

**SINGULAR DOMAIN PERTURBATION IN STATIONARY
P-LAPLACE EQUATIONS**

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This is to certify that the content of this thesis is my own work. This thesis has not been submitted for any other degree or purpose.

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Additionally I did not use any generative AI in the thesis.

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1. INTRODUCTION

1.1. Problem. In this thesis, we aim to investigate non-linear diffusion problems on domains with shrinking holes, and study the continuous dependence of the weak solutions. We start by defining exactly what is meant by a domain with shrinking holes. Throughout this introductory section, we are in the following setting.

Assumption 1.1. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz-continuous boundary, $d \in \{n \in \mathbb{N} \mid n \geq 2\}$, and $K := \{x_1, \dots, x_m\}$ be a set of m distinct points of Ω . For every $j = 1, \dots, m$, let $K_j \subset \mathbb{R}^d$ be a nonempty, compact, simply connected domains containing the origin $0 \in \mathbb{R}^d$ with at least a Lipschitz-continuous boundary ∂K_j . Let $(\varepsilon_n)_{n \geq 1}$ be a zero-sequence of positive numbers $\varepsilon_n \in (0, 1]$ so that for every $n \in \mathbb{N}$, and every $j \in \{1, \dots, m\}$, $x_j + \varepsilon_n K_j \subseteq \Omega$. Then, for every $n \in \mathbb{N}$, we set

$$H_n := \bigcup_{j=1}^m (x_j + \varepsilon_n K_j), \quad \Omega_n := \Omega \setminus H_n, \quad \text{and} \quad \Omega^* := \Omega \setminus K.$$

In this given setting, the set H_n represents the set of finitely many *holes* in Ω_n , which we have taken out of Ω . By construction, the sets H_n shrink as $n \rightarrow \infty$ to the set K of m distinct points, where shrinking means *converging in Hausdorff distance* (see Definition A.9). Therefore, we have that

$$\Omega_n \rightarrow \Omega^* \quad \text{as } n \rightarrow \infty.$$

Next, we introduce the elliptic boundary value problems driven by the *p-Laplace operator* Δ_p . The *p-Laplace operator* Δ_p is a non-linear 2nd-order differential operator defined by

$$(1.1) \quad \Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u) := \sum_{i=1}^d D_i(|\nabla u|^{p-2} D_i).$$

for sufficiently smooth functions $u : \Omega \rightarrow \mathbb{R}$. Note, for $p = 2$, the *p-Laplace operator* Δ_p reduces to the well-known linear *Laplace operator* $\Delta u := \sum_{i=1}^d D_{ii} u$.

We begin by stating the mixed-type boundary-value problem on the set Ω_n with holes.

Let $1 < p, p' < \infty$ so that $\frac{1}{p} + \frac{1}{p'} = 1$, $V \in L^\infty(\Omega)$ be a potential satisfying $V \geq v_0 > 0$ a.e. on Ω . Then for every $n \in \mathbb{N}$, let $f_n \in L^{p'}(\Omega_n)$, $\alpha_n > 0$, and u_n be a weak solution (see Definition 2.11) of

$$(1.2) \quad \begin{cases} -\Delta_p u_n + V(x) |u_n|^{p-2} u_n = f_n & \text{in } \Omega_n, \\ |\nabla u_n|^{p-2} \nabla u_n \cdot \nu + \alpha_n |u_n|^{p-2} u_n = 0 & \text{on } \partial H_n, \\ u_n = 0 & \text{on } \partial \Omega. \end{cases}$$

Here, ν denotes the outward pointing unit normal vector on ∂H_n .

Further, we consider the following Poisson problem on the limit set Ω^* . Furthermore, for given $f \in L^{p'}(\Omega)$, let u be a weak solution of the Poisson problem

$$(1.3) \quad \begin{cases} -\Delta_p u + V(x) |u|^{p-2} u = f & \text{in } \Omega^*, \\ u = 0 & \text{on } \partial \Omega. \end{cases}$$

The p -Laplacian is a famous equation that has a variety of applications due to its non-linear properties, especially when $1 \leq p < 2$. These of particular interest in the recent applications to machine learning [15], image processing [7], materials [3] and even ecological applications such as modelling populations [13].

In this framework, our aim is to establish the following convergence result. For this, we extend f_n by zero on $H_n = \Omega \setminus \Omega_n$, and recall

$$(1.4) \quad p^* := \begin{cases} \frac{pd}{d-p} & 1 \leq p < d, \\ \infty & p > d. \end{cases}$$

If $p = d$ then p^* takes on any value between $[d, +\infty)$.

Theorem 1.2 (Convergence on domains with shrinking holes). *Following the assumptions of this section, suppose that*

$$f_n \rightharpoonup f \quad \text{weakly in } L^{p'}(\Omega) \text{ as } n \rightarrow \infty$$

and $(\alpha_n)_{n \geq 1}$ be a zero-sequence in $(0, \infty)$. Let u_n be the unique weak solution of (1.2) for f_n . Then, there exist a weak solution u of (1.3) corresponding to f such that for every $1 \leq q \leq p^*$, one has that

$$\|u_n - u\|_{L^q(\Omega_n)} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Remark 1.3. We note that this convergence result is not yet complete as it does not provide the uniqueness of the weak solution u of the limiting problem (1.3). But we leave this task to future research.

1.2. Structure of thesis. The structure of the thesis is laid out as follows:

In the second section we establish the weak formulation of the boundary-value problem using the p -Laplacian. First, we establish the existence and uniqueness of solutions to the Dirichlet boundary problem, and show that it is well-posed. Then, we consider the motivating problem of the thesis with the mixed boundary condition problem (1.2).

The third section focuses on the measure-theoretic framework of transformations, establishing the preliminaries of embeddings used later in the following sections.

In the fourth section we seek to establish uniform Sobolev embeddings independent of the volume $|\Omega|$ of the domain Ω . We demonstrate this using so called ‘‘cone conditions’’. First, we establish domain independent on bounded domains, and then extended to unbounded domains.

The fifth section builds on the work of Daners [4] to establish uniform Sobolev estimates on shrinking domains. However we extend the results to $p \neq 2$ as in the pre-published work of Hauer& Lee [6].

The final section begins with an introduction to the Calderón operator. Then, the remaining components of the section are to the best of the author’s knowledge, original results. The first is establishing uniformly bounded families of extension operators for domains with shrinking holes. Then, due to the result of these bounded extension operators an alternative proof of the main result in Section 5 is provided. Next a Rellich-Kondarchev’s compactness result is establish for the same shrinking domains. Finally, convergence in the L^q norm is established for the mixed boundary condition problem (1.2). The results of this section will be published.

The appendix covers the background material use throughout the thesis for the sake of completion covering geometry, linear algebra, measure theory, functional analysis, mollifiers, and singular integrals.

2. WEAK FORMULATION OF BOUNDARY-VALUE PROBLEMS DRIVEN BY THE p -LAPLACIAN

The p -Laplace operator Δ_p is a non-linear 2nd-order differential operator, which as mentioned in section 1 has been used extensively in the literature to model power law based diffusion. This section will provide a comprehensive introduction to the background and setting. We start by formally defining the p -Laplace operator Δ_p . Let $\Omega \subset \mathbb{R}^d$ be an open subset with, $d \in \{n \in \mathbb{N} \mid n \geq 2\}$ and $1 \leq p < \infty$. Then

$$(2.1) \quad \Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u) := \sum_{i=1}^d D_i (|\nabla u|^{p-2} D_i u).$$

for sufficiently smooth functions $u : \Omega \rightarrow \mathbb{R}$.

2.1. Dirichlet p -Laplace operator. We start by recalling the definition of the Sobolev space $W^{1,p}(\Omega)$ as given in [2, Section 9.1].

Definition 2.1. Let $1 \leq p < \infty$ and $\Omega \subseteq \mathbb{R}^d$ be an open subset. Then, the first Sobolev space $W^{1,p}(\Omega)$ is defined by

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) \mid \exists g_1, \dots, g_d \in L^p(\Omega) \text{ satisfying (2.2)}\},$$

where

$$(2.2) \quad \int_{\Omega} u D_i \xi \, dx = - \int_{\Omega} g_i \xi \, dx \quad \forall \xi \in C_c^\infty(\Omega).$$

Notation 2.2. Since g_1, \dots, g_d are uniquely defined by the integral equation (2.2) and represent the distributional partial derivatives of u , one sets $D_i u := g_i$. We equip $W^{1,p}(\Omega)$ with the norm

$$\|u\|_{W^{1,p}(\Omega)} := \left(\|u\|_{L^p(\Omega)}^p + \|Du\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}}$$

for every $u \in W^{1,p}(\Omega)$.

The Sobolev space $W^{1,p}(\Omega)$ equipped with this norm $\|\cdot\|_{W^{1,p}(\Omega)}$ becomes a closed linear subspace of $L^p(\Omega)$ and hence is a Banach space.

It is natural to introduce a Sobolev space for domains Ω with zero boundary conditions which we label as $W_0^{1,p}(\Omega)$.

Definition 2.3. Let $1 \leq p < \infty$ and $\Omega \subseteq \mathbb{R}^d$ be an open subset. Then, we define the Sobolev space with zero boundary values as

$$W_0^{1,p}(\Omega) := \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{W^{1,p}(\Omega)}}$$

where $\overline{\{\cdot\}}^{\|\cdot\|_{W^{1,p}(\Omega)}}$ is compact support in the $W^{1,p}(\Omega)$ norm. We also have the associated dual space (see Definition B.3)

$$W^{-1,p'}(\Omega) := (W_0^{1,p}(\Omega))'$$

where p' denotes the Hölder conjugate defined by $\frac{1}{p} + \frac{1}{p'} = 1$.

The spaces of functions where the p -Laplace operator Δ_p is well-defined on depends on the boundary conditions of the differential operator. We start with the simple case of homogeneous Dirichlet boundary conditions.

Definition 2.4. Let $1 < p < \infty$ and $\Omega \subseteq \mathbb{R}^d$ be an open subset. Then, the *Dirichlet p -Laplace operator* $\Delta_p : W_0^{1,p}(\Omega) \rightarrow W^{-1,p'}(\Omega)$ is defined by

$$\langle \Delta_p u, v \rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)} := - \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla v dx,$$

for every $u, v \in W_0^{1,p}(\Omega)$.

In this section, we are mainly focusing on the following *Poisson problem* equipped with homogeneous Dirichlet boundary condition associated with the Dirichlet p -Laplace operator $\Delta_p u$:

$$(2.3) \quad \begin{cases} -\Delta_p u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

for given $f \in W^{-1,p'}(\Omega)$.

From Definition 2.4, we have a natural definition for the notion of *weak solutions* of the Poisson problem (2.3).

Definition 2.5. Let $1 < p < \infty$ and $\Omega \subseteq \mathbb{R}^d$ be an open subset. For given $f \in W^{-1,p'}(\Omega)$, a function $u \in W_0^{1,p}(\Omega)$ is called a *weak solution* of the Poisson problem (2.3) provided u satisfies

$$(2.4) \quad \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla v dx = \langle f, v \rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)}.$$

for all $v \in W_0^{1,p}(\Omega)$.

We will now show that the Dirichlet p -Laplace operator Δ_p is a well defined, continuous mapping.

Proposition 2.6. Let $1 \leq p < \infty$ and $\Omega \subseteq \mathbb{R}^d$ be an open subset. Then, the Dirichlet p -Laplace operator $\Delta_p^0 : W_0^{1,p}(\Omega) \rightarrow W^{-1,p'}(\Omega)$ is a continuous and well-defined mapping and the following hold:

$$(2.5) \quad \|\Delta_p u\|_{W^{-1,p'}(\Omega)} \leq \|u\|_{W_0^{1,p}(\Omega)}^{p-1} \quad \text{for all } u \in W_0^{1,p}(\Omega),$$

$$(2.6) \quad \langle -\Delta_p u, u \rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)} = \|u\|_{W_0^{1,p}(\Omega)}^p \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

For the sake of completeness we provide the outline of a proof of Proposition 2.6 based on Lecture 3 from the series [10].

Proof. We first show that (2.5) holds by utilising the definition of the $W_0^{1,p}(\Omega)$ norm $\|\cdot\|_{W^{-1,p'}(\Omega)}$, the definition of the Dirichlet p -Laplacian Δ_p (see (B.1) and

(2.1) respectively) and Hölder's inequality

$$\begin{aligned}
\|\Delta_p u\|_{W^{-1,p'}(\Omega)} &= \sup_{\|v\|_{W_0^{1,p}(\Omega)} \leq 1} |\langle \Delta_p u, v \rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)}| \\
&= \sup_{\|v\|_{W_0^{1,p}(\Omega)} \leq 1} \left| - \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla v \, dx \right| \\
&\leq \sup_{\|v\|_{W_0^{1,p}(\Omega)} \leq 1} \|\nabla u\|_{L^p(\Omega)}^{p-1} \|\nabla v\|_{L^p(\Omega)} \\
&\leq \sup_{\|v\|_{W_0^{1,p}(\Omega)} \leq 1} \|u\|_{W_0^{1,p}(\Omega)}^{p-1} \|v\|_{W_0^{1,p}(\Omega)} \\
&\leq \|u\|_{W_0^{1,p}(\Omega)}^{p-1}.
\end{aligned}$$

Thus showing that (2.5) holds and that the Dirichlet p -Laplacian $\Delta_p : W_0^{1,p}(\Omega) \rightarrow W^{-1,p'}(\Omega)$ is a well defined mapping. For equation (2.6) we use the definition of the p -Laplacian Δ_p (see (2.1)),

$$- \langle \Delta_p u, v \rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)} = \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla v \, dx = \| |\nabla u| \|_{L^p(\Omega)}$$

□

Before we finish proving Proposition 2.6 we first introduce the following lemma establishing the linear continuity of several operators.

Lemma 2.7. *For every $1 \leq p < \infty$ and for an open subset $\Omega \subseteq \mathbb{R}^d$ the following operators are linear continuous*

$$\begin{aligned}
\operatorname{div} : W^{1,p}(\Omega; \mathbb{R}^d) &\rightarrow L^p(\Omega), & \vec{u} := (u_1, \dots, u_d) &\mapsto \sum_{i=1}^d \frac{\partial u}{\partial x_i} \\
\operatorname{div} : L^{p'}(\Omega; \mathbb{R}^d) &\rightarrow W^{1,p}(\Omega), & \vec{u} &\mapsto \sum_{i=1}^d \frac{\partial u}{\partial x_i}.
\end{aligned}$$

We now continue the proof of Proposition 2.6 by showing that the p -Laplacian Δ_p is continuous.

Proof. In order to prove the continuity of $\Delta_p : W_0^{1,p}(\Omega) \rightarrow W^{-1,p'}(\Omega)$, let $(u_n)_{n \geq 1} \subseteq W_0^{1,p}(\Omega)$ be a sequence converging to some $u \in W_0^{1,p}(\Omega)$. We note by Theorem B.37 we have that for every convergent sequence in $L^p(\Omega)$ there is a subsequence which converges almost everywhere and is dominated by a function $g \in L^p(\Omega)$. We apply this to the sequence $(u_n)_{n \geq 1}$ in $L^p(\Omega; \mathbb{R}^d)$. That is for a function $g \in L^p(\Omega)$ we have that

$$(2.7) \quad |\nabla u_{n_k}(x)| \leq g(x) \quad \text{for all } n \geq 1 \text{ and almost everywhere for } x \in \Omega$$

and that $\nabla u_{n_k}(x) \rightarrow \nabla u(x)$ almost everywhere for $x \in \Omega$. For convenience we rewrite n_k as n . Next by the continuity of $x \mapsto |x|^{p-2}x$ on \mathbb{R}^d , we have the following

$$\begin{aligned}
|\nabla u_n(x)|^{p-2} \nabla u_n(x) &\rightarrow |\nabla u(x)|^{p-2} \nabla u(x) \\
|\nabla u_n(x)|^{p-2} \nabla u_n(x) &= |\nabla u_n(x)|^{p-1} \leq g^{p-1}(x)
\end{aligned}$$

for all $n \geq 1$ and almost everywhere for $x \in \Omega$. Thus each $|\nabla u_n(x)|^{p-2} \nabla u_n(x) \in L^{p'}(\Omega; \mathbb{R}^d)$ and by Lebesgue's dominated convergence theorem (see Theorem B.36) we have,

$$|\nabla u_n|^{p-2} \nabla u_n \rightarrow |\nabla u|^{p-2} \nabla u$$

in $L^{p'}(\Omega; \mathbb{R}^d)$. As $(u_n)_{n \geq 1}$ was an arbitrary sequence in $W_0^{1,p}(\Omega)$, we have shown that the mapping

$$\phi : \begin{cases} W_0^{1,p}(\Omega) \rightarrow L^{p'}(\Omega; \mathbb{R}^d) \\ u \mapsto \vec{\phi}(u) := |\nabla u|^{p-2} \nabla u \end{cases}$$

is continuous. As the p -Laplacian Δ_p can be defined as

$$\Delta_p u = \operatorname{div} \vec{\phi}(u) = (\operatorname{div} \circ \vec{\phi})(u)$$

by Lemma 2.7 we have that $\Delta_p : W_0^{1,p}(\Omega) \rightarrow W^{-1,p'}(\Omega)$ is continuous. \square

We now introduce an energy functional in order to determine the uniqueness of the weak solution to the Poisson problem (2.3).

Proposition 2.8. Let $1 \leq p < \infty$ and $\Omega \subseteq \mathbb{R}^d$ be an open subset. We consider the *energy functional* $\mathcal{E} : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ given by

$$(2.8) \quad \mathcal{E}(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx,$$

for all $u \in W_0^{1,p}(\Omega)$. Then, \mathcal{E} is strictly convex and continuously differentiable on $W_0^{1,p}(\Omega)$, and

$$(2.9) \quad \langle \mathcal{E}'(u), h \rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)} = \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla h dx,$$

for all $u, h \in W_0^{1,p}(\Omega)$.

We recall Definition 2.4 for the Dirichlet p -Laplace operator $\Delta_p : W_0^{1,p}(\Omega) \rightarrow W^{-1,p'}(\Omega)$ which is applied to the energy functional $\mathcal{E} : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ using equation (2.9) we have,

$$\langle \mathcal{E}'(u), h \rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)} = \langle -\Delta_p u, h \rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)},$$

for all $u, h \in W_0^{1,p}(\Omega)$. For every $f \in W^{-1,p'}(\Omega)$ we recall from the definition of weak solutions (see Definition 2.5) that $u \in W_0^{1,p}(\Omega)$ is a weak solution of the boundary value problem (2.3) if u satisfies equation (2.4). To establish well-posedness of the boundary value problem (2.4) we now consider 2 steps, determining existence and uniqueness and then determining continuous dependence. We start by establishing existence and uniqueness.

Theorem 2.9. Let $1 \leq p < \infty$ and $\Omega \subseteq \mathbb{R}^d$ be an open and bounded subset. Then, there exists a unique weak solution $u \in W_0^{1,p}(\Omega)$ to the non-linear Dirichlet boundary problem (2.3).

Proof. Let $1 \leq p < \infty$ and $\Omega \subseteq \mathbb{R}^d$ be an open and bounded subset and for a given $f \in W^{-1,p'}(\Omega)$ we define the functional $\mathcal{E}_f : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ by

$$\mathcal{E}_f(u) := \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \langle f, u \rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)},$$

for all $u, h \in W_0^{1,p}(\Omega)$. By Poincarè's inequality there exists a $c > 0$ so that

$$\|\nabla u\|_{L^p(\Omega)} \geq c \|u\|_{W_0^{1,p}(\Omega)},$$

for all $u \in W_0^{1,p}(\Omega)$. Then, we have the lower estimate

$$(2.10) \quad \mathcal{E}_f(u) \geq \frac{\min\{1, c^p\}}{2p} \|u\|_{W_0^{1,p}(\Omega)}^p - \|f\|_{W^{-1,p'}(\Omega)} \|u\|_{W_0^{1,p}(\Omega)}.$$

Using Young's inequality with $a = \|f\|_{W^{-1,p'}(\Omega)}$, $b = \|u\|_{W_0^{1,p}(\Omega)}$, $\varepsilon = \frac{\min\{1, c^p\}}{4p}$ then the estimate (2.10) is updated to

$$\mathcal{E}_f(u) \geq \frac{\min\{1, c^p\}}{4p} \|u\|_{W_0^{1,p}(\Omega)}^p - \frac{1}{p' \left(\frac{\min\{1, c^p\}}{4}\right)^{\frac{p'}{p}}} \|f\|_{W^{-1,p'}(\Omega)}^{p'}.$$

Thus for $\|u\|_{W_0^{1,p}(\Omega)} \rightarrow +\infty$ we have that $\mathcal{E}_f \rightarrow +\infty$ we conclude that \mathcal{E}_f is coercive. By Theorem B.34 for every $f \in W^{-1,p'}(\Omega)$ there is a unique $u \in W_0^{1,p}(\Omega)$ satisfying

$$\mathcal{E}_f(u) = \min_{v \in W_0^{1,p}(\Omega)} \mathcal{E}_f(v).$$

By Proposition B.17 this is equivalent to stating

$$(2.11) \quad 0 \in \partial \mathcal{E}_f(u).$$

Combining this with Proposition 2.8, \mathcal{E}_f is differentiable on $W_0^{1,p}(\Omega)$ and by Proposition B.16

$$\partial \mathcal{E}_f = \{\mathcal{E}'_f(u)\}.$$

So equation (2.11) yields that $\mathcal{E}_f = 0 \in W_0^{1,p}(\Omega)$ is equivalent to

$$\langle \mathcal{E}'_f(u), v \rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)} = \langle f, v \rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)},$$

for all $u \in W_0^{1,p}(\Omega)$.

Thus we conclude that that $u \in W_0^{1,p}(\Omega)$ is a weak solution of the boundary value problem (2.3) and due to the strict convexity of \mathcal{E}_f is unique. \square

2.2. Mixed boundary p -Laplace operator. We now focus on the existence and uniqueness of problem (1.2) which we recall is given as,

$$(2.12) \quad \begin{cases} -\Delta_p u_n + V(x) |u_n|^{p-2} u_n = f_n & \text{in } \Omega_n, \\ |\nabla u_n|^{p-2} \nabla u_n \cdot \nu + \alpha_n |u_n|^{p-2} u_n = 0 & \text{on } \partial H_n, \\ u_n = 0 & \text{on } \partial \Omega \end{cases}$$

we note that the p -Laplace operator Δ_p in this case is *not* using Dirichlet boundary conditions. We now base the following on the methods in [5, Section 2]. The first step is to introduce the notion of a weak solution and what space this weak solution belongs to.

Let $1 < p < \infty$ and Ω be a bounded domain in \mathbb{R}^d satisfying the Assumption 1.1. Then, for every $n \in \mathbb{N}$, let $(V^{1,p}(\Omega_n), \|\cdot\|_{V^{1,p}(\Omega_n)})$ be a Banach space such that

$$V^{1,p}(\Omega_n) \hookrightarrow W^{1,p}(\Omega_n),$$

so that

$$(2.13) \quad \left\{ u \in C_c^1(\bar{\Omega}_n \setminus \partial \Omega) \cap W^{1,p}(\Omega_n) \mid \int_{\partial H_n} |u|^p d\mathcal{H}_{d-1} < \infty \right\} \subseteq V^{1,p}(\Omega_n),$$

recalling \mathcal{H}_{d-1} denotes the $(d-1)$ dimensional Hausdorff measure restricted on ∂H_n . Here, we equip $V^{1,p}(\Omega_n)$ with the norm

$$(2.14) \quad \|u\|_{V^{1,p}(\Omega_n)} := \left(\|u\|_{L^p(\Omega_n)}^p + \|\nabla u\|_{L^p(\Omega_n)}^p + \left\| u\alpha_n^{1/p} \right\|_{L^p(\partial H_n)}^p \right)^{1/p}$$

for every $u \in V^{1,p}(\Omega_n)$, which is equivalent to the $W^{1,p}$ -norm

$$\|u\|_{W^{1,p}(\Omega_n)} := \left(\|u\|_{L^p(\Omega_n)}^p + \|\nabla u\|_{L^p(\Omega_n)}^p \right)^{1/p},$$

for all $u \in V^{1,p}(\Omega_n)$.

Remark 2.10. We also have that $V^{1,p}(\Omega_n)$ is uniformly convex and a reflexive space due to [5, Remark 2.2b].

Next, we introduce the notion of *weak solutions* to the mixed boundary-value problem (2.13) as given in [5, Definition 2.3].

Definition 2.11. Let

$$a_n(u, v) := \int_{\Omega_n} |\nabla u|^{p-2} \nabla u \nabla v dx + \int_{\Omega_n} V |u|^{p-2} u v dx + \alpha_n \int_{\partial H_n} |u|^{p-2} u v d\mathcal{H}_{d-1}$$

for every $u, v \in V^{1,p}(\Omega_n)$. Then, for given $f \in (V^{1,p}(\Omega_n))'$, we call $u \in V^{1,p}(\Omega_n)$ a *weak solution* of the Poisson problem (1.2) if

$$(2.15) \quad a_n(u, v) = \langle f, v \rangle_{(V^{1,p}(\Omega_n))', V^{1,p}(\Omega_n)},$$

for all $v \in V^{1,p}(\Omega_n)$.

Remark 2.12. We note that $u \mapsto \mathcal{E}(u) := \frac{1}{p} a_n(u, u)$ is convex and continuous on $V^{1,p}(\Omega_n)$. Additionally it is coercivity as $a_n(u, u) \rightarrow \infty$ as $\|u\| \rightarrow \infty$.

Recalling the methods in Section 2.1 by the convexity, coercivity, and continuity of \mathcal{E} on $V^{1,p}(\Omega_n)$, we then deduce that the Poisson problem (1.2) is well-posed.

Theorem 2.13. *For every $f \in (V^{1,p}(\Omega_n))'$, there exists a unique weak solution $u \in V^{1,p}(\Omega_n)$ to the non-linear boundary problem (1.2).*

2.3. The Poisson problem on the limiting domain. Finally, we come to the Poisson problem on the limiting domain.

For given $f \in L^{p'}(\Omega)$, let u be a weak solution of the Poisson problem

$$(2.16) \quad \begin{cases} -\Delta_p u + V(x) |u|^{p-2} u = f & \text{in } \Omega^*, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

where $V(x) \geq 0$. Thus we aim to show that for the weak solutions u_n of problem (2.12) and the weak solution u of problem (2.16), we have that

$$\lim_{n \rightarrow \infty} \|u_n - u\|_{L^q(\Omega_n)} = 0$$

for every $1 \leq q < p^*$.

We start by defining a suitable space for weak solutions to the Poisson problem (2.16).

