Path models and combinatorial representations of affine Hecke algebras

Eloise Kate Little

This thesis is submitted in fulfillment of
the requirements for the degree of
Doctor of Philosophy

Department of Mathematics and Statistics
University of Sydney

THE UNIVERSITY OF
SYDNEY

February, 2025

Contents

Acknowledgements

First and foremost I would like to thank my supervisor James Parkinson, without whom this thesis would not be possible. I am very grateful for his guidance, patience and support throughout this process. His ability to make seemingly impossible tasks feel achievable was invaluable throughout my honours year and PhD.

Thanks goes to Jérémie Guilhot and Nathan Chapelier-Laget, who collaborated with James and I throughout my candidature. The experience of working in a team was one of the most beneficial and unique experiences of my PhD. I am very grateful for this experience and for all that I have learned from Jérémie and Nathan over the past 3 years. I would also like to thank Jérémie for hosting me in Tours, France.

I would like to acknowledge my co-supervisor, Oded Yacobi, and members of the Department of Mathematics at the University of Sydney for helpful conversations and guidance. Thanks also to my peers in the department for not taking themselves too seriously and reminding me to do the same.

A special thanks goes to my partner for their support, useful conversations, help with code and their calming presence. I am glad that there are two normalisations of the Kazhdan-Lusztig basis so that we could argue over which is better and then fall in love.

Finally I would like to thank my family. To my grandparents, your continued support and encouragement has meant a great deal during this process. To my mum and sister, you both inspire me everyday, thank you for believing in me. I love you all dearly.

This PhD was funded by the Postgraduate Research Scholarship in Representation Theory under the ARC grant New Directions in Hecke Algebras (ARC discovery DP200100712). The fees were covered by the RTP fee offset from the Australian government.

Statement of Originality

This thesis contains no material which has been accepted for the award of any other degree or diploma. All work in this thesis, except where duly attributed to another person, is believed to be original.

Eloise Kate Little

Authorship attribution statement

The content of Part I is published as [21] in collaboration with Jérémie Guilhot and James Parkinson. The paper [21] includes both the reduced and non-reduced cases whereas Part I focuses solely on the reduced case. However, the proofs, examples and background are expanded to include more detail.

The content of Part II along with Appendix A is submitted for publication as [8] in collaboration with Nathan Chapelier-Laget, Jérémie Guilhot and James Parkinson. Again, the proofs and examples in this thesis are expanded from the paper for further understanding.

The research direction and mathematical analysis was contributed to jointly between the collaborators for both parts. I substantially contributed to the development of ideas, computation of examples and proof of results for both papers.

Eloise Kate Little
February 20th, 2025.

As supervisor for the candidature upon which this thesis is based, I can confirm that the authorship attribution statements above are correct.

James Parkinson
February 22nd, 2025.

Abstract

In this thesis we explore particular combinatorial representations of affine Hecke algebras with general parameters. We introduce the combinatorial model of J-folded alcove paths and show that the matrix entries of our representations are described by these alcove paths. We categorise which of the representations are bounded and give conjectures connecting the representations to Kazhdan-Lusztig theory and Opdam's Plancherel Theorem. Upon specialising to type $\widetilde{A}_n$, we show that the combinatorial representations form a balanced system of cell representations. For this type, we prove an asymptotic version of Opdam's Plancherel Theorem and develop a J-analogue to the classical Satake isomorphism. Using this asymptotic Plancherel Theorem we construct a new explicit description of Lusztig's asymptotic algebra in type $\widetilde{A}_n$ in terms of a ring of specialised matrices formed from the constructed representations.

Introduction

The Hecke algebra corresponding to a Coxeter group W is a particular q-deformation of the group algebra of W, where q is an indeterminant. In [25] Kazhdan and Lusztig defined a basis of the Hecke algebra that has become a fundamental tool in the representation theory of Hecke algebras and thus their underlying Coxeter groups. One of the main tools of Kazhdan-Lusztig theory is the theory of cells, which partition W and are defined from the multiplication between Kazhdan-Lusztig basis elements. In [35] Lusztig conjectured a list of properties of the theory known as $P1$-$P15$. These conjectures give properties of the cells as well as properties of Lusztig's $\mathbf{a}$-function and Lusztig's asymptotic algebra. Lusztig's $\mathbf{a}$-function (see [29]) is a function that assigns a positive integer to each Coxeter group element, defined by the maximal degree of q in the structure constants of the Kazhdan-Lusztig basis. Lusztig's asymptotic algebra (see [31]) is a simplification of the Hecke algebra that retains essential representation theory information, such as the cell structure. The asymptotic algebra, often denoted $\mathcal{J}$, has basis $(\mathsf{t}_w)_{w\in W}$ with multiplication defined using the coefficient of the highest degree of q in the structure constants of the Kahzdan-Lusztig basis.

Conjectures $P1$-$P15$ are proven in the equal parameters case but are still conjectural for general parameters. Geck formulated a method of proving these 15 conjectures for Hecke algebras defined from a spherical Coxeter group with arbitrary parameters, without using a geometric interpretation of the group (see [17]). The method centred on constructing a family of matrix representations of the Hecke algebra, indexed by Kazhdan-Lusztig cells, that satisfy a list of criteria. The matrix representation family is called a *balanced system of cell representations*. In [23] Guilhot and Parkinson extended this method to Hecke algebras defined from affine Coxeter groups. The criteria mirror information of the Hecke algebra, in particular the structure of $\mathcal{J}$ and Lusztig's $\mathbf{a}$-function. That is, the matrix representations are required to be bounded with respect to q-degree and when all criteria are met this bound is equivalent to Lusztig's $\mathbf{a}$-function. In addition, a ring formed from a specialisation of the matrices is isomorphic to $\mathcal{J}$ (see [23, §2.2]). Guilhot and Parkinson proved $P1$-$P15$ using such a family of matrix representations for the Coxeter groups $\widetilde{G}_2$ and $\widetilde{C}_2$ for arbitrary parameters in [23] and [22] respectively. For the matrix representations corresponding to infinite cells, they proved a combinatorial formula for the matrix entries using alcove paths which was instrumental in finding the bound of the matrix representations.

Alcove paths are intimately connected with the combinatorics of affine Hecke algebras. This connection was made clear by Ram in [42], who stated a formula describing the right action of the standard Hecke algebra basis onto the Berstein-Lusztig basis in terms of positively folded alcove walks (see also [44]). This work stems from the work of Littelmann and Gaussent on positively folded galleries, in turn an adaptation of the Littelmann path model (see [16]). In [23] and [22] Guilhot and Parkinson create path formulas describing the matrix entries of particular representations in terms of positively folded alcove paths, now restricting the paths to be contained within a subset of alcoves.

In Part I of this thesis we will generalise the path formulas of Guilhot and Parkinson to all irreducible affine Coxeter groups of reduced type with arbitrary parameters by defining *J-folded alcove paths*. We construct and analyse representations of affine Hecke algebras whose matrix entries are described by path formulas and then categorise which of the representations are bounded (in terms of the affine version of Geck's criteria on balanced systems of cell representations). Part II focuses on explicitly describing Lusztig's asymptotic algebra for type $\widetilde{A}_n$, using the path formula from Part I. The representations constructed in Part I, now restricting to the $\widetilde{A}_n$ case, are shown to form a balanced system of cell representations.

We now describe the setup in more detail. Let W be an irreducible affine Weyl group, let W_0 be the corresponding spherical subgroup and let Φ be the root system of W_0 over an n-dimensional Euclidean vector space V (note that Φ is assumed to be reduced). The simple reflections of W are $s_i = s_{\alpha_i}$, for $\alpha_1, \dots, \alpha_n$ the simple roots of Φ, and s_0 the affine generator. The set of affine roots is $\widetilde{\Phi} = \{\alpha + k\delta \mid \alpha \in \Phi, k \in \mathbb{Z}\}$ and the corresponding set of affine hyperplanes is $\mathbb{H} = \{H_{\alpha,k} \mid \alpha \in \Phi, k \in \mathbb{Z}\}$ where $H_{\alpha,k}$ is the set of points $x \in V$ that are perpendicular to $\alpha + k\delta$. The elements of W are in bijection with the set formed by the closure of the open connected components of $V \setminus \bigcup_{H \in \mathbb{H}} H$, the elements of which are called alcoves.

We will work in the context of the extended affine Weyl group associated to W, denoted $\widetilde{W}$. Set $L : \widetilde{W} \to \mathbb{N}$ to be a weight function on $\widetilde{W}$, that is $L(s_i) > 0$ for all $i \in \{0, 1, \dots, n\}$ and $L(uv) = L(u) + L(v)$ for $\ell(uv) = \ell(u) + \ell(v)$ (where $\ell(u)$ is the reduced length of the word $u \in \widetilde{W}$). Let $\widetilde{\mathcal{H}}$ be the (weighted) affine Hecke algebra corresponding to $\widetilde{W}$ defined over the ring $R = \mathbb{Z}[\mathsf{q}, \mathsf{q}^{-1}]$ and let $\mathsf{q}_i = \mathsf{q}^{L(s_i)}$.

The representations of $\widetilde{\mathcal{H}}$ constructed in Chapter 5 (see Definition 5.1.1) are inspired by modules of Deodhar (see [9, §2], and also [10], [11] and [12]) and Lusztig (see [34, Lemma 4.7]). The action is defined combinatorially using the different types of crossings between alcoves, and the representations are finite dimensional. The representations are denoted $\pi_{J,\mathsf{v}}$ as they are dependent on two pieces of data: a subset $J \subseteq I = \{1, 2, \dots, n\}$ and a *J-parameter system* v. The J-parameter system is a family $\mathsf{v} = (\mathsf{v}_\alpha)_{\alpha \in \Phi_J}$ such that $\mathsf{v}_\alpha = \mathsf{v}_\beta$ if $\beta \in W_J\alpha$ and $\mathsf{v}_{\alpha_j} \in \{\mathsf{q}_j, -\mathsf{q}_j^{-1}\}$ (where W_J is the parabolic subgroup of W_0 generated by $\{\alpha_j \mid j \in J\}$ with corresponding root system Φ_J).

After constructing these representations in Theorem 5.1.4 we prove a path formula for their matrix entries in Theorem 5.3.3. The path formula is one of the main results of Part I and describes the matrix entries in terms of J-folded alcove paths, paths that are contained to a subset of the alcoves depending on J (as was done by Guilhot and Parkinson in [23] and [22]).

For $J \subseteq I$, the *fundamental J-alcove* is the set

$$\mathcal{A}_J = \{x \in V \mid 0 \le \langle x, \alpha \rangle \le 1 \text{ for all } \alpha \in \Phi_J^+\}.$$

The J-folded alcove paths remain within the fundamental J-alcove by 'bouncing' on its boundary walls and are positively folded within the bounds of $\mathcal{A}_J$ (positively folded in the classical sense, see [42]). Figure 1 gives an example of a J-folded alcove path for type $\widetilde{G}_2$ with $J = \{1\}$. The path can fold at any point allowed by the classical definition of folds but must 'bounce' instead of crossing the walls of the fundamental J-alcove. In Figure 1 the second, eleventh and twenty-second steps of the path are bounces and the fifth and fifteenth steps are positive folds. The bounces play a different role in the path formula to the folds, one that is interlinked with the chosen J-parameter system.

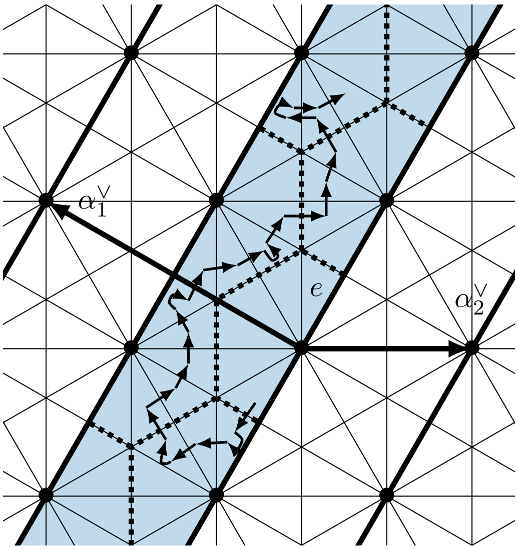


Figure 1: The fundamental J-alcove and a J-folded alcove path for type $\widetilde{G}_2$ and $J = \{1\}$.

The fundamental J-alcove plays an important role in the path formula, but is also fundamental to the results of Part II. In Chapter 2 the structure of the fundamental J-alcove is analysed. Two important features are the set of *pseudo-translations* that act as the translations of $\mathcal{A}_J$ and the subgroup of W_0 that stabilises $\mathcal{A}_J$, denoted G_J. The semi-direct product of G_J with the set of pseudo-translations is the subset of $\widetilde{W}$ that stabilises $\mathcal{A}_J$. Both the set of pseudo-translations and G_J are instrumental in the analysis of Lusztig's asymptotic algebra in type $\widetilde{A}_n$ in Part II.

The representations in [23] and [22] are induced from 1-dimenional representations of Levi subalgebras of $\widetilde{\mathcal{H}}$, however these are equivalent to representations of the type defined in this thesis. It is shown that, for all types, the representation $\pi_{J,\mathsf{v}}$ is isomorphic to the induction of a 1-dimensional representation of the Levi subalgebra corresponding to $J \subseteq I$ (Theorem 5.2.5). In fact, all representations of $\widetilde{\mathcal{H}}$ induced from 1-dimensional representations of Levi subalgebras can be realised by some $\pi_{J,\mathsf{v}}$.

The final main result of Part I is classifying the bounded representations of type $\pi_{J,\mathsf{v}}$. The matrix representation $\pi_{J,\mathsf{v}}$ is bounded if for all $w \in \widetilde{W}$ and all matrix entries of $\pi_{J,\mathsf{v}}(T_w)$ there is a uniform bound on the degree of q. This is the boundedness condition in the criteria to be a balanced system of cell representations. The *bound* $\mathbf{a}_{J,\mathsf{v}}$ of $\pi_{J,\mathsf{v}}$ is the maximum degree of q for all entries of $\pi_{J,\mathsf{v}}(T_w)$ and all $w \in \widetilde{W}$. In Theorem 6.1.4 and Proposition 6.1.5 the J and v that result in a bounded $\pi_{J,\mathsf{v}}$ are classified, noting any restriction on the parameters. It is conjectured that the set of elements of $w \in \widetilde{W}$ whose matrices $\pi_{J,\mathsf{v}}(T_w)$ reach the bound $\mathbf{a}_{J,\mathsf{v}}$ is contained in a Kazhdan-Lusztig cell and that these elements satisfy $\mathbf{a}(w) = \mathbf{a}_{J,\mathsf{v}}$ (where $\mathbf{a}(w)$ is Lusztig's $\mathbf{a}$-function on $w \in \widetilde{W}$).

To form a conjectural formula for $\mathbf{a}_{J,\mathsf{v}}$ we introduce the canonical trace function on $\widetilde{\mathcal{H}}$ (see [18]). In [40] Opdam proved a decomposition of this trace function as a sum of integrals over families of representations called the Plancherel formula. Restricting to elements of $\mathcal{H}_0$ (the Hecke algebra associated to W_0) the trace function decomposes as a sum over characters of irreducible representations of $\mathcal{H}_0$. Opdam's Plancherel formula is an analogue to this decomposition in the affine case. In the finite case the degree of the coefficients of the characters are linked to Lusztig's $\mathbf{a}$-function. It is conjectured that this also occurs in the affine analogue, and this was confirmed for the cases $\widetilde{G}_2$ and $\widetilde{C}_2$ in [23] and [22]. Due to this conjectured connection, in

Chapter 6 we give a conjectural formula for the bound of $\pi_{J,\mathsf{v}}$ in terms of Macdonalds c-function (see [36]), which is known to be linked to the coefficients of the integrals in Opdam's Plancherel formula.

There is also a canonical trace on $\mathcal{J}$, described in Chapter 3 following the construction made by Lusztig in [35, §20.1(b)]. Guilhot and Parkinson gave a decomposition of the inner product formed from this trace function in the $\widetilde{G}_2$ and $\widetilde{C}_2$ cases ([23] and [22]), forming asymptotic Plancherel formulas. In Part II we find an asymptotic Plancherel formula for type $\widetilde{A}_n$ and use it to describe Lusztig's asymptotic algebra.

In the equal parameters case (that is, $L(s_i) = L(s_j)$ for all $i, j \in I \cup \{0\}$), Lusztig gave a conjectural description of $\mathcal{J}$ for all affine types in [33]. This conjectural description was first confirmed in type $\widetilde{A}_n$ by Xi in [49] using the results of Shi [45] on chains and anti-chains. They proved that for all Kazdan-Lusztig two-sided cells of $\widetilde{A}_n$, the corresponding subalgebra of $\mathcal{J}$ is isomorphic to the representation ring of a particular maximal reductive subgroup of the centraliser of $\mathrm{SL}_{n+1}(\mathbb{C})$ with respect to a certain unipotent element. Another approach to constructing the asymptotic algebra in type $\widetilde{A}_n$ was given by Kim and Pylyavskyy in [26] using the affine matrix ball construction. In Part II we give a new description of the asymptotic algebra of type $\widetilde{A}_n$ in terms of the symmetry group of the fundamental J-alcove G_J.

The description of the Kazhdan-Lusztig cells for $\widetilde{W}$ corresponding to type $\widetilde{A}_n$ is known (see [45] and [30]) and the number of two-sided cells is in bijection with the partitions of $n+1$. Due to this, when applying the results of Part I to the $\widetilde{A}_n$ case we now index all notation by partitions. In particular, we only consider $J \subseteq I$ formed from a partition. For the parition λ let J_λ be the corresponding subset of I, let W_λ be the parabolic subgroup of W_0 with respect to J_λ and let w_λ to be the longest element of W_λ. Let Δ_λ be the two-sided cell that contains $\mathsf{w}_{\lambda'}$ (with λ' the transposed partition of λ). Thus, $(\Delta_\lambda)_{\lambda \vdash n+1}$ is the full set of two-sided Kazhdan-Lusztig cells and we have that $\Delta_\lambda \leq_{LR} \Delta_\mu$ (in the Kazhdan-Lusztig two-sided preordering) if and only if $\lambda \leq \mu$ (in the dominance ordering of partitions). Let π_λ be the representation $\pi_{J_\lambda,\mathsf{v}}$ with J_λ the subset of I corresponding to λ and v such that π_λ is bounded (this forces only one option for v). A consequence of the results of Part II is that the family of matrix representations $(\pi_\lambda)_{\lambda \vdash n+1}$ is a balanced system of cell representations (see Corollary 11.3.4).

As $\widetilde{\mathcal{H}}$ in the $\widetilde{A}_n$ case has equal parameters, it satisfies the positivity properties of Elias and Williamson [14] which state that the coefficients of the Kazhdan-Lusztig polynomials and the coefficients of the structure constants with respect to the Kazhdan-Lusztig basis are non-negative. This positivity implies that the conjectures $P1$-$P15$ of Lusztig hold (see [35]). In particular, it is known that Lusztig's $\mathbf{a}$-function is equal for all elements within two-sided cells and that $\mathbf{a}(z) = \mathbf{a}(\mathsf{w}_{\lambda'}) = \ell(\mathsf{w}_{\lambda'})$ for all $z \in \Delta_\lambda$.

A main result of Part II is the asymptotic Plancherel formula for type $\widetilde{A}_n$. Before constructing the formula, the decomposition of the trace on $\widetilde{\mathcal{H}}$ is described using the results of Aubert and Plymen [2]. The terms of the Plancherel formula for type $\widetilde{A}_n$ are in bijection with partitions and the degree of the coefficient is linked to $\mathbf{a}(\mathsf{w}_{\lambda'})$, confirming (in this case) the conjectured theory that the results of the finite trace decomposition extend to the affine Plancherel formula. In Chapter 10, using the Plancherel formula and the fact that the coefficient realises Lusztig's $\mathbf{a}$-function, the asymptotic Plancherel formula for type $\widetilde{A}_n$ is described. The terms of the formula, given in Theorem 10.2.2, are also indexed by partitions and the formula gives a decomposition of the trace function on $\mathcal{J}$. From this decomposition it is shown that the subalgebra of $\mathcal{J}$ formed from Δ_λ is isomorphic to the matrix algebra $\mathfrak{C}_\lambda$ with $\mathbb{Z}$-basis given by the *leading matrices* $\mathfrak{c}_\lambda(w)$ for $w \in \Delta_\lambda$, where $\mathfrak{c}_\lambda(w)$ is the specialisation of $\mathsf{q}^{-\mathbf{a}(\mathsf{w}_{\lambda'})}\pi_\lambda(T_w)$ at $\mathsf{q}^{-1} = 0$. This result plays an essential role in our description of Lusztig's asymptotic algebra.

As well as the Plancherel formula, the construction of a λ-relative version of the Satake isomorphism is used in our analysis. The classical Satake isomorphism is an isomorphism between $\mathbf{1}_0\widetilde{\mathcal{H}}\mathbf{1}_0$ and the ring of W_0 symmetric functions, where $\mathbf{1}_0$ is a normalisation of the Kazhdan-Lusztig basis element corresponding to w_0, the longest element of W_0 (see [38]). The λ-analogue, given in Theorem 9.0.10 is an isomorphism between $\pi_\lambda(\mathbf{1}_\lambda\widetilde{\mathcal{H}}\mathbf{1}_\lambda)$ and the ring of G_λ-symmetric functions, where $\mathbf{1}_\lambda$ is a normalisation of the Kazhdan-Lusztig basis element corresponding to $\mathsf{w}_{\lambda'}$ and $G_\lambda = G_{J_\lambda}$.

Let Γ_λ be the right Kazhdan-Lusztig cell that contains $\mathsf{w}_{\lambda'}$. In Chapter 11 the set $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$ is described explicitly as a set of maximal length double coset representatives in bijection with the set of dominant pseudo-translations. The subalgebra of $\mathcal{J}$ spanned by t_w for $w \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ is shown to be isomorphic to the ring of G_λ-symmetric functions by mapping the basis elements to λ-analogues of the Schur functions. Using this isomorphism it is shown that, up to conjugation, $\mathfrak{C}_\lambda$ is the full matrix algebra over the ring of G_λ-symmetric functions. As $\mathcal{J} = \bigoplus_{\lambda \vdash n+1} \mathcal{J}_\lambda$, this explicitly describes Lusztig asymptotic algebra recalling that the asymptotic Plancherel formula implies that $\mathcal{J}_\lambda \cong \mathfrak{C}_\lambda$.

It follows from the description of $\mathcal{J}$ and three other criteria that $(\pi_\lambda)_{\lambda \vdash n+1}$ is a balanced system of cell representations (in terms of the criteria in [23]). The remaining criteria are known as boundedness, the killing property and cell recognition. The boundedness and killing criteria are proved in Chapter 8. For boundedness, Theorem 8.2.1 states that the representation π_λ is bounded and has bound $\mathbf{a}(\mathsf{w}_{\lambda'}) = \ell(\mathsf{w}_{\lambda'})$. For the killing property, Theorem 8.1.3 states that if w is in a cell corresponding to a partition that is lower than or incomparable to λ (in the dominance ordering) then the Kazhdan-Lusztig basis element of w will be killed by π_λ. The final result required is that π_λ recognises the cell Δ_λ, meaning that the leading matrix of w is non-zero if and only if $w \in \Delta_\lambda$. This result is a consequence of the asymptotic Plancherel formula given in Chapter 10.

The structure of the thesis is as follows. In Chapter 1 we give a brief introduction to root systems and their corresponding Weyl groups. In particular, introducing affine Weyl groups, affine root systems and the visualisation of affine Weyl groups as a set of alcoves. In Chapter 2 we analyse the fundamental J-alcove, including the pseudo-translation set and the symmetry group G_J. In Chapter 3 we recall well known definitions and properties of Hecke algebras with general parameters and Kazhdan-Lusztig theory. The conjectures of Lusztig, $P1$-$P15$ are stated and Lusztig's asymptotic algebra is defined. In this chapter we also introduce the notion of balanced systems of cell representations, introduce the canonical trace functions on $\widetilde{\mathcal{H}}$ and $\mathcal{J}$ and introduce Opdam's Plancherel formula.

Chapter 4 focuses on alcove paths. Ram's theory of positively folded alcove paths and their connection to Hecke algebra combinatorics (see [42]) is described, before introducing the J-analogue in the form of J-folded alcove paths. Also in this chapter, J-parameter systems are defined and linked to path combinatorics in the form of the *path* v*-mass*, a property of paths that is the coefficient in the path formula.

The path combinatorics are then used in Chapter 5 to define a module of $\widetilde{\mathcal{H}}$ and the corresponding irreducible representations $\pi_{J,\mathsf{v}}$ (Theorem 5.1.4). In Theorem 5.2.5 this module is proven to be isomorphic to the induced module of $\widetilde{\mathcal{H}}$ formed from a 1-dimensional representation of the J-Levi subalgebra of $\widetilde{\mathcal{H}}$ and in Theorem 5.3.3 the path formula is proven, which explicitly describes the matrix entries of $\pi_{J,\mathsf{v}}$ in terms of J-folded alcove paths. In Chapter 6 the J and v for which the representation $\pi_{J,\mathsf{v}}$ is bounded are classified (Theorem 6.1.4 and Proposition 6.1.5). In Conjecture 6.2.4 we give a conjectural formula for the bound of $\pi_{J,\mathsf{v}}$ in terms of Macdonalds c-function and Opdam's Plancherel formula. With Conjecture 6.1.10 this gives a conjectural formula for Lusztig's **a**-function, and in this chapter we prove that these conjectures hold in the rank 2 and 3 cases.

In Part II we reduce to the $\widetilde{A}_n$ case. Chapter 7 applies the results of Part I to this case, now indexing notation by partitions. A dominance order is defined for the weights of the fundamental J_λ-alcove, as well as G_λ-symmetric functions, which are λ-analogues to the Schur functions. In Chapter 8 the killing and boundedness criteria (in terms of the criteria to be a balanced system of cell representations) are proven in Theorem 8.2.1 and Theorem 8.1.3. Chapter 9 focuses on the λ-analogue of the classical Satake isomorphism. Importantly, in Theorem 9.0.7 it is proven that the function $f_\lambda(h) = \chi_\lambda(hC_{\mathsf{w}_{\lambda'}})$ for $h \in \widetilde{\mathcal{H}}$ is a G_λ-symmetric function.

In Chapter 10 the Plancherel formula for type $\widetilde{A}_n$ is described following [2] and the asympototic analogue, the asymptotic Plancherel formula, is proven in Theorem 10.2.2. Two important consequences are proven, that π_λ recognises the cell Δ_λ in Theorem 10.2.3 and that $\mathcal{J}_\lambda$ is isomorphic to the ring of leading matrices $\mathfrak{C}_\lambda$ in Theorem 10.2.4. Lusztig's asymptotic algebra in type $\widetilde{A}_n$ is explicitly described in Chapter 11 using a result of Xi [49]. Particular maximal length double coset representatives m_γ are defined for each dominant weight γ in the fundamental J_λ-alcove, the set of which is shown to be exactly the set $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$ in Theorem 11.2.10. In Proposition 11.2.8 the leading matrices of these elements are shown to have a unique non-zero entry equal to a λ-Schur function, and in Theorem 11.2.10 it is proven that the subalgebra of $\mathcal{J}$ generated by $\{t_w \mid w \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}\}$ is isomorphic to the ring of G_λ-symmetric functions. Finally, the explicit description of Lusztig's asymptotic algebra is given in Theorem 11.3.2 and the matrix family $(\pi_\lambda)_{\lambda \vdash n+1}$ is proven to be a balanced system of cell representations in Corollary 11.3.4.

The results of Chapter 11 rely on a property of the length of the maximal length double coset representative m_γ, stated in Theorem 11.2.5. The proof of this property is quite technical so is completed in Appendix A. Appendix B defines folding tables, a method used in [23] and [22] to find J-folded alcove paths. Although the method was not used for the results of the thesis, MAGMA code of the method was written by the author and used throughout to test conjectures and compute examples.

Part I

The path formula

Chapter 1

Root systems and Weyl groups

In this chapter we set up the background of affine root systems and alcoves. In Section 1.1 we give the definition of a root system and its associated Weyl group. In Section 1.2 we direct our focus to affine Weyl groups. We describe the affine root system and give a description of the affine Weyl group W as a semi-direct product of its root lattice with the naturally chosen finite subgroup W_0 of W. Importantly, we define the extended affine Weyl group. This extension of the affine Weyl group will be the focus of our study for the majority of the subsequent chapters. In Section 1.3 we define a set of alcoves from the affine root system which give a geometric depiction of W. We will further study alcoves in Chapter 4, and they are an integral definition for the path formula in Section 5.3. Finally, in Section 1.4 we describe and prove some properties of parabolic subgroups of W_0. We will further study parabolic subgroups in Chapter 2. The definitions and results of this chapter are well studied. For further information on root systems and Weyl groups see [7, VI].

1.1 General root systems and their associated Weyl groups

Let $I = \{1, 2, 3, \ldots, n\}$ (with integer $n \geq 1$) and let V be a vector space with a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$. Let Φ to be an irreducible, reduced, crystallographic root system of rank n in V. That is, Φ is a subset of V such that

1. Φ is finite, does not contain 0, and spans V,
2. for $\alpha \in \Phi$, Φ is closed under reflection through the hyperplane perpendicular to α,
3. (Reduced) if $\alpha, k\alpha \in \Phi$ then $k = \pm 1$,
4. (Irreducible) no Φ_1, Φ_2 exist such that $\Phi = \Phi_1 \sqcup \Phi_2$ with $\langle \alpha, \beta \rangle = 0$ for all $\alpha \in \Phi_1$ and $\beta \in \Phi_2$,
5. (Crystallographic) if $\alpha, \beta \in \Phi$ then $\langle \beta, \alpha^\vee \rangle \in \mathbb{Z}$, where $\alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$.

The *dual root system* is denoted by $\Phi^\vee = \{\alpha^\vee \mid \alpha \in \Phi\}$ and the set of *simple roots* is $\{\alpha_i \mid i \in I\}$. Denote $\Phi^+ \subseteq \Phi$ to be the set of positive roots of Φ (the roots that are a positive linear combination of simple roots). Denote $\Phi^- = -\Phi^+$ to be the negative roots and so $\Phi = \Phi^+ \sqcup \Phi^-$. We write $\alpha > 0$ if $\alpha \in \Phi^+$ and $\alpha < 0$ if $\alpha \in \Phi^-$.

We adopt the conventions from Bourbaki [7] when labelling the simple roots. For $\alpha = \sum_{i \in I} a_i \alpha_i \in \Phi$ we define $\mathrm{ht}(\alpha) = \sum_{i \in I} a_i$ to be the *height* of α and denote the *highest root* of Φ by $\varphi \in \Phi$. Define integers $m_i \geq 1$ by $\varphi = m_1\alpha_1 + \cdots + m_n\alpha_n$.

The *fundamental coweights* of Φ are the elements $\omega_i \in V$ such that $\langle \omega_i, \alpha_j \rangle = \delta_{i,j}$ for $i, j \in I$. Define the following two $\mathbb{Z}$-lattices:

$$Q = \mathbb{Z}\alpha_1^\vee + \cdots + \mathbb{Z}\alpha_n^\vee \quad \text{and} \quad P = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_n,$$

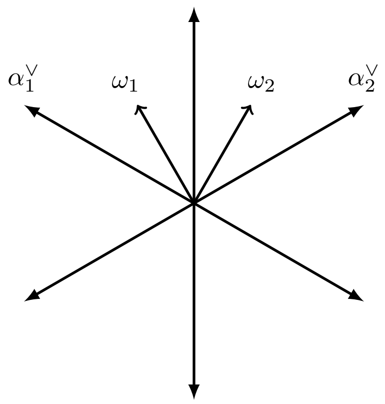
called the *coroot lattice* and the *coweight lattice* respectively. Note that $Q \subseteq P$. Denote the set of positive coweights as $P^+ = \mathbb{N}\omega_1 + \cdots + \mathbb{N}\omega_n$ and the set of positive coroots as $Q^+ = \mathbb{N}\alpha_1^\vee + \cdots + \mathbb{N}\alpha_n^\vee$.

As Φ is reduced, there are at most two roots lengths, referred to as the *long roots* and the *short roots*. If there is only one root length, called the *simply-laced case*, all roots are considered long.

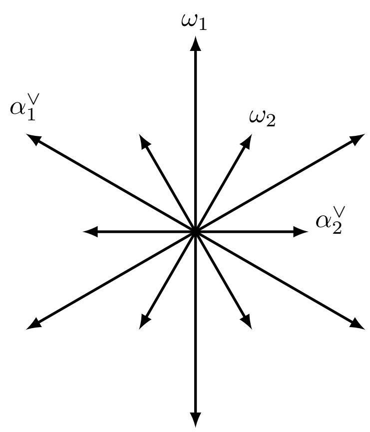
The irreducible, reduced and crystallographic root systems have been classified up to isomorphism. The isomorphism classes are A_n $(n \geq 1)$, B_n $(n \geq 2)$, C_n $(n \geq 2)$, D_n $(n \geq 4)$, F_4, G_2, E_6, E_7 and E_8 where only C_2 and B_2 are isomorphic. The coefficients of the highest root for each class are $(1,1,\dots,1)$, $(1,2,\dots,2)$, $(2,2,\dots,2,1)$, $(1,2,\dots,2,1,1)$, $(2,3,4,2)$, $(3,2)$, $(1,2,2,3,2,1)$, $(2,2,3,4,4,3,2,1)$ and $(2,3,4,6,5,4,3,2)$ respectively (in the form $(m_1, m_2, \dots, m_n)$). The classification and detailed root system descriptions can be found in [7, VI , §4].

Example 1.1.1. Let $V = \{[v_1, v_2, v_3] \in \mathbb{R}^3 \mid v_1 + v_2 + v_3 = 0\}$ and $e_i \in V$ be the vector with 1 in the i-th position and zeroes elsewhere. Let $\Phi = \pm\{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$ be the root system of type A_2, where $\alpha_1 = e_1 - e_2$ and $\alpha_2 = e_2 - e_3$. In this case the coroot system is equal to the root system and the fundamental coweights are $\omega_1 = \frac{2}{3}\alpha_1^\vee + \frac{1}{3}\alpha_2^\vee$ and $\omega_2 = \frac{1}{3}\alpha_1^\vee + \frac{2}{3}\alpha_2^\vee$ (as depicted in Figure 1.1). All roots in Φ are of the same length, and so are all considered long roots.

Now let $\Phi^+ = \pm\{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}$ be the root system of type G_2, where $\alpha_1 = e_1 - e_2$ and $\alpha_2 = -2e_1 + e_2 + e_3$. The dual root system is $\Phi^\vee = \pm\{\alpha_1^\vee, \alpha_2^\vee, \alpha_1^\vee + \alpha_2^\vee, \alpha_1^\vee + 2\alpha_2^\vee, \alpha_1^\vee + 3\alpha_2^\vee, 2\alpha_1^\vee + 3\alpha_2^\vee\}$ (as depicted in Figure 1.1). In this case we have $\omega_i \in \Phi^+$ for $i = 1, 2$, and hence $P = Q$. Explicitly, the fundamental coweights are $\omega_1 = 2\alpha_1^\vee + 3\alpha_2^\vee$ and $\omega_2 = \alpha_1^\vee + 2\alpha_2^\vee$. There are two root lengths, $\pm\{\alpha_2, 3\alpha_1 + 2\alpha_2, 3\alpha_1 + \alpha_2\}$ are the long roots and $\pm\{\alpha_1, 2\alpha_2 + \alpha_2, \alpha_1 + \alpha_2\}$ are the short roots.



The root system of A_2



The root system of G_2

Figure 1.1

For each $\alpha \in \Phi$, we define a hyperplane $H_\alpha = \{x \in V \mid \langle x, \alpha \rangle = 0\}$ perpendicular to α. Let s_α be the orthogonal reflection about H_α, thus $s_\alpha(x) = x - \langle x, \alpha \rangle \alpha^\vee$ for $x \in V$. For the simple roots α_i with $i \in I$, we write $s_i = s_{\alpha_i}$ to be the associated simple reflection. The *Weyl group* of Φ is the subgroup W_0 of $GL(V)$ generated by the simple reflections s_i, $i \in I$.

For $w \in W_0$ its *inversion set* is $\Phi(w) = \{\alpha \in \Phi^+ \mid w^{-1}\alpha \in \Phi^-\}$. Writing $w = s_{i_1}s_{i_2}\cdots s_{i_m}$ with a minimal number of generators we call m the *length* of w, denoted $\ell(w)$, and we have

$$\Phi(w) = \{\alpha_{i_1}, s_{i_1}\alpha_{i_2}, \ldots, s_{i_1}\cdots s_{i_{m-1}}\alpha_{i_m}\}. \tag{1.1.1}$$

Thus $\ell(w) = |\Phi(w)|$. The longest element of W_0, denoted w_0, is such that $\Phi(\mathsf{w}_0) = \Phi^+$.

1.2 Affine root systems and their associated affine Weyl groups

Let Φ be as above. The *affine root system* is the set

$$\widetilde{\Phi} = \{\alpha + k\delta \mid \alpha \in \Phi, k \in \mathbb{Z}\}$$

in the space $V \oplus \mathbb{R}\delta$ where we identify V with its dual and regard δ as the non-linear constant function $\delta : V \to \mathbb{R}$ with $\delta(v) = 1$ for all $v \in V$. For $\gamma \in V$, we write $\langle \gamma, \alpha + k\delta\rangle = \langle \gamma, \alpha\rangle + k$ (note that this is no longer bilinear). For $\alpha + k\delta \in \widetilde{\Phi}$ we call α the *linear root* associated to $\alpha + k\delta$.

For $\alpha \in \Phi^+$ and $k \in \mathbb{Z}$, we define the corresponding hyperplane to be

$$H_{\alpha,k} = \{x \in V \mid \langle x, \alpha - k\delta\rangle = 0\} = \{x \in V \mid \langle x, \alpha\rangle = k\}.$$

Denote the set of all hyperplanes as $\mathbb{H} = \{H_{\alpha,k} \mid \alpha \in \Phi^+, k \in \mathbb{Z}\}$. For $H_{\alpha,k} \in \mathbb{H}$ let $s_{\alpha,k}$ be the orthogonal reflection about $H_{\alpha,k}$, explicitly $s_{\alpha,k}(x) = x - (\langle x, \alpha\rangle - k)\alpha^\vee$ for $x \in V$. We then define the *affine Weyl group* as the group generated by $s_{\alpha,k}$, for all $\alpha \in \Phi$ and $k \in \mathbb{Z}$, and denote it by W. The group W is a subgroup of the group of affine transformations of V.

For $v \in V$, define the translation by v as $t_v(x) = x + v$ for $x \in V$. By definition, we have $t_{v_1}t_{v_2} = t_{v_1+v_2}$ and $wt_{v_1} = t_{wv_1}w$ for all $w \in W_0$ and $v_1, v_2 \in V$. It is clear that $s_{\alpha,k} = t_{k\alpha^\vee}s_\alpha$ and so for all $w \in W$ we can write $w = t_\gamma u$ for some $\gamma \in Q$ and $u \in W_0$. Thus, $W = Q \rtimes W_0$. The *extended affine Weyl group* corresponding to W is defined as $\widetilde{W} = P \rtimes W_0$ and as $Q \subseteq P$ we have $W \leq \widetilde{W}$. By definition every element of $\widetilde{W}$ can we written as a sequence of finite reflections and a translation by a weight in P. For $w \in \widetilde{W}$, we define the associated *translation coweight* $\mathrm{wt}(w)$ and *linear part* $\theta(w) \in W_0$ by

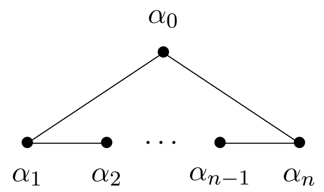
$$w = t_{\mathrm{wt}(w)}\theta(w). \tag{1.2.1}$$

The simple roots of the affine root system $\widetilde{\Phi}$ are $\alpha_0 = -\varphi + \delta$ and $\alpha_i + 0\delta$ with $i \in I$. Define $s_0 = s_{\varphi,1} = t_{\varphi^\vee}s_\varphi$ to be the reflection about the hyperplane associated to α_0. Then W is the Coxeter group generated by $S = \{s_i \mid i \in I\} \cup \{s_0\}$.

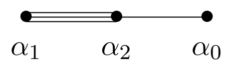
Define $\Sigma = P/Q$. Each $\sigma \in \Sigma$ induces a permutation of $I \cup \{0\}$ by $\sigma s_i \sigma^{-1} = s_{\sigma(i)}$. Hence, we can identify Σ with a subgroup of the group automorphisms of the affine Coxeter diagram associated with Φ. By definition $\widetilde{W} \cong W \rtimes \Sigma$. Note that, although W is a Coxeter group, in general, $\widetilde{W}$ is not.

Example 1.2.1. Consider the Coxeter diagram associated to the affine root system of A_n, (Figure 1.2). We have that $\Sigma = P/Q \cong \mathbb{Z}/(n+1)\mathbb{Z}$, generated by σ where $\sigma(i) = i + 1 \mod (n+1)$ for $i \in I \cup \{0\}$. This is the set of rotations of the cyclic diagram.

The Coxeter diagram of the affine root system of G_2 has no automorphisms. Therefore, $P = Q$ and Σ is trivial.



The affine Coxeter diagram of $\Phi = A_n$



The affine Coxeter diagram of $\Phi = G_2$

Figure 1.2

The set of positive affine roots is

$$\widetilde{\Phi}^+ = (\Phi^+ + \mathbb{Z}_{\geq 0}\delta) \cup (\Phi^- + \mathbb{Z}_{>0}\delta).$$

The action of $\widetilde{W}$ on the affine root system (given by the action on half spaces) is given by

$$w(\alpha + k\delta) = w\alpha + k\delta \text{ and } t_\gamma(\alpha + k\delta) = \alpha + (k - \langle \gamma, \alpha \rangle)\delta \text{ for } w \in W_0 \text{ and } \gamma \in P.$$

We record the following lemma connecting the roots and affine roots for future use.

Lemma 1.2.2. *For $\alpha \in \Phi$, the affine root $\alpha + k\delta$ is conjugate to α with respect to $\widetilde{W}$ for all $k \in \mathbb{Z}$.*

Proof. For $\alpha \in \Phi$ there exists some $i \in I$ and $u \in W_0$ such that $\alpha = u\alpha_i$. As $\langle \alpha_i, \omega_i \rangle = 1$ we have $\langle \alpha, u\omega_i \rangle = 1$. It follows that $\langle \alpha, ku\omega_i \rangle = k$ for all $k \in \mathbb{Z}$. Thus,

$$t_{ku\omega_i}(\alpha + k\delta) = \alpha + (k - \langle \alpha, ku\omega_i \rangle)\delta = \alpha + 0\delta$$

and as $t_{ku\omega_i} \in \widetilde{W}$ the result follows. □

For $w \in \widetilde{W}$ we define its inversion set as we did for W_0 in Section 1.1 as $\Phi(w) = \{\alpha \in \Phi^+ \mid w^{-1}\alpha \in \Phi^-\}$. We now define the *affine inversion set* of $w \in \widetilde{W}$ as

$$\widetilde{\Phi}(w) = \{\alpha + k\delta \in \widetilde{\Phi}^+ \mid w^{-1}(\alpha + k\delta) \in -\widetilde{\Phi}^+\}. \tag{1.2.2}$$

We also extend the definition of length from Section 1.1 to $\widetilde{W}$. Define the length function $\ell : W \to \mathbb{N}$ as the minimal number of generators required to write $w \in W$ as $w = s_{i_1}s_{i_2}\cdots s_{i_m}$ with $i_j \in I \cup \{0\}$. We extend this length function to $\widetilde{W}$ by setting $\ell(w\sigma) = \ell(w)$ for all $w \in W$ and $\sigma \in \Sigma$. Hence, $\Sigma = \{w \in \widetilde{W} \mid \ell(w) = 0\}$. A *reduced expression* of $w \in \widetilde{W}$ is a decomposition $w = s_{i_1}\cdots s_{i_{\ell(w)}}\sigma$ with $i_j \in I \cup \{0\}$ and $\sigma \in \Sigma$.

1.3 Alcoves

Let $H_{\alpha,k}$ be as in Section 1.2. The closure of the open connected components of $V \backslash (\bigcup_{\alpha,k} H_{\alpha,k})$ are called alcoves. Denote the set of alcoves by $\mathbb{A}$. In this section we will briefly describe alcoves and define an orientation on alcove walls. Alcoves are a helpful visual depiction of affine Weyl groups and will be further studied in Chapter 4.

The *fundamental alcove* is given by

$$A_0 = \{x \in V \mid 0 \leq \langle x, \alpha \rangle \leq 1 \text{ for all } \alpha \in \Phi^+\}$$

and the hyperplanes bounding A_0 are called the *walls* of A_0. Explicitly, the bounding hyperplanes are $H_{\alpha_i,0}$ with $i \in I$ and $H_{\varphi,1}$. The extreme points of the convex set A_0 are called the vertices of A_0, explicitly this is the set $\{x_i \mid i \in I \cup \{0\}\}$ with $x_0 = 0$ and $x_i = \frac{\omega_i}{m_i}$ (where m_i is as in Section 1.1). The action of $\sigma \in \Sigma$ on this set of vertices is given by $\sigma(x_i) = x_{\sigma(i)}$. Define a *panel* of A_0 to be a codimension-1 facet. A panel has type i if it is contained in $H_{\alpha_i,0}$ and type 0 if it is contained in $H_{\varphi,1}$.

The affine Weyl group W acts simply transitively on $\mathbb{A}$ with fundamental domain A_0. Using the action of W, we can transfer the notion of walls, panels, and panel types to all alcoves. Alcoves A and A' are called *i-adjacent* (written as $A \sim_i A'$) if $A \neq A'$ and A and A' share a panel of type i (with $i \in I \cup \{0\}$). For all $w \in W$ the alcoves wA_0 and ws_iA_0 are i-adjacent for all $i \in I \cup \{0\}$. Note that $\widetilde{W}$ also acts transitively on $\mathbb{A}$ but the action is no longer free.

Each hyperplane divides V into two half-spaces. For a hyperplane $H_{\alpha,k}$, we have a 'positive' half-space and a 'negative' half-space, determined by the inner product as follows:

$$H^+_{\alpha,k} = \{x \in V \mid \langle x, \alpha\rangle \geq k\} \quad \text{and} \quad H^-_{\alpha,k} = \{x \in V \mid \langle x, \alpha\rangle \leq k\}.$$

This notion of a positive side and a negative side is defined for all $H_{\alpha,k}$ with $\alpha \in \Phi^+$ and $k \in \mathbb{Z}$ and is called the *periodic orientation* of $\mathbb{H}$.

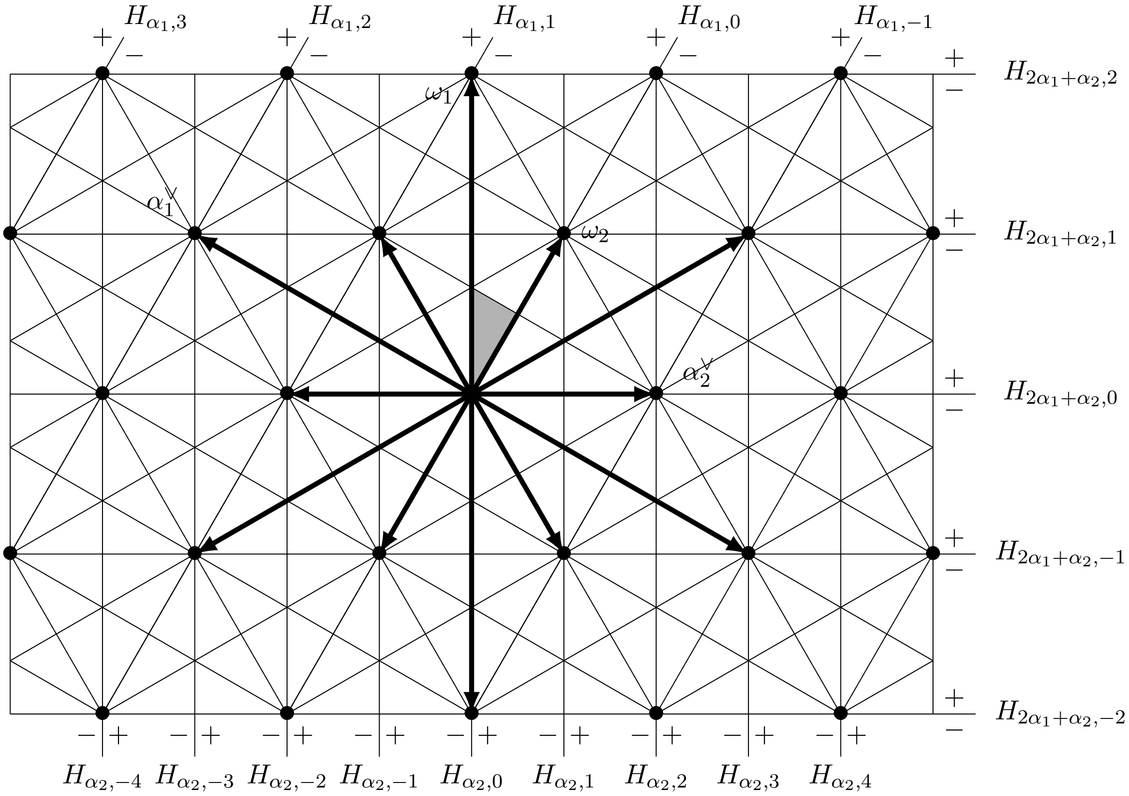


Figure 1.3: The hyperplanes and alcoves associated to $\Phi = G_2$

Example 1.3.1. The alcove set associated with the root system of type G_2 is illustrated in Figure 1.3. The points of $P = Q$ are drawn as bullets. In this diagram we can see how $\widetilde{W} = P \rtimes W_0$. To get to any alcove we reflect about the hyperplanes $H_{\alpha_i,0}$ for $i \in I$ and then translate by some $\gamma \in P$. In the figure, the periodic orientation is illustrated for the hyperplanes $H_{\alpha_1,k}$, $H_{\alpha_2,k}$ and $H_{2\alpha_1+\alpha_2,k}$ $(k \in \mathbb{Z})$.

For i-adjacent alcoves A and A', we notate the orientation information of their shared panel as

$$A^{-}|^{+}A'$$

if A is on the negative side and A' is of the positive side of the hyperplane containing their shared panel. Note that, for $w \in \widetilde{W}$, $\widetilde{\Phi}(w)$ is the set of affine roots such that their corresponding hyperplane separates A_0 and wA_0 (with $\widetilde{\Phi}(w)$ the affine inversion set of $w \in \widetilde{W}$). For $w \in \widetilde{W}$ and $i \in I \cup \{0\}$ we have that

$$wA_0{}^{-}|^{+}ws_iA_0 \text{ if and only if } w\alpha_i \in \Phi^{-} + \mathbb{Z}\delta. \tag{1.3.1}$$

1.4 Parabolic subgroups

Let $J \subseteq I$. The *J-parabolic subgroup* of W_0 is the subgroup generated by the set $\{s_j \mid j \in J\}$ and denoted W_J. The longest element of W_J is denoted by w_J and we have that $\ell(\mathsf{w}_J w) = \ell(w\mathsf{w}_J) = \ell(\mathsf{w}_J) - \ell(w)$ for all $w \in W_J$. Each coset $W_J w$ for $w \in W_0$ contains a unique representative of minimal length (this is well known, see for example [1, Proposition 2.20]). Let ${}^JW$ be the set of minimal length coset representatives for the coset $W_J \backslash W_0$. Then, for each $w \in W_0$ we have a unique decomposition

$$w = yu \text{ with } y \in W_J, u \in {}^JW.$$

In addition, for all $y \in W_J$ and $u \in {}^JW$, we have $\ell(yu) = \ell(y) + \ell(u)$.

The *support* of a root $\alpha = \sum_{i\in I} c_i\alpha_i \in \Phi$ is $\mathrm{supp}(\alpha) = \{i \in I \mid c_i \neq 0\}$. With this we define the *J-root system* $\Phi_J = \{\alpha \in \Phi \mid \mathrm{supp}(\alpha) \subseteq J\}$. In addition, we have a J-analogue to the inversion set, $\Phi_J(w) = \Phi(w) \cap \Phi_J$ for $w \in W_0$. The following lemma about the J-decomposition of $w \in W_0$ is well known (see, for example [24, Corollary 2.13]).

Lemma 1.4.1. *Let $J \subseteq I$. If $w = yu$ with $y \in W_J$ and $u \in {}^JW$ then $\Phi(y) = \Phi(w) \cap \Phi_J = \Phi_J(w)$. In particular, we have ${}^JW = \{u \in W \mid \Phi_J(u) = \emptyset\}$.*

Proof. Suppose there exists $\beta \in \Phi_J(w) \backslash \Phi(y)$. Since $\beta \notin \Phi(y)$ we have $y^{-1}\beta > 0$ and, hence, $\ell(s_\beta y) > \ell(y)$. Then, since $\beta \in \Phi_J(w)$ we have $\ell(s_\beta w) < \ell(w)$. As $\beta \in \Phi_J$ we have $s_\beta \in W_J$, but then the element $y' = s_\beta y \in W_J$ satisfies $\ell(y'u) = \ell(s_\beta w) < \ell(w) = \ell(y) + \ell(u) < \ell(s_\beta y) + \ell(u) = \ell(y') + \ell(u)$, contradicting the fact that $u \in {}^JW$. □

Recall the definition of $\theta(w)$ for $w \in \widetilde{W}$ from Section 1.2.

Definition 1.4.2. Let $w \in \widetilde{W}$. Define $\theta_J(w) \in W_J$ and $\theta^J(w) \in {}^JW$ by the equation $\theta(w) = \theta_J(w)\theta^J(w)$.

We record some basic facts of θ^J, θ_J and wt for future use.

Lemma 1.4.3. *Let $w, v \in \widetilde{W}$. Then*

(1) $\theta^J(wv) = \theta^J(\theta^J(w)v)$,

(2) $\theta_J(wv) = \theta_J(w)\theta_J(\theta^J(w)v)$,

(3) $\mathrm{wt}(wv) = \mathrm{wt}(w) + \theta_J(w)\mathrm{wt}(\theta^J(w)v)$.

