Skew-product graph of groups and their toolkits

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Statement of originality

This is to certify that to the best of my knowledge, the content of this thesis is my own work. This thesis has not been submitted for any degree or other purposes.

I certify that the intellectual content of this thesis is the product of my own work and that all the assistance received in preparing this thesis and sources have been acknowledged.

Chester Yau

Artificial Intelligence

No content produced by generative AI tools has been used in the preparation of this thesis.

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Abstract

The K-theory associated with a C^*-algebra plays a fundamental role in the classification and structural understanding of C^*-algebras. This powerful invariant, originally developed from algebraic topology, provides essential tools for distinguishing C^*-algebras up to isomorphism and has become central to the Elliott classification program. Elliott's groundbreaking classification theorem exemplifies this importance, establishing that two approximately finite-dimensional C^*-algebras are isomorphic if and only if their respective K_0-groups, ordered K_0-groups and identity classes are isomorphic. This theorem demonstrates how K-theory bridges the gap between abstract operator algebra theory and concrete computational methods.

An important class of C^*-algebras is those coming from directed graphs, and their K-theory is well understood. One way to calculate the K-theory of a graph algebra is to use to the skew-product graph; a construction originally developed in combinatorial graph theory by Gross and Tucker [1] as "voltage graphs", and later connected to crossed products by coactions in the work of Kaliszewski, Quigg and Raeburn [2].

This thesis investigates the C^*-algebras associated with graphs of groups, a rich mathematical structure first systematically developed by Bass [3] and Serre [4] in their foundational work on group actions on trees. We adapt and extend the skew-product construction for directed graphs to the graph of groups setting. Specifically, given a cocycle $c : \Gamma^1 \to H$ labelling the edges of a graph of groups $\mathscr{G}$ by a discrete group H, a definition of skew-product graphs of groups is provided. The main theoretical contribution demonstrates that there is a natural connection between the skew-product graph of groups C^*-algebra $C^*(\mathscr{G} \times_c H)$ and the crossed product by the induced coaction $C^*(\mathscr{G}) \rtimes_{\delta_c} H$. In addition, this definition of skew-product graphs of groups is shown to be consistent with the existing definition of skew-product graphs, in terms of the directed graph associated to graphs of groups $E_{\mathscr{G}}$. Finally, using the existing isomorphism between graphs of groups C^*-algebras and its fibred product groupoid algebra $C^*(F(\mathscr{G} * \partial W_{\mathscr{G}}))$, the isomorphism between the skew-product graph of groups C^*-algebra and the crossed product by the induced coaction is extended to the crossed product fibred product groupoid algebra by coaction $C^*(F(\mathscr{G} * \partial W_{\mathscr{G}})) \rtimes_\beta H$.

This thesis also includes a survey of K-theory for C^*-algebras, including the K-theory of graph algebras. The last chapter also acts as a literature review of the recent developments in K-theory for both graph of groups C^*-algebras and graph of groups actions on multitrees.

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Contents

Introduction

Operator algebras, a branch of functional analysis, have emerged as a powerful mathematical framework for studying spaces of operators on Hilbert spaces, providing profound connections between abstract algebra, topology, and quantum physics. The origins of operator algebras can be traced back to the early 20th century, where they were first developed to provide a rigorous mathematical foundation for quantum mechanics, addressing the need to understand observables and their algebraic relationships. The theory of operator algebras was pioneered by John von Neumann, who introduced von Neumann algebras (also known as W^*-algebras) in the 1930s as a generalisation of matrix algebras, motivated by the fundamental requirement to understand the spectral theory of self-adjoint operators in quantum mechanics. This was followed by the complementary development of C^*-algebras by Israel Gelfand and Mark Naimark, which broadened the scope of operator algebras to include a wider class of operator systems and established the foundation for modern noncommutative geometry.

The development of operator algebras has since grown into a rich and deep field of study, with applications that extend far beyond their initial motivations. One of the most significant applications of operator algebras is in quantum theory, where they provide the mathematical foundation for understanding the algebraic structure of quantum observables and the dynamics of quantum systems. C^*-algebras, in particular, have been used to model the algebras of observables in quantum mechanics, and they play a key role in the study of quantum statistical mechanics and quantum field theory. Moreover, the classification of operator algebras has become a central problem in mathematics, with connections to topology, geometry, and number theory.

One such example is the Cuntz-Krieger algebras, C^*-algebras that represent the structure of connected directed graphs by operators on a Hilbert space $\mathscr{H}$. To be exact, each vertex in the graph represents a projected subspace (or projection) of the Hilbert space, and each edge represents a map (or partial isometry) between those subspaces. These operators satisfy the Cuntz-Krieger relations, which gives the connection between the operators, hence the name. Cuntz-Krieger algebras were first studied by Cuntz and Krieger in the 1980s [5, 6].

In the late 1990s, the Cuntz-Krieger algebras were quickly recognised to be a rich supply of examples for operator algebraists. The graph algebra is, loosely speaking, the C^*-subalgebra of $\mathcal{B}(\mathscr{H})$ generated by these operators. Graph C^*-algebras were then heavily investigated by Kumjian, Pask, Raeburn and Renault [7]. The theory of graph C^*-algebras is then extended to other types of graph structures, such as C^*-algebras of arbitrary graphs (Chapter 5 of [8]), C^*-algebras of higher-rank graphs [9]. In his book [8], Iain Raeburn made deeper analysis on graph C^*-algebras, by laying out the two useful uniqueness theorems for graph C^*-algebras.

Graph C^*-algebras also have a direct application to dynamical systems. In chapter 7 of [8], the computation of K-theory of graph C^*-algebras is detailed, using several techniques in crossed products by actions and coactions, AF-algebras and the *dual Pimsner-Voiculescu sequence*

$$\begin{tikzcd}
K_0(A \rtimes_\alpha \mathbb{T}) \arrow[r, "\mathrm{id} - K_0(\hat{\alpha}_n^{-1})"] & K_0(A \rtimes_\alpha \mathbb{T}) \arrow[r] & K_0(A) \arrow[d] \\
K_1(A) \arrow[u] & K_1(A \rtimes_\alpha \mathbb{T}) \arrow[l] & K_1(A \rtimes_\alpha \mathbb{T}) \arrow[l, "\mathrm{id} - K_1(\hat{\alpha}_n^{-1})"']
\end{tikzcd}$$

a six-term exact sequence including the K-theory of C^*-algebras and their crossed products. More specifically, the concept of "skew-product" (also known as "voltage graphs") is used, a construction that traces its origins to classical combinatorial and topological graph theory of the 1970s, systematised by Gross and Tucker [1] as a tool for producing regular graph coverings. This classical construction was later connected to operator algebras through the work of Kaliszewski, Quigg and Raeburn [2], who established the fundamental relationship between skew-product graphs and crossed products by coactions. This creates a "bigger" graph using the original graph, and as it turns out, the crossed product of a graph C^*-algebra can be identified with the skew-product graph C^*-algebra, which is approximately finite dimensional by nature, thus making the corresponding K_0-group tractable. The identification is via the use of Theorem 2.4 from [2], which states that given a directed row-finite graph E, a discrete group G, and a labelling $c : E^1 \to G$, there is an induced coaction δ_c of G on $C^*(E)$ such that

$$C^*(E \times_c G) \cong C^*(E) \rtimes_{\delta_c} G.$$

This allows the replacement of the crossed products in the dual Pimsner-Voiculescu sequence, and the K-theory of the graph C^*-algebra can therefore be computed via a simplified version of the above exact sequence.

A more complex and fundamentally important graph-based structure is graphs of groups, a concept first systematically developed by Bass [3] and Serre [4] in their groundbreaking work on group actions on trees and covering theory. This structure enriches the combinatorial framework of ordinary graphs by assigning a group to each vertex and edge, with injective homomorphisms connecting edge groups to their range vertex groups. This construction provides a powerful tool for encoding complex geometric and algebraic relationships, bridging group theory, topology, and geometric group theory in profound ways. The transition from graphs to graphs of groups in the operator algebraic setting represents a significant leap in complexity and richness. In 2016, Brownlowe, Mundey, Pask, Spielberg, and Thomas [10] made tremendous progress in operator algebras by introducing and studying the universal C^*-algebra associated to graphs of groups. This work not only extended the successful theory of graph C^*-algebras to a more general setting but also established deep connections to the fundamental group of graphs of groups, thereby contributing to geometric group theory. However, the increased complexity of graphs of groups presents significant challenges for K-theory computations that do not arise in the ordinary graph case. This thesis is devoted to developing new tools and techniques to study the K-theory of graph of groups C^*-algebras, addressing this computational gap. The central innovation of this thesis is the introduction of skew-product graphs of groups, denoted $\mathscr{G} \times_c H$, which adapts the successful skew-product methodology from ordinary graphs to the graph of groups setting. This construction requires careful attention to the group-theoretic structure: unlike ordinary skew-product graphs, the labelling c must satisfy a cocycle condition to ensure that the underlying graph $E \times_c H$ remains well-defined as an undirected graph. This cocycle condition reflects the additional algebraic constraints imposed by the group structure attached to each vertex and edge. The main theoretical achievement demonstrates that this definition of skew-product graphs of groups is consistent with the existing definition of skew-product graphs when restricted to the directed graph associated to graphs of groups. Furthermore, we establish an analogue of the fundamental Theorem 2.4 from [2], proving that given a graph of groups $\mathscr{G} = (\Gamma, G)$ and a cocycle labelling $c : \Gamma^1 \to H$ to a discrete group H, the crossed product of $C^*(\mathscr{G})$ by the induced coaction δ_c is isomorphic to the skew-product graph of groups algebra $C^*(\mathscr{G} \times_c H)$. That is,

$$C^*(\mathscr{G} \times_c H) \cong C^*(\mathscr{G}) \rtimes_{\delta_c} H.$$

Remark 4.10 from [10] indicates that the C^*-algebra of the fibred product groupoid $C^*(F(\mathscr{G}) * \partial W_{\mathscr{G}})$ is isomorphic to $C^*(\mathscr{G})$. This fact is used to extend the result to

$$C^*(\mathscr{G} \times_c H) \cong C^*(\mathscr{G}) \rtimes_{\delta_c} H \cong C^*(F(\mathscr{G}) * \partial W_{\mathscr{G}}) \rtimes_{\beta} H.$$

where $\beta : C^*(F(\mathscr{G}) * \partial W_{\mathscr{G}}) \to C^*(F(\mathscr{G}) * \partial W_{\mathscr{G}}) \otimes C^*(H)$ is an induced coaction by δ_c.

This thesis is divided into two parts: Part 1 introduces C^*-algebras and explores various types, including crossed product algebras, graph algebras, and graph of groups algebras. Part 2 is dedicated to the study of the K-theory of C^*-algebras.

Chapter 1 provides the essential mathematical foundation required for the rest of the thesis, drawing primarily from the comprehensive treatment by Rørdam, Larsen and Laustsen [11] along with additional insights from homological algebra and category theory. It begins with a thorough review of the basic concepts of C^*-algebras, including their definition, fundamental examples, and core properties that will be utilised throughout the thesis. The chapter discusses short exact sequences, which are central to understanding extensions of C^*-algebras and provide crucial tools for K-theory computations. Additionally, there is an overview of matrices and their pivotal role in operator algebras, followed by a focused introduction to basic category theory, which provides the conceptual language and organisational framework for understanding the categorical structures encountered in later chapters. This chapter concludes with an examination of bounded operators in Hilbert spaces, establishing the concrete realisation of C^*-algebras as operator systems.

Chapter 2 focuses on group C^*-algebras and crossed products, fundamental constructions that bridge group theory and operator algebras. Following the comprehensive treatments by Sims [12] and Murphy [13], this chapter explores the universal C^*-algebra of a discrete group and the construction of crossed products by group actions and coactions. The chapter establishes the essential connection between dynamical systems and C^*-algebras, providing the foundational framework for understanding how group actions give rise to operator algebraic structures.

Chapter 3 introduces graph algebras, a class of C^*-algebras associated with row-finite directed graphs, following the authoritative treatment by Raeburn [8]. The construction of graph algebras through universal properties, their basic structural properties, and their significance in providing concrete examples of C^*-algebras with tractable K-theory are thoroughly explored. This chapter demonstrates how combinatorial graph-theoretic data can be translated into powerful operator algebraic objects.

Chapter 4 examines skew-product graphs and their associated C^*-algebras, continuing to follow Raeburn's comprehensive framework [8]. This chapter establishes the crucial connection between skew-product constructions and crossed products by coactions, providing the theoretical foundation that will be adapted to the graph of groups setting in later chapters. The techniques developed here demonstrate how graph-theoretic labellings can be used to construct larger, more tractable structures while preserving essential algebraic properties.

Chapter 5 is devoted to the study of graphs of groups, extending the ordinary graph framework to incorporate rich group-theoretic structure. Following the groundbreaking work of Brownlowe, Mundey, Pask, Spielberg, and Thomas [10], this chapter explores how the Bass-Serre theory of graphs of groups can be translated into the operator algebraic setting. The chapter discusses the universal C^*-algebra associated to a graph of groups, its fundamental properties, and its connections to geometric group theory, establishing the foundation for the novel constructions introduced in subsequent chapters.

Chapter 6 introduces the original mathematical structure called skew-product graphs of groups, which constitutes the primary contribution of this thesis. Building on techniques from Kaliszewski, Quigg, and Raeburn [2], this chapter develops the theoretical framework designed to facilitate K-theory computations for graph of groups C^*-algebras. The construction of this new structure requires careful adaptation of the classical skew-product methodology to accommodate the additional group-theoretic constraints inherent in graphs of groups. The chapter explores the properties of this construction and establishes its fundamental isomorphism with crossed products by coactions. Additionally, the chapter introduces groupoid algebras and their connection to graph of groups, extending the main results through the fibred product groupoid framework established in [10].

Chapter 7 focuses on the K_0-group, one of the two primary invariants in K-theory for C^*-algebras, following the systematic exposition by Rørdam, Larsen and Laustsen [11]. The construction of the K_0-group through equivalence classes of projections, its fundamental properties, and its central significance in the classification of C^*-algebras are thoroughly explored. The chapter emphasises the role of the K_0-group in understanding the structure of projections in a C^*-algebra and its essential connection to Murray-von Neumann equivalence, providing the foundation for computational techniques used in later chapters.

Chapter 8 is devoted to the study of the K_1-group, the second key invariant in K-theory, again following the comprehensive treatment in [11]. It examines the definition and construction of the K_1-group through equivalence classes of invertible elements, exploring its fundamental relationship with unitary elements in C^*-algebras. The chapter establishes the theoretical framework necessary for understanding how the K_1-group complements the K_0-group in providing complete K-theoretic information.

Chapter 9 is devoted to the study of inductive limits, which provide the categorical framework for understanding limiting processes in both C^*-algebras and their K-theory. Continuing to follow [11], this chapter focuses on the categories of C^*-algebras and abelian groups, exploring the continuity properties of both the K_0-functor and K_1-functor. These continuity results are essential for computing K-groups of inductive limits of C^*-algebras. The chapter concludes with the fundamental concept of approximately finite-dimensional (AF) algebras, which are infinite-dimensional C^*-algebras that can be approximated by finite-dimensional algebras, providing crucial examples where K-theory is completely computable.

Chapter 10 focuses on the K-theory of graph algebras, synthesising the theoretical foundations developed in previous chapters with the computational framework established by Raeburn [8].

The chapter introduces the fundamental Pimsner-Voiculescu exact sequence, which is central to computing the K-theory of graph algebras through their crossed product structure. By combining the skew-product methodology with the K-theoretic tools developed in Part 2, this chapter demonstrates the complete process of computing K-groups for graph C^*-algebras, providing the template that motivates the graph of groups extensions developed in this thesis.

Chapter 11 addresses the significant challenges encountered when attempting to replicate the successful K-theory computation methods of Chapter 10 in the graph of groups setting. The chapter provides a comprehensive literature review of recent developments in this area, focusing particularly on the innovative approach by Mundey and Rennie [14], which provides a Cuntz-Pimsner model for graph of groups C^*-algebras. Additionally, the chapter reviews the foundational work of Cuntz and Krieger [6] on C^*-algebras and topological Markov chains, and the recent advances by Brownlowe, Spielberg, Thomas, and Wu [15] on group actions on multitrees and their K-theory. This analysis highlights both the computational difficulties inherent in the graph of groups setting and the potential for the skew-product construction developed in this thesis to provide new computational pathways.

Part 1

C^*-Algebras

CHAPTER 1

Preliminaries

This chapter establishes the essential mathematical foundation required for the subsequent development of graph algebras, graph of groups algebras, and their K-theory. The material presented here provides the conceptual framework and technical tools that underpin the main contributions of this thesis. Drawing primarily from the comprehensive treatment by Rørdam, Larsen and Laustsen [11], along with insights from homological algebra and category theory, we present the fundamental structures and properties of C^*-algebras that will be utilised throughout our investigation. In particular, this chapter provides formal definitions for C^*-algebras and explores crucial related concepts including exact sequences, unitisation procedures, matrix algebras, and categorical frameworks that are essential for understanding the operator algebraic constructions in later chapters.

1.1 C^*-Algebras

C^*-algebras form the central mathematical objects of study in this thesis, providing the operator algebraic framework within which we will develop graph algebras, graph of groups algebras, and their associated K-theory. These structures, originally introduced by Gelfand and Naimark, represent a profound synthesis of algebraic and analytic concepts, offering both the flexibility of abstract algebra and the computational power of functional analysis. The fundamental properties and constructions presented in this section establish the theoretical foundation that will enable us to understand the more complex structures introduced in subsequent chapters. The material follows the comprehensive exposition in [11], supplemented with additional insights from operator theory.

DEFINITION 1.1. *A $*$**-algebra** is an associative algebra A with a $*$-operation (called involution) $a \mapsto a^*$ for every $a \in A$ such that*

$$(ab)^* = b^* a^*.$$

$*$-algebras are fundamental algebraic structures that provide the starting point for constructing C^*-algebras. They capture the essential algebraic properties required for operator algebras while remaining purely algebraic objects without any topological structure. The involution operation $a \mapsto a^*$ models the adjoint operation on operators, establishing the crucial connection between abstract algebra and functional analysis. In the construction of C^*-algebras, a $*$-algebra serves as the underlying algebraic framework: we begin with a $*$-algebra, equip it with a norm satisfying specific compatibility conditions with the algebraic operations, and then complete the resulting normed space to obtain a C^*-algebra.

DEFINITION 1.2. *A C^***-algebra** is a $*$-algebra A with a submultiplicative ($\|ab\| \leq \|a\|\|b\|$) norm $\|\cdot\| : A \to A$ that is complete and satisfies the C^***-identity**:*

$$\|a^* a\| = \|a\|^2,$$

and $\|a^*\| = \|a\|$ for every $a \in A$.

EXAMPLE 1.3 (Bounded operators of Hilbert spaces). *Let $\mathscr{H}$ be a Hilbert space. The set of bounded operators, denoted $\mathcal{B}(\mathscr{H})$, is a C^*-algebra by equipping the set with the operator norm $\|\cdot\|_{op}$:*

$$\|T\|_{op} = \sup_{\|\xi\| \leq 1} \|T\xi\|$$

for every $h \in \mathscr{H}$. Indeed, let $T, S \in \mathcal{B}(\mathscr{H})$ and $\alpha \in \mathbb{C}$, then for every $\xi \in \mathscr{H}$,

$$\begin{aligned} \|(\alpha T + S)\xi\| &\leq |\alpha|\|T\xi\| + \|S\xi\| \\ &\leq |\alpha|\|T\|_{op}\|\xi\| + \|S\|_{op}\|\xi\| \\ &= (|\alpha|\|T\|_{op} + \|S\|_{op})\|\xi\|. \end{aligned}$$

This shows the $\alpha T + S$ *is bounded, and closued under addition and scalar multiplicstion. In addtion,*

$$\|TS\|_{op} = \sup_{\|\xi\|\le 1} \|T(S\xi)\| \le \sup_{\|\xi\|\le 1} \|T\|_{op}\|S\xi\| = \sup_{\|\xi\|\le 1} \|T\|_{op}\|S\|_{op}\|\xi\| = \|T\|_{op}\|S\|_{op}.$$

So the muliplication is closed as well. The involution is defined by the following: For every $\xi, \eta \in H$,

$$\langle T\xi, \eta\rangle = \langle \xi, T^*\eta\rangle.$$

It is easy to see that the involution is conjugate-linear, and $\mathcal{B}(\mathscr{H})$ *becomes a* $*$*-algebra. A quick calculation shows that*

(1) $\|TS\|_{op} \le \|T_{op}\|\|S_{op}\|$ *for every* $T, S \in \mathcal{B}(\mathscr{H})$.

(2) *Let* $\{T_n\}$ *be a cauchy sequence, then for each fixed* $\xi \in \mathscr{H}$,

$$\|T_n\xi - T_m\xi\| \le \|T_n - T_m\|_{op}\|\xi\| \longrightarrow 0,$$

so $\{T_n\xi\}$ *is a Cauchy sequence in* $\mathscr{H}$. *Define a bounded operator* T *by* $T\xi := \lim_{n\to\infty} \xi$. *Note that* T *is bounded and for any fixed* $\xi \in \mathscr{H}$,

$$\begin{aligned}\|(T_n - T)\xi\| &= \|T_n\xi - T\xi\|\\ &= \|T_n\xi - \lim_{m\to\infty} T_m\xi\|\\ &\le \|T_n - \lim_{m\to\infty} T_m\|\|\xi\|\\ &\le \|T_n - \lim_{m\to\infty} T_m\| \longrightarrow 0.\end{aligned}$$

So $\{T_n\}$ *converges to* T *in the operator norm, and* $\mathcal{B}(\mathscr{H})$ *is complete.*

(3) *By submultiplicativity and the fact* $\|T^*\|_{op} = \|T\|_{op}$,

$$\|T^*T\|_{op} \;\le\; \|T^*\|_{op}\,\|T\|_{op} \;=\; \|T\|_{op}^2.$$

In addition,

$$\|T\|_{op}^2 = \sup_{\|\xi\|=1} \|T\xi\|^2 = \sup_{\|\xi\|=1} \langle T\xi,\, T\xi\rangle = \sup_{\|\xi\|=1} \langle \xi,\, T^*T\,\xi\rangle \;\le\; \|T^*T\|_{op}.$$

So $\|T^*T\|_{op} = \|T\|_{op}^2$.

Therefore, $\mathcal{B}(\mathscr{H})$ is a C^*-algebra.

A C^*-algebra A is **unital** if it contains the multiplicative identity 1_A. Note that not all C^*-algebras are unital, and if it is, the adjoint $1_A^* = 1_A$. To see that, let $a \in A$, then $1_A^* a = (a^* 1_A)^* = (a^*)^* = a$. We will see that being unital is quite important to us in later chapters.

DEFINITION 1.4. *A* ***$*$-homomorphism*** *between C^*-algebras A, B is an algebra homomorphism $\phi : A \to B$ that preserves involution, that is,*

$$\phi(a)^* = \phi(a^*)$$

for every $a \in A$.

LEMMA 1.5. *Suppose that $\phi : A \to B$ is a surjective $*$-homomorphism between unital A and B, then ϕ preserves identity. That is,*

$$\phi(1_A) = 1_B.$$

PROOF. Since ϕ is surjective, there exists some $a \in A$ such that $\phi(a) = 1_B$. Let $b \in B$ and $a' \in A$ such that $\phi(a') = b$. Then

$$\phi(a') = b = b1_B = \phi(a')\phi(a) = \phi(a'a)$$

and

$$\phi(a) - \phi(a'a) = 0 \Longrightarrow \phi(a - a'a) = 0 \Longrightarrow a - a'a = 0 \Longrightarrow a = a'a.$$

The right side is similar and this shows $a = 1_A$. ☐

One of the most useful fact in the theory of C^*-algebra is the fact that any C^*-algebra can be viewed as a subset of bounded operators in some Hilbert space, by the following theorem:

THEOREM 1.6 (Gelfand-Naimark). *Every C^*-algebras admit a faithful, non-degenerate representation into the bounded linear operators on a Hilbert space. That is, given any C^*-algebra A, there exists a Hilbert space $\mathscr{H}$ and injective $*$-homomorphism $\pi : A \to \mathcal{B}(\mathscr{H})$, so $\pi(A)$ becomes a subalgebra of the set of bounded operators.*

Refer to in [13, Theorem 3.4.1] to see the proof.

LEMMA 1.7. *Suppose that A, B are C^*-algebras. The direct sum $A \oplus B$, with the norm*

$$\|(a,b)\|_{A\oplus B} = \max(\|a\|_A, \|b\|_B)$$

becomes a C^*-algebra.

PROOF. Here we need to show that $A \oplus B$ is complete with the norm, the norm is submultiplicative and it satisfies the C^*-identity. First, suppose that $(a,b)_i$ is a Cauchy sequence in $A \oplus B$. Note that $(a,b)_i$ can be realized as (a_i, b_i), which is comprised of two Cauchy sequences (a_i) in A and (b_i) in B. Since A and B are complete, $(a_i) \to a \in A$ and $(b_i) \to b \in B$. Therefore, $(a,b)_i = (a_i, b_i) \to (a,b) \in A \oplus B$, making $A \oplus B$ complete. For sub-multiplicativity, let $(a,b), (a',b') \in A \oplus B$. Then

$$\begin{aligned}
\|(a,b)(a',b')\| &= \|(aa', bb')\| \\
&= \max\{\|aa'\|, \|bb'\|\} \\
&\leq \max\{\|a\|\|a'\|, \|b\|\|b'\|\} \\
&\leq \max\{\|a\|, \|b\|\} \max\{\|a'\|, \|b'\|\} \\
&= \|(a,b)\|\|(a',b')\|.
\end{aligned}$$

□

1.2 Exact Sequences

Exact sequences are fundamental tools from homological algebra that provide a powerful framework for understanding the relationships between C^*-algebras and their associated K-groups. In the context of C^*-algebras, exact sequences allow us to decompose complex algebraic structures into simpler, more manageable components, enabling us to study intricate objects by understanding their constituent parts and how they fit together. This approach is particularly valuable in K-theory, where exact sequences of C^*-algebras give rise to long exact sequences in K-theory, providing computational tools for determining K-groups of complex algebras from knowledge of simpler ones. The six-term exact sequences that arise in K-theory, such as the dual Pimsner-Voiculescu sequence central to our investigation of graph C^*-algebras, exemplify the power of this methodology. The theoretical framework presented here establishes the foundation for the exact sequence techniques that will be crucial in Part 2 of this thesis.

DEFINITION 1.8 (Exact Sequecnes). *Let $\phi : A \to B$, $\psi : B \to C$ be $*$-homomorphisms between C^*-algebras. We say the two-part sequence*

$$A \xrightarrow{\phi} B \xrightarrow{\psi} C$$

is exact at B if $\ker(\psi) = \operatorname{im}(\phi)$.

DEFINITION 1.9. *An exact sequence is a multi-part sequence $\{B_n, \phi_n\}_{i\in\mathbb{N}}$ of C^*-algebras and $*$-homomorphisms*

$$\cdots \xrightarrow{\phi_{n-1}} B_n \xrightarrow{\phi_n} B_{n+1} \xrightarrow{\phi_{n+1}} B_{n+2} \cdots ,$$

where each B_n is exact.

A **short exact sequence** is an four-part exact sequence starting and ending at the zero C^*-algebra

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

LEMMA 1.10. *Let*

$$0 \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow 0$$

be a short exact sequence. Then ϕ is injective and ψ is surjective.

PROOF. The short exact sequence is comprise of two parts: (1) $0 \longrightarrow A \xrightarrow{\phi} B$ and (2) $B \xrightarrow{\psi} C \longrightarrow 0$.

(1) This sequence is exact $\Rightarrow \{0\} = \operatorname{im} 0 = \ker \phi$. This shows injectivity of ϕ; and
(2) This sequence is exact $\Rightarrow \operatorname{im} \psi = \ker 0 = C$. This shows surjectivity of ψ.

□

EXAMPLE 1.11. *Let* $C_0((0,1))$ *denote the set of all functions* $f:(0,1)\to\mathbb{C}$ *that vanishes at infinity, and* $C([0,1])$ *denote the set of all continuous functions* $f:[0,1]\to\mathbb{C}$. *The sequence*

$$0 \longrightarrow C_0((0,1)) \xrightarrow{\iota} C([0,1]) \xrightarrow{\phi} \mathbb{C}\oplus\mathbb{C} \longrightarrow 0$$

where ι *is the inclusion and* $\phi(f)=(f(0),f(1))$ *is short exact. Exactness at* $C_0((0,1))$ *is shown as follows:*

$$\ker(\iota)=\{f\in C_0((0,1)):\iota(f)=0\}=\{f:f|_{(0,1)}=0\}=\{0\}.$$

For exactness at $C([0,1])$, *if* $g=\iota(f)$ *for some* $f\in C_0((0,1))$, *then* $\lim_{t\to 0^+}f(t)=\lim_{t\to 1^-}f(t)=0$. *Hence the continuous extension* g *satisfies* $g(0)=g(1)=0$, *so* $\phi(g)=(0,0)$. *If* $f\in C([0,1])$ *and* $\phi(f)=(0,0)$, *then* $f(0)=f(1)=0$. *Thus* $f|_{(0,1)}\in C_0((0,1))$ *and* $\iota(f|_{(0,1)})=f$. *To see that* ϕ *is surjective, let* $(a,b)\in\mathbb{C}\oplus\mathbb{C}$ *and define*

$$f(t)=(1-t)a+tb,\quad t\in[0,1].$$

Then $f\in C([0,1])$ *and* $\phi(f)=(a,b)$. *The combined results show that the sequence is short exact.*

DEFINITION 1.12 (Split Exact Sequences). *Let*

$$0\longrightarrow A\xrightarrow{\phi}B\xrightarrow{\psi}C\longrightarrow 0$$

be a short exact sequence. If there exists a $*$*-homomorphism* $\rho:C\to B$ *such that* $\psi\circ\rho=\mathrm{id}_C$, *then the sequence is called* ***split exact****, and* ρ *is called a lift.*

EXAMPLE 1.13. *In the previous example, the short exact sequence*

$$0 \longrightarrow C_0((0,1)) \xrightarrow{\iota} C([0,1]) \xrightarrow{\phi} \mathbb{C}\oplus\mathbb{C} \longrightarrow 0$$

does not split. To show that, suppose that there is $\rho:\mathbb{C}\oplus\mathbb{C}\to C([0,1])$ *such that* $\phi\circ\rho=\mathrm{id}_{\mathbb{C}\oplus\mathbb{C}}$. *Since* λ *is linear,*

$$\lambda(\alpha,\beta)=\alpha\lambda(1,0)+\beta\lambda(0,1),$$

so $\lambda(1,0), \lambda(0,1)$ *are the basis elements and* $\lambda(1,0) = f_1, \lambda(0,1) = f_2$ *for some* $f_1, f_2 \in C([0,1])$ *and*

$$\begin{aligned}(\alpha,\beta) &= \mathrm{id}_{\mathbb{C}\oplus\mathbb{C}}(\alpha,\beta)\\ &= \psi\circ\lambda(\alpha,\beta)\\ &= \alpha\psi(\lambda(1,0)) + \beta\psi(\lambda(0,1))\\ &= \alpha\psi(f_1) + \beta\psi(f_2)\\ &= \alpha(f_1(0), f_1(1)) + \beta(f_2(0), f_2(1)).\end{aligned}$$

So $f_1(0) = 1, f_1(1) = 0, f_2(0) = 0, f_2(1) = 1$. *Since* λ *is a homomorphism,*

$$f_1^2 = (\lambda((1,0)))^2 = \lambda((1,0)^2) = \lambda((1,0)) = f_1$$

and

$$f_2^2 = (\lambda((0,1)))^2 = \lambda((0,1)^2) = \lambda((0,1)) = f_2.$$

The only function that satisfies the above are $f(x) = 0$ *or* $f(x) = 1$. *But it does not satisfy the initial condition.*

1.3 Unitisation

The unitisation construction is essential for K-theory computations, as many fundamental results and computational techniques require the presence of a multiplicative identity. While not all C^*-algebras are naturally unital, the unitisation procedure allows us to extend any C^*-algebra to a unital one in a canonical way, preserving essential structural properties while enabling the application of unital K-theory techniques. This construction is particularly important for graph C^*-algebras, which are typically non-unital, and becomes crucial when computing their K-groups using exact sequence methods. For any C^*-algebra A, we define

$$\tilde{A} := \{a + \alpha 1_{\tilde{A}} : a \in A, \alpha \in \mathbb{C}\}.$$

Then $\tilde{A}$ is a C^*-algebra with identity $1_{\tilde{A}}$ (See [13, Theorem 2.1.6] for the proof). This is called the **unitisation** of A. This C^*-algebra $\tilde{A}$ contains A as an ideal, and inherits properties of A.

LEMMA 1.14. *Let the C^*-algebra A be unital. Then the unitisation $\tilde{A}$ is isomorphic to the direct sum $A \oplus \mathbb{C}$.*

PROOF. A is unital, then 1_A exists. Define $f := 1_{\tilde{A}} - 1_A$. Then for any $a \in A, \lambda \in \mathbb{C}$, $a + \lambda 1_{\tilde{A}} = (a + \lambda 1_A) + \lambda f$. Therefore

$$\tilde{A} = \{a + \lambda f : a \in A, \lambda \in \mathbb{C}\}.$$

Define a map $\mu : \tilde{A} \to A \oplus \mathbb{C}$ by $a + \lambda f \mapsto (a, \lambda)$. This is an isomorphism with inverse $\mu^{-1} : A \oplus \mathbb{C} \to \tilde{A}; (a, \lambda) \mapsto a + \lambda f$. □

Let $\phi : A \to B$ be a $*$-homomorphism between C^*-algebras, define map $\tilde{\phi} : \tilde{A} \to \tilde{B}$ by

$$\tilde{\phi}(a + \alpha 1_{\tilde{A}}) = \phi(a) + \alpha 1_{\tilde{B}}$$

for every $a \in A$. Then $\tilde{\phi}$ is well-defined, and $\tilde{\phi}(a + \alpha 1_{\tilde{A}})^* = \tilde{\phi}(a^* + \overline{\alpha} 1_{\tilde{A}})$. So $\tilde{\phi}$ is a $*$-homomorphism, and this is called the **unitisation** of ϕ.

REMARK 1.15. *Let A be a non-unital C^*-algebra. Define $\iota : A \to \tilde{A}; a \to a + 0 \cdot 1_{\tilde{A}} = a$ to be the inclusion of A, $\pi : \tilde{A} \to \mathbb{C}; a + \alpha 1_{\tilde{A}} \mapsto \alpha$ and $\lambda : \mathbb{C} \to \tilde{A}; \alpha \mapsto \alpha 1_{\tilde{A}}$. Then* $\mathrm{im}(\iota) = A = \ker(\pi)$ *and* $\pi \circ \lambda(\alpha) = \alpha \implies \pi \circ \lambda = \mathrm{id}_{\mathbb{C}}$. *So the sequence*

$$0 \to A \xhookrightarrow{\iota} \tilde{A} \underset{\lambda}{\overset{\pi}{\rightleftarrows}} \mathbb{C} \to 0$$

is split-exact.

1.4 Matrix Algebras

Matrix algebras over C^*-algebras play a fundamental role in K-theory, as the definition of K-groups inherently involves studying equivalence classes of projections and invertible elements within matrix algebras $M_n(A)$ for all positive integers n. This construction is essential because K-theory captures stability properties: two projections in A might not be equivalent within A itself, but they may become equivalent when viewed as projections in some larger matrix algebra $M_n(A)$. The matrix construction thus provides the stabilisation process that is central to the definition of K_0 and K_1 groups. Understanding the C^*-algebra structure of $M_n(A)$ and how it relates to the original algebra A is therefore crucial for the K-theory computations that will be performed in Part 2 of this thesis. In this section, we establish the C^*-algebra structure on the set $M_n(A)$ of $n \times n$ matrices with entries in a C^*-algebra A.

Suppose that A is a C^*-algebra, and denote

$$M_n(A) := \{(a_{ij})_{1\leq i,j\leq n} : a_{ij} \in A\}.$$

Set $(a_{ij})^* = (a_{ji}^*)$ and product to be the usual matrix multiplication. Then $M_n(A)$ becomes a $*$-algebra. We know from the Gelfand-Naimark-Segal representation theorem that every C^*-algebra admits a faithful(injective) representation $\pi : A \to B(H)$ to a Hilbert space H. If we set $\pi_n : M_n(A) \to B(H^n)$ to be the induced $*$-homomorphism and set the norm

$$\|(a_{ij})\| = \|\pi_n((a_{ij}))\|,$$

this norm satisfies the C^*-identity and $M_n(A)$ becomes a C^*-algebra. Let A, B be C^*-algebras and $\phi : A \to B$ be a $*$-homomorphism. Like the unitisation, there is an induced $*$-homomorphism $\phi_n : M_n(A) \to M_n(B)$ given by

$$\phi_n((a_{ij})) = (\phi(a_{ij})).$$

Indeed, let $(a_{ij}), (b_{ij}) \in M_n(A)$ and $\lambda \in \mathbb{C}$. Since ϕ is linear,

$$\begin{aligned}\phi_n((a_{ij}) + (b_{ij})) &= \phi_n((a_{ij} + b_{ij})) = (\phi(a_{ij} + b_{ij}))\\ &= (\phi(a_{ij}) + \phi(b_{ij})) = (\phi(a_{ij})) + (\phi(b_{ij}))\\ &= \phi_n((a_{ij})) + \phi_n((b_{ij})).\end{aligned}$$

Similarly, $\phi_n(\lambda(a_{ij})) = \phi_n((\lambda a_{ij})) = (\phi(\lambda a_{ij})) = (\lambda\phi(a_{ij})) = \lambda(\phi(a_{ij})) = \lambda\phi_n((a_{ij}))$. For matrix multiplication, we have

$$\begin{aligned}\phi_n((a_{ij})(b_{jk})) &= \phi_n\left(\left(\sum_{j=1}^{n} a_{ij}b_{jk}\right)\right) = \left(\phi\left(\sum_{j=1}^{n} a_{ij}b_{jk}\right)\right)\\ &= \left(\sum_{j=1}^{n} \phi(a_{ij})\phi(b_{jk})\right) = (\phi(a_{ij}))(\phi(b_{jk}))\\ &= \phi_n((a_{ij}))\phi_n((b_{jk})).\end{aligned}$$

Finally, for the involution property,

$$\begin{aligned}\phi_n((a_{ij})^*) &= \phi_n((a_{ji}^*)) = (\phi(a_{ji}^*)) = (\phi(a_{ji})^*)\\ &= (\phi(a_{ij}))^* = \phi_n((a_{ij}))^*.\end{aligned}$$

1.5 Elements in C^*-Algebras

In the setting of operator algebras, some elements are fundamental elements as they capture essential structural properties and enable a deeper understanding of these algebras' dynamics.

DEFINITION 1.16. *Let A be a C^*-algebra. An element $p \in A$ is called a* ***projection*** *if $p = p^* = p^2$. The set of projections in A is denoted $\mathcal{P}(A)$.*

DEFINITION 1.17. *Let A be a C^*-algebra. An element $v \in A$ is called a* ***partial isometry*** *if v^*v and vv^* is a projection.*

Result from the following lemma is frequently used during calculations:

LEMMA 1.18. *Let v be a partial isometry. Then v^* is also a partial isometry and $v = vv^*v$.*

PROOF. Let v be a partial isometry. Then v^*v is a projection.

$$\begin{aligned}\|vv^*v - v\|^2 &= \|(vv^*v - v)^*(vv^*v^* - v)\| \\ &= \|(v^*vv^* - v^*)(vv^*v - v)\| \\ &= \|(v^*v)^3 - (v^*v)^2 - (v^*v)^2 + v^*v\| \\ &= \|0\| \\ &= 0.\end{aligned}$$

So $v = vv^*v$. To show that v^* is a partial isometry, it suffice to show that $q = vv^*$ is a projection.

$$q^* = (vv^*)^* = vv^* = q,$$

$$q^2 = (vv^*)^2 = (vv^*v)v^* = vv^* = q.$$

□

$p = v^*v$ is called the **source projection** of v and $q = vv^*$ is called the **range projection** of v. Two projections p, q are **orthogonal** if $pq = 0$. A set of projections $\{p_i\}_{i\in I}$ is called mutually orthogonal if each projection is orthogonal to each other. If two projections are orthogonal, their sum is also a projection.