Notation 2.14. The space $W_{0,\partial\Omega}^{1,p}(\Omega^*)$ is given by

$$(2.17) \quad W_{0,\partial\Omega}^{1,p}(\Omega^*) := \{u \in W^{1,p}(\Omega^*) \mid \text{trace } u = 0 \text{ on } \partial\Omega\}.$$

Remark 2.15. (1) We note that for $1 < p < d$ and $d \geq 2$, the limit problem (2.16) coincides with Poisson problem on Ω , that is,

$$\begin{cases} -\Delta_p u + V(x) |u|^{p-2} u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

since the space $W_{0,\partial\Omega}^{1,p}(\Omega) = W_{0,\partial\Omega}^{1,p}(\Omega^*)$ this follows from [9, Lemma 3.2.3]. If we intend to describe limit problem (2.16) on Ω for $p \geq d$, then one would need to add zero-order terms similar to

$$\sum_{j=1}^m \text{Cap}_p(\{x_j\}) |u|^{p-2} u,$$

where $\text{Cap}_p(\{x_j\})$ denotes the p -capacity of the point $x_j \in \Omega$. We leave the details to a forthcoming research project.

(2) Further, we remark that the convergence result in Theorem 1.2 is not yet complete as it does not provide the uniqueness of the weak solution u of the limiting problem (1.3). But we leave this task to future research.

3. BOUNDED FAMILIES OF SOBOLEV EMBEDDINGS

3.1. Preliminaries. We begin by introducing the transformation operator $\mathbf{A}_{\phi^{-1}}$, which is central to establish Sobolev embeddings with upper bounds independent of the domain. Here, we follow [1, Section 3.34].

Definition 3.1. Let Ω and $G \subseteq \mathbb{R}^d$ be two given open subsets, we call a mapping $\phi : \Omega \rightarrow G$ a C^1 -*diffeomorphism* if ϕ is bijective and continuously differentiable on Ω and its inverse $\phi^{-1} : G \rightarrow \Omega$ is continuously differentiable on G .

The following notation is used throughout the thesis.

Notation 3.2. For a given measure space $(\Sigma, \mathcal{B}, \lambda)$, we denote by $M(\Sigma, \lambda)$ the set of λ -a.e. equivalence classes of measurable functions $u : \Sigma \rightarrow \mathbb{R}^d$, meaning that two functions $u, v : \Sigma \rightarrow \mathbb{R}^d$, belong to the same equivalence class provided

$$u(x) = v(x) \quad \lambda\text{-a.e. on } \Sigma.$$

If Σ is an open subset of \mathbb{R}^d and λ denotes a Lebesgue measure of \mathbb{R}^d restricted on Σ , then we write $M(\Sigma)$ instead of $M(\Sigma, \lambda)$.

Definition 3.3. Let Ω and $G \subseteq \mathbb{R}^d$ be two given open subsets and $\phi : \Omega \rightarrow G$ a continuous bijective mapping, we denote by $\mathbf{A}_{\phi^{-1}} : M(\Omega) \rightarrow M(G)$ the *composition operator*

$$(3.1) \quad (\mathbf{A}_{\phi^{-1}} u)(y) := u(\phi^{-1}(y)),$$

for every $u \in M(\Omega)$ and every $y \in G$.

Remark 3.4 (Inverse mapping). We recall, for a given open subset $\Omega \subseteq \mathbb{R}^d$ and continuous bijection $\phi : \mathbb{R}^d \rightarrow \mathbb{R}^d$, the composition $\mathbf{A}_{\phi^{-1}} u = u \circ \phi^{-1}$ belongs to $M(G)$ for every $u \in M(\Omega)$ (see [14, Section 1.11]). Furthermore, it is not difficult to see that the operator $\mathbf{A}_{\phi^{-1}} : M(\Omega) \rightarrow M(G)$ is linear. Now, Let $\mathbf{B}_\phi : M(G) \rightarrow M(\Omega)$ be the operator given by

$$(3.2) \quad (\mathbf{B}_\phi u)(x) := u(\phi(x))$$

for every $u \in M(G)$ and $x \in \Omega$. Then, one sees that for every $u \in M(G)$,

$$(\mathbf{A}_{\phi^{-1}} \circ \mathbf{B}_{\phi})u = \mathbf{A}_{\phi^{-1}}(u \circ \phi) = u(\phi^{-1} \circ \phi) = u.$$

Similarly, one verifies that $\mathbf{B}_{\phi} \circ \mathbf{A}_{\phi^{-1}} = I_{M(\Omega)}$. Hence, the composition operator $\mathbf{A}_{\phi^{-1}} : M(\Omega) \rightarrow M(G)$ as given by (3.1) is bijective and \mathbf{B}_{ϕ} given by (3.2) is its *inverse*, so we have $\mathbf{A}_{\phi^{-1}}^{-1} = \mathbf{B}_{\phi}$.

Notation 3.5. For a given continuous matrix-valued function $f = (f_{ij})_{i,j=1}^d : \Omega \rightarrow \mathbb{R}^{d \times d}$ defined on an open subset $\Omega \subseteq \mathbb{R}^d$, we set

$$|f|_{\inf, \Omega} := \inf_{x \in \Omega} |\det f(x)|$$

and

$$|f|_{\infty, \Omega} := \sup_{x \in \Omega} |\det f(x)|.$$

With this notation in mind, we now fix the following assumption.

Assumption 3.6. Let Ω and $G \subseteq \mathbb{R}^d$ be two open subsets, and $\phi : \Omega \rightarrow G$ be a \mathcal{C}^1 -diffeomorphism and $D\phi$ be its Jacobian. We assume that

$$0 < |D\phi|_{\inf, \Omega} \text{ and } |D\phi|_{\infty, \Omega} < \infty.$$

Then, one has that

$$0 < |D\phi|_{\inf, \Omega} \leq |\det D\phi(x)| \leq |D\phi|_{\infty, \Omega} < \infty$$

for every $x \in \Omega$.

We provide some motivating examples of transformations ϕ .

Example 3.7. For a given invertible matrix $\mathbf{A} \in \mathbb{R}^{d \times d}$ and vector $\mathbf{b} \in \mathbb{R}^d$, let $\phi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be given by

$$(3.3) \quad \phi(x) := \mathbf{A}x + \mathbf{b}$$

for all $x \in \mathbb{R}^d$. The *affine transformation* ϕ given by (3.3) is an \mathcal{C}^1 -diffeomorphism with inverse $\phi^{-1} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ given by

$$\phi^{-1}(y) := \mathbf{A}^{-1}(y + \mathbf{b})$$

for all $y \in \mathbb{R}^d$.

We also have that the derivative $D\phi = \mathbf{A}$. Several specific examples are given below.

- (1) The first example of ϕ is that of a *rotation transformation*

$$\phi(x) = \mathbf{R}x, \quad x \in \mathbb{R}^d,$$

which consists of a rotation matrix $\mathbf{R} \in SO(d, \mathbb{R})$ (see Definition A.6). We recall that \mathbf{R} is orthogonal and has determinant $\det \mathbf{R} = 1$.

- (2) The next example is the *shear transformation*,

$$\phi(x) = \mathbf{R}_s x. \quad x \in \mathbb{R}^d$$

which consists of a composition shear matrices $\mathbf{R}_s \in SL(d, \mathbb{R})$ (see Definition A.7). We recall that the volume is preserved and the determinate $\det \mathbf{R}_s = 1$.

(3) The final example is the *parallelepiped to cube transform*,

$$\phi(x) := \mathbf{SRR}_s x + \mathbf{b} \quad x \in \mathbb{R}^d$$

this transformation consists of a composition of a scaling matrix with non-zero scaling factors \mathbf{S} , a rotation matrix \mathbf{R} , a non-singular shearing matrix \mathbf{R}_s and a translation vector \mathbf{b} . The inverse mapping *cube to parallelepiped transform* is then given by

$$\phi^{-1}(y) := \mathbf{R}_s^{-1} \mathbf{R}^{-1} \mathbf{S}^{-1}(y - \mathbf{b}) \quad y \in \mathbb{R}^d.$$

We also recall that the determinate of the rotation and shear matrices are equal to 1 we have $\det \mathbf{SRR}_s = \det S$ and $\det \mathbf{R}_s^{-1} \mathbf{R}^{-1} \mathbf{S}^{-1} = \det \mathbf{S}^{-1}$ respectively.

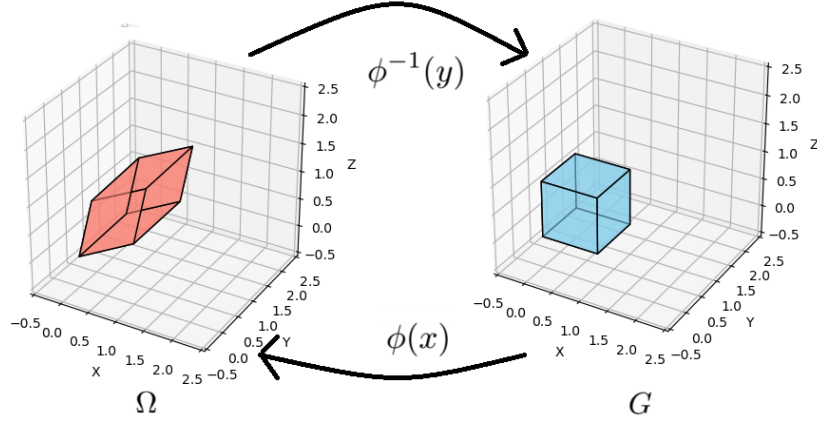


FIGURE 1. An example of a parallelepiped transforming into a cube.

Lemma 3.8. *Let $1 \leq p < \infty$, Ω and $G \subseteq \mathbb{R}^d$ be two open subsets, and $\phi : \Omega \rightarrow G$ a diffeomorphism satisfying Assumption 3.6. Then, the transformation operator*

$$\mathbf{A}_{\phi^{-1}} : L^p(\Omega) \rightarrow L^p(G) \quad \text{defined by (3.1)}$$

is a well-defined linear isomorphism. In particular, one has that

$$(3.4) \quad 0 < |D\phi|_{\inf, \Omega}^{1/p} \|u\|_{L^p(\Omega)} \leq \|\mathbf{A}_{\phi^{-1}} u\|_{L^p(G)} \leq |D\phi|_{\infty, \Omega}^{1/p} \|u\|_{L^p(\Omega)}$$

for every $u \in L^p(\Omega)$.

Proof. Let $u \in L^p(\Omega)$. Then, as mentioned in Remark 3.4, $\mathbf{A}_{\phi^{-1}} u : G \rightarrow \mathbb{R}$ is measurable. Moreover, by the definition (3.1) of $\mathbf{A}_{\phi^{-1}}$ and by using the substitution $y = \phi(z)$, we see that

$$\begin{aligned} \|\mathbf{A}_{\phi^{-1}} u\|_{L^p(G)}^p &= \int_G |u(\phi^{-1}(y))|^p dy \\ \|\mathbf{A}_{\phi^{-1}} u\|_{L^p(G)}^p &= \int_{\Omega} |u(z)|^p |\det D\phi(z)| dz \\ &\leq |D\phi|_{\infty, \Omega} \int_{\Omega} |u(z)|^p dz. \end{aligned}$$

Thus, $\mathbf{A}_{\phi^{-1}}u \in L^p(G)$. Now, to see that the lower bound of (3.4) also holds, we note that this is proven in the same way by using \mathbf{B}_ϕ defined in (3.2) and by recalling that $\mathbf{B}_\phi = \mathbf{A}_{\phi^{-1}}^{-1}$. \square

The following theorem shows that the transformation operator $\mathbf{A}_{\phi^{-1}}$ for a diffeomorphism ϕ remains an isomorphism between Sobolev spaces.

Theorem 3.9. *Let $1 \leq p < \infty$, Ω and G be two open subsets of \mathbb{R}^d , and $\phi : \Omega \rightarrow G$ a diffeomorphism satisfying Assumption 3.6. Then, the transformation operator $\mathbf{A}_{\phi^{-1}} : W^{1,p}(\Omega) \rightarrow W^{1,p}(G)$ given by (3.2) is a well-defined isomorphism. In particular, one has that*

$$(3.5) \quad 0 < C_1(\phi, \Omega) \|u\|_{W^{1,p}(\Omega)} \leq \|\mathbf{A}_{\phi^{-1}}u\|_{W^{1,p}(G)} \leq C_2(\phi, \Omega) \|u\|_{W^{1,p}(\Omega)}$$

for every $u \in W^{1,p}(\Omega)$, where

$$C_1(\phi, \Omega) = \min \left(|D\phi|_{\inf, \Omega}, \frac{|D\phi|_{\inf, \Omega}}{|D\phi|_{\infty, \Omega}} \right)^{1/p}$$

and

$$C_2(\phi, \Omega) = \max \left(|D\phi|_{\infty, \Omega}, \frac{|D\phi|_{\infty, \Omega}}{|D\phi|_{\inf, \Omega}} \right)^{1/p}.$$

Example 3.10. We now provide the computations of the constants $C_1(\phi, \Omega)$ and $C_2(\phi, \Omega)$ for the rotation transformation, the shear transformation, and the transformation from a parallelepiped to cube. (see Example 3.7).

- (1) *Rotation transformation $\phi(x) = \mathbf{R}x$:* Here, \mathbf{R} is a rotation/orthogonal matrix in $\mathbb{R}^{d \times d}$ with determinant $\det \mathbf{R} = 1$. Thus, the constants are given by

$$C_1(\phi, \Omega) = C_2(\phi, \Omega) = 1.$$

- (2) *Shear transformation $\phi(x) = \mathbf{R}_s x$:* Here, \mathbf{R} is a shear/special linear matrix in $\mathbb{R}^{d \times d}$ with determinant $\det \mathbf{R} = 1$. Thus, the constants are given by

$$C_1(\phi, \Omega) = C_2(\phi, \Omega) = 1.$$

- (3) *Parallelepiped to cube $\phi := \mathbf{SRR}_s + \mathbf{b}$:* As both the determinate of the rotation and shear matrices are equal to 1, we simply have

$$(3.6) \quad C_1(\phi, \Omega) = \min \left(1, |\mathbf{S}|^{1/p} \right).$$

$$(3.7) \quad C_2(\phi, \Omega) = \max \left(1, |\mathbf{S}|^{1/p} \right).$$

The proof of Theorem 3.9 following the methods in [1, Section 3.34] is outlined below.

Proof. Let $u \in W^{1,p}(\Omega)$. Then, by Lemma 3.8, $\mathbf{A}_{\phi^{-1}}u \in L^p(G)$. It remains to show that $\mathbf{A}_{\phi^{-1}}u \in W^{1,p}(G)$, that is, we need to show that there are $g_1, \dots, g_d \in L^p(G)$ such that

$$(3.8) \quad \int_G \mathbf{A}_{\phi^{-1}}u D_i \xi \, dx = - \int_G g_i D_i \xi \, dx$$

for every $\xi \in C_c^\infty(G)$ and every $i = 1, \dots, d$. Due to the Meyers and Serrin theorem (see Theorem B.6 in the Appendix), and so there exists a sequence

$$(u_n)_{n \geq 1} \subseteq C^1(G) \cap W^{1,p}(G)$$

satisfying

$$(3.9) \quad u_n \rightarrow u \quad \text{in } L^p(\Omega) \text{ as } n \rightarrow \infty,$$

$$(3.10) \quad \text{for every } i = 1, \dots, d, D_i u_n \rightarrow D_i u \text{ in } L^p(\Omega) \text{ as } n \rightarrow \infty.$$

Since the transformation ϕ is a diffeomorphism, we have for every $n \geq 1$ that $\mathbf{A}_{\phi^{-1}u_n}$ is continuously differentiable and

$$D_i [\mathbf{A}_{\phi^{-1}u_n}] (x) = D\phi^{-1}(x) D_i u_n(x),$$

for every $x \in G$. From this, we conclude that each $\mathbf{A}_{\phi^{-1}u_n}$ satisfies

$$(3.11) \quad \int_G \mathbf{A}_{\phi^{-1}u_n} D_i \xi \, dx = - \int_G D\phi^{-1} D_i u_n D_i \xi \, dx$$

for every $\xi \in C_c^\infty(G)$ and every $i = 1, \dots, d$. Moreover, since $D\phi^{-1}$ is continuous on G , we have for every compact subset $K \subseteq G$,

$$|D\phi^{-1}(x) D_i u_n(x)| \leq \|D\phi^{-1}\|_{L^\infty(K)} |D_i u_n(x)|$$

for every $x \in K$ and $n \geq 1$. Thus, and by (3.10), it follows that

$$\int_G D\phi^{-1}(x) D_i u_n(x) \xi \, dx \rightarrow \int_G D\phi^{-1}(x) D_i u \xi \, dx \quad \text{as } n \rightarrow \infty.$$

Furthermore, by (3.9) and by the continuity of $\mathbf{A}_{\phi^{-1}} : L^p(\Omega) \rightarrow L^p(G)$, we conclude that

$$\int_G \mathbf{A}_{\phi^{-1}u_n} D_i \xi \, dx \rightarrow \int_G \mathbf{A}_{\phi^{-1}u} D_i \xi \, dx \quad \text{as } n \rightarrow \infty.$$

Therefore, sending $n \rightarrow \infty$ in the integral equation (3.11) shows that $\mathbf{A}_{\phi^{-1}u}$ satisfies (3.8) with $g_i = D\phi^{-1}(x) D_i u$ for every $i = 1, \dots, d$.

To conclude this proof, we note that by (3.1) and by the substitution $x = \phi(z)$ for every $z \in \Omega$, one finds

$$\begin{aligned} \|\mathbf{A}_{\phi^{-1}u_n}\|_{L^p(G)}^p &= \int_G |\nabla u_n(\phi^{-1}(x))|^p \, dx \\ &= \int_G |\nabla u_n(z)|^p |\det D\phi^{-1}(z)| |\det D\phi(z)| \, dz \\ &\leq \frac{|D\phi|_{\infty, \Omega}}{|D\phi|_{\inf, \Omega}} \int_G |\nabla u_n(z)|^p \, dz. \end{aligned}$$

Thus, the limit (3.10) implies that

$$\|\mathbf{A}_{\phi^{-1}u}\|_{L^p(G)}^p \leq \frac{|D\phi|_{\infty, \Omega}}{|D\phi|_{\inf, \Omega}} \int_G |\nabla u(z)|^p \, dz.$$

From this and by the upper estimate in (3.4), we conclude that

$$\|\mathbf{A}_{\phi^{-1}u}\|_{W^{1,p}(G)} \leq C_2(\phi, \Omega) \|u\|_{W^{1,p}(\Omega)}$$

holds, establishing the upper bound of (3.5). The lower bound of (3.5) is proven in the same way and finally bounding the inverse is analogously proven by using \mathbf{B}_ϕ (recall $\mathbf{B}_\phi = \mathbf{A}_{\phi^{-1}}^{-1}$ in place of $\mathbf{A}_{\phi^{-1}}$ and performing the same steps with G and Ω swapped). This establishes the proof of Theorem 3.9. \square

With these preliminaries in mind, we now turn to our first main section on Sobolev embeddings with constants independent of the volume of the domain.

3.2. Sobolev embeddings independent of the volume of the domain. We recall the Sobolev embedding theorem for $\Omega = \mathbb{R}^d$ (see, for example [11, Chapter 9]).

Theorem 3.11 (Sobolev, Gagliardo and Nirenberg embedding theorem). *For every $1 \leq p < d$, one has that*

$$W^{1,p}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$$

for every $p \leq q \leq p^*$, we recall that $p^* = \frac{pd}{d-p}$ (see equation (1.4)). In the case that $p = d$, one has that

$$W^{1,p}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$$

for every $d \leq q \leq \infty$. Finally in the case where $p > d$, one has that

$$W^{1,p}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d).$$

In each case the Sobolev embedding constants only depend on p and d .

In this section, we develop Sobolev embeddings for more general domains Ω of \mathbb{R}^d . The aim is to determine conditions such that the Sobolev embedding constant is independent of the volume $|\Omega|$ of the domain Ω . In general this is not true as we show in Corollary 3.24. To ensure domain independence of the Sobolev embedding constant, one requires, for example some form of regularity of the boundary of the domain. Namely the so called ‘‘cone conditions’’. We demonstrate that this is a sufficient condition for the Sobolev embedding constant to be independent on the volume $|\Omega|$ of Ω . We start this section by introducing the concept of finite cones and the interior and exterior cone conditions following [1, Section 4.1 and 4.3].

Definition 3.12. Given a point $x \in \mathbb{R}^d$, let B_1 be an open ball centred at x and B_2 be a second open ball so that $x \notin B_2$. Then, a *finite cone* C_x at vertex $x \in \mathbb{R}^d$ is given by the set

$$C_x = B_1 \cap \{x + \lambda(y - x) : y \in B_2, \lambda > 0\}.$$

Definition 3.13. An open subset $\Omega \subseteq \mathbb{R}^d$ is said to satisfy the *interior cone condition* if every $x \in \Omega$ is the vertex of a finite cone C_x with its closure contained in Ω . Moreover, after possible rotation and translation C_x is represented by

$$(3.12) \quad C_x = \{x = (x_1, x_2, \dots, x_{d-1}, x_d) \in \mathbb{R}^d \mid x_1^2 + \dots + x_{d-1}^2 < bx_d^2 \ \& \ x_d \in (0, a)\}$$

for some $a, b > 0$. Furthermore, a closed set $G \subseteq \mathbb{R}^d$ is said to satisfy the *exterior cone condition* if the complement $G^c := \mathbb{R}^d \setminus G$ satisfies the interior cone condition.

Example 3.14. To better visualise the cone conditions, we provide several examples of open sets $\Omega \subseteq \mathbb{R}^d$ which satisfy the interior cone condition:

- (1) Ω is an open ball (see Definition A.4),
- (2) Ω is an open cube (see Definition A.5),
- (3) Ω is a parallelepiped (see Definition A.3),
- (4) Ω is a Lipschitz domain (see Definition A.2).

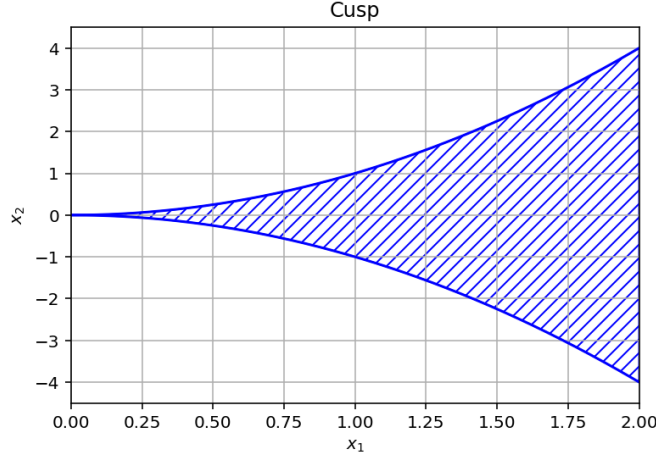
On the other hand, an example of Ω which does *not* satisfy the interior cone condition is the cusp, that is

$$\Omega = \{x = (x_1, \dots, x_{d-1}, x_d) \in \mathbb{R}^d \mid x_d > 0, |(x_1, \dots, x_{d-1})| < x_d^\alpha, 1 < \alpha\}.$$

An example of a cusp is demonstrated in Figure 2.

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For completeness we provide a proof for each of these examples.

FIGURE 2. An example of a cusp domain with $\alpha = 2$.

Proof. Open Ball: Let $\Omega \subset \mathbb{R}^d$ be an open ball with radius $r > 0$ and dimension $d \geq 2$. Then we choose an interior cone C_{ball} so that $a = \frac{r}{\sqrt{2}}$ and $b = 1$ (see equation (3.12)). By definition this satisfies the interior cone condition as

$$x_1^2 + x_2^2 + \cdots + x_d^2 < (1+b)x_d^2 < 2a^2 = r^2.$$

Open Cube: Let $\Omega \subset \mathbb{R}^d$ be an open cube with side length $c > 0$ and dimension $d \geq 2$. We then choose an interior cone C_{Cube} with constants $b = 1$ and $a = \frac{\sqrt{dc}}{2\sqrt{2}}$ (see equation (3.12)). By definition this satisfies the interior cone condition as

$$x_1^2 + x_2^2 + \cdots + x_d^2 < (1+b)x_d^2 < 2a^2 = \frac{ds^2}{4}$$

and as C_{Cube} is symmetric in x_1, \dots, x_{d-1} (after rotation and translation), we have that

$$|x_i| < \frac{\sqrt{b}}{\sqrt{d-1}}x_d < \frac{1}{\sqrt{d-1}} \frac{\sqrt{dc}}{2\sqrt{2}} < \frac{1}{2}c$$

for $d \geq 2$, which implies that C_{Cube} lies entirely inside the open cube with any vertex $x \in \Omega$.

Parallelepiped: Let Ω be a parallelepiped. Then, we use the fact that Ω is isomorphic to a cube after scaling, rotation and shearing (see Example 3.7, (3)). As the open cube satisfies the interior cone condition with cone C_{Cube} , we now determine an interior cone that also works for Ω with cone C_{Par} . Recall that the interior cones remain unchanged after rotation. Further, scaling changes the size of the cone. Thus, we first scale the entire cone C_{Cube} by the smallest non-zero factor in the scaling matrix S , defined as s_{\min} . If the smallest scaling factor is larger than 1, then the size of the cone remains unchanged.

Finally, when one shears the cone, the cone may widen, thus we shrink the aperture angle of the cone so that the face is bounded above by an open ball with radius equivalent to the radius of the minor axis of the sheared cone of the open cube. Thus the interior cone condition still holds.