Proof. As $w = t_{\mathrm{wt}(w)}\theta_J(w)\theta^J(w)$ and $v = t_{\mathrm{wt}(v)}\theta(v)$ we have that

$$\begin{aligned} wv &= t_{\mathrm{wt}(w)+\theta_J(w)\theta^J(w)\mathrm{wt}(v)}\theta_J(w)\theta^J(w)\theta(v) \\ &= t_{\mathrm{wt}(w)+\theta_J(w)\theta^J(w)\mathrm{wt}(v)}\theta_J(w)\theta_J(\theta^J(w)\theta(v))\theta^J(\theta^J(w)\theta(v)). \end{aligned}$$

As $\theta^J(w)v = \theta^J(w)t_{\mathrm{wt}(v)}\theta(v) = t_{\theta^J(w)\mathrm{wt}(v)}\theta^J(w)\theta(v)$ we have $\mathrm{wt}(\theta^J(w)v) = \theta^J(w)\mathrm{wt}(v)$ and the result follows. □

For $J \subseteq I$, we have the decomposition $V = V_J \oplus V^J$ with

$$V_J = \sum_{j\in J} \mathbb{R}\alpha_j^\vee \quad \text{and} \quad V^J = \sum_{i\in I\setminus J} \mathbb{R}\omega_i.$$

For $\gamma \in V$, denote γ^J to be the orthogonal projection of γ onto V^J. Let $P^J = \{\gamma^J \mid \gamma \in P\} \subseteq V^J$. Note that P^J is not in general a subset of P, see Example 1.4.4 below. Let $\{\overline{\omega_i} \mid i \in I \setminus J\}$ be an arbitrary choice of $\mathbb{Z}$-basis for P^J.

Example 1.4.4. Consider $\Phi = A_2$ with $J = \{1\}$, displayed in Figure 1.4. The elements of P^J are illustrated as blue bullets on $V^J = H_{\alpha_1,0}$ (the thicker hyperplane). In the figure we can see that $\omega_1^J = \frac{1}{2}\omega_2 \in P^J$ and that $\frac{1}{2}\omega_2 \notin P$. Choosing $\overline{\omega_2} = \frac{1}{2}\omega_2$, we have that $P^J = \mathbb{Z}\overline{\omega_2}$.

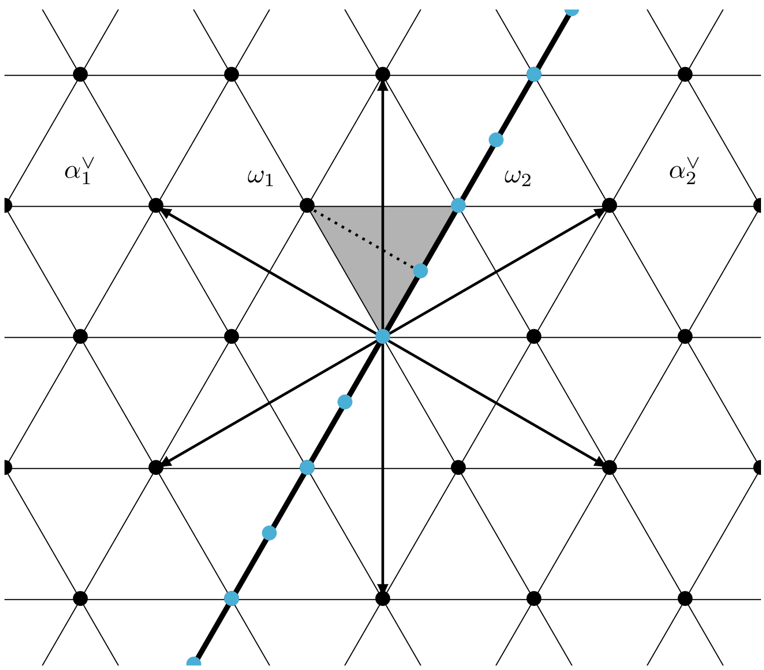


Figure 1.4: The set P^J for type $\Phi = A_2$ with $J = \{1\}$

For $J \subseteq I$, let $\Phi_{J,l}$ and $\Phi_{J,s}$ be the long and short roots of Φ_J, respectively. Note that in the simply-laced case, as all roots are long, we have $\Phi_{J,s} = \emptyset$. Define

$$\rho_J = \frac{1}{2}\sum_{\alpha\in\Phi_{J,l}^+} \alpha \quad \text{and} \quad \rho_J' = \frac{1}{2}\sum_{\alpha\in\Phi_{J,s}^+} \alpha.$$

For every $i \in J$ define $\tilde{\omega}_i$ such that $\langle \alpha_j^\vee, \tilde{\omega}_i\rangle = \delta_{i,j}$ for all $j \in J$. Then $\{\tilde{\omega}_j \mid j \in J\}$ is the basis of V_J dual to the basis $\{\alpha_j^\vee \mid j \in J\}$.

Lemma 1.4.5. *We have*

$$\rho_J = \sum_{\{j \in J \,|\, \alpha_j \in \Phi_{J,l}\}} \tilde{\omega}_j \quad \text{and} \quad \rho'_J = \sum_{\{j \in J \,|\, \alpha_j \in \Phi_{J,s}\}} \tilde{\omega}_j.$$

In particular ρ_J (respectively ρ'_J) is orthogonal to all short (respectively long) simple roots of Φ_J.

Proof. We follow the proof given in [7, VI, §1, Proposition 29] now extending to include the non-simply laced case. We can assume that J is one connected component (otherwise we consider the connected components separately). Consider $\alpha_j \in \Phi_{J,l}$. As s_j permutes the set of all positive long roots except α_j, we have

$$\begin{aligned} s_j\left(\frac{1}{2}\sum_{\alpha \in \Phi^+_{J,l}} \alpha\right) &= \frac{1}{2}\sum_{\alpha \in \Phi^+_{J,l}} s_j(\alpha) \\ &= \frac{1}{2}\sum_{\alpha \in \Phi^+_{J,l}\setminus\{\alpha_j\}} \alpha + \frac{1}{2} s_j(\alpha_j) \\ &= \rho_J - \alpha_j. \end{aligned}$$

By definition, we also have $s_j(\rho_J) = \rho_J - \langle \rho_J, \alpha_j^\vee \rangle \alpha_j$. Therefore, $\langle \rho_J, \alpha_j^\vee \rangle = 1$. As $\langle \tilde{\omega}_i, \alpha_j^\vee \rangle = \delta_{i,j}$, we have $\langle \sum_{\{i \in J | \alpha_i \in \Phi_{J,l}\}} \tilde{\omega}_i, \alpha_j^\vee \rangle = 1$ and so $\langle \rho_J - \sum_{\{i \in J | \alpha_i \in \Phi_{J,l}\}} \tilde{\omega}_j, \alpha_j^\vee \rangle = 0$ for all α_j long.

For $\alpha_{j'} \in \Phi_{J,s}$, as $s_{j'}$ permutes the set of all long roots, we have $s_{j'}(\rho_J) = \rho_J$. As $\langle \sum_{\{i \in J | \alpha_i \in \Phi_{J,l}\}} \tilde{\omega}_i, \alpha_{j'}^\vee \rangle = 0$, we have that $\langle \rho_J - \sum_{i \in J | \alpha_i \in \Phi_{J,l}\}} \tilde{\omega}_i, \alpha_{j'}^\vee \rangle = 0$ for all $\alpha_{j'}$ short.

Finally, $\langle \rho_J, \alpha_i^\vee \rangle = \langle \sum_{\{i \in J | \alpha_i \in \Phi_{J,l}\}} \tilde{\omega}_i, \alpha_k \rangle = 0$ for all $k \in I \setminus J$. Hence, $\rho_J = \sum_{\{i \in J | \alpha_i \in \Phi_{J,l}\}} \tilde{\omega}_i$. A similar argument gives the expression for ρ'_J. ☐

Chapter 2

The fundamental J-alcove

In Chapter 1 we defined a set of hyperplanes $\mathbb{H}$ and a corresponding set of alcoves $\mathbb{A}$. In this chapter we define a new set of 'alcoves' formed from a subset of $\mathbb{H}$ and then focus our study on the fundamental 'alcove' of the set. The new set is a partition of $\mathbb{A}$, the elements of which are called J-alcoves. In Section 2.1 we take the set of hyperplanes associated to roots of Φ_J^+ and, as was done in the definition of alcoves, set the J-alcoves to be the closure of the open connected components of V after removing this set of hyperplanes. We define the fundamental J-alcove to be the intersection of the points between $H_{\alpha,0}$ and $H_{\alpha,1}$ for all $\alpha \in \Phi_J^+$. This J-alcove is a fundamental domain for the action of the group generated by reflection about $H_{\alpha,k}$ for all $\alpha \in \Phi_J^+$ and $k \in \mathbb{Z}$. The fundamental J-alcove, denoted as $\mathcal{A}_J$, is the focus for the remainder of the chapter.

In Section 2.2 we explicitly describe the set of weights within $\mathcal{A}_J$. To each weight γ within this set we define an element τ_γ called a pseudo-translation by γ. These pseudo-translations are the subject of Section 2.3. The pseudo-translations can be thought of as translations within the fundamental J-alcove, in the sense that when $J = I$ all pseudo-translations are true translations. It is shown that the set of all pseudo-translations, denoted $\mathbb{T}_J$, is in bijection with the quotient group P/Q_J where Q_J is the $\mathbb{Z}$-span of the coroots of Φ_J. Furthermore, we show that $\mathbb{T}_J$ acts freely on the set of elements w of $\widetilde{W}$ such that $wA_0 \subseteq \mathcal{A}_J$ (denoted $\mathbb{W}^J$) with a fundamental domain of ${}^JW$. Therefore, for all $w \in \mathbb{W}^J$ we have a decomposition $w = \tau_\gamma u$ where γ is a weight within $\mathcal{A}_J$ and $u \in {}^JW$, giving a J-analogue to the weight and finite component decomposition from Section 1.2. Finally, in Section 2.4 we explore the elements of W_0 that stabilise the fundamental J-alcove and show that the semi-direct product of $\mathbb{T}_J$ with the set of these stabilising elements is the subgroup of $\widetilde{W}$ that stabilises $\mathcal{A}_J$.

2.1 J-alcoves and J-affine Weyl groups

In this section we define J-alcoves. They are defined similarly to how alcoves are defined in Section 1.3, however we now limit to considering only the hyperplanes associated to roots in the J-root system. Restricting to this hyperplane set we define a Weyl group, 'alcove' set and fundamental 'alcove' as we did for the complete hyperplane set in Chapter 1.

For $J \subseteq I$, let the set of hyperplanes associated to roots in the J-root system be

$$\mathbb{H}_J = \{H_{\alpha,k} \mid \alpha \in \Phi_J^+, k \in \mathbb{Z}\}.$$

If $J = I$ then $\mathbb{H}_J = \mathbb{H}$. Let the set of 'alcoves' associated to $\mathbb{H}_J$ be the closures of the open connected components of $V \setminus (\cup_{H \in \mathbb{H}_J} H)$. We denote this set $\mathbb{A}_J$, and call its elements *J-alcoves*.

Let

$$\mathcal{A}_J = \{x \in V \mid 0 \leq \langle x, \alpha \rangle \leq 1 \text{ for all } \alpha \in \Phi_J^+\}.$$

We call this set the *fundamental J-alcove*, it is clear from definition that $\mathcal{A}_J$ is a J-alcove.

The *J-affine Weyl group* is

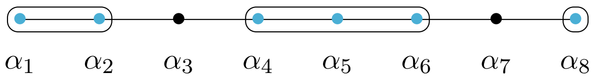
$$W_J^{\text{aff}} = \langle s_{\alpha,k} \mid \alpha \in \Phi_J^+, k \in \mathbb{Z} \rangle.$$

The generalisation of the results of Section 1.3 (see [7, V, §3]) gives that W_J^{aff} acts simply transitively on the set $\mathbb{A}_J$. In addition, $\mathcal{A}_J$ is a fundamental domain for the action of W_J^{aff} on V (see [7, V §3.3] for further information on fundamental domains).

Let $\mathcal{K}(J)$ be the set of connected components of J (connected in the sense of the Coxeter diagram associated with Φ^+, labelled using Bourbaki's conventions [7]). For a connected subset $K \subseteq J$, let φ_K be the highest root in Φ_K. We have that

$$\Phi_J = \bigsqcup_{K \in \mathcal{K}(J)} \Phi_K. \tag{2.1.1}$$

Example 2.1.1. Consider type A_8 with $J = \{1, 3, 4, 6\}$. The Coxeter diagram of A_8 is displayed below, colouring the vertices associated with α_j for $j \in J$ in blue:



We have that $\mathcal{K}(J) = \{\{1, 2\}, \{4, 5, 6\}, \{8\}\}$ and

$$\Phi_J := \pm\{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\} \cup \pm\{\alpha_4, \alpha_5, \alpha_6, \alpha_4 + \alpha_5, \alpha_5 + \alpha_6, \alpha_4 + \alpha_5 + \alpha_6\} \cup \pm\{\alpha_8\}.$$

Lemma 2.1.2. *Let $J \subseteq I$. The walls of the fundamental J-alcove $\mathcal{A}_J$ are $H_{\alpha_j,0}$ and $H_{\varphi_K,1}$ with $j \in J$ and $K \in \mathcal{K}(J)$. That is,*

$$\mathcal{A}_J = \{x \in V \mid \langle x, \alpha_j \rangle \geq 0 \textit{ for all } j \in J \textit{ and } \langle x, \varphi_K \rangle \leq 1 \textit{ for all } K \in \mathcal{K}(J)\}.$$

Proof. Let

$$\mathcal{A}'_J = \{x \in V \mid \langle x, \alpha_j \rangle \geq 0 \text{ for all } j \in J \text{ and } \langle x, \varphi_K \rangle \leq 1 \text{ for all } K \in \mathcal{K}(J)\}.$$

If $x \in \mathcal{A}_J$ then $0 \leq \langle x, \alpha \rangle \leq 1$ for all $\alpha \in \Phi_J^+$. In particular, this is true for $\alpha = \alpha_i$ for all $i \in I$ and $\alpha = \varphi_K$ for all $K \in \mathcal{K}(J)$. Therefore, $\mathcal{A}_J \subseteq \mathcal{A}'_J$.

Now suppose that $x \in \mathcal{A}'_J$. Let $\alpha \in \Phi_J$. Writing $\alpha = \sum_{j \in J} a_j \alpha_j$ we have $\langle x, \alpha \rangle = \sum_{j \in J} a_j \langle x, \alpha_j \rangle \geq 0$. Since $\alpha \in \Phi_J$ we have $\alpha \in \Phi_K$ for some $K \in \mathcal{K}(J)$, by (2.1.1). Since $\langle x, \alpha_j \rangle \geq 0$ for all $j \in J$ and $\varphi_K - \alpha$ is a nonnegative linear combination of roots α_k with $k \in K \subseteq J$ we have $\langle x, \varphi_K - \alpha \rangle \geq 0$, and so $\langle x, \alpha \rangle \leq \langle x, \varphi_K \rangle \leq 1$, hence $x \in \mathcal{A}_J$. $\square$

By Lemma 2.1.2, the set of reflections about the walls of $\mathcal{A}_J$ is composed of s_j for $j \in J$ and $s_{\varphi_K,1}$ for $K \in \mathcal{K}(J)$. Denote $s'_j = s_j$ for $j \in J$ and $s'_{0_K} = s_{\varphi_K,1}$ for $K \in \mathcal{K}(J)$ (where 0_K is a symbol). Let

$$J^{\text{aff}} = J \cup \{0_K \mid K \in \mathcal{K}(J)\}. \tag{2.1.2}$$

We have that $\{s'_j \mid j \in J^{\text{aff}}\}$ generates W_J^{aff},

$$W_J^{\text{aff}} = \prod_{K \in \mathcal{K}(J)} W_K^{\text{aff}}, \tag{2.1.3}$$

where W_K^{aff} is generated by $\{s'_k \mid k \in \{0_K\} \cup K\}$.

Example 2.1.3. Figure 2.1 displays the J-alcoves for W corresponding to the root system of type G_2.

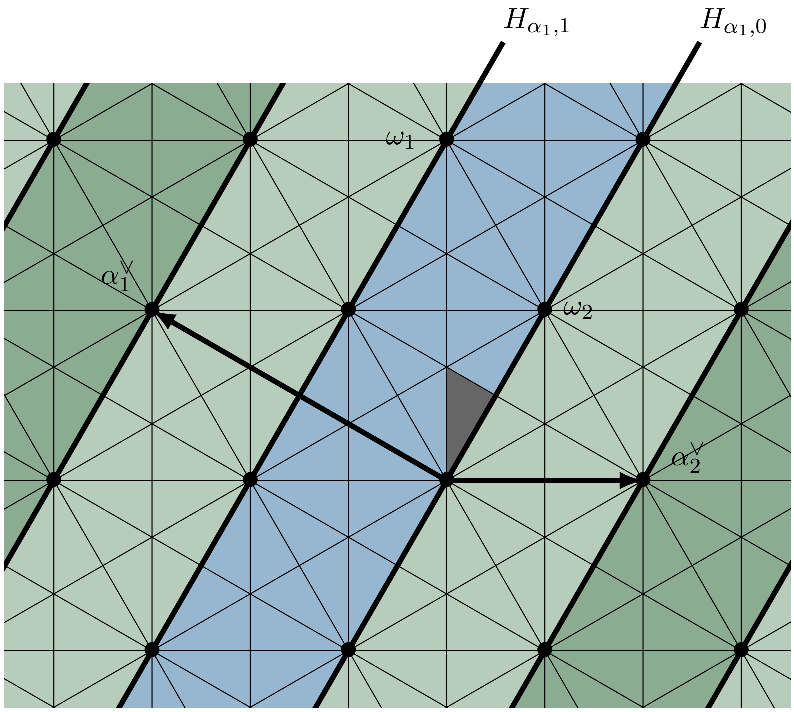


Figure 2.1: The J-alcoves for $\Phi = G_2$ and $J = \{1\}$

The set of hyperplanes associated to J is $\mathbb{H}_J = \{H_{\alpha_1,k} \mid k \in \mathbb{Z}\}$. In the figure these hyperplanes (all hyperplanes parallel to $H_{\alpha_1,0}$) are illustrated with a thicker line. The J-alcoves are the coloured sections, separated by the hyperplanes in $\mathbb{H}_J$. The fundamental J-alcove, $\mathcal{A}_J$, is the blue region, bounded by $H_{\alpha_1,0}$ and $H_{\alpha_1,1}$.

In this example there is only one connected component of J, given by $K = \{1\}$ with $\varphi_K = \alpha_1$. Hence, $W_J^{\text{aff}} = \langle s'_1, s'_{0_K} \rangle$ with $s'_1 = s_1$ and $s'_{0_K} = s_{\alpha_1,1}$.

2.2 The set $P^{(J)}$

This section describes the weights that lie within the fundamental J-alcove. Define

$$Q_J = \sum_{\alpha \in \Phi_J} \mathbb{Z}\alpha^\vee.$$

Lemma 2.2.1. *Let $J \subseteq I$ and let $\gamma \in P$. There exists a unique $\gamma^* \in (\gamma + Q_J) \cap \mathcal{A}_J$.*

Proof. Let $\gamma \in P$. Then $\gamma \in A$ for some J-alcove $A \in \mathbb{A}_J$. Since W_J^{aff} acts transitively on $\mathbb{A}_J$ there is some $w \in W_J^{\mathrm{aff}}$ such that $wA = \mathcal{A}_J$ (and in particular, $w\gamma \in \mathcal{A}_J$). Since w is a product of reflections in hyperplanes $H_{\alpha,k}$ with $\alpha \in \Phi_J^+$ and $k \in \mathbb{Z}$, and since $s_{\alpha,k}(\gamma) = \gamma - (\langle \gamma, \alpha \rangle - k)\alpha^\vee$ it follows that $w\gamma \in \gamma + Q_J$, proving the existence of γ^*. Uniqueness follows from the fact that $\mathcal{A}_J$ is a fundamental domain for the action of W_J^{aff} on V. □

Definition 2.2.2. Let $\gamma^{(J)}$ be the unique element $\gamma^* \in (\gamma + Q_J) \cap \mathcal{A}_J$.

Note that, in general, the orthogonal projection γ^J and $\gamma^{(J)}$ are distinct where γ^J is as in Section 1.4 (see Example 2.2.3). Denote

$$P^{(J)} = \mathcal{A}_J \cap P$$

to be the set of coweights in $\mathcal{A}_J$. In particular $\gamma^{(J)} \in P^{(J)}$ for all $\gamma \in P$. Note that, Lemma 2.2.1 gives that $P^{(J)}$ is in bijection with P/Q_J, by $\gamma \mapsto \gamma + Q_J$ for all $\gamma \in P^{(J)}$.

Example 2.2.3. Figure 2.2 shows the equivalence classes of P/Q_J for $\Phi = A_2$ with $J = \{1\}$. The fundamental alcove $\mathcal{A}_J$ is shaded blue. As $\Phi_J = \{\alpha_1, -\alpha_1\}$ we have $Q_J = \mathbb{Z}\alpha_1^\vee$. The equivalence classes of P/Q_J are displayed by dotted green lines, with an equivalence class consisting of all weights lying on one dotted line. The elements of $P^{(J)}$ are the weights on $H_{\alpha_1,0}$ and $H_{\alpha_1,1}$.

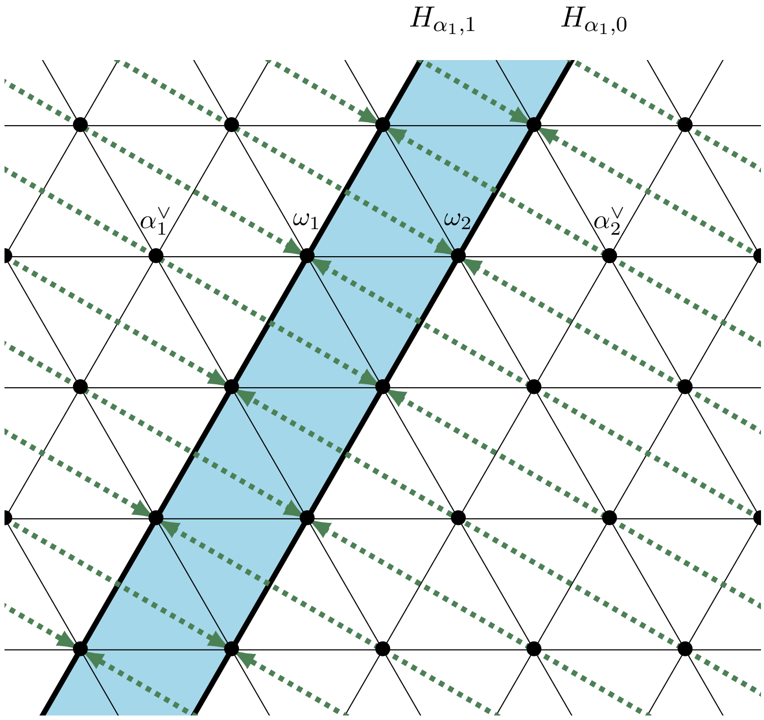


Figure 2.2: The quotient map $P \to P/Q_J$, $\gamma \mapsto \gamma + Q_J$ for $\Phi = A_2$ and $J = \{1\}$

The map $P \to P^{(J)}$ is displayed on the diagram by the dotted arrows. All weights on a dotted line go to the unique weight in $\mathcal{A}_J$ on that dotted line. For example, if $\gamma = 3\omega_1 - \omega_2$ then $\gamma - \alpha_1^\vee = \omega_1 \in P^{(J)}$. Therefore, $\omega_1 \in (\gamma + Q_J) \cap A_J$, and so $\gamma^{(J)} = \omega_1$. Comparing to Figure 1.4, we can see that P^J and $P^{(J)}$ are distinct. For example, $\omega_1 \notin P^J$.

The following lemma gives a useful characterisation of $P^{(J)}$. For $\gamma \in P$, let

$$J(\gamma) = \{j \in J \mid \langle \gamma, \alpha_j \rangle \neq 0\}. \tag{2.2.1}$$

Lemma 2.2.4. *Let $J \subseteq I$ and $\gamma \in P$. Then $\gamma \in \mathcal{A}_J$ if and only if the set $J(\gamma)$ has the following properties:*

(1) *if $j \in J(\gamma)$ then $\langle \gamma, \alpha_j \rangle = 1$,*
(2) *for each $K \in \mathcal{K}(J)$ we have $|J(\gamma) \cap K| \leq 1$,*
(3) *if $j \in J(\gamma)$ then $\langle \omega_j, \varphi_K \rangle = 1$ where $K \in \mathcal{K}(J)$ is the connected component of J containing j.*

Proof. Write $\gamma = \sum_{i \in I} c_i \omega_i$. If $\gamma \in P \cap \mathcal{A}_J$ then $0 \leq \langle \gamma, \alpha_j \rangle \leq 1$ for all $j \in J$, so $c_j \in \{0, 1\}$. For each $K \in \mathcal{K}(J)$ there is at most one $j \in K$ with $c_j = 1$, otherwise $\langle \gamma, \varphi_K \rangle \geq 2$. If $j \in K$ with $c_j = 1$ then for $\alpha \in \Phi_K^+$ we have $\langle \gamma, \alpha \rangle = \langle \omega_j, \alpha \rangle$ and $0 \leq \langle \omega_j, \alpha \rangle \leq 1$. In particular, taking $\alpha = \varphi_K \in \Phi_J$ we have $\langle \omega_j, \varphi_K \rangle = 1$. The converse is clear using Lemma 2.1.2. □

Lemma 2.2.4 categorises the elements of $P^{(J)}$ as precisely the elements

$$\gamma = \sum_{i \in I \setminus J} a_i \omega_j + \sum_{j \in J'} \omega_j \tag{2.2.2}$$

with $a_i \in \mathbb{Z}$ and where $J' \subseteq J$ is a set with $|J' \cap K| \leq 1$ for all $K \in \mathcal{K}(J)$ and if $j \in J' \cap K$ then the coefficient of α_j in φ_K is 1. These conditions imply that $J' = J(\gamma)$.

Example 2.2.5. Let $\Phi = G_2$ and $J = \{1\}$. Then all $\gamma \in P^{(J)}$ are of the form $\gamma = a\omega_2$ or $\gamma = a\omega_2 + \omega_1$ with $a \in \mathbb{Z}$. The former are on $H_{\alpha_1,0}$ and the latter are on $H_{\alpha_1,1}$ (see Figure 2.1).

2.3 The J-translation group

In this section we construct a bijection between $P^{(J)}$ (defined in Section 2.2) to a subset of $\widetilde{W}$. These elements, called pseudo-translations, act like translations of A_0 within the fundamental J-alcove. In particular, every $w \in \widetilde{W}$ such that $wA_0 \subseteq \mathcal{A}_J$ can be written in the form of a pseudo-translation and then an element of ${}^J W$. By doing this we gain a J-analogue to defining the coweight translation and linear part of an element of $\widetilde{W}$, now restricting to $\mathcal{A}_J$.

Definition 2.3.1. For $\gamma \in P^{(J)}$ let

$$\mathsf{y}_\gamma = \mathsf{w}_{J \setminus J(\gamma)} \mathsf{w}_J \quad \text{and} \quad \tau_\gamma = t_\gamma \mathsf{y}_\gamma,$$

where $J(\gamma)$ is as in (2.2.1). We call τ_γ a *pseudo-translation* by $\gamma \in P^{(J)}$.

Note that both y_γ and τ_γ depend on the subset J, however this dependence is suppressed in the notation for simplicity.

Lemma 2.3.2. *For $\gamma \in P^{(J)}$ we have $\Phi(\mathsf{y}_\gamma) = \Phi_J^+ \setminus \Phi_{J \setminus J(\gamma)}^+ = \{\alpha \in \Phi_J^+ \mid \langle \gamma, \alpha \rangle = 1\}$.*

Proof. It is clear from the definition of y_γ that $\Phi(\mathsf{y}_\gamma) = \Phi_J^+ \setminus \Phi_{J \setminus J(\gamma)}^+$. A root $\alpha \in \Phi_J^+$ does not lie in $\Phi_{J \setminus J(\gamma)}^+$ if and only if there exists $j \in J(\gamma)$ such that the coefficient of α_j in α is strictly positive. By Lemma 2.2.4 this occurs if and only if the coefficient of α_j in α is 1, and hence the result. □

Recall the classical formula $\ell(t_\gamma) = \sum_{\alpha \in \Phi^+} |\langle \gamma, \alpha \rangle|$ (obtained by counting hyperplanes crossed between A_0 and $t_\gamma A_0$). The following proposition gives an analogous formula for the pseudo-translations.

Proposition 2.3.3. *For $\gamma \in P^{(J)}$ we have*

$$\ell(\tau_\gamma) = \sum_{\alpha \in \Phi^+ \backslash \Phi_J} |\langle \gamma, \alpha \rangle|.$$

Proof. By [37, (2.4.1)] we have

$$\ell(t_\gamma w) = \sum_{\alpha \in \Phi^+} |\langle \gamma, \alpha \rangle - \chi^-(w^{-1}\alpha)|,$$

for $\gamma \in P$ and $w \in W_0$ where $\chi^-(\cdot)$ is the characteristic function of Φ^- (that is, $\chi^-(\alpha') = 1$ when $\alpha' \in \Phi^-$ and 0 otherwise). Let $\gamma \in P^{(J)}$. Consider the contribution of $|\langle \gamma, \alpha \rangle - \chi^-(\mathsf{y}_\lambda^{-1}\alpha)|$ to the sum from $\alpha \in \Phi^+$. If $\alpha \in \Phi_J^+$ then either $\langle \gamma, \alpha \rangle = 0$ in which case $\alpha \notin \Phi(\mathsf{y}_\gamma)$ and so $\chi^-(\mathsf{y}_\gamma^{-1}\alpha) = 0$, or $\langle \gamma, \alpha \rangle = 1$ in which case $\alpha \in \Phi(\mathsf{y}_\gamma)$ and so $\chi^-(\mathsf{y}_\gamma^{-1}\alpha) = 1$ (in both cases using Lemma 2.3.2). Thus if $\alpha \in \Phi_J^+$ then the contribution to the sum is 0. If $\alpha \in \Phi^+ \backslash \Phi_J$ then $\chi^-(\mathsf{y}_\gamma^{-1}\alpha) = 0$, hence the result. □

Lemma 2.3.4. *Let $\gamma, \mu \in P^{(J)}$. Then*
(1) $(\gamma + \mu)^{(J)} = \gamma + \mathsf{y}_\gamma \mu$ *and* $(-\gamma)^{(J)} = -\mathsf{y}_\gamma^{-1}\gamma$;
(2) $\mathsf{y}_\gamma \mathsf{y}_\mu = \mathsf{y}_{(\gamma+\mu)^{(J)}} = \mathsf{y}_\mu \mathsf{y}_\gamma$ *and* $\mathsf{y}_\gamma^{-1} = \mathsf{y}_{(-\gamma)^{(J)}}$;
(3) $\tau_\gamma \tau_\mu = \tau_{(\gamma+\mu)^{(J)}} = \tau_\mu \tau_\gamma$ *and* $\tau_\gamma^{-1} = \tau_{(-\gamma)^{(J)}}$.

Proof. (1) Since $\mathsf{y}_\gamma \in W_J$ we have $\gamma + \mathsf{y}_\gamma \mu \in \gamma + \mu + Q_J$, and so to prove that $(\gamma+\mu)^{(J)} = \gamma + \mathsf{y}_\gamma \mu$ it suffices, by the uniqueness in Lemma 2.2.1, to show that $\gamma + \mathsf{y}_\gamma \mu \in P^{(J)}$. To do this, let $\alpha \in \Phi_J^+$, and write $\beta = \mathsf{y}_\gamma^{-1}\alpha$. Then $\langle \gamma + \mathsf{y}_\gamma \mu, \alpha \rangle = \langle \gamma, \alpha \rangle + \langle \mu, \beta \rangle$. Since $\gamma \in P^{(J)}$ we have $\langle \gamma, \alpha \rangle \in \{0, 1\}$. If $\langle \gamma, \alpha \rangle = 0$ then $\alpha \notin \Phi(\mathsf{y}_\gamma)$ (by Lemma 2.3.2), so $\beta = \mathsf{y}_\gamma^{-1}\alpha \in \Phi_J^+$, and so $\langle \gamma + \mathsf{y}_\gamma \mu, \alpha \rangle = \langle \mu, \beta \rangle$, giving $0 \le \langle \gamma + \mathsf{y}_\gamma \mu, \alpha \rangle \le 1$ (as $\beta \in \Phi_J^+$ and $\mu \in \mathcal{A}_J$). If $\langle \gamma, \alpha \rangle = 1$ then $\alpha \in \Phi(\mathsf{y}_\gamma)$ and so $\beta \in -\Phi_J^+$. Thus $\langle \gamma + \mathsf{y}_\gamma \mu, \alpha \rangle = 1 + \langle \mu, \beta \rangle$ and so $0 \le \langle \gamma + \mathsf{y}_\gamma \mu, \alpha \rangle \le 1$ (as $\beta \in -\Phi_J^+$ and $\mu \in \mathcal{A}_J$ gives $-1 \le \langle \mu, \beta \rangle \le 0$). Hence $\gamma + \mathsf{y}_\gamma \mu \in P^{(J)}$.

Similarly, to show that $(-\gamma)^{(J)} = -\mathsf{y}_\gamma^{-1}\gamma$ we need that $-\mathsf{y}_\gamma^{-1}\gamma \in P^{(J)}$. Again letting $\alpha \in \Phi_J^+$, we have $\langle -\mathsf{y}_\gamma^{-1}\gamma, \alpha \rangle = -\langle \gamma, \mathsf{y}_\gamma \alpha \rangle$. If $\mathsf{y}_\gamma \alpha \in \Phi_J^+$ then $\mathsf{y}_\gamma \alpha \notin \Phi(\mathsf{y}_\gamma)$ (as $\alpha \in \Phi^+$) and as $\gamma \in P^{(J)}$ we have $\langle \gamma, \mathsf{y}_\gamma \alpha \rangle \in \{0, 1\}$. Therefore $-\langle \gamma, \mathsf{y}_\gamma \alpha \rangle = 0$. If $-\mathsf{y}_\gamma \alpha \in \Phi_J^+$ then, as α is positive, we have $-\mathsf{y}_\gamma \alpha \in \Phi(\mathsf{y}_\gamma)$. Thus $-\langle \gamma, \mathsf{y}_\gamma \alpha \rangle = \langle \gamma, -\mathsf{y}_\gamma \alpha \rangle = 1$ and so $-\mathsf{y}_\gamma^{-1}\gamma \in P^{(J)}$.

(2) By (1) it suffices to show that $\mathsf{y}_\gamma \mathsf{y}_\mu = \mathsf{y}_{\gamma + \mathsf{y}_\gamma \mu}$ and $\mathsf{y}_\gamma^{-1} = \mathsf{y}_{-\mathsf{y}_\gamma^{-1}\gamma}$. To prove the first statement we shall show that $\Phi(\mathsf{y}_{\gamma+\mathsf{y}_\gamma\mu}) = \Phi(\mathsf{y}_\gamma \mathsf{y}_\mu)$. It follows from Lemma 2.3.2 that

$$\Phi(\mathsf{y}_{\gamma+\mathsf{y}_\gamma\mu}) = \{\alpha \in \Phi_J^+ \mid \langle \gamma, \alpha \rangle = 0 \text{ and } \langle \mu, \mathsf{y}_\gamma^{-1}\alpha \rangle = 1, \text{ or } \langle \gamma, \alpha \rangle = 1 \text{ and } \langle \mu, \mathsf{y}_\gamma^{-1}\alpha \rangle = 0\}.$$

Suppose that $\alpha \in \Phi(\mathsf{y}_{\gamma+\mathsf{y}_\gamma\mu})$. If $\langle \gamma, \alpha \rangle = 1$ and $\langle \mu, \mathsf{y}_\gamma^{-1}\alpha \rangle = 0$ then $\alpha \in \Phi(\mathsf{y}_\gamma)$ and $-\mathsf{y}_\gamma^{-1}\alpha \notin \Phi(\mathsf{y}_\mu)$, and so $\alpha \in \Phi(\mathsf{y}_\gamma \mathsf{y}_\mu)$. If $\langle \gamma, \alpha \rangle = 0$ and $\langle \mu, \mathsf{y}_\gamma^{-1}\alpha \rangle = 1$ then $\alpha \in \Phi_J^+ \backslash \Phi(\mathsf{y}_\gamma)$ and $\mathsf{y}_\gamma^{-1}\alpha \in \Phi(\mathsf{y}_\mu)$, giving $\alpha \in \Phi(\mathsf{y}_\gamma \mathsf{y}_\mu)$. Conversely, suppose that $\alpha \in \Phi(\mathsf{y}_\gamma \mathsf{y}_\mu)$. Then $\alpha \in \Phi_J^+$ with $\mathsf{y}_\mu^{-1}\mathsf{y}_\gamma^{-1}\alpha < 0$, and there are two cases. If $\mathsf{y}_\gamma^{-1}\alpha > 0$ and $\mathsf{y}_\mu^{-1}(\mathsf{y}_\gamma^{-1}\alpha) < 0$ then $\langle \gamma, \alpha \rangle = 0$ (as $\alpha \notin \Phi(\mathsf{y}_\gamma)$) and $\langle \mu, \mathsf{y}_\gamma^{-1}\alpha \rangle = 1$, so $\alpha \in \Phi(\mathsf{y}_{\gamma+\mathsf{y}_\gamma\mu})$. If $\mathsf{y}_\gamma^{-1}\alpha < 0$ and $\mathsf{y}_\mu^{-1}(-\mathsf{y}_\gamma^{-1}\alpha) > 0$ then $\langle \gamma, \alpha \rangle = 1$ and $\langle \mu, -\mathsf{y}_\gamma^{-1}\alpha \rangle = 0$, and so again $\alpha \in \Phi(\mathsf{y}_{\gamma+\mathsf{y}_\gamma\mu})$. Hence $\mathsf{y}_\gamma \mathsf{y}_\mu = \mathsf{y}_{\gamma+\mathsf{y}_\gamma\mu} = \mathsf{y}_{(\gamma+\mu)^{(J)}} = \mathsf{y}_\mu \mathsf{y}_\gamma$.

To prove $\mathsf{y}_\gamma^{-1} = \mathsf{y}_{-\mathsf{y}_\gamma^{-1}\gamma}$ we need $\Phi(\mathsf{y}_{\mathsf{y}_\gamma^{-1}\gamma}) = \Phi(\mathsf{y}_\gamma^{-1})$. By Lemma 2.3.2

$$\Phi(\mathsf{y}_{-\mathsf{y}_\gamma^{-1}\gamma}) = \{\alpha \in \Phi_J^+ \mid \langle \gamma, -\mathsf{y}_\gamma \alpha \rangle = 1\}.$$

Assume $\alpha \in \Phi(\mathsf{y}_{-\mathsf{y}_\gamma^{-1}\gamma})$. Then $\langle \gamma, -\mathsf{y}_\gamma \alpha \rangle = 1$ implies that $-\mathsf{y}_\gamma \alpha > 0$ (as otherwise, by the arguments of the proof of (1), we have $\mathsf{y}_\gamma \alpha > 0$ which implies that $\langle \gamma, -\mathsf{y}_\gamma \alpha \rangle = 0$). Hence,

$\alpha \in \Phi(\mathsf{y}_\gamma^{-1})$. Conversely, assume that $\alpha \in \Phi(\mathsf{y}_\gamma^{-1}) \subseteq \Phi_J^+$ so $\mathsf{y}_\gamma\alpha < 0$. As $\gamma \in P^{(J)}$ and as $-\mathsf{y}_\gamma\alpha > 0$ we have $\langle\gamma, -\mathsf{y}_\gamma\alpha\rangle \in \{0,1\}$. As $-\mathsf{y}_\gamma^{-1}\mathsf{y}_\gamma\alpha = -\alpha < 0$ we have $-\mathsf{y}_\gamma\alpha \in \Phi(\mathsf{y}_\gamma)$ and thus $\langle\gamma, -\mathsf{y}_\gamma\alpha\rangle = 1$. Therefore, $\alpha \in \Phi(\mathsf{y}_{-\mathsf{y}_\gamma^{-1}\gamma})$.

(3) By (1) and (2) we have

$$\tau_\gamma\tau_\mu = t_\gamma\mathsf{y}_\gamma t_\mu\mathsf{y}_\mu = t_{\gamma+\mathsf{y}_\gamma\mu}\mathsf{y}_\gamma\mathsf{y}_\mu = t_{(\gamma+\mu)^{(J)}}\mathsf{y}_{(\gamma+\mu)^{(J)}} = \tau_{(\gamma+\mu)^{(J)}}.$$

Similarly, $\tau_\gamma^{-1} = \tau_{(-\gamma)^{(J)}}$. ☐

Define the *J-translation group* to be the set of all pseudo-translations denoted by

$$\mathbb{T}_J = \{\tau_\gamma \mid \gamma \in P^{(J)}\}. \tag{2.3.1}$$

Corollary 2.3.5. *We have $\mathbb{T}_J \cong P/Q_J$. In particular, $\mathbb{T}_J$ is an abelian group of rank $|I\backslash J|$.*

Proof. Lemma 2.3.4(3) shows that $\mathbb{T}_J$ is a group, and that the map $f : P \to \mathbb{T}_J$ given by $f(\gamma) = \tau_{\gamma^{(J)}}$ is a group homomorphism. This homomorphism is surjective (as $\gamma^{(J)} = \gamma$ for $\gamma \in P^{(J)}$) and $\ker(f) = Q_J$. ☐

Remark 2.3.6. By Corollary 2.3.5 the map $\mathbb{T}_J \to P/Q_J$ with $\tau_\gamma \to \gamma^{(J)} + Q_J$ is an isomorphism. In contrast, we note that the map $\mathbb{T}_J \to P^J$ with $\tau_\gamma \mapsto \gamma^J$ is not an isomorphism. It is a homomorphism by Lemma 2.3.4(3) and the fact that $(\gamma^{(J)})^J = \gamma^J$. In addition, it is surjective as for all $\gamma \in P^J$ we have $\tau_{\gamma^{(J)}} \mapsto \gamma^J$, again as $(\gamma^{(J)})^J = \gamma^J$. However, the homomorphism is not injective. Consider for example $J = I$, then $P^J = \{0\}$ is trivial but $\mathbb{T}_J = P/Q = \Sigma$.

By Corollary 2.3.5 and Lemma 2.2.1, for $\gamma \in P$, we now have the following three bijections

$$\begin{gathered}\mathbb{T}_J \longleftrightarrow P^{(J)} \longleftrightarrow P/Q_J \\ \tau_{\gamma^{(J)}} \longleftrightarrow \gamma^{(J)} \longleftrightarrow \gamma^{(J)} + Q_J\end{gathered}$$

allowing us to shift between these groups as necessary. Note that these groups interpolate between P and Σ. When $J = \emptyset$ we have $P/Q_J = P$ and when $J = I$ we have $P/Q_J = P/Q = \Sigma$.

We now show that these pseudo-translations act like translations when restricting to $\mathcal{A}_J$. First we define $\mathbb{W}^J \subseteq \widetilde{W}$ by

$$\mathbb{W}^J = \{w \in \widetilde{W} \mid wA_0 \subseteq \mathcal{A}_J\}. \tag{2.3.2}$$

Thus $\mathbb{W}^J \cap W$ is in bijection with the alcoves (defined in Section 1.3) contained in the fundamental J-alcove.

Recall the definition of J^{aff} from 2.1.2. In the following lemma we give three characterisations of $\mathbb{W}^J$. In particular, note that the third characterisation gives that $\mathbb{W}^J$ is an affine analogue of ${}^JW$ (see Lemma 1.4.1).

Lemma 2.3.7. *We have*

$$\begin{aligned}\mathbb{W}^J &= \{w \in \widetilde{W} \mid \ell(s_{\beta,k}w) > \ell(w) \textit{ for all } \beta + k\delta \in \Phi_J + \mathbb{Z}\delta\} \\ &= \{w \in \widetilde{W} \mid \ell(s'_j w) > \ell(w) \textit{ for all } j \in J^{\mathrm{aff}}\} \\ &= \{w \in \widetilde{W} \mid \widetilde{\Phi}(w) \cap (\Phi_J + \mathbb{Z}\delta) = \emptyset\}.\end{aligned}$$

Proof. We have $w \in \mathbb{W}^J$ if and only if the alcove wA_0 lies on the same side of $H_{\beta,k}$ as A_0 for all $(\beta, k) \in \Phi_J \times \mathbb{Z}$, if and only if $\ell(s_{\beta,k}w) > \ell(w)$. Hence the first equality. For the second equality, we have $w \in \mathbb{W}^J$ if and only if the alcove wA_0 lies on the same side of $H_{\alpha_j,0}$ and $H_{\varphi_K,1}$ as A_0 for all $j \in J$ and $K \in \mathcal{K}(J)$ (see Lemma 2.1.2). The third characterisation follows from the first characterisation and the fact that if $\beta - k\delta \in \widetilde{\Phi}^+$ then $\ell(s_{\beta,k}w) < \ell(w)$ if and only if $\beta + k\delta \in \widetilde{\Phi}(w)$. □

The following theorem gives an explicit decomposition of the elements of $\mathbb{W}^J$, into a pseudo-translation and a minimal coset representative.

Theorem 2.3.8. *We have* $\mathbb{W}^J = \{\tau_\gamma u \mid \gamma \in P^{(J)}$ *and* $u \in {}^J W\}$.

Proof. We first show that $\tau_\gamma \in \mathbb{W}^J$ for all $\gamma \in P^{(J)}$. The hyperplane in the parallelism class of $\alpha \in \Phi_J$ passing through γ is $H_{\alpha,\langle\gamma,\alpha\rangle}$. As $\gamma \in P^{(J)}$ we have $\langle\gamma,\alpha\rangle \in \{0, 1\}$. If $\langle\gamma,\alpha\rangle = 0$ then $t_\gamma A_0$ is on the same side of $H_{\alpha,\langle\gamma,\alpha\rangle}$ as A_0. However, when $\langle\gamma,\alpha\rangle = 1$, the hyperplane $H_{\alpha,\langle\gamma,\alpha\rangle}$ separates A_0 and $t_\gamma A_0$. Hence, the hyperplanes in the parallelism classes of Φ_J separating the alcove $t_\gamma A_0$ from $\mathcal{A}_J$ are precisely the hyperplanes $H_{\alpha,1}$ with $\alpha \in \Phi_J^+$ (with $\langle\gamma,\alpha\rangle = 1$). Let $v \in W_J$ be the word formed by crossing these hyperplanes, so $t_\gamma v A_0 \subseteq \mathcal{A}_J$. As translations preserve hyperplane and alcove orientation, the hyperplanes that separate A_0 and vA_0 are associated to the same roots as the hyperplanes separating $t_\gamma A_0$ from $t_\gamma v A_0$. The hyperplanes separating A_0 and vA_0 are $H_{\alpha,0}$ such that $\langle\gamma,\alpha\rangle = 1$. By Lemma 2.3.2 this gives that $v = \mathsf{y}_\gamma$. Hence $\tau_\gamma A_0 \subseteq \mathcal{A}_J$ and so $\tau_\gamma \in \mathbb{W}^J$.

Let $u \in {}^J W$ and suppose that $\tau_\gamma u \notin \mathbb{W}^J$. That is, $\tau_\gamma u A_0 \not\subseteq \mathcal{A}_J$. Since $\tau_\gamma A_0 \subseteq \mathcal{A}_J$ (from the previous paragraph) there exists a hyperplane $H_{\alpha,k}$ separating $t_\gamma \mathsf{y}_\gamma A_0$ from $t_\gamma \mathsf{y}_\gamma u A_0$ with $\alpha \in \Phi_J^+$ and $k \in \{0, 1\}$. Translating by $t_{-\gamma}$ implies that the hyperplane $H_{\alpha,0}$ separates y_γ from $\mathsf{y}_\gamma u$. However, then $\Phi_J(u) \neq \emptyset$ contradicting Lemma 1.4.1. Thus $\{\tau_\gamma u \mid \gamma \in P^{(J)}$ and $u \in {}^J W\} \subseteq \mathbb{W}^J$.

For the reverse containment, suppose that $w \in \mathbb{W}^J$. Write $w = t_\gamma u_1 u_2$ with $\gamma = \mathrm{wt}(w)$, $u_1 \in W_J$, and $u_2 \in {}^J W$. As $w(0) = \gamma$ and $0 \in A_0$, it is clear that $\gamma \in P^{(J)}$. By Lemma 1.4.1, $\Phi_J(u_2) = \emptyset$. There are no walls of $\mathcal{A}_J$ separating $t_\gamma u_1$ from $t_\gamma u_1 u_2$. Hence, as ${}^J W \subseteq \mathbb{W}^J$, we have $t_\gamma u_1 A_0 \subseteq \mathcal{A}_J$. Therefore, u_1 is the element of W_0 that takes $t_\gamma A_0$ into $\mathcal{A}_J$. By the arguments of the first paragraph, the hyperplanes that separate $t_\gamma A_0$ and $t_\gamma u_1 A_0$, corresponding to the roots in the inversion set of u_1, are $H_{\alpha,1}$ for $\alpha \in \Phi_J^+$ such that $\langle\gamma,\alpha\rangle = 1$. Thus, by Lemma 2.3.2 $u_1 = \mathsf{y}_\gamma$. So $w = \tau_\gamma u_2$ with $u_2 \in {}^J W$, completing the proof. □

Remark 2.3.9. The first paragraph of the proof of Theorem 2.3.8 shows that if $\gamma \in P^{(J)}$ then y_γ may be characterised as the unique element of W_J such that $t_\gamma \mathsf{y}_\gamma \in \mathbb{W}^J$.

Recall the definition of $\theta_J(w)$ and $\theta^J(w)$ from Definition 1.4.2.

Corollary 2.3.10. *If* $w \in \mathbb{W}^J$ *then* $\theta_J(w) = \mathsf{y}_{\mathrm{wt}(w)}$, *and so* $w = \tau_{\mathrm{wt}(w)}\theta^J(w)$.

Proof. By Theorem 2.3.8 $w = \tau_\gamma u = t_\gamma \mathsf{y}_\gamma u$ with $\gamma = \mathrm{wt}(w)$ and $u \in {}^J W$, hence the result. □

Corollary 2.3.11. *The group* $\mathbb{T}_J$ *acts freely on* $\mathbb{W}^J$ *with fundamental domain* ${}^J W$.

Proof. If $\gamma \in P^{(J)}$ and $w \in \mathbb{W}^J$ then by Theorem 2.3.8 we have $w = \tau_\mu u$ with $\mu = \mathrm{wt}(w) \in P^{(J)}$ and $u \in {}^J W$. By Lemma 2.3.4(3) we have $\tau_\gamma \cdot w = \tau_\gamma \tau_\mu u = \tau_{(\gamma+\mu)^{(J)}} u$, and hence $\tau_\gamma \cdot w \in \mathbb{W}^J$ by Theorem 2.3.8. Thus $\mathbb{T}_J$ acts on $\mathbb{W}^J$. It is clear that this action is free (only τ_0 fixes $w \in \mathbb{W}^J$), and that ${}^J W$ is a fundamental domain. □

Example 2.3.12. Figure 2.3 shows the fundamental J-alcove for $\Phi = G_2$ and $J = \{1\}$. The elements of $P^{(J)}$ are the black bullets and the alcoves shaded green represent $\tau_\gamma A_0$ with $\gamma \in P^{(J)}$. The identity alcove, $\tau_0 A_0$, is illustrated in blue. The fundamental domain ${}^J W$ for the action of $\mathbb{T}_J$ is illustrated on the right. The black dotted lines represent the decomposition of $\mathcal{A}_J$ described in Theorem 2.3.8 and Corollary 2.3.11.

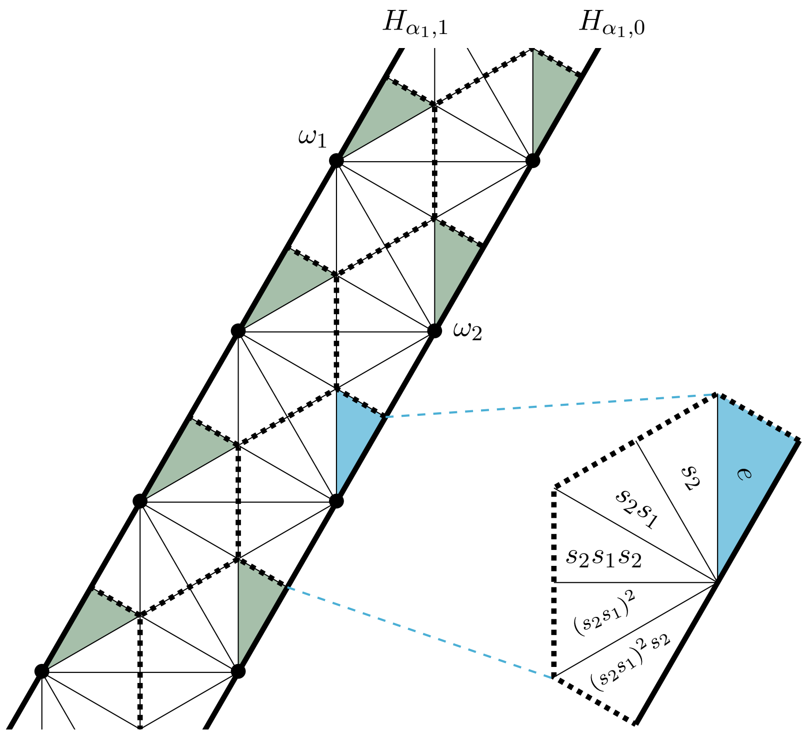


Figure 2.3: The fundamental J-alcove for $\Phi = G_2$ and $J = \{1\}$

As $P = Q$ for $\Phi = G_2$, $\mathbb{T}_J$ is a torsion free abelian group of rank $|I \setminus J| = 1$. Explicitly, the elements of $\mathbb{T}_J$ are either of the form $t_{k\omega_2+\omega_1}s_1$ or of the form $t_{k\omega_2}$ where $k \in \mathbb{Z}$.