REMARK 1.19. *Let v be a partial isometry in $\mathcal{B}(\mathscr{H})$. Then $v\mathscr{H} = vv^*(v\mathscr{H}) \subseteq vv^*\mathscr{H}$ and $vv^*\mathscr{H} = v(v^*\mathscr{H}) \subseteq v\mathscr{H}$. So $v\mathscr{H} = vv^*\mathscr{H}$ and v takes the source projected subspace $v^*v\mathscr{H}$ to the range projected subspace $vv^*\mathscr{H}$. (Recall that $v(v^*v\mathscr{H}) = v\mathscr{H}$).*

DEFINITION 1.20. *Let A be a unital C^*-algebra. An element $u \in A$ is a* ***unitary*** *if $u^*u = uu^* = 1_A$.*

1.6 Category Theory

Category theory provides the conceptual framework for understanding K-theory as a systematic assignment of algebraic invariants to C^*-algebras. Rather than studying individual C^*-algebras in isolation, category theory allows us to understand the relationships between different C^*-algebras and how these relationships are preserved under various constructions. This perspective is essential for K-theory, which assigns to each C^*-algebra a pair of abelian groups (K_0 and K_1) in a way that respects homomorphisms between algebras. Formally, K-theory defines functors from the category of C^*-algebras $\mathbf{C^*alg}$ to the category of abelian groups $\mathbf{Ab}$, as established in [11]. The functorial nature of K-theory is crucial for computational purposes: it ensures that exact sequences of C^*-algebras give rise to exact sequences in K-theory, enabling the calculation of K-groups for complex algebras from simpler ones. The categorical framework also clarifies how various constructions (such as crossed products, direct sums, and tensor products) behave with respect to K-theory, providing the organisational principle that underlies much of the theory developed in this thesis.

DEFINITION 1.21. *A **Category** $\mathcal{C}$ consists of two classes: objects, denoted* $\mathrm{ob}(\mathcal{C})$*, and for any* $X, Y \in \mathrm{ob}(\mathcal{C})$*, the set of arrows between* X *and* Y*, denoted* $\mathrm{Hom}(X, Y)$*, satisfying the following rules:*

(1) composition rule: Let $f : X \to Y$ *and* $g : Y \to Z$*. Then there is a map*

$$\mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) \to \mathcal{C}(X,Z);$$
$$(f,g) \mapsto g \circ f = gf;$$

(2) identity rule: for any $X \in ob(\mathcal{C})$*, the identity arrow, denoted* 1_X*, is in* $\mathrm{ob}(\mathcal{C})$*, and it satisfies*

$$f \circ 1_X = f = 1_Y \circ f$$

for every $f \in \mathrm{Hom}(X,Y)$*; and*

(3) associative rule: for $f : X \to Y, g : Y \to Z$ *and* $h : Z \to W$*,*

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

EXAMPLE 1.22. *Any group* G *can be regarded as a category of one object. This single object can be denoted* $*$*. Any group element* $g \in G$ *is regarded as a morphism* $g : * \to *$*. Composition of morphisms is given by the group operation in* G*. The identity morphism is the identity element* $e \in G$*.*

EXAMPLE 1.23. *The category of abelian groups, denoted* $\mathbf{Ab}$*, consists of all abelian groups as objects and homomorphisms between abelian groups as morphisms. The composition of morphismns is given by the composition of group homomorphisms, the identity morphism is the identity homomorphism between abelian groups.*

Two objects X, Y are **isomorphic** if there is $f : X \to Y$ and $g : Y \to X$ such that $g \circ f = 1_X$ and $f \circ g = 1_Y$.

DEFINITION 1.24. *Let* $\mathcal{C}, \mathcal{D}$ *be categories. A* ***functor*** $F : \mathcal{C} \to \mathcal{D}$ *maps* $\mathrm{ob}(\mathcal{C})$ *to* $\mathrm{ob}(\mathcal{D})$ *and* $\mathcal{C}(X, Y)$ *to* $\mathcal{D}(FX, FY)$ *for every* $X, Y \in \mathrm{ob}(\mathcal{C})$ *satisfying the following:*

(1) $F1_X = 1_{FX}$*; and*

(2) for every $f \in \mathcal{C}(X, Y)$ *and* $g \in \mathcal{C}(Y, Z)$*,*

$$F(g \circ f) = Fg \circ Ff.$$

EXAMPLE 1.25. *Fix an object* X *in* $\mathcal{C}$*. Assign to each object* $A \in \mathrm{ob}(\mathcal{C})$ *the set* $\mathcal{C}(X, A)$ *and to each arrow* $f : A \to B$ *the function* $f_* : \mathcal{C}(X, A) \to \mathcal{C}(X, B)$ *by*

$$f_*(h) = f \circ h$$

for every $h \in \mathcal{C}(X, A)$*. Then, for any* $f : A \to B$ *and* $g : B \to C$*,*

$$(g \circ f)_*(h) = (g \circ f) \circ h = g \circ (f \circ h) = g_*(f_*(h)) = (g_* \circ f_*)(h)$$

and

$$(1_A)_*(h) = 1_A \circ h = h = 1_{\mathcal{C}(X,A)}(h).$$

Call this map $\mathcal{C}(X, \_)$ *and it is a functor from* $\mathcal{C}$ *to* $\mathbf{Set}$*, the category of sets.*

DEFINITION 1.26. *Let* $\mathcal{C}$ *be a category. An object* Z *in* $\mathcal{C}$ *is called a* ***zero object*** *if for every* $X \in \mathrm{ob}(\mathcal{C})$*,* $\mathcal{C}(X, Z)$ *and* $\mathcal{C}(Z, X)$ *contains exactly one arrow.*

The zero object of the category of abelian groups is the trivial group $\{e\}$, where e is the identity element. Another example is the $\{0\}$, the zero C^*-algebra in the category of C^*-algebras. Zero objects are unique up to isomorphism, as shown in the following lemma:

LEMMA 1.27. *Suppose that Z, Z' are zero objects in a category $\mathcal{C}$, then Z is isomorphic to Z'.*

PROOF. By the definition of zero objects, $\mathcal{C}(Z, Z'), \mathcal{C}(Z', Z), \mathcal{C}(Z, Z), \mathcal{C}(Z', Z')$ contains exactly one element. Let $f \in \mathcal{C}(Z, Z')$ and $g \in \mathcal{C}(Z', Z)$. Then $g \circ f \in \mathcal{C}(Z, Z')$ and $f \circ g \in \mathcal{C}(Z', Z)$. By the definition of category, $g \circ f = 1_Z$ and $f \circ g = 1_{Z'}$ and thus $Z \cong Z'$. $\square$

CHAPTER 2

Group C^*-Algebras and Crossed Products

Crossed products are fundamental constructions in the theory of C^*-algebras that arise from group actions and coactions on C^*-algebras, providing essential tools for this thesis's investigation of graph algebras and their K-theory. These constructions play a crucial role in noncommutative geometry and dynamical systems, but their importance for our purposes lies in their connection to the skew-product methodology that will be central to our approach to graph of groups C^*-algebras.

For actions, the crossed product provides a mechanism to extend a C^*-algebra by incorporating the symmetry encoded by a group action. This process captures both the algebraic structure and the dynamics of the action, leading to new algebras that reflect the interplay between the original algebra and the group. In the context of graph C^*-algebras, crossed products by actions will allow us to realise certain graph constructions as crossed products, enabling the application of powerful K-theoretic techniques.

Coactions, on the other hand, involve a dual notion where the group is encoded within the algebra, providing a way to model quantum symmetries. The crossed product by a coaction yields an algebra that intertwines the structure of the original algebra with the dual group, leading to a rich interplay between algebraic and topological properties. Crucially for this thesis, the connection between skew-product graphs and crossed products by coactions will provide the theoretical foundation for extending these techniques to the graph of groups setting.

This chapter explores both types of crossed products, following the comprehensive treatments by Sims [12] and Murphy [13], establishing the theoretical framework that will enable the skew-product constructions developed in later chapters.

2.1 Group Algebras

The study of group C^*-algebras is essential for our later calculations involving crossed products and the skew-product constructions that form the core of this thesis. Group C^*-algebras provide the fundamental examples of how group structures can be encoded in operator algebraic terms, establishing the foundation for understanding more complex constructions like crossed products by actions and coactions. While the general theory applies to locally compact groups, we will focus exclusively on the case of discrete groups, as these are the groups that appear in graph of groups and in the labelling functions for skew-product constructions. This restriction allows us to avoid measure-theoretic complications while maintaining all the essential features needed for our applications.

2.2 Group C^*-Algebras of Discrete Groups

DEFINITION 2.1. *Let G be a discrete group. Define $C_c(G)$ to be the set of functions from G to $\mathbb{C}$ which has finite support. That is,*

$$C_c(G) = \{f : G \to \mathbb{C} : \textit{ All but finitely many } x \in G \textit{ such that } f(x) \neq 0\}.$$

One can see that with point-wise addition ($(f + g)(x) = f(x) + g(x)$), $C_c(G)$ becomes a vector space. We define the product of functions $f * g$ as the **convolution**

$$(f * g)(x) := \sum_{s\in G} f(s)g(s^{-1}x)$$

and the involution as

$$f^*(x) := \overline{f(x^{-1})}.$$

Since f, g have finite support, $f * g$ also have finite support so $f * g \in C_c(G)$, and with similar argument, $f^* \in C_c(G)$. If we define the involution of $f \mapsto f^*$ by $f^*(x) := \overline{f(x^{-1})}$, then $C_c(G)$ becomes a $*$-algebra.

Let $U : G \to U(\mathscr{H})$ be a unitary representation of G on $\mathscr{H}$. Then we can show that there is a $*$-representation $\pi_U : C_c(G) \to \mathcal{B}(\mathscr{H})$ by

$$\pi_U(f) = \sum_{s \in G} f(s) U_s$$

Now, for each $s \in G$, define a function $\delta_s : G \to \mathbb{C}$ by

$$\delta_s(x) := \begin{cases} 1, & x = s \\ 0, & \text{otherwise} \end{cases}.$$

Then it is apparent that δ_s has finite support and thus belong to $C_c(G)$. Now, for each $f \in C_c(G)$,

$$\sum_{s \in G} f(s)\delta_s(x) = f(x)\delta_s(x) = f(x).$$

Therefore, $\{\delta_s : s \in G\}$ is a basis for $C_c(G)$. One can also see that if $e \in G$ is the identity of G, then δ_e is the multiplicative identity of $C_c(G)$. To see this,

$$(\delta_e * f)(x) = \sum_{s \in G} \delta_e(s) f(s^{-1}x) = \delta_e(e) f(e^{-1}x) = f(x) = \cdots = (f * \delta_e)(x).$$

So $C_c(G)$ is a unital $*$-algebra. In addition,

$$(\delta_w * \delta_v)(x) = \sum_{s \in G} \delta_w(s)\delta(s^{-1}x) = \delta_w(w)\delta_v(w^{-1}x) = \begin{cases} 1, & v = w^{-1}x \\ 0, & \text{otherwise} \end{cases} = \delta_{wv}(x)$$

and

$$\delta_v^*(x) = \overline{\delta_v(x^{-1})} = \delta_v(x^{-1}) = \begin{cases} 1, & v = x^{-1} \\ 0, & \text{otherwise} \end{cases} = \delta_{v^{-1}}(x).$$

So $\delta_w * \delta_v = \delta_{wv}$ and $\delta_v^* = \delta_{v^{-1}}$. This implies δ_s is an unitary element in $C_c(G)$ for every $s \in G$, and we can define a homomorphism $\delta : G \to U(C_c(G))$ by $s \mapsto \delta_s$.

THEOREM 2.2. *Let G be a discrete group. let $C_c(G)$ be its group algebra. Let $\delta : G \to U(C_c(G))$ be the homomorphism defined as above and π_U be the corresponding $*$-representation defined as above. Then $U \mapsto \pi_U$ is a bijection from between the set of unitary representations of G to the set of unital $*$-representations of $C_c(G)$ on a Hilbert space $\mathscr{H}$.*

PROOF. We have already shown $U \to \pi_U$. We only need to show the other direction. Let G be a discrete group and let $\pi : C_c(G) \to \mathcal{B}(\mathscr{H})$ be a unital $*$-representation. Then

$$\pi \circ \delta(x) = \pi(\delta_x) = \sum_{s \in G} \delta_x(t) U_x = \delta_x(x) U_x = U_x.$$

So $U = \pi \circ \delta$ is the unitary representation defined earlier and completes the proof. □

Next, we complete the $*$-algebra to get a C^*-algebra. To do that, we need to first define a C^* norm.

DEFINITION 2.3. *Let $\pi : A \to \mathcal{B}(\mathscr{H})$ be a $*$-representation. The representation is called* ***cyclic*** *if there exists a $\xi \in \mathscr{H}$ such that $\mathscr{H} = \overline{\text{span}}\{\pi(a)\xi : a \in A\}$. Such element ξ is called a* ***cyclic vector****.*

That is, you can find an element in the space that spans the pre-Hilbert space when applied with $\pi(a)$ for all $a \in A$. The next lemma shows that every unital representation of a unital $*$-alegbra can be decomposed into cyclic representations.

Recall that given an operator X of Hilbert space $\mathscr{H}$, an **invariant subspace** $\mathscr{K}$ of X is a subspace of $\mathscr{H}$ such that $X(\mathscr{K}) \subseteq \mathscr{K}$.

LEMMA 2.4. *Let A be an unital $*$-algebra. Then every unital representation of A is equivalent to a direct sum of cyclic representations.*

PROOF. See proof from [16]. □

COROLLARY 2.5. *Let A be a $*$-algebra, and fix a Hillbert space $\mathscr{H}$ such that* $\dim(\mathscr{H}) \geq \dim(A)$. *Then for any $*$-representation π of A on $\mathscr{H}$,*

$$\|\pi(a)\| \leq \sup\{\|\rho(a)\| : \rho \text{ a cyclic representation of } A \text{ on } \mathscr{H}\}.$$

PROOF. See proof from [16]. $\square$

PROPOSITION 2.6. *let G be a discrete group. And let $\mathscr{H}$ be a Hilbert space such that* $\dim(\mathscr{H}) \geq |G|$. *Define the norm $\|\cdot\|_{C^*} : C_c(G) \to [0,\infty)$ by*

$$\|f\|_{C^*} := \sup\{\|\rho\| : \rho \text{ a cyclic representation of } C_c(G) \text{ on } \mathscr{H}\}.$$

Then the norm $\|\cdot\|_{C^*}$ is finite and it satisfies the C^*-identity and thus, the completion of $C_c(G)$ becomes a C^*-algebra.

PROOF. Let $A := C_c(G)$. Recall that a cyclic representation of A on $\mathscr{H}$ is a $*$-homomorphism $\rho : A \to B(\mathscr{H})$ together with a unit vector $\xi \in \mathscr{H}$ such that $\rho(A)\,\xi$ is dense in $\mathscr{H}$. First, we need to show that $\|f\|_{C^*} < \infty$. Every $f \in A = C_c(G)$ has finite support. That is,

$$f \;=\; \sum_{g\in G} f(g)\,\delta_g, \qquad \text{for finitely many } g \text{ such that } f(g) \neq 0.$$

Fix a $*$–representation ρ, the operators $\rho(\delta_g)$ are unitaries (since $\delta_g^* = \delta_{g^{-1}}$ and $\delta_g\,\delta_{g^{-1}} = \delta_e$), so

$$\|\rho(\delta_g)\| = 1.$$

Then

$$\|\rho(f)\| = \left\|\sum\nolimits_g f(g)\,\rho(\delta_g)\right\| \;\leq\; \sum\nolimits_g |f(g)|\,\|\rho(\delta_g)\| = \sum_{g\in G} |f(g)| < \infty.$$

Taking the supremum over all cyclic representations shows

$$\|f\|_{C^*} = \sup_{\rho \text{ cyclic}} \|\rho(f)\| \;\leq\; \sum_{g\in G} |f(g)| < \infty.$$

Next, we show that the norm $\|\cdot\|_{C^*}$ satisfies the C^*–identity $\|ff^*\|_{C^*} = \|f\|^2_{C^*}$. Let $\mathcal{R}$ be the family of all cyclic representations of A on $\mathscr{H}$. Define

$$\Pi := \bigoplus_{\rho\in\mathcal{R}} \rho : A \to \mathcal{B}\Big(\bigoplus_{\rho\in\mathcal{R}} \mathscr{H}\Big).$$

Since direct sums of cyclic representations remain a $*$–representation, Π is again a $*$–homomorphism. On top of that,

$$\|\Pi(f)\| = \sup_{\rho\in\mathcal{R}} \|\rho(f)\| = \|f\|_{C^*},$$

so $\|\cdot\|_{C^*}$ is exactly the operator norm induced by the faithful representation Π. But every $*$–homomorphism into $B(\mathscr{H})$ satisfies the C*–identity, that is,

$$\|\Pi(ff^*)\| = \|\Pi(f)\,\Pi(f^*)\| = \|\Pi(f)\|^2.$$

Therefore

$$\|ff^*\|_{C^*} = \|\Pi(ff^*)\| = \|\Pi(f)\|^2 = \|f\|^2_{C^*},$$

as required.

Hence $\|\cdot\|_{C^*}$ is a finite $*$–norm satisfying the C^*–identity. Its completion is the enveloping C^*–algebra of $C_c(G)$. □

This C^*-algebra is called $C^*(G)$, the **Group C^*-algebra** of G. It turns out $C^*(G)$ is the C^*-algebra generated by the unitary elements $\{\delta_x : x \in G\}$ with universal property: If there is a C^*-algebra A generated by unitary elements $\{\kappa_x : x \in G\}$, then there is a homomorphism $\delta_G : C^*(G) \to A$ such that $\kappa = \delta_G \circ \delta$. That is, $\delta_G(\delta_x) = \kappa_x$ for all $x \in G$.

EXAMPLE 2.7. *Let $G = \mathbb{Z}$. Then its group algebra $C^*(\mathbb{Z})$ is generated by $\{\delta_n : n \in \mathbb{Z}\}$. Since $n = 1 + 1 + \cdots 1$,*

$$\delta_n = \delta_{1+1+\cdots 1} = \delta_1\delta_1 \cdots \delta_1 = (\delta_1)^n.$$

So $C^*(\mathbb{Z})$ is generated by a cyclic unitary element δ_1. That is,

$$C^*(\mathbb{Z}) = \overline{\text{span}}\{\delta_1^n : n \in \mathbb{Z}\}.$$

EXAMPLE 2.8. *Recall the tensor product* $A \otimes B$ *between two* C^**-algebras* A *and* B *is also a* C^**-algebra. Let* G *be a group. Then the tensor product* $C^*(G) \otimes C^*(G)$ *is a* C^**-algebra* $C^*(\{\delta_s \otimes \delta_t : s, t \in G\})$*. By the universal property, there is a homomorphism* $\delta_G : C^*(G) \to C^*(G) \otimes C^*(G)$ *such that* $\delta_G(\delta_s) = \delta_s \otimes \delta_s$*. This fact will be used later.*

2.3 Crossed Product C^*-Algebras by Group actions

In this section we develop the theory of crossed product C^*-algebras arising from group actions on C^*-algebras, establishing one of the two fundamental constructions that will be essential for our later work with skew-product graphs and their connection to K-theory computations. These constructions, originally inspired by quantum mechanics and dynamical systems, provide a systematic way to incorporate group symmetries into the algebraic framework.

A C^*-dynamical system consists of a C^*-algebra A, a group G, and a group action α of G on A. The crossed product of the system, denoted $A \rtimes_\alpha G$, is a C^*-algebra that encodes both the structure of A and the dynamics of the action α. This crossed product C^*-algebra contains A as an ideal and provides a crucial tool for studying the K-theory of A, particularly when the K-groups are difficult to compute directly. The significance for this thesis lies in the fact that skew-product graph algebras can often be realised as crossed products, enabling the application of exact sequence techniques in K-theory.

Recall from the uniqueness theorems section that an action of a discrete (also for locally compact) group G on a C^*-algebra A is a homomorphism $\alpha : G \to \mathrm{Aut}(A)$. The guage action discussed in that section is an action of the compact group $\mathbb{T}$ on the graph algebra $C^*(E)$.

DEFINITION 2.9 (C^*-dynamical system). *Let* A *be a* C^**-algebra,* G *be a group and* $\alpha : G \to \mathrm{Aut}(A)$ *be an action of* G *on* A*. A* ***dynamical system*** *is a triple* (A, G, α).

EXAMPLE 2.10. *Recall any complex number* z *can be rotated from the origin by an angle* θ *by multiplying* $e^{i\theta}$*. If* $z \in \mathbb{T}$*, then* $e^{2n\pi i\theta} z \in \mathbb{T}$ *for any* $n \in \mathbb{N}$*. Therefore, there is an action*

$\alpha : \mathbb{Z} \to \mathrm{Aut}(C_0(\mathbb{T}))$ *by*

$$\alpha_n(f)(z) = f(e^{-2n\pi i\theta} z),$$

and $(C_0(\mathbb{T}), \mathbb{Z}, \alpha)$ *becomes a dynamical system.*

Notice in the above example, the action α is induced by action of $\alpha : \mathbb{Z} \to \mathrm{Aut}(\mathbb{T})$ by the rotation function $r_\theta : \mathbb{T} \to \mathbb{T}$ where

$$r_\theta(z) = e^{2n\pi i\theta} z.$$

Notice that $r_\theta \in \mathrm{Aut}(\mathbb{T})$ and $\alpha_n(f)(z) = f(r_{-\theta}(z)) = f(r_\theta^{-1}(z))$. We often often write the action α of G on the locally compact space X as $g \cdot x$ instead of $\alpha_g(x)$ for every $g \in G, x \in X$ and leave the notation to the induced action. In terms of the above example, we would then write

$$\alpha_n(f)(z) = f(-\theta \cdot z).$$

One can imagine an action on a space in classical mechanics, as the states M_t of a system at time t is a sub-space of the state space M. When time moves forward, the state M_0 at time 0 changes to state M_t at time t. So the time deduction of the system is described by an action of the group of real numbers $(\mathbb{R}, +)$ on the state space M.

The notation in the previous example is used in the following proposition, and it which generalizes the previous example.

PROPOSITION 2.11. *Let X be a locally compact Hausdorff space. For any dynamical system $(C_0(X), G, \alpha)$, there is an unique action of G on X such that $\alpha_g(f)(x) = f(g^{-1} \cdot x)$ for every $f \in C_0(X), g \in G, x \in X$.*

PROOF. The proof follows from Proposition 2.15 and Gelfand theory for commutative C^*-algebra $C_0(X)$ just as in [12]. □

Here it is important to mention about the multiplier algebra. Let A be a C^*-algebra. The multiplier algebra of A, denoted $\mathcal{M}(A)$, is defined to be the biggest unital C^*-algebra that contains A as an essential ideal. In order words, $\mathcal{M}(A)$ is the maximal unitisation of A. The most iconic example would be $A = \mathcal{K}(\mathscr{H})$, the C^*-algebra of compact operators on $\mathscr{H}$ and $\mathcal{M}(A) = \mathcal{B}(\mathscr{H})$, the C^*-algebra of all bounded linear operators on $\mathscr{H}$.

DEFINITION 2.12. *Let (A, G, α) be a dynamical system and B be a C^*-algebra. A* ***covariant homomorphism*** *of (A, G, α) to B, is a pair (π, U) of $*$-homomorphism $\pi : A \to \mathcal{M}(B)$ and strongly continuous homomorphism $U : G \to \mathcal{U}(\mathcal{M}(B))$ such that*

$$\pi(\alpha_g(a)) = U_g \pi(a) U_g^* \tag{2.1}$$

for every $a \in A$ and $g \in G$. If $B = \mathcal{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$, then (π, U) is called a covariant representation.

EXAMPLE 2.13. *Let G be a locally compact group. Then the left translation,*

$$g \cdot h = gh$$

induces an action $\mathrm{lt} : G \to \mathrm{Aut}(C_0(G))$ by

$$\mathrm{lt}_g(f)(h) = f(g^{-1} \cdot h) = f(g^{-1}h).$$

Let $L^2(G)$ be the Hilbert space. Define $\pi : C_0(G) \to L^2(G)$ by

$$\pi(f)\xi(h) = f(h)\xi(h)$$

and $\lambda : G \to \mathcal{U}(L^2(G))$ by

$$\lambda_g(\xi)(h) = \xi(g^{-1}h)$$

for every $g, h \in G, \xi_g \in L^2(G)$.

$$\begin{aligned}
\lambda_g \circ \pi(f) \circ \lambda_g^* \xi_h &= \lambda_g \circ \pi(f) \circ \lambda_{g^{-1}} \xi_h \\
&= \lambda_g \circ \pi(f) \xi_{gh} \\
&= \lambda_g (f(g^{-1}h) \xi_{gh}) \\
&= f(g^{-1}h) \lambda_g \xi_{gh} \qquad\qquad . \\
&= f(g^{-1}h) \xi_{g^{-1}gh} \\
&= f(g^{-1}h) \xi_h \\
&= \alpha_g(f) \xi_h = \pi(\alpha_g(f)) \xi_h
\end{aligned}$$

Then the pair (π, λ) *satisfies (2.1) and it becomes a covariant representation from* $(C_0(G), G, \mathrm{lt})$ *into* $L^2(G)$.

The following theorem, where the full proof can be found in [17], gives the existence of a unital C^*-algebra $A \rtimes_\alpha G$ that contains A as an ideal:

THEOREM 2.14. *Suppose that* (A, G, α) *a dynamical system. Then there exists a* C^**-algebra* $A \rtimes_\alpha G$ *and homomorphisms* $i_A : A \to A \rtimes_\alpha G$ *and* $i_G : G \to U(A \rtimes_\alpha G)$ *such that*

(1) $A \rtimes_\alpha G = \overline{\mathrm{span}}\{i_A(a) i_G(s) : a \in A, s \in G\}$,

(2) if (π, U) *is a covariant representation of* (A, G, α) *on a Hilbert space* $\mathscr{H}$*, then there exists a non-degenerate representation* $\pi \times U : A \rtimes_\alpha G \to \mathscr{H}$ *satisfying*

$$(\pi \times U) \circ i_A = \pi \quad \textit{and} \quad ((\pi \times U) \circ i_G) = U.$$

That is, the homomorphism $\pi \times U$ *extends* (i_A, i_G) *to* (π, U).

(3) $i_A(\alpha_s(a)) = i_G(s) i_A(a) i_G(s)^*$ *for every* $a \in A, s \in G$.

$\pi \times U$ is called the **integrated form** of (π, U), and the C^*-algebra $A \rtimes_\alpha G$ is called the **(full) crossed product** of A by G and as in the beginning of the section, $A \rtimes_\alpha G$ contains A as an ideal. The crossed product is usually constructed based on $C_c(G, A)$, the space of finitely supported functions $f : G \to A$.

PROPOSITION 2.15. *Suppose that* (A, G, α) *is a dynamical system. If* G *is a discrete group (a topological group with a discrete topology), then* $C_c(G, A)$ *becomes a* $*$*-algebra when equipped with the following:*

$$(f *_\alpha g)(x) := \sum_{s \in G} f(s)\alpha_s(g(s^{-1})x),$$
$$f^*(x) := \alpha_x(f(x^{-1})^*),$$

and for any covariant representation (π, U) *from* (A, G, α) *to* $\mathscr{H}$*, there is a unital* $*$*-representation* $\pi \times U : C_c(G, A) \to \mathscr{H}$ *such that*

$$(\pi \times U)(f) = \sum_{s \in G} \pi(f(s))u_s$$

for every $f \in C_c(G, A)$*.*

PROOF. This is proved in [12]. □

To go from representations of $C_c(G, A)$ to covariant representations of (A, G, α), we need copies of both G and A inside $C_c(G, A)$. Define $i_G : G \to C_c(G, A)$ by

$$i_G(s) := \delta_s 1_A,$$

where

$$\delta_s 1_A(x) := \begin{cases} 1_A, & x = s \\ 0_A, & x \neq s \end{cases}$$

for every $x \in G$, and define $i_A : A \in C_c(G, A)$ by

$$i_A(a) := \delta_e a,$$

where

$$\delta_e a(x) := \begin{cases} a, & x = s \\ 0_A, & x \neq s \end{cases}.$$

Then by some calculation, one can see that $i_G : G \to C_c(G, A)$ is a unitary homomorphism and $i_A : A \to C_c(G, A)$ is a homomorphism such that

$$i_A(\alpha_s(a)) = i_G(s)i_A(a)i_G(s)^*$$

for every $a \in A, s \in G$. In addition,

$$f = \sum_{s \in G} i_A(f(s))i_G(s)$$

for any $f \in C_c(G, A)$. The same construction in norm for $C^*(G)$ can be used to complete the algebra. Let $\mathscr{H}$ be a Hilbert space with $\dim(\mathscr{H}) \leq |G|$. Define $\|\cdot\| : C_c(G, A) \to [0, \infty)$ by

$$\|f\|_{C^*} := \sup \Big\{ \|\rho(f)\| : \rho \text{ is a cyclic representation of } C_c(G, A) \text{ on a subspace of } \mathscr{H} \Big\}.$$

With the above construction, the completion of $C_c(G, A)$ under the norm $\|\cdot\|_{C^*}$ becomes a C^*-algebra and satisfies the previous theorem. This crossed product with the homomorphisms is unqiue up to isomorphisms: If (B, j_A, j_G) is another triple satisfying (1), (2) and (3), then there is a unital isomorphism $\phi : A \rtimes_\alpha G \to B$ such that

$$\phi \circ i_A = j_A \quad \text{and} \quad \phi \circ i_G = j_G.$$

DEFINITION 2.16. *Let G be a locally compact abelian group. The* ***Pontryagin dual*** *of G, denoted $\hat{G}$, is the group of continuous group homomorphism from G to the unit group $\mathbb{T}$ under point-wise multiplication. That is,*

$$\hat{G} = \Big\{ f : G \to \mathbb{T} : f \textit{ continuous} \Big\}.$$

In Category theory's literature, $\hat{G}$ would be written as $\mathrm{Hom}(G, \mathbb{T})$.

It turns out there is a connection between the dual group and the original group, by the following lemma:

LEMMA 2.17. *Let* (A, G, α) *be a dynamical system where* G *is abelian. Then there is an action* $\hat{\alpha} : \hat{G} \to \operatorname{Aut}(A \rtimes_\alpha G)$ *such that*

$$\hat{\alpha}_\gamma(i_A(a)i_G(s)) = i_A(a)\gamma(s)i_G(s)$$

for every $\gamma \in \hat{G}, a \in A, s \in G$*. In addition,* $\hat{\alpha}$ *is continuous in the sense that, if* $\gamma_n \to \gamma$ *pointwise, then* $\hat{\alpha}_{\gamma_n}(a) \to \hat{\alpha}_\gamma(a)$ *for all* $a \in A \rtimes_\alpha G$*. This action is called the* ***dual action*** *and* $(A \times_\alpha G, \hat{G}, \hat{\alpha})$ *becomes a dynamical system.*

PROOF. Refer to [17, Proposition 5] for the proof. □

2.4 Crossed Product C^*-Algebras by Group Coactions

Coactions represent the dual perspective to actions in the theory of crossed products, and their associated crossed products will play a central role in this thesis through their connection to skew-product constructions. While actions describe how a group acts on a C^*-algebra from the outside, coactions encode the group structure within the algebra itself, providing a framework for understanding the algebraic structure that emerges from skew-product graphs. The theory developed in this section, following [17], establishes the foundation for the fundamental theorem connecting skew-product graph algebras to crossed products by coactions, which will be essential for our K-theory computations.

DEFINITION 2.18. *Let* A *be a* C^**-algebra. and G be a group. A* ***coaction*** *of G on* A *is a homomorphism* $\delta : A \to A \otimes C^*(G)$ *satisfying the* ***coaction identity****:*

$$(\delta \otimes \mathrm{id}_{C^*(G)}) \circ \delta = (\mathrm{id}_A \otimes \delta_G) \circ \delta,$$

where $\delta_G : C^*(G) \to C^*(G) \otimes C^*(G); u_g \mapsto u_g \otimes u_g$*. Here* $C^*(G) \otimes C^*(G)$ *is the maximal tensor product. Sometimes we denotes both* $\mathrm{id}_{C^*(G)}$ *and* id_A *by* ι *for convenience.*

There is an interesting duality between actions and coactions, as shown in the following example:

EXAMPLE 2.19. *Let Γ be a compact abelian group, and let $\alpha : \Gamma \to \mathrm{Aut}(A)$ by an action of Γ on the C^*-algebra A. Then the Pontryagin dual $\hat{\Gamma}$ is a discrete abelian group, and the Fourier tranform of G gives an isomorphism of $C^*(\hat{\Gamma})$ onto $C(\Gamma)$, and it carries $\delta_G : C^*(\hat{\Gamma}) \to C^*(\hat{\Gamma}) \otimes C^*(\hat{\Gamma})$ into the comultiplication*

$$\alpha_\Gamma : C(\Gamma) \to C(\Gamma) \otimes C(\Gamma) = C(\Gamma \times \Gamma) \text{ by } \alpha_\Gamma(f)(\gamma, \tau) = f(\gamma\tau).$$

Define a map $\delta : A \to C(\Gamma, A) = A \otimes C(\Gamma)$ by

$$\delta(a)(\gamma) = \alpha_\gamma(a).$$

Then

$$(\delta \otimes \iota)(\delta(a))(\gamma, \tau) = \alpha_\gamma(\alpha_\tau(a)) = \alpha_{\gamma\tau}(a) = (\iota \otimes \alpha_\Gamma)(\delta(a))(\gamma, \tau).$$

$\delta : A \to A \otimes C^*(\hat{\Gamma}) \cong A \otimes C(\Gamma)$ satisfies the coaction identity, and thus becomes a coaction of $\hat{\Gamma}$ on A.

If the group G is a discrete group, then we can understand the coaction of G as follows: Let δ be a coaction of G on a C^*-algebra A, and let $s \in G$. The **spectral subspace** is defined as

$$A_s := \Big\{ a \in A : \delta(a) = a \otimes u_s \Big\}.$$

Note that for $s, t \in G$, and let $a_s \in A_s$ and $a_t \in A_t$,

$$\delta(a_s a_t) = \delta(a_s)\delta(a_t) = (a_s \otimes u_s)(a_t \otimes u_t) = a_s a_t \otimes u_{st} \in A_s A_t,$$

and

$$\delta(a_s^*) = \delta(a_s)^* = (a_s \otimes u_s)^* = a_s^* \otimes u_{s^{-1}} \in A_{s^{-1}}.$$

So $A_s A_t \subset A_{st}$ and $A_s^* = A_{s^{-1}}$.

DEFINITION 2.20. *Let δ be a coaction of a discrete group G on a C^*-algebra A and let B be a C^*-algebra. The pair of homomorphisms $\pi : A \to B$ and $U : C_0(G) \to B$ ($C_0(G)$ is the set of complex-valued functions from G that vanishes at infinity) is called a* ***covariant homomorphism*** *of the dynamical system (B, G, δ) if (π, U) satisfies*

$$\pi(a_s)U(f) = U(\mathrm{lt}_s(f))\pi(a_s)$$

for every $a_s \in A_s$, *where* $\mathrm{lt}_s : C_0(G) \to C_0(G)$ *is the left translation by* $s \in G$.

With the covariant homomorphism provided above, the crossed product by coaction $A \rtimes_\delta G$ is the C^*-algebra generated by a universal covariant homomorphism (j_A, j_G), where $j_A : A \to A \rtimes_\delta G$ and $j_G : C_0(G) \to A \rtimes_\delta G$. In particular, the crossed product $A \rtimes_\delta G$ satisfies the following:

(1) $A \rtimes_\delta G = \overline{\mathrm{span}}\{j_A(a)j_G(f) : a \in A, f \in C_0(G)\}$;

(2) if (π, μ) is a covariant representation of (A, G, δ), then there exists a non-degenerate representation $\pi \times \mu$ of $A \rtimes_\delta G$ satisfying

$$(\pi \times \mu) \circ j_A = \pi \quad \text{and} \quad ((\pi \times \mu) \circ j_G) = \mu.$$

That is, the homomorphism $\pi \times U$ extends (j_A, j_G) to (π, μ); and

(3) for every non-degenerate representation ρ of B, the pair $(\rho \circ j_A, \rho \circ j_G)$ is a covariant representation of (A, G, δ).

The existence of coaction crossed product is proved in [18, Proposition 2.13]. Since G is discrete, any $f \in C_0(G)$ is of form $\sum_{s \in G} c_s \chi_s$, where $c_s \in \mathbb{C}$. So

$$A \rtimes_\delta G = \overline{\mathrm{span}}\Big\{ j_A(a)j_G(\chi_s) : a \in A, s \in G \Big\},$$

with the property $j_A(a_s)j_G(\chi_t) = j_G(\mathrm{lt}_s(\chi_t))j_A(a_s) = j_G(\chi_{st})j_A(a_s)$ for all $s, t \in G$.

EXAMPLE 2.21. *Suppose that* Γ *is a compact abelian group and* $\alpha : \Gamma \to \mathrm{Aut}(A)$ *be an action of* Γ *on the* C^**-alebrga* A*. As in example 2.19, there is a coaction* $\delta : A \to A \otimes C^*(\hat{\Gamma})$ *of the dual group* $\hat{\Gamma}$ *on* A*.*

Define $j_A := i_A : A \to A \rtimes_\alpha \Gamma$ *and* $j_{\hat{\Gamma}} : C_0(\hat{\Gamma}) \to A \rtimes_\alpha \Gamma$ *by* $j_{\hat{\Gamma}} := i_\Gamma \circ \mathcal{F}\{\cdot\}$, *where* $\mathcal{F}\{\cdot\} : C_0(\hat{\Gamma}) \to C(\Gamma)$ *is the fourier transform. The pair* $(j_A, j_{\hat{\Gamma}})$ *is a covariant homomorphism to* $A \rtimes_\alpha \Gamma$, *thus there exists an integrated form* $j_A \times j_{\hat{\Gamma}} : A \rtimes_\delta \hat{\Gamma} \to A \rtimes_\alpha \Gamma$. *Turns out this homomorphism is an isomorphisam, as illustrate in [8].*

CHAPTER 3

Graphs and their C^*-Algebras

Graphs and their associated C^*-algebras form a profound and elegant intersection of mathematics, where the combinatorial nature of graphs meets the deep structural properties of operator algebras. These C^*-algebras, known as graph C^*-algebras, provide both a rich source of examples in operator algebra theory and, crucially for this thesis, a template for understanding the more complex graph of groups C^*-algebras that form our primary focus. The success of K-theory computations for graph C^*-algebras, particularly through the use of skew-product constructions and the dual Pimsner-Voiculescu sequence, motivates our approach to developing similar techniques for graph of groups. The theory presented in this chapter, following Raeburn's authoritative treatment [8], establishes the fundamental techniques and computational methods that will be adapted and extended to the graph of groups setting in subsequent chapters. Understanding how graph structure translates into operator algebraic properties, and how these properties enable K-theory computations, provides the conceptual foundation for the novel constructions introduced later in this thesis.

3.1 Basic Graph Theory

The combinatorial foundation underlying graph C^*-algebras requires a precise understanding of directed graphs and their essential properties. The definitions presented here establish the graph-theoretic framework that will be translated into operator algebraic terms, ultimately enabling the construction of C^*-algebras that encode both the structure and dynamics of the underlying graphs. Here when we use the word "graph", we mostly refer to directed graphs. For undirected graphs, we will make clear of that.

DEFINITION 3.1. *A* ***graph*** *is a quadruple $E = (E^0, E^1, s, r)$, where E^0, E^1 are sets, and s, r are functions from E^1 to E^0. We call E^0 the* ***vertex set****, E^1 the* ***edge set****. The function s, r are called the* ***source map*** *and the* ***range map*** *respectively.*

One visualises elements of E^0 as points and elements of E^1 as arrows going from one point to another (can be from a point to itself). The two functions $s, r : E^1 \to E^0$ maps an arrow to its starting and ending point respectively. For example, in figure 3.1, the edge $s(e) = v_1, r(e) = v_2$.

DEFINITION 3.2. *A graph E is called* ***row-finite*** *if $|r^{-1}(v)| < \infty$ for every $v \in E^0$. That is, all vertices receive finite many edges.*

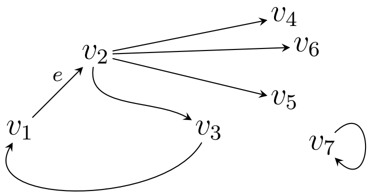


FIGURE 3.1: An example of a graph, which is not connected.