Lipschitz domain: Let Ω be a Lipschitz domain and x be a point on the boundary $\partial\Omega$ with neighbourhood $U_x \subseteq \mathbb{R}^d$. Further, let $f : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ be a Lipschitz-continuous function such that for every $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$, $f(\xi_1, \dots, \xi_{d-1}) = \xi_d$ which means that $\xi \in \partial\Omega$, moreover we recall that if ξ lies in the interior of the domain then one has that

$$f(\xi_1, \dots, \xi_{d-1}) < \xi_d,$$

and ξ lies in the exterior of the domain then one has that

$$f(\xi_1, \dots, \xi_{d-1}) > \xi_d,$$

For further details on this property we refer the reader to Remark 3.19. We recall the following estimate from Definition A.2 for every $\xi = (\xi_1, \dots, \xi_d) \in U_x$ near $0 \in \mathbb{R}^d$ in a local coordinate system centred on x such that the interior is given as Ω^c and the exterior as Ω , then one has that

$$(3.13) \quad |f(\xi_1, \dots, \xi_{d-1})| \leq M \|(\xi_1, \dots, \xi_{d-1})\|_2$$

for some Lipschitz constant $M > 0$. We now introduce a finite cone C_0 with vertex centred at 0, as defined in equation (3.12) with b such that $0 < b < \frac{1}{M^2}$. We have for every $\xi \in C_x \cap U_x$ that,

$$|f(\xi_1, \dots, \xi_{d-1})| < M \sqrt{\frac{1}{M^2}} \xi_d = \xi_d.$$

Thus $U_x \cap C_x \subseteq U_x \cap \Omega$ by definition. As x was an arbitrary point on the boundary we conclude that Ω^c satisfies the exterior cone condition and therefore Ω satisfies the interior cone condition. \square

Definition 3.15. An open subset $\Omega \subseteq \mathbb{R}^d$ is said to satisfy the *uniform interior cone condition* if there exists a family $\{U_j\}_{j \in \mathbb{N}}$ of open subsets $U_j \subseteq \mathbb{R}^d$ with each U_j having finite volume $|U_j|$ and $\{U_j\}_{j \in \mathbb{N}}$ covers the boundary $\partial\Omega$, that is,

$$\partial\Omega \subseteq \bigcup_{j \in \mathbb{N}} U_j.$$

Additionally we have a corresponding sequence $\{C_j\}_{j \in \mathbb{N}}$ of finite cones C_j , which are respectively congruent to a fixed cone C . We also have that $\{U_j\}_{j \in \mathbb{N}}$ and $\{C_j\}_{j \in \mathbb{N}}$ satisfy the following properties:

- (1) There is an $M > 0$ such that for every $j \geq 1$, the diameter (see Definition B.2)

$$\text{diam}(U_j) \leq M.$$

- (2) For every $\delta > 0$ sufficiently small, one has that

$$\Omega_\delta := \{x \in \Omega : \text{dist}(x, \partial\Omega) < \delta\} \subseteq \bigcup_{j=1}^{\infty} U_j.$$

- (3) For every $j \geq 1$, one has that

$$(3.14) \quad Q_j := \bigcup_{x \in \Omega \cap U_j} (x + C_j) \subset \Omega.$$

- (4) there is an $R \in \mathbb{N}$ such that every collection $\{Q_{j_1}, \dots, Q_{j_{R+1}}\}$ of $R+1$ sets Q_j defined in (3.14) satisfies

$$\bigcap_{j=1}^{R+1} Q_{j_i} = \emptyset.$$

Example 3.16. All of the examples provided for the interior cone condition (see Example 3.14) also satisfy a uniform interior cone condition, with the exception of the cusp which also does not satisfy the uniform interior cone condition.

It is possible to decompose a bounded domain satisfying the interior cone condition into a finite collection of open subsets with associated parallelepipeds.

Theorem 3.17. *Let $\Omega \subseteq \mathbb{R}^d$ be an open bounded subset satisfying the interior cone condition. Then, there exists a finite collection $\{\Omega_1, \dots, \Omega_m\}$ of open subsets Ω_j of Ω such that*

$$(3.15) \quad \Omega = \bigcup_{j=1}^m \Omega_j.$$

Additionally for each Ω_j , there exists a corresponding subset $A_j \subseteq \overline{\Omega}_j$ given by

$$(3.16) \quad A_j := \{x \in \overline{\Omega} : x + P_j \subset \Omega\},$$

where P_j is an open parallelepiped with a vertex at the origin associated to Ω_j and A_j respectively, such that

$$\Omega_j := \bigcup_{x \in A_j} (x + P_j).$$

We prove Theorem 3.17 following the methods in the first part of [1, Theorem 4.8].

Proof. Let C_0 be the finite cone with vertex at the origin $0 \in \mathbb{R}^d$ associated with the interior cone condition. Then for every $x \in \Omega$, x is the vertex of a finite cone $C_x \subseteq \Omega$, which is congruent to C_0 and contained wholly within Ω . We introduce a finite number of “sub”-cones C_1, \dots, C_k of which each have their vertices at zero (and each aperture angle is less than the aperture angle of C_0), so that any cone congruent to C_0 with vertex at the origin contains at least one of the cones C_j , for $1 \leq j \leq k$.

For each C_j we associate an open parallelepiped P_j with one vertex at the origin so that $P_j \subset C_j$. Then, for each $x \in \Omega$ there exists j , for $1 \leq j \leq k$ such that

$$x + P_j \subset x + C_j \subset C_x \subseteq \Omega.$$

Since Ω is open and $\overline{x + P_j}$ is compact, it follows that $y + P_j \subset \Omega$ for all y sufficiently close to x . Furthermore, for every $x \in \Omega$ we find $y \in \Omega$ such that $x \in y + P_j$ for some j , $1 \leq j \leq k$. Thus we conclude that Ω is expressed as a union of finitely many translates of parallelepipeds. We now have that the set A_j given by equation (3.16) is used to construct Ω_j by

$$\Omega_j = \bigcup_{x \in A_j} (x + P_j)$$

and by setting $m = k$ we have demonstrated equation (3.15). Thus completes the proof. \square

We aim to demonstrate that bounded domains satisfying the interior cone condition can be broken down into a finite collection of subsets each satisfying the strong local Lipschitz property. For the sake of completeness we define the strong local Lipschitz property as in Definition [1, Definition 4.5].

Definition 3.18. An open subset $\Omega \subseteq \mathbb{R}^d$ is said to satisfy the *strong local Lipschitz condition* if there exists constants $\delta, M > 0$, a locally finite open cover $\{U_j\}_{j \in \mathbb{N}}$ of $\partial\Omega$ and each $U_j \in \{U_j\}_{j \in \mathbb{N}}$ has an associated function $f_j : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$. Additionally the following conditions hold:

- (1) There exists a constant integer R such that every collection of $R + 1$ sets U_j has empty intersection.
- (2) For every pair of sufficiently close points

$$x, y \in \Omega_\delta := \{x \in \Omega : \text{dist}(x, \partial\Omega) < \delta\}$$

with $\text{dist}(x, y) < \delta$ there exists j such that,

$$x, y \in \mathcal{V}_j := \{x \in U_j : \text{dist}(x, \partial U_j) > \delta\}$$

- (3) Each function f_j satisfies the *Lipschitz condition* with constant M :

$$|f(\xi_1, \dots, \xi_{d-1}) - f(\eta_1, \dots, \eta_{d-1})| \leq M|(\xi_1 - \eta_1, \dots, \xi_{d-1} - \eta_{d-1})|.$$

- (4) For a fixed Cartesian coordinate system $(\xi_{j,1}, \dots, \xi_{j,d})$ in U_j the intersection of $U_j \cap \Omega$ is represented by

$$U_j \cap \Omega = \{\xi_j = (\xi_{j,1}, \dots, \xi_{j,d}) \in U_j | \xi_{j,d} < f_j(\xi_{j,1}, \dots, \xi_{j,d-1})\}$$

Remark 3.19. If $\Omega \subseteq \mathbb{R}^d$ is a bounded open set then it satisfies Definition 3.18 with the simplified requirement that the boundary $\partial\Omega$ is locally Lipschitz (see Definition A.2). We also note that the inequality relating $\xi_{j,d}$ to $f_j(\xi_{j,1}, \dots, \xi_{j,d-1})$ is also able to determine the boundary and exterior of the domain, that is

$$U_j \cap \Omega^c = \{\xi_j = (\xi_{j,1}, \dots, \xi_{j,d}) \in U_j | \xi_{j,d} > f_j(\xi_{j,1}, \dots, \xi_{j,d-1})\}$$

and

$$U_j \cap \partial\Omega = \{\xi_j = (\xi_{j,1}, \dots, \xi_{j,d}) \in U_j | \xi_{j,d} = f_j(\xi_{j,1}, \dots, \xi_{j,d-1})\}.$$

We now demonstrate the desired theorem relating bounded domains satisfying the interior cone condition, to the strong local Lipschitz property.

Theorem 3.20. *Let $\Omega \subseteq \mathbb{R}^d$ be a given bounded open subset satisfying the interior cone condition. Then, for every sufficiently small $\rho > 0$, there is a finite collection of subsets $\{U_1, \dots, U_m\}$ as defined in Theorem 3.17 such that*

$$\rho > \sup_{j \in \{1, \dots, m\}} \text{diam}(\overline{\Omega}_j)$$

then each $\Omega_j \in \{U_1, \dots, U_m\}$ has the strong local Lipschitz property.

Theorem 3.20 is proven following the methods in the second part of [1, Theorem 4.8].

Proof. We recall from Theorem 3.17 the set A_j is given by

$$A_j := \{x \in \overline{\Omega}_j : x + P_j \subset \Omega\}.$$

For each $j \in \{1, \dots, m\}$ there are 2 cases:

Case 1: If the diameter of A_j is less than ρ we directly utilise A_j .

Case 2: If the diameter of A_j is greater than ρ we split A_j into a finite union of sets with diameter less than ρ and relabel these accordingly for the new larger collection.

Thus after relabelling we now focus on an arbitrary $\Omega_j \subseteq \{U_1, \dots, U_m\}$ given by

$$\Omega_j = \bigcup_{x \in A_j} (x + P_j)$$

where $\text{diam}(A_j) \leq \rho$ and P_j is the parallelepiped associated to A_j . To demonstrate that Ω_j satisfies the strong Lipschitz condition (see Definition 3.18) for sufficiently small ρ , we introduce an infinite pyramid with a corner piece centred on each of the 2^d vertices v_l , $1 \leq l \leq 2^d$, of the fixed parallelepiped P_j defined as

$$Q_l := \{y = v_l + \lambda(x - v_l) : x \in P_j, \lambda > 0, 1 \leq l \leq 2^d\}.$$

P_j is expressed as the intersection of each of the pyramids,

$$P_j = \bigcap_{l=1}^{2^d} Q_l.$$

We set $\delta = \text{dist}(x_0, \partial P_j)$ where x_0 is the centre of P_j and introduce an open ball B with radius $\sigma = \frac{\delta}{2}$. Next, we suppose B is able to cross two opposite faces at $x + P_j$, $y + P_j$ for every $x, y \in A_j$, we now set two arbitrary points from each opposing face a and b such that $x + a \in B$ and $y + b \in B$, then the following inequality is established

$$\begin{aligned} \rho &\geq \text{dist}(x, y) = \text{dist}(x + b, y + b) \\ &\geq \text{dist}(x + b, x + a) - \text{dist}(x + a, y + b) \\ &\geq 2\delta - 2\sigma = \delta. \end{aligned}$$

Therefore if $\rho < \delta$, and $x \in \Omega_j$ is fixed then B cannot cross two opposite faces of the parallelepiped $x + P_j$. So we have a vertex $x + v_l$ that it is shared with all faces that intersect with B (if no faces intersect this is also trivially true) so the pyramid Q_l and parallelepiped P_j intersected with the ball are equal, that is

$$B \cap (x + P_j) = B \cap (x + Q_l)$$

which means that

$$B \cap \Omega_j = B \cap (\cup_{x \in A_j} (x + Q_l)).$$

Next we choose coordinates $\xi = (\xi_1, \dots, \xi_{d-1}, \xi_d)$ in B such that the ξ_d axis lies in the direction of the centre of P_j towards the vertex v_l . Then, $(x + Q_l) \cap B$ is represented by an equality of the form $\xi_d < f_x(\xi_1, \dots, \xi_{d-1})$, where f_x satisfies a Lipschitz condition with a constant independent of x . Finally we define $f : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ by

$$f(\xi_1, \dots, \xi_{d-1}) := \sup_{x \in A_j} f_x(\xi_1, \dots, \xi_{d-1})$$

which is also a Lipschitz function. So we have $\xi_d < f(\xi_1, \dots, \xi_{d-1})$ and as B was arbitrarily placed on $\partial \Omega_j$, Ω_j has the strong Lipschitz property for all $1 \leq j \leq m$. \square

We now use the above results to demonstrate a volume independent embedding for $W^{1,p}(\Omega)$ when the open subset $\Omega \subseteq \mathbb{R}^d$ is a bounded and satisfies the interior cone condition.

Theorem 3.21. *Let $1 \leq p < d$, $d \geq 2$, $p^* := dp/(d-p)$ and $\Omega \subseteq \mathbb{R}^d$ be an bounded open subset satisfying the interior cone condition. Then, we have that*

$$\|u\|_{L^{p^*}(\Omega)} \leq C_{Sob,p} \|u\|_{W^{1,p}(\Omega)}$$

for every $u \in W^{1,p}(\Omega)$ with uniform constant

$$C_{Sob,p} = \max \left\{ \sup_{s,Q} |\mathcal{S}_{s,Q}|^{-1/p}, 1 \right\} 2^{(p-1)/p} \frac{dp-p}{d-p},$$

where $\mathcal{S}_{s,Q}$ is a scaling factor for the parallelepiped to cube transformation on a subset of Ω and is dependent only on C , the cone associated with the interior cone condition and d .

Remark 3.22. We note that the constant $\mathcal{S}_{s,Q}$ is entirely determined by the cone C associated to the interior cone condition and the dimension d .

The proof of Theorem 3.21 is outlined following the methods in [1, Lemma 5.10].

Proof. Our goal is to demonstrate that for any $u \in W^{1,p}(\Omega)$,

$$(3.17) \quad \|u\|_{L^q(\Omega)} \leq K \|u\|_{W^{1,p}(\Omega)},$$

with some K dependent only on p, d and C .

By Theorem 3.17 we have that Ω is a union of finitely many subdomains with the strong local Lipschitz property (see Definition 3.18) and each subdomain is split into a union of corresponding sub parallelepipeds. As these depend only on d and C , therefore it is only required to establish (3.17) on one of these sub-domains, referred from this point onwards as Ω_s .

Next using Theorem 3.9 we apply a non-singular affine transformation of the parallelepipeds to cubes Q with edge length 2 and edges parallel to the coordinate axis. It is assumed that

$$\Omega_{s,Q} = \bigcup_{x \in A} (x + Q)$$

with $A \subseteq \Omega_{s,Q}$ defined in Theorem 3.20, and additionally we recall that $\Omega_{s,Q}$ has the segment property A.1.

By Theorem B.7 we only need to show estimate (3.17) for every $u \in C^\infty(\overline{\Omega_{s,Q}})$. For every $x \in \Omega_{s,Q}$ let $w_i(x)$ represent the intersection of $\Omega_{s,Q}$ with the straight line equation parallel to the x_i coordinate axis for every $1 \leq i \leq d$. As $w_i(x)$ contains a segment of unit length (due to the earlier transformation), we set $x + te_i$, $0 \leq t \leq 1$ where e_i denotes the unit vector along the x_i axis. Next we introduce the quantity $\gamma = (dp-p)/(d-p)$ so that $\gamma \geq 1$. By integration by parts for every $u \in C^\infty(\overline{\Omega_{s,Q}})$

$$(3.18) \quad \begin{aligned} & \int_0^1 |u(x + (1-t)e_i)|^\gamma dt \\ &= |u(x)|^\gamma - \gamma \int_0^1 t |u(x + (1-t)e_i)|^{\gamma-1} \frac{d}{dt} u(x + (1-t)e_i) dt. \end{aligned}$$

As x_i is fixed, we set $\hat{x}_i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d)$ we define $F_i : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ by

$$(3.19) \quad F_i(\hat{x}_i) = \sup_{y \in w_i(x)} |u(y)|^{p/(d-p)}.$$

Then, we apply (3.18) to equation (3.19) which gives

$$|F_i(\hat{x}_i)|^{d-1} \leq \int_{w_i(x)} |u(x)|^\gamma dx_i + \gamma \int_{w_i(x)} |u(x)|^{\gamma-1} |D_i u(x)| dx_i.$$

Next we integrate over the projection of $\Omega_{s,Q}$ onto the $d-1$ -dimensional plane $x_i = 0$, set as Ω_i ,

$$\int_{\Omega_i} |F_i(\hat{x}_i)|^{d-1} d\hat{x}_i \leq \int_{\Omega_{s,Q}} |u(x)|^\gamma dx + \gamma \int_{\Omega_{s,Q}} |u(x)|^{\gamma-1} |D_i u(x)| dx.$$

If $p > 1$, then one has that $\gamma > 1$ and applying Hölder's inequality

$$\begin{aligned} \|F_i\|_{L^{d-1}(\Omega_i)}^{d-1} &\leq \gamma \left[\int_{\Omega_{s,Q}} |u(x)|^{(\gamma-1)p'} dx \right]^{1/p'} \left[\int_{\Omega_{s,Q}} (|u(x)| |D_i u(x)|)^p dx \right]^{1/p} \\ &\leq 2^{(p-1)/p} \gamma \|u\|_{W^{1,p}(\Omega_{s,Q})} \|u\|_{L^q(\Omega_{s,Q})}^{q/p'} \end{aligned}$$

as $(\gamma-1)p' = q$ and Q is a cube with unit length 2. Finally using Lemma B.8 applied to F_i , for every $1 \leq i \leq d$, and noting $k = d-1$ for this Lemma we get that

$$\begin{aligned} \|u\|_{L^q(\Omega_{s,Q})}^q &= \int_{\Omega_{s,Q}} |u(x)|^{p^*} dx \\ &\leq \int_{\Omega_i} F_i(\bar{x}_i) dx \\ &\leq \prod_{i=1}^d \|F_i\|_{L^{d-1}(\Omega_i)} \\ &\leq \left(2^{(p-1)/p} \gamma \|u\|_{W^{1,p}(\Omega_{s,Q})} \|u\|_{L^q(\Omega_{s,Q})}^{q/p'} \right)^{d/(d-1)}. \end{aligned}$$

As $(d-1)q/d - q/p' = 1$ the desired estimate (3.17) is derived by cancelling the excess terms due to $\|u\|_{L^q(\Omega_{s,Q})}^{q/p'}$ is finite as $u \in C^\infty(\bar{\Omega}_{s,Q})$ and $\Omega_{s,Q}$ is bounded. Finally as $C^\infty(\bar{\Omega}_{s,Q})$ is dense in $W^{1,p}(\Omega_{s,Q})$ one has that estimate (3.17) extends to all of $W^{1,p}(\Omega_{s,Q})$. So one has that

$$\|u\|_{L^q(\Omega_{s,Q})} \leq 2^{(p-1)/p} (dp-p)/(d-p) \|u\|_{W^{1,p}(\Omega_{s,Q})}.$$

Now we convert this estimate back from a cube into the original parallelepiped, by estimate (3.6) we have

$$\|u\|_{L^q(\Omega_{s,Q})} \leq \max(|\mathbf{S}_{s,Q}|^{-1/p}, 1) 2^{(p-1)/p} (dp-p)/(d-p) \|u\|_{W^{1,p}(\Omega_s)},$$

where $\mathbf{S}_{s,Q}$ is the scaling matrix associated with the transformation between the cube and the parallelepiped on $\Omega_{s,Q}$. We note that this scaling matrix is ultimately dependent on both the dimension d and the cone C . As $C^\infty(\Omega)$ is dense in $W^{1,p}(\Omega)$ we extend this over the entirety of Ω to get

$$(3.20) \quad \|u\|_{L^q(\Omega)} \leq S_{d,C} 2^{(p-1)/p} (dp-p)/(d-p) \|u\|_{W^{1,p}(\Omega)},$$

where

$$S_{d,C} = \max \left(\sup_{s,Q} |\mathbf{S}_{s,Q}|^{-1/p}, 1 \right).$$

□

We now prove that we have the desired Sobolev embedding constant independent of the volume $|\Omega|$ for bounded open Ω satisfying the interior cone condition for any q bounded between $p \leq q \leq p^*$.

Lemma 3.23. *For a given bounded open subset $\Omega \subseteq \mathbb{R}^d$ satisfying the interior cone condition we have that if $p < d$, then*

$$W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$$

for every $p \leq q \leq p^*$. The Sobolev embedding constant only depend on d, p, q and the cone C associated with the interior cone condition.

The proof of Lemma 3.23 follows the methods in [1, Lemma 5.12].

Proof. We set $q_0 = p^*$ by Theorem 3.21 we have that $W^{1,p}(\Omega) \hookrightarrow L^{q_0}(\Omega)$. For every $p \leq q \leq q_0$ we now set $s = (q_0 - q)p/(q_0 - p)$ and $t = p/s = (q_0 - p)/(q_0 - q)$ we then apply Hölder's inequality so one has that

$$\begin{aligned} \|u\|_{L^q(\Omega)}^q &= \int_{\Omega} |u(x)|^s |u(x)|^{q-s} dx \\ &\leq \left[\int_{\Omega} |u(x)|^{st} dx \right]^{1/t} \left[\int_{\Omega} |u(x)|^{(q-s)t'} dx \right]^{1/t'} \\ &= \|u\|_{L^p(\Omega)}^{p/t} \|u\|_{L^{q_0}(\Omega)}^{q_0/t'} \leq \left(S_{d,C} 2^{(p-1)/p} (dp - p)/(d - p) \right)^{q_0/t'} \|u\|_{L^p(\Omega)}^q. \end{aligned}$$

Where $S_{d,C}$ is the constant from estimate (3.20), simplifying we have that

$$(3.21) \quad \|u\|_{L^q(\Omega)} \leq \left(S_{d,C} 2^{(p-1)/p} (dp - p)/(d - p) \right)^{q_0/(t'q)} \|u\|_{W^{1,p}(\Omega)}.$$

□

We also provide a simple corollary in the special case that $p = d$.

Corollary 3.24. If $p = d$, we have that $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $1 \leq q < \infty$.

The proof of Corollary 3.24 is briefly outlined following the methods in [1, Corollary 5.13].

Proof. We recall that $p' = p/(p - 1)$ and we have two cases to consider:

Case 1: $q \geq p'$

We set $q = ds/(d - s)$, where $s = pq/(p + q)$ satisfying $1 \leq s < p$. By Theorem B.10, $W^{1,p}(\Omega) \hookrightarrow W^{1,s}(\Omega)$ with an embedding constant depending on the volume of Ω . Finally as $s < d$,

$$W^{1,s}(\Omega) \hookrightarrow L^q(\Omega)$$

by Lemma 3.23.

Case 2: $p \leq q \leq p'$

For this case the Sobolev embedding is achieved by interpolating between

$$W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$$

and

$$W^{1,p}(\Omega) \hookrightarrow L^{p'}(\Omega)$$

as in estimate (3.17).

□

Remark 3.25. It should be noted that the Sobolev embedding constant for Corollary 3.24 can have a dependence on the volume $|\Omega|$ of Ω .

Now, we generalise the embedding independent of volume $|\Omega|$ to the case of unbounded domains Ω satisfying the interior cone condition.

Theorem 3.26 (Unbounded Sobolev, Gagliardo and Nirenberg). *For a given open subset $\Omega \subseteq \mathbb{R}^d$ satisfying the interior cone condition one has that if $p < d$, then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $p \leq q \leq p^*$. Additionally if $2 \leq p = d$ then*

$$W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$$

for every $p \leq q < \infty$. In particular, for each case, the Sobolev embedding constant only depend on p, d, q and volume of the cone.

We provide the proof of Theorem 3.26 following the methods in [1, Lemma 5.14]

Proof. \mathbb{R}^d is tessellated by cubes of unit length. If $\lambda = (\lambda_1, \dots, \lambda_d)$ is a d -tuple of integers, we define

$$H_\lambda := \{x \in \mathbb{R}^d : \lambda_j \leq x_j \leq \lambda_j + 1 \text{ for } 1 \leq j \leq d\}.$$

Then, $\mathbb{R}^d = \cup_\lambda H_\lambda$. As per Theorem 3.17 one has that Ω is written as a union of finitely many subdomains so that $\Omega = \cup_{j=1}^N \Omega_j$, with $\Omega_j = \cup_{x \in A_j} (x + P_j)$, where A_j, P_j are defined as in Theorem 3.17. The number N and dimensions of P_j are dependent on the dimension d and the cone C associated with the interior cone condition. For each λ and for every $1 \leq j \leq N$ let

$$\Omega_{\lambda,j} = \bigcup_{x \in A_j \cap H_\lambda} (x + P_j).$$

The bounded subdomains $\Omega_{\lambda,j}$ possess the following properties:

- (1) The domain Ω is the union of $\Omega_{\lambda,j}$ given by,

$$(3.22) \quad \Omega = \bigcup_{\lambda,j} \Omega_{\lambda,j}.$$

- (2) There exists a finite cone C' depending on P_1, \dots, P_N (i.e. only depend on d and C) so that $\Omega_{\lambda,j}$ has the interior cone property determined by C' .
- (3) There exists a constant integer R depending on d and C , so that any $R+1$ of the subdomains $\Omega_{\lambda,j}$ have empty intersections.
- (4) Finally there exists constants K' and K'' depending on d and C such that for each $\Omega_{\lambda,j}$ are lower and upper bounds on the volume of $\Omega_{\lambda,j}$.

We now have two cases to consider.

Case 1: $p < d$

Let $u \in W^{1,p}(\Omega)$, if $p \leq q \leq p^*$, then as Ω_λ is bounded and has the interior cone property a combined with Lemma 3.23 we have

$$(3.23) \quad \|u\|_{L^q(\Omega_{\lambda,j})} \leq K \|u\|_{W^{1,p}(\Omega_{\lambda,j})}$$

where

$$K = \left(S_{d,C} 2^{(p-1)/p} (dp - p) / (d - p) \right)^{q_0/(t'q)}$$

with $S_{d,C}$ defined as in Theorem 3.21. As this constant is independent of both λ and j , so by Property (3.22) and given that subdomains $\Omega_{\lambda,j}$ have empty intersections and for every $p \leq q$ we have that

$$\begin{aligned} \|u\|_{L^q(\Omega)}^q &\leq \sum_{\lambda,j} \|u\|_{L^q(\Omega_{\lambda,j})}^q \\ &\leq K^q \sum_{\lambda,j} \left[\|u\|_{W^{1,p}(\Omega_{\lambda,j})}^p \right]^{q/p} \\ &\leq K^q \left[\sum_{\lambda,j} \|u\|_{W^{1,p}(\Omega_{\lambda,j})}^p \right]^{q/p} \\ &\leq K^q R^{q/p} \|u\|_{W^{1,p}(\Omega)}^q. \end{aligned}$$

Therefore one has that $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ with an embedding constant of $KR^{1/q}$.

Case 2: $p = n$

Equation (3.23) holds for any q such that $p \leq q < \infty$ by Corollary 3.24 and using the fact that K is independent of Ω , from here the same proof as Case 1 is then used. \square

3.3. Bounded families of Sobolev embeddings. This section is dedicated to establish uniform bounds on local Sobolev embeddings for a given family $(\Omega_n)_{n \geq 1}$ of domains $\Omega_n \subseteq \mathbb{R}^d$ Uniform local Sobolev embeddings. We will generalise the work of Daners in [4] to the case where $p \neq 2$ using the ideas in the preprint [6] by Hauer and Lee.