2.4 Symmetries of the fundamental J-alcove

The set of elements of W_0 that stabilise the fundamental J-alcove, denoted G_J, is studied in this section. We show that the subgroup of $\widetilde{W}$ that stabilises the fundamental J-alcove is $\mathbb{T}_J \rtimes G_J$ where $\mathbb{T}_J$ is as in Section 2.3.

Definition 2.4.1. Let $G_J = \{g \in W_0 \mid g\mathcal{A}_J = \mathcal{A}_J\}$ be the subgroup of W_0 stabilising $\mathcal{A}_J$.

Theorem 2.4.2. *We have* $G_J = \{g \in W_0 \mid g\Phi_J^+ = \Phi_J^+\}$.

Proof. Let $g \in G_J$. Then for all $\alpha \in \Phi_J^+$ and all $x \in \mathcal{A}_J$ we have $0 \leq \langle gx, \alpha\rangle \leq 1$. We claim that $g^{-1}\Phi_J^+ = \Phi_J^+$. Assume the negation, that there exists $g \in G_J$ such that $g^{-1}\alpha \notin \Phi_J^+$. Then either $g^{-1}\alpha \in -\Phi_J^+$ or there exists $i \in \operatorname{supp}(g^{-1}\alpha)$ with $i \notin J$. In the former case, choose any $j \in \operatorname{supp}(g^{-1}\alpha) \subseteq J$ and let m_j^K be the coefficient of α_j in φ_K, where K is the connected component of J containing j. Let $x = \frac{1}{m_j^K}\omega_j$. For $\beta \in \Phi_J^+$, $0 \leq \langle x, \beta\rangle \leq \langle x, \varphi_K\rangle = 1$ (when $\beta \in \Phi_J^+ \setminus \Phi_K$ we have $\langle x, \beta\rangle = 0$). Hence, $0 \leq \langle x, \beta\rangle \leq 1$ for all $\beta \in \Phi_J^+$, and so $x \in \mathcal{A}_J$. As

$g^{-1}\alpha \in -\Phi_J^+$, $\langle x, g^{-1}\alpha\rangle < 0$, a contradiction. In the latter case, let $x = 2\omega_i \in \mathcal{A}_J$ where $i \notin J$ and $i \in \mathrm{supp}(g^{-1}\alpha)$. Then $|\langle x, g^{-1}\alpha\rangle| \geq 2$, again a contradiction. So, when $g \in G_J$ we have $g\Phi_J^+ = \Phi_J^+$. On the other hand, let g be such that $g\Phi_J^+ = \Phi_J^+$. For $x \in \mathcal{A}_J$ and all $\alpha \in \Phi_J^+$ we have $\langle gx, \alpha\rangle = \langle x, g^{-1}\alpha\rangle$. As $g^{-1}\alpha \in \Phi_J^+$ and $x \in \mathcal{A}_J$ we have that $0 \leq \langle x, g^{-1}\alpha\rangle \leq 1$, and so $g\mathcal{A}_J = \mathcal{A}_J$. □

Proposition 2.4.3. *For $g \in G_J$ and $\alpha \in \Phi_J^+$ we have $\mathrm{ht}(g\alpha) = \mathrm{ht}(\alpha)$. In particular g maps the simple roots of Φ_J to the simple roots of Φ_J, and hence induces a permutation of J. This permutation maps connected components of J to connected components of J.*

Proof. Let $g \in G_J$ and $\alpha \in \Phi_J^+$. By Theorem 2.4.2 we have that $g\Phi_J^+ = \Phi_J^+$. For $j \in \mathrm{supp}(\alpha)$, $\mathrm{ht}(g\alpha_j) \geq 1$. As g is linear, this implies that $\mathrm{ht}(g\alpha) \geq \mathrm{ht}(\alpha)$. As $g^{-1} \in G_J$ we also have $\mathrm{ht}(g^{-1}\beta) \geq \mathrm{ht}(\beta)$ for all $\beta \in \Phi_J^+$. By Theorem 2.4.2, we have $g\alpha = \beta$ for some $\beta \in \Phi_J^+$ and so $\mathrm{ht}(g^{-1}g\alpha) \geq \mathrm{ht}(g\alpha)$. Thus $\mathrm{ht}(g\alpha) = \mathrm{ht}(\alpha)$. In particular, simple roots of Φ_J are mapped to simple roots of Φ_J. This induces a permutation of J defined by $\alpha_{g(j)} = g\alpha_j$ for $j \in J$.

We now show that this permutation maps connected components of J to connected components of J. Let K be a connected component of J with highest root φ_K. We claim that $g\varphi_K = \varphi_{K'}$ where K' is a connected component of J. Assume the negation, there is a $i \in J$ such that $\alpha = g\varphi_K + \alpha_i$ is a root of Φ_J. Then $g^{-1}\alpha = \varphi_K + \alpha_{g^{-1}(i)}$ is a root of Φ_J. In particular, this is a root of Φ_K with $\mathrm{ht}(g^{-1}\alpha) \geq \mathrm{ht}(\varphi_K)$, a contradiction. Thus g maps highest roots to highest roots, and so g maps K to K'. □

Remark 2.4.4. For $J = \emptyset$, we have that $\mathcal{A}_J = V$ so $\mathbb{W}^J = \widetilde{W}$ and $G_J = W_0$. For $J = I$, we have that $\mathcal{A}_J = A_0$ and $\mathbb{W}^J = \Sigma$. In this case, e is the only element of W_0 that stabilises all roots in Φ^+, hence $G_J = \{e\}$.

Lemma 2.4.5. *If $\gamma \in P^{(J)}$ and $g \in G_J$ then*

$$g\mathsf{y}_\gamma g^{-1} = \mathsf{y}_{g\gamma}, \quad g\tau_\gamma g^{-1} = \tau_{g\gamma}, \quad \textit{and} \quad \ell(\tau_{g\gamma}) = \ell(\tau_\gamma).$$

Proof. To prove that $g\mathsf{y}_\gamma g^{-1} = \mathsf{y}_{g\gamma}$ it is sufficient to show that $\Phi(g\mathsf{y}_\gamma g^{-1}) = \Phi(\mathsf{y}_{g\gamma})$. If $\alpha \in \Phi(\mathsf{y}_{g\gamma})$ then $\alpha \in \Phi_J^+$ with $\langle g\gamma, \alpha\rangle = 1$. Thus $\langle \gamma, g^{-1}\alpha\rangle = 1$ and so $g^{-1}\alpha \in \Phi(\mathsf{y}_\gamma)$ as $g^{-1}\alpha \in \Phi_J^+$ by Theorem 2.4.2. Hence $\mathsf{y}_\gamma^{-1}g^{-1}\alpha \in -\Phi_J^+$ and it follows by Theorem 2.4.2 that $g\mathsf{y}_\gamma^{-1}g^{-1}\alpha \in -\Phi_J^+$. Therefore, $\Phi(\mathsf{y}_{g\gamma}) \subseteq \Phi(g\mathsf{y}_\gamma g^{-1})$.

On the other hand, suppose that $\alpha \in \Phi(g\mathsf{y}_\gamma g^{-1})$. Since g maps simple roots of Φ_J to simple roots of Φ_J (Theorem 2.4.2) we have $gs_jg^{-1} = s_{g(j)} \in W_J$ for all $j \in J$, and so $g\mathsf{y}_\gamma g^{-1} \in W_J$. Thus $\alpha \in \Phi_J^+$, and so $g^{-1}\alpha \in \Phi_J^+$. If $g^{-1}\alpha \notin \Phi(\mathsf{y}_\gamma)$ then $g\mathsf{y}_\gamma^{-1}g^{-1}\alpha \in \Phi_J^+$, a contradiction, and so $g^{-1}\alpha \in \Phi(\mathsf{y}_\gamma)$. Thus $\langle \gamma, g^{-1}\alpha\rangle = 1$, and so $\langle g\gamma, \alpha\rangle = 1$, giving $\alpha \in \Phi(\mathsf{y}_{g\gamma})$ as required.

It then follows that $g\tau_\gamma g^{-1} = t_{g\gamma}(g\mathsf{y}_\gamma g^{-1}) = t_{g\gamma}\mathsf{y}_{g\gamma} = \tau_{g\gamma}$ for all $\gamma \in P^{(J)}$. By Proposition 2.3.3 we have $\ell(\tau_{g\gamma}) = \sum_{\alpha\in\Phi^+\setminus\Phi_J} |\langle \gamma, g^{-1}\alpha\rangle| = \sum_{\alpha\in\Phi^+\setminus\Phi_J} |\langle \gamma, \alpha\rangle| = \ell(\tau_\gamma)$. □

Corollary 2.4.6. *The subgroup of $\widetilde{W}$ stabilising $\mathcal{A}_J$ is $\mathbb{T}_J \rtimes G_J$.*

Proof. Let $w \in \widetilde{W}$ and suppose that $w\mathcal{A}_J = \mathcal{A}_J$. Let $\gamma = \mathrm{wt}(w)$, and so $w(0) = \gamma \in P^{(J)}$. Then $\tau_\gamma^{-1}w(0) = 0$, and so $g = \tau_\gamma^{-1}w \in W_0$ with $g\mathcal{A}_J = \mathcal{A}_J$. Thus $g \in G_J$ and $w = \tau_\gamma g \in \mathbb{T}_J G_J$. □

Note that the group $\mathbb{T}_J \rtimes G_J$ plays the role of the "extended affine Weyl group" of $\mathcal{A}_J$ in the sense that if $J = \emptyset$ we have $\mathbb{T}_J \rtimes G_J = \widetilde{W}$.

Example 2.4.7. Let $\Phi = G_2$ and $J = \{1\}$. We have that $s_{3\alpha_1+2\alpha_2} = s_2s_1s_2s_1s_2$ is the reflection about the hyperplane $H_{3\alpha_1+2\alpha_2,0}$. In Figure 2.4 $\mathcal{A}_J$ is coloured blue and $H_{3\alpha_1+2\alpha_2,0}$ is labelled. It can be seen that $\mathcal{A}_J$ is fixed by reflecting about $H_{3\alpha_1+2\alpha_2,0}$. The fundamental J-alcove is not fixed by any other nontrivial elements of W_0 and so $G_J = \{e, s_2s_1s_2s_1s_2\}$. The elements of the subgroup of $\widetilde{W}$ that stabilises $\mathcal{A}_J$ are the reflections about the bold hyperplanes. The elements of $\mathbb{T}_JG_J$ are coloured green, with $\mathbb{T}_Js_2s_1s_2s_1s_2$ in light green and $\mathbb{T}_Je$ in dark green.

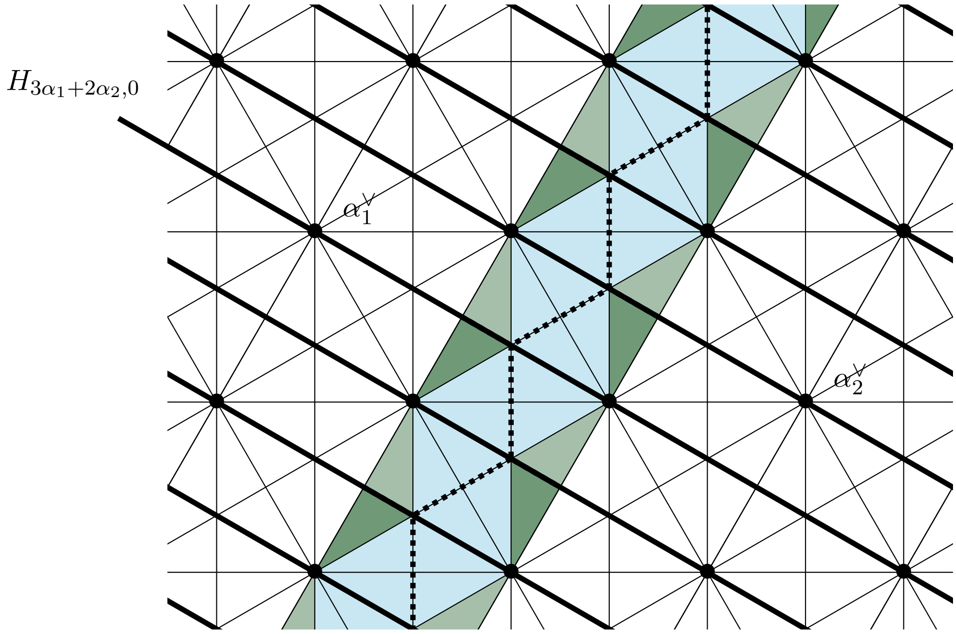


Figure 2.4: G_J for $\Phi = G_2$

Chapter 3

Hecke algebras and the Kazhdan-Lusztig basis

The representation theory of the Weyl group W can be investigated by exploring a particular q-deformation of its group algebra, where q is an indeterminate. This deformation is called the Hecke algebra associated to W. In [25], Kazhdan and Lusztig formed a basis for the Hecke algebra which yields rich representation theory. In this chapter we give a brief background on Hecke algebras, the Kazhdan-Lusztig basis and some results.

In Section 3.1, for the affine Weyl group W, we define the associated affine Hecke algebra with general parameters. In addition, we extend the definition to $\widetilde{W}$, called the extended Hecke algebra and denoted as $\widetilde{\mathcal{H}}$. We will then define the Kazhdan-Lusztig basis of $\widetilde{\mathcal{H}}$ in Section 3.2. The multiplication between these basis elements gives rise to a preorder, whose equivalence classes are called cells. From each cell a natural $\widetilde{\mathcal{H}}$-module arises. Also in this section, we state the definition of Lusztig's $\mathbf{a}$-function and the 15 conjectures posed by Lusztig in [35] about cells and the preorder. Finally, we will introduce Lusztig's asymptotic algebra.

In Section 3.3 we introduce the notion of a balanced system of cell representations. In [17] Geck gave a list of criteria for a family of matrix representations of the subalgebra of $\widetilde{\mathcal{H}}$ with basis indexed by elements of W_0 (the finite Hecke algebra, denoted $\mathcal{H}_0$). This list of criteria gives a method for proving the 15 conjectures of Lusztig for Hecke algebras with unequal parameters. In [23] and [22] Guilhot and Parkinson extended this criteria to the affine case. They called a family of matrix representations satisfying the criteria a *balanced system of cell representations*. Guilhot and Parkinson used a balanced system of cell representations along with the Plancherel Theorem (discussed in Section 3.4) to prove the Lusztig conjectures for the affine cases when $\Phi = G_2$ and $\Phi = C_2$, for arbitrary parameters. In Section 3.3 we state the definition of a balanced system of cell representations along with some results of Guilhot and Parkinson. The background of this section will be used in Part II when we show that a system of matrix representations for $\Phi = A_n$ is a balanced system of cell representations.

In the final section, Section 3.4, we define the canonical trace function on $\widetilde{\mathcal{H}}$ and briefly describe how it decomposes when restricting to elements of $\mathcal{H}_0$ and when looking at all elements of $\widetilde{\mathcal{H}}$ (in which the decomposition, found by Opdam [40], is called the Plancherel Theorem). We also define a trace function on Lusztig's asymptotic algebra which will be explored further for type A_n in Chapter 10. The definitions and results of this chapter are well studied, for further information see [5].

3.1 Hecke algebras

Let W be the affine Weyl group with generating set indexed by $I \cup \{0\}$ (see Section 1.2). A *weight function* related to W is defined as a function $L : W \to \mathbb{N}$ such that

1. $L(wu) = L(w) + L(u)$ for $\ell(wu) = \ell(w) + \ell(u)$, and
2. $L(s_i) > 0$ for all $i \in I \cup \{0\}$.

We have the following equivalent definition.

Proposition 3.1.1. *For weight function $L : W \to \mathbb{N}$ we have*

(1) $L(s_i) = L(s_j)$ *if s_i is conjugate to s_j in W,*

(2) $L(w) = L(s_{i_1}) + \cdots + L(s_{i_m})$ *for $w = s_{i_1} \cdots s_{i_m}$ reduced.*

Proof. (2) is clear as $w = s_{i_1} \cdots s_{i_m}$ is reduced. Consider (1). If s_i and s_j are conjugate in W then there exists some $w \in W$ such that $s_i w = w s_j$. If $\ell(s_i w) = \ell(w) + 1$ then $\ell(w s_j) = \ell(w) + 1$ and so by the definition of L we have

$$L(s_i) + L(w) = L(w) + L(s_j).$$

Thus, $L(s_i) = L(s_j)$ as required. If $\ell(s_i w) = \ell(w) - 1$ then $\ell(w s_j) = \ell(w) - 1$ and $\ell(s_i s_i w) = \ell(s_i w) + 1 = \ell(w s_j) + 1 = \ell(w s_j s_j)$. Thus,

$$L(w s_j) + L(s_j) = L(w s_j s_j) = L(w) = L(s_i s_i w) = L(s_i w) + L(s_i)$$

and so again $L(s_i) = L(s_j)$ as required. □

We say that we are in the *equal parameters case* when $L(s_i) = L(s_j)$ for all $i, j \in I \cup \{0\}$. For indeterminate q, let $\mathsf{q}_i = \mathsf{q}^{L(s_i)}$. By Proposition 3.1.1 we have that $\mathsf{q}_i = \mathsf{q}_j$ if s_i and s_j are conjugate in W. Let $\mathsf{R} = \mathbb{Z}[\mathsf{q}, \mathsf{q}^{-1}]$. The *(weighted) affine Hecke algebra* is the R-algebra $\mathcal{H}$ with basis $(T_w)_{w \in W}$ and multiplication given by

$$\begin{aligned} T_w T_u &= T_{wu} && \text{if } \ell(wu) = \ell(w) + \ell(u) \\ T_w T_{s_i} &= T_{w s_i} + (\mathsf{q}_i - \mathsf{q}_i^{-1}) T_w && \text{if } \ell(w s_i) = \ell(w) - 1 \end{aligned} \tag{3.1.1}$$

for $w, u \in W$ and $i \in I \cup \{0\}$. We call $\{T_w \mid w \in W\}$ the *standard basis* of $\mathcal{H}$.

For simplicity, we will write T_i in place of T_{s_i}. All T_i with $i \in I \cup \{0\}$ are invertible with $T_i^{-1} = T_i - (\mathsf{q}_i - \mathsf{q}_i^{-1}) T_e$. It follows that for all $w \in W$ each T_w is invertible. Let $\mathsf{q}_w = \mathsf{q}^{L(w)}$. By Proposition 3.1.1, for $w \in W$ we have $\mathsf{q}^{L(w)} = \mathsf{q}_{i_1} \cdots \mathsf{q}_{i_m}$ for $w = s_{i_1} \cdots s_{i_m}$ reduced.

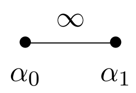
By the multiplication rules (3.1.1) we have that $T_i^2 = T_e + (\mathsf{q}_i - \mathsf{q}_i^{-1}) T_i$. Therefore, all 1-dimensional representations of $\mathcal{H}$ have T_i mapping to either q_i or $-\mathsf{q}_i^{-1}$. Let $\mathcal{H}_0$ denote the subalgebra of $\mathcal{H}$ generated by $\{T_w \mid w \in W_0\}$ (called the *finite Hecke algebra* associated to $\mathcal{H}$).

We extend this notion to the extended Weyl group $\widetilde{W}$. First we extend the weight function so that $L(\sigma) = 0$ for all $\sigma \in \Sigma$, and so $\mathsf{q}_\sigma = 1$. Define $\widetilde{\mathcal{H}}$ to be the *extended affine Hecke algebra*, the R-algebra with standard basis $(T_w)_{w \in \widetilde{W}}$. The relations of $\widetilde{\mathcal{H}}$ with respect to this basis are the relations (3.1.1) and $T_\sigma T_i = T_{\sigma(i)} T_\sigma$ for all $i \in I \cup \{0\}$ and $\sigma \in \Sigma$. We have that T_σ is invertible for all $\sigma \in \Sigma$ with $T_\sigma^{-1} = T_{\sigma^{-1}}$, and so for every $w \in \widetilde{W}$ we have that T_w is invertible. It is clear that $\mathcal{H}$ is a subalgebra of $\widetilde{\mathcal{H}}$.

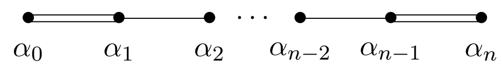
We will sometimes require the indeterminiants to be indexed by roots. To do this we note the following properties; if $\alpha \in W_0 \alpha_i \cap W_0 \alpha_j$ then s_i is conjugate to s_j, and for $\alpha + k\delta \in \widetilde{\Phi}$ there exists some $u \in \widetilde{W}$ such that $u\alpha = \alpha + k\delta$ by Lemma 1.2.2. Thus, for $\alpha \in \widetilde{\Phi}$ we define $\mathsf{q}_\alpha = \mathsf{q}_i$ if $\alpha \in \widetilde{W}\alpha_i$.

We state the following convention, and will assume its application from Chapter 4 onwards. Note that examples in this chapter will be completed without the restriction of this convention.

Convention 3.1.2. It is convenient to make the following convention: If $\Phi = A_1$ then $\mathsf{q}_0 = \mathsf{q}_1$ and if $\Phi = \mathsf{C}_n$ $(n \geq 2)$ then $\mathsf{q}_0 = \mathsf{q}_n$ (the case where $\mathsf{q}_0 \neq \mathsf{q}_n$ is completed in [21] through analysis of the non-reduced root system $\Phi = \mathsf{BC}_n$ for $n \geq 1$). For the remaining types, q_0 is set by the fact that $\mathsf{q}_i = \mathsf{q}_j$ when s_i and s_j are conjugate. With this convention we have $\mathsf{q}_{\sigma(i)} = \mathsf{q}_i$ for all $i \in \{0\} \cup I$ and $\sigma \in \Sigma$. For example, consider the affine Coxeter diagram of type $\Phi = C_n$ depicted in Figure 3.1. Recall that Σ is isomorphic to a subgroup of the automorphisms of the affine Coxeter diagram. In this case Σ has one non-trivial element corresponding to the only automorphism of the affine Coxeter diagram: $\alpha_0 \mapsto \alpha_n$ and $\alpha_i \mapsto \alpha_{n-i}$ for all $1 \leq i \leq n-1$. With $\mathsf{q}_0 = \mathsf{q}_n$ we have that $\mathsf{q}_{\sigma(i)} = \mathsf{q}_i$ for all $0 \leq i \leq n$ where σ is the function on indices associated to this automorphism. Similarly for $\Phi = A_1$, depicted in Figure 3.1.



The affine Coxeter diagram of $\Phi = A_1$



The affine Coxeter diagram of $\Phi = C_n$

Figure 3.1

3.2 The Kazhdan-Lusztig basis and Lusztig's conjectures

This section focuses on the basis of $\widetilde{\mathcal{H}}$ defined by Kazhdan and Lusztig in [25]. This basis gives a richer representation theory than the standard basis $(T_w)_{w\in\widetilde{W}}$ and has consequentially been well studied. In this section we define the basis and state some well known properties. We construct equivalence classes (called cells) from a preorder on group elements defined by the basis and then construct natural $\widetilde{\mathcal{H}}$-modules from these cells. The definition of Lusztig's $\mathbf{a}$-function will be stated along with the 15 conjectures Lusztig posed in [35] about cells and the preorder.

Let $\widetilde{W}$ be an extended affine Weyl group with associated extended Hecke algebra $\widetilde{\mathcal{H}}$ with standard basis $\{T_w \mid w \in \widetilde{W}\}$. Definitions for this section will be made on the extended affine Weyl group $\widetilde{W}$, however all definitions can be made similarly for W or W_0.

Let $\leq$ denote the *extended Bruhat order*. The *Bruhat order* is the preorder such that for all $w, u \in W$ we have $w \leq u$ if w is a subexpression of some reduce expression of u. In fact, if w is a subexpression of a reduced expression of u it will be a subexpression of all reduced expressions of u (see [3, Corollary 2.2.3]). We extend this definition to $\widetilde{W}$ by setting $w\sigma \leq u\sigma$ and $\sigma w \leq \sigma u$ for all $w \leq u$ and $\sigma \in \Sigma$.

The *bar involution* is the involution $^{-} : \mathsf{R} \to \mathsf{R}$ such that $\mathsf{q} \mapsto \mathsf{q}^{-1}$. We extend this involution to $\widetilde{\mathcal{H}}$, setting $\overline{T_w} = T_{w^{-1}}^{-1}$, so that

$$\overline{\sum_{w\in\widetilde{W}} a_w T_w} = \sum_{w\in\widetilde{W}} \overline{a_w} T_{w^{-1}}^{-1}.$$

In [25], Kazhdan and Lusztig proved the existence and uniqueness of a bar involution invariant basis of $\widetilde{\mathcal{H}}$. Let $(C_w)_{w\in\widetilde{W}}$ denote this basis. For all $w \in \widetilde{W}$, C_w satisfies

$$\overline{C_w} = C_w \quad \text{and} \quad C_w = T_w + \sum_{y<w} P_{y,w} T_y$$

where $P_{y,w} \in \mathsf{q}^{-1}\mathbb{Z}[\mathsf{q}^{-1}]$ are called the *Kazhdan-Lusztig polynomials.* We set $P_{w,w} = 1$ for all $w \in \widetilde{W}$ and $P_{y,w} = 0$ when $y \not< w$. Note that $C_e = T_e$ and that for all $i \in I \cup \{0\}$ we have $C_{s_i} = T_i + P_{e,s_i} T_e$ where $P_{e,s_i} = \mathsf{q}_i^{-1}$. As C_w is bar invariant we have

$$T_w + \sum_{y<w} P_{y,w} T_y = T_{w^{-1}}^{-1} + \sum_{y<w} \overline{P_{y,w}} T_{y^{-1}}^{-1}. \tag{3.2.1}$$

For $w \in W$, by expanding this expression and equating coefficients, $P_{y,w}$ can be calculated. Note that $P_{y,w}$ is dependent on L, shown in the following example.

Example 3.2.1. Consider W associated with $\Phi = A_1$, generated by s_1 and s_0. Let $L(s_1) = a$ and $L(s_0) = b$. Consider $w = s_1 s_0 s_1 \in \widetilde{W}$. Equating coefficients (and substituting in all calculated $P_{y,w}$ for all $y < w$) in 3.2.1 we have that

$$P_{s_1,s_1s_0s_1} - \overline{P_{s_1,s_1s_0s_1}} = -\mathsf{q}^{a+b} + \mathsf{q}^{a-b} - \mathsf{q}^{-a+b} + \mathsf{q}^{-a-b}$$

As $P_{s_1,s_1s_0s_1} \in \mathsf{q}^{-1}\mathbb{Z}[\mathsf{q}^{-1}]$ we have

$$P_{s_1,s_1s_0s_1} = \begin{cases} -\mathsf{q}^{b-a} + \mathsf{q}^{-a-b} & \text{if } a > b \\ \mathsf{q}^{-2a} & \text{if } a = b \\ \mathsf{q}^{a-b} + \mathsf{q}^{-a-b} & \text{if } a < b, \end{cases}$$

which is dependent on L. (Note that we do not assume Convention 3.1.2 here. With the convention, the only case we consider for $\Phi = A_1$ is $a = b$.)

Recall that w_J is the longest element of the parabolic subgroup W_J of W_0.

Proposition 3.2.2. *[5, §10.5] Let $J \subseteq I$. Then*

$$C_{w_J} = \sum_{w\in W_J} \mathsf{q}_{w_J}^{-1} \mathsf{q}_w T_w = \sum_{w\in W_J} \mathsf{q}_{w_J} \mathsf{q}_w^{-1} T_{w^{-1}}^{-1}.$$

Proof. Let $A = \sum_{w\in W_J} a_w T_w = \sum_{w\in W_J} \mathsf{q}_w \mathsf{q}_{w_J}^{-1} T_w$. For $i \in J$ we have

$$T_i A = \sum_{w\in W_J} (\mathsf{q}_w \mathsf{q}_{w_J}^{-1}) T_{s_i w} + \sum_{s_i w < w} (\mathsf{q}_w \mathsf{q}_{w_J}^{-1})(\mathsf{q}_i - \mathsf{q}_i^{-1}) T_w$$

For $x \in W_J$ with $s_i x < x$, the coefficient of T_x in the above sum is

$$(\mathsf{q}_i - \mathsf{q}_i^{-1})(\mathsf{q}_x \mathsf{q}_{w_J}^{-1}) + \mathsf{q}_{s_i x} \mathsf{q}_{w_J}^{-1} = \mathsf{q}_i \mathsf{q}_x \mathsf{q}_{w_J}^{-1} = \mathsf{q}_i a_x.$$

For $x \in W_J$ with $s_i x > x$, the coefficient is

$$\mathsf{q}_{s_i x} \mathsf{q}_{w_J}^{-1} = \mathsf{q}_i a_x.$$

Hence, $T_s A = \mathsf{q}_i A$. Applying the bar involution to both sides and using $T_i^{-1} = T_i - (\mathsf{q}_i - \mathsf{q}_i^{-1}) T_e$ we have $T_i \overline{A} = \mathsf{q}_i \overline{A}$. By defintion we have that $\overline{A} = \sum_{w\in W_J} b_w T_w$ with $b_{w_J} = 1$. We aim to prove that $A = \overline{A}$. We do so by a descending induction on the length of $x \in W_J$. We have $b_{w_J} = 1 = a_{w_J}$. Assume that $s_i x < x$ for $i \in J$. As $T_s\overline{A} = \mathsf{q}_i\overline{A}$ we have that

$$\sum_{w \in W_J} b_w T_{sw} + \sum_{s_i w < w} b_w(\mathsf{q}_i - \mathsf{q}_i^{-1})T_w = \sum_{w \in W_J} \mathsf{q}_i b_w T_w.$$

Equating the coefficients of T_x in the above equation, we have

$$b_x(\mathsf{q}_i - \mathsf{q}_i^{-1}) + b_{s_i x} = \mathsf{q}_i b_x.$$

Hence, $b_x = \mathsf{q}_i^{-1} b_x$ and so, by the inductive hypothesis, $b_x = \mathsf{q}_w \mathsf{q}_{w_J}^{-1}$. By the uniqueness of the Kazhdan-Lusztig basis, $C_{w_J} = A$. The second equality follows by applying the bar involution. □

It follows from Proposition 3.2.2 and (3.1.1) that $T_i C_{\mathsf{w}_J} = C_{\mathsf{w}_J} T_i = \mathsf{q}_i C_{\mathsf{w}_J}$ for all $j \in J$. By induction on length we have $T_w C_{\mathsf{w}_J} = C_{\mathsf{w}_J} T_w = \mathsf{q}_w C_{\mathsf{w}_J}$ for all $w \in W_J$. Thus,

$$C_{\mathsf{w}_J}^2 = \mathsf{q}_{\mathsf{w}_J}^{-1} \sum_{w \in W_J} \mathsf{q}_w T_w C_{\mathsf{w}_J} = \mathsf{q}_{\mathsf{w}_J}^{-1} \sum_{w \in W_J} \mathsf{q}_w^2 C_{\mathsf{w}_J} = \mathsf{q}_{\mathsf{w}_J}^{-1} W_J(\mathsf{q}^2) C_{\mathsf{w}_J}$$

where $W_J(\mathsf{q}^2) = \sum_{w \in W_J} \mathsf{q}_w^2$. Similarly

$$C_{\mathsf{w}_J}^2 = \mathsf{q}_{\mathsf{w}_J} W_J(\mathsf{q}^{-2}) C_{\mathsf{w}_J}$$

as $T_{w^{-1}}^{-1} C_{\mathsf{w}_J} = \mathsf{q}_w^{-1} C_{\mathsf{w}_J}$. By similar arguments, we have the following corollary.

Corollary 3.2.3. *If* $\ell(\mathsf{w}_J w) = \ell(w) - \ell(\mathsf{w}_J)$ *then*

$$C_{\mathsf{w}_J} C_w = \mathsf{q}^{-\ell(\mathsf{w}_J)} W_J(\mathsf{q}^2) C_w.$$

Proof. By assumption, $\ell(s_j w) < \ell(w)$ for all $j \in J$. By [35, Theorem 6.6], this implies that $C_{s_j} C_w = (\mathsf{q}_j + \mathsf{q}_j^{-1})C_w$. Thus, $(T_{s_j} + \mathsf{q}_j^{-1})C_w = (\mathsf{q}_j + \mathsf{q}_j^{-1})C_w$ and so $T_{s_j} C_w = \mathsf{q}_j C_w$. By induction on length $T_v C_w = \mathsf{q}_v C_w$ for all $v \in W_J$. The result follows by Proposition 3.2.2 since

$$C_{\mathsf{w}_J} C_w = \mathsf{q}_{\mathsf{w}_J}^{-1} \sum_{v \in W_J} \mathsf{q}_v T_v C_w = \mathsf{q}_{\mathsf{w}_J}^{-1} \sum_{v \in W_J} \mathsf{q}_v^2 C_w = \mathsf{q}_{\mathsf{w}_J}^{-1} W_J(\mathsf{q}^2) C_w.$$

□

For $w, u \in \widetilde{W}$ let

$$C_w C_u = \sum_{y \in \widetilde{W}} h_{w,u,y} C_y.$$

where the elements $h_{w,u,y} \in \mathsf{R}$ are called the *structure constants* of $\widetilde{\mathcal{H}}$. In [14] it was proved that for the equal parameters case we have a positivity condition on the structure constants and the Kazhdan-Lusztig polynomials; $P_{x,y} \in \mathbb{N}[\mathsf{q}^{-1}]$ and $h_{x,y,z} \in \mathbb{N}[\mathsf{q}, \mathsf{q}^{-1}]$ for all $x, y, z \in W$. This is not true in the unequal parameters case. We can see this in Example 3.2.1 when $a > b$.

Defined in [35, Chapter 13], the *Lusztig* **a**-*function* is the function $\mathbf{a} : \widetilde{W} \to \mathbb{N}$ such that, for $y \in \widetilde{W}$

$$\mathbf{a}(y) = \min\{n \in \mathbb{N} \mid \mathsf{q}^{-n} h_{w,u,y} \in \mathbb{Z}[\mathsf{q}^{-1}] \text{ for all } w, u \in \widetilde{W}\}. \tag{3.2.2}$$

For affine Weyl groups this function is well defined. In fact, by [35, Chapter 13] and [32, 7.2] we have $\mathbf{a}(y) \leq L(\mathsf{w}_0)$ for all $y \in \widetilde{W}$.

For $w, u \in \widetilde{W}$ write $w \leftarrow_R u$ if there exists some $y \in \widetilde{W}$ such that $h_{u,y,w} \neq 0$. Let $\leq_R$ be the preorder generated by the transitive closure of $\leftarrow_R$. Define the equivalence $\sim_R$ on $\widetilde{W}$ by $w \sim_R u$ if $w \leq_R u$ and $u \leq_R w$. The equivalence classes of this relation are called the *right cells* of $\widetilde{W}$ with respect to the Kazhdan-Lusztig basis. Similarly we can define *left cells*, by instead taking the order $w \leftarrow_L u$ if there exists some $y \in \widetilde{W}$ such that $h_{y,u,w} \neq 0$. The *left-right cells* or *two-sided cells* are also defined, now using the preorder $\leq_{LR}$ that is generated by the preorders $\leq_L$ and $\leq_R$. The partial orders $\leq_L$, $\leq_R$ and $\leq_{LR}$ induce partial orders on the equivalence classes generated by $\sim_L$, $\sim_R$ and $\sim_{LR}$ respectively.

Remark 3.2.4. The left and right descent sets of $w \in \widetilde{W}$ are defined as follows:

$$D_L(w) = \{s_i \mid i \in I \cup \{0\}, \ell(s_i w) < \ell(w)\} \text{ and } D_R(w) = \{s_i \mid i \in I \cup \{0\}, \ell(w s_i) < \ell(w)\}.$$

By [35, Lemma 8.6], if $x \leq_L y$ then $D_R(y) \subseteq D_R(x)$ and if $x \leq_R y$ then $D_L(y) \subseteq D_L(x)$. Due to this, $e \not\leq_L w$ and $e \not\leq_R w$ for all $w \in \widetilde{W} \backslash \{e\}$. Therefore, the left (right or two-sided) cell containing e only contains e. As $C_w C_e = C_w$ and $C_e C_w = C_w$ for all $w \in \widetilde{W}$, we also have that $w \leq_L e$ and $w \leq_R e$ for all $w \in \widetilde{W}$.

By definition, if $w \leftarrow_R u$ for $w, u \in \widetilde{W}$ then C_w is in the expansion of $C_u C_z$ with a non-zero coefficient. We have $C_\sigma C_x = C_{\sigma x}$ for all $x \in \widetilde{W}$ and $\sigma \in \Sigma$ as $T_\sigma C_x = T_{\sigma x} + \sum_{y<x} P_{y,x} T_{\sigma y} = C_{\sigma x}$ by the uniqueness of the Kazhdan-Lusztig basis. Similarly, $C_x C_\sigma = C_{x\sigma}$ for all $x \in \widetilde{W}$ and $\sigma \in \Sigma$. Thus, $x \sim_L x\sigma$ and $x \sim_R \sigma x$ for all $x \in \widetilde{W}$ and $\sigma \in \Sigma$ (as $C_x C_\sigma = C_{x\sigma}$, $C_{x\sigma} C_{\sigma^{-1}} = C_x$, $C_\sigma C_x = C_{\sigma x}$ and $C_{\sigma^{-1}} C_{\sigma x} = C_x$). Furthermore, if $w \leftarrow_R u$ then $C_{\sigma w}$ appears in the expansion of $C_{\sigma u} C_z$ for some z and so $\sigma w \leftarrow_R \sigma u$. By a similar argument we have that if $w \leftarrow_L u$ then $w\sigma \leftarrow_L u\sigma$ for all $w, u \in \widetilde{W}$ and $\sigma \in \Sigma$. Therefore, for all $w, u \in \widetilde{W}$ and $\sigma \in \Sigma$ if $w \sim_R u$ then $\sigma w \sim_R \sigma u$ and if $w \sim_L u$ then $w\sigma \sim_L u\sigma$. Therefore, left cells are invariant under multiplying by elements of Σ on the right and right cells are invariant under multiplying by elements of Σ on the left.

Example 3.2.5. Consider $\widetilde{W}$ associated to $\Phi = A_1$. As in Example 3.2.1 we set $L(s_1) = a$ and $L(s_0) = b$ where s_1 and s_0 are the generators of W and $s_1 s_0$ is of infinite order. Denote s_i^1 to be the elements of W that begin with s_1 of length i and s_i^0 to be the elements that begin with s_0 of length i.

Assume $a = b$. Let σ be the function such that $\sigma(1) = 2$ and $\sigma(2) = 1$. Extending to an action on the generators defined by $\sigma s_i = s_{\sigma(i)}$ we have that $\Sigma = \{e, \sigma\}$. Set

$$\begin{aligned}
\Gamma_0 &= \{e\}, \\
\Gamma_1 &= \{s_i^0 \mid i \geq 1\}, \\
\Gamma_2 &= \{s_i^1 \mid i \geq 1\}.
\end{aligned}$$

The right cells are $\Gamma_k \cup \sigma\Gamma_k$ for $0 \leq k \leq 2$. The $\mathbf{a}$-function is constant within each Γ_k so we can write $\mathbf{a}(\Gamma_k)$ for the $\mathbf{a}$-function value of elements of Γ_k. We have $\mathbf{a}(\Gamma_0) = 0$, $\mathbf{a}(\Gamma_1) = a$ and $\mathbf{a}(\Gamma_2) = a$. The left-right cells are $\Gamma_0 \cup \sigma\Gamma_0$ and $(\Gamma_1 \cup \Gamma_2) \cup \sigma(\Gamma_1 \cup \Gamma_2)$.

Now assume that $a > b$. In this case there is no Coxeter diagram automorphism and so

$\Sigma = \emptyset$. Let

$$\begin{aligned}\Gamma_0 &= \{e\}\\ \Gamma_1 &= \{s_0\}\\ \Gamma_2 &= \{s_i^0 \mid i \geq 2\}\\ \Gamma_3 &= \{s_i^1 \mid i \geq 1\}.\end{aligned}$$

The right cells are Γ_k for $0 \leq k \leq 3$ and we have $\mathbf{a}(\Gamma_0) = 0$, $\mathbf{a}(\Gamma_1) = b$, $\mathbf{a}(\Gamma_2) = a$ and $\mathbf{a}(\Gamma_3) = a$. The left-right cells are Γ_0, Γ_1 and $\Gamma_2 \cup \Gamma_3$.

For each cell there is a natural $\widetilde{\mathcal{H}}$-module. Let $* \in \{R, L, LR\}$. For a cell Γ (left, right or two-sided) define

$$\begin{aligned}\Gamma_{\leq_*} &= \{w \in \widetilde{W} \mid \text{ there exists } u \in \Gamma \text{ such that } w \leq_* u\},\\ \Gamma_{\geq_*} &= \{w \in \widetilde{W} \mid \text{ there exists } u \in \Gamma \text{ such that } w \geq_* u\}.\end{aligned} \tag{3.2.3}$$

For example, from Remark 3.2.4, when $\Gamma = \{e\}$ we have $\Gamma_{\leq_{LR}} = \widetilde{W}$ and $\Gamma_{\geq_{LR}} = \Gamma$. We also define $\Gamma_{<_*} = \Gamma_{\leq_*} \backslash \Gamma$ and set two R-modules as follows:

$$\widetilde{\mathcal{H}}^{\leq_*} = \langle C_w \mid w \in \Gamma_{\leq_*} \rangle \quad \text{and} \quad \widetilde{\mathcal{H}}^{<_*} = \langle C_w \mid w \in \Gamma_{<_*} \rangle. \tag{3.2.4}$$

When $* = R$ these modules are right ideals of $\mathcal{H}$, when $* = L$ they are left ideals and when $* = LR$ they are two-sided ideals (see [35, Lemma 8.2]). Hence, the quotient $\widetilde{\mathcal{H}}_\Gamma = \widetilde{\mathcal{H}}^{\leq_*}/\widetilde{\mathcal{H}}^{<_*}$ is a right module of $\widetilde{\mathcal{H}}$ when $* = R$, a left module when $* = R$ and a bi-module when $* = LR$. Each has basis $\{[C_w] \mid w \in \Gamma\}$ where $[C_w]$ is the equivalence class of C_w defined by the quotient. Therefore, for each cell we have a natural representation of the Hecke algebra.

In [35, Conjectures 14.2], Lusztig posed 15 conjectures about the cells and the $\mathbf{a}$-function. To state these conjectures we first define some notation. Let $\gamma_{x,y,z^{-1}}$ be the constant term of the polynomial $\mathsf{q}^{-\mathbf{a}(z)} h_{x,y,z}$ for $x, y, z \in \widetilde{W}$. Let $\deg : \mathsf{R} \to \mathbb{Z} \cup \{-\infty\}$ denote the degree of polynomials in q, for example $\deg(\mathsf{q}^{-3} + \mathsf{q}^{-4}) = -3$ and $\deg(0) = -\infty$. Define $n_z \in \mathbb{Z}\backslash\{0\}$ to be the coefficient of the maximal degree of q in $P_{e,z}$. Furthermore, define the set $\mathcal{D} = \{z \in W \mid \mathbf{a}(z) = -\deg(P_{e,z})\}$ whose elements are called *distinguished involutions*. We now state the conjectures; let $z, z' \in \widetilde{W}$.

- P1. For any $z \in \widetilde{W}$, $\mathbf{a}(z) \leq -\deg(P_{e,z})$.
- P2. If $d \in \mathcal{D}$ and $x, y \in \widetilde{W}$ with $\gamma_{x,y,d} \neq 0$, then $x = y^{-1}$.
- P3. If $y \in \widetilde{W}$, there exists a unique $d \in \mathcal{D}$ such that $\gamma_{y^{-1},y,d} \neq 0$.
- P4. If $z' \leq_{LR} z$ then $\mathbf{a}(z') \geq \mathbf{a}(z)$. Therefore, if $z' \sim_{LR} z$, then $\mathbf{a}(z') = \mathbf{a}(z)$.
- P5. If $d \in \mathcal{D}, y \in \widetilde{W}$ and $\gamma_{y,y^{-1},d} \neq 0$, then $\gamma_{y^{-1},y,d} = n_d = \pm 1$.
- P6. If $d \in \mathcal{D}$, then $d^2 = e$.
- P7. For any $x, y, z \in \widetilde{W}$, $\gamma_{x,y,z} = \gamma_{y,z,x}$.
- P8. For $x, y \in \widetilde{W}$ such that $\gamma_{x,y,z} \neq 0$ we have $x^{-1} \sim_R y, y^{-1} \sim_R z, z^{-1} \sim_R x$.

P9. $z' \leq_L z$ and $\mathbf{a}(z') = \mathbf{a}(z)$ implies $z' \sim_L z$.

P10. $z' \leq_R z$ and $\mathbf{a}(z') = \mathbf{a}(z)$ implies $z' \sim_R z$.

P11. $z' \leq_{LR} z$ and $\mathbf{a}(z') = \mathbf{a}(z)$ implies $z' \sim_{LR} z$.

P12. For $J \subseteq I \cup \{0\}$, if $y \in W_J$ then $\mathbf{a}(y)$ computed in W_J is equal to $\mathbf{a}(y)$ computed in $\widetilde{W}$.

P13. Any right cell Γ contains a unique element $d \in \mathcal{D}$ with $\gamma_{x,x^{-1},d} \neq 0$ for all $x \in \Gamma$.

P14. $z \sim_{LR} z^{-1}$.

P15. Let q' be a second indeterminant and let $h'_{x,y,z} \in \mathbb{Z}[\mathsf{q}', \mathsf{q}'^{-1}]$ be obtained from $h_{x,y,z}$ by $\mathsf{q} \mapsto \mathsf{q}'$. If $x, x', y, w \in \widetilde{W}$ such that $\mathbf{a}(w) = \mathbf{a}(y)$, then $\sum_{y'} h'_{w,x',y'} h_{x,y',y} = \sum_{y'} h_{x,w,y'} h'_{y',x',y}$.

Assuming the $\mathbf{a}$-function is well defined, Lusztig proved these conjectures for the equal parameters case in [35, §15] using the positivity properties proved by Elias and Williamson [14]. They also proved the quasi-split case in [35, §16], when W is derived from a larger group and the weight function is taken as the length function on the larger group and then restricted to W. The conjectures have also been proven true when W is the dihedral group, in the infinite case [35, §17] and in the finite case [17, Proposition 5.1]. In [46] the conjectures are proven for the case when the order of st is infinite for all generators s and t of W. Writing all definitions in terms of W_0, the conjectures have been proven for W_0 associated with $\Phi = F_4$ and for a specific case when W_0 is associated with $\Phi = B_n$ (see [17] and [6]). Finally, the conjectures have also been proved for hyperbolic Coxeter groups of rank 3 by Gao and Xie [15] and for Coxeter groups with a complete graph by Xie [50].

In [23] and [22], Guilhot and Parkinson proved the conjectures for the case when the extended affine Weyl group $\widetilde{W}$ is associated with $\Phi = G_2$ and $\Phi = C_2$. To do so they established a balanced system of cell representations (see Section 3.3) and constructed a combinatorial formula for the matrix entries of these representations. This combinatorial formula will be generalised to any affine Weyl group in Chapter 5.

Assuming that the conjectures are true, define $\mathcal{J}$ to be a free abelian group with generators $(\mathsf{t}_w)_{w\in\widetilde{W}}$ and multiplication defined by

$$\mathsf{t}_w \mathsf{t}_u = \sum_{y \in \widetilde{W}} \gamma_{w,u,y^{-1}} \mathsf{t}_y.$$

In [35, §18], Lusztig shows that this multiplication is associative. This ring is called *Lusztig's asymptotic algebra.*

For a two-sided cell Γ denote $\mathcal{J}_\Gamma$ to be the $\mathcal{J}$-subring with generators $(\mathsf{t}_w)_{w\in\Gamma}$.

3.3 Balanced systems of cell representations

In [17] Geck gives a list of conditions on a family of matrix representations of the Hecke algebra defined for finite W_0 which, when satisfied, can be used to check $P1$-$P15$. In [23] and [22] Guilhot and Parkinson extended this criteria to affine Weyl groups and used them, along with the Plancherel Theorem (see Section 3.4), to prove Lusztig's conjectures for $\Phi = G_2$ and $\Phi = C_2$ respectively. In this section we will state the criteria and give some of the results of Guilhot and Parkinson; that when the conditions are met the matrix bound aligns with the $\mathbf{a}$-function and that Lusztig's asymptotic algebra is isomorphic to a specific algebra constructed from the matrix representations.

Recall that $\mathsf{R} = \mathbb{Z}[\mathsf{q}, \mathsf{q}^{-1}]$. Let R' be a polynomial ring of R. Notate $\mathsf{R}'_{\leq 0}$ to be the associated $\mathbb{Z}[\mathsf{q}^{-1}]$ polynomial ring and R'_0 to be the associated $\mathbb{Z}$ polynomial ring.

Let W be a Weyl group with generating set $\{s_i \mid i \in I \cup \{0\}\}$ and associated extended Hecke algebra $\widetilde{\mathcal{H}}$. A *matrix representation* of $\widetilde{\mathcal{H}}$ is a triple (π, M, B) where $\pi : \widetilde{\mathcal{H}} \to M$, M is a right module over an R polynomial ring R' and B is a basis of M. We write $\pi(h)$ for the matrix representation of $h \in \widetilde{\mathcal{H}}$ and $[\pi(h)]_{u,v}$ for its (u, v)-th matrix entry with $u, v \in \mathsf{B}$.

The matrix representation is called *bounded* if there exists $n \in \mathbb{N}$ such that, for all $w \in \widetilde{W}$ and $u, v \in \mathsf{B}$, we have

$$\mathsf{q}^{-n}[\pi(T_w)]_{u,v} \in \mathsf{R}'_0. \tag{3.3.1}$$

The *bound* of (π, M, B) is the minimum n required to bound all matrix entries in this way, denoted $\mathbf{a}_{\pi,M,\mathsf{B}}$. That is

$$\mathbf{a}_{\pi,M,\mathsf{B}} = \min\{n \in \mathbb{N} \mid \mathsf{q}^{-n}[\pi(T_w)]_{u,v} \in \mathsf{R}'_0 \text{ for all } w \in \widetilde{W} \text{ and } u, v \in \mathsf{B}\}. \tag{3.3.2}$$

For a bounded representation (π, M, B) with bound $\mathbf{a}_{\pi,M,\mathsf{B}}$ the *cell recognised* by the representation is the subset of $\widetilde{W}$ defined by

$$\Gamma\pi, M, \mathsf{B} = \{w \in \widetilde{W} \mid \deg([\pi(T_w)]_{u,v}) = \mathbf{a}_{\pi,M,\mathsf{B}} \text{ for some } u, v \in \mathsf{B}\}.$$

Remark 3.3.1. Consider a finite dimensional matrix representation $\pi : \widetilde{\mathcal{H}} \to M$ and two bases B and B' for the representation. If (π, M, B) is bounded then (π, M, B') is also bounded as the finite transition matrix from B to B' has a bound on its entries q-degrees. Therefore, if the representation π is bounded for one basis it is bounded for all. Note, however, that $\mathbf{a}_{\pi,M,\mathsf{B}}$ is not necessarily equal to $\mathbf{a}_{\pi,M,\mathsf{B}'}$. The bound, and hence the recognised cell, are dependent on the chosen basis.

Denote $|_{\mathsf{q}^{-1}=0}$ to be the specialisation of an element of $\mathsf{R}'_{\leq 0}$ into R'_0 by taking $\mathsf{q}^{-1} = 0$. The *leading matrices* of (π, M, B) are

$$\mathfrak{c}_{\pi,M,\mathsf{B}}(w) = (\mathsf{q}^{-\mathbf{a}_{\pi,M,\mathsf{B}}}\pi(C_w))|_{\mathsf{q}^{-1}=0} \tag{3.3.3}$$

for all $w \in \widetilde{W}$. The non-trivial leading matrices are $\{\mathfrak{c}_{\pi,M,\mathsf{B}}(w) \mid w \in \Gamma_{\pi,M,\mathsf{B}}\}$.

Example 3.3.2. Let (π, M, B) be such that $\mathbf{a}_{\pi,M,\mathsf{B}} = 1$ and consider

$$\pi(T_w) = \begin{pmatrix} -\mathsf{q}^{-3} + \mathsf{q}^{-1} & -\mathsf{q}^{-1} & 0 & 0 \\ 0 & -\mathsf{q}^{-3} + 2\mathsf{q}^{-1} - \mathsf{q} & \mathsf{q}^{-2} & \mathsf{q}^{-2} + -1 \\ 0 & \mathsf{q}^{-2} + -1 & 0 & -\mathsf{q}^{-1} \\ \mathsf{q}^{-2} & 0 & 0 & 0 \end{pmatrix}$$

Then, the corresponding leading matrix is

$$\mathfrak{c}_{\pi,M,\mathsf{B}}(w) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The following lemma shows that we can replace C_w by T_w in the definition of the leading matrices.

Lemma 3.3.3. *For* (π, M, B) *bounded, we have* $\mathfrak{c}_{\pi,M,\mathsf{B}}(w) = (\mathsf{q}^{-\mathbf{a}_{\pi,M,\mathsf{B}}}\pi_{J,\mathsf{v}}(T_w))|_{\mathsf{q}^{-1}=0}$.

Proof. Recall from Section 3.2 that

$$C_w = T_w + \sum_{y<w} P_{y,w} T_y$$

with $P_{y,w} \in \mathsf{q}^{-1}\mathbb{Z}[\mathsf{q}^{-1}]$. Multiplying both sides by $\mathsf{q}^{-\mathbf{a}_{\pi,M,\mathsf{B}}}$ and applying π we have

$$\mathsf{q}^{-\mathbf{a}_{\pi,M,\mathsf{B}}}\pi(C_w) = \mathsf{q}^{-\mathbf{a}_{\pi,M,\mathsf{B}}}\pi(T_w) + \sum_{y<w} \mathsf{q}^{-\mathbf{a}_{\pi,M,\mathsf{B}}} P_{y,w}\pi(T_y).$$

As the maximum degree of the entries of $P_{y,w}\pi(T_y)$ is strictly bounded by $\mathbf{a}_{\pi,M,\mathsf{B}}$, the result follows upon specialisation. □

For each two-sided cell, suppose there exists a matrix representation of $\widetilde{\mathcal{H}}$ over an R-polynomial ring. We call this system of representations a *balanced system of cell representations* if the following hold: for two-sided cell Γ with matrix representation $(\pi_\Gamma, M_\Gamma, \mathsf{B}_\Gamma)$ we have

$B1$. (The killing property) If $w \notin \Gamma_{\geq LR}$ then $\pi_\Gamma(C_w) = 0$.