DEFINITION 3.3. *Let E be a row-finite graph. The* ***vertex matrix*** *of E, denoted A_E, is a $E^0 \times E^0$ matrix defined by*

$$A_E(v, w) = \#\Big\{e \in E^1 : r(e) = v, s(e) = w\Big\}.$$

The vertex matrix saves the structure of graphs, regardless of how it is presented. For the graph E in figure 3.1, its vertex matrix is

$$A_E = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

REMARK 3.4. *For a row-finite graph, since each vertex receives finite number of edges, every entry of the vertex matrix is finite.*

DEFINITION 3.5 (Paths). *A* ***path*** $\mu = e_1 e_2 \cdots e_n$ *is a sequence of edges such that* $r(e_i) = s(e_{i-1})$ *for all* $2 \leq i \leq n$. *We define the source of the path to be the source of the last edge and the range of the path to be the range of the first edge. That is,* $s(\mu) = s(e_n)$ *and* $r(\mu) = r(e_1)$. *The* ***length*** *of the path* $\mu = e_1 e_2 \cdots e_n$, *denoted* $|\mu|$, *is* n. *We denote the set of paths of length* n *to be* E^n, *and the set of all paths to be* E^*.

With the definition as above, it make sense why we denoted the set of vertices as E^0, because we consider vertices a path of length 0.

DEFINITION 3.6. *A path* μ *is called a* ***closed path*** *if* $r(\mu) = s(\mu)$. *A* ***cycle*** *is a closed path such that its length* $|\mu| \geq 1$ *and* $s(e_i) \neq s(e_j)$ *for* $i \neq j$.

Essentially, a closed path is a path that starts and ends at the same vertex. A cycle is a closed path such that it does not revisit the same vertex at any point. We allow an edge that is a "loop" ($s(e) = r(e)$) to become a cycle, in which $|e| = 1$.

3.2 Cuntz-Krieger Algebras

The construction of graph C^*-algebras begins with the fundamental insight that directed graphs can be represented by operators on a Hilbert space, where vertices correspond to orthogonal projections and edges correspond to partial isometries. This representation, known as a Cuntz-Krieger family, provides the bridge between combinatorial graph theory and operator algebra theory. The specific operators and the relations they satisfy encode the essential structural information of the graph while enabling the application of operator algebraic techniques. This approach, originally developed by Cuntz and Krieger [6] for subshifts, was later extended to arbitrary row-finite directed graphs.

DEFINITION 3.7. *Let E be a row-finite graph and $\mathscr{H}$ be a Hillbert space. A* ***Cuntz-Krieger E- family*** *on $\mathscr{H}$ is a set of mutually orthogonal projections $\{P_v : v \in E^0\}$ and a set of partial isometries $\{S_e : e \in E^1\}$ satisfying the Cuntz-Krieger relations:*

(1) $P_{s(e)} = S_e^* S_e$ for all $v \in E^0$; and
(2) $P_v = \sum_{\{e\in E^1 : r(e)=v\}} S_e S_e^*$ whenever v is not a source.

To make sense of the relation, one should view each vertex as a projected subspace of $\mathscr{H}$, and each arrow as a partial isometry bringing one subspace to another.

Notice that since S_e is a partial isometry, $S_eS_e^*$ is a projection, and we can see from the second CK relation that any projections $S_eS_e^*$ is being dominated by $P_r(e)$ ($S_eS_e^*\mathscr{H} \subset P_{r(e)}\mathscr{H}$). Therefore,

$$S_e = P_{r(e)} S_e = S_e P_{s(e)},$$

and by taking the adjoint,

$$S_e^* = S_e^* P_{r(e)} = P_{s(e)} S_e^*.$$

EXAMPLE 3.8. *Take $\mathscr{H} = \ell^2(\mathbb{N}) = \overline{\text{span}}\{e_i : i \geq 0\}$, and $S_e(e_i) = e_{2i}, S_f(e_i) = e_{2i+1}$. Then $S_e^*S_e = 1 = P_v$ and $S_eS_e^* + S_fS_f^* = 1 = P_v$. Therefore, $\{S, P\}$ is a Cuntz-Krieger family for the graph in figure 3.2.*

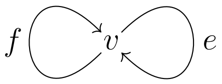


FIGURE 3.2: Graph for the Cuntz-Krieger E-family of $\ell^2(\mathbb{N})$

PROPOSITION 3.9. *Let E be a row-finite graph and $\{S, P\}$ be a Cuntz-Krieger E-family in a C^*-algebra B. Then*

(1) $\{S_eS_e^* : e \in E^1\}$ are mutually orthogonal;
(2) $S_e^*S_f \neq 0$ implies $e = f$;

(3) $S_eS_f \neq 0$ *implies* $s(e) = r(f)$*; and*

(4) $S_eS_f^* \neq 0$ *implies* $s(e) = s(f)$*.*

PROOF. (1) Recall $P_v = \sum_{\{e\in E^1:r(e)=v\}} S_eS_e^*$. This implies the projections $\{S_eS_e^*\}$ are mutually orthogonal when they have the same range. Let $e \neq f \in E^1$ such that $r(e) \neq r(f)$. Then

$$(S_eS_e^*)(S_fS_f^*) = S_e(S_e^*S_f)S_f^* = S_e(S_e^*P_{r(e)}P_{r(f)}S_f)S_f^* = S_eS_e^*0S_fS_f^* = 0.$$

(2)

$$S_e^*S_f = (S_e^*S_eS_e^*)(S_fS_f^*S_f) = S_e^*(S_eS_e^*)(S_fS_f^*)S_f.$$

Recall from (1), $(S_eS_e^*)(S_fS_f^*) \neq 0$ implies $e = f$.

(3)

$$S_eS_f = S_eP_{s(e)}P_{r(f)}S_f.$$

If it is not zero, this implies $P_{s(e)}P_{r(f)} \neq 0$ so $s(e) = r(f)$.

(4)

$$S_eS_f^* = S_eP_{s(e)}P_{s(f)}S_f^*.$$

If it is not zero, this implies $P_{s(e)}P_{s(f)} \neq 0$ so $s(e) = s(f)$.

□

Part (3) from Proposition 3.9 tells us that if ef is a path, then S_eS_f is non-zero. One can generalise this result to a path $\mu = e_1e_2\cdots e_n$ and write $S_\mu := S_{e_1}S_{e_2}\cdots S_{e_n}$. If μ is not a path but merely a set of edges, then $S_\mu = 0$ by Proposition 3.9. We also view vertices $v \in E^0$ as a path of length 0 and we write $S_v := P_v$.

COROLLARY 3.10. *Let* E *be a row-finite graph and* $\{S, P\}$ *be a Cuntz-Krieger* E*-family in a* C^**-algebra* B *Let* $\mu, \nu \in E^*$ *be a path.*

(1) If $|\mu| = |\nu|$*, and* $\mu \neq \nu$*, then* $(S_\mu S_\mu^*)(S_\nu S_\nu^*) = 0$*;*

(2) $S_\mu S_\nu^* = \begin{cases} S_{\mu'}^*, & \mu = \nu\mu' \textit{ for some } \nu' \in E^* \\ S_{\nu'}, & \nu = \mu\nu' \textit{ for some} \nu' \in E^* \\ 0, & \textit{otherwise} \end{cases}$*;*

(3) If $S_\mu S_\nu \neq 0$*, then* $\mu\nu$ *is a path and* $S_\mu S_\nu = S_{\mu\nu}$*; and*

(4) If $S_\mu S_\nu^* \neq 0$*, then* $s(\mu) = s(\nu)$*.*

PROOF. See proof at page 10 from [8]. □

CORCOLLARY 3.11. *Let* E *be a row-finite graph and* $\{S, P\}$ *is a Cuntz-Krieger* E*-family in a* C^**-algebra* B*. For paths* $\alpha, \beta, \mu, \nu \in E^*$*,*

$$(S_\mu S_\nu^*)(S_\alpha S_\beta^*) = \begin{cases} S_{\mu\alpha'} S_\beta^*, & \alpha = \nu\alpha' \\ S_\mu S_{\beta\nu'}^*, & \nu = \alpha\nu' \\ 0, & \textit{otherwise} \end{cases}$$

PROOF. See proof at page 10 from [8]. □

Corollary 3.11 tells us that any non-zero finite product of partial isometries can be written as $S_\mu S_\nu^*$ for some paths $\mu, \nu \in E^*$ having the same sources, which gives us the following corollary.

COROLLARY 3.12. *Let* $\{S, P\}$ *be a Cuntz-Krieger* E*-family. Then the* C^**-algebra generated by* $\{S_e, P_v : e \in E^0, v \in E^1\}$ *can be written as*

$$C^*(\{S, P\}) = \overline{\text{span}}\{S_\mu S_\nu^* : \mu, \nu \in E^*, s(\mu) = s(\nu)\}$$

PROOF. See proof at page 10 from [8]. □

PROPOSITION 3.13. *Let* E *be a row-finite graph with no cycles, and let* $\{v_i\}$ *be the set of vertices in* E *such that* v_i *is a source. Then*

$$C^*(S, P) \cong \bigoplus_{i=1}^{n} M_{|\{\mu \in E^* : s(\mu) = v_i\}|}(\mathbb{C}).$$

The above proposition tells you that given a graph E. If it contains no cycles, then the Cuntz-Krieger algebra $C^*(S, P)$ is isomorphic to the direct sum of complex matrices with dimension equal to the number of paths starting at that source.

PROOF. See proof at page 11 from [8]. ☐

3.3 Graph C^*-Algebras

Having established how graphs can be represented by Cuntz-Krieger families of operators, we now construct the universal C^*-algebra associated to a graph. This universality is crucial: while many different C^*-algebras may contain Cuntz-Krieger families satisfying the relations for a given graph E, there exists a unique universal algebra $C^*(E)$ from which all others arise as quotients. This universal property not only provides theoretical elegance but also enables computational techniques, particularly for K-theory, by ensuring that all structural information about graph representations is encoded in a single, canonical object.

PROPOSITION 3.14. *For any row-finite directed graph E, there is a C^*-algebra $C^*(E)$ generated by a Cuntz-Krieger E-family $\{s, p\}$ such that for any Cuntz-Krieger E-family $\{T, Q\}$ in a C^*-algebra A, there is a homomorphism $\pi_{T,Q} : C^*(E) \to A$ such that*

$$\pi_{T,Q}(s_e) = T_e \quad \textit{and} \quad \pi_{T,Q}(p_v) = Q_v$$

for all $e \in E^1$ and $v \in E^0$.

PROOF. The idea is to define a vector space of formal linear combinations that sastisfies the CK E-family, assign a norm based on a ast-representation on a Hilbert space and complete it via the norm topology. See proof at page 13 from [8]. ☐

This C^*-algebra $C^*(E)$ is unique up to isomorphism, as shown in the next corollary:

COROLLARY 3.15 (**Universal property of** $C^*(E)$). *Let E be a graph. Suppose B is a C^*-algebra generated by a Cuntz-Krieger E-family $\{w, r\}$, and for any Cuntz-Krieger E-family*

$\{T, Q\}$ in a C^*-algebra A, there is a homomorphism $\rho_{T,Q} : B \to A$ such that

$$\rho_{T,Q}(w_e) = T_e \quad \textit{and} \quad \rho_{T,Q}(r_v) = Q_v$$

for all $e \in E^1$ and $v \in E^0$. Then there is an isomorphism $\phi : C^*(E) \to B$ such that

$$\phi(s_p) = w_e \quad \textit{and} \quad \phi(p_v) = r_v$$

for all $e \in E^1$ and $v \in E^0$.

The corollary above is a very useful one. In practice, if we wish to find a homomorphism from $C^*(E)$ to a C^*-algebra A, all we need to do is find elements $\{S_e, P_v\}$ in A satisfying the Cuntz-Kreiger E-family, then the universal property above gives a homomorphism that takes s_e to S_e and p_v to P_v.

3.4 Uniqueness Theorems for Graph Algebras

The uniqueness theorems for graph algebras are fundamental results that determine when the universal C^*-algebra $C^*(E)$ can be characterised by specific structural properties of the underlying graph. These theorems, which form the theoretical cornerstone of graph C^*-algebra theory, establish conditions under which homomorphisms from $C^*(E)$ are either injective or surjective, thereby providing powerful tools for understanding the structure of these algebras. The insights gained from these theorems are essential for the K-theory computations that will be central to this thesis, as they enable the identification of graph algebras with more tractable structures. This section follows Chapter 2 of Raeburn's comprehensive treatment [8].

PROPOSITION 3.16. *Let E be a graph and $\{s, p\}$ be the generators of $C^*(E)$. Then there is an action $\gamma : \mathbb{T} \to \mathrm{Aut}(C^*(E))$ such that*

$$\gamma_z(s_e) = zs_e \quad \textit{and} \quad \gamma_z(p_e) = p_e$$

for every $z \in \mathbb{T}$, $e \in E^1$ and E^0.

PROOF. Fix $z \in \mathbb{T}$. Then for each $e \in E^1$ and $v \in E^0$,

$$p_v = \sum s_e s_e^* = \sum z\overline{z} s_e s_e^* = \sum (zs_e)(zs_e)^*$$

and

$$p_{s(e)} = s_e^* s_e = z\overline{z} s_e^* s_e = (zs_e)^*(zs_e).$$

Therefore, for any fixed $z \in \mathbb{T}$, $\{zs, p\}$ is a Cuntz-Krieger E-family in $C^*(E)$. To show that $\gamma_z \in \operatorname{Aut}(C^*(E))$, we need to show that γ_z is a isomorphism, which is easy. For any $z \in \mathbb{T}$, $\overline{z} \in \mathbb{T}$ and so $\{\overline{z}s_e, p_v\}$ is a Cuntz-Krieger E-family in $C^*(E)$.

$$\gamma_z \circ \gamma_{\overline{z}}(s_e) = \gamma_z(\overline{z}s_e) = z\overline{z}s_e = s_e = \overline{z}zs_e = \gamma_{\overline{z}} \circ \gamma_z(s_e)$$

and

$$\gamma_z\gamma_{\overline{z}}(p_v) = p_v.$$

□

γ is called the **gauge action** of $\mathbb{T}$ on $C^*(E)$, it is used extensively in later chapter for calculation.

THEOREM 3.17 (**The Gauge-Invariant Uniqueness Theorem**). *Let E be a row-finite directed graph. Let $\{T, Q\}$ be a Cuntz-Krieger E-family in a C^*-algebra B where each projections $Q_v \neq 0$. If there is a continuous action $\beta : \mathbb{T} \to \operatorname{Aut} B$ such that*

$$\beta_z(T_e) = zT_e \quad \textit{and} \quad \beta_z(Q_v) = Q_v$$

for every $e \in E^1$ and $v \in E^0$. Then $\pi_{T,Q} : C^*(E) \to C^*(T, Q)$ is isomorphic.

PROOF. Refer to [8, Chapter 3] for proof. □

This theorem tells us that if $\{T, Q\}$ is a Cuntz-Krieger E-family and there is a gauge action β such that $\beta_z(T_e) = zT_e$ and $\beta_z(Q_v) = Q_v$, then $\{T, Q\}$ generates a C^*-algebra isomorphic to $C^*(E)$.

DEFINITION 3.18. *Let E be a graph and let $\mu \in E^*$ be a cycle in E. An edge e is said to be an* ***entry*** *to the cycle μ if there exists i such that $r(e) = r(\mu_i)$ and $e \neq \mu_i$.*

THEOREM 3.19 (**The Cuntz-Krieger Uniqueness Theorem**). *Let E be a row-finite directed graph. If every cycles has an entry and $\{T, Q\}$ is a Cuntz-Krieger E-family in a C^*-algebra B where every Q_v is non-zero, then the homomorphism $\pi_{T,Q} : C^*(E) \to B$ is an isomorphism.*

PROOF. Refer to [8, Chapter 3] for proof. □

The Cuntz-Krieger uniqueness theorem shows that if there is an entry to every cycles in the graph E, then the existence of the gauge action is unnecessary and any Cuntz-Krieger E-family with nonzero P_v generates isomorphic C^*-algebras. The proof of the two uniqueness theorems are not elementary and takes a whole chapter. To see the full proof, visit chapter 3 from [8].

The following definition is essential for the next chapters, but we will state it here. Namely, we construct a new graph $\hat{E}$ based on an existing graph E, and state the property of its graph algbera.

DEFINITION 3.20. *Let E be a row-finite graph with no sources. The dual graph $\hat{E}$ has vertex set $\hat{E}^0 = E^1$, edge set $\hat{E}^1 = E^2$ and the maps $s_{\hat{E}}, r_{\hat{E}} : \hat{E}^1 \to \hat{E}^0$ is defined as follows:*

$$s_{\hat{E}}(ef) = f \quad \textit{and} \quad r_{\hat{E}}(ef) = e.$$

THEOREM 3.21. *Suppose E is a row-finite graph with no sources, and let $\hat{E}$ be its dual graph. Then the graph algebra $C^*(\hat{E})$ is isomorphic to $C^*(E)$.*

PROOF. Refer to [8, Corollary 2.6] for proof. □

CHAPTER 4

Skew-Product Graphs and their C^*-Algebras

Skew-product graphs represent a fundamental construction that bridges combinatorial graph theory with operator algebras, providing essential tools for this thesis's approach to computing K-theory of graph algebras and, ultimately, graph of groups algebras. Originally developed in combinatorial graph theory by Gross and Tucker [1] as "voltage graphs" for constructing graph coverings, this concept was later connected to operator algebras through the fundamental work of Kaliszewski, Quigg and Raeburn [2]. The construction involves creating a new graph from an original graph using a group-valued labelling function on the edges, resulting in a larger graph that encodes both the original combinatorial structure and the group action. The significance for this thesis lies in the fundamental connection between skew-product graph algebras and crossed products by coactions, which enables the application of powerful K-theoretic techniques through exact sequence methods. This chapter, following Raeburn's comprehensive treatment [8], establishes the theoretical foundation that will be extended to the graph of groups setting in Chapter 6.

4.1 Skew-Product Graphs

The construction of a skew-product graph begins with a labelling of the edges of the original graph by elements of a discrete group. This labelling function is crucial as it determines how the new graph's structure encodes the group action, controlling how edges in the expanded graph connect vertices. The resulting construction produces a graph that is typically larger than the original but retains essential structural properties while gaining new features that make it amenable to K-theory computations through its connection to crossed products.

DEFINITION 4.1. *Let E be a row-finite graph, G be a discrete group, and $c : E^1 \to G$ be a labelling function on the edges of E. The* ***skew-product graph*** *$E \times_c G$ is built upon on E by setting the edge and vertex set as follows:*

$$(E \times_c G)^0 = E^0 \times G \quad \textit{and} \quad (E \times_c G)^1 = E^1 \times G$$

The source and range map is defined as

$$s(e, g) = (s(e), g) \quad \textit{and} \quad r(e, g) = (r(e), c(e)g)$$

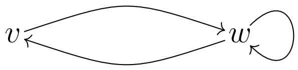
Let E be a row-finite graph without sources. Define the labelling function $c : E^1 \to \mathbb{Z}$ to be the function that takes any edges to -1. That is, for any $e \in E^1$, $c(e) = -1$. Then the skew-product graph $E \times_c \mathbb{Z}$ with the above labelling function is defined as follows:

$$(E \times_c \mathbb{Z})^0 = E^0 \times \mathbb{Z} \quad \text{and} \quad (E \times_c \mathbb{Z})^1 = E^1 \times \mathbb{Z}.$$

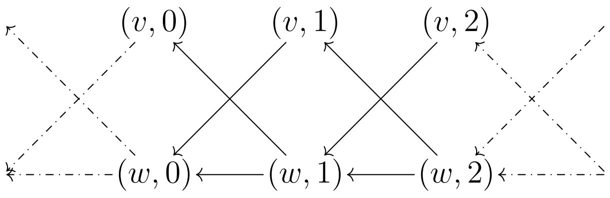
The source map and range map is then

$$s(e, n) = (s(e), n) \quad \text{and} \quad r(e, n) = (r(e), n - 1).$$

EXAMPLE 4.2. *Let E be a directed graph with the following structure:*



Let $G = \mathbb{Z}$. Define the labelling $c : E^1 \to \mathbb{Z}$ by $c(e) = -1$ for all $e \in E^1$. Then the skew-product graph $E \times_c \mathbb{Z}$ has the following structure:



Since $\mathbb{Z}$ *is an infinite group, so is* $E \times_c \mathbb{Z}$*. Turns out this labelling is special, and will be shown in an example in the future.*

4.2 Skew-Product Graph Algebras and Crossed Products by Coactions

This section establishes the fundamental connection between skew-product graph algebras and crossed products by coactions, which forms the theoretical heart of our approach to computing K-theory for graph C^*-algebras. The key insight is that every edge labelling on a graph induces a coaction on the corresponding graph C^*-algebra, and the resulting crossed product by this coaction is isomorphic to the skew-product graph algebra. This connection is crucial for K-theory computations because crossed products often have more tractable K-groups than the original algebras, particularly when exact sequence techniques can be applied. The isomorphism established here enables the use of the dual Pimsner-Voiculescu sequence and related tools for computing K-theory, providing the foundation for the techniques that will be extended to graph of groups in Chapter 6.

PROPOSITION 4.3. *Let* E *be a directed graph,* G *be a discrete group and let* $c : E^1 \to G$ *be a labelling. Then* c *induces a coaction* $\delta_c : C^*(E) \to C^*(E) \otimes C^*(G)$ *such that*

$$\delta_c(s_e) = s_e \otimes u_{c(e)} \quad \textit{and} \quad \delta_c(p_v) = p_v \otimes 1$$

for every $e \in E^1$ *and* $v \in E^0$*.*

PROOF. For $C^*(E) \otimes C^*(G)$, define $S_e := s_e \otimes u_c(e)$ and $P_v := p_v \otimes 1$. Then

(1) $S_e^* S_e = (s_e^* \otimes u_{c(e)}^*)(s_e \otimes u_{c(e)}) = s_e^* s_e \otimes u_1 = p_{s(e)} \otimes 1 = P_{s(e)}$; and

(2)

$$\begin{aligned}\sum_{r(e)=v} S_e S_e^* &= \sum_{r(e)=v} (s_e \otimes u_{c(e)})(s_e^* \otimes u_{c(e)}^*) = \sum_{r(e)=v} (s_e s_e^* \otimes 1) \\ &= (\sum_{r(e)=v} s_e s_e^*) \otimes 1 = p_v \otimes 1 = P_v.\end{aligned}$$

By the universal property of graph algebras, there exists a homomorphism $\delta_c : C^*(E) \to C^*(E) \otimes C^*(G)$ by $s_e \mapsto s_e \otimes u_{c(e)}$ and $p_v \mapsto p_v \otimes 1$.

It suffice to check the generators s_e and p_v on the coaction identity:

$$\begin{aligned}(\delta_c \otimes \iota) \circ \delta_c(s_e) &= (\delta_c \otimes \iota)(s_e \otimes u_{c(e)}) = s_e \otimes u_{c(e)} \otimes u_{c(e)} = (\iota \otimes \delta_G)(s_e \otimes u_{c(e)}) \\ &= (\iota \otimes \delta_G) \circ \delta_c(s_e), \text{ and} \\ (\delta_c \otimes \iota) \circ \delta_c(p_v) &= (\delta_c \otimes \iota)(p_v \otimes 1) = p_v \otimes 1 \otimes 1 = (\iota \otimes \delta_G)(p_v \otimes 1) \\ &= (\iota \otimes \delta_G) \circ \delta_c(p_v).\end{aligned}$$

□

The following result represents the cornerstone of the skew-product methodology for graph C^*-algebras. This theorem establishes the fundamental isomorphism that enables K-theory computations by connecting skew-product graph algebras to crossed products by coactions, thereby opening the door to exact sequence techniques.

PROPOSITION 4.4. *Let E be a row-finite directed graph. Let $c : E^1 \to G$ be a labelling function. Then there is an induced coaction δ_c of G on $C^*(E)$ such that the skew-product algebra $C^*(E \times_c G)$ is isomorphic to the crossed product $C^*(E) \times_{\delta_c} G$. In addition, this isomorphism converts the dual action of G into the action β defined on the Cuntz-Krieger E-family by*

$$\beta_s(s_{(e,t)}) = s_{(e,ts^{-1})} \quad \textit{and} \quad \beta_s(p_{(e,t)}) = p_{(e,ts^{-1})}.$$

PROOF. The overview of the proof involves defining the collection of elements $\{t, q\}$ in $C^*(E) \rtimes_{\delta_c} G$ by

$$t_{(f,t)} := j_{C^*(E)}(s_f) j_{\mathbb{T}}(\chi_t) \quad \text{and} \quad q_{(v,t)} := j_{C^*(E)}(p_v) j_{\mathbb{T}}(\chi_t).$$

Then it can be shown that they satisfy the Cuntz-Krieger E-family and thus a homomorphism $\pi : C^*(E \times_c G) \to C^*(E) \rtimes_{\delta_c} G$ exists, where it takes $s_{(f,t)}$ to $t_{(f,t)}$ and $p_{(v,t)}$ to $q_{(v,t)}$. Since there is a continuous action $\beta : \mathbb{T} \to \mathrm{Aut}(C^*(E) \rtimes_{\delta_c} G)$ by $\beta_z(t_{(f,t)}) = z t_{(f,t)}$ and

$\beta_z(q_{(v,t)}) = q_{(v,t)}$, The gauge-invariant uniqueness theorem says π is an isomorphism of $C^*(E \times_c G)$ onto $C^*(t,q) = C^*(E) \rtimes_{\delta_c} G$. Details can be found at [8] as proposition 6.7. $\square$

LEMMA 4.5. *Let E be a directed graph, and let c be the labelling as shown in exmaple 4.2. That is, $c : G^1 \to \mathbb{Z}$ by $c(e) = -1$ for all $e \in E^1$. Then there is an isomorphism $\phi : C^*(E \times_c \mathbb{Z}) \to C^*(E) \rtimes_\gamma \mathbb{T}$, such that*

$$\phi \circ \beta_m = \hat{\gamma}_m \circ \phi,$$

where $\beta : G \to \mathrm{Aut}(C^*(E))$ is the action from proposition 4.4.

PROOF. For a graph algebra $C^*(E)$, we have the gauge action $\gamma : \mathbb{T} \to \mathrm{Aut}(C^*(E))$. Notice that the dual group of $\mathbb{T}$ is $\mathbb{Z}$. By example 2.21, there is coaction δ of $\hat{\mathbb{T}} = \mathbb{Z}$ on $C^*(E)$. This coaction $\delta = \delta_c$. And so $C^*(E) \rtimes_\gamma \mathbb{T} \cong C^(E) \rtimes_{\delta_c} \mathbb{Z}$.

Recall from Proposition 4.4, there is an isomorphism between $C^*(E \times_c \mathbb{Z})$ and $C^*(E) \rtimes_{\delta_c} \mathbb{Z}$. So $C^*(E \times_c \mathbb{Z}) \cong C^*(E) \rtimes_\gamma \mathbb{T}$. The identification of this crossed product with the AF-algebra follows from Lemma 7.10 of [8], which shows that crossed products by gauge actions yield AF-algebras. $\square$

CHAPTER 5

Graphs of Groups and their C^*-Algebras

This chapter introduces graphs of groups, a rich mathematical structure that significantly generalises ordinary graphs by incorporating group-theoretic data at each vertex and edge. Originally developed by Bass [3] and Serre [4] in their foundational work on group actions on trees and covering theory, graphs of groups provide a powerful framework for understanding the relationship between combinatorial and algebraic structures. The recent work of Brownlowe, Mundey, Pask, Spielberg, and Thomas [10] successfully extended the theory of graph C^*-algebras to this setting, assigning a universal C^*-algebra to each graph of groups. These C^*-algebras form the primary focus of this thesis, as we develop new techniques for computing their K-theory through novel skew-product constructions that adapt and extend the successful methods from ordinary graph C^*-algebras.

5.1 Graphs of Groups

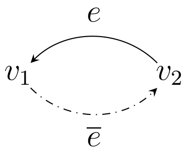
The definition of graphs of groups requires extending the notion of directed graphs to include reverse edges, creating a framework where each edge can be traversed in both directions. This bidirectional structure is essential for encoding the group-theoretic information that distinguishes graphs of groups from ordinary directed graphs. The construction builds upon the Bass-Serre theory, which demonstrates how combinatorial and algebraic structures can be elegantly combined to study fundamental groups and group actions on trees.

For any edge e in a graph, we define the reverse edge $\overline{e} \neq e$ such that

(1) $r(\overline{e}) = s(e)$;

(2) $s(\overline{e}) = r(e)$; and
(3) $\overline{\overline{e}} = e$.

This is better illustrated with a graph:



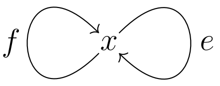
(3) implies the reverse of the reverse of an edge e is the e itself. The reverse edges are often not shown in a graph since it makes the graph messy. We remind ourselves that such edges exist.

The definition of a reverse edge is extended to paths. In particular, for any paths $p = e_1e_2 \cdots e_n$, the reverse of p is $\overline{p} = \overline{e}_n\, \overline{e}_{n-1} \cdots \overline{e_1}$.

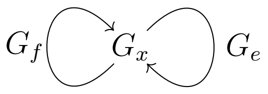
DEFINITION 5.1. *A* ***graph of groups*** $\mathscr{G}$ *is a pair* (Γ, G) *containing*

(1) a connected graph Γ*;*
(2) a collection of ***vertex groups*** G_x *for every* $x \in \Gamma^0$*;*
(3) a collection of ***edge groups*** G_e *for every* $e \in \Gamma^1$ *such that* $G_e = G_{\overline{e}}$*; and*
(4) an injective homomorphism $\alpha_e : G_e \to G_{r(e)}$ *for every* $e \in \Gamma^1$.

To make it clearer, a graph of groups is a graph where each vertex and edge is a group and for each edge there is an injective homomorphism from the edge group to the vertex group of its range. Here is an example: For a graph as follow:



A graph of group with the above graph would be like this:



Recall that we have the reverse edge hidden in the graph. So for any edge e, there is also an injective homomorphism $\alpha_{\overline{e}} : G_{\overline{e}} = G_e \to G_{r(\overline{e})} = G_{s(e)}$. One notice that since there are more than one group in a graph of groups, we need to be extra careful about any elements mentioned in any groups. For any vertices $x \in \Gamma^0$ or edges $e \in \Gamma^1$, we write 1_x or 1_e for the identity elements of that vertex or edge group respectively. For special instance, we might even write 1 for all the identities.

Recall for any subgroup $H \subseteq G$, the index $[G : H]$ of H is the number of cosets of H in G. A graph of group $\mathscr{G}$ is **locally finite** if

(1) the underlying graph is row-finite; and
(2) the subgroup index $[G_{r(e)} : \alpha_e(G_e)] < \infty$ for each edge $e \in E^1$.

It is called **nonsingular** if for any edges $e \in \Gamma^1$ such that $r^{-1}(r(e)) = \{e\}$, $[G_{r(e)} : \alpha_e(G_e)] = 1$. This means that for any edge e that has unique range $r(e)$, $\alpha_e(G_e)$ is a proper subgroup of $G_{r(e)}$. This two definitions are stated becuase only nonsingular locally finite graph of groups are considered in this whole thesis.

Recall a transversal for a subgroup $H \subset G$ is a set $S \subset G$ such that every cosets of H contains exactly one element of S. For example, let $G = \mathbb{Z}$ and $H = 5\mathbb{Z}$, then the set $S = \{0, 1, 2, 3, 4\}$ is a transversal of H. One notice that S is not the only transversal of H, as one can pick $S' = \{5, 6, 7, 8, 9\}$ and it still satisfies the criteria.

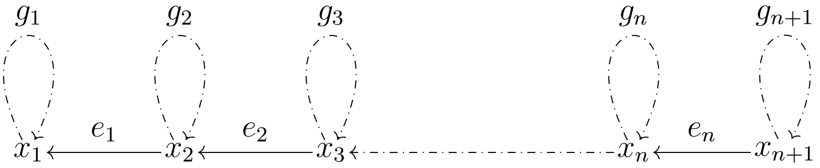
DEFINITION 5.2. *($\mathscr{G}$-words, $\mathscr{G}$-paths) Let $\mathscr{G} = (\Gamma, G)$ be a graph of group, and fix a transversal $\sum_e$ for each $e \in E^1$ such that $1_e \in \sum_e$.*

(1) A $\mathscr{G}$-word is a string of the form

$$g_1 \text{ or } g_1e_1g_2e_2\cdots g_ne_n \text{ or } g_1e_1g_2e_2\cdots g_ne_ng_{n+1},$$

where

- *(a) $s(e_i) = r(e_{i+1})$ for all $1 \le i \le n-1$. That is, $e_1e_2\cdots e_n$ is a path in Γ;*
- *(b) $g_j \in G_{r(e_j)}$ for all $1 \le i \le n$ and $g_{n+1} \in G_{s(e_n)}$. That is, the group elements belong to the vertex group that is either the range of following edge or the source of the edge that the group element is behind. This is better visualized with a graph:*



- *(c) The length of the $\mathscr{G}$-word is the length of the path $\mu = e_1e_2\cdots e_n$, denoted $|\mu|$. If the $\mathscr{G}$-word is a single vertex group element, then its length is 0.*

(2) A reduced $\mathscr{G}$-word is a $\mathscr{G}$-word such that

- *(a) $g_j \in \Gamma_{e_j}$ for all $1 \le j \le n$. That is, any group element belongs to the transversal.*
- *(b) if $e_i = \overline{e}_{i+1}$ for any $1 \le i \le n$, then $g_i + 1 \ne 1_{r(e_{i+1})}$. That is, no combinations of either $e_i 1 \overline{e}_i$ or $\overline{e}_i 1 e_i$ should be in a reduced $\mathscr{G}$-word.*
- *(c) The source and range of the $\mathscr{G}$-word is the source and range of the underlying path.*

(3) A $\mathscr{G}$-path is a reduced $\mathscr{G}$-word of form

$$g_1 = 1_{G_x} \text{ or } g_1e_1g_2e_2\cdots g_ne_n$$

for any $x \in E^0$. That is, a $\mathscr{G}$-path of length > 0 should always end with edges (e_n in the above form) and start with group elements (g_1 in the above form). In addition,

- *(a) The set of $\mathscr{G}$-paths of length n is denoted $\mathscr{G}^n$;*
- *(b) The set of all $\mathscr{G}$-paths is denoted $\mathscr{G}^*$; and*
- *(c) The set of $\mathscr{G}$-paths with range x is denoted $x\mathscr{G}^*$ and the set of $\mathscr{G}$-paths with source y is denoted $\mathscr{G}^*y$.*

5.2 Graph of Groups Algebras

The construction of C^*-algebras associated with graphs of groups represents a significant generalisation of graph C^*-algebra theory, successfully extending the Cuntz-Krieger approach to accommodate the rich group-theoretic structure inherent in graphs of groups. The C^*-algebra $C^*(\mathscr{G})$ associated with a graph of groups $\mathscr{G}$, introduced by Brownlowe, Mundey, Pask, Spielberg, and Thomas [10], is defined as the universal C^*-algebra generated by a $\mathscr{G}$-family of partial isometries and partial unitaries. This construction is analogous to graph algebras but requires additional complexity to handle the group structures at vertices and edges. Crucially for this thesis, there exists a natural connection between graph of groups algebras and ordinary graph algebras: for any graph of groups $\mathscr{G}$, there is an associated directed graph $E_{\mathscr{G}}$ such that $C^*(E_{\mathscr{G}})$ embeds naturally into $C^*(\mathscr{G})$, providing a bridge between the two theories that will be essential for developing our skew-product constructions.

DEFINITION 5.3. *A* ***partial unitary*** *in a C^*-algebra A is a partial isometry U such that $U^*U = UU^*$.*

Note that a partial unitary becomes a unitary operator at a specified subspace of a represented Hilbert space. That is, let $\phi : A \to \mathcal{B}(\mathscr{H})$ be a representation of A on $\mathcal{B}(\mathscr{H})$. $\phi(UU^*) = \phi(U^*U) = 1_{\mathcal{B}(\mathscr{K})}$ for a some subspace $\mathscr{K} \subset \mathscr{H}$.

DEFINITION 5.4. *For each $e \in \Gamma^1$, choose a transversal Γ_e for each subgroup $\alpha_e(G_e)$ of $G_{r(e)}$ that contains the identity element 1_e.*

A $(\mathscr{G}, \sum)$-family is a collection of partial isometries S_e for each $e \in \Gamma^1$ and $*$-representations $g \mapsto U_{x,g}$ of G_x by partial unitaries for each $x \in \Gamma^0$ such that

(1) $U_{x,1_x}U_{y,1_y} = 0$ *for every* $x, y \in \Gamma^0$ *such that* $x \neq y$*;*
(2) $U_{r(e),\alpha_e(g)}S_e = S_eU_{s(e),\alpha_{\overline{e}}(g)}$ *for every* $e \in \Gamma^1, g \in G_e$*;*
(3) $U_{s(e),1_{s(e)}} = S_e^*S_e + S_{\overline{e}}S_{\overline{e}}^*$ *for every* $e \in \Gamma^1$*; and*
(4)

$$S_e^*S_e = \sum_{r(f)=s(e),h\in\Gamma_f,hf\neq 1\overline{e}} (U_{s(e),h}S_f)(U_{s(e),h}S_f)^*$$

for every $e \in \Gamma^1$.

Notice that for any $x \in \Gamma^0$, $U^2_{x,1_x} = U_{x,1_x^2} = U_{x,1_x}$ and $U^*_{x,1_x} = U_{x,1_x^{-1}} = U_{x,1_x}$. So with (1), $\{U_{x,1_x} : x \in \Gamma^0\}$ is a set of mutually orthogonal projections. From (4), we see that for each $x \in \Gamma^0$, the collection $\{U_{x,h}S_f : hf \in x\mathscr{G}^1\}$ is a partial isometry and their final projections are mutually orthogonal. From (3), We see that the projection $U_{s(e),1}$ contains the initial projection $S^*_eS_e$ of S_e and the final projection $S_{\overline{e}}S^*_{\overline{e}}$ of $S_{\overline{e}}$ as subprojections, and so they are orthogonal (since $U_{s(e),1}$ is a projection), which means that $S_eU_{s(e),1} = S_e$ and $U_{s(e),1}S_{\overline{e}} = S_{\overline{e}}$. Replacing $\overline{e}$ with e leads to $U_{r(e),1}S_e = U_{s(\overline{e}),1}S_e = S_e$. Combing (3) and (4) and the above, we get

$$\begin{aligned}
U_{s(e),1} &= S^*_eS_e + S_{\overline{e}}S^*_{\overline{e}} \\
&= \sum_{r(f)=s(e),h\in\Gamma_f,hf\neq 1\overline{e}} (U_{s(e),h}S_f)(U_{s(e),h}S_f)^* + (U_{s(e),1}S_{\overline{e}})(U_{s(e),1}S_{\overline{e}})^* \\
&= \sum_{r(f)=s(e),h\in\Gamma_f} (U_{s(e),h}S_f)(U_{s(e),h}S_f)^*.
\end{aligned}$$

So for each $e \in \Gamma^1$, its projection $U_{s(e),1}$ is the sum of all final projections of $\{U_{s(e),h}S_f : hf \in s(e)\mathscr{G}^1\}$.