We now extend the calculations of the previous sections to domains with shrinking holes as in Assumption 1.1. We aim to establish the Sobolev constant

$$C_{Sob,p}(\Omega) := \sup \left\{ \|u\|_{L^{p^*}(\Omega)} \mid u \in W^{1,p}(\Omega), \|u\|_{W^{1,p}(\Omega)} = 1 \right\}$$

for a sequence of domains $(\Omega_n)_{n \geq 1}$ is bounded. Let $1 \leq p < \infty$ and the open subsets $\Omega^* \subseteq \Omega_n \subseteq \Omega$ satisfy Assumption 1.1. We recall

$$p^* := \begin{cases} \frac{pd}{d-p} & 1 \leq p < d, \\ \infty & p > d. \end{cases}$$

If $p = d$ then p^* takes on any value between $[d, +\infty)$.

If the family of domains $(\Omega_n)_{n \geq 1}$ satisfy an interior cone condition given by Definition 3.13 for the same cone, we show this is a sufficient condition to bound the Sobolev constant. From Lemma 3.26 one has that

$$C_{Sob,p}(\Omega) \leq c$$

for domains satisfying the interior cone condition. Next we formulate a proposition that provides a sufficient condition to provide an upper bound to the sequence of Sobolev constants $(C_{Sob,p}(\Omega_n))_{n \geq 1}$. The results are based on the work by Daners [4, Proposition 2.4] with the results generalised to $p \neq 2$ using the ideas in the preprint by Hauer & Lee [6]. First we start by giving a motivating example.

Example 3.27. Let $K = \{x_0\}$ for some $x_0 \in \Omega$ for every $d > 2$. Then, K is compact and let \bar{B}_R be a ball centred at x_0 for radius $R > 0$ chosen so that $\bar{B}_{2R} \subseteq \Omega$. Then, for a sequence $(\delta_n)_{n \geq 0} \subset (0, 1)$ so that $\delta_n \rightarrow 0$ as $n \rightarrow \infty$, set $H_n := \delta_n \bar{B}_R$ for every $n \geq 1$. Then, each H_n is a closed subset of Ω satisfying

Assumption 1.1 and satisfies the exterior cone condition with the same cone for all $n \geq 1$.

Proposition 3.28. Let $1 \leq p < \infty$, for a sequence of domains $(H_n)_{n \geq 1}$ satisfying the exterior cone condition uniformly in n and assuming that $C_{Sob,p}(\Omega)$ is finite then for $\Omega_n := \Omega \setminus H_n$, the sequence $(C_{Sob,p}(\Omega_n))_{n \geq 1}$ is bounded.

We prove Proposition 3.28 based on [4, Theorem 2.8] with p generalised to $p \neq 2$ as in the pre-published work by Hauer & Lee [6].

Proof. We choose a subset $\Omega' \subseteq \Omega$ satisfying the interior cone property so that $H_n \subseteq \Omega'$ for all $n \geq 1$. We also set $\Omega'_n := \Omega' \setminus H_n$ for all $n \geq 1$ and introduce a truncation function $\varphi \in C^\infty(\bar{\Omega})$ bounded between $0 \leq \varphi \leq 1$ and $\varphi = 0$ on $\bar{\Omega}'$, we note the existence of such a function is given by the Lemma of Uryson. For every $u \in W^{1,p}(\Omega_n)$ then $\varphi u \in W^{1,p}(\Omega)$ with support $\Omega \setminus \bar{\Omega}'$. Therefore we have

$$\begin{aligned} \|u\|_{L^{p^*}(\Omega_n)} &\leq \|\varphi u\|_{L^{p^*}(\Omega_n)} + \|(1-\varphi)u\|_{L^{p^*}(\Omega_n)} \\ &\leq \|\varphi u\|_{L^{p^*}(\Omega \setminus \bar{\Omega}')} + \|(1-\varphi)u\|_{L^{p^*}(\Omega'_n)} + \|(1-\varphi)u\|_{L^{p^*}(\Omega \setminus \bar{\Omega}')} \\ &\leq 2\|u\|_{L^{p^*}(\Omega \setminus \bar{\Omega}')} + \|(1-\varphi)u\|_{L^{p^*}(\Omega'_n)} \\ &\leq 2C_{Sob,p}(\Omega \setminus \bar{\Omega}') \|u\|_{W^{1,p}(\Omega \setminus \bar{\Omega}')} + \|u\|_{L^{p^*}(\Omega'_n)} \end{aligned}$$

By our initial assumption, H_n satisfies the exterior cone condition with the same cone for all $n \geq 1$. Thus $(\Omega'_n)_{n \geq 1}$ satisfies a interior cone condition with the same cone for all $n \geq 1$. So Theorem 3.26 implies that there is a constant $c > 0$ dependent only on p, N, p^* and the cone C so that $C_{Sob,p}(\Omega'_n) \leq c$ for all $n \geq 1$. So one has that

$$\begin{aligned} \|u\|_{L^{p^*}(\Omega_n)} &\leq 2C_{Sob,p}(\Omega \setminus \bar{\Omega}') \|u\|_{W^{1,p}(\Omega \setminus \bar{\Omega}')} + C_{Sob,p}(\Omega'_n) \|u\|_{W^{1,p}(\Omega'_n)} \\ &\leq 2C_{Sob,p}(\Omega \setminus \bar{\Omega}') \|u\|_{W^{1,p}(\Omega_n)} + c \|u\|_{W^{1,p}(\Omega'_n)} \end{aligned}$$

Therefore one has that

$$\sup_{n \geq 1} \sup_{u \in W^{1,p}(\Omega_n) \setminus \{0\}} \frac{\|u\|_{L^{p^*}(\Omega_n)}}{\|u\|_{W^{1,p}(\Omega_n)}} \leq 2C_{Sob,p}(\Omega \setminus \bar{\Omega}') + c.$$

□

Next consider the case where the holes $(H_n)_{n \geq 1}$ still shrink to a set K , but do not necessarily satisfy an exterior cone condition with the same cone for all $n \geq 1$.

Assumption 3.29. For every $1 \leq p < \infty$ and $\Omega \subseteq \mathbb{R}^d$ be an open measurable subset and $u \in L^1(\Omega)$. We suppose there exists a $R > 0$ so that a closed ball $\bar{B}_{2R} \subseteq \Omega$ and introduce a sequence $(\delta_n)_{n \geq 1} \subset (0, 1)$ so that $\delta_n \rightarrow 0$ as $n \rightarrow \infty$, such that $H_n \subseteq B_{\delta_n}$ for every $n \geq 1$. We also assume that there are constants $c_{Poi} > 0$ and $c_{Sob} > 0$ so as the holes are rescaled to $\mathcal{U}_n := B_{2R} \setminus \delta^{-1} H_n$ they additionally satisfy a *uniform Poincaré inequality*

$$(Poi) \quad \sup_{n \geq 1} \inf_{u \in W^{1,p}(\mathcal{U}_n) \setminus \{0\}} \frac{\|\nabla u\|_{L^p(\mathcal{U}_n)}}{\|u - \bar{u}_{\mathcal{U}_n}\|_{L^p(\mathcal{U}_n)}} \geq c_{Poi}$$

and a *uniform Sobolev inequality*

$$(Sob) \quad \sup_{n \geq 1} C_p(\mathcal{U}_n) \leq c_{Sob}.$$

where we recall $\bar{u}_{|\Omega}$ is mean the of u over Ω given by

$$\bar{u}_{|\Omega} = \frac{1}{|\Omega|} \int_{\Omega} u dx.$$

We provide 2 examples that satisfy these assumptions.

Example 3.30. The example given in Example 3.27 satisfies Assumption 3.29. Additionally *shrinking holes of a fixed shape* also satisfy Assumption 3.29. For an open set $H \subseteq B_R$ with a smooth boundary so that $B_{2R} \setminus H$ is connected, the sequence $(\delta_n)_{n \geq 0} \subset (0, R]$ so that $\delta_n \rightarrow 0$ as $n \rightarrow \infty$ set $H_n := \delta_n H$ for every $n \geq 1$. Then, for every $n \geq 1$ all of the rescaled holes are defined as $\mathcal{U}_n = B_{2R} \setminus H$ independent of the value n , and so estimates (Poi) and (Sob) are trivially satisfied.

Now we wish to show the main result of this section, demonstrating that the sequence of constants $(C_{Sob,p}(\Omega_n))_{n \geq 1}$ is bounded under Assumption 3.29.

Theorem 3.31. *Let $1 \leq p < \infty$ and Ω be a open subset of \mathbb{R}^d such that $C_p(\Omega)$ is finite and that Assumption 3.29 holds. Then, for $\Omega_n := \Omega \setminus H_n$, the sequence $(C_{Sob,p}(\Omega_n))_{n \geq 1}$ of Sobolev constants is bounded.*

We now prove Theorem 3.31, which is based off a generalisation of [4, Theorem 2.8] to $p \neq 2$, we base this off the ideas in [6, Theorem 1.3].

Proof. We assume that the sequence $(C_{Sob,p}(\Omega_n))_{n \geq 1}$ is unbounded and aim to reach a contradiction. For integer $k \geq 1$ there is an index $n_k \geq 1$ such that $C_{Sob,p}(\Omega_{n_k}) \geq k$. For convenience we relabel the index n_k to k , then for every $u_k \in W^{1,p}(\Omega_k)$ there is a sequence $(u_k)_{k \geq 1}$ satisfying $\|u_k\|_{W^{1,p}(\Omega_k)} = 1$ and

$$(3.24) \quad k \|u_k\|_{W^{1,p}(\Omega_k)} = k \leq \|u_k\|_{L^{p^*}(\Omega_k)},$$

for every integer $k \geq 1$. As we have that $B_{2\delta_k} \setminus H_k = (\Omega \setminus H_k) \setminus (\Omega \setminus B_{2\delta_k})$, so by the reverse triangle inequality and estimate (3.24) we have

$$\begin{aligned} \|u_k\|_{L^{p^*}(B_{2\delta_k} \setminus H_k)} &= \left\| u_k \mathbf{1}_{\Omega_k} - u_k \mathbf{1}_{\Omega \setminus B_{2\delta_k}} \right\|_{L^{p^*}(\Omega)} \\ &\geq \|u_k\|_{L^{p^*}(\Omega_k)} - \|u_k\|_{L^{p^*}(\Omega \setminus B_{2\delta_k})} \\ &\geq k \|u_k\|_{W^{1,p}(\Omega_k)} - \|u_k\|_{L^{p^*}(\Omega \setminus B_{2\delta_k})}. \end{aligned}$$

Taking advantage of the convexity of $t \mapsto t^p$ for every $0 \leq t$ we have the inequality

$$(3.25) \quad \frac{1}{2^{p'}}(t + s) \leq \sqrt[p]{t^p + s^p} \text{ for every } t, s \geq 0.$$

By assumption $C_{Sob,p}(\Omega)$ is finite and for each ball $B_{2\delta_k}$ satisfies the exterior cone condition with the same cone for all $k \geq 1$ thus Proposition 3.28 shows that the sequence $(C_{Sob,p}(\Omega \setminus B_{2\delta_k}))_{n \geq 1}$ is bounded. In particular there exists a $k_0 \geq 1$ so that

$$0 \leq \frac{k}{2^{p'}} - C_{Sob,p}(\Omega \setminus B_{2\delta_k})$$

for all $k_0 \leq k$. Combining this with estimate (3.25) and setting

$$\begin{aligned} t &= \|u_k\|_{W^{1,p}(B_{2\delta_k} \setminus H_k)} \\ s &= \|u_k\|_{W^{1,p}(\Omega \setminus B_{2\delta_k})} \end{aligned}$$

we have that

$$\begin{aligned}
\|u_k\|_{L^{p^*}(B_{2\delta_k}\setminus H_k)} &\geq k \|u_k\|_{W^{1,p}(\Omega_k)} - \|u_k\|_{W^{1,p}(\Omega\setminus B_{2\delta_k})} \\
&\geq k \|u_k\|_{W^{1,p}(\Omega_k)} - C_{Sob,p}(\Omega\setminus B_{2\delta_k}) \|u_k\|_{W^{1,p}(\Omega\setminus B_{2\delta_k})} \\
&= k \left[\|u_k\|_{W^{1,p}(B_{2\delta_k}\setminus H_k)}^p + \|u_k\|_{W^{1,p}(\Omega\setminus B_{2\delta_k})}^p \right]^{\frac{1}{p}} \\
&\quad - C_{Sob,p}(\Omega\setminus B_{2\delta_k}) \|u_k\|_{W^{1,p}(\Omega\setminus B_{2\delta_k})} \\
&\geq \frac{k}{2^{p'}} \left(\|u_k\|_{W^{1,p}(B_{2\delta_k}\setminus H_k)} + \|u_k\|_{W^{1,p}(\Omega\setminus B_{2\delta_k})} \right) \\
&\quad - C_{Sob,p}(\Omega\setminus B_{2\delta_k}) \|u_k\|_{W^{1,p}(\Omega\setminus B_{2\delta_k})} \\
&= \frac{k}{2^{p'}} \|u_k\|_{W^{1,p}(B_{2\delta_k}\setminus H_k)} \\
&\quad + \left(\frac{k}{2^{p'}} - C_{Sob,p}(\Omega\setminus B_{2\delta_k}) \right) \|u_k\|_{W^{1,p}(\Omega\setminus B_{2\delta_k})} \\
&\geq \frac{k}{2^{p'}} \|u_k\|_{W^{1,p}(B_{2\delta_k}\setminus H_k)}
\end{aligned}$$

for every $k \geq k_0$ so we have shown,

$$(3.26) \quad \frac{k}{2^{p'}} \|u_k\|_{W^{1,p}(B_{2\delta_k}\setminus H_k)} \leq \|u_k\|_{L^{p^*}(B_{2\delta_k}\setminus H_k)}.$$

By (Sob) and (Poi) one has that

$$(3.27) \quad \|v\|_{L^{p^*}(\mathcal{U}_k)} \leq c_0 \|\nabla v\|_{L^p(\mathcal{U}_k)},$$

for every $v \in W^{1,p}(\mathcal{U}_k)$ with mean $\bar{u}|_{\mathcal{U}} = 0$ where c_0 is a constant dependent only on c_{Poi} and c_{Sob} we emphasise is independent of k . As $\mathcal{U}_k = B_{2R}\setminus\delta_k^{-1}H_k$ we rescale $u_k \in W^{1,p}(B_{2\delta_k}\setminus H_k)$ to the domain \mathcal{U}_k by applying the substitution $v_k = \frac{R}{\delta_k} u_k(x-x_0)$. By estimate (3.27) and recalling $\|\nabla u_k\|_{L^p(\Omega_k)} \leq \|\nabla u_k\|_{W^{1,p}(\Omega_k)}$, we have that

$$\begin{aligned}
\|u_k\|_{L^{p^*}(B_{2\delta_k}\setminus H_k)} &= \left[\frac{\delta_k}{R} \right]^{\frac{d}{p^*}} \|v_k\|_{L^{p^*}(\mathcal{U}_k)} \\
&\leq c_0 \left[\frac{\delta_k}{R} \right]^{\frac{d}{p^*}} \|\nabla v_k\|_{L^p(\mathcal{U}_k)} \\
(3.28) \quad &= c_0 \left[\frac{\delta_k}{R} \right]^{\frac{d}{p^*} + 1 - \frac{d}{p}} \|\nabla u_k\|_{L^p(B_{2\delta_k}\setminus H_k)} \\
&\leq c_0 \left[\frac{\delta_k}{R} \right]^{\frac{d}{p^*} + 1 - \frac{d}{p}} \|u_k\|_{W^{1,p}(B_{2\delta_k}\setminus H_k)}.
\end{aligned}$$

Subtracting estimates (3.28) and (3.26) we have

$$0 \leq c_0 \left[\frac{\delta_k}{R} \right]^{\frac{d}{p^*} + 1 - \frac{d}{p}} - \frac{k}{2^{p'}} < 0$$

which leads to a contradiction for sufficiently large k . \square

4. UNIFORMLY BOUNDED FAMILIES OF EXTENSION OPERATORS

In this section, we aim to show that for a family of domains $(\Omega_\varepsilon)_{\varepsilon>0}$ the associated Calderón extension operators $\mathbb{E}_\varepsilon : W^{1,p}(\Omega_\varepsilon) \rightarrow W^{1,p}(\mathbb{R}^d)$ are uniformly bounded on domains Ω_ε with a finite number of shrinking holes as $\varepsilon \rightarrow 0^+$ (see Assumption 1.1). The main idea of this proof, namely for a given family $(\Omega_\varepsilon)_{\varepsilon>0}$ of domains Ω_ε containing a finitely many shrinking holes as $\varepsilon \rightarrow 0^+$ then the corresponding family $(\mathbb{E}_\varepsilon)_{\varepsilon>0}$ of Calderón operators $\mathbb{E}_\varepsilon : W^{1,p}(\Omega_\varepsilon) \rightarrow W^{1,p}(\mathbb{R}^d)$ remains uniformly bounded under the given assumptions, which will be specified in this section. This is based on an idea given by E.N. Dancer, which was already used by Daners [4] and more recently by Hauer & Lee [6] to establish the uniform Sobolev embedding on domains with shrinking holes (see Subsection 3.3).

4.1. The Caldéron Extension Operator. We start by introducing a modified version of the *uniform interior cone condition* (see Definition 3.15), which we take from [1, Theorem 4.32].

Definition 4.1. An open subset $\Omega \subseteq \mathbb{R}^d$ is said to satisfy the *modified uniform interior cone condition* provided it satisfies the uniform interior cone condition with the following changes:

- (1) There is a finite open cover $\mathcal{U}_{\partial\Omega} = \{U_1, \dots, U_N\}$ of the boundary $\partial\Omega$.
- (2) Each set $U_j \in \mathcal{U}_{\partial\Omega}$ does not need to be bounded.

That is we restrict the number of covers $U_j \in \mathcal{U}_{\partial\Omega}$ from potentially infinite to N elements, and allow each U_j to be unbounded.

Remark 4.2. As a consequence of property (2) of the uniform interior cone condition (see Definition 3.15) we express Ω as a combination of open covers $U_j \in \mathcal{U}_{\partial\Omega}$ and an open subset $U_0 \subset \Omega$ such that $\text{dist}(\Omega_0, \partial\Omega)$ is sufficiently large. One then has that

$$\Omega = \bigcup_{j=0}^N U_j,$$

which covers the entirety of Ω .

We now introduce a lemma that will be of use later in this subsection.

Lemma 4.3.

We now introduce the concept of an extension operator, which we recall from [11, Definition 12.16].

Definition 4.4. Let $1 \leq p < \infty$ and $\Omega \subseteq \mathbb{R}^d$ be an open subset. Then Ω is said to be a $(1, p)$ -*extension domain* or is said to admit the $(1, p)$ -*extension property*, if there exists a bounded linear operator

$$(4.1) \quad \mathbb{E} : W^{1,p}(\Omega) \rightarrow W^{1,p}(\mathbb{R}^d) \text{ defined by } \mathbb{E}(u)(x) = u(x)$$

for a.e. $x \in \Omega$ and every $u \in W^{1,p}(\Omega)$.

In other words, definition 4.4 is saying that a domain Ω admits the $(1, p)$ -extension property if there is an operator \mathbb{E} defined by (4.1) and a constant $K > 0$ such that the operator norm

$$(4.2) \quad \|\mathbb{E}\|_{\mathcal{L}(W^{1,p}(\Omega), W^{1,p}(\mathbb{R}^d))} \leq K.$$

Note, in general this constant K can be dependent on the volume $|\Omega|$ of Ω . In this thesis, in order to avoid potential singularities in our later applications, we are interested in conditions implying that this constant K might depend on p and the dimension d , but is independent of the volume $|\Omega|$ of the domain Ω . We specify domains Ω with such a property as follows (see [1, Section 4.24]).

Definition 4.5. An open subset $\Omega \subseteq \mathbb{R}^d$ is said to admit the *simple $(1, p)$ -extension property* if there exists a bounded linear operator $\mathbb{E} : W^{1,p}(\Omega) \rightarrow W^{1,p}(\mathbb{R}^d)$ satisfying (4.1) and there is a constant $K_{p,d} > 0$ independent of the volume $|\Omega|$ of Ω such that (4.2) holds. If Ω admits the simple $(1, p)$ -extension property, then we call the operator \mathbb{E} a *simple $(1, p)$ -extension operator*.

The following theorem demonstrates that if the domain Ω satisfies the modified uniform interior cone condition (see Definition 4.1), then Ω admits the simple $(1, p)$ -extension property with the Calderón extension operator \mathbb{E} .

Theorem 4.6 (Existence of the Calderón extension operator). *Let $\Omega \subset \mathbb{R}^d$ be an open subset satisfying the modified uniform interior cone condition. Then, for every $1 < p < \infty$, Ω admits the simple $(1, p)$ -extension property.*

Here, we outline the proof of Theorem 4.6 based on [1, Section 3.13]. The proof proceeds in several steps.

Proof of Theorem 4.6. Let $\Omega \subseteq \mathbb{R}^d$ be a domain in \mathbb{R}^d satisfying the modified uniform interior cone condition and $\mathcal{U}_{\partial\Omega} = \{U_1, \dots, U_N\}$ be a family of open subsets U_j covering the boundary $\partial\Omega$, and $\{C_1, \dots, C_N\}$ a corresponding family of cones C_j associated with U_j with vertices centred at the origin $0 \in \mathbb{R}^d$.

We now introduce several definitions and functions in order to construct the framework required to define the Calderón extension operator. To do this, we introduce several definitions. The first definition we consider is the C^∞ -partition of unity, based on [1, Section 3.13].

Definition 4.7. For a given open subset $\Omega \subseteq \mathbb{R}^d$, let \mathcal{U} be a collection of open subsets U of \mathbb{R}^d covering Ω , that is

$$\Omega \subseteq \bigcup_{U \in \mathcal{U}} U.$$

Then, one calls a collection Ψ of test functions $\psi \in C_c^\infty(\mathbb{R}^d)$ a C^∞ -*partition of unity* of Ω corresponding to \mathcal{U} if Ψ satisfies the following properties:

- (1) $0 \leq \psi \leq 1$ for every $\psi \in \Psi$.
- (2) For every relatively compact subset $K \subseteq \mathbb{R}^d$ satisfying $\overline{K} \subseteq \Omega$, all but possibly finitely many $\psi \in \Psi$ satisfy $\psi \equiv 0$ on K .
- (3) For every $\psi \in \Psi$, there is a $U \in \mathcal{U}$ such that $\text{supp}(\psi) \subseteq U$.
- (4) For every $x \in \Omega$, $\sum_{\psi \in \Psi} \psi(x) = 1$.

The next theorem provides the existence of a C^∞ -*partition of unity* as given in [1, Theorem 3.14].

Theorem 4.8. *For every open subset $\Omega \subseteq \mathbb{R}^d$ and every collection \mathcal{U} of open subsets $U \subset \mathbb{R}^d$ covering Ω , there exists a C^∞ -partition of unity Ψ of Ω corresponding to \mathcal{U} .*

For the construction of the Calderón operator \mathbb{E} we now introduce homogeneous functions of degree k following [1, Section 4.29].

Definition 4.9. A function $g : \mathbb{R}^d \rightarrow \mathbb{R}$ is said to be a *homogeneous function of degree k* on $B_R \setminus \{0\}$ if $g(tx) = t^k g(x)$ for all $x \in B_R \setminus \{0\}$, and $0 < t \leq 1$ where B_R denotes the ball of radius R .

Given a covering $\mathcal{U}_{\partial\Omega} = \{U_1, \dots, U_N\}$ of the boundary $\partial\Omega$, we introduce two associated collections $\{\phi_j\}_{j \in \{1, \dots, N\}}$ and $\{\psi_j\}_{j \in \{1, \dots, N\}}$ of auxiliary functions for the construction of the Calderón operator \mathbb{E} (see [1, Proof of Theorem 4.32]).

For every $j \in \{1, \dots, N\}$, let $\phi_j : \mathbb{R}^d \setminus \{0\} \rightarrow \mathbb{R}$ be a $C^\infty(\mathbb{R}^d \setminus \{0\})$ -function with the following properties;

- (1) $\phi_j \geq 0$;
- (2) the support $\text{supp}(\phi_j) \subseteq -C_j \cup \{0\}$, where we recall that $-C_j = \{-x \mid x \in C_j\}$ and C_j is the cone associated with U_j ;
- (3) there exists an $\varepsilon > 0$ such that ϕ_j restricted to the punctured ball $B_\varepsilon \setminus \{0\}$ is homogeneous of degree $1 - d$ (see Definition 4.9).

Associated with each ϕ_j , we define the function $\psi_j : \mathbb{R}^d \setminus \{0\} \rightarrow \mathbb{R}$ by setting

$$\psi_j(x) = \frac{\partial}{\partial r} [r^{d-1} \phi_j(rz)]$$

for every $y = rz \in \mathbb{R}^d \setminus \{0\}$ with $r = |y| > 0$, $z \in \mathbb{S}_{d-1}$, where \mathbb{S}_{d-1} denotes the unit sphere in \mathbb{R}^d . As ϕ_j is homogeneous of degree $1 - d$ on the punctured ball $B_\varepsilon(0) \setminus \{0\}$ we have that $g(x) := r^{d-1} \phi_j$ is homogeneous of degree 0 on the punctured ball. This means that we have that $g(rz) = g(z)$ and thus the derivative $\frac{\partial g}{\partial r} = 0$ on the punctured ball $B_R \setminus \{0\}$. Thus one has that $\psi_j \equiv 0$ on the punctured ball. Therefore, each ψ_j can be extended continuously to $x = 0$ by setting $\psi_j(0) = 0$ and we can conclude $\psi_j \in C_0^\infty(-C_j)$.

Next, for each $U_j \in \mathcal{U}_{\partial\Omega}$ the Calderón operator \mathbb{E} also requires so called sub-extension operators $\mathbb{E}_j u : U_j \cap \Omega \rightarrow \mathbb{R}^d$ (see [1, Theorem 4.32]). For every $x \in U_j \cap \Omega$, let

$$\begin{aligned} \mathbb{E}_j u(x) &= -K_{p,j} \left\{ \int_{\mathbb{S}_{d-1}} \int_0^\infty \phi_j(rz) r^{d-1} \frac{\partial u(x-rz)}{\partial r} dr d\sigma(z) \right. \\ &\quad \left. + \int_{\mathbb{S}_{d-1}} \int_0^\infty \psi_j(xz) u(x-rz) dr d\sigma(z) \right\} \\ &= -K_{p,j} (\hat{I}_1 + \hat{I}_2), \end{aligned} \tag{4.3}$$

for every $y = rz \in \mathbb{R}^d \setminus \{0\}$ with $r = |y| > 0$, $z \in \mathbb{S}_{d-1}$ and where the constant $K_{p,j}$ is given by

$$K_{p,j} := \lim_{r \rightarrow 0} \left(\int_{\mathbb{S}_{d-1}} r^{d-1} \phi_j(rz) d\sigma(z) \right)^{-1} \tag{4.4}$$

where the integral $\int_{\mathbb{S}_{d-1}} f(z) d\sigma(z)$ denotes the surface integral over the unit sphere \mathbb{S}_{d-1} for smooth functions f .