$B2$. (Boundedness) The representation $(\pi_\Gamma, M_\Gamma, \mathsf{B}_\Gamma)$ is bounded with bound $\mathbf{a}_{\pi_\Gamma,M_\Gamma,\mathsf{B}_\Gamma}$.

$B3$. $\mathfrak{c}_{\pi_\Gamma,M_\Gamma,\mathsf{B}_\Gamma}(w) \neq 0$ if and only if $w \in \Gamma$.

$B4$. The elements of $(\mathfrak{c}_{\pi_\Gamma,M_\Gamma,\mathsf{B}_\Gamma}(w))_{w\in\Gamma}$ are free over $\mathbb{Z}$.

$B5$. If $\Gamma' \leq_{LR} \Gamma$ then $\mathbf{a}_{\pi_{\Gamma'},M_{\Gamma'},\mathsf{B}_{\Gamma'}} \geq \mathbf{a}_{\pi_\Gamma,M_\Gamma,\mathsf{B}_\Gamma}$.

Remark 3.3.4. For a two-sided cell Γ, recall the definition of the $\widetilde{\mathcal{H}}$-module $\widetilde{\mathcal{H}}_\Gamma$ from Section 3.2. This representation of $\widetilde{\mathcal{H}}$ satisfies $B1$. To see this, consider C_w with $w \in \widetilde{W}$. If the right action of C_w on $\widetilde{\mathcal{H}}_\Gamma$ is non-trivial, then there exists $x, y \in \Gamma$ such that

$$C_x C_w = \sum_{z\in\widetilde{W}} h_{x,w,z} C_z$$

and $h_{x,w,y} \neq 0$. Therefore, $y \leq_L w$ and so $y \leq_{LR} w$ and $w \in \Gamma_{\geq LR}$ as required. (A similar argument shows that a left action would also satisfy $B1$, in which $h_{w,x,y}$ gives $y \leq_R w$.)

Note that $B2$ with $B3$ is equivalent to: there exists $\mathbf{a}_{\pi_\Gamma,M_\Gamma,\mathsf{B}_\Gamma} \in \mathbb{N}$ such that

$$\max\{\deg[\pi_\Gamma(C_w)]_{u,v} \mid u, v \in \mathsf{B}\} \leq \mathbf{a}_{\pi_\Gamma,M_\Gamma,\mathsf{B}_\Gamma}$$

for all $w \in \widetilde{W}$ with equality if and only if $w \in \Gamma$.

Define $\widetilde{\gamma}_{x,y,z^{-1}} \in \mathbb{Z}$ to be the coefficient of $\mathsf{q}^{\mathbf{a}_{\pi_\Gamma,M_\Gamma,\mathsf{B}_\Gamma}}$ in $h_{x,y,z}$ (note the similarity to the definition of $\gamma_{x,y,z^{-1}}$ from Section 3.2). We define another property:

$\widetilde{B}$. For every $z \in \Gamma$ there exists $x, y \in \Gamma$ such that $\widetilde{\gamma}_{x,y,z^{-1}} \neq 0$.

For a balanced system of cell representations satisfying property $\widetilde{B}$ it has been shown that the matrix bound is Lusztig's $\mathbf{a}$-function and so $\gamma_{x,y,z^{-1}} = \widetilde{\gamma}_{x,y,z^{-1}}$ ([23]). We will state the results and their proofs for completeness.

Let Λ denote the set of two-sided cells for $w \in \widetilde{W}$. Let $(\pi_\Gamma, M_\Gamma, \mathsf{B}_\Gamma)_{\Gamma\in\Lambda}$ be a system of representations satisfying $B1 - B5$ and $\widetilde{B}$.

Theorem 3.3.5. *[23, Proposition 2.4 and Theorem 2.6] For all $w \in \widetilde{W}$ we have $\mathbf{a}(w) = \mathbf{a}_{\pi_\Gamma,M_\Gamma,\mathsf{B}_\Gamma}$.*

Proof. We will first prove $\deg(h_{x,y,z}) \leq \mathbf{a}_{\pi_\Gamma,M_\Gamma,\mathsf{B}_\Gamma}$ by a downwards induction. Note first that, for $x, y \in \widetilde{W}$ we have

$$C_xC_y = \sum_{z\in\widetilde{W}} h_{x,y,z}C_z = \sum_{\Gamma'\in\Lambda}\sum_{z\in\Gamma'} h_{x,y,z}C_z. \tag{3.3.4}$$

For the base case, let $\Gamma = \{e\}$. Applying π_Γ to 3.3.4 we have

$$\pi_\Gamma(C_x)\pi_\Gamma(C_y) = \sum_{\Gamma'\in\Lambda}\sum_{z\in\Gamma'} h_{x,y,z}\pi_\Gamma(C_z) = h_{x,y,e}\pi_\Gamma(C_e). \tag{3.3.5}$$

by $B1$ and as $\Gamma_{\geq_{LR}} = \Gamma$ by Remark 3.2.4. The left side of 3.3.5 is bounded by $2\mathbf{a}_{\pi_\Gamma,M_\Gamma,\mathsf{B}_\Gamma}$. The right hand side is bounded by $\deg(h_{x,y,e}) + \mathbf{a}_{\pi_\Gamma,M_\Gamma,\mathsf{B}_\Gamma}$, proving the base case.

Let Γ be a two-sided cell and suppose that for all two-sided cells Γ' such that $\Gamma' >_{LR} \Gamma$ we have that $\deg(h_{x,y,z}) \leq \mathbf{a}_{\pi_{\Gamma'},M_{\Gamma'},\mathsf{B}_{\Gamma'}}$ for all $z \in \Gamma$. Applying π_Γ to 3.3.4 and using $B1$ we have

$$\begin{aligned}\pi_\Gamma(C_x)\pi_\Gamma(C_y) &= \sum_{\substack{\Gamma'\in\Lambda,\\ \Gamma'>_{LR}\Gamma}}\sum_{z\in\Gamma'} h_{x,y,z}\pi_\Gamma(C_z) + \sum_{\substack{\Gamma'\in\Lambda,\\ \Gamma'\leq_{LR}\Gamma}}\sum_{z\in\Gamma'} h_{x,y,z}\pi_\Gamma(C_z)\\ &= \sum_{\substack{\Gamma'\in\Lambda,\\ \Gamma'>_{LR}\Gamma}}\sum_{z\in\Gamma'} h_{x,y,z}\pi_\Gamma(C_z) + \sum_{z\in\Gamma} h_{x,y,z}\pi_\Gamma(C_z)\end{aligned} \tag{3.3.6}$$

The q degree of the left hand side of 3.3.6 is bounded by $2\mathbf{a}_{\pi_\Gamma,M_\Gamma,\mathsf{B}_\Gamma}$. The degree of the double sum on the right hand side is strictly bounded by $\mathbf{a}_{\pi_{\Gamma'},M_{\Gamma'},\mathsf{B}_{\Gamma'}} + \mathbf{a}_{\pi_\Gamma,M_\Gamma,\mathsf{B}_\Gamma}$ by the inductive hypothesis and $\deg(\pi_\Gamma(C_z)) < \mathbf{a}_{\pi_\Gamma,M_\Gamma,\mathsf{B}_\Gamma}$ for all $z \in \widetilde{W}\backslash\Gamma$ using $B2$ and $B3$. In addition, by $B5$ we have that $\mathbf{a}_{\pi_{\Gamma'},M_{\Gamma'},\mathsf{B}_{\Gamma'}} + \mathbf{a}_{\pi_\Gamma,M_\Gamma,\mathsf{B}_\Gamma} \leq 2\mathbf{a}_{\pi_\Gamma,M_\Gamma,\mathsf{B}_\Gamma}$.

Let $m = \max\{\deg(h_{x,y,z}) \mid z \in \Gamma\}$. By $B3$, for $z \in \Gamma$ such that $\deg(h_{x,y,z}) = m$ we have that

$$\pi_\Gamma(C_z) = \mathsf{q}^{\mathbf{a}_{\pi_\Gamma,M_\Gamma,\mathsf{B}_\Gamma}}\mathfrak{c}_{\pi_\Gamma,M_\Gamma,\mathsf{B}_\Gamma}(z) + \text{ lower degree } \mathsf{q} \text{ terms}$$

with $\mathfrak{c}_{\pi_\Gamma,M_\Gamma,\mathsf{B}_\Gamma}(z) \neq 0$. Therefore, the right hand side of 3.3.6 becomes

$$\mathsf{q}^{m+\mathbf{a}_{\pi_\Gamma,M_\Gamma,\mathsf{B}_\Gamma}}\sum_z \widetilde{\gamma}_{x,y,z^{-1}}\mathfrak{c}_{\pi_\Gamma,M_\Gamma,\mathsf{B}_\Gamma}(z) + \text{ lower degree } \mathsf{q} \text{ terms}$$

with the sum over $z \in \Gamma$ such that $\deg(h_{x,y,z}) = m$ and $\widetilde{\gamma}_{x,y,z^{-1}} \in \mathbb{Z}$. By $B4$, the leading matrices $\mathfrak{c}_{\pi_\Gamma,M_\Gamma,\mathsf{B}_\Gamma}(w)$ for $w \in \Gamma$ are free over $\mathbb{Z}$, so the coefficient of $\mathsf{q}^{m+\mathbf{a}_{\pi_\Gamma,M_\Gamma,\mathsf{B}_\Gamma}}$ in the above sum is non-negative. Due to the bound on the left hand side of 3.3.4, we have that $m + \mathbf{a}_{\pi_\Gamma,M_\Gamma,\mathsf{B}_\Gamma} \leq 2\mathbf{a}_{\pi_\Gamma,M_\Gamma,\mathsf{B}_\Gamma}$, so $m \leq \mathbf{a}_{\pi_\Gamma,M_\Gamma,\mathsf{B}_\Gamma}$ as required.

Therefore, $\mathbf{a}(w) \leq \mathbf{a}_{\pi_\Gamma,M_\Gamma,\mathsf{B}_\Gamma}$, so all that remains to prove is that the maximum degree of $h_{x,y,z}$ does reach $\mathbf{a}_{\pi_\Gamma,M_\Gamma,\mathsf{B}_\Gamma}$. This follows immediately from $\widetilde{B}$. $\square$

Let $\widetilde{\mathcal{J}}_\Gamma$ be the subset of M_Γ generated by $(\mathfrak{c}_{\pi_\Gamma, M_\Gamma, \mathsf{B}_\Gamma}(w))_{w\in\Gamma}$.

Corollary 3.3.6. *[23, Corollary 2.5 and Corollary 2.7] For $\Gamma \in \Lambda$, the set $\widetilde{\mathcal{J}}_\Gamma$ is a ring and is isomorphic $\mathcal{J}_\Gamma$.*

Proof. Let $x, y \in \Gamma$ and let $\mathbf{a}_{\pi_\Gamma} = \mathbf{a}_{\pi_\Gamma, M_\Gamma, \mathsf{B}_\Gamma}$. Multiplying both sides of 3.3.6 by $\mathsf{q}^{-\mathbf{a}_{\pi_\Gamma}}$, we have

$$\mathsf{q}^{-\mathbf{a}_{\pi_\Gamma}}\pi_\Gamma(C_x)\mathsf{q}^{-\mathbf{a}_{\pi_\Gamma}}\pi_\Gamma(C_y) = \sum_{\substack{\Gamma'\in\Lambda,\\ \Gamma'>_{LR}\Gamma}}\sum_{z\in\Gamma'}\mathsf{q}^{-\mathbf{a}_{\pi_\Gamma}}h_{x,y,z}\mathsf{q}^{-\mathbf{a}_{\pi_\Gamma}}\pi_\Gamma(C_z) + \sum_{z\in\Gamma}\mathsf{q}^{-\mathbf{a}_{\pi_\Gamma}}h_{x,y,z}\mathsf{q}^{-\mathbf{a}_{\pi_\Gamma}}\pi_\Gamma(C_z)$$

Upon specialising, the double sum vanishes by $B3$ and the equation becomes

$$\mathfrak{c}_{\pi_\Gamma, M_\Gamma, \mathsf{B}_\Gamma}(x)\mathfrak{c}_{\pi_\Gamma, M_\Gamma, \mathsf{B}_\Gamma}(y) = \sum_{z\in\Gamma}\widetilde{\gamma}_{x,y,z^{-1}}\mathfrak{c}_{\pi_\Gamma, M_\Gamma, \mathsf{B}_\Gamma}(z). \tag{3.3.7}$$

By Theorem 3.3.5, we have that $\mathbf{a}_{\pi_\Gamma} = \mathbf{a}(w)$ for all $w \in \Gamma$. By the definition of $\gamma_{x,y,z^{-1}}$ and $\widetilde{\gamma}_{x,y,z^{-1}}$, this implies that $\gamma_{x,y,z^{-1}} = \widetilde{\gamma}_{x,y,z^{-1}}$. Let ψ denote the $\mathbb{Z}$-linear map such that $\psi(\mathsf{t}_w) = \mathfrak{c}_{\pi_\Gamma, M_\Gamma, \mathsf{B}_\Gamma}(w)$. By (3.3.7) and as $\gamma_{x,y,z^{-1}} = \widetilde{\gamma}_{x,y,z^{-1}}$ we have that ψ is a surjective homomorphism of unital rings. Furthermore as the leading matrices are free over $\mathbb{Z}$ (by $B3$), if $\psi\left(\sum_{w\in\Gamma} a_w \mathsf{t}_w\right) = \sum_{w\in\Gamma} a_w \mathfrak{c}_{\pi_\Gamma, M_\Gamma, \mathsf{B}_\Gamma}(w) = 0$ then $a_w = 0$ for all $w \in \Gamma$. Thus, ψ is an isomorphism, completing the proof. □

3.4 The trace function and the Plancherel Theorem

There exists a canonical trace function on $\widetilde{\mathcal{H}}$. This trace function restricted to elements in $\mathcal{H}_0$ decomposes as a sum over characters of irreducible representations of $\mathcal{H}_0$. In [40] Opdam constructed an affine analogue to this decomposition called the Plancherel Theorem. In this section we define the trace function, explore its properties and briefly introduce the Plancherel formula (which will be further described in Section 6.2). We also define a trace function on Lusztig's asymptotic algebra $\mathcal{J}$, of which we will find a decomposition (the asymptotic Plancherel Theorem) for type A_n in Chapter 10.

Let $\mathrm{Tr} : \widetilde{\mathcal{H}} \to \mathsf{R}$ denote the canonical trace function on $\widetilde{\mathcal{H}}$, the R-linear extension of $\mathrm{Tr}(T_w) = \delta_{w,e}$. That is, Tr is defined as

$$\mathrm{Tr}\left(\sum_{w\in\widetilde{W}} a_w T_w\right) = a_e.$$

Denote $*$ to be the anti-involution on $\widetilde{\mathcal{H}}$ defined as follows:

$$\left(\sum_{w\in\widetilde{W}} a_w T_w\right)^* = \sum_{w\in\widetilde{W}} a_w T_{w^{-1}}.$$

We have the following property of the trace function.

Lemma 3.4.1. *For $u, v \in \widetilde{W}$ we have $\mathrm{Tr}(T_u T_v^*) = \delta_{u,v}$ and thus $\mathrm{Tr}(h_1 h_2) = \mathrm{Tr}(h_2 h_1)$ for all $h_1, h_2 \in \widetilde{\mathcal{H}}$.*

Proof. To prove $\mathrm{Tr}(T_uT_v^*) = \delta_{u,v}$ we proceed by induction on $\ell(v)$. When $\ell(v) = 0$ the statement is true by the definition of the trace function. Assume that $\mathrm{Tr}(T_uT_v^*) = \delta_{u,v}$ for $\ell(v)$ and all $u \in \widetilde{W}$. Let $v' = vs$ for some $s \in S$ such that $vs > v$. Then

$$\mathrm{Tr}(T_uT_{v'}^*) = \mathrm{Tr}(T_uT_sT_{v^{-1}}) = \begin{cases} \mathrm{Tr}(T_{us}T_{v^{-1}}) & \text{if } us > u \\ \mathrm{Tr}(T_{us}T_{v^{-1}}) + (\mathsf{q}_s - \mathsf{q}_s^{-1})T_uT_{v^{-1}}) & \text{if } us < u \end{cases}$$

by (3.1.1). If $u = v$ then $us \not< u$ and so $\mathrm{Tr}(T_{us}T_{v^{-1}}) = \mathrm{Tr}(T_{us}T_{v^{-1}}) = \delta_{us,v} = \delta_{u,vs}$ by the inductive hypothesis.

If $u \neq v$ then $\mathrm{Tr}(T_uT_{v^{-1}}) = 0$ and so, again, $\mathrm{Tr}(T_{us}T_{v^{-1}}) = \mathrm{Tr}(T_{us}T_{v^{-1}}) = \delta_{us,v} = \delta_{u,vs}$ by the inductive hypothesis.

Let $h_1 = \sum_{w\in\widetilde{W}} a_wT_w$ and $h_2 = \sum_{u\in\widetilde{W}} b_uT_u$. Then

$$\begin{aligned}\mathrm{Tr}(h_1h_2) &= \mathrm{Tr}\left(\sum_{u,w\in\widetilde{W}} a_wb_uT_wT_u\right) \\ &= \sum_{u,w\in\widetilde{W}} a_wb_u\mathrm{Tr}(T_wT_u) \\ &= \sum_{u,w\in\widetilde{W}} a_wb_u\delta_{w,u^{-1}} \\ &= \sum_{u,w\in\widetilde{W}} b_ua_w\delta_{u,w^{-1}} = \mathrm{Tr}(h_2h_1).\end{aligned}$$

□

Define $\langle\cdot,\cdot\rangle : \widetilde{\mathcal{H}} \times \widetilde{\mathcal{H}} \to \mathsf{R}$ to be the symmetric bilinear form defined by $\langle h_1, h_2\rangle = \mathrm{Tr}(h_1h_2^*)$ for $h_1, h_2 \in \widetilde{\mathcal{H}}$. As

$$\left\langle \sum_{w\in\widetilde{W}} a_wT_w, \sum_{u\in\widetilde{W}} a_uT_u \right\rangle = \sum_{w,u\in\widetilde{W}} a_wa_u\mathrm{Tr}(T_wT_{u^{-1}}) = \sum_{u,w\in\widetilde{W}} a_wa_u\delta_{w,u} = \sum_{w\in\widetilde{W}} a_w^2$$

$\langle\cdot,\cdot\rangle$ is an inner product on $\widetilde{\mathcal{H}}$.

We note the following extrapolation of degree in q to non-zero rational functions of q. The function $f(\mathsf{q}) = \frac{a(\mathsf{q})}{b(\mathsf{q})} \neq 0$ can be rewritten as $f(\mathsf{q}) = \mathsf{q}^N\frac{a'(\mathsf{q}^{-1})}{b'(\mathsf{q}^{-1})}$ where $a'(\mathsf{q}^{-1})$ and $b'(\mathsf{q}^{-1})$ are polynomials in q^{-1} with non-zero constant term and $N \in \mathbb{Z}$ unique. We call N the *degree* of $f(\mathsf{q})$, denoted as $\deg f(\mathsf{q}) = N$. For example,

$$\deg\left(\frac{(\mathsf{q}^2+1)(\mathsf{q}^3+1)}{\mathsf{q}^7-\mathsf{q}+1}\right) = -2.$$

Note that when $b(\mathsf{q}) = 1$, $\deg f(\mathsf{q})$ agrees with the usual definition of degree.

Recall that $\mathcal{H}_0$ is the finite Hecke algebra associated with Φ, that is the subalgebra of $\mathcal{H}$ generated by $\{T_w \mid w \in W_0\}$. The trace function (and thus the inner product) restricted to $\mathcal{H}_0$ decomposes as

$$\mathrm{Tr}(h) = \sum_{\pi\in\mathrm{Irrep}(\mathcal{H}_0)} m_\pi\chi_\pi(h) \tag{3.4.1}$$

for all $h \in \mathcal{H}_0$, where Irrep($\mathcal{H}_0$) are the irreducible representations of $\mathcal{H}_0$ and the values m_π are called the *generic degrees* of $\mathcal{H}_0$ (see [18, §8.1.8 and Chapter 11]). This decomposition can be particularly useful as there is a connection between the generic degrees and Lusztig's **a**-function; for $w \in \widetilde{W}$ in the cell associated to π we have $-\deg m_\pi = 2\mathbf{a}(\mathbf{w})$ (see [17]). In [17] Geck used this to prove $P1$-$P15$ for the finite Hecke algebra associated with $\Phi = F_4$.

Example 3.4.2. Let $\Phi = B_2$ and let W_0 denote the associated finite Weyl group with generating set $\{s_1, s_2\}$. We set $L(s_1) = b$ and $L(s_2) = a$, so $\mathsf{q}_1 = \mathsf{q}^b$ and $\mathsf{q}_2 = \mathsf{q}^a$.

Let $\mathcal{H}_0$ be the finite Hecke algebra generated by

$$\{T_w \mid w \in W_0\} = \{T_e, T_1, T_2, T_{12}, T_{21}, T_{121}, T_{121}, T_{1212}\}$$

(for simplicity we denote elements of W_0 by their indices, for example $s_1s_2s_1$ as 121). The following are the irreducible representations of $\mathcal{H}_0$:

$$\begin{aligned}
&\pi_0 : T_1 \mapsto \mathsf{q}^b,\ T_2 \mapsto \mathsf{q}^a \\
&\pi_1 : T_1 \mapsto \mathsf{q}^b,\ T_2 \mapsto -\mathsf{q}^{-a} \\
&\pi_2 : T_1 \mapsto -\mathsf{q}^{-b},\ T_2 \mapsto \mathsf{q}^a \\
&\pi_3 : T_1 \mapsto -\mathsf{q}^{-b},\ T_2 \mapsto -\mathsf{q}^{-a} \\
&\pi_4 : T_1 \mapsto \begin{bmatrix} -\mathsf{q}^{-b} & 0 \\ -c_{12}\mathsf{q}^{-b} & \mathsf{q}^b \end{bmatrix},\ T_2 \mapsto \begin{bmatrix} \mathsf{q}^a & -c_{21}\mathsf{q}^{-a} \\ 0 & -\mathsf{q}^{-a} \end{bmatrix}
\end{aligned}$$

where $c_{12}, c_{21} \in \mathsf{R}$ such that $c_{12}c_{21} = \mathsf{q}^{2a} + \mathsf{q}^{2b}$. See [18, §8.1.9] for the construction and proof of the irreducibility of π_4, but note that Geck and Pfeiffer use a different normalisation of the Hecke algebra where the quadratic relation is $\hat{T_s}^2 = v_s\hat{T_s} + (v_s - 1)\hat{T_e}$ with $\hat{T_w}$ the generators of their Hecke algebra (to change between the two normalisations take $v_s \mapsto q_s^2$ and $\hat{T_w} \mapsto q_wT_w$).

Following [18, §8.1.8], define

$$\langle \chi_\pi, \chi_\mu \rangle = \sum_{w \in W_0} \chi_\pi(T_w)\chi_\mu(T_{w^{-1}})$$

for $\pi, \mu \in \mathrm{Irrep}(\mathcal{H}_0)$. Importantly, we have that $\langle \chi_\pi, \chi_\mu \rangle = 0$ unless $\chi_\pi = \chi_\mu$. Hence,

$$\chi_\mu(T_e) = \sum_{w \in W_0} \mathrm{Tr}(T_w)\chi_\mu(T_{w^{-1}}) = \langle \mathrm{Tr}, \chi_\mu \rangle = \sum_{\pi \in \mathrm{Irrep}(\mathcal{H}_0)} m_\pi \langle \chi_\pi, \chi_\mu \rangle = m_\mu \langle \chi_\mu, \chi_\mu \rangle$$

and the generic degrees are given by

$$m_\mu = \frac{\chi_\mu(T_e)}{\langle \chi_\mu, \chi_\mu \rangle}. \tag{3.4.2}$$

Denote $m_i = m_{\pi_i}$. By direct calculation using 3.4.2 we have

$$\begin{aligned}
m_0 &= \frac{1}{\mathsf{q}^{4a+4b}(\mathsf{q}^{-4a-4b} + \mathsf{q}^{-2a-4b} + \mathsf{q}^{-4a-2b} + 2\mathsf{q}^{-2a-2b} + \mathsf{q}^{-2b} + \mathsf{q}^{-2a} + 1)} \\
m_1 &= \frac{1}{\mathsf{q}^{2b}(\mathsf{q}^{-2b} + 1 + \mathsf{q}^{-2a-2b} + 2\mathsf{q}^{-2a} + \mathsf{q}^{-4a} + \mathsf{q}^{-2a+2b} + \mathsf{q}^{-4a+2b})} \\
m_2 &= \frac{1}{\mathsf{q}^{2a}(\mathsf{q}^{-2a} + 1 + \mathsf{q}^{-2b-2a} + 2\mathsf{q}^{-2b} + \mathsf{q}^{-4b} + \mathsf{q}^{2a-2b} + \mathsf{q}^{2a-4b})} \\
m_3 &= \frac{1}{1 + \mathsf{q}^{-2a} + \mathsf{q}^{-2b} + 2\mathsf{q}^{-2a-2b} + \mathsf{q}^{-2a-4b} + \mathsf{q}^{-4a-2b} + \mathsf{q}^{-4a-4b}} \\
m_4 &= \frac{2}{\mathsf{q}^{2a}(4\mathsf{q}^{-2a} + \mathsf{q}^{2b-2a} + \mathsf{q}^{-2b-2a} + 1 + \mathsf{q}^{-4a} + 2\mathsf{q}^{-a+b} - 2\mathsf{q}^{-a-b} - 2\mathsf{q}^{b-3a} + 2\mathsf{q}^{-3a-b})} \\
&= \frac{2}{\mathsf{q}^{2b}(4\mathsf{q}^{-2b} + 1 + \mathsf{q}^{-4b} + \mathsf{q}^{2a-2b} + \mathsf{q}^{-2a-2b} + 2\mathsf{q}^{a-b} - 2\mathsf{q}^{a-3b} - 2\mathsf{q}^{-b-a} + 2\mathsf{q}^{-a-3b})}
\end{aligned}$$

and so the decomposition of the trace function is

$$\mathrm{Tr}(h) = \sum_{i=0}^{4} m_i \chi_{\pi_i}.$$

The degree of each generic degree depends on the chosen weight function. In this example, for all $a, b \in \mathbb{N}$ we have $\deg m_0 = -4a - 4b$ and $\deg m_3 = 0$. The remaining generic degrees split into cases;

$$\deg m_1 = \begin{cases} -2b & \text{if } a \geq b \\ -4b + 2a & \text{if } a < b \end{cases}$$

$$\deg m_2 = \begin{cases} 4a - 2b & \text{if } a > b \\ 2a & \text{if } a \leq b \end{cases}$$

$$\deg m_4 = \begin{cases} 2a & \text{if } a \geq b \\ 2b & \text{if } a \leq b \end{cases}$$

We want an analogue for this trace decomposition in the affine case, for the trace function on all $h \in \widetilde{\mathcal{H}}$. This analogue comes in the form of the Plancherel formula of Opdam [40]. The summation is replaced with an integral and the generic degrees become a measure. Let $\widetilde{\mathcal{H}}_{\mathbb{C}}$ be the extension of $\widetilde{\mathcal{H}}$ so that scalars are now in $\mathbb{C}$ and we specialise q to a real number $q > 1$. Thus, $\widetilde{\mathcal{H}}_{\mathbb{C}}$ is now an algebra over $\mathbb{C}$. In addition, we redefine the anti-involution $*$ so that

$$\left(\sum_{w \in \widetilde{W}} a_w T_w \right)^* = \sum_{w \in \widetilde{W}} \overline{a_w} T_{w^{-1}}$$

where $\overline{a_w}$ denotes complex conjugation.

Following [41, §2.4], let $||h||_2 = \sqrt{\langle h, h \rangle}$ for all $h \in \widetilde{\mathcal{H}}_{\mathbb{C}}$, where $\langle \cdot, \cdot \rangle$ is the inner product defined by the trace function now extended to $\widetilde{\mathcal{H}}_{\mathbb{C}}$. We have that $\widetilde{\mathcal{H}}_{\mathbb{C}}$ acts on itself by left multiplication. The corresponding operator norm is defined by

$$||h|| = \sup\{||hx||_2 \mid x \in \widetilde{\mathcal{H}}_{\mathbb{C}} \text{ and } ||x||_2 \leq 1\}.$$

Let $\overline{\mathcal{H}}_{\mathbb{C}}$ denote the completion of $\widetilde{\mathcal{H}}_{\mathbb{C}}$ with respect to this operator norm. Thus, $\overline{\mathcal{H}}_{\mathbb{C}}$ is a non-commutative C^*-algebra.

The irreducible representations of $\widetilde{\mathcal{H}}_{\mathbb{C}}$ are the irreducible representations of $\widetilde{\mathcal{H}}$, extending scalars and specialising $\mathsf{q} \mapsto q$. The irreducible representations of $\overline{\mathcal{H}}_{\mathbb{C}}$ are the extensions of the irreducible representations of $\widetilde{\mathcal{H}}_{\mathbb{C}}$ that are continuous with respect to the ℓ_2-operator norm. By [40, Corollary 6.2] these are the irreducible tempered representations of $\widetilde{\mathcal{H}}$ (see [40, §2.6] for the definition of tempered).

It is well known that every irreducible representation of $\widetilde{\mathcal{H}}$ is finite (specifically all irreducible representations have degree at most $|W_0|$). By the theory of C^*-algebras (see [13, §8.8]) there exists a unique positive Borel measure μ such that

$$\mathrm{Tr}(h) = \int_{\mathrm{Irrep}(\overline{\mathcal{H}}_{\mathbb{C}})} \chi_\pi(h)d\mu(\pi) \tag{3.4.3}$$

for all $h \in \overline{\mathcal{H}}_{\mathbb{C}}$. The measure μ is called the *Plancherel measure* and is the affine analogue of the generic degrees in (3.4.1). See Example 6.2.2 for an explicit example of this decomposition.

Recall from Section 3.2 that $\mathcal{J}$ denotes Lusztig's asymptotic algebra, a free abelian group with generators $(\mathsf{t}_w)_{w\in\widetilde{W}}$. Also from Section 3.2, recall the definitions of the set of distinugished involutions $\mathcal{D}$, the integers n_d (for $d \in \mathcal{D}$) and the constant term $\gamma_{x,y,z^{-1}}$ (for $x, y, z \in \widetilde{W}$). We write $d(w)$ for the unique element of $\mathcal{D}$ in the right cell containing w (see $P13$ from Section 3.2). By $P5$ we have $n_{d(x)} = \gamma_{x,x^{-1},d(x)} = \pm 1$.

Definition 3.4.3. Define a linear map $\mathrm{Tr}^\infty : \mathcal{J} \to \mathbb{Z}$ by

$$\mathrm{Tr}^\infty\left(\sum_{w\in\widetilde{W}} a_w T_w\right) = \sum_{d\in\mathcal{D}} n_d a_d.$$

Define a bilinear form $\langle\cdot,\cdot\rangle^\infty : \mathcal{J} \times \mathcal{J} \to \mathbb{Z}$ by

$$\langle A, B\rangle^\infty = \mathrm{Tr}^\infty(AB^*)$$

where $A, B \in \mathcal{J}$ and * is defined by $\left(\sum_{w\in\widetilde{W}} a_w \mathsf{t}_w\right)^* = \sum_{w\in\widetilde{W}} a_w \mathsf{t}_{w^{-1}}$.

Theorem 3.4.4. *We have the following properties:*

(1) *The linear map* Tr^∞ *is a trace functional on* $\mathcal{J}$.
(2) *The bilinear form* $\langle\cdot,\cdot\rangle^\infty$ *is an inner product and* $(\mathsf{t}_w)_{w\in W}$ *is an orthonormal basis of* $\mathcal{J}$.
(3) *For* $A, B, C \in \mathcal{J}$ *we have* $\langle AB, C\rangle^\infty = \langle B, A^*C\rangle^\infty$.

Proof. By definition $\mathsf{t}_x\mathsf{t}_y^* = \sum_{z\in\widetilde{W}} \gamma_{x,y^{-1},z^{-1}}\mathsf{t}_z$, and so

$$\mathrm{Tr}^\infty(\mathsf{t}_x\mathsf{t}_y^*) = \sum_{d\in\mathcal{D}} n_d\gamma_{x,y^{-1},d}.$$

If $x \neq y$, then by $P2$, $\gamma_{x,y^{-1},d} = 0$ for all $d \in \mathcal{D}$ and so $\mathrm{Tr}^\infty(\mathsf{t}_x\mathsf{t}_y^*) = 0$. If $x = y$ then by $P3$, $P5$ and $P13$ we have

$$\mathrm{Tr}^\infty(\mathsf{t}_x\mathsf{t}_y^*) = n_{d(x)}\gamma_{x,x^{-1},d(x)} = n_{d(x)}^2 = 1.$$

Thus, $\mathrm{Tr}^\infty(\mathsf{t}_x\mathsf{t}_y^*) = \delta_{x,y}$ for all $x, y \in \widetilde{W}$.

Let $A, B \in \mathcal{J}$ with $A = \sum_{x\in\widetilde{W}} a_x \mathsf{t}_x$ and $B = \sum_{y\in\widetilde{W}} b_y \mathsf{t}_y$. Then

$$AB^* = \sum_{z\in\widetilde{W}} \sum_{x,y\in\widetilde{W}} a_x b_y \gamma_{x,y^{-1},z^{-1}} t_z.$$

Applying the trace and using $\mathrm{Tr}^{\infty}(\mathsf{t}_x\mathsf{t}_y^*) = \delta_{x,y}$ we have

$$\begin{aligned}\mathrm{Tr}^{\infty}(AB^*) &= \sum_{d\in\mathcal{D}} n_d \sum_{x,y\in\widetilde{W}} a_x b_y \gamma_{x,y^{-1},d} \\ &= \sum_{x,y\in\widetilde{W}} a_x b_y \sum_{d\in\mathcal{D}} n_d \gamma_{x,y^{-1},d} \\ &= \sum_{x,y\in\widetilde{W}} a_x b_y \mathrm{Tr}^{\infty}(\mathsf{t}_x\mathsf{t}_y^*) \\ &= \sum_{x\in\widetilde{W}} a_x b_x \\ &= \mathrm{Tr}^{\infty}(B^*A)\end{aligned}$$

as $\mathrm{Tr}^{\infty}(\mathsf{t}_x\mathsf{t}_y^*) = \delta_{x,y} = \delta_{x^{-1},y^{-1}} = \mathrm{Tr}^{\infty}(\mathsf{t}_y^*\mathsf{t}_x)$. Therefore, Tr^{∞} is a trace functional and as

$$\langle A, A\rangle^{\infty} = \sum_{w\in\widetilde{W}} a_w^2$$

the bilinear form $\langle\cdot,\cdot\rangle^{\infty}$ is an inner product. As $\mathrm{Tr}^{\infty}(t_x t_y^*) = \delta_{x,y}$ we have that $(\mathsf{t}_w)_{w\in\widetilde{W}}$ is an orthonomoral basis of $\mathcal{J}$.

For (3), let $x, y, z \in \widetilde{W}$. We have

$$\begin{aligned}\langle \mathsf{t}_x\mathsf{t}_y, \mathsf{t}_z\rangle^{\infty} &= \left\langle \sum_{w\in\widetilde{W}} \gamma_{x,y,w^{-1}} \mathsf{t}_w, \mathsf{t}_z \right\rangle^{\infty} \\ &= \sum_{w\in\widetilde{W}} \gamma_{x,y,w^{-1}} \langle \mathsf{t}_w, \mathsf{t}_z\rangle^{\infty} \\ &= \gamma_{x,y,z^{-1}}\end{aligned}$$

Similarly, we have that $\langle \mathsf{t}_y, \mathsf{t}_x^*\mathsf{t}_z\rangle = \left\langle \mathsf{t}_y, \sum_{w\in\widetilde{W}} \gamma_{x^{-1},z,w^{-1}} \mathsf{t}_w \right\rangle^{\infty} = \gamma_{x^{-1},y,z^{-1}}$. The two are equal since

$$\gamma_{x^{-1},z,y^{-1}} = \gamma_{z,y^{-1},x^{-1}} = \gamma_{y^{-1},x^{-1},z} = \gamma_{x,y,z^{-1}}$$

by $\gamma_{x,y,z} = \gamma_{x^{-1},y^{-1},z^{-1}}$ (as $h_{x,y,z} = h_{x^{-1},y^{-1},z^{-1}}$, by expanding $(C_xC_y)^* = C_y^*C_x^*$ on either side) and $\gamma_{x,y,z} = \gamma_{y,z,x}$ by $P7$. The result follows by linear extension. □

The Plancherel Theorem gives a spectral decomposition of $\langle\cdot,\cdot\rangle$ (stated in (3.4.3)). We want an analogue to this decomposition for the inner product $\langle\cdot,\cdot\rangle^{\infty}$ on Lusztig's asymptotic algebra. This decomposition is called the *asymptotic Plancherel Theorem* and was introduced by Guilhot and Parkinson in [23, §8.3, §8.6] and [22, §7.3, §7.6]. See Section 10.2 for this decomposition for $\Phi = A_n$.

Chapter 4

Alcove paths

In [42] Ram defined alcove walks, a sequence of crossings between alcoves in $\mathbb{A}$ and describes how these paths between alcoves are connected to the elements of the Berstein-Lusztig basis $\{X_w \mid w \in \widetilde{W}\}$ of $\widetilde{\mathcal{H}}$. Ram explicitly described the multiplication between elements of this basis with the standard basis $\{T_w \mid w \in \widetilde{W}\}$ using positively folded alcove paths (positive in terms of hyperplane orientation). In [23] and [22], Guilhot and Parkinson defined a J-analogue to positively folded alcove paths for the cases when $\Phi = G_2$ and $\Phi = C_2$ respectively, and in each case proved a J-analogue to the multiplication formula of Ram. In this chapter we describe positively folded alcove paths and their J-analogue in preparation to prove the path formula (the J-analogue of Rams multiplication formula) in Chapter 5 for any reduced root system Φ. Section 4.1 describes the background including the definition of positively folded alcove paths, the Berstein-Lusztig basis of $\widetilde{\mathcal{H}}$ and the multiplication formula of Ram. In Section 4.2 we define the J-analogue of positively folded alcove paths for general reduced Φ as a sequence of crossings between alcoves restricted to the fundamental J-alcove $\mathcal{A}_J$ from Chapter 2. These paths are called J-folded alcove paths.

The multiplication formula for X_u and T_w of Ram gives the multiplication as a sum, over a set of positively folded paths, of Berstein-Lusztig elements with coefficients $\mathcal{Q}(p)$, dependent on the positively folded path p. In Section 4.3 we construct a J-analogue to $\mathcal{Q}(p)$ that will be the coefficient in the path formula of Chapter 5, where we will now sum over J-folded alcove paths. First we construct a family of parameters v from the elements of Φ_J. With this parameter family we define the new coefficient called the v-mass of a J-folded alcove path p, denoted $\mathcal{Q}_{J,\mathsf{v}}(p)$. The remainder of the section consists of stating and proving properties of v and $\mathcal{Q}_{J,\mathsf{v}}(p)$. These properties will be used to construct a right $\widetilde{\mathcal{H}}$-module (Theorem 5.1.4) and to prove the J-analogue of Rams multiplication formula (Theorem 5.3.2) in Chapter 5.

4.1 Positively folded alcove paths and the Berstein-Lusztig basis

In this section we introduce the notion of positively folded paths between alcoves of $\mathbb{A}$, following Rams alcove walk model in [42]. With the concept of alcove paths we then introduce the Berstein-Lusztig presentation of $\widetilde{\mathcal{H}}$ accompanied with the Berstein-Lusztig basis $\{X_w \mid w \in \widetilde{W}\}$. Finally, we explicitly describe the multiplication of $X_u T_w$ ($u, w \in \widetilde{W}$) in terms of positively folded alcove paths.

Let $\widetilde{W}$ be as in Section 1.2 with root system Φ. Recall that A_0 denotes the fundamental alcove in the set of alcoves $\mathbb{A}$ (see Section 1.3). In addition, recall the orientation notation between alcoves from Section 1.3. For $A, A' \in \mathbb{A}$, the notation $A^-|^+A'$ dictates that A is on the negative side and A' is on the positive side of the hyperplane containing the shared panel of A and A'.

Definition 4.1.1. Let $\vec{w} = s_{i_1}s_{i_2}\dots s_{i_\ell}\sigma$ be an expression for $w \in \widetilde{W}$ (not necessarily reduced) with $\sigma \in \Sigma$ and $i_1, \dots, i_l \in I \cup \{0\}$. A *positively folded alcove path* of type $\vec{w}$ starting at $v_0 \in \widetilde{W}$ is a sequence $p = (v_0, v_1, \dots, v_\ell, v_\ell\sigma)$ where $v_1, \dots v_\ell \in \widetilde{W}$, satisfying the following conditions: for $1 \leq k \leq l$

(1) $v_k \in \{v_{k-1}, v_{k-1}s_{i_k}\}$, and

(2) if $v_{k-1} = v_k$ then $v_{k-1}A_0{}^+|^-v_{k-1}s_{i_k}A_0$.

A positively folded path is a sequence of elements of $\widetilde{W}$ but can also be interpreted as a sequence of alcoves. Let $p = (v_0, v_1, \dots, v_\ell, v_\ell\sigma)$ be a positively folded path, then $(v_0A_0, v_1A_0, \dots, v_lA_0, v_\ell\sigma A_0)$ is the corresponding alcove sequence where v_kA_0 is either adjacent or equal to $v_{k-1}A_0$ for all $1 \leq k \leq l$. Due to this correspondence, we can visualise alcove paths as a path of arrows between alcoves in $\mathbb{A}$. To do so we define the following terms:

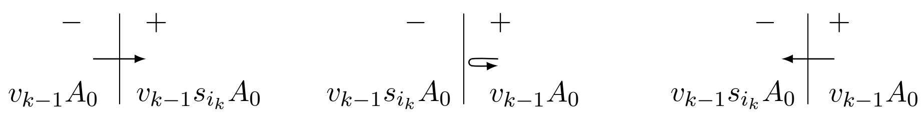
Let $\vec{w} = s_{i_1}\dots s_{i_\ell}\sigma$ and let $p = (v_0, \dots, v_\ell\sigma)$ be a positively folded alcove path of type $\vec{w}$. The index $k \in \{1, 2, \dots, l\}$ is called

(1) a *positive i_k-crossing* if $v_k = v_{k-1}s_{i_k}$ and $v_{k-1}A_0{}^-|^+v_kA_0$,

(2) a *negative i_k-crossing* if $v_k = v_{k-1}s_{i_k}$ and $v_{k-1}A_0{}^+|^-v_kA_0$,

(3) an *i_k-fold* if $v_k = v_{k-1}$ (in which case it is forced that $v_{k-1}A_0{}^+|^-v_{k-1}s_{i_k}A_0$).

The corresponding arrows defined between alcoves are the following:



positive i_k-crossing i_k-fold   negative i_k-crossing

With this notation, the path p can be visualised as a string of arrows, joined tip to tail, between alcoves. Each arrow goes between $v_{k-1}A_0$ and v_kA_0 for some $1 \leq k \leq l$.

For $w, v \in \widetilde{W}$, denote

$$\mathcal{P}(\vec{w}, v) = \{\text{all positively folded paths of type } \vec{w} \text{ starting at } v\},$$

dependent on the choice of expression $\vec{w}$ of w.

From Section 1.2 recall the definitions of $\mathrm{wt}(w)$ and $\theta(w)$ for $w \in \widetilde{W}$. We define the following properties of a path $p = (v_0, v_1, v_2, \dots, v_\ell, v_\ell\sigma) \in \mathcal{P}(\vec{w}, v_0)$:

– the *start* of p is $\mathrm{start}(p) = v_0$.

– the *end* of p is $\mathrm{end}(p) = v_\ell\sigma$.

– the *length* of p is ℓ.

– the *coweight* of p is $\mathrm{wt}(p) = \mathrm{wt}(\mathrm{end}(p))$.

– the *final direction* of p is $\theta(p) = \theta(\text{end}(p))$.

For $p \in \mathcal{P}(\vec{w}, v)$ we notate the number of type i folds for each $i \in I \cup \{0\}$ as $f_i(p)$. Then define

$$f(p) = \#(\text{folds in } p) = \sum_{i=0}^{n} f_i(p).$$

If $f(p) = 0$ then p is a *straight path*.

Example 4.1.2. Two positively folded alcove paths are displayed in Figure 4.1, for $\Phi = G_2$. The green path p is of type $\vec{w} = s_2s_1s_0s_2s_0s_2s_1s_2s_1s_2s_1s_0s_2s_1s_0s_1s_2s_1s_0$ starting at $v = s_2s_1$. The path has $f_1 = 2$, $f_2 = 0$ and $f_0 = 1$. Hence, $f(p) = 3$ and the length of p is 19. We have $\text{wt}(p) = 2\alpha_1^\vee + 2\alpha_2^\vee$ and $\theta(p) = s_1s_2s_1s_2$.

The blue path p' is of type $\vec{w}' = s_0s_1s_2s_1s_2s_0s_2s_0s_2s_1$ starting at $v' = e$. As $f(p') = 0$ the path is straight. We have that its length is 10, $\text{wt}(p') = \alpha_2^\vee$ and $\theta(p') = 21$.

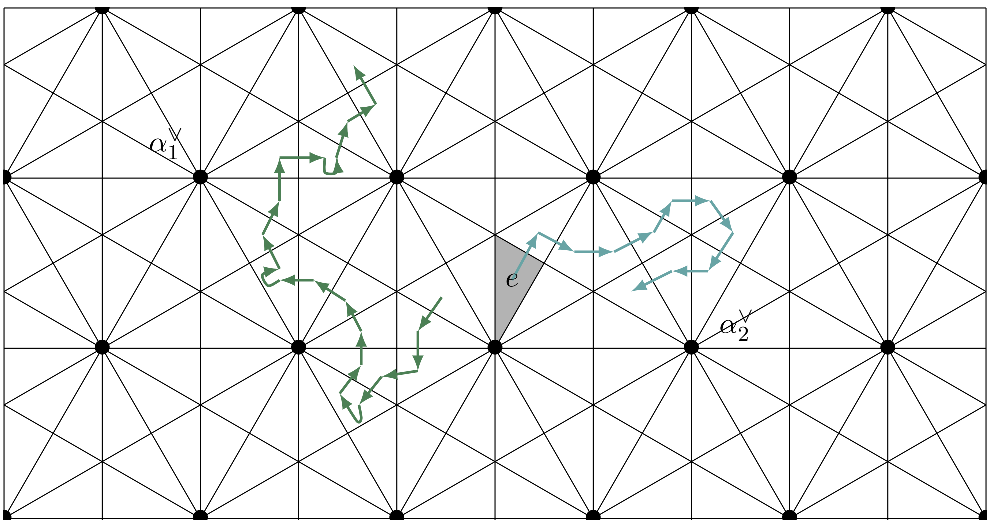


Figure 4.1: Two positively folded paths for $\widetilde{W}$ associated with $\Phi = G_2$

Let $\widetilde{\mathcal{H}}$ be the extended Hecke algebra of $\widetilde{W}$, as defined in Section 3.1. Let $\vec{w} = s_{i_1} \dots s_{i_\ell}\sigma$ be an expression of $w \in \widetilde{W}$ and let $p = (e, s_{i_1}, \dots, s_{i_1} \cdots s_{i_\ell})$ be the straight alcove path of type $s_{i_1} \dots s_{i_\ell}$ starting at e. Let $\epsilon_1, \dots, \epsilon_\ell$ be the sequence such that

$$\epsilon_k = \begin{cases} +1 & \text{if } v_{k-1}A_0{}^-|^+v_kA_0, \\ -1 & \text{if } v_{k-1}A_0{}^+|^-v_kA_0. \end{cases}$$

This sequence corresponds to the sequence of the signs of the crossings in the straight path, positive for a positive crossing and negative for a negative crossing. Define the element

$$X_w = T_{i_1}^{\epsilon_1} \cdots T_{i_\ell}^{\epsilon_\ell} T_\sigma \in \widetilde{\mathcal{H}}. \tag{4.1.1}$$

This element does not depend on the chosen expression for w (see [19, Theorem 1.1.1]).

Example 4.1.3. Let $w' = s_0s_1s_2s_1s_2s_0s_2s_0s_2s_1 \in \widetilde{W}$ as in Example 4.1.2, for $\widetilde{W}$ associated with $\Phi = G_2$. Let $\widetilde{\mathcal{H}}$ be the corresponding Hecke algebra, then

$$X_{w'} = T_{s_0}^{\epsilon_1}T_{s_1}^{\epsilon_2}T_{s_2}^{\epsilon_3}T_{s_1}^{\epsilon_4}T_{s_2}^{\epsilon_5}T_{s_0}^{\epsilon_6}T_{s_2}^{\epsilon_7}T_{s_0}^{\epsilon_8}T_{s_2}^{\epsilon_9}T_{s_1}^{\epsilon_{10}} = T_{s_0}T_{s_1}^{-1}T_{s_2}^{-1}T_{s_1}T_{s_2}T_{s_0}T_{s_2}^{-1}T_{s_0}^{-1}T_{s_2}^{-1}T_{s_1}^{-1}$$

where the sequence $(\epsilon_i)_{1\leq i\leq 10}$ is the sequence of signs of the crossings in Figure 4.1.

By the relations of the affine Hecke algebra (3.1.1), we have that $X_w - T_w$ is a linear combination of terms T_v where $v < w$ in the extended Bruhat order. Therefore, $\{X_w \mid w \in \widetilde{W}\}$ is a basis of $\widetilde{\mathcal{H}}$.

For $\gamma \in P$ we set

$$X^\gamma = X_{t_\gamma}.$$

It is clear that $X^\gamma X^\mu = X^\mu X^\gamma = X^{\gamma+\mu}$ for all $\gamma, \mu \in P$. By (1.2.1) we can decompose $w \in \widetilde{W}$ into $w = t_{\mathrm{wt}(w)}\theta(w)$ with $\mathrm{wt}(w) \in P$ and $\theta(w) \in W_0$. Therefore,

$$X_w = X_{t_{\mathrm{wt}(w)}\theta(w)} = X^{\mathrm{wt}(w)}X_{\theta(w)} = X^{\mathrm{wt}(w)}T^{-1}_{\theta(w)^{-1}} \tag{4.1.2}$$

as $t_{\mathrm{wt}(w)}A_0$ is on the positive side of each hyperplane through $\mathrm{wt}(w)$ and $X_u = T^{-1}_{u^{-1}}$ for $u \in W_0$. Hence, we now have a basis $\{X_w \mid w \in \widetilde{W}\} = \{X^\gamma T^{-1}_{u^{-1}} \mid \gamma \in P, u \in W_0\}$ of $\widetilde{\mathcal{H}}$ that reflects the semi-direct product definition of $\widetilde{W}$, allowing us to again think in terms of translations and finite components. This basis is called the *Bernstein-Lusztig basis*.

By [32, Proposition 3.6], for $i \in I$ and $\gamma \in P$ we have

$$T_i X^\gamma = X^{s_i\gamma}T_i + (\mathsf{q}_i - \mathsf{q}_i^{-1})\frac{X^\gamma - X^{s_i\gamma}}{1 - X^{-\alpha_i^\vee}} \tag{4.1.3}$$

called the *Bernstein-Lusztig relation*. As $s_i\gamma = \gamma - \langle \gamma, \alpha_i\rangle\alpha_i^\vee$, we have

$$\frac{X^\gamma - X^{s_i\gamma}}{1 - X^{-\alpha_i^\vee}} = \frac{X^\gamma - X^\gamma X^{-\langle\gamma,\alpha_i\rangle\alpha_i^\vee}}{1 - X^{-\alpha_i^\vee}} = \frac{X^\gamma(1 - X^{-\langle\gamma,\alpha_i\rangle\alpha_i^\vee})}{1 - X^{-\alpha_i^\vee}},$$

and so, as $\langle\gamma, \alpha_i\rangle \in \mathbb{Z}$, the expression in (4.1.3) is in $\widetilde{\mathcal{H}}$.

The relations of $\widetilde{\mathcal{H}}$ in the Bernstein-Lusztig basis are the Bernstein-Lusztig relation, $X^\gamma X^\mu = X^{\gamma+\mu}$ (for $\gamma, \mu \in P$) and the relations between the T_u for $u \in W_0$. These are

$$\begin{cases} T_i^2 = T_e + (\mathsf{q}_i - \mathsf{q}_i^{-1})T_i \\ T_iT_jT_i\cdots = T_jT_iT_j\cdots \quad (m_{ij} \text{ terms on each side}) \end{cases}$$

for distinct $i, j \in I$ with m_{ij} the order of s_is_j in $\widetilde{W}$.

As $s_0 = t_{\varphi^\vee}s_\varphi$, and by (4.1.2), we have

$$T_0 = X^{\varphi^\vee}T^{-1}_{s_\varphi}. \tag{4.1.4}$$

In the case of equal parameters, Ram showed that the combinatorics of positively folded paths give a description of the transition matrix between the standard basis $(T_w)_{w\in\widetilde{W}}$ and the Bernstein-Lusztig basis (see [42, Theorem 3.3]). In [23, Proposition 3.2] this idea was extended to the unequal parameters case.

