, z), (m, x, z+1), \ldots, (m, x, z+e-1)\}$$

for some $x \in \{r, r+1\}$ and some $z \in \mathbb{Z}_{>0}$, such that $\operatorname{res}(m, x, z) = \operatorname{res}(A)$.

Let $k = k^A$ be the number of row A-bricks contained in $\mathscr{B}^A$. We always list these bricks as $B_1^A, B_2^A, \dots, B_k^A$ in the following order: first, the bricks contained in row $r+1$ of the Garnir belt $\mathscr{B}^A$, ordered from left to right; then the bricks contained in row r of the Garnir belt $\mathscr{B}^A$, ordered from left to right.

Definition 1.2.23. *Let $A \in [\boldsymbol{\lambda}]$ be a Garnir node and write $\mathscr{B}^A$ for its Garnir belt. Let $B_1^A, \dots, B_{k^A}^A$ be the row A-bricks in $\mathscr{B}^A$, listed as above. For each $1 \le t \le k^A$, set $n_t^A := \min\{\, G^A(x) \mid x \in B_t^A \,\}$. For $1 \le t < k^A$, define the* brick transposition *$w_t^A \in \mathfrak{S}_n$ by*

$$w_t^A := \prod_{a=0}^{e-1} \big(n_t^A + a,\ n_t^A + e + a\big). \tag{1.2.24}$$

The brick permutation group *of A is the subgroup $\mathfrak{S}^A := \langle\, w_1^A, \dots, w_{k^A-1}^A \,\rangle \le \mathfrak{S}_n$, with the convention $\mathfrak{S}^A = \{1\}$ if $k^A \le 1$.*

The group $\mathfrak{S}^A$ acts on the set of $\boldsymbol{\lambda}$-tableaux by permuting entries, and hence it acts on the Garnir tableau G^A. Define the *Garnir set* and the associated residue sequence by

$$\mathrm{Gar}^A := \{\, w \cdot G^A \mid w \in \mathfrak{S}^A \,\}, \qquad \mathbf{i}^A := \mathbf{i}^{G^A} \in I^n.$$

Lemma 1.2.25. *Suppose that $\boldsymbol{\lambda} \in \mathscr{P}_\alpha^\kappa$ and that $A \in [\boldsymbol{\lambda}]$ is a Garnir node. Then*

$$\mathrm{Gar}^A \setminus \{G^A\} = \Big\{ T \in \mathrm{Std}(\boldsymbol{\lambda}) \ \Big|\ T \trianglerighteq G^A \text{ and } \mathbf{i}^T = \mathbf{i}^A \Big\}.$$

Proof. This is [**KMR12**, Lemma 5.5]. □

In particular, with respect to the partial order $\trianglerighteq$ defined in Subsection 1.2.6, G^A is the unique minimal element in $\mathrm{Gar}\,A$. There is also a unique maximal element T^A in $\mathrm{Gar}\,A$, obtained by rearranging the row A-bricks in $\mathscr{B}^A$ in row-reading order.

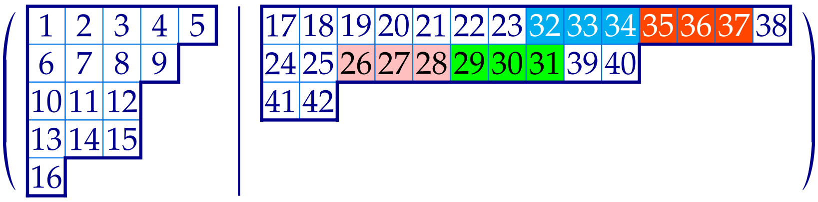
Let $f = f^A$ be the number of row A-bricks in row r of the Garnir belt $\mathscr{B}^A$. Define $\mathscr{D}^A$ to be the set of minimal-length left coset representatives of $\mathfrak{S}_f \times \mathfrak{S}_{k-f}$ in $\mathfrak{S}^A \cong \mathfrak{S}_k$. Note that $\mathfrak{S}^A \le \mathfrak{S}_n$, hence $\mathscr{D}^A \subseteq \mathfrak{S}_n$ and its elements act on $\boldsymbol{\lambda}$-tableaux. Moreover,

$$\mathrm{Gar}^A = \{\, w \cdot T^A \mid w \in \mathscr{D}^A \,\},$$

Example 1.2.26. *Continue with Example 1.2.21 and take $A = (2,1,8)$. Fix type $A_2^{(1)}$ and the charge $(0,1)$. The Young diagram $[\boldsymbol{\lambda}]$, filled with residues, is as follows:*

$$\left(\ \begin{array}{|c|c|c|c|c|}\hline 0&1&2&0&1\\\hline 2&0&1&2\\\cline{1-4} 1&2&0\\\cline{1-3} 0&1&2\\\cline{1-3} 2\\\cline{1-1}\end{array}\ \middle|\ \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&2&0&1&2&0&1&2&0&1&2&0&1&2\\\hline 0&1&2&0&1&2&0&1&2&0\\\cline{1-10} 2&0\\\cline{1-2}\end{array}\ \right)$$

The row A-bricks are colored as below. The bricks $B_1^A, B_2^A, B_3^A, B_4^A$ are colored pink, green, cyan, and orange, respectively.

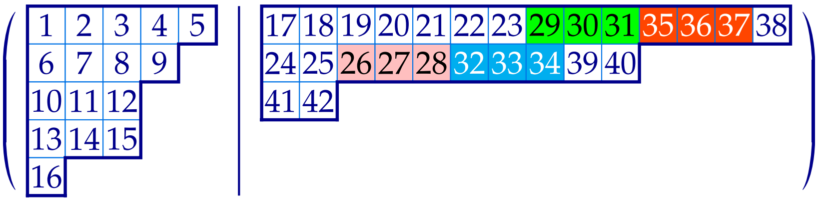


$$\left(\begin{array}{ccccc}1&2&3&4&5\\6&7&8&9\\10&11&12\\13&14&15\\16\end{array}\;\middle|\;\begin{array}{ccccccccccccccc}17&18&19&20&21&22&23&32&33&34&35&36&37&38\\24&25&26&27&28&29&30&31&39&40\\41&42\end{array}\right)$$

So $k^A = 4$ *and* $f^A = 2$. *For* $1 \le i \le 3$, *let* w_i^A *be the brick transposition. Then the brick permutation group is* $\mathfrak{S}^A = \langle w_1^A, w_2^A, w_3^A \rangle$, *and* $\mathscr{D}^A$ *is*

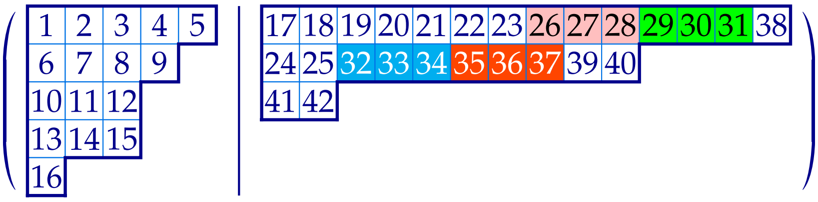
$$\mathscr{D}^A = \{1, w_2^A, w_1^A w_2^A, w_3^A w_2^A, w_1^A w_3^A w_2^A, w_2^A w_1^A w_3^A w_2^A\}$$

Apply w_2^A *to the Garnir tableau, we get the following:*



$$\left(\begin{array}{ccccc}1&2&3&4&5\\6&7&8&9\\10&11&12\\13&14&15\\16\end{array}\;\middle|\;\begin{array}{ccccccccccccccc}17&18&19&20&21&22&23&29&30&31&35&36&37&38\\24&25&26&27&28&32&33&34&39&40\\41&42\end{array}\right)$$

The unique maximal element T^A *in* Gar^A *is the following:*



$$\left(\begin{array}{ccccc}1&2&3&4&5\\6&7&8&9\\10&11&12\\13&14&15\\16\end{array}\;\middle|\;\begin{array}{ccccccccccccccc}17&18&19&20&21&22&23&26&27&28&29&30&31&38\\24&25&32&33&34&35&36&37&39&40\\41&42\end{array}\right)$$

The Garnir set is $\mathrm{Gar}^A = \{w \cdot T^A \mid w \in \mathscr{D}^A\}$. *In particular,* $G^A = (w_2^A w_1^A w_3^A w_2^A) \cdot T^A$. ◇

1.3. Hecke Algebras

This thesis focuses on the representation theory and related combinatorics of Hecke algebras and their generalizations. In this section, we briefly review the definitions and basic results needed later.

1.3.1. Coxeter systems and Hecke algebras. A *Coxeter system* is a pair (W, S) where W is a group generated by a set S subject to relations

$$(st)^{m_{st}} = 1 \qquad (s, t \in S),$$

determined by a matrix of exponents $M = (m_{st})_{s,t\in S}$. The entries satisfy $m_{ss} = 1$ and $m_{st} = m_{ts} \in \{2, 3, \ldots, \infty\}$ for $s \neq t$. If $m_{st} = \infty$, no relation is imposed between s and t. Let ℓ denote the length function and $\le$ the Bruhat order on W; see [Hum90, Chapter 5] for standard theory.

Associated to a Coxeter system (W, S) is its Coxeter diagram $\Gamma(W, S)$: this is the graph with vertex set S, in which two distinct vertices $s, t \in S$ are joined by an edge if $m_{st} \ge 3$. By convention, no label is written on an edge when $m_{st} = 3$, and an edge is labeled by m_{st} when $m_{st} \ge 4$; the case $m_{st} = 2$ corresponds to no edge. We refer to the isomorphism class of $\Gamma(W, S)$ as the *type* of (W, S), and call (W, S) *irreducible* if $\Gamma(W, S)$

is connected. In particular, when $\Gamma(W,S)$ is a Dynkin diagram such as A_{r-1} or $A_{e-1}^{(1)}$, defined explicitly in Subsection 1.1.1 and Subsection 1.1.2, we say that (W,S) is of type A_{r-1} or $A_{e-1}^{(1)}$, respectively.

Following the notation of [**Soe97**], the *Hecke algebra* $\mathcal{H}(W)$ over $\mathbb{Z}[v,v^{-1}]$ is generated by $\{H_s \mid s \in S\}$ with the quadratic relations

$$(H_s + v)(H_s - v^{-1}) = 0 \qquad (s \in S),$$

and the braid relations

$$\underbrace{H_sH_tH_s\cdots}_{m_{st}\text{ factors}} = \underbrace{H_tH_sH_t\cdots}_{m_{st}\text{ factors}} \qquad (s,t \in S,\ m_{st} < \infty).$$

For any element $w \in W$, it can be written as a product of elements of S. Such an expression is *reduced* if it has minimal length. Reduced expressions are generally not unique. For a reduced expression $w = s_1\cdots s_r$, set $H_w = H_{s_1}\cdots H_{s_r}$. It is a consequence of Matsumoto's theorem that H_w is independent of the choice of reduced expression of w. Moreover, $\{H_w \mid w \in W\}$ is a basis of $\mathcal{H}(W)$. See [**Hum90**, Chapter 7] for more details.

We define the *bar-involution* on $\mathcal{H}(W)$ by $\overline{v} = v^{-1}$ and $\overline{H_w} = H_{w^{-1}}^{-1}$, and extend it linearly. The *Kazhdan–Lusztig basis* $\{\underline{H}_w \mid w \in W\}$ is the unique bar-invariant basis satisfying

$$\underline{H}_w = \sum_{y\leq w} h_{y,w}(v)\,H_y, \qquad h_{w,w} = 1, \quad h_{y,w}(v) \in v\,\mathbb{Z}[v]\ (y < w).$$

The coefficients $h_{y,w}$ are called *Kazhdan–Lusztig polynomials*.

We remark that these notations differ from the original notations used in [**KL79**]; see [**Soe97**, Section 2] for a discussion of these conventions.

If a Coxeter group is of Dynkin type $*$, for example of type $A_{e-1}^{(1)}$, then the corresponding Hecke algebra is simply called the Hecke algebra of type $*$.

1.3.2. Parabolic Kazhdan–Lusztig polynomials.

While the parabolic Kazhdan–Lusztig theory was originally developed by Deodhar [**Deo87**], we adopt the framework and notation of Soergel [**Soe97**, Section 3] for consistency.

Let (W,S) be a Coxeter system and take a subset $I \subseteq S$. The subgroup W_I generated by I is called a *parabolic subgroup*. The corresponding *parabolic subalgebra* $\mathcal{H}_I \subseteq \mathcal{H}(W)$ is the subalgebra generated by $\{H_s \mid s \in I\}$.

Let ${}^IW$ denote the set of minimal length representatives for the *right cosets* $W_I\backslash W$:

$${}^IW = \{w \in W \mid \ell(sw) > \ell(w) \text{ for all } s \in I\}.$$

Note that each $w \in {}^IW$ is the unique element of minimal length in the coset W_Iw.

Let sgn_I be the 1–dimensional right $\mathcal{H}_I$–module where H_s acts by multiplication by $-v$ for all $s \in I$. The *anti-spherical module* is the induced right $\mathcal{H}(W)$-module:

$$\mathcal{N}_I \;=\; \mathrm{sgn}_I \otimes_{\mathcal{H}_I} \mathcal{H}(W).$$

This module has a standard basis given by $N_x = 1 \otimes H_x$ for $x \in {}^I W$. The bar-involution on $\mathcal{H}(W)$ extends to a bar-involution on $\mathcal{N}_I$, denoted $n \mapsto \overline{n}$, which is determined by the properties:

$$\overline{N_{\mathrm{id}}} = N_{\mathrm{id}} \quad \text{and} \quad \overline{n \cdot h} = \overline{n} \cdot \overline{h} \quad (n \in \mathcal{N}_I, h \in \mathcal{H}(W)).$$

By [Deo87, Proposition 3.2] or [Soe97, Theorem 3.1], there exists a unique bar-invariant basis $\{\underline{N}_x \mid x \in {}^I W\}$ of $\mathcal{N}_I$ such that

$$\underline{N}_x \;=\; N_x \;+ \sum_{\substack{y \in {}^I W \\ y < x}} \mathfrak{n}_{y,x}(v)\, N_y, \qquad \mathfrak{n}_{y,x}(v) \in v\,\mathbb{Z}[v].$$

The coefficients $\mathfrak{n}_{y,x}(v)$ are called the *anti-spherical* (or *parabolic*) *Kazhdan–Lusztig polynomials.* [1]

1.3.3. Symmetric Groups and Iwahori-Hecke Algebra.

Let n be a positive integer. The *symmetric group* $\mathfrak{S}_n$ acts on the set $\{1, 2, \ldots, n\}$ from the left by permuting its elements. For $1 \leq i \leq n-1$, let $\sigma_i = (i\; i+1)$ be the *simple transposition* swapping i and $i+1$ (and fixing all other elements). It is well known that $\{\sigma_i \mid 1 \leq i \leq n-1\}$ generates $\mathfrak{S}_n$. Equivalently, $\mathfrak{S}_n$ is a Coxeter group with Coxeter generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{if } |i-j| > 1, \qquad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad (1 \leq i \leq n-2).$$

Take $S = \{\sigma_i \mid 1 \leq i \leq n-1\}$. It is easy to verify that the Coxeter diagram of $(\mathfrak{S}_n, S)$ is the Dynkin diagram of type A_{n-1}.

If R is a commutative ring with 1, the group algebra of $\mathfrak{S}_n$ over R is denoted $R\mathfrak{S}_n$. In this thesis, we primarily consider $R = \mathbb{Z}[v, v^{-1}]$ or a field R.

The *Iwahori–Hecke algebra* $\mathcal{H}_n$ is the Hecke algebra of the symmetric group, in the sense of Subsection 1.3.1. Equivalently, the Iwahori–Hecke algebra is simply the Hecke algebra of type A_{n-1}.

There is another natural presentation of $\mathcal{H}_n$, see [Mat99, Equation 1.10], as follows: $\mathcal{H}_n := \mathcal{H}_{\Bbbk,q}(\mathfrak{S}_n)$[2] is the unital associative $\Bbbk$-algebra with generators $T_1, \cdots, T_{n-1}$ subject

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[1] The reader is encouraged to visit Lievis to see the pattern of anti-spherical KL polynomials and many other interactive visualizations.

[2] Some authors omit $\Bbbk$ and write $\mathcal{H}_q(\mathfrak{S}_n)$ instead.

only to the following relations

$$(T_i - q)(T_i + 1) = 0, \quad \text{for } i = 1, 2, \ldots, n-1$$

$$T_iT_j = T_jT_i, \quad \text{for } |i-j| > 1$$

$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \quad \text{for } i = 1, 2, \ldots, n-2$$

Note that the two presentations are related by $T_i = v^{-1}H_{\sigma_i}$ and $q = v^{-2}$.

For further details, we refer to [**JK84**, **CSST10**] for the representation theory of symmetric groups, and to [**Mat99**] for the representation theory of Iwahori–Hecke algebras.

1.3.4. KLR algebras.

The Iwahori–Hecke algebras discussed in Subsection 1.3.3 are the level one case of another important class of algebras, namely the *cyclotomic Hecke algebras*, or *Ariki–Koike algebras*. We omit the details here and refer to [**GJ11**, Chapter 5] for details.

Instead, we turn to the following class of algebras, namely the *cyclotomic KLR algebras*, which are isomorphic to the corresponding cyclotomic Hecke algebras via the Brundan–Kleshchev isomorphism; see [**BK09a**, Theorem 1.1]. The key point is that cyclotomic KLR algebras are naturally graded. Hence, via this isomorphism, cyclotomic Hecke algebras—and in particular Iwahori–Hecke algebras—carry a natural grading, allowing us to study their graded structures. See Subsection 1.3.5 for details of the graded module categories used in this thesis.

Definition 1.3.1 ([**KL09**, **KL10**, **Rou08**]). *Let $\Bbbk$ be a field, and fix $\alpha \in Q^+$ such that* $\mathrm{ht}(\alpha) = n$. *The **KLR algebra** R_α of type $A^{(1)}_{e-1}$ $(e > 2)$ is the unital $\Bbbk$-algebra generated by the elements:*

(1.3.2) $$\{e(\mathbf{i}) \mid \mathbf{i} \in I^\alpha\} \cup \{y_1, \cdots, y_n\} \cup \{\psi_1, \cdots, \psi_{n-1}\}$$

subject only to the following relations:

(1.3.3) $$e(\mathbf{i})e(\mathbf{j}) = \delta_{\mathbf{i},\mathbf{j}}e(\mathbf{i}), \sum_{\mathbf{i}\in I^\alpha} e(\mathbf{i}) = 1$$

(1.3.4) $$y_re(\mathbf{i}) = e(\mathbf{i})y_r, \psi_re(\mathbf{i}) = e(\sigma_r\mathbf{i})\psi_r$$

(1.3.5) $$y_ry_s = y_sy_r$$

(1.3.6) $$\psi_ry_s = y_s\psi_r \quad \textit{if } s \neq r, r+1$$

(1.3.7) $$\psi_r\psi_s = \psi_s\psi_r \quad \textit{if } |r-s| > 1$$

(1.3.8) $$\psi_ry_{r+1}e(\mathbf{i}) = (y_r\psi_r + \delta_{\mathbf{i}_r,\mathbf{i}_{r+1}})e(\mathbf{i})$$

(1.3.9) $$y_{r+1}\psi_re(\mathbf{i}) = (\psi_ry_r + \delta_{\mathbf{i}_r,\mathbf{i}_{r+1}})e(\mathbf{i})$$

(1.3.10) $$\psi_r^2e(\mathbf{i}) = Q_{\mathbf{i}_r,\mathbf{i}_{r+1}}(y_r, y_{r+1})e(\mathbf{i})$$

(1.3.11) $$\psi_r\psi_{r+1}\psi_re(\mathbf{i}) = \psi_{r+1}\psi_r\psi_{r+1}e(\mathbf{i}) + Q_{\mathbf{i}_r,\mathbf{i}_{r+1},\mathbf{i}_{r+2}}(y_r, y_{r+1}, y_{r+2})e(\mathbf{i})$$

where

$$Q_{\mathbf{i}_r,\mathbf{i}_{r+1}}(y_r,y_{r+1}) = \begin{cases} 0 & \textit{if } \mathbf{i}_r = \mathbf{i}_{r+1} \\ 1 & \textit{if } \mathbf{i}_{r+1} \neq \mathbf{i}_r, \mathbf{i}_r \pm 1 \\ y_{r+1} - y_r & \textit{if } \mathbf{i}_r \to \mathbf{i}_{r+1} \\ y_r - y_{r+1} & \textit{if } \mathbf{i}_r \leftarrow \mathbf{i}_{r+1} \end{cases}$$

and

$$Q_{\mathbf{i}_r,\mathbf{i}_{r+1},\mathbf{i}_{r+2}}(y_r,y_{r+1},y_{r+2}) = \begin{cases} 1 & \textit{if } \mathbf{i}_r = \mathbf{i}_{r+2} \to \mathbf{i}_{r+1} \\ -1 & \textit{if } \mathbf{i}_r = \mathbf{i}_{r+2} \leftarrow \mathbf{i}_{r+1} \\ 0 & \textit{else} \end{cases}$$

Given any dominant weight $\Lambda \in P^+$, the corresponding *cyclotomic KLR algebra* R_α^Λ is generated by the same elements (1.3.2) subject only to the above relations with the additional cyclotomic relations

$$y_1^{(\Lambda|\alpha_{\mathbf{i}_1})} e(\mathbf{i}) = 0 \qquad \text{for all } \mathbf{i} = (\mathbf{i}_1, \dots, \mathbf{i}_n) \in I^\alpha. \tag{1.3.12}$$

The (cyclotomic) KLR algebras are also called (cyclotomic) *quiver Hecke algebras*. Most importantly, R_α and R_α^Λ have $\mathbb{Z}$-gradings determined by

$$\deg e(\mathbf{i}) = 0, \qquad \deg\big(y_r e(\mathbf{i})\big) = 2, \qquad \deg\big(\psi_r e(\mathbf{i})\big) = -a_{\mathbf{i}_r,\mathbf{i}_{r+1}}.$$

for all admissible r and $\mathbf{i} \in I^\alpha$.

Take $w \in \mathfrak{S}_n$ and a reduced expression $w = \prod_{1\le j\le m} \sigma_{i_j}$, and define $\psi_w := \psi_{i_1} \cdots \psi_{i_m}$. However, this definition depends on the choice of reduced expression for w, since (1.3.11) breaks the usual braid relation. We therefore fix, for each $w \in \mathfrak{S}_n$, a choice of reduced expression, and define ψ_w with respect to that choice.

Definition 1.3.13. *Fix quiver type $A_{e-1}^{(1)}$ with vertex set I, and let $\Lambda \in P^+$ and $\alpha \in Q^+$. Let $\boldsymbol{\kappa} = (\kappa_1, \cdots, \kappa_\ell) \in I^\ell$ be a charge of Λ. Take $\boldsymbol{\lambda} \in \mathscr{P}_\alpha^{\boldsymbol{\kappa}}$ and $T \in \mathrm{RStd}(\boldsymbol{\lambda})$ a row-standard $\boldsymbol{\lambda}$-tableau. Define $\psi^T := \psi_{w^T}$ for the fixed choice of reduced expression of w^T, where w^T is defined in (1.2.14).*

The degree of standard tableaux defined in Subsection 1.2.7 is compatible with the grading of KLR algebras in the following sense:

Lemma 1.3.14 ([**BKW11**, Corollary 3.14]). *With the same setting as in Definition 1.3.13, let $T \in \mathrm{Std}(\boldsymbol{\lambda})$ be a standard $\boldsymbol{\lambda}$-tableau, then*

$$\deg\big(\psi^T e(\mathbf{i}^{\boldsymbol{\lambda}})\big) = \deg T - \deg T^{\boldsymbol{\lambda}}$$

Remark 1.3.15. *The definition of the degree of standard tableaux extends to row-standard tableaux by Lemma 1.3.14.* ◇

We define the KLR algebra of rank n to be

$$R_n := \bigoplus_{\alpha \in Q_n^+} R_\alpha, \qquad \text{where } Q_n^+ = \{\alpha \in Q^+ \mid \mathrm{ht}(\alpha) = n\}.$$

Theorem 1.3.16 ([KL09, Theorem 2.5]). *The algebra R_n is free as a $\Bbbk$-module with basis $\{\psi_w y_1^{m_1} \dots y_n^{m_n} 1_{\mathbf{i}} \mid w \in \mathfrak{S}_n, m_1, \dots, m_n \in \mathbb{Z}_{\geq 0}, \mathbf{i} \in I^n\}$.*

Fix a charge $\boldsymbol{\kappa}$ of Λ. For each ℓ-partition $\boldsymbol{\lambda} \in \mathscr{P}_\alpha^{\boldsymbol{\kappa}}$, define the idempotent $e_{\boldsymbol{\lambda}}$ by

$$e_{\boldsymbol{\lambda}} := e(\mathbf{i}^{\boldsymbol{\lambda}}) \in R_\alpha \tag{1.3.17}$$

By definition, the elements $\{e(\mathbf{i}) \mid \mathbf{i} \in I^\alpha\}$ are pairwise orthogonal idempotents, and the KLR algebra R_α is unital with identity element

$$1_\alpha \ := \sum_{\mathbf{i} \in I^\alpha} e(\mathbf{i}).$$

In particular, for any R_α-module M one has the decomposition as vector spaces:

$$M = \bigoplus_{\mathbf{i} \in I^\alpha} e(\mathbf{i}) M.$$

1.3.5. Module category. In this thesis, we work with $\mathbb{Z}$-graded modules and homogeneous homomorphisms (of degree 0).

Fix a field $\Bbbk$ and let $A = \bigoplus_{a \in \mathbb{Z}} A_a$ be a $\mathbb{Z}$-graded $\Bbbk$-algebra. A left A-module M is *$\mathbb{Z}$-graded* if it decomposes as $M = \bigoplus_{d \in \mathbb{Z}} M_d$ and, for every homogeneous $x \in A_a$, one has

$$x\, M_d \subseteq M_{a+d} \qquad (a, d \in \mathbb{Z}).$$

For $k \in \mathbb{Z}$, a homomorphism $f : M \to N$ of graded A-modules is *homogeneous of degree k* if

$$f(M_d) \subseteq N_{d+k} \qquad (d \in \mathbb{Z}).$$

If $M = \bigoplus_{d \in \mathbb{Z}} M_d$ is a graded A-module and $k \in \mathbb{Z}$, the *grading shift* $M\langle k\rangle$ is the graded A-module with

$$(M\langle k\rangle)_d := M_{d-k} \qquad (d \in \mathbb{Z}).$$

Equivalently, $\deg_{M\langle k\rangle}(m) = \deg_M(m) + k$ for every homogeneous $m \in M$.

Let A be a finite-dimensional graded algebra and let M be a finite-dimensional graded A-module. For a simple graded A-module N, let

$$[M : N]$$

denote the multiplicity of N as a composition factor of M in a graded composition series.

Specializing to $A = R_\alpha$, we write R_α-mod for the category of finite-dimensional graded left R_α-modules with homogeneous homomorphisms of degree 0.

Let $\alpha, \beta \in Q^+$. Set $R_{\alpha,\beta} := R_\alpha \otimes_{\Bbbk} R_\beta$, viewed as a graded algebra in the usual way. If M is a graded R_α-module and N is a graded R_β-module, their *outer tensor product* is

$$M \boxtimes N := M \otimes_{\Bbbk} N,$$

which is a graded $R_{\alpha,\beta}$-module via

$$(x \otimes y)(m \otimes n) = (xm) \otimes (yn) \qquad \text{for } x \in R_\alpha, y \in R_\beta, m \in M, n \in N.$$

There is an injective homogeneous (non-unital) algebra homomorphism

$$\iota_{\alpha,\beta} : R_{\alpha,\beta} \hookrightarrow R_{\alpha+\beta} \qquad e(\mathbf{i}) \otimes e(\mathbf{j}) \longmapsto e(\mathbf{ij}),$$

where $\mathbf{ij}$ denotes concatenation of sequences. The image of the identity element of $R_{\alpha,\beta}$ is the idempotent

$$e_{\alpha,\beta} := \iota_{\alpha,\beta}(1_\alpha \otimes 1_\beta) = \sum_{\mathbf{i} \in I^\alpha,\ \mathbf{j} \in I^\beta} e(\mathbf{ij}) \in R_{\alpha+\beta}.$$

Definition 1.3.18. *Define induction and restriction functors by*

$$\mathrm{Ind}_{\alpha,\beta}^{\alpha+\beta} := R_{\alpha+\beta} e_{\alpha,\beta} \otimes_{R_{\alpha,\beta}} - \ : \ R_{\alpha,\beta}\text{-mod} \to R_{\alpha+\beta}\text{-mod},$$

$$\mathrm{Res}_{\alpha,\beta}^{\alpha+\beta} := e_{\alpha,\beta} R_{\alpha+\beta} \otimes_{R_{\alpha+\beta}} - \ : \ R_{\alpha+\beta}\text{-mod} \to R_{\alpha,\beta}\text{-mod}.$$

Remark 1.3.19. *For any graded $R_{\alpha+\beta}$-module L, multiplication induces a canonical isomorphism*

$$e_{\alpha,\beta} R_{\alpha+\beta} \otimes_{R_{\alpha+\beta}} L \xrightarrow{\sim} e_{\alpha,\beta} L, \qquad x \otimes v \mapsto xv.$$

Hence $\mathrm{Res}_{\alpha,\beta}^{\alpha+\beta}$ can be interpreted as left multiplication by the idempotent $e_{\alpha,\beta}$. In particular, $\mathrm{Res}_{\alpha,\beta}^{\alpha+\beta}$ is exact and sends finite-dimensional modules to finite-dimensional modules. Moreover, $\mathrm{Ind}_{\alpha,\beta}^{\alpha+\beta}$ is also exact, by [KL09, Proposition 2.16]. ◇

These constructions have obvious generalizations to $n \geq 2$ factors. Given $\beta_1, \ldots, \beta_n \in Q^+$, set

$$R_{\beta_1,\ldots,\beta_n} := R_{\beta_1} \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} R_{\beta_n}, \qquad \beta := \beta_1 + \cdots + \beta_n.$$

Under the injective degree 0 homogeneous (non-unital) map $R_{\beta_1,\ldots,\beta_n} \hookrightarrow R_\beta$, the image of $1_{\beta_1} \otimes \cdots \otimes 1_{\beta_n}$ is the idempotent

$$e_{\beta_1,\ldots,\beta_n} := \sum_{\mathbf{i}^{(1)} \in I^{\beta_1}, \ldots, \mathbf{i}^{(n)} \in I^{\beta_n}} e\big(\mathbf{i}^{(1)} \cdots \mathbf{i}^{(n)}\big) \in R_\beta.$$

Define

$$\mathrm{Ind}_{\beta_1,\ldots,\beta_n}^{\beta} := R_\beta e_{\beta_1,\ldots,\beta_n} \otimes_{R_{\beta_1,\ldots,\beta_n}} - \ : \ R_{\beta_1,\ldots,\beta_n}\text{-mod} \to R_\beta\text{-mod},$$

$$\mathrm{Res}_{\beta_1,\ldots,\beta_n}^{\beta} := e_{\beta_1,\ldots,\beta_n} R_\beta \otimes_{R_\beta} - \ : \ R_\beta\text{-mod} \to R_{\beta_1,\ldots,\beta_n}\text{-mod}.$$

Finally, if $M_a \in R_{\beta_a}$-mod for $a = 1, \ldots, n$, we write the induced product as

$$M_1 \circ \cdots \circ M_n := \mathrm{Ind}_{\beta_1,\ldots,\beta_n}^{\beta}(M_1 \boxtimes \cdots \boxtimes M_n).$$

1.3.6. Cellular structure. It is well known that the Iwahori–Hecke algebras are cellular algebras in the sense of [**GL96**], and admit several different cellular bases; see [**Mat99**, Chapter 2] for the general theory of cellular algebras and [**Mat99**, Chapter 3] for the construction of cellular bases for Iwahori–Hecke algebras.

Since cyclotomic KLR algebras are graded generalizations of the Iwahori–Hecke algebras, one expects cyclotomic KLR algebras to admit homogeneous cellular bases compatible with the grading. The first such graded cellular basis was given in [**HM10**]. Later, in [**Bow21**], a family of diagrammatic cellular bases was constructed using the KLRW algebras introduced by Webster; see [**Web19**, **Web17**].

In this thesis, the explicit construction of (graded) cellular bases does not play a central role, so we instead record some important properties and refer to the references above for further details.

We first consider the Iwahori–Hecke algebra $\mathcal{H}_n = \mathcal{H}_{\Bbbk,q}(\mathfrak{S}_n)$, where q is an eth root of unity. A partition $\lambda = (\lambda_1, \cdots, \lambda_r)$ is called *e-regular* if it has no e equal parts. A partition is called *e-restricted* if its conjugate λ' is e-regular. For any partition $\lambda \in \mathscr{P}_n$, one can construct the (dual) Specht module S^λ, which coincides with the cell module coming from a suitable cellular basis of the algebra. In particular, as a standard consequence of cellular theory, the set $\{D^\lambda \mid \lambda \in \mathscr{P}_n \text{ is } e\text{-regular}\}$, where D^λ is the head of S^λ, is a complete and irredundant list of simple modules of $\mathcal{H}_n$. We note that the cellular basis depends on the choice of partial order on partitions: by choosing a different order, one can replace e-regular partitions of n by e-restricted partitions, which is the convention in [**Mat99**]. The simple modules D^λ are also called *James modules* by some authors.

Returning to the cyclotomic KLR algebras of type $A^{(1)}_{e-1}$, let R^Λ_α with $\mathrm{ht}(\alpha) = n$. Let $I = \{0, 1, \cdots, e-1\}$ be the vertex set of the quiver and fix a charge $\boldsymbol{\kappa}$ of Λ. For each multipartition $\boldsymbol{\lambda} \in \mathscr{P}^{\boldsymbol{\kappa}}_\alpha$, one can construct the Specht module $S^{\boldsymbol{\lambda}}$ in several ways: (1) via graded cellular bases as in [**HM10**]; (2) as a graded lift of Specht modules for cyclotomic Hecke algebras, see [**BKW11**]; (3) via the highest-weight presentation in [**KMR12**]. We will give a short but detailed account of the third approach in Subsection 1.3.8, since it plays an essential role in this thesis.

As in the level-1 case (the Iwahori–Hecke algebra), a subset of multipartitions, called Kleshchev partitions and admitting a purely combinatorial description, indexes a complete and irredundant set of simple graded R^Λ_α-modules. We refer to [**HM10**] for related results and to [**Mat14**] for a good survey of the representation theory of cyclotomic KLR algebras of type $A^{(1)}_{e-1}$.

1.3.7. Graded decomposition numbers. Let $\Bbbk$ be a field. Fix $e \in \{2, 3, \dots\} \cup \{\infty\}$, let $I = \{0, 1, \dots, e-1\}$ (or $I = \mathbb{Z}$ if $e = \infty$), and let R^Λ_α be the cyclotomic KLR algebra of type $A^{(1)}_{e-1}$ (or A_∞) with $\Lambda \in P^+$ and $\alpha \in Q^+$. Choose a charge $\boldsymbol{\kappa}$ of Λ. For each multipartition $\boldsymbol{\lambda} \in \mathscr{P}^{\boldsymbol{\kappa}}_\alpha$ there is a graded Specht (cell) module $S^{\boldsymbol{\lambda}}$, and for each Kleshchev multipartition $\boldsymbol{\mu} \in \mathscr{P}^{\boldsymbol{\kappa}}_\alpha$ there is a graded simple module $D^{\boldsymbol{\mu}}$, as discussed in Subsection 1.3.6.

Definition 1.3.20. *For $\boldsymbol{\lambda} \in \mathscr{P}^{\kappa}_{\alpha}$ and Kleshchev $\boldsymbol{\mu} \in \mathscr{P}^{\kappa}_{\alpha}$, the* graded decomposition number *is the Laurent polynomial*

$$d^{e}_{\boldsymbol{\lambda}\boldsymbol{\mu}}(v) \;=\; \sum_{d\in\mathbb{Z}} \big[S^{\boldsymbol{\lambda}} : D^{\boldsymbol{\mu}}\langle d\rangle\big]\, v^{d} \;\in\; \mathbb{Z}[v,v^{-1}],$$

where $\langle d\rangle$ denotes the grading shift. Evaluating at $v = 1$ recovers the ordinary (ungraded) decomposition number:

$$d^{e}_{\boldsymbol{\lambda}\boldsymbol{\mu}}(1) = \big[S^{\boldsymbol{\lambda}} : D^{\boldsymbol{\mu}}\big].$$

We refer to [**BK09b**, **BKW11**] for graded Specht modules and graded decomposition numbers in this generality.

When $\ell = 1$ (so multipartitions are partitions) and $q \in \Bbbk^{\times}$ has finite multiplicative order e, the cyclotomic KLR algebra R^{Λ}_{α} (with $|\alpha| = n$) identifies with the Iwahori–Hecke algebra $\mathcal{H}_n = \mathcal{H}_{\Bbbk,q}(\mathfrak{S}_n)$ via the Brundan–Kleshchev isomorphism. Under this identification, S^{λ} is the usual Specht module and the simples D^{μ} are indexed by e-regular partitions μ.[3] To emphasize the graded structure, let $\widetilde{S}^{\lambda}$ and $\widetilde{D}^{\mu}$ be the corresponding graded lifts of S^{λ} and D^{μ}. Hence

$$d^{e}_{\lambda\mu}(v) = \sum_{d\in\mathbb{Z}} \big[\widetilde{S}^{\lambda} : \widetilde{D}^{\mu}\langle d\rangle\big]\, v^{d}, \qquad \text{with} \qquad d^{e}_{\lambda\mu}(1) = \big[S^{\lambda} : D^{\mu}\big].$$

For the ungraded theory, see [**Mat99**, Section 2.2 & Chapter 6].

We say that two partitions lie in the same *block* if they have the same e-core and e-weight; see Subsection 1.2.4. It is a classical result that for $\mathcal{H}_n$ one has $d^{e}_{\lambda\mu}(v) = 0$ unless λ and μ lie in the same block; see [**Mat99**, Corollary 5.38].

1.3.8. Universal Specht module.

In this section we recall the universal graded (row) Specht modules following [**KMR12**]. The key point is that the permutation modules are naturally defined *by induction* from one-dimensional segment modules ([**KMR12**, §3.6]), and then identified with a cyclic (highest-weight) presentation ([**KMR12**, §5.3–§5.4]). We then impose the homogeneous Garnir relations to obtain the universal Specht module.

Fix the quiver of type A_{∞} (i.e. $e = \infty$) or $A^{(1)}_{e-1}$ (i.e. $2 \le e < \infty$), with vertex set $I = \mathbb{Z}$ or $I = \mathbb{Z}/e\mathbb{Z}$, respectively. Fix $\Lambda \in P^{+}$ and a charge $\boldsymbol{\kappa}$ of Λ. Fix $\alpha \in Q^{+}$ and set $d = \mathrm{ht}(\alpha)$. Let R_{α} and R^{Λ}_{α} be the KLR algebra and its cyclotomic quotient, respectively. For $\boldsymbol{\lambda} \in \mathscr{P}^{\kappa}_{\alpha}$, let $T^{\boldsymbol{\lambda}}$ be the initial tableau and $\mathbf{i}^{\boldsymbol{\lambda}} := \mathbf{i}^{T^{\boldsymbol{\lambda}}} \in I^{d}$ its residue sequence.

1.3.8.1. *Permutation modules.* For $i \in I$ and $N \in \mathbb{Z}_{>0}$, define the *segment*

$$s(i,N) := (i, i+1, \dots, i+N-1) \in I^{N},$$

---

[3] As discussed in Subsection 1.3.6, the subset of (multi)partitions that index the simple modules depends on the choice of partial order. In particular, in the level one case, we always choose the partial order such that the set of e-regular partitions indexes the simple modules.

where entries are taken modulo e if $e > 0$. Let $\alpha_{i,N} \in Q^+$ be the positive root associated to this segment, that is, $\alpha_{i,N} = \alpha\big(s(i,N)\big)$; see (1.1.2).

Let $s = s(i,N)$ and set $\alpha_s = \alpha_{i,N}$. The *segment module* $M(s)$ is the rank-one graded R_{α_s}-module generated by $m(s)$ in degree 0, with action

$$e(\mathbf{j})m(s) = \delta_{\mathbf{j},s}\, m(s), \qquad y_r m(s) = 0, \qquad \psi_r m(s) = 0,$$

for all admissible $\mathbf{j}$ and r.

Let $\boldsymbol{s} = (s(1), \dots, s(n))$ be an ordered tuple of segments. Write α_r for the positive root associated to $s(r)$ and set $\alpha := \alpha_1 + \cdots + \alpha_n$. Define

$$M(\boldsymbol{s}) := M(s(1)) \circ \cdots \circ M(s(n)) := \operatorname{Ind}^{R_\alpha}_{R_{\alpha_1} \otimes \cdots \otimes R_{\alpha_n}} \big(M(s(1)) \boxtimes \cdots \boxtimes M(s(n))\big),$$

with cyclic generator

$$m(\boldsymbol{s}) := 1 \otimes m(s(1)) \otimes \cdots \otimes m(s(n)).$$

Let $j(\boldsymbol{s}) := s(1) \cdots s(n) \in I^d$ be the concatenation of the segments.

Now fix $\boldsymbol{\lambda} \in \mathscr{P}^{\boldsymbol{\kappa}}_\alpha$. List the nonempty rows of $\boldsymbol{\lambda}$ as $R_1, \dots, R_g$ (from the first component to the ℓth component, and within each component from top to bottom). If R_a has length N_a and the leftmost node of R_a has residue i_a, set

$$r(a) := s(i_a, N_a).$$

Define the associated row tuple

$$\boldsymbol{r}(\boldsymbol{\lambda}) := (r(1), \dots, r(g)).$$

Definition 1.3.21. *The* row permutation module *is defined by*

$$M^{\boldsymbol{\lambda}} := M\big(\boldsymbol{r}(\boldsymbol{\lambda})\big) \langle \deg(T^{\boldsymbol{\lambda}}) \rangle, \tag{1.3.22}$$

and we write $m^{\boldsymbol{\lambda}}$ for its (shifted) cyclic generator.

1.3.8.2. *Homogeneous Garnir relations.* Fix a Garnir node $A \in [\boldsymbol{\lambda}]$. Let $\mathcal{B}^A$ be the Garnir belt of A, decomposed into A-bricks as in Subsection 1.2.8. Let $k = k^A$ be the number of bricks in $\mathcal{B}^A$, and let $\mathfrak{S}^A \cong \mathfrak{S}_k$ be the brick permutation group, with Coxeter generators $w^A_1, \dots, w^A_{k-1}$ defined as (1.2.24) swapping adjacent bricks. Let $f = f^A$ be the number of bricks lying in the first-row part of $\mathcal{B}^A$, so that $\mathfrak{S}_f \times \mathfrak{S}_{k-f} \le \mathfrak{S}^A$. Let $\mathscr{D}^A$ be the set of minimal-length left coset representatives of $\mathfrak{S}_f \times \mathfrak{S}_{k-f}$ in $\mathfrak{S}^A$.

Let T^A be the unique maximal row-standard tableau in the Garnir set Gar^A. Set

$$\mathbf{i}^A := \mathbf{i}^{T^A} \in I^d, \qquad m^A := \psi^{T^A} m^{\boldsymbol{\lambda}} \in M^{\boldsymbol{\lambda}}.$$

Following [KMR12, §5.4], we define the brick operators as follows.

Definition 1.3.23. *For $1 \le r \le k-1$, set*

$$\sigma^A_r := \psi_{w^A_r}\, e(\mathbf{i}^A), \qquad \tau^A_r := (\sigma^A_r + 1)\, e(\mathbf{i}^A).$$

If $u \in \mathfrak{S}^A$ *and* $u = w^A_{r_1} \cdots w^A_{r_a}$ *is a chosen reduced expression, define*

$$\sigma^A_u := \sigma^A_{r_1} \cdots \sigma^A_{r_a}, \qquad \tau^A_u := \tau^A_{r_1} \cdots \tau^A_{r_a}.$$

Lemma 1.3.24 ([**KMR12**, §5.4]). *For every* $u \in \mathscr{D}^A$*, the element* τ^A_u *is independent of the choice of reduced expression of* u*.*

Definition 1.3.25. *The homogeneous Garnir element associated to* A *is*

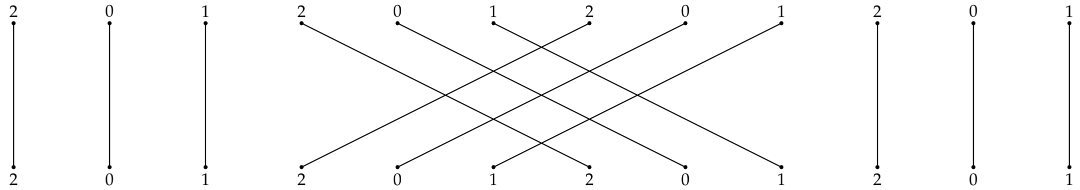
$$g^A := \sum_{u \in \mathscr{D}^A} \tau^A_u \, \psi^{T^A} \in R_\alpha.$$

Equivalently, the corresponding ***Garnir relation*** *in* M^λ *may be written as*

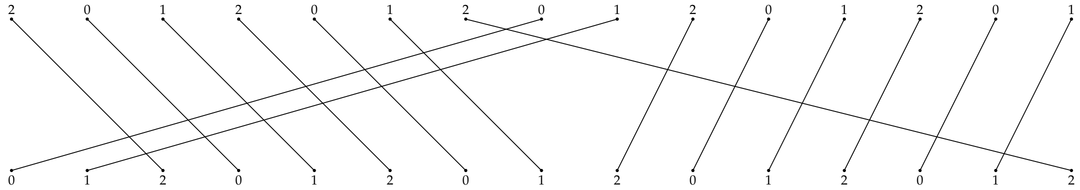
$$g^A m^\lambda = \sum_{u \in \mathscr{D}^A} \tau^A_u \, m^A.$$

The KLR algebras admit a diagrammatic description; see [**KL09**, Section 2]. We briefly introduce this in Subsection 4.1.5. We end this section with an example using this diagrammatic description to describe σ^A_r and $\psi_{w^{T^A}}$. The multiplication is from the bottom to the top.

Example 1.3.26. *Continue with Example 1.2.26,* $\sigma^A_2 := \psi_{w^A_r} e(\mathbf{i}^A)$ *is the following string diagram:*



These strands correspond to the entries from 26 *to* 37 *in* T^λ*, while all undrawn strands are taken to be vertical straight strands. Similarly,* $\psi_{w^{T^A}}$ *is as follows:*



These strands correspond to the entries from 24 *to* 38 *in* T^λ *and* T^A*, while, as before, all undrawn strands are taken to be vertical straight strands.* ◇

Remark 1.3.27. *If* $\mathscr{D}^A = \{1\}$*, then* $g^A := \psi^{G^A}$*, so the Garnir relation reduces to the "trivial" relation* $\psi^{G^A} z^\lambda = 0$*. In particular, for partitions of hook shape in type* $A^{(1)}_{e-1}$ *and arbitrary partitions in type* A_∞*, any Garnir node has the trivial Garnir relation.* ◇

1.3.8.3. *Universal Specht modules.* We define the universal graded Specht modules via a highest-weight-module presentation:

Definition 1.3.28. *Let* $\boldsymbol{\lambda} \in \mathscr{P}^{\boldsymbol{\kappa}}_\alpha$ *and* $d = \mathrm{ht}(\alpha)$*. The* universal graded (row) Specht module S^λ *is the graded* R_α*-module generated by* z^λ*, which is homogeneous of degree* $\deg(T^\lambda)$*, subject to the relations*

(a) $e(\mathbf{j})z^{\boldsymbol{\lambda}} = \delta_{\mathbf{j},\mathbf{i}^{\boldsymbol{\lambda}}}\, z^{\boldsymbol{\lambda}}$ *for all* $\mathbf{j} \in I^{\alpha}$;
(b) $y_r z^{\boldsymbol{\lambda}} = 0$ *for* $r = 1, \dots, d$;
(c) $\psi_r z^{\boldsymbol{\lambda}} = 0$ *whenever* $r \longrightarrow_{T^{\boldsymbol{\lambda}}} r+1$;
(d) $g^A z^{\boldsymbol{\lambda}} = 0$ *for every Garnir node* $A \in [\boldsymbol{\lambda}]$.

Remark 1.3.29. *Let M be the cyclic module generated by $m^{\boldsymbol{\lambda}}$ subject only to the relations (1)–(3) in Definition 1.3.28. Then $M \cong M^{\boldsymbol{\lambda}}$ as graded R_α-modules and there is a natural surjection $M^{\boldsymbol{\lambda}} \twoheadrightarrow S^{\boldsymbol{\lambda}}$ sending $m^{\boldsymbol{\lambda}} \mapsto z^{\boldsymbol{\lambda}}$, and*

$$S^{\boldsymbol{\lambda}} \;\cong\; M^{\boldsymbol{\lambda}} \Big/ \Big\langle\, g^A m^{\boldsymbol{\lambda}} \;\Big|\; \forall A \in [\boldsymbol{\lambda}] \text{ is a Garnir node} \Big\rangle.$$

◇

In later sections, we call $m^{\boldsymbol{\lambda}}$ and $z^{\boldsymbol{\lambda}}$ the *standard cyclic generators* of the modules $M^{\boldsymbol{\lambda}}$ and $S^{\boldsymbol{\lambda}}$, respectively.

1.3.8.4. *Bases.* For completeness, we record the standard basis results.

Theorem 1.3.30 ([**KMR12**, Theorem 5.6]). *The permutation module $M^{\boldsymbol{\lambda}}$ has a $\Bbbk$-basis*

$$\{\, \psi^T m^{\boldsymbol{\lambda}} \mid T \in \mathrm{RStd}(\boldsymbol{\lambda}) \,\}$$

Corollary 1.3.31. *Let ℓ, ℓ' be two positive integers, and take two charges $\boldsymbol{\kappa} \in I^{\ell}$ and $\boldsymbol{\kappa}' \in I^{\ell'}$. Let $\boldsymbol{\lambda} \in \mathscr{P}^{\boldsymbol{\kappa}}_{\alpha}$ and $\boldsymbol{\mu} \in \mathscr{P}^{\boldsymbol{\kappa}'}_{\alpha}$. Assume that $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ have the same ordered list of row segments (i.e. $\boldsymbol{r}(\boldsymbol{\lambda}) = \boldsymbol{r}(\boldsymbol{\mu})$). Then there is a canonical graded R_α-module isomorphism*

$$\Phi :\; M^{\boldsymbol{\lambda}} \xrightarrow{\;\sim\;} M^{\boldsymbol{\mu}}, \qquad \Phi(r\, m^{\boldsymbol{\lambda}}) = r\, m^{\boldsymbol{\mu}} \;\; (r \in R_\alpha).$$

In particular, splitting any component into several consecutive components by cutting off rows (and preserving the row order) does not change the isomorphism class of the corresponding permutation module.

The following result shows that the highest-weight presentation Definition 1.3.28 indeed defines the graded Specht module.

Theorem 1.3.32 ([**KMR12**, Corollary 6.24]). *There is a homogeneous degree 0 isomorphism between $S^{\boldsymbol{\lambda}}$ and the graded cell module of R^{Λ}_{α} constructed in* [**HM10**]. *Moreover, $S^{\boldsymbol{\lambda}}$ has a $\Bbbk$-basis $\{\, \psi^T z^{\boldsymbol{\lambda}} \mid T \in \mathrm{Std}(\boldsymbol{\lambda}) \,\}$.*

Theorem 1.3.33. *Suppose that $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(\ell)}) \in \mathscr{P}^{\boldsymbol{\kappa}}_{\alpha}$. Then*

$$S^{\boldsymbol{\lambda}} \;\cong\; S^{\lambda^{(1)}} \circ \cdots \circ S^{\lambda^{(\ell)}} \langle d_{\boldsymbol{\lambda}} \rangle, \tag{1.3.34}$$

where

$$d_{\boldsymbol{\lambda}} \;:=\; \deg(T^{\boldsymbol{\lambda}}) - \deg(T^{\lambda^{(1)}}) - \cdots - \deg(T^{\lambda^{(\ell)}}).$$

as graded R_α-modules. In particular, $S^{\lambda^{(1)}} \circ \cdots \circ S^{\lambda^{(\ell)}} \langle d_{\boldsymbol{\lambda}} \rangle$ factors through the surjection $R_\alpha \twoheadrightarrow R^{\Lambda}_{\alpha}$, and the isomorphism (1.3.34) is also an isomorphism of graded R^{Λ}_{α}-modules.

1.4. Alcove geometry

The alcove geometry will be needed in Chapter 3. In this section, we briefly recall the definitions, specialized to type $A_{e-1}^{(1)}$.

1.4.1. Affine hyperplanes and alcoves. Follow the notation for type A_{r-1} from Subsection 1.1.1. Let Q be the root lattice embedded in the Euclidean space E. The lattice Q acts on E by translations.

The *(finite) Weyl group* of type A_{r-1}, denoted by W_0, is the symmetric group $\mathfrak{S}_r$, acting on E by permuting the coordinates.

The *affine Weyl group* is

$$W \;=\; W_0 \ltimes Q,$$

The affine Weyl group is a Coxeter group with generators $s_0, s_1, \cdots, s_{r-1}$ where $\langle s_1, \cdots, s_{r-1}\rangle \cong W_0$; See [**Hum90**, Chapter 4] for more details.

For $\alpha \in R$ and $k \in \mathbb{Z}$, define the *affine hyperplane*

$$H_{\alpha,k} = \{x \in E \mid \langle \alpha^\vee, x\rangle = k\}.$$

This hyperplane separates E into two open half-spaces: the *positive side* $H^+_{\alpha,k}$ and the *negative side* $H^-_{\alpha,k}$, defined by:

$$\begin{aligned} H^+_{\alpha,k} &= \{x \in E \mid \langle \alpha^\vee, x\rangle > k\},\\ H^-_{\alpha,k} &= \{x \in E \mid \langle \alpha^\vee, x\rangle < k\}. \end{aligned}$$

Let $\mathcal{H} = \{H_{\alpha,k} \mid \alpha \in R^+, k \in \mathbb{Z}\}$ be the set of those affine hyperplanes. The *alcoves* are the connected components of the complement of the union of these hyperplanes:

$$E \setminus \bigcup_{H \in \mathcal{H}} H.$$

A *facet* of an alcove $\mathcal{A}$ is a codimension-one face of its closure $\overline{\mathcal{A}}$. Geometrically, it is the intersection of $\overline{\mathcal{A}}$ with a single hyperplane $H \in \mathcal{H}$ that forms a boundary of the alcove.

Let $\mathcal{R} \subseteq E$ be a connected region defined as the union of the closures of a finite collection of alcoves. The *boundary* of $\mathcal{R}$, denoted $\partial\mathcal{R}$, is the set of points $x \in \mathcal{R}$ such that every open neighborhood of x intersects both $\mathcal{R}$ and its complement $E \setminus \mathcal{R}$. Combinatorially, $\partial\mathcal{R}$ is the union of the walls that serve as a facet for exactly one alcove in the collection. The *interior* of $\mathcal{R}$ is defined as $\mathcal{R}^\circ = \mathcal{R} \setminus \partial\mathcal{R}$.

The *fundamental alcove* is

$$\mathcal{A}_0 = \left\{x \in E \,\middle|\, 0 < \langle \alpha_i^\vee, x\rangle < 1 \; (1 \le i \le r-1),\; \langle \theta^\vee, x\rangle < 1\right\},$$

where θ is the highest root defined in (1.1.1).

The *dominant chamber* is

$$C^+ = \{x \in E \mid \langle \alpha_i^\vee, x\rangle \geq 0 \text{ for all } i\}.$$

An alcove contained in the dominant chamber is called a *dominant alcove*.

We identify E with $P_{\mathbb{R}} = P \otimes_{\mathbb{Z}} \mathbb{R}$. Writing elements of W as $w = w_0 t_\gamma$ with $w_0 \in W_0$ and $\gamma \in Q$, the standard action of W on E is given by

$$w \cdot x \;=\; w_0(x) + \gamma, \qquad x \in E. \tag{1.4.1}$$

This action consists of affine isometries and acts simply transitively on the set of alcoves [**Hum90**, Theorem 4.5].

1.4.2. The level-e action. Let $e > 2$ be a positive integer. The *level-e action* of W on E is defined by dilating the translation part by a factor of e:

$$w \star_e \lambda \;=\; w_0(\lambda) + e\gamma, \qquad \lambda \in E,\ w = w_0 t_\gamma.$$

Note that the case $e = 1$ recovers the standard action in (1.4.1). This action preserves the set $\mathcal{H}^{(e)}$, which is also called *e-alcove arrangement* and consists of the e-dilated hyperplanes:

$$\mathcal{H}^{(e)} \;:=\; \{H_{\alpha,k}^{(e)} \mid \alpha \in R^+, k \in \mathbb{Z}\}, \quad \text{where } H_{\alpha,k}^{(e)} = \{x \in E \mid \langle \alpha^\vee, x\rangle = ke\}.$$

The connected components of the complement $E \setminus \bigcup \mathcal{H}^{(e)}$ are called *e–alcoves* and an e-alcove contained in the dominant chamber is called a *dominant e-alcove*. The *level-e fundamental alcove* is

$$\mathcal{A}_0^{(e)} = \left\{x \in E \,\middle|\, 0 < \langle \alpha_i^\vee, x\rangle < e \ (1 \leq i \leq r-1),\ \langle \theta^\vee, x\rangle < e\right\}.$$

The level-e action induces a simply transitive action on the set of e-alcoves.

1.4.3. Shi coefficients. A convenient way to encode the position of an alcove is via *Shi coefficients*, which were introduced in [**Shi87**]. Recall that for any real number c, the floor function $\lfloor c \rfloor$ denotes the greatest integer less than or equal to c.

For any e-alcove $\mathcal{A}$ and positive root $\alpha \in R^+$, the value $\lfloor \langle \alpha^\vee, x\rangle / e \rfloor$ is constant for all $x \in \mathcal{A}$. We define the *level-e Shi coefficient* of $\mathcal{A}$ with respect to α as:

$$k_\alpha^{(e)}(\mathcal{A}) \;:=\; \left\lfloor \frac{\langle \alpha^\vee, x\rangle}{e} \right\rfloor \qquad (x \in \mathcal{A}).$$

The collection of integers $\{k_\alpha^{(e)}(\mathcal{A}) \mid \alpha \in R^+\}$ uniquely determines the alcove $\mathcal{A}$. Specifically, two dominant weights λ, μ lying in the interiors of e-alcoves belong to the same alcove if and only if

$$\left\lfloor \frac{\langle \alpha^\vee, \lambda\rangle}{e} \right\rfloor \;=\; \left\lfloor \frac{\langle \alpha^\vee, \mu\rangle}{e} \right\rfloor \quad \text{for all } \alpha \in R^+. \tag{1.4.2}$$

1.4.4. Reindexing the anti-spherical KL polynomials. Recall from Subsection 1.3.2 that the anti-spherical Kazhdan–Lusztig polynomials $n_{y,x}(v)$ are indexed by elements

$x, y \in {}^I W$. The map

$$x \longmapsto \mathcal{A}_x := x \star_e \mathcal{A}_0^{(e)}$$

defines a bijection between the set of minimal representatives ${}^I W$ and the set of e-alcoves contained in the dominant chamber C^+; see [**Soe97**, Section 4]. Thus, we may view the polynomials $n_{y,x}(v)$ as being indexed by pairs of dominant e-alcoves via the identification $n_{\mathcal{A}_y,\mathcal{A}_x}(v) := n_{y,x}(v)$.

We further relabel these polynomials using dominant weights. Following [**GW98**, Section 5], we associate a unique dominant e-alcove $\mathcal{A}(\lambda)$ to every dominant weight $\lambda \in P^+$ as follows:

- If λ lies in the interior of a dominant e–alcove $\mathcal{A}$ (i.e., $\lambda \in \mathcal{A}$), set $\mathcal{A}(\lambda) = \mathcal{A}$.
- If λ lies on some hyperplanes $H_{\alpha,k}^{(e)}$, let $\mathcal{A}(\lambda)$ be the unique dominant e–alcove such that λ lies in its closure $\overline{\mathcal{A}(\lambda)}$ and λ lies on the *positive side* of every hyperplane separating $\mathcal{A}(\lambda)$ from the origin.

Finally, for any pair of dominant weights $\mu, \lambda \in P^+$, we define the *level-e anti-spherical Kazhdan–Lusztig polynomial* denoted by $n_{\mu,\lambda}^e(v)$:

$$n_{\mu,\lambda}^e(v) := \begin{cases} n_{\mathcal{A}(\mu),\mathcal{A}(\lambda)}(v) & \text{if } \mu \text{ and } \lambda \text{ are in the same } W\text{-orbit under } \star_e, \\ 0 & \text{otherwise.} \end{cases}$$

This allows us to work with anti-spherical Kazhdan–Lusztig polynomials indexed directly by dominant weights. We include the superscript e to emphasize that the value depends on the level of the affine action used to define the e-alcoves. See Figure 8 and Figure 9 for examples of associating a dominant weight to the corresponding e-alcove; we have shaded the e-alcove using a lighter shade of the color used for the point representing the dominant weight.

CHAPTER 2

Generalized Specht Filtration

This chapter is a copy of [**Qin25**] with minor changes. We briefly summarize the main results. In type $A^{(1)}_{e-1}$, for a partition λ of hook shape, we construct a Specht filtration of the permutation module M^λ (see Definition 1.3.21) in a very explicit way; see Theorem 2.1.2. In type A_∞, for any partition λ, we construct a generalized Specht filtration of the permutation module M^λ; see Theorem 2.3.2. A base case for the second result is the two-row case in type A_∞, where the permutation module M^λ admits a Specht filtration; see Theorem 2.2.9.

In this chapter, when the order of the components of an ℓ-partition is fixed and the leading residues are already determined as $(i_1, \cdots, i_\ell) \in I^\ell$, we always fix a charge κ satisfying $\kappa_1 > \kappa_2 > \cdots > \kappa_\ell$ and $\overline{\kappa_j} \equiv i_j \pmod{e}$ for each j. By abuse of notation, we often write $\mathscr{P}^\Lambda_\alpha := \mathscr{P}^\kappa_\alpha$ in this case.