With this notation and setting in mind, we can now define the *Calderón extension operator* $\mathbb{E} : W^{1,p}(\Omega) \rightarrow W^{1,p}(\mathbb{R}^d)$.

Definition 4.10. For a given open subset Ω in \mathbb{R}^d satisfying the modified uniform interior cone condition, the *Calderón extension operator* $\mathbb{E} : W^{1,p}(\Omega) \rightarrow W^{1,p}(\mathbb{R}^d)$ is defined by

$$\mathbb{E}u = \omega_0 u + \sum_{j=1}^N \omega_j \mathbb{E}_j u$$

for every $u \in W^{1,p}(\Omega)$, and where $\{\omega_0, \omega_1, \dots, \omega_N\}$ is a C^∞ -partition of unity of Ω associated with $\{U_0, U_1, \dots, U_N\}$ consisting of a finite covering $\mathcal{U}_{\partial\Omega} = \{U_1, \dots, U_N\}$ of the boundary $\partial\Omega$ and $U_0 \subset \Omega$ which is an open subset with ∂U_0 having positive distance to the boundary $\partial\Omega$, that is $\text{dist}(\partial\Omega, \mathcal{U}_0) > 0$.

Now, we complete the proof of Theorem 4.6.

Continuation of the proof of Theorem 4.6. In order to prove that $\mathbb{E}u$ is a simple $(1,p)$ -extension operator it is sufficient to show that for each $j \in \{1, \dots, N\}$ we have that $\mathbb{E}_j u$ satisfies

$$(4.5) \quad \mathbb{E}_j u(x) = u(x) \text{ for all } x \in U_j \cap \Omega,$$

$$(4.6) \quad \|\mathbb{E}_j u\|_{W^{1,p}(\mathbb{R}^d)} \leq K_{p,j} \|u\|_{W^{1,p}(\Omega)},$$

where $K_{p,j}$ is a constant dependent only on p and j defined by equation (4.4).

As $C^\infty(\Omega)$ is dense in the Banach space $W^{1,p}(\Omega)$, we first establish (4.6) for functions $u \in C^\infty(\Omega)$. Since the extension operator \mathbb{E}_j is Lipschitz continuous with the constant $K_{p,j}$ it can then be uniquely extended on $W^{1,p}(\Omega)$. Hence, let $u \in C^\infty(\Omega)$. By the Lemma 4.3 we have that $u(x - rz)$ is infinitely differentiable for $rz \in \text{supp}(\phi_j)$. By using integration by parts and noting that ϕ is homogeneous of degree $1 - d$ in the punctured ball of ε radius $B_\varepsilon \setminus \{0\}$, one has the following

$$-\int_0^\infty \phi_j(rz) r^{d-1} \frac{\partial u(x - rz)}{\partial r} dr = \lim_{r \rightarrow 0} r^{d-1} \phi_j(rz) u(y) + \int_0^\infty \psi_j(rz) u(x - rz) dr.$$

So therefore we have that equation (4.3) is expressed as

$$\begin{aligned} \mathbb{E}_j u(y) &= \left(\lim_{r \rightarrow 0} \int_{\mathbb{S}^{d-1}} r^{d-1} \phi_j(rz) d\sigma(z) \right)^{-1} u(y) \lim_{r \rightarrow 0} \int_{\mathbb{S}^{d-1}} r^{d-1} \phi_j(rz) d\sigma(z) \\ &= u(y) \end{aligned}$$

for x almost everywhere in $U_j \cap \Omega$. This shows that (4.5) holds. Next we must show that estimate (4.6) is satisfied, this is equivalent to,

$$\|D^\alpha \mathbb{E}_j u\|_{L^p(\mathbb{R}^d)} \leq K_\alpha \|u\|_{W^{1,p}(\Omega)}$$

for any multi-index α satisfying $|\alpha| \leq 1$. We now consider the two integrals in equation (4.3), \hat{I}_1 and \hat{I}_2 separately.

For \hat{I}_2 we directly apply Young's theorem (see D.7) with

$$\theta(x) = \psi_j(x) |x|^{1-d} \in L^1(\mathbb{R}^d)$$

For an appropriately smooth approximation of u we have,

$$\begin{aligned} \|D^\alpha(\theta(x) * u)\|_{L^p(\mathbb{R}^d)} &= \|\theta(x) * D^\alpha u\|_{L^p(\mathbb{R}^d)} \\ &\leq \|\theta(x)\|_{L^p(\mathbb{R}^d)} \|u\|_{L^p(\Omega)}. \end{aligned}$$

Finally we consider \hat{I}_1 . We now convert the integral back into Cartesian coordinates with $\frac{\partial}{\partial r} = \sum_{i=1}^d z_i \frac{\partial}{\partial y_i}$, where $z = (z_1, \dots, z_d) \in \mathbb{S}_{d-1}$ and $y = (y_1, \dots, y_d) \in \mathbb{R}^d$ respectively. This yields

$$(4.7) \quad \int_{\mathbb{S}_{d-1}} \int_0^\infty \phi_j(rz) r^{d-1} \frac{\partial u(x-rz)}{\partial r} dr d\sigma(z) = \sum_{i=1}^d \int_{\mathbb{R}^d} z_i \phi_j(y) \frac{\partial u(x-y)}{\partial y_i} dy.$$

It remains to show that all terms of the sum in equation (4.7) are bounded. We note that $z_i \phi_j(x)$ is homogeneous of degree $1-d$ in the punctured ball $B_\varepsilon \setminus \{0\}$ and belongs to C^∞ . Recalling our assumption that $u \in C^\infty(\Omega)$, we have

$$\begin{aligned} \int_{\mathbb{R}^d} z_i \phi_j(y) \frac{\partial u(x-y)}{\partial y_i} dy &= \lim_{\delta \rightarrow 0^+} \int_{\mathbb{R}^d \setminus B_\delta} z_i \phi_j(y) \frac{\partial u(x-y)}{\partial y_i} dy \\ &= \lim_{\delta \rightarrow 0^+} \left([z_i \phi_j(y) u(x-y)]_{\mathbb{R}^d \setminus B_\delta} + \int_{\mathbb{R}^d \setminus B_\delta} u(x-y) \frac{\partial z_i \phi_j(y)}{\partial y_i} dy \right). \end{aligned}$$

As this is a surface integral over the sphere S_δ of the form $u(y - \cdot)$ this tends to $Ku(y)$ for some K independent of Ω . Noting that $\frac{\partial z_i \phi_j(y)}{\partial y_i}$ is homogeneous of degree $-d$ in the ball B_ε we have that $\frac{\partial z_i \phi_j(y)}{\partial y_i}$ satisfies all the requirements of the singular kernel g in the Calderón-Zygmund theorem inequality (see Theorem D.24) for a convolution operator $\|g * u\|_{L^p(\mathbb{R}^d)}$, so one has that (assuming $u \in L^p(\Omega)$ is identically zero outside of Ω),

$$\|z_i \phi_j * u\|_{L^p(\mathbb{R}^d)} \leq K_{i,j,p} \|u\|_{L^p(\Omega)}.$$

where we note $K_{i,j,p}$ is independent of Ω . \square

4.2. Uniformly Bounded Extension Operators on Domains with Shrinking Holes. Now that we have established the suitability of the Calderón operator \mathbb{E} as a simple $(1, p)$ -extension operator we introduce the framework to consider uniform extension operator bounds.

Assumption 4.11. For every $1 \leq p < \infty$ and given an open subset $\Omega \subseteq \mathbb{R}^d$ with the origin $0 \in \Omega$ and $u \in L^1(\Omega)$ and Ω satisfies Assumption 1.1. We suppose there exists a $R > 0$ so that a closed ball $\overline{B}_{2R} \subseteq \Omega$ and introduce a sequence $(\delta_n)_{n \geq 1} \subset (0, 1)$ so that $\delta_n \rightarrow 0$ as $n \rightarrow \infty$, such that $H_n \subseteq B_{\delta_n}$ for every $n \geq 1$. We also assume that there exists a constant $c_{Poi} > 0$ so as the holes are rescaled to $\mathcal{U}_n := B_{2R} \setminus \delta^{-1} H_n$ they satisfy a *uniform Poincaré inequality*

$$(4.8) \quad \sup_{n \geq 1} \inf_{u \in W^{1,p}(\mathcal{U}_n) \setminus \{0\}} \frac{\|\nabla u\|_{L^p(\mathcal{U}_n)}}{\|u - \bar{u}_{|\mathcal{U}_n}\|_{L^p(\mathcal{U}_n)}} \geq c_{Poi}$$

which holds for all $u \in W^{1,p}(\mathcal{U}_n)$, and recall $\bar{u}_{|\Omega}$ is mean the of u over Ω given by

$$\bar{u}_{|\Omega} = \frac{1}{|\Omega|} \int_{\Omega} u dx.$$

We start with the simplified example where Ω is an open ball with a single shrinking hole that is also a smaller open ball of radius ε_n centred at $x = 0 \in \mathbb{R}^d$ that is, $H_n := B_{\delta_n}$. This will show the concepts on a model geometry with the limiting case of Ω^* as the punctured ball.

Theorem 4.12. *For every $1 \leq p < \infty$ and given an open ball $\Omega \subset \mathbb{R}^d$ and a shrinking hole $H_n = B_{\varepsilon_n}$ centred at $0 \in \mathbb{R}^d$. Thus, for $\Omega_n := \Omega \setminus H_n$, the sequence of associated extension operators $(\mathbb{E}_n)_{n \geq 1}$ is bounded, that is*

$$\mathcal{C}_E := \sup_{n \geq 1} \|\mathbb{E}_n\|_{\mathcal{L}(W^{1,p}(\Omega_n), W^{1,p}(\mathbb{R}^d))} < \infty,$$

where $(\mathbb{E}_n)_{n \geq 1}$ is a sequence of Calderón operators associated to Ω_n given by Definition 4.10.

We now provide an original proof for the simplified case of punctured ball.

Proof. We choose the cover $(U_{n,j})_{j \in \{1, \dots, N\}}$ for \mathbb{E}_n such that the outer boundary $\partial\Omega$ is fully covered by $\cup_{j=1}^{N-1} U_{n,j}$, and $U_{n,0}$ is chosen in the same manner as Definition 4.10. The remaining components is chosen such that $U_{n,N}$ fully covers H_k we choose $U_{n,N}$ to be a ball of radius $2\delta_{K_i}$ centred at the hole. As

$$\mathbb{E}_n u = \omega_0^{(n)} u + \sum_{j=1}^N \omega_j^{(n)} \mathbb{E}_{n,j} u,$$

we see that the first $N - 1$ terms

$$\omega_0^{(n)} u + \sum_{j=1}^{N-1} \omega_j^{(n)} \mathbb{E}_{n,j} u$$

do not become singular as n increases due to the fact that the outer boundary remains constant for all $n \geq 1$, thus we focus on the remaining cover.

We now assume that

$$\sup_{n \geq 1} \left\| \omega_j^{(n)} \mathbb{E}_{n,N} \right\|_{\mathcal{L}(W^{1,p}(U_N \setminus H_{n_k}), W^{1,p}(\mathbb{R}^d))} = +\infty.$$

Then, for every $k \geq 1$, there exists an $n_k \geq 1$ such that

$$\left\| \omega_j^{(n_k)} \mathbb{E}_{n_k,N} \right\|_{\mathcal{L}(W^{1,p}(U_N \setminus H_{n_k}), W^{1,p}(\mathbb{R}^d))} \geq k$$

and hence, there is a $u_{n_k} \in W^{1,p}(U_{n,N} \setminus H_{n_k})$ satisfying

$$\|u_{n_k}\|_{W^{1,p}(U_{n,N} \setminus H_{n_k})} = 1$$

and

$$\left\| \omega_N^{(n_k)} \mathbb{E}_{n_k,N} u_{n_k} \right\|_{\mathcal{L}(W^{1,p}(U_N \setminus H_{n_k}), W^{1,p}(\mathbb{R}^d))} \geq k.$$

For convenience we relabel n_k as k . We note that $\|u_k\|_{W^{1,p}(U_{k,N} \setminus H_k)} = 1$ yields that

$$(4.9) \quad \|\nabla u_k\|_{L^p(U_{k,N} \setminus H_k)} \leq 1.$$

Next we note that Ω satisfies Assumption 4.11 without the need for a transformation and we then use the uniform Poincaré inequality 4.8 combined with the continuity of the extension operator, for every $v \in W^{1,p}(U_{k,N} \setminus H_k)$ with mean zero one has that,

$$(4.10) \quad \begin{aligned} \|\mathbb{E}_{N,k} v\|_{W^{1,p}(U_{k,N} \setminus H_k)} &\leq K_N \|v\|_{W^{1,p}(U_{k,N} \setminus H_k)} \\ &\leq K_N (1 + c_{Poi}) \|\nabla v\|_{L^p(U_{k,N} \setminus H_k)} \end{aligned}$$

where K_N and c_{Poi} are both independent of k . Then, by estimate (4.10)

$$\begin{aligned}
k &\leq \left\| \omega_N^{(k)} \mathbb{E}_{k,N} u_k \right\|_{W^{1,p}(U_{k,N} \setminus H_k)} \\
&\leq \left\| \omega_N^{(k)} \mathbb{E}_{k,N} u_k \right\|_{L^p(U_{k,N} \setminus H_k)} + \left\| \nabla(\omega_N^{(k)} \mathbb{E}_{k,N} u_k) \right\|_{L^p(U_{k,N} \setminus H_k)} \\
&< (2 + \|\omega_N\|_{L^\infty(\Omega)}) \left(\|\mathbb{E}_{N,k} u_k\|_{L^p(U_{k,N} \setminus H_k)} \right. \\
&\quad \left. + \|\nabla \mathbb{E}_{N,k} u_k\|_{L^p(U_{k,N} \setminus H_k)} \right)
\end{aligned}$$

for convenience we define a constant

$$C_{\omega_N} = 2 + \|\omega_N\|_{L^\infty(\Omega)}$$

and recalling estimate (4.9),

$$\begin{aligned}
k &\leq C_{\omega_N} K_N (1 + c_{Poi}) \|\nabla u_k\|_{L^p(U_{k,N} \setminus H_k)} \\
&\leq C_{\omega_N} K_N (1 + c_{Poi}),
\end{aligned}$$

which leads to a contradiction for sufficiently large k as $C_{\omega_N} K_N (1 + c_{Poi})$ is a constant independent of k . \square

We now consider a more general sequence of domains with shrinking holes.

Theorem 4.13. *For every $1 \leq p < \infty$ and given an open subset $\Omega \subseteq \mathbb{R}^d$ that satisfies Assumption 4.11. Thus, for $\Omega_n := \Omega \setminus H_n$, the sequence of associated extension operators $(\mathbb{E}_n)_{n \geq 1}$ is bounded, that is*

$$C_E := \sup_{n \geq 1} \|\mathbb{E}_n\|_{\mathcal{L}(W^{1,p}(\Omega_n), W^{1,p}(\mathbb{R}^d))} < \infty,$$

where $(\mathbb{E}_n)_{n \geq 1}$ is a sequence of Calderón operators associated to Ω_n given by Definition 4.10.

We now provide an original proof that builds off the ideas in [1, Theroem 4.32] and the simplified case in Theorem 4.12.

Proof. We choose the cover $(U_{n,j})_{j \in \{1, \dots, N\}}$ for \mathbb{E}_n such that the outer boundary $\partial\Omega$ is fully covered by $\cup_{j=1}^M U_{n,j}$, and $U_{n,0}$ is chosen in the same manner as Definition 4.10. The remaining components are chosen such that

$$\cup_{j=M+1}^N U_{n,j}$$

fully covers H_k we choose these $U_{n,j}$ to be balls of radius $2\delta_{K_i}$ centred at each hole K_i . We also assume that each hole is covered by a single cover. As

$$\mathbb{E}_n u = \omega_0^{(n)} u + \sum_{j=1}^N \omega_j^{(n)} \mathbb{E}_{n,j} u,$$

we see that the first M terms

$$\omega_0^{(n)} u + \sum_{j=1}^M \omega_j^{(n)} \mathbb{E}_{n,j} u$$

do not become singular as n increases due to the fact that the outer boundary remains constant for all $n \geq 1$, thus we focus on the remaining elements $j \in \{M+1, \dots, N\}$.

We now assume that there exists a $j \in \{M+1, \dots, N\}$ such that

$$\sup_{n \geq 1} \left\| \omega_j^{(n)} \mathbb{E}_{n,j} \right\|_{\mathcal{L}(W^{1,p}(U_j \setminus H_{n_k}), W^{1,p}(\mathbb{R}^d))} = +\infty.$$

Then, for every $k \geq 1$, there exists an $n_k \geq 1$ such that

$$\left\| \omega_j^{(n_k)} \mathbb{E}_{n_k,j} \right\|_{\mathcal{L}(W^{1,p}(U_j \setminus H_{n_k}), W^{1,p}(\mathbb{R}^d))} \geq k$$

and hence, there is a $u_{n_k} \in W^{1,p}(U_{n,j} \setminus H_{n_k})$ satisfying

$$\|u_{n_k}\|_{W^{1,p}(U_{n,j} \setminus H_{n_k})} = 1$$

and

$$\left\| \omega_j^{(n_k)} \mathbb{E}_{n_k,j} u_{n_k} \right\|_{\mathcal{L}(W^{1,p}(U_j \setminus H_{n_k}), W^{1,p}(\mathbb{R}^d))} \geq k.$$

For convenience we relabel n_k as k . We note that $\|u_k\|_{W^{1,p}(U_{n,j} \setminus H_k)} = 1$ yields that

$$(4.11) \quad \|\nabla u_k\|_{L^p(U_{k,j} \setminus H_k)} \leq 1.$$

Next we note that on the domain $\mathcal{U}_k = B_{2R} \setminus \delta_k^{-1} H_k$ we use the uniform Poincaré inequality 4.8 combined with the continuity of the extension operator, for every $v \in W^{1,p}(\mathcal{U}_k)$ with mean zero one has that,

$$(4.12) \quad \begin{aligned} \left\| \hat{\mathbb{E}}_{j,k} v \right\|_{\mathcal{L}(W^{1,p}(\mathcal{U}_k), W^{1,p}(\mathbb{R}^d))} &\leq \hat{K}_j \|v\|_{W^{1,p}(\mathcal{U}_k)} \\ \left\| \hat{\mathbb{E}}_{j,k} v \right\|_{\mathcal{L}(W^{1,p}(\mathcal{U}_k), W^{1,p}(\mathbb{R}^d))} &\leq \hat{K}_j (1 + c_{Poi}) \|\nabla v\|_{L^p(\mathcal{U}_k)} \end{aligned}$$

where \hat{K}_j and c_{Poi} are both independent of k .

We now rescale the functions $u_k \in W^{1,p}(U_{n,j} \setminus H_k)$ to $v_k \in W^{1,p}(\mathcal{U}_k)$ with mean $v_k|_{\mathcal{U}} = 0$ by the substitution $y = R/\delta_k(x - x_0)$, $v_k(y) = u_k(x)$, $\hat{\mathbb{E}}_{j,k} v_k = \mathbb{E}_{k,j} u_k$ and $\hat{\omega}_j(y) = \omega_j^{(k)}(x)$. Then, by estimate (4.12)

$$\begin{aligned} k &\leq \left\| \omega_j^{(k)} \mathbb{E}_{k,j} u_k \right\|_{W^{1,p}(U_{k,j} \setminus H_k)} \\ &\leq \left\| \omega_j^{(k)} \mathbb{E}_{k,j} u_k \right\|_{\mathcal{L}(L^p(U_j \setminus H_k), L^p(\mathbb{R}^d))} + \left\| \nabla(\omega_j^{(k)} \mathbb{E}_{k,j} u_k) \right\|_{\mathcal{L}(L^p(U_j \setminus H_k), L^p(\mathbb{R}^d))} \\ &\leq \left(\frac{\delta_k}{R} \right)^{d/p-1} \left(\frac{\delta_k}{R} \left\| \hat{\omega}_j \mathbb{E}_{k,j} v_k \right\|_{\mathcal{L}(L^p(\mathcal{U}_k), L^p(\mathbb{R}^d))} \right. \\ &\quad \left. + \left\| \nabla(\omega_j^{(k)} \mathbb{E}_{k,j} v_k) \right\|_{\mathcal{L}(L^p(\mathcal{U}_k), L^p(\mathbb{R}^d))} \right) \\ &< (2 + \|\hat{\omega}_j\|_{L^\infty(\Omega)}) \left(\frac{\delta_k}{R} \right)^{d/p-1} \left(\left\| \hat{\mathbb{E}}_{j,k} v_k \right\|_{\mathcal{L}(L^p(\mathcal{U}_k), L^p(\mathbb{R}^d))} \right. \\ &\quad \left. + \left\| \nabla \hat{\mathbb{E}}_{j,k} v_k \right\|_{\mathcal{L}(L^p(\mathcal{U}_k), L^p(\mathbb{R}^d))} \right) \end{aligned}$$

for convenience we define a constant

$$C_{\hat{\omega}_j} = 2 + \|\hat{\omega}_j\|_{L^\infty(\Omega)}$$

and recalling estimate (4.11),

$$\begin{aligned} k &\leq C_{\hat{\omega}_j} \hat{K}_j (1 + c_{Poi}) \left(\frac{\delta_k}{R} \right)^{d/p-1} \|\nabla v_k\|_{L^p(\mathcal{U}_k)} \\ &= C_{\hat{\omega}_j} \hat{K}_j (1 + c_{Poi}) \|\nabla u_k\|_{L^p(U_{n,j} \setminus H_k)} \\ &\leq C_{\hat{\omega}_j} \hat{K}_j (1 + c_{Poi}), \end{aligned}$$

which leads to a contradiction for sufficiently large k as $C_{\hat{\omega}_j} \hat{K}_j (1 + c_{Poi})$ is a constant independent of k . \square

Remark 4.14. We note that the proof is similar if a finite number of elements of the cover $(U_{n,j})_{j \in \{1, \dots, N\}}$ contain the i -th hole, $H_{k,i}$. The union of the elements is still transformed to the same uniform ball (see Assumption 4.11). Thus we replace \hat{K}_j with $\sup_{j: U_{k,j} \cap H_{k,i} \neq \emptyset} \hat{K}_j$ and $C_{\hat{\omega}_j}$ by $\sup_{j \in \{U_{k,j} \cap H_{k,i} \neq \emptyset\}} C_{\hat{\omega}_j}$. The proof then continues as before.

For the sake of clarity we provide a diagram of the process of creating the uniform cover \mathcal{U}_k , in Figure 3.

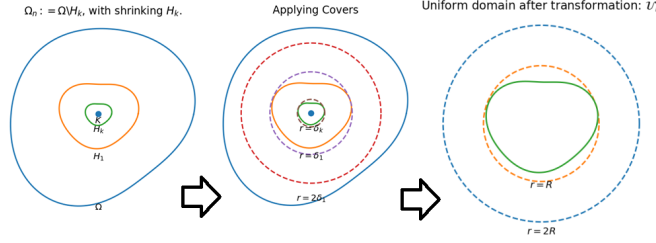


FIGURE 3. An example of the process used in the proof of Theorem 4.13.

Utilising Theorem 4.13, we now give an alternative proof of Theorem 3.31.

Alternative proof of Theorem 3.31. Let $1 \leq p < \infty$, $\Omega \subseteq \mathbb{R}^d$ be an open set satisfying Assumption 4.11, and set $\Omega_n := \Omega \setminus H_n$ for every $n \geq 1$. Furthermore, let $\mathbb{E}_n : W^{1,p}(\Omega_n) \rightarrow W^{1,p}(\mathbb{R}^d)$ be the associated extension operator to $W^{1,p}(\Omega_n)$ and $u_n \in W^{1,p}(\Omega_n)$. Thus, by the classical Sobolev embedding theorem (see Lemma 3.26), one has that

$$\begin{aligned} \|u_n\|_{L^{p^*}(\Omega_n)} &= \|\mathbb{E}_n u_n\|_{L^{p^*}(\Omega_n)} \\ &\leq \|\mathbb{E}_n u_n\|_{L^{p^*}(\mathbb{R}^d)} \\ &\leq C_{d,p} \|\mathbb{E}_n u_n\|_{W^{1,p}(\mathbb{R}^d)} \\ &\leq C_{d,p} \mathcal{C}_E \|u_n\|_{W^{1,p}(\Omega)}, \end{aligned}$$

showing that the sequence $(C_{Sob,p}(\Omega_n))_{n \geq 1}$ of Sobolev constants is bounded above by $C_{d,p} \mathcal{C}_E$. \square

In addition, we get the following version of the classical Rellich-Kondarchev's compactness result (see, for instance [11, Theorem 12.18]) for moving domains with shrinking holes. To the best of our knowledge, this result seems to be new in the literature.

Corollary 4.15 (Rellich-Kondarchev on moving domains). Let $1 < p < \infty$, Ω be an open domain in \mathbb{R}^d with finite Lebesgue measure, and assume the Assumption 4.11 holds. We set $\Omega_n := \Omega \setminus H_n$ for every $n \geq 1$. Then, every bounded sequence $(u_n)_{n \geq 1} \subseteq W^{1,p}(\Omega_n)$ admits a subsequence $(u_{l_n})_{n \geq 1}$ and a $u \in W^{1,p}(\mathbb{R}^d)$ such that

$$(4.13) \quad \lim_{n \rightarrow \infty} \|u_{l_n} - u\|_{L^p(\Omega_{l_n})} = 0.$$

Moreover, if $1 \leq p < \infty$, then for every $1 \leq q < p^*$, one has that

$$(4.14) \quad \lim_{n \rightarrow \infty} \|u_{l_n} - u\|_{L^q(\Omega_{l_n})} = 0.$$

To prove Corollary 4.15, we make use of the following auxiliary results. The first lemma is taken from [11, Lemma 12.20]. We omit its proof.

Lemma 4.16. *Let $1 \leq p < \infty$ and let $u \in W^{1,p}(\mathbb{R}^d)$ and $J_{1/k}$ for every $k > 0$ be the standard mollifier as defined in Definition C.2. Then, there exists a constant $C := C(d, p) > 0$ such that*

$$(4.15) \quad \int_{\mathbb{R}^d} |(J_{1/k} * u)(x) - u(x)|^p dx \leq \frac{C}{k^p} \int_{\mathbb{R}^d} |\nabla u(x)|^p dx$$

for every $k > 0$.

Our proof of Corollary 4.15 is based on an idea by Leoni [11] to prove the classic Rellich-Kondrachov compactness theorem.