Proposition 4.1.4. *[42, Theorem 3.3], [23, Proposition 3.2] Let $w, u \in \widetilde{W}$ and let $\vec{w}$ be a reduced expression for w. Then*

$$X_uT_w = \sum_{p\in\mathcal{P}(\vec{w},u)} \mathcal{Q}(p)X_{\mathrm{end}(p)} \quad \textit{where} \quad \mathcal{Q}(p) = \prod_{i\in I\cup\{0\}} (\mathsf{q}_i - \mathsf{q}_i^{-1})^{f_i(p)}.$$

Proof. We proceed via an increasing induction on the length of w. For $w = e$, we have

$$X_u T_e = \sum_{p \in \mathcal{P}(\vec{e},u)} \mathcal{Q}(p) X_{\mathrm{end(p)}} = X_u$$

as required. Assume the statement is true for $w \in \widetilde{W}$. Let $i \in I \cup \{0\}$ such that $\ell(ws_i) = \ell(w)+1$. Then

$$X_u T_{ws_i} = X_u T_w T_{s_i} = \sum_{p \in \mathcal{P}(\vec{w},u)} \mathcal{Q}(p) X_{\mathrm{end}(p)} T_{s_i}$$

Consider a path $p \in \mathcal{P}(\vec{w}, u)$. There are two cases:

(i) $\mathrm{end}(p)A_0{}^-|^+\mathrm{end}(p)s_iA_0$, or

(ii) $\mathrm{end}(p)A_0{}^+|^-\mathrm{end}(p)s_iA_0$.

In case (i), we have $X_{\mathrm{end}(p)}T_{s_i} = X_{\mathrm{end}(p)s_i} = X_{\mathrm{end}(pc_i^+)}$ where pc_i^+ is the concatenation of path p with a positive crossing of type i.

In case (ii), we have $X_{\mathrm{end}(p)s_i} = X_{\mathrm{end}(p)}T_{s_i}^{-1}$. As $T_{s_i} = T_{s_i}^{-1} + (\mathsf{q}_i - \mathsf{q}_i^{-1})$ we have

$$X_{\mathrm{end}(p)}T_{s_i} = X_{\mathrm{end}(p)s_i} + (\mathsf{q}_i - \mathsf{q}_i^{-1})X_{\mathrm{end}(p)} = X_{\mathrm{end}(pc_i^-)} + (\mathsf{q}_i - \mathsf{q}_i^{-1})X_{\mathrm{end}(pf_i^+)}$$

where pc_i^- is the concatenation of the path p with a negative crossing of type i and pf_i^+ is the concatenation of the path p with a fold of type i. □

4.2 J-folded alcove paths

In Definition 4.1.1 we defined positively folded alcove paths in V (between alcoves in $\mathbb{A}$). In this section we define folded paths restricted to the fundamental J-alcove (between alcoves in $\mathbb{A}_J$), called J-folded paths. These J-folded paths are crucial in creating the J-analogue to Proposition 4.1.4, proved in Theorem 5.3.2.

Let $\mathbb{W}^J$ be as defined in (2.3.2), and $\mathcal{A}_J$ be as defined in Section 2.1.

Definition 4.2.1. Let $\vec{w} = s_{i_1}s_{i_2}\cdots s_{i_\ell}\sigma$ be an expression for $w \in \widetilde{W}$ (not necessarily reduced) with $\sigma \in \Sigma$ and $i_1, \dots i_\ell \in I \cup \{0\}$. A *$J$-folded alcove path* of type $\vec{w}$ starting at $v_0 \in \mathbb{W}^J$ is a sequence $p = (v_0, v_1, \dots, v_\ell, v_\ell\sigma)$ where $v_1, \dots, v_\ell \in \mathbb{W}^J$, satisfying following conditions: for $1 \le k \le \ell$

(1) $v_k \in \{v_{k-1}, v_{k-1}s_{i_k}\}$

(2) if $v_{k-1} = v_k$ then either

  (i) $v_{k-1}s_{i_k}A_0 \subseteq \mathcal{A}_J$ and $v_{k-1}A_0{}^+|^-v_{k-1}s_{i_k}A_0$, or

  (ii) $v_{k-1}s_{i_k}A_0 \nsubseteq \mathcal{A}_J$

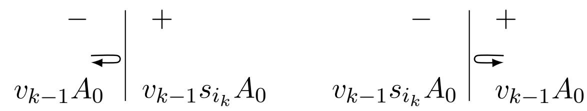
Path p has length ℓ and we define $\mathrm{end}(p) = v_\ell\sigma$ and $\mathrm{start}(p) = v_0$.

As for Definition 4.1.1 a J-folded path can be interpreted as a sequence of alcoves, now with the restriction that each alcove in the sequence is a subset of $\mathcal{A}_J$. For $p = (v_0, v_1, \dots, v_\ell, v_\ell\sigma)$ this sequence of alcoves is $(v_0A_0, \dots, v_\ell\sigma A_0)$ where $v_kA_0 \subseteq \mathcal{A}_J$ and v_kA_0 is either adjacent or equal to $v_{k-1}A_0$ for all $1 \le k \le \ell$. So, we can again visualise these paths as a path of arrows between alcoves. We define the following terms: the index $k \in \{1, 2, \dots, \ell\}$ is called

(1) a *positive bounce* if $v_{k-1}s_{i_k}A_0 \not\subseteq \mathcal{A}_J$ and $v_{k-1}A_0{}^+|^-v_{k-1}s_{i_k}A_0$,

(2) a *negative bounce* if $v_{k-1}s_{i_k}A_0 \not\subseteq \mathcal{A}_J$ and $v_{k-1}A_0{}^-|^+v_{k-1}s_{i_k}A_0$.

The corresponding arrows are



negative bounce positive bounce

The paths can be thought of as a string of arrows of the two types above along with the arrows for positively folded alcove paths (Section 4.1).

For a bounce $v_k = v_{k-1}$ we say that index k *occurs* on the hyperplane separating $v_{k-1}A_0$ and $v_{k-1}s_{i_k}A_0$. By Lemma 2.1.2, a positive bounce k occurs on $H_{\alpha_i,0}$ for some $i \in J$ and a negative bounce k occurs on $H_{\varphi_K,1}$ for some $K \in \mathcal{K}(J)$.

For $v \in \mathbb{W}^J$ and $w \in \widetilde{W}$, denote

$$\mathcal{P}_J(\vec{w}, v) = \{\text{all } J\text{-folded alcove paths of type } \vec{w} \text{ starting at } v\}.$$

If $J = \emptyset$ then $\mathbb{W}^J = \widetilde{W}$ and $\mathcal{A}_J = V$. In this case the J-folded alcove paths are the same as the positively folded paths and so $\mathcal{P}_J(\vec{w}, v) = \mathcal{P}(\vec{w}, v)$ for all $w, v \in \widetilde{W}$.

Recall the definition of the set of pseudo-translations $\mathbb{T}_J$ from 2.3.1. Let F be a fundamental domain for the action of $\mathbb{T}_J$ on $\mathbb{W}^J$. It was shown in Corollary 2.3.10 that ${}^JW$ is such a fundamental domain. We will keep the definitions of this section for a general F as in some cases a different fundamental domain can be easier to work with (an example of this is described in [21, Section 6]). For each $w \in \mathbb{W}^J$ we have a unique decomposition

$$w = \tau_\gamma u \quad \text{with } \gamma \in P^{(J)} \text{ and } u \in \mathsf{F}.$$

Define the *coweight of w relative to* F and the *final direction of w relative to* F by

$$\mathrm{wt}(w, \mathsf{F}) = \gamma \quad \text{and} \quad \theta(w, \mathsf{F}) = u.$$

When $\mathsf{F} = {}^JW$ we have $\mathrm{wt}(w, {}^JW) = \mathrm{wt}(w)$ and $\theta(w, {}^JW) = \theta^J(w)$.

Let $w \in \widetilde{W}$ and $v \in \mathbb{W}^J$, for $p \in \mathcal{P}_J(\vec{w}, v)$ we define the *coweight of p relative to* F and the *final direction of p relative to* F as

$$\mathrm{wt}(p, \mathsf{F}) = \mathrm{wt}(\mathrm{end}(p), \mathsf{F}) \quad \text{and} \quad \theta(p, \mathsf{F}) = \theta(\mathrm{end}(p), \mathsf{F}). \tag{4.2.1}$$

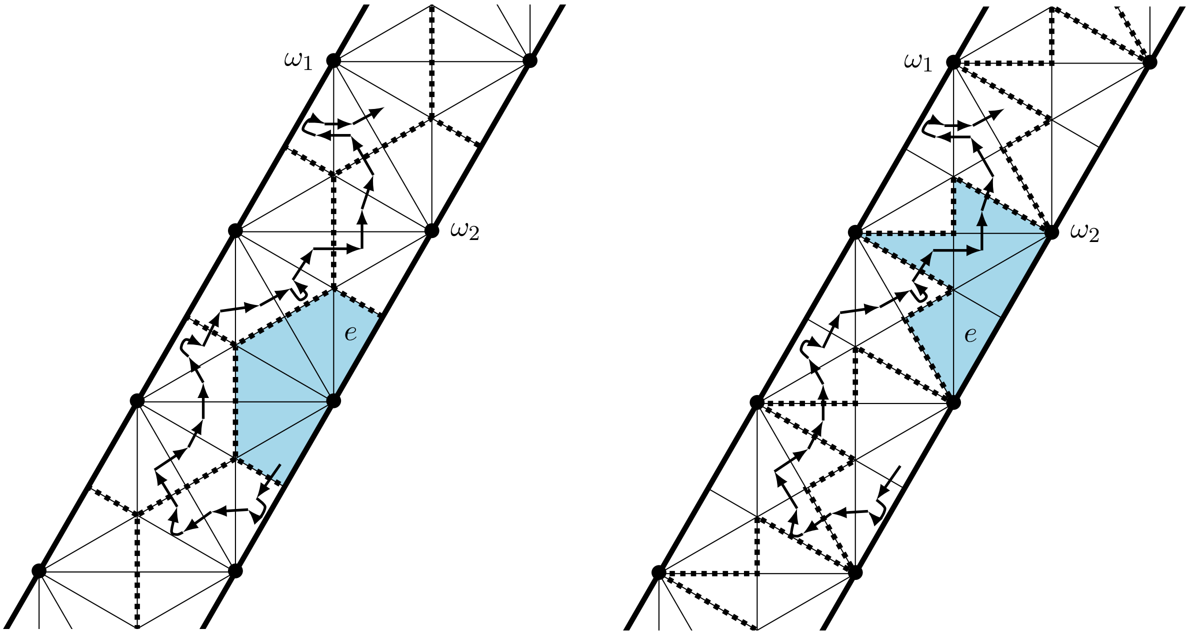
When $\mathsf{F} = {}^JW$, we notate $\mathrm{wt}(p) = \mathrm{wt}(\mathrm{end}(p))$ (as in Section 4.1) and $\theta^J(p) = \theta^J(\mathrm{end}(p))$.

As for positively folded paths, for $p \in \mathcal{P}_J(\vec{w}, v)$ we define $f_i(p)$ to be the number of folds of type i in path p and $f(p) = \sum_{i=0}^n f_i(p)$. Also, as with positively folded paths, a J-folded path p is called straight if $f(p) = 0$, noting that these straight J-folded alcove paths can still contain bounces. For each $\alpha \in \Phi_J^+$ we define

$$\begin{aligned}
b_\alpha^+(p) &= \#(\text{positive bounces in } p \text{ occurring on } H_{\alpha,0}) \\
b_\alpha^-(p) &= \#(\text{negative bounces in } p \text{ occurring on } H_{\alpha,1}) \\
b_\alpha(p) &= b_\alpha^+(p) + b_\alpha^-(p).
\end{aligned}$$

By Lemma 2.1.2, $b_\alpha^-(p) = 0$ unless $\alpha = \varphi_K$ for some $K \in \mathcal{K}(J)$ and $b_\alpha^+(p) = 0$ unless $\alpha = \alpha_i$ for some $i \in J$.

Example 4.2.2. Let $\vec{w} = s_0s_1s_2s_1s_2s_0s_1s_2s_1s_2s_1s_0s_2s_1s_0s_2s_0s_1s_2s_0s_2s_1s_2s_1 \in \widetilde{W}$ with $\Phi = G_2$ and $J = \{1\}$. Set $v = s_2s_1s_2s_1s_2$ and let p be the path in $\mathcal{P}_J(\vec{w}, v)$ depicted in Figure 4.2.



(a) $\mathsf{F} = {}^JW$ (b) $\mathsf{F} = \{e, s_2, s_0s_2s_0, s_0s_2s_1, s_0s_2, s_0\}$

Figure 4.2: The J-folded alcove path for $\Phi = G_2$ with $J = \{1\}$ and fundamental domain F

We have that $f_1(p) = 0$, $f_2(p) = 1$ and $f_0(p) = 1$, so $f(p) = 2$. In addition, $b^+_{\alpha_1}(p) = 1$ and $b^-_{\alpha_1}(p) = 2$, so $b_{\alpha_1}(p) = 3$. In Figure 4.2(a) we set the fundamental domain for the action of $\mathbb{T}_J$ on $\mathbb{W}^J$ to be ${}^JW$, depicted by dotted lines. With this fundamental domain we have $\mathrm{wt}(p, {}^JW) = \omega_1$ and $\theta(p, {}^JW) = \theta^J(p) = s_2s_1s_2$.

In Figure 4.2(b) we set the fundamental domain to be the set $\mathsf{F} = \{e, s_2, s_0s_2s_0, s_0s_2s_1, s_0s_2, s_0\}$ (again depicted with dotted lines). Now $\mathrm{wt}(p, \mathsf{F}) = \omega_1 - \omega_2$ and $\theta(p, \mathsf{F}) = s_0s_2s_1$.

Example 4.2.3. Let $\Phi = A_2$ and $J = \{1\}$. Let $\sigma \in \Sigma$ be defined by $\sigma(i) = (i + 1) \mod 3$. Set $\mathsf{F} = \Sigma = \{e, \sigma, \sigma^2\}$ as the fundamental domain of the action of $\mathbb{T}_J$ on $\mathbb{W}^J$. Figure 4.3 displays the fundamental J-alcove. The automorphism σ can be thought of as jumping to a different sheet of alcoves, in which the types between the faces of the alcoves are acted on by σ^{-1}. That is, if between two alcoves of $\mathbb{A}$ we have a face of type i, the corresponding alcoves in the sheet $\sigma\mathbb{A}$ will have a type $\sigma^{-1}(i)$ face between them (similarly for $\sigma^2\mathbb{A}$ with $\sigma^{-2}(i)$). The three sheets restricted to the fundamental domain, $\mathcal{A}_J$, $\sigma\mathcal{A}_J$ and $\sigma^2\mathcal{A}_J$, are displayed in Figure 4.3 with the faces coloured red, blue and green for types 1, 0 and 2 respectively. A path $p \in \mathcal{P}_J(s_2s_0s_1s_0s_2s_1s_0s_1s_0\sigma, e)$ is displayed in $\mathcal{A}_J$. It can be seen in the diagram that $\tau_{-2\omega_2} = s_2s_1s_3s_2\sigma^2$. Hence, we have that $\mathrm{wt}(p, \mathsf{F}) = -2\omega_2$ and $\theta(p, \mathsf{F}) = \sigma^2$. Now consider a path p' of type $s_1s_2s_0s_2s_1s_0s_2s_0s_2\sigma$ beginning at σ (pictured in $\sigma\mathcal{A}_J$ of the figure). We have $\mathrm{wt}(p', \mathsf{F}) = -2\omega_2$ and $\theta(p', \mathsf{F}) = e$. Finally, consider $p'' \in \mathcal{P}(s_0s_1s_2s_1s_0s_2s_1s_2s_1\sigma, \sigma^2)$ displayed in the $\sigma^2\mathcal{A}_J$ sheet of the figure. In this case $\mathrm{wt}(p''', \mathsf{F}) = -2\omega_2$ and $\theta(p''', \mathsf{F}) = \sigma$. All three paths have one fold and two bounces, note that these bounces and the fold will be of different types for each path (types corresponding to the colours at which they occur).

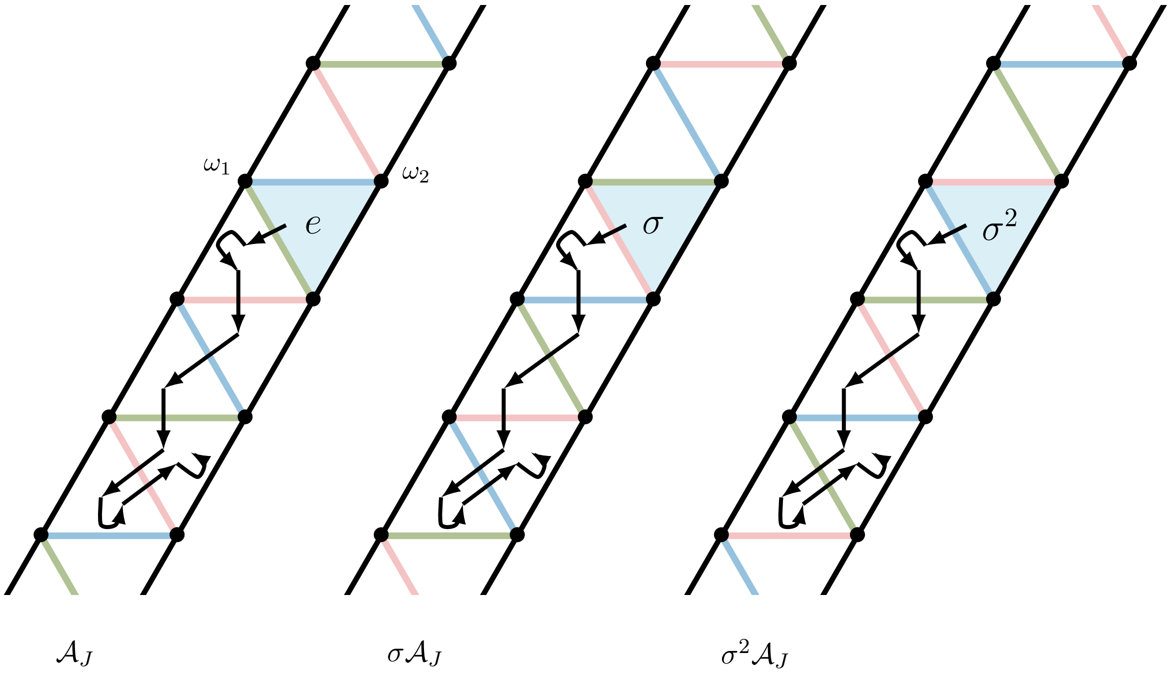


Figure 4.3: J-folded alcove path for $\Phi = A_2$ with $J = \{1\}$

Recall the definition of $\gamma^{(J)}$ from Definition 2.2.2. There is a natural action of pseudo-translations on J-folded alcove paths, described below.

Lemma 4.2.4. *If $p = (v_0, v_1, \ldots, v_\ell, v_\ell\sigma)$ is a J-folded alcove path of type $\vec{w} = s_{i_1} \cdots s_{i_\ell}\sigma$ and $\gamma \in P^{(J)}$ then*

$$\tau_\gamma \cdot p = (\tau_\gamma v_0, \tau_\gamma v_1, \ldots, \tau_\gamma v_\ell, \tau_\gamma v_\ell \sigma)$$

is a J-folded alcove path of type $\vec{w}$, and folds are mapped to folds and bounces to bounces. For any fundamental domain F we have

$$\mathrm{wt}(\tau_\gamma \cdot p, \mathsf{F}) = (\gamma + \mathrm{wt}(p, \mathsf{F}))^{(J)} \quad \textit{and} \quad \theta(\tau_\gamma \cdot p, \mathsf{F}) = \theta(p, \mathsf{F}).$$

Proof. As $\mathbb{T}_J$ acts on $\mathbb{W}^J$ we have that $\tau_\gamma v_0, \ldots \tau_\gamma v_\ell, \tau_\gamma v_\ell \sigma \in \mathbb{W}^J$. Let k be the index of a bounce or a fold, so $v_{k-1} = v_k$. We have that $v_{k-1}s_{i_k}A_0 \subseteq \mathcal{A}_J$ if and only if $\tau_\gamma v_{k-1}s_{i_k}A_0 \subseteq \mathcal{A}_J$ by the action of $\mathbb{T}_J$. Hence, if index k is a bounce in p then $\tau_\gamma v_{k-1}s_{i_k}A_0 \nsubseteq \mathcal{A}_J$ and so $\tau_\gamma \cdot p$ has a bounce at index k. If k is a fold in p then $\tau_\gamma v_{k-1}s_{i_k}A_0 \subseteq \mathcal{A}_J$. It remains to prove that this fold is positively oriented, that is, $\tau_\gamma v_{k-1}A_0{}^{+}|^{-}\tau_\gamma v_{k-1}s_{i_k}A_0$. This is equivalent to $\mathsf{y}_\gamma\theta(v_{k-1})A_0{}^{+}|^{-}\mathsf{y}_\gamma\theta(v_{k-1})s_{i_k}A_0$ as translations preserve orientation. As k is a fold in p we have $\theta(v_{k-1})A_0{}^{+}|^{-}\theta(v_{k-1})s_{i_k}A_0$. By (1.3.1) we have that $\alpha = \theta(v_{k-1})\alpha_{i_k} \in \Phi^+$. As k is a fold and not a bounce we have that $\alpha \notin \Phi_J$ and so, as $\mathsf{y}_\gamma \in W_J$, we have that $\mathsf{y}_\gamma\alpha \in \Phi^+$. By (1.3.1) we have that $\mathsf{y}_\gamma\theta(v_{k-1})A_0{}^{+}|^{-}\mathsf{y}_\gamma\theta(v_{k-1})s_{i_k}A_0$ as required. Therefore, τ_γ acting on p maps bounces to bounces and folds to folds. The coweight and the final direction follow immediately from Lemma 2.3.4. □

If $p \in \mathcal{P}_J(\vec{w}, v)$ for $w \in \widetilde{W}$ and $v \in \mathbb{W}^J$, the associated *J-straightened alcove path* p_J is the path obtained by straightening all the bounces of p (but not the folds). Formally, let $p = (v, v_1, v_2, \ldots, v_\ell, v_\ell\sigma)$ with bounces occurring at the indices $1 \leq k_1 < k_2 < \cdots < k_r \leq \ell$ on the hyperplanes $H_{\beta_1,\nu_1}, \ldots, H_{\beta_r,\nu_r}$ with $\beta_1, \ldots \beta_r \in \Phi_J^+$ and $\nu_1, \ldots, \nu_r \in \{0, 1\}$. Write $p =$

$p_0 \cdot p_1 \cdot p_2 \cdots p_r$ where $p_0 = (v, v_1, \dots, v_{k_1-1})$, $p_j = (v_{k_j}, \dots, v_{k_{j+1}-1})$ for $1 \leq j \leq r-1$, and $p_r = (v_{k_r}, \dots, v_\ell, v_\ell\sigma)$. Then $p_J = p_0 \cdot (s_{\beta_1,\nu_1}p_1) \cdot (s_{\beta_1,\nu_1}s_{\beta_2,\nu_2}p_2) \cdots (s_{\beta_1,\nu_1} \cdots s_{\beta_r,\nu_r}p_r)$.

Proposition 4.2.5. *Let $w \in \widetilde{W}$ and let $\vec{w}$ be any expression for w (not necessarily reduced). Let $v \in \mathbb{W}^J$ and let $p \in \mathcal{P}_J(\vec{w}, v)$. The J-straightening map $p \mapsto p_J$ is a bijection from $\mathcal{P}_J(\vec{w}, v)$ to the set $\{p' \in \mathcal{P}(\vec{w}, v) \mid\ p'$ has no folds on hyperplanes $H_{\alpha,k}$ with $\alpha \in \Phi_J^+$ and $k \in \mathbb{Z}\}$.*

Proof. First consider the effect of straightening a bounce. Let $\alpha \in \Phi_J^+$ be the root associated to the hyperplane on which the bounce occurs. Let $\beta \in \Phi^+ \backslash \Phi_J$ be the root associated with a wall on which a later fold occurs. After straightening the bounce this fold will now occur on a wall with associated linear root $s_\alpha\beta$. As $\beta \notin \Phi_J$ and as $s_\alpha \in W_J$ we have that $s_\alpha\beta \in \Phi^+\backslash\Phi_J$ (as s_α permutes the set Φ_J). Therefore, the reflected fold will be positively oriented (see (1.3.1)) and will not occur on a Φ_J-wall (that is, it will not occur on a hyperplane of type $H_{\alpha',k}$ with $\alpha' \in \Phi_J$ and $k \in \mathbb{Z}$). Therefore, $p_J \in \mathcal{P}(\vec{w}, v)$ has no folds on the Φ_J-walls.

Now consider the reverse map. Let $p' \in \mathcal{P}(\vec{w}, v)$ with no folds on Φ_J-walls. Let $\alpha \in \Phi_J^+$ be the root associated to a crossing on a Φ_J wall and let $\beta \in \Phi^+\backslash\Phi_J$ be the root associated with the wall of a future fold of p'. After removing the crossing and replacing it with a bounce, the fold will occur on the wall associated with $s_\alpha\beta$. Again we have $s_\alpha\beta \in \Phi^+\backslash\Phi_J$ and so it is positively oriented. Applying this procedure to all Φ_J-wall crossings gives a path in $\mathcal{P}_J(\vec{w}, v)$.

These operations are mutually inverse procedures, hence the bijection. □

Example 4.2.6. Let p be the path depicted in Figure 4.2. The J-straightening of p is depicted in Figure 4.4.

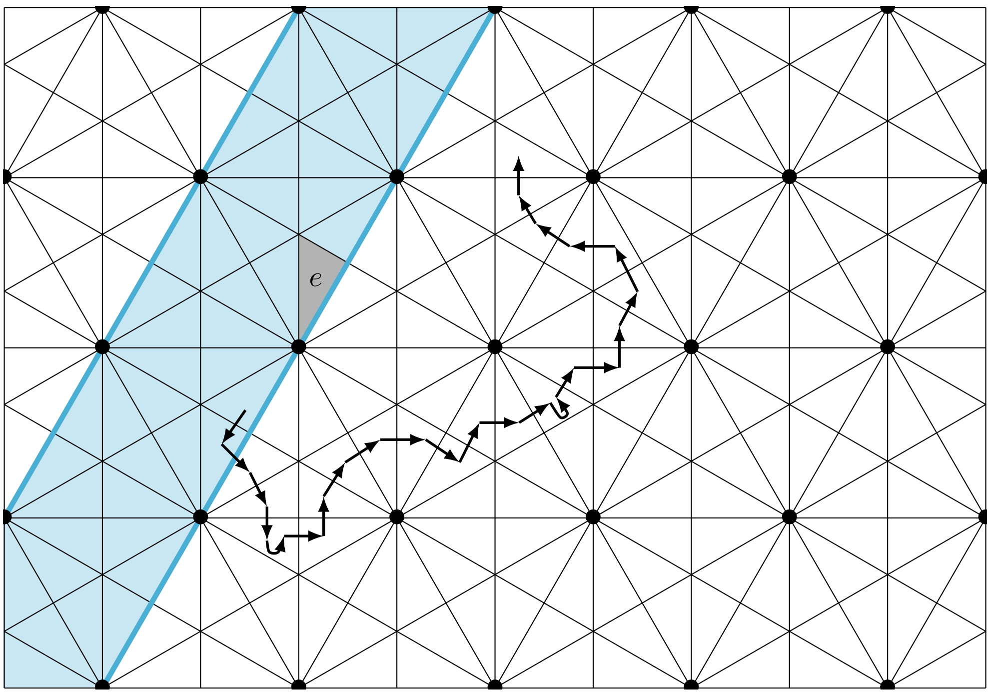


Figure 4.4: J-straightening of the path in Figure 4.2

Let $\vec{w} = s_{i_1} \cdots s_{i_l}\sigma$ be an expression of $w \in \widetilde{W}$. We define the associated *reversed expression* by

$$\mathrm{rev}(\vec{w}) = s_{\sigma^{-1}i_l} \cdots s_{\sigma^{-1}i_1}\sigma^{-1}.$$

If $\vec{w}$ is a reduced expression of w, then $\mathrm{rev}(\vec{w})$ is a reduced expression for w^{-1}.

Let $p = (v_0, \ldots, v_l, v_l\sigma) \in \mathcal{P}_J(\vec{w}, v_0)$ with $\mathrm{wt}(p) = \gamma$. The *inverse alcove path* is

$$p^{-1} = (\tau_\gamma^{-1}v_l\sigma, \tau_\gamma^{-1}v_{l-1}\sigma, \ldots, \tau_\gamma^{-1}v_1\sigma, \tau_\gamma^{-1}v_0\sigma, \tau_\gamma^{-1}v_0).$$

Roughly speaking, the involution $p \mapsto p^{-1}$ is given by "reading the path p backwards".

Proposition 4.2.7. *Let $p \in \mathcal{P}(\vec{w}, u)$ with $\theta^J(p) = v$ and $\mathrm{wt}(p) = \gamma$. Then p^{-1} is a J-folded alcove path of type $\mathrm{rev}(\vec{w})$ starting at $v \in {}^JW$ with $\theta^J(p^{-1}) = u$ and $\mathrm{wt}(p^{-1}) = (-\gamma)^{(J)}$. Moreover, the map $p \mapsto p^{-1}$ is a bijection from $\{p \in \mathcal{P}_J(\vec{w}, u) \mid \theta^J(p) = v\} \to \{p \in \mathcal{P}_J(\mathrm{rev}(\vec{w}), v) \mid \theta^J(p) = u\}$.*

Proof. Let $\vec{w} = s_{i_1} \cdots s_{i_l}\sigma$ and $p = (v_0, \ldots, v_l, v_l\sigma)$. Write $p^{-1} = (v'_0, \ldots, v'_l, v'_l\sigma^{-1})$, and so $v'_k = \tau_\gamma^{-1}v_{l-k}\sigma$ for $1 \le k \le l$. We have that

$$v'^{-1}_{k-1}v'_k = (\tau_\gamma^{-1}v_{\ell-k+1}\sigma)^{-1}(\tau_\gamma^{-1}v_{\ell-k}\sigma) = \sigma^{-1}v_{\ell-k+1}^{-1}v_{\ell-k}\sigma \in \{1, s_{\sigma^{-1}(i_{\ell-k})}\}$$

as $v_{k-1}^{-1}v_k \in \{e, s_{i_k}\}$ by Definition 4.2.1. Thus, p^{-1} is a path of type $\mathrm{rev}(\vec{w})$. We now show that p^{-1} is J-folded.

By Definition 4.2.1 we have that $v_kA_0 \in \mathcal{A}_J$. As $\sigma A_0 = A_0$ and as τ_γ^{-1} preserves $\mathcal{A}_J$ by Corollary 2.4.6 we have $v'_kA_0 \in \mathcal{A}_J$ for all $1 \le k \le l$ and so p^{-1} stays within $\mathcal{A}_J$.

As τ_γ^{-1} preserves the boundary of $\mathcal{A}_J$, if (v_{k-1}, v_k) is a bounce in p then (v'_{l-k}, v'_{l-k+1}) is a bounce in p^{-1}. Thus to show that p^{-1} is J-folded, it remains to prove that positive folds are mapped to positive folds. Let (v_{k-1}, v_k) be a (necessarily positive) fold in p, so $v_{k-1} = v_k$ and $v_{k-1}A_0{}^+|^- v_{k-1}s_{i_k}A_0$. Thus $v'_{l-k} = \tau_\gamma^{-1}v_k\sigma = \tau_\gamma^{-1}v_{k-1}\sigma$ and so $v'_{l-k}A_0 = \tau_\gamma^{-1}v_{k-1}A_0$. By Lemma 2.3.4 and Corollary 2.3.10 we have $\tau_\gamma^{-1} = t_{(-\gamma)^{(J)}}\mathsf{y}_{(-\gamma)^{(J)}}$ with $\mathsf{y}_{(-\gamma)^{(J)}} \in W_J$. Since $v_{k-1}s_{i_k}A_0 \subseteq \mathcal{A}_J$ the hyperplane on which the fold in p occurs has associated linear root $\alpha \in \Phi^+\backslash\Phi_J$. Thus, the hyperplane separating $v'_{l-k}A_0$ and $v'_{l-k}s_{\sigma^{-1}i_k}A_0 = \tau_\gamma^{-1}v_{k-1}s_{i_k}\sigma A_0$ has associated linear root $t_{(-\gamma)^{(J)}}\mathsf{y}_{(-\gamma)^{(J)}}\alpha \in \Phi^+\backslash\Phi_J + \mathbb{Z}\delta$. It follows by (1.3.1) that $v'_{l-k}A_0{}^+|^- v'_{l-k}s_{\sigma^{-1}i_k}A_0$ and so (v'_{l-k}, v'_{l-k+1}) is a positive fold in p^{-1}.

Finally, as $v_l\sigma = \tau_\gamma v$ we have $v'_0 = v$ and as $v'_l\sigma^{-1} = \tau_\gamma^{-1}v_0 = \tau_{(-\gamma)^{(J)}}u$ we have $\mathrm{wt}(p^{-1}) = (-\gamma)^{(J)}$ and $\theta^J(p^{-1}) = u$ as required. It is clear that $(p^{-1})^{-1} = p$ and so the map $p \mapsto p^{-1}$ is an involution, and hence bijective. □

Example 4.2.8. Let p be the path shown in Figure 4.2. The path p^{-1} is depicted in Figure 4.5. It is clear in the figure that $\mathrm{wt}(p^{-1}) = \omega_1 - 3\omega_2 = (-\omega_1)^{(J)}$ and that p^{-1} starts at $\theta^J(p) = s_2s_1s_2$ and ends at $s_2s_1s_2s_1s_2$.

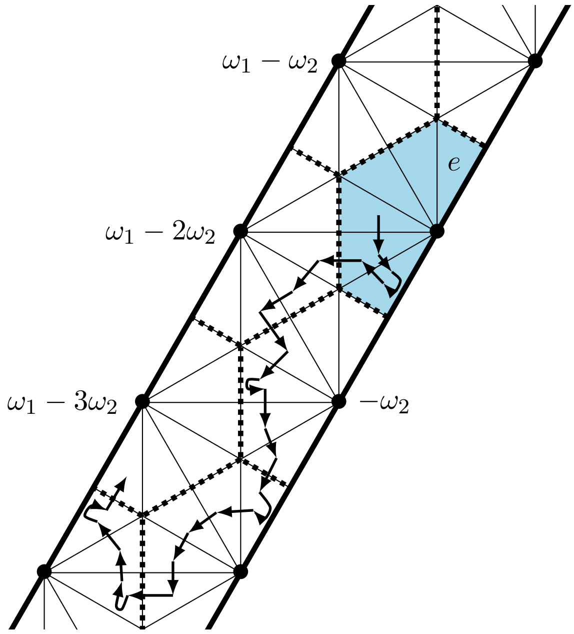


Figure 4.5: The inverse alcove path to the path in Figure 4.2

4.3 J-parameter systems and path v-mass

In the formula for $X_u T_w$ in Proposition 4.1.4, $\mathcal{Q}(p)$ is the coefficient of $X_{\mathrm{end}(p)}$ when summing over positively folded paths p. In this section we construct the J-analogue to $\mathcal{Q}(p)$, which will become the coefficient term in the J-analogue of Proposition 4.1.4 proved in Theorem 5.3.2. First, to Φ_J, we construct a family of parameters v called the J-parameter system. Using this system of parameters we construct the J-analogue coefficient $\mathcal{Q}_{J,\mathsf{v}}(p)$ for a J-folded path p. The remainder of the section proves properties of v and $\mathcal{Q}_{J,\mathsf{v}}(p)$ which will be required in Chapter 5.

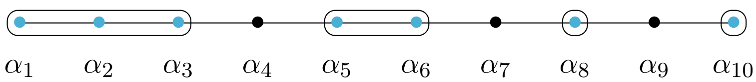
Definition 4.3.1. For $J \subseteq I$, a *J-parameter system* is a family $\mathsf{v} = (\mathsf{v}_\alpha)_{\alpha \in \Phi_J}$ such that;

(1) $\mathsf{v}_\alpha = \mathsf{v}_\beta$ if $\beta \in W_J \alpha$,

(2) $\mathsf{v}_{\alpha_j} \in \{\mathsf{q}_j, -\mathsf{q}_j^{-1}\}$.

Note that we set $\mathsf{v}_\alpha = 1$ for all $\alpha \notin \Phi_J$. If $K \in \mathcal{K}(J)$ there are at most two root lengths in Φ_K. As K is connected, all roots of the same length are conjugate in W_K. By (1) of Definition 4.3.1 $\mathsf{v}_\alpha = \mathsf{v}_{\alpha'}$ for all $\alpha, \alpha' \in \Phi_{K,l}$ and $\mathsf{v}_\beta = \mathsf{v}_{\beta'}$ for all $\beta, \beta' \in \Phi_{K,s}$.

Example 4.3.2. Let $\Phi = A_{10}$ and $J = \{1, 2, 3, 5, 6, 8, 10\}$, depicted in the following diagram.

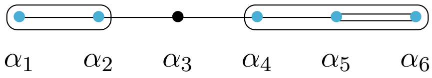


Let $K \in \mathcal{K}(J)$. By (1) in Definition 4.3.1, we have that $\mathsf{v}_\alpha = \mathsf{v}_{\alpha'}$ for all $\alpha, \alpha' \in \Phi_K$. The possible J-parameter systems are characterised as follows:

$$\begin{cases} \mathsf{v}_\alpha = \mathsf{v}_{\alpha_1} \in \{\mathsf{q}_1, \mathsf{q}_1^{-1}\} \text{ for all } \alpha \in \Phi_{\{1,2,3\}} \\ \mathsf{v}_\alpha = \mathsf{v}_{\alpha_5} \in \{\mathsf{q}_5, \mathsf{q}_5^{-1}\} \text{ for all } \alpha \in \Phi_{\{5,6\}} \\ \mathsf{v}_{\alpha_8} \in \{\mathsf{q}_8, \mathsf{q}_8^{-1}\} \text{ and } \mathsf{v}_{\alpha_{10}} \in \{\mathsf{q}_{10}, \mathsf{q}_{10}^{-1}\} \end{cases}$$

Therefore, there are $2^4 = 16$ distinct J-parameter systems.

Now let $\Phi = B_6$ with $J = \{1, 2, 4, 5, 6\}$, depicted in the following diagram.



The connected component $K = \{1, 2\}$ of type $\Phi_K = A_2$ has $\mathsf{v}_\alpha = \mathsf{v}_{\alpha_1}$ for all $\alpha \in \Phi_K$. The remaining connected component $K' = \{4, 5, 6\}$ of type $\Phi_{K'} = B_3$ has J-parameter system values as follows:

$$v_\alpha = v_{\alpha_4} \text{ for all } \alpha \in \Phi_{K',l}$$
$$v_\alpha = v_{\alpha_6} \text{ for all } \alpha \in \Phi_{K',s}.$$

Hence, there are $2^3 = 8$ distinct J-parameter systems.

Definition 4.3.3. Let $\mathsf{v} = (\mathsf{v}_\alpha)_{\alpha \in \Phi_J}$ be a J-parameter system. The v-*mass* of a J-folded alcove path p is

$$\mathcal{Q}_{J,\mathsf{v}}(p) = \prod_{\alpha \in \Phi_J^+} \mathsf{v}_\alpha^{b_\alpha(p)} \prod_{i \in I \cup \{0\}} (\mathsf{q}_i - \mathsf{q}_i^{-1})^{f_i(p)}.$$

This definition will be fundamental in defining the J-analogue to Proposition 4.1.4. If $J = \emptyset$ then the J-parameter system is vacuous and for a J-folded alcove path p we have $\mathcal{Q}_{J,\mathsf{v}}(p) = \mathcal{Q}(p)$ where $\mathcal{Q}(p)$ is as in Proposition 4.1.4.

Example 4.3.4. Let p be the path depicted in Figure 4.2, then

$$\mathcal{Q}_{J,v}(p) = \mathsf{v}_{\alpha_1}^3 (\mathsf{q}_2 - \mathsf{q}_2^{-1})(\mathsf{q}_0 - \mathsf{q}_0^{-1})$$

with $\mathsf{v}_{\alpha_1} \in \{\mathsf{q}_1, \mathsf{q}_1^{-1}\}$.

Recall the action of $\mathbb{T}_J$ on J-folded alcove paths, as described in Lemma 4.2.4. We now show that $\mathcal{Q}_{J,\mathsf{v}}(p)$ is invariant under this action.

Lemma 4.3.5. *If p is a J-folded alcove path and $\gamma \in P^{(J)}$ then $\mathcal{Q}_{J,\mathsf{v}}(\tau_\gamma \cdot p) = \mathcal{Q}_{J,\mathsf{v}}(p)$.*

Proof. Lemma 4.2.4 showed that, under the action of $\mathbb{T}_J$, bounces are mapped to bounces and folds are mapped to folds. Let index k be a bounce (respectively fold) of type i_k. The bounce (respectively fold) at index k of $\tau_\gamma \cdot p$ is of type $\sigma(i_k)$ for some $\sigma \in \Sigma$. By Convention 3.1.2 we have $\mathsf{q}_{\sigma(i)} = \mathsf{q}_i$ and the result follows. $\square$

We extend the definition of a J-parameter system $\mathsf{v} = (\mathsf{v}_\alpha)_{\alpha \in \Phi_J}$ to the affine root system as such:

$$\mathsf{v}_{\alpha+k\delta} = \mathsf{v}_\alpha \quad \text{if } \alpha \in \Phi_J \text{ and } k \in \mathbb{Z} \tag{4.3.1}$$

and $\mathsf{v}_{\alpha+k\delta} = 1$ if $\alpha + k\delta \notin \Phi_J + \mathbb{Z}\delta$. Recall from Section 4.2 that the J-straightening of a path p is p_J.

Proposition 4.3.6. *Let p be a J-folded alcove path and let $\mathsf{v} = (\mathsf{v}_\alpha)_{\alpha \in \Phi_J}$ be a J-parameter system. Then*

$$\mathcal{Q}_{J,\mathsf{v}}(p) = \mathcal{Q}(p_J) \prod_{\alpha - k\delta \in \widetilde{\Phi}^+} \mathsf{v}_{\alpha-k\delta}^{c_{\alpha,k}(p_J)}$$

where $c_{\alpha,k} = \#$(crossing in p that occur on the hyperplane $H_{\alpha,k}$) and $\mathcal{Q}(\cdot)$ is as in Proposition 4.1.4 (note that the product has finitely many terms $\neq 1$).

Proof. By Proposition 4.2.5, under $p \mapsto p_J$ folds are mapped to folds. Furthermore, the J-affine Weyl group W_J^{aff} is type preserving, so i-folds are mapped to i-folds. This shows that

$$\prod_{i\in\{0\}\cup I}(\mathsf{q}_i - \mathsf{q}_i^{-1})^{f_i(p)} = \prod_{i\in\{0\}\cup I}(\mathsf{q}_i - \mathsf{q}_i^{-1})^{f_i(p_J)} = \mathcal{Q}(p_J).$$

It remains to prove that for $p \in \mathcal{P}_J(\vec{w}, v)$ we have

$$\prod_{\alpha\in\Phi_J^+} \mathsf{v}_\alpha^{b_\alpha(p)} = \prod_{\alpha-k\delta\in\widetilde{\Phi}^+} \mathsf{v}_{\alpha-k\delta}^{c_{\alpha,k}(p_J)}.$$

Each crossing of a $\Phi_J + \mathbb{Z}\delta$ wall in p_J corresponds to a bounce on a wall of $\mathcal{A}_J$ in p. If the crossing occurs on the hyperplane $H_{\beta,k}$ for $\beta \in \Phi_J$ then the bounce will occur on $vH_{\beta,k} = H_{\beta',r}$ for some $v \in W_J^{\text{aff}}$ and $\beta' \in \Phi_J$ with β' the linear root of $v\beta$. The result follows from Definition 4.3.1(1) and (4.3.1). □

Recall the definition of the inverse alcove path of p, denoted p^{-1}, from Section 4.2 (before Proposition 4.2.7).

Proposition 4.3.7. *If p is a J-folded alcove path, then $\mathcal{Q}_J(p) = \mathcal{Q}_J(p^{-1})$.*

Proof. As in the proof of Proposition 4.2.7, set $\vec{w} = s_{i_1}\cdots s_{i_l}\sigma$, $p = (v_0, \dots, v_l, v_l\sigma)$ and $p^{-1} = (v_0', \dots, v_l', v_l'\sigma^{-1})$ so $v_k' = \tau_\gamma^{-1} v_{l-k}\sigma$ for $1 \le k \le l$. Assume that (v_{k-1}, v_k) is a bounce in p, so $v_k = v_{k-1}$. Thus, $v_{l-k}' = \tau_\gamma^{-1} v_k\sigma = \tau_\gamma^{-1} v_{k-1}\sigma$. Let $\alpha \in \Phi_J + \mathbb{Z}\delta$ be the affine root associated to the hyperplane between $v_{k-1}A_0$ and $v_{k-1}s_kA_0$. The hyperplane between $v_{l-k}'A_0$ and $v_{l-k}'s_{\sigma^{-1}(i_k)}A_0 = \tau_\gamma^{-1}v_{k-1}s_{i_k}\sigma A_0 = \tau_\gamma^{-1}v_{k-1}s_{i_k}A_0$ has associated root

$$\tau_\gamma^{-1}\alpha = t_{(-\gamma)^{(J)}}\mathsf{y}_{(-\gamma)^{(J)}}\alpha \in \Phi_J + \mathbb{Z}\delta$$

by Lemma 2.3.4 and Corollary 2.3.10, since $\mathsf{y}_{(-\gamma)^{(J)}} \in W_J$. By (4.3.1) and Definition 4.3.3 we have that $\mathsf{v}_\alpha = \mathsf{v}_{\tau_\gamma^{-1}\alpha}$.

Now assume that (v_{k-1}, v_k) is a positive fold in p. This positive fold is of type i_k. The corresponding fold (v_{l-k}', v_{l-k+1}') in p^{-1} is of type $\sigma^{-1}(i_k)$. By Convention 3.1.2 we have that $\mathsf{q}_i = \mathsf{q}_{\sigma(i)}$. Thus, $\mathcal{Q}_J(p) = \mathcal{Q}_J(p^{-1})$. □

Let $\mathsf{v} = (\mathsf{v}_\alpha)_{\alpha\in\Phi_J}$ be a J-parameter system. For $\gamma \in P$ and $y \in W_J$, define

$$\mathsf{v}^\gamma = \prod_{\alpha\in\Phi_J^+}\mathsf{v}_\alpha^{\langle\gamma,\alpha\rangle} \quad\text{and}\quad \mathsf{v}(y) = \prod_{\alpha\in\Phi(y)}\mathsf{v}_\alpha. \tag{4.3.2}$$

Note that for $\gamma \in P \cap V^J$ we have $\mathsf{v}^\gamma = 1$ as $\langle\gamma, \alpha\rangle = 0$ for all $\alpha \in \Phi_J^+$.

To prove the J-analogue to Proposition 4.1.4, stated in Theorem 5.3.2, we will need the following properties of v^γ and $\mathsf{v}(y)$.

Lemma 4.3.8. *Let v be a J-parameter system. If $\gamma \in P^{(J)}$ and $y \in W_J$ then*

$$\mathsf{v}(y\mathsf{y}_\gamma) = \mathsf{v}^{y\gamma}\mathsf{v}(y).$$

Proof. From the definition of $\mathsf{v}(\cdot)$ we have

$$\frac{\mathsf{v}(y\mathsf{y}_\gamma)}{\mathsf{v}(y)} = \frac{\prod_{\alpha\in\Phi(y\mathsf{y}_\gamma)}\mathsf{v}_\alpha}{\prod_{\alpha\in\Phi(y)}\mathsf{v}_\alpha} = \prod_{\alpha\in\Phi_J^+}\mathsf{v}_\alpha^{\sigma_\alpha}, \quad\text{where}\quad \sigma_\alpha = \begin{cases} 1 & \text{if } \alpha\in\Phi(y\mathsf{y}_\gamma)\backslash\Phi(y) \\ -1 & \text{if } \alpha\in\Phi(y)\backslash\Phi(y\mathsf{y}_\gamma) \\ 0 & \text{otherwise.}\end{cases}$$

Recall that $J(\gamma) = \{j \in J \mid \langle \gamma, \alpha_j \rangle \neq 0\}$, and so $J \backslash J(\gamma) = \{j \in J \mid \langle \gamma, \alpha_j \rangle = 0\}$. It follows from Lemma 2.2.4 that

$$\mathsf{v}^{y\gamma} = \prod_{\alpha \in \Phi_J^+} \mathsf{v}_\alpha^{\langle \gamma, y^{-1}\alpha \rangle} = \prod_{\alpha \in \Phi_J^+} \mathsf{v}_\alpha^{\sigma'_\alpha} \quad \text{where} \quad \sigma'_\alpha = \begin{cases} 1 & \text{if } y^{-1}\alpha \in \Phi_J^+ \backslash \Phi_{J\backslash J(\gamma)} \\ -1 & \text{if } y^{-1}\alpha \in (-\Phi_J^+) \backslash \Phi_{J\backslash J(\gamma)} \\ 0 & \text{if } y^{-1}\alpha \in \Phi_{J\backslash J(\gamma)}. \end{cases}$$

By Lemma 2.3.2 $\Phi(\mathsf{y}_\gamma) = \Phi_J^+ \backslash \Phi_{J\backslash J(\gamma)}$. For $\alpha \in \Phi_J^+$ we have

$$\begin{aligned} \alpha \in \Phi(y\mathsf{y}_\gamma) \backslash \Phi(y) &\iff y^{-1}\alpha > 0 \text{ and } \mathsf{y}_\gamma^{-1} y^{-1}\alpha < 0 \\ &\iff y^{-1}\alpha \in \Phi(\mathsf{y}_\gamma). \end{aligned}$$

Furthermore,

$$\begin{aligned} \alpha \in \Phi(y) \backslash \Phi(y\mathsf{y}_\gamma) &\iff y^{-1}\alpha < 0 \text{ and } \mathsf{y}_\gamma^{-1} y^{-1}\alpha > 0 \\ &\iff -y^{-1}\alpha \in \Phi(\mathsf{y}_\gamma) \\ &\iff y^{-1}\alpha \in (-\Phi_J^+) \backslash \Phi_{J\backslash J(\gamma)}. \end{aligned}$$

Thus, $\sigma_\alpha = \sigma'_\alpha$ and the result follows. ☐

Lemma 4.3.9. *Let* v *be a* J*-parameter system, if* $j \in J$ *then* $\mathsf{v}^{\alpha_j^\vee} = \mathsf{v}_{\alpha_j}^2$.

Proof. Let $K \in \mathcal{K}(J)$ be the connected component with $j \in K$. Let v_s be the constant value of v_β for $\beta \in \Phi_{K,s}$ and v_l be the constant value of $\mathsf{v}_{\beta'}$ for $\beta' \in \Phi_{K,l}$. Then, we have

$$\mathsf{v}^{\alpha_j^\vee} = \prod_{\alpha \in \Phi_J^+} \mathsf{v}_\alpha^{\langle \alpha_j^\vee, \alpha \rangle} = \prod_{\alpha \in \Phi_K^+} \mathsf{v}_\alpha^{\langle \alpha_j^\vee, \alpha \rangle} = \mathsf{v}_s^{\langle \alpha_j^\vee, 2\rho'_K \rangle} \mathsf{v}_l^{\langle \alpha_j^\vee, 2\rho_K \rangle},$$

where ρ_K, ρ'_K are as in Section 1.4. The result follows from Lemma 1.4.5. ☐

Lemma 4.3.10. *Let* $K \in \mathcal{K}(J)$ *and let* $\alpha \in \Phi_K^+$ *be a long root of* Φ_K *(with all roots long if* Φ_K *is simply-laced). Then* $\mathsf{v}^{\alpha^\vee} \mathsf{v}(s_\alpha)^{-1} = \mathsf{v}_\alpha$.

Proof. Let $w = s_{j_1} \cdots s_{j_\ell} \in W_K$ be of minimal length subject to $w^{-1}\alpha = \alpha_k$ for some $k \in K$. Let $\beta_0 = \alpha$ and define $\beta_1, \ldots, \beta_\ell \in \Phi_K$ by $\beta_r = s_{j_r}\beta_{r-1}$ for $1 \leq r \leq \ell$, and so $\beta_\ell = \alpha_k$. If $\beta_r \in \Phi_K^-$ then there exists $1 \leq r' < r$ such that $\beta_{r'} = -\alpha_{j_{r'}}$. As $\alpha, \alpha_k \in \Phi_K^+$ there also exists $r < r'' \leq \ell$ such that $\beta_{r''} = \alpha_{j_{r''}}$ and $\beta_{r''-1} = -\alpha_{j_{r''}}$. Let $w' = s_{j_1} s_{j_2} \cdots \hat{s}_{j_{r'}} \cdots \hat{s}_{j_{r''}} \cdots s_{j_\ell}$ be the expression formed by removing $s_{j_{r'}}$ and $s_{j_{r''}}$ from $w = s_{j_1} \cdots s_{j_\ell}$. Then $w'^{-1}\alpha = \alpha_k$ contradicts the minimality of w. Hence, $\beta_1, \ldots, \beta_\ell \in \Phi_K^+$.

Since each β_r is a long root of Φ_K we have $\langle \beta_{r-1}^\vee, \alpha_{j_r} \rangle \in \{-1, 0, 1\}$ (see [7, VI, §1, Proposition 8]). As $\beta_r \in \Phi_K^+$ we have $\langle \beta_{r-1}^\vee, \alpha_{j_r} \rangle = 1$ and so

$$\beta_r^\vee = s_{j_r}\beta_{r-1}^\vee = \beta_{r-1}^\vee - \langle \beta_{r-1}^\vee, \alpha_{j_r} \rangle \alpha_{j_r}^\vee = \beta_{r-1}^\vee - \alpha_{j_r}^\vee.$$

We claim that

$$\mathsf{v}^{\beta_r^\vee} \mathsf{v}(s_{\beta_r})^{-1} = \mathsf{v}_\alpha \quad \text{for } 0 \leq r \leq \ell.$$

We argue by downward induction on r. Let $r = \ell$ then $\beta_\ell = \alpha_k$. By Lemma 4.3.9 we have

$$\mathsf{v}^{\alpha_k^\vee} \mathsf{v}_{\alpha_k}^{-1} = \mathsf{v}_{\alpha_k}$$

and the result follows as $\mathsf{v}(s_k) = \mathsf{v}_{\alpha_k}$ and $\mathsf{v}_\alpha = \mathsf{v}_{\alpha_k}$ (by (1) of Definition 4.3.1).