2.1. Specht Filtration in type $A^{(1)}_{e-1}$ for Hook Partitions

In this section, we fix the quiver $A^{(1)}_{e-1}$ with $e > 2$, and take a positive root $\alpha \in Q^+$ and a dominant weight $\Lambda \in P^+$. Since our construction of the Specht filtration is component-by-component, see Theorem 1.3.33, we may assume $\Lambda = \Lambda_x$ for some $0 \leq x \leq e-1$.

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathscr{P}^\Lambda_\alpha$, we define the corresponding permutation module M^λ. In general, there could be many Specht filtrations of M^λ. For example, let $\mu_i = (\lambda_i) \in \mathscr{P}^{\Lambda_{\mathrm{res}(i,1)}}$ for each $1 \leq i \leq r$ and $\boldsymbol{\mu} = (\mu_1 | \cdots | \mu_r)$, then (see Subsection 1.3.5)

$$M^\lambda \cong M^{\mu_1} \circ \cdots \circ M^{\mu_r} \cong S^{\mu_1} \circ \cdots \circ S^{\mu_r} \cong S^{\boldsymbol{\mu}}$$

by Corollary 1.3.31 and Theorem 1.3.33. However, this approach is not natural and proves to be unhelpful for our purposes. The reason is as follows:

Although M^λ is an R_α-module, it is primarily used to construct the Specht module S^λ over the cyclotomic algebra R^Λ_α, where Λ is fixed. Hence, a meaningful Specht filtration of M^λ should be one with head S^λ. In other words, we aim to construct a Specht filtration for the submodule of M^λ generated by the Garnir relations associated to $[\lambda]$.

In general, constructing such a filtration is a difficult problem. However, for hook partitions, the situation simplifies considerably. One of the key reasons is that in this case, all Garnir relations are trivial as mentioned in Remark 1.3.27.

Throughout this section, we fix a hook partition $\lambda = (k, 1^r)$. For each $1 \le i \le r$, let $A_i = (i, 1) \in [\lambda]$. One checks easily that $\{A_i \mid 1 \le i \le r\}$ is the set of all Garnir nodes in $[\lambda]$. We set

$$\psi^{A_i} := \psi^{G^{A_i}} \quad \text{for all } 1 \le i \le r.$$

Lemma 2.1.1. *Let v be the standard cyclic generator of the permutation module M^λ over R_α, where $\alpha = \alpha_\lambda$. Then:*

$$\psi^{A_i} v = \begin{cases} \psi_1\psi_2 \cdots \psi_k e(\mathbf{i}^\lambda)v & \text{if } i = 1, \\ \psi_{k+i-1} e(\mathbf{i}^\lambda)v & \text{if } 2 \le i \le r \end{cases}$$

Theorem 2.1.2 (**Specht Filtration**). *Fix the quiver $A^{(1)}_{e-1}$ $(e > 2)$ or A_∞, take $\alpha \in Q^+$, $\Lambda = \Lambda_x$ and a hook partition $\lambda = (k, 1^r)$ such that $\alpha_\lambda = \alpha$. A Specht filtration of the permutation module M^λ is given by the following chain of R_α-modules:*

$$M^\lambda = M_0 \supsetneq M_1 \supsetneq M_2 \supsetneq \cdots \supsetneq M_r \supsetneq M_{r+1} = 0$$

where, for $1 \le i \le r$, the module M_i is the submodule of M^λ generated by $\{\psi^{A_i}v, \dots, \psi^{A_r}v\}$. Moreover,

$$M_i/M_{i+1} \cong S^{\lambda_i}$$

where

$$\lambda_i = \begin{cases} \lambda = (k, 1^r) & \text{if } i = 0, \\ (k+1, 1^{r-1}) & \text{if } i = 1, \\ (k|\underbrace{1|1|\cdots|1}_{i-2 \text{ times}}|(2, 1^{r-i})) & \text{if } 2 \le i \le r, \end{cases}$$

and S^{λ_i} is the Specht module over $R^{\Lambda(i)}_\alpha$ where $\Lambda(i)$ is determined by the charge:

$$\kappa(i) = \begin{cases} x & \text{if } i = 0, \\ x - 1 & \text{if } i = 1, \\ (x, x-1, \cdots, x-i+2, x-i) & \text{if } 2 \le i \le r \end{cases}$$

Before giving the proof, we show the construction in an example.

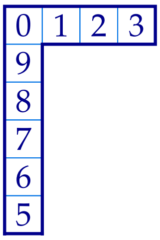
Example 2.1.3. *Take quiver $A^{(1)}_9$ with vertex set $\{0, 1, 2, \cdots, 9\}$ and $\Lambda = \Lambda_0$, consider*

$$\beta = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \alpha_9$$

and $\lambda = (4, 1^5)$.

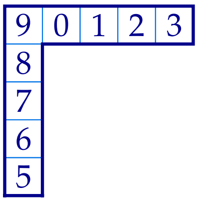
We know $\dim M^\lambda = 15120$. *The partitions that appear in the Specht filtration of M^λ are listed below, with their corresponding Young diagrams filled with residues.*

(a). $\lambda_0 = \lambda = (4, 1^5)$



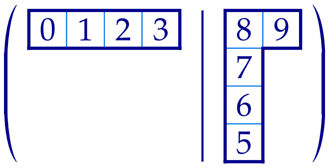
$\dim S^{\lambda_0} = 56 \qquad \Lambda(0) = \Lambda_0$

(b). $\lambda_1 = (5, 1^4)$



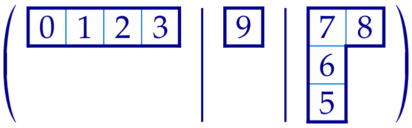
$\dim S^{\lambda_1} = 70 \qquad \Lambda(1) = \Lambda_9$

(c). $\lambda_2 = (4|2, 1^3)$



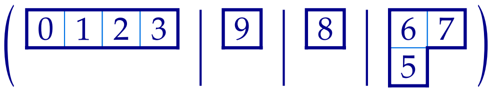
$\dim S^{\lambda_2} = 504 \qquad \Lambda(2) = \Lambda_0 + \Lambda_8$

(d). $\lambda_3 = (4|1|2, 1^2)$



$\dim S^{\lambda_3} = 1890 \qquad \Lambda(3) = \Lambda_0 + \Lambda_9 + \Lambda_7$

(e). $\lambda_4 = (4|1|1|2, 1)$



$\dim S^{\lambda_4} = 5040 \qquad \Lambda(4) = \Lambda_0 + \Lambda_9 + \Lambda_8 + \Lambda_6$

(f). $\lambda_5 = (4|1|1|1|2)$

, $\dim S^{\lambda_5} = 7560, \quad \Lambda(5) = \Lambda_0+\Lambda_9+\Lambda_8+\Lambda_7+\Lambda_5$

It is easy to verify that: $\sum\limits_{0 \le i \le 5} \dim S^{\lambda_i} = 56+70+504+1890+5040+7560 = 15120 = \dim M^{\lambda}$.

◇

The equality of dimensions plays a crucial role in our proof, so we begin by establishing this fact.

LEMMA 2.1.4. *In type $A^{(1)}_{e-1}$ or type A_∞, suppose $\lambda = (k, 1^r)$ is a hook partition, then* $\dim M^\lambda = \sum\limits_{i=0}^{r} \dim S^{\lambda_i}$.

PROOF. We use the *hook length formula* to compute the dimension of Specht modules; see, for instance, [**CSST10**, Theorem 4.2.14]. Then

$$\dim M^\lambda = \frac{(k+r)!}{k!}$$

$$\dim S^{\lambda_0} = \dim S^{\lambda} = \frac{(k+r)!}{(k+r)r!(k-1)!} = \frac{(k+r)!k}{(k+r)r!k!}$$
$$\dim S^{\lambda_1} = \frac{(k+r)!}{(k+r)k!(r-1)!} = \frac{(k+r)!r}{(k+r)k!r!}$$

For $2 \leq i \leq r-1$, it is easy to see:

$$\begin{aligned}
\dim S^{\lambda_i} &= \binom{k+r}{k}\binom{r}{1}\binom{r-1}{1}\cdots\binom{r-i+3}{1}(r-i+1)\\
&= \frac{(k+r)!}{k!r!}\frac{r(r-1)\cdots(r-i+3)(r-i+2)(r-i+1)}{r-i+2}\\
&= \frac{(k+r)!}{k!r!}\frac{r!}{(r-i)!(r-i+2)}\\
&= \frac{(k+r)!}{k!(r-i)!(r-i+2)}\\
&= \frac{(k+r)!(r-i+1)}{k!(r-i+2)!}
\end{aligned}$$

For S^{λ_r}, we know:

$$\dim S^{\lambda_r} = \frac{(k+r)!}{k!2}$$

Hence we have the following:

$$\begin{aligned}
\sum_{0\leq i\leq r} \dim S^{\lambda_i} &= \frac{(k+r)!}{k!}\left(\frac{k}{(k+r)r!} + \frac{r}{(k+r)r!} + \sum_{2\leq i\leq r-1}\frac{r-i+1}{(r-i+2)!} + \frac{1}{2}\right)\\
&= \frac{(k+r)!}{k!}\left(\frac{1}{r!} + \sum_{2\leq i\leq r-1}\frac{r-i+2}{(r-i+2)!} - \sum_{2\leq i\leq r-1}\frac{1}{(r-i+2)!} + \frac{1}{2}\right)\\
&= \frac{(k+r)!}{k!}\left(\frac{1}{r!} + \sum_{2\leq t\leq r-1}\frac{1}{t!} - \sum_{3\leq t\leq r}\frac{1}{t!} + \frac{1}{2}\right)\\
&= \frac{(k+r)!}{k!} = \dim M^{\lambda}.
\end{aligned}$$

□

Lemma 2.1.4 shows that the filtration of Theorem 2.1.2 is well-behaved at the level of vector spaces. From this point onward, we fix the notation as in Theorem 2.1.2 and assume $\lambda = (k, 1^r)$ with $k > 1$ and $r > 1$ to avoid trivial cases. The case $r = 1$ is analogous, and can be handled by the same argument as in Lemma 2.1.6.

Lemma 2.1.5. *Let v be the standard cyclic generator of M^{λ} and $\mathbf{i} = \mathbf{i}^{\lambda} = \mathrm{res}(T^{\lambda})$, we have:*

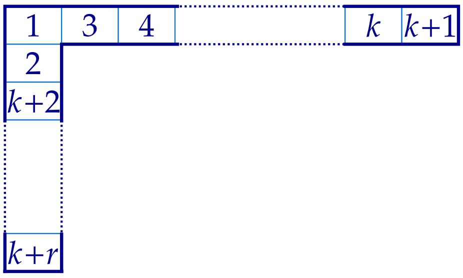
(a) $y_j\left(\psi_1\psi_2\cdots\psi_k e(\mathbf{i})v\right) = 0,$ *for* $1 \leq j \leq k+r,$

(b) $y_j\left(\psi_{k+i-1} e(\mathbf{i})v\right) = 0,$ *for* $2 \leq i \leq r,\ 1 \leq j \leq k+r,$

(c) $\psi_j\left(\psi_1\psi_2\cdots\psi_k e(\mathbf{i})v\right) = 0,$ *for* $1 \leq j \leq k,$

(d) $\psi_j\left(\psi_{k+i-1} e(\mathbf{i})v\right) = 0,$ *for* $2 \leq i \leq r,\ 1 \leq j \leq k-1.$

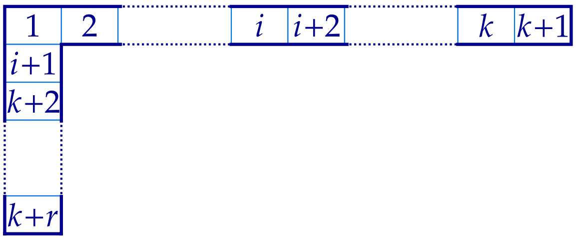
Proof. To show (a), if $j = 1$, then the tableau of $\sigma_2\sigma_3\cdots\sigma_k T^\lambda$ is the following:



Hence we have:

$$y_1\psi_1\big(\psi_2\cdots\psi_k e(\mathbf{i})v\big) = \psi_1 y_2\big(\psi_2\cdots\psi_k e(\mathbf{i})v\big)$$

For each $2 \le i \le k-1$, the tableau $\sigma_{i+1}\sigma_{i+2}\cdots\sigma_k T^\lambda$ is the following:



let $\delta_i = \delta_{\mathrm{res}(1,i),\mathrm{res}(2,1)} \in \{0,1\}$, then we have:

$$\begin{aligned}
&\psi_1\cdots\psi_{i-1}\big(y_i\psi_i\big)\psi_{i+1}\cdots\psi_k v\\
={}&\psi_1\cdots\psi_{i-1}\big(\psi_i y_{i+1}-\delta_i\big)\psi_{i+1}\cdots\psi_k e(\mathbf{i})v\\
={}&\psi_1\cdots\psi_{i-1}\psi_i y_{i+1}\psi_{i+1}\cdots\psi_k e(\mathbf{i})v-\delta_i\psi_1\cdots\psi_{i-1}\psi_{i+1}\cdots\psi_k e(\mathbf{i})v\\
={}&\psi_1\cdots\psi_{i-1}\psi_i y_{i+1}\psi_{i+1}\cdots\psi_k e(\mathbf{i})v-\delta_i\psi_1\cdots\psi_{i-2}\psi_{i+1}\cdots\psi_k\psi_{i-1} e(\mathbf{i})v\\
={}&\psi_1\cdots\psi_{i-1}\psi_i y_{i+1}\psi_{i+1}\cdots\psi_k e(\mathbf{i})v.
\end{aligned}$$

Hence we have:

$$\begin{aligned}
y_1\psi_1\psi_2\cdots\psi_k e(\mathbf{i})v &= \psi_1 y_2\psi_2\cdots\psi_k e(\mathbf{i})v\\
&= \psi_1\psi_2\cdots y_i\psi_i\cdots\psi_k e(\mathbf{i})v\\
&= \psi_1\psi_2\cdots y_k\psi_k e(\mathbf{i})v\\
&= \psi_1\psi_2\cdots\psi_{k-1}\big(\psi_k y_{k+1}-\delta_k\big)e(\mathbf{i})v\\
&= \psi_1\psi_2\cdots\psi_{k-1}\psi_k y_{k+1}e(\mathbf{i})v = 0
\end{aligned}$$

If $1 < j \le k+1$, we have:

$$\begin{aligned}
y_j\psi_1\psi_2\cdots\psi_k e(\mathbf{i})v &= \psi_1\psi_2\cdots\psi_{j-2}y_j\psi_{j-1}\cdots\psi_k e(\mathbf{i})v\\
&= \psi_1\psi_2\cdots\psi_{j-2}(\psi_{j-1}y_{j-1}+\delta_{j-1})\psi_j\cdots\psi_k e(\mathbf{i})v\\
&= \psi_1\psi_2\cdots\psi_{j-2}\psi_{j-1}y_{j-1}\psi_j\cdots\psi_k e(\mathbf{i})v\\
&= \psi_1\psi_2\cdots\psi_{j-2}\psi_{j-1}\psi_j\cdots\psi_k y_{j-1}e(\mathbf{i})v = 0.
\end{aligned}$$

If $j > k+1$, then y_j commutes with $\psi_1\psi_2\cdots\psi_k$, and hence (a) is trivial.

To show (b), if $j \neq k+i-1, k+i$, then y_j commutes with ψ_{k+i-1} and it follows. As $\mathbf{i}_{k+i-1} = \mathrm{res}(i,1) \leftarrow \mathrm{res}(i+1,1) = \mathbf{i}_{k+i}$, we have:

$$y_{k+i-1}\psi_{k+i-1}e(\mathbf{i})v = \psi_{k+i-1}y_{k+i}e(\mathbf{i})v = 0,$$
$$y_{k+i}\psi_{k+i-1}e(\mathbf{i})v = \psi_{k+i-1}y_{k+i-1}e(\mathbf{i})v = 0.$$

To show (c), if $j = 1$,

$$\begin{aligned}
\psi_1^2\psi_2\cdots\psi_k e(\mathbf{i})v &= (y_1 - y_2)\psi_2\cdots\psi_k e(\mathbf{i})v \\
&= -y_2\psi_2\cdots\psi_k e(\mathbf{i})v + \psi_2\cdots\psi_k y_1 e(\mathbf{i})v \\
&= -y_2\psi_2\cdots\psi_k e(\mathbf{i})v \\
&= -\psi_2 y_3\psi_3\cdots\psi_k e(\mathbf{i})v \\
&= 0
\end{aligned}$$

The last equality holds by above argument in (a). The second last equality holds because $\delta_2 = \delta_{\mathrm{res}(1,2),\mathrm{res}(2,1)} = 0$ since $e > 2$.

If $j = 2$, we use (1.3.11) and that $\delta_1 = \delta_{\mathrm{res}(1,1),\mathrm{res}(2,1)} = 0$:

$$\begin{aligned}
\psi_2\psi_1\psi_2\psi_3\cdots\psi_k e(\mathbf{i})v &= (\psi_1\psi_2\psi_1 - \delta_1)\psi_3\cdots\psi_k e(\mathbf{i})v \\
&= \psi_1\psi_2\psi_1\psi_3\cdots\psi_k e(\mathbf{i})v \\
&= \psi_1\psi_2\psi_3\cdots\psi_k\psi_1 e(\mathbf{i})v \\
&= 0
\end{aligned}$$

If $2 < j \leq k$, then we use (1.3.11):

$$\begin{aligned}
\psi_j\psi_1\cdots\psi_k e(\mathbf{i})v &= \psi_1\cdots\psi_j\psi_{j-1}\psi_j\cdots\psi_k e(\mathbf{i})v \\
&= \psi_1\cdots(\psi_{j-1}\psi_j\psi_{j-1} - \delta_{j-1})\cdots\psi_k e(\mathbf{i})v \\
&= \psi_1\cdots\psi_{j-1}\psi_j\psi_{j-1}\cdots\psi_k e(\mathbf{i})v - \delta_{j-1}\psi_1\cdots\psi_{j-2}\psi_{j+1}\cdots\psi_k e(\mathbf{i})v \\
&= \psi_1\cdots\psi_{j-1}\psi_j\cdots\psi_k\psi_{j-1} e(\mathbf{i})v - \delta_{j-1}\psi_1\cdots\psi_{j+1}\cdots\psi_k\psi_{j-2} e(\mathbf{i})v \\
&= 0
\end{aligned}$$

To show (d), notice that ψ_j commutes with ψ_{k+i-1} and $\psi_j e(\mathbf{i})v = 0$ for any $2 \leq i \leq r, 1 \leq j \leq k-1$. □

LEMMA 2.1.6. *There is a surjective R_α-homomorphism from S^{λ_1} to M_1/M_2, which maps the standard cyclic generator w of S^{λ_1} to $\psi^{A_1}v + M_2$.*

Proof. Let w be the standard cyclic generator of the Specht module S^{λ_1}. It has the following presentation:

$$e(\mathbf{j}')w = \delta_{\mathbf{j},\mathbf{j}'}e(\mathbf{j})w, \qquad \text{where } \mathbf{j} = \operatorname{res}(T^{\lambda_1}) \tag{1}$$

$$y_j e(\mathbf{j})w = 0, \qquad \text{for } 1 \le j \le k+r \tag{2}$$

$$\psi_j e(\mathbf{j})w = 0, \qquad \text{for } 1 \le j \le k \tag{3}$$

$$\psi_1\psi_2\cdots\psi_{k+1}e(\mathbf{j})w = 0 \tag{4}$$

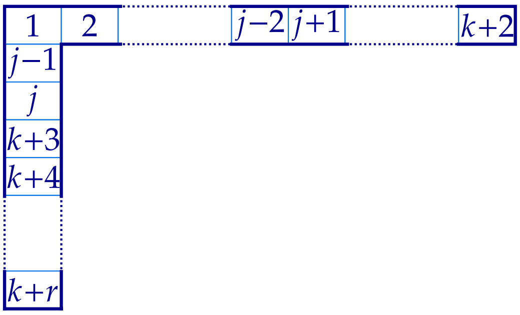
$$\psi_{k+i}e(\mathbf{j})w = 0, \qquad \text{for } 2 \le i \le r-1 \tag{5}$$

We verify that M_1/M_2 with the standard cyclic generator $\psi^{A_1}v + M_2 = \psi_1\psi_2\cdots\psi_k e(\mathbf{i}^\lambda)v + M_2$ satisfies these relations, hence there is a surjective R_α-homomorphism ϕ_1 from S^{λ_1} to M_1/M_2 mapping w to $\psi^{A_1}v + M_2$.

The (1) relation is clear since $\operatorname{res}_{\Lambda_x}(G^{A_1}) = \operatorname{res}_{\Lambda_x}(\sigma_1\sigma_2\cdots\sigma_k T^\lambda) = \operatorname{res}_{\Lambda_{x-1}}(T^{\lambda_1})$.

The (2) and (3) relations hold by (a) and (c) from Lemma 2.1.5.

For (4), notice that $T_j := \sigma_j\sigma_{j+1}\cdots\sigma_{k+1}(\sigma_{j-1}\sigma_j\cdots\sigma_k T^\lambda)$ $(j \ge 3)$ is the following tableau:



Let $\mathbf{j}' := \operatorname{res}(T_j)$. It is easy to see $\operatorname{res}_{j-2}(T_j) = \operatorname{res}_j(T_j) \to \operatorname{res}_{j-1}(T_j)$ if and only if $\operatorname{res}(1, j-2) = \operatorname{res}(3,1)$ and this is the only case for which $Q_{\mathbf{j}'_{j-2},\mathbf{j}'_{j-1},\mathbf{j}'_j}(y_{j-2}, y_{j-1}, y_j) \neq 0$. Let $\delta^j = \delta_{\operatorname{res}(1,j-2),\operatorname{res}(3,1)}$, then we have:

$$Q_{\mathbf{j}'_{j-2},\mathbf{j}'_{j-1},\mathbf{j}'_j}(y_{j-2}, y_{j-1}, y_j) = \delta^j$$

Notice that $\delta^3 = 0$. We keep applying the relations (1.3.11):

$$\begin{aligned}
&\psi_1\psi_2\psi_3\psi_4\cdots\psi_{k+1}\big(\psi_1\psi_2\cdots\psi_k v + M_2\big)\\
&= (\psi_1\psi_2\psi_1)\psi_3\psi_4\cdots\psi_{k+1}\big(\psi_2\psi_3\cdots\psi_k v + M_2\big)\\
&= (\psi_2\psi_1\psi_2)\psi_3\psi_4\cdots\psi_{k+1}\big(\psi_2\psi_3\cdots\psi_k v + M_2\big)\\
&= \psi_2\psi_1(\psi_2\psi_3\psi_2)\psi_4\cdots\psi_{k+1}\big(\psi_3\cdots\psi_k v + M_2\big)\\
&= (\psi_2\psi_1)(\psi_3\psi_2\psi_3 + \delta^4)\psi_4\cdots\psi_{k+1}\big(\psi_3\cdots\psi_k v + M_2\big)\\
&= (\psi_2\psi_1)(\psi_3\psi_2\psi_3)\psi_4\cdots\psi_{k+1}\big(\psi_3\cdots\psi_k v + M_2\big)\\
&= (\psi_2\psi_1)(\psi_3\psi_2)\cdots(\psi_{j-1}\psi_j\psi_{j-1})\psi_{j+1}\cdots\psi_{k+1}\big(\psi_j\cdots\psi_k e(\mathbf{i})v + M_2\big)\\
&= (\psi_2\psi_1)(\psi_3\psi_2)\cdots(\psi_j\psi_{j-1}\psi_j + \delta^{j+1})\psi_{j+1}\cdots\psi_{k+1}\big(\psi_j\cdots\psi_k e(\mathbf{i})v + M_2\big)\\
&= (\psi_2\psi_1)(\psi_3\psi_2)\cdots(\psi_j\psi_{j-1}\psi_j)\psi_{j+1}\cdots\psi_{k+1}\big(\psi_j\cdots\psi_k e(\mathbf{i})v + M_2\big)\\
&= \cdots\\
&= (\psi_2\psi_1)(\psi_3\psi_2)\cdots(\psi_{k-1}\psi_{k-2})(\psi_k\psi_{k-1})\psi_k\psi_{k+1}\psi_k e(\mathbf{i})v + M_2\\
&= (\psi_2\psi_1)(\psi_3\psi_2)\cdots(\psi_{k-1}\psi_{k-2})(\psi_k\psi_{k-1})(\psi_{k+1}\psi_k\psi_{k+1} + \delta^{k+2})e(\mathbf{i})v + M_2\\
&= (\psi_2\psi_1)(\psi_3\psi_2)\cdots(\psi_{k-1}\psi_{k-2})(\psi_k\psi_{k-1})\psi_{k+1}\psi_k\psi_{k+1}e(\mathbf{i})v + M_2\\
&= 0 + M_2.
\end{aligned}$$

All the expressions with δ^j $(4 \le j \le k+2)$ vanish because the earlier term ψ_{j-3} commutes with the terms to its right and kill v. The last equality holds because the Garnir relation $\psi^{A_2}v = \psi_{k+1}e(\mathbf{i}^\lambda)v \in M_2$.

For (5), as already noted in Lemma 2.1.1, we have

$$\psi_{k+i}\psi^{A_1}v = \psi^{A_1}\psi_{k+i}v = \psi^{A_1}\psi^{A_{i+1}}v \in M_2,$$

where the first equality holds because ψ_{k+i} commutes with $\psi_1\cdots\psi_k$ for $2 \le i \le r-1$. □

Lemma 2.1.7. *For each $2 \le i \le r-1$ there is a canonical surjective R_α-homomorphism from S^{λ_i} to M_i/M_{i+1}, which maps the standard cyclic generator w of S^{λ_i} to $\psi^{A_i}v + M_{i+1}$.*

Proof. Let w be the standard cyclic generator of the Specht module S^{λ_i} and let $\mathbf{i} := \mathbf{i}^\lambda$. Then it has the following presentation:

(i) $e(\mathbf{i}')w = \delta_{\mathbf{j},\mathbf{i}'}w,$ where $\mathbf{j} = \mathrm{res}(T^{\lambda_i})$

(ii) $y_j e(\mathbf{j})w = 0,$ for $1 \le j \le k+r$

(iii) $\psi_j e(\mathbf{j})w = 0,$ for $1 \le j \le k-1$ or $j = k+i-1$

(iv) $\psi_{k+i-1}\psi_{k+i}e(\mathbf{j})w = 0$

(v) $\psi_{k+i+j}e(\mathbf{j})w = 0,$ for $1 \le j \le r-1-i$

We verify that M_i/M_{i+1} with the standard cyclic generator $\psi^{A_i}v+M_{i+1} = \psi_{k+i-1}e(\mathbf{i})v+M_{i+1}$ satisfies these relations, hence there is a surjective R_α- homomorphism ϕ_i from S^{λ_i} to M_i/M_{i+1} mapping w to $\psi^{A_i}v + M_{i+1}$.

(i) can be shown by the same argument as in Lemma 2.1.6.

(ii) and (iii) are satisfied by (b) and (d) from Lemma 2.1.5.

(iv) can be verified by applying the relation (1.3.11):

$$\psi_{k+i-1}\psi_{k+i}e(\mathbf{j})\big(\psi_{k+i-1}e(\mathbf{i})v + M_{i+1}\big) = \psi_{k+i}\psi_{k+i-1}\psi_{k+i}e(\mathbf{i})v + M_{i+1} = 0 + M_{i+1}$$

The first equality holds because $\mathbf{i}_{k+i-1} \neq \mathbf{i}_{k+i+1}$ since $e > 2$.

For the remaining relations from (v), use the fact ψ_{k+i-1} commutes with ψ_{k+i+j} for $1 \leq j \leq r-1-i$:

$$\psi_{k+i+j}e(\mathbf{j})\big(\psi_{k+i-1}v + M_{i+1}\big) = \psi_{k+i-1}\psi_{k+i+j}v + M_{i+1}$$

As $R_\alpha\{\psi_{k+i+1}v, \cdots, \psi_{k+r-1}v\} = M_{i+1}$, the conclusion follows. □

Lemma 2.1.8. *There is a canonical surjective R_α-homomorphism from S^{λ_r} to M_r, which maps the standard cyclic generator w of S^{λ_r} to $\psi^{A_r}v$.*

Proof. The proof is almost the same as Lemma 2.1.7, but easier because there is no Garnir relation this time. □

Now we can prove the main theorem.

Proof of Theorem 2.1.2. The $i = 0$ case is trivial since M_1 is generated by the Garnir relations and we know that

$$M^\lambda/M_1 \cong S^\lambda$$

as R_α-modules.

For $1 \leq i \leq r$, we have constructed a surjective R_α-homomorphism from S^{λ_i} to M_i/M_{i+1} by Lemma 2.1.6, Lemma 2.1.7 and Lemma 2.1.8. By Lemma 2.1.4, since the sum of the dimensions of S^{λ_i} for $0 \leq i \leq r$ equals the dimension of M^λ, all these surjective homomorphisms must be isomorphisms. Hence, the proof is complete. □

Remark 2.1.9. *See Subsection 1.2.7 for the definition of degree of tableaux. One might expect that the Specht filtration satisfies the property that the isomorphism between S^{λ_i} and M_i/M_{i+1} is homogeneous of degree zero. However, this is not true in general. Indeed, it is easy to see that, as graded R_α-modules, we have: $S^{\lambda_i}\langle -\deg T^{\lambda_i} + \deg T^\lambda + \deg \psi^{A_i}e(\mathbf{i})\rangle \cong M_i/M_{i+1}$.* ◇

Remark 2.1.10. *The Specht filtration in Theorem 2.1.2 is generated by the Garnir relations in top-to-bottom order. By reversing this order, one obtains a skew Specht filtration of M^λ whose factors are the skew Specht modules introduced in [**Mut15**].* ◇

2.2. Two-Row Partition Case in Type A_∞

In this section, we fix a partition $\lambda = (k, r)$ with $k \geq r$ and prove that, in the linear quiver A_∞ case (or in type $A^{(1)}_{e-1}$ with e large enough), all Garnir relations of λ are generated by the first one.

With the data above, the initial tableau T^λ is the following:

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\hline 1 & 2 & \cdots & s & s+1 & \cdots & r & r+1 & \cdots & k-1 & k \\ \hline k+1 & k+2 & \cdots & k+s & k+s+1 & \cdots & k+r & & & & \\ \hline \end{array}$$

Let $B_i := (1, i)$ for $1 \leq i \leq r$. Clearly they are all the Garnir nodes of $[\lambda]$. Set $\psi^{B_i} := \psi^{G^{B_i}}$ for each $1 \leq i \leq r$, and $\mathbf{i} := \mathbf{i}^\lambda = \mathrm{res}(T^\lambda)$.

Lemma 2.2.1. *For $1 \leq s \leq r$, we have:*

$$\psi^{B_s} = (\psi_{2s-1} \cdots \psi_{k+s-2}\psi_{k+s-1}) \cdots (\psi_{s+1} \cdots \psi_k\psi_{k+1})(\psi_s \cdots \psi_{k-1}\psi_k)e(\mathbf{i}) \tag{2.2.2}$$

Proof. If $s = 1$, the permutation $\sigma_1 \cdots \sigma_k$ transforms the initial tableau T^λ into the following Garnir tableau G^{B_1}:

$$\begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 2 & 3 & 4 & \cdots & r+1 & r+2 & \cdots & k & k+1 \\ \hline 1 & k+2 & k+3 & \cdots & k+r & & & & \\ \hline \end{array}$$

If $s > 1$, the permutation $\sigma_s \cdots \sigma_{k-1}\sigma_k$ transforms the initial tableau into the following:

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1 & 2 & 3 & \cdots & s+1 & s+2 & \cdots & r+1 & r+2 & \cdots & k & k+1 \\ \hline s & k+2 & k+3 & \cdots & k+s & k+s+1 & \cdots & k+r & & & & \\ \hline \end{array}$$

Similarly, after applying $(\sigma_{s+1} \cdots \sigma_k\sigma_{k+1})(\sigma_s\sigma_{k-1} \cdots \sigma_k)e(\mathbf{i})$ to T^λ, we obtain the following tableau:

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1 & 2 & 3 & \cdots & s+2 & s+3 & \cdots & r+2 & r+3 & \cdots & k+1 & k+2 \\ \hline s & s+1 & k+3 & \cdots & k+s & k+s+1 & \cdots & k+r & & & & \\ \hline \end{array}$$

Continuing this procedure s times, we obtain the Garnir tableau G^{B_s}:

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1 & 2 & 3 & \cdots & 2s & 2s+1 & \cdots & s+r & s+r+1 & \cdots & k+s-1 & k+s \\ \hline s & s+1 & s+2 & \cdots & 2s-1 & k+s+1 & \cdots & k+r & & & & \\ \hline \end{array}$$

Let

$$w := (\sigma_{2s-1} \cdots \sigma_{k+s-2}\sigma_{k+s-1}) \cdots (\sigma_{s+1} \cdots \sigma_k\sigma_{k+1})(\sigma_s \cdots \sigma_{k-1}\sigma_k) \in \mathfrak{S}_{k+r}.$$

We have shown that $wT^\lambda = G^{B_s}$. It is classical (see, for instance, [**Mat99**, Proposition 3.3]) that the map

$$\{\text{row-standard } \lambda\text{-tableaux}\} \longrightarrow W^\lambda \subset \mathfrak{S}_{k+r}, \qquad T \longmapsto w^T,$$

is a bijection onto the set W^λ of minimal-length representatives of $\mathfrak{S}_{k+r}/(\mathfrak{S}_k \times \mathfrak{S}_r)$, characterized by $w^T T^\lambda = T$. Therefore, since $wT^\lambda = G^{B_s}$, we must have $w = w^{G^{B_s}}$, and hence $w \in W^\lambda$. In particular, w has minimal length in its coset and therefore the expression is reduced.

Moreover, by [**KMR12**, Lemma 3.17], w is *fully commutative*, so any two reduced expressions for w are related by commuting braid moves. Consequently, ψ^w is independent of the choice of reduced expression, and the stated equality follows. □

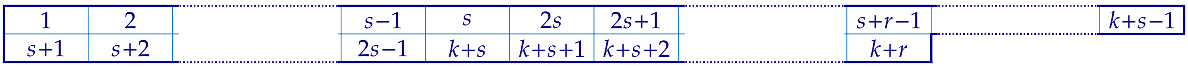
Lemma 2.2.3. *For* $1 \le s \le r-1$, *take the Garnir node* B_{s+1} *and let* v *be the standard cyclic generator of* M^λ. *Then*

$$\psi_{2s}\psi^{B_{s+1}}e(\mathbf{i})v = 0$$

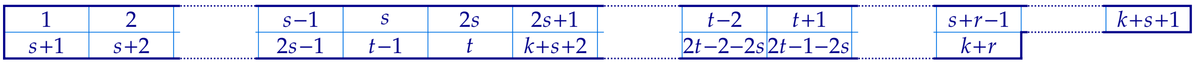
Proof. If $s > 1$, set $\psi' := (\psi_{2s-1}\cdots\psi_{k+s-2})\cdots(\psi_{s+1}\cdots\psi_k)$. By Lemma 2.2.1, we have:

$$\psi^{B_{s+1}} = (\psi_{2s+1}\cdots\psi_{k+s})(\psi_{2s}\cdots\psi_{k+s-1})\psi'$$

Let $T_1 := (\sigma_{2s-1}\cdots\sigma_{k+s-2})\cdots(\sigma_{s+1}\cdots\sigma_k)T^\lambda$, then it is of the following form:



Let $T_2 := (\sigma_t\sigma_{t+1}\cdots\sigma_{k+s})(\sigma_{t-1}\cdots\sigma_{k+s-1})T_1$ for $2s+2 \le t < k+s$, then it is of the following form:

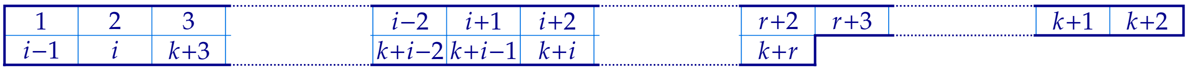


It is easy to see $\mathrm{res}_{T_2}(t-2) \neq \mathrm{res}_{T_2}(t)$ and $\mathrm{res}_{T_1}(2s) \neq \mathrm{res}_{T_1}(2s+2)$ since the quiver is A_∞ (or $A^{(1)}_{e-1}$ with $e \gg 0$). We can apply (1.3.11):

$$\begin{aligned}
\psi_{2s}\psi^{B_{s+1}}e(\mathbf{i})v &= \psi_{2s}(\psi_{2s+1}\psi_{2s+2}\cdots\psi_{k+s})(\psi_{2s}\psi_{2s+1}\cdots\psi_{k+s-1})\psi'e(\mathbf{i})v \\
&= \big(\psi_{2s}\psi_{2s+1}\psi_{2s}\big)(\psi_{2s+2}\cdots\psi_{k+s})(\psi_{2s+1}\cdots\psi_{k+s-1})\psi'e(\mathbf{i})v \\
&= \big(\psi_{2s+1}\psi_{2s}\psi_{2s+1}\big)(\psi_{2s+2}\cdots\psi_{k+s})(\psi_{2s+1}\cdots\psi_{k+s-1})\psi'e(\mathbf{i})v \\
&= (\psi_{2s+1}\psi_{2s})\big(\psi_{2s+1}\psi_{2s+2}\psi_{2s+1}\big)(\psi_{2s+3}\cdots\psi_{k+s})(\psi_{2s+3}\cdots\psi_{k+s-1})\psi'e(\mathbf{i})v \\
&= \cdots \\
&= (\psi_{2s+1}\psi_{2s})(\psi_{2s+2}\psi_{2s+1})\cdots(\psi_{k+s-1}\psi_{k+s-2})\psi_{k+s-1}\psi_{k+s}\psi_{k+s-1}\psi'e(\mathbf{i})v \\
&= (\psi_{2s+1}\psi_{2s})(\psi_{2s+2}\psi_{2s+1})\cdots(\psi_{k+s-1}\psi_{k+s-2})\psi_{k+s}\psi_{k+s-1}\psi_{k+s}\psi'e(\mathbf{i})v \\
&= (\psi_{2s+1}\psi_{2s})(\psi_{2s+2}\psi_{2s+1})\cdots(\psi_{k+s-1}\psi_{k+s-2})\psi_{k+s}\psi_{k+s-1}\psi'\psi_{k+s}e(\mathbf{i})v = 0
\end{aligned}$$

The second last equality holds because ψ_{k+s} commutes with ψ' and it kills v.

If $s = 1$, by Lemma 2.2.1, $\psi^{B_2} = (\psi_3\cdots\psi_{k+1})(\psi_2\cdots\psi_k)e(\mathbf{i})$. For $4 \le i \le k+1$, the tableau $T := (\sigma_i\cdots\sigma_{k+1})(\sigma_{i-1}\cdots\sigma_k)T^\lambda$ is as follows:



In particular, $\mathrm{res}_T(i-2) \neq \mathrm{res}_T(i)$. We can apply (1.3.11) and compute:

$$\begin{aligned}
\psi_2\psi^{B_2}e(\mathbf{i})v &= \psi_2(\psi_3\cdots\psi_{k+1})(\psi_2\cdots\psi_k)e(\mathbf{i})v\\
&= (\psi_2\psi_3\psi_2)(\psi_4\cdots\psi_{k+1})(\psi_3\cdots\psi_k)e(\mathbf{i})v\\
&= (\psi_3\psi_2\psi_3)(\psi_4\cdots\psi_{k+1})(\psi_3\cdots\psi_k)e(\mathbf{i})v\\
&= (\psi_3\psi_2)(\psi_3\psi_4\psi_3)(\psi_5\cdots\psi_{k+1})(\psi_4\cdots\psi_k)e(\mathbf{i})v\\
&= (\psi_3\psi_2)(\psi_4\psi_3\psi_4)(\psi_5\cdots\psi_{k+1})(\psi_4\cdots\psi_k)e(\mathbf{i})v\\
&= \cdots\\
&= (\psi_3\psi_2)(\psi_4\psi_3)\cdots(\psi_k\psi_{k-1})\psi_k\psi_{k+1}\psi_k e(\mathbf{i})v\\
&= (\psi_3\psi_2)(\psi_4\psi_3)\cdots(\psi_k\psi_{k-1})\psi_{k+1}\psi_k\psi_{k+1} e(\mathbf{i})v\\
&= 0
\end{aligned}$$

□

We are now in a position to prove the main result of this section:

Theorem 2.2.4. *Let v be the standard cyclic generator of M^λ, for $1 \leq s \leq r-1$, we have:*

$$(\psi_s\psi_{s+1}\cdots\psi_{k+s})\psi^{B_s}e(\mathbf{i})v = -\psi^{B_{s+1}}e(\mathbf{i})v \tag{2.2.5}$$

Proof. We compute as follows:

$$\begin{aligned}
&(\psi_s\psi_{s+1}\cdots\psi_{k+s})\psi^{B_s}e(\mathbf{i})v\\
={}&(\psi_s\psi_{s+1}\cdots\psi_{k+s})(\psi_{2s-1}\cdots\psi_{k+s-1})\cdots(\psi_{s+1}\cdots\psi_{k+1})\\
&\quad\cdot(\psi_s\cdots\psi_k)e(\mathbf{i})v\\
={}&(\psi_s\psi_{s+1}\cdots\psi_{k+s})(\psi_{2s-1}\psi_{2s-2}\cdots\psi_s)(\psi_{2s}\cdots\psi_{k+s-1})\cdots(\psi_{s+2}\cdots\psi_{k+1})\\
&\quad\cdot(\psi_{s+1}\cdots\psi_k)e(\mathbf{i})v\\
={}&(\psi_s\psi_{s+1}\cdots\psi_{2s-1}\psi_{2s}\psi_{2s+1}\cdots\psi_{k+s})(\psi_{2s-1}\psi_{2s-2}\cdots\psi_s)\\
&\quad\cdot(\psi_{2s}\cdots\psi_{k+s-1})\cdots(\psi_{s+2}\cdots\psi_{k+1})(\psi_{s+1}\cdots\psi_k)e(\mathbf{i})v\\
={}&(\psi_s\psi_{s+1}\cdots\psi_{2s-1}\psi_{2s}\psi_{2s-1}\psi_{2s-2}\cdots\psi_s)(\psi_{2s+1}\cdots\psi_{k+s})\\
&\quad\cdot(\psi_{2s}\cdots\psi_{k+s-1})\cdots(\psi_{s+2}\cdots\psi_{k+1})(\psi_{s+1}\cdots\psi_k)e(\mathbf{i})v\\
={}&(\psi_s\psi_{s+1}\cdots\psi_{2s-1}\psi_{2s}\psi_{2s-1}\psi_{2s-2}\cdots\psi_s)\,\psi^{B_{s+1}}e(\mathbf{i})v.
\end{aligned}$$

Notice $T^{2s-2} := (\sigma_{2s-2}\cdots\sigma_s)G^{B_{s+1}}$ is the following tableau:

1	2	3		$s-1$	$2s-1$	$2s+2$	$2s+3$		$s+r+1$	$s+r+2$		$k+s$	$k+s+1$
s	$s+1$	$s+2$		$2s-2$	$2s$	$2s+1$	$k+s+2$		$k+r$				

Since $\mathrm{res}_{T^{2s-2}}(2s-1)=\mathrm{res}_{T^{2s-2}}(2s+1)\leftarrow\mathrm{res}_{T^{2s-2}}(2s)$, we get:

$$\begin{aligned}
&(\psi_s\psi_{s+1}\cdots\psi_{2s-1}\psi_{2s}\psi_{2s-1}\psi_{2s-2}\cdots\psi_s)\psi^{B_{s+1}}e(\mathbf{i})v\\
&=(\psi_s\psi_{s+1}\cdots\psi_{2s-2}\big(\psi_{2s}\psi_{2s-1}\psi_{2s}-1\big)\psi_{2s-2}\cdots\psi_s)\psi^{B_{s+1}}e(\mathbf{i})v\\
&=(\psi_s\psi_{s+1}\cdots\psi_{2s-2}\big(\psi_{2s}\psi_{2s-1}\psi_{2s}\big)\psi_{2s-2}\cdots\psi_s)\psi^{B_{s+1}}e(\mathbf{i})v\\
&\quad-(\psi_s\psi_{s+1}\cdots\psi_{2s-2}\psi_{2s-2}\cdots\psi_s)\psi^{B_{s+1}}e(\mathbf{i})v
\end{aligned}$$

It is clear the first term vanishes by Lemma 2.2.3 as ψ_{2s} commutes with the remaining $\psi_{2s-2}\cdots\psi_s$.

Let $T^r:=(\sigma_r\cdots\sigma_s)G^{B_{s+1}}$ for $s<r<2s-2$, which is of the following form:

1	2		$r-s+1$	$r-s+2$		$r+1$	$2s+2$		$s+r+1$	$s+r+2$		$k+s+1$
s	$s+1$		r	$r+2$		$2s$	$2s+1$		$k+r$			

As $\mathrm{res}_{T^r}(r+1)$ and $\mathrm{res}_{T^r}(r+2)$ are not adjacent or equal, apply (1.3.10):

$$\begin{aligned}
(\psi_s\psi_{s+1}\cdots\psi_{2s-2}\psi_{2s-2}\cdots\psi_s)\psi^{B_{s+1}}e(\mathbf{i})v&=(\psi_s\psi_{s+1}\cdots\psi_{2s-3}\psi_{2s-3}\cdots\psi_s)\psi^{B_{s+1}}e(\mathbf{i})v\\
&=\cdots\\
&=\psi^{B_{s+1}}e(\mathbf{i})v
\end{aligned}$$

□

Corollary 2.2.6. *The submodule of M^λ generated by*

$$\{\psi^{B_1}e(\mathbf{i})v,\psi^{B_2}e(\mathbf{i})v,\dots,\psi^{B_r}e(\mathbf{i})v\}$$

is cyclic, with generator $\psi^{B_1}e(\mathbf{i})v$.

Corollary 2.2.7. *Fix quiver A_∞ or $A^{(1)}_{e-1}$ with $e\gg 0$. Fix $\Lambda\in P^+$ a fundamental weight and $\alpha\in Q^+$ a positive root. Let $\lambda=(\lambda_1,\cdots,\lambda_r)\in\mathscr{P}^\Lambda_\alpha$ and $\mathbf{i}:=\mathbf{i}^\lambda$, and form the permutation module M^λ with standard cyclic generator v, then the submodule M_1 of M^λ generated by the Garnir relations has the following form:*

$$M_1=R_\alpha\{\psi^{A_1}e(\mathbf{i})v,\cdots,\psi^{A_{r-1}}e(\mathbf{i})v\}$$

where $\{A_i=(i,1)|1\le i\le r-1\}$ is the set of Garnir nodes in the first column of $[\lambda]$.

Proof. Each Garnir relation only involves two rows, hence we only need to show any Garnir relation relating row i and $i+1$ is generated by the Garnir relation corresponding to the Garnir node A_i, which is just Corollary 2.2.6. □

We are now ready to construct a Specht filtration of M^λ. As a first step, we compare the dimensions, analogous to Lemma 2.1.4.

Lemma 2.2.8. *Let $\lambda_0=\lambda$ and $\lambda_1=(k+1|r-1)$. Then* $\dim M^\lambda=\dim S^{\lambda_0}+\dim S^{\lambda_1}$.

Proof. The dimension of M^λ is

$$\frac{(k+r)!}{k!r!}$$

and the dimension of S^{λ_0} is

$$\frac{(k+r)!}{\big((k+1)\cdots(k-r+2)\big)(k-r)!r!}=\frac{(k+r)!(k-r+1)}{(k+1)!r!}$$

The dimension of S^{λ_1} is:

$$\binom{k+r}{k+1}=\frac{(k+r)!}{(k+1)!(r-1)!}=\frac{(k+r)!r}{(k+1)!r!}$$

It is immediate to get the desired equality. □

Theorem 2.2.9. *Suppose the quiver is A_∞ or $A^{(1)}_{e-1}$ with $e \gg 0$. Fix $\alpha \in Q^+$ and $\Lambda_i \in P^+$. Let $\lambda = (k, r)$ with $k \geq r$ be a two-row partition such that $\alpha_\lambda = \alpha$. Let v be the standard cyclic generator of the permutation module M^λ, and let $\mathbf{i} := \mathbf{i}^\lambda$. Then M^λ admits a Specht filtration:*

$$M^\lambda = M_0 \supsetneq M_1 \supsetneq 0,$$

where $M_1 = R_\alpha \cdot \psi^{B_1} e(\mathbf{i})v$, and we have:

$$M_0/M_1 \cong S^\lambda, \qquad M_1 \cong S^{\lambda_1},$$

where $\lambda_1 = (k+1 \mid r-1)$ and S^{λ_1} is the Specht module over R^Λ_α with Λ determined by the charge $\kappa = (i-1, i)$.

Proof. By Corollary 2.2.6, the only part that remains to be proven is the isomorphism between M_1 and S^{λ_1}. Using essentially the same argument as in Lemma 2.1.6, we can show that the cyclic generator $\psi^{B_1}e(\mathbf{i})v$ satisfies all the defining relations of the standard cyclic generator w of S^{λ_1}. Hence, there exists a surjective homomorphism from S^{λ_1} onto M_1. The conclusion then follows from Lemma 2.2.8. □

At the end of this section, we record the following result:

Lemma 2.2.10. *Let v be the standard cyclic generator of M^λ. For $2 \leq s \leq r$, we have:*

$$(\psi_{k+s-1}\cdots\psi_s\psi_{s-1})\,\psi^{B_s}e(\mathbf{i})v = -\psi^{B_{s-1}}e(\mathbf{i})v.$$

Proof. The proof is analogous to that of Theorem 2.2.4. We briefly state the key procedures:

$$\begin{aligned}
&(\psi_{k+s-1}\cdots\psi_s\psi_{s-1})\,\psi^{B_s}e(\mathbf{i})v\\
&=(\psi_{k+s-1}\cdots\psi_s\psi_{s-1})\,(\psi_{2s-1}\cdots\psi_{k+s-2}\psi_{k+s-1})\cdots(\psi_{s+1}\cdots\psi_k\psi_{k+1})(\psi_s\cdots\psi_{k-1}\psi_k)e(\mathbf{i})v\\
&=(\psi_{k+s-1}\cdots\psi_{2s-2})(\psi_{2s-1}\cdots\psi_{k+s-2}\psi_{k+s-1})\psi^{B_{s-1}}e(\mathbf{i})v\\
&=(\psi_{k+s-1}\cdots\psi_{2s-2}\psi_{2s-1}\psi_{2s-2}\cdots\psi_{k+s-2}\psi_{k+s-1})\psi^{B_{s-1}}e(\mathbf{i})v\\
&=\big(\psi_{k+s-1}\cdots\psi_{2s-3}(\psi_{2s-1}\psi_{2s-2}\psi_{2s-1}-1)\psi_{2s-3}\cdots\psi_{k+s-2}\psi_{k+s-1}\big)\psi^{B_{s-1}}e(\mathbf{i})v
\end{aligned}$$

Then we need to prove $\psi_{2s-2}\psi^{B_{s-1}}e(\mathbf{i})v = 0$ and the conclusion follows. □

Lemma 2.2.10, together with Theorem 2.2.4, shows that the choice of Garnir relation in Corollary 2.2.6 between any two adjacent rows is be arbitrary. In other words, for each pair of adjacent rows i and $i+1$, we may choose any Garnir node $(i,j) \in [\lambda]$ and use the corresponding Garnir relation. This single relation suffices to generate all the Garnir relations between the two rows in $[\lambda]$.

2.3. General Partition Case in Type A_∞

One might expect that Theorem 2.2.9 naturally extends to arbitrary partitions, particularly given Corollary 2.2.7. However, this extension encounters a fundamental obstacle: the dimension equality established in Lemma 2.2.8 fails to hold for general partitions. In fact, the surjective homomorphism from S^{λ_i} to M_i/M_{i+1} is not an isomorphism in general. Instead, we construct a finite Specht resolution of M_i/M_{i+1}.

Example 2.3.1. *Fix Λ_0 and consider the partition $\lambda = \lambda_0 = (5,5,4,2,2)$.*

$$\begin{array}{|c|c|c|c|c|}\hline 0&1&2&3&4\\\hline -1&0&1&2&3\\\hline -2&-1&0&1\\\cline{1-4} -3&-2\\\cline{1-2} -4&-3\\\cline{1-2}\end{array}$$

We have $\dim M^\lambda = 4631346720$ *and* $\dim S^\lambda = 4594590$.

The 'filtration' generated by the Garnir relations contains the following Specht modules:

$\lambda_1 = (4|(6,4,2,2))$:

$$\left(\begin{array}{|c|c|c|c|}\hline 0&1&2&3\\\hline\end{array}\;\middle|\;\begin{array}{|c|c|c|c|c|c|}\hline -1&0&1&2&3&4\\\hline -2&-1&0&1\\\cline{1-4} -3&-2\\\cline{1-2} -4&-3\\\cline{1-2}\end{array}\right)$$

$\lambda_2 = (5|3|(6,2,2))$:

$$\left(\begin{array}{|c|c|c|c|c|}\hline 0&1&2&3&4\\\hline\end{array}\;\middle|\;\begin{array}{|c|c|c|}\hline -1&0&1\\\hline\end{array}\;\middle|\;\begin{array}{|c|c|c|c|c|c|}\hline -2&-1&0&1&2&3\\\hline -3&-2\\\cline{1-2} -4&-3\\\cline{1-2}\end{array}\right)$$

$\lambda_3 = (5|5|1|(5,2))$:

$$\left(\begin{array}{|c|c|c|c|c|}\hline 0&1&2&3&4\\\hline\end{array}\;\middle|\;\begin{array}{|c|c|c|c|c|}\hline -1&0&1&2&3\\\hline\end{array}\;\middle|\;\begin{array}{|c|}\hline -2\\\hline\end{array}\;\middle|\;\begin{array}{|c|c|c|c|c|}\hline -3&-2&-1&0&1\\\hline -4&-3\\\cline{1-2}\end{array}\right)$$

$\lambda_4 = (5|5|4|1|3)$:

$$\left(\begin{array}{|c|c|c|c|c|}\hline 0&1&2&3&4\\\hline\end{array}\;\middle|\;\begin{array}{|c|c|c|c|c|}\hline -1&0&1&2&3\\\hline\end{array}\;\middle|\;\begin{array}{|c|c|c|c|}\hline -2&-1&0&1\\\hline\end{array}\;\middle|\;\begin{array}{|c|}\hline -3\\\hline\end{array}\;\middle|\;\begin{array}{|c|c|c|}\hline -4&-3&-2\\\hline\end{array}\right)$$

By computation, we get:

$$\dim S^{\lambda_1} = 128648520$$

$$\dim S^{\lambda_2} = 551350800$$

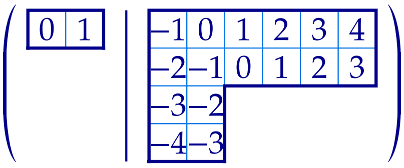
$$\dim S^{\lambda_3} = 1235025792$$
$$\dim S^{\lambda_4} = 3087564480$$

However, unlike the case of hook or two-row partitions, this time we have:

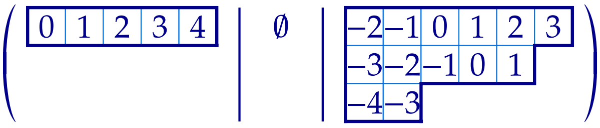
$$\sum_{0\le i\le 4} \dim S^{\lambda_i} \neq \dim M^{\lambda}.$$

The problem arises because, the surjection $S^{\lambda_i} \twoheadrightarrow M_i/M_{i+1}$ *is not an isomorphism in general. Indeed, the kernels of these maps are themselves Specht modules corresponding to the following partitions:*

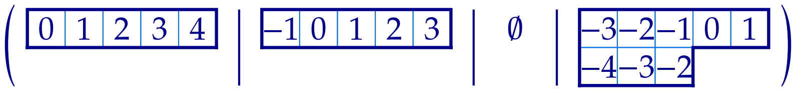
$\mu_1 = \big(2|(6,6,2,2)\big)$*:*



$\mu_2 = \big(5|\emptyset|(6,5,2)\big)$*:*



$\mu_3 = \big(5|5|\emptyset|(5,3)\big)$*:*



and the dimensions are: $\dim S^{\mu_1} = 22972950$, $\dim S^{\mu_2} = 44108064$ *and* $\dim S^{\mu_3} = 308756448$.

It is not hard to see that:

$$\sum_{0\le i\le 4} \dim S^{\lambda_i} - \sum_{1\le i\le 3} \dim S^{\mu_i} = \dim M^{\lambda}.$$

◇

Our main result in this section is the following:

THEOREM 2.3.2. *Suppose the quiver is* A_∞ *or* $A^{(1)}_{e-1}$ *with* $e \gg 0$. *Fix* $\alpha \in Q^+$ *and let* $\Lambda := \Lambda_x$ *be a fundamental weight. Take* $\lambda = (\lambda_1, \cdots, \lambda_r) \in \mathscr{P}^{\Lambda}_{\alpha}$. *Set*

$$k_i = k_i(\lambda) := \begin{cases} \max\{1 \le j \le r-i \mid \lambda_{i+j} - j \ge 0\}, & \text{if } 1 \le i \le r-1, \\ 1, & \text{if } i = 0. \end{cases}$$

Let v *be the standard cyclic generator of the permutation module* M^λ, *and let* $\mathbf{i} := \mathbf{i}^\lambda$. *Define* $B_i := (i, \lambda_{i+1}) \in [\lambda]$ *for* $1 \le i \le r-1$. *Then* M^λ *admits a generalized Specht filtration in the following sense:*

$$M^\lambda = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_{r-1} \supsetneq M_r = 0$$

such that for each $0 \le i \le r-1$, *there exists an exact sequence of* R_α*-modules:*

$$0 \longrightarrow S^{\mu_{i,k_i}} \xrightarrow{\phi_{i,k_i}} S^{\mu_{i,k_i-1}} \cdots \xrightarrow{\phi_{i,2}} S^{\mu_{i,1}} \xrightarrow{\phi_{i,1}} M_i/M_{i+1} \longrightarrow 0, \tag{2.3.3}$$

where

$$M_i = R_\alpha\{\psi^{B_i}v, \cdots, \psi^{B_{r-1}}v\}, 1 \le i \le r-1.$$

The multipartitions $\mu_{i,j}$ *is given by*

$$\mu_{i,j} = \big(\lambda_1|\cdots|\lambda_{i-1}|\lambda_{i+j}-j|(\lambda_i+1,\cdots,\lambda_{i+j-1}+1,\lambda_{i+j+1},\cdots,\lambda_r)\big), \quad 1 \le i \le r-1,$$

and

$$\mu_{0,1} = \lambda.$$

Here, $S^{\mu_{i,j}}$ *is the Specht module associated with* $\mu_{i,j}$ *over the cyclotomic KLR algebra* $R_\alpha^{\Lambda(i)}$*, where* $\Lambda(i)$ *is determined by the charge* $\kappa = (x, x-1, \cdots, x-i+1, x-i)$.

The resolution for each M_i/M_{i+1} is called a **Specht resolution** and the filtration is called a **generalized Specht filtration**.

The following proof relies on results established later in this section. We present it first because it offers greater clarity.

PROOF OF THEOREM 2.3.2. The proof proceeds by induction on the length r of the partition $\lambda = (\lambda_1, \dots, \lambda_r)$. The base cases, $r = 1$ is trivial and $r = 2$ was established in Theorem 2.2.9. Assume $r \ge 3$, and let $\nu = (\lambda_2, \dots, \lambda_r)$. By the induction hypothesis, assume that Theorem 2.3.2 holds for any partition of length less than or equal to $r-1$. In particular, M^ν possesses the desired generalized Specht filtration:

$$M^\nu = N_0 \supsetneq N_1 \supsetneq \cdots \supsetneq N_{r-2} \supsetneq N_{r-1} = 0 \tag{2.3.4}$$

Let $\beta = \alpha_{\lambda_1}$ and define $F := \mathrm{Ind}_{R_\beta \otimes R_{\alpha-\beta}}^{R_\alpha}$. According to [**KL09**, Proposition 2.16], F is an exact functor. Let $S^\beta = L_\beta$ be the one-dimensional Specht module associated with (λ_1) over $R_\beta^{\Lambda_x}$. The functor $L_\beta \boxtimes - := L_\beta \otimes_{\Bbbk} -$ is also an exact functor (since L_β is free over $\Bbbk$), mapping the category of finite-dimensional $R_{\alpha-\beta}$-modules to the category of finite-dimensional $R_\beta \otimes R_{\alpha-\beta}$-modules. For modules D_1 over R_β and D_2 over $R_{\alpha-\beta}$, respectively, define $D_1 \circ D_2 := F(D_1 \boxtimes D_2)$. In particular, if $D_1 = S^\beta$, we consider the module $S^\beta \circ D_2$. By Theorem 1.3.33, if ν' is a partition such that $\alpha_{\nu'} = \alpha - \beta$, then $S^\beta \circ S^{\nu'} \cong S^{\mu'}$, where $\mu' = (\lambda_1|\nu')$.