By (3),

$$\begin{aligned}
& U_{s(e),1_{s(e)}} = S^*_eS_e + S_{\overline{e}}S^*_{\overline{e}} \\
\Longrightarrow\quad & U_{s(e),1_{s(e)}}S_{\overline{e}} = S^*_eS_eS_{\overline{e}} + S_{\overline{e}}S^*_{\overline{e}}S_{\overline{e}} \\
\Longrightarrow\quad & S_{\overline{e}} = S^*_eS_eS_{\overline{e}} + S_{\overline{e}} \\
\Longrightarrow\quad & S^*_eS_eS_{\overline{e}} = 0 \\
\Longrightarrow\quad & S_eS^*_eS_eS_{\overline{e}} = 0 \\
\Longrightarrow\quad & S_eS_{\overline{e}} = 0.
\end{aligned}$$

This is where the the graph of groups relation differ from the the Cuntz-Krieger relation, since $e\overline{e}$ is a path in Γ, so the corresponding $S_{e\overline{e}}$ non-zero in a Cuntz-Krieger family. One also notice that the choice of transversal does not matter. To see this, for each $e \in \Gamma^1$, let

Γ'_e be another transversal containing the identity but differing from Γ_e. Let $f \in \Gamma^1$ such that $r(f) = s(e)$. For each $h' \in \Gamma_f$, we can write $h' = h\alpha_f(g)$ for some $h \in \Gamma_f$ and $h \in \Gamma_f$. Then

$$\begin{aligned}
(U_{s(e),h'}S_f)(U_{s(e),h'}S_f)^* &= U_{s(e),h'}S_fS_f^*U^*_{s(e),h'} \\
&= U_{s(e),h\alpha_f(g)}S_fS_f^*U^*_{s(e),h\alpha_f(g)} \\
&= U_{s(e),h}U_{s(e),\alpha_f(g)}S_fS_f^*U^*_{s(e),\alpha_f(g)}U^*_{s(e),h} \\
&= U_{s(e),h}S_fU_{s(f),\alpha_{\overline{f}}(g)}U^*_{s(f),\alpha_{\overline{f}}(g)}S_f^*U^*_{s(e),h} \\
&= U_{s(e),h}S_fU_{s(f),\alpha_{\overline{f}}(1)}S_f^*U^*_{s(e),h} \\
&= U_{s(e),h}S_fU_{s(f),1}S_f^*U^*_{s(e),h} \\
&= U_{s(e),h}S_fS_f^*U^*_{s(e),h} \\
&= (U_{s(e),h}S_f)(U_{s(e),h}S_f)^*.
\end{aligned}$$

Given the fact that the choice of transversal does not matter, we don't need to specify which set of transversal we are using, and so we can remove $\sum$ from the definition 5.4, and the $(\mathscr{G}, \sum)$-family is named a $\mathscr{G}$-family instead.

DEFINITION 5.5. *Let $\mathscr{G}$ be a locally finite nonsingular graph of groups. The C^***-algebra of graph of groups**, denoted $C^*(\mathscr{G})$, is the C^*-algebra generated by the $\mathscr{G}$-family $\{u_x, s_e : x \in \Gamma^0, e \in \Gamma^1\}$.*

This $\mathscr{G}$-family is universal: Let B be a C^*-algebra. If there is a $\mathscr{G}$-family $\{U_x, S_e : x \in \Gamma^0, e \in \Gamma^1\}$ in B, there there is a unique $*$-homomorphism $\phi : C^*(\mathscr{G}) \to B$ such that $\phi(u_{x,g}) = U_{x,g}$ for every $g \in G_x, x \in \Gamma^0$ and $\phi(s_e) = S_e$ for every $e \in \Gamma^1$.

Given a graph of group $\mathscr{G}$, one would be curious about the relationships between graph of groups algebras and graph algebras. The following defines a directed graph associated to a graph of groups.

DEFINITION 5.6. *let $\mathscr{G}$ be a locally finite nonsingular graph of groups. Define a graph $E_{\mathscr{G}}$ as follows:*

$$E_{\mathscr{G}} := (E^0_{\mathscr{G}} := \mathscr{G}^1, E^1_{\mathscr{G}} := \mathscr{G}^2, r_{E_{\mathscr{G}}}, s_{E_{\mathscr{G}}})$$

where $r_{E_{\mathscr{G}}}, s_{E_{\mathscr{G}}} : \mathscr{G}^2 \to \mathscr{G}^1$ *by*

$$r_{E_{\mathscr{G}}}(g_1e_1g_2e_2) = g_1e_1 \quad \textit{and} \quad s_{E_{\mathscr{G}}}(g_1e_1g_2e_2) = g_2e_2.$$

This graph treats each path in $\mathscr{G}$ with length 1 as vertices and each path in $\mathscr{G}$ with length 2 as edges. Notice that for any $g_1e_1 \in \mathscr{G}^1$, there are $\mathscr{G}$-paths $g_1e_1g_2\overline{e}_1$ where $1_{s(e)} \neq g_2 \in G_{s(e)}$. These paths all have range g_1e_1 so $E_{\mathscr{G}}$ is a graph with no source.

DEFINITION 5.7. *Suppose* $\mu = g_1e_1g_2e_2 \cdots g_ne_n$ *is a* $\mathscr{G}$*-path. Define*

$$S_\mu := U_{r(e_1),g_1}S_{e_1}U_{r(e_2),g_2}S_{e_2} \cdots U_{r(e_n),g_n}S_{e_n}$$

LEMMA 5.8. *Let* T, S *be partial isometries such that* TT^* *is a subprojection of* S^*S. *Then* ST *is also a partial isometry.*

PROOF. If TT^* is a subprojection of S^*S, then $TT^*S^*S = TT^* = S^*STT^*$. In addition,

$$(STT^*S^*)^* = STT^*S^*, and$$

$$(STT^*S^*)^2 = STT^*S^*STT^*S^* = STT^*TT^*S^* = STT^*S^*.$$

so STT^*S^* is a projection. Similar calcluations can be done on $(ST)^*(ST) = T^*S^*ST$ so $(ST)^*(ST)$ is also a projeciton. □

For each $1 < i \leq n$,

$$(U_{r(e_i),g_i}S_{e_i})^*(U_{r(e_i),g_i}S_{e_i}) = S_{e_i}^*U_{r(e_i),g_i}^*U_{r(e_i),g_i}S_{e_i} = S_{e_i}^*U_{r(e_i),1}S_{e_i} = S_{e_i}^*S_{e_i}.$$

Thus, from (4), we see that the final projection of $U_{r(e_i),g_i}S_{e_i}$ is a subprojection of the initial projection of $U_{r(e_{i-1}),g_{i-1}}S_{e_{i-1}}$, and by Lemma 5.8 S_μ becomes a partial isometry.

THEOREM 5.9. *Suppose that* $\mathscr{G}$ *is a locally finite nonsingular graph of groups and let* $E_{\mathscr{G}}$ *be the graph defined as 5.6. Let* $\{q, t\}$ *be the Cuntz-Krieger* $E_{\mathscr{G}}$*-family generating* $C^*(E_{\mathscr{G}})$. *Then there is an embedding* $\phi : C^*(E_{\mathscr{G}}) \to C^*(\mathscr{G})$ *such that*

$$\phi(q_\nu) = s_\nu s_\nu^* \quad \textit{and} \quad \phi(t_\mu) = s_\mu s_{s_{E_{\mathscr{G}}}(\mu)}^*$$

for every $\nu \in E^0_{\mathscr{G}}, \mu \in E^1_{\mathscr{G}}$. *That is,* ϕ *is an injective homomorphism, and the image* $\phi(C^*(E_{\mathscr{G}}))$ *has a Cuntz-Krieger* $E_{\mathscr{G}}$*-family structure and is generated by* $s_\nu s^*_\nu$ *and* $s_\mu s^*_{s_{E_{\mathscr{G}}}(\mu)}$.

In addition, if the edge group G_e *is trivial for every* $e \in \Gamma^1$, *then the map* ϕ *becomes an isomorphism.*

PROOF. Refer to [10, Theorem 3.6] for proof. □

A further calculation shows that the inverse $\phi^{-1} : C^*(\mathscr{G}) \to C^*(E_{\mathscr{G}})$ is defined as below:

$$\phi^{-1}(s_e) = \sum_{\substack{\mu \in E_{\mathscr{G}} \\ r_E(\mu) = 1e}} t_\mu \quad \text{and} \quad \phi^{-1}(u_{x,g}) = \sum_{\substack{\mu,\nu \in E^1_{\mathscr{G}} \\ s_E(\mu) = s_E(\nu) \\ r_E(\nu) = hf \\ r_E(\mu) = (gh)f \\ r(f) = x}} t_\mu t^*_\nu$$

LEMMA 5.10. *Suppose that* $\mathscr{G}$ *is a nonsingular locally finite graph of countable groups and* $E_{\mathscr{G}}$ *is the associated directed graph. Then*

$$C^*(E_{\mathscr{G}}) = \overline{\operatorname{span}}\{s_\mu s^*_\nu : \mu, \nu \in \mathscr{G}^*, s(\mu) = s(\nu)\}, \textit{and}$$

$$C^*(\mathscr{G}) = \overline{\operatorname{span}}\{s_\mu u_{s(\mu),g} s^*_\nu : \mu, \nu \in \mathscr{G}^*, s(\mu) = s(\nu), g \in G_{s(\mu)}\}.$$

PROOF. The proof can be found in [10, Lemma 3.11]. □

CHAPTER 6

New Construction For Skew-Product Graphs of Groups and their C^*-Algebras

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This chapter presents the central original contribution of this thesis: the introduction and development of skew-product graphs of groups and their associated C^*-algebras. Building upon the successful skew-product methodology for ordinary graphs established in Chapter 4, we extend this construction to accommodate the rich group-theoretic structure inherent in graphs of groups. This extension requires careful attention to the algebraic constraints imposed by the vertex and edge groups, necessitating a cocycle condition on the labelling function that ensures the resulting structure remains well-defined. The main theoretical achievement of this chapter is establishing an analogue of the fundamental theorem connecting skew-product graph algebras to crossed products by coactions, thereby enabling the application of K-theoretic techniques to graph of groups C^*-algebras. This work provides new computational tools that address the previously intractable problem of computing K-theory for these algebras.

6.1 Skew-Product Graphs of Groups

The construction of skew-product graphs of groups must account for the fundamental difference between ordinary directed graphs and the underlying graphs of graphs of groups. In a graph of groups $\mathscr{G} = (\Gamma, G)$, each edge $e \in \Gamma$ has a corresponding reverse edge $\overline{e}$ with opposite range and source, making Γ effectively an undirected graph. This bidirectional structure requires a more sophisticated approach to the skew-product construction than the directed case, as we must ensure that the group labelling respects the involutive structure of edge reversal. The key insight is that the labelling function must satisfy a cocycle condition that ensures compatibility with the reverse edge operation.

DEFINITION 6.1. *An* ***undirected graph*** *is a graph* $\Gamma = (\Gamma^0, \Gamma^1, r, s, b)$ *such that* $b : \Gamma^1 \to \Gamma^1$ *is a map satisfying*

(1) $s(b(e)) = r(e)$*;*
(2) $r(b(e)) = s(e)$*;*
(3) $b(e) \neq e$*; and*
(4) $b^2(e) = e$*.*

for each $e \in \Gamma$*.*

The map $b : \Gamma^1 \to \Gamma^1$ refers to the "backtracking" of an edge, so $b(e) = \overline{e}$ for all edges $e \in \Gamma^1$. For each undirected graph Γ and each discrete group G, we can construct a new undirected graph $\Gamma \times_c G$.

DEFINITION 6.2. *Let* $\Gamma = (\Gamma^0, \Gamma^1, r, s, b)$ *be an undirected graph,* G *be a discrete group and* $c : \Gamma^1 \to G$ *be a labelling with the cocyle property:*

$$c(b(e)) = c(e)^{-1} \text{ (or } c(\overline{e}) = c(e)^{-1}\text{).}$$

The skew-product undirected graph*, denoted* $\Gamma \times_c G = ((\Gamma \times_c G)^0, (\Gamma \times_c G)^1, r_c, s_c, b_c)$*, is a graph with* $(\Gamma \times_c G)^0 = \Gamma^0 \times G$*,* $(\Gamma \times_c G)^1 = \Gamma^1 \times G$*,* $r_c(e, g) = (r(e), c(e)g)$*,* $s_c(e, g) = (s(e), g)$ *and* $b_c(e, g) = (b(e), c(e)g)$ *(or* $\overline{(e, g)} = (\overline{e}, c(e)g)$*) for every* $e \in \Gamma, g \in G$*.*

Notice that the requirement of c being cocyle was not a requirement for the labelling of a directed graph in the earlier chapter. This cocycle property is to ensure that $\Gamma \times_c G$ is also an undirected graph. To see that, one can see that all the four conditions in the definition of undirected graph is satisfied:

(1) $s_c(b_c(e, g)) = s_c(b(e), c(e)g) = (s(b(e)), c(e)g) = (r(e), c(e)g) = r(e, g)$;
(2) $r_c(b_c(e, g)) = r_c(b(e), c(e)g) = (r(b(e)), c(b(e))c(e)g) = (rb((e)), c(e)^{-1}c(e)g) = (s(e), g) = s(e, g)$;

(3) $b(e,g) = (b(e), c(e)g) \neq (e,g)$; and

(4) $b_c^2(e,g) = b_c(b(e), c(e)g) = (b^2(e), c(b(e))c(e)g) = (b^2(e), c(e)^{-1}c(e)g) = (e,g)$.

Therefore, $\Gamma \times_c G$ is an undirected graph. Inspired from the skew-product graphs, we are looking to define a skew-product graph of groups that aligns with the definition of the skew-product graphs:

DEFINITION 6.3. *Let $\mathscr{G} = (\Gamma, G)$ be a graph of groups. Suppose that $c : \Gamma^1 \to H$ is a* ***cocycle*** *labelling of discrete group H. Then the* ***skew-product graph of groups****, denoted $\mathscr{G} \times_c H$, is a graph of groups with*

(1) underlying graph $\Gamma \times_c H = (\Gamma^0 \times H, \Gamma^1 \times H, r_c, s_c, b_c)$, where s_c, r_c and b_c and defined as defined above;

(2) a collection of vertex groups $G_{(x,h)}$, where $G_{(x,h)} = G_x$ for each $h \in H$;

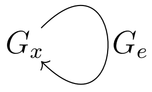
(3) a collection of edge groups $G_{(e,h)}$, where $G_{(e,h)} = G_e$ for each $h \in H$; and

(4) a collection of injective homomorphisms

$$\alpha_{(e,h)} : G_{(e,h)} = G_e \to G_{r(e,h)} = G_{(r(e),c(e)h)} = G_{r(e)}$$

for each $h \in H$, and we set $\alpha_{(e,h)} = \alpha_e$.

EXAMPLE 6.4 (Skew-product with finite vertices). *Let $\mathscr{G} = (\Gamma, G)$ be a graph of groups with single vertex x and a single edge e from x to itself.*



Let $H = \mathbb{Z}_3$ and define the labelling $c : \Gamma^1 \to H$ by $c(e) = \overline{1}$ and $c(\overline{e}) = \overline{2}$. Then c is a cocycle and the skew-product graph of groups $\mathscr{G} \times_c H$ has graph structure:

$$\begin{tikzcd}(x,\overline{0}) \arrow[r, "{(e,\overline{0})}"] & (x,\overline{1}) \arrow[r, "{(e,\overline{1})}"] & (x,\overline{2}) \arrow[ll, bend left, "{(e,\overline{2})}"]\end{tikzcd}$$

where $G_{(x,\bar{0})} = G_{(x,\bar{1})} = G_{(x,\bar{2})} = G_x$ *and* $G_{(e,\bar{0})} = G_{(e,\bar{1})} = G_{(e,\bar{2})} = G_e$. *Also,* $\overline{(e,\overline{0})} = (\overline{e},\overline{1}), \overline{(e,\overline{1})} = (\overline{e},\overline{2}), \overline{(e,\overline{2})} = (\overline{e},\overline{0})$.

EXAMPLE 6.5 (Skew-product with countably infinite vertices). *Let* $\mathscr{G} = (\Gamma, G)$ *be the graph of groups in the above example. Let* $H = \mathbb{Z}$ *and define the labelling* $c : \Gamma^1 \to H$ *by* $c(e) = 1$ *and* $c(\overline{e}) = -1$. *Then the skew-product graph of groups* $\mathscr{G} \times_c H$ *has graph structure:*

$$\begin{tikzcd}\cdots \arrow[r, "{(e,-2)}"] & (x,-1) \arrow[r, "{(e,-1)}"] & (x,0) \arrow[r, "{(e,0)}"] & (x,1) \arrow[r, "{(e,1)}"] & \cdots \arrow[r, "{(e,2)}"] & (x,n) \arrow[r, "{(e,n)}"] & \cdots\end{tikzcd},$$

where $s(e,n) = (x,n), r(e,n) = (x,n+1), G_{(x,n)} = G_x, G_{(e,n)} = G_e$ *for any* $n \in \mathbb{Z}$.

We will show that this is a natural way of defining the underlying graph of the skew-product graph of groups in regards to the well-defined skew-product graphs:

DEFINITION 6.6. *Let* $E = (E^0, E^1, r_E, s_E), F = (F^0, F^1, r_F, s_F)$ *be directed graphs. A* ***graph isomorphism*** *between E and F, is a pair of bijections* $\phi^0 : E^0 \to F^0$ *and* $\phi^1 : E^1 \to F^1$ *satisfying*

(1) $\phi^0 \circ s_E = s_F \circ \phi^1$*; and*
(2) $\phi^0 \circ r_E = r_F \circ \phi^1$.

A graph isomorphism keeps the graph structure by preserving the sources and ranges.

LEMMA 6.7. *Let* $\mathscr{G} = (\Gamma, G)$ *be a graph of groups and* H *be a discrete group. Then for every cocycle labelling* $c : \Gamma^1 \to H$*, there is an induced skew-product graph labelling* $c_E : E^1_{\mathscr{G}} \to H$ *by* $c_E(g_1 e_1 g_2 e_2) = c(e_2)$ *such that*

$$E_{\mathscr{G} \times_c H} \cong E_{\mathscr{G}} \times_{c_E} H,$$

via graph isomorphism

$$\phi^0 : (E_{\mathscr{G}\times_c H})^0 \to (E_{\mathscr{G}} \times_{c_E} H)^0 \,;\, \phi^0(g_1(e_1,h_1)) = (g_1e_1,h_1), and$$

$$\phi^1 : (E_{\mathscr{G}\times_c H})^1 \to (E_{\mathscr{G}} \times_{c_E} H)^1 \,;\, \phi^1(g_1(e_1,h_1)g_2(e_2,h_2)) = (g_1e_1g_2e_2,h_2)$$

PROOF. To show that the graphs $E_{\mathscr{G}\times_c H} = ((\mathscr{G}\times_c H)^1, (\mathscr{G}\times_c H)^2, r_c, s_c)$ and $E_{\mathscr{G}}\times_{c_*} H = (\mathscr{G}^1 \times H, \mathscr{G}^2 \times H, r_{c_*}, s_{c_*})$ are isomorphic, we need to show the following:

(1) The maps (ϕ^0,ϕ^1) are bijections; and
(2) The maps (ϕ^0,ϕ^1) satisfies
    (a) $\phi^0 \circ r_c = r_{c_*} \circ \phi^1$; and
    (b) $\phi^0 \circ s_c = s_{c_*} \circ \phi^1$.

Let $g(e,h) \neq g'(e',h') \in E_{\mathscr{G}\times_c H}$. Then at least one of

(1) $g = g'$;
(2) $e = e'$; and
(3) $h = h'$.

is not met. Therefore we have $\phi^0(g(e,h)) = (ge,h) \neq (g'e',h') = \phi^0(g'(e',h'))$. So ϕ^0 is injective. Let $(ge,h) \in (E_{\mathscr{G}} \times_{c_*} H)^1$. Then $\phi(g(e,n)) = (ge,h)$, showing surjectivity. For ϕ^1, suppose that $g_1(e_1,h_1)g_2(e_2,h_2) \neq g_1'(e_1',h_1')g_2'(e_2',h_2') \in (E_{\mathscr{G}\times_c H})^1$. Then one of the following

(1) $g_1 = g_1'$;
(2) $e_1 = e_1'$;
(3) $h_1 = h_1'$;
(4) $g_2 = g_2'$;
(5) $e_2 = e_2'$; and
(6) $h_2 = h_2'$.

is not met. Recall that $r(e_2, h_2) = s(e_1, h_1)$, this means $c(e_2)h_2 = h_1$, so (3) becomes $c(e_2)h_2 = c(e'_2)h'_2$, which can be omitted. Then we have

$$\phi^1(g_1(e_1, h_1)g_2(e_2, h_2)) = (g_1e_1g_2e_2, h_2) \neq (g'_1e'_1g'_2e'_2, h'_2) = g'_1(e'_1, h'_1)g'_2(e'_2, h'_2).$$

So ϕ^0 is injective. Let $(g_1e_1g_2e_2, h) \in (E_{\mathscr{G}} \times_{c_E} H)^1$, then we have $\phi(g_1(e_1, c(e_2)hg_2(e_2, h)))$, showing surjectivity. Finally, for the graph range/source conditions. Let $g_1(e_1, h_1)g_2(e_2, h_2) \in E_{\mathscr{G}\times_c H}$. Note that in order for $g_1(e_1, h_1)g_2(e_2, h_2)$ to be a $(\mathscr{G} \times_c H)$-path, we need 1) $e_1e_2 \in \Gamma^1$ and 2) $h_1 = c(e_2)h_2$. Then

$$\begin{aligned}\phi^0 \circ r_c(g_1(e_1, h_1)g_2(e_2, h_2)) &= \phi^0(g_1(e_1, h_1)) = (g_1e_1, h_1)\\ &= (g_1e_1, c(e_2)h_2) = (r_E(g_1e_1g_2e_2), c_*(g_1e_1g_2e_2)h_2)\\ &= r_{c_*}(g_1e_1g_2e_2, h_2) = r_{c_*} \circ \phi^1(g_1(e_1, h_1)g_2(e_2, h_2))\end{aligned}$$

and

$$\begin{aligned}\phi^0 \circ s_c(g_1(e_1, h_1)g_2(e_2, h_2)) &= \phi^0(g_2(e_2, h_2)) = (g_2e_2, h_2)\\ &= (s_E(g_1e_1g_2e_2), h_2) = s_{c_*}(g_1e_1g_2e_2, h_2)\\ &= s_{c_*} \circ \phi(g_1(e_1, h_1)g_2(e_2, h_2))\end{aligned}$$

□

The above lemma says that the following diagram

$$\begin{tikzcd}
\mathscr{G} \arrow[r] \arrow[d, "c"] & E_{\mathscr{G}} \arrow[d, "c_E"] \\
\mathscr{G} \times_c H \arrow[r] & E_{\mathscr{G}\times_c H} \cong E_{\mathscr{G}} \times_{c_E} H
\end{tikzcd}$$

commutes. This means the definition of skew-product graph of graphs we defined agrees with the definition of skew-product graph in the sense of underlying graph of graph of groups: Given a graph of groups, it does not matter if we take the underlying graph first or build the skew-product graph of groups first, it leads to the same skew-product graph at the end. In the next section, we examine the skew-product graph of groups.

6.2 Skew-Product Graph of Groups C^*-Algebras and Crossed Products

This section establishes the main theoretical result of this thesis by proving that the fundamental connection between skew-product constructions and crossed products by coactions, established by Kaliszewski, Quigg and Raeburn [2] for ordinary directed graphs, extends to the graph of groups setting. The significance of this result cannot be overstated: it provides the essential bridge that enables the application of K-theoretic techniques developed for crossed products to the previously intractable setting of graph of groups C^*-algebras. By establishing that skew-product graph of groups algebras are isomorphic to crossed products by coactions, we open the door to exact sequence methods and other computational tools that have proven so successful in the ordinary graph case.

The following theorem represents the cornerstone achievement of this thesis, establishing that the skew-product methodology successfully extends from ordinary graphs to graphs of groups:

THEOREM 6.8. *Let $\mathcal{G} = (\Gamma, G)$ be a graph of groups, H be a discrete group, and $c : \Gamma^1 \to H$ be a labelling with the cocycle property. Then there exists a coaction $\delta_c : C^*(\mathcal{G}) \to \mathcal{M}(C^*(\mathcal{G}) \otimes C^*(H))$ such that*

$$C^*(\mathcal{G}) \rtimes_{\delta_c} H \cong C^*(\mathcal{G} \times_c H).$$

This theorem establishes the graph of groups analogue of the fundamental skew-product theorem from [2], which was instrumental in computing K-theory for graph algebras by realising crossed products as skew-product graph algebras with more tractable structure. The present result serves the same crucial purpose for graph of groups C^*-algebras: it enables the replacement of difficult-to-analyse graph of groups algebras with crossed products by coactions, which are amenable to exact sequence techniques in K-theory. This provides the theoretical foundation for applying the dual Pimsner-Voiculescu sequence and related computational tools to graph of groups C^*-algebras, addressing a longstanding computational challenge in this area.

PROOF. First, define $\delta_c : C^*(\mathscr{G}) \to \mathcal{M}(C^*(\mathscr{G}) \otimes C^*(H))$ by

$$s_e \mapsto s_e \otimes u_{c(e)} \quad \text{and} \quad u_{x,g} \mapsto u_{x,g} \otimes 1.$$

To see that δ_c is an coaction. We need to show that it satisfies the coaction identity on the generator elements (s_e and $u_{x,g}$).

$$\begin{aligned}(\delta_c \otimes \mathrm{id}_{C^*(H)}) \circ \delta_c(s_e) &= (\delta_c \otimes \mathrm{id}_{C^*(H)})(s_e \otimes u_{c(e)}) = s_e \otimes (u_{c(e)} \otimes u_{c(e)}) \\ &= (\mathrm{id}_{C^*(\mathscr{G})} \otimes \delta_H)(s_e \otimes u_{c(e)}) = (\mathrm{id}_{C^*(\mathscr{G})} \otimes \delta_H) \circ \delta_c(s_e), \text{ and}\end{aligned}$$

$$\begin{aligned}(\delta_c \otimes \mathrm{id}_{C^*(H)}) \circ \delta_c(u_{x,g}) &= (\delta_c \otimes \mathrm{id}_{C^*(H)})(u_{x,g} \otimes u_1) = u_{x,g} \otimes (u_1 \otimes u_1) \\ &= (\mathrm{id}_{C^*(\mathscr{G})} \otimes \delta_H)(u_{x,g} \otimes u_1) = (\mathrm{id}_{C^*(\mathscr{G})} \otimes \delta_H) \circ \delta_c(u_{x,g})\end{aligned}$$

This shows that δ_c is a coaction. Next, we go on and show that the crossed product generated by δ_c is isomorphic to the skew-product graph of groups C^*-algebra. To make the steps easier, we lay out some of the facts about the generators of $C^*(\mathscr{G})$. Note that $s_e, s_{\overline{e}}^* \in C^*(\mathscr{G})_{c(e)}$, $s_e^*, s_{\overline{e}} \in C^*(\mathscr{G})_{c(e)^{-1}} = C^*(\mathscr{G})_{c(\overline{e})}$. Since H is a discrete group, any function $f \in C_0(H)$ can be expressed as

$$f(h) = \sum_{t \in H} f(t)\chi_h(t).$$

That is, the set $\{\chi_t\}_{t \in H}$ spans a dense subset of $C_0(H)$ and we have the following:

$$\begin{aligned}j_{C*(\mathscr{G})}(s_e) j_H(\chi_t) &= j_H(\chi_{c(e)t}) j_{C*(\mathscr{G})}(s_e), \text{ and} \\ j_{C*(\mathscr{G})}(u_{x,g}) j_H(\chi_t) &= j_H(\chi_t) j_{C*(\mathscr{G})}(u_{x,g})\end{aligned}$$

The similar goes for $s_{\overline{e}}^*, s_e^*, s_{\overline{e}}$. Now, we define a $\mathscr{G} \times_c H$-family in $C^*(\mathscr{G}) \rtimes_{\delta_c} H$. Let

$$\begin{aligned}S_{(e,t)} &:= j_{C*(\mathscr{G})}(s_e) j_H(\chi_t), \text{ and} \\ U_{(x,t),g} &:= j_{C*(\mathscr{G})}(u_{x,g}) j_H(\chi_t)\end{aligned}$$

We will show that the set of elements defined above satisfies the four relations in the $\mathscr{G}\times_c H$-family.

(1) $U_{(x,t),1}U_{(y,t'),1}=0$ for every $(x,t)\neq(y,t')$:

Suppose that $(x,t)\neq(y,t')$. Then we have at least one of the following situation:

(a) $x\neq y$; and

(b) $t\neq t'$. Then

$$\begin{aligned}U_{(x,t),1}U_{(y,t'),1}&=j_{C^*(\mathscr{G})}(u_{x,1})j_H(\chi_t)j_{C^*(\mathscr{G})}(u_{y,1})j_H(\chi_{t'})\\&=j_{C^*(\mathscr{G})}(u_{x,1}u_{y,1})j_H(\chi_t\chi_{t'})\\&=\begin{cases}j_{C^*(\mathscr{G})}(u_{x,1})j_H(\chi_t) & ,x=y\quad\text{and}\quad t=t'\\0 & ,\ \text{otherwise}\end{cases}\\&=\begin{cases}U_{(x,t),1} & ,(x,t)=(y,t')\\0 & ,\ \text{otherwise.}\end{cases}\end{aligned}$$

(2) $U_{r_c(e,t),\alpha_{(e,t)}(g)}S_{(e,t)}=S_{(e,t)}U_{s_c(e,t),\alpha_{\overline{(e,t)}}(g)}$:

$$\begin{aligned}U_{r_c(e,t),\alpha_{(e,t)}(g)}S_{(e,t)}&=U_{(r(e),c(e)t),\alpha_e(g)}S_{(e,t)}\\&=j_{C^*(\mathscr{G})}(u_{r(e),\alpha_e(g)})j_H(\chi_{c(e)t})j_{C^*(\mathscr{G})}(s_e)j_H(\chi_t)\\&=j_{C^*(\mathscr{G})}(u_{r(e),\alpha_e(g)}s_e)j_H(\chi_t)j_H(\chi_t)\\&=j_{C^*(\mathscr{G})}(s_e)j_{C^*(\mathscr{G})}(u_{s(e),\alpha_{\overline{e}}(g)})j_H(\chi_t)j_H(\chi_t)\\&=j_{C^*(\mathscr{G})}(s_e)j_{C^*(\mathscr{G})}(u_{s(e),\alpha_{\overline{(e,t)}}(g)})j_H(\chi_t)j_H(\chi_t)\\&=j_{C^*(\mathscr{G})}(s_e)j_H(\chi_t)j_{C^*(\mathscr{G})}(u_{s(e),\alpha_{\overline{(e,t)}}(g)})j_H(\chi_t)\\&=S_{(e,t)}U_{(s(e),t),\alpha_{\overline{(e,t)}}(g)}\\&=S_{(e,t)}U_{s_c(e,t),\alpha_{\overline{(e,t)}}(g)}.\end{aligned}$$

(3) $U_{s_c(e,t),1} = S^*_{(e,t)}S_{(e,t)} + S_{\overline{(e,t)}}S^*_{\overline{(e,t)}}$ for every $(e,t) \in (\Gamma \times_c H)^1$:

$$\begin{aligned}
S^*_{(e,t)}S_{(e,t)} + S_{\overline{(e,t)}}S^*_{\overline{(e,t)}} &= S^*_{(e,t)}S_{(e,t)} + S_{(\overline{e},c(e)t)}S^*_{(\overline{e},c(e)t)} \\
&= j_H(\chi_t)j_{C^*(\mathscr{G})}(s_e^*s_e)j_H(\chi_t) + j_{C^*(\mathscr{G})}(s_{\overline{e}})j_H(\chi_{c(e)t})j_{C^*(\mathscr{G})}(s^*_{\overline{e}}) \\
&= j_{C^*(\mathscr{G})}(s_e^*s_e)j_H(\chi_t) + j_{C^*(\mathscr{G})}(s_{\overline{e}}s^*_{\overline{e}})j_H(\chi_t) \\
&= j_{C^*(\mathscr{G})}(s_e^*s_e + s_{\overline{e}}s^*_{\overline{e}})j_H(\chi_t) \\
&= j_{C^*(\mathscr{G})}(u_{s(e),1})j_H(\chi_t) \\
&= U_{(s(e),t),1} = U_{s_c(e,t),1}.
\end{aligned}$$

(4)

$$S^*_{(e,t)}S_{(e,t)} = \sum_{\substack{r_c(f,s)=s_c(e,t) \\ h\in\sum_{(f,s)} \\ h(f,s)\neq 1\overline{(e,t)}}} S_{h(f,s)}S^*_{h(f,s)}$$

for every $(e,t) \in (\Gamma \times_c H)^1$:

Note that

(a) $(r(f), c(f)s) = r_c(f,s) = s_c(e,t) = (s(e),t)$ implies $r(f) = s(e)$ and $s = c(f)^{-1}t$; and

(b) $h(f,s) = h(f, c(f)^{-1}t) \neq 1\overline{(e,t)} = 1(\overline{e}, c(e)t)$ implies either i) $h \neq 1$ or ii) $(f, c(f)^{-1}t) \neq (\overline{e}, c(e)t)$. For ii), it implies either $f \neq \overline{e}$ or $c(f)^{-1}t \neq c(e)t$, and both of them leads to $f \neq \overline{e}$. So the condition becomes either $h \neq 1$ or $f \neq \overline{e}$, which is equivalent to $hf \neq 1\overline{e}$. Therefore,

$$\begin{aligned}
&\sum_{r_c(f,s)=s_c(e,t),h\in\sum_{(f,s)},h(f,s)\neq 1\overline{(e,t)}} S_{h(f,s)}S^*_{h(f,s)} \\
&= \sum_{r(f)=s(e),h\in\sum_f,hf\neq 1\overline{e}} S_{h(f,c(f)^{-1}t)}S^*_{h(f,c(f)^{-1}t)} \\
&= \sum_{r(f)=s(e),h\in\sum_f,hf\neq 1\overline{e}} j_{C^*(\mathscr{G})}(u_{(r(f),h)})j_H(\chi_t)j_{C^*(\mathscr{G})}(s_f)j_H(\chi_{c(f)^{-1}t}) \\
&j_H(\chi_{c(f)^{-1}t})j_{C^*(\mathscr{G})}(s_f^*)j_H(\chi_t)j_{C^*(\mathscr{G})}(u^*_{r(f),h}) \\
&= j_H(\chi_t)j_{C^*(\mathscr{G})}(\sum_{r(f)=s(e),h\in\sum_f,hf\neq 1\overline{e}} s_{hf}s^*_{hf})j_H(\chi_t) \\
&= j_H(\chi_t)j_{C^*(\mathscr{G})}(s_e^*s_e)j_H(\chi_t) \\
&= (j_{C^*(\mathscr{G})}(s_e)j_H(\chi_t^*))^*(j_{C^*(\mathscr{G})}(s_e)j_H(\chi_t^*)) \\
&= S^*_{(e,t)}S_{(e,t)}.
\end{aligned}$$

By the universal property of graph of groups algebra, there is a homomorphism

$$\begin{aligned}
\Phi : C^*(\mathscr{G}\times_c H) \to C^*(\mathscr{G}) &\rtimes_{\delta_c} H; \\
s_{(e,t)} &\mapsto j_{C*(\mathscr{G})}(s_e)j_H(\chi_t) \\
u_{(x,t),g} &\mapsto j_{C*(\mathscr{G})}(u_{x,g})j_H(\chi_t)
\end{aligned}$$

Next, Consider the following elements:

$$S_e := \sum_{t\in H} s_{(e,t)} \quad \text{and} \quad U_{x,g} := \sum_{t\in H} u_{(x,t),g}$$

They both belong to the multiplier algebra $\mathcal{M}(C^*(\mathscr{G}\times_c H))$. This is apparent when H is finite, but not so when H is infinite. To show that $\sum_{t\in H} s_{(e,t)}, \sum_{t\in H} u_{(x,t),g}$ belong to the multiplier algebra, we need to show that $\sum_{t\in H} s_{(e,t)}, \sum_{t\in H} u_{(x,t),g}$ converges strictly in $\mathcal{M}(C^*(\mathscr{G}\times_c H))$.

Let $F_1 \subset F_2 \subset \cdots \subset H$ be an strictly increasing finite subset of H. Define $x_i := \sum_{f\in F_i} s_{(e,t)}$, then $\{x_i\}_{i\in\mathbb{N}\setminus\{\infty\}}$ becomes a net. Since the C^*-algebra $C^*(\mathscr{G}\times_c H)$ is generated by $s_{(e,t)}, s^*_{(e,t)}, u_{(x,t),g}, u^*_{(x,t),g}$, it suffice to show that $\{x_i a\}$ and $\{a x_i\}$ converges to xa

and ax respectively with a as any of the above. We can omit $u^*_{(x,t),g}$ since it is just a form of $u_{(x,t),g}$.

Recall that

(1) $s_{(e,t)}s_{(f,h)} = 0$ if $r_c(f,h) \neq s_c(e,t)$ or $(f,h) = \overline{(e,t)}$;
(2) $s_{(e,t)}s^*_{(f,h)} = 0$ if $s_c(e,t) \neq s_c(f,h)$; and
(3) $s_{(e,t)}u_{(x,h),g} = 0$ if $s_c(e,t) \neq (x,h)$.

Let $a := s_{(f,h)}$, then

$$x_i a = \sum_{t\in F_i} s_{(e,t)}s_{(f,h)} = \begin{cases} s_{(e,t)}s_{(f,h)} & , (s(e) \neq r(f) \text{ or } h \neq c(f)^{-1}t) \text{ or } f,h) = \overline{(e,t)} \\ 0 & , \text{otherwise.} \end{cases}$$

Therefore, the net $\{x_i a\}$ is a set of 0's until the above condition is met, and therefore converges. The other two elements are similar in calculations. The other way is also similar. Thus, $\sum_{t\in H} s_{(e,t)}, \sum_{t\in H} u_{(x,t),g} \in \mathcal{M}(C^*(\mathscr{G} \times_c H))$.

We will show that the above elements form a $\mathscr{G}$-family:

(1) $U_{x,1}U_{y,1} = 0$ for every $x, y \in \Gamma^0$ such that $x \neq y$:

$$\begin{aligned} U_{x,1}U_{y,1} &= \sum_{t\in H} u_{(x,t),1} \sum_{ts\in H} u_{(y,s),1} = \sum_{t,s\in H} u_{(x,t),1}u_{(y,s),1} = \sum_{t\in H} u_{(x,t),1}u_{(y,y),1} \\ &= \begin{cases} \sum_{t\in H} u_{(x,t),1} & , x = y \\ 0 & , x \neq y \end{cases} = \begin{cases} U_{x,1} & , x = y \\ 0 & , x \neq y. \end{cases} \end{aligned}$$

(2) $U_{r(e),\alpha_e(g)}S_e = S_e U_{s(e),\alpha_{\overline{e}}(g)}$ for every $e \in \Gamma^1, g \in G_e$:

$$\begin{aligned}
U_{r(e),\alpha_e(g)}S_e &= \sum_{t\in H} u_{(r(e),t),\alpha_e(g)} \cdot \sum_{s\in H} s_{(e,s)} = \sum_{t,s\in H} u_{(r(e),t),\alpha_e(g)} s_{(e,s)} \\
&= \sum_{t\in H} u_{(r(e),t),\alpha_e(g)} s_{(e,c(e)^{-1}t)} = \sum_{t\in H} u_{r_c(e,c(e)^{-1}t),\alpha_e(g)} s_{(e,c(e)^{-1}t)} \\
&= \sum_{s\in H} u_{(r_c(e,s),\alpha_{(e,s)}(g)} s_{(e,s)} = \sum_{s\in H} s_{(e,s)} u_{(s_c(e,s),\alpha_{\overline{(e,s)}}(g)} \\
&= \sum_{s\in H} s_{(e,s)} u_{(s_c(e,s),\alpha_{\overline{e}}(g)} = \sum_{s,t\in H} s_{(e,s)} u_{(s(e),t),\alpha_{\overline{e}}(g)} \\
&= \sum_{s\in H} s_{(e,s)} \cdot \sum_{t\in H} u_{(s(e),t),\alpha_{\overline{e}}(g)} = S_e U_{s(e),\alpha_{\overline{e}}(g)}.
\end{aligned}$$

(3) $U_{s(e),1_{s(e)}} = S_e^* S_e + S_{\overline{e}} S_{\overline{e}}^*$ for every $e \in \Gamma^1$:

$$\begin{aligned}
S_e^* S_e + S_{\overline{e}} S_{\overline{e}}^* &= \sum_{t\in H} s^*_{(e,t)} \sum_{s\in H} s_{(e,s)} + \sum_{t\in H} s_{(\overline{e},t)} \sum_{s\in H} s^*_{(\overline{e},s)} \\
&= \sum_{t,s\in H} s^*_{(e,t)} s_{(e,s)} + \sum_{t,s\in H} s_{(\overline{e},t)} s^*_{(\overline{e},s)} \\
&= \sum_{t\in H} s^*_{(e,t)} s_{(e,t)} + \sum_{t\in H} s_{(\overline{e},t)} s^*_{(\overline{e},t)} = \sum_{t\in H} s^*_{(e,t)} s_{(e,t)} + \sum_{s\in H} s_{(\overline{e},c(e)s)} s^*_{(\overline{e},c(e)s)} \\
&= \sum_{t\in H} (s^*_{(e,t)} s_{(e,t)} + s_{\overline{(e,t)}} s^*_{\overline{(e,t)}}) = \sum_{t\in H} u_{s_c(e,t),1} = \sum_{t\in H} u_{s(e),t),1} = U_{s(e),1}.
\end{aligned}$$

(4) $S_e^* S_e = \sum_{r(f)=s(e),h\in\Gamma_f,hf\neq 1\overline{e}} (U_{s(e),h}S_f)(U_{s(e),h}S_f)^*$ for every $e \in \Gamma^1$:

Recall that

(a) $r_c(f,s) = s_c(e,t) \Longrightarrow (r(f), c(f)s) = (s(e), t)$ implies

 (i) $r(f) = s(e)$; and

 (ii) $s = c(f)^{-1}t$.