As $\beta_r = s_{j_r}\beta_{r-1}$ we have that $s_{\beta_{r-1}} = s_{j_r}s_{\beta_r}s_{j_r}$ (with length adding). Hence, with $\beta_{r-1}^\vee = \beta_r^\vee + \alpha_{j_r}^\vee$, we have

$$\mathsf{v}^{\beta_{r-1}^\vee}\mathsf{v}(s_{\beta_{r-1}})^{-1} = \mathsf{v}^{\beta_r^\vee}\mathsf{v}^{\alpha_{j_r}^\vee}\mathsf{v}(s_{j_r}s_{\beta_r}s_{j_r})^{-1}.$$

By (1.1.1), $\Phi(s_{j_r}s_{\beta_r}s_{j_r}) = \{\alpha_{j_r}\} \cup s_{j_r}\Phi(s_{\beta_r}) \cup \{s_{j_r}s_{\beta_r}\alpha_{j_r}\}$ and by Definition 4.3.1(1), $\mathsf{v}_{\alpha_{j_r}} = \mathsf{v}_{s_{j_r}s_{\beta_r}\alpha_{j_r}}$ and $\mathsf{v}_\beta = \mathsf{v}_{s_{j_r}\beta}$ for all $\beta \in \Phi(\beta_r)$. Hence,

$$\mathsf{v}(s_{j_r}s_{\beta_r}s_{j_r}) = \mathsf{v}(s_{j_r})^2\mathsf{v}(s_{\beta_r}) = \mathsf{v}_{\alpha_{j_r}}^2\mathsf{v}(s_{\beta_r}).$$

Then

$$\mathsf{v}^{\beta_r^\vee}\mathsf{v}^{\alpha_{j_r}^\vee}\mathsf{v}(s_{j_r}s_{\beta_r}s_{j_r})^{-1} = \mathsf{v}^{\beta_r^\vee}\mathsf{v}^{\alpha_{j_r}^\vee}\mathsf{v}_{\alpha_{j_r}}^{-2}\mathsf{v}(s_{\beta_r})^{-1} = \mathsf{v}^{\beta_r^\vee}\mathsf{v}(s_{\beta_r})^{-1}$$

where the second equality follows from Lemma 4.3.9. The result follows by induction. □

Lemma 4.3.11. *Let* v *be a* J*-parameter system. If* $K \in \mathcal{K}(J)$ *and* $y \in W_J$ *then*

$$\frac{\mathsf{v}(y)}{\mathsf{v}(ys_{\varphi_K})}\mathsf{v}^{y\varphi_K^\vee} = \begin{cases} \mathsf{v}_{\varphi_K} & \text{if } y\varphi_K \in \Phi_J^+ \\ \mathsf{v}_{\varphi_K}^{-1} & \text{if } y\varphi_K \in -\Phi_J^+. \end{cases}$$

Proof. The generators s_i for $i \in K$ are commutative with all s_j for $j \in J\backslash K$. Therefore, we can write $y = y'y_k$ where $y_k \in W_k$ and $y' \in W_J/W_k$. Then

$$\frac{\mathsf{v}(y)}{\mathsf{v}(ys_{\varphi_K})} = \frac{\prod_{\alpha\in\Phi(y')}\mathsf{v}_\alpha\prod_{\alpha\in\Phi(y_k)}\mathsf{v}_\alpha}{\prod_{\alpha\in\Phi(y')}\mathsf{v}_\alpha\prod_{\alpha\in\Phi(y_ks_{\varphi_k})}\mathsf{v}_\alpha} = \frac{\prod_{\alpha\in\Phi(y_k)}\mathsf{v}_\alpha}{\prod_{\alpha\in\Phi(y_ks_{\varphi_k})}\mathsf{v}_\alpha}$$

and so we only need to consider $y \in W_K$.

Let $\alpha \in \Phi_K^+$. If $y^{-1}\alpha \neq \pm\varphi_K$ and $y^{-1}\alpha \in \Phi_K^+$ then $\langle\varphi_K^\vee, y^{-1}\alpha\rangle \in \{0, 1\}$ by [7, VI, §1.8, Proposition 25(iv)]. Then, since $s_{\varphi_K}(y^{-1}\alpha) = y^{-1}\alpha - \langle\varphi_K^\vee, y^{-1}\alpha\rangle\varphi_K$, we have that $y^{-1}\alpha \in \Phi(s_{\varphi_K})$ if and only if $\langle\varphi_K^\vee, y^{-1}\alpha\rangle = 1$. If $y^{-1}\alpha \in -\Phi_K^+$ then the same argument follows with $-y^{-1}\alpha$ and we have

$$\langle\varphi_K^\vee, y^{-1}\alpha\rangle = \begin{cases} 1 & \text{if } y^{-1}\alpha \in \Phi(s_{\varphi_K}) \\ -1 & \text{if } y^{-1}\alpha \in -\Phi(s_{\varphi_K}) \\ 0 & \text{otherwise.} \end{cases}$$

Let $\epsilon = 1$ when $y\varphi_K \in \Phi_K^+$ and $\epsilon = -1$ when $y\varphi_K \in -\Phi_K^+$. Then,

$$\mathsf{v}^{y\varphi_K^\vee} = \prod_{\alpha\in\Phi_K^+}\mathsf{v}_\alpha^{\langle\varphi_K^\vee, y^{-1}\alpha\rangle} = \mathsf{v}_{\varphi_K}^\epsilon\prod_{\alpha\in\Phi_K^+}\mathsf{v}_\alpha^{\sigma_\alpha} \quad \text{where} \quad \sigma_\alpha = \begin{cases} 1 & \text{if } y^{-1}\alpha \in \Phi(s_{\varphi_K}) \\ -1 & \text{if } y^{-1}\alpha \in -\Phi(s_{\varphi_K}) \\ 0 & \text{otherwise.} \end{cases}$$

as $\langle\varphi_K^\vee, \varphi_K\rangle = 2$.

On the other hand, as in the proof of Lemma 4.3.8 we have

$$\frac{\mathsf{v}(ys_{\varphi_K})}{\mathsf{v}(y)} = \prod_{\alpha\in\Phi_K^+}\mathsf{v}_\alpha^{\sigma'_\alpha}, \quad \text{where} \quad \sigma'_\alpha = \begin{cases} 1 & \text{if } \alpha \in \Phi(ys_{\varphi_K})\backslash\Phi(y) \\ -1 & \text{if } \alpha \in \Phi(y)\backslash\Phi(ys_{\varphi_K}) \\ 0 & \text{otherwise.} \end{cases}$$

The results follows by:

$$\alpha \in \Phi(ys_{\varphi_K})\backslash\Phi(y) \iff y^{-1}\alpha \in \Phi_K^+ \text{ and } s_{\varphi_K}^{-1}y^{-1}\alpha \in -\Phi_K^+ \iff y^{-1}\alpha \in \Phi(s_{\varphi_K}),$$
$$\alpha \in \Phi(y)\backslash\Phi(ys_{\varphi_K}) \iff y^{-1}\alpha \in -\Phi_K^+ \text{ and } s_{\varphi_K}^{-1}y^{-1}\alpha \in \Phi_K^+ \iff y^{-1}\alpha \in -\Phi(s_{\varphi_K}).$$

□

Lemma 4.3.12. *Let $w \in \widetilde{W}$ and $i \in I \cup \{0\}$. Let $u = \theta^J(w)$. Then*

$$\frac{\mathsf{v}(\theta_J(w))}{\mathsf{v}(\theta_J(ws_i))}\mathsf{v}^{\mathrm{wt}(ws_i)-\mathrm{wt}(w)} = \begin{cases} 1 & \text{if } us_i \in \mathbb{W}^J \\ \mathsf{v}_{u\alpha_i} & \text{if } us_i \notin \mathbb{W}^J \text{ and } wA_0{}^-|^+ws_iA_0 \\ \mathsf{v}_{u\alpha_i}^{-1} & \text{if } us_i \notin \mathbb{W}^J \text{ and } wA_0{}^+|^-ws_iA_0. \end{cases}$$

Proof. The expression on the left hand side of the equation is invariant under replacing w with $t_\gamma w$ for any $\gamma \in P$, and orientation is preserved under translations. Therefore we can assume without loss of generality that $w \in W_0$.

So $w = yu \in W_0$ with $y = \theta_J(w)$ and $u = \theta^J(w)$. Suppose first that $i \in I$. If $us_i \in \mathbb{W}^J$ then $us_i \in {}^JW$ and so $\theta^J(ws_i) = us_i$, and hence $\theta_J(ws_i) = y$. The result follows as $\mathrm{wt}(ws_i) = \mathrm{wt}(w) = 0$. If $us_i \notin \mathbb{W}^J$ then $us_i = s_ju$ for a unique $j \in J$, and so $\theta_J(ws_i) = ys_j$. If $wA_0{}^-|^+ws_iA_0$ then $\ell(ws_i) = \ell(w) - 1$ (as $w \in W_0$) which implies that $\ell(ys_j) = \ell(y) - 1$. On the other hand, if $wA_0{}^+|^-ws_iA_0$ then $\ell(ws_i) = \ell(w) + 1$ which implies that $\ell(ys_j) = \ell(y) + 1$. By (1.1.1) we have $\Phi(ys_j) = \Phi(y) \cup \{y\alpha_j\}$ if $\ell(ys_j) = \ell(y) + 1$ and $\Phi(ys_j) = \Phi(y)\backslash\{ys_j\alpha_j\}$ if $\ell(ys_j) = \ell(y) - 1$. Thus, as $\mathrm{wt}(ws_i) = \mathrm{wt}(w) = 0$,

$$\frac{\mathsf{v}(\theta_J(w))}{\mathsf{v}(\theta_J(ws_i))}\mathsf{v}^{\mathrm{wt}(ws_i)-\mathrm{wt}(w)} = \frac{\mathsf{v}(y)}{\mathsf{v}(ys_j)} = \frac{\mathsf{v}(y)}{\mathsf{v}(y)\mathsf{v}_{y\alpha_j}^{\epsilon}} = v_{y\alpha_j}^{-\epsilon}$$

where $\epsilon = 1$ if $\ell(ws_i) = \ell(w) + 1$ and $\epsilon = -1$ if $\ell(ws_i) = \ell(w) - 1$. By Definition 4.3.1(1) and $us_iu^{-1} = s_j$ we have that $\mathsf{v}_{y\alpha_j} = \mathsf{v}_{\alpha_j} = \mathsf{v}_{u\alpha_i}$, and the result follows.

Suppose now that $i = 0$. If $us_0 \in \mathbb{W}^J$ then $t_{u\varphi^\vee}us_\varphi \in \mathbb{W}^J$ and so $u\varphi^\vee \in P^{(J)}$. By Corollary 2.3.10, we then have $\theta_J(us_0) = \mathsf{y}_{u\varphi^\vee}$. Thus $\theta_J(ws_0) = y\mathsf{y}_{u\varphi^\vee}$ and so, with $\mathrm{wt}(ws_0) = yu\varphi^\vee$, we have

$$\frac{\mathsf{v}(\theta_J(w))}{\mathsf{v}(\theta_J(ws_i))}\mathsf{v}^{\mathrm{wt}(ws_i)-\mathrm{wt}(w)} = \frac{\mathsf{v}(y)}{\mathsf{v}(y\mathsf{y}_{u\varphi^\vee})}\mathsf{v}^{yu\varphi^\vee},$$

and since $u\varphi^\vee \in P^{(J)}$ the result follows by Lemma 4.3.8.

If $us_0 \notin \mathbb{W}^J$ then the wall between uA_0 and us_0A_0 will be a Φ_J-wall. Therefore, $u\alpha_0 \in \Phi_J + \mathbb{Z}\delta$. As $\alpha_0 = -\varphi + \delta$ we have that $u\varphi \in \Phi_J$.

As $ws_0 = t_{yu\varphi^\vee}ys_{u\varphi}u$ we have that $\mathrm{wt}(ws_0) = w\varphi^\vee = yu\varphi^\vee$ and $\theta_J(ws_0) = ys_{u\varphi}$ (as $s_{u\varphi} \in W_J$ as $u\varphi \in \Phi_J$). Thus,

$$\frac{\mathsf{v}(\theta_J(w))}{\mathsf{v}(\theta_J(ws_i))}\mathsf{v}^{\mathrm{wt}(ws_i)-\mathrm{wt}(w)} = \frac{\mathsf{v}(y)}{\mathsf{v}(ys_{u\varphi})}\mathsf{v}^{yu\varphi^\vee}.$$

By Lemma 2.1.2 we have that $u\varphi = \varphi_K$ for some $K \in \mathcal{K}(J)$. Moreover $wA_0{}^-|^+ws_0A_0$ if and only if $w\varphi \in \Phi_J^+$, if and only if $yu\varphi = y\varphi_K \in \Phi_J^+$. The result now follows from Lemma 4.3.11. □

Chapter 5

$\widetilde{\mathcal{H}}$-modules and the path formula

In Chapter 2 we introduced the fundamental J-alcove $\mathcal{A}_J$ and its translation group $\mathbb{T}_J$, and in Chapter 4 we defined paths that remain within this J-alcove, and for each such path we assigned a v-mass. In this chapter we use these properties to construct finite dimensional $\widetilde{\mathcal{H}}$ representations $(\pi_{J,\mathsf{v}}, M_{J,\mathsf{v}})$, for each J-parameter system with $J \subseteq I$, and develop a combinatorial formula for finding their matrix entries, the *path formula*. In [23, Theorem 7.2] and [22, Theorem 4.3], Parkinson and Guilhot constructed path formulas for types G_2 and $C_2 = B_2$ where the modules were induced representations of $\widetilde{\mathcal{H}}$ from 1-dimensional representations of Levi subalgebras. They used these path formulas to prove Lusztig's conjectures $P1$-$P15$ (see Section 3.2) for the affine cases G_2 and C_2 by proving that the corresponding representations formed a balanced system of cell representations. In this chapter, we; construct modules inspired by the work of Lusztig [34, Lemma 4.7] and Deodhar [9, Corollary 2.3], prove a path formula for the matrix entries of these representations, and prove that they are isomorphic to the representations of $\widetilde{\mathcal{H}}$ induced from particular 1-dimensional representations of Levi subalgebras (and thus, equivalent to the representations of Parkinson and Guilhot in type G_2 and C_2). In Chapter 11 we show that these representations form a balanced system of cell representations for $\Phi = A_n$. Furthermore, in Chapter 6 we use this combinatorial module to classify when $(\pi_{J,\mathsf{v}}, M_{J,\mathsf{v}})$ is a bounded matrix representation of $\widetilde{\mathcal{H}}$.

In Section 5.1 we define $M_{J,\mathsf{v}}$ and prove that it is in fact a module of $\widetilde{\mathcal{H}}$. In Section 5.2 we define Levi subalgebras associated to $J \subseteq I$ and set up 1-dimensional representations of these subalgebras. We then induce to a representation of $\widetilde{\mathcal{H}}$ and show that the action of this induced module agrees with the action defined for $M_{J,\mathsf{v}}$. The path formula is proven in Section 5.3. The formula, in Theorem 5.3.3, describes each matrix entry of $\pi_{J,\mathsf{v}}$ as a sum over J-folded alcove paths with coefficients given by the v-mass of each path. Finally, in Section 5.4 we prove the irreducibility of the modules $M_{J,\mathsf{v}}$ by exploring their weight spaces.

To motivate the construction of $M_{J,\mathsf{v}}$ recall Lusztig's periodic Hecke module $M = \bigoplus_{A \in \mathbb{A}} \mathsf{R}A$ (from [34, §3.2]) with basis indexed by alcoves of $\mathbb{A}$ (see Section 1.3) and $\widetilde{\mathcal{H}}$-action given by

$$(wA_0) \cdot T_i = \begin{cases} ws_iA_0 & \text{if } wA_0{}^-|^+ ws_iA_0 \\ ws_iA_0 + (\mathsf{q}_i - \mathsf{q}_i^{-1})wA_0 & \text{if } wA_0{}^+|^- ws_iA_0 \end{cases}$$

for $i \in I \cup \{0\}$. The construction of $M_{J,\mathsf{v}}$ in Definition 5.1.1 takes this combinatorial construction and incorporates the action of $\mathbb{T}_J$ on $\mathcal{A}_J$, thus making the module finite dimensional.

5.1 The module $M_{J,\mathsf{v}}$

Let $\{\zeta_i \mid i \in I\}$ be a family of commuting invertible indeterminants. For $\gamma = \sum_{i\in I} a_i\omega_i \in P$ we write $\zeta^\gamma = \prod_{i\in I}\zeta_i^{\alpha_i}$. For $J \subseteq I$ let $\mathcal{I}_J$ denote the ideal of the Laurent polynomial ring $\mathsf{R}[\{\zeta_i^{\pm 1} \mid i \in I\}]$ generated by $\zeta^{\alpha_j^\vee} - 1$ for all $j \in J$. Let $\zeta_J^\gamma = \zeta^\gamma + \mathcal{I}_J$. In particular, note that $\zeta_J^\gamma = 1$ for all $\gamma \in Q_J$, and so $\zeta_J^\gamma = \zeta_J^{\gamma^{(J)}}$ for all $\gamma \in P$. We write

$$\mathsf{R}[\zeta_J] = \mathsf{R}[\{\zeta_i^{\pm 1} \mid i \in I\}]/\mathcal{I}_J.$$

Recall from (4.3.1) that we extend the J-parameter family from Φ to $\widetilde{\Phi}$ by $\mathsf{v}_{\alpha+k\delta} = \mathsf{v}_\alpha$ for $\alpha \in \Phi$ and $k \in \mathbb{Z}$. For $u \in \widetilde{W}$ and $i \in I \cup \{0\}$ we denote $u \to us_i$ as the crossing from uA_0 to us_iA_0 ('crossing' in terms of paths, see Section 4.1). We now define a module $M_{J,\mathsf{v}}$ motivated by Lusztig's periodic Hecke module but now incorporating the action of $\mathbb{T}_J$ on $\mathcal{A}_J$.

Definition 5.1.1. Let $J \subseteq I$ and let v be a J-parameter system. Denote $M_{J,\mathsf{v}}$ to be the module over the ring $\mathsf{R}[\zeta_J]$ with basis $\{\mathbf{m}_u \mid u \in {}^JW\}$, and for $i \in I \cup \{0\}$ and $\sigma \in \Sigma$ we define

$$\mathbf{m}_u \cdot T_i = \begin{cases} \zeta_J^{\mathrm{wt}(us_i)}\mathbf{m}_{\theta^J(us_i)} & \text{if } u \to us_i \text{ is positive and } us_i \in \mathbb{W}^J \\ \zeta_J^{\mathrm{wt}(us_i)}\mathbf{m}_{\theta^J(us_i)} + (\mathsf{q}_i - \mathsf{q}_i^{-1})\mathbf{m}_u & \text{if } u \to us_i \text{ is negative and } us_i \in \mathbb{W}^J \\ \mathsf{v}_{u\alpha_i}\mathbf{m}_u & \text{if } us_i \notin \mathbb{W}^J \text{ (hence } u\alpha_i \in \Phi_J + \mathbb{Z}\delta) \end{cases}$$

$$\mathbf{m}_u \cdot T_\sigma = \zeta_J^{\mathrm{wt}(u\sigma)}\mathbf{m}_{\theta^J(u\sigma)}.$$

We will show that the defined action in Definition 5.1.1 extends to a right action of $\widetilde{\mathcal{H}}$ on $M_{J,\mathsf{v}}$. To do so we define a linear map $\varpi_{J,v} : \widetilde{\mathcal{H}} \to M_{J,\mathsf{v}}$ by linearly extending the definition

$$\varpi_{J,\mathsf{v}}(X_w) = (\mathsf{v}\zeta_J)^{\mathrm{wt}(w)}\mathsf{v}(\theta_J(w))^{-1}\mathbf{m}_{\theta^J(w)} \tag{5.1.1}$$

for $w \in \widetilde{W}$.

We will use the following lemma to prove Proposition 5.1.3.

Lemma 5.1.2. *Let $w \in \widetilde{W}$ and $i \in \{0\} \cup I$. Write $u = \theta^J(w)$, and suppose that $us_i \in \mathbb{W}^J$. Then $w \to ws_i$ is a positive crossing if and only if $u \to us_i$ is a positive crossing.*

Proof. Since $us_i \in \mathbb{W}^J$ we have $u\alpha_i \notin \Phi_J + \mathbb{Z}\delta$. Suppose that $w \to ws_i$ is positive, and so $w\alpha_i = -\alpha + k\delta \in -\Phi^+ + \mathbb{Z}\delta$ (recall (1.3.1)). Write $w = t_\gamma yu$ with $\gamma = \mathrm{wt}(w)$, $y = \theta_J(w)$, and $u = \theta^J(w)$. Then

$$u\alpha_i = y^{-1}t_{-\gamma}(w\alpha_i) = y^{-1}t_{-\gamma}(-\alpha + k\delta) \in -y^{-1}\alpha + \mathbb{Z}\delta.$$

As $u\alpha_i \notin \Phi_J + \mathbb{Z}\delta$ we have that $y^{-1}\alpha \notin \Phi_J$. Then, as $y \in W_J$ we have $\alpha \notin \Phi_J$. This implies that $y^{-1}\alpha \in \Phi^+$ as $y \in W_J$ cannot invert a root in $\Phi^+\backslash\Phi_J$. Hence, $u \to us_i$ is also positive.

Conversely, suppose that $u \to us_i$ is positive. So $u\alpha_i = -\beta + k\delta$ with $\beta \in \Phi^+$, and since $us_i \in \mathbb{W}^J$ we have $\beta \notin \Phi_J$. Then

$$w\alpha_i = t_\gamma yu\alpha_i = t_\gamma(-y\beta + k\delta) \in -y\beta + \mathbb{Z}\delta,$$

and since $\beta \notin \Phi_J$ and $y \in W_J$ we have $y\beta > 0$, hence the result. ☐

Proposition 5.1.3. *For $i \in I \cup \{0\}$ and $\sigma \in \Sigma$ we have*

$$\varpi_{J,\mathsf{v}}(hT_i) = \varpi_{J,\mathsf{v}}(h) \cdot T_i \quad \textit{and} \quad \varpi_{J,\mathsf{v}}(hT_\sigma) = \varpi_{J,\mathsf{v}}(h) \cdot T_\sigma$$

for all $h \in \widetilde{\mathcal{H}}$ (with $\mathbf{m} \cdot T_i$ and $\mathbf{m} \cdot T_\sigma$ as in Definition 5.1.1 for $\mathbf{m} \in M_{J,\mathsf{v}}$).

Proof. By linearity it is sufficient to prove that $\varpi_{J,\mathsf{v}}(X_wT_i) = \varpi_{J,\mathsf{v}}(X_w)\cdot T_i$ and $\varpi_{J,\mathsf{v}}(X_wT_\sigma) = \varpi_{J,\mathsf{v}}(X_w)\cdot T_\sigma$ for all $w \in \widetilde{W}$, $i \in I \cup \{0\}$ and $\sigma \in \Sigma$.

Consider the second formula. Let $\mu \in P$ be such that $t_\mu w \in \mathbb{W}^J$. Then $\theta(t_\mu w) = \theta(w)$ and so by Corollary 2.3.10 and Lemma 4.3.8 we have

$$\mathsf{v}(\theta_J(w)) = \mathsf{v}(\theta_J(t_\mu w)) = \mathsf{v}(\mathsf{y}_{\mathrm{wt}(t_\mu w)}) = \mathsf{v}^{\mathrm{wt}(t_\mu w)} = \mathsf{v}^{\mu+\mathrm{wt}(w)}.$$

As $\sigma A_0 = A_0$ we have that $t_\mu w\sigma \in \mathbb{W}^J$ and a similar argument shows that $\mathsf{v}(\theta_J(w\sigma)) = \mathsf{v}^{\mu+\mathrm{wt}(w\sigma)}$. Hence

$$\mathsf{v}^{\mathrm{wt}(w\sigma)}\mathsf{v}(\theta_J(w\sigma))^{-1} = \mathsf{v}^{-\mu} = \mathsf{v}^{\mathrm{wt}(w)}\mathsf{v}(\theta_J(w))^{-1}.$$

Using this formula, along with the definition of $\varpi_{J,\mathsf{v}}$, we have

$$\begin{aligned}\varpi_{J,\mathsf{v}}(X_wT_\sigma) = \varpi_{J,\mathsf{v}}(X_{w\sigma}) &= (\mathsf{v}\zeta_J)^{\mathrm{wt}(w\sigma)}\mathsf{v}(\theta_J(w\sigma))^{-1}\mathbf{m}_{\theta^J(w\sigma)}\\ &= \zeta_J^{\mathrm{wt}(w\sigma)-\mathrm{wt}(w)}(\mathsf{v}\zeta_J)^{\mathrm{wt}(w)}\mathsf{v}(\theta_J(w))^{-1}\mathbf{m}_{\theta^J(w\sigma)}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\varpi_{J,\mathsf{v}}(X_w)\cdot T_\sigma &= (\mathsf{v}\zeta_J)^{\mathrm{wt}(w)}\mathsf{v}(\theta_J(w))^{-1}\mathbf{m}_{\theta^J(w)}\cdot T_\sigma\\ &= \zeta_J^{\mathrm{wt}(\theta^J(w)\sigma)}(\mathsf{v}\zeta_J)^{\mathrm{wt}(w)}\mathsf{v}(\theta_J(w))^{-1}\mathbf{m}_{\theta^J(\theta^J(w)\sigma)}.\end{aligned}$$

By Lemma 1.4.3(3) $\mathrm{wt}(w\sigma) = \mathrm{wt}(w) + \theta_J(w)\mathrm{wt}(\theta^J(w)\sigma)$. As $\theta_J(w) \in W_J$ we have $\mathrm{wt}(w\sigma) = \mathrm{wt}(w) + \mathrm{wt}(\theta^J(w)\sigma) + \gamma$ for some $\gamma \in Q_J$ and we have $\zeta^{\mathrm{wt}(w\sigma)-\mathrm{wt}(w)} = \zeta_J^{\mathrm{wt}(\theta^J(w)\sigma)}$ as $\zeta_J^\gamma = 1$. Then, as $\theta^J(\theta^J(w)\sigma) = \theta^J(w\sigma)$, it follows that $\varpi_{J,\mathsf{v}}(X_wT_\sigma) = \varpi_{J,\mathsf{v}}(X_w)\cdot T_\sigma$.

We now prove the first formula, $\varpi_{J,\mathsf{v}}(X_wT_i) = \varpi_{J,\mathsf{v}}(X_w)\cdot T_i$. Write $u = \theta^J(w)$. By Lemma 1.4.3(1) we have

$$\theta^J(ws_i) = \theta^J(\theta_J(w)us_i) = \theta^J(\theta^J(\theta_J(w))us_i) = \theta^J(us_i).$$

Since $T_i = T_i^{-1} + (\mathsf{q}_i - \mathsf{q}_i^{-1})$ we have $X_wT_i = X_{ws_i}$ if $w \to ws_i$ is positive, and $X_wT_i = X_{ws_i} + (\mathsf{q}_i - \mathsf{q}_i^{-1})X_w$ if $w \to ws_i$ is negative. Thus,

$$\varpi_{J,\mathsf{v}}(X_wT_i) = \begin{cases}(\mathsf{v}\zeta_J)^{\mathrm{wt}(ws_i)}\mathsf{v}(\theta_J(ws_i))^{-1}\mathbf{m}_{\theta^J(us_i)} & \text{if } \epsilon = 1\\ (\mathsf{v}\zeta_J)^{\mathrm{wt}(ws_i)}\mathsf{v}(\theta_J(ws_i))^{-1}\mathbf{m}_{\theta^J(us_i)} + (\mathsf{q}_i - \mathsf{q}_i^{-1})(\mathsf{v}\zeta_J)^{\mathrm{wt}(w)}\mathsf{v}(\theta_J(w))^{-1}\mathbf{m}_u & \text{if } \epsilon = -1\end{cases}$$

where $\epsilon \in \{-1, 1\}$ is the sign of the crossing $w \to ws_i$.

We now compute $\varpi_{J,\mathsf{v}}(X_w)\cdot T_i$. There are various cases to consider. We first note the following universal fact: by Lemma 1.4.3(3) $\mathrm{wt}(ws_i) = \mathrm{wt}(w)+\theta_J(w)\mathrm{wt}(us_i) = \mathrm{wt}(w)+\mathrm{wt}(us_i)+\gamma$ for some $\gamma \in Q_J$ as $\theta_J(w) \in {}^JW$ and so $\zeta_J^{ws_i} = \zeta_J^{\mathrm{wt}(w)}\zeta_J^{\mathrm{wt}(us_i)}$ (as $\zeta_J^\gamma = 1$).

Case 1: Suppose that $u \to us_i$ is positive and $us_i \in \mathbb{W}^J$. By Lemma 5.1.2 $w \to ws_i$ is then necessarily positive. We have that

$$\begin{aligned}\varpi_{J,\mathsf{v}}(X_w)\cdot T_i &= (\mathsf{v}\zeta_J)^{\mathrm{wt}(w)}\mathsf{v}(\theta_J(w))^{-1}\mathbf{m}_u\cdot T_i\\ &= (\mathsf{v}\zeta_J)^{\mathrm{wt}(w)}\zeta_J^{\mathrm{wt}(us_i)}\mathsf{v}(\theta_J(w))^{-1}\mathbf{m}_{\theta^J(us_i)}\\ &= \mathsf{v}^{\mathrm{wt}(w)}\zeta_J^{\mathrm{wt}(ws_i)}\mathsf{v}(\theta_J(w))^{-1}\mathbf{m}_{\theta^J(us_i)}.\end{aligned}$$

By Lemma 4.3.12

$$\mathsf{v}^{\mathrm{wt}(w)}\mathsf{v}(\theta_J(w))^{-1} = \mathsf{v}^{\mathrm{wt}(ws_i)}\mathsf{v}(\theta_J(ws_i))^{-1} \tag{5.1.2}$$

and the result, $\varpi_{J,\mathsf{v}}(X_w)\cdot T_i=\varpi_{J,\mathsf{v}}(X_wT_i)$, follows.

Case 2: Suppose that $u\to us_i$ is negative and $us_i\in\mathbb{W}^J$. Then Lemma 5.1.2 gives that $w\to ws_i$ is also negative. We compute

$$\begin{aligned}\varpi_{J,\mathsf{v}}(X_w)\cdot T_i&=(\mathsf{v}\zeta_J)^{\mathrm{wt}(w)}\mathsf{v}(\theta_J(w))^{-1}\mathbf{m}_u\cdot T_i\\&=(\mathsf{v}\zeta_J)^{\mathrm{wt}(w)}\mathsf{v}(\theta_J(w))^{-1}\big[\zeta_J^{\mathrm{wt}(us_i)}\mathbf{m}_{\theta^J(us_i)}+(\mathsf{q}_i-\mathsf{q}_i^{-1})\mathbf{m}_u\big]\\&=\mathsf{v}^{\mathrm{wt}(w)}\mathsf{v}(\theta_J(w))^{-1}\zeta_J^{\mathrm{wt}(ws_i)}\mathbf{m}_{\theta^J(us_i)}+(\mathsf{v}\zeta_J)^{\mathrm{wt}(w)}\mathsf{v}(\theta_J(w))^{-1}\big(\mathsf{q}_i-\mathsf{q}_i^{-1})\mathbf{m}_u\end{aligned}$$

and the result follows from (5.1.2).

Case 3: Suppose that $us_i\notin\mathbb{W}^J$. Then we have

$$\varpi_{J,\mathsf{v}}(X_w)\cdot T_i=(\mathsf{v}\zeta_J)^{\mathrm{wt}(w)}\mathsf{v}(\theta_J(w))^{-1}\mathsf{v}_{u\alpha_i}\mathbf{m}_u.$$

If $i\in I$, as $us_i\notin\mathbb{W}^J$ there exists a unique $j\in J$ such that $s_ju=us_i$ and so $\theta^J(us_i)=u$. If $i=0$ then $us_0=t_{u\varphi^\vee}s_{u\varphi}u$ and as $u\alpha_i\in\Phi_J+\mathbb{Z}\delta$ we have $s_{u\varphi}\in W_J$ and so again $\theta^J(us_i)=u$. As $us_i\notin\mathbb{W}^J$, we have $u\alpha_i\in\Phi_J+\mathbb{Z}\delta$. If $u\to us_i$ is positive then this crossing occurs across a hyperplane $H_{\varphi_K,1}$ for some connected component $K\in\mathcal{K}(J)$. As $u\in W_0$, $\mathrm{wt}(us_i)=\varphi_K^\vee\in Q_J$. If $u\to us_i$ is negative then this crossing occurs across a hyperplane $H_{\alpha_j,0}$ for some $j\in J$ and so $\mathrm{wt}(us_i)=0$. In both cases we have $\zeta_J^{\mathrm{wt}(ws_i)}=\zeta_J^{\mathrm{wt}(w)}\zeta_J^{\mathrm{wt}(us_i)}=\zeta_J^{\mathrm{wt}(w)}$. Thus the calculation of $\varpi_{J,\mathsf{v}}(X_wT_i)$ from above becomes

$$\varpi_{J,\mathsf{v}}(X_wT_i)=\begin{cases}\mathsf{v}^{\mathrm{wt}(ws_i)}\zeta_J^{\mathrm{wt}(w)}\mathsf{v}(\theta_J(ws_i))^{-1}\mathbf{m}_u&\text{if }\epsilon=1\\\big[\mathsf{v}^{\mathrm{wt}(ws_i)}\zeta_J^{\mathrm{wt}(w)}\mathsf{v}(\theta_J(ws_i))^{-1}+(\mathsf{q}_i-\mathsf{q}_i^{-1})(\mathsf{v}\zeta_J)^{\mathrm{wt}(w)}\mathsf{v}(\theta_J(w))^{-1}\big]\mathbf{m}_u&\text{if }\epsilon=-1,\end{cases}$$

where, as before, ϵ is the sign of the crossing $w\to ws_i$.

For $\epsilon=1$ we have $\varpi_{J,\mathsf{v}}(X_w)\cdot T_i=\varpi_{J,\mathsf{v}}(X_wT_i)$ as

$$\mathsf{v}^{\mathrm{wt}(w)}\mathsf{v}(\theta_J(w))^{-1}\mathsf{v}_{u\alpha_i}=\mathsf{v}^{\mathrm{wt}(ws_i)}\mathsf{v}(\theta_J(ws_i))^{-1}.$$

by Lemma 4.3.12.

Finally, consider $\epsilon=-1$. We claim that $\mathsf{v}_{u\alpha_i}-\mathsf{q}_i+\mathsf{q}_i^{-1}=\mathsf{v}_{u\alpha_i}^{-1}$. As $u\in{}^JW$ but $us_i\notin\mathbb{W}^J$ we have that either $u\alpha_i=\alpha_j$ for some $j\in J$ or $u\alpha_i=\varphi_K+\delta$ for some $K\in\mathcal{K}(J)$. In the first case, by Proposition 3.1.1 we have that $\mathsf{q}_i=\mathsf{q}_j$ and so $\mathsf{v}_{u\alpha_i}\in\{\mathsf{q}_i,-\mathsf{q}_i^{-1}\}$ and the claim holds. In the latter case there exists $k\in K$ and $y\in W_K$ such that $y\alpha_k=\varphi_K$. Thus, by Lemma 1.2.2, α_i is conjugate to $\alpha_k+\delta$ which is conjugate to α_k. Thus, $\mathsf{v}_{u\alpha_i}\in\{\mathsf{q}_i,\mathsf{q}_i^{-1}\}$ and the claim also holds in this case.

By Lemma 4.3.12 we have

$$\mathsf{v}^{\mathrm{wt}(w)}\mathsf{v}(\theta_J(w))^{-1}\mathsf{v}_{u\alpha_i}^{-1}=\mathsf{v}^{\mathrm{wt}(ws_i)}\mathsf{v}(\theta_J(ws_i))^{-1}$$

and so

$$\begin{aligned}\mathsf{v}^{\mathrm{wt}(w)}sv(\theta_J(w))^{-1}\mathsf{v}_{u\alpha_i}&=\mathsf{v}^{\mathrm{wt}(w)}\mathsf{v}(\theta_J(w))^{-1}(\mathsf{v}_{u\alpha_i}^{-1}+(q_i-q_i^{-1}))\\&=\mathsf{v}^{\mathrm{wt}(ws_i)}\mathsf{v}(\theta_J(ws_i))^{-1}+\mathsf{v}^{\mathrm{wt}(w)}\mathsf{v}(\theta_J(w))^{-1}(q_i-q_i^{-1}).\end{aligned}$$

The required result, $\varpi_{J,\mathsf{v}}(X_w)\cdot T_i=\varpi_{J,\mathsf{v}}(X_wT_i)$, follows. □

Theorem 5.1.4. *The action in Definition 5.1.1 extends to a right action of $\widetilde{\mathcal{H}}$ on the module $M_{J,\mathsf{v}}$.*

Proof. It is sufficient to check that the relations (for $i, j \in I \cup \{0\}$, $\sigma, \sigma' \in \Sigma$)

$$\begin{cases} T_i T_j T_i \cdots = T_j T_i T_j \cdots \text{ with } m_{i,j} \text{ terms on either side} \\ T_i^2 = T_e + (\mathsf{q}_i - \mathsf{q}_i^{-1}) T_i \\ T_\sigma T_i = T_{\sigma(i)} T_\sigma \\ T_\sigma T_{\sigma'} = T_{\sigma\sigma'} \end{cases}$$

are respected by the proposed action (where $m_{i,j}$ is the order of $s_i s_j$ in W). First note that for $u \in {}^J W$ we have $\varpi_{J,\mathsf{v}}(X_u) = \mathbf{m}_u$ as $\mathrm{wt}(u) = 0$ and $\theta_J(u) = e$. For the first relation (the braid relation) we have

$$\varpi_{J,\mathsf{v}}(X_u T_i T_j T_i \cdots) = (\cdots(((\mathbf{m}_u \cdot T_i) \cdot T_j) \cdot T_i) \cdots)$$
$$\varpi_{J,\mathsf{v}}(X_u T_j T_i T_j \cdots) = (\cdots(((\mathbf{m}_u \cdot T_j) \cdot T_i) \cdot T_j) \cdots)$$

by Proposition 5.1.3. The braid relation is then satisfied by the action as $\varpi_{J,\mathsf{v}}(X_u T_i T_j T_i \cdots) = \varpi_{J,\mathsf{v}}(X_u T_j T_i T_j \cdots)$.

The remaining relations follow similarly: For $T_i^2 = T_e + (q_i - q_i^{-1}) T_i$, as $\varpi_{J,\mathsf{v}}$ is linear, we have

$$\begin{aligned} ((\mathbf{m}_u \cdot T_i) \cdot T_i) &= \varpi_{J,\mathsf{v}}(X_u T_i T_i) \\ &= \varpi_{J,\mathsf{v}}(X_u T_e) + (q_i - q_i^{-1}) \varpi_{J,\mathsf{v}}(X_u T_i) \\ &= \mathbf{m}_u \cdot T_e + (q - q^{-1}) \mathbf{m}_u \cdot T_i. \end{aligned}$$

For $T_\sigma T_i = T_{\sigma(i)} T_i$ we have

$$(\mathbf{m}_u \cdot T_\sigma) \cdot T_i = \varpi_{J,\mathsf{v}}(X_u T_\sigma T_i) = \varpi_{J,\mathsf{v}}(X_u T_{\sigma(i)} T_\sigma) = (\mathbf{m}_u \cdot T_{\sigma(i)}) \cdot T_\sigma.$$

Lastly, for $T_\sigma T_{\sigma'} = T_{\sigma\sigma'}$ we have

$$(\mathbf{m}_u \cdot T_\sigma) \cdot T_{\sigma'} = \varpi_{J,\mathsf{v}}(X_u T_\sigma T_{\sigma'}) = \varpi_{J,\mathsf{v}}(X_u T_{\sigma\sigma'}) = \mathbf{m}_u \cdot T_{\sigma\sigma'}.$$

□

Corollary 5.1.5. *The map $\varpi_{J,\mathsf{v}} : \widetilde{\mathcal{H}} \to M_{J,\mathsf{v}}$ satisfies*

$$\varpi_{J,\mathsf{v}}(hh') = \varpi_{J,\mathsf{v}}(h) \cdot h' \quad \text{for all } h, h' \in \widetilde{\mathcal{H}}.$$

Proof. This is immediate from Theorem 5.1.4. □

5.2 Levi subalgebras and induced representations

In this section we define Levi subalgebras, form $\widetilde{\mathcal{H}}$-modules induced from 1-dimensional representations of these subalgebras and show that the induced representations are isomorphic to the modules of the form $M_{J,\mathsf{v}}$ constructed in Section 5.1. In fact, we show that all representations of $\widetilde{\mathcal{H}}$ induced from 1-dimensional representations of Levi subalgebras can be linked to modules of the form $M_{J,\mathsf{v}}$.

The *J-Levi subalgebra* $\mathcal{L}_J$ is the subalgebra of the extended Hecke algebra $\widetilde{\mathcal{H}}$ generated by T_j, $j \in J$, and X^γ, $\gamma \in P$.

Let $\varpi_{J,\mathsf{v}}$ be as in Section 5.1. By Corollary 5.1.5, we have that $\varpi_{J,\mathsf{v}}(X_w) = \mathbf{m}_e \cdot X_w$ for all $w \in \widetilde{W}$. Particularly, we have $\mathbf{m}_e \cdot X^\gamma = (\mathsf{v}\zeta_J)^\gamma \mathbf{m}_e$ for all $\gamma \in P$ and $\mathbf{m}_e \cdot T_j = \mathsf{v}_{\alpha_j}\mathbf{m}_e$ for all $j \in J$. Therefore, $\mathsf{R}[\zeta_J]\mathbf{m}_e$ is stable under the action of $\mathcal{L}_J$ and we can construct a map $\psi_{J,\mathsf{v}} : \mathcal{L}_J \to \mathsf{R}[\zeta_J]$ by

$$\mathbf{m}_e \cdot h = \psi_{J,\mathsf{v}}(h)\mathbf{m}_e \quad \text{for } h \in \mathcal{L}_J.$$

Recall the definition of τ_γ and y_γ for $\gamma \in P$ from Definition 2.3.1.

Proposition 5.2.1. *We have the following.*

(1) *The map $\psi_{J,\mathsf{v}}$ is a 1-dimensional representation of $\mathcal{L}_J$.*
(2) $\psi_{J,\mathsf{v}}(T_j) = \mathsf{v}_{\alpha_j} \in \{\mathsf{q}_j, -\mathsf{q}_j^{-1}\}$ *for all $j \in J$.*
(3) $\psi_{J,\mathsf{v}}(T_y) = \mathsf{v}(y)$ *for all $y \in W_J$.*
(4) $\psi_{J,\mathsf{v}}(X^\gamma) = \mathsf{v}^\gamma\zeta_J^\gamma$ *for all $\gamma \in P$.*
(5) $\psi_{J,\mathsf{v}}(X^{\alpha_j^\vee}) = \psi_{J,\mathsf{v}}(T_j)^2$.
(6) $\psi_{J,\mathsf{v}}(T_{\mathsf{y}_\gamma}) = \mathsf{v}^\gamma$ *for all $\gamma \in P^{(J)}$.*
(7) $\psi_{J,\mathsf{v}}(X_{\tau_\gamma}) = \zeta_J^\gamma$ *for all $\gamma \in P^{(J)}$.*

Proof. (2) and (4) were shown above (as $\mathsf{v}_{\alpha_j} \in \{\mathsf{q}_j, \mathsf{q}_j^{-1}\}$ by 4.3.1). (1) follows from Corollary 5.1.5 as the relations between generators of $\mathcal{L}_J$ will be satisfied by the action. For (3) let $y = s_{i_1}s_{i_2}\cdots s_{i_\ell}$ with $i_j \in J$. Then, by the definition of the action we have $\mathbf{m}_e \cdot T_y = \prod_{1\le j\le \ell} \mathsf{v}_{\alpha_{i_j}}\mathbf{m}_e$. As $y \in W_0$ we have that $\Phi(y) = \{\alpha_{i_1}, s_{i_1}\alpha_{i_2}, \ldots, s_{i_1}\cdots s_{i_{\ell-1}}\alpha_{i_\ell}\}$ (by (1.1.1)) and by Definition 4.3.1(1) $\mathsf{v}_{\alpha_{i_j}} = \mathsf{v}_{s_{i_1}\cdots s_{i_{j-1}}\alpha_{i_j}}$. Therefore

$$\mathbf{m}_e \cdot T_y = \prod_{1\le j\le \ell} \mathsf{v}_{\alpha_{i_j}}\mathbf{m}_e = \mathsf{v}(y)\mathbf{m}_e$$

and the result follows.

For (5), the Berstein relation (see 4.1.3) gives that

$$T_jX^{\omega_j} = X^{\omega_j-\alpha_j^\vee}T_j + (\mathsf{q}_j - \mathsf{q}_j^{-1})X^{\omega_j}$$

and so, as $T_j^{-1} = T_j - (\mathsf{q}_j - \mathsf{q}_j^{-1})T_e$, we have $T_j^{-1}X^{\omega_j} = X^{\omega_j-\alpha_j^\vee}T_j$. By (1), this then implies that $\psi_{J,\mathsf{v}}(X^{\alpha_j^\vee}) = \psi_{J,\mathsf{v}}(T_j)^2$.

(6) follows from (3) and the fact that $\mathsf{v}(\mathsf{y}_\gamma) = \mathsf{v}^\gamma$ (by Lemma 4.3.8 setting $y = e$). For (7),

$$\mathbf{m}_e \cdot X_{\tau_\gamma} = \varpi(X_{\tau_\gamma}) = (\mathsf{v}\zeta_J)^\gamma\mathsf{v}(\mathsf{y}_\gamma)^{-1}\mathbf{m}_e = \zeta_J^\gamma\mathbf{m}_e$$

as $\mathsf{v}(\mathsf{y}_\gamma) = \mathsf{v}^\gamma$. ☐

Lemma 5.2.2. *Let $K \in \mathcal{K}(J)$ and let $\alpha \in \Phi_K^+$ be a long root of Φ_K (with all roots long if Φ_K is simply-laced). Then $\psi_{J,\mathsf{v}}(X^{\alpha^\vee}T_{s_\alpha}^{-1}) = \mathsf{v}_\alpha$.*

Proof. By (3) and (4) of Proposition 5.2.1 we have

$$\psi_{J,\mathsf{v}}(X^{\alpha^\vee}T_{s_\alpha}^{-1}) = \mathsf{v}^{\alpha^\vee}\zeta_J^{\alpha^\vee}\mathsf{v}(s_\alpha)^{-1}.$$

The result follows by Lemma 4.3.10 and as $\zeta_J^{\alpha^\vee} = 1$ (as $\alpha \in \Phi_J$). ☐

From Section 2.1, recall the definitions of W_J^{aff} and its generating set $\{s'_j \mid j \in J_{\text{aff}}\} = \{s_j \mid j \in J\} \cup \{s_{\varphi_K,1} \mid K \in \mathcal{K}(J)\}$. For this generating set, we notate the corresponding elements of $\widetilde{\mathcal{H}}$ by

$$T'_j = T_j \quad \text{and} \quad T'_{0_K} = X^{\varphi_K^\vee} T^{-1}_{s_{\varphi_K}}$$

for $j \in J$ and $K \in \mathcal{K}(J)$ and denote the subalgebra of $\widetilde{\mathcal{H}}$ generated by these elements as $\mathcal{H}_J^{\text{aff}}$. By 2.1.3, we have that $\mathcal{H}_J^{\text{aff}} = \prod_{K \in \mathcal{K}(J)} \mathcal{H}_K^{\text{aff}}$ where $\mathcal{H}_K^{\text{aff}}$ is the affine Hecke algebra of type W_K^{aff}. As $s_{\varphi_K} \in W_J$ for all $K \in \mathcal{K}(J)$, we have that $\mathcal{H}_J^{\text{aff}}$ is a subalgebra of $\mathcal{L}_J$.

Corollary 5.2.3. *The restriction of $\psi_{J,\mathsf{v}}$ to $\mathcal{H}_J^{\text{aff}}$ is a 1-dimensional representation of $\mathcal{H}_J^{\text{aff}}$ where*

$$\psi_{J,\mathsf{v}}(T'_j) = \mathsf{v}_{\alpha_j} \qquad \textit{and} \qquad \psi_{J,\mathsf{v}}(T'_{0_K}) = \mathsf{v}_{\varphi_K}$$

for $j \in J$ and $K \in \mathcal{K}(J)$.

Proof. This follows directly from the fact that $\mathcal{H}_J^{\text{aff}}$ is a subalgebra of $\mathcal{L}_J$, Proposition 5.2.1 and Lemma 5.2.2. □

Let $\xi_{J,\mathsf{v}}$ be the generator of the 1-dimensional $\mathcal{L}_J$ module $\mathsf{R}[\zeta_J]\xi_{J,\mathsf{v}}$ such that

$$\xi_{J,\mathsf{v}} \cdot h = \psi_{J,\mathsf{v}}(h)\xi_{J,\mathsf{v}}$$

for all $h \in \mathcal{L}_J$. Notate

$$M'_{J,\mathsf{v}} = \operatorname{Ind}_{\mathcal{L}_J}^{\widetilde{\mathcal{H}}}(\psi_{J,\mathsf{v}}) = (\mathsf{R}[\zeta_J]\xi_{J,\mathsf{v}}) \otimes_{\mathcal{L}_J} \widetilde{\mathcal{H}}$$

to be the $\widetilde{\mathcal{H}}$ module induced from $\mathsf{R}[\zeta_J]\xi_{J,\mathsf{v}}$.

Proposition 5.2.4. *The module $M'_{J,\mathsf{v}}$ has basis $\{\xi_{J,\mathsf{v}} \otimes X_u \mid u \in W^J\}$.*

Proof. Since $\{X_w \mid w \in \widetilde{W}\}$ is a basis of $\widetilde{\mathcal{H}}$ the set $\{\xi_{J,\mathsf{v}} \otimes X_w \mid w \in \widetilde{W}\}$ spans $M'_{J,\mathsf{v}}$.

By 4.1.2, for $w \in \widetilde{W}$ we have $X_w = X^\gamma T^{-1}_{u^{-1}}$ where $\gamma = \text{wt}(w)$ and $u = \theta(w)$. Furthermore, letting $u_1 = \theta_J(w)$ and $u_2 = \theta^J(w)$ we have

$$\xi_{J,\mathsf{v}} \otimes X_w = \xi_{J,\mathsf{v}} \otimes X^\gamma T^{-1}_{u_1^{-1}} T^{-1}_{u_2^{-1}} = \psi_{J,\mathsf{v}}(X^\gamma T^{-1}_{u_1^{-1}})(\xi_{J,\mathsf{v}} \otimes X_{u_2})$$

as $X^\gamma T^{-1}_{u_1^{-1}} \in \mathcal{L}_J$. Thus $M'_{J,\mathsf{v}}$ is spanned by $\{\xi_{J,\mathsf{v}} \otimes X_u \mid u \in W^J\}$, and these elements are linearly independent as $\{X^\gamma T^{-1}_{v^{-1}} T^{-1}_{u^{-1}} \mid \gamma \in P,\, v \in W_J,\, u \in W^J\}$ is a basis of $\widetilde{\mathcal{H}}$. □

Let $(\pi_{J,\mathsf{v}}, M_{J,\mathsf{v}})$ denote the representation of $\widetilde{\mathcal{H}}$ defined in Definition 5.1.1 and $(\pi'_{J,\mathsf{v}}, M'_{J,\mathsf{v}})$ denote the representation of $\widetilde{\mathcal{H}}$ induced from the 1-dimensional representation of $\mathcal{L}_J$ with multiplicative character $\psi_{J,\mathsf{v}}$. We write $[\pi_{J,\mathsf{v}}(h)]_{u,v}$ for the (u,v)-th entry (with $u, v \in {}^JW$) of the matrix $\pi_{J,\mathsf{v}}(h)$ with respect to the basis $\{\mathbf{m}_u \mid u \in {}^JW\}$, similarly for $[\pi'_{J,\mathsf{v}}(h)]_{u,v}$ with respect to the basis $\{\xi_{J,\mathsf{v}} \otimes X_u \mid u \in {}^JW\}$.

Theorem 5.2.5. *We have $M_{J,\mathsf{v}} \cong M'_{J,\mathsf{v}}$. Moreover*

$$[\pi_{J,\mathsf{v}}(h)]_{u,v} = [\pi'_{J,\mathsf{v}}(h)]_{u,v} \quad \textit{for all } h \in \widetilde{\mathcal{H}} \textit{ and } u, v \in W^J,$$

where $M_{J,\mathsf{v}}$ and $M'_{J,\mathsf{v}}$ are endowed with the bases $\{\mathbf{m}_u \mid u \in W^J\}$ and $\{\xi_{J,\mathsf{v}} \otimes X_u \mid u \in W^J\}$ respectively.

Proof. We will show that the action of T_i ($i \in I \cup \{0\}$) and T_σ ($\sigma \in \Sigma$) on $M'_{J,\mathsf{v}}$ with respect to the basis $\{\xi_{J,\mathsf{v}} \otimes X_u \mid u \in {}^JW\}$ agrees with the action defined in Definition 5.1.1. First we will consider the action of T_σ. As $\sigma A_0 = A_0$ we have that $u\sigma \in \mathbb{W}^J$. Thus by Corollary 2.3.10 we have that

$$(\xi_{J,\mathsf{v}} \otimes X_u) \cdot T_\sigma = \xi_{J,\mathsf{v}} \otimes X_{u\sigma} = \xi_{J,\mathsf{v}} \otimes X_{\tau_{\mathrm{wt}(u\sigma)}} X_{\theta^J(u\sigma)} = \psi_{J,\mathsf{v}}(X_{\tau_{\mathrm{wt}(u\sigma)}})\xi_{J,\mathsf{v}} \otimes X_{\theta^J(u\sigma)}$$

and the result follows by Proposition 5.2.1.