Hence, the generalized Specht filtration (2.3.4) of M^ν yields the following sequence of $R_\beta \otimes R_{\alpha-\beta}$-modules:

$$S^\beta \boxtimes M^\nu = S^\beta \boxtimes N_0 \supsetneq S^\beta \boxtimes N_1 \supsetneq \cdots \supsetneq S^\beta \boxtimes N_{r-2} \supsetneq S^\beta \boxtimes N_{r-1} = 0$$

Applying the exact functor F, we obtain the following filtration of R_α-modules:

$$S^\beta \circ M^\nu = S^\beta \circ N_0 \supsetneq S^\beta \circ N_1 \supsetneq \cdots \supsetneq S^\beta \circ N_{r-2} \supsetneq S^\beta \circ N_{r-1} = 0 \tag{2.3.5}$$

By Lemma 2.3.10, $M_{i+1} \cong S^\beta \circ N_i$ for $1 \leq i \leq r-2$, and $M^\lambda = S^\beta \circ M^\nu$. This filtration (2.3.5) can therefore be rewritten as:

$$M^\lambda = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_{r-1} \supsetneq M_r = 0$$

Suppose that for each i, the Specht resolution of the quotient module N_i/N_{i+1} is given by:

$$0 \longrightarrow S^{\nu_{i,t_i}} \longrightarrow \cdots \longrightarrow S^{\nu_{i,1}} \longrightarrow N_i/N_{i+1} \longrightarrow 0 \tag{2.3.6}$$

where $t_i = k_i(\nu)$. Let $k_i = k_i(\lambda)$. By our construction in Theorem 2.3.2, it is clear that $t_i = k_{i+1}$ and $\mu_{i+1,j} = (\lambda_1|\nu_{i,j})$. Applying the exact functor $F(S^\beta \boxtimes -) = S^\beta \circ -$ to this resolution (2.3.6) yields:

$$\begin{tikzcd}
0 \arrow[r] & S^\beta \circ S^{\nu_{i,t_i}} \arrow[r] \arrow[d, "\cong"] & \cdots \arrow[r] & S^\beta \circ S^{\nu_{i,1}} \arrow[r] \arrow[d, "\cong"] & S^\beta \circ (N_i/N_{i+1}) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & S^{\mu_{i+1,k_{i+1}}} \arrow[r] & \cdots \arrow[r] & S^{\mu_{i+1,1}} \arrow[r] & M_{i+1}/M_{i+2} \arrow[r] & 0
\end{tikzcd}$$

For $2 \leq i \leq r-1$, this gives to the desired Specht resolution for M_i/M_{i+1} from Theorem 2.3.2. The remaining task is to demonstrate that for the submodule M_1 (where $M_1 \subsetneq M_0$, and M_1 is generated by all Garnir relations), the quotient M_1/M_2 admits the following desired Specht resolution:

$$0 \longrightarrow S^{\mu_{1,k_1}} \xrightarrow{\phi_{1,k_1}} \cdots \longrightarrow S^{\mu_{1,1}} \xrightarrow{\phi_{1,1}} M_1/M_2 \longrightarrow 0 \tag{2.3.7}$$

We construct this resolution by applying results from [**HM15**]. First, all the 2-partitions $\mu_{1,j}$ $(1 \leq j \leq k_1)$ are Kleshchev by Lemma 2.3.15. Thus, each $S^{\mu_{1,j}}$ has a unique irreducible head $D^{\mu_{1,j}}$.

Let $d_{\lambda,\mu}$ be the decomposition number $[S^\lambda : D^\mu]$, where μ is a Kleshchev multipartition. By Corollary 2.3.18, we have

$$d_{\mu_{1,j},\mu} = \begin{cases} 1 & \text{if } \mu = \mu_{1,j} \text{ or } \mu_{1,j+1}, \\ 0 & \text{otherwise.} \end{cases}$$

This implies that each $S^{\mu_{1,j}}$ $(1 \leq j < k_1)$ has a composition series:

$$0 \subsetneq D^{\mu_{1,j+1}} \subsetneq S^{\mu_{1,j}},$$

such that

$$S^{\mu_{1,j}}/D^{\mu_{1,j+1}} \cong D^{\mu_{1,j}},$$

with $S^{\mu_{1,k_1}} \cong D^{\mu_{1,k_1}}$ itself irreducible.

Using this structure, the resolution can now be constructed explicitly. The map ϕ_{1,k_1} is the canonical embedding of $S^{\mu_{1,k_1}}$ into the submodule $D^{\mu_{1,k_1}}$ of $S^{\mu_{1,k_1-1}}$. For $2 \leq j \leq k_1 - 1$, define the maps $\phi_{1,j}$ by sending the submodule $D^{\mu_{1,j+1}}$ to zero, thereby inducing:

$$S^{\mu_{1,j}}/D^{\mu_{1,j+1}} \cong D^{\mu_{1,j}} \hookrightarrow S^{\mu_{1,j-1}}.$$

The map $\phi_{1,1}$ is constructed similarly to Theorem 2.2.9, by sending the standard cyclic generator $u_{1,1}$ of $S^{\mu_{1,1}}$ to the element $\psi^{B_1}v + M_2$. It is routine (and analogous to the arguments in Lemma 2.1.6 and Lemma 2.1.8) to verify that $\psi^{B_1}v + M_2$ satisfies all defining relations for $u_{1,1}$. Hence, the map $\phi_{1,1}$ is a surjective homogeneous homomorphism. The composition series ensures that $S^{\mu_{1,1}}$ has exactly one proper non-trivial submodule $D^{\mu_{1,2}}$. By Lemma 2.3.20 the kernel of $\phi_{1,1}$ is nonzero if $k_1 \neq 1$. Therefore, they coincide. If $k_1 = 1$, then $S^{\mu_{1,1}}$ is simple and isomorphic to M_1/M_2.

Thus, we have constructed an exact sequence given by (2.3.7), completing the proof by induction. □

Corollary 2.3.8. *Assume the same conditions as in Theorem 2.3.2, and define the modules M_i as in the generalized Specht filtration there. For each $1 \le i \le r-1$, set $\nu_i := \big(\lambda_{i+1} - 1|(\lambda_i + 1, \lambda_{i+2}, \cdots, \lambda_r)\big)$. Then $M_i/M_{i+1} \cong S^{(\lambda_1)} \circ \cdots \circ S^{(\lambda_{i-1})} \circ D^{\nu_i}$. In particular, $M_{r-1} \cong S^{\mu_{r-1,1}}$ and $M_1/M_2 \cong D^{\mu_{1,1}}$.*

Proof. By the proof of Theorem 2.3.2, the statement is true for $i = 1$ since $M_1/M_2 \cong D^{\mu_{1,1}}$ and $\nu_1 = \mu_{1,1}$. Then the general case follows by induction. □

Corollary 2.3.8 illustrates how far our generalized Specht filtration deviates from an actual Specht filtration: rather than each factor M_i/M_{i+1} being isomorphic to a Specht module, it is isomorphic to an "almost-Specht" module. We use the term "almost-Specht" to emphasize that $S^{\beta_1} \circ \cdots \circ S^{\beta_{i-1}} \circ S^{\nu_i}$ is a Specht module, and the difference lies only in the final term.

Corollary 2.3.9. *Assume the same conditions as in Theorem 2.3.2, and define the partitions $\mu_{1,j}$ as in the generalized Specht filtration there, for each $1 \le j \le k_1$. Then:*

$$\dim D^{\mu_{1,j}} = \sum_{j \le s \le k_1} (-1)^{s-j} \dim S^{\mu_{1,s}}$$

Proof. In the proof of Theorem 2.3.2, we observe that

$$\dim S^{\mu_{1,j}} = \dim D^{\mu_{1,j}} + \dim D^{\mu_{1,j+1}} \quad \text{for } j \neq k_1(\lambda),$$

and

$$\dim S^{\mu_{1,k_1(\lambda)}} = \dim D^{\mu_{1,k_1(\lambda)}}.$$

The stated equality then follows immediately. □

We remind the reader that the dimensions of simple modules in the representation theory of KLR algebras are generally difficult to compute. In our setting, however, the dimensions of the Specht modules $S^{\mu_{1,j}}$ can be readily determined via the hook length formula. Thus, the last corollary provides a convenient method for computing the dimensions of the simple modules $D^{\mu_{1,j}}$. Furthermore, in conjunction with Corollary 2.3.8, it is straightforward to compute the dimension of M_i/M_{i+1} for any $1 \le i \le r-1$.

The remainder of this section is devoted to proving the auxiliary results used in the proof of Theorem 2.3.2. For convenience, we adopt the notation and assumptions of Theorem 2.3.2 throughout, unless stated otherwise. We recall and fix some of them here for clarity. For $\lambda = (\lambda_1, \cdots, \lambda_r) \in \mathscr{P}^\Lambda_\alpha$, we set $\beta = \alpha_{(\lambda_1)} \in Q^+$ and $\nu = (\lambda_2, \dots, \lambda_r)$. The module $S^\beta := S^{(\lambda_1)}$ is the one-dimensional Specht module associated with (λ_1) over R^Λ_β. Assume M^ν has a generalized Specht filtration as in (2.3.4) where N_i is the i-th stage.

Lemma 2.3.10. *For $1 \le i \le r-2$, we have $M_{i+1} \cong S^\beta \circ N_i$ and $M^\lambda = S^\beta \circ M^\nu$.*

Proof. The last statement follows directly from the definition. Recall that N_i is an $R_{\alpha-\beta}$-submodule of M^ν.

For $1 \le i \le r-2$, let $B_{i+1} = (i+1, \lambda_{i+2}) \in [\lambda]$ and $C_i = (i, \lambda_{i+2}) \in [\nu]$ denote the last Garnir nodes in the $(i+1)$-st row of $[\lambda]$ and the i-th row of $[\nu]$, respectively. Then, by definition, we have

$$M_{i+1} = R_\alpha\{\psi^{B_{i+1}}v, \dots, \psi^{B_{r-1}}v\} \subset M^\lambda$$

and

$$N_i = R_{\alpha-\beta}\{\psi^{C_i}v, \dots, \psi^{C_{r-2}}v\} \subset M^\nu.$$

Let v_β and v_N denote the standard cyclic generators of $L_\beta = S^\beta$ and M^ν, respectively. Let $v_M := v_\beta \otimes v_N$ be the standard cyclic generator of M^λ.

Applying the exact functor $L_\beta \circ - := \mathrm{Ind}^{R_\alpha}_{R_{\beta,\alpha-\beta}}(L_\beta \otimes -)$ to $N_i \subset M^\nu$, we obtain

$$L_\beta \circ N_i \subset L_\beta \circ M^\nu \cong M^\lambda,$$

where the isomorphism on the right maps $v_\beta \otimes v_N$ to v_M.

By the standard inclusion $R_\beta \otimes R_{\alpha-\beta} \hookrightarrow R_\alpha$, we know that

$$v_\beta \otimes \psi^{C_i}v_N = \psi^{B_{i+1}}(v_\beta \otimes v_N)$$

for each $1 \le i \le r-2$. Hence,

$$\begin{aligned} L_\beta \circ N_i &= R_\alpha\{v_\beta \otimes \psi^{C_i}v_N, \dots, v_\beta \otimes \psi^{C_{r-2}}v_N\} \\ &= R_\alpha\{\psi^{B_{i+1}}(v_\beta \otimes v_N), \dots, \psi^{B_{r-1}}(v_\beta \otimes v_N)\} \\ &\cong R_\alpha\{\psi^{B_{i+1}}v_M, \dots, \psi^{B_{r-1}}v_M\} \\ &= M_{i+1}, \end{aligned}$$

as desired. □

For simplicity, from now on, we simplify the notations and write $k := k_1$, $\mu_j := \mu_{1,j}$ and $\phi_j := \phi_{1,j}$ where $1 \le j \le k$.

Recall that for a cyclotomic KLR algebra R^Λ_α, where Λ is of level ℓ, we can associate to each ℓ-partition $\boldsymbol{\lambda} \in \mathscr{P}^\Lambda_\alpha$ a cell module $C^{\boldsymbol{\lambda}}$, which is (graded) isomorphic to the Specht module $S^{\boldsymbol{\lambda}}$. There exists a distinguished subset of $\mathscr{P}^\Lambda_\alpha$ called the set of *Kleshchev partitions*. If $\boldsymbol{\lambda}$ is a Kleshchev partition, then $S^{\boldsymbol{\lambda}}$ is indecomposable and $D^{\boldsymbol{\lambda}}$ is its unique irreducible head. Moreover, the set

$$\{D^{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \mathscr{P}_\alpha^\Lambda \text{ is Kleshchev}\}$$

is a complete set of irreducible modules for R_α^Λ. These results can be found in [**HM10**].

For our purposes, we do not need the recursive definition of Kleshchev partitions; instead, we record the following results.

Proposition 2.3.11 ([**HM15**, Corollary 3.23]). *Suppose that $e = 0$ or $e > n$, $\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_\ell$ and $\mu \in \mathscr{P}_n^\Lambda$. Then $\mu = (\mu^{(1)}, \dots, \mu^{(\ell)})$ is Kleshchev if and only if*

$$\mu^{(l)}_{r+\kappa_l-\kappa_{l+1}} \leq \mu^{(l+1)}_r, \quad \textit{for } 1 \leq l < \ell \textit{ and } r \geq 1.$$

Definition 2.3.12. *Let $\mathrm{Std}^{\boldsymbol{\mu}}(\boldsymbol{\lambda})$ be the set $\{\mathbf{s} \in \mathrm{Std}(\boldsymbol{\lambda}) | \mathbf{s} \trianglerighteq T^{\boldsymbol{\mu}}$ and $\mathrm{res}(\mathbf{s}) = \mathbf{i}^{\boldsymbol{\mu}}\}$.*

Definition 2.3.13. *Suppose that $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}_n^\Lambda$. Define the graded decomposition number to be*

$$d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q) = [S^{\boldsymbol{\lambda}} : D^{\boldsymbol{\mu}}]_q = \sum_{d \in \mathbb{Z}} [S^{\boldsymbol{\lambda}} : D^{\boldsymbol{\mu}}\langle d \rangle]\, q^d.$$

where $[M : L]$ is the graded multiplicity of L in M for any graded simple module L and graded module M.

Proposition 2.3.14 ([**HM15**, Appendix B]). *Fix a linear quiver A_∞ or $A^{(1)}_{e-1}$ with $e \gg 0$. Suppose Λ is of level 2, then $\#\mathrm{Std}^{\boldsymbol{\mu}}(\boldsymbol{\lambda}) \leq 1$. If the equality holds, let $t^{\boldsymbol{\mu}}_{\boldsymbol{\lambda}}$ be the unique element in $\mathrm{Std}^{\boldsymbol{\mu}}(\boldsymbol{\lambda})$. Moreover, (suppose $\boldsymbol{\mu}$ is a Kleshchev partition), we have:*

$$d_{\boldsymbol{\lambda}\boldsymbol{\mu}}(q) = \begin{cases} q^{\deg t^{\boldsymbol{\mu}}_{\boldsymbol{\lambda}} - \deg T^{\boldsymbol{\mu}}} & \textit{if } \#\mathrm{Std}^{\boldsymbol{\mu}}(\boldsymbol{\lambda}) = 1 \\ 0 & \textit{else} \end{cases}$$

Lemma 2.3.15. *For $1 \leq j \leq k_1$, μ_j is a Kleshchev partition.*

Proof. By Proposition 2.3.11, we only need to verify $(\mu_j)^{(1)}_{r+1} \leq (\mu_j)^{(2)}_r$ for each $r \geq 1$. By construction in Theorem 2.3.2, we know this is true since ${\mu_j}^{(1)}$ consists of one row. □

We introduce two useful quantities for a partition $\lambda = (\lambda_1, \cdots, \lambda_r)$. For all admissible i, set

$$n_i := \sum_{1 \leq j \leq i} \lambda_j, \qquad d_i = \lambda_i - \lambda_{i+1} + 1. \tag{2.3.16}$$

Lemma 2.3.17. *The Specht resolution in (2.3.3) when $i = 1$ are just the 2-partitions μ in $\mathscr{P}_\alpha^{\Lambda(1)}$ such that $\mu \trianglelefteq \mu_1$ listed in the dominance order of partitions, i.e. $\mu_j \triangleleft \mu_i$ if and only if $j > i$.*

Proof. Since we are working in type A_∞ or $A^{(1)}_{e-1}$ with $e \gg 0$, each diagonal of $[\lambda]$ has a distinct residue. The only possible way to move a removable node up or down while preserving the residue is to move it along a diagonal, thereby keeping the content unchanged. It is then easy to see that, in order to maintain the structure of a 2-partition, it is impossible to move any node within a single component.

Hence, the only way to construct 2-partitions strictly smaller than $\mu_1 = (\lambda_2 - 1 \mid \nu_1)$—where $\nu_1 = (\lambda_1 + 1, \lambda_3, \cdots, \lambda_r)$—is to move nodes from the first component to the second. Under our chosen residue sequence, the only such possibility is to move the last d_1 nodes into the second row of ν_1, which yields μ_2.

Similarly, for each j, the partition μ_{j+1} is the unique 2-partition in $\mathscr{P}_\alpha^{\Lambda(i)}$ that lies immediately below μ_j with respect to the dominance order. The number k is the maximal index such that there does not exist any 2-partition in $\mathscr{P}_\alpha^{\Lambda(i)}$ lying strictly below μ_k. □

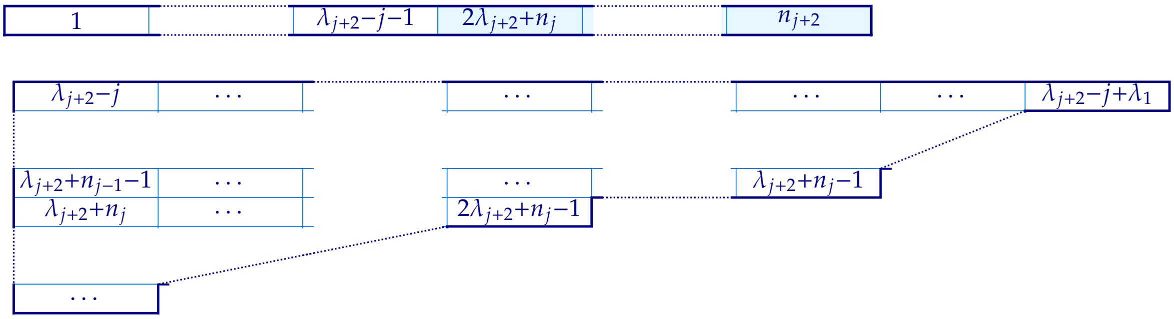
Corollary 2.3.18. *For $\mu_j, 1 \le j \le k$, we have:*

$$d_{\mu_j\nu}(q) = \begin{cases} q^{\deg t^\nu_{\mu_j} - \deg T^\nu} & \text{if } \nu = \mu_j \text{ or } \mu_{j+1} \\ 0 & \text{else} \end{cases}$$

Proof. By Proposition 2.3.14, we only need to show that $\#\mathrm{Std}^\nu(\mu_j) = 1$ if and only if $\nu = \mu_j$ or $\nu = \mu_{j+1}$.

If $\nu = \mu_j$, this is clear by taking $t^\nu_{\mu_j} = T^{\mu_j}$.

If $\nu = \mu_{j+1}$, set n_j and d_j as in (2.3.16). It is straightforward to verify that $t^\nu_{\mu_j}$ is obtained from $T^{\mu_{j+1}}$ by moving the last d_j nodes from the $(j+1)$-st row to the first component, concatenating them with the first row. In other words, $t^\nu_{\mu_j}$ is the following tableau:



By the standard theory of cellular algebras (see [**HM10**], for example), we have $d_{\mu_j,\nu} \neq 0$ only if $\mu_j \trianglerighteq \nu$. Hence, by Lemma 2.3.17, it suffices to verify that there is no element in $\mathrm{Std}^\nu(\mu_j)$ for $\nu = \mu_s$ with $s \ge j+2$.

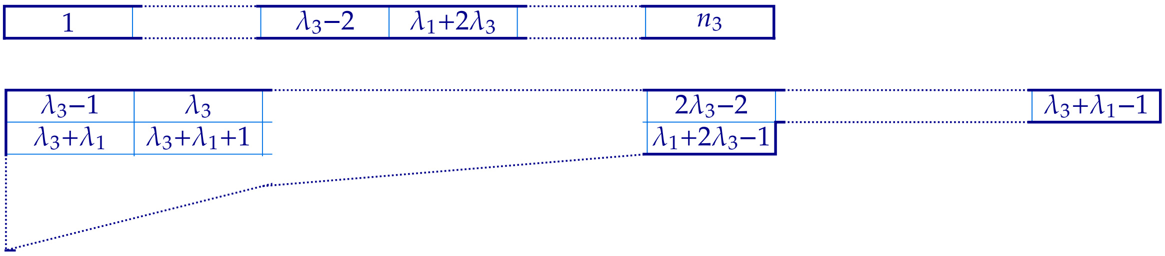
Suppose, for contradiction, that there exists $T \in \mathrm{Std}^\nu(\mu_j)$. Then the first $\lambda_{s+1} - s$ entries must be $1, 2, \ldots, \lambda_{s+1} - s$. Furthermore, by the condition $\mathbf{i}^T = \mathbf{i}^{T^\nu}$, the first j rows of the second component must coincide with T^ν as well: entries increase (in row-reading order) from $\lambda_{s+1} - s + 1$ to $\lambda_{s+1-s+n_j+j}$. The only possible difference begins at the last d_{j+1} nodes of the $(j+1)$-st row of T^ν: in the second component, μ_j has λ_{j+2} nodes in the $(j+1)$-st row, whereas ν has $\lambda_{j+2} + d_{j+1}$ nodes. In this row of T, the first λ_{j+2} entries are identical to those in T^ν. However, to satisfy $T \trianglerighteq T^\nu$, the next node must lie in the first component. But the node with this residue is not adjacent to the $(\lambda_{s+1} - s)$-th node unless $s = j + 1$. Therefore, the desired tableau cannot be standard. □

Corollary 2.3.19. *The module S^{μ_k} is irreducible.*

Proof. Since μ_k is a minimal element in $\mathscr{P}_\alpha^{\Lambda(1)}$, this can be verified either using Corollary 2.3.18, or deduced from the standard theory of cellular algebras together with Lemma 2.3.17. □

Lemma 2.3.20. *If $k \neq 1$, then $\psi^{t_{\mu_1}^{\mu_2}} \psi^{B_1} v \in M_2$.*

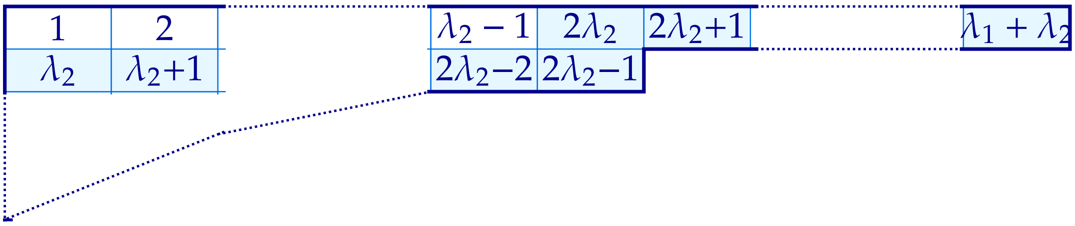
Proof. The condition $k \neq 1$ is equivalent to $\lambda_3 \geq 2$. Set n_i and d_i as in (2.3.16). Let $T := t_{\mu_1}^{\mu_2}$, by Corollary 2.3.18, we know T is of the following form:



Hence (for simplicity, we omit the idempotent in the expression of ψ^T)

$$\psi^T = (\psi_{n_3-d_2} \cdots \psi_{\lambda_3-1}) \cdots (\psi_{n_3-1} \cdots \psi_{\lambda_2-1})$$

The Garnir tableau of B_1 is of the following form:



and

$$\psi^{B_1} = (\psi_{2\lambda_2-1} \cdots \psi_{n_2-1}) \cdots (\psi_{\lambda_2} \cdots \psi_{\lambda_1}) e(\mathbf{i})$$

Set $A_i = (1, i) \in [\lambda]$ for $1 \le i \le \lambda_2$, then A_i are all the Garnir nodes between the first two rows and $B_1 = A_{\lambda_2}$. We will keep applying Lemma 2.2.10 (modulo ± signs):

$$\begin{aligned}
&\psi^T \psi^{B_1} v \\
&= (\psi_{n_3-d_2} \cdots \psi_{\lambda_3-1}) \cdots (\psi_{n_3-1} \cdots \psi_{\lambda_2-1}) \psi^{B_1} v \\
&= (\psi_{n_3-d_2} \cdots \psi_{\lambda_3-1}) \cdots (\psi_{n_3-1} \cdots \psi_{\lambda_2-1}) \psi^{A_{\lambda_2}} v \\
&= (\psi_{n_3-d_2} \cdots \psi_{\lambda_3-1}) \cdots (\psi_{n_3-1} \cdots \psi_{\lambda_1+\lambda_2})(\psi_{\lambda_1+\lambda_2-1} \cdots \psi_{\lambda_2-1}) \psi^{A_{\lambda_2}} v \\
&= (\psi_{n_3-d_2} \cdots \psi_{\lambda_3-1}) \cdots (\psi_{n_3-1} \cdots \psi_{\lambda_1+\lambda_2}) \psi^{A_{\lambda_2-1}} v \\
&= (\psi_{n_3-d_2} \cdots \psi_{\lambda_3-1}) \cdots (\psi_{n_3-2} \cdots \psi_{\lambda_2-2})(\psi_{n_3-1} \cdots \psi_{\lambda_1+\lambda_2}) \psi^{A_{\lambda_2-1}} v \\
&= (\psi_{n_3-d_2} \cdots \psi_{\lambda_3-1}) \cdots (\psi_{n_3-2} \cdots \psi_{\lambda_1+\lambda_2-1})(\psi_{\lambda_1+\lambda_2-2} \cdots \psi_{\lambda_2-2})(\psi_{n_3-1} \cdots \psi_{\lambda_1+\lambda_2}) \psi^{A_{\lambda_2-1}} v \\
&= (\psi_{n_3-d_2} \cdots \psi_{\lambda_3-1}) \cdots (\psi_{n_3-2} \cdots \psi_{\lambda_1+\lambda_2-1})(\psi_{n_3-1} \cdots \psi_{\lambda_1+\lambda_2})(\psi_{\lambda_1+\lambda_2-2} \cdots \psi_{\lambda_2-2}) \psi^{A_{\lambda_2-1}} v \\
&= (\psi_{n_3-d_2} \cdots \psi_{\lambda_3-1}) \cdots (\psi_{n_3-2} \cdots \psi_{\lambda_1+\lambda_2-1})(\psi_{n_3-1} \cdots \psi_{\lambda_1+\lambda_2}) \psi^{A_{\lambda_2-2}} v \\
&= \cdots \\
&= (\psi_{n_3-d_2} \cdots \psi_{\lambda_3+\lambda_1}) \cdots (\psi_{n_3-1} \cdots \psi_{\lambda_1+\lambda_2}) \psi^{A_{\lambda_3-1}} v \\
&= (\psi_{n_3-d_2} \cdots \psi_{n_3-1}) \cdots (\psi_{\lambda_3+\lambda_1} \cdots \psi_{\lambda_1+\lambda_2}) \psi^{A_{\lambda_3-1}} v \\
&= \psi^{B_2} \psi^{A_{\lambda_3-1}} v \\
&= \psi^{A_{\lambda_3-1}} \psi^{B_2} v \in M_2
\end{aligned}$$

Note $B_2 = (2, \lambda_3) \in [\lambda]$ and ψ^{B_2} commutes with $\psi^{A_{\lambda_3}-1}$ because the two Garnir belts do not intersect and hence all the ψ_i in the two expressions commute. □

2.4. Higher Levels and Skew Specht Filtrations

2.4.1. Higher Level Case. In this section, we briefly discuss how to construct a (generalized) Specht filtration of $M^{\boldsymbol{\lambda}}$ for $\boldsymbol{\lambda}$ an ℓ-partition with $\ell > 1$.

Let $\boldsymbol{\lambda} \in \mathscr{P}^{\Lambda}_{\alpha}$, where Λ is a dominant weight of level ℓ. By definition, we have

$$M^{\boldsymbol{\lambda}} \cong M^{\lambda^{(1)}} \circ \cdots \circ M^{\lambda^{(\ell)}}.$$

For each $1 \le s \le \ell$, we have constructed a (generalized) Specht filtration of $M^{\lambda^{(s)}}$ in various cases, as described in Section 2.1, Section 2.2, and Section 2.3. Suppose the length of $\boldsymbol{\lambda}^{(s)}$ is r_s, with the filtration of $M^{\lambda^{(s)}}$ given by:

$$M^{\lambda^{(s)}} = M_0^s \supsetneq M_1^s \supsetneq \cdots \supsetneq M_{r_s-1}^s \supsetneq M_{r_s}^s = 0.$$

For each $1 \le i \le r_s$, there exists a Specht resolution:

$$0 \longrightarrow S^{\mu^s_{ik_{si}}} \longrightarrow \cdots \longrightarrow S^{\mu^s_{i1}} \twoheadrightarrow M_i^s / M_{i+1}^s \longrightarrow 0,$$

where $k_{si} = k_i(\boldsymbol{\lambda}^{(s)})$.

Define

$$M_i := M_i^1 \circ M_i^2 \circ \cdots \circ M_i^\ell,$$

so that we obtain a filtration of M^λ:

$$M^\lambda = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_{r-1} \supsetneq M_r = 0,$$

where $r = \max\{r_s \mid 1 \le s \le \ell\}$, and we set $M_i^s := \Bbbk$ to be the trivial module whenever $i > r_s$.

Since the external tensor product commutes with taking quotients, we have:

$$M_i/M_{i+1} \cong M_i^1/M_{i+1}^1 \circ M_i^2/M_{i+1}^2 \circ \cdots \circ M_i^\ell/M_{i+1}^\ell.$$

Hence, for each i, we obtain the following Specht resolution:

$$0 \longrightarrow S^{\mu_{ik_i}^1} \circ S^{\mu_{ik_i}^2} \circ \cdots \circ S^{\mu_{ik_i}^\ell} \longrightarrow \cdots \longrightarrow S^{\mu_{i1}^1} \circ S^{\mu_{i1}^2} \circ \cdots \circ S^{\mu_{i1}^\ell} \twoheadrightarrow M_i/M_{i+1} \longrightarrow 0,$$

where $k_i = \max\{k_{si} \mid 1 \le s \le \ell\}$. For any $j > k_{si}$, we set $\mu_{ij}^s := \emptyset$ to be the empty partition, so that $S^{\mu_{ij}^s}$ is the trivial module.

For each $1 \le j \le k_i$, define

$$\nu_{ij} := (\mu_{ij}^1 \mid \mu_{ij}^2 \mid \cdots \mid \mu_{ij}^\ell) \in \mathscr{P}_\alpha^\Lambda.$$

Then, by Theorem 1.3.33, the above Specht resolution becomes

$$0 \longrightarrow S^{\nu_{ik_i}} \longrightarrow \cdots \longrightarrow S^{\nu_{i1}} \twoheadrightarrow M_i/M_{i+1} \longrightarrow 0.$$

2.4.2. Skew Specht filtration. As mentioned in Remark 2.1.10, we can construct a skew Specht filtration by reversing the order of the Garnir relations. To be precise, suppose we are working in type A_∞, and let $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathscr{P}_\alpha^\Lambda$ be a partition. Define $C_i := (i, 1) \in [\lambda]$ for $1 \le i \le r-1$ to be the set of Garnir nodes in the first column. By Theorem 2.2.4, we know that

$$M_1 = R_\alpha\{\psi^{C_1}v, \ldots, \psi^{C_{r-1}}v\}$$

is the submodule generated by all Garnir relations. Let v be the standard cyclic generator of M^λ. Instead of defining

$$M_i := R_\alpha\{\psi^{C_i}v, \ldots, \psi^{C_{r-1}}v\} \quad \text{for } 1 \le i \le r-1,$$

as in Theorem 2.3.2, we define

$$M_i' := R_\alpha\{\psi^{C_1}v, \ldots, \psi^{C_{r-i}}v\}.$$

Then the filtration

$$M^\lambda = M_0' \supsetneq M_1' \supsetneq \cdots \supsetneq M_{r-1}' \supsetneq M_r' = 0$$

has the following property: for each $1 \le i \le r-1$, there exists a resolution of M_i'/M_{i+1}' by skew Specht modules as introduced in [Mut15]. The explicit formulas for the skew partitions that appear in this resolution, as well as the detailed proof, are omitted.

CHAPTER 3

Stingray Patterns of Dominant Weights

This chapter is based on [**Qin26a**], with minor changes. Throughout the chapter, we work with the e-abacus with finitely many beads, as introduced in Subsection 1.2.1. Thus, for any partition λ, we always choose an integer $r \geq \ell(\lambda)$ and consider the abacus $\mathrm{Ab}_e^r(\lambda)$ with r beads.

We briefly summarize the main ideas and results. For any partition λ with at most r rows, there is a canonical way to identify it with a dominant weight of $\mathfrak{sl}_r$. We introduce this construction in Section 3.1 and write the resulting map as Ω. The importance of this map is that, by Theorem 3.3.4, it interprets the graded decomposition numbers $d^e_{\lambda,\mu}(q)$ (see Subsection 1.3.7) as the anti-spherical Kazhdan–Lusztig polynomials $\mathfrak{n}^e_{\Omega(\lambda),\Omega(\mu)}(q)$ (see Subsection 1.3.2 and Subsection 1.4.4). For the latter, the independence of the quantum characteristic e is transparent, whereas it is difficult to see directly for the former.

In combinatorial terms, the independence of e for graded decomposition numbers is reflected in the abacus by the runner-removal theorems. In particular, the first such theorem is the empty-runner-removal theorem of James and Mathas; see Theorem 3.3.1. By a remark of Goodman, it is possible to deduce Theorem 3.3.1 from the above identification in a more transparent way than using Fock space calculations; see Section 3.3. In short, the map Ω behaves well under empty-runner addition (or removal), preserving the alcove and hence preserving the anti-spherical Kazhdan–Lusztig polynomials.

However, something mysterious happens: the other runner-removal theorems, such as Theorem 3.3.2, do not interact well with the map Ω. Finding an explicit description of this interaction is still unclear to the author.[1] Nevertheless, we consider a closely related and intrinsically interesting question.

The blocks of Iwahori–Hecke algebras can be described in terms of partitions via the e-core and e-weight; see Subsection 1.3.7. For any $r \geq 1$, the domain of the map Ω is the set of partitions with at most r rows. These partitions can certainly have different e-cores and e-weights. Fix an e-weight w, and consider the set of partitions with at most r rows and e-weight w, denoted by $\mathscr{P}_{r,e,w}$. Our aim is to study the image $\mathcal{W}_{r,e,w}$ of $\mathscr{P}_{r,e,w}$ under the map Ω.

When $e < r$, the situation is rather singular and depends on e, so we restrict to the more regular case $e > r$. In this setting, based on abacus combinatorics, one can deduce

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[1] We are working on this using AI, and hope to say more in the future.

that $\mathcal{W}_{r,e,w}$ admits a simplicial decomposition; see Theorem 3.2.5 for the case $w = 0$ and Theorem 3.2.9 for general w. As a corollary, we obtain a closed counting formula for $|\mathcal{W}_{r,e,w}|$; see Corollary 3.2.13 for $w = 0$ and Theorem 3.2.14 for general w.

From Figure 10, which depicts $\mathcal{W}_{3,8,w}$ for $0 \le w \le 9$, one observes an interesting boundary pattern that resembles stingrays; when w is large, one also sees hexagons appearing in the interior. We study these phenomena, which we call the stingray pattern and the regular pattern, respectively; see Example 3.2.35 for a colouring of the stingrays. It turns out that the stingray pattern appears for every $e > r$ once w is sufficiently large; see Subsection 3.2.3 for a complete treatment. We also treat the regular pattern when $r = 3$, mainly because for $r > 3$ some unexpected holes appear.

Finally, we study the $(r-1)$-simplices appearing in $\mathcal{W}_{r,e,w}$, which correspond to weak compositions; see Subsection 3.2.4. This provides a way to index a subset of the alcoves, and this indexing is compatible with the action of the affine Weyl group on abaci and on alcoves.

Recall from Subsection 1.2.1 that, for a partition λ and a positive integer $a > \ell(\lambda)$, we define the a-beta numbers $\beta_i = \beta_i(\lambda) := \lambda_i + a - i$ for $1 \le i \le a$ and the e-abacus of λ with a beads. Since each bead has position β_i, we identify the bead with its position and refer to this bead as β_i throughout this chapter. In an e-abacus, the runners are labeled $0, 1, \cdots, e-1$ from left to right. In this chapter, for convenience, we often refer to a runner as the k-th runner; by convention, this means the k-th runner when counting from the left.

3.1. The geometric map Ω

We now introduce the central map connecting partitions to the weight lattice. Recall that for a partition λ with $\ell(\lambda) \le r$, the r-beta numbers are $\beta_i(\lambda) = \lambda_i + r - i$. Define the map Ω sending a partition λ to a dominant weight in P^+ by:

$$\Omega(\lambda) = \sum_{i=1}^{r-1} (\lambda_i - \lambda_{i+1} + 1)\Lambda_i.$$

Substituting the definition of the beta numbers, we observe that the coefficients correspond precisely to the consecutive differences of the beta sequence:

$$\beta_i(\lambda) - \beta_{i+1}(\lambda) = (\lambda_i + r - i) - (\lambda_{i+1} + r - (i+1)) = \lambda_i - \lambda_{i+1} + 1.$$

Thus, the map can be written in terms of beta numbers as:

$$\Omega(\lambda) = \sum_{i=1}^{r-1} (\beta_i(\lambda) - \beta_{i+1}(\lambda))\Lambda_i.$$

This map allows us to visualize partitions as points in the dominant chamber C^+.

3.2. Patterns for dominant weights

In Section 3.1, we defined the map Ω which sends partitions to the dominant weight lattice P^+. Since the blocks of the Hecke algebra are parametrized by pairs consisting of an e–core and an e–weight (see Subsection 1.2.4 and Subsection 1.3.7), it is natural to investigate the geometric pattern of the dominant weights corresponding to partitions with a fixed e-weight. In this section, we describe and analyze these patterns under the assumption that $e > r \geq 3$.

For any triple $(r, e, w) \in \mathbb{N}^3$ with $e \geq 2$, let $\mathscr{P}_{r,e,w}$ denote the set of partitions with at most r parts and e–weight w. In particular, $\mathscr{P}_{r,e,0}$ consists of the e–core partitions of length at most r.

Let P^+ be the dominant weight lattice of $\mathfrak{sl}_r$. The corresponding set of dominant weights of $\mathscr{P}_{r,e,w}$ under Ω is:

$$\mathcal{W}_{r,e,w} := \Omega(\mathscr{P}_{r,e,w}) \subset P^+.$$

Our primary goal is to characterize the set $\mathcal{W}_{r,e,w}$.

Example 3.2.1. *We draw $\mathcal{W}_{3,10,8}$ in the dominant weight lattice in Figure 1, where each black ball represents a dominant weight in $\mathcal{W}_{3,10,8}$.* ◇

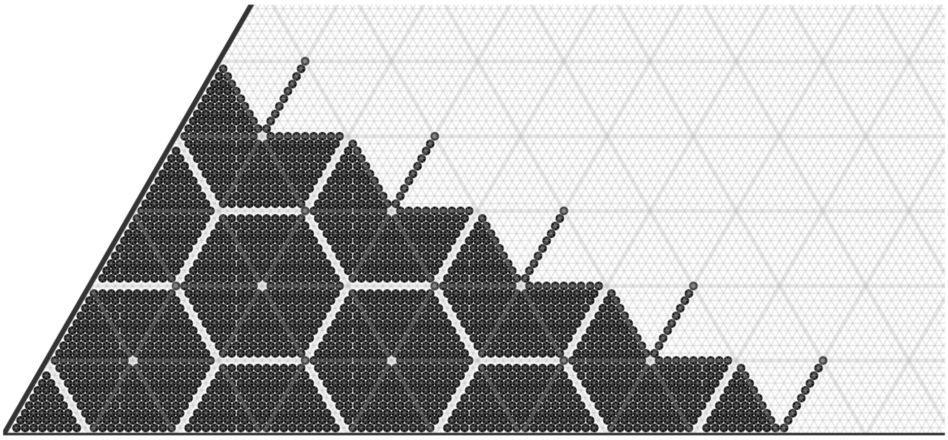


Figure 1. $r = 3$, $e = 10$, $w = 8$

3.2.1. Simplicial structure of e-cores.

We start with the case of e-cores (i.e., $w = 0$). To simplify notation, we define:

$$\mathscr{P}_{r,e} := \mathscr{P}_{r,e,0} \quad \text{and} \quad \mathcal{W}_{r,e} := \mathcal{W}_{r,e,0}.$$

Thus $\mathscr{P}_{r,e}$ is the set of e-core partitions with at most r parts, and $\mathcal{W}_{r,e} = \Omega(\mathscr{P}_{r,e})$ is the corresponding set of dominant weights. We assume throughout that $r < e$.

For each composition $\mu = (\mu_1, \ldots, \mu_j)$ of r (this implies $j \leq r$), we define a subset of partitions $\mathscr{P}_{r,e}(\mu) \subset \mathscr{P}_{r,e}$ as follows. Consider the family of e–abaci obtained by choosing j distinct runners $0 \leq s_1 < \cdots < s_j \leq e - 1$ and placing μ_i beads on the

s_i–runner (as high as possible) for each $i = 1, \dots, j$, while leaving the remaining runners empty. Each such abacus corresponds to a unique e-core partition; let $\mathscr{P}_{r,e}(\mu)$ be the set of all such partitions and define the corresponding set of dominant weights by:

$$\mathcal{W}_{r,e}(\mu) \;:=\; \Omega(\mathscr{P}_{r,e}(\mu)).$$

Since every e-core partition with at most r parts corresponds to an e-abacus with exactly r beads placed on some flush runners, we have the following decomposition:

$$\mathscr{P}_{r,e} \;=\; \bigsqcup_{\mu \models r} \mathscr{P}_{r,e}(\mu) \quad \text{and} \quad \mathcal{W}_{r,e} \;=\; \bigsqcup_{\mu \models r} \mathcal{W}_{r,e}(\mu),$$

Before stating the main result of this subsection, we formalize our geometric terminology.

Definition 3.2.2. *Let $d \geq 0$ and $L > 0$ be integers. We define the* standard d-dimensional simplex of dilation factor L *to be the set:*

$$\Delta^d(L) \;=\; \left\{(x_0, \dots, x_d) \in \mathbb{R}^{d+1}_{\geq 0} \;\middle|\; x_0 + \cdots + x_d = L\right\}.$$

The lattice points *of $\Delta^d(L)$ are the points in $\Delta^d(L) \cap \mathbb{Z}^{d+1}$.*

Definition 3.2.3. *Let V and W be vector spaces. A map $\phi : V \longrightarrow W$ is called* affine linear *if there exists a linear transformation $T : V \longrightarrow W$ and a constant vector $c \in W$ such that*

$$\phi(v) \;=\; T(v) + c \qquad \textit{for all } v \in V.$$

In the context of the weight lattice P, we say that a subset $S \subset P$ forms the *lattice points of a d-dimensional simplex of dilation factor L* if there exists an injective affine linear map $\phi : \mathbb{R}^{d+1} \longrightarrow P \otimes \mathbb{R}$ such that $S = \phi(\Delta^d(L) \cap \mathbb{Z}^{d+1})$.

We require the following lemma to handle the translation-invariance of the abacus under Ω.

Lemma 3.2.4. *Let $\beta = (\beta_1, \dots, \beta_r)$ and $\gamma = (\gamma_1, \dots, \gamma_r)$ be beta numbers. The map Ω, viewed as a function of the beta numbers via $\Omega(\beta) = \sum_{i=1}^{r-1}(\beta_i - \beta_{i+1})\Lambda_i$, is only invariant under global shift. That is, $\Omega(\beta) = \Omega(\gamma)$ if and only if there exists an integer $k \in \mathbb{Z}$ such that $\gamma_i = \beta_i + k$ for all $1 \leq i \leq r$.*

Proof. If $\gamma_i = \beta_i + k$ for all i, then $\gamma_i - \gamma_{i+1} = (\beta_i + k) - (\beta_{i+1} + k) = \beta_i - \beta_{i+1}$. Since Ω depends only on these differences, $\Omega(\gamma) = \Omega(\beta)$.

Conversely, if $\Omega(\gamma) = \Omega(\beta)$, then the coefficients in the fundamental weight basis must match:

$$\gamma_i - \gamma_{i+1} \;=\; \beta_i - \beta_{i+1} \quad (1 \leq i \leq r-1).$$

This implies $\gamma_i - \beta_i = \gamma_{i+1} - \beta_{i+1}$ for all i. The difference $\gamma_i - \beta_i$ is a constant k independent of i, proving the claim. □

Theorem 3.2.5. *Fix integers $e > r > 0$, and let μ be a composition of r of length j. Then $\mathcal{W}_{r,e}(\mu)$ forms the lattice points of a $(j-1)$–dimensional simplex of dilation factor $e - j$.*

Proof. Fix the composition μ. Partitions in $\mathscr{P}_{r,e}(\mu)$ are determined by the positions of the j non-empty runners, denoted $0 \leq s_1 < s_2 < \cdots < s_j \leq e-1$.

We define the *gap variables* $g_0, \dots, g_j$ representing the spacing between non-empty runners:

$$\begin{aligned} g_0 &= s_1, \\ g_k &= s_{k+1} - s_k - 1 \quad \text{for } 1 \leq k \leq j-1, \\ g_j &= e - 1 - s_j. \end{aligned}$$

These variables satisfy $g_k \in \mathbb{Z}_{\geq 0}$ and $\sum_{k=0}^{j} g_k = e - j$.

By Lemma 3.2.4, the map Ω is invariant under global shift. Hence we may shift all beads to the left by s_1, which does not change the image. With this normalization, $s_1 = 0 = g_0$.

The remaining variables $\mathbf{g} = (g_1, \dots, g_j)$ satisfy $\sum_{k=1}^{j} g_k = e - j$, corresponding to the lattice points of the standard simplex $\Delta^{j-1}(e-j)$. It remains to show that Ω induces an injective, affine linear map on this domain.

Let $\kappa \in \mathscr{P}_{r,e}(\mu)$ be the core partition corresponding to $\mathbf{g}$ and let β_i be the corresponding beta numbers of κ. The corresponding dominant weight is $\Omega(\kappa) = \sum_{i=1}^{r-1} (\beta_i - \beta_{i+1}) \Lambda_i$. Suppose the beads β_i and β_{i+1} lie on the chosen runners indexed by s_u and s_v, respectively, where $1 \leq u, v \leq j$. Then

$$\beta_i - \beta_{i+1} = \begin{cases} u - v + \sum\limits_{v \leq t < u} g_t & \text{if } u > v \text{ (same rows)}, \\ j + u - v + \sum\limits_{v \leq t \leq j} g_t + \sum\limits_{1 \leq t < u} g_t & \text{if } u \leq v \text{ (different rows)}. \end{cases} \tag{3.2.6}$$

In both cases, $\beta_i - \beta_{i+1}$ is an affine linear function of the variables $\mathbf{g}$. Thus, Ω is an affine linear map in the sense of Definition 3.2.3.

To verify injectivity, consider two distinct elements $(g_0, g_1, \dots, g_j) = \mathbf{g} \neq \mathbf{g}' = (g'_0, g'_1, \dots, g'_j)$ in the domain. We may normalize as above, and hence $g_0 = g'_0 = 0$. Let $k = \min\{1 \leq t \leq j \mid g_t \neq g'_t\}$. Consider the top beads located on the k-th non-empty runner and the $k+1$-th non-empty runner. By (3.2.6), the difference between their values is $g_k + 1$ and $g'_k + 1$, respectively. Thus the coordinate associated with this pair differs, implying $\Omega(\mathbf{g}) \neq \Omega(\mathbf{g}')$. □

Remark 3.2.7. *We do not need the assumption $r < e$ to show that $\mathcal{W}_{r,e}(\mu)$ is a simplex. However, without this condition the claim that $\mathcal{W}_{r,e}(\mu)$ has dimension $j-1$ may fail. For example, if $\ell(\mu) = e < r$, then $\mathcal{W}_{r,e}(\mu)$ collapses to a single point; see Example 3.2.8.* ◇

Example 3.2.8. *Consider the case $\mathfrak{sl}_3$ (so $r = 3$) with $e = 2$. Then $|\mathcal{W}_{3,2}(\mu)| = 1$ for $\mu \in \{(2,1),(1,2)\}$, and $\mathcal{W}_{3,2}(1,1,1) = \varnothing$. See Figure 2. Similarly, see Figure 3 for the case $\mathfrak{sl}_3$ with $e = 3$ where $|\mathcal{W}_{3,3}(1,1,1)| = 1$.*

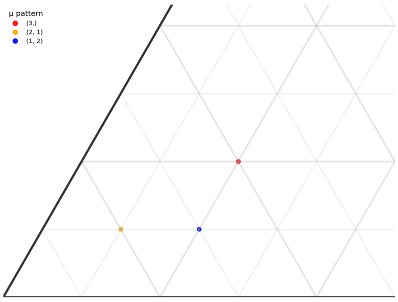


FIGURE 2. $r = 3, e = 2, w = 0$

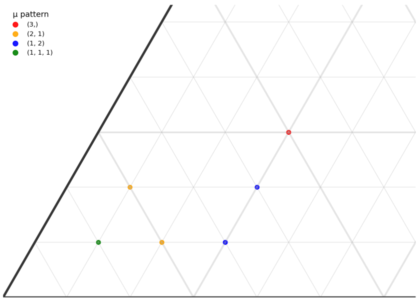


FIGURE 3. $r = 3, e = 3, w = 0$

◇

3.2.2. Simplicial Decomposition of $\mathcal{W}_{r,e,w}$. We now generalize from $w = 0$ to arbitrary weight $w \in \mathbb{N}$. For a fixed composition $\mu \models r$ of length j, define a subset of $\mathscr{P}_{r,e,w}$ as follows:

$$\mathscr{P}_{r,e,w}(\mu) := \left\{\lambda \in \mathscr{P}_{r,e,w} \mid \operatorname{core}_e(\lambda) \in \mathscr{P}_{r,e}(\mu)\right\}.$$

For any partition $\lambda \in \mathscr{P}_{r,e,w}(\mu)$, the e-quotient (see (1.2.11)) encodes the vertical moves of beads relative to the e-core to have e-weight w. Since the core lies in $\mathscr{P}_{r,e}(\mu)$, the beads of λ are supported on exactly j runners. Consequently, the e-quotient has empty entries corresponding to the other empty runners. We define the *restricted e-quotient* of λ to be the subsequence of the e-quotient corresponding to these j non-empty runners, denoted by the tuple $(\lambda^{(1)}, \ldots, \lambda^{(j)})$. Since the i-th non-empty runner (counting from left to right) carries exactly μ_i beads, the corresponding partition $\lambda^{(i)}$ must satisfy the length constraint $\ell(\lambda^{(i)}) \leq \mu_i$.

Recall from Definition 1.2.7 that these tuples correspond to the j-partitions of w of type μ, and $A(\mu; w)$ is the number of such j-partitions. Set $\mathcal{W}_{r,e,w}(\mu) := \Omega\big(\mathscr{P}_{r,e,w}(\mu)\big)$. Then we have the decomposition:

$$\mathscr{P}_{r,e,w} = \bigsqcup_{\mu \models r} \mathscr{P}_{r,e,w}(\mu) \quad \text{and} \quad \mathcal{W}_{r,e,w} = \bigsqcup_{\mu \models r} \mathcal{W}_{r,e,w}(\mu).$$

THEOREM 3.2.9 (**Simplicial Decomposition**). *Fix positive integers $e > r$ and $w \in \mathbb{N}$. Let μ and ν be distinct compositions of r. Then*

(a). $\mathcal{W}_{r,e,w}(\mu) \cap \mathcal{W}_{r,e,w}(\nu) = \varnothing$.

(b). *The set $\mathcal{W}_{r,e,w}(\mu)$ is a disjoint union of copies of the simplex $\mathcal{W}_{r,e}(\mu)$, and the number of copies is exactly $A(\mu; w)$.*

PROOF. **(a)** Take $\lambda \in \mathscr{P}_{r,e,w}(\mu)$ and $\eta \in \mathscr{P}_{r,e,w}(\nu)$ with $\Omega(\lambda) = \Omega(\eta)$. Let

$$\beta(\lambda) = (\beta_1, \ldots, \beta_r), \qquad \beta(\eta) = (\beta'_1, \ldots, \beta'_r)$$

be the corresponding r-beta numbers in decreasing order. By Lemma 3.2.4, there exists an integer k such that $\beta'_i = \beta_i + k$ for all $1 \leq i \leq r$.

This corresponds to a global shift of all runners to the right by k (or to the left by $-k$), which does not change the number of non-empty runners. Hence $\ell(\mu) = \ell(\nu)$, and ν is obtained from μ by a cyclic permutation of its parts:

$$\nu = (\mu_i, \mu_{i+1}, \dots, \mu_j, \mu_1, \dots, \mu_{i-1})$$

for some $1 \leq i \leq j$.

If $i = 1$ then $\nu = \mu$, as desired. Suppose $i \neq 1$. A nontrivial cyclic shift moves at least one non-empty runner from the rightmost to the left, which strictly increases the the e-weight. Since both λ and η lie in $\mathscr{P}_{r,e,w}$ and have the same e-weight w, this is impossible. Therefore $i = 1$ and $\mu = \nu$.

(b) Fix the composition μ of length j. Then by Theorem 3.2.5 $\mathcal{W}_{r,e}(\mu)$ is a $(j-1)$-dimensional simplex where the degrees of freedom come from choosing the runner gaps. For $w \geq 0$, any $\lambda \in \mathscr{P}_{r,e,w}(\mu)$ is obtained from some core $\lambda^0 \in \mathscr{P}_{r,e}(\mu)$ by moving beads downwards on the chosen j runners, and the moves are recorded by the restricted e-quotient, which is a j-partition of w of type μ.

Thus each j-partition of w of type μ determines a copy of the simplex $\mathcal{W}_{r,e}(\mu)$ inside $\mathcal{W}_{r,e,w}(\mu)$. Since there are $A(\mu; w)$ such j-partitions, there are $A(\mu; w)$ such copies.

To see they are disjoint, let $\lambda, \eta \in \mathscr{P}_{r,e,w}(\mu)$ define the same dominant weight. Let $\beta_i(\lambda)$ and $\beta_i(\eta)$ be their beta numbers. By Lemma 3.2.4, there exists some integer k such that $\beta_i(\eta) = \beta_i(\lambda) + k$ for all $1 \leq i \leq r$. Write $k = qe + s$ with $0 \leq s < e$. We may assume $q \geq 0$.

First, consider the term qe. Increasing a beta number by e corresponds to sliding a bead down one row on its runner. If we increase every bead $\beta_i(\lambda)$ by qe, then the row index of every bead increase by q. Since there are r beads in total, this operation increases the total e-weight by $q \cdot r$. Since λ and η both have fixed e-weight w, we must have $q = 0$.

Now we are left with the term $k = s$ where $0 \leq s < e$. This corresponds to a cyclic shift of the runner indices by s. However, by definition of the simplex $\mathcal{W}_{r,e,0}(\mu)$, we have fixed the runner configuration (specifically, we normalized the first non-empty runner to be at position 0 or fixed the gap variables). As argued in part (a), such a shift strictly changes the e-weight unless $s = 0$. Thus $k = 0$ and $\lambda = \eta$. Therefore, the copies are pairwise disjoint. □

Example 3.2.10. *In Figure 10, we plot the sets $\mathcal{W}_{3,10,w}$ for weights w ranging from 0 to 9. This figure is drawn in the dominant weight lattice of $\mathfrak{sl}_3$, where each colored ball represents a dominant weight in $\mathcal{W}_{3,10,w}$. We use distinct colors to distinguish the subsets $\mathcal{W}_{3,10,w}(\mu)$ arising from the simplicial decomposition, corresponding to the types $\mu \in \{(3), (2,1), (1,2), (1,1,1)\}$.* ◇

EXAMPLE 3.2.11. *For* $(r, e, w) = (3, 8, 8)$, *consider* $W_{r,e,w}(\mu)$ *with* $\mu = (1, 1, 1)$. *We label each copy of* $W_{r,e}(\mu)$ *by the corresponding* 3*–partition of type* μ, *see* Figure 4. *Similarly, see* Figure 5 *for the case* $(r, e, w) = (3, 12, 10)$. ◇

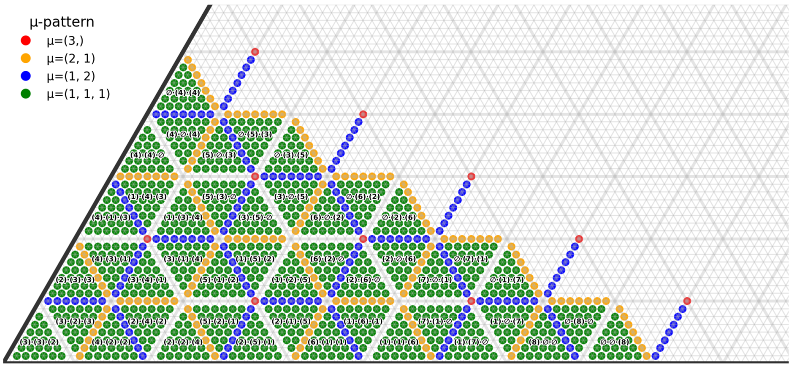


FIGURE 4. $r = 3$, $e = 8$, $w = 8$

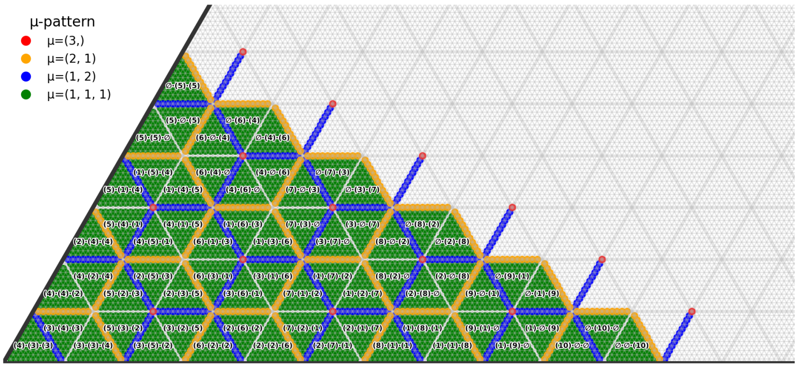


FIGURE 5. $r = 3$, $e = 12$, $w = 10$

COROLLARY 3.2.12. *The number of lattice points in a* $(j-1)$*-dimensional simplex of dilation factor* $e - j$ *is given by the binomial coefficient* $\binom{e-1}{j-1}$.

PROOF. By Theorem 3.2.5, the set of weights $\mathcal{W}_{r,e,0}(\mu)$ corresponds bijectively to the set of gap variable tuples $\mathbf{d} = (d_1, \ldots, d_j)$ satisfying $d_i \in \mathbb{Z}_{\geq 0}$ and $\sum_{i=1}^{j} d_i = e - j$.

This counts the number of non-negative integer solutions to a linear equation, which is a standard stars-and-bars problem. Alternatively, recall that these tuples parametrise the choice of j distinct runners $0 = s_1 < s_2 < \cdots < s_j \leq e - 1$. Since the first runner is fixed at 0, we must choose the remaining $j-1$ distinct indices from the set $\{1, \ldots, e-1\}$. The number of such choices is exactly $\binom{e-1}{j-1}$. □

COROLLARY 3.2.13. *Assume* $r < e$. *Then*

$$|\mathcal{W}_{r,e}| = \binom{e+r-2}{r-1}.$$

Proof. By Theorem 3.2.9, $\mathcal{W}_{r,e}$ is the disjoint union of the sets $\mathcal{W}_{r,e}(\mu)$ as μ ranges over all compositions of r. If μ has length j, Corollary 3.2.12 states that $|\mathcal{W}_{r,e}(\mu)| = \binom{e-1}{j-1}$.

The number of compositions of r into exactly j parts is $\binom{r-1}{j-1}$. Summing over all possible lengths $1 \le j \le r$, we obtain:

$$\begin{aligned} |\mathcal{W}_{r,e,0}| &= \sum_{j=1}^{r} \binom{r-1}{j-1}\binom{e-1}{j-1} \\ &= \sum_{k=0}^{r-1} \binom{r-1}{k}\binom{e-1}{k} \\ &= \sum_{k=0}^{r-1} \binom{r-1}{r-1-k}\binom{e-1}{k} \qquad \text{(symmetry of binomial coeffs)} \\ &= \binom{(r-1)+(e-1)}{r-1}, \end{aligned}$$

where the last equality follows from Vandermonde's Identity. □

Theorem 3.2.14. *Fix integers $r \ge 3$, $e > r$, and $w \ge 0$. Then*

$$|\mathcal{W}_{r,e,w}| = \sum_{\mu \models r} A(\mu; w)\binom{e-1}{\ell(\mu)-1}, \tag{3.2.15}$$

Proof. By Theorem 3.2.9(a), $\mathcal{W}_{r,e,w}$ is the disjoint union of the sets $\mathcal{W}_{r,e,w}(\mu)$ as μ runs over all compositions of r. Thus,

$$|\mathcal{W}_{r,e,w}| = \sum_{\mu \models r} |\mathcal{W}_{r,e,w}(\mu)|.$$

Fix a composition $\mu = (\mu_1, \dots, \mu_j) \models r$. By Corollary 3.2.12, the size of the simplex $\mathcal{W}_{r,e}$ is $\binom{e-1}{j-1}$. By Theorem 3.2.9(b), $\mathcal{W}_{r,e,w}(\mu)$ is a disjoint union of copies of this simplex. The number of copies is exactly $A(\mu; w)$. Therefore:

$$|\mathcal{W}_{r,e,w}(\mu)| = A(\mu; w)\binom{e-1}{j-1}. \tag{3.2.16}$$

Summing (3.2.16) over all compositions $\mu \models r$ yields the desired formula. □

Corollary 3.2.17. *Suppose $r = 3$ and $r < e$. Then $|\mathcal{W}_{3,e,w}|$ is given by the quadratic polynomial in e:*

$$|\mathcal{W}_{3,e,w}| = a_w e^2 + b_w e + c_w, \tag{3.2.18}$$

where the coefficients depend only on w:

$$a_w = \frac{(w+1)(w+2)}{4}, \qquad b_w = \left\lfloor \frac{w+2}{2} \right\rfloor - a_w, \qquad c_w = p_3(w).$$

Here, $p_3(w) = \lfloor (w^2+3)/12 \rfloor$ is the number of partitions of w into exactly 3 parts.