Proof. Let $(u_n)_{n \geq 1} \subseteq W^{1,p}(\Omega_n)$ be a bounded sequence. Then, due to Theorem 4.13, each u_n admits an extension $\mathbb{E}_n u_n \in W^{1,p}(\mathbb{R}^d)$ and the sequence $(\mathbb{E}_n u_n)_{n \geq 1}$ is bounded in $W^{1,p}(\mathbb{R}^d)$. Since for every $1 < p < \infty$, $W^{1,p}(\mathbb{R}^d)$ is reflexive and since for reflexive Banach spaces the closed unit ball is relatively weakly compact (see Theorem B.32), there is a $u \in W^{1,p}(\mathbb{R}^d)$ and subsequence $(u_{l_n})_{n \geq 1}$ of $(u_n)_{n \geq 1}$ such that

$$(4.16) \quad \mathbb{E}_{l_n} u_{l_n} \rightharpoonup u \quad \text{weakly in } W^{1,p}(\mathbb{R}^d) \text{ as } n \rightarrow \infty.$$

We claim that the limit (4.13) holds.

Since the sequence $(\nabla \mathbb{E}_n u_n)_{n \geq 1}$ is bounded in $L^p(\mathbb{R}^d; \mathbb{R}^d)$, we conclude from inequality (4.15) of Lemma 4.16 that

$$\sup_{n \geq 1} \|J_{1/k} * \mathbb{E}_n u_n - \mathbb{E}_n u_n\|_{L^p(\mathbb{R}^d)} \leq \frac{c_1}{k}$$

where $c_1 := C \sup_{n \geq 1} \|\nabla \mathbb{E}_n u_n\|_{L^p(\mathbb{R}^d)}$. Hence, taking the limit for $k \rightarrow \infty$ in the previous inequality yields that

$$(4.17) \quad \lim_{k \rightarrow \infty} \sup_{n \geq 1} \|J_{1/k} * \mathbb{E}_n u_n - \mathbb{E}_n u_n\|_{L^p(\mathbb{R}^d)} = 0.$$

On the other hand, by the triangle inequality of the $L^p(\Omega)$ -norm, one sees that

$$\begin{aligned}
& \|u_{l_n} - u\|_{L^p(\Omega_{l_n})} \\
&= \|u_{l_n} - (J_{1/k} * \mathbb{E}_{l_n} u_{l_n}) + (J_{1/k} * \mathbb{E}_{l_n} u_{l_n}) - (J_{1/k} * u) + (J_{1/k} * u) - u\|_{L^p(\Omega_{l_n})} \\
&\leq \| \mathbb{E}_{l_n} u_{l_n} - (J_{1/k} * \mathbb{E}_{l_n} u_{l_n}) \|_{L^p(\Omega_{l_n})} + \| (J_{1/k} * \mathbb{E}_{l_n} u_{l_n}) - (J_{1/k} * u) \|_{L^p(\Omega_{l_n})} \\
&\quad + \| (J_{1/k} * u) - u \|_{L^p(\Omega_{l_n})} \\
&\leq \| \mathbb{E}_{l_n} u_{l_n} - (J_{1/k} * \mathbb{E}_{l_n} u_{l_n}) \|_{L^p(\mathbb{R}^d)} + \| J_{1/k} * (\mathbb{E}_{l_n} u_{l_n} - u) \|_{L^p(\Omega)} \\
&\quad + \| (J_{1/k} * u) - u \|_{L^p(\mathbb{R}^d)}.
\end{aligned}$$

By (4.17) and in particular (4.15), holds for the weak limit u given by (4.16), for every given $\varepsilon > 0$, there exists an index $k_0 > 0$ such that

$$\| \mathbb{E}_{l_n} u_{l_n} - (J_{1/k_0} * \mathbb{E}_{l_n} u_{l_n}) \|_{L^p(\mathbb{R}^d)} < \frac{\varepsilon}{3} \quad \text{and} \quad \| (J_{1/k_0} * u) - u \|_{L^p(\mathbb{R}^d)} < \frac{\varepsilon}{3}$$

for all $n \geq 1$. Therefore, in order to establish the limit (4.13), it remains to show that there is an $N_0 \in \mathbb{N}$ such that

$$(4.18) \quad \| J_{1/k_0} * (\mathbb{E}_{l_n} u_{l_n} - u) \|_{L^p(\Omega)} < \frac{\varepsilon}{3}$$

for all $n \geq N_0$. Owing to (4.16), $u_n \rightharpoonup u$ weakly in $L^p(\mathbb{R}^d)$ as $n \rightarrow \infty$ and since for every $x \in \mathbb{R}^d$, $J_{1/k_0}(x - \cdot) \in L^{p'}(\mathbb{R}^d)$, it follows that

$$(J_{1/k_0} * \mathbb{E}_{l_n} u_{l_n})(x) = \int_{\mathbb{R}^d} J_{1/k_0}(x - y) \mathbb{E}_{l_n} u_{l_n}(y) dy \rightarrow (J_{1/k_0} * u)(x) \quad \text{in } \mathbb{R}$$

as $n \rightarrow \infty$ for every $x \in \mathbb{R}^d$, or equivalently,

$$(4.19) \quad \lim_{n \rightarrow \infty} (J_{1/k_0} * (\mathbb{E}_{l_n} u_{l_n} - u))(x) = 0,$$

for every $x \in \mathbb{R}^d$.

Further, as the mollifier J_{1/k_0} is defined by

$$J_{1/k_0}(x) = k_0^d J(k_0 x)$$

for every $x \in \mathbb{R}^d$, and for some non-negative bump function $J \in C_c^\infty(\mathbb{R}^d)$ with support $\text{supp}(J)$ contained in the closed unit ball $\bar{B}_1(0)$, and since $(\mathbb{E}_{l_n} u_{l_n})_{n \geq 1}$ is bounded in $L^p(\mathbb{R}^d)$, there is a constant $C_2 > 0$ such that

$$\begin{aligned}
(4.20) \quad & |J_{1/k_0} * (\mathbb{E}_{l_n} u_{l_n} - u)(x)| \\
& \leq k_0^d \int_{\mathbb{R}^d} J_{1/k_0}(k_0(x - y)) |\mathbb{E}_{l_n} u_{l_n}(y) - u(y)| dy \\
& \leq k_0^d \|J\|_{L^\infty(\mathbb{R}^d)} \int_{B_{1/k_0}(0)} |\mathbb{E}_{l_n} u_{l_n}(y) - u(y)| dy \\
& \leq k_0^d C_2
\end{aligned}$$

for every $x \in \mathbb{R}^d$ and every $n \in \mathbb{N}$. Now, thanks to equation (4.19) and equation (4.20) and since we recall Ω admits finite Lebesgue measure, it follows from Lebesgue's dominated convergence theorem (see Theorem B.36) that $J_{1/k_0} * (\mathbb{E}_{l_n} u_{l_n} - u) \rightarrow 0$ in $L^p(\Omega)$ as $n \rightarrow \infty$. From this, we know that there is an index $N_0 \in \mathbb{N}$ such that (4.18) holds. This proves the limit (4.13). To see that the second limit (4.14) holds as well, one proceeds as before, but one applies the classical Sobolev embedding $W^{1,p}(\mathbb{R}^d) \hookrightarrow L^{p^*}(\mathbb{R}^d)$ (see Unbounded Sobolev, Gagliardo and

Nirenberg theorem 3.26) to the sub-sequence $(E_{l_n} u_{l_n})_{n \geq 1}$, and then one establishes first the limit (4.14) for $q = p^*$. This limit for $q = p^*$ implies that, in particular, equation (4.14) holds for all $1 \leq q \leq p^*$ due to the continuous embedding $L^{p^*}(\Omega_{l_n}) \hookrightarrow L^q(\Omega_{l_n})$, which holds uniformly due to $\Omega_{l_n} \subseteq \Omega$ for all $n \geq 1$ and Ω has finite Lebesgue measure. \square

5. PROOF OF CONVERGENCE ON DOMAINS WITH SHRINKING HOLES

In this section, we outline the proof of our main Theorem 1.2 stated in Section 1.

We use the compactness result stated in Corollary 4.15 to obtain $L^q(\Omega_n)$ convergence of the weak solutions u_n of the mixed boundary-value problem (1.2) which we recall briefly here. To do this, we assume in this section that the domain Ω satisfies Assumption 1.1.

Let $1 < p < \infty$ and $p' = p/(p-1)$. For given $f_n \in L^{p'}(\Omega_n)$ and $\alpha_n > 0$, let $u_n \in V^{1,p}(\Omega_n)$ be the unique weak solution of

$$(5.1) \quad \begin{cases} -\Delta_p u_n + V(x) |u_n|^{p-2} u_n = f_n & \text{in } \Omega_n, \\ |\nabla u_n|^{p-2} \nabla u_n \cdot \nu + \alpha_n |u_n|^{p-2} u_n = 0 & \text{on } \partial H_n, \\ u_n = 0 & \text{on } \partial \Omega. \end{cases}$$

Further, for given $f \in L^{p'}(\Omega)$, let $u \in W_{0,\partial\Omega}^{1,p}(\Omega^*)$ (see equation (2.17)) be a weak solution of the Poisson problem

$$(5.2) \quad \begin{cases} -\Delta_p u + V(x) |u|^{p-2} u = f & \text{in } \Omega^*, \\ u = 0 & \text{on } \partial \Omega. \end{cases}$$

We extend f_n by zero on Ω_n^c , and suppose that

$$f_n \rightharpoonup f \quad \text{weakly in } L^{p'}(\Omega) \text{ as } n \rightarrow \infty,$$

and $(\alpha_n)_{n \geq 1} \subset (0, \infty)$ be a zero-sequence.

Proof of Theorem 1.2. We multiply the differential equation in (5.1) by u_n . Then, we obtain that

$$\int_{\Omega_n} |\nabla u_n|^p dx + \int_{\Omega_n} V |u_n|^p dx + \alpha_n \int_{\partial H_n} |u_n|^p d\mathcal{H}_{d-1} = \int_{\Omega_n} f_n u_n dx.$$

Since $V \geq v_0 > 0$, and since by Hölder's and Youngs-inequality,

$$\begin{aligned} \left| \int_{\Omega_n} f_n u_n dx \right| &\leq \|f_n\|_{L^{p'}(\Omega_n)} \|u_n\|_{L^p(\Omega_n)} \\ &\leq \|f_n\|_{L^{p'}(\Omega)} \|u_n\|_{L^p(\Omega_n)} \\ &\leq C \|f_n\|_{L^{p'}(\Omega)}^{p'} + \frac{v_0}{2} \|u_n\|_{L^p(\Omega_n)}^p \end{aligned}$$

for some constant $C > 0$, it follows that

$$(5.3) \quad \int_{\Omega_n} |\nabla u_n|^p dx + \frac{v_0}{2} \int_{\Omega_n} |u_n|^p dx + \alpha_n \int_{\partial H_n} |u_n|^p d\mathcal{H}_{d-1} \leq C \|f_n\|_{L^{p'}(\Omega)}^{p'}.$$

Since weak convergence of $(f_n)_{n \geq 1}$ in $L^{p'}(\Omega)$ implies that $(f_n)_{n \geq 1}$ is bounded in $L^{p'}(\Omega)$ and since $\alpha_n \geq 0$, we conclude from (5.3) that for the sequence $(u_n)_{n \geq 1}$ one has that

$$\sup_{n \geq 1} \|u_n\|_{W^{1,p}(\Omega)} \leq C^{1/p} \|f_n\|_{L^{p'}(\Omega)}^{1/(p-1)}.$$

Thus, due to the *Rellich-Kondarchev result on moving domains* (Corollary 4.15), it follows that there is a function $u \in W^{1,p}(\mathbb{R}^d)$ and a sub-sequence $(u_{k_n})_{n \geq 1}$ of $(u_n)_{n \geq 1}$ such that

$$\lim_{n \rightarrow \infty} \|u_{k_n} - u\|_{L^q(\Omega_{k_n})} = 0.$$

In addition, one has that

$$u_{k_n} \rightharpoonup u \quad \text{weakly in } W^{1,p}(O) \text{ as } n \rightarrow \infty,$$

for every open and bounded subset O such that the closure $\bar{O} \subset \Omega^* \cup \partial\Omega$. Thus, multiplying the differential equation in (5.1) by a test function $\xi \in C_c^\infty(\Omega^*)$ and subsequently sending $n \rightarrow \infty$ yields that u is a weak solution of

$$-\Delta_p \tilde{u} + V |u|^{p-2} u = f \quad \text{in } \Omega^*.$$

To see that the trace $u = 0$ on $\partial\Omega$, one uses that the trace $u_n = 0$ on $\partial\Omega$ and that $(u_n)_{n \geq 1}$ is bounded in $W^{1,p}(\Omega_1)$, where $\Omega_1 = \Omega \setminus H_1$. \square

6. CONCLUDING REMARKS

In conclusion, we have rigorously established that families of shrinking domains $(\Omega_\varepsilon)_{\varepsilon > 0}$ have embedding constants independent of the domain under certain conditions, and that we can establish that the associated family of Calderón operators $(\mathbb{E}_\varepsilon)_{\varepsilon > 0}$ are also uniformly bounded. This led to 3 new results, an alternative proof for Theorem 3.31, Rellich-Kondarchev on moving domains (see Corollary 4.15), and finally obtaining $L^q(\Omega_n)$ convergence of the weak solutions u_n of the mixed boundary-value problem (1.2).

Looking beyond this thesis there are several areas to be improved, and natural extensions to be made. First the uniqueness in the limit problem should be given further thought. Next we should consider both extending these results to the non-local problem as well as include capacity, to allow for more complicated “holes”. Finally it would be of interest to add numerical simulations to extend these results.

APPENDIX A. GEOMETRY AND LINEAR ALGEBRA

In order to ensure domain independent Sobolev embeddings in Subsection 3.2 we used several key geometric and linear algebraic ideas. In this section we lay out these ideas. We start by introducing two notions of regularity. Firstly we recall the segment property [1, Definitions 4.2].

Definition A.1. An open subset $\Omega \subseteq \mathbb{R}^d$ is said to satisfy the *segment property* if there exists a locally finite open cover $\{U_j\}$ of $\partial\Omega$ and a corresponding sequence of non-zero vectors $\{y_j\}$ such that, for each j , whenever $x \in \bar{\Omega} \cap U_j$, it holds that $x + ty_j \in \Omega$ for all $0 < t < 1$.

We also recall the locally Lipschitz property.

Definition A.2. An open subset $\Omega \subseteq \mathbb{R}^d$ is said to have a *locally Lipschitz boundary* $\partial\Omega$ if for each $x \in \partial\Omega$ there is a neighbourhood U_x such that $U_x \cap \partial\Omega$ is the graph of a Lipschitz continuous function.

A Key geometric object that was used throughout this document, is the d -dimensional parallelepiped (see [1, Section 4.1]).

Definition A.3. Given linearly independent vectors $y_1, \dots, y_d \in \mathbb{R}^d$, the set

$$P = \left\{ \sum_{j=1}^d \lambda_j y_j : 0 < \lambda_j < 1 \right\}$$

forms a d -dimensional parallelepiped with a vertex placed at the origin. $x + P$ similarly has a vertex placed at x . We also recall that $x + P$ contains a finite cone with a vertex at x in addition to being contained by a finite cone with a vertex at x . We see an example of a parallelepiped in Figure 1.

We also recall the definition of an open d -dimensional ball and open d -dimensional cube.

Definition A.4. Given a scalar $r > 0$ we say the open d -dimensional ball centred at vertex $x_0 \in \mathbb{R}^d$ is given by the set

$$B(r, x_0) = \{x \in \mathbb{R}^d : |x - x_0| < r\}.$$

Definition A.5. Given a scalar $c > 0$ we say the open d -dimensional cube centred at vertex $0 \in \mathbb{R}^d$ is given by the set

$$Q_0 = \left\{ x = (x_1, \dots, x_d) \in \mathbb{R}^d \mid -\frac{c}{2} < x_j < \frac{c}{2}, 1 \leq j \leq d \right\}$$

The open cube centred at $x_0 \in \mathbb{R}^d$ is given by the translation $Q_{x_0} = x_0 + Q_0$.

We now recall some linear algebra concepts that were used in this thesis. The following groups are defined in [8, Section 1.2.2 and Section 1.2.1]

Definition A.6. The *special orthogonal group* $SO(d, \mathbb{R})$ also known as the *rotational group* each element $\mathbf{R} \in SO(d, \mathbb{R})$ satisfies the following properties:

- (1) $\det(\mathbf{R}) = 1$,
- (2) $\mathbf{R}\mathbf{R}' = \mathbf{R}'\mathbf{R} = I$,
- (3) \mathbf{R} is invertible,

where \mathbf{R} is a square d -dimensional matrix with each element in \mathbb{R} .

Definition A.7. The *special linear group* $SL(d, \mathbb{R})$ also known as the *shear group* with each element $\mathbf{R}_s \in SL(d, \mathbb{R})$ satisfies the following properties:

- (1) $\det(\mathbf{R}_s) = 1$,
- (2) \mathbf{R}_s is invertible,

where \mathbf{R}_s is a square d -dimensional matrix with each element in \mathbb{R} .

To conclude this section of the appendix, we recall the definition of the *Hausdorff complementary topology* on open sets. For this we recall the following notation.

Notation A.8. First, for a given subset $A \subseteq \mathbb{R}^d$,

$$\text{dist}(x, A) := \inf_{y \in A} |x - y|,$$

where one sets $\text{dist}(x, \emptyset) = \infty$ and $\sup \emptyset = 0$. Now, for every two closed subsets A_1, A_2 of \mathbb{R}^d , the *Hausdorff metric*

$$d_{\mathcal{H}}(A_1, A_2) := \max \left\{ \sup_{x \in A_1} \text{dist}(x, A_2), \sup_{x \in A_2} \text{dist}(x, A_1) \right\},$$

where one sets $d_{\mathcal{H}}(\emptyset, F) = 0$ if $F = \emptyset$ and $d_{\mathcal{H}}(\emptyset, F) = \infty$ otherwise.

With this notation in mind, we now give the definition of the Hausdorff complementary topology.

Definition A.9. A sequence $(\Omega_n)_{n \geq 1}$ of open sets Ω_n in \mathbb{R}^d converges to the open set Ω in \mathbb{R}^d in the Hausdorff complementary topology if for every closed ball B in \mathbb{R}^d , one has that $d_{\mathcal{H}}(B \cap \Omega_n^c, B \cap \Omega^c) \rightarrow 0$ as $n \rightarrow \infty$. We write this as $\Omega_n \rightarrow \Omega$ in \mathcal{H}^c .

APPENDIX B. MEASURE THEORY AND FUNCTIONAL ANALYSIS

Throughout the thesis, several elementary measure-theoretic and functional analytic concepts were utilised. These are included below for the sake of completeness.

Specifying where a function is non-zero is essential for analysis, particularly in the context of zero-boundary condition problems. The support of a function u is introduced, as defined in [1, Section 1.2].

Definition B.1. Let $\Omega \subseteq \mathbb{R}^d$ be an open set and u be a function u defined on Ω . Then, we define the *support* of u as

$$\text{supp } u = \overline{\{x \in \Omega : u(x) \neq 0\}}.$$

We also recall the diameter of a set.

Definition B.2. For a subset $\Omega \subseteq \mathbb{R}^d$ the diameter of the set is given by

$$\text{diam}(\Omega) := \sup_{x, y \in \Omega} |x - y|.$$

We recall the definition of dual space based on [10].

Definition B.3. For a Banach space $(X, \|\cdot\|_X)$ we denote X' as the *dual space* of X , defined by

$$X' := \{x' \in X : x' \text{ is linear and continuous}\}.$$

equipped with the norm

$$(B.1) \quad \|x'\|_{X'} := \sup_{\|x\|_X \leq 1} |\langle x', x \rangle_{X', X}|$$

for all $x \in X, x' \in X'$.

We introduce a method to decompose \mathbb{R}^d into two domains, the first the region where f is bounded and a union of cubes containing the larger values of f , this is based off [16, Theorem 1.4].

Theorem B.4. Let f be a non-negative integral function on \mathbb{R}^d and let $\alpha > 0$ be a constant. Then, there exists a decomposition of \mathbb{R}^d such that:

- (1) $\mathbb{R}^d = F \cup G, F \cap G = \emptyset,$
- (2) $f(x) \leq \alpha$ a.e. on $F.$
- (3) G is the union of cubes, $G = \bigcup_k Q_k$ with disjoint interiors and for each Q_k

$$\alpha < \int \frac{1}{|Q_k|} \int_{Q_k} f(x) dx \leq 2^d \alpha.$$

We recall the interpolation theorem to extend norm estimates from from $L^1(\mathbb{R}^d)$ and $L^r(\mathbb{R}^d)$ to $1 < p < r$ as given in [16, Theorem 1.5].

Theorem B.5 (Interpolation). Let $1 < r < \infty$, if T is sub-additive from $L^1(\mathbb{R}^d) + L^r(\mathbb{R}^d)$ to the space on \mathbb{R}^d . If T satisfies the following:

- (1) $|T(f+g)(x)| \leq |Tf(x)| + |Tg(x)|.$
(2) $|\{x : |Tf(x)| > \alpha\}| \leq \frac{A_1}{\alpha} \|f\|_{L^1(\mathbb{R}^d)}, f \in L^1(\mathbb{R}^d).$
(3) $|\{x : |Tf(x)| > \alpha\}| \leq \left(\frac{A_2}{\alpha} \|f\|_{L^r(\mathbb{R}^d)}\right) f \in L^r(\mathbb{R}^d).$

Then, we have that,

$$\|T(f)\|_{L^p(\mathbb{R}^d)} \leq A_p \|f\|_{L^p(\mathbb{R}^d)} \quad f \in L^p(\mathbb{R}^d)$$

for all $1 < p < r$ where A_p depends only on A_1, A_2, p and r .

In Subsection 3.2, we used the fact that $C^\infty(\Omega)$ is dense in $W^{1,p}(\Omega)$ by the famous result of Meyers and Serrin (see [12, Theorem 3.18]).

Theorem B.6 (Meyers and Serrin). *If $1 \leq p < \infty$, then*

$$H^{1,p}(\Omega) = W^{1,p}(\Omega).$$

For the interested reader, we briefly outline the proof of Theorem B.6, where we follow [1, Theorem 3.16].

Proof. We recall that $H^{1,p}(\Omega)$ is the completion of

$$\{u \in C^1(\Omega) : \|u\|_{W^{1,p}(\Omega)}\}.$$

By definition one has that $H^{1,p}(\Omega) \subseteq W^{1,p}(\Omega)$. It remains to be shown that $W^{1,p}(\Omega) \subseteq H^{1,p}(\Omega)$ i.e. $H^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega)$. If $u \in W^{1,p}(\Omega)$ and $\varepsilon > 0$, we show that there exists a $\varphi \in C^\infty(\Omega)$ such that $\|u - \varphi\|_{W^{1,p}(\Omega)} < \varepsilon$. For every $k = 1, 2, \dots$ let

$$\Omega_k := \{x \in \Omega : |x| < k \text{ and } \text{dist}(x, \partial\Omega) > 1/k\}$$

and set $\Omega_0 = \emptyset$. Then, we have,

$$\mathcal{O} = \{U_k : U_k = \Omega_{k+1} \cap \overline{\Omega_{k-1}}^c, k = 1, 2, \dots\}$$

is a collection of open subsets of Ω that cover Ω . Let Ψ be a C^∞ -partition of unity for Ω that is subordinate to \mathcal{O} . Let ψ_k denote the sum of finitely many functions $\psi \in \Psi$ whose support are within U_k . Then, $\psi_k \in C_0^\infty(U_k)$ and

$$\sum_{k=1}^{\infty} \psi_k(x) = 1 \text{ on } \Omega.$$

If $0 < \varepsilon < \frac{1}{(k+1)(k+2)}$, then $J_\varepsilon * (\psi_k u)$ has support in $\Omega_{k+2} \cap (\Omega_{k-2})^c = \mathcal{V}_k \subset \subset \Omega$ where J_ε is a mollifier function given by Definition C.2. Since $\psi_k u \in W^{1,p}(\Omega)$ we select ε_k such that $0 < \varepsilon_k < 1/(k+1)(k+2)$, and we get that

$$\|J_{\varepsilon_k} * (\psi_k u) - \psi_k u\|_{W^{1,p}(\Omega)} = \|J_{\varepsilon_k} * (\psi_k u) - \psi_k u\|_{1,p,\mathcal{V}_k} < \varepsilon/2k.$$

Let $\varphi = \sum_{k=1}^{\infty} J_{\varepsilon_k} * (\psi_k u)$. On any $\Omega' \subset \subset \Omega$ all but a finite number of terms are non-zero. Therefore $\varphi \in C^\infty(\Omega)$ and we have for every $x \in \Omega_k$ we have

$$u(x) = \sum_{j=1}^{k+2} \psi_j(x)u(x), \quad \varphi(x) = \sum_{j=1}^{k+2} J_{\varepsilon_j} * (\psi_j(x)u(x)).$$

Thus

$$\|u - \varphi\|_{1,p,\Omega_k} \leq \sum_{j=1}^{k+2} \|J_{\varepsilon_j} * (\psi_j u) - \psi_j u\|_{W^{1,p}(\Omega)} < \varepsilon.$$

By the monotone convergence Theorem, $\|u - \varphi\|_{W^{1,p}(\Omega)} < \varepsilon$. \square

The following theorem shows that $C_0^\infty(\mathbb{R}^d)$ is dense in $W^{1,p}(\Omega)$ with certain conditions as given in [1, Theorem 3.18].

Theorem B.7. *Let $\Omega \subseteq \mathbb{R}^d$ be an open set with the segment property (A.1). Then, the set of restrictions to Ω of functions in $C_0^\infty(\mathbb{R}^d)$ is dense in $W^{1,p}(\Omega)$ for every $1 \leq p < \infty$.*

Next we introduce a useful estimate given in [1, Lemma 5.9] which was used in the proof of Theorem 3.21.

Lemma B.8. *Let $\Omega \subseteq \mathbb{R}^d$ be a given open subset where $d \geq 2$. We set k to be an integer satisfying $1 \leq k \leq d$, and let $\kappa = (\kappa_1, \dots, \kappa_k)$ denote a k -tuple of integers satisfying $1 \leq \kappa_1 \leq \dots \leq \kappa_k \leq d$. We additionally define S to be the set of $\binom{d}{k}$ k -tuples and given $x \in \mathbb{R}^d$, let x_κ denote the point*

$$(x_{\kappa_1}, \dots, x_{\kappa_k}) \in \mathbb{R}^k$$

and

$$dx_\kappa = dx_{\kappa_1} \dots dx_{\kappa_k}.$$

For a given $\kappa \in S$ let E_κ be the k -dimensional plane in \mathbb{R}^d spanned by the coordinate axis corresponding to the components of x_κ given by

$$E_\kappa = \{x \in \mathbb{R}^d : x_j = 0 \text{ if } j \notin \kappa, 1 \leq j \leq k\},$$

and for any set $G \subset \mathbb{R}^d$ set G_k as the projection of G onto E_k , in particular

$$\Omega_\kappa = \{x \in E_\kappa : \exists y \in \Omega \text{ so that } y_\kappa = x_\kappa\}.$$

Let F_κ be a function depending on on the k components of x_κ and belonging to $L^\lambda(\Omega_\kappa)$, where $\lambda = \binom{d-1}{k-1}$. Then, the function F defined on Ω by

$$F(x) = \prod_{\kappa \in S} F_\kappa(x_\kappa)$$

is contained in $L^1(\Omega)$, and $\|F\|_{L^1(\Omega)} \leq \prod_{\kappa \in S} \|F_\kappa\|_{L^\lambda(\Omega_\kappa)}$ that is,

$$(B.2) \quad \left[\int_{\Omega} |F(x)| dx \right]^\lambda \leq \prod_{\kappa \in S} \int_{\Omega_\kappa} |F_\kappa(x_\kappa)|^\lambda dx_\kappa.$$

Remark B.9. Lemma B.8 is proven by induction, using integration and Hölder's inequality, we omit the full details of this proof.