Now we consider the action of T_i with $i \in I \cup \{0\}$. There are three cases:

Case 1: Suppose that $uA_0{}^-|^+ us_iA_0$ with $us_i \in \mathbb{W}^J$. As $u \to us_i$ is a positive crossing, by definition $X_uT_i = X_{us_i}$ (see 4.1.1) and so

$$(\xi_{J,\mathsf{v}} \otimes X_u) \cdot T_i = \xi_{J,\mathsf{v}} \otimes X_{us_i}.$$

For $i \in I$ this gives the result. Consider $i = 0$. By Corollary 2.3.10 we have $us_0 = \tau_{\mathrm{wt}(us_0)}\theta^J(us_0)$ where $\mathrm{wt}(us_0) = u\varphi^\vee$. Hence,

$$\xi_{J,\mathsf{v}} \otimes X_{us_0} = \xi_{J,sv} \otimes X_{\tau_{u\varphi^\vee}} X_{\theta^J(us_0)} = \psi_{J,\mathsf{v}}(X_{\tau_{u\varphi^\vee}})\xi_{J,\mathsf{v}} \otimes X_{\theta^J(us_0)}$$

and $\psi_{J,\mathsf{v}}(X_{\tau_{u\varphi^\vee}}) = \zeta_J^{u\varphi^\vee}$ by Proposition 5.2.1 completes this case.

Case 2: Suppose that $uA_0{}^+|^- us_iA_0$ with $us_i \in \mathbb{W}^J$. As $u \to us_i$ is a negative crossing we have $X_uT_i^{-1} = X_{us_i}$. Therefore, as $T_i = T_i^{-1} + (q_i - q_i^{-1})T_e$

$$\begin{aligned}(\xi_{J,\mathsf{v}} \otimes X_u) \cdot T_i &= (\xi_{J,\mathsf{v}} \otimes X_u) \cdot T_i^{-1} + (q_i - q_i^{-1})(\xi_{J,\mathsf{v}} \otimes X_u)\\ &= \xi_{J,\mathsf{v}} \otimes X_{us_i} + (q_i - q_i^{-1})\xi_{J,\mathsf{v}} \otimes X_u.\end{aligned}$$

If $i \in I$ the result is proved. If $i = 0$ then, as in the previous case,

$$\xi_{J,\mathsf{v}} \otimes X_{us_0} = \zeta_J^{u\varphi^\vee}\xi_{J,\mathsf{v}} \otimes X_{\theta^J(us_0)}$$

and the result is proved for this case.

Case 3: Suppose that $us_i \notin \mathbb{W}^J$. If $i \in I$ there exists a unique $j \in J$ such that $s_ju = us_i$, the crossing $u \to us_i$ occurs across the hyperplane $H_{\alpha_j,0}$ and $u \to us_i$ is negative. Hence,

$$\begin{aligned}(\xi_{J,\mathsf{v}} \otimes X_u) \cdot T_i &= \xi_{J,\mathsf{v}} \otimes X_{us_i} + (q_i - q_i^{-1})\xi_{J,\mathsf{v}} \otimes X_u\\ &= \psi_{J,\mathsf{v}}(T_j^{-1})\xi_{J,\mathsf{v}} \otimes X_u + (q_i - q_i^{-1})\xi_{J,\mathsf{v}} \otimes X_u\\ &= (\mathsf{v}_{\alpha_j}^{-1} + q_i - q_i^{-1})\xi_{J,\mathsf{v}} \otimes X_u\\ &= \mathsf{v}_{\alpha_j}\xi_{J,\mathsf{v}} \otimes X_u\end{aligned}$$

and the result follows as $u\alpha_i = \alpha_j$.

If $i = 0$ then $u\alpha_0 = \varphi_k + \delta$ for some $k \in \mathcal{K}(J)$ and $u \to us_i$ is positive. We have that $us_0 = s_{\varphi_K,1}u = t_{\varphi_K^\vee}s_{\varphi_K}u$ and so

$$X_uT_0 = X_{us_0} = X_{t_{\varphi_K^\vee}s_{\varphi_K}u} = X^{\varphi_K}X_{s_{\varphi_K}u}.$$

We claim that $X_{s_{\varphi_K}u} = T_{s_{\varphi_K}}^{-1}X_u$. Let p be the straight path of type $\vec{u}$ from e to u where $\vec{u}$ is a reduced expression. As $u \in W_0$ all crossings in p are negative. Then $s_{\varphi_K}p$ is a path from s_{φ_K} to $s_{\varphi_K}u$. As $u \in \mathbb{W}^J$, path p does not cross any hyperplanes of the type $H_{\alpha,k}$ with $\alpha \in \Phi_J^+$ and

$k \in \mathbb{Z}$. As $\Phi(s_{\varphi_K}) \subseteq \Phi_J^+$, the negative crossings in p are mapped to negative crossings in $s_{\varphi_K}p$ (by (1.3.1)). Therefore, as $s_{\varphi_K} \in W_0$ we have

$$X_{s_{\varphi_K}u} = X_{s_{\varphi_K}}X_u = T_{s_{\varphi_K}}^{-1}X_u.$$

Since φ_K is long in Φ_K, with Lemma 5.2.2 we have

$$\begin{aligned}\xi_{J,\mathsf{v}} \otimes X_u \cdot T_0 &= \xi_{J,\mathsf{v}} \otimes X^{\varphi_K^\vee} T_{s_{\varphi_K}}^{-1} X_u \\ &= \psi_{J,\mathsf{v}}(X^{\varphi_K^\vee} T_{s_{\varphi_K}}^{-1})\xi_{J,\mathsf{v}} \otimes X_u \\ &= \mathsf{v}_{\varphi_K}\xi_{J,\mathsf{v}} \otimes X_u\end{aligned}$$

and the result follows from (4.3.1) and as $u\alpha_0 = \varphi_K + \delta$. □

Remark 5.2.6. If $J = \emptyset$, then v is vacuous and $(\pi'_{\emptyset,\mathsf{v}}, M'_{\emptyset,\mathsf{v}})$ is the *principal series representation* of $\widetilde{\mathcal{H}}$ over the ring $\mathsf{R}[\zeta_1^{\pm 1}, \ldots, \zeta_n^{\pm 1}]$ with central character ζ (note that $\zeta_J^\gamma = \zeta^\gamma$ for all $\gamma \in P$).

Remark 5.2.7. Up to specialising the 'variables' ζ_i and extending scalars, all representations of $\widetilde{\mathcal{H}}$ induced from 1-dimensional representations of Levi subalgebras can be realised by the construction given in Definition 5.1.1. To see this let $\psi : \mathcal{L}_J \to \mathsf{R}'$ be a 1-dimensional representation of $\mathcal{L}_J$ over an integral domain R' that contains R. For $\alpha \in \Phi_J$ define $\mathsf{v}_\alpha = \psi(T_j)$ when $\alpha \in W_J\alpha_j$ for $j \in J$. Furthermore, define $z : P \to \mathsf{R}'$ by $z^\gamma = \mathsf{v}^{-\gamma}\psi(X^\gamma)$ where $\mathsf{v}^{-\gamma}$ is as in (4.3.2). The relations of $\widetilde{\mathcal{H}}$ (see (3.1.1)) force $\psi(T_j) \in \{\mathsf{q}_j, -\mathsf{q}_j^{-1}\}$. If $\alpha, \beta \in \Phi_J$ are conjugate then there exists some $u \in W_J$ such that $us_\alpha = s_\beta u$. Thus,

$$\psi(u)\psi(s_\alpha) = \psi(s_\beta)\psi(u) \implies \psi(s_\alpha) = \psi(s_\beta)$$

and so v is a J-parameter system (see Definition 4.3.1). Furthermore, by Proposition 5.2.1 and as $\zeta_J^\gamma = 1$ for all $\gamma \in Q_J$ we have that $z^\gamma = 1$ for all $\gamma \in Q_J$.

After extending R' to a ring R'' if necessary, one can choose a specialisation $\zeta_i \mapsto z_i \in \mathsf{R}''$ such that $\zeta_J^\gamma = z^\gamma$ for all $\gamma \in P$. By Theorem 5.2.5 we have that $M_{J,\mathsf{v}}$ (created using the construction in Definition 5.1.1) specialises to $\mathrm{Ind}_{\mathcal{L}_J}^{\widetilde{\mathcal{H}}}(\psi)$ (extending scalars if necessary).

5.3 The path formula

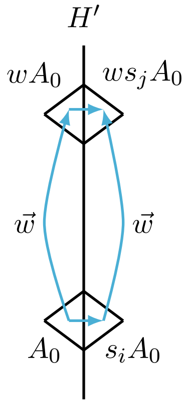
In this section we construct an explicit combinatorial formula for the matrix entries of $\pi_{J,\mathsf{v}} : \widetilde{\mathcal{H}} \to M_{J,\mathsf{v}}$. The formula is a sum over J-folded paths from Section 4.2, of ζ^γ terms, with path v-mass coefficients (see Section 4.3). The formula is the culmination of Chapter 2, Chapter 4, Section 5.1 and Section 5.2. We prove the formula for the basis $\{\mathbf{m}_u \mid u \in {}^JW\}$ and then expand to prove it for bases of the form $\mathsf{B}_\mathsf{F} = \{\varpi_{J,\mathsf{v}}(X_u) \mid u \in \mathsf{F}\}$ where F is a fundamental domain for the action of $\mathbb{T}_J$ on $\mathbb{W}^J$.

Recall from 5.2 that $(\pi_{J,\mathsf{v}}, M_{J,\mathsf{v}})$ denotes the representation of $\widetilde{\mathcal{H}}$ defined in Definition 5.1.1 and, for $h \in \widetilde{\mathcal{H}}$, $[\pi_{J,\mathsf{v}}(h)]_{u,v}$ denotes the (u, v)-th entry of the matrix of $\pi_{J,\mathsf{v}}(h)$ with respect to the basis $\{\mathbf{m}_u \mid u \in {}^JW\}$.

We will need the following lemma to prove Theorem 5.3.2.

Lemma 5.3.1. *If a hyperplane $H \in \mathbb{H}$ contains panels of type i and j then $\mathsf{q}_i = \mathsf{q}_j$.*

Proof. Let $v, w \in W$ such that $vA_0 \cap vs_iA_0$ and $wA_0 \cap ws_jA_0$ are panels of H of type i and j respectively. It is equivalent to instead consider the hyperplane, H', that contains panels $A_0 \cap s_iA_0$ and $uA_0 \cap us_jA_0$ where $u = v^{-1}w$ (applying v^{-1} to v and w). In addition, we can assume that eA_0 and wA_0 lie on the same side of H' (otherwise we could switch the roles of w and ws_j). Therefore, $\ell(ws_j) = \ell(w) + 1$ and we are in the following situation:



Hence, $ws_j = s_iw$ and so s_i is conjugate to s_j in $\widetilde{W}$. The result follows by Proposition 3.1.1. □

We now state the J-analogue of the multiplication formula of Ram, Proposition 4.1.4. This Theorem is the main step to proving the path formula in Theorem 5.3.3.

Theorem 5.3.2. *For $u \in {}^JW$ and $w \in \widetilde{W}$ we have*

$$\varpi_{J,\mathsf{v}}(X_uT_w) = \sum_{p \in \mathcal{P}_J(\vec{w},u)} \mathcal{Q}_{J,\mathsf{v}}(p)\varpi_{J,\mathsf{v}}(X_{\mathrm{end}(p)}),$$

where $\vec{w}$ is any reduced expression for w.

Proof. As in the proof of Proposition 4.1.4, we will induct on $\ell(w)$. For the base case let $\ell(w) = 0$, that is $w = \sigma$ for some $\sigma \in \Sigma$. There is only one path, denote it p. By Definition 5.1.1, Proposition 5.1.3 and (5.1.1) we have

$$\varpi_{J,\mathsf{v}}(X_uT_\sigma) = \varpi(X_u) \cdot T_\sigma = \mathbf{m}_u \cdot T_\sigma = \zeta_J^{\mathrm{wt}(u\sigma)}\mathbf{m}_{\theta^J(u\sigma)}$$

Furthermore, by (5.1.1) we have

$$\varpi_{J,\mathsf{v}}(X_{\mathrm{end}(p)}) = \varpi_{J,\mathsf{v}}(X_{u\sigma}) = \zeta_{J,\mathsf{v}}^{\mathrm{wt}(u\sigma)}\mathsf{v}^{\mathrm{wt}(u\sigma)}\mathsf{v}(\theta_J(u\sigma))^{-1}\mathbf{m}_{\theta^J(u\sigma)}$$

and the base case result follows as $\mathsf{v}^{\mathrm{wt}(u\sigma)}\mathsf{v}(\theta_J(u\sigma))^{-1} = \mathsf{v}^{\mathrm{wt}(u)}\mathsf{v}(\theta_J(u))^{-1}$ by the proof of Proposition 5.1.3 and the fact that $\mathsf{v}^{\mathrm{wt}(u)}\mathsf{v}(\theta_J(u))^{-1} = 1$ as $u \in {}^JW$.

Suppose that $k \in I \cup \{0\}$ such that $\ell(ws_k) = \ell(w) + 1$. By the inductive hypothesis

$$\varpi_{J,\mathsf{v}}(X_uT_{ws_k}) = \varpi_{J,\mathsf{v}}(X_uT_wT_k) = \varpi_{J,\mathsf{v}}(X_uT_w) \cdot T_k = \sum_{p \in \mathcal{P}_J(\vec{w},u)} \mathcal{Q}_{J,\mathsf{v}}(p)\varpi_{J,\mathsf{v}}(X_{\mathrm{end}(p)}T_k).$$

Let $p \in \mathcal{P}_J(\vec{w}, u)$ and $v = \mathrm{end}(p)$. There are 4 cases to consider for the crossing $v \to vs_k$.

Case 1: Let $vA_0{}^-|^+vs_kA_0$ and $vs_k \in \mathbb{W}^J$. Then $v \to vs_k$ is positive and so $X_vT_k = X_{vs_k}$. Let pc_k^+ be the path p concatenated with the positive crossing $v \to vs_k$. We have

$$\mathcal{Q}_{J,\mathsf{v}}(pc_k^+) = \mathcal{Q}_{J,\mathsf{v}}(p) \quad \text{and} \quad \mathrm{end}(pc_k^+) = vs_k$$

and so,

$$\mathcal{Q}_{J,\mathsf{v}}(p)\varpi_{J,\mathsf{v}}(X_vT_k) = \mathcal{Q}_{J,\mathsf{v}}(pc_k^+)\varpi_{J,\mathsf{v}}(X_{\mathrm{end}(pc_k^+)})$$

completing this case.

Case 2: Suppose $vA_0{}^+|^-vs_kA_0$ and $vs_k \in \mathbb{W}^J$. As $v \to vs_k$ is negative we have $X_{vs_k} = X_vT_k^{-1}$ and so

$$X_vT_k = X_{vs_k} + (q_k - q_k^{-1})X_v.$$

Let pc_k^- be the path p concatenated with the negative crossing $v \to vs_k$ and pf_k be the path p concatenated with a k-fold on the panel $vA_0 \cap vs_kA_0$. By the definition of the v-mass we have

$$\mathcal{Q}_{J,\mathsf{v}}(pf_k) = (q_k - q_k^{-1})\mathcal{Q}_{J,\mathsf{v}}(p) \quad \text{and} \quad \mathcal{Q}_{J,\mathsf{v}}(pc_k^-) = \mathcal{Q}_{J,\mathsf{v}}(p)$$

and so,

$$\begin{aligned}\mathcal{Q}_{J,\mathsf{v}}(p)\varpi_{J,\mathsf{v}}(X_vT_k) &= \mathcal{Q}_{J,\mathsf{v}}(p)\varpi_{J,\mathsf{v}}(X_{vs_k}) + (q_k - q_k^{-1})\mathcal{Q}_{J,\mathsf{v}}(p)\varpi_{J,\mathsf{v}}(X_v) \\ &= \mathcal{Q}_{J,\mathsf{v}}(pc_k^-)\varpi_{J,\mathsf{v}}(X_{\mathrm{end}(pc_k^-)}) + \mathcal{Q}_{J,\mathsf{v}}(pf_k)\varpi_{J,\mathsf{v}}(X_{\mathrm{end}(pf_k)})\end{aligned}$$

as $\mathrm{end}(pc_k^-) = vs_k$ and $\mathrm{end}(pf_k) = v$.

Case 3: Let $vA_0{}^-|^+vs_kA_0$ and $vs_k \notin \mathbb{W}^J$. By Lemma 2.1.2 the panel $vA_0 \cap vs_kA_0$ is on a hyperplane of type $H_{\varphi_K,1}$ with $K \in \mathcal{K}(J)$. Then $vs_k = s_{\varphi_K,1}v$, and as $v \to vs_k$ is positive we have

$$X_vT_k = X_{vs_k} = X_{t_{\varphi_K^\vee}s_{\varphi_K}v} = X^{\varphi_K^\vee}X_{s_{\varphi_K}v}.$$

By a similar argument as in *Case 3* of the proof of Theorem 5.2.5 we have that $X_{s_{\varphi_K}v} = T_{s_{\varphi_K}}^{-1}X_v$ (this time also noting that positive crossings of p are mapped to positive crossings of $s_{\varphi_K}p'$ where p' is a path from e to v of type $\vec{v}$, reduced). Let $pb_{\varphi_K}^-$ be the path p concatenated with the negative bounce occurring on $H_{\varphi_K,1}$, then

$$\mathcal{Q}_{J,\mathsf{v}}(pb_{\varphi_K}^-) = \mathsf{v}_{\varphi_K}\mathcal{Q}_{J,\mathsf{v}}(p) \quad \text{and} \quad \mathrm{end}(pb_{\varphi_K}^-) = v.$$

As φ_K is long in Φ_K and as $X^{\varphi_K^\vee}T_{s_{\varphi_K}}^{-1} \in \mathcal{L}_J$, Lemma 5.2.2 gives

$$\begin{aligned}\mathcal{Q}_{J,\mathsf{v}}(p)\varpi_{J,\mathsf{v}}(X_vT_k) &= \mathcal{Q}_{J,\mathsf{v}}(p)\varpi_{J,\mathsf{v}}(X^{\varphi_K^\vee}T_{s_{\varphi_K}}^{-1}X_v) \\ &= \mathcal{Q}_{J,\mathsf{v}}(p)\psi_{J,\mathsf{v}}(X^{\varphi_K^\vee}T_{s_{\varphi_K}}^{-1}\varpi_{J,\mathsf{v}}(X_v) \\ &= \mathcal{Q}_{J,\mathsf{v}}(p)\mathsf{v}_{\varphi_K}\varpi_{J,\mathsf{v}}(X_v) \\ &= \mathcal{Q}_{J,\mathsf{v}}(pb_{\varphi_K}^-)\varpi_{J,\mathsf{v}}(X_{\mathrm{end}(pb_{\varphi_K}^-)}).\end{aligned}$$

Case 4: Let $vA_0{}^+|^-vs_kA_0$ and $vs_k \notin \mathbb{W}^J$. By Lemma 2.1.2 the panel $vA_0 \cap vs_kA_0$ is on a hyperplane $H_{\alpha_j,0}$ for the unique $j \in J$ such that $vs_k = s_jv$. As $v \to vs_k$ is a negative crossing we have that $X_vT_k = X_{vs_k} + (q_k - q_k^{-1})X_v = X_{s_jv} + (q_k - q_k^{-1})X_v$. We claim that $X_{s_jv} = T_{s_j}^{-1}X_v$. Let $\vec{v}$ be a reduced expression of v and let p be the path of type $\vec{v}$ from e to v. The reflection of p in s_j, s_jp, is a path from s_j to s_jv. As $v \in \mathbb{W}^J$, path p does not contain any crossings on hyperplanes of type $H_{\alpha,k}$ for $\alpha \in \Phi_J$ and $k \in \mathbb{Z}$. Furthermore, as $\Phi(s_j) = \{\alpha_j\} \subseteq \Phi_J^+$ a negative (respectively positive) crossing in p is mapped to a negative (respectively positive) crossing in s_jp (see (1.3.1)). Hence, $X_{s_jv} = X_{s_j}X_v = T_j^{-1}X_v$ as $s_j \in W_0$.

Let $pb^+_{\alpha_j}$ be the path p concatenated with the positive bounce occurring on $H_{\alpha_j,0}$, then

$$\mathcal{Q}_{J,\mathsf{v}}(pb^+_{\alpha_j}) = \mathsf{v}_{\alpha_j}\mathcal{Q}_{J,\mathsf{v}}(p) \quad \text{and} \quad \mathrm{end}(pb^+_{\alpha_i}) = v.$$

Therefore, as $T_j^{-1} \in \mathcal{L}_J$ we have

$$\varpi_{J,\mathsf{v}}(X_vT_k) = \varpi_{J,\mathsf{v}}(X_{s_jv}) + (q_k - q_k^{-1})\varpi_{J,\mathsf{v}}(X_v) = (\psi_{J,\mathsf{v}}(T_j)^{-1} + q_k - q_k^{-1})\varpi_{J,\mathsf{v}}(X_v).$$

By Proposition 5.2.1 and Lemma 5.3.1 (as panels of type j and type k occur on $H_{\alpha_j,0}$) we have

$$\psi_{J,\mathsf{v}}(T_j)^{-1} + q_k - q_k^{-1} = \mathsf{v}_{\alpha_j} + q_j - q_j^{-1} = \mathsf{v}_{\alpha_j}.$$

Hence,

$$\mathcal{Q}_{J,\mathsf{v}}(p)\varpi_{J,\mathsf{v}}(X_vT_k) = \mathsf{v}_{\alpha_j}\mathcal{Q}_{J,\mathsf{v}}(p)\varpi(X_v) = \mathcal{Q}_{J,\mathsf{v}}(pb^+_{\alpha_j})\varpi_{J,\mathsf{v}}(X_{\mathrm{end}(pb^+_{\alpha_j})}).$$

The result follows by induction. □

Theorem 5.3.3. *Let $w \in \widetilde{W}$. The matrix entries of $\pi_{J,\mathsf{v}}(T_w)$ with respect to the basis $\{\mathbf{m}_u \mid u \in {}^JW\}$ are*

$$[\pi_{J,\mathsf{v}}(T_w)]_{u,v} = \sum_{\{p\in\mathcal{P}_J(\vec{w},u)\,|\,\theta^J(p)=v\}} \mathcal{Q}_{J,\mathsf{v}}(p)\zeta_J^{\mathrm{wt}(p)} \quad \textit{for } u,v \in {}^JW,$$

where $\vec{w}$ is any choice of reduced expression for w.

Proof. By Corollary 2.3.10 and (4.1.2) we have that

$$X_{\mathrm{end}(p)} = X^{\mathrm{wt}(p)}T^{-1}_{\mathsf{y}^{-1}_{\mathrm{wt}(p)}}T^{-1}_{\theta^J(p)^{-1}}.$$

Then by Theorem 5.3.2, Proposition 5.2.1 and (5.1.1)

$$\begin{aligned}
\mathbf{m}_u \cdot T_w = \varpi_{J,\mathsf{v}}(X_u)\cdot T_w &= \sum_{p\in\mathcal{P}_J(\vec{w},u)} \mathcal{Q}_{J,\mathsf{v}}(p)\varpi_{J,\mathsf{v}}(X^{\mathrm{wt}(p)}T^{-1}_{\mathsf{y}^{-1}_{\mathrm{wt}(p)}}T^{-1}_{\theta^J(p)^{-1}})\\
&= \sum_{p\in\mathcal{P}_J(\vec{w},u)} \mathcal{Q}_{J,\mathsf{v}}(p)\psi_{J,\mathsf{v}}(X^{\mathrm{wt}(p)}T^{-1}_{\mathsf{y}^{-1}_{\mathrm{wt}(p)}})\varpi_{J,\mathsf{v}}(T^{-1}_{\theta^J(p)^{-1}})\\
&= \sum_{p\in\mathcal{P}_J(\vec{w},u)} \mathcal{Q}_{J,\mathsf{v}}(p)\zeta_J^{\mathrm{wt}(p)}\varpi_{J,\mathsf{v}}(X_{\theta^J(p)})\\
&= \sum_{p\in\mathcal{P}_J(\vec{w},u)} \mathcal{Q}_J(p)\zeta_J^{\mathrm{wt}(p)}\mathbf{m}_{\theta^J(p)}
\end{aligned}$$

as $X^{\mathrm{wt}(p)}T^{-1}_{\mathsf{y}^{-1}_{\mathrm{wt}(p)}} \in \mathcal{L}_J$, completing the proof. □

It is sometimes useful to work with a different fundamental domain F for the action of $\mathbb{T}_J$ onto $\mathbb{W}^J$. In the following proposition and theorem we will show that changing the fundamental domain still gives a basis of $M_{J,\mathsf{v}}$ and that the path formula still holds.

Proposition 5.3.4. *If F is a fundamental domain for the action of $\mathbb{T}_J$ on $\mathbb{W}^J$ then*

$$\mathsf{B}_\mathsf{F} = \{\varpi_{J,\mathsf{v}}(X_u) \mid u \in \mathsf{F}\}$$

is a basis of $M_{J,\mathsf{v}}$.

Proof. When $\mathsf{F} = {}^J W$ the statement is true by the construction of $M_{J,\mathsf{v}}$ in Definition 5.1.1. Now consider a general fundamental domain F for the action of $\mathbb{T}_J$ on $\mathbb{W}^J$. By Corollary 2.3.10 and as $u \in \mathbb{W}^J$, for $u \in \mathsf{F}$ we have $u = \tau_{\mathrm{wt}(u)}\theta^J(u)$.

By Proposition 5.2.1 and Proposition 5.1.3

$$\begin{aligned}\varpi_{J,\mathsf{v}}(X_u) &= \varpi_{J,\mathsf{v}}(X_{\tau_{\mathrm{wt}(u)}}X_{\theta^J(u)})\\ &= \mathbf{m}_e \cdot X_{\tau_{\mathrm{wt}(u)}}X_{\theta^J(u)}\\ &= \zeta_J^{\mathrm{wt}(u)}\mathbf{m}_e \cdot X_{\theta^J(u)}\\ &= \zeta_J^{\mathrm{wt}(u)}\varpi_{J,\mathsf{v}}(X_{\theta^J(u)})\end{aligned}$$

and so, B_F is a basis of $M_{J,\mathsf{v}}$. □

Recall the definitions of $\theta(p,\mathsf{F})$ and $\mathrm{wt}(p,\mathsf{F})$ from (4.2.1).

Theorem 5.3.5. *Let F be a fundamental domain for the action of $\mathbb{T}_J$ on $\mathbb{W}^J$. With respect to the basis B_F of $M_{J,\mathsf{v}}$ from Proposition 5.3.4, the matrix entries of $\pi_{J,\mathsf{v}}(T_w)$, with $w \in \widetilde{W}$, are*

$$[\pi_{J,\mathsf{v}}(T_w)]_{u,v} = \sum_{\{p\in\mathcal{P}_J(\vec{w},u)\,|\,\theta(p,\mathsf{F})=v\}} \mathcal{Q}_{J,\mathsf{v}}(p)\zeta_J^{\mathrm{wt}(p,\mathsf{F})} \quad \text{for } u,v \in \mathsf{F},$$

where $\vec{w}$ is any choice of reduced expression for w.

Proof. As F is a fundamental domain of the action of $\mathbb{T}_J$ on $\mathbb{W}^J$, for all $u \in \mathsf{F}$ we can write $u = \tau_{\mathrm{wt}(u)}\theta^J(u)$ (see Corollary 2.3.10). In addition, by the proof of Proposition 5.3.4 we have that $\varpi_{J,\mathsf{v}}(X_u) = \zeta_J^{\mathrm{wt}(u)}\varpi_{J,\mathsf{v}}(X_{\theta^J(u)})$. Changing bases in Theorem 5.3.3 gives

$$[\pi_{J,\mathsf{v}}(T_w)]_{u,v} = \sum_{\{p\in\mathcal{P}_J(\vec{w},\theta^J(u))|\theta(p,{}^JW)=\theta^J(v)\}} \mathcal{Q}_{J,\mathsf{v}}(p)\zeta_J^{\mathrm{wt}(p)+\mathrm{wt}(u)-\mathrm{wt}(v)}.$$

Using Lemma 4.2.4 and Lemma 4.3.5 we have that $\mathcal{Q}_{J,\mathsf{v}}(\tau_{\mathrm{wt}(u)}\cdot p) = \mathcal{Q}_{J,\mathsf{v}}(p)$ and $\zeta_J^{\mathrm{wt}(\tau_{\mathrm{wt}(u)}\cdot p)} = \zeta_J^{(\mathrm{wt}(u)+\mathrm{wt}(p))^{(J)}} = \zeta_J^{\mathrm{wt}(u)+\mathrm{wt}(p)}$. It then follows that

$$\begin{aligned}[\pi_{J,\mathsf{v}}(T_w)]_{u,v} &= \sum_{\{p\in\mathcal{P}_J(\vec{w},\theta^J(u))|\theta(p,{}^JW)=\theta^J(v)\}} \mathcal{Q}_{J,\mathsf{v}}(\tau_{\mathrm{wt}(u)}\cdot p)\zeta_J^{\mathrm{wt}(\tau_{\mathrm{wt}(u)}\cdot p)-\mathrm{wt}(v)}\\ &= \sum_{\{p\in\mathcal{P}_J(\vec{w},u)\,|\,\theta(p,\mathsf{F})=v\}} \mathcal{Q}_{J,\mathsf{v}}(p)\zeta_J^{\mathrm{wt}(p)-\mathrm{wt}(v)},\end{aligned}$$

taking $\tau_{\mathrm{wt}(u)}\cdot p \mapsto p$ in the sum for the second equality. The result follows as $\mathrm{wt}(p,\mathsf{F}) = \mathrm{wt}(p) - \mathrm{wt}(v)$ when $v = \theta(p,\mathsf{F})$. □

Example 5.3.6. Let $\Phi = G_2$ and $J = \{1\}$, so $\Phi_J^+ = \{\alpha_1\}$. Let $L(s_1) = a$ and $L(s_2) = L(s_0) = b$ and so by Definition 4.3.1 the options for the J-parameter system are $\mathsf{v}_{\alpha_1} \in \{\mathsf{q}^a, -\mathsf{q}^{-a}\}$. As $\alpha_1^\vee = 2\omega_1 - 3\omega_2$, we have

$$\mathsf{R}[\zeta_J] = \mathsf{R}[\zeta_1^{\pm 1},\zeta_2^{\pm 1}]/(\zeta_1^2\zeta_2^{-3} - 1).$$

By Example 2.3.12 the elements of $P^{(J)}$ are of the form $k\omega_2 + \omega_1$ and $k\omega_2$ for all $k \in \mathbb{Z}$. Let $z = \zeta_1^{-1}\zeta_2^2 + \mathcal{I}_J = \zeta_1\zeta_2^{-1} + \mathcal{I}_J$, and thus $\zeta_J^{k\omega_2} = z^{2k}$ and $\zeta_J^{k\omega_2+\omega_1} = z^{2k+3}$. Consider

$\vec{w} = s_0s_2s_1s_2s_1 \in \widetilde{W}$ and $u = s_2s_1s_2s_1s_2$. Figure 5.1 displays the paths in the set $\mathcal{P}_J(\vec{w}, u)$. From these paths and Theorem 5.3.3 we have

$$\begin{aligned}
[\pi_{J,\mathsf{v}}(T_w)]_{u,e} &= (\mathsf{q}^b - \mathsf{q}^{-b})^2\mathsf{v}_{\alpha_1}^2 z^{-2} + (\mathsf{q}^a - \mathsf{q}^{-a})\mathsf{v}_{\alpha_1} z^{-2} \\
[\pi_{J,\mathsf{v}}(T_w)]_{u,s_2} &= (\mathsf{q}^b - \mathsf{q}^{-b})\mathsf{v}_{\alpha_1}^2 z^{-2} \\
[\pi_{J,\mathsf{v}}(T_w)]_{u,s_2s_1} &= (\mathsf{q}^b - \mathsf{q}^{-b})\mathsf{v}_{\alpha_1} z^{-2} + (\mathsf{q}^b - \mathsf{q}^{-b})\mathsf{v}_{\alpha_1} \\
[\pi_{J,\mathsf{v}}(T_w)]_{u,s_2s_1s_2} &= \mathsf{v}_{\alpha_1} z^{-2} \\
[\pi_{J,\mathsf{v}}(T_w)]_{u,s_2s_1s_2s_1} &= [\pi_{J,\mathsf{v}}(T_w)]_{u,s_2s_1s_2s_1s_2} = 0
\end{aligned}$$

This gives the 6-th row of $\pi_{J,\mathsf{v}}(T_w)$. Paths of type $\vec{w}$ beginning at the remaining $u \in {}^JW$ can similarly be found. The complete matrix of $\pi_{J,\mathsf{v}}(T_w)$ is

$$\begin{bmatrix}
0 & 0 & \mathsf{v}_{\alpha_1} z^2 & 0 & 0 & 0 \\
(\mathsf{q}^a - \mathsf{q}^{-a})z & 0 & (\mathsf{q}^b - \mathsf{q}^{-b})z & 0 & \mathsf{v}_{\alpha_1} z & 0 \\
z & 0 & 0 & 0 & 0 & 0 \\
\begin{matrix}(\mathsf{q}^b - \mathsf{q}^{-b})z^{-1}+ \\ (\mathsf{q}^b - \mathsf{q}^{-b})(\mathsf{q}^a - \mathsf{q}^{-a})+ \\ (\mathsf{q}^a - \mathsf{q}^{-a})(\mathsf{q}^b - \mathsf{q}^{-b})\mathsf{v}_{\alpha_1} z^{-1}\end{matrix} & (\mathsf{q}^a - \mathsf{q}^{-a})\mathsf{v}_{\alpha_1} z^{-1} & \begin{matrix}(\mathsf{q}^a - \mathsf{q}^{-a})z^{-1}+ \\ (\mathsf{q}^b - \mathsf{q}^{-b})^2\end{matrix} & 0 & \begin{matrix}(\mathsf{q}^b - \mathsf{q}^{-b})z^{-1}+ \\ (\mathsf{q}^b - \mathsf{q}^{-b})\mathsf{v}_{\alpha_1}\end{matrix} & z^{-1} \\
\begin{matrix}(\mathsf{q}^b - \mathsf{q}^{-b})\mathsf{v}_{\alpha_1} z^{-1}+ \\ (\mathsf{q}^b - \mathsf{q}^{-b})\end{matrix} & \mathsf{v}_{\alpha_1} z^{-1} & 0 & 0 & 0 & 0 \\
\begin{matrix}(\mathsf{q}^b - \mathsf{q}^{-b})^2\mathsf{v}_{\alpha_1}^2 z^{-2}+ \\ (\mathsf{q}^a - \mathsf{q}^{-a})\mathsf{v}_{\alpha_1} z^{-2}\end{matrix} & (\mathsf{q}^b - \mathsf{q}^{-b})\mathsf{v}_{\alpha_1}^2 z^{-2} & \begin{matrix}(\mathsf{q}^b - \mathsf{q}^{-b})\mathsf{v}_{\alpha_1} z^{-2}+ \\ (\mathsf{q}^b - \mathsf{q}^{-b})\mathsf{v}_{\alpha_1}\end{matrix} & \mathsf{v}_{\alpha_1} z^{-2} & 0 & 0
\end{bmatrix}$$

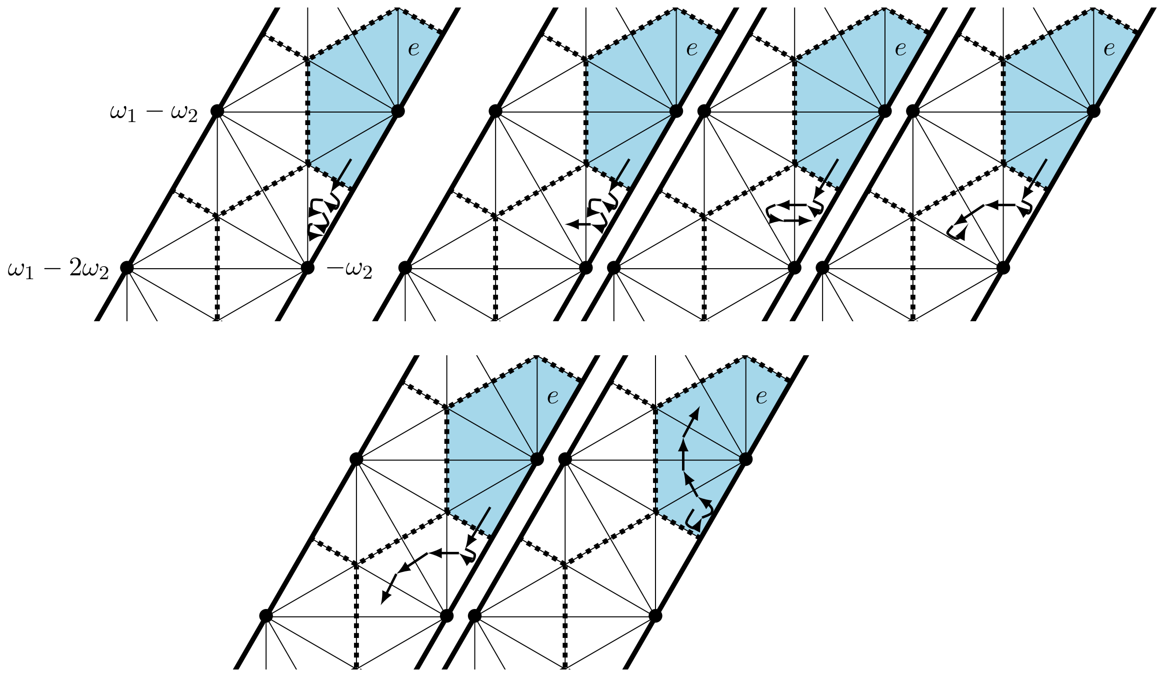


Figure 5.1: The set $\mathcal{P}_J(s_0s_1s_2s_1s_2, s_2s_1s_2s_1s_2)$ for $\Phi = G_2$ and $J = \{1\}$

Define a conjugation map conj : $\mathsf{R}[\zeta_J] \to \mathsf{R}[\zeta_J]$ by linearly extending $\mathrm{conj}(\zeta_J^\gamma) = \zeta_J^{-\gamma}$. We can then define an anti-involution $*$ on $\pi_{J,\mathsf{v}}(\widetilde{\mathcal{H}})$ by transposing the matrix $\pi_{J,\mathsf{v}}(h)$ and performing conjugation entry-wise, where $h \in \widetilde{\mathcal{H}}$. Recall that the usual anti-involution on $\widetilde{\mathcal{H}}$ (defined in Section 3.4) is the R-linear extension of $T_w^* = T_{w^{-1}}$. The following lemma, that explains the relation between the usual anti-involution and the anti-involution on our matrices, will be used in Part II.

Lemma 5.3.7. *For all $h \in \widetilde{\mathcal{H}}$ we have $\pi_{J,\mathsf{v}}(h^*) = \pi_{J,\mathsf{v}}(h)^*$.*

Proof. It is equivalent to prove $\pi_{J,\mathsf{v}}(h) = \pi_{J,\mathsf{v}}(h^*)^*$, and it is sufficient to prove this for $h = T_w$ with $w \in \widetilde{W}$. By Theorem 5.3.3, Proposition 4.2.7 and Proposition 4.3.7 we have

$$[\pi_{J,\mathsf{v}}(T_w)]_{u,v} = \sum_{\{p \in \mathcal{P}_J(\vec{w},u) | \theta^J(p) = v\}} \mathcal{Q}_J(p)\zeta_J^{\mathrm{wt}(p)} = \sum_{\{p \in \mathcal{P}_J(\mathrm{rev}(\vec{w}),v) | \theta^J(p) = u\}} \mathcal{Q}_J(p)\zeta_J^{-\mathrm{wt}(p)},$$

and the latter sum equals $[\pi_{J,\mathsf{v}}(T_w^*)^*]_{v,u}$. □

5.4 Intertwiners and the irreducibility of $M_{J,\mathsf{v}}$

In this final section of Chapter 5 we prove the irreducibility of the modules $M_{J,\mathsf{v}}$ constructed in Section 5.1 by exploring their weight spaces. To do this we introduce intertwiners which are rational expressions of Hecke algebra elements that have useful properties.

For each $1 \leq i \leq n$ define an *intertwiner* by

$$U_i = T_i - \frac{\mathsf{q}_i - \mathsf{q}_i^{-1}}{1 - X^{-\alpha_i^\vee}} T_e.$$

Note that this is not an element of $\widetilde{\mathcal{H}}$. These elements are called intertwiners as they 'intertwine' the weight spaces of the $\widetilde{\mathcal{H}}$-modules (see Proposition 5.4.6).

Proposition 5.4.1. *We have the following:*

(1) *For $i, j \in I$ such that $(s_i s_j)^m = e$ we have that $(U_i U_j)^m = T_e$, and hence the element $U_w = U_{i_1} \cdots U_{i_k}$ is independent of the chosen reduced expression $w = s_{i_1} \cdots s_{i_k} \in W_0$.*
(2) *$U_w X^\gamma = X^{w\gamma} U_w$ for all $w \in W_0$ and $\gamma \in P$.*
(3) *$U_i^2 = \mathsf{q}_i^2 \frac{(1-\mathsf{q}_i^{-2}X^{-\alpha_i^\vee})(1-\mathsf{q}_i^{-2}X^{\alpha_i^\vee})}{(1-X^{-\alpha_i^\vee})(1-X^{\alpha_i^\vee})}$.*
(4) *If $u, v \in W_0$ then $U_u U_v = b_{u,v}(X) U_{uv}$ for a rational function $b_{u,v}(X)$.*

Proof. By (4.1.3) we have

$$\begin{aligned} U_i X^\gamma &= T_i X^\gamma - \frac{\mathsf{q}_i - \mathsf{q}_i^{-1}}{1 - X^{-\alpha_i^\vee}} X^\gamma \\ &= X^{s_i\gamma} T_i + (\mathsf{q}_i - \mathsf{q}_i^{-1}) \frac{X^\gamma - X^{s_i\gamma}}{1 - X^{-\alpha_i^\vee}} - \frac{\mathsf{q}_i - \mathsf{q}_i^{-1}}{1 - X^{-\alpha_i^\vee}} X^\gamma \\ &= X^{s_i\gamma}\left(T_i - \frac{\mathsf{q}_i - \mathsf{q}_i^{-1}}{1 - X^{-\alpha_i^\vee}}\right) = X^{s_i\gamma} U_i. \end{aligned}$$

(1) We prove the case when $m = 2$, the remaining cases follow similarly. Let $(s_i s_j)^2 = e$. As $U_i X^\gamma = X^{s_i\gamma} U_i$ and as $s_i \alpha_j^\vee = \alpha_j^\vee$ we have

$$\begin{aligned} U_i U_j &= U_i \left(T_j - \frac{\mathsf{q}_j - \mathsf{q}_j^{-1}}{1 - X^{-\alpha_j^\vee}} \right) \\ &= U_i T_j - \frac{\mathsf{q}_j - \mathsf{q}_j^{-1}}{1 - X^{-\alpha_j^\vee}} U_i \\ &= T_i T_j - \frac{\mathsf{q}_i - \mathsf{q}_i^{-1}}{1 - X^{-\alpha_i^\vee}} T_j - \frac{\mathsf{q}_j - \mathsf{q}_j^{-1}}{1 - X^{-\alpha_j^\vee}} T_i + \frac{(\mathsf{q}_j - \mathsf{q}_j^{-1})(\mathsf{q}_i - \mathsf{q}_i^{-1})}{(1 - X^{-\alpha_j^\vee})(1 - X^{-\alpha_i^\vee})} \end{aligned}$$

As $T_i T_j = T_j T_i$ we have that $U_i U_j = U_j U_i$ as required.

(2) follows from induction on $\ell(w)$ using (1) and the fact that $U_i X^\gamma = X^{s_i\gamma} U_i$.

For (3), as $T_i^2 = T_e + (\mathsf{q}_i - \mathsf{q}_i^{-1}) T_i$ by (3.1.1), we have

$$\begin{aligned} U_i^2 &= U_i \left(T_i - \frac{\mathsf{q}_i - \mathsf{q}_i^{-1}}{1 - X^{-\alpha_i^\vee}} \right) \\ &= T_e + (\mathsf{q}_i - \mathsf{q}_i^{-1}) T_i - \frac{\mathsf{q}_i - \mathsf{q}_i^{-1}}{1 - X^{-\alpha_i^\vee}} T_i - \frac{\mathsf{q}_i - \mathsf{q}_i^{-1}}{1 - X^{\alpha_i^\vee}} T_i + \frac{(\mathsf{q}_i - \mathsf{q}_i^{-1})^2}{(1 - X^{-\alpha_i^\vee})(1 - X^{\alpha_i^\vee})} \\ &= \frac{(1 - X^{-\alpha_i^\vee})(1 - X^{\alpha_i^\vee}) + (\mathsf{q}_i - \mathsf{q}_i^{-1})}{(1 - X^{-\alpha_i^\vee})(1 - X^{\alpha_i^\vee})} \\ &\qquad + \frac{(\mathsf{q}_i - \mathsf{q}_i^{-1})((1 - X^{-\alpha_i^\vee})(1 - X^{\alpha_i^\vee}) - (1 - X^{-\alpha_i^\vee}) - (1 - X^{\alpha_i}))T_i}{(1 - X^{-\alpha_i^\vee})(1 - X^{\alpha_i^\vee})} \\ &= \frac{(\mathsf{q}_i^2 - X^{-\alpha_i^\vee})(1 - \mathsf{q}_i^{-2} X^{\alpha_i^\vee})}{(1 - X^{-\alpha_i^\vee})(1 - X^{\alpha_i^\vee})}. \end{aligned}$$

(4) follows by induction on $\ell(v)$. For $v = e$ the result is clear as

$$U_u U_e = U_u$$

by (1). Assume that $v = v's$ with $v > v'$, so

$$U_u U_v = U_u U_{v'} U_s = b_{u,v'}(X) U_{uv'} U_s$$

by (1) and the inductive hypothesis. There are two cases. If $uv's > uv'$ then $U_{uv'} U_s = U_{uv}$ and the result follows. On the other hand, if $uv's < uv'$, let w be such that $uv' = ws$ and $ws > w$. Then $U_{uv'} U_s = U_{ws} U_s = U_w U_s^2$ and the result follows from (3) and (2). □

Recall from Section 3.1 that for all $\alpha \in \widetilde{\Phi}$, $\mathsf{q}_\alpha = \mathsf{q}_i$ if $\alpha \in \widetilde{W}\alpha_i$. The following important theorem will be used in Part II.

Theorem 5.4.2. *For $J \subseteq I$ we have*

$$C_{\mathsf{w}_J} = \mathsf{q}_{\mathsf{w}_J} \sum_{w \in W_J} \mathsf{q}_w^{-1} c_w^J(X) U_w \quad \textit{where} \quad c_w^J(X) = \prod_{\beta \in \Phi_J^+ \setminus \Phi(w)} \frac{1 - \mathsf{q}_\beta^{-2} X^{-\beta^\vee}}{1 - X^{-\beta^\vee}}.$$

Proof. By the definition of U_w we have that $U_w = T_w +$ *lower terms* (lower than w in terms of the Bruhat order). This triangularity implies that $C_{\mathsf{w}_J} = \sum_{w \in W_J} a_w(X) U_w$ where $a_w(X)$ are rational functions with $a_{\mathsf{w}_J} = 1$, and the expression is unique. We have that $C_{\mathsf{w}_J} T_j = \mathsf{q}_j C_{\mathsf{w}_J}$ (see Section 3.2). Therefore,

$$\begin{aligned}
C_{\mathsf{w}_J} U_j &= C_{\mathsf{w}_J}\left(\mathsf{q}_j - \frac{\mathsf{q}_j - \mathsf{q}_j^{-1}}{1 - X^{-\alpha_j^\vee}}\right) \\
&= \sum_{w \in W_J} a_w(X) U_w \left(\mathsf{q}_j - \frac{\mathsf{q}_j - \mathsf{q}_j^{-1}}{1 - X^{-\alpha_j^\vee}}\right) \\
&= \sum_{w \in W_J} a_w(X) U_w \left(\mathsf{q}_j + \frac{\mathsf{q}_j X^{\alpha_j^\vee} - \mathsf{q}_j^{-1} X^{\alpha_j^\vee}}{1 - X^{\alpha_j^\vee}}\right) \\
&= \sum_{w \in W_J} a_w(X) \mathsf{q}_j U_w \left(\frac{1 - \mathsf{q}_j^{-2} X^{\alpha_j^\vee}}{1 - X^{\alpha_j^\vee}}\right) \\
&= \sum_{w \in W_J} a_w(X) \mathsf{q}_j \left(\frac{1 - \mathsf{q}_j^{-2} X^{w\alpha_j^\vee}}{1 - X^{w\alpha_j^\vee}}\right) U_w
\end{aligned}$$

by Proposition 5.4.1(2).

On the other hand, by Proposition 5.4.1(3) we have

$$\begin{aligned}
C_{\mathsf{w}_J} U_j &= \sum_{w \in W_J} a_w(X) U_w U_j \\
&= \sum_{\substack{w \in W_J \\ ws_j < w}} a_w(X) U_w U_j + \sum_{\substack{w \in W_J \\ ws_j > w}} a_w(X) U_w U_j \\
&= \sum_{\substack{v \in W_J \\ vs_j > v}} \mathsf{q}_j^2 a_{vs_j}(X) \frac{(1 - \mathsf{q}_j^{-2} X^{-v\alpha_j^\vee})(1 - \mathsf{q}_j^{-2} X^{v\alpha_j^\vee})}{(1 - X^{-v\alpha_j^\vee})(1 - X^{v\alpha_j^\vee})} U_v + \sum_{\substack{v \in W_J \\ vs_j < v}} a_{vs_j}(X) U_v
\end{aligned}$$

by substituting $v = ws_j$ in the third equality. Comparing coefficients, for $vs_j < v$ we have

$$a_{vs_j}(X) = \mathsf{q}_j a_v(X) \frac{1 - \mathsf{q}_j^{-2} X^{v\alpha_j^\vee}}{1 - X^{v\alpha_j^\vee}}.$$

For $w \in W_J$ we can write $w = \mathsf{w}_J s_{i_k} \cdots s_{i_1}$ with $i_1, \ldots, i_k \in J$ and $\ell(w) = \ell(\mathsf{w}_J) - k$. Then, using the above recursion, we have that

$$a_w(X) = \mathsf{q}_{\mathsf{w}_J} \mathsf{q}_w^{-1} a_{\mathsf{w}_J}(X) \prod_\alpha \frac{1 - \mathsf{q}_\alpha^{-2} X^{-w\alpha^\vee}}{1 - X^{-w\alpha^\vee}}$$

where the sum is over $\alpha \in \{\alpha_{i_1}, s_{i_1}\alpha_{i_2}, \ldots, s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k}\} = \Phi(w^{-1}\mathsf{w}_J)$ and where $\mathsf{q}_\alpha = \mathsf{q}_{j_\ell}$ for $\alpha = s_{i_1} \cdots s_{i_{l-1}} \alpha_{i_l}$.

The result is true if $w\Phi(w^{-1}\mathsf{w}_J) = \Phi_J^+ \backslash \Phi(w)$. If $\alpha \in \Phi_J^+ \backslash \Phi(w)$ then $w^{-1}\alpha > 0$ and so $\mathsf{w}_J w w^{-1} \alpha < 0$ as $\Phi(\mathsf{w}_J) = \Phi_J^+$. On the other hand, if $\alpha \in w\Phi(w^{-1}\mathsf{w}_J) \subseteq \Phi_J$ then there exists $\beta \in \Phi(w^{-1}\mathsf{w}_J)$ such that $\alpha = w\beta$ and $\mathsf{w}_J w\beta < 0$. As $\Phi(\mathsf{w}_J) = \Phi_J^+$ we have $w\beta > 0$, and the proof is complete as $w^{-1}w\beta > 0$. □

For each $1 \le i \le n$ we define a normalisation of U_i by

$$U_i' = (1 - X^{-\alpha_i^\vee})T_i - (\mathsf{q}_i - \mathsf{q}_i^{-1})T_e.$$

This element is now in $\widetilde{\mathcal{H}}$. By similar arguments to the proof of Proposition 5.4.1 we have the following properties:

(1) $U_w' = U_{i_1}' \cdots U_{i_k}'$ is independent of the chosen reduced expression of $w = s_{i_1} \cdots s_{i_k} \in W_0$.
(2) $U_i'^2 = \mathsf{q}_i^2(1 - \mathsf{q}_i^{-2}X^{-\alpha_i^\vee})(1 - \mathsf{q}^{-2}X^{\alpha_i^\vee})$.
(3) $U_w' X^\gamma = X^{w\gamma} U_w'$ for all $w \in W_0$ and $\gamma \in P$.

We will require the following lemma to prove, in Proposition 5.4.4, that we can construct a basis of $M_{J,\mathsf{v}}$ from these normalised intertwiners.