Proof. We apply Theorem 3.2.14 to the case $r = 3$. The compositions of 3 are (3), (2, 1), (1, 2), and (1, 1, 1). We compute the terms $A(\mu; w)$ for each case:

- For $\mu = (1, 1, 1)$, we count tuples of 3-partitions $(\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)})$ where $\ell(\lambda^{(i)}) \leq 1$. This corresponds to finding three non-negative integers summing to w. Thus $A((1, 1, 1); w) = \binom{w+2}{2}$ by stars-and-bars.
- For $\mu = (2, 1)$, we sum over the size k of the second partition (which has length ≤ 1, so it is just a non-negative integer). The first partition has size $w - k$ and length ≤ 2. Let $p_{\leq 2}(n)$ denote the number of partitions of n with at most 2 parts. Then $A((2, 1); w) = \sum_{k=0}^{w} p_{\leq 2}(w - k)$. By symmetry, $A((1, 2); w)$ yields the same count.
- For $\mu = (3)$, we count partitions of w with length at most 3. Thus $A((3); w) = p_{\leq 3}(w)$.

Substituting these into the general formula (3.2.15) yields:

$$|\mathcal{W}_{3,e,w}| = \binom{w+2}{2}\binom{e-1}{2} + 2\left(\sum_{k=0}^{w} p_{\leq 2}(k)\right)\binom{e-1}{1} + p_{\leq 3}(w).$$

The algebraic verification that this expression simplifies to $a_w e^2 + b_w e + c_w$ relies on standard partition identities, which is Lemma 3.4.5. □

3.2.3. Stingray and Regular patterns. In this section, we analyze the geometric structure of the weight sets $\mathcal{W}_{r,e,w}$. While our primary motivation is to explain the visual patterns observed in the case $r = 3$ (such as the "stingray" and "hexagon" shapes seen in Figure 10), many of the underlying structural results hold for arbitrary r. We therefore formulate our results for general $r \geq 3$ wherever possible.

We first establish a recurrence relation connecting patterns of weight w to those of weight $w + 1$. This explains how the pattern "grows" as the e-weight increases.

Lemma 3.2.19. *Assume $e > r$. The set $\mathcal{W}_{r,e,w}$ embeds into $\mathcal{W}_{r,e,w+1}$ via a translation by $e\Lambda_1$. Specifically, a point $\sum_{i=1}^{r-1} a_i\Lambda_i$ lies in $\mathcal{W}_{r,e,w}$ if and only if the point $(a_1+e)\Lambda_1+\sum_{i=2}^{r-1} a_i\Lambda_i$ lies in $\mathcal{W}_{r,e,w+1}$. Geometrically, this image of $\mathcal{W}_{r,e,w}$ is precisely $\mathcal{W}_{r,e,w+1} \cap H^{+}_{\alpha_1,e}$.*

Proof. Let $\lambda \in \mathscr{P}_{r,e,w}$ be a partition such that $\Omega(\lambda) = \sum a_i\Lambda_i$. Consider the e-abacus of λ with r beads. The coefficients of $\Omega(\lambda)$ are given by the differences of beta numbers: $a_i = \beta_i(\lambda) - \beta_{i+1}(\lambda)$.

We modify the abacus by sliding down the first bead β_1 (the largest beta number) by one. This operation replaces β_1 with $\beta_1' = \beta_1 + e$, leaving all other beads fixed ($\beta_j' = \beta_j$ for $j > 1$). Since $\beta_1 > \beta_2$, clearly $\beta_1' > \beta_2$, so the new sequence of beta numbers is still strictly decreasing and corresponds to a new partition λ'. Write $\Omega(\lambda') = \sum a_i'\Lambda_i$. The new coefficients a_i' are:

$$\begin{aligned} a_1' &= \beta_1' - \beta_2' = (\beta_1 + e) - \beta_2 = a_1 + e, \\ a_i' &= \beta_i' - \beta_{i+1}' = \beta_i - \beta_{i+1} = a_i \quad \text{for } i \geq 2. \end{aligned}$$

By the definition of e-weight, $w_e(\lambda') = w_e(\lambda) + 1 = w + 1$. Thus $\Omega(\lambda') \in \mathcal{W}_{r,e,w+1}$.

Conversely, suppose $\sum b_i\Lambda_i \in \mathcal{W}_{r,e,w+1}$ with $b_1 > e$. Suppose that this dominant weight arises from a partition η whose beta numbers are $\gamma_1, \dots, \gamma_r$. Then the condition $b_1 > e$ implies $\gamma_1 - \gamma_2 > e$. Thus $\gamma_1 - e$ is strictly greater than γ_2, and in the abacus of η we can slide the first bead γ_1 up by one, replacing γ_1 with $\gamma_1 - e$. This yields a partition of e-weight w mapping to $\sum(b_i - \delta_{i,1}e)\Lambda_i \in \mathcal{W}_{r,e,w}$. □

We next study the occurrence of 0–dimensional simplices in $\mathcal{W}_{r,e,w}$.

Lemma 3.2.20. *Let $a_1, \dots, a_{r-1}$ be non-negative integers. The point $P = \sum_{i=1}^{r-1} a_i e\Lambda_i$ lies in $\mathcal{W}_{r,e,w}$ if and only if*

$$w \geq \sum_{i=1}^{r-1} i(a_i - 1) \quad \textit{and} \quad w \equiv \sum_{i=1}^{r-1} i(a_i - 1) \ (\textit{mod}\ r)\,. \tag{3.2.21}$$

Moreover, such points correspond precisely to the component $\mathcal{W}_{r,e,w}\big((r)\big)$ in the simplicial decomposition; that is, they arise from abacus configurations where all r beads lie on a single runner.

Proof. Suppose that $\lambda \in \mathscr{P}_{r,e,w}$ maps to P, and let β_i be the beta numbers of λ. Then $\beta_i - \beta_{i+1} = a_i e$ for all $1 \leq i \leq r-1$. This implies $\beta_i \equiv \beta_{i+1} \pmod e$ for all i. Consequently, all r beads lie on the same runner of the e-abacus. Hence the partition λ belongs to the set $\mathscr{P}_{r,e,w}\big((r)\big)$, where $\mu = (r)$ is the composition of length 1.

Let the single non-empty runner be the j-th runner ($0 \leq j \leq e-1$). The positions of the beads are determined by the gaps a_i between them. Specifically, if the last bead β_r is at row k (position $ke + j$), then the bead β_i is at position:

$$\beta_i \;=\; \beta_r + \sum_{m=i}^{r-1}(\beta_m - \beta_{m+1}) \;=\; (ke + j) + e\sum_{m=i}^{r-1} a_m.$$

This shows the bead β_i is on row $k + \sum_{m=i}^{r-1} a_m$. By sliding all beads up as high as possible, we see that the bead β_i will be put on row $r - i$, hence the e-weight of λ is:

$$w_e(\lambda) \;= k + \sum_{i=1}^{r-1}\Big(k + \sum_{m=i}^{r-1} a_m - (r-i)\Big) = \; kr + \sum_{i=1}^{r-1} i(a_i - 1),$$

This is equivalent to the desired statement. □

We now formalize the patterns observed in Figure 10.

Definition 3.2.22. *A point $v \in C^+ \cap P$ is called an* affine vertex *(of the e-alcove geometry) if it lies in the intersection of the affine hyperplanes corresponding to all positive roots. That is, for each $\alpha \in R^+$, there exists a positive integer k_α such that:*

$$\langle \alpha^\vee, v\rangle \;=\; k_\alpha e.$$

Fix integers r, e, w with $e > r \geq 3$ and $w \geq 0$ as above. An affine vertex $v = \sum_{i=1}^{r-1} a_i \Lambda_i \in \mathcal{W}_{r,e,w}$ is called a *boundary affine vertex* of $\mathcal{W}_{r,e,w}$ if there is no distinct point $Q = \sum_{i=1}^{r-1} b_i \Lambda_i \in \mathcal{W}_{r,e,w}$ such that $b_i \geq a_i$ for all $1 \leq i \leq r-1$.

An affine vertex $u = \sum_{i=1}^{r-1} c_i \Lambda_i \in C^+ \cap P$ is called an *interior affine vertex* of $\mathcal{W}_{r,e,w}$ if there exists a boundary affine vertex $v = \sum_{i=1}^{r-1} a_i \Lambda_i$ of $\mathcal{W}_{r,e,w}$ such that $c_i \leq a_i$ for all i and $c_j < a_j$ for at least one index j. Note that we do not require an interior affine vertex of $\mathcal{W}_{r,e,w}$ to lie in $\mathcal{W}_{r,e,w}$ itself.

Example 3.2.23. *Let $(r, e, w) = (3, 8, 3)$ and consider $\mathcal{W}_{3,8,3}$; see Figure 10d. The boundary affine vertices are*

$$2e\Lambda_1 + 2e\Lambda_2 \quad \text{and} \quad 4e\Lambda_1 + e\Lambda_2,$$

and the interior affine vertices are

$$e\Lambda_1 + e\Lambda_2, \quad e\Lambda_1 + 2e\Lambda_2, \quad 2e\Lambda_1 + e\Lambda_2, \quad 3e\Lambda_1 + e\Lambda_2.$$

◇

Lemma 3.2.24. *Fix integers $r \geq 3$ and $e > r$. There exists an integer w_{in} such that for all $w \geq w_{in}$, the set $\mathcal{W}_{r,e,w}$ contains an affine vertex $v = \sum_{i=1}^{r-1} a_i e \Lambda_i$. Specifically, one may take $w_{in} = (r-1)^2$.*

Proof. By Lemma 3.2.20, an affine vertex $P = \sum_{i=1}^{r-1} a_i e \Lambda_i$ exists in $\mathcal{W}_{r,e,w}$ if and only if

$$w \geq \sum_{i=1}^{r-1} i(a_i - 1) \quad \text{and} \quad w \equiv \sum_{i=1}^{r-1} i(a_i - 1) \pmod{r}.$$

We restrict our search to the vertices such that $a_1 = \cdots = a_{r-2} = 1$. The condition reduces to find an integer $x_{r-1} := a_{r-1} - 1 \geq 0$ such that:

$$w \geq (r-1)x_{r-1} \quad \text{and} \quad w \equiv (r-1)x_{r-1} \pmod{r}.$$

Since $r - 1 \equiv -1 \pmod{r}$, the congruence becomes $-x_{r-1} \equiv w \pmod{r}$. This linear congruence has a unique solution x_{r-1} in the range $\{0, 1, \ldots, r-1\}$. For this minimal solution, the required weight is $(r-1)x_{r-1} \leq (r-1)^2$. Thus, for any $w \geq (r-1)^2$, there exists a valid choice of x_{r-1} (and hence $a_{r-1} \geq 1$, with $a_i = 1$ otherwise) satisfying the condition. □

For any real number c, the *ceiling function* $\lceil c \rceil$ is the least integer greater than or equal to c. For instance, $\lceil 5.2 \rceil = 6$.

Lemma 3.2.25. *Let λ be a partition such that $\ell(\lambda) \leq r$ and $\beta_1 > \beta_2 > \cdots > \beta_r$ are its r-beta numbers. Set $x_i := \beta_i - \beta_{i+1}$ and $m_i := \left\lceil \frac{x_i}{e} \right\rceil$ $(1 \leq i \leq r-1)$. Then the e–weight of λ satisfies*

$$w_e(\lambda) \geq \sum_{i=1}^{r-1} i\,(m_i - 1). \tag{3.2.26}$$

Proof. By definition, the e-weight $w_e(\lambda)$ is the total number of upward moves of beads required to reach the e-core. By definition of m_i, we can write $x_i = (m_i - 1)e + s_i$, where $1 \leq s_i \leq e$.

Consider a bead β_i. Since there are no other beads strictly between β_i and β_{i+1}, and the gap $x_i > (m_i - 1)e$, we can slide this bead up $m_i - 1$ steps without crossing β_{i+1}. Then the new position of this bead is $\beta_i' = \beta_i - (m_i - 1)e = \beta_{i+1} + s_i$. Since $s_i \geq 1$, we have $\beta_i' > \beta_{i+1}$, so the relative order of these beads is preserved.

Performing these moves on β_i creates empty space below β_i'. This allows all beads β_k with $k < i$ (which are larger than β_i) to also slide up by at least $(m_i - 1)$ while maintaining the strict decreasing order of the beta numbers. Thus, the gap associated with m_i contributes at least $m_i - 1$ upward moves to bead β_i, and consequently at least $m_i - 1$ moves to every bead β_k below it ($k < i$).

Summing these contributions over $i = 1, \ldots, r-1$, the total number of upward moves is at least $\sum_{i=1}^{r-1} i(m_i - 1)$. As the e-weight accounts for all necessary moves to reach its core, we obtain the desired inequality (3.2.26). □

Lemma 3.2.27. *Let $e > r \geq 3$, and $w \geq 0$. Suppose $Q = \sum_{i=1}^{r-1} x_i \Lambda_i \in \mathcal{W}_{r,e,w}$, then there exists an affine vertex $v = \sum_{i=1}^{r-1} a_i e \Lambda_i \in \mathcal{W}_{r,e,w}$ such that $a_i e \geq x_i$ for all $1 \leq i \leq r-1$.*

Proof. Set $m_i = \lceil x_i / e \rceil$, so $m_i e \geq x_i$ for each i. Let λ be a partition such that $\Omega(\lambda) = Q$. By Lemma 3.2.25 we have $w = w_e(\lambda) \geq \sum_{i=1}^{r-1} i\,(m_i - 1)$. Define $v = \sum_{i=1}^{r-1} m_i e\, \Lambda_i + \big(w - \sum_{i=1}^{r-1} i\,(m_i-1)\big) e\, \Lambda_1$, that is, $a_i = m_i$ for $i \geq 2$ and $a_1 = m_1 + \big(w - \sum_{i=1}^{r-1} i\,(m_i - 1)\big)$. Then $\sum_{i=1}^{r-1} i\,(a_i - 1) = w$ and by Lemma 3.2.20, we have $v \in \mathcal{W}_{r,e,w}$. □

Corollary 3.2.28. *An affine vertex $v = \sum_{i=1}^{r-1} a_i e \Lambda_i$ is a boundary affine vertex of $\mathcal{W}_{r,e,w}$ if and only if $v \in \mathcal{W}_{r,e,w}$ and there does not exist a distinct affine vertex $u = \sum_{i=1}^{r-1} b_i e \Lambda_i \in \mathcal{W}_{r,e,w}$ such that $b_i \geq a_i$ for all i.*

Proof. This follows from the definition of a boundary affine vertex and Lemma 3.2.27. □

Proposition 3.2.29. *Let $e > r \geq 3$, and $w \geq 0$. An affine vertex $v = \sum_{i=1}^{r-1} a_i e \Lambda_i$ is a boundary affine vertex of $\mathcal{W}_{r,e,w}$ if and only if $w = \sum_{i=1}^{r-1} i(a_i - 1)$.*

Proof. By Lemma 3.2.20, $v \in \mathcal{W}_{r,e,w}$ if and only if $w = \sum_{i=1}^{r-1} i(a_i - 1) + kr$ for some non-negative integer k. It suffices to show that v is a boundary affine vertex if and only if $k = 0$.

Suppose $k > 0$. We can construct another affine vertex $v' = v + (kr)e\Lambda_1 = (kr + a_1)e\Lambda_1 + \sum_{i=2}^{r-1} a_i e \Lambda_i$. Note that

$$1 \cdot (kr + a_1 - 1) + \sum_{i=2}^{r-1} i(a_i - 1) = kr + \sum_{i=1}^{r-1} i(a_i - 1) = w.$$

Thus, $v' \in \mathcal{W}_{r,e,w}$ by Lemma 3.2.20. Since the coordinates of v' are strictly greater than or equal to those of v and the first coordinate is strictly larger, v cannot be a boundary affine vertex.

Conversely, suppose $k = 0$. If there exists another affine vertex $v'' = \sum_{i=1}^{r-1} b_i e\Lambda_i$ in $\mathcal{W}_{r,e,w}$ such that $b_i \geq a_i$ for each i, then

$$\sum_{i=1}^{r-1} i(b_i - 1) \geq \sum_{i=1}^{r-1} i(a_i - 1) = w.$$

However, since $v'' \in \mathcal{W}_{r,e,w}$, Lemma 3.2.20 implies that $\sum_{i=1}^{r-1} i(b_i - 1) = w - k'r \leq w$ for some $k' \geq 0$. These inequalities force the sum to be exactly w, which implies $b_i = a_i$ for all i. Hence, by Corollary 3.2.28, v is a boundary affine vertex. □

For two affine vertices $v_1 = \sum_{i=1}^{r-1} a_i e\Lambda_i$ and $v_2 = \sum_{i=1}^{r-1} b_i e\Lambda_i$, we say

- v_1 and v_2 are *adjacent* if there exists an index j with $1 \leq j \leq r-1$ such that $a_i = b_i \pm \delta_{ij}$ for all i.
- (v_1, v_2) is a *bad pair* of $\mathcal{W}_{r,e,w}$ if v_1 is an interior affine vertex, v_2 is a boundary affine vertex, and $a_i + \delta_{i,r-1} = b_i$ for all i.
- (v_1, v_2) is a *good pair* of $\mathcal{W}_{r,e,w}$ if both v_1 and v_2 are interior affine vertices, $v_1 \notin \mathcal{W}_{r,e,w} \ni v_2$, and $a_i + \delta_{i,r-1} = b_i$ for all i.

In particular, both a bad pair and a good pair consist of adjacent affine vertices.

Example 3.2.30. *Let $(r, e, w) = (3, 8, 3)$ and consider $\mathcal{W}_{3,8,3}$; see Figure 10d. In this case the only bad pair is*

$$(2e\Lambda_1 + e\Lambda_2,\ 2e\Lambda_1 + 2e\Lambda_2),$$

and there is no good pair.

Now let $(r, e, w) = (3, 8, 5)$ and consider $\mathcal{W}_{3,8,5}$; see Figure 10f. The bad pairs are

$$(2e\Lambda_1 + 2e\Lambda_2,\ 2e\Lambda_1 + 3e\Lambda_2) \quad \textit{and} \quad (4e\Lambda_1 + e\Lambda_2,\ 4e\Lambda_1 + 2e\Lambda_2),$$

while the unique good pair is

$$(e\Lambda_1 + e\Lambda_2,\ e\Lambda_1 + 2e\Lambda_2).$$

◇

For simplicity, in the rest of this section we write $\mathcal{A}$ instead of $\mathcal{A}^{(e)}$ for an e–alcove.

For each affine vertex $v \in C^+ \cap P$, let $\mathcal{S}_v$ be the set of e–alcoves $\mathcal{A}$ such that v lies in the closure of $\mathcal{A}$, that is,

$$\mathcal{S}_v := \{\mathcal{A} \in \mathcal{H}^{(e)} \mid v \in \overline{\mathcal{A}}\}.$$

The *cell* $\mathfrak{C}_v$ centred at v is the union of the closures of all e–alcoves in $\mathcal{S}_v$, namely

$$\mathfrak{C}_v := \bigsqcup_{\mathcal{A} \in \mathcal{S}_v} \overline{\mathcal{A}}.$$

Its interior $\mathfrak{C}_v^\circ$ is called the *open cell* centred at v.

Then

$$\mathfrak{C}_v \cap P = \{w \in C^+ \cap P : |\langle w - v, \alpha^\vee \rangle| \leq e \text{ for all } \alpha \in R^+\}$$

and

$$\mathfrak{C}_v^\circ \cap P = \{w \in C^+ \cap P : |\langle w - v, \alpha^\vee \rangle| < e \text{ for all } \alpha \in R^+\}. \tag{3.2.31}$$

Definition 3.2.32. (a). *For each good pair* (v_1, v_2) *of* $\mathcal{W}_{r,e,w}$*, the* ***regular pattern*** Reg_{v_1,v_2} *associated to* (v_1, v_2) *is the set* $\mathfrak{C}_{v_1}^\circ \cap \mathcal{W}_{r,e,w}$*.*

(b). *For each bad pair* (v_1, v_2) *of* $\mathcal{W}_{r,e,w}$*, the* ***stingray pattern*** Stin_{v_1,v_2} *associated to* (v_1, v_2) *is the set* $\left(\mathfrak{C}_{v_1}^\circ \cap \mathcal{W}_{r,e,w}\right) \sqcup \{v_2\}$*.*

For any bad pair (v_1, v_2), define L_{v_1,v_2} to be the (discrete) segment connecting v_1 and v_2, that is,

$$\mathrm{L}_{v_1,v_2} = \{v_1 + k\Lambda_{r-1} \mid 1 \leq k \leq e\} = \{v_2 - k\Lambda_{r-1} \mid 0 \leq k \leq e-1\}.$$

We remind the readers that in this definition, v_1 is not included in this segment while v_2 is. The advantage of this definition is the following:

Lemma 3.2.33. *For any bad pair* (v_1, v_2) *of* $\mathcal{W}_{r,e,w}$*, the segment* L_{v_1,v_2} *is contained in* $\mathcal{W}_{r,e,w}$*.*

Proof. An affine vertex in $\mathcal{W}_{r,e,w}$ corresponds to a copy of $\mathcal{W}_{r,e}\big((r)\big)$. This in turn corresponds to abaci with a single non-empty runner carrying all r beads. We may assume this non-empty runner is the $(e-1)$–runner. Recall that $v_2 = \sum_{i=1}^{r-1}(\beta_i - \beta_{i+1})\Lambda_i$, where the β_i are the beta numbers corresponding to this abacus. We want to construct abaci with the same e–weight as this one whose images under Ω trace out the segment L_{v_1,v_2}, and hence give points in $\mathcal{W}_{r,e,w}$.

Shift the last bead β_r to the left by k positions, where $1 \leq k \leq e-1$, and then slide each of the other beads β_i up by one step. This operation does not change the e–weight, and it preserves the differences $\beta_i - \beta_{i+1}$ for all $i \neq r-1$. Since β_r decreases by k and β_{r-1} decreases by e, the difference $\beta_{r-1} - \beta_r$ decreases by $e-k$. Hence the resulting dominant weight is $v_2 - (e-k)\Lambda_{r-1}$, which ranges over $\mathrm{L}_{v_1,v_2} \setminus \{v_2\}$ as k varies. Adding the endpoint v_2 recovers the full segment L_{v_1,v_2}, as required. □

As this segment is contained in the cell $\mathfrak{C}_{v_1}$, we deduce the following:

Corollary 3.2.34. *For any bad pair* (v_1, v_2) *of* $\mathcal{W}_{r,e,w}$*, the segment* L_{v_1,v_2} *is contained in the associated stingray pattern* Stin_{v_1,v_2}*.*

Because of Corollary 3.2.34, we call L_{v_1,v_2} the *tail* of the stingray (or the *stingray tail*), and we call $\mathrm{Stin}_{v_1,v_2} \setminus \mathrm{L}_{v_1,v_2}$ the *body* of the stingray (or the *stingray body*).

Example 3.2.35. *We continue with Example 3.2.30 and consider the case* $\mathcal{W}_{3,8,5}$*; see Figure 6, where the stingray patterns are drawn in red and the regular pattern in blue. We also distinguish the stingray tail and the stingray body by using different shades of red.* ◇

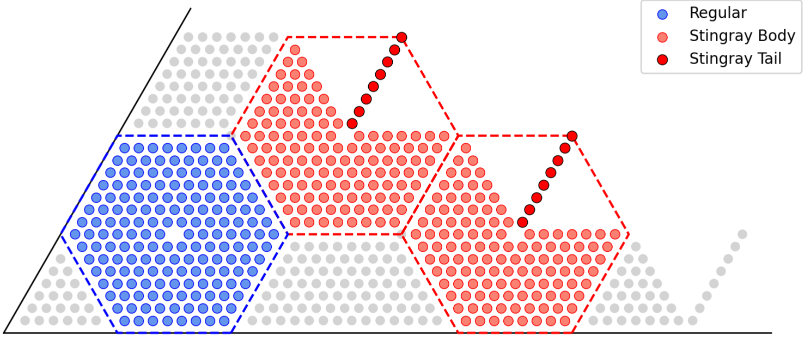


FIGURE 6. Two stingray patterns and one regular pattern for $(r, e, w) = (3, 8, 5)$.

Our next goal is to show that a stingray tail is isolated in an appropriate sense, which explains why we regard it as a "tail": it should be separated from the body, as in Figure 6.

LEMMA 3.2.36. *Let λ be a partition and let β_i be its r-beta numbers. Form the e-abacus of λ with r beads, and suppose that each runner has at most one bead on it. Then the e–weight of λ is given by*

$$w_e(\lambda) = \sum_{i=1}^{r} \lfloor \beta_i / e \rfloor$$

PROOF. A bead β_i lies in row $q_i = \lfloor \beta_i / e \rfloor$. Sliding it up to the row 0 makes that runner flush and contributes exactly q_i upward moves. Summing over all beads shows that the total number of upward moves, hence the e–weight, is $\sum_{i=1}^{r} \lfloor \beta_i / e \rfloor$, as claimed. □

LEMMA 3.2.37. *Let (v_1, v_2) be a bad pair of affine vertices of $\mathcal{W}_{r,e,w}$, and let L_{v_1,v_2} be the stingray tail. Then for an e–alcove $\mathcal{A}$,*

$$\mathrm{L}_{v_1,v_2} \subset \overline{\mathcal{A}} \quad \Longleftrightarrow \quad \mathcal{A} \in \mathcal{S}_{v_1} \cap \mathcal{S}_{v_2}.$$

PROOF. If $\mathcal{A} \in \mathcal{S}_{v_1} \cap \mathcal{S}_{v_2}$ then $v_1, v_2 \in \overline{\mathcal{A}}$ by definition, and since $\overline{\mathcal{A}}$ is convex it contains the whole segment connecting v_1 and v_2, hence in particular the discrete subset L_{v_1,v_2}.

Conversely, assume $\mathrm{L}_{v_1,v_2} \subset \overline{\mathcal{A}}$. The closure $\overline{\mathcal{A}}$ is convex and its intersection with the line through v_1 and v_2 is a closed interval. The tail L_{v_1,v_2} consists of all intermediate lattice points between v_1 and v_2, so the smallest closed interval in that line containing L_{v_1,v_2} has endpoints v_1 and v_2. Hence $v_1, v_2 \in \overline{\mathcal{A}}$, i.e. $\mathcal{A} \in \mathcal{S}_{v_1} \cap \mathcal{S}_{v_2}$. □

PROPOSITION 3.2.38. *Fix $e > r \geq 3$, $w \geq 0$. Let Stin_{v_1,v_2} be a stingray pattern contained in $\mathcal{W}_{r,e,w}$ and let L_{v_1,v_2} be the tail of this stingray. Then for any e–alcove $\mathcal{A} \in \mathcal{S}_{v_1}$ such that $\mathrm{L}_{v_1,v_2} \subset \overline{\mathcal{A}}$, we have $\mathcal{A} \cap \mathcal{W}_{r,e,w} = \emptyset$.*

PROOF. By Lemma 3.2.37, it suffices to prove the case when $\mathcal{A} \in \mathcal{S}_{v_2}$. Write $v_2 = \sum_{i=1}^{r-1} b_i e \, \Lambda_i$. Since (v_1, v_2) is a bad pair, v_2 is a boundary affine vertex, so by

Proposition 3.2.29 the e–weight (of any partition whose image under Ω is v_2) satisfies

$$w = \sum_{i=1}^{r-1} i\,(b_i - 1). \tag{3.2.39}$$

Suppose, for a contradiction, that there exists a point $Q = \sum_{i=1}^{r-1} x_i \Lambda_i \in \mathcal{A} \cap \mathcal{W}_{r,e,w}$.

Assume first that $x_j > b_j e$ for some j. By Lemma 3.2.27, there is an affine vertex $v' = \sum_{i=1}^{r-1} c_i e \Lambda_i \in \mathcal{W}_{r,e,w}$ with $c_i e \geq x_i$ for all i, and in particular $c_j e \geq x_j > b_j e$, so $c_j > b_j$. The distance condition (3.2.31) for each simple root α_i gives $x_i > (b_i - 1)e$, which forces $c_i \geq b_i$ for all i. Thus v_2 cannot be a boundary affine vertex by Corollary 3.2.28, contradicting our assumption. Hence we must have

$$x_i \leq b_i e \qquad \text{for all } 1 \leq i \leq r-1. \tag{3.2.40}$$

As $Q \in \mathcal{A}$, there exists some i such that the inequality in (3.2.40) is strict.

Let λ be a partition with $\Omega(\lambda) = Q$ and let $(\beta_1, \ldots, \beta_r)$ be its r–beta numbers. Then

$$\beta_i - \beta_r = \sum_{k=i}^{r-1} (\beta_k - \beta_{k+1}) = \sum_{k=i}^{r-1} x_k \qquad (1 \leq i \leq r-1). \tag{3.2.41}$$

Since Q lies in the interior of an e–alcove, by Lemma 3.2.52 we know that $Q \in \mathcal{W}_{r,e,w}(1^r)$. Therefore, by Lemma 3.2.36,

$$w_e(\lambda) = \sum_{i=1}^{r} \left\lfloor \frac{\beta_i}{e} \right\rfloor \geq \sum_{i=1}^{r-1} \left\lfloor \frac{\beta_r + \sum_{k=i}^{r-1} x_k}{e} \right\rfloor. \tag{3.2.42}$$

Because Q lies in an alcove whose closure contains the boundary vertex v_2, the distance condition (3.2.31) applied to the root $\alpha_i + \cdots + \alpha_{r-1}$ shows that, for each i,

$$\sum_{k=i}^{r-1} b_k e - e < \sum_{k=i}^{r-1} x_k < \sum_{k=i}^{r-1} b_k e, \tag{3.2.43}$$

where the right-hand inequality follows from (3.2.40). Dividing (3.2.43) by e gives

$$\sum_{k=i}^{r-1} b_k - 1 < \frac{1}{e} \sum_{k=i}^{r-1} x_k < \sum_{k=i}^{r-1} b_k,$$

so

$$\left\lfloor \frac{1}{e} \sum_{k=i}^{r-1} x_k \right\rfloor = \sum_{k=i}^{r-1} b_k - 1. \tag{3.2.44}$$

Thus

$$\left\lfloor \frac{\beta_r + \sum_{k=i}^{r-1} x_k}{e} \right\rfloor = \left\lfloor \frac{\beta_r}{e} + \frac{1}{e} \sum_{k=i}^{r-1} x_k \right\rfloor \geq \sum_{k=i}^{r-1} b_k - 1. \tag{3.2.45}$$

Substituting (3.2.45) into (3.2.42) yields

$$w_e(\lambda) \geq \sum_{i=1}^{r-1} \Big(\sum_{k=i}^{r-1} b_k - 1 \Big)$$

$$= \sum_{k=1}^{r-1} k\, b_k - (r-1). \tag{3.2.46}$$

On the other hand, from (3.2.39),

$$w = \sum_{k=1}^{r-1} k\,(b_k - 1) = \sum_{k=1}^{r-1} k\, b_k - \sum_{k=1}^{r-1} k$$

$$= \sum_{k=1}^{r-1} k\, b_k - \frac{r(r-1)}{2}. \tag{3.2.47}$$

For $r \geq 3$ we have $\frac{r(r-1)}{2} > r-1$, so combining (3.2.46) and (3.2.47) gives

$$w_e(\lambda) \geq \sum_{k=1}^{r-1} k\, b_k - (r-1) > \sum_{k=1}^{r-1} k\, b_k - \frac{r(r-1)}{2} = w.$$

Thus $w_e(\lambda) > w$, contradicting the assumption that $Q \in \mathcal{W}_{r,e,w}$. Therefore no such Q can exist, and we conclude that $\mathcal{A} \cap \mathcal{W}_{r,e,w} = \emptyset$. □

In contrast, a regular pattern does not exhibit this separation behavior. The following result explains the terminology.

Proposition 3.2.48. *Fix $e > r = 3$ and $w \geq 0$. Let Reg_{v_1,v_2} be a regular pattern contained in $\mathcal{W}_{r,e,w}$, associated to a good pair (v_1, v_2). Then for each e–alcove $\mathcal{A} \in \mathcal{S}_{v_1}$ all lattice points in the interior of $\mathcal{A}$ lie in $\mathcal{W}_{r,e,w}$, that is*

$$\mathcal{A}^\circ \cap P^+ \subset \mathcal{W}_{r,e,w}.$$

Proof. When $r = 3$, the set $\mathcal{S}_{v_1}$ consists of six e–alcoves. To prove the statement, we construct an explicit point in each of these alcoves; the result then follows from Lemma 3.2.52. Write $v_1 = a_1 e\Lambda_1 + a_2 e\Lambda_2$. One checks that

$$P_1 = (a_1e{+}1)\Lambda_1{+}(a_2e{+}1)\Lambda_2, \quad P_2 = (a_1e{+}2)\Lambda_1{+}(a_2e{-}1)\Lambda_2, \quad P_3 = (a_1e{+}1)\Lambda_1{+}(a_2e{-}2)\Lambda_2,$$

$$P_4 = (a_1e{-}1)\Lambda_1{+}(a_2e{-}1)\Lambda_2, \quad P_5 = (a_1e{-}2)\Lambda_1{+}(a_2e{+}1)\Lambda_2, \quad P_6 = (a_1e{-}1)\Lambda_1{+}(a_2e{+}2)\Lambda_2$$

lie in the six alcoves adjacent to v_1, hence in the six elements of $\mathcal{S}_{v_1}$. It remains to show that these six points lie in $\mathcal{W}_{3,e,w}$.

By assumption (v_1, v_2) is a good pair. Write $v_2 = b_1 e\Lambda_1 + b_2 e\Lambda_2$, so $b_1 = a_1$ and $b_2 = a_2 + 1$. Consider the abacus configuration of a partition λ with $\Omega(\lambda) = v_2$, and let $\beta_1, \beta_2, \beta_3$ be its 3–beta numbers (which we also view as the three beads). Then the three beads lie on a single runner. If β_3 is on row k, then β_2 is on row $k + b_2$ and β_1 is on row $k + b_2 + b_1$.

By Lemma 3.2.20 and Proposition 3.2.29, we have $(b_1 - 1) + 2(b_2 - 1) + 3k = w$ with $k > 0$. We explain the construction for P_5; the other points are handled similarly.

For the abacus of v_2, we may choose the unique non-empty runner to be the $(e-1)$–runner (recall $e > r = 3$). Now shift the bead β_1 two steps to the left and shift the bead β_3 one step to the left. The resulting abacus represents the point P_5. Its e–weight is

$$w' = k + (k + b_2 + b_1) + (k + b_2)$$

by Lemma 3.2.36, so $w' = w + 3$. Finally, slide all three beads up by one step which is possible because $k > 0$. This does not change the image under Ω, and it reduces the e–weight by 3, so we obtain an abacus of e–weight w still mapping to P_5. Hence $P_5 \in \mathcal{W}_{3,e,w}$. The same argument applies to P_1, P_2, P_3, P_4, P_6. □

Remark 3.2.49. *One might expect Proposition 3.2.48 to hold for all $r \geq 3$, but this is not the case. A counterexample occurs for $(r, e, w) = (5, 6, 15)$, with the good pair $v_1 = (42, 6, 6, 6)$ and $v_2 = (42, 6, 6, 12)$. It would be interesting to find a finer notion of a "perfect pair" for which Proposition 3.2.48 remains valid and this should be related to the Bruhat order.* ◇

Remark 3.2.50. *We refer the reader to github repository for more figures in $\mathfrak{sl}_3$, interactive figures in $\mathfrak{sl}_4$, and the code used to generate them.* ◇

3.2.4. Index the e-alcoves in $\mathfrak{sl}_r$.

From Figure 4 and Figure 5, one observes that the green simplices (that is, the copies of $\mathcal{W}_{3,e}(1,1,1)$) are precisely the interiors of the e-alcoves in the fundamental chamber of $\mathfrak{sl}_3$. These simplices are in bijection with the 3-partitions of w of type $(1,1,1)$, so we may use such 3-partitions to label the corresponding alcoves. More generally, the next result shows that one can index a subset of the e-alcoves in the fundamental chamber of $\mathfrak{sl}_r$ by the set of r-partitions of w of type (1^r).

Recall from Subsection 1.2.1 that a composition with r parts is a finite sequence of r nonzero non-negative integers. A *weak composition* with no more than r parts is a finite sequence of r non-negative integers, in which zeros are permitted. For example, $(5, 0, 1, 3)$ is a weak composition with no more than 4 parts but is not a composition.

For any r and w, the r-partitions of w of type (1^r) are in bijection with the weak compositions of w with at most r parts: given such an r-partition, take the lengths of its components as the parts of the corresponding composition. From now on we do not distinguish between these two sets and we often write weak compositions to simplify notation.

Lemma 3.2.51. *The set of dominant weights in the interior of the fundamental e–alcove, $\mathcal{A}_0^{(e)\circ} \cap P^+$, is the lattice points of an $(r-1)$-dimensional simplex of dilation factor $e - r$ in the sense of Definition 3.2.2.*

PROOF. By Subsection 1.4.2, a dominant weight $v = \sum_{i=1}^{r-1} a_i\Lambda_i$ lies in $\mathcal{A}_0^{(e)\circ}$ if and only if

$$a_i \in \mathbb{Z}_{>0} \quad \text{for all } 1 \le i \le r-1, \qquad \sum_{i=1}^{r-1} a_i \le e-1.$$

Let $b_i := a_i - 1 (1 \le i \le r-1)$, define integers

$$x_0 := e - r - \sum_{i=1}^{r-1} b_i, \quad x_i := b_i \ (1 \le i \le r-1).$$

It is straightforward to check that this construction gives a bijection between $\mathcal{A}_0^{(e)\circ} \cap P^+$ and the lattice points of the standard simplex $\Delta^{r-1}(e-r) \cap \mathbb{Z}^r$, with inverse map $(x_0, \dots, x_{r-1}) \mapsto \sum_{i=1}^{r-1}(x_i+1)\Lambda_i$.

Hence $\mathcal{A}_0^{(e)\circ} \cap P^+$ is the lattice point set of an $(r-1)$–simplex of dilation factor $e-r$, as claimed. □

LEMMA 3.2.52. *In $\mathcal{W}_{r,e,w}(1^r)$, each copy of $\mathcal{W}_{r,e}(1^r)$ (corresponding to an r–partition of w) is exactly the set of dominant weight lattice points in the interior of some e–alcove.*

PROOF. By Theorem 3.2.5 and Theorem 3.2.9, each copy of $\mathcal{W}_{r,e}(1^r)$ is the set of lattice points of a simplex in $C^+ \cap P$.

First, no point of such a simplex lies on the boundary of an e–alcove. Recall that a point $v = \sum_{i=1}^{r-1} a_i\Lambda_i$ lies on some affine hyperplane if and only if there exist indices $i \le j$ such that

$$a_i + a_{i+1} + \cdots + a_j \equiv 0 \ (\mathrm{mod}\ e),$$

equivalently, $\langle v, \alpha_i + \cdots + \alpha_j \rangle \equiv 0 \ (\mathrm{mod}\ e)$. Let v belong to a copy of $\mathcal{W}_{r,e}(1^r)$ and choose λ such that $\Omega(\lambda) = v$. Consider an e–abacus for λ with r–beta numbers $\beta_1 > \cdots > \beta_r$. If $a_i + \cdots + a_j \equiv 0 \ (\mathrm{mod}\ e)$ for some $i \le j$, then

$$(\beta_i - \beta_{i+1}) + \cdots + (\beta_j - \beta_{j+1}) = \beta_i - \beta_{j+1} \equiv 0 \ (\mathrm{mod}\ e),$$

so the beads β_i and β_{j+1} lie on the same runner. Hence some runner contains at least two beads. By Theorem 3.2.9, this excludes v from the component $\mathcal{W}_{r,e,w}(1^r)$, which consists precisely of points whose abaci have at most one bead on each runner. Thus no point of a copy of $\mathcal{W}_{r,e}(1^r)$ lies on any affine hyperplane, so every such point lies in the interior of some e–alcove.

Let S be a fixed copy of $\mathcal{W}_{r,e}(1^r)$. If S intersects any e–alcove, then by the above argument and the convexity of a simplex, S is contained in the interior of that e–alcove.

By Lemma 3.2.51 and the translation invariance of the e–alcoves, the number of dominant lattice points in the interior of any e–alcove is equal to the number of points of $\mathcal{W}_{r,e}(1^r)$. Since $S \subset \mathcal{A}_0^\circ \cap P^+$ and the cardinalities agree, we must have $S = \mathcal{A}_0^\circ \cap P^+$. □

Example 3.2.53. *Figure 4 and Figure 5 illustrate how to assign labels to the simplices in* $W_{3,e,w}(1,1,1)$*. In Figure 7 we make these labels more clear from Figure 5 by removing the green points. Moreover, we have simplified these labels to use the weak compositions of* $w = 10$ *with at most* 3 *parts, corresponding to the* 3*-partitions of* w *of type* $(1,1,1)$. ◇

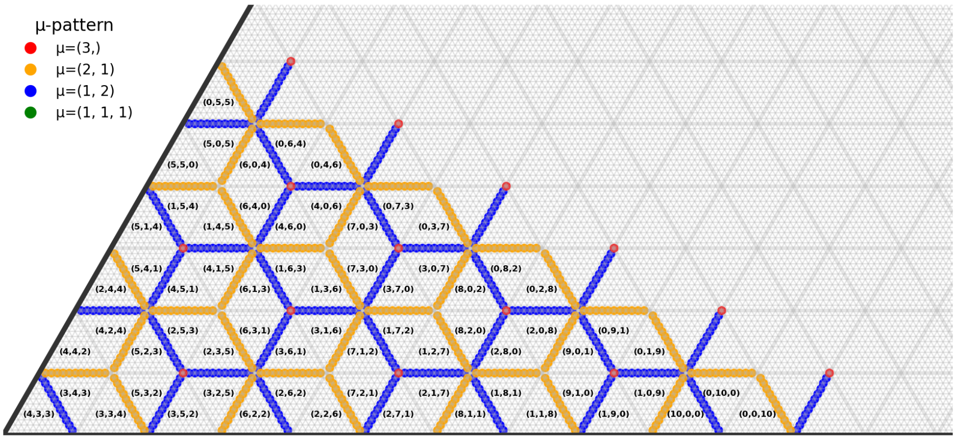


Figure 7. $r = 3$, $e = 12$, $w = 10$

We now examine these labels in Example 3.2.53 more closely.

First, in Figure 7, the fundamental e-alcove carries the label $(4,3,3)$. In general, Lemma 3.2.55 shows that the label of the fundamental e-alcove is given by the "most balanced" partition of w of length r.

Second, how the labels change when cross the walls between adjacent alcoves is related to the action of affine Weyl group on abaci. In Figure 7, there are three types of walls: blue, yellow, and gray. The gray walls correspond to empty walls, that is, walls not contained in $W_{3,e,w}$ (see Figure 5). Suppose an alcove has label (a,b,c) with $a+b+c = w = 10$. Then one observes:

- crossing a blue wall sends (a,b,c) to (b,a,c) (swapping the first two coordinates);
- crossing an empty (gray) wall sends (a,b,c) to (a,c,b) (swapping the last two coordinates);
- crossing a yellow wall sends (a,b,c) to $(c+1,b,a-1)$.

For readers familiar with the action of the affine Weyl group on abaci, these transformations agree with the case of three runners with one bead on each runner at rows a,b,c, respectively. The blue, gray, and yellow walls correspond to the Coxeter generators s_1, s_2, and s_0. This also explains why $(4,3,3)$ labels the fundamental alcove: it is characterized by the property that the action of s_2 and s_0 fixes it, so its label changes under only one wall-crossing.

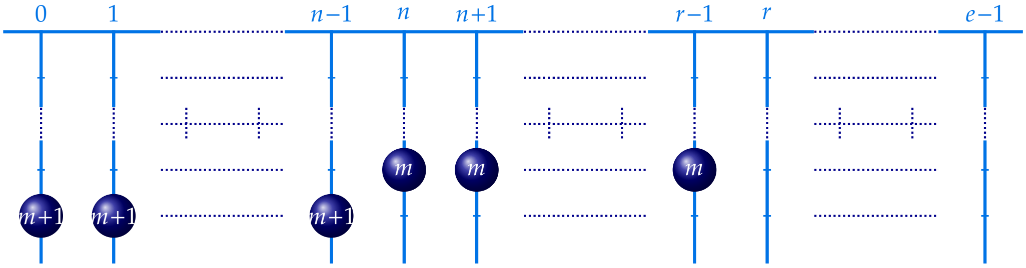
Fix $e > r \geq 3$ and $w \in \mathbb{N}$, and write $w = rm + n$ with $m, n \in \mathbb{N}$ and $0 \leq n \leq r-1$. Set

$$\lambda_\emptyset = \big(\overbrace{m+1, \dots, m+1}^{n \text{ terms}}, \overbrace{m, \dots, m}^{(r-n) \text{ terms}}\big). \tag{3.2.54}$$

Lemma 3.2.55. *For $\mathfrak{sl}_r$, in the picture of $W_{r,e,w}$, the level-e fundamental alcove $\mathcal{A}_0^{(e)}$ corresponds to the weak composition $\lambda_\emptyset$ defined in (3.2.54).*

Proof. By Lemma 3.2.52, it suffices to show that there exists a point in the copy indexed by $\lambda_\emptyset$ that lies in the fundamental alcove.

Consider the e–abacus with e runners, and choose the first r runners. For each runner $0 \leq i \leq n-1$ place a bead at row $m+1$, and for each runner $n \leq i \leq r-1$ place a bead at row m.

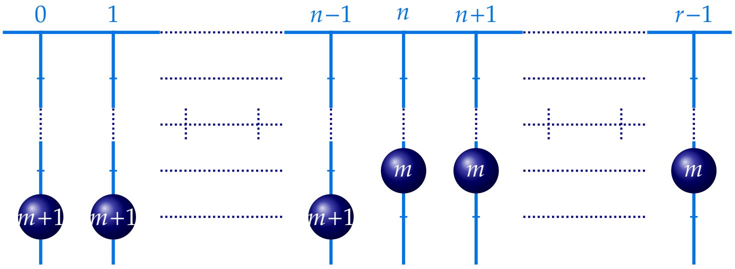


By definition, this abacus corresponds to a point in the copy of $W_{r,e,w}(1^r)$ indexed by $\lambda_\emptyset$. Let $(\beta_1, \dots, \beta_r)$ be the associated beta-numbers. The corresponding dominant weight is $\sum_{i=1}^{r-1} a_i \Lambda_i$, where $a_i = \beta_i - \beta_{i+1}$.

From the abacus configuration we have $\beta_i - \beta_{i+1} = 1$ for $i \neq n$ and $\beta_n - \beta_{n+1} = e - r + 1$. Hence $a_i = 1$ for $i \neq n$ and $a_n = e - r + 1 < e$, so this point lies in the fundamental e–alcove. □

The next observation shows that this choice is indeed natural, depending only on r and w.

Consider the sub-abacus consisting of the r non-empty runners:



The total number of gaps above the beads is $n(m+1) + (r-n)m = rm + n = w$, so this abacus corresponds to the partition (w^r). Equivalently, it is obtained from the empty partition by shifting each bead to the right by w steps in the abacus with r runners and r beads.

3.3. Proof of the Empty Runner Removal Theorem

We now state the runner removal theorems.[2] Let e, n be positive integers. Suppose λ and μ are partitions of size n lying in the same block (i.e., they share the same e-core and e-weight). Form the abaci $\mathrm{Ab}^r_e(\lambda)$ and $\mathrm{Ab}^r_e(\mu)$.

Fix $j \in \{0, 1, \dots, e-1\}$ and insert a new runner immediately to the left of the j–runner.

- If the new runner contains no beads, we call it an *empty* runner.
- If the new runner is flush and the first gap on this runner is on a row strictly greater than the row index of any bead on the existing runners, we call it a *full* runner.

In either case, adding the same number of beads to the new runner for both configurations yields new partitions λ^+ and μ^+.

Theorem 3.3.1 ([**JM02**], Theorem 3.2). *In characteristic* 0*, suppose* λ *and* μ *lie in the same block and* μ *is e–regular. If* λ^+ *and* μ^+ *are obtained by inserting an empty runner, then*

$$d^e_{\lambda,\mu}(v) = d^{e+1}_{\lambda^+,\mu^+}(v).$$

Theorem 3.3.2 ([**Fay07**], Theorem 4.1). *In characteristic* 0*, suppose* λ *and* μ *lie in the same block and* μ *is e–regular. If* λ^+ *and* μ^+ *are obtained by inserting a full runner, then*

$$d^e_{\lambda,\mu}(v) = d^{e+1}_{\lambda^+,\mu^+}(v).$$

Remark 3.3.3. *The condition that* μ *is e-regular is necessary only in the context of the Iwahori–Hecke algebra (where simple modules are indexed by e-regular partitions). These theorems were originally established in the context of q-Schur algebras, where simple modules exist for all partitions and the regularity condition can be dropped.* ◇

As noted in [**JM02**], Goodman suggested that Theorem 3.3.1 can be deduced from the following theorem.

Theorem 3.3.4 ([**GW98**]). *In characteristic* 0*, let* λ *and* μ *be two partitions of the same size with no more than r rows, and assume* μ *is e-regular. Then the graded decomposition number is given by the anti-spherical Kazhdan–Lusztig polynomial evaluated at the corresponding weights:*

$$d^e_{\lambda\mu}(v) = n^e_{\Omega(\lambda),\Omega(\mu)}(v).$$

In [**Fay07**, Section 2.3], the author briefly describes the idea of how to realize Goodman's remark. However, a detailed proof is not provided there. In this section, we provide the proof of Theorem 3.3.1.

The strategy is to show that the combinatorial operation of inserting an empty runner corresponds to a geometric stability of the weights. Specifically, we prove that

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[2] In the next chapter, we will restate the two runner removal theorems in a unified way; see Section 4.4.

the dominant weight $\Omega(\lambda)$ in the level-e arrangement occupies the exact same relative alcove as the weight $\Omega(\lambda^+)$ does in the level-$(e+1)$ arrangement.

Proof of Theorem 3.3.1. Let λ be a partition and μ be a e-regular partition, both of which have no more than r parts. Fix $k \in \{0, \dots, e-1\}$. Write the beta numbers of λ in the form $\beta_i(\lambda) = a_i e + b_i$ with $0 \leq b_i < e$. Inserting an empty runner before the k-runner gives the beta numbers of λ^+:

$$\beta_i(\lambda^+) \;=\; a_i(e+1) + b_i + \delta_{b_i \geq k}.$$

By Theorem 3.3.4, $d^e_{\lambda,\mu}(v) = \mathfrak{n}^e_{\Omega(\lambda),\Omega(\mu)}(v)$. Thus, to prove the theorem, it suffices to show that

$$\mathfrak{n}^e_{\Omega(\lambda),\Omega(\mu)}(v) \;=\; \mathfrak{n}^{e+1}_{\Omega(\lambda^+),\Omega(\mu^+)}(v).$$

Recall the level-e anti-spherical polynomials in Subsection 1.4.4, this equality holds if $\Omega(\lambda)$ and $\Omega(\lambda^+)$ define the same element of the affine Weyl group. Using the Shi coefficient criterion (1.4.2), we prove that for every positive root $\alpha \in R^+$:

$$k^{(e)}_\alpha\big(\Omega(\lambda)\big) \;=\; k^{(e+1)}_\alpha\big(\Omega(\lambda^+)\big). \tag{3.3.5}$$

Any positive root $\alpha \in R^+$ is a sum of consecutive simple roots: $\alpha = \alpha_i + \cdots + \alpha_{j-1}$ for $1 \leq i < j \leq r$. Since $\Omega(\lambda) = \sum_{k=1}^{r-1} \big(\beta_k(\lambda) - \beta_{k+1}(\lambda)\big)\Lambda_k$, using the duality $\langle \Lambda_k, \alpha_m^\vee \rangle = \delta_{km}$, we have:

$$\begin{aligned}
\langle \alpha^\vee, \Omega(\lambda) \rangle \;&=\; \left\langle \sum_{m=i}^{j-1} \alpha_m^\vee, \sum_{k=1}^{r-1} \big(\beta_k(\lambda) - \beta_{k+1}(\lambda)\big)\Lambda_k \right\rangle \\
&=\; \sum_{k=i}^{j-1} \big(\beta_k(\lambda) - \beta_{k+1}(\lambda)\big) \\
&=\; \beta_i(\lambda) - \beta_j(\lambda).
\end{aligned}$$

Using $\beta_i(\lambda) = a_i e + b_i$ as above, we have:

$$\langle \alpha^\vee, \Omega(\lambda) \rangle \;=\; (a_i - a_j)e + (b_i - b_j).$$

Similarly, for the partition λ^+, we have:

$$\langle \alpha^\vee, \Omega(\lambda^+) \rangle \;=\; (a_i - a_j)(e+1) + (b_i - b_j) + (\delta_{b_i \geq k} - \delta_{b_j \geq k}).$$

Let $A = a_i - a_j$ and let $\Delta := \delta_{b_i \geq k} - \delta_{b_j \geq k} \in \{-1, 0, 1\}$. We compute the floor functions in two cases.

Case 1: $b_i \geq b_j$**.** Then $0 \leq b_i - b_j < e$ and the level-e Shi coefficient $k^{(e)}_\alpha\big(\Omega(\lambda)\big)$ is:

$$\left\lfloor \frac{Ae + (b_i - b_j)}{e} \right\rfloor \;=\; A + \left\lfloor \frac{b_i - b_j}{e} \right\rfloor \;=\; A.$$

For level $e+1$, we examine the remainder term $R := (b_i - b_j) + \Delta$. Since $b_i \geq b_j$, we cannot have $b_j \geq k > b_i$, so $\Delta \in \{0, 1\}$ and $0 \leq R < e+1$. Thus, the level-$(e+1)$ Shi coefficient $k_\alpha^{(e+1)}\big(\Omega(\lambda^+)\big)$ is

$$\left\lfloor \frac{A(e+1)+R}{e+1} \right\rfloor \;=\; A.$$

Case 2: $b_i < b_j$**.** Then $-e < b_i - b_j < 0$. Rewrite $\langle \alpha^\vee, \Omega(\lambda)\rangle$ as $(A-1)e + (e + b_i - b_j)$. The term $(e + b_i - b_j)$ is strictly between 0 and e. Thus:

$$k_\alpha^{(e)}(\Omega(\lambda)) = \left\lfloor \frac{\langle \alpha^\vee, \Omega(\lambda)\rangle}{e} \right\rfloor \;=\; A-1.$$

For level $e+1$, rewrite $\langle \alpha^\vee, \Omega(\lambda^+)\rangle$ as $(A-1)(e+1)+R'$, where $R' = (e+1)+(b_i-b_j)+\Delta$. We must show $0 \le R' < e+1$. Since $b_i < b_j$, we cannot have $b_i \ge k > b_j$, so $\Delta \in \{0,-1\}$. Hence, we have $1 < b_i - b_j + e + 1 < e+1$ and $0 < R' < e+1$. Thus, the level-$(e+1)$ Shi coefficient is also $A-1$.

We have shown that for all $\alpha \in R^+$, the Shi coefficients coincide:

$$k_\alpha^{(e)}\big(\Omega(\lambda)\big) \;=\; k_\alpha^{(e+1)}\big(\Omega(\lambda^+)\big).$$

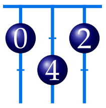
This implies that $\Omega(\lambda)$ and $\Omega(\lambda^+)$ correspond to the same element in the affine Weyl group (relative to their respective levels). The same argument applies to μ and μ^+. Moreover, μ^+ is clearly $(e+1)$-regular. Consequently,

$$\mathfrak{n}^{e}_{\Omega(\lambda),\Omega(\mu)}(v) \;=\; \mathfrak{n}^{e+1}_{\Omega(\lambda^+),\Omega(\mu^+)}(v),$$

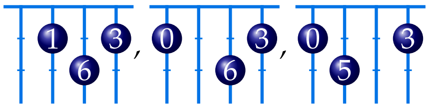
and therefore $d^{e}_{\lambda,\mu}(v) = d^{e+1}_{\lambda^+,\mu^+}(v)$, as desired. □

We end this section with a graphical explanation of the above proof, by drawing the alcoves corresponding to λ and λ^+ in the level-e and level-$(e+1)$ arrangements, respectively.

Example 3.3.6. *In type $A_2^{(1)}$, let $r = 3$. Take $\lambda = (2,1)$, its abacus with 3-beads is:*

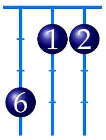


By adding an empty runner to the left of i-th runner where $i = 0,1,2$, we get λ^{+i}:

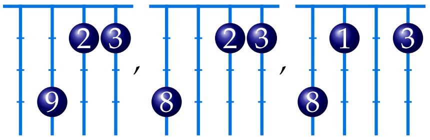


It is easy to see $\lambda^{+0} = (4,2,1)$, $\lambda^{+1} = (4,2)$ and $\lambda^{+2} = (3,2)$. Identifying them with the dominant weights in sl_3, we have: $\lambda+\rho = 2\Lambda_1+2\Lambda_2$, $\lambda^{+0}+\rho = 3\Lambda_1+2\Lambda_2$, $\lambda^{+1}+\rho = 3\Lambda_1+3\Lambda_2$ and $\lambda^{+2}+\rho = 2\Lambda_1+3\Lambda_2$. ◇

Example 3.3.7. *In type $A_2^{(1)}$, let $r = 3$. Take $\mu = (4,1,1)$, its abacus with 3-beads is:*

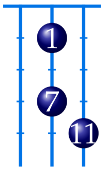


By adding an empty runner to the left of i-th runner where $i = 0, 1, 2$, we get μ^{+i}:

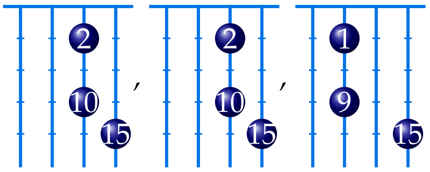


It is easy to see $\mu^{+0} = (7, 2, 2)$, $\mu^{+1} = (6, 2, 2)$ and $\mu^{+2} = (6, 2, 1)$. Identifying them with the dominant weights in sl_3, we have: $\mu + \rho = 4\Lambda_1 + \Lambda_2$, $\mu^{+0} + \rho = 6\Lambda_1 + \Lambda_2$, $\mu^{+1} + \rho = 5\Lambda_1 + \Lambda_2$ and $\mu^{+2} + \rho = 5\Lambda_1 + 2\Lambda_2$. ◇

EXAMPLE 3.3.8. *In type $A_2^{(1)}$, let $r = 3$. Take $\nu = (9, 6, 1)$, its abacus with 3-beads is:*



By adding an empty runner to the left of i-th runner where $i = 0, 1, 2$, we get ν^{+i}:



It is easy to see $\nu^{+0} = \lambda^{+1} = (13, 9, 2)$ and $\nu^{+2} = (13, 8, 1)$. Identifying them with the dominant weights in sl_3, we have: $\nu + \rho = 4\Lambda_1 + 6\Lambda_2$, $\nu^{+0} + \rho = \nu^{+1} + \rho = 5\Lambda_1 + 8\Lambda_2$ and $\nu^{+2} + \rho = 6\Lambda_1 + 8\Lambda_2$. ◇

EXAMPLE 3.3.9. *Consider Example 3.3.6. We plot $\lambda + \rho$ in the level-3 alcove (Figure 8) and $\lambda^{+i} + \rho$ in the level-4 alcove (Figure 9), both in blue. Similarly, for μ and ν from Example 3.3.7 and Example 3.3.8, we plot the corresponding points in red and green, respectively.*

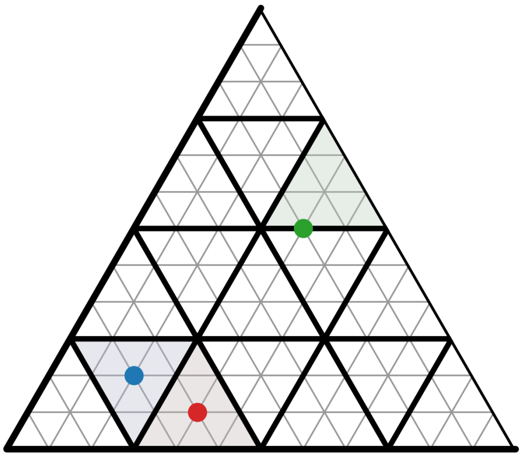


FIGURE 8. Level 3

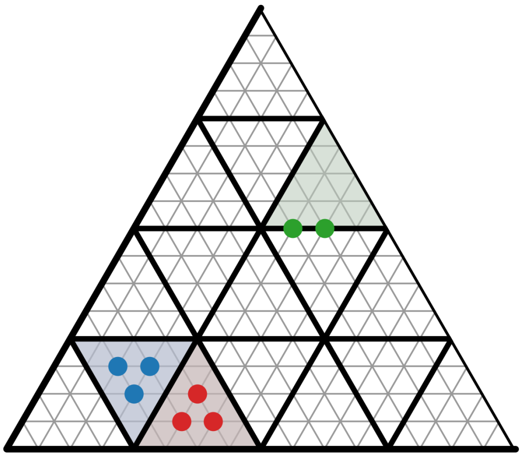


FIGURE 9. Level 4

◇

3.4. Combinatorial Identities

Let $\lfloor\cdot\rfloor$ be the floor function. For two positive integers w, r, we define $p_r(w)$ to be the number of partitions of w having exactly r positive parts. It is well-known that $p_2(w) = \lfloor \frac{w}{2} \rfloor$ and $p_3(w) = \left\lfloor \frac{w^2+3}{12} \right\rfloor$.

For each w and e, we define the following functions:

$$a_w = \frac{(w+1)(w+2)}{4}, \qquad b_w = \left\lfloor \frac{w+2}{2} \right\rfloor - a_w, \qquad c_w = p_3(w). \tag{3.4.1}$$

Lemma 3.4.2. *For every integer $w \geq 0$,*

$$2\sum_{i=1}^{w} \left\lfloor \frac{i}{2} \right\rfloor + 2w + 2 = \binom{w+2}{2} + \left\lfloor \frac{w+2}{2} \right\rfloor. \tag{3.4.3}$$

Proof. Write $w = 2m$ $(m > 0)$ or $w = 2m+1$ $(m \geq 0)$.

