README: ORBIT DATA OF Q-DIFFERENCE PAINLEVÉ ONE OVER FINITE FIELDS

PIETER ROFFELSEN

Description

This data set contains all (reduced) orbit lengths of the dynamical system

$$qP_{\mathrm{I}}: \quad (x,y,t) \mapsto (\overline{x},\overline{y},\bar{t}), \qquad \begin{cases} \overline{x} = \dfrac{t}{x - s^{-1}y}, \\ \overline{y} = \dfrac{s\,x}{y}, \\ \bar{t} = s\,t, \end{cases}$$

over finite fields $\mathbb{F}_q$, with the prime power q ranging between 2 and 499. It further contains code to generate and analyse the data.

Files

File Names.

- *qPI_orbit_length_data.zip*
- *qPI_data_analysis.nb*
- *qPI_genus_fibres.ipynb*
- *qPI_main.ipynb*

File Descriptions.

- All the data is collected in *qPI_orbit_length_data.zip*.
- All the data was created using Magma notebook *qPI_main.ipynb*.
- The data is analysed in Mathematica notebook *qPI_data_analysis.nb*.
- The Magma notebook *qPI_genus_fibres.ipynb* contains code to construct integrals of motion of qPI and compute the genera of their fibres.

Data Structure

Data Files. For each prime power q between 2 and 499, there is a separate .txt file, titled *qPI_data_q=?.txt* with question mark the relevant prime power, that contains all reduced orbit lengths of qP_{I} over $\mathbb{F}_q$.

Each .txt file consists of a nested list,

$$\{\{r_1, L_1\}, \{r_2, L_2\}, \ldots, \{r_k, L_k\}\},$$

where

- $k \geq 1$ is the number of divisors of $q-1$;
- $r_1, r_2, \ldots, r_k$ are the divisors of $q-1$ in increasing order;
- for every index $1 \leq i \leq k$, the list L_i is made out of tuples $\{l, f_l\}$, where l an occurring reduced orbit length and f_l its absolute frequency, for all choices of $s \in \mathbb{F}_q^*$ with multiplicative order $\mathrm{ord}(s) = r_i$.

The lists are denoted using curly brackets following the notational convention for lists in Mathematica.

This research was supported by NISDRG Grant #NI240100145.

Example. Consider the file *qPI_data_q=5.txt* Its content reads

```
{
{ 1 , {{ 1, 3 },{ 4, 6 },{ 5, 5 },{ 6, 3 },
       { 7, 3 },{ 8, 2 },{ 9, 3 },{ 10, 1 }}} ,
{ 2 , {{ 1, 1 },{ 4, 3 },{ 5, 2 },{ 6, 1 },
       { 7, 1 },{ 8, 1 },{ 9, 2 },{ 10, 1 }}} ,
{ 4 , {{ 4, 4 },{ 5, 2 },{ 6, 2 },{ 8, 2 },{ 9, 2 }}}
}
```

The divisors of $5-1=4$ are $1,2,4$, so $k=3$ and $(r_1,r_2,r_3)=(1,2,4)$ in the notation above. For each divisor, we have a corresponding list

$$L_1 = \{\{1,3\},\{4,6\},\{5,5\},\{6,3\},\{7,3\},\{8,2\},\{9,3\},\{10,1\}\},$$
$$L_2 = \{\{1,1\},\{4,3\},\{5,2\},\{6,1\},\{7,1\},\{8,1\},\{9,2\},\{10,1\}\},$$
$$L_3 = \{\{4,4\},\{5,2\},\{6,2\},\{8,2\},\{9,2\}\}.$$

The entry $\{5,2\}$ in L_3 encodes that there are precisely 2 orbits of reduced length 5, counted with s ranging over the set $\{s\in\mathbb{F}_5^*:\mathrm{ord}(s)=r_3\}=\{2,3\}$.

Similarly, the entry $\{4,3\}$ in L_2 encodes that there are precisely 3 orbits of reduced length 4, with $s=4$, since the only element $s\in\mathbb{F}_5^*$ with multiplicative order $r_2=2$ is $s=4$.

Technical Requirements

Software. Mathematica 14.2 and Magma V2.28-23 plus Jupyter notebook extension.

School of Mathematics and Statistics F07, The University of Sydney, NSW 2006, Australia

Email address: pieter.roffelsen@sydney.edu.au