The following theorem [1, Theorem 2.8] establishes several embedding results, but its application is dependent on the volume of the domain, Ω . For this reason, it is used as a stepping stone to the domain independent results.

Theorem B.10. *Let $1 \leq p \leq q \leq \infty$ and $\Omega \subseteq \mathbb{R}^d$ be an open subset and suppose $|\Omega| < \infty$. If $u \in L^q(\Omega)$ then $u \in L^p(\Omega)$ and*

$$\|u\|_{L^p(\Omega)} \leq (|\Omega|)^{1/p-1/q} \|u\|_{L^q(\Omega)}.$$

So that

$$L^p(\Omega) \hookrightarrow L^q(\Omega).$$

Additionally If $u \in L^\infty(\Omega)$, then we have

$$(B.3) \quad \lim_{p \rightarrow \infty} \|u\|_{L^p(\Omega)} = \|u\|_{L^\infty(\Omega)}.$$

For every $1 \leq p < \infty$ and $u \in L^p(\Omega)$ if there is a constant K such that for all p

$$(B.4) \quad \|u\|_{L^p(\Omega)} \leq K,$$

then $u \in L^\infty(\Omega)$ and

$$(B.5) \quad \|u\|_{L^\infty(\Omega)} \leq K.$$

We outline a summary of the proof of Theorem B.10 following the methods in [1, Section 2].

Proof. Part 1: estimate (B.10)

Estimate (B.10) trivially applies for every $p = q$, if $1 \leq p < q \leq \infty$ and $u \in L^q(\Omega)$, by Hölder's inequality we have

$$\int_{\Omega} |u(x)|^p dx \leq \left\{ \int_{\Omega} |u(x)|^q dx \right\}^{p/q} \left\{ \int_{\Omega} 1 dx \right\}^{1-p/q}.$$

and estimate (B.10) follows immediately.

Part 2: equation (B.3)

If $u \in L^\infty(\Omega)$, applying estimate (B.10) and taking the limit of $p \rightarrow \infty$ we have

$$(B.6) \quad \limsup_{p \rightarrow \infty} \|u\|_{L^p(\Omega)} \leq \|u\|_{L^\infty(\Omega)}.$$

Additionally for any $\varepsilon > 0$ there exists a set $A \subset \Omega$ with positive measure $\lambda(A)$ so that

$$|u(x)| \geq \|u\|_{L^\infty(\Omega)} - \varepsilon.$$

So raising this to the power of p and integrating over the domain we get

$$\int_{\Omega} |u(x)|^p dx \geq \int_A |u(x)|^p dx \geq \lambda(A) (\|u\|_{L^\infty(\Omega)} - \varepsilon)^p.$$

$$(B.7) \quad \lim_{\varepsilon \rightarrow 0^+} \liminf_{p \rightarrow \infty} \|u\|_{L^p(\Omega)} \geq \|u\|_{L^\infty(\Omega)}.$$

Equation (B.3) is obtained by combining bounds (B.6) and (B.7).

Part 3 : estimate (B.5)

Finally suppose if estimate (B.4) holds and $u \notin L^\infty(\Omega)$ or estimate (B.5) does not hold, then there exists a constant $K_1 > K$ and a $A \subseteq \Omega$ with $\lambda(A) > 0$ such that $x \in A$, $|u(x)| \geq K_1$. Using the same logic used to derive estimate (B.7) demonstrates that

$$\lim_{\varepsilon \rightarrow 0^+} \liminf_{p \rightarrow \infty} \|u\|_{L^p(\Omega)} \geq K_1$$

which contradicts estimate (B.4). \square

B.1. Derivatives. This subsection recalls various generalised derivative definitions for normed spaces based off [10].

Definition B.11 (Fréchet derivatives). Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces and $U \subseteq X$ be an open subset. A function $f : U \rightarrow Y$ is *Fréchet differentiable* at $x_0 \in U$ if there exists a linear, continuous mapping $T : X \rightarrow Y$ such that for all $h \in X$ sufficiently small, we have that

$$(B.8) \quad f(x_0 + h) = f(x_0) + Th + o(h)$$

and

$$(B.9) \quad \lim_{h \rightarrow 0} \frac{\|o(h)\|_Y}{\|h\|_X} = 0.$$

If both equations (B.8) and (B.9) holds then we set $f'(x_0) := T$ to be the *derivative of f at x_0* .

Definition B.12 (Gateaux derivative (directional derivative)). Let $(X, \|\cdot\|_X)$, and $(Y, \|\cdot\|_Y)$ be two normed spaces and $U \subseteq X$ be an open subset. A function $f : U \rightarrow Y$ is *Gateaux differentiable* (*Directional differentiable*) at some $x_0 \in U$ if there exists linear, continuous mapping $T_{x_0} : X \rightarrow Y$ such that for all $\mathbf{v} \in X : \|\mathbf{v}\| = 1$ one has that

$$(B.10) \quad \lim_{t \rightarrow 0} \frac{f(x_0 + t\mathbf{v}) - f(x_0)}{t} = T_{x_0}\mathbf{v}.$$

If equation (B.10) holds then we set $D_{\mathbf{v}}f(x_0) := T_{x_0}\mathbf{v}$ as the *directional derivative of f at x_0 in direction \mathbf{v}* .

Lemma B.13. *Let $(X, \|\cdot\|_X)$ be a normed space, $U \subseteq X$ be an open subset and additionally if $f : u \in U$. If there is an $x_0 \in U$ such that $f(x_0) = \min_U f$ then $f'(x_0) = 0$.*

Definition B.14 (Proper functions and effective domains). Let $(X, \|\cdot\|_X)$ be a normed space, and $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$. Then, we call the set

$$D(f) := \{x \in X | f(x) < +\infty\}$$

the *effective domain* of f . Additionally if $D(f) \neq \emptyset$ then f is called a *proper function*.

Definition B.15 (Sub-differential). Let $(X, \|\cdot\|_X)$ be a normed space and $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$. Then, we define the *sub-differential* of f as the potentially multi-value mapping $\partial f : X \rightarrow 2^{X'}$ where $(2^{X'})$ denotes the powerset of X' with *domain*

$$D(\partial f) := \left\{ x \in D(f) \mid \exists x' \in X' : \liminf_{t \rightarrow 0^+} \frac{f(x + t\mathbf{v}) - f(x)}{t} \geq \langle x', x \rangle_{X', X} \quad \forall \mathbf{v} \in X \right\}$$

and *value-set*

$$\partial f(x) := \left\{ x' \in X' \mid \liminf_{t \rightarrow 0^+} \frac{f(x + t\mathbf{v}) - f(x)}{t} \geq \langle x', x \rangle_{X', X} \quad \text{for all } \mathbf{v} \in X \right\}.$$

Proposition B.16. Let $(X, \|\cdot\|_X)$ be a normed space and $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$. If the following statements hold:

- $f : X \rightarrow \mathbb{R}$ is Gateau-differentiable, then $\partial f : X \rightarrow X'$ is a well-defined single-valued mapping with $\partial f = \{Df(x)\}$.
- $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is proper and convex.

Then, we have the following properties:

$$D(\partial f) := \{x \in D(f) | \exists x' \in X' \text{ so that } f(x + t\mathbf{v}) - f(x) \geq \langle x', x \rangle_{X', X} \quad \forall \mathbf{v} \in X\},$$

and

$$\partial f(x) := \{x' \in X' | f(x + \mathbf{v}) - f(x) \geq \langle x', x \rangle_{X', X} \quad \text{for all } \mathbf{v} \in X\}.$$

We recall the generalised idea of critical points in normed spaces as given in [10].

Proposition B.17. Let $(X, \|\cdot\|_X)$ be a normed space and $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be convex. Then, for every $x_0 \in D(f)$, one has that

$$f(x_0) = \inf_{x_0 \in G} f \Leftrightarrow O_{X'} \in \partial f(x_0).$$

B.2. Minimisation. The framework for minimising convex functions is now provided for completeness based off [10]. We start by introducing several required definitions.

Definition B.18. A real-extended-function $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ on a normed space $(X, \|\cdot\|_X)$ is *convex* if for every $x, y \in X$ and every $\lambda \in [0, 1]$,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

Furthermore, f is *strictly convex* if the estimate (B.18) is replaced with a strictly less than requirement,

$$f(\lambda x + (1 - \lambda)y) < \lambda f(x) + (1 - \lambda)f(y).$$

Definition B.19. A real-extended-function $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ on a normed space $(X, \|\cdot\|_X)$ is *lower semi-continuous* if for every $x \in X$ and every sequence $(x_n)_{n \geq 1} \subseteq X$,

$$(B.11) \quad x_n \rightarrow x \in X \Rightarrow f(x) \leq \liminf_{n \rightarrow \infty} f(x_n).$$

Conversely one has that *upper semi-continuous* if the inequality in estimate (B.11) is converted to

$$x_n \rightarrow x \in X \Rightarrow f(x) \geq \liminf_{n \rightarrow \infty} f(x_n).$$

Definition B.20. Let $X \subseteq \mathbb{R}^d$ be a set. The *sublevel set* is given by

$$E_c := \{x \in X \mid f(x) \leq c\}.$$

Lemma B.21. A real-extended-function $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ on a normed space $(X, \|\cdot\|_X)$ is *lower semi-continuous* (see Definition B.19) if and only if for all $c \in \mathbb{R}$ the sublevel set E_c is closed in X .

Definition B.22. Let $(X, \|\cdot\|_X)$ be a normed space. A sequence $(x_n)_{n \geq 1}$ *converges weakly* to some $x \in X$ denoted as $x_n \rightharpoonup x$ weakly in X as $n \rightarrow \infty$ if

$$\langle x', x_n \rangle_{X', X} \rightarrow \langle x', x \rangle_{X', X} \quad n \rightarrow \infty,$$

for all $x' \in X'$.

Definition B.23. A real-extended-function $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ on a normed space $(X, \|\cdot\|_X)$ is *weakly lower semi-continuous* if for every $x \in X$ and every sequence $(x_n)_{n \geq 1} \subseteq X$,

$$(B.12) \quad x_n \rightharpoonup x \in X \Rightarrow f(x) \leq \liminf_{n \rightarrow \infty} f(x_n).$$

Conversely we have that *weakly upper semi-continuous* if the inequality in estimate (B.12) is converted to

$$x_n \rightharpoonup x \in X \Rightarrow f(x) \geq \liminf_{n \rightarrow \infty} f(x_n).$$

Corollary B.24. Let $(X, \|\cdot\|_X)$ be a normed space and $K \subseteq X$ be a closed convex set. If the sequence $(x_n)_{n \geq 1} \subseteq K$ converges weakly to $x \in X$, then $x \in K$.

Corollary B.25. Let $(X, \|\cdot\|_X)$ be a normed space and $K \subseteq X$ be a closed convex set. If $f : K \rightarrow \mathbb{R} \cup \{+\infty\}$ is lower semi-continuous and convex then f is weakly lower semi-continuous (see Definition B.23).

Corollary B.26. Let $(X, \|\cdot\|_X)$ be a normed space. There exists $x' \in X'$ such that $\|x'\|_{X'} = 1$ and $\langle x', x_0 \rangle_{X', X} = \|x_0\|_X$. For every $x_0 \in X \setminus \{O_x\}$.

Definition B.27. Let $(X, \|\cdot\|_X)$ be a normed space, the set

$$X'' := (X')' := \{x'' : X' \rightarrow \mathbb{C}\}$$

is the *bidual* of X .

Lemma B.28. Let $(X, \|\cdot\|_X)$ be a normed space. Then, the mapping \mathcal{F} is well-defined and a linear isometry. Where \mathcal{F} is defined as,

$$(B.13) \quad \mathcal{F} : \begin{cases} X \rightarrow X'' \\ x \mapsto \mathcal{F}_x \end{cases}$$

and

$$\mathcal{F}_x : \begin{cases} X' \rightarrow \mathbb{C} \\ x' \mapsto \langle \mathcal{F}_x, x' \rangle_{X'', X'} := \langle x', x \rangle_{X', X} \end{cases}$$

Definition B.29. A Banach space $(X, \|\cdot\|_X)$ is *reflexive* if the isometry defined in equation (B.13) is surjective. In other words this means that a Banach space X is reflexive if for all $x'' \in X''$ there exists $x \in X$ so that

$$\langle x'', x' \rangle_{X'', X'} = \langle X', x \rangle_{X', X},$$

for all $x' \in X'$.

Theorem B.30. Every closed linear sub-space V of a reflexive Banach space $(X, \|\cdot\|_X)$ is reflexive.

Remark B.31. For every $1 \leq p < \infty$ and let $\Omega \subseteq \mathbb{R}^d$ be an open subset. Then, we have that $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ are both reflexive.

Theorem B.32. Let $(X, \|\cdot\|_X)$ be a reflexive Banach space. Every bounded sequence admits a weakly convergent subsequence.

Definition B.33. Let $(X, \|\cdot\|_X)$ be a Banach space, and $K \subseteq X$ be a closed subset and $f : K \rightarrow \mathbb{R} \cup \{+\infty\}$, f is defined as *weakly coercive* if

$$\lim_{\|x\|_X \rightarrow \infty} f(x) = \infty, \quad ,$$

for all $x \in K$.

Theorem B.34. Let $(X, \|\cdot\|_X)$ be a reflexive Banach space, $K \subseteq X$ be a closed, convex, non-empty subset, and $f : K \rightarrow \mathbb{R} \cup \{+\infty\}$ is a lower semi-continuous, convex and weakly coercive (see Definition B.33). Then, there exists $x_0 \in K$ so that

$$(B.14) \quad f(x_0) = \min_{x \in K} f(x) > -\infty.$$

Lemma B.35 (Uniqueness). Let $(X, \|\cdot\|_X)$ be a normed space, $K \subseteq X$ be a convex set. If $f : K \rightarrow \mathbb{R} \cup \{+\infty\}$ is strictly convex then there is at most one solution to equation (B.14).

B.3. Convergence. To identify the limit of a sequence of measurable functions that are bounded by a Lebesgue integrable function we call the Lebesgue's dominated convergence theorem (see [11, Theorem B.48, p. 661]).

Theorem B.36 (Lebesgue's dominated convergence). *Let $(\Sigma, \mathcal{B}, \lambda)$ be a measure space, $u_n : \Sigma \rightarrow [-\infty, \infty]$ be measurable functions be a sequence of measurable functions so that $\lim_{n \rightarrow \infty} u_n(x) = u(x)$ λ -a.e. $x \in \Sigma$. If there exists a Lebesgue integrable function v so that $|u_n(x)| \leq v(x)$ for λ -a.e. $x \in \Sigma$ and for all $n \in \mathbb{N}$, then u is Lebesgue integrable and*

$$\lim_{n \rightarrow \infty} \int_{\Sigma} |u_n - u| d\lambda = 0$$

and

$$\lim_{n \rightarrow \infty} \int_{\Sigma} u_n d\lambda = \int_{\Sigma} u d\lambda.$$

We also introduce a theorem that bounds sub-sequences by a function (see [2]).

Theorem B.37. *Let $\Omega \subseteq \mathbb{R}^d$ be an open subset for all $1 \leq p < \infty$. Then for a convergent sequence $(f_n)_{n \geq 1} \in L^p(\Omega)$ converging to $L^p(\Omega)$ there exists a subsequence $(f_{n_k})_{n_k \geq 1}$ and a function $h \in L^p(\Omega)$ such that*

- (1) f_{n_k} converges to f almost everywhere $x \in \Omega$;
- (2) $|f_{n_k}| \leq h(x)$ almost everywhere $x \in \Omega$.

In order to utilise the reflexivity of L^{p^*} we recall the Riesz's representation theorem in L^p (see [11, Theorem B.92]).

Theorem B.38 (Riesz's representation theorem in L^p). *Let $(\Sigma, \mathcal{B}, \lambda)$ be a measure space, $1 \leq p < \infty$. Then, every bounded linear functional $L : L^p(\Sigma) \rightarrow \mathbb{R}$ is represented by a unique $v \in L^{p'}(\Sigma)$ i.e.*

$$(B.15) \quad L(u) = \int_{\Sigma} uv d\lambda,$$

for every $u \in L^p(\Sigma)$. Additionally for every functional of the form matching equation (B.15) with $v \in L^{p'}(\Sigma)$, is a bounded linear functional on $L^p(\Sigma)$. Therefore the dual of $L^p(\Sigma)$ is $L^{p'}(\Sigma)$ and $L^p(\Sigma)$ is reflexive.

The following definition and theory will be used to show that $(u_n)_{n \geq 1}$ converges to u in L^p based off [11, Definition B.100 and Theorem B.101].

Definition B.39. Let $(\Sigma, \mathcal{B}, \lambda)$ be a measure space and $1 \leq p < \infty$. A family \mathcal{F} of measurable functions $u : \Sigma \rightarrow [-\infty, \infty]$ is p -*equi-integrable* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_E |u|^p d\lambda \leq \varepsilon$$

for all $u \in \mathcal{F}$ and for every measurable set $E \subseteq \Sigma$ with $\lambda(E) \leq \delta$.

Theorem B.40 (Vitali's Convergence). *Let $(\Sigma, \mathcal{B}, \lambda)$ be a measure space with finite measure, $1 \leq p < \infty$, and let $u_n, u \in L^p(\Sigma)$, $n \in \mathbb{N}$. Then, $(u_n)_{n \geq 1}$ converges to u in $L^p(\Sigma)$ if and only if the following conditions hold:*

- (1) $(u_n)_{n \geq 1}$ converges to u in measure.
- (2) $(u_n)_{n \geq 1}$ is p -*equi-integrable*.

APPENDIX C. MOLLIFIERS

Throughout this document, mollifier functions are used to provide smooth approximations via regularisation. Firstly we introduce a key theorem that is used to prove the properties of mollifier functions as given in [1, Theorem 2.13].

Theorem C.1. *Let $1 \leq p < \infty$ and Ω be an open subset $\Omega \subseteq \mathbb{R}^d$ then $C_0(\Omega)$ is dense in $L^p(\Omega)$.*

Now we introduce mollifier functions as defined in [1, Definition 2.17].

Definition C.2. For every $x \in \mathbb{R}^d$. We define the non-negative function $J(x) \in C_0^\infty(\mathbb{R}^d)$ as $J : \mathbb{R}^d \rightarrow \mathbb{R}$ with the following properties:

$$\begin{aligned} J(x) &= 0 \text{ if } |x| \geq \varepsilon, \\ \int_{\mathbb{R}^d} J(x) dx &= 1. \end{aligned}$$

Then, we define the *mollifier function* $J_\varepsilon \in C_0^\infty(\mathbb{R}^d)$ as $J_\varepsilon : \mathbb{R}^d \rightarrow \mathbb{R}$ defined as $J_\varepsilon(x) := \varepsilon^{-d} J(x/\varepsilon)$ inherits the following properties:

- (1) $J_\varepsilon(x)$ is non-negative.
- (2) $J_\varepsilon(x) = 0$ if $|x| \geq \varepsilon$.
- (3) $\int_{\mathbb{R}^d} J_\varepsilon(x) dx = 1$.

We also have that for functions $u : L^1(\mathbb{R}^d) \rightarrow \mathbb{R}$ the convolution

$$(C.1) \quad J_\varepsilon * u(x) = \int_{\mathbb{R}^d} J_\varepsilon(x-y)u(y)dy,$$

referred to as the *mollification* of u .

Remark C.3. We note the existence of the integral of $J_\varepsilon * u$ follows from Young's inequality (see [2, Theorem 4.15.]),

A simple example of a bump function is provided for illustrative purposes.

Example C.4. The *bump function*, for $\varepsilon > 0$, $J_\varepsilon(x) : \mathbb{R}^d \rightarrow \mathbb{R}$ is defined by

$$J_\varepsilon(x) := \begin{cases} \varepsilon^{-d} k e^{-1/(1-|x/\varepsilon|^2)}, & \text{if } |x| < \varepsilon \\ 0, & \text{if } |x| \geq \varepsilon \end{cases}$$

where k is the constant such that $\int_{\mathbb{R}^d} J(x) dx = 1$, we note that this has no closed form solution but is roughly $k \approx 2.25$ for $d = 1$. We also provide a small illustration to visualise this for $d = 1$ in Figure 4.

We provide several useful properties of mollifier functions.

Lemma C.5. *Let u be a function which is defined on \mathbb{R}^d and vanishes identically outside of the domain Ω and for $\varepsilon > 0$, the mollifier function J_ε defined as per Definition C.2, has the following properties:*

$$(C.2) \quad \text{If } u \in L_{loc}^1(\overline{\Omega}), \text{ then } J_\varepsilon * u \in C_0^\infty(\mathbb{R}^d).$$

$$(C.3) \quad \text{If } \text{supp}(u) \subset\subset \Omega, \text{ then } J_\varepsilon * u \in C_0^\infty(\mathbb{R}^d) \text{ for } \varepsilon < \text{dist}(\text{supp}(u), \partial\Omega).$$

$$(C.4) \quad \text{For every } 1 \leq p < \infty \text{ If } u \in L^p(\Omega) \text{ then } J_\varepsilon * u \in L^p(\Omega) \text{ and additionally,}$$

$$(C.5) \quad \|J_\varepsilon * u\|_{L^p(\Omega)} \leq \|u\|_{L^p(\Omega)}$$

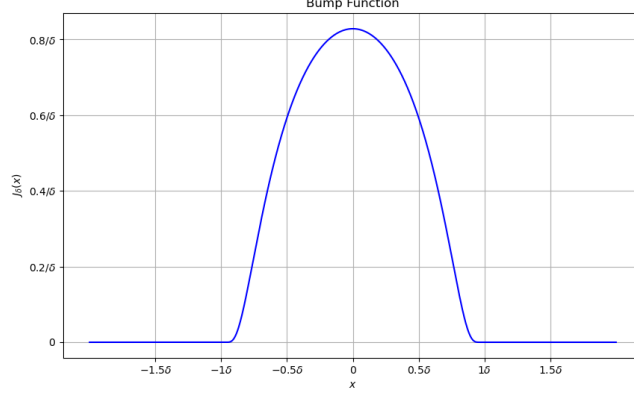


FIGURE 4. An example of a bump function.

and

$$(C.6) \quad \lim_{\varepsilon \rightarrow 0^+} \|J_\varepsilon * u - u\|_{L^p(\Omega)} = 0.$$

We prove Lemma C.5 following the methods in [1, Lemma 2.18].

Proof. As $J_\varepsilon(x - y)$ is infinitely differentiable and vanishes for $|x - y| \leq \varepsilon$, for every multi-index α and every function u that is integrable on compact sets in \mathbb{R}^d we have that

$$D^\alpha(J_\varepsilon * u) = \int_{\mathbb{R}^d} D_x^\alpha J_\varepsilon(x - y)u(y)dy,$$

So both properties (C.2) and (C.3) follow immediately. For Property C.4 let $u \in L^p(\Omega)$ with $1 < p < \infty$, we also use the Hölder conjugate exponent $p' = p/(p - 1)$ and then by Hölder's inequality we have

$$\begin{aligned} |J_\varepsilon * u(x)| &= \left| \int_{\mathbb{R}^d} J_\varepsilon(x - y)u(y)dy \right| \\ &\leq \left\{ \int_{\mathbb{R}^d} J_\varepsilon(x - y)dy \right\}^{1/p'} \left\{ \int_{\mathbb{R}^d} J_\varepsilon(x - y)|u(y)|^p dy \right\}^{1/p} \\ &= \left\{ \int_{\mathbb{R}^d} J_\varepsilon(x - y)|u(y)|^p dy \right\}^{1/p}. \end{aligned}$$

By Fubini's Theorem we see that

$$(C.7) \quad \begin{aligned} \int_{\Omega} |J_\varepsilon * u(x)|^p dx &\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J_\varepsilon(x - y)|u(y)|^p dy dx \\ &= \int_{\mathbb{R}^d} J_\varepsilon(x - y)dx \int_{\mathbb{R}^d} |u(y)|^p dy = \|u\|_{L^p(\Omega)}^p. \end{aligned}$$

By Theorem C.1 we also have for $\eta > 0$, that there exists a $\varphi \in C_0(\Omega)$ so that $\|u - \varphi\|_{L^p(\Omega)} < \eta/3$. We also have by (C.7) that

$$\|J_\varepsilon * u - J_\varepsilon * \varphi\|_{L^p(\Omega)} < \eta/3.$$

Next we see that

$$(C.8) \quad \begin{aligned} |J_\varepsilon * \varphi(x) - \varphi(x)| &= \left| \int_{\mathbb{R}^d} J_\varepsilon(x-y)(\varphi(y) - \varphi(x)) \right| \\ &\leq \sup_{|y-x| < \varepsilon} |\varphi(y) - \varphi(x)|. \end{aligned}$$

As φ is uniformly continuous on Ω so (C.8) tends to zero as $\varepsilon \rightarrow 0^+$. As $\text{supp } \varphi$ is compact, one has that $\|J_\varepsilon * \varphi - \varphi\|_{L^p(\Omega)} < \eta/3$ for sufficiently small ε . So now one has that

$$\begin{aligned} \|J_\varepsilon * u - u\|_{L^p(\Omega)} &= \|J_\varepsilon * u - u + J_\varepsilon * \varphi - J_\varepsilon * \varphi + \varphi - \varphi\|_{L^p(\Omega)} \\ &\leq \|J_\varepsilon * u - J_\varepsilon * \varphi\|_{L^p(\Omega)} + \|\varphi - u\|_{L^p(\Omega)} + \|J_\varepsilon * \varphi - \varphi\|_{L^p(\Omega)} \leq \eta. \end{aligned}$$

Thus satisfying Property C.4. We note for $p = 1$ we directly compute (C.7) from (C.1) and then repeat the remaining steps in the same fashion. \square

In order to determine the difference between a function and its mollifier with respect to $L^p(\Omega)$ norms i.e. $\lim_{\varepsilon \rightarrow 0^+} \|J_\varepsilon \cdot -\cdot\|_{L^p(\Omega)}$ we introduce the following theorem (see [11, Theorem C.16]).