Case $w = 2m$.

$$\sum_{i=1}^{2m} \left\lfloor \frac{i}{2} \right\rfloor = \sum_{k=1}^{m}\left(\left\lfloor \frac{2k-1}{2} \right\rfloor + \left\lfloor \frac{2k}{2} \right\rfloor \right) = \sum_{k=1}^{m} \big((k-1)+k\big) = \sum_{k=1}^{m}(2k-1) = m^2. \tag{3.4.4}$$

Hence the left-hand side of (3.4.3) equals $2m^2 + 4m + 2$. The right-hand side equals

$$\binom{2m+2}{2} + \left\lfloor \frac{2m+2}{2} \right\rfloor = \frac{(2m+2)(2m+1)}{2} + (m+1) = 2m^2 + 4m + 2.$$

Case $w = 2m+1$. Using the even case up to $2m$ and adding $\lfloor (2m+1)/2 \rfloor = m$,

$$\sum_{i=1}^{2m+1} \left\lfloor \frac{i}{2} \right\rfloor = m^2 + m,$$

so the left-hand side of (3.4.3) is $2(m^2+m)+2(2m+1)+2 = 2m^2+6m+4$. The right-hand side is

$$\binom{2m+3}{2} + \left\lfloor \frac{2m+3}{2} \right\rfloor = \frac{(2m+3)(2m+2)}{2} + (m+1) = 2m^2 + 6m + 4.$$

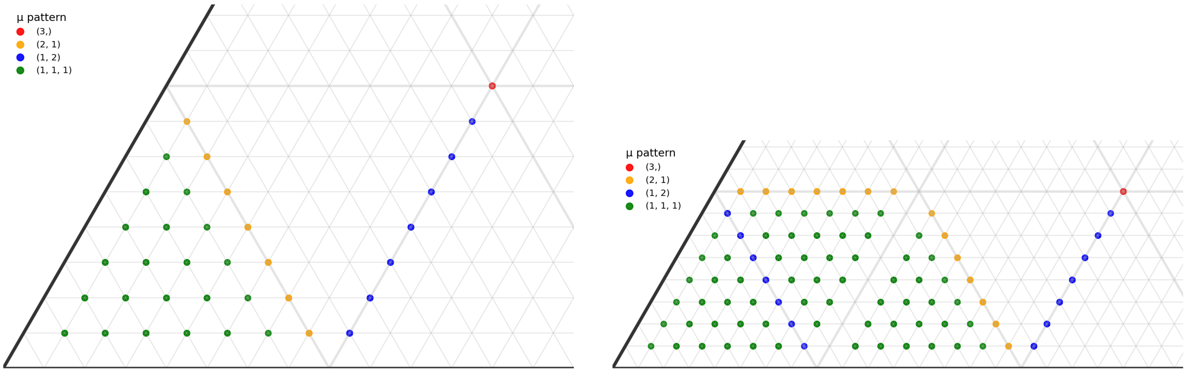
Both cases agree, proving (3.4.3). □

Lemma 3.4.5. *For integers $e \geq 3$ and $w \geq 0$,*

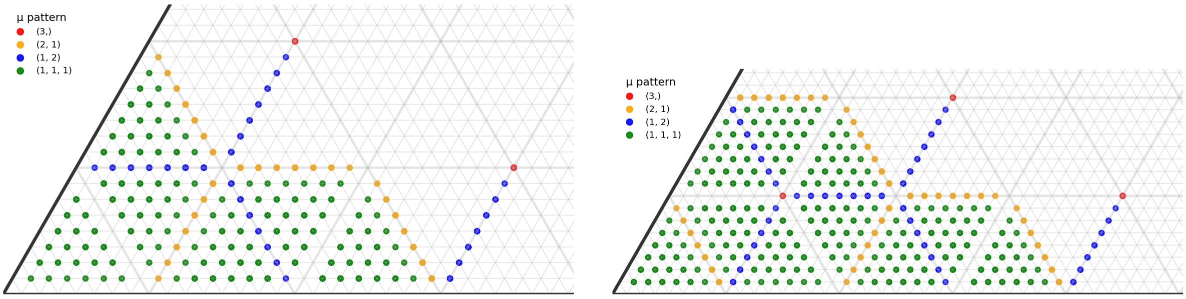
$$\binom{w+2}{2}\binom{e-1}{2} + 2\binom{e-1}{1}\left(\sum_{i=1}^{w} p_2(i) + w + 1\right) + p_3(w) + p_2(w) + 1 = a_w e^2 + b_w e + c_w, \tag{3.4.6}$$

Proof. Set $\alpha = \binom{w+2}{2}$ and $S = \sum_{i=1}^{w} p_2(i)$. Using

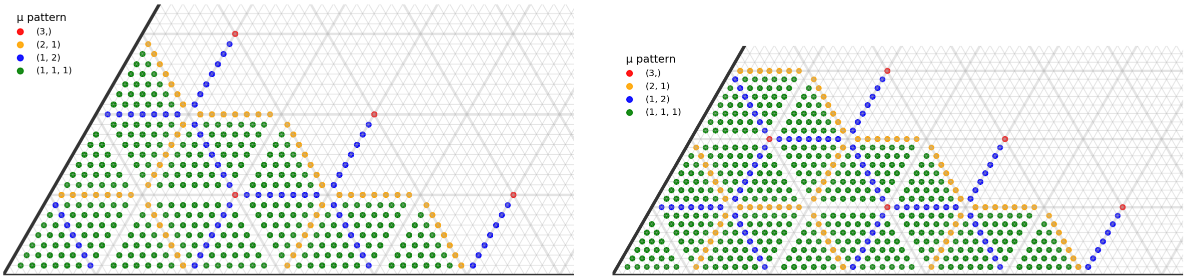
$$\binom{e-1}{2} = \tfrac{1}{2}(e^2 - 3e + 2), \qquad \binom{e-1}{1} = e - 1, \tag{3.4.7}$$



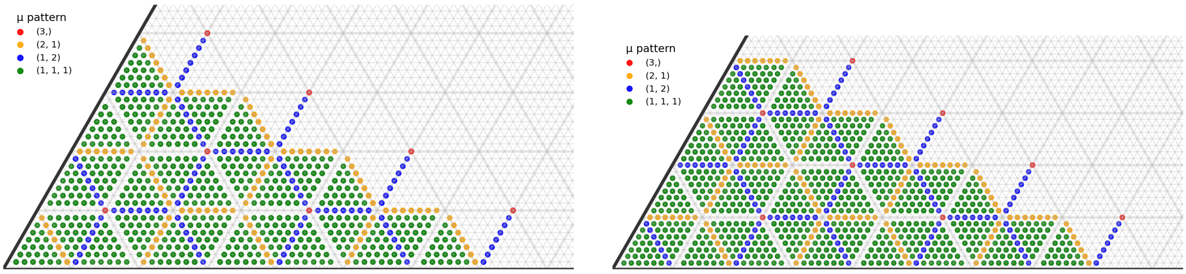
(A) $r = 3, \ e = 8, \ w = 0$ (B) $r = 3, \ e = 8, \ w = 1$



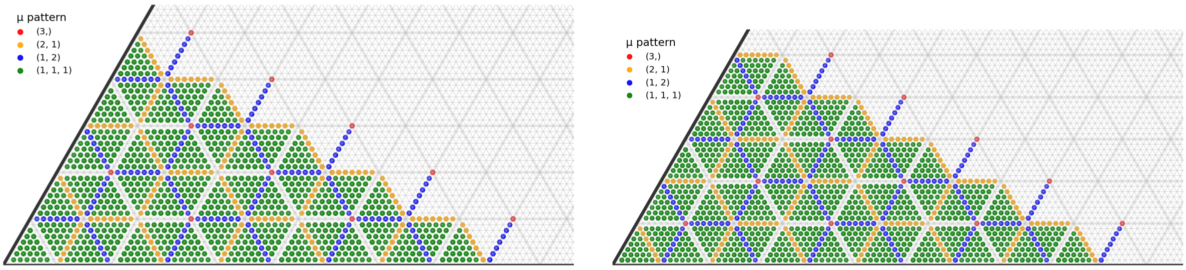
(C) $r = 3, \ e = 8, \ w = 2$ (D) $r = 3, \ e = 8, \ w = 3$



(E) $r = 3, \ e = 8, \ w = 4$ (F) $r = 3, \ e = 8, \ w = 5$



(G) $r = 3, \ e = 8, \ w = 6$ (H) $r = 3, \ e = 8, \ w = 7$



(I) $r = 3, \ e = 8, \ w = 8$ (J) $r = 3, \ e = 8, \ w = 9$

FIGURE 10. $\mathfrak{sl}_3$ dominant weight pictures for $e = 8$ and $w = 0, 1, \ldots, 9$.

the left-hand side of (3.4.6) becomes

$$E(e,w)=\frac{\alpha}{2}e^2+\Big(2(S+w+1)-\frac{3\alpha}{2}\Big)e+\Big(\alpha-2(S+w+1)+p_3(w)+p_2(w)+1\Big). \tag{3.4.8}$$

Thus the e^2–coefficient is $a_w=\alpha/2=\frac{(w+1)(w+2)}{4}$.

By Lemma 3.4.2,

$$2(S+w+1)=\alpha+\left\lfloor\frac{w+2}{2}\right\rfloor. \tag{3.4.9}$$

Therefore the e–coefficient equals

$$b_w=\Big(\alpha+\left\lfloor\frac{w+2}{2}\right\rfloor\Big)-\frac{3\alpha}{2}=\left\lfloor\frac{w+2}{2}\right\rfloor-\frac{\alpha}{2}=\left\lfloor\frac{w+2}{2}\right\rfloor-a_w,$$

and, using (3.4.9) together with $\lfloor\frac{w+2}{2}\rfloor=p_2(w)+1$, the constant term is

$$c_w=\alpha-2(S+w+1)+p_3(w)+p_2(w)+1=\alpha-\Big(\alpha+\left\lfloor\frac{w+2}{2}\right\rfloor\Big)+p_3(w)+p_2(w)+1=p_3(w).$$

This identifies all coefficients in (3.4.1) and proves (3.4.6). □

CHAPTER 4

Subdivision and Runner Removal Theorem

This chapter is based on the two papers [**Qin24**] and [**Qin26b**]. We give a short introduction here.

The subdivision map for KLR algebras of type $A^{(1)}_{e-1}$ was introduced by Maksimau in [**Mak18**] to relate categorical representations of $\widehat{\mathfrak{sl}_e}$ and $\widehat{\mathfrak{sl}_{e+1}}$. Mathas and Tubbenhauer [**MT23**] subsequently extended this construction to KLRW algebras, thereby generalizing Maksimau's result. In [**Qin24**], we attempted to generalize this map to cyclotomic KLRW algebras. In [**Qin26b**], we further extend the combinatorial framework of [**Qin24**] to full generality.

More precisely, the KLR algebras have three types of generators: idempotents $e(\mathbf{i})$, polynomial generators (or dots) y_i, and permutation generators ψ_i; see Subsection 1.3.4. In [**Qin24**] we define the subdivision map on the set of partitions in order to describe the image of idempotents under the subdivision map for KLR algebras. This map is given in two combinatorial ways: via Young diagrams and via James' abaci, and we show that these two definitions are equivalent. In [**Qin26b**], we simplify the combinatorial construction by distinguishing two kinds of strips in the Young-diagram description; see Subsection 4.2.1, and by using finite abaci rather than infinite abaci in the abacus description; see Subsection 4.2.2. We also show that these two simplified definitions are equivalent; see Theorem 4.2.44. Moreover, we extend the construction to the set of (row-)standard tableaux and show that it preserves the degrees of standard tableaux; see Subsection 4.2.4.

Any cyclotomic KLR algebra has a distinguished family of modules, called Specht modules. There is a subfamily of Specht modules whose heads give all irreducible modules of the corresponding cyclotomic KLR algebra; see [**HM10**] for more details. One motivation for both two papers [**Qin24**, **Qin26b**] is that the subdivision map for KLR(W) algebras may relate the Specht(Weyl) modules of two KLR(W) algebras and hence provide information about the corresponding graded decomposition numbers. In [**Qin26b**], we show that, under a certain condition, the Specht modules are related in the most natural way; see Subsection 4.3.3. The proof relies on the highest-weight presentation of Specht modules from [**KMR12**], and, as an intermediate step, we also prove that the permutation modules are related naturally; see Subsection 4.3.2.

It was suggested by Andrew Mathas that the subdivision of KLR(W) algebras should be connected to the runner-removal theorems of James and Mathas [**JM02**] and of Fayers [**Fay07**, **Fay09**] for Hecke and q-Schur algebras. We confirm this connection in both papers by showing that the abacus combinatorics used to construct the subdivision map on partitions can be interpreted as runner addition (see Subsection 4.2.2), in agreement with the constructions in [**Fay07**, **Del24**, **DP25**]; see Section 4.4.

Recently, Dell'Arciprete and Putignano extended the empty runner removal theorem [**JM02**] and the full runner removal theorem [**Fay07**] to higher levels, i.e. to Ariki–Koike algebras; see [**Del24**, **DP25**]. However, the general runner removal theorem of [**Fay09**] has not yet been generalised. In [**Qin26b**], we formulate some natural conjectures extending this result and present some evidence; see Conjecture 4.4.10, Conjecture 4.4.11 and Example 4.4.12.

In view of the categorical properties of the subdivision maps, whose combinatorial construction agrees with that of the runner-removal theorems, the subdivision map can be viewed as a categorification of the latter. Moreover, the proofs of the runner-removal theorems above are based on calculations in the Fock space and therefore work only in characteristic zero, and do not readily extend to positive characteristic. The advantage of the subdivision approach is that it has the potential to work in positive characteristic, as in [**CM10**]. However, we do not address conditions under which subdivision is exact as a functor on module categories, which is a key ingredient for extending [**CM10**]. We expect that this is related to [**MT17**].

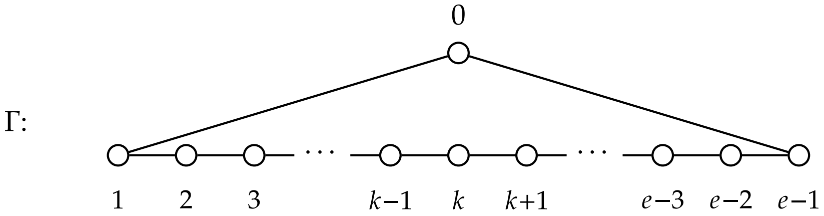
In this chapter, the structure is as follows: In Section 4.1 we give a self-contained definition of the subdivision map Φ_k on the affine quiver and the associated data (positive roots, words, dominant weights, KLR algebras, and their cyclotomic quotients), providing a base and extended version of [**Mak18**]. In Section 4.2 we develop the corresponding subdivision on partitions in two parallel languages—Young diagrams and e-abaci—and prove that the two definitions agree. Furthermore, we extend the map to the set of standard tableaux. In Section 4.3 we translate these combinatorial results back to the algebraic setting by analyzing the images of idempotents under Φ_k. Moreover, we relate permutation modules and (universal) Specht modules in the expected way (Subsection 4.3.2 and Subsection 4.3.3), giving our first categorical consequences, while leaving exactness issues for future work. Finally, in Section 4.4 we connect subdivision with the runner-removal theorems, treating both the level-one and higher-level cases, and we end with two conjectures on equalities of decomposition numbers in higher level; an example records explicit canonical-basis expansions that provide computational evidence.

4.1. Algebraic Subdivision

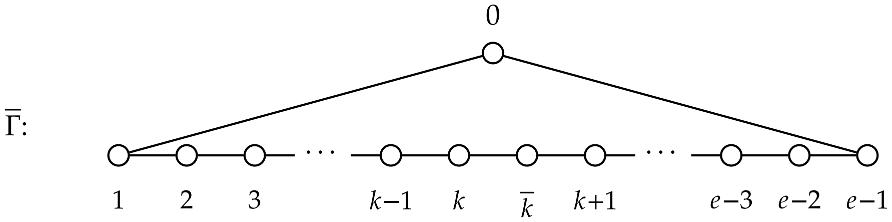
In this section, we introduce the subdivision maps for KLR algebras following [**Mak18**], with minor notational changes adapted to our combinatorial framework.

Since we will define subdivision maps on several kinds of objects—including quivers, positive roots, words, dominant weights, and (cyclotomic) KLR algebras—we will, by abuse of notation, use the same symbol Φ for all of them. When we apply these maps later, it will be clear from the objects involved which version of Φ is intended. We will, however, verify along the way that these definitions are compatible with one another.

4.1.1. Subdivision on quiver. Assume that the quiver $\Gamma = A_{e-1}^{(1)}$ has vertex set $I = \mathbb{Z}/e\mathbb{Z}$, which we identify with $\{0, 1, \ldots, e-1\}$ throughout the chapter.



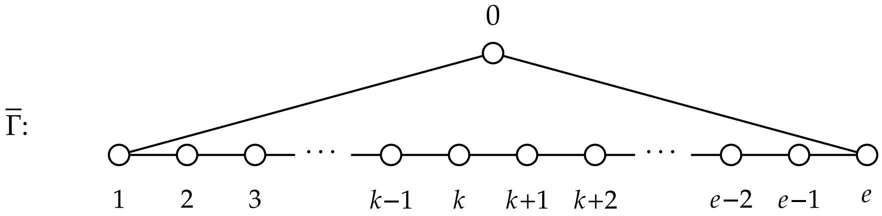
Fix an integer $k \in I$ and identify e with 0. The subdivision at the edge $k \longrightarrow k+1$ replaces this single edge by the two-step path $k \longrightarrow \overline{k} \longrightarrow k+1$, while leaving all other edges unchanged. We denote the resulting subdivision map by Φ_k, and write $\overline{\Gamma} := \Phi_k(\Gamma)$ for the resulting quiver:



As we will apply combinatorial arguments later, it is convenient to label the new quiver $\overline{\Gamma}$, which is a cyclic quiver with $e+1$ vertices, using the standard convention for type $A_e^{(1)}$. We therefore relabel the vertices as follows:

(4.1.1)
- keep the label i unchanged for $0 \leq i \leq k$;
- replace the label i by $i+1$ for $k+1 \leq i \leq e-1$;
- replace the label $\overline{k}$ by $k+1$.

With this convention, the quiver $\overline{\Gamma}$ becomes:



By abuse of notation, we will continue to write this relabelled quiver as $\overline{\Gamma}$. The vertex set of $\overline{\Gamma}$ is $\overline{I} = \mathbb{Z}/(e+1)Z$ which is identified with $\{0, 1, \cdots, e\}$.

Remark 4.1.2. *In [Mak18], there is no relabelling step: the new vertex is kept as a distinguished symbol* $\overline{k}$. *Our relabelling identifies this new vertex with* $k+1 \in \overline{I}$ *and shifts the old labels* $k+1, \ldots, e-1$ *up by one, so the two conventions differ only by this canonical relabelling.* ◇

4.1.2. Subdivision on positive roots.

Let $Q^+ = Q^+_I$ be the positive cone of the root lattice of type $A^{(1)}_{e-1}$, with simple roots $\alpha_0, \dots, \alpha_{e-1}$. For $\beta \in Q^+$ write

$$\beta = \sum_{i=0}^{e-1} x_i \alpha_i \qquad \text{with } x_i \in \mathbb{Z}_{\geq 0}.$$

For $k \in I = \{0, 1, \cdots, e-1\}$, the subdivision map Φ_k on Q^+ at the edge $k \to k+1$ is defined by

$$\Phi_k(\beta) \;=\; \sum_{i=0}^{k-1} x_i \alpha_i \;+\; x_k(\alpha_k + \alpha_{k+1}) \;+\; \sum_{i=k+1}^{e-1} x_i \alpha_{i+1}. \tag{4.1.3}$$

This is compatible with the subdivision map on quivers described in Subsection 4.1.1. Indeed, for $0 \leq i \leq k-1$ we have $\Phi_k(\alpha_i) = \alpha_i$, for $k+1 \leq i \leq e-1$ we have $\Phi_k(\alpha_i) = \alpha_{i+1}$, and $\Phi_k(\alpha_k) = \alpha_k + \alpha_{k+1}$.

The map Φ_k yields a bijection

$$\Phi_k : Q^+_I \to \left\{ \beta = \sum_{i \in \bar{I}} x_i \alpha_i \in Q^+_{\bar{I}} \;\middle|\; x_k = x_{k+1} \right\}, \qquad \alpha \mapsto \Phi_k(\alpha). \tag{4.1.4}$$

If there is no ambiguity in k, we write $\overline{\beta} = \Phi(\beta) = \Phi_k(\beta)$.

4.1.3. Subdivision on words.

Fix quiver type $A^{(1)}_{e-1}$ and let I be its vertex set. Let $\mathbf{i} \in I^\alpha$ with $n = \mathrm{ht}(\alpha)$, and fix $k \in I$. Let $\bar{I}$ be the vertex set of the quiver $A^{(1)}_e$, which is the image of $A^{(1)}_{e-1}$ under the subdivision map constructed in Subsection 4.1.1. Define $\Phi_k(\mathbf{i})$ to be the sequence with entries in $\bar{I}$ obtained from $\mathbf{i} = (\mathbf{i}_1, \dots, \mathbf{i}_n)$ by:

- keeping $\mathbf{i}_j$ unchanged if $0 \leq \mathbf{i}_j < k$;
- replacing $\mathbf{i}_j$ by $\mathbf{i}_j + 1$ if $k < \mathbf{i}_j \leq e-1$;
- replacing each occurrence of $\mathbf{i}_j = k$ by two consecutive entries $k, k+1$.

Example 4.1.5. *Fix type* $A^{(1)}_2$ *and* $k = 1$, *let* $\mathbf{i}^\lambda = (1200122012010)$ *from Example 1.2.15, then* $\Phi_k(\mathbf{i}^\lambda) = (12300123301230120)$. ◇

This definition is compatible with the subdivision map on positive roots in the following sense. Recall, for each word $\mathbf{i}$, we can define the associated positive root $\alpha(\mathbf{i})$ by (1.1.2), then we have

$$\Phi_k\big(\alpha(\mathbf{i})\big) = \alpha\big(\Phi_k(\mathbf{i})\big)$$

More explicitly, write $\alpha(\mathbf{i}) = \sum_{j=1}^{n} \alpha_{\mathbf{i}_j} \in Q^+$ and let m be the number of entries equal to k in $\mathbf{i}$. Then

$$\sum_{j=1}^{n+m} \alpha_{(\Phi_k(\mathbf{i}))_j} \;=\; \Phi_k\big(\alpha(\mathbf{i})\big). \tag{4.1.6}$$

Similarly to (4.1.4), Φ_k gives the following bijection

$$\Phi_k : I^\alpha \to \{\mathbf{j} \in I^{\Phi_k(\alpha)} \mid \mathbf{j}_t = k \text{ if and only if } t < N \text{ and } \mathbf{j}_{t+1} = k+1\}, \quad \text{where } N = \mathrm{ht}\big(\Phi_k(\alpha)\big) \tag{4.1.7}$$

To trace the position of a letter in a word after applying subdivision, we introduce the following *position-tracing function*.

DEFINITION 4.1.8. *Fix $k \in I$ and let $\mathbf{i} = (i_1, \dots, i_n) \in I_\alpha$. Write $m := \#\{t \mid i_t = k\}$. The position-tracing function associated to $(k, \mathbf{i})$ is the strictly increasing map*

$$\phi_{\mathbf{i}} : \{1, \dots, n\} \longrightarrow \{1, \dots, n+m\}$$

defined by

$$\phi_{\mathbf{i}}(t) := t + \#\{1 \le j < t \mid i_j = k\}.$$

Equivalently, $\phi_{\mathbf{i}}(1) = 1$ *and* $\phi_{\mathbf{i}}(t+1) = \phi_{\mathbf{i}}(t) + 1 + \delta_{i_t,k} \qquad (1 \le t < n).$

The following result is immediate from the definition.

LEMMA 4.1.9. *With the same notations as in Definition 4.1.8,*

$$\big(\Phi_k(\mathbf{i})\big)_{\phi_{\mathbf{i}}(t)} = \begin{cases} \mathbf{i}_t & \text{if } 0 \le \mathbf{i}_t < k, \\ k & \text{if } \mathbf{i}_t = k, \\ \mathbf{i}_t + 1 & \text{if } k+1 \le \mathbf{i}_t \le e-1. \end{cases}$$

Moreover, if $\big(\Phi_k(\mathbf{i})\big)_{\phi_{\mathbf{i}}(t)} = k$, *then* $\big(\Phi_k(\mathbf{i})\big)_{\phi_{\mathbf{i}}(t)+1} = k+1$. □

4.1.4. Subdivision on dominant weights. Let P^+ be the (integral) dominant weight lattice of type $A^{(1)}_{e-1}$, and let I be the vertex set of the corresponding $A^{(1)}_{e-1}$ quiver. We define the subdivision map on fundamental weights by

$$\Phi\Lambda_i = \begin{cases} \Lambda_i, & \text{if } 0 \le i \le k, \\ \Lambda_{i+1}, & \text{if } i > k. \end{cases} \tag{4.1.10}$$

For an arbitrary (integral dominant) weight $\Lambda = \sum_{i=0}^{e-1} x_i \Lambda_i$ with $x_i \ge 0$ for all i, we extend this map linearly by

$$\Phi(\Lambda) := \sum_{i=0}^{e-1} x_i \Phi(\Lambda_i).$$

In type $A^{(1)}_{e-1}$ (or type A_∞), this map preserves the *level* of the dominant weight. The compatibility of this definition with the ones in Subsection 4.1.2 is shown in the following results:

LEMMA 4.1.11. *Fix $\alpha \in Q^+$, $\Lambda \in P^+$, and $k \in I$. Let $\mathbf{i} \in I^\alpha$, and let $\Phi = \Phi_k$ be the subdivision map on positive roots or dominant weights. Then we have:*

$$\big(\Phi(\Lambda) \mid \alpha_{\Phi(\mathbf{i})_1}\big) = \big(\Lambda \mid \alpha_{\mathbf{i}_1}\big).$$

Proof. It suffices to verify the equality for any fundamental weight Λ_j; the general case follows by linearity. Recall that $\left(\Lambda_j \mid \alpha_{\mathbf{i}_1}\right) = \delta_{j,\mathbf{i}_1}$. Set $\overline{\Lambda} := \Phi(\Lambda)$ and $\bar{\mathbf{i}} := \Phi(\mathbf{i})$. Then

$$\left(\overline{\Lambda} \mid \alpha_{\bar{\mathbf{i}}_1}\right) = \begin{cases} \left(\Lambda_j \mid \alpha_{\mathbf{i}_1}\right) = \delta_{j,\mathbf{i}_1} & \text{if } j \le k,\ \mathbf{i}_1 \le k, \\ \left(\Lambda_j \mid \alpha_{\mathbf{i}_1+1}\right) = 0 & \text{if } j \le k,\ \mathbf{i}_1 > k, \\ \left(\Lambda_{j+1} \mid \alpha_{\mathbf{i}_1+1}\right) = \delta_{j+1,\mathbf{i}_1+1} & \text{if } j > k,\ \mathbf{i}_1 > k, \\ \left(\Lambda_{j+1} \mid \alpha_{\mathbf{i}_1}\right) = 0 & \text{if } j > k,\ \mathbf{i}_1 \le k. \end{cases} = \delta_{j,\mathbf{i}_1}$$

Thus, the desired equality holds in all cases. □

Lemma 4.1.12. *Fix $\alpha \in Q^+$, $\Lambda \in P^+$, and $k \in I$. Let $\Phi = \Phi_k$ be the subdivision map on positive roots or dominant weights. Then the pairing is preserved:*

$$(\Phi(\Lambda) \mid \Phi(\alpha)) = (\Lambda \mid \alpha).$$

Proof. By linearity, it suffices to verify the equality for fundamental weights Λ_j and simple roots α_i, i.e., to show $(\Phi(\Lambda_j) \mid \Phi(\alpha_i)) = \delta_{j,i}$. We verify the cases based on the index i:

(a). Case $i < k$: Then $\Phi(\alpha_i) = \alpha_i$. If $j \le k$, then $\Phi(\Lambda_j) = \Lambda_j$ and the pairing is $\delta_{j,i}$. If $j > k$, then $\Phi(\Lambda_j) = \Lambda_{j+1}$, and $(\Lambda_{j+1} \mid \alpha_i) = 0$ (since $j+1 > i$), matching $\delta_{j,i} = 0$.

(b). Case $i = k$: Then $\Phi(\alpha_k) = \alpha_k + \alpha_{k+1}$.
  - If $j < k$: $(\Lambda_j \mid \alpha_k + \alpha_{k+1}) = 0$.
  - If $j = k$: $(\Lambda_k \mid \alpha_k + \alpha_{k+1}) = 1 + 0 = 1$.
  - If $j > k$: $(\Lambda_{j+1} \mid \alpha_k + \alpha_{k+1}) = 0$.

  In all subcases, the result matches $\delta_{j,k}$.

(c). Case $i > k$: Then $\Phi(\alpha_i) = \alpha_{i+1}$. If $j \le k$, $(\Lambda_j \mid \alpha_{i+1}) = 0$. If $j > k$, $(\Lambda_{j+1} \mid \alpha_{i+1}) = \delta_{j+1,i+1} = \delta_{j,i}$.

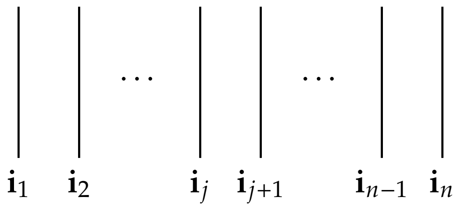
Thus, the equality holds for all basis elements. □

If there is no ambiguity in k, we write $\overline{\Lambda} = \Phi(\Lambda) = \Phi_k(\Lambda)$ without further explanation.

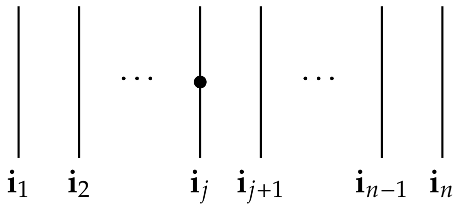
4.1.5. Subdivision on KLR algebras. Following [Mak18], one may define the subdivision map on KLR algebras. In this section, we give a diagrammatic definition of the map (see [Mak18, Remark 2.13] and [MT23, Section 4]), which we think is more intuitive and clearer. For the algebraic description, see [Mak18, Section 2].

Fix a quiver Γ of type $A_{e-1}^{(1)}$ with vertex set I, and let $\alpha \in Q^+$ have height n. Recall that the KLR algebra R_α is generated by elements subject to certain relations; see Definition 1.3.1. Concretely, it is generated by $e(\mathbf{i})$, $\psi_j e(\mathbf{i})$, and $y_i e(\mathbf{i})$, where $1 \le i \le n$, $1 \le j \le n-1$, and $\mathbf{i} \in I^\alpha$. These generators admit a diagrammatic interpretation as in [**KL09**]:

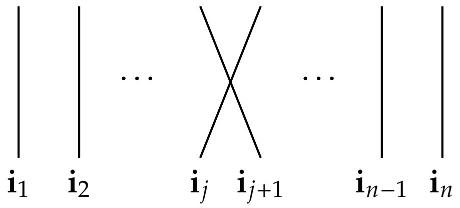
$e(\mathbf{i})$:



$y_j e(\mathbf{i})$:

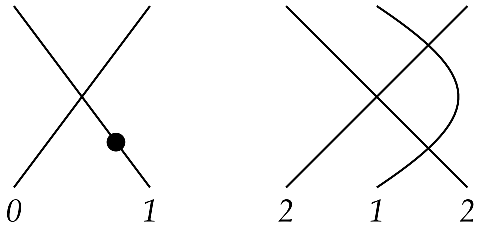


$\psi_j e(\mathbf{i})$:



In this string-diagrammatic presentation, a string labeled by $i \in I$ is called an i-string. The multiplication is given by vertical concatenation of diagrams.

Example 4.1.13. *The following string diagram represents the element* $\psi_1\psi_4\psi_3\psi_4 y_2 e(01212)$*:*



◇

Let $\overline{\Gamma} = \Phi_k(\Gamma)$ and $\overline{\alpha} = \Phi_k(\alpha)$. We construct a subdivision map $\Phi = \Phi_k$ from the KLR algebra $R_\alpha = R_\alpha(\Gamma)$ over $\Gamma = A^{(1)}_{e-1}$ to the KLR algebra $R_{\overline{\alpha}} = R_{\overline{\alpha}}(\overline{\Gamma})$ over $\overline{\Gamma} = A^{(1)}_e$. This map will be compatible with the subdivision maps on the quiver, positive roots, and dominant weights defined above.

It follows from Theorem 1.3.16 that the elements $\psi_w y_1^{a_1} \cdots y_n^{a_n} e(\mathbf{i})$, where $a_i \geq 0$, $w \in \mathfrak{S}_n$, and $\mathbf{i} \in I^\alpha$, form a basis of R_α. We first define a map Θ_k on these basis elements:

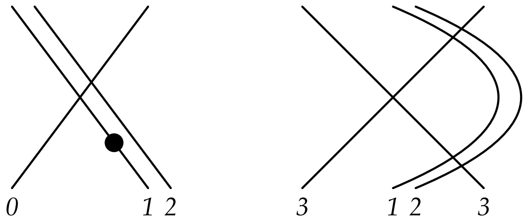
(a). The element $\Theta_k\big(e(\mathbf{i})\big)$ is obtained by adding a new string labeled $\overline{k}$ immediately to the right of each k-string.

(b). The newly added $\overline{k}$-string carries no dots even if the k-string carries dots.

(c). For a string diagram of the form $\psi_w\, e(\mathbf{i})$, first add the new strings as in (1), and then extend each new string so that it follows the shape of the corresponding k-string.

(d). Relabel the strings as follows: replace i by $i+1$ for $k+1 \leq i \leq e-1$, replace $\overline{k}$ by $k+1$, and leave all other labels unchanged.

Example 4.1.14. *The image of the string diagram from Example 4.1.13 under subdivision Φ_1 is the following string diagram:*



◇

We caution the reader that the map Θ is defined only on basis elements and does not extend to a $\Bbbk$-algebra homomorphism. Indeed, Θ does not preserve the defining relations of KLR algebras. As described below, after quotienting by a suitable ideal and applying an idempotent truncation, one obtains a well-defined homomorphism, which is in fact an isomorphism.

To describe this isomorphism, we introduce some additional notation. The notation below is taken from [**Mak18**, Section 2]. The reader is encouraged to consult that source for the more general definitions.

Recall that we subdivide at the edge $k \to k+1$. Let $d = \mathrm{ht}(\overline{\alpha})$. A sequence $\mathbf{i} = (\mathbf{i}_1, \ldots, \mathbf{i}_d) \in \overline{I}^{\overline{\alpha}}$ is called:

- *unordered* if there exists an index $r \in \{1, 2, \ldots, d\}$ such that the number of occurrences of $k+1$ in $(\mathbf{i}_1, \ldots, \mathbf{i}_r)$ is strictly greater than the number of occurrences of k in $(\mathbf{i}_1, \ldots, \mathbf{i}_r)$;
- *ordered*[1] if, for every index a with $\mathbf{i}_a = k$, we have $a < d$ and $\mathbf{i}_{a+1} = k+1$;
- *almost-ordered* if there exist an ordered sequence $\mathbf{j} \in \overline{I}^{\overline{\alpha}}$ and an index $r \in \{1, 2, \ldots, d-1\}$ such that $\mathbf{j}_r = k$ and $\mathbf{i} = \sigma_r(\mathbf{j})$.

We write $\overline{I}^{\overline{\alpha}}_{\mathrm{ord}}$, $\overline{I}^{\overline{\alpha}}_{\mathrm{un}}$, and $\overline{I}^{\overline{\alpha}}_{\mathrm{al}}$ for the subsets of ordered, unordered, and almost-ordered sequences in $\overline{I}^{\overline{\alpha}}$. Every almost-ordered sequence is unordered, so $\overline{I}^{\overline{\alpha}}_{\mathrm{al}} \subseteq \overline{I}^{\overline{\alpha}}_{\mathrm{un}}$. By the definition of subdivision on words, for any $\mathbf{i} \in \overline{I}^{\overline{\alpha}}$, the number of indices j with $\mathbf{i}_j = k$ is the same as the number with $\mathbf{i}_j = k+1$. In particular, the definition of an ordered sequence can be restated as follows: for each admissible a, we have $\mathbf{i}_a = k$ if and only if $a < d$ and $\mathbf{i}_{a+1} = k+1$. In other words, all occurrences of k and $k+1$ appear in adjacent pairs. In view of (4.1.7), Φ_k gives a bijection between I^{α} and $\overline{I}^{\overline{\alpha}}_{\mathrm{ord}}$.

---

[1] In [**Mak18**], the terminology *well-ordered* is used; we prefer *ordered* here, since *well-ordered* already has a standard meaning.

We define the *truncation idempotent* by

$$\mathfrak{e} = \sum_{\mathbf{i} \in I^{\overline{\alpha}}_{\mathrm{ord}}} e(\mathbf{i}) \in R_{\overline{\alpha}}(\overline{\Gamma}). \tag{4.1.15}$$

Definition 4.1.16 ([**Mak18**, Definition 2.6]). *The **balanced KLR algebra** is the algebra*

$$S_{\overline{\alpha}}(\overline{\Gamma}) = \mathfrak{e}R_{\overline{\alpha}}(\overline{\Gamma})\mathfrak{e} \Big/ \sum_{\mathbf{j} \in I^{\overline{\alpha}}_{un}} \mathfrak{e}R_{\overline{\alpha}}(\overline{\Gamma})e(\mathbf{j})R_{\overline{\alpha}}(\overline{\Gamma})\mathfrak{e}.$$

For simplicity, we omit the quiver $\overline{\Gamma}$ from the notation, and write $S_{\overline{\alpha}}$ for $S_{\overline{\alpha}}(\overline{\Gamma})$. Let

$$\mathfrak{J} = \sum_{\mathbf{j} \in I^{\overline{\alpha}}_{\mathrm{un}}} R_{\overline{\alpha}}e(\mathbf{j})R_{\overline{\alpha}}$$

be the two-sided ideal of $R_{\overline{\alpha}}$, called the *bad ideal*. Then

$$S_{\overline{\alpha}} = \mathfrak{e}R_{\overline{\alpha}}\mathfrak{e}/\mathfrak{e}\mathfrak{J}\mathfrak{e} \cong \mathfrak{e}(R_{\overline{\alpha}}/\mathfrak{J})\mathfrak{e}. \tag{4.1.17}$$

It is possible to reduce the number of idempotents appearing in the definition of $\mathfrak{e}\mathfrak{J}\mathfrak{e}$.

Lemma 4.1.18 ([**Mak18**, Lemma 3.7]). $\mathfrak{e}\mathfrak{J}\mathfrak{e} = \sum_{\mathbf{j} \in I^{\overline{\alpha}}_{al}} \mathfrak{e}R_{\overline{\alpha}}e(\mathbf{j})R_{\overline{\alpha}}\mathfrak{e}$.

Theorem 4.1.19 ([**Mak18**, Theorem 2.12]). *Fix a quiver Γ of type $A^{(1)}_{e-1}$, and let $\alpha \in Q^+$ and $k \in \{0, 1, \ldots, e-1\}$. Then there is a graded $\Bbbk$-algebra isomorphism*

$$\Phi_k : R_\alpha \longrightarrow S_{\overline{\alpha}},$$

induced by the map Θ on the basis of R_α, as described above.

Proof. By [**Mak18**, Theorem 2.12], Φ_k is a $\Bbbk$-algebra isomorphism. It remains to show that Φ_k preserves the grading. This follows from a straightforward diagrammatic check that Θ_i is homogeneous on the generators. □

4.1.6. Subdivision on cyclotomic KLR algebras. By the results of Subsection 4.1.5, we have the following (commutative) diagram:

$$\begin{tikzcd}
 & \mathfrak{e}R_{\overline{\alpha}}\mathfrak{e} \arrow[d, "\pi"] \\
R_\alpha \arrow[ur, dashed, "\Theta"] \arrow[r, "\Phi", "\cong"'] & S_{\overline{\alpha}}
\end{tikzcd}$$

Since Θ is defined only on basis elements of R_α, we draw it as a dashed arrow to emphasize this point. Let π be the canonical quotient map from $\mathfrak{e}R_{\overline{\alpha}}\mathfrak{e}$ to $S_{\overline{\alpha}} = \mathfrak{e}R_{\overline{\alpha}}\mathfrak{e}\big/\mathfrak{e}\mathfrak{J}\mathfrak{e}$, and let $\Phi = \Phi_k$ be the subdivision isomorphism from Theorem 4.1.19. As usual, we write $\overline{\alpha} = \Phi(\alpha)$, $\overline{\Lambda} = \Phi(\Lambda)$, and $\overline{\mathbf{i}} = \Phi(\mathbf{i})$ for $\alpha \in Q^+$, $\Lambda \in P^+$, and $\mathbf{i} \in I^\alpha$.

The cyclotomic ideal J^Λ_α of R_α is the two-sided ideal generated by the elements

$$\left\{ y_1^{(\Lambda|\alpha_{\mathbf{i}_1})} e(\mathbf{i}) \;\middle|\; \mathbf{i} \in I^\alpha \right\}.$$

Equivalently, set

$$e^{\Lambda}(\alpha) := \sum_{\mathbf{i}\in I^{\alpha}} y_1^{(\Lambda|\alpha_{\mathrm{i}_1})} e(\mathbf{i}),$$

so that $J_{\alpha}^{\Lambda} = R_{\alpha}\, e^{\Lambda}(\alpha)\, R_{\alpha}$.

Similarly, the cyclotomic ideal $J_{\overline{\alpha}}^{\overline{\Lambda}}$ of $R_{\overline{\alpha}}$ is the two-sided ideal generated by the elements

$$\left\{ y_1^{(\overline{\Lambda}|\alpha_{\mathrm{j}_1})} e(\mathbf{j}) \;\middle|\; \mathbf{j} \in I^{\overline{\alpha}} \right\}.$$

Equivalently, set

$$e^{\overline{\Lambda}}(\overline{\alpha}) := \sum_{\mathbf{j}\in \overline{I}^{\overline{\alpha}}} y_1^{(\overline{\Lambda}|\alpha_{\mathrm{j}_1})} e(\mathbf{j}),$$

so that $J_{\overline{\alpha}}^{\overline{\Lambda}} = R_{\overline{\alpha}}\, e^{\overline{\Lambda}}(\overline{\alpha})\, R_{\overline{\alpha}}$.

The image of J_{α}^{Λ} under Φ is the two-sided ideal of $S_{\overline{\alpha}}$ generated by $\Phi\big(e^{\Lambda}(\alpha)\big)$. In $S_{\overline{\alpha}}$ we have

$$\Phi\big(e^{\Lambda}(\alpha)\big) = \sum_{\mathbf{i}\in I^{\alpha}} \Phi(y_1)^{(\Lambda|\alpha_{\mathrm{i}_1})}\, \Phi\big(e(\mathbf{i})\big) = \sum_{\mathbf{i}\in I^{\alpha}} y_1^{(\Lambda|\alpha_{\mathrm{i}_1})} e(\Phi(\mathbf{i})) + \mathfrak{e}\mathfrak{J}\mathfrak{e} = \sum_{\mathbf{i}\in I^{\alpha}} y_1^{(\overline{\Lambda}|\alpha_{\overline{\mathrm{i}}_1})} e(\overline{\mathbf{i}}) + \mathfrak{e}\mathfrak{J}\mathfrak{e},$$

where the second equality follows from the definition of subdivision on KLR algebras and on positive roots, and the last equality follows from Lemma 4.1.11. On the other hand,

$$\mathfrak{e}e^{\overline{\Lambda}}(\overline{\alpha})\mathfrak{e} = \mathfrak{e}\sum_{\mathbf{j}\in \overline{I}^{\overline{\alpha}}} y_1^{(\overline{\Lambda}|\alpha_{\mathrm{j}_1})} e(\mathbf{j})\mathfrak{e} = \sum_{\mathbf{j}\in \overline{I}^{\overline{\alpha}}_{\mathrm{ord}}} y_1^{(\overline{\Lambda}|\alpha_{\mathrm{j}_1})} e(\mathbf{j}).$$

Since the subdivision map induces a bijection between I^{α} and $\overline{I}^{\overline{\alpha}}_{\mathrm{ord}}$, it follows that

$$\Phi\big(e^{\Lambda}(\alpha)\big) = \mathfrak{e}e^{\overline{\Lambda}}(\overline{\alpha})\mathfrak{e} + \mathfrak{e}\mathfrak{J}\mathfrak{e}.$$

Since $\Phi : R_{\alpha} \xrightarrow{\sim} S_{\overline{\alpha}}$ is an algebra isomorphism, it follows that

$$\Phi(J_{\alpha}^{\Lambda}) = S_{\overline{\alpha}}\,\Phi\big(e^{\Lambda}(\alpha)\big)\,S_{\overline{\alpha}} = S_{\overline{\alpha}}\big(\mathfrak{e}e^{\overline{\Lambda}}(\overline{\alpha})\mathfrak{e} + \mathfrak{e}\mathfrak{J}\mathfrak{e}\big)S_{\overline{\alpha}}.$$

Viewing ideals in $S_{\overline{\alpha}} = \mathfrak{e}R_{\overline{\alpha}}\mathfrak{e}/\mathfrak{e}\mathfrak{J}\mathfrak{e}$, this is the same as

$$\Phi(J_{\alpha}^{\Lambda}) = \big(\mathfrak{e}J_{\overline{\alpha}}^{\overline{\Lambda}}\mathfrak{e} + \mathfrak{e}\mathfrak{J}\mathfrak{e}\big)\big/\mathfrak{e}\mathfrak{J}\mathfrak{e} \subseteq \mathfrak{e}R_{\overline{\alpha}}\mathfrak{e}\big/\mathfrak{e}\mathfrak{J}\mathfrak{e}.$$

Therefore Φ induces an isomorphism on cyclotomic quotients:

$$R_{\alpha}^{\Lambda} = R_{\alpha}/J_{\alpha}^{\Lambda} \xrightarrow{\sim} S_{\overline{\alpha}}/\Phi(J_{\alpha}^{\Lambda}) \xrightarrow{\sim} \mathfrak{e}R_{\overline{\alpha}}\mathfrak{e}\big/\mathfrak{e}\big(\mathfrak{J} + J_{\overline{\alpha}}^{\overline{\Lambda}}\big)\mathfrak{e}.$$

Now the subdivision isomorphism induces the following isomorphism between the cyclotomic KLR algebra and a (cyclotomic) quotient of balanced KLR algebra:

$$\overline{\Phi} : R_{\alpha}^{\Lambda} \xrightarrow{\sim} \mathfrak{e}R_{\overline{\alpha}}^{\overline{\Lambda}}\mathfrak{e}/\mathfrak{e}\big(\mathfrak{J} + J_{\overline{\alpha}}^{\overline{\Lambda}}\big)\mathfrak{e}. \tag{4.1.20}$$

This isomorphism will be called the **cyclotomic subdivision isomorphism**.

4.1.7. Defect invariance. It is well known that the blocks of cyclotomic KLR algebras of type $A^{(1)}_{e-1}$ are the algebras R^{Λ}_{β} for $\beta \in Q^+$ and $\Lambda \in P^+$. In particular, they are indecomposable.

Definition 4.1.21. *Fix $\beta \in Q^+$ and $\Lambda \in P^+$. The defect of β is the integer:*

$$\operatorname{def}_{\Lambda} \beta = (\Lambda \mid \beta) - \frac{1}{2}(\beta \mid \beta) \tag{4.1.22}$$

The defect of a nonzero block (i.e. $R^{\Lambda}_{\beta} \neq 0$) is non-negative (see [**Kac83**, Lemma 11.13.2]) and measures the complexity of the corresponding block. We now show that the subdivision map preserves the defect of the block.

Proposition 4.1.23. *Fix the quiver of type $A^{(1)}_{e-1}$ with vertex set I and fix $k \in I$. Take $\alpha \in Q^+$ and $\Lambda \in P^+$. Set $\overline{\alpha} := \Phi_k(\alpha)$ and $\overline{\Lambda} := \Phi_k(\Lambda)$, where Φ_k is the corresponding subdivision map. Then $\operatorname{def}_{\Lambda} \beta = \operatorname{def}_{\bar{\Lambda}} \bar{\beta}$.*

Proof. Assume $\beta = \sum_{0 \le i \le e-1} x_i \alpha_i$. Then, recall from Subsection 4.1.2 that $\bar{\beta}$ is of the form:

$$\bar{\beta} = \sum_{0 \le i \le k} x_i \alpha_i + x_k \alpha_{k+1} + \sum_{k+1 \le i \le e-1} x_i \alpha_{i+1}.$$

Calculating the inner product $(\beta \mid \beta)$ gives:

$$\begin{aligned}(\beta \mid \beta) = (2x_0^2 - x_0 x_1 - x_0 x_{e-1}) \quad &+ \sum_{1 \le i \le k-1} (2x_i^2 - x_i x_{i-1} - x_i x_{i+1}) \\ + (2x_k^2 - x_k x_{k+1} - x_k x_{k-1}) \quad &+ \sum_{k+1 \le i \le e-2} (2x_i^2 - x_i x_{i-1} - x_i x_{i+1}) \\ + (2x_{e-1}^2 - x_{e-1} x_{e-2} - x_{e-1} x_0).&\end{aligned}$$

whereas, calculating the inner product $(\bar{\beta} \mid \bar{\beta})$ gives:

$$\begin{aligned}(\bar{\beta} \mid \bar{\beta}) &= (2x_0^2 - x_0 x_1 - x_0 x_{e-1}) \quad + \sum_{1 \le i \le k-1} (2x_i^2 - x_i x_{i-1} - x_i x_{i+1}) \\ &+ (2x_k^2 - x_k^2 - x_k x_{k-1}) + (2x_k^2 - x_k^2 - x_k x_{k+1}) \quad + \sum_{k+2 \le i \le e-1} (2x_{i-1}^2 - x_{i-2} x_{i-1} - x_i x_{i-1}) \\ &+ (2x_{e-1}^2 - x_{e-1} x_{e-2} - x_{e-1} x_0).\end{aligned}$$

Simplifying the middle terms, we observe $(\beta \mid \beta) = (\bar{\beta} \mid \bar{\beta})$. It remains to show $(\Lambda \mid \beta) = (\bar{\Lambda} \mid \bar{\beta})$, which follows by Lemma 4.1.12. □

4.2. Combinatorial Subdivision

In this section, we first define the subdivision map on partitions, which we use in Subsection 4.3.1 to describe the image of idempotents in a KLR algebra under the subdivision map Φ_k. We give two equivalent definitions, using Young diagrams (Subsection 4.2.1) and abaci (Subsection 4.2.2), since each is convenient for different arguments.

We then extend the construction to row-standard tableaux (Subsection 4.2.4). Its restriction to standard tableaux sends standard tableaux to standard tableaux, and we show that it preserves degree (Theorem 4.2.53). This construction is used for describing the bases of permutation modules and Specht modules under the isomorphism induced by the subdivision on KLR algebras; see Subsection 4.3.2 and Subsection 4.3.3.

A preliminary partition-level sketch in a restricted specialization appeared in [**Qin24**] (namely $k = 0$ in the subdivision datum; see Definition 4.2.1). The present section is self-contained and substantially extends this to a complete picture, and the tableaux-level construction is brand new.

DEFINITION 4.2.1. *A* ***subdivision datum*** *is a tuple* $(e, I, \Lambda, \alpha, \boldsymbol{\kappa}, k)$, *where* $e \in \mathbb{Z}_{\geq 3}$, I *is the vertex set of the quiver* $A^{(1)}_{e-1}$ *(identified with* $\{0, 1, \ldots, e-1\}$*),* $\Lambda \in P^+$ *is a dominant weight of level* $\ell \in \mathbb{Z}_{\geq 1}$, $\alpha \in Q^+$ *is a positive root,* $\boldsymbol{\kappa} \in I^\ell$ *is a charge of* Λ, *and* $k \in I$.

Fix a subdivision datum $(e, I, \Lambda, \alpha, \boldsymbol{\kappa}, k)$, let $\Phi = \Phi_k$ be the subdivision map (on quivers, positive roots, dominant weights) introduced in the earlier sections, and set $\overline{\Gamma} = \Phi(\Gamma)$, $\overline{\alpha} = \Phi(\alpha)$, and $\overline{\Lambda} = \Phi(\Lambda)$. Let $\overline{\boldsymbol{\kappa}} \in \overline{I}^\ell$ be such that $\Phi_k(\Lambda_{\kappa_i}) = \Lambda_{\overline{\kappa}_i}$ for each $1 \leq i \leq \ell$.[2]

Recall that $\mathscr{P}^{\boldsymbol{\kappa}}_{\alpha}$ is the set of ℓ-partitions of residue content α with respect to the charge $\boldsymbol{\kappa}$. Similarly, $\mathscr{P}^{\overline{\boldsymbol{\kappa}}}_{\overline{\alpha}}$ is the set of ℓ-partitions of residue content $\overline{\alpha}$ with respect to the charge $\overline{\boldsymbol{\kappa}}$. Our aim is to construct a subdivision map $\Phi_k : \mathscr{P}^{\boldsymbol{\kappa}}_{\alpha} \to \mathscr{P}^{\overline{\boldsymbol{\kappa}}}_{\overline{\alpha}}$ that is compatible with other subdivision maps defined in the earlier sections.

To define $\Phi_k(\boldsymbol{\lambda})$, we first define $\Phi_k(\lambda)$ for an arbitrary partition λ, and then apply the construction componentwise. More precisely, if $\boldsymbol{\lambda} = (\boldsymbol{\lambda}^{(1)}, \ldots, \boldsymbol{\lambda}^{(\ell)}) \in \mathscr{P}^{\boldsymbol{\kappa}}_{\alpha}$, then

$$\Phi_k(\boldsymbol{\lambda}) := \left(\Phi_k(\boldsymbol{\lambda}^{(1)}), \ldots, \Phi_k(\boldsymbol{\lambda}^{(\ell)})\right) \in \mathscr{P}^{\overline{\boldsymbol{\kappa}}}_{\overline{\alpha}}$$

For this reason, it suffices to consider the case $\Lambda = \Lambda_i$ for some $i \in I$.

4.2.1. Subdivision on partitions via Young diagrams.

Fix a subdivision datum $(e, I, \Lambda_i, \alpha, i, k)$[3] and let $\Lambda := \Lambda_i$. Recall from Subsection 1.2.2 that $[\lambda]$ is the Young diagram of a partition λ, and that $[\lambda]_\Lambda$ is the Young diagram filled with residues from I. As usual, since the choice of $\Lambda = \Lambda_i$ is clear from the context, we abuse notation and write $[\lambda]$ for the Young diagram filled with residues.

DEFINITION 4.2.2. *Fix a subdivision datum* $(e, I, \Lambda = \Lambda_i, \alpha, i, k)$ *and* $\lambda \in \mathscr{P}^{\Lambda}_{\alpha}$. *A* $\boldsymbol{(k, k+1)}$***-strip of length*** $\boldsymbol{m}$ *in* $[\lambda]$ *is a finite sequence of nodes in* $[\lambda]$, $(A_1, A_2, \ldots, A_m)$, *such that:*

(a). $\operatorname{res}(A_{2j+1}) = k$ *and* $\operatorname{res}(A_{2j}) = k + 1$ *for all admissible* j;
(b). *if* A_{2j+1} *lies in row* r *and column* c, *then* A_{2j+2} *lies in row* r *and column* $c + 1$;
(c). *if* A_{2j} *lies in row* r *and column* c, *then* A_{2j+1} *lies in row* $r + 1$ *and column* c.

---

[2] The reader should be cautious that this $\overline{\kappa}$ does not coincide with the subdivision of the word $\kappa \in I^\ell$.

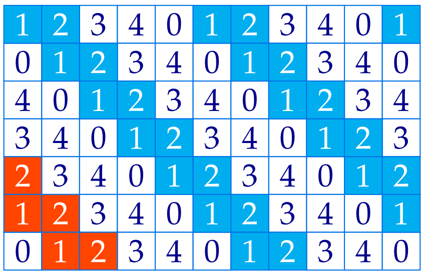
[3] Strictly speaking, it should be written as $(e, I, \Lambda_i, \alpha, (i), k)$, but we abuse notation and identify (i) with i.

Similarly, a ***$(k+1,k)$-strip of length m*** *in $[\lambda]$ is a finite sequence of nodes in $[\lambda]$, $(A_1, A_2, \dots, A_m)$, such that:*

(a). $\operatorname{res}(A_{2j+1}) = k+1$ *and* $\operatorname{res}(A_{2j}) = k$ *for all admissible j;*
(b). *if A_{2j+1} lies in row r and column c, then A_{2j+2} lies in row $r+1$ and column c;*
(c). *if A_{2j} lies in row r and column c, then A_{2j+1} lies in row r and column $c+1$.*

A $(k,k+1)$-strip or $(k+1,k)$-strip is called ***maximal*** *if it cannot be extended to a strip of greater length. If the length $m=1$, we call the strip* ***trivial****. We call A_1 the* ***initial node*** *and A_m the* ***terminal node****.*

Example 4.2.3. *Take the subdivision datum to be $(e, I, \Lambda, \alpha, \boldsymbol{\kappa}, k) = \big(5, I, \Lambda_1, 16(\alpha_0 + \alpha_1) + 15(\alpha_2+\alpha_3+\alpha_4), 1, 1\big)$ and $\lambda = (11^7)$, we form the Young diagram $[\lambda]$ and color the maximal $(1,2)$-strips in cyan, color the maximal $(2,1)$-strip in orange:*



1	2	3	4	0	1	2	3	4	0	1
0	1	2	3	4	0	1	2	3	4	0
4	0	1	2	3	4	0	1	2	3	4
3	4	0	1	2	3	4	0	1	2	3
2	3	4	0	1	2	3	4	0	1	2
1	2	3	4	0	1	2	3	4	0	1
0	1	2	3	4	0	1	2	3	4	0

◇

Since we work in type $A_{e-1}^{(1)}$, the following observation is immediate.

Lemma 4.2.4. *Every maximal $(k,k+1)$-strip in $[\lambda]$ has its initial node in the first row of $[\lambda]$, and every maximal $(k+1,k)$-strip in $[\lambda]$ has its initial node in the first column of $[\lambda]$.* ☐

The definition in Definition 4.2.2 extends in a natural way to triples $(k, k+1, k+2)$ and $(k+2, k+1, k)$. Indeed, we will define the image of a partition λ under subdivision by replacing its maximal $(k,k+1)$-strips and maximal $(k+1,k)$-strips with $(k,k+1,k+2)$-strips and $(k+2,k+1,k)$-strips, respectively. We make this precise as follows:

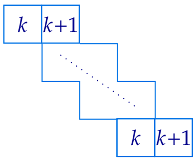
Definition 4.2.5. *Fix a subdivision datum $(e, I, \Lambda_i, \alpha, i, k)$ and take a partition $\lambda \in \mathscr{P}_\alpha^{\Lambda_i}$. For each maximal $(k,k+1)$-strip S in $[\lambda]$, insert a node labelled with $\overline{k}$ to the right of every k-node in S. Similarly, for each maximal $(k+1,k)$-strip S in $[\lambda]$, insert a node labelled with $\overline{k}$ above every k-node in S. After performing these two procedures for all maximal strips, apply the relabelling from (4.1.1) to each node. The resulting Young diagram is denoted by $\Phi_k^Y([\lambda])$.*

There are two points that are not immediate from the definition: (1) the resulting diagram is the Young diagram of a partition; (2) the new label on each node is exactly its residue with respect to the new charge (or dominant weight) for $A_e^{(1)}$. We now prove both statements.

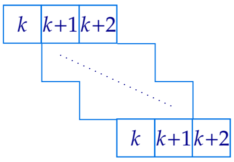
By Definition 4.2.5, all four kinds of maximal strips (except for the trivial $(k+1,k)$-strip, that is, a single $(k+1)$-node, which simply becomes a $(k+2)$-node) are treated as follows:

For any maximal $(k, k+1)$-strip,

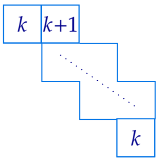
(i) if it is of the form



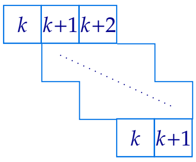
then replace it by



(ii) if it is of the form

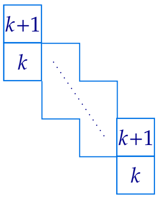


then replace it by

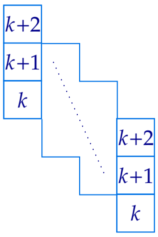


For any non-trivial maximal $(k+1, k)$-strip:

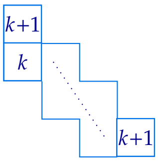
(a). if it is of the form



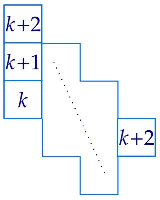
then replace it by



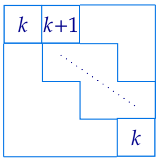
(b). if it is of the form



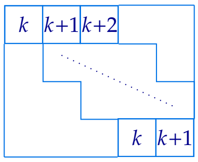
then replace it by



To show that $\Phi_k^Y[\lambda]$ is the Young diagram of a partition, it suffices to check that each of the four steps still produces the Young diagram of a partition. Consider a maximal $(k, k+1)$-strip of the form:



As λ is a partition, this strip must lie within a minimal rectangle (as shown in the figure above) in $[\lambda]$. Hence, by replacing the strip, we obtain the following:



This transformation still results in a partition by modifying such a rectangle. The other three operations can be verified in a similar manner. Hence, after replacing every maximal $(k, k+1)$-strip and $(k+1, k)$-strip, we obtain a new partition.

Let $\Phi_k^Y(\lambda)$ be the partition such that $[\Phi_k^Y(\lambda)] = \Phi_k^Y([\lambda])$. In this way, Φ_k^Y is realized as a map on the set of partitions (rather than merely on Young diagrams) via Young diagrams.

We next consider residues. Assume $\lambda \in \mathscr{P}_\alpha^{\Lambda_i}$. For any node $A = (r, c) \in [\lambda]$, the residue of A is $i + c - r \pmod{e}$. We call the node $A = (1, 1)$ the *first node* of $[\lambda]$. In particular, the residue of the first node is i, corresponding to the dominant weight Λ_i. If $0 \leq i \leq k$, then by our construction in Definition 4.2.5, the first node in $\Phi_k^Y([\lambda])$ has residue i; if $k+1 \leq i \leq e-1$, the first node in $\Phi_k^Y([\lambda])$ has residue $i+1$. This agrees with the definition of the subdivision map on dominant weights in (4.1.10). Hence the new dominant weight is $\Phi_k(\Lambda_i)$, and the new partition $\Phi_k^Y(\lambda)$ lies in $\mathscr{P}^{\Phi_k(\Lambda_i)}$.