(b) $h(f, c(f)^{-1}t) \neq 1\overline{(e,t)} = 1(\overline{e}, c(e)t) \Longrightarrow h(f, c(f)^{-1}t) \neq 1(\overline{e}, c(e)t)$ implies

 (i) $f \neq \overline{e}$; and

 (ii) $c(f)^{-1}t \neq c(e)t \Longrightarrow c(f)^{-1} \neq c(e) \Longrightarrow f \neq \overline{e}$.

$$\text{L.H.S} = (\sum_{t\in H} S_{(e,t)})^* (\sum_{s\in H} S_{(e,s)}) = \sum_{t,s\in H} S^*_{(e,t)} S_{(e,s)} = \sum_{t\in H} S^*_{(e,t)} S_{(e,t)}$$

$$= \sum_{t \in H} \Big(\sum_{\substack{r_c(f,s)=s_c(e,t) \\ h \in \sum_{(f,s)} \\ h(f,s) \neq 1\overline{(e,t)}}} u_{r_c(f,s),h} s_{(f,s)} s^*_{(f,s)} u^*_{r_c(f,s),h} \Big)$$

$$= \sum_{t \in H} \Big(\sum_{\substack{r(f)=s(e) \\ h \in \sum_f \\ hf \neq 1\overline{e}}} u_{r_c(f,c(f)^{-1}t),h} s_{(f,c(f)^{-1}t)} s^*_{(f,c(f)^{-1}t)} u^*_{r_c(f,c(f)^{-1}t),h} \Big)$$

$$= \sum_{s \in H} \Big(\sum_{\substack{r(f)=s(e) \\ h \in \sum_f \\ hf \neq 1\overline{e}}} u_{r_c(f,s),h} s_{(f,s)} s^*_{(f,s)} u^*_{r_c(f,s),h} \Big)$$

$$\text{R.H.S} = \sum_{\substack{r(f)=s(e) \\ h \in \sum_f \\ hf \neq 1\overline{e}}} U_{r(f),h} S_f S_f^* U^*_{r(f),h}$$

$$= \sum_{\substack{r(f)=s(e) \\ h \in \sum_f \\ hf \neq 1\overline{e}}} \Big(\sum_{t \in H} u_{(r(f),t),h} \sum_{t' \in H} s_{(f,t')} \sum_{t'' \in H} s^*_{(f,t'')} \sum_{t''' \in H} u^*_{(r(f),t'''),h} \Big)$$

$$= \sum_{\substack{r(f)=s(e) \\ h \in \sum_f \\ hf \neq 1\overline{e}}} \Big(\sum_{t,t',t'',t''' \in H} u_{(r(f),t),h} s_{(f,t')} s^*_{(f,t'')} u^*_{(r(f),t'''),h} \Big)$$

$$= \sum_{\substack{r(f)=s(e) \\ h \in \sum_f \\ hf \neq 1\overline{e}}} \Big(\sum_{t \in H} u_{(r(f),t),h} s_{(f,c(f)^{-1}t)} s^*_{(f,c(f)^{-1}t)} u^*_{(r(f),t),h} \Big)$$

$$= \sum_{t \in H} \Big(\sum_{\substack{r(f)=s(e) \\ h \in \sum_f \\ hf \neq 1\overline{e}}} u_{(r(f),t),h} s_{(f,c(f)^{-1}t)} s^*_{(f,c(f)^{-1}t)} u^*_{(r(f),t),h} \Big)$$

$$= \sum_{s \in H} \Big(\sum_{\substack{r(f)=s(e) \\ h \in \sum_f \\ hf \neq 1\overline{e}}} u_{(r(f),c(f)s),h} s_{(f,s)} s^*_{(f,s)} u^*_{(r(f),c(f)s),h} \Big)$$

$$= \sum_{s \in H} \Big(\sum_{\substack{r(f)=s(e) \\ h \in \sum_f \\ hf \neq 1\overline{e}}} u_{r_c(f,s),h} s_{(f,s)} s^*_{(f,s)} u^*_{(r_c(f,s),h)} \Big) = \text{L.H.S.}$$

Therefore, the universal property of graph algebra gives a homomorphism

$$\begin{aligned}\pi : C^*(\mathscr{G}) &\to \mathcal{M}(C^*(\mathscr{G} \times_c H));\\ s_e &\mapsto \sum_{t\in H} s_{(e,t)}\\ u_{x,g} &\mapsto \sum_{t\in H} u_{(x,t),g}\end{aligned}$$

Now, fix $t \in H$, consider the element $\sum_{x\in\Gamma^0} u_{(x,t),1}$. Apparently it belongs to $\mathcal{M}(C^*(\mathscr{G}\times_c H))$ when there are finitely many vertices in the underlying graph, but not so when there are infinitely many vertices. To show that $\sum_{x\in\Gamma^0} u_{(x,t),1}$ belongs to the multiplier algebra, we need to show that $\sum_{x\in\Gamma^0} u_{(x,t),1}$ converges strictly in $\mathcal{M}(C^*(\mathscr{G} \times_c H))$. This is the same as above.

Define a function

$$\begin{aligned}U : C_0(H) &\to \mathcal{M}(C^*(\mathscr{G} \times_c H));\\ \chi_t &\mapsto \sum_{x\in\Gamma^0} u_{(x,t),1}.\end{aligned}$$

This is indeed a $*$-homomorphism: Let $t, h \in H$, then

$$\begin{aligned}U(\chi_t)U(\chi_h) &= \sum_{x\in\Gamma^0} u_{(x,t),1} \sum_{x\in\Gamma^0} u_{(x,h),1} = \sum_{x,y\in\Gamma^0} u_{(x,t),1}u_{(x,h),1}\\ &= \begin{cases}\sum_{x\in\Gamma^0} u_{(x,t),1} & ,t = h\\ 0 & ,x \neq y\end{cases}\\ &= \begin{cases}U(\chi_t) & t = h\\ 0, & t \neq h\end{cases}\\ &= U(\chi_t\chi_h), and\end{aligned}$$

$$U(\chi_t)^* = (\sum_{x\in\Gamma^0} u_{(x,t),1})^* = \sum_{x\in\Gamma^0} u_{(x,t),1} = U(\chi_t) = U(\chi_t^*).$$

Now, we will show that the pair (π, U) is a covariant homomorphism from the dynamical system $(C^*(\mathscr{G}), \delta_c, H)$ to $\mathcal{M}(C^*(\mathscr{G} \times_c H))$. To show that, we need to show that for all the generators $s_e, u_{x,g} \in C^*(\mathscr{G})$ and $\chi_t \in C^0(H)$, $\pi(a_s)U(\chi_t) = U(\chi_{st})\pi(a_s)$. Recall that $s_e \in C^*(\mathscr{G})_{c(e)}, u_{x,g} \in C^*(\mathscr{G})_{1_H}$. So we need to show that

$$\pi(s_e)U(\chi_t) = U(\chi_{c(e)t})\pi(s_e) \quad \text{and} \quad \pi(u_{x,g})U(\chi_t) = U(\chi_t)\pi(u_{x,g}).$$

By calculation, we have

$$\begin{aligned}
\pi(s_e)U(\chi_t) &= \sum_{h\in H} s_{(e,h)} \sum_{x\in\Gamma^0} u_{(x,t),1} = \sum_{\substack{h\in H\\ x\in\Gamma^0}} s_{(e,h)}u_{(x,t),1} = s_{(e,t)}u_{(s(e),t),1} = s_{(e,t)}u_{s_c(e,t),1}\\
&= s_{(e,t)}, \text{ and}\\
U(\chi_{c(e)t})\pi(s_e) &= \sum_{x\in\Gamma^0} u_{(x,c(e)t),1} \sum_{h\in H} s_{(e,h)} = \sum_{\substack{h\in H\\ x\in\Gamma^0}} u_{(x,c(e)t),1}s_{(e,h)} = u_{(r(e),c(e)t),1}s_{(e,t)}\\
&= s_{(e,t)}.
\end{aligned}$$

The same calculation can be applied to $u_{x,g}$.Since (π, U) is covariant, there exists an integrated form

$$\begin{aligned}
&\pi \times U : C^*(\mathscr{G}) \rtimes_{\delta_c} H \to \mathcal{M}(C^*(\mathscr{G} \times_c H)) \text{ such that}\\
&(\pi \times U) \circ j_{\mathscr{G}} = \pi\\
&(\pi \times U) \circ j_H = U.
\end{aligned}$$

That is,

$$\begin{aligned}
(\pi \times U)(j_{\mathscr{G}}(s_e)) &= \sum_{t\in H} s_{(e,t)}\\
(\pi \times U)(j_{\mathscr{G}}(u_{x,g})) &= \sum_{t\in H} u_{(x,t),g}
\end{aligned}$$

$$(\pi \times U)(j_H(\chi_t)) = \sum_{x\in\Gamma^0} u_{(x,t),1}.$$

Finally, we will show that the two homomorphisms $\Phi : C^*(\mathscr{G} \times_c H) \to C^*(\mathscr{G}) \rtimes_{\delta_c} H$ and $\pi \times U : C^*(\mathscr{G}) \rtimes_{\delta_c} H \to \mathcal{M}(C^*(\mathscr{G} \times_c H))$ are inverses of each other. Notice it suffice to show this at their respective generating elements.

$$\begin{aligned}
(\pi \times U) \circ \Phi(s_{(e,t)}) &= (\pi \times U)(j_{\mathscr{G}}(s_e)j_H(\chi_t)) = (\pi \times U)(j_{\mathscr{G}}(s_e))(\pi \times U)(j_H(\chi_t)) \\
&= \sum_{s\in H} s_{(e,s)} \sum_{t\in H} u_{(x,t),1} = \sum_{s,t\in H} s_{(e,s)}u_{(x,t),1} = s_{(e,t)}u_{(s(e),t),1} \\
&= s_{(e,t)}u_{s_c(e,t),1} = s_{(e,t)}; \\
(\pi \times U) \circ \Phi(u_{(e,t),g}) &= (\pi \times U)(j_{\mathscr{G}}(u_{x,g})j_H(\chi_t)) = (\pi \times U)(j_{\mathscr{G}}(u_{x,g})(\pi \times U)(j_H(\chi_t)) \\
&= \sum_{s\in H} u_{(x,s),g} \sum_{y\in\Gamma_0} u_{(y,t),1} = \sum_{\substack{s\in H\\ y\in\Gamma_0}} u_{(x,s),g}u_{(x,t),1} = u_{(x,t),g}u_{(x,t),1} \\
&= u_{(x,t),g}; \\
\Phi \circ (\pi \times U)(j_{\mathscr{G}}(s_e)) &= \Phi(\sum_{t\in H} s_{(e,t)}) = \sum_{t\in H} \Phi(s_{(e,t)}) = \sum_{t\in H} j_{\mathscr{G}}(s_e)j_H(\chi_t) \\
&= j_{\mathscr{G}}(s_e) \sum_{t\in H} j_H(\chi_t) = j_{\mathscr{G}}(s_e)j_H(\sum_{t\in H} \chi_t) = j_{\mathscr{G}}(s_e); \\
\Phi \circ (\pi \times U)(j_{\mathscr{G}}(u_{x,g})) &= \Phi(\sum_{t\in H} u_{(x,t),g}) = \sum_{t\in H} \Phi(u_{(x,t),g}) = \sum_{t\in H} j_{\mathscr{G}}(u_{x,g})j_H(\chi_t) \\
&= j_{\mathscr{G}}(u_{x,g}) \sum_{t\in H} j_H(\chi_t) = j_{\mathscr{G}}(u_{x,g})j_H(\sum_{t\in H} \chi_t) = j_{\mathscr{G}}(u_{x,g}); \text{ and} \\
\Phi \circ (\pi \times U)(j_H(\chi_t)) &= \Phi(\sum_{x\in\Gamma_0} u_{(x,t),1}) = \sum_{x\in\Gamma_0} \Phi(u_{(x,t),1}) = (\sum_{x\in\Gamma_0} j_{\mathscr{G}}(u_{x,1}))j_H(\chi_t) \\
&= j_{\mathscr{G}}(\sum_{x\in\Gamma^0} u_{x,1})j_H(\chi_t) = j_H(\chi_t).
\end{aligned}$$

We have shown that both $\Phi \circ (\pi \times U)$, $(\pi \times U) \circ \Phi$ takes elements to itself. Completing the proof.

□

Notice from our definition of skew-product graph of groups, there wasn't any interactions between the vertex groups, the edge groups and the injective homomotphisms with the group H. The objects mentioned above are merely a duplicate of the original objects and are extended along the elements of the group H. To make it more interesting, we have a general definition for the skew-product graph of groups, where some flavour of H is being injected into the objects:

DEFINITION 6.9. *Let $\mathscr{G} = (\Gamma, G)$ be a graph of groups. Suppose that $c : \Gamma^1 \to H$ is a cocycle labelling of a discrete group H and let $\varphi : H \to H$ be an injective homomorphism. Then the* ***general skew-product graph of groups****, denoted $\mathscr{G} \times_{c,\varphi} H$ is a graph of groups with*

1. *underlying graph $\Gamma \times_c H = (\Gamma^0 \times H, \Gamma^1 \times H, r_c, s_c, b_c)$, where s_c, r_c and b_c and defined as defined above;*
2. *a collection of vertex groups $G_{(x,h)}$, where $G_{(x,h)} = G_x \times H$ for each $h \in H$;*
3. *a collection of edge groups $G_{(e,h)}$, where $G_{(e,h)} = G_e \times H$ for each $h \in H$; and*
4. *a collection of injective homomorphisms $\alpha_{(e,h)} : G_{(e,h)} = G_e \times H \to G_{r(e,h)} = G_{(r(e),c(e)h)} = G_{r(e)} \times H$ for each $h \in H$, where $\alpha_{(e,h)} = \alpha_e \times \varphi$.*

The reason we set φ to be injective became apparent: without this condition, $\alpha_{(e,h)}$ may fail to be injective. Notice that if for a locally-finite graph of groups, the general skew-product graph of groups need not be locally-finite. This is apparent when the discrete group H is infinite since

$$\begin{aligned}[G_{r_c(e,t)} : \alpha_{(e,t)}(G_{(e,t)})] &= [G_{r(e)} \times H : (\alpha_e \times \varphi)(G_e \times H)] = [G_{r(e)} \times H : \alpha_e(G_e) \times \varphi(H)] \\ &= [G_{r(e)} : \alpha_e(G_e)] \cdot [H : \varphi(H)].\end{aligned}$$

If H is infinite and φ is injective, we cannot guarantee $[H : \varphi(H)]$ to be finite. To create a locally-finite general skew-product graph of groups from a locally-finite graph of groups, the only way to guarantee that is to have φ be surjective, and hence isomorphic. Then $[H : \varphi(H)] = 1$ and the immediate consequence is $[G_{r_c(e,t)} : \alpha_{(e,t)}(G_{(e,t)})] = [G_{r(e)} : \alpha_e(G_e)]$. This brings us to the question of the choice of isomorphisms φ.

DEFINITION 6.10. *Suppose that* $\mathscr{G} = (\Gamma_{\mathscr{G}}, G)$ *and* $\mathscr{H} = (\Gamma_{\mathscr{H}}, H)$ *are graph of groups with their injective homomorphisms named* α *and* β *respectively. A graph of groups isomorphism between* $\mathscr{G}$ *and* $\mathscr{H}$ *consists of*

1. *a graph isomorphism* $\phi : \Gamma_{\mathscr{G}} \to \Gamma_{\mathscr{H}}$*;*
2. *a vertex group isomorphism* $\Phi_x : G_x \to H_{\phi^0(x)}$ *for every* $x \in \Gamma_{\mathscr{G}}$*; and*
3. *an edge group isomorphism* $\Phi_e : G_e \to H_{\phi^1(e)}$ *for every* $e \in \Gamma_{\mathscr{G}}$ *defined to be*

$$\Phi_e = \beta^{-1}_{\phi^1(e)} \circ \Phi_{r(e)}|_{\alpha_e(G_e)} \circ \alpha_e.$$

Notice from the definition of the edge group isomorphisms, it implies that the following diagram

$$\begin{tikzcd}
G_e \arrow[r, "\Phi_e"] \arrow[d, "\alpha_e"] & H_{\phi^1(e)} \arrow[d, "\beta_{\phi^1(e)}"] \\
G_{r(e)} \arrow[r, "\Phi_{r(e)}"] & H_{\phi^0(r(e))} = H_{r(\phi^1(e))}
\end{tikzcd}$$

commutes. The following lemma shows that the choice of φ does not affect the skew-product graph of groups structure, which leads to isomophic skew-product graph of groups algebras:

LEMMA 6.11. *Let* $\mathscr{G}$ *be a graph of groups. Let* $c : \Gamma^1 \to H$ *be a cocyle labelling of a discrete group* H*. Suppose that* $\varphi, \psi : H \to H$ *are distinct group isomorphisms. Then*

$$\mathscr{G} \times_{c,\varphi} H \cong \mathscr{G} \times_{c,\psi} H.$$

PROOF. To prove that the graph of groups are isomorphic, we need the following:

1. a graph isomorphism $\phi : \Gamma \times_c H \to \Gamma \times_c H$; and
2. a vertex group isomorphism $\Phi_{(x,t)} : G_{(x,t)} \to G_{(x,t)}$ for each vertex $(x, t) \in (\Gamma \times_c H)^0$ such that the induced edge group homomorphisms $\Phi_e : G_e \to H_{\phi^1(e)}$ is isomorphic for every edges.

Since the underlying graphs are the same for both skew-product graph of groups, the only graph isomorphism from a graph to itself, regardless of its graph structure, is the identity map

$\mathrm{id} : \Gamma \times_c H \to \Gamma \times_c H$. For each $(x,t) \in (\Gamma \times_c H)^0$, define

$$\begin{aligned}\Phi_{(x,t)} : G_{(x,t)} = G_x \times H \to G_{(x,t)} = G_x \times H;\\ \Phi_{(x,t)} := \mathrm{id} \times (\psi \circ \varphi^{-1}), \text{ and}\\ \Phi_{(e,t)} : G_{(e,t)} = G_e \times H \to G_{(e,t)} = G_e \times H;\\ \Phi_{(e,t)} := \mathrm{id} \times \mathrm{id}\,.\end{aligned}$$

Then for each edge (e,t),

$$\begin{aligned}\Phi_{(r(e))} \circ \alpha_{(e,t)} &= (\mathrm{id} \times (\psi \circ \varphi^{-1})) \circ (\alpha_e \times \varphi)\\ &= (\mathrm{id} \circ \alpha_e) \times (\psi \circ \varphi^{-1}) \circ \varphi)\\ &= (\alpha_e \circ \mathrm{id}) \times (\psi \circ \mathrm{id})\\ &= (\alpha_e \times \psi) \circ (\mathrm{id} \times \mathrm{id})\\ &= \beta_{(e,t)} \circ \Phi_{(e,t)}.\end{aligned}$$

□

We have shown that the choice of φ does no change the graph of groups structure.

THEOREM 6.12. *Let $\mathscr{G} = (\Gamma, G)$ be a graph of groups, H be a discrete group, and $c : \Gamma^1 \to H$ be a labelling with the cocycle property. Let δ_c be the coaction defined as above. Then*

$$C^*(\mathscr{G} \times_{c,\varphi} H) \cong (C^*(\mathscr{G}) \rtimes_{\delta_c} H) \otimes C^*(H).$$

In order to prove that, we need the following from [19, Theorem B.27 (b)]:

THEOREM 6.13. *Let $f : A \to C$ and $g : B \to C$ be two $*$-homomorphisms between C^*-algebras. If*

$$f(A)g(B) = g(B)f(A),$$

then there exists $f \otimes g : A \otimes B \to C$ such that

$$(f \otimes g)(a \otimes b) = f(a)g(b) \quad \textit{for every } a \in A, b \in B.$$

PROOF OF THEOREM 6.12. The idea of defining a homomorphism from the skew-product graph of groups algebras to the tensor product is the same as before: We want to find a $\mathscr{G} \times_{c,\varphi} H$-family in $(C^*(\mathscr{G}) \rtimes_{\delta_c} H) \otimes C^*(H)$. Let

$$S_{(e,t)} := j_{\mathscr{G}}(s_e) j_H(\chi_t) \otimes 1 \quad \text{and} \quad U_{(x,t),(g,h)} := j_{\mathscr{G}}(u_{x,g}) j_H(\chi_t) \otimes u_h$$

(1) $U_{(x,t),(1,1)} U_{(y,t'),(1,1)} = 0$ for every $(x,t) \neq (y,t')$:

$$\begin{aligned}
U_{(x,t),(1,1)} U_{(y,t'),(1,1)} &= (j_{\mathscr{G}}(u_{x,1}) j_H(\chi_t) \otimes 1)(j_{\mathscr{G}}(u_{y,1}) j_H(\chi_{t'}) \otimes 1) \\
&= j_{\mathscr{G}}(u_{x,1}) j_{\mathscr{G}}(u_{y,1}) j_H(\chi_t) j_H(\chi_{t'}) \otimes 1 \\
&= j_{\mathscr{G}}(u_{x,1} u_{y,1}) j_H(\chi_t \chi_{t'}) \otimes 1 \\
&= \begin{cases} j_{\mathscr{G}}(u_{x,1}) j_H(\chi_t) \otimes 1, & (x,t) = (y,t) \\ 0 \otimes 1, & (x,t) \neq (y,t) \end{cases} \\
&= \begin{cases} U_{(x,t),(1,1)} & (x,t) = (y,t), \\ 0, & (x,t) \neq (y,t). \end{cases}
\end{aligned}$$

Relationships (2), (3) and (4) follows that same proceedure as above in the skew-product graph of groups proof.

Now, for the other direction. Recall that we can define the following covariant homomorphism

$$\begin{gathered}
\pi_\varphi : C^*(\mathcal{G}) \to \mathcal{M}(C^*(\mathscr{G} \times_{c,\varphi} H)); \\
s_e \mapsto \sum_{t \in H} s_{(e,t)} \\
u_{x,g} \mapsto \sum_{t \in H} u_{(x,t),(g,1)} \\
U_\varphi : C_0(H) \to \mathcal{M}(C^*(\mathscr{G} \times_{c,\varphi} H)); \\
\chi_t \mapsto \sum_{x \in \Gamma^0} u_{(x,t),(1,1)}
\end{gathered}$$

which induces the integraded form

$$\begin{aligned} \pi_\varphi \times U_\varphi : C^*(\mathscr{G}) \rtimes_{\delta_c} H &\to \mathcal{M}(C^*(\mathscr{G} \times_{c,\varphi} H)); \\ j_{C^*(\mathcal{G})}(s_e) &\mapsto \sum_{t \in H} s_{(e,t)} \\ j_{C^*(\mathcal{G})}(u_{x,g}) &\mapsto \sum_{t \in H} u_{(x,t),(g,1)} \\ j_H(\chi_t) &\mapsto \sum_{x \in \Gamma^0} u_{(x,t),(1,1)} \end{aligned}$$

Define $\kappa : C^*(H) \to \mathcal{M}(C^*(\mathscr{G} \times_{c,\varphi} H))$ by $u_h \mapsto \sum_{\substack{x \in \Gamma^0 \\ t \in H}} u_{(x,t),(1,h)}$, which is indeed a $*$-homomorphism.

We want to show that the images of the two homomorphisms intertwine. Lets have a look at the generators $j_{C^*(\mathcal{G})}(s_e) j_H(\chi_t), j_{C^*(\mathcal{G})}(u_{x,g}) j_H(\chi_t), u_h$:

$$\begin{aligned} (\pi_\varphi \times U_\varphi)(j_{C^*(\mathcal{G})}(s_e) j_H(\chi_t))\kappa(u_h) &= \sum_{s \in H} s_{(e,s)} \sum_{x \in \Gamma^0} u_{(x,t),(1,1)} \sum_{\substack{y \in \Gamma^0 \\ t' \in H}} u_{(y,t'),(1,h)} \\ &= s_{(e,t)} \sum_{\substack{y \in \Gamma^0 \\ t' \in H}} u_{(y,t'),(1,h)} \\ &= s_{(e,t)} u_{(s(e),t),(1,h)} \\ &= s_{(e,t)} u_{s_c(e,t),(1,h)}, \text{ and} \\ \kappa(u_h)(\pi_\varphi \times U_\varphi)(j_{C^*(\mathcal{G})}(s_e) j_H(\chi_t)) &= \sum_{\substack{y \in \Gamma^0 \\ t' \in H}} u_{(y,t'),(1,h)} \sum_{s \in H} s_{(e,s)} \sum_{x \in \Gamma^0} u_{(x,t),(1,1)} \\ &= u_{(r(e),c(e)t),(1,h)} s_{(e,t)} u_{(s(e),t),(1,1)} \\ &= u_{r_c(e,t),(1,h)} s_{(e,t)} u_{s_c(e,t),(1,1)} \\ &= u_{r_c(e,t),(1,h)} s_{(e,t)} = u_{r_c(e,t),\alpha_{(e,t)}(1,\varphi^{-1}(h))} s_{(e,t)} \\ &= s_{(e,t)} u_{s_c(e,t),\alpha_{\overline{(e,t)}}(1,\varphi^{-1}(h))} = s_{(e,t)} u_{s_c(e,t),(1,h)}. \end{aligned}$$

The similar goes to the generators $j_{C^*(\mathcal{G})}(u_{x,g})j_H(\chi_t), u_h$. Therefore, we have shown that $(\pi_\varphi \times U_\varphi)(C^*(\mathscr{G}) \rtimes_{\delta_c} H)\kappa(C^*(H)) = \kappa(C^*(H)(\pi_\varphi \times U_\varphi)(C^*(\mathscr{G}) \rtimes_{\delta_c} H)$ and by theorem 6.13, there exists a homomorphism

$$(\pi_\varphi \times U_\varphi) \otimes \kappa : C^*(\mathscr{G}) \rtimes_{\delta_c} H \otimes C^*(H) \to \mathcal{M}(C^*(\mathscr{G} \times_{c,\varphi} H));$$

$$j_{C^*(\mathcal{G})}(s_e) \otimes u_h \mapsto \sum_{t\in H} s_{(e,t)} \sum_{\substack{x\in\Gamma^0\\ s\in H}} u_{(x,s),(1,h)} = \sum_{\substack{t,s\in H\\ y\in\Gamma^0}} s_{(e,t)}u_{(x,s),(1,h)} = \sum_{t\in H} s_{(e,t)}u_{s_c(e,t),(1,h)}, \text{ and}$$

$$j_{C^*(\mathcal{G})}(u_{x,g}) \otimes u_h \mapsto \sum_{t\in H} u_{(x,t),(g,1)} \sum_{\substack{y\in\Gamma^0\\ s\in H}} u_{(y,s),(1,h)} = \sum_{\substack{t,s\in H\\ y\in\Gamma^0}} u_{(x,t),(g,1)}u_{(y,s),(1,h)} = \sum_{t\in H} u_{(x,t),(g,h)}.$$

We will show that $(\pi_\varphi \times U_\varphi) \otimes \kappa$ and ϕ are inverses of each other.

$$\begin{aligned}
(\phi \circ (\pi_\varphi \times U_\varphi) \otimes \kappa)(j_{C^*(\mathcal{G})}(s_e) \otimes u_h) &= \phi(\sum_{t\in H} s_{(e,t)}u_{s_c(e,t),(1,h)}) \\
&= \sum_{t\in H} \phi(s_{(e,t)})\phi(u_{(s(e),t),(1,h)}) \\
&= \sum_{t\in H} (j_{C^*(\mathcal{G})}(s_e)j_H(\chi_t)j_{C^*(\mathcal{G})}(u_{s(e),1})j_H(\chi_t) \otimes u_h) \\
&= \sum_{t\in H} j_{C^*(\mathcal{G})}j_{C^*(\mathcal{G})}(s_e)j_H(\chi_t) \otimes u_h \\
&= j_{C^*(\mathcal{G})}(s_e) \otimes u_h; \\
(\phi \circ (\pi_\varphi \times U_\varphi) \otimes \kappa)(j_{C^*(\mathcal{G})}(u_{x,g}) \otimes u_h) &= \phi(\sum_{t\in H} u_{(x,t),(g,h)}) \\
&= \sum_{t\in H} j_{C^*(\mathcal{G})}(u_{x,g})j_H(\chi_t) \otimes u_h \\
&= j_{C^*(\mathcal{G})}(u_{x,g}) \otimes u_h, \text{ and} \\
(\phi \circ (\pi_\varphi \times U_\varphi) \otimes \kappa)(j_H(\chi_t) \otimes u_h) &= \phi(\sum_{x\in\Gamma_0} u_{(x,t),(1,h)}) \\
&= \sum_{x\in\Gamma_0} \phi(u_{(x,t),(1,h)}) \\
&= \sum_{x\in\Gamma^0} j_{C^*(\mathcal{G})}(u_{x,1})j_H(\chi_t) \otimes u_h \\
&= j_H(\chi_t) \otimes u_h.
\end{aligned}$$

Also,

$$\begin{aligned}
((\pi_\varphi \times U_\varphi) \otimes \kappa) \circ \phi(s_{(e,t)}) &= ((\pi_\varphi \times U_\varphi) \otimes \kappa)(j_{C^*(\mathcal{G})}(s_e) j_H(\chi_t) \otimes u_1) \\
&= s_{(e,t)} \sum_{\substack{x \in \Gamma^0 \\ s \in H}} u_{(x,s),(1,1)} \\
&= \sum_{\substack{x \in \Gamma^0 \\ s \in H}} s_{(e,t)} u_{(x,s),(1,1)} \\
&= s_{(e,t)} u_{s_c(e,t),(1,1)} = s_{(e,t)}, \text{ and} \\
((\pi_\varphi \times U_\varphi) \otimes \kappa) \circ \phi(u_{(x,t),(g,h)}) &= ((\pi_\varphi \times U_\varphi) \otimes \kappa)(j_{C^*(\mathcal{G})}(u_{x,g}) j_H(\chi_t) \otimes u_h) \\
&= u_{(x,t),(g,1)} \sum_{\substack{y \in \Gamma^0 \\ s \in H}} u_{(y,s),(1,h)} = u_{(x,t),(g,h)}.
\end{aligned}$$

□

This section talks about groupoids generated by graph of groups. It is discovered that there is a unique relationship between the C^*-algebra of graph of groups and the C^*-algebra of some groupoid induced by the graph of groups, called the fundamental groupoid.

6.3 Groupoids

A groupoid can be defined in two ways: The algebraic or category way. Recall from earlier chapters, a category $\mathcal{C}$ consists of the morphism set $\hom(\mathcal{C})$ and the object set $ob(\mathcal{C})$. Recall for any morphism in $\mathcal{C}$, its inverse need not exist. A groupoid is a category where any morphism $f : X \to Y$ has an inverse $f^{-1} : Y \to X$. For the algebraic version, we follow the definition from [20].

DEFINITION 6.14 (Groupoid in algebraic sense). *A **groupoid** is a set $\mathcal{G}$ together with a distinguished subset $\mathcal{G}^{(2)} \subseteq \mathcal{G} \times \mathcal{G}$, a multiplication map $(\alpha, \beta) \mapsto \alpha\beta$ from $\mathcal{G}^{(2)}$ to $\mathcal{G}$ and an inverse map $\gamma \mapsto \gamma^{-1}$ from $\mathcal{G} \times \mathcal{G}$ such that*

(1) $(\gamma^{-1})^{-1} = \gamma$ for all $\gamma \in \mathcal{G}$;

(2) *if* (α, β) *and* (β, γ) *belong to* $\mathcal{G}^{(2)}$*, then* $(\alpha\beta, \gamma)$ *and* $(\alpha, \beta\gamma)$ *belong to* $\mathcal{G}^{(2)}$*, and* $(\alpha\beta)\gamma = \alpha(\beta\gamma)$*; and*

(3) $(\gamma, \gamma^{-1}) \in \mathcal{G}^{(2)}$ *for all* $\gamma \in \mathcal{G}$*, and for all* $(\gamma, \eta) \in \mathcal{G}^{(2)}$*, we have* $\gamma^{-1}(\gamma\eta) = \eta$ *and* $(\gamma\eta)\eta^{-1} = \gamma$.

The *unit space* of $\mathcal{G}$ is $\mathcal{G}^{(0)} := \{\gamma^{-1}\gamma : \gamma \in \mathcal{G}\}$. Elements in the unit space are called *units*. The definition implies $\mathcal{G}^{(0)} = \{\gamma\gamma^{-1} : \gamma \in \mathcal{G}\}$. Define $r, s : \mathcal{G} \to \mathcal{G}^{(0)}$ by $r(\gamma) = \gamma\gamma^{-1}$ and $s(\gamma) = \gamma^{-1}\gamma$.

For a groupoid $\mathcal{G}$, the algebraic and categorical definition align as follows:

(1) The unit space is the set of objects: $\mathcal{G}^{(0)} = ob(\mathcal{G})$;

(2) Each element $\gamma \in \mathcal{G} \setminus \mathcal{G}^{(0)}$ is a function from its source to its range. That is, for $\gamma \in \mathcal{G} \setminus \mathcal{G}^{(0)}$ with $s(\gamma) = x, r(\gamma) = y$, $\gamma \in hom(x, y)$;

(3) $(\alpha, \beta) \in \mathcal{G}^{(2)}$ means α, β are composable in the morphism sense;

(4) rule (2) from the groupoid definition states that if the pair (α, β) and (β, γ) are composable pairs, then we $\alpha\beta$ and γ are composable pairs, as well as α and $\beta\gamma$; and

(5) rule (3) from the groupoid definition states that for morphisms $\gamma : X \to Y$, $\gamma\gamma^{-1} = 1_Y$ and $\gamma^{-1}\gamma = 1_X$.

EXAMPLE 6.15. *A group* G *is considered a groupoid where the unit space* $G^{(0)}$ *is the identity set* $\{1_G\}$ *and every elements* $g, h \in G$ *are composable with* $r(g) = s(g) = 1_G$.

6.4 Fundamental Groupoids and Étale Groupoids

In this section, the definition for a fundamental groupoid of a graph of groups is mentioned. Then the fibred product groupoid, an action groupoid that stores the action of graph of groups on its boundary. This groupoid has a special connection with the graph of groups.

A groupoid needs to have a topological structure before a C^*-algebra of the groupoid can be considered. The étale groupoid is an analogue of discrete groups.

DEFINITION 6.16. *Let* $\mathscr{G} = (\Gamma, G)$ *be a graph of groups. The fundamental groupoid, denoted* $\mathcal{F}(\mathscr{G})$*, has edge set* $\Gamma^1 \cup (\bigcup_{x \in \Gamma^0} G_x)$*, where* $e : s(e) \to r(e)$ *for any* $e \in \Gamma^1$ *and* $g : x \to x$ *for any* $g \in G_x$*, together with the relationship:*

(1) $\overline{e} = e^{-1}$ *for all* $e \in \Gamma^1$*; and*
(2) $e\alpha_{\overline{e}}(g)\overline{e} = \alpha_e(g)$ *for all* $e \in \Gamma^1$ *and all* $g \in G_e = G_{\overline{e}}$*.*

One can understand the groupoid by considering the vertices in $\mathscr{G}$ as spaces, and the edges and group elements are functions from one to other. Rule (1) states that the "ghost" edge $\overline{e}$ is considered the inverse of the edge e, and the same vice versa. Rule (2) shows that any representation of G-words in groupoid can be reduced, just as in the graph of groups.

DEFINITION 6.17 (**Topological groupoids**). *A topological groupoid is a groupoid* $\mathcal{G}$ *equipped with a locally compact topology under which* $\mathcal{G}^{(0)} \subseteq \mathcal{G}$ *is Hausdorff in the relative topology, the maps* r, s *and inverse map* $\gamma \mapsto \gamma^{-1}$ *are continuous, and the map* $(g, h) \mapsto gh$ *is continuous with respect to the relative topology on* $\mathcal{G}^{(2)}$ *as a subset of* $\mathcal{G} \times \mathcal{G}$*.*

EXAMPLE 6.18 (Discrete groupoids). *Every groupoid is a topological groupoid in the discrete topology.*

DEFINITION 6.19 (**Étale groupoids**). *A topological groupoid* $\mathcal{G}$ *is* ***étale*** *if the range map* $r : \mathcal{G} \to \mathcal{G}^{(0)}$ *is a local homeomorphism.*

EXAMPLE 6.20 (Discrete groupoids). *Every groupoid is an étale groupoid.*

EXAMPLE 6.21 (Graph groupoids). *Let* E *be a row-finite directed graph with no sources. That is, for any vertex* $v \in E^0$*,* $0 < |r^{-1}(v)| < \infty$*. Then the set of infinite paths* E^∞ *can be equipped with the topology inherited from the product space* $\prod_{i=1}^{\infty} E^1$ *and* E^∞ *becomes a totally-disconnected locally compact Hausdorff space. For any finite path* $\mu \in E^*$*, the* ***cylinder set*** *of* μ*, denoted* $Z(\mu)$*, is the set of infinite paths in* E *that extends* μ*. That is,* $Z(\mu) = \{\mu x : x \in E^\infty, r(x) = s(\mu)\}$*. Then the collection of cylinder sets* $\{Z(\mu) : \mu \in E^*\}$ *form a base of compact open sets for the topology. The map* $\sigma : E^\infty \to E^\infty$ *given by* $\sigma(x)_i = x_{i+1}$ *is a local homeomorphism, and it induces an action of* $\mathbb{N}$ *by local homeomorphisms. The* *associated Deaconu-Renault groupoid* $\mathcal{G}_E = \{(x, m-n, y) : \sigma^m(x) = \sigma^n(y)\}$ *is called the graph groupoid of E.*

A very useful tool to understand the structure of an étale groupoid is via its *bisection* and is used heavily when identifying its C^*-algebra:

DEFINITION 6.22. *A* ***bisection*** *of an étale groupoid* $\mathscr{G}$ *is a subset* B *such that there is an open set* U *containing* B *such that* $r : U \to r(U)$ *and* $s : U \to s(U)$ *are both homeomorphisms onto open subsets of* $\mathscr{G}$*.*

Define $\partial W_{\mathscr{G}}$ to be the boundary of $\mathscr{G}$: the set of infinite $\mathscr{G}$-paths in $\mathscr{G}$ and $x\partial W_{\mathscr{G}}$ be the set of infinite $\mathscr{G}$-paths in $\mathscr{G}$ with range x. Notice that for a G-word τ and a $\mathscr{G}$-paths ξ such that $s(\tau) = r(\xi)$, they form a $\mathscr{G}$-path by concatenating them. Therefore, the set of $\mathscr{G}$-words in $\mathscr{G}$ naturally form an action on the boundary set, called the *groupoid actions*.