Theorem C.6. *Let $1 \leq p < \infty$, $\Omega \subseteq \mathbb{R}^d$ be an open subset and $u \in L^p(\Sigma)$ and J_ε be a mollifier function defined in Definition C.2. Then*

$$(C.9) \quad \lim_{\varepsilon \rightarrow 0^+} \|J_\varepsilon * u - u\|_{L^p(\Sigma)} = 0.$$

APPENDIX D. SINGULAR INTEGRALS

This section explores integrals of singular kernels that are used to the to analyse the Calderón operator. We start by adding some required background information. The first being the Convolution operator, here we follow the definition in [16, Section 1.1].

Definition D.1. For every $1 < p, q, r < \infty$ and $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$ where $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}$. Then, the *convolution* $f * g : L^p(\mathbb{R}^d) \times L^q(\mathbb{R}^d) \rightarrow L^r(\Omega)$ is defined by

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)dy.$$

The Marcinkiewicz integral is a special type of integral is used to help evaluate our singular kernel problems. The following definition and theorem is based on [16, Section 2.3].

Definition D.2. For a closed set $F \subseteq \mathbb{R}^d$ along with a function $\delta(x) : \mathbb{R}^d \rightarrow \mathbb{R}$ which denotes the distance from x to F . The *Marcinkiewicz integral* is defined as

$$I(x) = \int_{|y| \leq 1} \frac{\delta(x+y)}{|y|^{d+1}} dy.$$

Theorem D.3. *For every $x \in \mathbb{R}^d$ and a closed set $F \subseteq \mathbb{R}^d$ with $I(x)$ as given by D.2 has the following properties*

$$\begin{cases} I(x) = \infty & \text{if } x \in F^c, \\ I(x) < \infty & \text{a.e. if } x \in F. \end{cases}$$

We recall the Schwartz space to take advantage of a generalised idea of a mollifier function. We base our definition on [17, Section 6.2].

Definition D.4. The *Schwartz space* $\mathcal{S}(\mathbb{R}^d)$ is all indefinitely differential functions f on \mathbb{R}^d that so that f and it's derivatives are rapidly vanishing. In other words

$$\sup_{x \in \mathbb{R}^d} |x^\alpha D_\beta f(x)| < \infty,$$

for every multi-index α and β .

Remark D.5. We note that mollifiers (see Definition C.2) satisfy the requirements for Definition D.4.

We recall an important result central to this section, the Plancherel theorem given in [17, Corollary 2.4].

Theorem D.6. For a function $f \in \mathcal{S}(\mathbb{R}^d)$ and its associated Fourier transform $\hat{f} \in \mathcal{S}(\mathbb{R}^d)$, we have the following identity

$$\int_{\mathbb{R}^d} |f(\eta)|^2 d\eta = \int_{\mathbb{R}^d} |\hat{f}(\eta)|^2 d\eta.$$

We also introduce a key inequality used throughout the remaining parts of thesis, Young's theorem, we use the definition given by Theorem A.2 [16, Section A.2].

Theorem D.7. For, $1 < p, q, r < \infty$ and $f \in L^p(\mathbb{R}^d), g \in L^q(\mathbb{R}^d)$ and given the convolution $f * g$ in Definition D.1 and where $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}$. Then, we have that

$$\|f * g\|_{L^r(\mathbb{R}^d)} \leq \|f\|_p \|g\|_q,$$

The (p, q) -type and (p, q) -weak types are used throughout this section are defined as in [16, Section 1.4.1].

Definition D.8. Let $1 < p, q \leq \infty$ from the mapping $T : L^p(\mathbb{R}^d) \rightarrow L^q(\mathbb{R}^d)$ and $f \in L^p(\mathbb{R}^d)$, then T is of *type* (p, q) if

$$\|T(f)\|_{L^q(\mathbb{R}^d)} \leq A \|f\|_{L^p(\mathbb{R}^d)},$$

where A is independent of f .

Definition D.9. Let $1 < p, q < \infty$ from the mapping $T : L^p(\mathbb{R}^d) \rightarrow L^q(\mathbb{R}^d)$ and $f \in L^p(\mathbb{R}^d)$, then T is of *weak-type* (p, q) if

$$|\{x : |Tf(x)| > \alpha\}| \leq A \|f\|_{L^p(\mathbb{R}^d)},$$

where A is independent of f and α .

Remark D.10. If $q = \infty$ then T is said to also be of weak type (p, q) if it is of (p, q) type.

We aim to show that Convolutions involving the kernel $g(x)$ are bounded in $L^p(\mathbb{R}^d)$. We start by examining a simplified case and then work towards the required singular kernels by the end of this section. First we introduce the assumptions of the most basic case which are to be the core of our following theorems. We follow the relevant proofs in Chapter 2 of [16, Chapter 2].

Assumption D.11. For every $K \in L^2(\mathbb{R}^d)$, we assume the following conditions hold.

(1) The Fourier transform of K is essentially bounded

$$(D.1) \quad |\hat{g}(x)| \leq B.$$

(2) For some $B > 0$ independent of g and d . Additionally $K \in C^1$ outside of the origin and

$$(D.2) \quad \int_{|x| \geq 2|y|} |g(x-y) - g(x)| dx \leq B, .$$

We now introduce a convolution operator for a dense subset of L^p .

Definition D.12. For $f \in L^1 \cap L^p$ we set

$$(D.3) \quad (Tf)(x) = \int_{\mathbb{R}^d} g(x-y)f(y)dy.$$

Before we state theorem, we include a small remark is conceptually used within the proof.

Remark D.13. For essentially large integrals we make an effective substitution. We start with the one dimensional integral

$$(D.4) \quad \int_{=L}^L \frac{b(y)dy}{x-y}$$

We approximate (D.4) by

$$(D.5) \quad \int_{=L}^L \left[\frac{1}{x-y} - \frac{1}{x} \right] b(y)dy$$

This is valid so long as $\int_{=L}^L b(y)dy = 0$. Next we observe that $\left[\frac{1}{x-y} - \frac{1}{x} \right] \sim \frac{L}{x^2}$ for every x sufficiently far from the interval $[-1, L]$. We also have that

$$\int_{|x| > 2L} \frac{1}{x^2} dx \leq 1.$$

Combining these components we ensure that the singularities are avoided if we ensure that the integrals are zero in the interval. We now extend this idea from \mathbb{R} to \mathbb{R}^d we ensure that the integral over each cube is fixed to zero.

The first theorem for the simplified case is demonstrated based on the methods of [16, Theorem 2.1].

Theorem D.14. For every $1 < p < \infty$, and let $g(x)$ be a kernel satisfying Assumption D.11 and given the operator defined in Definition D.12. Then, there exists a constant A_p , so that

$$\|T(f)\|_{L^p(\mathbb{R}^d)} \leq A_p \|f\|_{L^p(\mathbb{R}^d)}$$

where A_p depends only on B, p and d .

We now present a proof of Theorem D.14 based on [16, Theorem 2.1].

Proof. We recall Definition D.9 and initially consider two cases.

T is of weak type $(2, 2)$

By taking the Fourier transform of $(Tf)(y)$ for $f \in L^1 \cap L^2$ we then use estimate (D.1) combined with the Plancherel theorem D.6 to get that

$$(D.6) \quad \|T(f)\|_{L^2(\mathbb{R}^d)} \leq B \|f\|_{L^2(\mathbb{R}^d)} .$$

From (D.6) we extend T uniquely over $L^2(\mathbb{R}^d)$ using Theorem B.5 we obtain our desired estimate.

T is of weak type $(1, 1)$

We split the function f into two components $f = f_1 + f_2$. We set f_1 as the component that is equal to f when f is essentially small, and we set f_2 as the component equal to f when f is essentially large. For the f_1 component, we now see that this satisfies the $L^2(\mathbb{R}^d)$ requirements. Thus we have the required estimate from estimate (D.6). For the f_2 component, we consider Remark D.13 and use this simplified integral to evaluate the f_2 component.

We now split \mathbb{R}^d into two components, $\mathbb{R}^d = F \cup Q$ as in Theorem B.4. From this we define f_1 a.e. as

$$f_1 = \begin{cases} f(x), & \text{for } x \in F \\ \frac{1}{|Q_j|} \int_{Q_j} f(x) dx, & \text{for } x \in Q_j^0. \end{cases}$$

We combine this with the fact that $f = f_1 + f_2$ to determine that f_2 has the following properties

- (1) $b(x) = 0$ for every $x \in F$,
- (2) $\int_{Q_j} b(x) dx = 0$ for all cubes Q_j .

Thus we have for $Tf = Tf_1 + Tf_2$ and we have that

$$|\{x : |Tf(x)| \geq \alpha\}| \leq |\{x : |Tf_1(x)| \geq \alpha/2\}| + |\{x : |Tf_2(x)| \geq \alpha/2\}|$$

We now make estimates for $Tf_1(x)$ and $Tf_2(x)$. To estimate f_1 we first establish that $f_1 \in L^2(\mathbb{R}^d)$.

$$\begin{aligned} \|f_1\|_{L^2(\mathbb{R}^d)}^2 &= \int_{\mathbb{R}^d} |f_1|^2 dx = \int_F |f_1|^2 dx + \int_Q |f_1|^2 dx \\ &\leq \int_F \alpha |f| dx + C^2 \alpha^2 |Q| \\ &\leq (C^2 A + 1) \alpha \|f\|_{L^1(\mathbb{R}^d)}. \end{aligned}$$

Thus we directly use estimate (D.6) and formulate an estimate for f_2 . First we introduce a function b_j

$$b_j = \begin{cases} f_2(x), & \text{for } x \in Q_j \\ 0, & \text{for } x \notin Q_j. \end{cases}$$

Then, one has that

$$\begin{aligned} f_2(x) &= \sum_j b_j(x) \\ (D.7) \quad (Tb_j)(x) &= \int_{Q_j} g(x-y) b_j(y) dy \\ (Tf_2)(x) &= \sum_j (Tb_j)(x). \end{aligned}$$

For each cube Q_j , we introduce the associated cube Q_j^* which shares the same centre point y_j but has the edges rescaled by $2\sqrt{d}$. These cubes Q_j have several properties that should be noted

- (1) Let $Q_j \subseteq Q_j^*$, then $Q^* = \cup_j Q_j^*$ so that $Q \subseteq Q^*$ and $|Q^*| \leq (2\sqrt{d})^d |Q|$.

(2) Let $F^* = (Q^*)^c$ then $F^* \subseteq F$.

(3) For every $x \notin Q_j^*$, one has that $|x - y_j| \geq 2|y - y_j|$ for all $y \in Q_j$.

Now we add a zero valued function to $(Tb_j)(x)$ in equation (D.7)

$$(Tb_j)(x) = \int_{Q_j} [g(x - y) - g(x - y_j)]b_j(y)dy$$

as $\int_{Q_j} b_j(y)dy = 0$. Using property (3) we have for every $y \in Q_j$

$$(D.8) \quad \int_{Q_j^*} |g(x - y) - g(x - y_j)|dx \leq \int_{|x| \geq 2|y|} |g(x - y) - g(x)|dx \leq B.$$

Using estimate (D.8) we have that

$$\begin{aligned} \int_{F^*} |Tf_2(x)|dx &\leq \sum_j \int_{x \notin Q_j^*} \int_{y \in Q_j} |g(x - y) - g(x - y_j)|b_j(y)dy \\ &\leq B \sum_j \int_{Q_j} |b_j(y)|dy \\ &\leq C \|f\|_{L^1(\mathbb{R}^d)}. \end{aligned}$$

Thus we immediately have

$$(D.9) \quad |\{x \in F : |Tf_2(x)| > \alpha/2\}| \leq \frac{C}{\alpha} \|f\|_{L^1(\mathbb{R}^d)}.$$

combining this with estimate (D.6) we have our required estimate for weak type $(1, 1)$.

Now we look to determine estimates for every $p \neq 2$. For every $1 < p < 2$, we use the interpolation theorem B.5 and combining this with the weak $(1, 1)$ and $(2, 2)$ estimates we have that

$$\|Tf\|_{L^p(\mathbb{R}^d)} \leq A_p \|f\|_{L^p(\mathbb{R}^d)}$$

for every $1 < p < 2$ and for all $f \in L^p(\mathbb{R}^d)$. Next we use the duality between $L^p(\mathbb{R}^d)$ and $L^q(\mathbb{R}^d)$ using $\frac{1}{p} + \frac{1}{p'} = 1$, with $2 < p < \infty$, $1 < p' < 2$. For all continuous functions φ with compact support having $\|\varphi\|_{L^q(\mathbb{R}^d)} \leq 1$, for a locally integrable function ψ let

$$\sup_{\varphi} \int \psi \varphi dx = A < \infty$$

for some constant $A > 0$ then $\psi \in L^p(\mathbb{R}^d)$ and $\|\psi\|_{L^p(\mathbb{R}^d)} = A$. For $f \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, due to $g \in L^2(\mathbb{R}^d)$ and by Hölder's inequality we have that

$$\int_{\mathbb{R}^d} (Tf)\varphi dx \leq A_q \|f\|_{L^q(\mathbb{R}^d)}.$$

So taking the supremum of φ we have $\|Tf\|_{L^p(\mathbb{R}^d)} \leq A_q \|f\|_{L^p(\mathbb{R}^d)}$ completing our proof. \square

Remark D.15. We naturally extend Theorem D.14 to the uniform estimate of T to all of $L^p(\mathbb{R}^d)$ by continuity.

Remark D.16. Theorem D.14 covers the specific case of a bounded gradient where Property D.2 is replaced by,

$$(D.10) \quad |\nabla g(x)| \leq B'|x|^{-d}$$

where B' is a different constant replacing B in Assumption D.11.

As we assumed $g(x)$ was bounded in $L^2(\mathbb{R}^d)$ rather than have it be a consequence of the properties of $g(x)$ we now introduce some assumptions that ensure the singular kernel has a cancellation property.

Assumption D.17. We now update the assumptions of the kernel $g(x)$ by,

$$(D.11) \quad \begin{aligned} |g(x)| &\leq B|x|^{-d} & 0 < |x|, \\ \int_{|x| \geq 2|y|} |g(x-y) - g(x)| dx &\leq B & 0 < |y| \end{aligned}$$

where the constant B is the same as in assumption D.11. We also introduce a cancellation property,

$$(D.12) \quad \int_{R_1 < |x| < R_2} K(x) dx = 0, \quad 0 < R_1 < R_2 < \infty.$$

Now we add the definition of a convolution operator with an epsilon ball removed.

Definition D.18. For $f \in L^p(\mathbb{R}^d)$, $\varepsilon > 0$ and for all $1 < p < \infty$, let

$$T_\varepsilon(f)(x) = \int_{|y| \geq \varepsilon} f(x-y)g(y)dy.$$

Below is a lemma that we use to prove the boundedness of the $L^2(\mathbb{R}^d)$ case.

Lemma D.19. *Let $g(x)$ be a kernel satisfying Assumption D.17 with bound B . We define $g_\varepsilon(x)$ for $\varepsilon > 0$*

$$(D.13) \quad g_\varepsilon(x) = \begin{cases} g(x) & \text{if } |x| \geq \varepsilon, \\ 0 & \text{if } |x| < \varepsilon. \end{cases}$$

Then, we define $g_\varepsilon \in L^2(\mathbb{R}^d)$ for the Fourier transforms we have the estimate for $\varepsilon > 0$

$$(D.14) \quad \sup_{y \in \mathbb{R}^d} |\hat{g}_\varepsilon(y)| \leq CB,$$

where C depends only on d .

We prove Lemma D.19 following the methods in [11, Section 2]

Proof. First we consider the case $\varepsilon = 1$ for the estimate D.14. We see that $K_1(x)$ satisfies the both conditions (D.12) and (D.11) with B replaced by CB where C depends only on d . Next we take the Fourier transform of K_1 to get

$$\begin{aligned} \hat{g}(x) &= \lim_{R \rightarrow \infty} \int_{|x| \leq R} e^{2\pi i x \cdot y} g_1(x) dx \\ &= \int_{|x| \leq \frac{1}{|y|}} e^{2\pi i x \cdot y} g_1(x) dx + \lim_{R \rightarrow \infty} \int_{\frac{1}{|y|} \leq |x| \leq R} e^{2\pi i x \cdot y} g_1(x) dx \\ &= I_1 + I_2. \end{aligned}$$

We now estimate I_1 by taking advantage of the cancellation property (D.12) to add a zero term

$$(D.15) \quad \begin{aligned} I_1 &= \int_{|x| \leq \frac{1}{|y|}} e^{2\pi i x \cdot y} g_1(x) dx \\ &= \int_{|x| \leq \frac{1}{|y|}} (e^{2\pi i x \cdot y} - 1) g_1(x) dx. \end{aligned}$$

Now we bound equation (D.15) using estimate (D.11)

$$\begin{aligned} |I_1| &\leq C|y| \int_{x \leq \frac{1}{|y|}} |x| |g_1(x)| dx \\ &\leq C' B. \end{aligned}$$

To estimate I_2 we choose $z = z(y)$ so that $e^{2i\pi y \cdot z} = -1$, we set $z = \frac{1}{2} \frac{y}{|y|^2}$, with $|z| = \frac{1}{2|y|}$. Now we evaluate

$$\int_{\mathbb{R}^d} g_1(x) e^{2i\pi y \cdot z} dx = \frac{1}{2} \int_{\mathbb{R}^d} [g_1(x) - g_1(x - z)] e^{2i\pi z \cdot y} dx,$$

so we now have that

$$\begin{aligned} &\lim_{R \rightarrow \infty} \int_{\frac{1}{|y|} < |x| \leq R} g_1(x) e^{2i\pi x \cdot y} \\ &= \frac{1}{2} \lim_{R \rightarrow \infty} \int_{\frac{1}{|y|} < |x| \leq R} [g_1(x) - g_1(x - z)] e^{2i\pi z \cdot y} dx \\ (D.16) \quad &- \frac{1}{2} \int_{\frac{1}{|y|} < |x+z|} g_1(x) e^{2i\pi z \cdot y} dx. \\ &= I_3 + I_4. \end{aligned}$$

For the integral I_4 this integral is contained within a spherical shell and is bounded by since $|g_1(x)| \leq B|x|^{-d}$. I_3 is majorised by

$$\frac{1}{2} \int_{\frac{1}{|y|} \leq |x|} |g_1(x - z) - g_1(x)| dx.$$

However, as $z = \frac{1}{2|y|}$, then the analogous Property (D.11) is bounded by CB . Now, combining all of these separate estimates we satisfy the lemma for $\varepsilon = 1$.

To extend this to a general $\varepsilon > 0$ we add a dilation factor τ_ε so that

$$(\tau_\varepsilon f)(x) = f(\varepsilon x).$$

Let $T = \varphi * f$ be a convolution operator where φ is a kernel. Then, $\tau_\varepsilon T \tau_\varepsilon$ is a convolution operator with the kernel $\varphi_\varepsilon = \varepsilon^{-d} \varphi(\varepsilon^{-1} x)$. For a kernel φ that satisfies Assumption D.17 we have that φ_ε also satisfies the same assumptions with the same bounds. Now using the same g given in our lemma we let $g' = \varepsilon^d g(\varepsilon x)$ then g' also satisfies the requirements of the lemma and if

$$g'_1(x) = \begin{cases} g(x) & \text{if } |x| \geq \varepsilon, \\ 0 & \text{if } |x| < \varepsilon. \end{cases}$$

we know that $|g'_1(\hat{x})| \leq CB$ and the Fourier transform of $\varepsilon^{-d} g'_1(\varepsilon^{-1} x)$ is $\hat{g}'_1(\varepsilon y)$ is also bounded by CB , but we know that this $\varepsilon^{-d} g'_1(\varepsilon^{-1} x) = g_\varepsilon(x)$, thus we have proven the lemma. \square

We now introduce a theorem analogous to D.14 for the cancelling case based off [16, Theorem 2.2].

Theorem D.20. *Let $g(x)$ be a kernel satisfying Assumption D.17 and given the operator defined in Definition D.18. Then, for all $1 < p < \infty$ and $\varepsilon > 0$ there exists a constant A_p , so that*

$$(D.17) \quad \|T_\varepsilon(f)\|_{L^p(\mathbb{R}^d)} \leq A_p \|f\|_{L^p(\mathbb{R}^d)}$$

where A_p depends only on B, p and d and is independent of f and ε .

For each $f \in L^p(\mathbb{R}^d)$, $\lim_{\varepsilon \rightarrow 0^+} T_\varepsilon(f) = T(f)$. Additionally, one has that the operator T defined in Definition D.12 also satisfies estimate (D.17).

The following proof Theorem D.20 is based on [16, Section 2.3.4].

Proof. Using Lemma D.19 one has that if $g(x)$ satisfies the Properties (D.12) and (D.11) then K_ε also satisfies them so combining this with Corollary D.14 we have that (D.17) is satisfied.

Next we split an arbitrary $f \in L^p(\mathbb{R}^d)$ into two components $f = f_1 + f_2$ where f_1 is continuous with compact support and a continuous derivative and $f_2 \in L^p(\mathbb{R}^d)$ is a function with a sufficiently small $L^p(\mathbb{R}^d)$ norm. Now, for the function f_1 we have by the cancellation condition (D.12) that,

$$\begin{aligned} T_\varepsilon(f_1)(x) &= \int_{y \geq \varepsilon} g(y) f_1(x-y) dy \\ &= \int_{y \geq 1} g(y) f_1(x-y) dy + \int_{1 \geq y \geq \varepsilon} g(y) [f_1(x-y) - f_1(x-y)] dy \\ &= I_1 + I_2. \end{aligned}$$

The first integral I_1 is in $L^p(\mathbb{R}^d)$ as it is a convolution of an $L^p(\mathbb{R}^d)$ ($g(x)$) with an $L^1(\mathbb{R}^d)$ function (f_1). Next we note that the second integral I_2 also converges uniformly as $\varepsilon \rightarrow 0$ as

$$(D.18) \quad |f_1(x-y) - f_2(x)| \leq A|y|$$

for some constant A due to the differentiability of f_1 . Thus we conclude that $T_\varepsilon(x)$ converges in L^p norm as $\varepsilon \rightarrow 0$.

For the function f_2 we now apply estimate (D.17) to f_2 and see that $\lim_{\varepsilon \rightarrow 0} T_\varepsilon$ exists in $L^p(\mathbb{R}^d)$ space. We also have that the limiting operator T also satisfies estimate (D.17). \square

We now restrict the kernels to singular integral operators that are of particular interest to our work.

Assumption D.21. The kernel $g(x)$ is defined as,

$$g(x) = \frac{G(x)}{|x|^d},$$

where $G(x) : \mathbb{R}^d \rightarrow \mathbb{R}$ is homogeneous of degree 0 in $B_R \setminus \{0\}$ for some $R > 0$ and $G(x)$ is completely determined by its restriction to this sphere. Now we restrict the conditions of Assumptions D.11 and D.17 to these specific functions. The cancellation property (D.12) is then reduced to

$$\int_{\mathbb{S}_{d-1}} G(x) d\sigma(z) = 0.$$

Property (D.11) is difficult to directly express in terms of $G(x)$ but it is satisfied if $G(x)$ possesses a ‘‘Dini’’-type condition based on Property (D.11) which is satisfied if

$$\omega(\delta) = \sup_{x-x', |x|=|x'|=1} |G(x) - G(x')|$$

then

$$(D.19) \quad \int_0^1 \frac{\omega(\delta)d\delta}{\delta} < \infty.$$

Remark D.22. We note that if $G(x)$ is C^1 then it satisfies property (D.19) directly.

We now introduce one last operator definition for this section.

Definition D.23. The operator $T_\varepsilon(f)$ is defined by

$$(D.20) \quad T_\varepsilon(f)(x) = \int_{|y| \geq \varepsilon} \frac{G(y)}{|y|^d} f(x-y) dy.$$

The main result of the section is the following theorem which is used in defining the properties of the Calderón operator in Section 4 based off [16, Theorem 2.3]).

Theorem D.24 (Calderón-Zygmund theorem inequality). *Let $g(x)$ be a kernel satisfying Assumption D.21 and given the operator defined in Definition D.20. Then, for all $1 < p < \infty$ and $\varepsilon > 0$ there exists a constant A_p , so that*

$$(D.21) \quad \|T_\varepsilon(f)\|_{L^p(\mathbb{R}^d)} \leq A_p \|f\|_{L^p(\mathbb{R}^d)}$$

where A_p depends only on B, p and d and is independent of f and ε . For each $f \in L^p(\mathbb{R}^d)$, $\lim_{\varepsilon \rightarrow 0^+} T_\varepsilon(f) = T(f)$.

For the interested reader we now provide a proof of Calderón-Zygmund theorem inequality (Theorem D.24) based on [16, Section 2.4.2].

Proof. This theorem holds immediately by Theorem D.20 after we demonstrate $g(x)$ of the form $\frac{G(x)}{|x|^d}$ satisfies Property (D.11) if $G(x)$ while satisfying Property (D.19). We note that

$$g(x-y) - g(x) = \left(\frac{G(x-y) - G(x)}{|x-y|^d} \right) + G(x) \left[\frac{1}{|x-y|^d} - \frac{1}{|x|^d} \right].$$

Focusing on the second term $G(x) \left[\frac{1}{|x-y|^d} - \frac{1}{|x|^d} \right]$ it is immediately clear that is bounded as both $G(x)$ is bounded and

$$\int_{|x| \geq 2|y|} \left| \frac{1}{|x-y|^d} - \frac{1}{|x|^d} \right| dx \leq C$$

for some C independent of $G(x), p, d$. Next we consider the first term

$$\frac{G(x-y) - G(x)}{|x-y|^d}.$$

To estimate this we take advantage of the ‘‘Dini’’ style continuity and note that the distance between the projections of $x-y$ and x on the unit sphere,

$$\left| \frac{x-y}{|x-y|} - \frac{x}{|x|} \right|$$

is bounded by $C \frac{|y|}{|x|}$ for $|x| > 2|y|$. So the integral of this first group of terms is dominated by

$$C' \int_{|x| \geq 2|y|} \Omega \left(C \frac{|y|}{|x|} \right) \frac{dx}{|x|^d} = C'' \int_0^{c/2} \frac{\omega(\delta)}{\delta} < \infty.$$

Where C', C'', C are constants independent of g, G, p and d . \square

We note that the constant A_p is independent of f, T, p and d as shown in [16, Further result 6.2] should be emphasised

Remark D.25. We note that the constants A_p which appear in the preceding theorems are for every $1 < p \leq 2$

$$A_p \leq \frac{A}{p-1}$$

and for every $2 \leq p < \infty$

$$A_p \leq Ap$$

where A is the constant based on Definitions D.8 and D.9 and are independent of f, T, p and d .

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