Lemma 4.2.6. *Fix quiver type* $A_{e-1}^{(1)}$*, let* $\lambda \in \mathscr{P}^{\Lambda_i}$ *and let* $[\lambda]$ *be its Young diagram. A function* $f : [\lambda] \longrightarrow \mathbb{Z}/e\mathbb{Z}$ *coincides with the residue function on* $[\lambda]$ *with respect to* Λ_i *if and only if it satisfies the following two conditions:*

- *(i)* $f(1, 1) = i$;
- *(ii) for any two horizontally adjacent nodes* $(r, c), (r, c+1) \in [\lambda]$ *one has* $f(r, c+1) = f(r, c) + 1$*, and for any two vertically adjacent nodes* $(r, c), (r+1, c) \in [\lambda]$ *one has* $f(r+1, c) = f(r, c) - 1$*, with both equalities taken in* $\mathbb{Z}/e\mathbb{Z}$*.*

Proof. If f is the residue function, then the two properties follow immediately from definition. Conversely, assume f satisfies *(i)* and *(ii)*. We prove by induction on $r+c$ that $f(r,c) \equiv i+c-r \pmod{e}$ for all $(r,c) \in [\lambda]$. The base case $(1,1)$ is *(i)*. For $(r,c) \neq (1,1)$, either $(r,c-1) \in [\lambda]$ (if $c>1$) or $(r-1,c) \in [\lambda]$ (if $r>1$), and then *(ii)* gives

$$f(r,c) \equiv f(r,c-1)+1 \pmod{e} \quad \text{or} \quad f(r,c) \equiv f(r-1,c)-1 \pmod{e}$$

Applying the induction hypothesis to $(r,c-1)$ or $(r-1,c)$ yields $f(r,c) \equiv i+c-r \pmod{e}$, as required. Hence f agrees with the residue function. □

To show that the label associated to each node in $[\Phi_k^Y(\lambda)]$ coincides with the residue of that node with respect to $\Lambda_j := \Phi_k(\Lambda_i)$, note that we already know the first node of $[\Phi_k^Y(\lambda)]$ has residue j. Thus it suffices to show that *(ii)* in Lemma 4.2.6 holds for type $A_e^{(1)}$.

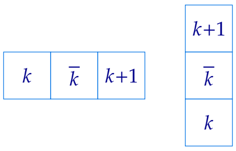
For each of the four steps above applied to a maximal strip, the diagram is modified only inside the minimal rectangle containing that strip. Before the relabeling procedure, the labels on all original nodes are unchanged (hence still equal to their residues in $[\lambda]$), and the only new labels that appear are the labels $\overline{k}$ on the nodes inserted to subdivide the strip: these are inserted either immediately to the right of a k-node (for a maximal $(k,k{+}1)$-strip) or immediately above a k-node (for a maximal $(k{+}1,k)$-strip).

Let L be the labeling on the set of nodes of $[\Phi_k^Y(\lambda)]$ after performing the local modifications and the final relabeling. Consider any pair of horizontally adjacent nodes $(r,c),(r,c{+}1) \in [\Phi_k(\lambda)]$ (respectively, vertically adjacent nodes $(r,c),(r{+}1,c) \in [\Phi_k(\lambda)]$). If both endpoints are original nodes of $[\lambda]$, then their labels were not altered during the local modification inside the supporting rectangles, and the global relabeling acts compatibly on both labels; hence the relations

$$\mathrm{L}(r,c+1) \equiv \mathrm{L}(r,c)+1 \pmod{e+1} \qquad \text{and} \qquad \mathrm{L}(r+1,c) \equiv \mathrm{L}(r,c)-1 \pmod{e+1}$$

continue to hold.

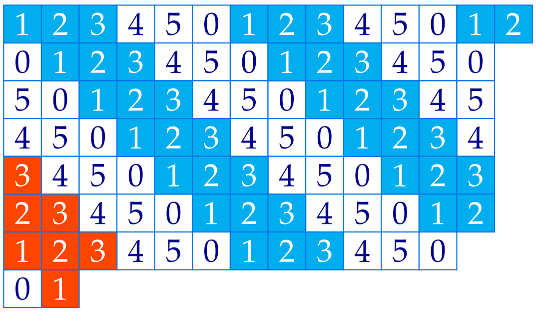
Therefore it remains only to check the horizontal and vertical adjacency relations for pairs of nodes in which at least one node is newly inserted. By construction, every inserted node is labeled $\overline{k}$ before relabeling and lies on a modified maximal strip. Such a node is inserted only in the following local situation: a node labeled k is adjacent (horizontally or vertically, according to the type of the strip) to another node on the strip, and the insertion places $\overline{k}$ immediately to the right of that k-node (in the horizontal case) or immediately above that k-node (in the vertical case). When the adjacent node on the strip exists and is labeled $k+1$, the insertion replaces the original adjacency by a chain of two adjacencies



respectively. Near an endpoint of the strip, the corresponding neighboring $k+1$-node may be absent; in that case $\overline{k}$ has only one adjacent neighbor along the strip, and there is no second adjacency relation to verify there.

After applying the relabeling, k and $k+1$ are sent to k and $k+2$, respectively, while $\overline{k}$ is sent to $k+1$. Consequently, every horizontal (respectively, vertical) adjacency involving an inserted node satisfies the required increasing (respectively, decreasing) relation modulo $e+1$ as well. Hence *(ii)* of Lemma 4.2.6 holds for L in type $A_e^{(1)}$. Since the first node has residue j, it follows from Lemma 4.2.6 that L coincides with the residue function on $[\Phi_k^Y(\lambda)]$ with respect to Λ_j.

Example 4.2.7. *Continue with Example 4.2.3, the image of* $\lambda = (11)^7$ *under the subdivision map* Φ_1^Y *is the one corresponding to the following Young diagram:*



1	2	3	4	5	0	1	2	3	4	5	0	1	2
0	1	2	3	4	5	0	1	2	3	4	5	0	
5	0	1	2	3	4	5	0	1	2	3	4	5	
4	5	0	1	2	3	4	5	0	1	2	3	4	
3	4	5	0	1	2	3	4	5	0	1	2	3	
2	3	4	5	0	1	2	3	4	5	0	1	2	
1	2	3	4	5	0	1	2	3	4	5	0		
0	1												

The $(1,2)$*-strips become the* $(1,2,3)$*-strips and the* $(2,1)$*-strips become the* $(3,2,1)$*-strips. The partition* $\Phi_1^Y(\lambda) = (14, 13^5, 12, 2)$ ◇

4.2.2. Subdivision on partitions via abaci.

The definition of the subdivision map on partitions in Subsection 4.2.1 is natural, but it is difficult to compute in practice for large partitions, since it requires working with the entire Young diagram. In this section, we give a second, more computable description of the map, and in Subsection 4.2.3 we show that the two definitions agree.

We use the abacus combinatorics from Subsection 1.2.1 and Subsection 1.2.2.

Definition 4.2.8. *Fix a subdivision datum* $(e, I, \Lambda_i, \alpha, i, k)$. *For each partition* $\lambda \in \mathscr{P}_\alpha^{\Lambda_i}$, *an* abacus subdivision datum *for* λ *is a tuple* $(a, c, d, a') \in \mathbb{Z}^4$ *such that*

$$a \geq \max\{k, \ell(\lambda)\}, \qquad a \equiv i \ (\mathit{mod}\ e), \qquad a + d = ce + k, \qquad d \in I, \qquad a' = a + c.$$

The subdivision map on the set of partitions is easier to define in terms of abaci combinatorics:

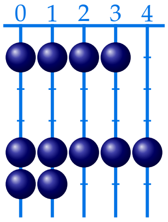
Definition 4.2.9. *Fix a subdivision datum* $(e, I, \Lambda_i, \alpha, i, k)$ *and let* $\lambda \in \mathscr{P}_\alpha^{\Lambda_i}$, *choose an abacus subdivision datum* (a, c, d, a') *for* λ. *Form the* e*-abacus of* λ *with* a *beads. Insert a new runner immediately to the left of the* k*-runner, and place* c *beads on this runner in the top rows. Finally, relabel the runners from left to right by* $0, 1, \ldots, e-1, e$. *The partition corresponding to the resulting* $(e+1)$*-abacus with* a' *beads is defined to be* $\Phi_k^A(\lambda)$.

It is useful to record the positions of the beads added in Definition 4.2.9. Define the set $\mathscr{T}_{e,k,c}$ of non-negative integers by

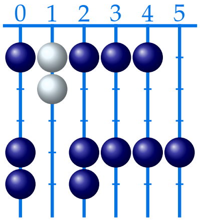
$$\mathscr{T}_{e,k,c} := \{ie + k \mid 0 \leq i \leq c - 1\}. \tag{4.2.10}$$

Let $(e, I, \Lambda_i, \alpha, i, k)$ be a subdivision datum and let (a, c, d, a') be an abacus subdivision datum for $\lambda \in \mathscr{P}_\alpha^{\Lambda_i}$. Then, in the resulting $(e+1)$-abacus for $\Phi_k^A(\lambda)$, the positions of the newly added beads are exactly the set $\mathscr{T}_{e+1,k,c}$.

Example 4.2.11. *Returning to Example 4.2.3, the subdivision datum is* $(e, I, \Lambda, \alpha, \boldsymbol{\kappa}, k) = (5, I, \Lambda_1, \alpha, 1, 1)$ *and* $\lambda = (11^7)$. *Choose an abacus subdivision datum for* λ *to be* $(a, c, d, a') = (11, 2, 0, 13)$. *Form the e-abacus of* λ *with* 11 *beads:*



Apply Φ_k^A *to* λ*: insert a new runner immediately to the left of the* 1*-runner, and place two beads on this runner as high as possible. The resulting* $(e+1)$*-abacus with* 13 *beads is:*



This abacus corresponds to the partition $(14, 13^5, 12, 2)$. *Comparing with Example 4.2.7, we see that the two definitions agree. The two newly added beads are located at* $\{1, 7\}$, *which is precisely* $\mathscr{T}_{6,1,2}$. ◇

Lemma 4.2.12. *Take* $\lambda \in \mathscr{P}_\alpha^\Lambda$ *e-regular, then* $\Phi_k^A(\lambda)$ *is* $(e+1)$*-regular.*

Proof. More generally, inserting a flush runner preserves e-regularity. Recall that an e-abacus display of λ is e-regular if and only if it contains no string of e consecutive beads with no gap between them. Suppose that $\Phi_k^A(\lambda)$ is not $(e+1)$-regular. Then the $(e+1)$-abacus display of $\Phi_k^A(\lambda)$ contains $(e+1)$ consecutive beads with no gap between them. Exactly one of these beads lies on the newly inserted runner, so the remaining e beads come from the original e-abacus display and form e consecutive beads with no gap between them. This shows that λ is not e-regular, a contradiction. □

In view of the discussion in Section 4.4, Lemma 4.2.12 is the same as [**Del24**, Lemma 4.18].

4.2.3. Equivalence of the two definitions. In this section, we prove that for a partition λ, the map $\Phi_k^Y(\lambda)$ defined in Subsection 4.2.1 using Young diagrams coincides with $\Phi_k^A(\lambda)$ defined in Subsection 4.2.2 using abaci. Consequently, in later sections we may simply write $\Phi_k(\lambda)$ and refer to it as the subdivision map on partitions.

Definition 4.2.13. *Fix a subdivision datum* $(e, I, \Lambda = \Lambda_x, \alpha, x, k)$ *and take* $\lambda \in \mathscr{P}_\alpha^\Lambda$. *Let* $k(\lambda)$ *be the number of non-trivial maximal* $(k+1, k)$*-strips in the Young diagram* $[\lambda]$. *By Lemma 4.2.4, such strips correspond to adjacent pairs in the first column* $(r, 1)$ *and* $(r+1, 1)$ *with residues* $k+1$ *and* k *respectively. Thus:*

$$k(\lambda) := \#\{1 \le r < \ell(\lambda) \mid \operatorname{res}(r,1) \equiv k+1 \pmod{e}\}.$$

In Definition 4.2.13, the maximal $(k+1, k)$-strip is required to be non-trivial. Indeed, by Definition 4.2.5, a trivial $(k+1, k)$-strip (i.e. a single $(k+1)$-node) simply becomes a $(k+2)$-node under Φ_k^Y and hence does not change the Young diagram under subdivision.

The case $k(\lambda) = 0$ is of key importance in Subsection 4.2.4 and Section 4.3. For later use, we extend this terminology to multipartitions as follows.

Definition 4.2.14. *Fix a subdivision datum* $(e, I, \Lambda, \alpha, \boldsymbol{\kappa}, k)$, *an* ℓ*-partition* $\boldsymbol{\lambda} \in \mathscr{P}^{\boldsymbol{\kappa}}$ *is* ***k-horizontal*** *if each component* $\boldsymbol{\lambda}^{(m)}$ *satisfies* $k(\boldsymbol{\lambda}^{(m)}) = 0$, *equivalently, if in every component of* $[\boldsymbol{\lambda}]$ *there is no non-trivial maximal* $(k+1, k)$*-strip.*

We now study some properties of $k(\lambda)$. Let $\lfloor \bullet \rfloor$ be the floor function, i.e. for any real number x, $\lfloor x \rfloor$ is the maximal integer N such that $x \ge N$.

Lemma 4.2.15. *Fix a subdivision datum* $(e, I, \Lambda_x, \alpha, x, k)$ *and take* $\lambda \in \mathscr{P}_\alpha^{\Lambda_x}$. *Let* $\rho \in I$ *be the unique integer satisfying* $\rho \equiv x - k \pmod{e}$. *Then*

$$k(\lambda) = \max\Big(0, \Big\lfloor \frac{\ell(\lambda) - 1 - \rho}{e} \Big\rfloor + 1\Big). \tag{4.2.16}$$

Proof. In the first column of $[\lambda]$, $\operatorname{res}(r,1) \equiv x + 1 - r \pmod{e}$. Hence $\operatorname{res}(r,1) \equiv k+1 \pmod{e}$ is equivalent to

$$x + 1 - r \equiv k + 1 \pmod{e} \quad \Longleftrightarrow \quad r \equiv x - k \pmod{e}. \tag{4.2.17}$$

We count integers r with $1 \le r \le \ell(\lambda) - 1$ satisfying (4.2.17). The smallest positive solution is ρ, and every solution is of the form $\rho + te$ for $t \in \mathbb{Z}_{\ge 0}$. Therefore $k(\lambda)$ is the number of non-negative integers t such that $\rho + te \le \ell(\lambda) - 1$.

If $\ell(\lambda) - 1 < \rho$, there are no solutions and $k(\lambda) = 0$. Otherwise, the largest admissible t is $\lfloor (\ell(\lambda) - 1 - \rho)/e \rfloor$, so the number of solutions is $\lfloor (\ell(\lambda) - 1 - \rho)/e \rfloor + 1$. This gives (4.2.16). □

Lemma 4.2.18. *Fix a subdivision datum* $(e, I, \Lambda_x, \alpha, x, k)$ *and take* $\lambda \in \mathscr{P}_\alpha^{\Lambda_x}$. *Then*

$$\ell\big(\Phi_k^Y(\lambda)\big) = \ell(\lambda) + k(\lambda).$$

Proof. The length of a partition is the number of nodes in the first column of its Young diagram. Maximal $(k+1, k)$-strips start from the first column. By Definition 4.2.5, the map Φ_k^Y replaces:

(a). A non-trivial maximal $(k+1,k)$-strip $\begin{array}{|c|}\hline k+1\\\hline k\\\hline\end{array}$ with a vertical triple $\begin{array}{|c|}\hline k+2\\\hline k+1\\\hline k\\\hline\end{array}$. This increases the column height by 1.

(b). A trivial maximal $(k+1,k)$-strip (single $k+1$) with a single $k+2$. This preserves height.

Thus, the total length increases by exactly the number of non-trivial maximal $(k+1,k)$-strips, $k(\lambda)$. □

Lemma 4.2.19. *Fix a subdivision datum* $(e, I, \Lambda_x, \alpha, x, k)$ *and take* $\lambda \in \mathscr{P}_\alpha^{\Lambda_x}$. *Let* $\rho \in I$ *be the unique integer satisfying* $\rho \equiv x - k \pmod{e}$. *Then* $k(\lambda) = 0$ *if and only if* $\ell(\lambda) \leq \rho$.

Proof. By Lemma 4.2.15, the condition $k(\lambda) = 0$ is equivalent to

$$\frac{\ell(\lambda) - 1 - \rho}{e} < 0.$$

Since $e > 0$, this simplifies to $\ell(\lambda) - 1 - \rho < 0$, or $\ell(\lambda) \leq \rho$. □

The following corollary will be useful later.

Corollary 4.2.20. *Fix a subdivision datum* $(e, I, \Lambda_x, \alpha, x, k)$ *and take* $\lambda \in \mathscr{P}_\alpha^{\Lambda_x}$. *Take an abacus subdivision datum* (a, c, d, a') *for* λ. *Then* $k(\lambda) = 0$ *if and only if* $\ell(\lambda) \leq e - d$.

Proof. Let $\rho \in I$ be the unique integer satisfying $\rho \equiv x - k \pmod{e}$. By definition, $a + d = ce + k$ and $a \equiv x \pmod{e}$. Hence $x - k \equiv ce - d \equiv -d \pmod{e}$, and it follows that $\rho = e - d$. The statement then follows from Lemma 4.2.19. □

To compare Φ_k^A and Φ_k^Y explicitly, we analyze the row lengths. We introduce three useful functions, which will be used to describe the subdivision map in terms of beta numbers. We fix a subdivision datum $(e, I, \Lambda_x, \alpha, x, k)$ for convenience.

Definition 4.2.21. *Take* $M \in \mathbb{Z}_{\geq 0}$ *and let* $N_k(M)$ *be the number of integers* y *such that* $0 \leq y \leq M$ *and* $y \equiv k \pmod{e}$. *An explicit formula is*

$$N_k(M) = \left\lfloor \frac{M-k}{e} \right\rfloor + 1. \tag{4.2.22}$$

Definition 4.2.23. *For* $u \in I = \{0, 1, \ldots, e-1\}$, *define the step function on* I *by*

$$\varepsilon_u(r) := \begin{cases} 0, & 0 \leq r < u, \\ 1, & u \leq r \leq e-1, \end{cases} \qquad (r \in I). \tag{4.2.24}$$

Definition 4.2.25. *For* $u \in I = \{0, 1, \ldots, e-1\}$, *define a map* $\iota_u : \mathbb{Z}_{\geq 0} \longrightarrow \mathbb{Z}_{\geq 0}$ *as follows. For* $n \in \mathbb{Z}_{\geq 0}$, *write* $n = qe + r$ *with* $q \in \mathbb{Z}_{\geq 0}$ *and* $r \in I$, *and set*

$$\iota_u(n) := q(e+1) + r + \varepsilon_u(r). \tag{4.2.26}$$

The subdivision map Φ_k^A acts on the corresponding beta set essentially by applying ι_k. More precisely, we have the following.

Lemma 4.2.27. *Fix a subdivision datum* $(e, I, \Lambda_x, \alpha, x, k)$. *Take* $\lambda \in \mathscr{P}_\alpha^{\Lambda_x}$ *and an abacus subdivision datum* (a, c, d, a') *for* λ. *Let* $B(\lambda; a)$ *and* $B(\Phi_k^A(\lambda); a')$ *be the corresponding beta sets (see (1.2.2)). Set* $\mathscr{T} := \mathscr{T}_{e+1,k,c}$ *as in (4.2.10). Then*

$$B(\Phi_k^A(\lambda); a') = \iota_k\big(B(\lambda; a)\big) \sqcup \mathscr{T}.$$

Proof. This is a direct translation of Definition 4.2.9, using the equivalence between the e-abacus with a' beads and the a'-beta numbers of the corresponding partition. □

Lemma 4.2.28. *Write* $M = qe + r$ *with* $0 \le r < e$, *then* $N_k(M) = q + \varepsilon_k(r)$.

Proof. By definition, $N_k(M) = \lfloor \frac{M-k}{e} \rfloor + 1$. Substituting $M = qe + r$:

$$N_k(M) = q + \left\lfloor \frac{r-k}{e} \right\rfloor + 1.$$

If $r \ge k$, the floor term is 0, yielding $q + 1$. If $r < k$, the floor term is -1, yielding q. This matches the definition of $q + \varepsilon_k(r)$. □

Corollary 4.2.29. *Take* $n \in \mathbb{Z}_{\ge 0}$ *and* $k \in \{0, 1, \cdots, e-1\}$, *we have* $\iota_k(n) = n + N_k(n)$. □

For $\lambda \in \mathscr{P}_\alpha^{\Lambda_x}$, define $m_r := m_{r,k}(\lambda)$ to be the number of k-nodes in the r-th row of $[\lambda]$. Take an abacus subdivision datum (a, c, d, a') for λ. Then, since $a \equiv x \pmod e$, m_r equals the number of integers in $\{a - r + 1, a - r + 2, \cdots, a - r + \lambda_r = \beta_r\}$ that are congruent to k modulo e. Thus,

$$m_r = N_k(\beta_r) - N_k(a - r). \tag{4.2.30}$$

We split the proof of the equivalence of the two definitions (Theorem 4.2.44) into two parts, depending on whether $k(\lambda) = 0$.

4.2.3.1. *The case* $k(\lambda) = 0$. Throughout this section, we fix a subdivision datum $(e, I, \Lambda_x, \alpha, x, k)$.

Lemma 4.2.31. *Take* $\lambda \in \mathscr{P}_\alpha^{\Lambda_x}$ *and assume* $k(\lambda) = 0$. *Let* (a, c, d, a') *be an abacus subdivision datum for* λ. *For* $1 \le r \le \ell(\lambda)$, *let* $\beta_r = \lambda_r - r + a = q_r e + t_r$ *with* $0 \le t_r < e$. *Then* $m_r = q_r - c + \varepsilon_k(t_r)$.

Proof. The residues in $[\lambda]_r$ are $\{a - r + 1, \cdots, a - r + \lambda_r = \beta_r\}$ mod e. Hence the number of k-nodes is $m_r = N_k(\beta_r) - N_k(a - r)$. By Lemma 4.2.28, the first term is $N_k(\beta_r) = q_r + \varepsilon_k(t_r)$.

For the second term $N_k(a - r)$, since $a - k = ce - d$, we have:

$$\frac{a - r - k}{e} = \frac{ce - d - r}{e} = c + \frac{-(d + r)}{e}.$$

Since $k(\lambda) = 0$, by Corollary 4.2.20, this implies $\ell(\lambda) \le e - d$. Since $1 \le r \le \ell(\lambda)$, we have $d + 1 \le d + r \le e$. Consequently,

$$-1 \le \frac{-(d+r)}{e} < 0 \implies \left\lfloor \frac{-(d+r)}{e} \right\rfloor = -1.$$

Thus by (4.2.22),

$$N_k(a-r) = (c-1)+1 = c.$$

Substituting these back into the expression for m_r proves the lemma. □

Lemma 4.2.32. *Take $\lambda \in \mathscr{P}_\alpha^{\Lambda_x}$ and an abacus subdivision datum (a,c,d,a') for λ. Let $\{\beta_r\}_{r\geq 1}$ and $\{\beta'_r\}_{r\geq 1}$ be the sets of a-beta (respectively a'-beta) numbers for λ and $\Phi_k^A(\lambda)$. If $k(\lambda)=0$, then*

$$\beta'_r = \iota_k(\beta_r), \qquad 1 \leq r \leq \ell(\lambda).$$

Proof. By Lemma 4.2.27, the set of beta numbers $\{\beta'_j\}_{j\geq 1}$ for $\Phi_k^A(\lambda)$ is the union of the image set $S = \{\iota_k(\beta_r)\}_{r\geq 1}$ and the set $\mathscr{T} := \mathscr{T}_{e+1,k,c}$ defined in (4.2.10), which corresponds to the beads on the newly inserted runner. We only need to prove that the order of the beta numbers is preserved.

There are two trivial cases. If $c=0$, then $\mathscr{T}=\emptyset$, so the result holds. If $c=1$ and $k=0$, then $\mathscr{T}=\{0\}$ and the conclusion holds naturally.

Assume now that we are not in these trivial cases. Let $X=(c-1)e+k-1 \geq 0$ and set $r_0 = a - X$. Since $ce = a+d-k$, we have:

$$\begin{aligned} r_0 &= a - X \\ &= a - (a+d-k-e+k-1) \\ &= a-a-d+e+1 \\ &= e-d+1. \end{aligned}$$

By Corollary 4.2.20, $k(\lambda)=0$ implies $\ell(\lambda) \leq e-d$. Thus $\ell(\lambda) < r_0$. Since $r_0 > \ell(\lambda)$, the corresponding part λ_{r_0} is zero. Therefore, the beta number $\beta_{r_0} = \lambda_{r_0}+a-r_0 = a-r_0 = X$.

Since the sequence of beta numbers is strictly decreasing, for any $1 \leq r \leq \ell(\lambda)$, we have $r < r_0$, which implies:

$$\beta_r > \beta_{r_0} = (c-1)e+k-1.$$

Therefore, $\beta_r \geq (c-1)e+k$. Applying the strictly increasing map ι_k, we get:

$$\iota_k(\beta_r) \geq \iota_k((c-1)e+k) = (c-1)(e+1)+k+1 > \max(\mathscr{T}).$$

This shows that the images of the first $\ell(\lambda)$ beta numbers are strictly larger than any element in $\mathscr{T}$. Consequently, they occupy the first $\ell(\lambda)$ positions in the sorted set $\{\beta'_j\}$, proving $\beta'_r = \iota_k(\beta_r)$ for these indices. □

Corollary 4.2.33. *Take $\lambda \in \mathscr{P}_\alpha^{\Lambda_x}$ and assume $k(\lambda)=0$. Then $\ell(\Phi_k^A(\lambda)) = \ell(\lambda)$.*

Proof. Set up as in the proof of Lemma 4.2.32. Define the values $X=(c-1)e+k-1$ and $Y=(c-1)(e+1)+k$. Let $B=\{\beta_r\}_{1\leq r\leq a}$ be the set of a-beta numbers for λ. We partition B into a "head" $B_{>X} = \{\beta \in B \mid \beta > X\}$ and a "tail" $B_{\leq X} = \{\beta \in B \mid \beta \leq X\}$. Similarly, let $B' = \{\beta'_r\}_{1\leq r\leq a'}$ be the set of a'-beta numbers for $\mu := \Phi_k^A(\lambda)$, partitioned into $B'_{>Y}$ and $B'_{\leq Y}$.

Let $\mathscr{T} = \mathscr{T}_{e+1,k,c}$ as in (4.2.10). By definition, the map ι_k induces a bijection between $\{0,\dots,X\}$ and $\{0,\dots,Y\} \setminus \mathscr{T}$, while $\mathscr{T}$ fills the gaps. Thus, $\iota_k(\{0,\dots,X\}) \cup \mathscr{T} = \{0,\dots,Y\}$.

First, consider the tail. Let $r_0 = e - d + 1$. In the proof of Lemma 4.2.32, we established that $X = a - r_0$. Since $k(\lambda) = 0 \implies \ell(\lambda) < r_0$, we have $\lambda_r = 0$ for all $r \geq r_0$ and thus $\beta_r = a - r \leq X$. Therefore, $B_{\leq X}$ is exactly the set of integers $\{0,\dots,X\}$ and

$$B'_{\leq Y} = \iota_k(B_{\leq X}) \cup \mathscr{T} = \iota_k(\{0,\dots,X\}) \cup \mathscr{T} = \{0,\dots,Y\}.$$

Since $Y = a' - r_0$, this set $\{0,\dots,a'-r_0\}$ corresponds exactly to the beta numbers $a - r$ for indices $r \geq r_0$. Thus, $\mu_r = 0$ for all $r \geq r_0$.

Next, consider the head. For $r < r_0$, we have $\beta_r \in B_{>X}$. By Lemma 4.2.32, $\beta'_r = \iota_k(\beta_r)$ for $1 \leq r \leq \ell(\lambda)$. We remark that the proof shows that this is actually true for all $1 \leq r \leq r_0 - 1$.

Since $1 \leq r \leq e - d$, we have $d + 1 \leq d + r \leq e$, which implies $\lfloor -(d+r)/e \rfloor = -1$. Thus $N_k(a - r) = c - 1 + 1 = c$. By Corollary 4.2.29, and using $a' = a + c$:

$$\iota_k(a - r) = (a - r) + N_k(a - r) = a - r + c = a' - r.$$

Using the fact that ι_k is strictly increasing:

$$\lambda_r > 0 \iff \beta_r > a - r \iff \iota_k(\beta_r) > \iota_k(a - r) \iff \beta'_r > a' - r \iff \mu_r > 0.$$

This equivalence holds for all $r < r_0$. Since $\ell(\lambda) < r_0$, this confirms that exactly the first $\ell(\lambda)$ parts of $\Phi^A_k(\lambda)$ are non-zero. Thus $\ell(\Phi^A_k(\lambda)) = \ell(\lambda)$. ☐

Proposition 4.2.34. *Take $\lambda \in \mathscr{P}^{\Lambda_x}_\alpha$ and assume $k(\lambda) = 0$. Then $\Phi^A_k(\lambda) = \Phi^Y_k(\lambda)$.*

Proof. First, consider $\Phi^Y_k(\lambda)$. Since $k(\lambda) = 0$, by Lemma 4.2.18, $\ell(\Phi^Y_k(\lambda)) = \ell(\lambda)$. Moreover, let m_r be the number of k-nodes in the r-th row of $[\lambda]$, it follows directly from Definition 4.2.5 that $\Phi^Y_k(\lambda)_r = \lambda_r + m_r$ for $1 \leq r \leq \ell(\lambda)$.

By Corollary 4.2.33, we also have $\ell(\Phi^A_k(\lambda)) = \ell(\lambda)$. It remains to show that the parts agree for $1 \leq r \leq \ell(\lambda)$.

Fix an abacus subdivision datum (a, c, d, a') for λ. Let $\mu = \Phi^A_k(\lambda)$. The r-th part is $\mu_r = \beta'_r + r - a'$. By Lemma 4.2.32, $\beta'_r = \iota_k(\beta_r)$ for $1 \leq r \leq \ell(\lambda)$. Since $a' = a + c$, we have

$$\mu_r - \lambda_r = \big(\iota_k(\beta_r) + r - (a + c)\big) - (\beta_r + r - a) = \iota_k(\beta_r) - \beta_r - c.$$

Write $\beta_r = q_r e + t_r$. By Definition 4.2.25, $\iota_k(\beta_r) - \beta_r = q_r + \varepsilon_k(t_r)$. Therefore,

$$\mu_r - \lambda_r = q_r + \varepsilon_k(t_r) - c.$$

By Lemma 4.2.31, this is exactly m_r. Hence $\mu_r = \lambda_r + m_r = \Phi^Y_k(\lambda)_r$ for $1 \leq r \leq \ell(\lambda)$, and so $\Phi^A_k(\lambda) = \Phi^Y_k(\lambda)$. ☐

4.2.3.2. *The case $k(\lambda) > 0$.*

Throughout this section, we fix a subdivision datum $(e, I, \Lambda_x, \alpha, x, k)$ and a partition $\lambda \in \mathscr{P}^{\Lambda_x}_\alpha$. We assume $s := k(\lambda) > 0$. Hence, the Young diagram $[\lambda]$ contains s

non-trivial maximal $(k+1,k)$-strips. By Lemma 4.2.4, these strips start in the first column.

For a non-trivial maximal $(k+1,k)$-strip S, let k_S and r_S be the row indices of the initial node and the terminal node of S, respectively. Non-triviality of S is equivalent to $r_S > k_S$.

Let $S_1, \dots, S_s$ be the non-trivial maximal $(k+1,k)$-strips in $[\lambda]$. Set $k_i := k_{S_i}$ and $r_i := r_{S_i}$, and reorder the strips so that

$$k_1 < k_2 < \cdots < k_s \qquad \text{and} \qquad r_1 \le r_2 \le \cdots \le r_s.$$

Definition 4.2.35. *We define the strip-tracing functions* $f, g : \mathbb{Z}_{\ge 1} \longrightarrow \mathbb{Z}_{\ge 0}$ *by:*

$$g(j) = \#\{i \mid r_i < j\} \tag{4.2.36}$$

$$f(j) = \begin{cases} 0 & \text{if } 1 \le j \le k_1, \\ i & \text{if } k_i < j \le k_{i+1}, \end{cases} \tag{4.2.37}$$

where we set $k_{s+1} = \infty$.

By definition, $g(j)$ is the number of maximal $(k+1,k)$-strips in $[\lambda]$ that end strictly before row j. Recall also that m_j denotes the number of k-nodes in the j-th row of $[\lambda]$ (as defined in subsubsection 4.2.3.1).

By Lemma 4.2.18, the subdivision map Φ_k^Y increases the number of rows by $s = k(\lambda)$. The rows of the new diagram $[\Phi_k^Y(\lambda)]$ can be naturally classified by the residue of the last node in each row:

- The rows that **do not** end with residue k correspond to the original rows of λ.
- The rows that **do** end with residue k correspond to the new rows created by enlarging the strips.

Lemma 4.2.38. *Take* $\lambda \in \mathscr{P}_\alpha^{\Lambda_x}$ *and let* $\lambda^+ = \Phi_k^Y(\lambda)$. *For* $1 \le j \le \ell(\lambda)$, *the following hold:*

(a). *The row in* λ^+ *that does not end with residue* k *is located at index* $j + g(j)$ *and has length*

$$\lambda^+_{j+g(j)} = \lambda_j + m_j + g(j) - f(j). \tag{4.2.39}$$

and we say this row is corresponding to the original j*-th row of* λ.

(b). *The* s *new rows that end with residue* k *are located at indices* $r_i + i$ *for* $1 \le i \le s$.

Proof. We first record the only way that a row of $[\lambda^+]$ can end with residue k. Let A be the last node of a row of $[\lambda]$. Tracking A by the local rules of Φ_k^Y in Definition 4.2.5, one checks:

(i) if $\operatorname{res}(A) \notin \{k, k+1\}$ then its image has residue $\Phi_k\big(\operatorname{res}(A)\big) \ne k$;

(ii) if A is the terminal node of a maximal $(k, k+1)$-strip and $\operatorname{res}(A) = k$, then A expands to a horizontal pair $\boxed{k}\boxed{k+1}$, so the row ends with residue $k+1$;

(iii) if $\mathrm{res}(A) = k+1$ and A is the terminal node of a maximal $(k, k+1)$-strip or a maximal $(k+1, k)$-strip, then A is replaced by a single $(k+2)$-node, so the row ends with residue $k+2$;

(iv) if A is the terminal node of a non-trivial maximal $(k+1, k)$-strip and $\mathrm{res}(A) = k$, then A is replaced by a vertical pair $\begin{array}{|c|}\hline k{+}1\\\hline k\\\hline\end{array}$, which creates a new row whose last node has residue k.

Hence the rows of λ^+ ending with residue k are exactly the new rows created from the terminal nodes of non-trivial maximal $(k+1, k)$-strips.

For (b), consider the ith non-trivial maximal $(k+1, k)$-strip S_i. Its terminal node lies in row r_i of $[\lambda]$, and by (iv) it produces exactly one new row. The order of the original rows is preserved, and the new row is inserted immediately after the image of row r_i. Since exactly $i-1$ new rows are inserted strictly above row r_i, the image of row r_i sits at index $r_i + (i-1)$, so the new row sits at index $r_i + i$.

For (a), fix $1 \le j \le \ell(\lambda)$. The image of the original jth row is shifted downward by the number of new rows inserted strictly above it, which is $g(j)$ by definition (4.2.36), so it is located at index $j + g(j)$.

To compute the length of the image of the jth row. We firstly ignore the maximal $(k+1, k)$-strips for a moment, every k-node in row j would expand to a horizontal pair $(k, k+1)$ in the same row, so the length would increase by 1 for each k-node. This gives the tentative value $\lambda_j + m_j$.

However, this tentative value $\lambda_j + m_j$ is too large, because not every k-node in row j expands horizontally. The problematic ones are those k-nodes in row j that lie on a non-trivial maximal $(k{+}1, k)$-strip that passes through row j. Indeed, for each such non-trivial maximal $(k{+}1, k)$-strip S one has $k_S < j \le r_S$, and $S \cap [\lambda]_j$ contains exactly one k-node (it is the left node of a horizontal $\begin{array}{|c|c|}\hline k & k{+}1\\\hline\end{array}$ if $k_S < j < r_S$, and it is the single terminal k-node if $j = r_S$).

In the construction of Φ_k^Y by Definition 4.2.5, this particular k-node does not contribute an extra node to the image of row j: as $\begin{array}{|c|}\hline k{+}1\\\hline k\\\hline\end{array}$ is replaced by $\begin{array}{|c|}\hline k{+}2\\\hline k{+}1\\\hline k\\\hline\end{array}$, the k-node in this row is replaced by a single $(k+1)$-node

Hence one must subtract 1 for each such strip that passes through row j. By Definition 4.2.35, the number of such strips is $f(j) - g(j)$. Therefore

$$\lambda^+_{j+g(j)} = (\lambda_j + m_j) - (f(j) - g(j)) = \lambda_j + m_j + g(j) - f(j),$$

as required. □

The row indices k_i can be computed explicitly as follows. Take an abacus subdivision datum for λ. Then, by Definition 4.2.13 and Corollary 4.2.20, we have $k_1 = e - d$ and hence $k_j = je - d$ for $1 \le j \le s = k(\lambda)$.

Moreover, $f(t)$ can be written in terms of the function $N_k(M)$ as follows. Suppose that $f(t) = j$. Then, by Definition 4.2.35, we have $je - d < t \leq (j+1)e - d$, and hence $je < t + d \leq (j+1)e$. We compute

$$N_k(a-t) = \left\lfloor \frac{a-t-k}{e} \right\rfloor + 1 = \left\lfloor \frac{ce-(d+t)}{e} \right\rfloor + 1 = c + \left\lfloor \frac{-(t+d)}{e} \right\rfloor + 1 = c-(j+1)+1 = c-j.$$

Hence,

$$f(t) = c - N_k(a-t), \quad (t \geq 1). \tag{4.2.40}$$

Remark 4.2.41. *Consider the e-abacus of λ with a beads, and the Young diagram $[\lambda]$. For $1 \leq i \leq \ell(\lambda)$, the bead corresponding to the beta number $\beta_i = \lambda_i + a - i$ lies on the k-runner if and only if $\lambda_i + a - i \equiv k \pmod e$. The last node of the i-th row in $[\lambda]$ has residue $a + \lambda_i - i \pmod e$. Hence the runner label agrees with the residue of the last node in the corresponding row.* ◇

Proposition 4.2.42. *Take $\lambda \in \mathscr{P}_\alpha^{\Lambda_x}$ and assume $k(\lambda) > 0$. Then $\Phi_k^A(\lambda) = \Phi_k^Y(\lambda)$.*

Proof. Fix an abacus subdivision datum (a, c, d, a') for λ. Let $\lambda^+ = \Phi_k^Y(\lambda)$ and $\mu = \Phi_k^A(\lambda)$.

For $1 \leq r \leq a$, set $\beta_r := \lambda_r + a - r$, and let $B(\lambda; a) = \{\beta_r \mid 1 \leq r \leq a\}$ be the a-beta set of λ. For $1 \leq r \leq a'$, set $\beta_r^+ := \lambda_r^+ + a' - r$, and let $B(\lambda^+; a') = \{\beta_r^+ \mid 1 \leq r \leq a'\}$ be the a'-beta set of λ^+. Similarly, $B(\mu; a')$ is the a'-beta set of μ.

To prove $\lambda^+ = \mu$, it suffices to show their a'-beta sets $B(\mu; a')$ and $B(\lambda^+; a')$ are equal. By Lemma 4.2.27, we have

$$B(\mu; a') = \iota_k\big(B(\lambda; a)\big) \cup \mathscr{T},$$

where $\mathscr{T} := \mathscr{T}_{e+1,k,c} = \{q(e+1) + k \mid 0 \leq q \leq c-1\}$ as in (4.2.10).

Step 1. We first show $\iota_k\big(B(\lambda; a)\big) \subseteq B(\lambda^+; a')$.

Fix $1 \leq j \leq a$. By Lemma 4.2.38, the image of the original jth row lies in row $j + g(j)$ of $[\lambda^+]$ and has length $\lambda_j + m_j + g(j) - f(j)$. Hence

$$\beta_{j+g(j)}^+ = \big(\lambda_j + m_j + g(j) - f(j)\big) + a' - (j + g(j)) = \beta_j + m_j - f(j) + c.$$

Using (4.2.30) (applied with $r = j$) gives $m_j = N_k(\beta_j) - N_k(a-j)$, so

$$\beta_{j+g(j)}^+ = \beta_j + N_k(\beta_j) - N_k(a-j) - f(j) + c.$$

By (4.2.40), $N_k(a-j) = c - f(j)$. Substituting this yields

$$\beta_{j+g(j)}^+ = \beta_j + N_k(\beta_j) = \iota_k(\beta_j).$$

where the last equality follows from Corollary 4.2.29. Thus every element of $\iota_k(B(\lambda; a))$ occurs among the beta numbers of λ^+, so $\iota_k(B(\lambda; a)) \subseteq B(\lambda^+; a')$.

Step 2. Secondly, we show that $\mathscr{T} \subseteq B(\lambda^+; a')$.

Let $M := a' - \ell(\lambda^+) - 1$. Since $\lambda^+_r = 0$ for all $r > \ell(\lambda^+)$, we have

$$\{\beta^+_r \mid \ell(\lambda^+) < r \leq a'\} = \{a' - r \mid \ell(\lambda^+) < r \leq a'\} = \{0, 1, \ldots, M\}.$$

Hence every element of $\mathscr{T}$ that is at most M lies in $B(\lambda^+; a')$.

It remains to consider the elements of $\mathscr{T}$ that are larger than M. Let $t = q(e+1)+k \in \mathscr{T}$ and assume $t > M$. So t cannot come from a zero row of λ^+ and must occur as β^+_r for some $1 \leq r \leq \ell(\lambda^+)$. Moreover, by Remark 4.2.41, t can correspond only to a row whose last node has residue k.

By Lemma 4.2.38, the rows of $[\lambda^+]$ ending with residue k are precisely the newly inserted rows, and the ith inserted row occurs at index $r_i + i$. Moreover, this row is created from the terminal k-node of some non-trivial maximal $(k+1, k)$-strip: this k-node is replaced by a vertical pair with residues $k+1$ above k. Hence the inserted row $r_i + i$ has the same length as the row immediately above it (the image of row r_i), and therefore its a'-beta number is one less than the a'-beta number of that row. Since $g(r_i) = i - 1$, the row above $r_i + i$ is $r_i + i - 1$, and Step 1 gives $\beta^+_{r_i+i-1} = \iota_k(\beta_{r_i})$. Thus

$$\beta^+_{r_i+i} = \beta^+_{r_i+i-1} - 1 = \iota_k(\beta_{r_i}) - 1.$$

Because the strip ends at a k-node, the last node of row r_i has residue k in $[\lambda]$, so $\beta_{r_i} \equiv k \pmod{e}$. Writing $\beta_{r_i} = qe + k$ gives $\iota_k(\beta_{r_i}) = q(e+1) + k + 1$, and therefore

$$\beta^+_{r_i+i} = q(e+1) + k \in \mathscr{T}.$$

Hence every inserted row contributes an element of $\mathscr{T}$ to $B(\lambda^+; a')$. In total, there are $s = k(\lambda)$ such elements. It remains to prove that there are exactly s elements of the form $t = q(e+1) + k > M$ where $0 \leq q \leq c - 1$. We have:

$$q(e+1)+k > M = a' - \ell(\lambda^+) - 1 = (a+c) - \big(\ell(\lambda)+s\big) - 1 = (ce+k-d) + c - \big(\ell(\lambda)+s\big) - 1$$

which is equivalent to

$$X := \frac{d + \ell(\lambda) + s + 1}{e+1} > c - q \tag{4.2.43}$$

Let $\rho \in I$ be the unique integer satisfying $\rho \equiv x - k \pmod{e}$. Then by Lemma 4.2.15, we have:

$$(s-1)e \leq \ell(\lambda) - 1 - \rho < se$$

and, by the proof of Corollary 4.2.20, $\rho = e - d$. Hence

$$s(e+1) \leq \ell(\lambda) - 1 + d + s < s(e+1) + e$$

which implies

$$s(e+1) + 2 \leq \ell(\lambda) + 1 + d + s \leq s(e+1) + e + 1$$

which implies

$$s + \frac{2}{e+1} \leq \frac{d + \ell(\lambda) + s + 1}{e+1} \leq s + 1.$$

This means $X \in (s, s+1]$. In particular, since $1 \le c - q \le c$ and $c \ge s$ (as the inserted rows correspond to a subset of $\mathscr{T}$), there are exactly $s = k(\lambda)$ values satisfying (4.2.43), as desired. Hence $\mathscr{T} \subset B(\lambda^+; a')$.

Step 3 We have shown that $\iota_k(B(\lambda; a)) \subseteq B(\lambda^+; a')$ and $\mathscr{T} \subseteq B(\lambda^+; a')$ by the last two steps. By Lemma 4.2.27,

$$B(\mu; a') = \iota_k(B(\lambda; a)) \sqcup \mathscr{T} \subseteq B(\lambda^+; a').$$

Comparing cardinalities gives equality of the two beta sets, and consequently $\lambda^+ = \mu$. ☐

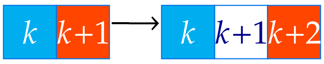
We can now state the main theorem of this section.

THEOREM 4.2.44. *Definition 4.2.5 and Definition 4.2.9 are equivalent.*

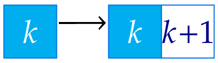
PROOF. This follows from Proposition 4.2.34 and Proposition 4.2.42. ☐

4.2.4. Subdivision on standard tableaux. In this section, we show that if a multipartition is k-horizontal (see Definition 4.2.14), then the subdivision on partitions in Definition 4.2.5 extends to a map on the set of (row)-standard tableaux. Moreover, it preserves the degree of a standard tableau; see Theorem 4.2.53.

Fix a subdivision datum $(e, I, \Lambda, \alpha, \boldsymbol{\kappa}, k)$ and a k-horizontal ℓ-partition $\boldsymbol{\lambda} \in \mathscr{P}_\alpha^{\boldsymbol{\kappa}}$. Let $\boldsymbol{\mu} := \Phi_k(\boldsymbol{\lambda})$ be the image. By Definition 4.2.5, in this case $\boldsymbol{\mu}$ is obtained by inserting an extra node immediately to the right of every k-node in $[\boldsymbol{\lambda}]$. That is, if a k-node is not the last node in its row, then



and otherwise,



We distinguish the nodes of $[\boldsymbol{\mu}]$ coming from $[\boldsymbol{\lambda}]$ from those inserted in the construction. Consider the initial tableaux $T^{\boldsymbol{\lambda}}$ and $T^{\boldsymbol{\mu}}$, and set $\mathbf{i}^{\boldsymbol{\lambda}} := \mathbf{i}^{T^{\boldsymbol{\lambda}}}$. Let $\phi := \phi_{\mathbf{i}^{\boldsymbol{\lambda}}}$ be the position-tracing function associated to $(k, \mathbf{i}^{\boldsymbol{\lambda}})$; see Definition 4.1.8. For each node $A \in [\boldsymbol{\lambda}]$ with $T^{\boldsymbol{\lambda}}(A) = t$ (where $1 \le t \le d = \mathrm{ht}(\alpha)$), define $A^+ \in [\boldsymbol{\mu}]$ by $T^{\boldsymbol{\mu}}(A^+) = \phi(t)$.

The nodes of the form A^+ for some $A \in [\boldsymbol{\lambda}]$ are called *old nodes*, and the remaining nodes of $[\boldsymbol{\mu}]$ are *new nodes*. Every new node is a $(k+1)$-node, lying immediately to the right of an old k-node; if B is such a new node and A^+ is the old k-node immediately to its left, write $B = A^\sharp$. In particular, if $A \in [\boldsymbol{\lambda}]$ is a k-node with $T^{\boldsymbol{\lambda}}(A) = t$, then $A^\sharp$ satisfies $T^{\boldsymbol{\mu}}(A^\sharp) = \phi(t) + 1$.

We can be more precise. Let $A = (m, r, c) \in [\boldsymbol{\lambda}]$. When we apply Φ_k, since the only non-trivial maximal strips are $(k, k+1)$-strips, the row index of every node is unchanged, and the column index increases by the number of k-nodes in the same row that occur strictly before A. This number is $\phi(t) - \phi(t_{m,r})$ where

$$t_{m,r} := T^{\lambda}(m,r,1) = 1 + \sum_{1\le i\le m-1} \left|\lambda^{(i)}\right| + \sum_{1\le i\le r-1} \lambda_i^{(m)}.$$

Hence

$$A^{+} = (m, r, c + \phi(t) - \phi(t_r)) \in [\boldsymbol{\mu}].$$

Let $\overline{\alpha} := \Phi_k(\alpha) = \alpha(\boldsymbol{\mu})$ be the residue content of $\boldsymbol{\mu}$ and let $d' := \mathrm{ht}(\overline{\alpha})$. Take any $T \in \mathrm{RStd}(\boldsymbol{\lambda})$, let $\mathbf{i}^T$ be the residue sequence of T and $\phi^T := \phi_{\mathbf{i}^T}$ be the position-tracing function associated to $(k, \mathbf{i}^T)$. We construct a map T' from $[\boldsymbol{\mu}]$ to $\{1, \cdots, d'\}$ as follows:

(a). if $B \in [\boldsymbol{\mu}]$ is a old node of the form A^{+} for $A \in [\boldsymbol{\lambda}]$. Assume $T(A) = t$, define $T'(B) := \phi^T(t)$.

(b). if $B \in [\boldsymbol{\mu}]$ is a new node of the form $A^{\sharp}$ for $A \in [\boldsymbol{\lambda}]$. Assume $T(A) = t$, define $T'(B) := \phi^T(t) + 1$.

Lemma 4.2.45. *Fix a subdivision datum* $(e, I, \Lambda, \alpha, \boldsymbol{\kappa}, k)$, *k-horizontal* $\boldsymbol{\lambda} \in \mathscr{P}^{\boldsymbol{\kappa}}_{\alpha}$. *Take* $T \in \mathrm{RStd}(\boldsymbol{\lambda})$ *and let* $\boldsymbol{\mu} := \Phi_k(\boldsymbol{\lambda})$, *then* T' *is a* $\boldsymbol{\mu}$*-tableau.*

Proof. It suffices to show that T' is a bijection from $[\boldsymbol{\mu}]$ to $\{1, \ldots, d'\}$. This follows from Definition 4.1.8 and the definition of T'. Indeed, ϕ^T is an embedding from $\{1, \cdots, d\}$ to $\{1, \cdots, d'\}$, and the elements of $\{1, \cdots, d'\}$ not in its image are precisely those of the form $\phi^T(t) + 1$ with $\mathbf{i}^T_t = k$. $\square$

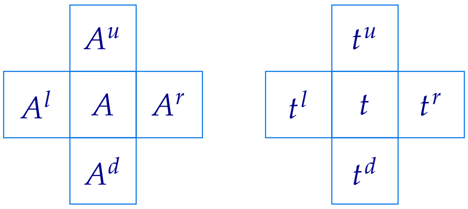
The following is an easy consequence of Definition 4.1.8.

Lemma 4.2.46. *Fix a subdivision datum* $(e, I, \Lambda, \alpha, \boldsymbol{\kappa}, k)$, *k-horizontal* $\boldsymbol{\lambda} \in \mathscr{P}^{\boldsymbol{\kappa}}_{\alpha}$. *Take* $T \in \mathrm{RStd}(\boldsymbol{\lambda})$, *if* $\mathbf{i}^T_t = k$, *then* $\phi^T(t+1) = \phi^T(t) + 2$. *In particular, for any* $t' > t$, *we have* $\phi^T(t') > \phi^T(t) + 1$. $\square$

Proposition 4.2.47. *Fix a subdivision datum* $(e, I, \Lambda, \alpha, \boldsymbol{\kappa}, k)$. *Let* $\boldsymbol{\lambda} \in \mathscr{P}^{\boldsymbol{\kappa}}_{\alpha}$ *be k-horizontal, and let* $T \in \mathrm{RStd}(\boldsymbol{\lambda})$. *Set* $\boldsymbol{\mu} := \Phi_k(\boldsymbol{\lambda})$. *Then* T' *is row-standard, that is,* $T' \in \mathrm{RStd}(\boldsymbol{\mu})$. *Moreover, if* $T \in \mathrm{Std}(\boldsymbol{\lambda})$, *then* T' *is standard, that is,* $T' \in \mathrm{Std}(\boldsymbol{\mu})$.

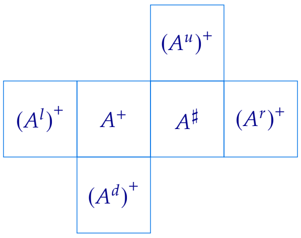
Proof. By Lemma 4.2.45, T' is a $\boldsymbol{\mu}$-tableau. It remains to show that T' is row-standard, and, if T is column-standard, that T' is column-standard as well. Both properties can be checked locally.

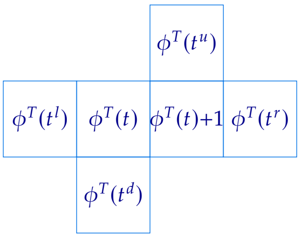
Let $A \in [\boldsymbol{\lambda}]$ be a node such that $T(A) = t$, and suppose that the surrounding nodes (some of which may not exist) are as follows (we draw the nodes in $[\boldsymbol{\lambda}]$ on the left and the corresponding entries of T on the right):



where $T(A^*) = t^*$ for $* \in \{l, r, u, d\}$. Apply the construction of $[\mu]$ and T', there are several cases:

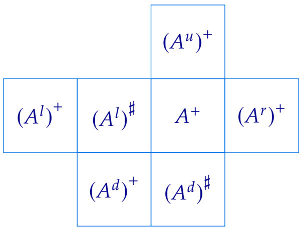
(a). If $\operatorname{res}(A) = k$, then the corresponding nodes in $[\mu]$ and the entries of T' are as follows:

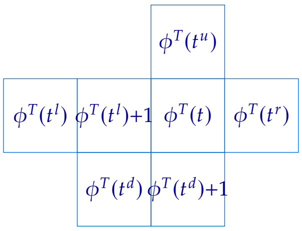




Since T is row-standard, we have $t^l < t < t^r$. The inequality $\phi^T(t^l) < \phi^T(t)$ follows from the fact that ϕ^T is strictly increasing, and $\phi^T(t)+1 < \phi^T(t^r)$ follows from Lemma 4.2.46 together with the assumption $\operatorname{res}(A) = \mathbf{i}^T_t = k$. If, moreover, T is standard, then $t^u < t < t^d$. Both $\phi^T(t) < \phi^T(t^d)$ and $\phi^T(t^u) < \phi^T(t) + 1$ follow from the fact that ϕ^T is strictly increasing.

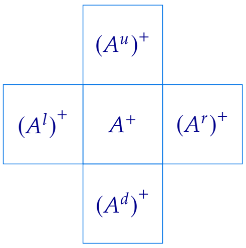
(b). If $\operatorname{res}(A) = k + 1$, then $\operatorname{res}(A^l) = \operatorname{res}(A^d) = k$, and the corresponding nodes in $[\mu]$ and the entries of T' are as follows:

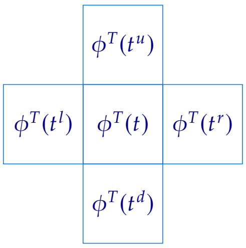




Since T is row-standard, we have $t^l < t < t^r$. The inequality $\phi^T(t) < \phi^T(t^r)$ follows from the fact that ϕ^T is strictly increasing, and $\phi^T(t^l)+1 < \phi^T(t)$ follows from Lemma 4.2.46 together with the condition $\operatorname{res}(A^l) = \mathbf{i}^T_{t^l} = k$. If, moreover, T is standard, then $t^u < t < t^d$. Both $\phi^T(t^u) < \phi^T(t)$ and $\phi^T(t) < \phi^T(t^d) + 1$ follow from the fact that ϕ^T is strictly increasing, while $\phi^T(t^l) + 1 < \phi^T(t^d)$ again follows from Lemma 4.2.46.

(c). If $\operatorname{res}(A) \neq k, k + 1$, then the corresponding nodes in $[\mu]$ and the entries of T' are as follows:





In this case, all inequalities follow immediately from the fact that ϕ^T is strictly increasing.

Since every node in $[\mu]$ is either of the form A^+ or of the form $A^\sharp$, the cases above exhaust all possibilities. Hence T' is a row-standard μ-tableau, and is standard if T is. □

By Proposition 4.2.47, we define the subdivision map on row-standard tableaux by

$$\Phi_k : \mathrm{RStd}(\boldsymbol{\lambda}) \longrightarrow \mathrm{RStd}(\boldsymbol{\mu}), \qquad T \mapsto \Phi_k(T) := T',$$

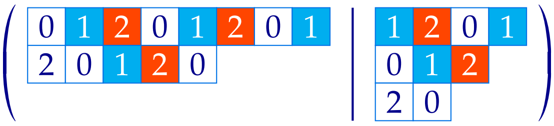
which restricts to the set of standard tableaux:

$$\Phi_k : \mathrm{Std}(\boldsymbol{\lambda}) \longrightarrow \mathrm{Std}(\boldsymbol{\mu}), \qquad T \mapsto \Phi_k(T) := T'.$$

By construction, $\Phi_k(T^{\boldsymbol{\lambda}}) = T^{\boldsymbol{\mu}}$ always holds and hence $\Phi_k(\mathbf{i}^{\boldsymbol{\lambda}}) = \mathbf{i}^{\boldsymbol{\mu}}$.

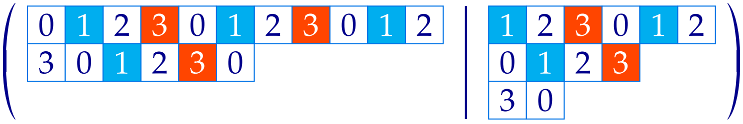
Example 4.2.48. *Let the subdivision datum be* $(e, I, \Lambda, \alpha, \boldsymbol{\kappa}, k) = (3, I, \Lambda_0+\Lambda_1, \alpha, (0,1), 1)$, *and consider the multipartition* $\boldsymbol{\lambda} = (8,5 \mid 4,3,2) \in \mathscr{P}^{\boldsymbol{\kappa}}$. *Then* $\boldsymbol{\lambda}$ *is k-horizontal, and* $\boldsymbol{\mu} := \Phi_k(\boldsymbol{\lambda}) = (11,6 \mid 6,4,2)$. *The Young diagrams filled with residues are as follows. We draw the k-nodes and the* $(k+1)$*-nodes in the maximal* $(k,k+1)$*-strips of* $[\boldsymbol{\lambda}]$ *in cyan and orange, respectively. Similarly, we draw the corresponding nodes in* $[\boldsymbol{\mu}]$ *in the same colors:*

$[\boldsymbol{\lambda}]$:



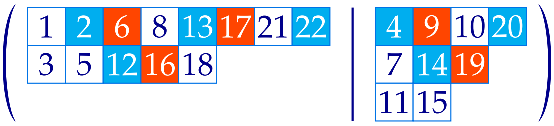
$$\left(\begin{array}{cccccccc}0&1&2&0&1&2&0&1\\2&0&1&2&0\end{array}\;\middle|\;\begin{array}{cccc}1&2&0&1\\0&1&2\\2&0\end{array}\right)$$

$[\boldsymbol{\mu}]$:



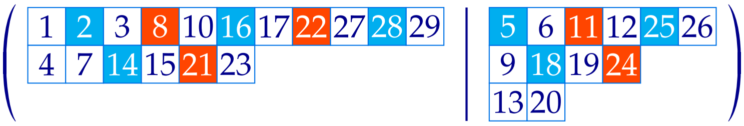
$$\left(\begin{array}{ccccccccccc}0&1&2&3&0&1&2&3&0&1&2\\3&0&1&2&3&0\end{array}\;\middle|\;\begin{array}{cccccc}1&2&3&0&1&2\\0&1&2&3\\3&0\end{array}\right)$$

Take $T \in \mathrm{Std}(\boldsymbol{\lambda})$ *to be the following standard tableau:*



$$\left(\begin{array}{cccccccc}1&2&6&8&13&17&21&22\\3&5&12&16&18\end{array}\;\middle|\;\begin{array}{cccc}4&9&10&20\\7&14&19\\11&15\end{array}\right)$$

Applying Φ_k, *we obtain the following tableau* $\Phi_k(T)$*:*



$$\left(\begin{array}{ccccccccccc}1&2&3&8&10&16&17&22&27&28&29\\4&7&14&15&21&23\end{array}\;\middle|\;\begin{array}{cccccc}5&6&11&12&25&26\\9&18&19&24\\13&20\end{array}\right)$$

It is easy to check (for example, using SageMath) that $\deg T = 12 = \deg \Phi_k(T)$. ◇

We want to prove that the subdivision map we have just constructed preserves the degree of a standard $\boldsymbol{\lambda}$-tableau. Before stating the theorem, we first show that removable nodes and addable nodes behave well under subdivision (on partitions).

Lemma 4.2.49. *Fix a subdivision datum* $(e, I, \Lambda, \alpha, \boldsymbol{\kappa}, k)$. *Let* $\boldsymbol{\lambda} \in \mathscr{P}^{\boldsymbol{\kappa}}$ *be k-horizontal and* $\boldsymbol{\mu} := \Phi_k(\boldsymbol{\lambda})$. *Let* $A \in [\boldsymbol{\lambda}]$ *be a node.*

(a). *If* $\mathrm{res}(A) = i \neq k$, *let* $j := \Phi_k(i)$, *where* $\Phi_k(i)$ *is the subdivision on words defined in Subsection 4.1.3. Then A is a removable i-node of* $[\boldsymbol{\lambda}]$ *if and only if* A^+ *is a removable j-node of* $[\boldsymbol{\mu}]$.