The action groupoid, called the fibred product groupoid, originated from definition 2.5 in [21] describes the above action:

DEFINITION 6.23 (**Fibred product groupoids**). *Let* $\mathscr{G}$ *be a graph of groups. The fibred product groupoid, denoted* $F(\mathscr{G}) * \partial W_{\mathscr{G}}$*, is the set*

$$\{(\tau, \xi) : \tau \text{ is a reduced } \mathscr{G}\text{-word, } \xi \text{ is an infinite } \mathscr{G}\text{-path, } s(\tau) = r(\xi)\}$$

such that

(1) $(\tau, \xi)^{-1} = (\tau^{-1}, \tau\xi)$*;*
(2) $((\tau, \xi), (\lambda, \eta)) \in (F(\mathscr{G}) * \partial W_{\mathscr{G}})^{(2)}$ *if and only if* $\lambda\eta = \xi$*. If that is the case,*

$$(\tau, \xi)(\lambda, \eta) = (\tau\lambda, \eta); \text{ and}$$

(3) $r(\tau, \xi) = (1_{r(\tau)}, \tau\xi)$ *and* $s(\tau, \xi) = (1_{r(\xi)}, \xi)$*.*

LEMMA 6.24. *Let* $\mathscr{G}$ *be a graph of groups and* $F(\mathscr{G}) * \partial W_{\mathscr{G}}$ *be its fibred product groupoid. Then* $F(\mathscr{G}) * \partial W_{\mathscr{G}}$ *is an étale groupoid.*

PROOF. The fundamental groupoid $F(\mathscr{G})$ is a topological space with the discrete topology and the sets $\mathcal{B}_{F(\mathscr{G})} := \{\{\mu\} : \mu$ a reduced $\mathscr{G}$-word$\}$ forms a base for the discrete topology. Just as in the graph groupoid example, define the cylinder set of a $\mathscr{G}$-path μ in $\mathscr{G}$ as $Z(\mu) := \{\mu x : \mu x$ an infinite $\mathscr{G}$-path$\}$. Then again, the collection of cylinder sets forms a base of compact open sets for the topology, and $\partial W_{\mathscr{G}}$ becomes topological space. The product space $F(\mathscr{G}) \times \partial W_{\mathscr{G}}$ becomes a topological space with the product topology generated by the basis $\mathcal{B}_{F(\mathscr{G}) \times \partial W_{\mathscr{G}}} := \{\{\mu\} \times Z(\nu) :$ μ a reduced $\mathscr{G}$-word, ν an infinite $\mathscr{G}$-path$\}$. Notice that the fibred product groupoid $F(\mathscr{G}) * \partial W_{\mathscr{G}}$ is a subset of $F(\mathscr{G}) \times \partial W_{\mathscr{G}}$. Define $\mathcal{B}_{F(\mathscr{G}) * \partial W_{\mathscr{G}}} := \{\{\mu\} \times Z(\nu) :$ μ a reduced $\mathscr{G}$-word, ν an infinite $\mathscr{G}$-path, $s(\mu) = r(\nu)\}$. Then $F(\mathscr{G}) * \partial W_{\mathscr{G}}$ becomes a topological space with base $\mathcal{B}_{F(\mathscr{G}) * \partial W_{\mathscr{G}}}$. The range map $r : F(\mathscr{G}) * \partial W_{\mathscr{G}} \to (F(\mathscr{G}) * \partial W_{\mathscr{G}})^{(0)}$ is indeed a local homeomorphism. To see that, let $(\mu, \xi) \in F(\mathscr{G}) * \partial W_{\mathscr{G}}$ such that $\xi = \nu\zeta$ for some $\mathscr{G}$-path ν and infinite $\mathscr{G}$-path ζ. Then the image $r((\mu, Z(\nu)))$ is open in $(F(\mathscr{G}) * \partial W_{\mathscr{G}})^{(0)}$ and $r|_{(\mu, Z(\nu))} : (\mu, Z(\nu)) \to r(\mu, Z(\nu))$ is homeomorphic. So r is a local homeomorphism and $F(\mathscr{G}) * \partial W_{\mathscr{G}}$ becomes an étale groupoid. □

In the next section, we will show that the the C^*-algebra of the graph of groups and the C^*-algebra of the associated fibred product groupoid is isomorphic.

6.5 Graph of Groups Algebras and Fibred Product Groupoid Algebras

It is noted in Remark 4.10 from [10] that the semidirect product groupoid $\pi_1(\mathscr{G}, v) \ltimes v\partial X_{\mathscr{G}}$ and $F(\mathscr{G}) * \partial W_{\mathscr{G}}$ are equivalent groupoids, and hence have stably isomorphic C^*-algebras. This means $C^*(\mathscr{G})$ is isomorphic to $C^*(F(\mathscr{G}) * \partial W_{\mathscr{G}})$. Here we use this fact to show the isomorphism of two crossed products.

The construction of C^*-algebra of an étale groupoid $\mathscr{G}$ is done by the following steps: Consider the the complex vector space $C_c(\mathscr{G})$, the set of continuous compact supported functions from $\mathscr{G}$ to $\mathbb{C}$. Equip $C_c(\mathscr{G})$ with the *convolution* and *involution*:

$$(f * g)(\tau, \xi) = \sum_{(\alpha,\lambda)(\beta,\eta)=(\tau,\xi)} f(\alpha, \lambda) g(\beta, \eta) \text{ and } f^*(\tau, \xi) = \overline{f(\tau, \xi)^{-1}}$$

for some $f, g \in C_c(\mathscr{G})$ and $(\tau, \xi) \in F(\mathscr{G}) * \partial W_{\mathscr{G}}$. Then $C_c(\mathscr{G})$ becomes a $*$-algebra. Next, equip $C_c(\mathscr{G})$ with a norm, then the C^*-algebra $C^*(\mathscr{G})$ is the norm-completed of $C_c(\mathscr{G})$. More details can be found in [20].

LEMMA 6.25. *Let $\mathscr{G} = (\Gamma, G)$ be a graph of groups, H be a discrete group, and let $c : \Gamma^1 \to H$ be a labelling with cocycle property. Then the induced coaction δ_c induces a coaction β of H on $C^*(F(\mathscr{G}) * \partial W_{\mathscr{G}})$ such that*

$$C^*(\mathscr{G}) \rtimes_{\delta_c} H \cong C^*\big(F(\mathscr{G}) * \partial W_{\mathscr{G}}\big) \rtimes_{\beta} H.$$

PROOF. Let $\phi : C^*(\mathscr{G}) \to C^*(F(\mathscr{G}) * \partial W_{\mathscr{G}})$ be the isomorphism. Define $\beta : C^*(F(\mathscr{G}) * \partial W_{\mathscr{G}}) \to C^*(F(\mathscr{G}) * \partial W_{\mathscr{G}}) \otimes C^*(H)$ by

$$\beta := \big(\phi \otimes \mathrm{id}_{C^*(H)}\big) \circ \delta_c \circ \phi^{-1}$$

Then β is indeed a coaction:

$$\begin{aligned}
\Big(\beta \otimes \mathrm{id}_{C^*(H)}\Big) \circ \beta &= (((\phi \otimes \mathrm{id}_{C^*(H)}) \circ \delta_c \circ \phi^{-1}) \otimes \mathrm{id}_{C^*(H)}) \circ (\phi \otimes \mathrm{id}_{C^*(H)}) \circ \delta_c \circ \phi^{-1} \\
&= (((\phi \otimes \mathrm{id}_{C^*(H)}) \circ \delta_c) \otimes \mathrm{id}_{C^*(H)}) \circ \delta_c \circ \phi^{-1} \\
&= ((\phi \otimes \mathrm{id}_{C^*(H)}) \otimes \mathrm{id}_{C^*(H)}) \circ (\delta_c \otimes \mathrm{id}_{C^*(H)}) \circ \delta_c \circ \phi^{-1} \\
&= ((\phi \otimes \mathrm{id}_{C^*(H)}) \otimes \mathrm{id}_{C^*(H)}) \circ (\mathrm{id}_{C^*(\mathscr{G})} \otimes \delta_H) \circ \delta_c \circ \phi^{-1} \\
&= ((\phi \otimes (\mathrm{id}_{C^*(H)} \otimes \mathrm{id}_{C^*(H)})) \circ (\mathrm{id}_{C^*(\mathscr{G})} \otimes \delta_H) \circ \delta_c \circ \phi^{-1} \\
&= (\phi \otimes \delta_H) \circ \delta_c \circ \phi^{-1} \\
&= (\mathrm{id} \otimes \delta_H) \circ (\phi \otimes \mathrm{id}) \circ \delta_c \circ \phi^{-1} \\
&= \Big(\mathrm{id} \otimes \delta_H\Big) \circ \beta.
\end{aligned}$$

Recall that for dynamical system (B, H, δ), the crossed product $B \rtimes_{\delta} H$ is generated by a universal covariant representation (j_B, j_G). For crossed products $C^*(\mathscr{G}) \rtimes_{\delta_c} H$ and $C^*(F(\mathscr{G}) *$

$\partial W_{\mathscr{G}}) \rtimes_\beta H$, let $(j_{C^*(\mathscr{G})}, j_H)$ and $(i_{C^*(F(\mathscr{G})*\partial W_{\mathscr{G}})}, i_H)$ be the respective universal covariant representation.

Define $\pi : C^*(\mathscr{G}) \to C^*(F(\mathscr{G}) * \partial W_{\mathscr{G}}) \rtimes_\beta H$ and $U : C_0(H) \to C^*(F(\mathscr{G}) * \partial W_{\mathscr{G}}) \rtimes_\beta H$ by

$$\pi = i_{C^*(F(\mathscr{G})*\partial W_{\mathscr{G}})} \circ \phi \text{ and } U = i_H.$$

Then (π, U) is indeed a covariant homomorphism: First, suppose that $a_s \in C^*(\mathscr{G})_s$ for some $s \in H$, that is, $\delta_c(a_s) = a_s \otimes u_s$, then

$$\begin{aligned}\beta(\phi(a_s)) &= (\phi \otimes \mathrm{id}_{C^*(H)}) \circ \delta(a_s)\\ &= (\phi \otimes \mathrm{id}_{C^*(H)})(a_s \otimes u_s)\\ &= \phi(a_s) \otimes u_s.\end{aligned}$$

So $\phi(a_s) \in C^*(F(\mathscr{G}) * \partial W_{\mathscr{G}})_s$. Then

$$\pi(a_s)U(\chi_t) = i_{C^*(F(\mathscr{G})*\partial W_{\mathscr{G}})}(\psi(a_s))i_H(\chi_t) = i_H(\chi_{st})i_{C^*(F(\mathscr{G})*\partial W_{\mathscr{G}})}(\psi(a_s)).$$

By the universal property, there exists an integrated form $\pi \times U : C^*(\mathscr{G}) \rtimes_{\delta_c} H \to C^*(F(\mathscr{G}) * \partial W_{\mathscr{G}}) \rtimes_\beta H$ such that

$$(\pi \times U) \circ j_{C^*(\mathscr{G})} = \pi = i_{C^*(F(\mathscr{G})*\partial W_{\mathscr{G}})} \circ \phi \text{ and } (\pi \times U) \circ j_H = U = i_H.$$

Define $\pi' : C^*(F(\mathscr{G}) * \partial W_{\mathscr{G}}) \to C^*(\mathscr{G}) \rtimes_{\delta_c} H$ and $U' : C_0(H) \to C^*(\mathscr{G}) \rtimes_{\delta_c} H$ by

$$\pi' = j_{C^*(\mathscr{G})} \circ \phi^{-1} \text{ and } U' = j_H.$$

Then (π', U') is indeed a covariant homomorphism: First, suppose that $b_s \in C^*(F(\mathscr{G}) * \partial W_{\mathscr{G}})_s$, that is, $\beta(b_s) = b_s \otimes u_s$. Then

$$\begin{aligned}b_s \otimes u_s = \beta(b_s) &= (\phi \otimes \mathrm{id}_{C^*(H)}) \circ \delta_c(\phi^{-1}(b_s))\\ &= (\phi \otimes \mathrm{id}_{C^*(H)})(\phi^{-1}(b_s) \otimes u_g)\\ &= b_s \otimes u_g\end{aligned}$$

for some $g \in H$. This means $g = s$ and $\phi^{-1}(b_s) \in C^*(\mathscr{G})_s$. Then

$$\pi'(b_s)U'(\chi_t) = j_{C^*(\mathscr{G})}(\phi^{-1}(b_s))j_H(\chi_t) = j_H(\chi_s t)j_{C^*(\mathscr{G})}(\phi^{-1}(b_s)).$$

By the universal property, there exists an integrated form $\pi' \times U' : C^*(F(\mathscr{G}) * \partial W_{\mathscr{G}}) \rtimes_\beta H \to C^*(\mathscr{G}) \rtimes_{\delta_c} H$ such that

$$(\pi' \times U') \circ i_{C^*(F(\mathscr{G})*\partial W_{\mathscr{G}})} = \pi' = j_{C^*(\mathscr{G})} \circ \phi^{-1} \text{ and } (\pi' \times U') \circ i_H = U' = j_H.$$

We will show that $\pi \times U$ and $\pi' \times U'$ are inverses of each other.

$$\begin{aligned}
(\pi' \times U') \circ (\pi \times U)(j_{C^*(\mathscr{G})}(a)) &= (\pi' \times U')(i_{C^*(F(\mathscr{G})*\partial W_{\mathscr{G}})}(\phi(a))) \\
&= j_{C^*(\mathscr{G})} \circ \phi^{-1}(\phi(a)) \\
&= j_{C^*(\mathscr{G})}(a); \\
(\pi' \times U') \circ (\pi \times U)(j_H(\chi_t)) &= (\pi' \times U')(i_H(\chi_t)) \\
&= j_H(\chi_t); \\
(\pi \times U) \circ (\pi' \times U')(i_{C^*(F(\mathscr{G})*\partial W_{\mathscr{G}})}(a)) &= (\pi \times U)(j_{C^*(\mathscr{G}}(\phi^{-1}(a))) \\
&= i_{C^*(F(\mathscr{G})*\partial W_{\mathscr{G}})} \circ \phi(\phi^{-1}(a)) \\
&= i_{C^*(F(\mathscr{G})*\partial W_{\mathscr{G}})}(a); \text{ and} \\
(\pi \times U) \circ (\pi' \times U')(i_H(\chi_t)) &= (\pi \times U)(j_H(\chi_t)) \\
&= i_H(\chi_t).
\end{aligned}$$

We have shown that $(\pi' \times U') \circ (\pi \times U) = \mathrm{id}_{C^*(\mathscr{G}) \rtimes_{\delta_c} H}$ and $(\pi \times U) \circ (\pi' \times U') = \mathrm{id}_{C^*(F(\mathscr{G})*\partial W_{\mathscr{G}}) \rtimes_\beta H}$, completing the proof. □

Given the above, we have:

THEOREM 6.26. *Let $\mathscr{G} = (\Gamma, G)$ be a graph of groups, H be a discrete group, and $c : \Gamma^1 \to H$ be a labelling with the cocycle property. Then*

$$C^*(\mathscr{G} \times_c H) \cong C^*(\mathscr{G}) \rtimes_{\delta_c} H \cong C^*(F(\mathscr{G}) * \partial W_{\mathscr{G}}) \rtimes_\beta H,$$

where $\beta = \left(\phi \otimes \mathrm{id}_{C^*(H)}\right) \circ \delta_c \circ \phi^{-1}$.

PROOF. Use the above lemma and the theorem 6.8. ☐

Initially in the above theorem, we were looking to include another isomorphism of C^*-algebra to it. This C^*-algebra is regarding to the concept of **skew-product groupoids**, which are analogues of skew-product graphs and skew-proudct graph of groups. For more details, refer to [2].

DEFINITION 6.27. *Let* Q *be an étale groupoid and* G *be a discrete group. We can realise* G *as a groupoid. Suppose that* $d : Q \to G$ *is a continuous groupoid homomorphism. The* ***skew-product groupoid****, denoted* $Q \times_c G$ *is the set* $Q \times G$ *with the induced product topology and operations given for* $(x, y) \in Q^2$ *and* $s \in G$ *by*

$$(x, c(y)s)(y, s) = (xy, s) \quad \textit{and} \quad (x, s)^{-1} = (x^{-1}, c(x)s)$$

and

$$s(x, s) = (s(x), s) \quad \textit{and} \quad r(x, s) = (r(x), c(x)s).$$

For each $s \in G$, define $D_s = \{f \in C_c(Q) : \operatorname{supp} f \subseteq d^{-1}(s)\}$. and put $\mathcal{D} = \bigcup_{s \in G} D_s$. Then with the opeartion from $C_c(Q)$, $\mathcal{D}$ becomes a fiber bundle over G and $D_s D_t \subseteq D_{st}$ and $C_s^* = C_{s^{-1}}$. We have $\operatorname{span}_{s \in G} D_s = C_c(Q)$. This leads to the following lemma:

LEMMA 6.28. *Let d be a countinuous groupoid homomorphism of an étale groupoid Q into a discrete group G. Then there is a coaction* Δ_d *of G on* $C^*(Q)$ *such that*

$$\Delta_d(f_s) = f_s \otimes u_s$$

for all $s \in G, f_s \in D_s$.

PROOF. The proof can be found in lemma 4.2 from [2]. ☐

THEOREM 6.29. *Let d be a continuous groupoid homomorphism of an étale groupoid Q to a discrete group G, and let Δ be the coaction in the pervious lemma, Then*

$$C^*(Q) \rtimes_{\Delta_d} G \cong C^*(Q \times G).$$

PROOF. The proof can be found in Theorem 4.3 from [2]. □

Given the above, it might be an illusion to assume the crossed product $C^*(F(\mathscr{G}) * \partial W_{\mathscr{G}}) \rtimes_\beta H$ in Theorem 6.26 has the same structure as the crossed product in the above theorem. This is not true, even if it is, the proof is not trivial: One can see that the spectral subspaces are different, since the spectral subspaces by β inherits from δ_c, as we saw earlier. in particular, if $a_s \in C^*(\mathscr{G})_s$, then $\phi(a_s) \in C^*(F(\mathscr{G}) * \partial W_{\mathscr{G}})_s$. However, the spectral subspace D_s, as we have seen in the above, is based on the support of the functions in $C_c(F(\mathscr{G}) * \partial W_{\mathscr{G}})$. In order to show that the coaction β is a an induced coaction Δ_d of some groupoid homomorphism $d : F(\mathscr{G}) * \partial W_{\mathscr{G}} \to H$. We need to show that $D_s = C^*(F(\mathscr{G}) * \partial W_{\mathscr{G}})_s$, which is not trival.

If the above is satisfied, and we managed to find the underlying groupoid homomorphism d, we will have the following:

CONJECTURE 6.30. *Let $\mathscr{G} = (\Gamma, G)$ be a graph of groups, H be a discrete group, and $c : \Gamma^1 \to H$ be a labelling with the cocycle property. Then there exists an induced continuous groupoid homomorphism $d : F(\mathscr{G}) * \partial W_{\mathscr{G}} \to H$ such that*

$$C^*(\mathscr{G} \times_c H) \cong C^*(\mathscr{G}) \rtimes_{\delta_c} H \cong C^*(F(\mathscr{G}) * \partial W_{\mathscr{G}}) \rtimes_\beta H \cong C^*((F(\mathscr{G}) * \partial W_{\mathscr{G}}) \times_d H),$$

where $\beta = \big(\phi \otimes \mathrm{id}_{C^*(H)}\big) \circ \delta_c \circ \phi^{-1}$.

We suspect that the induced groupoid homomorphism $d : F(\mathscr{G}) * \partial W_{\mathscr{G}} \to H$ is

$$d(g_1 e_1 \cdots g_n e_n g_{n+1}, \nu) = \prod_{i=1}^{n} c(e_i).$$

Part 2

K-Theory

CHAPTER 7

The K_0-Group

Part 2 of this thesis develops the K-theory foundations essential for computing the K-groups of graph of groups C^*-algebras using the skew-product techniques introduced in Part 1. The K-theory of a C^*-algebra consists of two primary invariants: the K_0-group, which captures information about projections and their equivalences, and the K_1-group, which encodes information about unitary elements and their homotopy classes. These invariants are particularly powerful because they are functorial, meaning that homomorphisms between C^*-algebras induce homomorphisms between their K-groups, enabling the use of exact sequence techniques.

This chapter focuses on the K_0-group, establishing the theoretical framework that will enable us to compute K_0-groups for skew-product graph of groups algebras through their connection to crossed products by coactions. The material follows the systematic exposition by Rørdam, Larsen and Laustsen [11], emphasising the computational aspects that will be crucial for our applications to graph of groups C^*-algebras.

7.1 Equivalence Classes of Projections

The construction of the K_0-group begins with understanding how projections in a C^*-algebra can be organised into a semigroup structure through the operation of direct sum. However, to achieve the full power of K-theory as a computational tool, we must pass from projections to equivalence classes of projections, where the equivalence relation is defined through homotopy. This approach captures the essential algebraic information while providing the stability properties necessary for functoriality and exact sequence techniques.

DEFINITION 7.1. *Let A be a C^*-algebra. The two elements $a, b \in A$ are called* ***homotopic*** *in A, denoted $a \sim_h b$, if there's a continuous function $\nu : [0,1] \to A$ such that $\nu(0) = a$ and $\nu(1) = b$. ν is called a continuous path.*

One imagines there's a "continuous" line of points in A that "connects" a and b. The homotopy defined above is an equivalence relation. That is, if $a \sim_h b$ and $b \sim_h c$, then it satisfies

(1) $a \sim_h a$;
(2) $b \sim_h a$; and
(3) $a \sim_h c$.

This is not hard to see. The continuous path from a to itself would be $\nu(t) = a$ for all $t \in [0,1]$. Suppose that there is a continuous path $\nu : [0,1] \to A$ such that $\nu(0) = a$ and $\nu(1) = b$. Define a new path $\mu : [0,1] \to A$ by $\mu(t) = \nu(1-t)$. Then μ is a continuous path and satisfies $\mu(0) = b$ and $\mu(1) = a$. For part (3). Suppose that μ, ν are continuous paths in A from A to B and $B \to C$ respectively. Define path $w(t) = (1-t)\nu(t) + t\mu(t)$. Then $w(0) = a$ and $w(1) = c$.

The following lemma is often used in calculation:

LEMMA 7.2. *Suppose that A is a unital C^*-algebra and let $u, v \in A$ be unitary elements. Then*

$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \sim_h \begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} vu & 0 \\ 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}$$

in $\mathcal{U}(M_2(A))$. In addition,

$$\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \sim_h \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

PROOF. See [11, Lemma 2.1.5] for proof. □

DEFINITION 7.3 (*Murray-von Neumann equivalence*). *Let p, q be projections in A. We say p and q are* ***Murray-von Neumann equivalent****, denoted*

$p \sim_{mv} q$ *if there is a partial isometry $v \in A$ such that $p = v^*v$ and $q = vv^*$.*

This definition can be extended to projections of different dimension. In particular,

DEFINITION 7.4. *Let p be projections in $M_n(A)$ and q be projections in $M_m(A)$. We say p and q are **equivalent**, denoted $p \sim q$, if there is a partial isometry $v \in M_{m,n}(A)$ such that $p = v^*v$ and $q = vv^*$.*

Here v is a $m \times n$ matrix with entries in A. Notice if p, q are projections in the same $M_n(A)$ and $p \sim q$, then $p \sim_{mv} q$ also.

DEFINITION 7.5 (*unitary equivalence*). *Let p, q be projections in A. We say p and q are **unitary equivalent**, denoted $p \sim_u q$, if there is a unitary element $u \in \tilde{A}$ such that $q = upu^*$*

One can show that the equivalence relations imply others. In particular:

PROPOSITION 7.6. *Let p, q be projections in a unital C^*-algebra A. Then*

(1) $p \sim_h q \Longrightarrow p \sim_u q$,
(2) $p \sim_u q \Longrightarrow p \sim_{mv} q$.

In addition, $p \sim_{mv} q$ and $1_A - p \sim 1_A - q$ if and only if $p \sim_u q$.

PROOF. This is Proposition 2.2.6 and 2.2.7 from [11]. Proofs can be found there. ☐

PROPOSITION 7.7. *Let p, q be projections in a C^*-algebra A. Then*

(1) $p \sim_{mv} q \Longrightarrow \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \sim_u \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$ *in* $M_2(A)$,
(2) $p \sim_u q \Longrightarrow \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \sim_h \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$ *in* $M_2(A)$.

PROOF. For (1), we have $p = v^*v \sim_{mv} vv^* = q$ for some $v \in A$. Define

$$w_1 = \begin{pmatrix} v & 1-q \\ 1-p & v^* \end{pmatrix} \quad \text{and} \quad w_2 = \begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix},$$

then $w_1, w_2 \in \mathcal{U}_2(\tilde{A})$. Let $u = w_1 w_2$. Then

$$u \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} u^* = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}.$$

For (2), suppose that $q = upu^*$ for some $u \in \mathcal{U}(\tilde{A})$. Recall that

$$\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \sim_h \begin{pmatrix} uu^* & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

so there is a continuous path $t \mapsto w_t$ such that

$$w_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad w_1 = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}.$$

Let $w_t' = w_t \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} w_t^*$. Then $w_t' \in \mathcal{P}(M_2(A))$ for every $t \in [0,1]$ and the map $t \mapsto w_t'$ is continuous. Finally,

$$w_0' = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad w_1' = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}.$$

□

7.2 Monoid of Projection Classes

Recall in the last part we have considered not only projections in A but also projections in $M_n(A)$. Denote

$$\mathcal{P}_n(A) = \mathcal{P}(M_n(A)) \quad \text{and} \quad \mathcal{P}_\infty(A) = \bigcup_{n=1}^{\infty} \mathcal{P}_n(A).$$

So $\mathcal{P}_n(A)$ is the set of projections in $M_n(A)$ and $\mathcal{P}_\infty(A)$ is the union of all projections in any $\mathcal{P}_n(A)$. One might ask, with different dimensions, how is this set helpful? Recall in the previous section, two projections with different dimensions are equivalent if they meet the $\sim$ condition, so we are extending the Murray-von Neumann equivalence relation to make it not a problem.

The equivalence relation $\sim$ partitions $P_\infty(A)$ into different classes $[p]_{\mathcal{D}}$. The set

$$\mathcal{D}(A) := \mathcal{P}_\infty(A)/\sim$$

becomes a semigroup when defining a operation $\oplus$ on $\mathcal{D}(A)$ by

$$p \oplus q = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} = \operatorname{diag}(p, q),$$

and so

$$[p]_{\mathcal{D}} + [q]_{\mathcal{D}} = [p \oplus q]_{\mathcal{D}}.$$

Notice that for a projection $p \in M_n(A)$, $p \sim p \oplus 0_m \in M_{n+m}(A)$. Then $[0]_{\mathcal{D}} + [p]_{\mathcal{D}} = [p]_{\mathcal{D}} = [p]_{\mathcal{D}} + [0]_{\mathcal{D}}$ for every $p \in \mathcal{P}(A)$. So $[0]_{\mathcal{D}}$ is the identity in $\mathcal{D}(A)$ and it becomes a monoid. Also, since $[p]_{\mathcal{D}} + [q]_{\mathcal{D}} = [q]_{\mathcal{D}} + [p]_{\mathcal{D}}$, $\mathcal{D}(A)$ is also abelian.

7.3 The Grothendieck Construction

Having constructed abelian monoids from projections in matrix algebras over C^*-algebras, we now employ the Grothendieck construction to obtain groups from these monoids. This construction is fundamental to K-theory as it provides the mechanism for passing from the natural additive structure of projections to the group structure required for homological techniques and exact sequences.

The Grothendieck construction provides a universal way to embed any abelian monoid into a group, analogous to how the integers are constructed from the natural numbers. The key idea is to represent group elements as formal differences of monoid elements, with an appropriate equivalence relation that ensures the construction yields a group. In the context of K-theory, this process transforms the monoid of projection classes into the K_0-group, enabling the application of powerful algebraic techniques.

DEFINITION 7.8 (Grothendieck construction). *Let $(M, +)$ be an abelian monoid. Define an equivalence relation on the set $M \times M$ by $(x_1, y_1) \sim (x_2, y_2)$ if there is $m \in M$ such that*

$$x_1 + y_2 + m = x_2 + y_1 + m.$$

Define an operation $+$ *on* $\mathcal{G}(M) = M \times M/\sim$ *by*

$$\langle x_1, y_1\rangle + \langle x_2, y_2\rangle = \langle x_1 + x_2, y_1 + y_2\rangle,$$

where $\langle x, y\rangle = [(x, y)]$ *for every* $x, y \in M$*. Then* $\mathcal{G}(M)$ *becomes a group with identity* $\langle x, x\rangle$ *and inverse* $-\langle x, y\rangle = \langle y, x\rangle$*.*

EXAMPLE 7.9. *The pairs* $(2, 5), (3, 6)$ *are equivalent as*

$$2 + 6 + n = 8 + n = 3 + 5 + n$$

for every $n \in \mathbb{N}$*. This is the same for every pair* $(n, m) \in \mathbb{N} \times \mathbb{N}$ *and so* $\mathcal{G}(\mathbb{N}_0) \cong \mathbb{Z}$ *as stated earlier.*

DEFINITION 7.10. *(Grothendieck map) Let* M *be a monoid. The* ***Grothendieck map****, is the map* $\gamma_M : M \to \mathcal{G}(M)$ *by* $x \mapsto \langle x + y, y\rangle$*, where* $y \in M$*.*

Notice that the choice of y does not matter since

$$\langle x + y, y\rangle = \langle x, 0\rangle + \langle y, y\rangle = \langle x, 0\rangle = \langle x, 0\rangle + \langle z, z\rangle = \langle x + z, z\rangle.$$

Usually we use the identity element in M (0 in this case) so $\gamma_M(x) = \langle x, 0\rangle$. The Grothendieck map is used to make the K_0 group look more neat. The following lemma is later used to prove some K_0-related theorems.

LEMMA 7.11. *(1) Let* M *be a monoid,* H *be an abelian group and* $f : M \to H$ *be an additive map (*$f(x) + f(y) = f(x + y)$*). Then there is exactly one group homomorphism* $\psi : \mathcal{G}(M) \to H$ *such that* $\psi \circ \gamma_M = f$*. That is, the diagram*

$$\begin{tikzcd} H & \\ M \arrow[u, "f"] \arrow[r, "\gamma_M"] & \mathcal{G}(M) \arrow[ul, "\psi"'] \end{tikzcd}$$

commutes.

(2) Let M, N *be monoids and* $\phi : M \to N$ *be additive maps, then there is exactly one group homomorphism* $\mathcal{G}(\phi) : \mathcal{G}(M) \to \mathcal{G}(N)$ *making the diagram*

$$\begin{tikzcd}
M \arrow[r, "\phi"] \arrow[d, "\gamma_M"'] & N \arrow[d, "\gamma_N"'] \\
\mathcal{G}(M) \arrow[r, "\mathcal{G}(\phi)"] & \mathcal{G}(N)
\end{tikzcd}$$

commute.

(3) $\mathcal{G}(M) = \{\gamma_M(x) - \gamma_M(y) : x, y \in M\}$

(4) $\gamma_M(x) = \gamma_M(y)$ *if and only if* $x + z = y + z$ *for some* $z \in M$.

PROOF. Refer to [11, section 3.1.2] for proof. □

(1) is called the universal property and (2) is called the functoriality.

7.4 The K_0-Group of Unital C^*-Algebras

It is apparent that the K_0-group of A is defined as the Grothendieck group of the monoid $\mathcal{D}(A)$. That is, the $K_0(A) = \mathcal{G}(\mathcal{D}(A)) = ((\mathcal{D}(A) \times \mathcal{D}(A))/\sim, +)$. At this point the elements in $K_0(A)$ looks extremely messy, since there are equivalence classes on top of equivalence classes on top of equivalence classes.

EXAMPLE 7.12. *Let* $p, q \in \mathcal{P}_n(\mathbb{C})$. *Recall the rank of* p *is the dimension of the projected subspace* $p\mathbb{C}^n$. *Two projected subspaces* $p\mathbb{C}^n, q\mathbb{C}^n$ *are isomorphic if and only if they have the same dimension, that is, the rank of* p *and* q *is the same. Hence the isomorphism* $v : p\mathbb{C}^n \to q\mathbb{C}^n$ *is a partial isometry in* $M_n(\mathbb{C})$ *such that* $v^*v = p$ *and* $vv^* = q$. *Then we have* $[p]_{\mathcal{D}} = [q]_{\mathcal{D}} \iff \operatorname{rank}(p) = \operatorname{rank}(q)$. *Thus* $\mathcal{D}(\mathbb{C}) = \mathbb{N}$ *and* $K_0(\mathbb{C}) = \mathbb{Z}$.

Define $[\cdot]_0 : \mathcal{P}_\infty(A) \to K_0(A)$ by

$$[p]_0 = \gamma([p]_{\mathcal{D}}) = \langle [p]_{\mathcal{D}}, [0]_{\mathcal{D}} \rangle,$$

where γ is the Grothendieck map from $\mathcal{D}(A)$ to $\mathcal{G}(\mathcal{D}(A)) = K_0(A)$.

DEFINITION 7.13. *The* ***stable equivalence*** *in $\mathcal{P}_\infty(A)$ is an equivalence relation $\sim_s$ defined as:*

$$p \sim_s q \quad \text{if and only if} \quad p \oplus r \sim q \oplus r \text{ for some } r \in \mathcal{P}_\infty(A).$$

The stable equivalence is equivalent to $p \oplus 1_n \sim q \oplus 1_n$, where 1_n is the identity element of $M_n(A)$, when A is unital. The following proposition describes the elements of $K_0(A)$, the structure of the group and its relationships with $\mathcal{P}_\infty(A)$:

PROPOSITION 7.14 (**Standard picture of K_0-groups for unital C^*-algebras**). *Let A be a unital C^*-algebra. Then*

$$\begin{aligned} K_0(A) &= \{[p]_0 - [q]_0 : p, q \in P_\infty(A)\} \\ &= \{[p]_0 - [q]_0 : p, q \in P_n(A)\}, \end{aligned}$$

where

(1) $[p]_0 + [q]_0 = [p \oplus q]_0$ *for all projections in* $p, q \in \mathcal{P}_\infty(A)$.
(2) $[0_A]$ *is the identity in* $K_0(A)$.
(3) If $p \sim q$, *then* $[p]_0 = [q]_0$.
(4) If p, q *are mutually orthogonal projections, then* $[p]_0 + [q]_0 = [p + q]_0$.
(5) $[p]_0 = [p]_0$ *if and only if* $p \sim_s q$.

PROOF. (1) $[p \oplus q]_0 = \gamma([p \oplus q]_{\mathcal{D}}) = \gamma([p]_{\mathcal{D}} + q]_{\mathcal{D}}) = \gamma([p]_{\mathcal{D}}) + \gamma([q]_{\mathcal{D}}) = [p]_0 + [q]_0$.

(3) $p \sim_h q \implies p \sim_{mv} q \implies p \sim q \implies [p]_{\mathcal{D}} = [q]_{\mathcal{D}} \implies [p]_0 = [q]_0$.

(2) Recall that

$$\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} p \\ 0 \end{pmatrix} \begin{pmatrix} p & 0 \end{pmatrix} \sim \begin{pmatrix} p & 0 \end{pmatrix} \begin{pmatrix} p \\ 0 \end{pmatrix} = p.$$

By (1) and (3), we have $[p]_0 + [0]_0 = [p \oplus 0]_0 = [p]_0 = [0]_0 + [p]_0$.

(4) Let p, q be orthogonal projections. Let $v = \begin{pmatrix} p \\ q \end{pmatrix}$. Then

$$p \oplus q = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} \begin{pmatrix} p & q \end{pmatrix} \sim \begin{pmatrix} p & q \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = p + q,$$

so $p \oplus q \sim p + q \implies [p \oplus q]_0 \sim [p + q]_0$.

(5) Let $[p]_0 = [q]_0$. Then $[p \oplus r]_{\mathcal{D}} = [p]_{\mathcal{D}} + [r]_{\mathcal{D}} = [q]_{\mathcal{D}} + [r]_{\mathcal{D}} = [q \oplus r]_{\mathcal{D}}$ for some $r \in \mathcal{P}_\infty(A)$ so $p \oplus r \sim q \oplus r \implies p \sim_s q$. Conversely, Let $p \sim_s q$. Then $p \oplus r \sim q \oplus r$ for some $r \in \mathcal{P}_\infty(A)$. So we have $[p]_0 + [r]_0 = [p \oplus r]_0 = [q \oplus r]_0 = [q]_0 + [r]_0 \implies [p]_0 = [q]_0$.

□

EXAMPLE 7.15. *$K_0(M_n(\mathbb{C})) \cong \mathbb{Z}$ for all $n \in \mathbb{N}$. To see this, let Tr be the standard trace on $M_n(\mathbb{C})$. Let p, q be projections in $M_n(\mathbb{C})$ such that $p \sim q$. Then there is a partial isometry $v \in M_n(\mathbb{C})$ such that $p = v^*v \sim vv^* = q$. Then*

$$\mathrm{Tr}(p) = \mathrm{Tr}(v^*v) = \mathrm{Tr}(vv^*) = tr(q).$$

Recall that any projection in $M_n(\mathbb{C})$ is diagonalisable with eigenvalues 0 or 1. Therefore,

$$\mathrm{Tr}(r) = \mathrm{rank}(r) = \dim\big(r(\mathbb{C}^n)\big),$$

which gives $dim(p\mathbb{C}^n) = dim(q\mathbb{C}^n)$. Suppose that $dim(p\mathbb{C}^n) = dim(q\mathbb{C}^n) = k$, fix an orthonormal basis $\{e_1, \dots, e_k\}$ and $\{f_1, \dots, f_k\}$ for $p\mathbb{C}^n$ and $q\mathbb{C}^n$ respectively. Define $v : \mathbb{C}^n \to \mathbb{C}^n$ by $e_i \mapsto f_i$ for every $1 \leq i \leq k$ and $v(p\mathbb{C}^n)^\perp = 0$. One checks that $v^*v = p$ and $vv^* = q$, so $p \sim q$. Thus the following are equivalent:

(1) $p \sim q$.
(2) $\mathrm{Tr}(p) = \mathrm{Tr}(q)$.
(3) $dim(p\mathbb{C}^n) = dim(q\mathbb{C}^n)$.

Projections in $M_n(\mathbb{C})$ are classified by their rank, as shown above. That is, for projections $p, q \in M_n(\mathbb{C})$, $p \sim q$ if and only if they have the same rank. Therefore,

$$\mathcal{D}(\mathbb{C}) \cong \{0, 1, 2, \dots\} = \mathbb{Z}^+,$$

and $K_0(\mathbb{C}) \cong \mathbb{Z}$. Example 3.3.2 from [11] shows that

$$K_0(\mathrm{Tr}) : K_0(M_n(\mathbb{C})) \to K_0(\mathbb{C}) = \mathbb{Z}$$

is an isomorphism, and the cyclic group $K_0(M_n(\mathbb{C}))$ is generated by $[e]_0$, where $e \in \mathcal{P}_n(\mathbb{C})$ is any one-dimensional projection.

PROPOSITION 7.16 (**Universal property of** K_0). *Suppose that A is a unital C^*-algebra. Let G be an abelian group. If there is a map $\nu : \mathcal{P}_\infty(A) \to G$ such that*

(1) $\nu(p \oplus q) = \nu(p) + \nu(q)$ for all $p, q \in \mathcal{P}_\infty(A)$;
(2) $\nu(0_A) = 0$; and
(3) If $p, q \in \mathcal{P}_n(A)$ and $p \sim_h q$ in $\mathcal{P}_n(A)$, then $\nu(p) = \nu(q)$.

Then there exists a unique group homomorphism $\alpha : K_0(A) \to G$ satisfying $\alpha([p]_0) = \nu(p)$ for all $p \in \mathcal{P}_\infty(A)$.

PROOF. Refer to proposition 3.1.8 from [11]. ☐

LEMMA 7.17. *Suppose that A, B are unital C^*-algebras and let $\psi : A \to B$ be a $*$-homomorphism. Then there exists a group homomorphism $K_0(\psi) : K_0(A) \to K_0(B)$ such that the following diagram*

$$\begin{tikzcd}
\mathcal{P}_\infty(A) \arrow[r, "\psi"] \arrow[d, "{[\cdot]_0}"] & \mathcal{P}_\infty(B) \arrow[d, "{[\cdot]_0}"] \\
K_0(A) \arrow[r, "K_0(\psi)"] & K_0(A)
\end{tikzcd}$$

commutes.

PROOF. Recall for a $*$-homomorphism $\psi : A \to B$, we can extend to a $*$-homomorphism $\psi : M_n(A) \to M_n(B)$ for all $n \in \mathbb{N}$ that takes projections to projections. so $\psi(\mathcal{P}_\infty)(A) \subseteq \mathcal{P}_\infty(B)$. Define $\nu : \mathcal{P}_\infty(A) \to K_0(B)$ by $\nu(p) = [\psi(p)]_0$. Then

(1) $\nu(p \oplus q) = [\psi(p \oplus q)]_0 = [\psi(p) \oplus \psi(q)]_0 = [\psi(p)]_0 + [\psi(q)]_0 = \nu(p) + \nu(q)$;
(2) $\nu(0_A) = [\psi(0_A)] = [0_B] = 0$; and
(3) let $p \sim_h q$ in $\mathcal{P}_\infty(A)$, then $\psi(p) \sim_h \psi(q)$ in $\mathcal{P}_\infty(B)$ and $\nu(p) = [\psi(p)]_0 = [\psi(q)]_0 = \nu(q)$.

Therefore there exists a group homomorphism $K_0(\psi) : K_0(A) \to K_0(B)$ such that

$$K_0(\psi)([p]_0) = [\psi(p)]_0.$$

That is, $K_0(\psi) \circ [\cdot]_0 = [\cdot]_0 \circ \psi$. ☐

PROPOSITION 7.18 (**Functoriality of K_0 for for unital C^*-algebras**). *K_0 is a functor from* $\mathbf{C^*alg}$ *to* $\mathbf{Ab}$. *More specifically,*

(1) $K_0(\mathrm{id}_A) = \mathrm{id}_{K_0(A)}$ for any unital C^*-algebra A;
(2) Let A, B, C be unital C^*-algebras and $\varphi : A \to B$ and $\psi : B \to C$ be $*$-homomorphisms, then $K_0(\psi \circ \varphi) = K_0(\psi) \circ K_0(\varphi)$;
(3) $K_0(\{0\}) = \{0\}$; and
(4) $K_0(0_{B,A}) = 0_{K_0(B),K_0(A)}$.

PROOF. We have seen from the above lemma that for any unital $*$-homomorphism $\psi : A \to B$, there exists a group homomorphism $K_0(\psi) : K_0(A) \to K_0(B)$ by $K_0(\psi)([p]_0) = [\psi(p)]_0$ for every $p \in \mathcal{P}_\infty(A)$.

(1) $K_0(\mathrm{id}_A)([p]_0) = [\mathrm{id}_A(p)]_0 = [p]_0 \implies K_0(\mathrm{id}_A) = \mathrm{id}_{K_0(A)}$ for every $p \in \mathcal{P}_\infty(A)$; and
(2) $K_0(\psi \circ \varphi)([p]_0) = [\psi \circ \varphi(p)]_0 = K_0(\psi)([\varphi(p)]_0) = K_0(\psi) \circ K_0(\varphi)([p]_0)$.

Then by the standard picture, the above holds for any $g \in K_0(A)$. For (3), we have $\mathcal{P}_n(\{0\}) = \{0_n\}$. So $K_0(\{0\}) = G(\{0\}) = \{0\}$. For (4), since $0_{B,A} = 0_{B,0} \circ 0_{0,A}$, it follows from (2) and (3). ☐

DEFINITION 7.19. *Suppose that A, B are C^*-algebras and let $\varphi, \psi : A \to B$ be $*$-homomorphisms. φ, ψ are called **homotopic**, denoted by $\varphi \sim_h \psi$, if there exists a collection of $*$-homomorphisms $\varphi_t : A \to B$ where $t \in [0,1]$ such that the map $t \mapsto \varphi_t(a)$ is continuous for each $a \in A$ such that $\varphi_0 = \varphi$ and $\varphi_1 = \psi$.*

*The two C^*-algebras A, B are called **homotopy equivalent** if there are $*$-homomorphisms $\varphi : A \to B$ and $\psi : B \to A$ such that $\psi \circ \varphi \sim_h \mathrm{id}_A$ and $\varphi \circ \psi \sim_h \mathrm{id}_B$.*

PROPOSITION 7.20. *Let A, B be unital C^*-algebras. Then*

(1) $K_0(\varphi) = K_0(\psi)$ for any homotopic $*$-homomorphisms $\varphi, \psi : A \to B$.
(2) Let A, B be homotopy equivalent, then $K_0(A) \cong K_0(B)$. That is, if $\varphi : A \to B$ and $\psi : B \to A$ are homotopic, then $K_0(\psi) : K_0(A) \to K_0(B)$ and $K_0(\psi) : K_0(B) \to K_0(A)$ are inverses of each other.

PROOF. (1) Let $\varphi_t : A \to B$ be a continuous path with $\varphi_0 = \varphi$ and $\varphi_1 = \psi$. Extend the paths to $\varphi_t : M_n(A) \to M_n(B)$ and observe that the map $t \mapsto \varphi_t(p)$ is continuous for all $p \in \mathcal{P}_n(A)$, so $\varphi(p) \sim_h \psi(p)$ and we have

$$K_0(\varphi)([p]_0) = [\varphi(p)]_0 = [\psi(p)]_0 = K_0(\psi)([p]_0).$$

By the standard picture of K_0, we have $K_0(\varphi) = K_0(\psi)$.

(2) (2) follows from (1) and functoriality of K_0.

☐

LEMMA 7.21. *For every unital C^*-algebra A, the split exact sequence*

$$0 \to A \xrightarrow{\iota} \tilde{A} \underset{\lambda}{\overset{\pi}{\rightleftarrows}} \mathbb{C} \to 0$$

induces a split exact sequence

$$0 \to K_0(A) \xrightarrow{K_0(\iota)} K_0(\tilde{A}) \underset{K_0(\lambda)}{\overset{K_0(\pi)}{\rightleftarrows}} K_0(\mathbb{C}) \to 0.$$

PROOF. Refer to lemma 3.2.7 and lemma 3.2.8 from [11]. □

7.5 The K_0-Group of C^*-algebras

The same definition of K_0-group works for non-unital C^*-algebras. In [22], the notation is $K_{00}(A)$ for the Grothendieck group of the monoid $\mathcal{D}(A)$ for both unital and non-unital C^*-algebra A. One would assume that $K_0(A) = K_{00}(A)$. For a unital C^*-algebra A it is, but this is not the case when A is non-unital. In particular, the functor K_{00} is not half exact.

For this chapter, the definition of K_0-group for C^*-algebras is defined. It will be shown when a C^*-algebra is unital, this version of K_0-group is equivalent to the version defined in the previous version.

Let A be a non-unital C^*-algebra. Recall that there is a split exact sequence

$$0 \to A \xrightarrow{\iota} \tilde{A} \underset{\lambda}{\overset{\pi}{\rightleftarrows}} \mathbb{C} \to 0.$$

For the homomorphism $\pi : \tilde{A} \to \mathbb{C}$, there is $K_0(\psi) : K_0(\tilde{A}) \to K_0(\mathbb{C})$. Let $K_0(A) = \ker(K_0(\psi))$, that is, define $K_0(A)$ to be the kernel of $K_0(\psi)$. Then $K_0(A)$ is a subgroup of $K_0(\tilde{A})$.

Let $p \in \mathcal{P}_\infty(A) \subset \mathcal{P}_\infty(\tilde{A})$, and consider $[p]_0 \in K_0(\tilde{A})$. Then

$$K_0(\pi)([p]_0) = [\pi(p)]_0 = [0_A]_0 = 0 \implies [p]_0 \in K_0(A),$$

so there is a map $[\cdot]_0 : \mathcal{P}_\infty(A) \to K_0(A)$. If A is unital, then there is a map $K_0(\iota) : K_0(A) \to K_0(\tilde{A})$. When A is non-unital, then there is an inclusion $\ker(K_0(\psi)) = K_0(A) \to K_0(\tilde{A})$. Either way, we obtain a short exact sequence

$$0 \longrightarrow K_0(A) \longrightarrow K_0(\tilde{A}) \xrightarrow{K_0(\pi)} K_0(\mathbb{C}) \longrightarrow 0.$$

REMARK 7.22. *When A is unital, then $K_0(A) \cong \operatorname{im}(K_0(\iota))$, and $K_0(\iota)$ maps $[p]_0 \in K_0(A)$ to $[p]_0 \in K_0(\tilde{A})$. Therefore, $K_0(A) = \ker(K_0(\pi))$ for both unital and non-unital C^*-algebra A. So the definition of $K_0(A)$ aligns for all C^*-algebras.*

REMARK 7.23. *Let $\varphi : A \to B$ be a $*$-homomorphism and let $\tilde{\varphi} : \tilde{A} \to \tilde{B}$ be its unitisation. functoriality of K_0 for unital C^*-algebras shows that is a group homomorphism $K_0(\tilde{\varphi}) : K_0(\tilde{A}) \to K_0(\tilde{B})$. Define $K_0(\varphi)$ to be the restriction of $K_0(\tilde{\varphi})$ to $K_0(A) = \ker(K_0(\pi))$. Then $\operatorname{im}(K_0(\varphi)) \subseteq K_0(B)$ so $K_0(\varphi) : K_0(A) \to K_0(B)$ by $K_0(\varphi)([p]_0) = [\varphi(p)]_0$, which aligns with the unital case of K_0.*

7.6 The Functor K_0

PROPOSITION 7.24 (**Functoriality of** K_0). *K_0 is a functor from* **C*alg** *to* **Ab**. *More specifically,*

(1) $K_0(\mathrm{id}_A) = \mathrm{id}_{K_0(A)}$ for all C^*-algebra A;
(2) For $*$-homomorphisms $\varphi : A \to B$ and $\psi : B \to C$, $K_0(\psi \circ \varphi) = K_0(\psi) \circ K_0(\varphi)$;
(3) $K_0(\{0\}) = \{0\}$; and
(4) $K_0(0_{B,A}) = 0_{K_0(B),K_0(A)}$ for every C^*-algebras A, B.

PROOF. Remark 7.23 shows that K_0 takes a $*$-homomorphism $\varphi : A \to B$ to the restriction of $K_0(\tilde{\varphi}) : K_0(A) \to K_0(B)$ to its kernel. (1) and (2) follows from (1) and (2) of functoriality of K_0 for the unital C^*-algebras, since $\mathrm{id}_A^\sim = \mathrm{id}_{\tilde{A}}$ and $(\psi \circ \varphi)^\sim = \tilde{\psi} \circ \tilde{\varphi}$. For (3), let $A = \{0\}$, then $\tilde{A} = \mathbb{C}$ and the short exact sequence becomes

$$0 \to 0 \to \mathbb{C} \xrightarrow{\pi} \mathbb{C} \to 0,$$

where $\pi = \mathrm{id}_{\mathbb{C}}$. So $K_0(\{0\}) = \ker(K_0(\pi)) = \{0\}$. Finally, (4) follows the proof of the functoriality of K_0 for unital C^*-algebras. □

PROPOSITION 7.25. *Let A, B be unital C^*-algebras. Then*

(1) $K_0(\varphi) = K_0(\psi)$ for any homotopic $*$-homomorphisms $\varphi, \psi : A \to B$.

(2) Let A, B be homotopy equivalent, then $K_0(A) \cong K_0(B)$. That is, if $\varphi : A \to B$ and $\psi : B \to A$ are homotopic, then $K_0(\psi) : K_0(A) \to K_0(B)$ and $K_0(\psi) : K_0(B) \to K_0(A)$ are inverses of each other.

PROOF. (1) Let $\varphi \sim_h \psi$, then $\tilde{\varphi} \sim_h \tilde{\psi}$ and so $K_0(\tilde{\varphi}) = K_0(\tilde{\psi})$. Then the restrictions $K_0(\varphi)$ and $K_0(\psi)$ and the same.

(2) This follows from the functoriality of K_0.

□

Given all the above, we have a new standard picture for K_0-group of all C^*-algebras. Before that, there is one tool we need.

DEFINITION 7.26. *With the split exact sequence*

$$0 \to A \xrightarrow{\iota} \tilde{A} \underset{\lambda}{\overset{\pi}{\rightleftarrows}} \mathbb{C} \to 0,$$

define $s := \lambda \circ \pi : \tilde{A} \to \tilde{A}$. That is, $s(a + \alpha 1_{\tilde{A}}) = \alpha 1_{\tilde{A}}$. s is called a ***scalar mapping****.*

PROPOSITION 7.27. *(**Standard picture of K_0-groups for C^*-algebras**) Let A be a C^*-algebra. Then*

$$K_0(A) = \{[p]_0 - s([p]_0) : p \in \mathcal{P}_\infty(\tilde{A})\},$$

where

(1) For $p, q \in \mathcal{P}_\infty(\tilde{A})$, the following are equivalent:
  (a) $[p]_0 - [s(p)]_0 = [q]_0 - [s(q)]_0$.
  (b) There exist natural numbers k, l such that $p \oplus 1_k \sim_0 q \oplus 1_l$ in $\infty(\tilde{A})$.
  (c) There exist scalar projections r_1, r_2 such that $p \oplus r_1 \sim_0 q \oplus r_2$.

(2) For $p \in \mathcal{P}_\infty(\tilde{A})$ such that $[p]_0 - [s(p)]_0 = 0$, there is a natural number m such that $p \oplus 1_m \sim s(p) \oplus 1_m$; and

(3) Let $\psi : A \to B$ be a $*$-homomorphism, then

$$K_0(\psi)([p]_0 - [s(p)]_0) = [\tilde{\psi}(p)]_0 - [s(\tilde{\psi}(p))]_0.$$

PROOF. Refer to proposition 4.2.2 of [11] ☐

PROPOSITION 7.28 (**Direct sums of** K_0). *For any C^*-algebras A, B, we have*

$$K_0(A \oplus B) \cong K_0(A) \oplus K_0(B).$$

In particular, we have two inclusion mappings $\iota_A : A \to A \oplus B$ and $\iota_B : B \to A \oplus B$ and there is an isomorphism

$$K_0(\iota_A) \oplus K_0(\iota_B) : K_0(A \oplus B) \to K_0(A) \oplus K_0(B);$$

$$(g, h) \mapsto K_0(\iota_A)(g) + K_0(\iota_B)(h).$$

PROOF. Let $\alpha : K_0(A) \to K_0(A) \oplus K_0(B)$ by $g \to (g, 0)$ and $\beta : K_0(A) \oplus K_0(B) \to K_0(B)$ by $(g, h) \mapsto h$. Define $\pi_B : A \oplus B \to B$ by $(a, b) \mapsto b$. Then both

$$0 \to K_0(A) \xrightarrow{\alpha} K_0(A) \oplus K_0(B) \xrightarrow{\beta} K_0(B \to 0$$

and

$$0 \to K_0(A) \xrightarrow{K_0(\iota_A)} K_0(A) \oplus K_0(B) \xrightarrow{K_0(\pi_B} K_0(B \to 0$$

are exact with $K_0(\iota_A) \oplus K_0(\iota_B) \circ \alpha = K_0(\iota)$ and $\beta = K_0(\pi_B) \circ K_0(\iota_A) \oplus K_0(\iota_B)$ since $\pi_B \circ \iota_A = 0$ and $\pi_B \circ \iota_B = \mathrm{id}_B$. This shows that $K_0(\iota_A) \oplus K_0(\iota_B)$ is an isomorphism. ☐

CHAPTER 8

The K_1-Group

The K_1-group forms the second fundamental component of the K-theory invariants that will enable our computation of the K-theory for skew-product graph of groups algebras. While the K_0-group, discussed in the previous chapter, captures information about projections and their equivalences, the K_1-group encodes the homotopy classification of unitary elements in the algebra. This dual structure is essential for this thesis because exact sequence techniques, particularly the dual Pimsner-Voiculescu sequence that will be crucial for our skew-product computations, require both K_0 and K_1 components to provide complete K-theoretic information.

The significance of K_1-groups for our approach lies in their behaviour under crossed product constructions. When we establish the connection between skew-product graph of groups algebras and crossed products by coactions in Chapter 6, the resulting exact sequences will involve both K_0 and K_1 groups, making a thorough understanding of K_1 essential for computing the complete K-theory invariants. The material in this chapter follows the systematic treatment of Rørdam, Larsen and Laustsen [11], with additional insights from Murphy [13], focusing on the computational aspects that will prove crucial for our applications to graph of groups C^*-algebras.

8.1 Unitaries

The construction of the K_1-group mirrors the approach for K_0, but focuses on unitary elements rather than projections. Just as projections in matrix algebras provide the building blocks for K_0, unitary elements in matrix algebras form the foundation for K_1. This parallel construction is crucial for maintaining the functorial properties that make K-theory so powerful for our skew-product computations, ensuring that the exact sequences arising from crossed product constructions have the necessary algebraic structure to enable computation.

DEFINITION 8.1. *Let A be a unital C^*-algebra. The set of unitary elements in A is denoted $\mathcal{U}(A)$. The set of unitary elements in $M_n(A)$ is denoted $\mathcal{U}_n(A)$ and $\mathcal{U}_\infty(A) = \bigcup_{n=1}^{\infty} \mathcal{U}_n(A)$ is the union of unitaries in $M_n(A)$.*

Define a binary operation $\oplus$ on $\mathcal{U}_\infty(A)$ by

$$u \oplus v = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} = \operatorname{diag}(u, v),$$

and define a relation $\sim_1$ on $\mathcal{U}_\infty(A)$: Let $u \in \mathcal{U}_n(A)$ and $v \in \mathcal{U}_m(A)$, write $u \sim_1 v$ if there exists a natural number $k \geq \max\{m, n\}$ such that $u \oplus 1_{k-n} \sim_h v \oplus 1_{k-m}$ in $\mathcal{U}_k(A)$. Note that by the above setting, we have $u \oplus 1_0 = u$.

LEMMA 8.2. *Let A be a unital C^*-algebra. Then*

(1) $\sim_1$ is an equivalence relation on $\mathcal{U}_\infty(A)$;
(2) $u \sim_1 u \oplus 1_{n'}$ for all $u \in \mathcal{U}_\infty(A)$ and $n' \in \mathbb{N}$;
(3) $u \oplus v \sim_1 v \oplus u$ for all $u, v \in \mathcal{U}_\infty(A)$;
(4) Let $u, u', v, v' \in \mathcal{U}_\infty(A)$ such that $u \sim_1 u'$ and $v \sim_1 v'$. Then $u \oplus v \sim_1 u' \oplus v'$;
(5) Suppose that $u, v \in \mathcal{U}_n(A)$ for some $n \in \mathbb{N}$, then $uv \sim_1 vu \sim_1 u \oplus v$; and
(6) $(u \oplus v) \oplus w = u \oplus (v \oplus w)$ for all $u, v, w \in \mathcal{U}_\infty(A)$.

PROOF. Refer to Lemma 8.1.2 in [11]. □

8.2 The K_1-Group of C^*-Algebras

Having established the equivalence relation for unitaries, we now construct the K_1-group through a quotient construction that parallels the development of K_0 but captures fundamentally different topological information. The K_1-group encodes the homotopy classes of unitaries, providing invariants that are preserved under the crossed product constructions central to our approach. This preservation property is essential for our skew-product methodology, as it ensures that the K_1-invariants of graph of groups algebras can be computed from the K_1-invariants of the underlying graph algebras through exact sequence techniques.

The K_1 group of a C^*-algebra A is defined as $K_1(A) = \mathcal{U}_\infty(\tilde{A})/\sim_1$ and the following gives what $K_1(A)$ looks like:

$$K_1(A) = \{[u]_1 \in \mathcal{U}_\infty(\tilde{A})\},$$

where

(1) $[u \oplus v]_1 = [u]_1 + [v]_1$,

(2) $[1_{\tilde{A}}]_1$ is the identity.

PROPOSITION 8.3. *(**Standard picture of K_1-groups for C^*-algebras**) Let A be a C^*-algebra. Then*

$$K_1(A) = \{[u]_1 : u \in \mathcal{U}_\infty(\tilde{A})\},$$

and the map $[\cdot]_1 : \mathcal{U}_\infty(\tilde{A}) \to K_1(A)$ has the following properties:

(1) $[u \oplus v]_1 = [u]_1 + [v]_1$;

(2) $[1]_1 = 0$;

(3) $u, v \in \mathcal{U}_n(\tilde{A})$ with $u \sim_h v$ implies $[u]_1 = [v]_1$;

(4) $[uv]_1 = [vu]_1 = [u]_1 + [v]_1$ for every $u, v \in \mathcal{U}_n(\tilde{A})$; and

(5) $[u]_1 = [v]_1 \Leftrightarrow u \sim_1 v$ for every $u, v \in \mathcal{U}_\infty(\tilde{A})$.

PROOF. The format of $K_1(A)$ and properties (1), (2) and (5) are immediate consequences of the definition of $K_1(A) = \mathcal{U}_\infty(\tilde{A})/\sim_1$. (3) follows from (5). For (4), it follows from the fact that $uv \sim_1 vu \sim_1 u \oplus v$. □

EXAMPLE 8.4. *$K_1(M_n(\mathbb{C})) = 0$ for every $n \in \mathbb{N}$. To see this, fix $u \in \mathcal{U}_n(\mathbb{C})$. Choose $k \geq n$. Using the identification*

$$M_k(M_n(\mathbb{C})) \cong M_{kn}(\mathbb{C}),$$

By the spectral theorem for normal matrices, there exists a unitary $V \in \mathcal{U}_{kn}(\mathbb{C})$ and real numbers $\theta_1, \dots, \theta_{kn}$ such that

$$u \oplus 1_{k-n} = V \operatorname{diag}\big(e^{i\theta_1}, \dots, e^{i\theta_{kn}}\big)\, V^*.$$

Define a path $f : [0,1] \to \mathcal{U}_{kn}(\mathbb{C})$ by

$$f(t) \;=\; V \operatorname{diag}\big(e^{\,i(1-t)\theta_1}, \dots, e^{\,i(1-t)\theta_{kn}}\big)\, V^*.$$

Then we have

$$f(0) = u \oplus 1_{k-n}, \quad \textit{and} \quad f(1) = V\, I\, V^* = I_{kn},$$

and $f(t)$ is continuous in t. Hence

$$u \oplus 1_{k-n} \sim_h I_{kn} \quad \Longrightarrow \quad u \sim_1 I.$$

This shows that any unitary is equivalent to the identity matrix, so $K_1(M_n(\mathbb{C}))$ is the trivial group for any $n \in \mathbb{N}$.

PROPOSITION 8.5 (**Universal property of** K_1). *Let A be a C^*-algebra, and let G be an abelian group, and let $\nu : \mathcal{U}_\infty(\tilde{A}) \to G$ be a map satisfying*

(1) $\nu(u \oplus v) = \nu(u) + \nu(v)$,
(2) $\nu(1) = 0$, and
(3) $u \sim_h v$ for some $u, v \in \mathcal{U}_n(\tilde{A}) \implies \nu(u) = \nu(v)$.

Then there exists a unique group homomorphism $\alpha : K_1(A) \to G$ such that

$$\nu(u) = \alpha([u]_1).$$

PROOF. Suppose that $u \in \mathcal{U}_n(\tilde{A})$ and $v \in \mathcal{U}_m(\tilde{A})$ such that $u \sim_1 v$. Fix an integer $k \geq \max\{n, m\}$ with $u \oplus 1_{k-n} \sim_h v \oplus 1_{k-m}$ in $\mathcal{U}_k(\tilde{A})$. Then (1) and (2) shows that $\nu(1_r) = 0$ for all $r \in \mathbb{N}$. (1) and (3) imply

$$\nu(u) = \nu(u \oplus 1_{k-n}) = \nu(v \oplus 1_{k-m}) = \nu(v).$$

Therefore, there is a map $\alpha : K_1(A) \to G$ such that $\nu(u) = \alpha([u]_1)$ for all $u \in \mathcal{U}_\infty(\tilde{A})$. In addition,

$$\alpha([u]_1 + [v]_1) = \alpha([u \oplus v]_1) = \nu(u \oplus v) = \nu(u) + \nu(v) = \alpha([u]_1) + \alpha([v]_1).$$

So α is a group homomorphism, and the uniqueness of α follows from the fact that $[\cdot]_1$ is surjective. □

The definition of K_1 group above works for all C^*-algebras, but for unital C^*-algebras A, it is more natural to define the group $K_1(A)$ to be $\mathcal{U}_\infty(A)/\sim_1$ instead of $\mathcal{U}_\infty(\tilde{A})/\sim_1$. Recall that for a unital C^*-algebra A, $\tilde{A} = A + \mathbb{C}f$, where $f = 1_{\tilde{A}} - 1_A$. Define a map $\mu : \tilde{A} \to A$ by $\mu(a + \alpha f) = a$ for all $a \in A, \alpha \in \mathbb{C}$. Then $\mu(1_{\tilde{A}}) = \mu(1_A + f) = 1_A$, so μ is a unital $*$-homomorphism. The map μ can be extended to $\mu : M_n(\tilde{A}) \to M_n(A)$ for each $n \in \mathbb{N}$, and since μ takes unitaries to unitaries, we have a map $\mu : \mathcal{U}_\infty(\tilde{A}) \to \mathcal{U}_\infty(A)$.

PROPOSITION 8.6. *Let A be a unital C^*-algebra. Then there is an isomorphism $\rho : K_1(A) \to \mathcal{U}_\infty(A)/\sim_1$ such that the following diagram*

$$\begin{tikzcd}
\mathcal{U}_\infty(\tilde{A}) \arrow[r, "\mu"] \arrow[d, "{[\cdot]_1}"] & \mathcal{U}_\infty(A) \arrow[d] \\
K_1(A) \arrow[r, "\rho"] & \mathcal{U}_\infty(A)/\sim_1
\end{tikzcd}$$

commutes, where $\mu : \mathcal{U}_\infty(\tilde{A}) \to \mathcal{U}_\infty(A)$ is the map defined above.

PROOF. Note that $\mu : \mathcal{U}_\infty(\tilde{A}) \to \mathcal{U}_\infty(A)$ is surjective and $\mu(u \oplus v) = \mu(u) \oplus \mu(v)$ for all $u, v \in \mathcal{U}_\infty(\tilde{A})$. Hence it suffice to show that $\mu(u) \sim_h \mu(v) \iff u \sim_h v$ for all $u, v \in \mathcal{U}_n(\tilde{A})$.

Suppose that $u \sim_h v$, then the continuous μ gives $\mu(u) \sim_h \mu(v)$. Conversely, suppose that $\mu(u) \sim_h \mu(v)$ in $\mathcal{U}_n(A)$. Since μ takes $a + \alpha f$ to a, there are $u_0, v_0 \in \mathcal{U}_n(\mathbb{C}f)$ such that $u = \mu(u) + u_0$ and $v = \mu(v) + v_0$. So $u_0 \sim_h v_0$ in $\mathcal{U}_n(\mathbb{C}f)$, which shows that $u = \mu(u) + u_0 \sim_h \mu(v) + v_0 = v$ in $\mathcal{U}_n(\tilde{A})$. □

Given the above proposition, whenever A is unital, we can identify $K_1(A)$ with $\mathcal{U}_\infty(A)/\sim_1$. This also implies that, unital or not,

$$K_1(A) \cong K_1(\tilde{A})$$

for every C^*-algebra A.

8.3 The Functor K_1

The functorial properties of K_1 are crucial for the exact sequence techniques that form the heart of our computational approach to graph of groups K-theory. Just as with K_0, the functor K_1 preserves the algebraic structure needed for the dual Pimsner-Voiculescu sequence and related tools. When we establish that skew-product graph of groups algebras arise as crossed products by coactions, the functoriality of K_1 ensures that the resulting exact sequences provide a systematic method for computing K_1-groups of these complex algebras from the K_1-groups of their simpler components.

Just like K_0, K_1 is a functor from the category $\mathbf{C^*alg}$ to the category $\mathbf{Ab}$, by the following proposition:

LEMMA 8.7. *Let $\varphi : A \to B$ be a $*$-homomorphism, then there exists a group homomorphism $K_1(\varphi) : K_1(A) \to K_1(B)$ such that*

$$K_1(\varphi)([u]_1) = [\tilde{\varphi}(u)]_1$$

for every $u \in \mathcal{U}_\infty(A)$. In addition, if A, B are unital C^*-algebras and φ is a unital $*$-homomorphism, then

$$K_1(\varphi)([u]_1) = [\varphi(u)]_1$$

for all $u \in \mathcal{U}_\infty(A)$.

PROOF. Let $\varphi : A \to B$ be a $*$-homomorphism between C^*-algebras. Then there is an induced unital $*$-homomorphism $\tilde{\varphi} : \tilde{A} \to \tilde{B}$, which extends to $\tilde{\varphi} : M_n(\tilde{A}) \to M_n(\tilde{B})$.

Define $\nu : \mathcal{U}_\infty(\tilde{A}) \to K_1(B)$ by

$$\nu(u) = [\tilde{\varphi}(u)]_1$$

for all $u \in \mathcal{U}_\infty(A)$. Then

(1) $\nu(u \oplus v) = [\tilde{\varphi}(u \oplus v)]_1 = [\tilde{\varphi}(u) \oplus \tilde{\varphi}(v)]_1 = [\tilde{\varphi}(u)]_1 + [\tilde{\varphi}(v)]_1 = \nu(u) + \nu(v)$;

(2) $\nu(1_{\tilde{A}}) = [\tilde{\varphi}(1_{\tilde{A}})]_1 = [1_{\tilde{B}}]_1 = 0$; and

(3) Suppose that $u \sim_h v$ in $\mathcal{U}_\infty(\tilde{A})$. Then the continuity of $\tilde{\varphi}$ shows that $\tilde{\varphi}(u) \sim_h \tilde{\varphi}(v)$. So

$$\nu(u) = [\tilde{\varphi}(u)]_1 = [\tilde{\varphi}(v)]_1 = \nu(v).$$

By the universal property of K_1, there is a group homomorphism $K_1(\varphi) : K_1(A) \to K_1(B)$ such that

$$K_1(\varphi)([u]_1) = [\tilde{\varphi}(u)]_1$$

for all $u \in \mathcal{U}_\infty(\tilde{A})$. Suppose that both A, B are unital C^*-algebras and $\varphi : A \to B$ is unital. For any $u \in \mathcal{U}_n(A)$, recall that $K_1(A) \cong \mathcal{U}_\infty(A)/\sim_1$, then

$$K_1(\varphi)([u]_1) = [\tilde{\varphi}(u)]_1 = [\tilde{\varphi}(u + 0f)]_1 = [\varphi(u)]_1.$$

□

PROPOSITION 8.8 (**Functoriality of** K_1). *Let A, B be C^*-algebras. Then*

(1) $K_1(\mathrm{id}_A) = \mathrm{id}_{K_1(A)}$*;*

(2) $K_1(\psi \circ \varphi) = K_1(\psi) \circ K_1(\varphi)$ *for any $*$-homomorphisms $\varphi : A \to B$ and $\psi : B \to C$;*

(3) $K_1(\{0\}) = \{0\}$*;*

(4) $K_1(0_{B,A}) = 0_{K_1(B),K_1(A)}$*;*

(5) For homotopic $*$-homomorphisms $\varphi, \psi : A \to B$, $K_1(\varphi) = K_1(\psi)$; and

(6) If A, B are homotopy equivalent via $\varphi : A \to B$ and $\psi : B \to A$, then $K_1(A) \cong K_1(B)$ via $K_1(\varphi)$ and $K_1(\varphi)^{-1} = K_1(\psi)$.

PROOF. Recall that $K_1(\varphi)([u]_1) = [\tilde{\varphi}(u)]_1$ and $(\mathrm{id}_A)^\sim$ and $(\psi \circ \varphi)^\sim = \tilde{\psi} \circ \tilde{\varphi}$. Then (1) and (2) follow. For (3), since $K_1(A) \cong K_1(\tilde{A})$ for every C^*-algebra A, we have $K_1(\{0\}) \cong K_1(\{0\}^\sim) = K_1(\mathbb{C}) = \{0\}$. For (4) and (5), refer to Proposition 8.2.2 in [11]. □

PROPOSITION 8.9 (**Half exactness of K_1**). *Suppose that*

$$0 \to I \overset{\varphi}{\to} A \overset{\psi}{\to} B \to 0$$

is a short exact sequence of C^*-algebras. Then the sequence

$$K_1(I) \xrightarrow{K_1(\varphi)} K_1(A) \xrightarrow{K_1(\psi)} K_1(B) \tag{8.1}$$

is exact.

PROOF. Recall that from (4) of Proposition 8.8 that K_1 maps zero morphisms of C^*-algebras to zero morphisms of corresponding K_1-groups. Exactness at A means $\psi \circ \varphi = 0$. So we have

$$K_1(\psi) \circ K_1(\varphi) = K_1(\psi \circ \varphi) = K_1(0_{B,I}) = 0_{K_1(B),K_1(I)}.$$

So $\operatorname{im}(K_1(\varphi)) \subseteq \ker(K_1(\psi))$. For the reverse inclusion, refer to [11, Proposition 8.2.4]. Therefore, the sequence in 8.1 is exact. □

PROPOSITION 8.10 (**Direct sums of K_1**). *For any C^*-algebras A, B, we have*

$$K_1(A \oplus B) \cong K_1(A) \oplus K_1(B).$$

In particular, we have two inclusion mappings $\iota_A : A \to A \oplus B$ and $\iota_B : B \to A \oplus B$ and there is an isomorphism

$$K_1(\iota_A) \oplus K_1(\iota_B) : K_1(A \oplus B) \to K_1(A) \oplus K_1(B);$$
$$(g, h) \mapsto K_1(\iota_A)(g) + K_1(\iota_B)(h).$$

PROOF. Refer to Proposition 8.2.4 in [11]. □

CHAPTER 9

Inductive Limits in K-Theory

Inductive limits provide a crucial technique for understanding the K-theory of infinite-dimensional C^*-algebras by approximating them through sequences of simpler, finite-dimensional algebras. This approach is fundamental to our computational strategy for graph of groups K-theory because many of the algebras we encounter, particularly skew-product graph algebras, naturally arise as inductive limits of increasingly complex finite-dimensional subalgebras. The power of inductive limits lies in the continuity properties of K-theory: the K-groups of the limit algebra can be computed as the inductive limit of the K-groups of the approximating algebras, transforming difficult infinite-dimensional computations into manageable finite-dimensional ones.

For this thesis, inductive limits are particularly significant because skew-product graph algebras, which form the foundation of our approach to graph of groups K-theory, are often approximately finite-dimensional (AF) algebras. This property enables the systematic computation of their K-theory through the techniques developed in this chapter, providing the computational tools necessary for our novel approach to graph of groups C^*-algebras. The material follows the treatment in Rørdam, Larsen and Laustsen [11], emphasising the aspects most relevant to our applications.

9.1 Inductive Limits

DEFINITION 9.1 (**Inductive sequences in a category**). *Let $\mathcal{C}$ be a category. An* ***inductive sequence*** *is a sequence $\{A_n\}_{n=1}^{\infty}$ of objects in $\mathcal{C}$ with a collection of morphisms $\varphi_n : A_n \to$*

A_{n+1} *in $\mathcal{C}$. The inductive sequence is usually written as*

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots .$$

For every $m > n$, we write

$$\varphi_{m,n} : A_n \to A_m \quad \text{by} \quad \varphi_{m,n} = \varphi_{m-1} \circ \varphi_{m-2} \circ \cdots \circ \varphi_{n+1} \circ \varphi_n.$$

The collection of morphisms $\{\varphi_n\}$ are called ***connecting maps****.*

EXAMPLE 9.2. *The chapter about skew-product graphs in part 1 gives examples of inductive sequences. Let E be a directed row-finite graph, with the labelling $c : E^1 \to \mathbb{Z}$ in 4.2. That is, $c(e) = -1$ for all $e \in E^1$. Then the graph C^*-algebra $C^*(E \times_c \mathbb{Z})$ is generated by $\{s_{(e,n)}, p_{(v,n)} : e \in E^1, v \in E^0, n \in \mathbb{Z}\}$.*

Let F_n be the subgraph of $E \times_c \mathbb{Z}$ constructed by edges (e, k) and vertices (v, k) where $k \leq n$. Then the graph C^*-algebra $C^*(F_n)$ is generated by $\{s_{(k,n)}, p_{(v,k)} : e \in E^1, v \in E^0, k \leq n\}$. Observe that all the projections are non-zero since the sources in F_n are the vertices (v, n), and so the Cuntz-Krieger uniqueness theorem gives a homomorphism $\pi_n : C^*(F_n) \to C^*(F_{n+1})$ by

$$\pi_n(s_{(e,k)}) = q_{(e,k)} \quad \text{and} \quad \pi_n(p_{(v,k)}) = t_{(v,k)}$$

for all $n \in \mathbb{N}$. Then we have an inductive sequence of graph C^*-algebras

$$C^*(F_1) \xrightarrow{\pi_1} C^*(F_2) \xrightarrow{\pi_2} C^*(F_3) \xrightarrow{\pi_3} \cdots .$$

DEFINITION 9.3 (**Inductive limits**). *Suppose that*

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots$$

is an inductive sequence in a category $\mathcal{C}$. The ***inductive limit*** *of the sequence is a system $(A, \{\mu_n\}_{n=1}^{\infty})$, denoted $\varinjlim A_n$, where A is an object in $\mathcal{C}$, and each $\mu_n : A_n \to A$ is a morphism in $\mathcal{C}$ such that*

(1) $\mu_n = \mu_{n+1} \circ \varphi_n$; and

(2) *if* $(B, \{\lambda_n\}_{n=1}^{\infty})$ *is a system, where* $B \in \mathrm{ob}(\mathcal{C})$ *and* $\lambda_n = \lambda_{n+1} \circ \varphi_n$ *for all* $n \in \mathbb{N}$, *then there is one and only one morphism* $\lambda : A \to B$ *such that* $\lambda_n = \lambda \circ \mu_n$ *for each* $n \in \mathbb{N}$.

for each $n \in \mathbb{N}$.

Essentially, the inductive limit of a sequence of objects $\{A_i\}$ is an object A where for each A_i in the sequence, there exists a map between the object and the limit. The second condition shows that the inductive limit is unique up to isomorphism.

EXAMPLE 9.4. *The inductive limit of the sequence*

$$\mathbb{C} \xrightarrow{\varphi_1} M_2(\mathbb{C}) \xrightarrow{\varphi_2} M_3(\mathbb{C}) \xrightarrow{\varphi_3} \cdots ,$$

where

$$\varphi_n : M_n(\mathbb{C}) \to M_{n+1}(\mathbb{C}); m \mapsto \begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix},$$

is isomorphic to $\mathcal{K}$, *the* C^**-algebra of compact operators on a separable infinite dimensional Hilbert space. This is because each* $M_n(\mathbb{C})$ *can be realised as the* C^**-algebra of compact operators on a n dimensional Hilbert space.*

9.2 Inductive Limits of Sequences of C^*-Algebras and Abelian Groups

PROPOSITION 9.5 (Inductive limits of C^*-algebras). *Every inductive sequence of* C^**-algebras*

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots$$

has an inductive limit $(A, \{\mu_n\})$. *In addition,*

(1) $A = \overline{\bigcup_{n=1}^{\infty} \mu_n(A_n)}$;
(2) $\|\mu_n(a)\| = \lim_{m\to\infty} \|\varphi_{m,n}(a)\|$ *for every* $n \in \mathbb{N}$ *and* $a \in A_n$;
(3) $\ker(\mu_n) = \{a \in A_n : \lim_{m\to\infty} \|\varphi_{m,n}(a)\| = 0\}$; *and*

(4) *if* $(B, \{\lambda_n\})$ *is a system and* $\lambda : A \to B$ *as in (2) of inductive limit definition, then*
   (a) $\ker(\mu_n) \subseteq \ker(\lambda_n)$ *for all* $n \in \mathbb{N}$*;*
   (b) λ *is injective if and only if* $\ker(\lambda_n) \subseteq \ker(\mu_n)$ *for all* $n \in \mathbb{N}$*; and*
   (c) λ *is surjective if and only if* $B = \overline{\bigcup_{n=1}^{\infty} \lambda_n(A_n)}$*.*

PROOF. Refer to Proposition 6.2.4 in [11]. □

EXAMPLE 9.6. *Recall from example 9.2, there is an inductive sequence of graph* C^**-algebras*

$$C^*(F_1) \xrightarrow{\pi_1} C^*(F_2) \xrightarrow{\pi_2} C^*(F_3) \xrightarrow{\pi_3} \cdots ,$$

and they are increasing C^**-subalgebras of the skew-product graph* C^**-algebra* $C^*(E \times_c \mathbb{Z})$*. For each* $C^*(F_n)$*, there is an injective* $*$*-homomorphism* $\mu_n : C^*(F_n) \to C^*(E \times_c \mathbb{Z})$ *such that*

$$\nu_n = \nu_{n+1} \circ \pi_n,$$

and unsurprisingly, $(C^*(E \times_c \mathbb{Z}), \mu_n)$ *is the inductive limit of the sequence.*

PROPOSITION 9.7 (Inductive limits of abelian groups). *Every inductive sequence of abelian groups*

$$G_1 \xrightarrow{\alpha_1} G_2 \xrightarrow{\alpha_2} G_3 \xrightarrow{\alpha_3} \cdots$$

has an inductive limit $(G, \{\beta_n\})$*. In addition,*

(1) $G = \overline{\bigcup_{n=1}^{\infty} \beta_n(G_n)}$*;*
(2) $\ker(\beta_n) = \bigcup_{m=n+1}^{\infty} \ker(\alpha_{m,n})$*; and*
(3) *if* $(H, \{\gamma_n\})$ *is a system and* $\gamma : G \to H$ *as in (2) of inductive limit definition, then*
   (a) $\ker(\mu_n) \subseteq \ker(\lambda_n)$ *for all* $n \in \mathbb{N}$*;*
   (b) γ *is injective if and only if* $\ker(\gamma_n) \subseteq \ker(\beta_n)$ *for all* $n \in \mathbb{N}$*; and*
   (c) γ *is surjective if and only if* $H = \overline{\bigcup_{n=1}^{\infty} \gamma_n(G_n)}$*.*

9.3 Continuity of K-Theory

The continuity of K-theory functors under inductive limits represents one of the most powerful computational tools in K-theory, enabling the calculation of K-groups for complex infinite-dimensional algebras through systematic approximation by simpler finite-dimensional ones. This property is essential for our approach to graph of groups K-theory because it allows us to compute the K-groups of skew-product graph algebras, which naturally arise as inductive limits, by computing the K-groups of their finite-dimensional approximants. The continuity results ensure that the challenging problem of computing K-theory for infinite-dimensional graph of groups algebras can be reduced to manageable finite-dimensional computations, providing the computational foundation for our novel techniques.