(b). *If* $\mathrm{res}(A) = k$, *then A is a removable k-node of* $[\boldsymbol{\lambda}]$ *if and only if* $A^\sharp$ *is a removable* $(k+1)$*-node of* $[\boldsymbol{\mu}]$ *and* A^+ *is a removable k-node of* $[\boldsymbol{\mu}] \setminus \{A^\sharp\}$.

Proof. Assume first that $\mathrm{res}(A) = i \neq k$, and set $j := \Phi_k(i)$. Let $A = (m, r, c)$, and let $A^+ = (m, r, c^+) \in [\boldsymbol{\mu}]$ be the (unique) j-node corresponding to A.

Recall that A is removable in $[\lambda]$ if and only if $c = \lambda_r^{(m)}$ and either $r = \ell(\lambda^{(m)})$ or $\lambda_r^{(m)} > \lambda_{r+1}^{(m)}$. Similarly, A^+ is removable in $[\mu]$ if and only if $c^+ = \mu_r^{(m)}$ and either $r = \ell(\mu^{(m)})$ or $\mu_r^{(m)} > \mu_{r+1}^{(m)}$.

Let p_t^m denote the number of k-nodes in the t-th row of $[\lambda^{(m)}]$, for $1 \leq t \leq \ell(\lambda^{(m)})$. By construction of Φ_k and since λ is k-horizontal, we have

$$\mu_t^{(m)} = \lambda_t^{(m)} + p_t^m \qquad \text{for all } t.$$

Since A is not a k-node and λ is k-horizontal, A is the last node of $[\lambda_r^{(m)}]$ if and only if A^+ is the last node of $[\mu_r^{(m)}]$. In other words, $c^+ = \mu_r^{(m)}$ if and only if $c = \lambda_r^{(m)}$. Thus removability of A and A^+ reduces to comparing adjacent row lengths. In particular, if $r = \ell(\lambda^{(m)})$, there is nothing to prove. So we assume $r < \ell(\lambda^{(m)})$.

($\Leftarrow$) Suppose that A^+ is removable. Then $\mu_r^{(m)} > \mu_{r+1}^{(m)}$ and hence

$$\lambda_r^{(m)} - \lambda_{r+1}^{(m)} = \left(\mu_r^{(m)} - \mu_{r+1}^{(m)}\right) - \left(p_r^m - p_{r+1}^m\right).$$

Since $p_r^m - p_{r+1}^m \geq 0$, it follows that $\lambda_r^{(m)} - \lambda_{r+1}^{(m)} \geq 1 - (p_r^m - p_{r+1}^m)$. If $p_r^m - p_{r+1}^m = 0$, then $\lambda_r^{(m)} - \lambda_{r+1}^{(m)} \geq 1$, so $\lambda_r^{(m)} > \lambda_{r+1}^{(m)}$ and A is removable. If $p_r^m - p_{r+1}^m > 0$, assume that $\lambda_r^{(m)} = \lambda_{r+1}^{(m)}$. Since the last node A of row r of $[\lambda^{(m)}]$ is not a k-node, this implies that the number of k-nodes in rows r and $r+1$ must be the same, which contradicts $p_r^m > p_{r+1}^m$. Hence $\lambda_r^{(m)} > \lambda_{r+1}^{(m)}$ and A is removable in all cases.

($\Rightarrow$) If A is removable then $\lambda_r^{(m)} > \lambda_{r+1}^{(m)}$, using $p_r^m \geq p_{r+1}^m$, we get

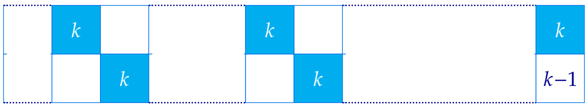
$$\mu_r^{(m)} - \mu_{r+1}^{(m)} = \left(\lambda_r^{(m)} - \lambda_{r+1}^{(m)}\right) + \left(p_r^m - p_{r+1}^m\right) \geq 1,$$

hence A^+ is removable.

If $\mathrm{res}(A) = k$, then under subdivision the node $A = (m, r, c)$ corresponds to a horizontally adjacent pair in $[\mu_r^{(m)}]$: an old k-node A^+ and a new $(k+1)$-node $A^\sharp$ immediately to the right of A^+. We want to prove that A is removable in $[\lambda]$ if and only if $A^\sharp$ is removable in $[\mu]$ and A^+ is removable in $[\mu] \setminus \{A^\sharp\}$.

Since $A^\sharp$ is immediately to the right of A^+ in the same row, it is the last node in $[\mu_r^{(m)}]$ if and only if A is the last node of $[\lambda_r^{(m)}]$. By the same argument as in the last case, it suffices to compare the adjacent row lengths in $[\lambda^{(m)}]$ and in $[\mu^{(m)}]$. We assume $r < \ell(\lambda^{(m)})$.

($\Rightarrow$) If A is removable, so $c = \lambda_r^{(m)}$. Since A is a k-node, the node immediately below A has residue $k-1$, in other words, it is of the form:



as a result, whether or not this $(k-1)$-node lies in $[\boldsymbol{\lambda}]$, we always have $p_r^m > p_{r+1}^m$. Hence

$$\mu_r^{(m)} - \mu_{r+1}^{(m)} = \lambda_r^{(m)} - \lambda_{r+1}^{(m)} + p_r^m - p_{r+1}^m > 1 + 0 = 1.$$

Hence B is removable of $[\boldsymbol{\mu}]$ and A^+ is removable of $[\boldsymbol{\mu}] \setminus \{A^\sharp\}$.

($\Leftarrow$) Conversely, if $[\boldsymbol{\mu}] \setminus \{A^\sharp, A^+\}$ is still (the Young diagram of) an ℓ-partition, then $\mu_r^{(m)} \geq \mu_{r+1}^{(m)} + 2$. If $p_r^m > p_{r+1}^m + 1$, then at least the node immediately below A is not contained in $[\boldsymbol{\lambda}]$, and hence A is removable. Otherwise,

$$\lambda_r^{(m)} - \lambda_{r+1}^{(m)} = \left(\mu_r^{(m)} - \mu_{r+1}^{(m)}\right) - \left(p_r^m - p_{r+1}^m\right) \geq 2 - 1 = 1,$$

and hence A is also removable. □

We also need to prove the analogous statements for addable nodes of $[\boldsymbol{\lambda}]$ and $[\boldsymbol{\mu}]$. To do this, we extend the definition of subdivision on partitions.

Fix a subdivision datum $(e, I, \Lambda, \alpha, \boldsymbol{\kappa}, k)$. For each node $A = (m, r, c)$, we define the residue $\operatorname{res}(A) := \operatorname{res}_{\boldsymbol{\kappa}}(A)$ as in Subsection 1.2.2; if $\operatorname{res}(A) = i$, we refer to A as an i-node. Equivalently, we may choose an ℓ-partition $\boldsymbol{\lambda} \in \mathscr{P}^{\boldsymbol{\kappa}}$ such that $A \in [\boldsymbol{\lambda}]$ and then define the residue of A in the usual way. By definition of the residue function, it is independent of the choice of $\boldsymbol{\lambda}$ containing A.

Let $p_k(A)$ be the number of k-nodes of the form (m, r, x) such that $1 \leq x < c$. We define the node $A^+ = \big(m, r, c + p_k(A)\big)$. If there is a k-horizontal ℓ-partition $\boldsymbol{\lambda}$ containing A and A is an i-node, then under the subdivision map, $A^+ \in [\Phi_k(\boldsymbol{\lambda})]$ is the corresponding $\Phi_k(i)$-node, as before.

Lemma 4.2.50. *Fix a subdivision datum $(e, I, \Lambda, \alpha, \boldsymbol{\kappa}, k)$. Let $\boldsymbol{\lambda} \in \mathscr{P}^{\boldsymbol{\kappa}}$ be k-horizontal and set $\boldsymbol{\mu} := \Phi_k(\boldsymbol{\lambda})$. Let A be a node, not necessarily in $[\boldsymbol{\lambda}]$.*

(a). *If $\operatorname{res}(A) = i \neq k$, let $j := \Phi_k(i)$. Then A is an addable i-node of $[\boldsymbol{\lambda}]$ if and only if A^+ is an addable j-node of $[\boldsymbol{\mu}]$.*

(b). *If $\operatorname{res}(A) = k$, then A is an addable k-node of $[\boldsymbol{\lambda}]$ if and only if A^+ is an addable k-node of $[\boldsymbol{\mu}]$ and $A^\sharp$ is an addable $(k+1)$-node of $[\boldsymbol{\mu}] \cup \{A^+\}$.*

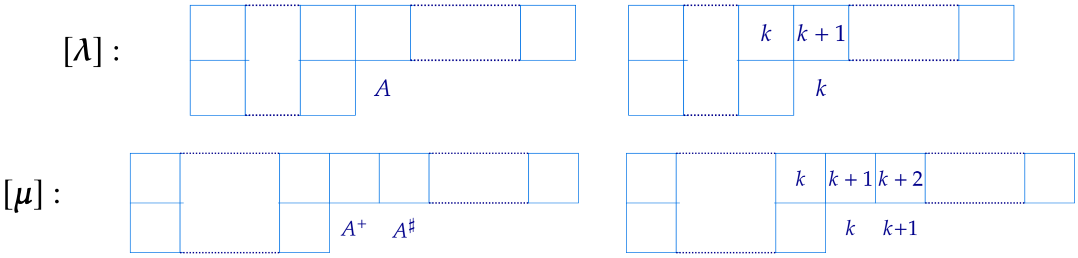
Proof. Let $A = (m, r, c)$ be a node with $\operatorname{res}(A) = i \neq k$. We want to prove that A is an addable node of $[\boldsymbol{\lambda}]$ if and only if A^+ is an addable node of $[\boldsymbol{\mu}]$.

By definition, A is addable of $[\boldsymbol{\lambda}]$ if and only if $\lambda_r^{(m)} + 1 = c$ and $\lambda_{r-1}^{(m)} \geq c$ whenever $r \neq 1$. Similarly, $A^+ = (m, r, c^+)$ is addable of $[\boldsymbol{\mu}]$ if and only if $\mu_r^{(m)} + 1 = c^+$ and $\mu_{r-1}^{(m)} \geq c^+$ whenever $r \neq 1$. Let p_t^m denote the number of k-nodes in the t-th row of $[\lambda^{(m)}]$, for $1 \leq t \leq \ell(\lambda^{(m)})$.

Suppose that A is addable. Then $c^+ = c + p_r^m = \lambda_r^{(m)} + 1 + p_r^m = 1 + \mu_r^{(m)}$, as required. Moreover, if $r \neq 1$, then $\mu_{r-1}^{(m)} - c^+ = \left(\lambda_{r-1}^{(m)} + p_{r-1}^m\right) - \left(c + p_r^m\right) = \left(\lambda_{r-1}^{(m)} - c\right) + \left(p_{r-1}^m - p_r^m\right) \geq 0$, so $\mu_{r-1}^{(m)} \geq c^+$. Hence A^+ is addable in $[\boldsymbol{\mu}]$.

Conversely, suppose that A^+ is addable in $[\boldsymbol{\mu}]$. Then $c^+ = \mu_r^{(m)} + 1 = \lambda_r^{(m)} + p_r^m + 1$. Since the number of k-nodes before c^+ is the same as the number of k-nodes in $[\mu_r^{(m)}]$, which is the same as the number of k-nodes in $[\boldsymbol{\lambda}_r^{(m)}]$, this number is p_r^m by definition. Hence $c^+ = c + p_r^m$ and $c = \boldsymbol{\lambda}_r^{(m)} + 1$, as required. Now assume $r \neq 1$. If $p_{r-1}^m > p_r^m$, then $\boldsymbol{\lambda}_{r-1}^{(m)} > \boldsymbol{\lambda}_r^{(m)} = c - 1$ since $\mathrm{res}(A) \neq k$, and hence $\boldsymbol{\lambda}_{r-1}^{(m)} \geq c$. Otherwise, $p_{r-1}^m = p_r^m$ and $\boldsymbol{\lambda}_{r-1}^{(m)} - c = \big(\boldsymbol{\mu}_{r-1}^{(m)} - p_{r-1}^m\big) - c = \big(\boldsymbol{\mu}_{r-1}^{(m)} - c^+\big) - \big(p_{r-1}^m - p_r^m\big) \geq 0$, as desired. Therefore A is addable in $[\boldsymbol{\lambda}]$, and the claim follows.

Let $A = (m, r, c)$ be a k-node. Under the subdivision map, there is a unique k-node $A^+ = (m, r, c^+)$ and a unique $(k+1)$-node $A^\sharp = (m, r, c^+ + 1)$ corresponding to A. Namely, it is of the following form (we draw the node labels on the left and their residues on the right; the same convention applies to all figures below in the proof of Theorem 4.2.53):



We want to prove that A is an addable node of $[\boldsymbol{\lambda}]$ if and only A^+ is an addable node of $[\boldsymbol{\mu}]$ and $A^\sharp$ is an addable node of $[\boldsymbol{\mu}] \cup \{A^+\}$.

Suppose that A is an addable k-node of $[\boldsymbol{\lambda}]$. Then $c = \boldsymbol{\lambda}_r^{(m)} + 1$ and $\boldsymbol{\lambda}_{r-1}^{(m)} \geq c$ whenever $r \neq 1$. By construction, $c^+ = p_r^m + c = p_r^m + \boldsymbol{\lambda}_r^{(m)} + 1 = \boldsymbol{\mu}_r^{(m)} + 1$. It remains to show that $\boldsymbol{\mu}_{r-1}^{(m)} \geq c^+ + 1$. Since $\mathrm{res}(A) = k$ and $A \notin [\boldsymbol{\lambda}]$, the maximal $(k, k+1)$-strip containing A has no intersection with $[\boldsymbol{\lambda}_r^{(m)}]$ but does intersect $[\boldsymbol{\lambda}_{r-1}^{(m)}]$. In particular, $p_{r-1}^m > p_r^m$. Hence

$$\boldsymbol{\mu}_{r-1}^{(m)} - c^+ = \boldsymbol{\lambda}_{r-1}^{(m)} + p_{r-1}^m - c - p_r^m \geq 1.$$

Conversely, suppose that $[\boldsymbol{\mu}] \cup \{A^+, A^\sharp\}$ is still the Young diagram of an ℓ-partition. Then $c^+ = \boldsymbol{\mu}_r^{(m)} + 1 = \boldsymbol{\lambda}_r^{(m)} + p_r^m + 1$ and $\boldsymbol{\mu}_{r-1}^{(m)} \geq c^+ + 1$. Since the number of k-nodes before A^+ in $[\boldsymbol{\mu}_r^{(m)}]$ is the same as the number of k-nodes before A in $[\boldsymbol{\lambda}_r^{(m)}]$, which is p_r^m, we have $c^+ = p_r^m + c$ and hence $c = \boldsymbol{\lambda}_r^{(m)} + 1$. It remains to show that $\boldsymbol{\lambda}_{r-1}^{(m)} \geq c$. If $p_{r-1}^m > p_r^m + 1$, then $\boldsymbol{\lambda}_{r-1}^{(m)} > \boldsymbol{\lambda}_r^{(m)} = c - 1$. Otherwise $p_{r-1}^m = p_r^m + 1$, and $\boldsymbol{\lambda}_{r-1}^{(m)} - c = \boldsymbol{\mu}_{r-1}^{(m)} - p_{r-1}^m - c^+ + p_r^m = \boldsymbol{\mu}_{r-1}^{(m)} - c^+ - 1 \geq 0$, as required. □

For an ℓ-partition $\boldsymbol{\lambda} = (\boldsymbol{\lambda}^{(1)}, \cdots, \boldsymbol{\lambda}^{(\ell)})$, we say that the component $\boldsymbol{\lambda}^{(i)}$ is earlier than $\boldsymbol{\lambda}^{(j)}$ if $i < j$, and we call $\boldsymbol{\lambda}^{(i)}$ an *earlier component* of $\boldsymbol{\lambda}$. Equivalently, we say that $\boldsymbol{\lambda}^{(j)}$ is later than $\boldsymbol{\lambda}^{(i)}$, and we call $\boldsymbol{\lambda}^{(j)}$ a *later component* of $\boldsymbol{\lambda}$.

For $\boldsymbol{\lambda} \in \mathscr{P}_n^\kappa$ and $T \in \mathrm{Std}(\boldsymbol{\lambda})$, the *last node* of T is the node $A \in [\boldsymbol{\lambda}]$ such that $T(A) = n$. In particular, since T is standard, A is a removable node of $[\boldsymbol{\lambda}]$. Hence we can define $d_A(\boldsymbol{\lambda})$ as in (1.2.17).

Lemma 4.2.51. *Fix a subdivision datum* $(e, I, \Lambda, \alpha, \boldsymbol{\kappa}, k)$. *Take* $\boldsymbol{\lambda} \in \mathscr{P}_\alpha^{\boldsymbol{\kappa}}$ *to be* k*-horizontal and set* $\boldsymbol{\mu} := \Phi_k(\boldsymbol{\lambda})$. *Let* A *be the last node of* T. *Suppose that* $\operatorname{res}(A) = k$. *Then*

$$d_A(\boldsymbol{\lambda}) = d_{A^\sharp}(\boldsymbol{\mu}) + d_{A^+}([\boldsymbol{\mu}] \setminus \{A^\sharp\}).$$

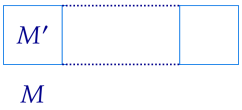
Proof. By the definition of $d_\bullet(\bullet)$ in Subsection 1.2.7, it is enough to prove the following two statements: (1) the number of addable k-nodes below A of $[\boldsymbol{\lambda}]$ is equal to the sum of the number of addable $(k+1)$-nodes below $A^\sharp$ of $[\boldsymbol{\mu}]$ and the number of addable k-nodes below A^+ of $[\boldsymbol{\mu}] \setminus \{A^\sharp\}$; (2) the number of removable k-nodes below A of $[\boldsymbol{\lambda}]$ is equal to the sum of the number of removable $(k+1)$-nodes below $A^\sharp$ of $[\boldsymbol{\mu}]$ and the number of removable k-nodes below A^+ of $[\boldsymbol{\mu}] \setminus \{A^\sharp\}$.

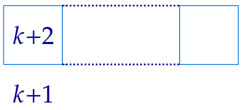
The subdivision map on ℓ-partitions is defined componentwise and hence preserves the component index of every node. Since $\boldsymbol{\lambda}$ is k-horizontal, the row index of each node is also preserved. As a result, the order (above/below) of nodes defined in Subsection 1.2.7 is preserved. Let $\bar{I}$ be the vertex set of the new quiver after subdivision.

We firstly treat the addable node case. Consider any addable $(k+1)$-node M of $[\boldsymbol{\mu}]$ below $A^\sharp$, there are three possibilities:

(i) In some row below $A^\sharp$, the last node M' of that row in $[\boldsymbol{\mu}]$ is a k-node, M is immediately to the right of M', and $[\boldsymbol{\mu}] \cup \{M\}$ is (the Young diagram of) an ℓ-partition. In other words, it is of the following form:

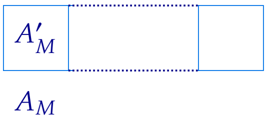
(ii) In some component $[\mu^{(i)}]$ that is no earlier than the component containing $A^\sharp$, the first node M' of the last row has residue $k+2$, and the node M lies immediately below M'. In other words, it is of the following form:

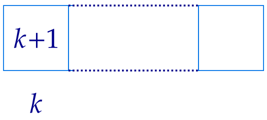




(iii) M is the first node of a component later than the component containing $A^\sharp$, and the charge of this component is $k+1 \in \bar{I}$.

The first case is impossible, since after subdivision any k-node appears with a $(k+1)$-node immediately to its right, and vice versa. The third case is also impossible because, under subdivision, no component can have charge $k+1$; see Subsection 4.1.4. For each such node M in the second case, it corresponds uniquely to an addable k-node A_M in $[\boldsymbol{\lambda}]$ below A, of the following form:





Any other addable k-node A_0 of $[\boldsymbol{\lambda}]$ below A is of one of the following forms:

(i) A_0 is the first node of a later component than the component containing A, and the charge of this component is k.

(ii) In some row below A, the last node A_0' of that row in $[\lambda]$ is a $(k-1)$-node, A_0 is immediately to the right of A_0', and $[\lambda] \cup \{A_0\}$ is an ℓ-partition. In other words, it is of the following form:

On the other hand, any addable k-node M_0 below A^+ of $[\mu] \setminus \{A^\sharp\}$ is of one of the following forms:

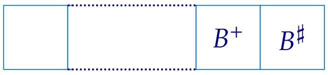
(i) M_0 is the first node of a later component than the component containing A^+ and the charge of this component is k.

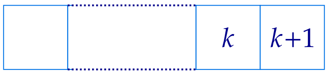
(ii) In some row below A^+, the last node M_0' of that row in $[\mu] \setminus \{A^\sharp\}$ is a $(k-1)$-node, M_0 is immediately to the right of M_0', and $[\mu] \setminus \{A^\sharp\} \cup \{M_0\}$ is an ℓ-partition. In other words, it is of the following form:

Under the subdivision map Φ_k, and by the assumption that λ is k-horizontal, the correspondence $M_0 \leftrightarrow A_0$ is one-to-one in each case.

By Lemma 4.2.50, the statement for the number of addable nodes follows.

We next consider the removable-node case. Every removable $(k+1)$-node N of $[\mu]$ below $A^\sharp$ is a new node $B^\sharp$ immediately to the right of a k-node B^+ for some k-node B in $[\lambda]$, and $B^\sharp$ is the last node of some row below $A^\sharp$. In other words, it is of the following form:





Every removable k-node B of $[\lambda]$ below A is the last node of some row below A, of the form:

Under the subdivision map, as discussed at the beginning of this section, $B \leftrightarrow B^\sharp$ is a one-to-one correspondence. On the other hand, there is no removable k-node in $[\mu] \setminus \{A^\sharp\}$ below A^+, since every k-node in $[\mu]$ appears to the left of a $(k+1)$-node and hence cannot be removable. Therefore, by Lemma 4.2.49, the desired equality for the number of removable nodes also holds.

Combining the addable-node case and the removable-node case then yields the desired equality $d_A(\lambda) = d_{A^\sharp}(\mu) + d_{A^+}([\mu] \setminus \{A^\sharp\})$. □

The proof of the next result is analogous to that of Lemma 4.2.51.

Lemma 4.2.52. *Fix a subdivision datum* $(e, I, \Lambda, \alpha, \boldsymbol{\kappa}, k)$. *Take* $\boldsymbol{\lambda} \in \mathscr{P}_\alpha^{\boldsymbol{\kappa}}$ *to be* k*-horizontal and set* $\boldsymbol{\mu} := \Phi_k(\boldsymbol{\lambda})$. *Let* A *be the last node of* T. *Suppose that* $\operatorname{res}(A) \neq k$. *Then*

$$d_A(\boldsymbol{\lambda}) = d_{A^+}(\boldsymbol{\mu}).$$

Proof. Let $j = \Phi_k(i)$, where Φ_k is the subdivision map on words introduced in Subsection 4.1.3. Then A^+ is a j-node in $[\boldsymbol{\mu}]$. It is enough to prove that the number of addable i-nodes of $[\boldsymbol{\lambda}]$ below A is equal to the number of addable j-nodes of $[\boldsymbol{\mu}]$ below A^+, and that the number of removable i-nodes of $[\boldsymbol{\lambda}]$ below A is equal to the number of removable j-nodes of $[\boldsymbol{\mu}]$ below A^+. Let $\bar{I}$ be the vertex set of the new quiver after subdivision.

Any addable i-nodes M of $[\boldsymbol{\lambda}]$ below A is of the following form:

(i) In some row below A, the last node M' of that row in $[\boldsymbol{\lambda}]$ is a k-node, M is immediately to the right of M', and $[\boldsymbol{\lambda}] \cup \{M\}$ is an ℓ-partition.
(ii) In some component $\boldsymbol{\lambda}^{(m)}$ that is no earlier than the component containing A, the first node M' of the last row has residue $i+1$, and the node M lies immediately below M'.
(iii) M is the first node of later component than the component containing A and the charge of this component is $i \in I$.

While any addable j-node A_M is of the following form:

(i) In some row below A^+, there exists a k-node $A'_M \in [\boldsymbol{\mu}]$ such that A'_M is the last node in that row, A_M is immediately to the right of A'_M, and $[\boldsymbol{\mu}] \cup \{A_M\}$ is an ℓ-partition.
(ii) In some component $\boldsymbol{\mu}^{(m)}$ that is no earlier than the component containing A^+, the first node A'_M of the last row has residue $j+1$, and the node A_M lies immediately below A'_M.
(iii) A_M is the first node of later component than the component where A^+ lives and the charge of this component is $j \in I$.

Under the subdivision Φ_k, $M \leftrightarrow A_M$ is thus a natural one-to-one correspondence and $A_M = M^+$ in our generalized definition of subdivision above Lemma 4.2.49. Hence by Lemma 4.2.50, the desired equality of addable nodes holds.

Similarly, applying Lemma 4.2.49, one can show that the removable i-nodes of $[\boldsymbol{\lambda}]$ below A are in one-to-one correspondence with the removable j-nodes of $[\boldsymbol{\mu}]$ below A^+.

Combining these two statements gives the desired equality $d_A(\boldsymbol{\lambda}) = d_{A^+}(\boldsymbol{\mu})$. □

Theorem 4.2.53. *Fix a subdivision datum* $(e, I, \Lambda, \alpha, \boldsymbol{\kappa}, k)$, $\boldsymbol{\lambda} \in \mathscr{P}_\alpha^{\boldsymbol{\kappa}}$ *and* $T \in \operatorname{Std}(\boldsymbol{\lambda})$. *Suppose* $\boldsymbol{\lambda}$ *is* k*-horizontal, then* $\deg T = \deg \Phi_k(T)$.

Proof. Let $\boldsymbol{\mu} := \Phi_k(\boldsymbol{\lambda})$. Set $\operatorname{ht}(\alpha) = d$ and $\operatorname{ht}\left(\Phi_k(\alpha)\right) = d'$. Let $A = T^{-1}(d)$ be the last node of T. We proceed by induction on d. If $d = 0$, then $\boldsymbol{\lambda} = \emptyset = \boldsymbol{\mu}$ and the claim holds by definition.

For $d > 0$, by definition of the degree of standard tableaux, we have

$$\deg T = d_A(\lambda) + \deg\big(T \downarrow (d-1)\big)$$

By inductive hypothesis, we have:

$$\deg\big(T \downarrow (d-1)\big) = \deg\Big(\Phi_k\big(T \downarrow (d-1)\big)\Big)$$

There are two cases, depending on whether $\operatorname{res}(A) = k$. We compute the degree in those two cases.

(a). If $\operatorname{res}(A) = k$, then applying the subdivision map to $[\lambda]$ produces, in $[\mu]$, an old node A^+ together with a new $(k+1)$-node $A^\sharp$ immediately to the right of A^+. In this case, by our definition of $\Phi_k(T)$, we have $\big(\Phi_k(T)\big)(A^+) = d' - 1$ and $\big(\Phi_k(T)\big)(A^\sharp) = d'$. Removing these two nodes from $[\mu]$, it follows that $[\mu] \setminus \{A^\sharp, A^+\} = \operatorname{Shape}\Big(\Phi_k\big(T \downarrow (d-1)\big)\Big)$ and $\Phi_k(T) \downarrow (d'-2) = \Phi_k\big(T \downarrow (d-1)\big)$. Hence:

$$\begin{aligned}\deg\big(\Phi_k(T)\big) &= d_{A^\sharp}(\mu) + d_{A^+}([\mu] \setminus \{A^\sharp\}) + \deg\Big(\Phi_k(T) \downarrow (d'-2)\Big)\\ &= d_{A^\sharp}(\mu) + d_{A^+}([\mu] \setminus \{A^\sharp\}) + \deg\Big(\Phi_k\big(T \downarrow (d-1)\big)\Big)\end{aligned}$$

It suffices to show that $d_A(\lambda) = d_{A^\sharp}(\mu) + d_{A^+}([\mu] \setminus \{A^\sharp\})$, which follows from Lemma 4.2.51.

(b). If $\operatorname{res}(A) = i \neq k$, then applying the subdivision map gives a unique old node $A^+ \in [\mu]$ such that $\big(\Phi_k(T)\big)(A^+) = d'$, and we have $[\mu] \setminus \{A^+\} = \operatorname{Shape}\Big(\Phi_k\big(T \downarrow (d-1)\big)\Big)$ and $\Phi_k(T) \downarrow (d'-1) = \Phi_k\big(T \downarrow (d-1)\big)$. Hence:

$$\begin{aligned}\deg\big(\Phi_k(T)\big) &= d_{A^+}(\mu) + \deg\Big(\Phi_k(T) \downarrow (d'-1)\Big)\\ &= d_{A^+}(\mu) + \deg\Big(\Phi_k\big(T \downarrow (d-1)\big)\Big)\end{aligned}$$

It suffices to show that $d_A(\lambda) = d_{A^+}(\mu)$, which follows from Lemma 4.2.52.

Hence, $\deg T = \deg \Phi_k(T)$ follows by induction. □

4.3. Categorical Subdivision

In this section, we use the combinatorial framework of the subdivision maps on partitions and standard tableaux to prove some categorical results concerning the subdivision map on KLR algebras. The main results are Theorem 4.3.3, Theorem 4.3.9, and Theorem 4.3.19.

4.3.1. Image of Idempotents.

Fix a subdivision datum $(e, I, \Lambda, \alpha, \kappa, k)$. Let R_α be the corresponding KLR algebra. For any $\lambda \in \mathscr{P}^{\kappa}_{\alpha}$, the idempotent element $e_\lambda \in R_\alpha$ is defined in (1.3.17). The aim of this section is to describe $\Phi_k(e_\lambda)$. For simplicity, in this section we write $\overline{I}$, $\overline{\alpha}$, $\overline{\Lambda}$, and $\overline{\boldsymbol{\lambda}}$ for the images of I, α, Λ, and $\boldsymbol{\lambda}$ under the corresponding subdivision map Φ_k. Let $\mathfrak{e} \in R_{\overline{\alpha}}$ be the truncation idempotent defined in (4.1.15).

LEMMA 4.3.1. *Fix a subdivision datum* $(e, I, \Lambda_x, \alpha, x, k)$ *and take* $\lambda \in \mathscr{P}_\alpha^{\Lambda_x}$*. Then* $e_{\overline{\lambda}}\mathfrak{e} = 0$ *if and only* λ *is not* k*-horizontal.*

PROOF. By the definition of e_λ, the residue sequence $\mathbf{i}^\lambda$ corresponds to the standard tableau T^λ. Suppose λ is k-horizontal, i.e. there is no non-trivial maximal $(k+1, k)$-strip. Then every k-node in $[\lambda]$ lies in a maximal $(k, k+1)$-strip. By Definition 4.2.5, each k-node is replaced by a horizontal pair $\begin{array}{|c|c|}\hline k & k{+}1 \\ \hline\end{array}$ in $[\overline{\lambda}]$. The corresponding residue sequence is locally of the form $\cdots, k, k+1, \cdots$, and thus $\mathbf{i}^{\overline{\lambda}} \in \overline{I}^{\overline{\alpha}}_{\mathrm{ord}}$. In particular, $e(\mathbf{i}^{\overline{\lambda}})\mathfrak{e} = e(\mathbf{i}^{\overline{\lambda}}) \neq 0$.

Conversely, suppose that λ is not k-horizontal and that there exists a non-trivial maximal $(k+1, k)$-strip. Let S be the top such strip, and let A_1 be its initial node of residue $k+1$ (so A_1 lies above the initial nodes of all other non-trivial maximal $(k+1, k)$-strips). Let A_2 be the k-node immediately below A_1. By Definition 4.2.9, this vertical pair $\begin{array}{|c|}\hline A_1 \\ \hline A_2 \\ \hline\end{array}$ is replaced by the triple $\begin{array}{|c|}\hline k{+}2 \\ \hline k{+}1 \\ \hline k \\ \hline\end{array}$. In $[\overline{\lambda}]$, the $k+1$-node inside this triple is precisely the initial node of the top non-trivial maximal $(k+1, k)$-strip. Therefore, every k-node above this $(k+1)$-node lies in a maximal $(k, k+1)$-strip and, in particular, occurs in a horizontal adjacent pair $\begin{array}{|c|c|}\hline k & k{+}1 \\ \hline\end{array}$. Now assume that this $(k+1)$-node is the m-th node of $[\overline{\lambda}]$ in row-reading order, and write $\mathbf{i}^{\overline{\lambda}} = (\mathbf{i}'_1, \mathbf{i}'_2, \ldots)$. Then the initial segment $(\mathbf{i}'_1, \mathbf{i}'_2, \ldots, \mathbf{i}'_m)$ contains one more $k+1$ than k, so $\mathbf{i}^{\overline{\lambda}} \in \overline{I}^{\overline{\alpha}}_{\mathrm{un}}$ by definition and hence $e_{\overline{\lambda}}\mathfrak{e} = 0$. □

COROLLARY 4.3.2. *Fix a subdivision datum* $(e, I, \Lambda_x, \alpha, x, k)$*. If* $x \neq k$*, then all partitions* $\lambda \in \mathscr{P}_\alpha^{\Lambda_x}$ *such that* $\mathfrak{e}e_{\overline{\lambda}} \neq 0$ *are* e*-regular.*

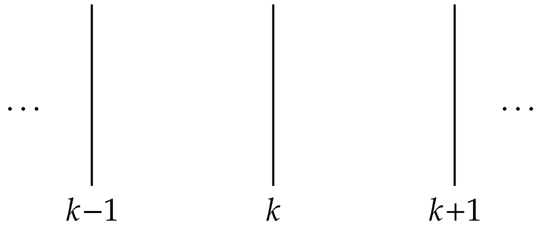
PROOF. Let (a, c, d, a') be an abacus subdivision datum for λ. Since $a + d = ce + k$ and $a \equiv x \pmod{e}$, we have $x + d \equiv k \pmod{e}$. Since $x, k, d \in I$, if $x \neq k$, then $d \not\equiv 0 \pmod{e}$, and hence $e - d < e$. By Corollary 4.2.20, λ is k-horizontal (i.e. $k(\lambda) = 0$) if and only if $\ell(\lambda) \leq e - d$, and hence $\ell(\lambda) < e$. By Lemma 4.3.1, the conclusion follows. □

Recall from Theorem 4.1.19 that Φ_k is an isomorphism between R_α and the balanced KLR algebra $S_{\overline{\alpha}}$. Since $e_\lambda \in R_\alpha$ is a non-zero idempotent, it must be mapped to a non-zero idempotent in $S_{\overline{\alpha}}$. By Lemma 4.3.1, in the level one case, if λ is not k-horizontal, then $0 = \mathfrak{e}e_{\overline{\lambda}}\mathfrak{e} + \mathfrak{e}\mathfrak{J}\mathfrak{e} \in S_{\overline{\alpha}}$ and thus $\Phi_k(e_\lambda) \neq e_{\overline{\lambda}}$. By contrast, in the k-horizontal case, we have the following result.

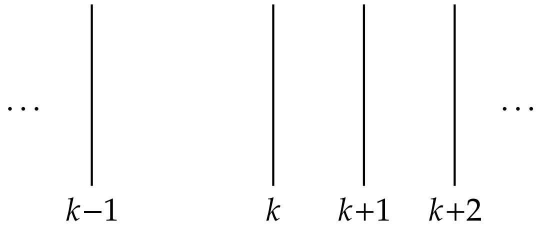
THEOREM 4.3.3. *Fix a subdivision datum* $(e, I, \Lambda, \alpha, \boldsymbol{\kappa}, k)$*, and let* $\boldsymbol{\lambda} \in \mathscr{P}_\alpha^{\boldsymbol{\kappa}}$ *be a* k*-horizontal multipartition. Then*

$$\Phi_k(e_{\boldsymbol{\lambda}}) = e_{\Phi_k(\boldsymbol{\lambda})} + \mathfrak{e}\mathfrak{J}\mathfrak{e}.$$

Proof. The definition of e_λ and the subdivision maps Φ_k on partitions and KLR algebras are all defined componentwise. Therefore, it suffices to prove the level-one case. By Definition 4.2.5, we only need to consider the local pattern of k-nodes as all other nodes are just relabeled in a compatible way. As λ is k-horizontal, k-nodes only appears in maximal $(k, k+1)$-strips in $[\lambda]$. Consider the idempotent string diagram e_λ, where locally we have the pattern:



Under the subdivision, it is mapped to:



Translating this back to the partition, by Definition 4.2.5, this corresponds to: the $(k, k+1)$-strip becoming the $(k, k+1, k+2)$-strip. Since the other strings only change their labels accordingly, the idempotent string diagram corresponds to $\Phi_k(\lambda)$. Hence, we obtain $\Phi_k(e_\lambda) = e_{\Phi_k(\lambda)} + \mathfrak{e}\mathfrak{J}\mathfrak{e}$. □

4.3.2. Subdivision of Permutation Modules.

In this section, we present our first categorical result on subdivision. We first introduce a new map on the set of partitions (and extend it naturally to multipartitions), inspired by the level-up phenomenon in [**Qin25**]: the Specht modules appearing in the generalized Specht filtration of a permutation module M^λ, where λ is a partition, are typically indexed by multipartitions.

Throughout this subsection, we fix a subdivision datum $(e, I, \Lambda_x, \alpha, x, k)$, unless stated otherwise.

Take a partition $\lambda = (\lambda_1, \dots, \lambda_{\ell(\lambda)}) \in \mathscr{P}_\alpha^{\Lambda_x}$. As in subsubsection 4.2.3.2, let $s = k(\lambda)$ be the number of non-trivial maximal $(k+1, k)$-strips in $[\lambda]$, and list them as $S_1, \dots, S_s$ so that the row containing the initial node of S_i, written $k_i := k_{S_i}$, satisfies

$$k_1 < k_2 < \cdots < k_s.$$

For convenience set $k_0 := 0$ and $k_{s+1} := \ell(\lambda)$. Note that $(k_i, 1) \in [\lambda]$ is a $(k+1)$-node for each $1 \le i \le s$, and $(k_i + 1, 1) \in [\lambda]$ is a k-node (in the first column) for each $1 \le i \le s$.

Set $\boldsymbol{\kappa} = (x, \underbrace{k, \cdots, k}_{s \text{ times}}) \in I^{s+1}$. Define the *splitting map*

$$\Psi_k : \mathscr{P}_\alpha^{\Lambda_x} \longrightarrow \mathscr{P}_\alpha^{\boldsymbol{\kappa}}$$

by

$$\Psi_k(\lambda) = (\lambda^0 \mid \cdots \mid \lambda^s), \qquad \lambda^i := (\lambda_{k_i+1}, \lambda_{k_i+2}, \ldots, \lambda_{k_{i+1}}) \quad (0 \le i \le s).$$

Equivalently, $\Psi_k(\lambda)$ is obtained by cutting the Young diagram $[\lambda]$ into consecutive blocks of rows, with cuts immediately below rows $k_1, \ldots, k_s$, and regarding each block as a separate component.

The map Ψ_k is well-defined because, with the fixed choice $\boldsymbol{\kappa}$, the natural identification of nodes between $[\lambda]$ and $[\Psi_k(\lambda)]$ preserves residues. Indeed, Ψ_k only cuts $[\lambda]$ into consecutive blocks of rows and reinterprets each block as a separate component, without changing the column positions of any nodes. The 0th component inherits charge x, which matches the residue of the first node of $[\lambda]$. For each $1 \le i \le s$, the first node of the ith component corresponds to the first node of row $k_i + 1$ of $[\lambda]$, and by construction this is a k-node; assigning charge k to that component therefore ensures that residues in this component agree with the residues of the corresponding nodes in $[\lambda]$. Consequently every node keeps the same residue under the identification, so the residue content is preserved and $\Psi_k(\lambda) \in \mathscr{P}_\alpha^{\boldsymbol{\kappa}}$.

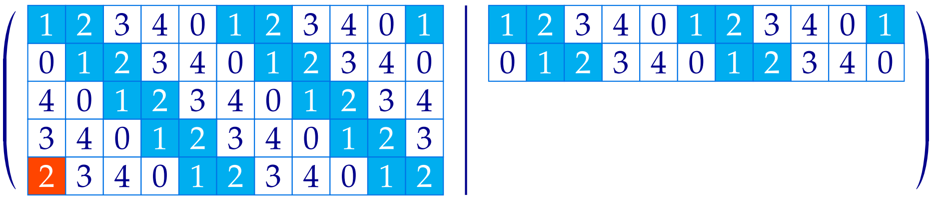
Take a charge $\varkappa = (\varkappa_1, \cdots, \varkappa_\ell) \in I^\ell$ and consider $\boldsymbol{\lambda} \in \mathscr{P}_\alpha^{\varkappa}$. We define $\Psi_k(\boldsymbol{\lambda})$ componentwise: if $\boldsymbol{\lambda} = (\boldsymbol{\lambda}^{(1)}, \cdots, \boldsymbol{\lambda}^{(\ell)})$, then we set

$$\Psi_k(\boldsymbol{\lambda}) = \left(\Psi_k\left(\boldsymbol{\lambda}^{(1)}\right), \cdots, \Psi_k\left(\boldsymbol{\lambda}^{(\ell)}\right)\right)$$

If the number of non-trivial maximal $(k+1, k)$-strips in $\boldsymbol{\lambda}^{(i)}$ is s_i for $1 \le i \le \ell$, then it is straightforward to check that $\Psi_k(\boldsymbol{\lambda}) \in \mathscr{P}_\alpha^{\boldsymbol{\varkappa}}$, where

$$\boldsymbol{\varkappa} = (\varkappa_1, \underbrace{k, \cdots, k}_{s_1 \text{ times}}, \varkappa_2, \underbrace{k, \cdots, k}_{s_2 \text{ times}}, \cdots, \varkappa_\ell, \underbrace{k, \cdots, k}_{s_\ell \text{ times}}) \in I^{s+\ell}, \quad s = \sum_{1 \le i \le \ell} s_i.$$

Example 4.3.4. *Again consider Example 4.2.3, the image $\Psi_k(\lambda)$ has the following Young diagram:*



$$\left(\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&2&3&4&0&1&2&3&4&0&1\\\hline 0&1&2&3&4&0&1&2&3&4&0\\\hline 4&0&1&2&3&4&0&1&2&3&4\\\hline 3&4&0&1&2&3&4&0&1&2&3\\\hline 2&3&4&0&1&2&3&4&0&1&2\\\hline\end{array}\;\middle|\;\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&2&3&4&0&1&2&3&4&0&1\\\hline 0&1&2&3&4&0&1&2&3&4&0\\\hline\end{array}\right)$$

◇

There is a special case in which the splitting map does nothing.

Lemma 4.3.5. *Take a k-horizontal partition $\lambda \in \mathscr{P}_\alpha^{\Lambda_x}$. Then $\Psi_k(\lambda) = \lambda$.*

Proof. This follows directly from the definition. □

As we can observe in Example 4.3.4, the advantage of the splitting map is the following:

Lemma 4.3.6. *Take any partition $\lambda \in \mathscr{P}_\alpha^{\Lambda_x}$. Then $\Psi_k(\lambda)$ is k-horizontal.*

Proof. This is obvious by construction. □

Corollary 4.3.7. *Take a partition $\lambda \in \mathscr{P}_\alpha^{\Lambda_x}$ and set $\boldsymbol{\mu} = \Phi_k\big(\Psi_k(\lambda)\big)$. Then for each k-node $(m, r, c) \in [\boldsymbol{\mu}]$, we have $(m, r, c+1) \in [\boldsymbol{\mu}]$, and this node has residue $k+1$. In particular, $\mathbf{i}^{\mu} \in \overline{I}^{\overline{\alpha}}_{ord}$.*

Proof. By Lemma 4.3.6, $\Psi_k(\lambda)$ is k-horizontal. Hence, when apply Φ_k, a k-node A is replaced by a horizontal pair $(A^+, A^\sharp)$ where A^+ is a k-node and $A^\sharp$ is a $(k+1)$-node. □

Another direct consequence of Lemma 4.3.6 is the following.

Corollary 4.3.8. *Take a partition $\lambda \in \mathscr{P}_\alpha^{\Lambda_x}$. Then*

$$\Phi_k\big(e_{\Psi_k(\lambda)}\big) = e_{\Phi_k\left(\Psi_k(\lambda)\right)} + \mathfrak{e}\mathfrak{J}\mathfrak{e}. \qquad \square$$

Our main theorem in this section is the following.

Theorem 4.3.9. *Fix a subdivision datum $(e, I, \Lambda, \alpha, \boldsymbol{\kappa}, k)$ and take $\boldsymbol{\lambda} \in \mathscr{P}_\alpha^{\Lambda}$. Then there is an isomorphism of graded R_α-modules*

$$M^{\Psi_k(\lambda)} \;\cong\; \mathfrak{e}M^{\Phi_k\left(\Psi_k(\lambda)\right)} \big/ \mathfrak{e}\mathfrak{J}\mathfrak{e}\, M^{\Phi_k\left(\Psi_k(\lambda)\right)}.$$

Proof. Let $\boldsymbol{\mu} = \Psi_k(\boldsymbol{\lambda})$ and $\boldsymbol{\nu} = \Phi_k\big(\Psi_k(\boldsymbol{\lambda})\big)$, and set

$$N^{\boldsymbol{\nu}} := \mathfrak{e}M^{\boldsymbol{\nu}} \big/ \mathfrak{e}\mathfrak{J}\mathfrak{e}\, M^{\boldsymbol{\nu}}.$$

Let $\mathrm{ht}(\alpha) = d$. Let $T^{\boldsymbol{\mu}}$ and $T^{\boldsymbol{\nu}}$ be the initial tableaux of shapes $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$, respectively, and let $\mathbf{i}^{\boldsymbol{\mu}}$ and $\mathbf{i}^{\boldsymbol{\nu}}$ be the corresponding residue sequences. As usual, let $\overline{I}$ and $\overline{\alpha}$ be the images of I and α under the corresponding subdivision maps Φ_k. Let $d' = \mathrm{ht}(\overline{\alpha})$.

For $\boldsymbol{\mu}$ and α, set $K_\alpha^{\boldsymbol{\mu}}$ to be the ideal of R_α generated by the following set of elements:

$$\{e(\mathbf{i}) - \delta_{\mathbf{i},\mathbf{i}^{\mu}}, y_j, \psi_t \mid \mathbf{i} \in I^\alpha, 1 \le j \le d, t \rightarrow_{T^{\mu}} t+1\} \tag{4.3.10}$$

then $M^{\boldsymbol{\mu}} \cong \big(R_\alpha / K_\alpha^{\boldsymbol{\mu}}\big)\langle \deg T^{\boldsymbol{\mu}} \rangle$ induced by the canonical projection $R_\alpha \twoheadrightarrow M^{\boldsymbol{\mu}}, r \mapsto r \cdot m^{\boldsymbol{\mu}}$.

Similarly, for $\boldsymbol{\nu}$ and $\overline{\alpha}$, set $K_{\overline{\alpha}}^{\boldsymbol{\nu}}$ to be the ideal of $R_{\overline{\alpha}}$ generated by the following set of elements:

$$\{e(\mathbf{j}) - \delta_{\mathbf{j},\mathbf{i}^{\nu}}, y_j, \psi_t \mid \mathbf{j} \in I^{\overline{\alpha}}, 1 \le j \le d', t \rightarrow_{T^{\nu}} t+1\}$$

then $M^{\boldsymbol{\nu}} \cong \big(R_{\overline{\alpha}} / K_{\overline{\alpha}}^{\boldsymbol{\nu}}\big)\langle \deg T^{\boldsymbol{\nu}} \rangle$ induced by the canonical projection $R_{\overline{\alpha}} \twoheadrightarrow M^{\boldsymbol{\nu}}, r \mapsto r \cdot m^{\boldsymbol{\nu}}$.

Let $\mathfrak{e} \in R_{\overline{\alpha}}$ be the truncation idempotent, defined in (4.1.15). Then it follows that

$$\big(\mathfrak{e}R_{\overline{\alpha}} / \mathfrak{e}K_{\overline{\alpha}}^{\boldsymbol{\nu}}\big)\langle \deg T^{\boldsymbol{\nu}} \rangle \cong \mathfrak{e}M^{\boldsymbol{\nu}}.$$

which is induced from the canonical projection $\mathfrak{e}R_{\overline{\alpha}} \twoheadrightarrow \mathfrak{e}M^{\boldsymbol{\nu}}, \mathfrak{e}r \mapsto \mathfrak{e}r \cdot m^{\boldsymbol{\nu}}$.

Since $\mathbf{i}^{\nu} \in \overline{I}^{\overline{\alpha}}_{\mathrm{ord}}$ by Corollary 4.3.7, we know $\mathfrak{e}m^{\nu} = m^{\nu}$. Hence the map $\mathfrak{e}R_{\overline{\alpha}}\mathfrak{e} \longrightarrow \mathfrak{e}M^{\nu}$ given by $\mathfrak{e}r\mathfrak{e} \mapsto \mathfrak{e}r\mathfrak{e} \cdot \mathfrak{e}m^{\nu}$ is surjective as $\mathfrak{e}r\mathfrak{e} \cdot \mathfrak{e}m^{\nu} = \mathfrak{e}r\mathfrak{e}m^{\nu} = \mathfrak{e}rm^{\nu}$. Therefore, it follows that:

$$\big(\mathfrak{e}R_{\overline{\alpha}}\mathfrak{e}/\mathfrak{e}K^{\nu}_{\overline{\alpha}}\mathfrak{e}\big)\langle \deg T^{\nu}\rangle \cong \mathfrak{e}M^{\nu}.$$

Equivalently, $\mathfrak{e}M^{\nu}$ is a cyclic $\mathfrak{e}R_{\overline{\alpha}}\mathfrak{e}$-module with cyclic generator $\mathfrak{e}m^{\nu}$ annihilated by

$$\{\mathfrak{e}\big(e(\mathbf{j}) - \delta_{\mathbf{j},\mathbf{i}^{\nu}}\big)\mathfrak{e}, \mathfrak{e}y_j\mathfrak{e}, \mathfrak{e}\psi_t\mathfrak{e} \mid \mathbf{j} \in I^{\overline{\alpha}}, 1 \le j \le d', t \rightarrow_{T^{\nu}} t+1\}$$

By definition of $\mathfrak{e}$, it can be simplified as the following:

$$\{e(\mathbf{j}) - \delta_{\mathbf{j},\mathbf{i}^{\nu}}\mathfrak{e}, \mathfrak{e}y_j\mathfrak{e}, \mathfrak{e}\psi_t\mathfrak{e} \mid \mathbf{j} \in \overline{I}^{\overline{\alpha}}_{\mathrm{ord}}, 1 \le j \le d', t \rightarrow_{T^{\nu}} t+1\}$$

Let $S_{\overline{\alpha}}$ be the balanced KLR algebra $\mathfrak{e}R_{\overline{\alpha}}\mathfrak{e}/\mathfrak{e}\mathfrak{J}\mathfrak{e}$ which is isomorphic to R_{α} by the subdivision map, see Theorem 4.1.19. The module N^{ν} is naturally a cyclic $S_{\overline{\alpha}}$-module with cyclic generator $w^{\nu} := \mathfrak{e}m^{\nu} + \mathfrak{e}\mathfrak{J}\mathfrak{e}$, which has the same degree as m^{ν}. Moreover, we have:

$$\begin{aligned}
\mathfrak{e}M^{\nu}/\mathfrak{e}\mathfrak{J}\mathfrak{e}M^{\nu} &\cong \Big(\big(\mathfrak{e}R_{\overline{\alpha}}\mathfrak{e}/\mathfrak{e}K^{\nu}_{\overline{\alpha}}\mathfrak{e}\big)/\big(\mathfrak{e}\mathfrak{J}\mathfrak{e}(\mathfrak{e}R_{\overline{\alpha}}\mathfrak{e}/\mathfrak{e}K^{\nu}_{\overline{\alpha}}\mathfrak{e})\big)\Big)\langle \deg T^{\nu}\rangle \\
&\cong \Big(\big(\mathfrak{e}R_{\overline{\alpha}}\mathfrak{e}/\mathfrak{e}K^{\nu}_{\overline{\alpha}}\mathfrak{e}\big)/\big((\mathfrak{e}\mathfrak{J}\mathfrak{e} + \mathfrak{e}K^{\nu}_{\overline{\alpha}}\mathfrak{e})/\mathfrak{e}K^{\nu}_{\overline{\alpha}}\mathfrak{e}\big)\Big)\langle \deg T^{\nu}\rangle \\
&\cong \Big(\mathfrak{e}R_{\overline{\alpha}}\mathfrak{e}/(\mathfrak{e}\mathfrak{J}\mathfrak{e} + \mathfrak{e}K^{\nu}_{\overline{\alpha}}\mathfrak{e})\Big)\langle \deg T^{\nu}\rangle \\
&\cong \Big(\big(\mathfrak{e}R_{\overline{\alpha}}\mathfrak{e}/\mathfrak{e}\mathfrak{J}\mathfrak{e}\big)/\big((\mathfrak{e}\mathfrak{J}\mathfrak{e} + \mathfrak{e}K^{\nu}_{\overline{\alpha}}\mathfrak{e})/\mathfrak{e}\mathfrak{J}\mathfrak{e}\big)\Big)\langle \deg T^{\nu}\rangle \\
&= \Big(S_{\overline{\alpha}}/\big((\mathfrak{e}\mathfrak{J}\mathfrak{e} + \mathfrak{e}K^{\nu}_{\overline{\alpha}}\mathfrak{e})/\mathfrak{e}\mathfrak{J}\mathfrak{e}\big)\Big)\langle \deg T^{\nu}\rangle
\end{aligned}$$

Set $\overline{K}^{\nu}_{\overline{\alpha}} := (\mathfrak{e}\mathfrak{J}\mathfrak{e} + \mathfrak{e}K^{\nu}_{\overline{\alpha}}\mathfrak{e})/\mathfrak{e}\mathfrak{J}\mathfrak{e}$. Then it is generated by the following set of elements:

$$\{e(\mathbf{j}) - \delta_{\mathbf{j},\mathbf{i}^{\nu}}\mathfrak{e} + \mathfrak{e}\mathfrak{J}\mathfrak{e}, \mathfrak{e}y_j\mathfrak{e} + \mathfrak{e}\mathfrak{J}\mathfrak{e}, \mathfrak{e}\psi_t\mathfrak{e} + \mathfrak{e}\mathfrak{J}\mathfrak{e} \mid \mathbf{j} \in \overline{I}^{\overline{\alpha}}_{\mathrm{ord}}, 1 \le j \le d', t \rightarrow_{T^{\nu}} t+1\} \tag{4.3.11}$$

and N^{ν} is the $S_{\overline{\alpha}}$-module with cyclic generator w^{ν} annihilated by this set of elements.

We construct two maps:

$$f : R_{\alpha} \longrightarrow N^{\nu}, \qquad x \mapsto \Phi_k(x)w^{\nu}$$

and

$$g : S_{\overline{\alpha}} \longrightarrow M^{\mu}, \qquad y \mapsto \Phi_k^{-1}(y)m^{\mu}$$

We want to show that

$$f(K^{\mu}_{\alpha}) = 0, \qquad g(\overline{K}^{\nu}_{\overline{\alpha}}) = 0 \tag{4.3.12}$$

If this is true, then f and g induce surjective maps:

$$\overline{f} : M^{\mu} \twoheadrightarrow N^{\nu}, \quad xm^{\mu} \mapsto \Phi_k(x)w^{\nu}$$

and

$$\overline{g} : N^{\nu} \twoheadrightarrow M^{\mu}, \quad yw^{\nu} \mapsto \Phi_k^{-1}(y)m^{\mu}.$$

By construction, we have $\overline{f}\circ\overline{g}=\mathrm{id}$ and $\overline{g}\circ\overline{f}=\mathrm{id}$. Moreover, by Theorem 4.2.53, $\deg T^{\mu}=\deg T^{\nu}$. Hence $\overline{f}$ is a degree-0 homogeneous isomorphism of R_{α}-modules between M^{μ} and N^{ν}.

To prove the (4.3.12), we proceed case by case, according to the type of generators in (4.3.10) and (4.3.11).

Verification of $e(\mathbf{i})$*-relations.* For $\mathbf{i}\in I^{\alpha}$, we have

$$
\begin{aligned}
f\left(e(\mathbf{i})-\delta_{\mathbf{i},\mathbf{i}^{\mu}}\right) &= \Phi_k\left(e(\mathbf{i})-\delta_{\mathbf{i},\mathbf{i}^{\mu}}\right)w^{\nu}\\
&= \Big(e\left(\Phi_k(\mathbf{i})\right)-\delta_{\mathbf{i},\mathbf{i}^{\mu}}\Phi_k(1)+\mathfrak{e}\mathfrak{J}\mathfrak{e}\Big)w^{\nu}\\
&= \Big(e(\bar{\mathbf{i}})-\delta_{\mathbf{i},\mathbf{i}^{\mu}}\mathfrak{e}+\mathfrak{e}\mathfrak{J}\mathfrak{e}\Big)w^{\nu}\\
&= \Big(e(\bar{\mathbf{i}})-\delta_{\bar{\mathbf{i}},\mathbf{i}^{\nu}}\mathfrak{e}+\mathfrak{e}\mathfrak{J}\mathfrak{e}\Big)w^{\nu}=0
\end{aligned}
$$

where the second equality follows from Corollary 4.3.8, the third equality follows from the isomorphism $\Phi_k : R_{\alpha}\longrightarrow S_{\overline{\alpha}}$ sending 1 to $\mathfrak{e}$ and the notation $\bar{\mathbf{i}}:=\Phi_k(\mathbf{i})$, and the fourth equality follows from (4.1.7): the map $\mathbf{i}\mapsto\bar{\mathbf{i}}$ gives a bijection $I^{\alpha}\longrightarrow\overline{I}^{\overline{\alpha}}_{\mathrm{ord}}$, and, by construction in Subsection 4.2.4, $\Phi_k(T^{\mu})=T^{\nu}$; hence $\bar{\mathbf{i}}=\mathbf{i}^{\nu}$ if and only if $\mathbf{i}=\mathbf{i}^{\mu}$.

Conversely, for $\mathbf{j}\in\overline{I}^{\overline{\alpha}}_{\mathrm{ord}}$, again by (4.1.7), there exists unique $\mathbf{i}\in I^{\alpha}$ such that $\Phi_k(\mathbf{i})=\mathbf{j}$. Hence, reversing the argument above, we have:

$$
\begin{aligned}
g\Big(e(\mathbf{j})-\delta_{\mathbf{j},\mathbf{i}^{\nu}}\mathfrak{e}+\mathfrak{e}\mathfrak{J}\mathfrak{e}\Big) &= \Big(\Phi_k^{-1}\left(e(\mathbf{j})+\mathfrak{e}\mathfrak{J}\mathfrak{e}\right)-\delta_{\mathbf{j},\mathbf{i}^{\nu}}\Phi_k^{-1}\left(\mathfrak{e}+\mathfrak{e}\mathfrak{J}\mathfrak{e}\right)\Big)m^{\mu}\\
&= \Big(e\left(\Phi_k^{-1}(\mathbf{j})\right)-\delta_{\mathbf{j},\mathbf{i}^{\nu}}\Big)m^{\mu}\\
&= \Big(e(\mathbf{i})-\delta_{\mathbf{j},\mathbf{i}^{\nu}}\Big)m^{\mu}\\
&= \Big(e(\mathbf{i})-\delta_{\mathbf{i},\mathbf{i}^{\mu}}\Big)m^{\mu}=0
\end{aligned}
$$

Verification of ψ*-relations.* Let $\phi:=\phi_{\mathbf{i}^{\mu}}$ be the position-tracing function defined in Definition 4.1.8. For $1\le t\le d-1$ such that $t\longrightarrow_{T^{\mu}}t+1$, we have

$$
f(\psi_t)=\Phi_k(\psi_t)w^{\nu}=\left(\mathfrak{e}\psi_{\phi(t)}\mathfrak{e}+\mathfrak{e}\mathfrak{J}\mathfrak{e}\right)w^{\nu}
$$

Let $A,B\in[\boldsymbol{\mu}]$ be the two nodes such that $T^{\mu}(A)=t$ and $T^{\mu}(B)=t+1$. By our construction of subdivision on standard tableaux, applied to T^{μ}, we have the following two cases.

- If $\operatorname{res}(A)\neq k$, then A^{+} and B^{+} are horizontally adjacent nodes in $[\boldsymbol{\nu}]$ such that $T^{\nu}(A^{+})=\phi(t)$ and $T^{\nu}(B^{+})=\phi(t+1)=\phi(t)+1$.
- If $\operatorname{res}(A)=k$, then A^{+} and $A^{\sharp}$ are the corresponding horizontally adjacent nodes in $[\boldsymbol{\nu}]$ such that $T^{\nu}(A^{+})=\phi(t)$ and $T^{\nu}(A^{\sharp})=\phi(t)+1$.

Hence, in any case, $\mathfrak{e}\psi_{\phi(t)}\mathfrak{e}+\mathfrak{e}\mathfrak{J}\mathfrak{e}\in\overline{K}^{\boldsymbol{\nu}}_{\overline{\alpha}}$ and $f(\psi_t)=0$.