THEOREM 9.8 (Continuity of K_0). *For each inductive sequence of C^*-algebras*

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots ,$$

the K_0 group of the inductive limit is isomorphic to the inductive limit of the sequence of K_0 groups of the inductive sequence. That is, $K_0(\varinjlim A_n) = \varinjlim K_0(A_n)$. More specifically, let $(A, \{\mu_n\})$ be the inductive limit of the sequence above, and let $G, \{\beta_n\}$ be the inductive limit of the sequence of groups below

$$K_0(A_1) \xrightarrow{K_0(\varphi_1)} K_0(A_2) \xrightarrow{K_0(\varphi_2)} K_0(A_3) \xrightarrow{K_0(\varphi_3)} \cdots ,$$

then there is a unique group isomorphism $\gamma : G \to K_0(A)$ such that

$$K_0(\mu_n) = \gamma \circ \beta_n$$

for every $n \in \mathbb{N}$.

PROOF. Recall that $\mu_n = \mu_{n+1} \circ \varphi_n$ for every $n \in \mathbb{N}$. Functorality of K_0 yields group homomorphisms $K_0(\mu_n) = K_0(\mu_{n+1}) \circ K_0(\varphi_n)$. Using the definition of inductive limits, there exists a unique group homomorphism $\gamma : G \to K_0(A)$ such that $K_0(\mu_n) = \gamma \circ \beta_n$. So γ is surjective.

Now, let $g \in \ker(\gamma)$. Find $n \in \mathbb{N}$ and $k \in K_0(A_n)$ such that $g = \beta_n(h)$. Then $0 = \gamma(g) = K_0(\mu_n)(h)$ and so $K_0(\varphi_{m,n})(h) = 0$ for some integer $m > n$, where $g = (\beta_m \circ K_0(\varphi_{m,n}))(h) = 0$, showing injectivity of γ. □

THEOREM 9.9 (**Continuity of K_1**). *For each inductive sequence of C^*-algebras*

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots ,$$

the K_1 group of the inductive limit is isomorphic to the inductive limit of the sequence of K_1 groups of the inductive sequence. That is, $K_1(\varinjlim A_n) = \varinjlim K_1(A_n)$. More specifically, let $(A, \{\mu_n\})$ be the inductive limit of the sequence above, and let $G, \{\beta_n\}$ be the inductive limit of the sequence of groups below

$$K_1(A_1) \xrightarrow{K_1(\varphi_1)} K_1(A_2) \xrightarrow{K_1(\varphi_2)} K_1(A_3) \xrightarrow{K_1(\varphi_3)} \cdots ,$$

then there is a unique group isomorphism $\gamma : G \to K_1(A)$ such that

$$K_1(\mu_n) = \gamma \circ \beta_n$$

for every $n \in \mathbb{N}$.

PROOF. The proof is out-of-scope for this thesis. The main idea is to use the fact that the K_1-group of A is isomorphic to the K_0-group of SA, the suspension of A. Refer to [11]. □

9.4 Approximately Finite-Dimensional Algebras

Approximately finite-dimensional (AF) algebras play a central role in our approach to computing K-theory for graph of groups algebras because they represent a class of infinite-dimensional C^*-algebras whose K-theory is completely computable through inductive limit techniques. The significance of AF algebras for this thesis lies in the fundamental result that skew-product graph algebras with appropriate edge labellings are AF algebras, making their K-theory fully accessible through the continuity properties established in the previous section. This connection provides the key computational tool for our approach: by showing that graph of groups algebras can be realised through skew-product constructions involving AF algebras, we gain access to explicit K-theory computations that would otherwise be intractable.

DEFINITION 9.10. *An AF-algebra is a C^*-algebra that is isomorphic to the inductive limit of a sequence of finite dimensional C^*-algebras.*

EXAMPLE 9.11. *Theorem 2.4 in [7] says that a directed graph E has no cycles if and only if $C^*(E)$ is an AF-algebra. To summerise, if the graph E has no loops, then is it shown that every any finite set of generators $s_\alpha s_\beta^*$ lives inside the subalgebra generated by a finite subgraph $H \subset E$, whereas $C^*(H)$ is finite-dimensional, it follows that $C^*(E)$ is AF. Then, it is shown that if E has a loop, then $C^*(E)$ cannot be AF. Combining the above gives the desired result.*

REMARK 9.12. *Recall from example 9.6 that given a directed row-finite graph E and the labelling $c : E^1 \to \mathbb{Z}; c(e) = -1$ for all $e \in E^1$, $(C^*(E \times_c \mathbb{Z}, \mu_n)$ is the inductive limit of the sequence*

$$C^*(F_1) \xrightarrow{\pi_1} C^*(F_2) \xrightarrow{\pi_2} C^*(F_3) \xrightarrow{\pi_3} \cdots .$$

Observe that $E \times_c \mathbb{Z}$ has no cycles, regardless of the structure of E. By the previous example, $C^*(E \times_c \mathbb{Z})$ is an AF-algebra.

REMARK 9.13. *Suppose that A is an AF-algebra, then $K_1(A) = 0$. To see that, realise that A is the inductive limit of a sequence of finite-dimensional C^*-algebras. That is, there exists an inductive sequence*

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots ,$$

where A_i is a finite-dimensional C^*-algebras and $\varinjlim(A_i) = A$. Since each A_i is finite-dimensional,

$$A_i = M_{k_{i_1}}(\mathbb{C}) \oplus M_{k_{i_2}}(\mathbb{C}) \oplus \cdots \oplus M_{k_{i_r}}(\mathbb{C})$$

for some positive integer r and $k_{i_1}, k_{i_2}, \ldots, k_{i_r}$. From previous example we know that $K_1(M_n(\mathbb{C})) \cong 0$, so $K_1(A_i) \cong 0 \oplus 0 \oplus \cdots 0 \cong 0$. So We have

$$K_1(A) = K_1(\varinjlim A_n) \cong \varinjlim K_1(A_n) \cong \varinjlim 0 = 0.$$

EXAMPLE 9.14. *Let* $\mathscr{H}$ *be a separable Hilbert space and consider the* C^**-algebra of compact operators* $\mathcal{K}(\mathscr{H})$. *Let* $e_n : n \in \mathbb{N}$ *be an orthonormal basis for* $\mathscr{H}$. *Then we can write* $\mathcal{K}(\mathscr{H}) = \overline{\bigcup_n M_n(\mathbb{C})}$.

Then the inclusions $i_n : M_n(\mathbb{C}) \to M_{n+1}(\mathbb{C}); a \mapsto a \oplus 0$ *preserves the matrix rank.*

CHAPTER 10

K-Theory for Graph Algebras

This chapter represents a crucial bridge between the foundational K-theory techniques developed in previous chapters and their application to graph of groups algebras, which forms the central contribution of this thesis. By computing the K-theory of graph C^*-algebras $C^*(E)$, we establish the computational template that will be extended to the more complex setting of graph of groups in Chapter 11. The techniques presented here—particularly the use of the dual Pimsner-Voiculescu exact sequence and the realisation of graph algebras through skew-product constructions involving AF algebras—provide the methodological foundation for our novel approach to graph of groups K-theory.

The significance of this chapter lies not only in the explicit computation of $K_0(C^*(E))$ and $K_1(C^*(E))$ in terms of the vertex matrix A_E, but more importantly in demonstrating how exact sequence techniques can systematically reduce complex K-theoretic computations to linear algebraic problems. This reduction principle, achieved through the connection between graph algebras and crossed products by the gauge action, will be central to our approach for graph of groups algebras, where similar but more sophisticated exact sequence arguments will enable K-theory computations for these significantly more complex structures.

10.1 The Pimsner-Voiculescu Exact Sequence and its dual

The Pimsner-Voiculescu exact sequence represents one of the most powerful computational tools in K-theory, providing the mechanism for computing K-groups of complex algebras through their connections to crossed products. For this thesis, the dual Pimsner-Voiculescu sequence is particularly crucial because it enables the computation of graph algebra K-theory through the gauge action, which naturally leads to skew-product constructions involving AF algebras. This approach provides the template for our extension to graph of groups algebras, where analogous exact sequence techniques will enable K-theory computations for these more complex structures. The exact sequence framework ensures that the algebraic complexity of these computations reduces to manageable linear algebraic problems involving vertex matrices.

THEOREM 10.1 (Pimsner-Voiculescu Exact Sequence). *Let A be a C^*-algebra and $\alpha : \mathbb{Z} \to \mathrm{Aut}(A)$ be an action of $\mathbb{Z}$ on A. Then there is a cyclic six-term exact sequence*

$$\begin{tikzcd}
K_0(A) \arrow[r, "\mathrm{id} - K_0(\alpha_z)"] & K_0(A) \arrow[r, "K_0(\iota)"] & K_0(A \rtimes_\alpha \mathbb{Z}) \arrow[d] \\
K_1(A \rtimes_\alpha \mathbb{Z}) \arrow[u] & K_1(A) \arrow[l, "K_1(\iota)"'] & K_1(A) \arrow[l, "\mathrm{id} - K_1(\alpha_z)"']
\end{tikzcd} .$$

where $\iota : A \to A \rtimes_\alpha \mathbb{Z}$ is the embedding of A into $A \rtimes_\alpha \mathbb{Z}$.

PROOF. Refer to [23, Theorem 2.4] for proof. □

This provides a way to obtain information about the crossed products. A variant of the Pimsner-Viculescu exact sequence is another six-term sequence for the crossed products by action of $\mathbb{T}$, obtained by the following theorem:

THEOREM 10.2 (Connes' Thom Isomorphism). *Let A be a C^*-algebra and $\alpha : \mathbb{R} \to \mathrm{Aut}(A)$ be an action. Then*

$$K_0(A \rtimes_\alpha \mathbb{R}) \cong K_1(A) \quad \textit{and} \quad K_1(A \rtimes_\alpha \mathbb{R}) \cong K_0(A).$$

PROOF. Refer to [24, Theorem 2] for the proof. □

The Dual Pimsner-Voiculescu Exact Sequence. Let A be a C^*-algebra and let α be an action of the circle group $\mathbb{T}$ on A. If we regard α as an action of $\mathbb{R}$, and identify $K_0(A \rtimes_\alpha \mathbb{R}) \cong K_1(A)$

and $K_1(A \rtimes_\alpha \mathbb{R}) \cong K_0(A)$ by the Connes' Thom Isomorphism Theorem, we have obtained the Dual Pimsner-Voiculescu Exact Sequence

$$\begin{tikzcd}
K_0(A \rtimes_\alpha \mathbb{T}) \arrow[r, "\mathrm{id} - K_0(\hat{\alpha}_n^{-1})"] & K_0(A \rtimes_\alpha \mathbb{T}) \arrow[r] & K_0(A) \arrow[d] \\
K_1(A) \arrow[u] & K_1(A \rtimes_\alpha \mathbb{T}) \arrow[l] & K_1(A \rtimes_\alpha \mathbb{T}) \arrow[l, "\mathrm{id} - K_1(\hat{\alpha}_n^{-1})"']
\end{tikzcd}.$$

where

$$\hat{\alpha}_n(i_A(a)) = i_A(a) \quad \text{and} \quad \hat{\alpha}_n(i_{\mathbb{T}}(z)) = z^n i_{\mathbb{T}}(z)$$

for every $a \in A, z \in \mathbb{T}, n \in \mathbb{Z}$.

The existence of both the exact sequences is established in [22] (Chapter 10.6 for the dual exact sequence), and there are no elementary proofs for the above theorems.

Recall from 4.5, there is an isomorphism between $C^*(E) \rtimes_\gamma \mathbb{T}$ and $C^*(E \times_1 \mathbb{Z})$. Since skew-product graphs have no cycles, example 9.11 shows that $C^*(E \rtimes_1 \mathbb{Z})$ is an AF-algebra. 9.13 shows that $K_1(C^*(E) \rtimes_\gamma \mathbb{T}) \cong K_1(C^*(E \rtimes_1 \mathbb{Z})) = 0$. Therefore, the dual Pimsner-Voiculescu exact sequence collapses to

$$0 \to K_1(C^*(E)) \to K_0(C^*(E) \rtimes_\gamma \mathbb{T}) \xrightarrow{\mathrm{id} - K_0(\hat{\gamma}_1^{-1})} K_0(C^*(E) \rtimes_\gamma \mathbb{T}) \to K_0(C^*(E)) \to 0.$$

Since we have an isomorphism $C^*(E \times_1 \mathbb{Z}) \cong C^*(E) \rtimes_\gamma \mathbb{T}$, which induces an isomorphism of their K-theory, we can replace $C^*(E) \rtimes_\gamma \mathbb{T}$ with $C^*(E \times_1 \mathbb{Z})$ and the action $\hat{\gamma}$ with β. So the above exact sequence becomes

$$0 \to K_1(C^*(E)) \to C^*(E \times_1 \mathbb{Z}) \xrightarrow{\mathrm{id} - K_0(\beta_1^{-1})} C^*(E \times_1 \mathbb{Z}) \to K_0(C^*(E)) \to 0.$$

Thus, we have the following result:

LEMMA 10.3. *$K_1(C^*(E))$ and $K_0(C^*(E))$ is the kernel and cokernel of the homomorphism*

$$\mathrm{id} - K_0(\beta_1^{-1}) : C^*(E \times_1 \mathbb{Z}) \to C^*(E \times_1 \mathbb{Z})$$

respectively.

PROOF. Name the homomorphism $K_1(C^*(E)) \to C^*(E \times_1 \mathbb{Z})$ to be f and the homomorphism $C^*(E \times_1 \mathbb{Z}) \to K_0(C^*(E))$ to be g. Then we have

$$0 \to K_1(C^*(E)) \xrightarrow{f} C^*(E \times_1 \mathbb{Z}) \xrightarrow{\operatorname{id} - K_0(\beta_1^{-1})} C^*(E \times_1 \mathbb{Z}) \xrightarrow{g} K_0(C^*(E)) \to 0.$$

With the above exact sequence being started and ended by the zero group (trivial group), we have the following fact:

(1) the map $f : K_1(C^*(E)) \to C^*(E \times_1 \mathbb{Z})$ is injective, and
(2) the map $g : C^*(E \times_1 \mathbb{Z}) \to K_0(C^*(E))$ is surjective.

Extending (1), we have

$$K_1(C^*(E)) \cong \operatorname{im}(f) = \ker(\operatorname{id} - K_0(\beta_1^{-1})).$$

For (2), by the first isomorphism theorem, we have

$$K_0(C^*(E)) = \operatorname{im}(g) \cong C^*(E \times_1 \mathbb{Z})/\ker(g) = C^*(E \times_1 \mathbb{Z})/\operatorname{im}(\operatorname{id} - K_0(\beta_1^{-1})).$$

So we have

$$\operatorname{coker}(\operatorname{id} - K_0(\beta_1^{-1})) = C^*(E \times_1 \mathbb{Z})/\operatorname{im}(\operatorname{id} - K_0(\beta_1^{-1})) \cong K_0(C^*(E)).$$

□

10.2 Computations of the K-groups

This section demonstrates the computational power of combining the dual Pimsner-Voiculescu exact sequence with inductive limit techniques, showing how the K-theory of graph algebras reduces to linear algebraic computations involving the vertex matrix. This computational approach is fundamental to our methodology because it provides the template for extending these techniques to graph of groups algebras, where similar but more sophisticated matrix computations will enable K-theory calculations for significantly more complex structures. The key insight is that the exact sequence framework transforms topological K-theory problems into concrete linear algebra, making previously intractable computations accessible.

Recall from Remark 9.12, we can create a sequence of subgraphs $\{F_n\}_{n\in\mathbb{N}}$ of $E \times_1 \mathbb{Z}$ such that $C^*(F_1) \subseteq C^*(F_2) \subseteq C^*(F_3) \subseteq \cdots$. The inductive limit of such inductive sequence is $C^*(E \times_1 \mathbb{Z})$.

LEMMA 10.4. *For each $n \in \mathbb{N}$, $K_0(F_n)$ is the free abelian group generated by $\{[p_{(v,n)}]_0 : v \in E^0\}$.*

PROOF. Fix $n \in \mathbb{N}$. Recall that

$$C^*(E) = \overline{\text{span}}\{s_\mu s_\nu^* : s(\mu) = s(\nu), \mu, \nu \in E^*\}.$$

Consider an element $s_\mu s_\nu^*$ in $C^*(F_n)$, so $s(\mu) = s(\nu) = (w, k)$ with $k \leq n$. Since every path of length $n-k$ with range (w,k) begins at a source, we can apply the Cuntz-Krieger relations $n-k$ times to $s_\mu s_\nu^*$ to get a finite sum of $s_\alpha s_\beta^*$ such that $s(\alpha) = s(\beta) = (v, n)$. This means

$$C^*(F_n) = \overline{\text{span}}\{s_\mu s_\nu^* : s(\mu) = s(\nu) = (v,n) \text{ for some } v \in E^0\}.$$

For each fixed (v, n), the elements $s_\mu s_\nu^*$ with $s_\mu = s_\nu = (v, n)$ form a family of matrix units, therefore

$$A_{(v,n)} := \overline{\text{span}}\{s_\mu s_\nu^* : s_\mu = s_\nu = (v,n)\}$$

is a C^*-algebra that is isomorphic to the C^*-algebra of compact operators on the the Hilbert space

$$\ell^2(\{\mu \in F_n^* : s(\mu) = (v,n)\}).$$

For vertices $v \neq w$, the subalgebras $A_{(v,n)}$ satisfies $A_{(w,n)}A_{(v,n)} = 0$ since $s_\nu^* s_\alpha = 0$ whenever $s(\nu) \neq s(\alpha)$. Thus we have

$$C^*(F_n) \cong \bigoplus_{v\in E^0} A_{(v,n)}.$$

Since $p_{(v,n)}$ is a rank-one projection in $A_{(v,n)}$, $K_0(A_{(v,n)})$ is the free abelain group generaed by $[p_{(v,n)}]_1$, and so

$$K_0(C^*(F_n)) \cong K_0(\bigoplus_{v\in E^0} A_{(v,n)}) \cong \bigoplus_{v\in E^0} K_0(A_{(v,n)}) \cong \bigoplus_{v\in E^0} \mathbb{Z}[p_{(v,n)}]_1.$$

☐

COROLLARY 10.5. *$K_0(E \times_1 \mathbb{Z})$ is isomorphic to the inductive limit $(\mathbb{Z}^{E^0}, A_E^T)$, where the map between the K_0-groups is the multiplication by the transpose of the vertex matrix A_E of E.*

PROOF. The Cuntz-Krieger relation at (v,n) implies

$$\begin{aligned}[p_{(v,n)}]_0 &= [\sum_{r(e,k)=(v,n)} s_{(e,k)}s^*_{(e,k)}]_0 = \sum_{r(e,k)=(v,n)} [s_{(e,k)}s^*_{(e,k)}]_0 \\ &= \sum_{r(e,k)=(v,n)} [s^*_{(e,k)}s_{(e,k)}]_0 = \sum_{r(e,k)=(v,n)} [p_{(s(e),n+1)}]_0 \\ &= \sum_{r(e)=v} [p_{s(e),n+1}]_0.\end{aligned}$$

Recall that the vertex matrix is defined by

$$A_E(v,w) = \#\{e \in E^1 : r(e) = v, s(e) = w\}.$$

So the above becomes

$$[p_{(v,n)}]_0 = \sum_{w\in E^0} A_E(v,w)[p_{(w,n+1)}]_0.$$

Therefore, the K_0-group of $C^*(F_n)$ depends on the vertices of E and every $K_0(C^*(F_n))$ is isomorphic to a subset of the direct sum $\mathbb{Z}^{E^0}$, and the inductive limit is isomorphic to $\varinjlim(\mathbb{Z}, A_E^t)$ of

$$K_0(F_1) \xrightarrow{K_0(\pi_1)} K_0(F_2) \xrightarrow{K_0(\pi_2)} K_0(F_3) \xrightarrow{K_0(\pi_3} \cdots .$$

☐

THEOREM 10.6 (**K-groups of graph C^*-algebras**). *Let E be a row-finite graph with no source, and let A_E be the vertex matrix. Then with*

$$1 - A_E^t : \mathbb{Z}^{E^0} \to \mathbb{Z}^{E^0},$$

$K_1(C^*(E)) \cong \ker(1 - A_E^t)$ *and* $K_0(C^*(E)) \cong \operatorname{coker}(1 - A_E^t)$.

To proof this, we need the following lemma:

LEMMA 10.7. *Let V be a countable set and T be a column-finite $V \times V$ matrix with integer entries. Define the following inductive sequence*

$$G_1 = \mathbb{Z}^V \xrightarrow{i_1} G_2 = \mathbb{Z}^V \xrightarrow{i_3} G_3 \to \cdots .$$

where $i_n : G_n \to G_{n+1} by i_n(A) = Ta$. For each $n \in \mathbb{N}$, denote by i^n the canonical map of G_n into the inductive limit $G := \varinjlim(G_n, i_n)$. Let $\alpha : G \to G$ such that

$$\alpha(i^n(a)) = i^n(Ta)$$

for every $a \in G_n$. Then the map i^1 is an isomorphism of $\ker(1 - T)$ *onto* $\ker(\operatorname{id} - \alpha)$ *and induces an isomorphism of* $\operatorname{coker}(1 - T)$ *onto* $\operatorname{coker}(\operatorname{id} - \alpha)$.

PROOF. Refer to Lemma 7.17 in [8]. □

With the above lemma, we can deliver the proof:

PROOF OF THEOREM 10.6. Since E is row-finite, we have $|r^{-1}(v)| < \infty$ for all $v \in E^0$. This implies sum of each row in the vertex matrix A_E is finite. Then the sum of each column in the transpose of the vertex matrix A_E^T is finite also, and thus gives a well-defined map on the direct sum

$$\mathbb{Z}^{E^0} := \big\{(n_v)_{v \in E^0} : \text{all but finitely many } n_v \text{ are zero}\big\}.$$

Since $K_0(E \times_1 \mathbb{Z}) \cong \varinjlim(\mathbb{Z}^{E^0}, A_E^T)$, we want to see how $K_0(\beta_1^{-1})$ is converted. Recall that

$$\beta_1^{-1}(p_{(v,n)}) = p_{(v,n-1)} \quad \text{and} \quad \beta_1^{-1}(s_{(e,n)}) = s_{(e,n-1)}.$$

We have

$$\beta_1^{-1}(p_{(v,n)}) = p_{(v,n-1)} = \sum_{r(e,k)=(v,n-1)} s_{(e,k)}s^*_{(e,k)} = \sum_{r(e)=v} s_{(e,n)}s^*_{(e,n)}.$$

So β_1^{-1} maps each $C^*(F_n)$ to itself. Applying the K_0 functor to β_1^{-1} gives us

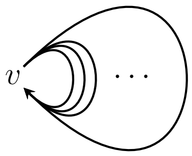
$$\begin{aligned} K_0(\beta_1^{-1})([p_{(v,n)}]_0) &= \sum_{r(e)=v} [s_{(e,n)}s^*_{(e,n)}]_0 = \sum_{r(e)=v} [s^*_{(e,n)}s_{(e,n)}]_0 \\ &= \sum_{r(e)=v} [p_{(s(e),n)}]_0 = \sum_{w\in E^0} A_E(v,w)[p_{(w,n)}]_0. \end{aligned}$$

Thus the homomorphism

$$K_0(\beta_1^{-1}) : K_0(C^*(F_n)) \cong \mathbb{Z}^{E^0} \to K_0(C^*(F_n)) \cong \mathbb{Z}^{E^0}$$

is multiplication by A_E^T. Lemma 10.3 says the K_1-group and K_0-group of $C^*(E)$ are the kernel and cokernel of the homomorphism $\mathrm{id} - K_0(\beta_1^{-1})$ respectively. Take $T = A_E^T$ and $V = E^0$, then applying lemma 10.7 gives isomorphisms of $\ker(\mathrm{id} - K_0(\beta_1^{-1}))$ to $\ker(1 - A_E^T)$ and $\operatorname{coker}(\mathrm{id} - K_0(\beta_1^{-1}))$ to $\operatorname{coker}(1 - A_E^T)$. □

EXAMPLE 10.8. *The Cuntz algebra $\mathcal{O}_n$ is the universal C^*-algebra generated by n isometries of an infinite-dimensional Hilbert space. It is the universal C^*-algebra of the following graph*



The vertex matrix A_E is just an integer $[n]$, since there is only one vertex v, with n edges from and to itself. So we have the map $1 - A_E^T : \mathbb{Z} \to \mathbb{Z}; m \mapsto (1-n)m$ for every $m \in \mathbb{Z}$. The kernel of this map is $\{0\}$, so $K_1(\mathcal{O}_n)$ is just the trivial group. The image of the map is

$(1-n)\mathbb{Z} = (n-1)\mathbb{Z}$. *So the cokernel of the map is* $\mathbb{Z}/(n-1)\mathbb{Z} = \mathbb{Z}_{n-1}$, *and so* $K_0(\mathcal{O}_n)$ *is isomorphic to the group of integers modulo* n.

CHAPTER 11

Towards a K-Theory for Graph of Groups C^*-Algebras

This chapter represents the culmination of this thesis, bringing together the skew-product methodology developed in Part 1 with the K-theory foundations established in Part 2 to advance the computation of K-theory for graph of groups C^*-algebras. While our original approach—extending the successful skew-product techniques from graph algebras to graph of groups algebras—encounters certain technical obstacles related to approximate finite-dimensionality, this chapter demonstrates how alternative computational methods can address these challenges and provides a foundation for future developments in the field.

The central insight of this thesis lies in recognising that the rich group-theoretic structure of graph of groups, while making them significantly more complex than ordinary graphs, also provides additional computational tools through correspondences and exact sequence techniques. The skew-product graph of groups construction introduced in Chapter 6, though not immediately yielding AF algebras as in the graph case, establishes a crucial connection between graph of groups algebras and crossed products by coactions. This connection, combined with recent advances in Cuntz-Pimsner correspondence methods and multitree techniques, opens new pathways for systematic K-theory computations.

The significance of this work extends beyond the specific computations presented here: it demonstrates how the successful methodologies from graph C^*-algebra theory can be adapted and extended to more complex structures, providing a template for future investigations into K-theory for even more sophisticated algebraic constructions involving group actions and combinatorial data.

While the skew-product approach originally envisioned for this thesis encounters technical challenges in the graph of groups setting, recent advances in Cuntz-Pimsner correspondence theory have provided alternative computational pathways. The breakthrough work of Mundey, Pask and Spielberg [14] demonstrates how graph of groups C^*-algebras can be realised as Cuntz-Pimsner algebras of carefully constructed correspondences, enabling systematic K-theory computations through six-term exact sequences. This approach, while different from our skew-product methodology, validates the central premise of this thesis: that K-theory computations for graph of groups algebras can be made tractable through sophisticated extensions of techniques successful for graph algebras.

In essence, section 3.1 of [14] constructs a C^*-correspondence for a graph of groups by the following steps:

(1) Let $\mathscr{G} = (\Gamma, G)$ be a graph of groups. For each edge $e \in \Gamma^1$, define $B_e := C^*(G_e)$ to be the group C^*-algebra of the edge group G_e.

(2) For each edge e, define a right B_e-module F_e with $C_c(G_e)$ equipped with the convolution product.

(3) Define a right action of $a \in C_c(G_e)$ on $\xi \in C_c(G_{r(e)})$ by

$$(\xi \cdot a)(h) := \sum_{g \in G_e} \xi(h\alpha_e(g))a(g^{-1}),$$

and define $C_c(G_e)$-valued inner product on $\xi, \eta \in C_c(G_{r(e)})$ by

$$(\xi|\eta)_{C_c(G_e)}(h) := \sum_{g \in G_{(r(e))}} \overline{\xi(g)}\eta(g\alpha_e(h)) = \sum_{\mu \in \sum_e} \sum_{k \in G_e} \overline{\xi(\mu\alpha_e(k))}\eta(\mu\alpha_e(kh)).$$

(4) A norm on $C_c(G_e)$ is given by $\|\xi\| = \|(\xi|\xi)_{C_c(G_e)}(h)\|_{full}^{1/2}$, where $\|\cdot\|$ denotes the full C^*-norm of $C_c(G_e)$. Then $C_c(G_{r(e)})$ can be completed into a right B_e-module.

(5) The constuction of the modles F_e can be framed with the context of *Green's Imprimitivity Theorem*. Each F_e is shown to be a full and faithful imprimitivity bimodule between $C(\sum_e)$ and $C^*(G_e)$.

(6) Denote

$$B := \bigoplus_{e \in \Gamma^1} B_e \quad \text{and} \quad F := \bigoplus_{e \in \Gamma^1} F_e,$$

and the direct sum of all the modules

$$D = \bigoplus_{fe\in\Gamma^2} D_{fe} \quad \text{with} \quad D_{fe} := \bigoplus_{\mu\in\sum_e \Delta_{fe}} B_e \quad \text{where} \quad \Delta_{fe} := \begin{cases} \emptyset & \text{, if } f \neq \overline{e} \\ \{1_{G_{r(e)}}\} & \text{, if } f = \overline{e} \end{cases}.$$

D is called the graph of groups module associated to $\mathscr{G}$.

(7) Define $\overline{\varphi}_{fe} : B_f \to \operatorname{End}^0_{B_e}(D_{fe})$ satifying

$$\overline{\varphi}_{fe}(a)\xi(\mu, h) = \sum_{k\in G_f} a(k^{-1})\xi(\alpha_{\overline{f}}(k)\cdot\mu, c_{fe}(k,\mu)h)$$

for all $a \in C_c(G_f)$ and $\xi \in C_c((\sum_e \Delta_{fe}) \times G)e)$.

(8) The C^*-correspondences $(\overline{\varphi}_{fe}, D_{fe})$ is then assembled into a single C^*-correspondence $(\overline{\varphi}, D)$, which is the *graph of groups correspondence associated to* $\mathscr{G}$.

(9) the amplified graph of groups correspondence (φ, E) is the extension to the graph of groups correspondence, where

$$E \cong F \otimes_B D \otimes_B F^* \quad \text{and} \quad \varphi : A \to \operatorname{End}_A(E) \quad \text{by} \quad \varphi = \oplus_{fe\in\Gamma^2}\varphi_{fe}.$$

COROLLARY 11.1. *Suppose that G is a countable group acting without inversions on a locally finite nonsingular tree X, and let τ denote the induced action on $C(\partial X)$. Let $\mathscr{G} = (\Gamma, G)$ denote the quotient graph of groups associated to the action of G on X. Then with $(\overline{\varphi}, D)$ the graph of groups correspondence for $\mathscr{G}$ and (φ, E) as the amplified graph of groups correspondence,*

$$\mathcal{O}_D \sim_{me} \mathcal{O} \cong C^*(\mathscr{G}) \sim_{me} C(\partial X_{\mathscr{G}}) \rtimes_\tau \pi_1(\mathscr{G}) \cong C(\partial X) \rtimes_\tau G,$$

where $\sim_{me}$ denotes Morita equivalence.

THEOREM 11.2 (Main theorm of [14]). *Let $\mathscr{G} = (\Gamma, G)$ be a locally finite nonsingular graph of countable groups. For $i = 0, 1$ let $\Lambda_i = \cdot \otimes_A (\operatorname{id} - [D]) : K_i(B) \to K_i(B)$, Let $m : K_*(\mathcal{O}_D) \to K_*(C^*(\mathscr{G}))$ denote the isomorphism induced by the Morita equivalence of Corollary 11.1 and let $i_B : B \hookrightarrow \mathcal{O}_D$ denote the universal inclusion. Then the following six-term sequence of abelian groups is exact:*

$$\begin{tikzcd}
\bigoplus_{e\in\Gamma^1} K_0(C^*(G_e)) \arrow[r, "\Lambda_0"] & \bigoplus_{e\in\Gamma^1} K_0(C^*(G_e)) \arrow[r, "m\circ(i_B)_*"] & K_0(C^*(\mathscr{G})) \arrow[d, "\partial"] \\
K_1(C^*(\mathscr{G})) \arrow[u, "\partial"] & \bigoplus_{e\in\Gamma^1} K_1(C^*(G_e)) \arrow[l, "m\circ(i_B)_*"'] & \bigoplus_{e\in\Gamma^1} K_1(C^*(G_e)) \arrow[l, "\Lambda_1"']
\end{tikzcd}.$$

PROOF. This follows immediately from Theorem 4.9 in [6]. □

EXAMPLE 11.3. *Fix $m, n \geq 2$ and let $\mathscr{G} = (\Gamma, G)$ be an edge of groups (a graph of groups consisting of a single edge) with $G_v = \mathbb{Z}/n\mathbb{Z}, G_w = \mathbb{Z}/m\mathbb{Z}, G_e = \{0\}$, and injective homomorphisms α_e and $\alpha_{\overline{e}}$ being the inclusion of $\{0\}$. Then $|\sum_e| = n$ and $|\sum_{\overline{e}}| = m$. In this case $C^*(G_e) = C^*(\overline{e}) \cong \mathbb{C}$ so*

$$K_0(B) = \mathbb{Z}[1_e]_0 \oplus \mathbb{Z}[1_{\overline{e}}]_0 \quad \text{and} \quad K_1(B) = 0.$$

Applying the six-term sequence above, it follows that $K_0(C^*(\mathscr{G})) \cong \operatorname{coker}(\Lambda_0)$ and $K_1(C^*(\mathscr{G})) \cong \ker(\Lambda_0)$ where $\Lambda_0 : \mathbb{Z}[1_e]_0 \oplus \mathbb{Z}[1_{\overline{e}}]_0 \to \mathbb{Z}[1_e]_0 \oplus \mathbb{Z}[1_{\overline{e}}]_0$ is the $\mathbb{Z}$-linear map given by

$$(x, y) \otimes_B (\operatorname{id}_* -[D]) = (x - (|\textstyle\sum_e| - 1)y, y - (|\textstyle\sum_{\overline{e}}|)z).$$

Treating $[1_e]_0$ as the column $(1 \quad 0)^T$ and $[1_{\overline{e}}]_0$ as the column $(1 \quad 0)^T$ the map Λ_0 has matrix representation

$$\begin{pmatrix} 1 & n-1 \\ m-1 & 1 \end{pmatrix} \quad \textit{with Smith normal form} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1-(n-1)(m-1) \end{pmatrix}.$$

It now follows that

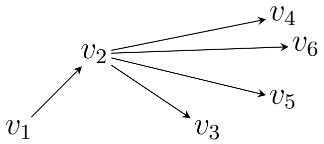
$$K_0(C^*(\mathscr{G})) \cong \frac{\mathbb{Z}}{(1-(n-1)(m-1))\mathbb{Z}} \quad \text{and} \quad K_1(C^*(\mathscr{G})) \cong \begin{cases} \mathbb{Z} & \text{if} \quad m = n = 2 \\ 0 & \text{otherwise} \end{cases}.$$

Since all the edge groups are trivial, Theorem 5.9 states that $C^*(\mathscr{G})$ is isomorphic to $C^*(E_{\mathscr{G}})$. Since we can compute the K-groups of C^*-algebras of directed groups. One can verify that the K-groups computed above agrees.

The complementary approach developed by Brownlowe, Ramagge and Robertson [15] provides another perspective on K-theory computations for structures related to graph of groups algebras. Their work focuses on group actions on multitrees, which possess a directed graph structure that connects to the Bass-Serre theory underlying graph of groups. The main result is a six-term exact sequence in K-theory for the reduced crossed product $C_0(\partial E) \rtimes_r G$ induced from the action of a countable discrete group G on a row-finite, finitely-aligned multitree E with no sources. This approach demonstrates another pathway for computing K-theory when graph of groups algebras act on geometric structures, providing additional tools for the systematic computational program envisioned in this thesis.

DEFINITION 11.4. *A **multitree** is a directed graph E in which there is at most one direct path between any two vertices.*

An example of a multitree would be like the following:



THEOREM 11.5 (Main result of [15]). *Let G be a countable discrete group acting on a row-finite, finitely aligned multitree E with no sources. Suppose that the action $G \curvearrowright \Omega$ is amenable. Then for $\alpha_i := \sum_{[e]\in G\backslash E^1}(\theta_e)_{*,i}$, $i = 0, 1$, we have the following six-term sequence*

$$\begin{tikzcd}
\bigoplus_{[v]\in G\backslash E^0} K_0(C_r^*(G_v)) \arrow[r, "\mathrm{id} - \alpha_0"] & \bigoplus_{[v]\in G\backslash E^0} K_0(C_r^*(G_v)) \arrow[r] & K_0(C_0(\partial E) \rtimes_{\tau_\partial, r} G) \arrow[d] \\
K_1(C_0(\partial E) \rtimes_{\tau_\partial, r} G) \arrow[u] & \bigoplus_{[v]\in G\backslash E^0} K_1(C_r^*(G_v)) \arrow[l] & \bigoplus_{[v]\in G\backslash E^0} K_1(C_r^*(G_v)) \arrow[l, "\mathrm{id} - \alpha_1"']
\end{tikzcd}.$$

The formulas for the K-theory of $C_0(\partial E) \rtimes_r G$ is provided. Through not strictly about graphs of groups, graph of groups algebras acts on multitrees, as described in [10], and the six-term exact sequence for the multitrees can be applied to compute the K-theory of graph of groups C^*-algebras, in the case of free actions or when vertex stabilisers are infinite cyclic.

11.1 Conclusions and Future Directions

This thesis has demonstrated that the fundamental approach of extending successful K-theoretic techniques from graph C^*-algebras to graph of groups C^*-algebras is both viable and fruitful, despite encountering certain technical obstacles along the way. The skew-product graph of groups construction introduced in Chapter 6, while not immediately yielding the AF algebra property that makes graph algebra K-theory so tractable, establishes crucial structural connections that enable alternative computational approaches.

The key achievements of this work include:

(1) The development of skew-product constructions for graph of groups, providing a systematic method for realising graph of groups algebras as crossed products by coactions;
(2) The establishment of fundamental connections between ordinary graph algebras and graph of groups algebras through these skew-product constructions;
(3) The demonstration that, while direct AF approximation may not be available for graph of groups algebras, alternative exact sequence techniques can provide systematic computational pathways for their K-theory.

The recent advances in Cuntz-Pimsner correspondence theory, as exemplified by the work of Mundey, Pask and Spielberg, validate the central premise of this thesis: that sophisticated extensions of techniques successful for graph algebras can indeed make K-theory computations for graph of groups algebras tractable. The correspondence approach, while different from our original skew-product methodology, demonstrates that multiple computational pathways can emerge from the foundational structures established in this work.

Future research directions suggested by this thesis include:

(1) Investigation of conditions under which skew-product graph of groups algebras might possess approximate finite-dimensionality or related structural properties that enable inductive limit techniques;

(2) Development of hybrid approaches combining skew-product constructions with Cuntz-Pimsner correspondence methods to optimise computational efficiency;
(3) Extension of these techniques to even more complex algebraic structures involving higher-dimensional combinatorial data and group actions;
(4) Applications of these K-theoretic tools to classification problems in the Elliott program for graph of groups algebras.

The techniques and constructions developed in this thesis thus provide both immediate computational tools and a foundation for continued advancement in the systematic computation of K-theory for complex operator algebraic structures arising from combinatorial and group-theoretic data.

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