Conversely, for $t \longrightarrow_{T^{\boldsymbol\nu}} t+1$. There are three cases:

- If $\mathrm{res}_{T^{\boldsymbol\nu}}(t) = k$, then there exists $A \in [\boldsymbol\mu]$ such that $T^{\boldsymbol\nu}(A^+) = t$ and $T^{\boldsymbol\nu}(A^\sharp) = t+1$. However, in this case, since $\mathrm{res}(A^\sharp) = k+1$, we have $\psi_t e(\mathbf{i}^{\boldsymbol\nu}) = e\big(\sigma_t(\mathbf{i}^{\boldsymbol\nu})\big)\psi_t$ and $\sigma_t(\mathbf{i}^{\boldsymbol\nu}) \in \overline{I}^{\overline{\alpha}}_{\mathrm{un}}$. Hence $\mathfrak{e}\psi_t\mathfrak{e} \in \mathfrak{e}\mathfrak{J}\mathfrak{e}$.
- If $\mathrm{res}_{T^{\boldsymbol\nu}}(t) = k+1$, then there exist two horizontally adjacent nodes $A, B \in [\boldsymbol\mu]$ such that $T^{\boldsymbol\nu}(A^\sharp) = t$ and $T^{\boldsymbol\nu}(B^+) = t+1$. Set $t' := \phi^{-1}(t+1)$; then $T^{\boldsymbol\mu}(A) = t'-1$ and $T^{\boldsymbol\mu}(B) = t'$, i.e. $(t'-1) \longrightarrow_{T^{\boldsymbol\mu}} t'$. Hence we have:
$$g(\mathfrak{e}\psi_t\mathfrak{e} + \mathfrak{e}\mathfrak{J}\mathfrak{e}) = \Phi_k^{-1}\big(\mathfrak{e}\psi_t\mathfrak{e} + \mathfrak{e}\mathfrak{J}\mathfrak{e}\big)m^{\boldsymbol\mu} = \psi_{t'-1}m^{\boldsymbol\mu} = 0.$$
- If $\mathrm{res}_{T^{\boldsymbol\nu}}(t) \neq k, k+1$, then there exist two horizontally adjacent nodes $A, B \in [\boldsymbol\mu]$ such that $T^{\boldsymbol\nu}(A^+) = t$ and $T^{\boldsymbol\nu}(B^+) = t+1$. Let $t' = \phi^{-1}(t)$; then $T^{\boldsymbol\mu}(A) = t'$ and $T^{\boldsymbol\mu}(B) = \phi^{-1}(t+1) = t'+1$, i.e. $t' \longrightarrow_{T^{\boldsymbol\mu}} (t'+1)$. Hence we have:
$$g(\mathfrak{e}\psi_t\mathfrak{e} + \mathfrak{e}\mathfrak{J}\mathfrak{e}) = \Phi_k^{-1}\big(\mathfrak{e}\psi_t\mathfrak{e} + \mathfrak{e}\mathfrak{J}\mathfrak{e}\big)m^{\boldsymbol\mu} = \psi_{t'}m^{\boldsymbol\mu} = 0.$$

So, in any case, we have $g(\mathfrak{e}\psi_t\mathfrak{e} + \mathfrak{e}\mathfrak{J}\mathfrak{e}) = 0$.

Verification of y-relations. Let $\phi := \phi_{\mathbf{i}^{\boldsymbol\mu}}$ be the position-tracing function defined in Definition 4.1.8. Take $1 \le j \le d$, then
$$f(y_j) = \Phi_k(y_j)w^{\boldsymbol\nu} = \big(\mathfrak{e}y_{\phi(j)}\mathfrak{e} + \mathfrak{e}\mathfrak{J}\mathfrak{e}\big)w^{\boldsymbol\nu} = 0$$

Conversely, take $1 \le j \le d'$. If there exists $1 \le j' \le d$ such that $\phi(j') = j$, then
$$g(\mathfrak{e}y_j\mathfrak{e} + \mathfrak{e}\mathfrak{J}\mathfrak{e}) = \Phi_k^{-1}\big(\mathfrak{e}y_j\mathfrak{e} + \mathfrak{e}\mathfrak{J}\mathfrak{e}\big)m^{\boldsymbol\mu} = y_{j'}m^{\boldsymbol\mu} = 0$$
Else, there exists some k-node $A \in [\boldsymbol\mu]$ such that $T^{\boldsymbol\nu}(A^\sharp) = j$ and $T^{\boldsymbol\nu}(A^+) = j-1$. Then
$$0+\mathfrak{e}\mathfrak{J}\mathfrak{e} = \mathfrak{e}\psi_{j-1}^2\mathfrak{e}+\mathfrak{e}\mathfrak{J}\mathfrak{e} = \mathfrak{e}\psi_{j-1}^2 e(\mathbf{i}^{\boldsymbol\nu})+\mathfrak{e}\mathfrak{J}\mathfrak{e} = \mathfrak{e}Q_{k,k+1}(y_{j-1},y_j)e(\mathbf{i}^{\boldsymbol\nu})+\mathfrak{e}\mathfrak{J}\mathfrak{e} = \mathfrak{e}(y_j-y_{j-1})\mathfrak{e}+\mathfrak{e}\mathfrak{J}\mathfrak{e}$$
where the first equality follows from the ψ-relations check in the last step, the third equality follows from (1.3.10) in Definition 1.3.1, and we use $e(\mathbf{i}^{\boldsymbol\nu}) + \mathfrak{e}\mathfrak{J}\mathfrak{e} = \mathfrak{e} + \mathfrak{e}\mathfrak{J}\mathfrak{e}$ in $S_{\overline{\alpha}}$, which follows from the presentation (4.3.11). Hence
$$g\big(\mathfrak{e}y_j\mathfrak{e} + \mathfrak{e}\mathfrak{J}\mathfrak{e}\big) = g\big(\mathfrak{e}y_{j-1}\mathfrak{e} + \mathfrak{e}\mathfrak{J}\mathfrak{e}\big) = 0.$$

After verifying the three types of generators, we have proved that (4.3.10) and (4.3.11) correspond to each other under f and g, which completes the proof. □

Corollary 4.3.13. *Fix a subdivision datum $(e, I, \Lambda, \alpha, \boldsymbol\kappa, k)$ and take $\boldsymbol\lambda \in \mathscr{P}^{\Lambda}_{\alpha}$. Then*
$$M^{\boldsymbol\lambda} \cong \mathfrak{e}M^{\Phi_k\left(\Psi_k(\boldsymbol\lambda)\right)}/\mathfrak{e}\mathfrak{J}\mathfrak{e}M^{\Phi_k\left(\Psi_k(\boldsymbol\lambda)\right)}$$

Proof. This follows from Corollary 1.3.31, which gives $M^{\boldsymbol\lambda} \cong M^{\Psi_k(\boldsymbol\lambda)}$, together with Theorem 4.3.9. □

4.3.3. Subdivision of Specht Modules. We extend Theorem 4.3.9 to the Specht modules by checking that the Garnir relations are preserved, thereby extending Theorem 4.3.3.

We follow the definitions and notation from Subsection 4.2.4, such as old nodes A^+ and new nodes $A^\sharp$.

Lemma 4.3.14. *Fix a subdivision datum* $(e, I, \Lambda, \alpha, \boldsymbol{\kappa}, k)$*, and let* $\boldsymbol{\lambda} \in \mathscr{P}^{\boldsymbol{\kappa}}_{\alpha}$*. Let* $\phi := \phi_{\mathbf{i}^{\boldsymbol{\lambda}}}$ *be the position-tracing function associated with* $(k, \mathbf{i}^{\boldsymbol{\lambda}})$*. Set* $\boldsymbol{\mu} := \Phi_k(\boldsymbol{\lambda})$*, and let* $T^{\boldsymbol{\lambda}}$ *and* $T^{\boldsymbol{\mu}}$ *be the corresponding initial tableaux. If* $[\boldsymbol{\lambda}]$ *is* k*-horizontal, then for each* $1 \le t \le \operatorname{ht}(\alpha)$*,* $A = (T^{\boldsymbol{\lambda}})^{-1}(t)$ *is a Garnir node of* $[\boldsymbol{\lambda}]$ *if and only if* $A^+ = (T^{\boldsymbol{\mu}})^{-1}(\phi(t))$ *is a Garnir node of* $[\boldsymbol{\mu}]$*.*

Proof. Since Φ_k is defined componentwise, it suffices to treat the case of an ordinary partition. Let λ be a partition and set $\mu = \Phi_k(\lambda)$. By the above arguments, for the node $A = (a, c) = (T^\lambda)^{-1}(t) \in [\lambda]$, the corresponding node A^+ in $[\mu]$ is

$$A^+ = (a,\ \phi(t) - \phi(t_a) + c). \tag{4.3.15}$$

Let $B := (a+1, c)$ and, if $B \in [\lambda]$, set $s := T^\lambda(B)$, so that $t \downarrow_{T^\lambda} s$. We show that

$$B \in [\lambda] \quad \Longleftrightarrow \quad \text{there exists a node } B' \in [\mu] \text{ with } A^+ \downarrow_{T^\mu} B'. \tag{4.3.16}$$

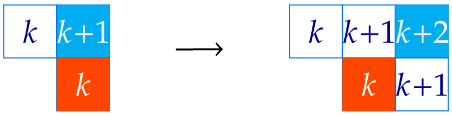
Since A is a Garnir node of $[\lambda]$ if and only if $B \in [\lambda]$, and A^+ is a Garnir node of $[\mu]$ if and only if there exists B' with $A^+ \downarrow_{T^\mu} B'$, this will prove the lemma.

(⇒). Assume $B \in [\lambda]$ and keep the notation above. Let B^+ be the *old* node of $[\mu]$ corresponding to B, i.e. the unique node with $T^\mu(B^+) = \phi(s)$. As in (4.3.15), if $t_{a+1} := 1 + \sum_{1 \le i \le a} \lambda_i$ is the first entry of row $a+1$, then

$$B^+ = (a+1,\ \phi(s) - \phi(t_{a+1}) + c). \tag{4.3.17}$$

We distinguish two cases.

Case 1: $\operatorname{res}_{T^\lambda}(t) = k+1$. Then $\operatorname{res}_{T^\lambda}(s) = k$. In this situation, B is a k-node and, because $k(\lambda) = 0$, the subdivision rule for maximal $(k, k+1)$-strips inserts exactly one new node in row $a+1$ immediately to the *right* of the old node B^+ (this is the new node carrying the label $\phi(s) + 1$, which by definition is not in the image of ϕ).



Moreover, comparing the horizontal shifts in rows a and $a+1$ up to column c, we have

$$\phi(t) - \phi(t_a) = \phi(s) - \phi(t_{a+1}) + 1,$$

so (4.3.15)–(4.3.17) give that B^+ lies one column to the *left* of A^+. Therefore the newly inserted node immediately to the right of B^+ lies in the *same column* as A^+ and is in row $a+1$. Denote this node by B^{++}; then $A^+ \downarrow_{T^\mu} B^{++}$.

Case 2: $\operatorname{res}_{T^\lambda}(t) \ne k+1$. In this case, no new node is inserted between the images of A and B in the relevant column, and the horizontal shifts in rows a and $a+1$ up to column c coincide:

$$\phi(t) - \phi(t_a) = \phi(s) - \phi(t_{a+1}).$$

Hence (4.3.15)–(4.3.17) imply that B^+ lies directly below A^+ in the same column, so $A^+ \downarrow_{T^\mu} B^+$.

Thus in either case there exists $B' \in [\mu]$ with $A^+ \downarrow_{T^\mu} B'$, proving the forward implication of (4.3.16).

($\Leftarrow$). Conversely, assume that there exists $B' \in [\mu]$ with $A^+ \downarrow_{T^\mu} B'$, so B' lies in row $a+1$ and in the same column as A^+.

First suppose that $T^\mu(B') \in \mathrm{im}(\phi)$. Then there exists s such that $T^\mu(B') = \phi(s)$, and hence $B' = (T^\mu)^{-1}(\phi(s)) = B^+$ for the node $B := (T^\lambda)^{-1}(s) \in [\lambda]$. We claim that necessarily $\mathrm{res}_{T^\lambda}(t) \neq k+1$. Indeed, if $\mathrm{res}_{T^\lambda}(t) = k+1$, then by Case 1 above the unique node of $[\mu]$ lying in row $a+1$ and directly below A^+ is the inserted node B^{++}, whose label is $\phi(s)+1 \notin \mathrm{im}(\phi)$; this contradicts $T^\mu(B') \in \mathrm{im}(\phi)$. Therefore $\mathrm{res}_{T^\lambda}(t) \neq k+1$, and then Case 2 applies: the node of $[\mu]$ in row $a+1$ directly below A^+ is exactly B^+, where $B = (a+1, c)$. Since $B' = B^+$, we conclude that $B = (a+1, c) \in [\lambda]$.

Now suppose that $T^\mu(B') \notin \mathrm{im}(\phi)$. By Definition 4.2.5 and the assumption that $k(\lambda) = 0$, labels not in $\mathrm{im}(\phi)$ occur precisely on the nodes inserted in Case 1. In particular, the only way to have a node in row $a+1$ directly below A^+ with label outside $\mathrm{im}(\phi)$ is that we are in Case 1, and then the defining construction of B^{++} forces the existence of $B = (a+1, c) \in [\lambda]$.

Thus in either case the existence of $B' \in [\mu]$ with $A^+ \downarrow_{T^\mu} B'$ implies that $B = (a+1, c) \in [\lambda]$, which completes the reverse implication of (4.3.16).

□

Lemma 4.3.18. *Under the hypothesis of Lemma 4.3.14, let $A \in [\boldsymbol{\lambda}]$ be a Garnir node such that $T^{\boldsymbol{\lambda}}(A) = t$, and let $A^+ \in [\boldsymbol{\mu}]$ be the corresponding Garnir node such that $T^{\boldsymbol{\mu}}(A^+) = \phi(t)$. Let $g^A \in R_\alpha$ and $g^{A^+} \in R_{\overline{\alpha}}$ be the Garnir elements defined in Definition 1.3.25. Then, in the balanced KLR algebra $S_{\overline{\alpha}} = \mathfrak{e}R_{\overline{\alpha}}\mathfrak{e}/\mathfrak{e}\mathfrak{J}\mathfrak{e}$, we have*

$$\Phi_k(g^A) \;=\; \mathfrak{e}\, g^{A^+}\, \mathfrak{e} \;+\; \mathfrak{e}\mathfrak{J}\mathfrak{e}.$$

Proof. By Lemma 4.3.14, A^+ is a Garnir node whenever A is. Write $\mathcal{B}^A$ and $\mathcal{B}^{A^+}$ for the Garnir belts in $[\boldsymbol{\lambda}]$ and $[\boldsymbol{\mu}]$ respectively, and let

$$B_1^A, \dots, B_{k^A}^A \qquad \text{and} \qquad B_1^{A^+}, \dots, B_{k^{A^+}}^{A^+}$$

be the corresponding lists of row-bricks, so that the brick transpositions $w_r^A \in \mathfrak{S}_d$ and $w_r^{A^+} \in \mathfrak{S}_{d'}$ are defined by

$$w_r^A = \prod_{a=0}^{e-1} \big(n_r^A + a,\ n_r^A + e + a\big), \qquad w_r^{A^+} = \prod_{a=0}^{e} \big(n_r^{A^+} + a,\ n_r^{A^+} + e + 1 + a\big),$$

with $n_r^A = \min\{G^A(x) \mid x \in B_r^A\}$ and $n_r^{A^+} = \min\{G^{A^+}(x) \mid x \in B_r^{A^+}\}$.

Let $i := \mathrm{res}_{n_1^A}(G^A)$. Then any row brick B_j^A has residue sequence of one of the following forms:

(a). $\begin{array}{|c|c|c|c|c|c|c|c|c|}\hline i & i{+}1 & \cdots & k & \cdots & e{-}1 & 0 & \cdots & i{-}1 \\ \hline\end{array}$ if $i \leq k$;

(b). $\begin{array}{|c|c|c|c|c|c|c|c|c|}\hline i & i{+}1 & \cdots & e{-}1 & 0 & \cdots & k & \cdots & i{-}1 \\ \hline\end{array}$ if $i > k$.

After applying the subdivision map Φ_k, we obtain the corresponding consecutive nodes in a row of $\mathscr{B}^{A^+}$, with residue sequence

(a). $\begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline i & i{+}1 & \cdots & k & k{+}1 & \cdots & e & 0 & \cdots & i{-}1 \\ \hline\end{array}$ if $i \leq k$;

(b). $\begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline i{+}1 & i{+}2 & \cdots & e & 0 & \cdots & k & k{+}1 & \cdots & i \\ \hline\end{array}$ if $i > k$.

In either case, this is the row brick $B_j^{A^+}$. Thus we obtain a bijection of bricks $B_j^A \leftrightarrow B_j^{A^+}$, and in particular $k^A = k^{A^+}$. Let f^A and f^{A^+} be the numbers of row-bricks in the top rows of the corresponding Garnir belts. Then $f^A = f^{A^+}$ as well. Consequently, we obtain an identification of brick permutation groups

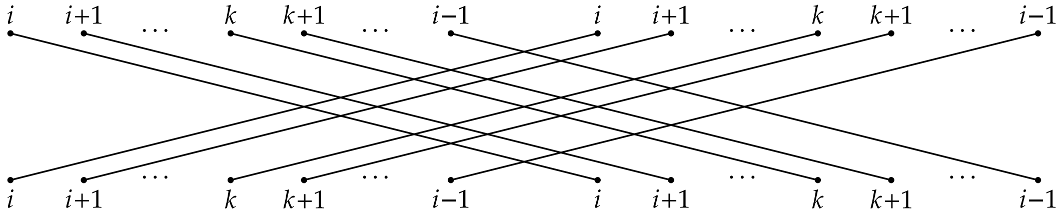
$$S^A = \langle w_1^A, \ldots, w_{k^A-1}^A \rangle \cong \langle w_1^{A^+}, \ldots, w_{k^{A^+}-1}^{A^+} \rangle = S^{A^+},$$

sending $w_r^A \mapsto w_r^{A^+}$. This also sends the minimal coset representatives $\mathscr{D}^A \subseteq S^A$ bijectively onto $\mathscr{D}^{A^+} \subseteq S^{A^+}$. For $u \in \mathscr{D}^A$, write $u^+ \in \mathscr{D}^{A^+}$ for its image under this bijection.

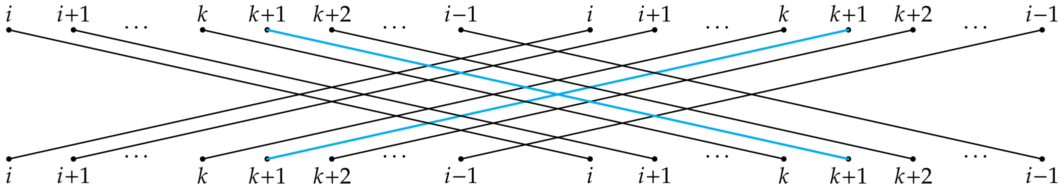
Recall the brick operators in Definition 1.3.23:

$$\sigma_r^A = \psi_{w_r^A}\, e(\mathbf{i}^A), \qquad \tau_r^A = (\sigma_r^A + 1)\, e(\mathbf{i}^A),$$

and similarly for A^+. The diagrammatic presentation of $\psi_{w_r^A}$ is as follows (with all undrawn strands taken to be vertical straight strands):



By the construction of the subdivision map Φ_k (see Subsection 4.1.5), the image corresponds to the following string diagram(with all undrawn strands taken to be vertical straight strands):



This string diagram corresponds to the element $\psi_{w_r^{A^+}} \in R_{\overline{\alpha}}$. Moreover, it is easy to see

$$\Phi_k\big(e(\mathbf{i}^A)\big) = \mathfrak{e}\, e\big(\mathbf{i}^{A^+}\big)\, \mathfrak{e} + \mathfrak{e}\mathfrak{J}\mathfrak{e}.$$

Since Φ_k is an R_α-algebra homomorphism, we have:

$$\Phi_k(\sigma_r^A) \;=\; \mathfrak{e}\,\sigma_r^{A^+}\,\mathfrak{e} \;+\; \mathfrak{e}\mathfrak{J}\mathfrak{e} \qquad \text{in } S_{\overline{\alpha}}.$$

and

$$\Phi_k(\tau_r^A) = \Phi_k\big((\sigma_r^A + 1)e(\mathbf{i}^A)\big) = \big(\Phi_k(\sigma_r^A) + 1\big)\Phi_k\big(e(\mathbf{i}^A)\big) = \mathfrak{e}\,\tau_r^{A^+}\,\mathfrak{e} \;+\; \mathfrak{e}\mathfrak{J}\mathfrak{e}.$$

If $u \in \mathscr{D}^A$ has a reduced expression $u = w_{r_1}^A \cdots w_{r_a}^A$, then

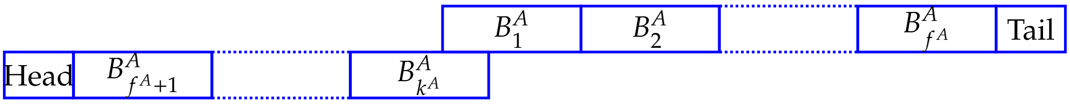
$$\tau_u^A = \tau_{r_1}^A \cdots \tau_{r_a}^A, \qquad \tau_{u^+}^{A^+} = \tau_{r_1}^{A^+} \cdots \tau_{r_a}^{A^+},$$

and hence multiplicativity gives

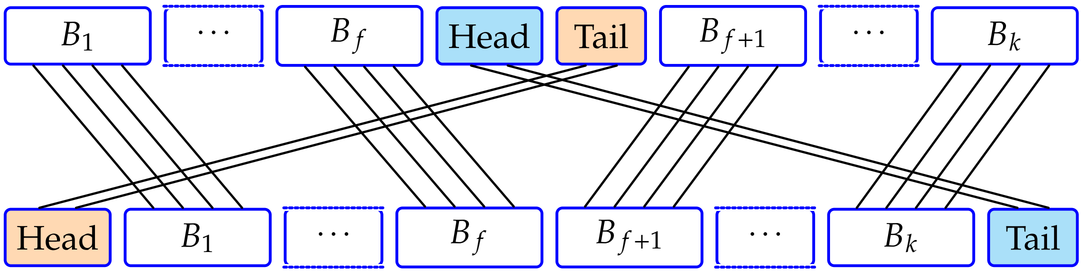
$$\Phi_k(\tau_u^A) = \mathfrak{e}\,\tau_{u^+}^{A^+}\,\mathfrak{e} \;+\; \mathfrak{e}\mathfrak{J}\mathfrak{e} \qquad (u \in \mathscr{D}^A).$$

Let T^A and T^{A^+} be the maximal tableaux in the Garnir sets Gar^A and Gar^{A^+} respectively. Recall T^A is got by rearranging the row bricks of G^A in $\mathcal{B}^A$ by row-reading-order and similarly for T^{A^+}, hence by the previous correspondence on bricks $B_j^A \leftrightarrow B_j^A$, it is immediately that T^{A^+} is obtained from T^A by replacing each row-brick B_j^A by $B_j^{A^+}$.

Recall outside the Garnir belts $\mathcal{B}^A$ and $\mathcal{B}^{A^+}$, T^A and T^{A^+} coincide with the initial tableaux T^λ and T^μ. This means, when compute the permutations $w^{T^A} \in \mathfrak{S}_d$ such that $w^{T^A} \cdot T^\lambda = T^A$ and $w^{T^{A^+}} \in \mathfrak{S}_{d'}$ such that $w^{T^{A^+}} \cdot T^\mu = T^{A^+}$, we only need the consider the changes in Garnir belts. Now the Garnir belt $\mathcal{B}^A$ in T^A is the following form:



Here the head and the tail refer to the parts of the Garnir belt that do not lie in any row A-brick. In view of the initial tableau T^λ and the maximal tableau T^A, the string-diagram presentation of $\psi_{w^{T^A}}$ is of the form (with all undrawn strands taken to be vertical straight strands):



By definition, T^{A^+} and $\psi_{w^{T^{A^+}}}$ admit analogous descriptions. By the correspondence of row bricks and the definition of subdivision via string diagrams, we have:

$$\Phi_k(\psi^{T^A}) \;=\; \mathfrak{e}\,\psi^{T^{A^+}}\,\mathfrak{e} \;+\; \mathfrak{e}\mathfrak{J}\mathfrak{e} \qquad \text{in } S_{\overline{\alpha}}.$$

Recall the Garnir elements are of the form:

$$g^A = \sum_{u \in \mathscr{D}^A} \tau_u^A \, \psi^{T^A}, \qquad g^{A^+} = \sum_{v \in \mathscr{D}^{A^+}} \tau_v^{A^+} \, \psi^{T^{A^+}}.$$

Computing in $S_{\overline{\alpha}}$:

$$\begin{aligned}
\Phi_k(g^A) &= \sum_{u \in \mathscr{D}^A} \Phi_k(\tau_u^A)\,\Phi_k(\psi^{T^A}) \\
&= \sum_{u \in \mathscr{D}^A} \left(\mathfrak{e}\,\tau_{u^+}^{A^+}\,\mathfrak{e} + \mathfrak{e}\mathfrak{J}\mathfrak{e}\right)\left(\mathfrak{e}\,\psi^{T^{A^+}}\,\mathfrak{e} + \mathfrak{e}\mathfrak{J}\mathfrak{e}\right) \\
&= \mathfrak{e}\Big(\sum_{u^+ \in \mathscr{D}^{A^+}} \tau_{u^+}^{A^+}\,\psi^{T^{A^+}}\Big)\mathfrak{e} \;+\; \mathfrak{e}\mathfrak{J}\mathfrak{e} \\
&= \mathfrak{e}\,g^{A^+}\,\mathfrak{e} \;+\; \mathfrak{e}\mathfrak{J}\mathfrak{e}.
\end{aligned}$$

□

We can now state the main theorem of this section:

Theorem 4.3.19. *Fix a subdivision datum* $(e, I, \Lambda, \alpha, \boldsymbol{\kappa}, k)$ *and take* $\boldsymbol{\lambda} \in \mathscr{P}_\alpha^\Lambda$. *Then there is an isomorphism of graded* R_α*-modules*

$$S^{\Psi_k(\boldsymbol{\lambda})} \;\cong\; \mathfrak{e}S^{\Phi_k\left(\Psi_k(\boldsymbol{\lambda})\right)} \big/ \mathfrak{e}\mathfrak{J}\mathfrak{e}\,S^{\Phi_k\left(\Psi_k(\boldsymbol{\lambda})\right)}.$$

Proof. Let $\boldsymbol{\mu} = \Psi_k(\boldsymbol{\lambda})$ and $\boldsymbol{\nu} = \Phi_k(\boldsymbol{\mu})$, and set

$$\widetilde{S}^{\boldsymbol{\nu}} := \mathfrak{e}S^{\boldsymbol{\nu}} \big/ \mathfrak{e}\mathfrak{J}\mathfrak{e}\,S^{\boldsymbol{\nu}}.$$

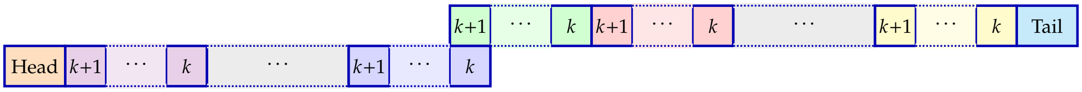
Write $\widetilde{z}^{\boldsymbol{\nu}} := \mathfrak{e}z^{\boldsymbol{\nu}} + \mathfrak{e}\mathfrak{J}\mathfrak{e}\,S^{\boldsymbol{\nu}} \in \widetilde{S}^{\boldsymbol{\nu}}$ for the image of the standard cyclic generator $z^{\boldsymbol{\nu}}$. Then $\widetilde{S}^{\boldsymbol{\nu}}$ is a cyclic $S_{\overline{\alpha}}$-module generated by $\widetilde{z}^{\boldsymbol{\nu}}$.

By Theorem 4.3.9 we already have an isomorphism of graded R_α-modules

$$M^{\boldsymbol{\mu}} \;\cong\; \mathfrak{e}M^{\boldsymbol{\nu}} \big/ \mathfrak{e}\mathfrak{J}\mathfrak{e}\,M^{\boldsymbol{\nu}}.$$

induced by sending the cyclic generator $m^{\boldsymbol{\mu}}$ of $M^{\boldsymbol{\mu}}$ to the cyclic generator $w^{\boldsymbol{\nu}} := \mathfrak{e}m^{\boldsymbol{\nu}} + \mathfrak{e}\mathfrak{J}\mathfrak{e}$ of $\mathfrak{e}M^{\boldsymbol{\nu}} / \mathfrak{e}\mathfrak{J}\mathfrak{e}\,M^{\boldsymbol{\nu}}$. Since $S^{\boldsymbol{\mu}}$ (resp. $S^{\boldsymbol{\nu}}$) is obtained from $M^{\boldsymbol{\mu}}$ (resp. $M^{\boldsymbol{\nu}}$) by imposing the Garnir relations, to check the map $S^{\boldsymbol{\mu}} \longrightarrow \widetilde{S}^{\boldsymbol{\nu}}$ given by $z^{\boldsymbol{\mu}} \mapsto \widetilde{z}^{\boldsymbol{\nu}}$ is an isomorphism, it remains to check that the Garnir relations correspond under Φ_k.

By Lemma 4.3.14 and Lemma 4.3.18, let $\phi := \phi_{\mathbf{i}^{\boldsymbol{\mu}}}$ be the position-tracing function associated to $(k, \mathbf{i}^{\boldsymbol{\mu}})$. Then the subdivision map Φ_k induces a bijection between Garnir nodes $A \in [\boldsymbol{\mu}]$ and the corresponding nodes $A^+ \in [\boldsymbol{\nu}]$, and it matches the Garnir elements g^A with $\mathfrak{e}g^{A^+}\mathfrak{e} + \mathfrak{e}\mathfrak{J}\mathfrak{e}$. Recall that the nodes $B \in [\boldsymbol{\mu}]$ of the form $B = A^+$ are precisely those satisfying $T(B) = \phi(t)$ for some $1 \leq t \leq d$. It therefore remains to consider those nodes X that are not of the form A^+ for any $A \in [\boldsymbol{\mu}]$. These new nodes all have residue $k+1$ and occur immediately to the right of a k-node. In particular, the Garnir belt $\mathcal{B}^X$, filled with residues, has the following form:



In particular, outside the Garnir belt $\mathcal{B}^X$, the last entry in T^ν (and hence in T^X and G^X) before any entry of $\mathcal{B}^X$ has residue k. Then the residue sequence $\mathbf{i}^X$ of the Garnir tableau G^X (and hence the residue sequence of every tableau in the Garnir set Gar X) is:

$$\cdots, k, \mathrm{res}(\mathrm{Head}), \underbrace{k{+}1, \cdots, e, 0, \cdots, k}_{k^X \text{ times}}, \mathrm{res}(\mathrm{Tail}), \cdots$$

Here res(Head) and res(Tail) are the residue sequences corresponding to the head and tail parts of the Garnir belt. Note that the first node in Head cannot have residue $k+1$: by assumption there are only non-trivial $(k, k+1)$-strips, so every $(k+1)$-node must occur immediately to the right of a k-node. In particular, $\mathbf{i}^X \notin I^{\overline{\alpha}}_{\mathrm{ord}}$, since there is an occurrence of k that is not immediately followed by $k+1$. Therefore $e(\mathbf{i}^X)\mathfrak{e} = \mathfrak{e}e(\mathbf{i}^X) = 0$, and hence:

$$\begin{aligned}
\mathfrak{e}g^X\mathfrak{e} + \mathfrak{e}\mathfrak{J}\mathfrak{e} &= \mathfrak{e}\sum_{u\in\mathscr{D}^X} \tau^X_u\, e(\mathbf{i}^X)\,\psi^{T^X}\mathfrak{e} + \mathfrak{e}\mathfrak{J}\mathfrak{e}\\
&= \mathfrak{e}\sum_{u\in\mathscr{D}^X} \tau^X_{r^u_1}\cdots\tau^X_{r^u_{\ell(u)}}\, e(\mathbf{i}^X)\,\psi^{T^X}\mathfrak{e} + \mathfrak{e}\mathfrak{J}\mathfrak{e}\\
&= \mathfrak{e}\sum_{u\in\mathscr{D}^X} e(\mathbf{i}^X)\,\tau^X_{r^u_1}\cdots\tau^X_{r^u_{\ell(u)}}\, e(\mathbf{i}^X)\,\psi^{T^X}\mathfrak{e} + \mathfrak{e}\mathfrak{J}\mathfrak{e}\\
&= 0 \in S_{\overline{\alpha}}.
\end{aligned}$$

where the second-to-last equality holds because $\tau^X_i = (\sigma^X_i + 1)e(\mathbf{i}^X)$ and σ^X_i only permutes the row bricks inside the Garnir belt $\mathcal{B}^X$, and hence does not change the residue sequence $\mathbf{i}^X$. Therefore, we have shown that the Garnir element g^X vanishes in $S_{\overline{\alpha}}$ whenever X is not of the form A^+ for any $A \in [\boldsymbol{\mu}]$. Now the theorem follows from Theorem 4.3.9. □

Corollary 4.3.20. *Fix a subdivision datum $(e, I, \Lambda, \alpha, \boldsymbol{\kappa}, k)$, and take $\boldsymbol{\lambda} \in \mathscr{P}^{\boldsymbol{\kappa}}_\alpha$ to be k-horizontal. Then there is an isomorphism of graded R_α-modules*

$$S^{\boldsymbol{\lambda}} \;\cong\; \mathfrak{e}S^{\Phi_k(\boldsymbol{\lambda})} \big/ \mathfrak{e}\mathfrak{J}\mathfrak{e}\, S^{\Phi_k(\boldsymbol{\lambda})}.$$

Proof. This follows from Lemma 4.3.5 and Theorem 4.3.19. □

4.4. Connection with Runner Removal Theorems

In this section, we explain that the combinatorics of the subdivision map developed in the previous section coincides with the runner removal theorems, thereby providing a natural categorification.

4.4.1. Level 1 Case.

Fix a subdivision datum $(e, I, \Lambda = \Lambda_x, \alpha, x, k)$ such that $\mathrm{ht}(\alpha) = n$. By the Brundan–Kleshchev isomorphism [**BK09a**, Theorem 1.1], the level 1 cyclotomic KLR algebras are isomorphic to the corresponding Iwahori–Hecke algebras $\mathcal{H}_q(\mathfrak{S}_n)$, where q is a primitive e-th root of unity. Hence, in this section, the Specht modules and simple modules can be viewed as modules over the classical Iwahori–Hecke algebras, equipped with a grading coming from the KLR grading. See also [**BKW11**, **HM10**, **KMR12**] for three different approaches to construct graded Specht modules.

Following [**Fay07**], we define an operation + on partitions. Fix an integer $d \in \mathbb{Z}$. For a partition $\lambda \in \mathscr{P}_\alpha^\Lambda$, choose a positive integer $a \geq \ell(\lambda)$ such that $a + d \geq 0$. Let c, k be the unique non-negative integers satisfying

$$a + d = ce + k, \quad k \in I = \{0, 1, \cdots, e-1\}.$$

Form the e-abacus of λ with a beads, and add a flush runner with c beads on the left of k-runner. This new $(e+1)$-abacus with $a + c$ beads corresponds uniquely to a partition, which is denoted by $\lambda^+ := \lambda^{+d}$. It is not hard to verify that this definition of λ^+ is independent of a.

The importance of this construction is illustrated by the following theorem.

THEOREM 4.4.1 (Runner Removal Theorem). *Assume that the base field* $\Bbbk$ *of the Hecke algebras has characteristic* 0. *Suppose that* $\lambda, \mu \in \mathscr{P}_\alpha^\Lambda$ *and that* λ *is e-regular. Fix* $d \in \mathbb{Z}$. *If one of the following holds:*

- $d \geq \lambda_1$ *(cf.* [**Fay07**, Theorem 3.1]*);*
- $d < -\ell(\lambda)$ *(cf.* [**JM02**, Theorem 4.5] *and* [**DP25**, Theorem 4.1]*).*

Then $d^e_{\mu,\lambda}(q) = d^{e+1}_{\mu^{+d},\lambda^{+d}}(q)$.

The e-regular assumption is not necessary, since they work with q-Schur algebras. In fact, they prove a stronger statement in terms of the canonical basis of the Fock space. Let $\mathcal{F}_e(\Lambda)$ be the (level-1) q-Fock space: the free $\mathbb{Q}(q)$-vector space with standard basis $\{\,|\lambda\rangle \mid \lambda \in \mathscr{P}^\Lambda\,\}$. It carries an integrable $U_q(\widehat{\mathfrak{sl}}_e)$-module structure, and the submodule generated by the vacuum vector $|\varnothing\rangle$ is isomorphic to the irreducible highest-weight module $V(\Lambda)$.

The bar involution on $U_q(\widehat{\mathfrak{sl}}_e)$ induces a bar involution on $V(\Lambda)$, which extends to a bar involution on the whole of $\mathcal{F}_e(\Lambda)$ by work of [**LT96**]. Consequently, for every partition $\mu \in \mathscr{P}^\Lambda$ there is a unique bar-invariant vector

$$G_e(\mu) = \sum_{\lambda \in \mathscr{P}^\Lambda} d^e_{\lambda\mu}(q)\,|\lambda\rangle, \tag{4.4.2}$$

such that $d^e_{\mu\mu}(q) = 1$ and $d^e_{\lambda\mu}(q) \in q\mathbb{Z}[q]$ for $\lambda \neq \mu$; moreover $d^e_{\lambda\mu}(q) = 0$ unless $\lambda \trianglelefteq \mu$ in dominance order of partitions. The set $\{G_e(\mu) \mid \mu \in \mathscr{P}^\Lambda\}$ is called the *canonical basis* of $\mathcal{F}_e(\Lambda)$.

When μ is e-regular, the element $G_e(\mu)$ lies in $V(\Lambda)$ and coincides with the canonical basis element in $V(\Lambda)$ indexed by μ. If the corresponding Iwahori–Hecke algebra is defined over a field of characteristic 0 and is specialised at a primitive eth root of unity, then Ariki's categorification theorem [**Ari96**, Theorem 4.4] identifies $d^e_{\lambda\mu}(1)$ with the decomposition number $[S^\lambda : D^\mu]$ (with μ e-regular). Thanks to the Brundan–Kleshchev isomorphism [**BK09a**], Brundan and Kleshchev lift Ariki's categorification theorem to the graded case in [**BK09b**, Section 5.5]: $d^e_{\lambda\mu}(q)$ is the graded decomposition number $[S^\lambda : D^\mu]_q$.

Their proof of Theorem 4.4.1 proceeds by extending $\lambda \mapsto \lambda^{+d}$ to a linear operator on the whole Fock space and then establishing

$$G_{e+1}(\lambda^{+d}) = G_e(\lambda)^{+d}. \tag{4.4.3}$$

For standard references on level 1 Fock spaces, decomposition numbers, quantum groups, and related topics, see [**Mat99**, Chapter 6].

The construction of λ^{+d} is very similar to the subdivision on partitions in Definition 4.2.9, but it has some different features. In Definition 4.2.9 one starts with a fixed integer $k \in I$, which specifies the position in the abacus where the new runner is inserted, whereas the construction of λ^{+d} starts with the parameter $d \in \mathbb{Z}$. The two parameters are linked by the equation $a + d = ce + k$.

In Definition 4.2.9, we require $a \equiv x \pmod{e}$, while this is not required in the definition of λ^{+d}. This is because, in level 1, we have $V(\Lambda_0) \cong V(\Lambda_x)$ for any $x \in I$. For convenience, one often also chooses $a \equiv 0 \pmod{e}$ in the construction of λ^{+d}. Moreover, in Definition 4.2.9, since $k \in I$, we necessarily have $d \in I$, whereas in the construction of λ^{+d} the parameter d can be any integer. Thus Definition 4.2.9 can be viewed as a special case of the construction of λ^{+d}, with an equivalent description in Young diagrams given by Definition 4.2.5.[4]

The most important feature of this special case (namely $d \in I$) is that the hypotheses of Theorem 4.4.1 are rarely satisfied. In other words, for a general partition λ, the interval $[0, e-1]$ is an exceptional set on which Theorem 4.4.1 does not apply.

Nevertheless, there is a general runner removal theorem that applies in this situation. To state it, we first need an object called the extended beta multiset.

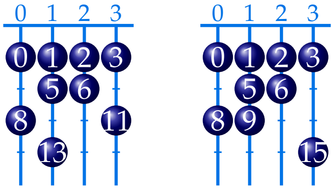
Recall that the a-beta set $B(\lambda; a)$ of a partition is the set of a-beta numbers of λ, $B(\lambda; a) = \{\beta^a_i(\lambda) \mid 1 \le i \le a\}$. The extended beta multiset can be viewed as a multiplicity-counting version of the beta set. For each $z \in \mathbb{Z}_{\ge 0}$, define the multiplicity mul_z to be $\#\big(B(\lambda; a) \cap \{z, z+e, z+2e, \ldots\}\big)$. The extended beta multiset $\mathfrak{X}^e_a(\lambda)$ is the union of $\underbrace{\{z, \cdots, z\}}_{\mathrm{mul}_z}$ over all $z \in \mathbb{Z}_{\ge 0}$, taken as a multiset.

---

[4] It would be interesting to find a general description of λ^{+d} in terms of Young diagrams.

For each fixed integer $d \in \mathbb{Z}$, define $\epsilon_d(\lambda)$ to be the number of elements $z \in \mathfrak{X}^e_a(\lambda)$, counted with multiplicity, such that $z \geq a + d$. It is an exercise to show that $\epsilon_d(\lambda)$ does not depend on the choice of a.

Example 4.4.4. *Fix type* $A^{(1)}_3$ *and* $d = 4$. *Let* $\lambda = (5, 4, 2, 1, 1)$ *and* $\mu = (7, 2, 2, 1, 1)$. *Take* $a = 9$. *Then* $a + d = 13 = 4 \cdot 3 + 1$, *so* $c = 3$ *and* $k = 1$. *The e-abaci with* 9 *beads for* λ *and* μ *are as follows:*



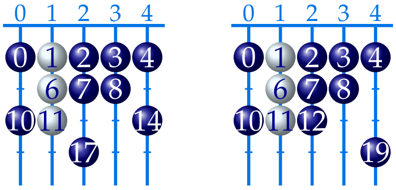
The beads are labelled by the corresponding beta numbers. The extended beta multisets of λ *and* μ *are*

$$\mathfrak{X}^e_a(\lambda) = \{0, 0, 1, 1, 1, 2, 2, 3, 3, 4, 5, 5, 6, 7, 8, 9, 11, 13\}, \tag{4.4.5}$$

$$\mathfrak{X}^e_a(\mu) = \{0, 0, 1, 1, 1, 2, 2, 3, 3, 4, 5, 5, 6, 7, 8, 9, 11, 15\}. \tag{4.4.6}$$

Since $a + d = 13$, *we have* $\epsilon_d(\lambda) = 1$ *and* $\epsilon_d(\mu) = 1$. *The partitions* λ^{+d} *and* μ^{+d} *have the following abacus displays:*

(4.4.7)



Hence $\lambda^{+d} = (6, 4, 2, 2, 1, 1, 1)$ *and* $\mu^{+d} = (8, 2, 2, 2, 1, 1, 1)$. ◇

Now we are able to state the *general runner removal theorem*.

Theorem 4.4.8 ([**Fay09**, Theorem 3.4]). *Assume that the base field* $\Bbbk$ *of the Hecke algebras has characteristic* 0. *Take* $d \in \mathbb{Z}$ *and let* $\lambda, \mu \in \mathscr{P}^\Lambda_\alpha$ *with* λ *e-regular. Set* $\lambda^+ := \lambda^{+d}$ *and* $\mu^+ := \mu^{+d}$. *If* $\epsilon_d(\lambda) = \epsilon_d(\mu)$, *then* $d^e_{\mu,\lambda}(q) = d^{e+1}_{\mu^+,\lambda^+}(q)$.

We remark, following [**Fay09**, Section 4.3], that Theorem 4.4.8 actually *contains* Theorem 4.4.1 as a special case; this is why it is called the general runner removal theorem. Another key feature is that the condition in Theorem 4.4.8 is a relation between the two partitions under consideration, rather than an absolute requirement as in Theorem 4.4.1.

Unlike Theorem 4.4.1, Theorem 4.4.8 imposes no restriction on the parameter d. By specialising to $0 \leq d \leq e - 1$, it applies to our subdivision map on partitions. In view of Theorem 4.3.19, the subdivision isomorphism Φ_k *categorifies* this runner removal theorem over a field of characteristic 0.

Remark 4.4.9. *There are several differences between the setting of [**Fay09**] and ours, and we briefly discuss some of them here. Fayers studies runner removal in [**Fay09**], whereas this* *chapter treats runner addition. In other words, we pass from* λ *to* λ^{+d}*, that is, from an* e*-abacus to an* $(e+1)$*-abacus, while Fayers passes from* λ *to* λ^{-d}*, that is, from an* $(e+1)$*-abacus to an* e*-abacus.*

For the reader's convenience, we clarify the notation. Let $\overline{d}$ *be the unique integer in* $[0,e]$ *such that* $\overline{d} \equiv d \pmod{e+1}$*. Then* λ^{+d} *is* $\overline{d}$*-empty and* $\lambda = (\lambda^{+d})^{-\overline{d}}$ *in Fayers' notation. It is also straightforward to check that* $\epsilon_d(\lambda) = \mathfrak{L}_{\overline{d}}(\lambda^{+d})$*, where* $\mathfrak{L}_{\overline{d}}(-)$ *is defined in [**Fay09**, Definition 3.3].*

The reader should be careful that we use the equation $a+d = ce+k$*, whereas [**Fay09**] uses* $a+k = ce+d$*. Thus, when comparing results, the parameters* k *and* d *should be swapped.* ◇

4.4.2. Higher Level Case. The discussion of Fock spaces and canonical bases in Subsection 4.4.1 has an analogous, but more complex, higher-level theory; see [**BK09b**] or [**GJ11**, Chapter 6] for a general discussion of graded decomposition numbers of cyclotomic KLR algebras of type $A^{(1)}_{e-1}$ and related topics, including higher-level Fock spaces.

Recently, Theorem 4.4.1 has been generalized to higher levels in [**DP25**, **Del24**] via computations in higher-level Fock spaces, whereas Theorem 4.4.8 has not yet been extended. Nevertheless, we formulate a conjecture below concerning our subdivision.

Fix a subdivision datum $(e, I, \Lambda, \alpha, \boldsymbol{\kappa}, k)$ and take an ℓ-partition $\boldsymbol{\lambda} = (\boldsymbol{\lambda}^{(1)}, \cdots, \boldsymbol{\lambda}^{(\ell)}) \in \mathscr{P}^{\boldsymbol{\kappa}}_{\alpha}$. For each $1 \le i \le \ell$, take an abacus subdivision datum $(a, c, d, a') = (a_i, c_i, d_i, a'_i)$ for $\boldsymbol{\lambda}^{(i)}$.

We then extend the level-1 construction of λ^{+d} to higher levels by defining it componentwise:

$$\boldsymbol{\lambda}^{+\boldsymbol{d}} := \left(\left(\boldsymbol{\lambda}^{(1)}\right)^{+d_1}, \dots, \left(\boldsymbol{\lambda}^{(\ell)}\right)^{+d_\ell} \right).$$

Set

$$\epsilon_{\boldsymbol{d}}(\boldsymbol{\lambda}) := \left(\epsilon_{d_1}(\boldsymbol{\lambda}^{(1)}), \dots, \epsilon_{d_\ell}(\boldsymbol{\lambda}^{(\ell)})\right), \quad \left|\epsilon_{\boldsymbol{d}}(\boldsymbol{\lambda})\right| := \sum_{i=1}^{\ell} \epsilon_{d_i}(\boldsymbol{\lambda}^{(i)}).$$

Recall from Section 4.2 that the subdivision map on partitions is defined componentwise. As a result, by the discussion in Subsection 4.4.1, if $d_i \in I$ for each $1 \le i \le \ell$, then $(\cdot)^{+\boldsymbol{d}}$ coincides with the subdivision map, that is, $\Phi_k(\boldsymbol{\lambda}) = \boldsymbol{\lambda}^{+\boldsymbol{d}}$. In view of Theorem 4.3.19, the subdivision provides a categorification of this case.

We can now state the following two conjectures:

Conjecture 4.4.10. *Over a characteristic* 0 *field, let* $\boldsymbol{d} = (d_1, \cdots, d_\ell) \in I^\ell$ *such that* $0 \le a_i + d_i = c_i e + k$ *for each* $1 \le i \le \ell$*, take* $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}^{\kappa}_{\alpha}$*, if* $\epsilon_{\boldsymbol{d}}(\boldsymbol{\lambda}) = \epsilon_{\boldsymbol{d}}(\boldsymbol{\mu})$*, then* $d^{e}_{\boldsymbol{\mu},\boldsymbol{\lambda}}(q) = d^{e+1}_{\boldsymbol{\mu}^{+\boldsymbol{d}},\boldsymbol{\lambda}^{+\boldsymbol{d}}}(q)$*.*

Conjecture 4.4.11. *Over a characteristic* 0 *field, let* $\boldsymbol{d} = (d_1, d_2) \in I^2$ *such that* $0 \le a_i + d_i = c_i e + k$ *for each* $1 \le i \le 2$*, take* $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}^{\kappa}_{\alpha}$*, if* $|\epsilon_{\boldsymbol{d}}(\boldsymbol{\lambda})| = |\epsilon_{\boldsymbol{d}}(\boldsymbol{\mu})|$*, then* $d^{e}_{\boldsymbol{\mu},\boldsymbol{\lambda}}(q) = d^{e+1}_{\boldsymbol{\mu}^{+\boldsymbol{d}},\boldsymbol{\lambda}^{+\boldsymbol{d}}}(q)$*.*

We remark that the hypothesis of Conjecture 4.4.11 is much weaker than that of Conjecture 4.4.10. However, it can only reasonably be expected to hold in level two, since analogous statements admit counterexamples in higher levels. We end this section with an example supporting Conjecture 4.4.11.

EXAMPLE 4.4.12. *Fix quiver* $A_2^{(1)}$. *Fix* $k = 1$ *and the charge* $\boldsymbol{\kappa} = (0, 1)$. *Take a* 2*-partition* $\boldsymbol{\lambda} = \big((6), (5, 1, 1)\big)$. *By choosing suitable* $\mathbf{a}$ *and the corresponding* $\boldsymbol{d}$ *(for example, take* $\mathbf{a} = (3, 4)$ *and* $\boldsymbol{d} = (1, 0)$*), one easily verifies that* $\boldsymbol{\lambda}^{+\boldsymbol{d}} = \big((8), (7, 1, 1)\big)$ *and* $\epsilon_{\boldsymbol{d}}(\boldsymbol{\lambda}) = (2, 2)$.

The canonical basis element $G_e(\boldsymbol{\lambda})$ *is given in Figure 1. The terms coloured red correspond to* 2*-partitions* $\boldsymbol{\mu}$ *such that* $\boldsymbol{\mu}^{+\boldsymbol{d}}$ does not *occur in the expansion of* $G_{e+1}(\boldsymbol{\lambda}^{+\boldsymbol{d}})$. *The terms coloured cyan are those that* do *occur and satisfy* $|\epsilon_{\boldsymbol{d}}(\boldsymbol{\lambda})| = |\epsilon_{\boldsymbol{d}}(\boldsymbol{\mu})| = 4$. *The remaining terms likewise occur in the expansion of* $G_{e+1}(\boldsymbol{\lambda}^{+\boldsymbol{d}})$ *but satisfy* $|\epsilon_{\boldsymbol{d}}(\boldsymbol{\lambda})| \neq |\epsilon_{\boldsymbol{d}}(\boldsymbol{\mu})|$.

The canonical basis element $G_{e+1}(\boldsymbol{\lambda}^{+\boldsymbol{d}})$ *is given in Figure 2: the cyan terms in* $G_{e+1}(\boldsymbol{\lambda}^{+\boldsymbol{d}})$ *correspond bijectively to the cyan terms in* $G_e(\boldsymbol{\lambda})$. *The reader can verify that for all cyan* $\boldsymbol{\mu}$*, we have* $d^{e}_{\boldsymbol{\mu},\boldsymbol{\lambda}}(q) = d^{e+1}_{\boldsymbol{\mu}^{+\boldsymbol{d}},\boldsymbol{\lambda}^{+\boldsymbol{d}}}(q)$. *For simplicity, we have omitted terms* $\boldsymbol{\mu}$ *that are not of the form* $\boldsymbol{\nu}^{+\boldsymbol{d}}$ *for some* $\boldsymbol{\nu}$ *appearing in* $G_e(\boldsymbol{\lambda})$. ◇

Canonical Basis Expansion: $G_e(\lambda)$

$$
\begin{aligned}
G_e(\boldsymbol{\lambda}) = {} & |[6],[5,1^2]\rangle \\
& + q\,|[5,1],[5,1^2]\rangle \\
& + q\,|[4,1],[5,2,1]\rangle \\
& + q^2\,|[4,1],[4,3,1]\rangle \\
& + (q^3+q)\,|[4],[5,3,1]\rangle \\
& + q^2\,|[3,2,1],[3^2,1]\rangle \\
& + q^3\,|[3,2],[5,1^3]\rangle \\
& + q^4\,|[3,2],[3^2,1^2]\rangle \\
& + q^2\,|[3,1^2],[3^2,2]\rangle \\
& + (q^3+q)\,|[3,1],[6,2,1]\rangle \\
& + q^2\,|[3,1],[5,2^2]\rangle \\
& + q^2\,|[3,1],[4^2,1]\rangle \\
& + q^4\,|[3,1],[3^3]\rangle \\
& + q^4\,|[3,1],[3^2,1^3]\rangle \\
& + 2q^2\,|[3],[6,3,1]\rangle \\
& + (q^4+q^2)\,|[3],[5,3,2]\rangle \\
& + q^2\,|[2^3],[5,1^2]\rangle \\
& + (q^4+q^2)\,|[2^2],[5,3,1]\rangle \\
& + (2q^3+q)\,|[2,1],[6,3,1]\rangle \\
& + (q^5+2q^3+q)\,|[2,1],[5,3,2]\rangle \\
& + (q^4+q^2)\,|[2,1],[3^2,2^2]\rangle \\
& + q^2\,|[1^3],[6,3,1]\rangle \\
& + (q^4+q^2)\,|[1^3],[5,3,2]\rangle \\
& + q^3\,|[1^3],[3^2,2^2]\rangle \\
& + q^2\,|[1^2],[8,2,1]\rangle \\
& + 2q^3\,|[1^2],[7,3,1]\rangle \\
& + q^4\,|[1^2],[5^2,1]\rangle \\
& + (q^5+q^3)\,|[1^2],[5,3^2]\rangle \\
& + (q^5+q^3)\,|[1^2],[5,3,1^3]\rangle \\
& + q^4\,|[1^2],[5,2,1^4]\rangle \\
& + q^5\,|[1^2],[4,3,1^4]\rangle \\
& + (q^4+q^2)\,|[1],[8,3,1]\rangle \\
& + q^2\,|\emptyset,[11,1^2]\rangle \\
& + q^3\,|\emptyset,[8,3,2]\rangle \\
& + q^4\,|\emptyset,[5,3,2^2,1]\rangle \\
& + q^4\,|\emptyset,[3^3,2,1^2]\rangle
\end{aligned}
$$

$$
\begin{aligned}
& + q\,|[6],[3^2,1]\rangle \\
& + q^2\,|[5,1],[3^2,1]\rangle \\
& + q^2\,|[4,1],[5,1^3]\rangle \\
& + q^3\,|[4,1],[3^2,1^2]\rangle \\
& + q\,|[3,2,1],[5,1^2]\rangle \\
& + q^2\,|[3,2],[5,2,1]\rangle \\
& + q^3\,|[3,2],[4,3,1]\rangle \\
& + q\,|[3,1^2],[6,1^2]\rangle \\
& + q^2\,|[3,1],[7,1^2]\rangle \\
& + (q^4+q^2)\,|[3,1],[6,1^3]\rangle \\
& + q^3\,|[3,1],[5,1^4]\rangle \\
& + 2q^3\,|[3,1],[4,3,2]\rangle \\
& + (q^5+q^3)\,|[3,1],[3^2,2,1]\rangle \\
& + q\,|[3],[8,1^2]\rangle \\
& + q^3\,|[3],[5,4,1]\rangle \\
& + q^3\,|[3],[3^2,2^2]\rangle \\
& + q^3\,|[2^3],[3^2,1]\rangle \\
& + q^2\,|[2,1],[8,1^2]\rangle \\
& + (q^4+q^2)\,|[2,1],[5,4,1]\rangle \\
& + q^2\,|[2,1],[5,1^5]\rangle \\
& + q^3\,|[2,1],[3^2,1^4]\rangle \\
& + q^3\,|[1^3],[5,4,1]\rangle \\
& + q^3\,|[1^3],[5,1^5]\rangle \\
& + q^4\,|[1^3],[3^2,1^4]\rangle \\
& + q^3\,|[1^2],[8,1^3]\rangle \\
& + (2q^4+q^2)\,|[1^2],[6,3,1^2]\rangle \\
& + (q^5+q^3)\,|[1^2],[5,4,1^2]\rangle \\
& + (q^6+2q^4+q^2)\,|[1^2],[5,3,2,1]\rangle \\
& + q^3\,|[1^2],[5,2^3]\rangle \\
& + 2q^4\,|[1^2],[4,3,2^2]\rangle \\
& + q^5\,|[1^2],[3^3,2]\rangle \\
& + (q^5+q^3)\,|[1],[5,3,2^2]\rangle \\
& + q^3\,|\emptyset,[9,3,1]\rangle \\
& + q^4\,|\emptyset,[6,3,2^2]\rangle \\
& + q^3\,|\emptyset,[4,3,2^2,1^2]\rangle \\
& + q^5\,|\emptyset,[3^2,2^3,1]\rangle
\end{aligned}
$$

FIGURE 1. Expansion of $G_3((6),(5,1,1))$.

Canonical Basis Expansion: $G_{e+1}(\lambda^+)$

$$
\begin{aligned}
G_{e+1}(\lambda^+) = {} & |[8],[7,1^2]\rangle && + q\,|[8],[4^2,1]\rangle \\
& + q\,|[7,1],[7,1^2]\rangle && + q^2\,|[7,1],[4^2,1]\rangle \\
& + q\,|[5,1],[7,3,1]\rangle && + q^2\,|[5,1],[6,4,1]\rangle \\
& + (q^3+q)\,|[5],[7,4,1]\rangle && + q\,|[4,3,1],[7,1^2]\rangle \\
& + q^2\,|[4,3,1],[4^2,1]\rangle && + q^2\,|[4,2],[7,3,1]\rangle \\
& + q^2\,|[4,2],[7,1^4]\rangle && + q^3\,|[4,2],[6,4,1]\rangle \\
& + q^3\,|[4,2],[4^2,1^3]\rangle && + q^2\,|[4,1],[10,1^2]\rangle \\
& + (q^3+q)\,|[4,1],[8,3,1]\rangle && + q^3\,|[4,1],[8,1^4]\rangle \\
& + q^2\,|[4,1],[7,3,2]\rangle && + q^2\,|[4,1],[6,5,1]\rangle \\
& + 2q^3\,|[4,1],[6,4,2]\rangle && + q^4\,|[4,1],[4^3]\rangle \\
& + q^4\,|[4,1],[4^2,2,1^2]\rangle && + q\,|[4],[11,1^2]\rangle \\
& + 2q^2\,|[4],[8,4,1]\rangle && + q^3\,|[4],[7,5,1]\rangle \\
& + (q^4+q^2)\,|[4],[7,4,2]\rangle && + (q^4+q^2)\,|[3,2],[7,4,1]\rangle \\
& + q^2\,|[3,1],[11,1^2]\rangle && + (2q^3+q)\,|[3,1],[8,4,1]\rangle \\
& + (q^4+q^2)\,|[3,1],[7,5,1]\rangle && + (q^5+2q^3+q)\,|[3,1],[7,4,2]\rangle \\
& + q^3\,|[3,1],[4^2,2^2,1]\rangle && + q^2\,|[1^2],[11,3,1]\rangle \\
& + 2q^3\,|[1^2],[10,4,1]\rangle && + q^3\,|[1^2],[8,4,1^3]\rangle \\
& + q^4\,|[1^2],[7^2,1]\rangle && + q^4\,|[1^2],[7,5,1^3]\rangle \\
& + (q^5+q^3)\,|[1^2],[7,4^2]\rangle && + (q^5+q^3)\,|[1^2],[7,4,2,1^2]\rangle \\
& + q^4\,|[1^2],[7,3,2^2,1]\rangle && + (q^5+q^3)\,|[1^2],[6,4,2^2,1]\rangle \\
& + q^4\,|[1^2],[4^3,2,1]\rangle && + (q^4+q^2)\,|[1],[11,4,1]\rangle \\
& + q^4\,|[1],[7,4,2^2,1]\rangle && + q^2\,|\emptyset,[15,1^2]\rangle \\
& + q^3\,|\emptyset,[12,4,1]\rangle && + q^3\,|\emptyset,[11,4,2]\rangle + \dots
\end{aligned}
$$

Figure 2. Expansion of $G_4\big((8),\ (7,1,1)\big)$.

Acknowledgements

This is not a comprehensive list of acknowledgements, as I am not very good at writing this kind of thing. If I have forgotten to mention you here, please forgive me—your help is sincerely appreciated. (Also, I did not spend as much time writing this as I probably should have.)

To start, I am most grateful to my supervisor Andrew Mathas for introducing me to this area of research, for many insightful and helpful discussions, for his careful reading of my manuscripts, and for his constant support throughout my PhD. I also thank my secondary supervisor Dani Tubbenhauer for the many Zoom meetings during my first year, and for providing valuable advice, book recommendations, and answers to many of my questions in this subject.

My thanks also go to my colleagues in Carslaw office 807—Nick Bridger, James Gabor, Sean Skinner, and Damian Lin—for their collegiality and for contributing to a supportive and enjoyable working environment. I also thank the other colleagues there for many valuable conversations and encouragement.

I have been fortunate to be in a School with many excellent researchers working in representation theory and related areas. I thank Kevin Coulembier and Alex Sherman for organising several learning seminars, which helped me understand many aspects of the subject. I also thank Oded Yacobi and Gus Lehrer for helpful conversations.

During the final six months of my study, I thank SMRI for providing an excellent working environment and for facilitating frequent interactions with colleagues and visiting researchers. In particular, I thank Finn Klein, Tom Goertzen, Tasman Fell, Connor Simpson, Geordie Williamson, Thomas Le Fils, Bailey Whitbread and others for many stimulating discussions, and for many enjoyable lunches together.

I thank Jun Hu for hosting me for several months at Beijing Institute of Technology while I was waiting for my visa. I am also grateful to his group of students and postdocs, especially Shixuan Wang and Huang Lin, for introducing me to classical results on quiver Hecke algebras and for many helpful discussions.

I also thank the referees of the paper [**Qin25**], which forms Chapter 2, for their careful reading and constructive comments, which identified numerous typographical errors and improved the exposition.

Finally, I thank my family, friends, and partner for their continued support throughout (and before) my PhD. The research reported in this thesis was supported by the award of a Postgraduate Research Stipend Scholarship in Graded Representation Theory (SC4222).

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