Lemma 5.4.3. *If $\alpha \in \Phi \backslash \Phi_J$ then $\psi_{J,\mathsf{v}}(X^{\alpha^\vee}) \notin \mathsf{R}$.*

Proof. For $\alpha \in \Phi$, if $\psi_{J,\mathsf{v}}(X^{\alpha^\vee}) = (\mathsf{v}\zeta_J)^{\alpha^\vee} \in \mathsf{R}$ then $\zeta_J^{\alpha^\vee} = 1$ and so $(\alpha^\vee)^J = 0$. This implies that $\alpha^\vee \in V_J$ and so $\alpha \in \Phi_J$. ☐

Proposition 5.4.4. *The module $M_{J,\mathsf{v}}$ has basis $\{\varpi_{J,\mathsf{v}}(U_u') \mid u \in {}^JW\}$.*

Proof. Let $u = s_{i_1} \cdots s_{i_\ell}$ be a reduced expression for $u \in W_0$. By (4.1.3) we have

$$\begin{aligned} U_u' &= (1 - X^{-\alpha_{i_1}^\vee})T_{i_1}(1 - X^{-\alpha_{i_2}^\vee})T_{i_2} \cdots (1 - X^{-\alpha_{i_\ell}^\vee})T_{i_\ell} + \text{lower terms} \\ &= (1 - X^{-\alpha_{i_1}^\vee})(1 - X^{-s_{i_1}\alpha_{i_2}^\vee}) \cdots (1 - X^{-s_{i_1}\cdots s_{i_{\ell-1}}\alpha_{i_\ell}^\vee})T_u + \text{lower terms} \\ &= \left[\prod_{\alpha \in \Phi(u)} (1 - X^{-\alpha^\vee})\right] T_u + \text{lower terms,} \end{aligned}$$

where 'lower terms' refers to a linear combination of terms $p_v(X)T_v$ where $v < u$ in the Bruhat preorder and $p_v(X)$ is a function of X. In addition, we have $T_u = X_u +$ lower terms, where lower terms refers to R-linear combinations of X_v for $v < u$. By (5.1.1) we have that $\varpi_{J,\mathsf{v}}(X_u) = \mathbf{m}_u$ for all $u \in {}^JW$ and so, for $u \in {}^JW$ we have

$$\varpi_{J,\mathsf{v}}(U_u') = \left[\prod_{\alpha \in \Phi(u)} (1 - \psi_{J,\mathsf{v}}(X^{\alpha^\vee})^{-1})\right] \mathbf{m}_u + \text{lower terms.} \tag{5.4.1}$$

By Lemma 1.4.1, $\Phi_J(u) = \emptyset$ for $u \in {}^JW$. Therefore, by Lemma 5.4.3 $\psi_{J,\mathsf{v}}(X^{\alpha^\vee}) \neq 1$ for all $\alpha \in \Phi(u)$. Hence, the coefficient of $\mathbf{m}_u$ in (5.4.1) does not vanish, and the result follows as $\{\mathbf{m}_u \mid u \in {}^JW\}$ is a basis of $M_{J,\mathsf{v}}$ by definition. ☐

Remark 5.4.5. We can extend the definition of $\varpi_{J,\mathsf{v}}$ so that

$$\varpi_{J,\mathsf{v}}\left(\frac{1}{1 - X^{-\alpha_i^\vee}}\right) = \frac{1}{1 - \psi_{J,\mathsf{v}}(X^{-\alpha_i^\vee})}\mathbf{m}_e$$

for $1 \le i \le n$. As $1 - X^{-\alpha_i^\vee}$ does not act by zero on $M_{J,\mathsf{v}}$ we can consider $\{\varpi_{J,\mathsf{v}}(U_u) \mid u \in {}^JW\}$ as a 'basis' of $M_{J,\mathsf{v}}$ where we now extend scalars to rational functions of ζ_J. We make the following comments about $\pi_{J,\mathsf{v}}$ with respect to the basis $B' = \{\varpi_{J,\mathsf{v}}(U_u) \mid u \in {}^JW\}$ (similar comments can be made about the representation with respect to the normalised basis $\{\varpi_{J,\mathsf{v}}(U_u') \mid u \in {}^JW\}$). We notate $\pi_{J,\mathsf{v}}(h; B')$ for the matrix representation of $h \in \widetilde{\mathcal{H}}$ with respect to basis B'.

(1) the matrix of $\pi_{J,\mathsf{v}}(X^\gamma; B')$ is diagonal for all $\gamma \in P$, with entries $\psi_{J,\mathsf{v}}(X^{u\gamma}) = \mathsf{v}^{u\gamma}\zeta_J^{u\gamma}$ for $u \in {}^JW$.
(2) the matrix for $\pi_{J,\mathsf{v}}(U_w; B')$, for $w \in W_0$, has at most one non-zero entry in each row and column. Specifically, if $u \in {}^JW$ and $w \in W_0$ then the u-th row of $\pi_{J,\mathsf{v}}(U_w; B')$ is zero if $uw \notin {}^JW$ and has an entry only in the uw-th column if $uw \in {}^JW$.

The first comment follows directly from Proposition 5.4.1(2). For the second comment, first note that as $\psi_{J,\mathsf{v}}(T_j) = \mathsf{v}_{\alpha_i}$ and $\psi_{J,\mathsf{v}}(X^{\alpha_j^\vee}) = \mathsf{v}_{\alpha_j}^2$ for $j \in J$ (by Proposition 5.2.1) we have that $\psi_{J,\mathsf{v}}(U_j) = 0$ for all $j \in J$. This then implies that $\varpi_{J,\mathsf{v}}(U_u) \cdot U_i = 0$ for $i \in I$ such that $us_i \notin {}^JW$ as $s_j u = us_i$ for a unique $j \in J$. If $us_i \in {}^JW$ then when $us_i > u$ we have $\varpi_{J,\mathsf{v}}(U_u) \cdot U_{s_i} = \varpi_{J,\mathsf{v}}(U_{us_i})$ and when $us_i < u$ we have

$$\varpi_{J,\mathsf{v}}(U_u) \cdot U_i = \varpi_{J,\mathsf{v}}(U_{us_i} U_i^2) = \psi_{J,\mathsf{v}}(U_i^2)\varpi_{J,\mathsf{v}}(U_{us_i}).$$

Result (2) then follows by induction on the length of w and by Lemma 5.4.3. Note that this result implies that $\chi_{J,\mathsf{v}}(U_w) = 0$ if $w \neq e$ where $\chi_{J,\mathsf{v}}$ is the character of $\pi_{J,\mathsf{v}}$.

The following proposition gives the decomposition of $M_{J,\mathsf{v}}$ into weight spaces.

Proposition 5.4.6. *Let $J \subseteq I$ and let v be a J-parameter system. For $u \in W_0$ let*

$$M_u = \{\mathbf{m} \in M_{J,\mathsf{v}} \mid \mathbf{m} \cdot X^\gamma = \psi_{J,\mathsf{v}}(X^{u\gamma})\mathbf{m} \text{ for all } \gamma \in P\}.$$

(1) *If $u, us_i \in W^J$ then the map $\tilde{U}_i' : M_u \to M_{us_i}$ with $\tilde{U}_i'(\mathbf{m}) = \mathbf{m} \cdot U_i'$ is bijective.*
(2) *For $u \in W^J$ we have $M_u = \{r\varpi_{J,\mathsf{v}}(U_u') \mid r \in \mathsf{R}[\zeta_J]\}$, and*

$$M_{J,\mathsf{v}} = \bigoplus_{u \in W^J} M_u.$$

Proof. (1) For $u \in W_0$ let $z_u : P \to \mathsf{R}[\zeta_J]$ be the map $z_u^\gamma = \psi_{J,\mathsf{v}}(X^{u\gamma})$. Then $z_u^\gamma = z^{u\gamma}$ with $z = z_e$. With this notation we have that

$$M_u = \{\mathbf{m} \in M_{J,\mathsf{v}} \mid \mathbf{m} \cdot X^\gamma = z_u^\gamma \mathbf{m} = z^{u\gamma}\mathbf{m} \text{ for all } \gamma \in P\}.$$

For any $u \in W_0$ and $i \in I$, if $\mathbf{m} \in M_u$ then as $U_w' X^\gamma = X^{w\gamma} U_w'$ for all $w \in W_0$ and $\gamma \in P$ we have that

$$(\mathbf{m} \cdot U_i')X^\gamma = (\mathbf{m} \cdot X^{s_i\gamma})U_i' = z_u^{s_i\gamma}(\mathbf{m} \cdot U_i') = z_{us_i}^\gamma(\mathbf{m} \cdot U_i'),$$

and so $\mathbf{m} \cdot U_i \in M_{us_i}$. Similarly, if $\mathbf{m} \in M_{us_i}$ then

$$(\mathbf{m} \cdot U_i')X^\gamma = (\mathbf{m} \cdot X^{s_i\gamma})U_i' = z_{us_i}^{s_i\gamma}(\mathbf{m} \cdot U_i') = z_u^\gamma(\mathbf{m} \cdot U_i'),$$

and so $\mathbf{m} \cdot U_i \in M_u$. Therefore, we can define operators $\tilde{U}_i' : M_u \to M_{us_i}$ and $\tilde{U}_i' : M_{us_i} \to M_u$ such that $\tilde{U}_i'(\mathbf{m}) = \mathbf{m} \cdot U_i$. Thus $\tilde{U}_i'^2 : M_u \to M_u$ with

$$\tilde{U}_i'^2(\mathbf{m}) = \mathbf{m} \cdot U_i'^2 = \mathsf{q}_i^2(1 - \mathsf{q}_i^{-2} z^{-u\alpha_i^\vee})(1 - \mathsf{q}_i^{-2} z^{u\alpha_i^\vee})\mathbf{m}.$$

If $z^{u\alpha_i^\vee} \neq \mathsf{q}_i^{\pm 2}$ then the operators $\tilde{U}_i' : M_u \to M_{us_i}$ and $\tilde{U}_i' : M_{us_i} \to M_u$ are bijective. By Lemma 5.4.3, when $u, us_i \in {}^JW$ we have $z^{u\alpha_i^\vee} \notin \mathsf{R}$ as $u\alpha_i \notin \Phi_J$ and (1) follows.

(2) For $u \in {}^JW$, $\gamma \in P$ and $r \in \mathsf{R}[\zeta_J]$ we have

$$r\varpi_{J,\mathsf{v}}(U_u') \cdot X^\gamma = r\varpi_{J,\mathsf{v}}(X^{u\gamma}U_u') = r\psi_{J,\mathsf{v}}(X^{u\gamma})\varpi_{J,\mathsf{v}}(U_u') \in M_u$$

and so $\{r\varpi_{J,\mathsf{v}}(U_u) \mid r \in \mathsf{R}[\zeta_J]\} \subseteq M_u$. Therefore, by Proposition 5.4.4 the weight spaces M_u (for $u \in {}^JW$) span $M_{J,\mathsf{v}}$. Thus, to prove (2) it is sufficient to show that each M_u is distinct, meaning $M_{u_1} = M_{u_2}$ forces $u_1 = u_2$ for $u_1, u_2 \in {}^JW$.

Let $u_1, u_2 \in {}^JW$ with $M_{u_1} = M_{u_2}$. This implies that $\psi_{J,\mathsf{v}}(X^{u_1\gamma}) = \psi_{J,\mathsf{v}}(X^{u_2\gamma})$ for all $\gamma \in P$, which then gives that $\psi_{J,\mathsf{v}}(X^{u_1\gamma - u_2\gamma}) = 1$ for all $\gamma \in P$. Replacing γ with $u_2^{-1}\gamma$ we have $\psi_{J,\mathsf{v}}(X^{u_1u_2^{-1}\gamma - \gamma}) = 1$ for all $\gamma \in P$. This then implies that $(u_1u_2^{-1}\gamma - \gamma)^J = 0$ for all $\gamma \in P$, and so $u_1u_2^{-1}\gamma \in \gamma + V_J$ for all $\gamma \in P$. Setting $\gamma = \alpha^\vee$ we have that $u_1u_2^{-1}\alpha^\vee \in \Phi_J^\vee$ for all $\alpha^\vee \in \Phi_J^\vee$. Therefore, $u_1u_2^{-1} \in W_J$ and so $u_1 \in W_Ju_2$. As $u_1, u_2 \in {}^JW$, this forces $u_1 = u_2$ as required. □

Corollary 5.4.7. *Let $J \subseteq I$ and let v be a J-parameter system. The representation $(\pi_{J,\mathsf{v}}, M_{J,\mathsf{v}})$ is irreducible.*

Proof. Let N be a nonzero $\widetilde{\mathcal{H}}$-invariant submodule of $M_{J,\mathsf{v}}$. As N is $\widetilde{\mathcal{H}}$-invariant it is also invariant under the action of the elements X^γ for $\gamma \in P$. Thus N is also a module of the commutative algebra spanned by $\{X^\gamma \mid \gamma \in P\}$, and thus N contains a simple module of this algebra. As all representations of commutative algebras are 1-dimensional, there exists $v \in N$ such that $v \cdot X^\gamma$ is equal to a multiple of v. Hence, by Proposition 5.4.6(2), there is some $u \in {}^JW$ such that $N \cap M_u \neq \emptyset$. Since M_u is 1-dimensional we have $M_u \subseteq N$ and as N is $\widetilde{\mathcal{H}}$ invariant, $\mathbf{m} \cdot U_i' \in N$ for all $i \in I$. Thus, $M_{us_i} \subseteq N$ for all $i \in I$ by Proposition 5.4.6(1) which forces $N = M_{J,\mathsf{v}}$ as required. □

Chapter 6

Boundedness and $\mathbf{a}_{J,\mathsf{v}}$

Recall from Section 3.3 that a matrix representation π is bounded if, for all $w \in \widetilde{W}$, the degree of q in all entries of the matrix $\pi(T_w)$ are bounded. In this chapter we determine when the representations of $\widetilde{\mathcal{H}}$ defined in Chapter 5 are bounded. That is, we give a complete classification of the J-parameter systems such that $(\pi_{J,\mathsf{v}}, M_{J,\mathsf{v}})$ is bounded, noting any restrictions on parameters. In addition, we give a conjectural formula for Lusztig's $\mathbf{a}$-function on the elements $w \in W$ whose matrix $\pi_{J,\mathsf{v}}(T_w)$ realises the highest q degree, inspired by the Plancherel formula of Opdam [40] (defined in Section 3.4).

The classification of bounded representations is given in Section 6.1. Theorem 6.1.4 determines that boundedness of $(\pi_{J,\mathsf{v}}, M_{J,\mathsf{v}})$ is equivalent to boundedness of an associated 1-dimensional representation of $\mathcal{H}_J^{\mathrm{aff}}$, and Proposition 6.1.5 classifies all bounded 1-dimensional representations. Thus, Theorem 6.1.4 with Proposition 6.1.5 gives the full classification. Also in this section, it is conjectured that the set of elements of $\widetilde{W}$ who realise our matrix bound is contained within a two-sided Kazhdan-Lusztig cell of $\widetilde{W}$ and that the matrix bound is equal to Lusztig's $\mathbf{a}$-function for these elements.

In Section 3.4 it was shown that the canonical trace of $\widetilde{\mathcal{H}}$ when restricted to elements of $\mathcal{H}_0$ decomposes as a sum of characters of irreducible representations of $\mathcal{H}_0$, with coefficients related to Lusztig's $\mathbf{a}$-function. In Section 6.2 we form a conjectural expression for Lusztig's $\mathbf{a}$-function using the decomposition of the trace for elements of $\widetilde{\mathcal{H}}$, the Plancherel formula (defined in Section 3.4). The remainder of this section proves links between our conjectural bound and the conjectures made in Section 6.1 and proves that the conjectures hold for all $\widetilde{\mathcal{H}}$ of rank 2 and 3 and for all $\widetilde{\mathcal{H}}$ when $J = \emptyset$.

6.1 Classification of boundedness

Let $\widetilde{\mathcal{H}}$ be the extended affine Hecke algebra associated to the root system Φ with weight function L, as defined in Section 3.1. Denote $(\pi_{J,\mathsf{v}}, M_{J,\mathsf{v}}, \mathsf{B}_{^JW})$ to be the matrix representation of $\widetilde{\mathcal{H}}$ defined in Chapter 5 where $\mathsf{B}_{^JW} = \{\varpi_{J,\mathsf{v}}(X_u) \mid u \in {}^JW\} = \{\mathbf{m}_u \mid u \in {}^JW\}$. Recall from Section 3.3 that a matrix representation $\pi : \widetilde{\mathcal{H}} \to M$ with basis B is called bounded if $\deg([\pi(T_w)]_{u,v})$ is bounded from above for all $w \in \widetilde{W}$ and $u, v \in \mathsf{B}$. In this section we classify the J-parameter systems such that our representation $(\pi_{J,\mathsf{v}}, M_{J,\mathsf{v}}, \mathsf{B}_{^JW})$ is bounded and give some conjectures about the minimal bound value.

Recall from Section 3.3 that the bound of $(\pi_{J,\mathsf{v}}, M_{J,\mathsf{v}}, \mathsf{B}_{^JW})$ is denoted $\mathbf{a}_{\pi,M,\mathsf{B}_{^JW}}$ and the set of elements recognised by this representation is denoted $\Gamma_{\pi_{J,\mathsf{v}},M_{J,\mathsf{v}},\mathsf{B}_{^J W}}$. The following example describes the bound and cell recognised for the most well known case.

Example 6.1.1. If $J=\emptyset$ then $(\pi_{\emptyset,\mathsf{v}},M_{\emptyset,\mathsf{v}},\mathsf{B}_W)=(\pi,M,\mathsf{B})$ is the principle series representation (see Remark 5.2.6). By [23, Lemma 6.2] we have that $\deg\mathcal{Q}(p)\le L(\mathsf{w}_0)$ for all positively folded paths p of reduced type. Thus, by Theorem 5.3.5 (π,M,B) is bounded with bound $L(\mathsf{w}_0)$.

For each $\gamma\in P$ let $W_\gamma=\{w\in\widetilde{W}\mid w\cdot\gamma=\gamma\}$ be the stabiliser of γ in $\widetilde{W}$ and let w_γ be the longest element of W_γ. Let $P_0\subseteq P$ be such that $\gamma\in P_0$ if and only if $W_\gamma\cong W_0$ and $L(\mathsf{w}_\gamma)=L(\mathsf{w}_0)$. By an analysis similar to the proof of [23, Theorem 6.6] the elements of $\widetilde{W}$ recognised by π are

$$\Gamma=\{w\in\widetilde{W}\mid w=w_1\cdot\mathsf{w}_\gamma\cdot w_2,\ w_1,w_2\in\widetilde{W},\gamma\in P_0\}$$

where the notation $u\cdot v$ means uv such that $\ell(uv)=\ell(u)+\ell(v)$. We give a brief explanation of why this is the case (see [23, Theorem 6.6] for a more concrete explanation). If $w\in\widetilde{W}$ such that $[\pi(T_w)]_{u,v}$ has degree $L(\mathsf{w}_0)$ for some $u,v\in\mathsf{B}$ then there exists a path $p\in\mathcal{P}(\vec{w},u)$ whose path v-mass has degree $L(\mathsf{w}_0)$ (where $\vec{w}$ is reduced). This then implies that the straight path from u to uw of type $\vec{w}$ crosses every hyperplane direction and so uw lies in the anti-dominant sector based at some $\gamma\in P_0$. Thus there exists a reduced expression of w of the form $u^{-1}t_\gamma\cdot\mathsf{w}_\gamma\cdot v'$ for some $v'\in\widetilde{W}$ so $w\in\Gamma$. Conversely, if $w=w_1\cdot\mathsf{w}_\gamma\cdot w_2\in\Gamma$ for some $\gamma\in P_0$ and $w_1,w_2\in\widetilde{W}$ then, as the lengths are increasing, there exists some $u\in\mathsf{B}$ such that there exists a path beginning at the alcove uw_1A_0 of type $\vec{\mathsf{w}_\gamma}$ that is all folds. This path concatenated with the straight path from $uw_1\mathsf{w}_\gamma$ to $uw_1\mathsf{w}_\gamma w_2$ of type $\vec{w_2}$ has degree $L(\mathsf{w}_0)$, and so the result follows from Theorem 5.3.3.

Let F and F' be fundamental domains for the action of $\mathbb{T}_J$ onto $\mathbb{W}^J$ (see Section 2.3 for the definitions of $\mathbb{T}_J$ and $\mathbb{W}^J$). Recall that B_F and $\mathsf{B}_{\mathsf{F}'}$ are bases of $M_{J,\mathsf{v}}$ by Proposition 5.3.4. By Remark 3.3.1, $(\pi_{J,\mathsf{v}},M_{J,\mathsf{v}},\mathsf{B}_\mathsf{F})$ is bounded if and only if $(\pi_{J,\mathsf{v}},M_{J,\mathsf{v}},\mathsf{B}_{\mathsf{F}'})$ is bounded. The connection between the boundedness of these two representations is stronger than this, as shown in the following proposition.

Proposition 6.1.2. *Let F and F' be fundamental domains for the action of $\mathbb{T}_J$ on $\mathbb{W}^J$ such that the associated matrix representations, $(\pi_{J,\mathsf{v}},M_{J,\mathsf{v}},\mathsf{B}_\mathsf{F})$ and $(\pi_{J,\mathsf{v}},M_{J,\mathsf{v}},\mathsf{B}_{\mathsf{F}'})$, are bounded. Then these matrix representations have the same bound, and recognise the same cell.*

Proof. Let $u\in\mathsf{F}$. By Corollary 2.3.10, we have the following decomposition $u=\tau_{\mathrm{wt}(u)}\theta^J(u)$. By Proposition 5.2.1 and (5.1.1) we have $\varpi(X_u)=\zeta_J^{\mathrm{wt}(u)}\mathbf{m}_{\theta^J(u)}$. Therefore, the change of basis matrix from B_F to $\mathsf{B}_{^J W}$ is a monomial matrix with entries independent of q. This forces $\mathbf{a}_{\pi_{J,\mathsf{v}},M_{J,\mathsf{v}},\mathsf{B}_\mathsf{F}}=\mathbf{a}_{\pi_{J,\mathsf{v}},M_{J,\mathsf{v}},\mathsf{B}_{\mathsf{F}'}}$ and $\Gamma_{\pi_{J,\mathsf{v}},M_{J,\mathsf{v}},\mathsf{B}_\mathsf{F}}=\Gamma_{\pi_{J,\mathsf{v}},M_{J,\mathsf{v}},\mathsf{B}_{\mathsf{F}'}}$ as required. □

By Proposition 6.1.2 the bound is independent on the chosen fundamental domain F. Hence, we can simplify notation to

$$\mathbf{a}_{J,\mathsf{v}}=\mathbf{a}_{\pi_{J,\mathsf{v}},M_{J,\mathsf{v}},\mathsf{B}_\mathsf{F}}\text{ and }\Gamma_{J,\mathsf{v}}=\Gamma_{\pi_{J,\mathsf{v}},M_{J,\mathsf{v}},\mathsf{B}_\mathsf{F}}$$

for all fundamental domains F for the action of $\mathbb{T}_J$ on $\mathbb{W}^J$.

Remark 6.1.3. The main motivation behind the path formula (Theorem 5.3.3) was to understand boundedness for the $(\pi_{J,\mathsf{v}},M_{J,\mathsf{v}})$ representations combinatorially. In [23] and [22] the J-folded alcove paths were effectively used to determine boundedness and the bounds for these representations. However, things become more complicated when advancing beyond rank 3 Hecke algebras.

If the J-folded alcove paths v-mass degree was bounded from above then Theorem 5.3.3 would immediately give boundedness. Stated explicitly, for $\vec{w} \in \widetilde{W}$ reduced and all $u, v \in {}^J W$, if there exists some $N > 0$ such that $\deg \mathcal{Q}_{J,\mathsf{v}}(p) \leq N$ for all $p \in \mathcal{P}_J(\vec{w}, u)$ with $\theta^J(p) = v$ then, by Theorem 5.3.3, the degree of the matrix entries of $\pi_{J,\mathsf{v}}(T_w)$ are bounded. However, the reverse direction is more subtle and complicates the theory. If the degree of the matrix entries of $\pi_{J,\mathsf{v}}(T_w)$ are bounded by $\mathbf{a}_{J,\mathsf{v}}$, it is not true that the J-folded alcove paths of type $\vec{w}$ are also bounded by $\mathbf{a}_{J,\mathsf{v}}$, since there may be cancellations.

For example, let $\Phi = A_3$ and $J = \{1\}$. Let v be the J-parameter system such that $\mathsf{v}_{\alpha_1} = -\mathsf{q}^{-1}$ where $\mathsf{q} = \mathsf{q}_1$. Consider w_0, the longest element of W_0, with set reduced expression $\vec{\mathsf{w}_0} = 323123$ (writting i instead of s_i). The following is all paths within the set $\{p \in \mathcal{P}_J(\vec{\mathsf{w}_0}, e) \mid \theta^J(p) = e\}$, their relative v-masses and their degrees ($\hat{i}$ denoting a fold and $\check{i}$ denoting a bounce):

$p_1 = \hat{3}\hat{2}\hat{3}\check{1}\hat{2}\hat{3}$	$\mathcal{Q}_{J,\mathsf{v}}(p_1) = -\mathsf{q}^{-1}(\mathsf{q} - \mathsf{q}^{-1})^5$	$\deg(\mathcal{Q}_{J,\mathsf{v}}(p_1)) = 4$
$p_2 = \hat{3}2\hat{3}\hat{1}2\hat{3}$	$\mathcal{Q}_{J,\mathsf{v}}(p_2) = (\mathsf{q} - \mathsf{q}^{-1})^4$	$\deg(\mathcal{Q}_{J,\mathsf{v}}(p_2)) = 4$
$p_3 = \hat{3}\hat{2}3\check{1}\hat{2}3$	$\mathcal{Q}_{J,\mathsf{v}}(p_3) = -\mathsf{q}^{-1}(\mathsf{q} - \mathsf{q}^{-1})^3$	$\deg(\mathcal{Q}_{J,\mathsf{v}}(p_3)) = 2$
$p_4 = 3\hat{2}3\check{1}\hat{2}\hat{3}$	$\mathcal{Q}_{J,\mathsf{v}}(p_4) = -\mathsf{q}^{-1}(\mathsf{q} - \mathsf{q}^{-1})^3$	$\deg(\mathcal{Q}_{J,\mathsf{v}}(p_4)) = 2$
$p_5 = 32\hat{3}\hat{1}23$	$\mathcal{Q}_{J,\mathsf{v}}(p_5) = (\mathsf{q} - \mathsf{q}^{-1})^2$	$\deg(\mathcal{Q}_{J,\mathsf{v}}(p_5)) = 2$

By Theorem 5.3.3 we then have

$$[\pi_{J,\mathsf{v}}(T_{\mathsf{w}_0})]_{e,e} = \sum_{i=1}^{5} \mathcal{Q}_{J,\mathsf{v}}(p_i) = \mathsf{q}^{-4}(\mathsf{q} - \mathsf{q}^{-1})^2,$$

which has a degree of -2. It is well known that $\mathbf{a}_{J,\mathsf{v}} = \ell(\mathsf{w}_{J'}) = 3$ with $J' = \{1, 2\}$. Therefore, the maximal degree of the paths is above both the degree of the matrix entry but also the maximal degree bound for the representation $\pi_{J,\mathsf{v}}$. In the sum the leading q terms have cancelled. In the lower dimensional cases these cancellations can be controlled (see [23] and [22]). However as the rank increases the cancellations become more difficult to deal with. Understanding these cancellations in a general sense would greatly increase understanding of boundedness, and thus could lead to a better understanding of the cell recognised by these representations (with implications in Kazhdan-Lusztig theory as can be seen in [23] and [22]).

The cancellations mentioned in Remark 6.1.3 make it difficult to determine $\mathbf{a}_{J,\mathsf{v}}$ combinatorially with paths. However, paths and the path formula can still be used to make useful comments on boundedness. In Theorem 6.1.4 we classify the J-parameter systems for which the corresponding representation $\pi_{J,\mathsf{v}}$ is bounded.

Recall the definition of $\mathcal{H}_J^{\mathrm{aff}}$, from Section 5.2, as the subalgebra of $\widetilde{\mathcal{H}}$ generated by $\{T_j \mid j \in J\} \cup \{X^{\varphi_K^\vee} T_{s_{\varphi_K}}^{-1} \mid K \in \mathcal{K}(J)\}$. By Corollary 5.2.3 we have that $\psi_{J,\mathsf{v}}$ restricts to a 1-dimensional representation of $\mathcal{H}_J^{\mathrm{aff}}$.

Theorem 6.1.4. *Let v be a weighted J-parameter system. The following are equivalent.*

(1) *The representation $(\pi_{J,\mathsf{v}}, M_{J,\mathsf{v}})$ is bounded.*
(2) *We have $\deg \mathsf{v}^{\gamma} \leq 0$ for all $\gamma \in P^+$.*
(3) *We have $\deg \mathsf{v}^{\omega_j} \leq 0$ for all $j \in J$.*
(4) *We have $\deg \mathsf{v}^{u\gamma} \leq 0$ for all $\gamma \in P^+$ and all $u \in W^J$.*
(5) *The associated 1-dimensional representation $\psi_{J,\mathsf{v}}$ of $\mathcal{H}_J^{\mathrm{aff}}$ is bounded.*

(6) *There is a uniform bound* $\deg \mathcal{Q}_{J,\mathsf{v}}(p) \leq N$ *for all* J*-folded alcove paths of reduced type.*

Proof. First note that, by Remark 3.3.1, if $\pi_{J,\mathsf{v}}$ is bounded for one basis it is bounded for all bases. In this proof we will use both $\mathsf{B}_{^JW}$ and the basis described in Proposition 5.4.4.

(1)$\Longrightarrow$(2). Assume $(\pi_{J,\mathsf{v}}, M_{J,\mathsf{v}})$ is bounded. Let $\gamma \in P^+$, by Proposition 5.2.1

$$\mathbf{m}_e \cdot X^\gamma = \psi_{J,\mathsf{v}}(X^\gamma)\mathbf{m}_e = \mathsf{v}^\gamma \zeta_J^\gamma \mathbf{m}_e$$

and so $[\pi_{J,\mathsf{v}}(X^\gamma)]_{e,e} = \mathsf{v}^\gamma\zeta_J^\gamma$. It is then forced that $\deg \mathsf{v}^\gamma \leq 0$ as otherwise $[\pi_{J,\mathsf{v}}(X^{N\gamma})]_{e,e} = \mathsf{v}^{N\gamma}\zeta_J^{N\gamma}$ would be unbounded for $N \in \mathbb{N}$.

(2)$\Longrightarrow$(3) as $\omega_j \in P^+$ for all $j \in J$.

(3)$\Longrightarrow$(4). By the definition of $\mathsf{v}^{u\gamma}$ (see (4.3.2)), the fact that $\alpha = \sum_{i\in I}\langle \omega_i, \alpha\rangle\alpha_i$ and as $\mathsf{v}^{\omega_i} = 1$ for $i \in I\backslash J$ we have that

$$\begin{aligned}\mathsf{v}^{u\gamma} &= \prod_{\alpha\in\Phi_J^+} \mathsf{v}_\alpha^{\langle u\gamma,\alpha\rangle} = \prod_{\alpha\in\Phi_J^+} \mathsf{v}_\alpha^{\sum_{i\in I}\langle u\gamma, \langle\omega_i,\alpha\rangle\alpha_i\rangle} = \prod_{\alpha\in\Phi_J^+} \mathsf{v}_\alpha^{\sum_{i\in I}\langle\omega_i,\alpha\rangle\langle u\gamma,\alpha_i\rangle} \\ &= \prod_{i\in I}\prod_{\alpha\in\Phi_J^+} \mathsf{v}_\alpha^{\langle u\gamma,\alpha_i\rangle\langle\omega_i,\alpha\rangle} = \prod_{i\in I} \mathsf{v}^{\langle u\gamma,\alpha_i\rangle\omega_i} = \prod_{j\in J}\mathsf{v}^{\langle u\gamma,\alpha_j\rangle\omega_j}\end{aligned}$$

If $u \in {}^JW$ we have that $u^{-1}\alpha \in \Phi^+$ for all $\alpha \in \Phi_J^+$ by Lemma 1.4.1. Therefore, $\langle u\gamma, \alpha\rangle = \langle \gamma, u^{-1}\alpha\rangle \geq 0$ for all $\gamma \in P^+$, $u \in {}^JW$ and $\alpha \in \Phi_J^+$. Thus $\deg \mathsf{v}^{u\gamma} \leq 0$ as $\deg \mathsf{v}^{\omega_i} \leq 0$ for all $i \in I$.

(4)$\Longrightarrow$(1). Assume $\deg \mathsf{v}^{u\gamma} \leq 0$ for all $\gamma \in P^+$ and $u \in {}^JW$. For $\gamma \in P^+$, denote m_γ to be the unique minimal length element of $W_0 t_\gamma W_0$. We can write $t_\gamma = m_\gamma w_\gamma$ where $w_\gamma \in W_0$ such that $\ell(t_\gamma) = \ell(m_\gamma) + \ell(w_\gamma)$. Furthermore, every $w \in \widetilde{W}$ can be written as $w = um_\gamma v$ for some $\gamma \in P^+$ and some $u, v \in W_0$ with $\ell(w) = \ell(u) + \ell(m_\gamma) + \ell(v)$. Thus,

$$T_w = T_u T_{m_\gamma} T_v = T_u T_{t_\gamma} T_{w_\gamma}^{-1} T_v = T_u X^\gamma T_{w_\gamma}^{-1} T_v.$$

Denote the basis of $M_{J,\mathsf{v}}$ described in Proposition 5.4.4 by B'. By Remark 5.4.5 we have that $\pi_{J,\mathsf{v}}(X^\gamma)$ with respect to B' is a diagonal matrix with entries $\psi_{J,\mathsf{v}}(X^{u\gamma}) = \mathsf{v}^{u\gamma}\zeta_J^{u\gamma}$ for $u \in {}^JW$. So the entries of $\pi(X^\gamma; \mathsf{B}')$ are bounded by assumption. As $u, v, w_\gamma \in W_0$ the degree of the entries of the matrices $\pi_{J,\mathsf{v}}(T_u; \mathsf{B}')$, $\pi_{J,\mathsf{v}}(T_v; \mathsf{B}')$ and $\pi_{J,\mathsf{v}}(T_{w_\gamma}; \mathsf{B}')$ are bounded. Hence, the degree of the entries of $\pi_{J,\mathsf{v}}(T_w; \mathsf{B}')$ are bounded for all $w \in \widetilde{W}$. This gives the result (as boundedness is independent of the chosen basis, see Remark 3.3.1).

(3)$\Longleftrightarrow$(5). For the case when $J = I$ we have, by the equivalence of (1) and (3), that a 1-dimensional representation $\psi_{J,\mathsf{v}}$ of $\widetilde{\mathcal{H}}$ is bounded if and only if $\deg \mathsf{v}^{\omega_i} \leq 0$ for all $i \in I$. Applying this rationale to the subalgebra $\mathcal{H}_J^{\text{aff}}$ of $\widetilde{\mathcal{H}}$ we have that the 1-dimensional representation $\psi_{J,\mathsf{v}}$ of $\mathcal{H}_J^{\text{aff}}$ is bounded if and only if $\deg \mathsf{v}^{\omega_j} \leq 0$ for all $j \in J$.

(1)$\Longleftrightarrow$(6). Theorem 5.3.3 gives (6)$\Longrightarrow$(1). For the reverse implication suppose that $(\pi_{J,\mathsf{v}}, M_{J,\mathsf{v}})$ is bounded. We require a bound on the degree of path v-mass. As described in Remark 6.1.3 there is no obvious bound (the bound is not the same as the bound of $\pi_{J,\mathsf{v}}$) and cancellations do occur. We will construct a bound as follows. Let p be a J-folded path of reduced type and let p_J denote its J-straightening (see Section 4.2). By Proposition 4.2.5 p_J is positively folded. By [23, Lemma 6.2] or [29, Lemma 7.7] we have that $f(p_J) \leq \ell(\mathsf{w}_0)$ where $f(p_J)$ is the number of folds in p_J. Decompose p_J into the following; $p_J = p_0 \cdot f_1 \cdot p_1 \cdot f_2 \cdots f_k \cdot p_k$ where $f_1, \ldots, f_k$

are the folds occurring in p_J and $p_0, p_1, \dots, p_k$ are the straight paths between the folds. Note that as $f(p_J) \leq \ell(\mathsf{w}_0)$ we have that $k \leq \ell(\mathsf{w}_0)$. Consider the straight path p_j $(0 \leq j \leq k)$. Let $x, y \in W_J^{\mathrm{aff}}$ be such that p_j begins in $x\mathcal{A}_J$ and ends in $y\mathcal{A}_J$. By Proposition 4.3.6 the contribution to $\mathcal{Q}_{J,\mathsf{v}}(p_J)$ of the path p_j is

$$\prod_{\alpha+k\delta \in \widetilde{\Phi}(x^{-1}y)} \mathsf{v}_{\alpha+k\delta}.$$

We claim that this contribution is equal to $\psi_{J,\mathsf{v}}(T'_{x^{-1}y})$, where $T'_{x^{-1}y}$ is in the basis of $\mathcal{H}_J^{\mathrm{aff}}$. Let $T'_{x^{-1}y} = T'_{i_1} \cdots T'_{i_l}$ be a minimal length expression for $T'_{x^{-1}y}$ in the generators of $\mathcal{H}_J^{\mathrm{aff}}$. By Corollary 5.2.3 we have that $\psi_{J,\mathsf{v}}(T'_j) = \mathsf{v}_{\alpha_j}$ and $\psi_{J,\mathsf{v}}(T'_{0_K}) = \mathsf{v}_{\varphi_K}$ for $j \in J$ and $K \in \mathcal{K}(J)$. Hence,

$$\psi_{J,\mathsf{v}}(T'_{x^{-1}y}) = \prod_{i=1}^{l} \mathsf{v}_{\alpha'_j} = \prod_{\alpha+k\delta \in \widetilde{\Phi}(x^{-1}y)} \mathsf{v}_{\alpha+k\delta}$$

where $\alpha'_j = \alpha_{i_j}$ if $i_j \in J$ and $\alpha'_j = \varphi_K + \delta$ if $i_j = 0_K$ for some $K \in \mathcal{K}(J)$, and the last equality follows by Definition 4.3.1(1) and (4.3.1). As (1) implies (5) we have that the associated 1-dimensional representation of $\mathcal{H}_J^{\mathrm{aff}}$ is bounded. Thus, there exists N' such that $\deg \psi_{J,\mathsf{v}}(T'_w) \leq N'$ for all $w \in W_J^{\mathrm{aff}}$. Therefore, by Proposition 4.3.6

$$\begin{aligned} \deg(\mathcal{Q}_{J,\mathsf{v}}(p)) &= \deg\left(\mathcal{Q}(p_J) \prod_{\alpha+k\delta \in \widetilde{\Phi}^+} \mathsf{v}_{\alpha+k\delta}^{c_{\alpha+k\delta}}\right) \\ &\leq \sum_{i \in I \cup \{0\}} L(s_i) f_i(p) + (k+1)N' \\ &\leq \sum_{i \in I \cup \{0\}} L(s_i) f_i(p) + (\ell(\mathsf{w}_0)+1)N' \end{aligned}$$

where the first sum is coming from the folds of p. □

Theorem 6.1.4 shows that determining the boundedness of $(\pi_{J,\mathsf{v}}, M_{J,\mathsf{v}})$ is equivalent to determining the boundedness of the associated 1-dimensional representation of $\mathcal{H}_J^{\mathrm{aff}}$. Therefore, we can classify the J-parameter systems that result in a bounded associated representation $\pi_{J,\mathsf{v}}$ by finding all J-parameter systems that result in a bounded 1-dimensional representation $\psi_{J,\mathsf{v}}$ of $\mathcal{H}_J^{\mathrm{aff}}$. As irreducible components of $\mathcal{H}_J^{\mathrm{aff}}$ interact separately, it suffices to consider the 1-dimensional representations of $\widetilde{\mathcal{H}}$ with an irreducible root system Φ.

Denote $L(s_i) = a$ if α_i is a short root and $L(s_i) = b$ if α_i is a long root. These are the only weight labels we need for a reduced irreducible root system as there are two root lengths and roots of the same length are conjugate (see Proposition 3.1.1). Therefore, by Definition 4.3.1 we have that $\mathsf{v}_s \in \{\mathsf{q}^a, -\mathsf{q}^{-a}\}$ and $\mathsf{v}_l \in \{\mathsf{q}^b, -\mathsf{q}^{-b}\}$ where $\mathsf{v}_s = \mathsf{v}_\alpha$ for all α short and $\mathsf{v}_l = \mathsf{v}_\beta$ for all β long.

We now classify all 1-dimensional representations of $\widetilde{\mathcal{H}}$ associated with a reduced irreducible root system Φ. In [21, Proposition 5.11] this classification is given for all irreducible Φ, reduced or non-reduced. Recall that simply-laced refers to the case where the roots of Φ are all of the same length, in this case they are all considered long roots. Also, recall that Convention 3.1.2 implies that $\mathsf{q}_0 = \mathsf{q}_n$ for the $\Phi = C_n$ case.

Proposition 6.1.5. *The bounded* 1*-dimensional representations of* $\widetilde{\mathcal{H}}(L)$ *are the maps* $\psi_{I,\mathsf{v}}$ *where* $\mathsf{v} = (\mathsf{v}_\alpha)_{\alpha\in\Phi}$ *is an* I*-parameter system appearing in the list below.*

(1) *If* Φ *is simply-laced then* $\mathsf{v}_\alpha = -\mathsf{q}^{-b}$ *for all* $\alpha \in \Phi$*.*

(2) *If* Φ *is reduced and not simply-laced then the possible values of* $(\mathsf{v}_s, \mathsf{v}_l)$ *are as follows, with the stated constraints on* a, b*:*

$(\mathsf{v}_s, \mathsf{v}_l)$	B_n	C_n	F_4	G_2
$(-\mathsf{q}^{-a}, -\mathsf{q}^{-b})$	$a, b \geq 1$	$a, b \geq 1$	$a, b \geq 1$	$a, b \geq 1$
$(\mathsf{q}^{a}, -\mathsf{q}^{-b})$	$a/b \leq n-1$	$a/b \leq 1/(n-1)$	$a/b \leq 6/5$	$a/b \leq 3/2$
$(-\mathsf{q}^{-a}, \mathsf{q}^{b})$	$a/b \geq 2(n-1)$	$a/b \geq 2/(n-1)$	$a/b \geq 5/3$	$a/b \geq 2$

Proof. By Theorem 6.1.4 for $(\pi_{I,\mathsf{v}}, M_{I,\mathsf{v}})$ to be bounded it suffices to show that $\deg \mathsf{v}^{\omega_i} \leq 0$ for all $i \in I$. Let $\rho = \rho_I$ and $\rho' = \rho'_I$ be as in Section 1.4. If Φ is simply-laced then $\mathsf{v}_\alpha = \mathsf{v} \in \{\mathsf{q}^b, -\mathsf{q}^{-b}\}$ is constant for all $\alpha \in \Phi$. Hence,

$$\mathsf{v}^{\omega_i} = \prod_{\alpha\in\Phi^+} \mathsf{v}_\alpha^{\langle\omega_i,\alpha\rangle} = \prod_{\alpha\in\Phi^+} \mathsf{v}^{\langle\omega_i,\alpha\rangle} = \mathsf{v}^{\langle\omega_i,2\rho\rangle}$$

for all $i \in I$. As $\langle\omega_i, 2\rho\rangle > 0$ this forces $\mathsf{v} = -\mathsf{q}^{-b}$.

Now consider the case when Φ is not simply-laced and so has two root lengths. Now we have

$$\mathsf{v}^{\omega_i} = \prod_{\alpha\in\Phi^+_{I,s}} \mathsf{v}_\alpha^{\langle\omega_i,\alpha\rangle} \prod_{\beta\in\Phi^+_{I,l}} \mathsf{v}_\beta^{\langle\omega_i,\beta\rangle} = \prod_{\alpha\in\Phi^+_{I,s}} \mathsf{v}_s^{\langle\omega_i,\alpha\rangle} \prod_{\beta\in\Phi^+_{I,l}} \mathsf{v}_l^{\langle\omega_i,\beta\rangle} = \mathsf{v}_s^{\langle\omega_i,2\rho'\rangle}\mathsf{v}_l^{\langle\omega_i,2\rho\rangle}.$$

If $\mathsf{v}_s = -\mathsf{q}^{-a}$ and $\mathsf{v}_l = -\mathsf{q}^{-b}$ then, as $\langle\omega_i, 2\rho'\rangle > 0$ and $\langle\omega_i, 2\rho\rangle > 0$, it is clear that the representation is bounded if and only if $a, b \geq 1$.

If $\mathsf{v}_s = \mathsf{q}^a$ and $\mathsf{v}_l = -\mathsf{q}^{-b}$ then the representation is bounded if and only if

$$\deg \mathsf{v}^{\omega_i} = -\mathsf{q}^{a\langle\omega_i,2\rho'\rangle - b\langle\omega_i,2\rho\rangle} \leq 0$$

for all $i \in I$. Thus, the representation is bounded if and only if $\frac{a}{b} \leq \frac{\langle\omega_i,2\rho\rangle}{\langle\omega_i,2\rho'\rangle}$ for all $i \in I$. If $\mathsf{v}_s = -\mathsf{q}^{-a}$ and $\mathsf{v}_l = \mathsf{q}^b$, by a similar argument, the representation is bounded if and only if $\frac{a}{b} \geq \frac{\langle\omega_i,2\rho\rangle}{\langle\omega_i,2\rho'\rangle}$ for all $i \in I$. We now consider each case.

If $\Phi = B_n$ then in the Bourbaki conventions from [7] we have that $2\rho = 2(n-1)e_1 + 2(n-2)e_2 + \cdots + 2e_{n-1}$ and $2\rho' = e_1 + e_2 + \cdots + e_n$. Furthermore, $\omega_i = e_1 + e_2 + \cdots + e_i$ for all $i \in I$ so $\langle\omega_i, 2\rho'\rangle = i$ for all $i \in I$,

$$\langle\omega_i, 2\rho\rangle = 2(n-1) + \cdots 2(n-i) = 2ni - 2\sum_{j=1}^{i} j = i(2n-i-1)$$

for $i < n$ and $\langle\omega_n, 2\rho\rangle = \langle\omega_{n-1}, 2\rho\rangle = n(2n-n-1)$. If $\mathsf{v}_s = \mathsf{q}^a$ and $\mathsf{v}_l = -\mathsf{q}^{-b}$ then $\frac{a}{b} \leq 2n-i-1$ for all $1 \leq i \leq n$ and so $\frac{a}{b} \leq n-1$. If $\mathsf{v}_s = -\mathsf{q}^{-a}$ and $\mathsf{v}_l = \mathsf{q}^b$ then $\frac{a}{b} \geq 2n-i-1$ for all $1 \leq i \leq n$ and so $\frac{a}{b} \geq 2(n-1)$ as required.

If $\Phi = C_n$ then $2\rho = 2(e_1 + \cdots + e_n)$ and $2\rho' = 2(n-1)e_1 + 2(n-2)e_2 + \cdots + 2e_{n-1}$. We have that $\omega_i = e_1 + e_2 + \cdots + e_i$ for $i \leq n-1$ and $\omega_n = \frac{1}{2}(e_1 + \cdots + e_n)$. If $i \leq n-1$ then $\frac{\langle\omega_i,2\rho\rangle}{\langle\omega_i,2\rho'\rangle} = \frac{2}{2n-i-1}$ and if $i = n$ then $\frac{\langle\omega_n,2\rho\rangle}{\langle\omega_n,2\rho'\rangle} = \frac{2}{n-1}$. If $\mathsf{v}_s = \mathsf{q}^a$ and $\mathsf{v}_l = -\mathsf{q}^{-b}$ then $\frac{a}{b} \leq \frac{1}{n-1}$ and if $\mathsf{v}_s = -\mathsf{q}^{-a}$ and $\mathsf{v}_l = \mathsf{q}^b$ then $\frac{a}{b} \geq \frac{2}{n-1}$ as required.

If $\Phi = F_4$ then $2\rho = 6e_1 + 4e_2 + 2e_3$ and $2\rho' = 5e_1 + e_2 + e_3 + e_4$. We have that $\omega_i = e_1 + e_2$, $\omega_2 = 2e_1 + e_2 + e_3$, $\omega_3 = 3e_1 + e_2 + e_3 + e_4$ and $\omega_4 = 2e_1$. By direct calculation we have

$$\frac{\langle \omega_1, 2\rho\rangle}{\langle \omega_1, 2\rho'\rangle} = \frac{10}{6} \qquad \frac{\langle \omega_3, 2\rho\rangle}{\langle \omega_3, 2\rho'\rangle} = \frac{24}{18}$$
$$\frac{\langle \omega_2, 2\rho\rangle}{\langle \omega_2, 2\rho'\rangle} = \frac{18}{12} \qquad \frac{\langle \omega_4, 2\rho\rangle}{\langle \omega_4, 2\rho'\rangle} = \frac{12}{10}$$

and the result follows.

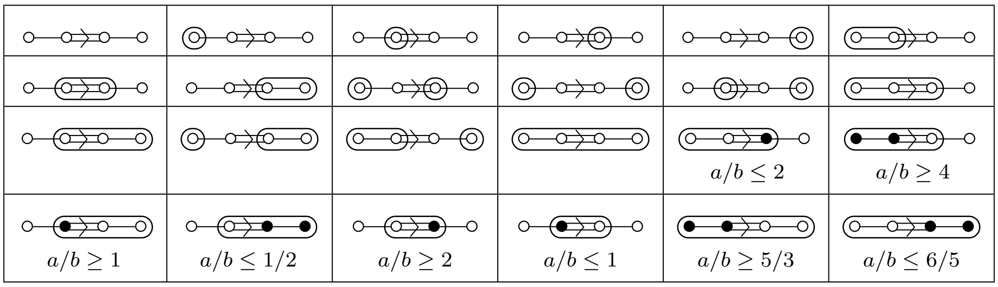
If $\Phi = G_2$ then $2\rho = -e_1 - e_2 + 2e_3$ and $2\rho' = -e_2 + e_3$. We have that $\omega_1 = -e_2 + e_3$ and $\omega_2 = \frac{1}{3}(-e_1 - e_2 + 2e_3)$ and so by direct calculation,

$$\frac{\langle \omega_1, 2\rho\rangle}{\langle \omega_1, 2\rho'\rangle} = \frac{3}{2} \quad \text{and} \quad \frac{\langle \omega_2, 2\rho\rangle}{\langle \omega_2, 2\rho'\rangle} = \frac{2}{1}.$$

The result follows. □

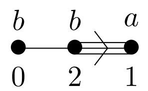
Theorem 6.1.4 together with Proposition 6.1.5 give an explicit classification for the bounded modules $M_{J,\mathsf{v}}$ (see the following example).

Example 6.1.6. Let $\Phi = F_4$. Using Theorem 6.1.4 and Proposition 6.1.5 we classify the bounded representations $(\pi_{J,\mathsf{v}}, M_{J,\mathsf{v}})$ in the below table. We display the classification using the Coxeter diagram of finite F_4 with Bourbaki labelling (see [7]) and where the nodes $j \in J$ are encircled. For $j \in J$ the associated node is black if $\mathsf{v}_{\alpha_j} = \mathsf{q}^{L(s_j)}$ and white if $\mathsf{v}_{\alpha_j} = -\mathsf{q}^{-L(s_j)}$. For the generators of W, let the weight function be denoted by $L(s_1) = L(s_2) = b$ and $L(s_3) = L(s_4) = a$. If there are constraints on $\frac{a}{b}$ for $(\pi_{J,\mathsf{v}}, M_{J,\mathsf{v}})$ to be bounded they are written below the corresponding diagram.



Although we can classify the bounded representations, finding the bound $\mathbf{a}_{J,\mathsf{v}}$ of these representations in general is still a mystery. The following example demonstrates that, even for a 1-dimensional representation, this bound and the elements that reach this bound are difficult to find.

Example 6.1.7. Let $\Phi = G_2$ with $L(s_1) = a$ and $L(s_2) = l(s_0) = b$.



Let $\pi_{I,\mathsf{v}} = \psi_{I,\mathsf{v}}$ be the 1-dimensional representation of $\widetilde{\mathcal{H}}$ with I-parameter system defined by $\mathsf{v}_{\alpha_1} = \pi_{I,\mathsf{v}}(T_1) = \mathsf{q}^a$ and $\mathsf{v}_{\alpha_2} = \pi_{I,\mathsf{v}}(T_2) = \pi_{I\mathsf{v}}(T_0) = -\mathsf{q}^{-b}$. By Proposition 6.1.5 we have that $\pi_{I,\mathsf{v}}$ is bounded if and only if $a/b \leq 3/2$. We claim that the bound of $\pi_{I,\mathsf{v}}$ and the elements recognised by $\pi_{I,\mathsf{v}}$ are as follows: (notating s_i as i)

(i) if $a/b < 1$ then $\mathbf{a}_{I,\mathsf{v}} = a$ and $\Gamma_{I,\mathsf{v}} = \{1\}$,
(ii) if $a/b = 1$ then $\mathbf{a}_{I,\mathsf{v}} = a = 3a-2b$ and $\Gamma_{I,\mathsf{v}} = \{1, 121, 12121\}$,
(iii) if $1 < a/b < 3/2$ then $\mathbf{a}_{I,\mathsf{v}} = 3a-2b$ and $\Gamma_{I,\mathsf{v}} = \{12121\}$,
(iv) if $a/b = 3/2$ then $\mathbf{a}_{I,\mathsf{v}} = 3a-2b$ and $\Gamma_{I,\mathsf{v}} = \{(12121)(02121)^k \mid k \geq 0\}$.

We will prove (iii) and (iv) (cases (i) and (ii) are similar).

Let $1 < a/b \leq 3/2$, and denote $\mathbf{a} = \mathbf{a}_{I,\mathsf{v}}$ and $\Gamma = \Gamma_{I,\mathsf{v}}$. As $\deg(\pi_{I,\mathsf{v}}(T_{12121})) = 3a-2b > 0$ we have that $\mathbf{a} \geq 3a-2b$. Let $w \in \Gamma$, and so $\deg(\pi_{I,\mathsf{v}}(T_w)) = \mathbf{a}$. Recall the definition of the left descent set of w, denoted $D_L(w)$, from Remark 3.2.4. If $s \in D_L(w)$ then there exists w_1 such that $w = s \cdot w_1$ (recall that $u \cdot v$ implies that $\ell(uv) = \ell(u) + \ell(v)$). Therefore, $\deg(\pi_{I,\mathsf{v}}(T_{w_1})) = \mathbf{a} - \deg(\pi_{I,\mathsf{v}}(T_s))$. If $s = s_0$ or $s = s_2$ then $\deg(\pi_{I,\mathsf{v}}(T_{w_1}) = \mathbf{a} + b > \mathbf{a}$, a contradiction. It is thus forced that $D_L(w) = \{s_1\}$ and so $w = s_1 \cdot w_1$. As $\deg(\pi_{I,\mathsf{v}}(T_1)) = a < 3a-2b$, we have $s_1 \notin \Gamma$ and so $w_1 \neq e$. We continue in this pattern as follows:

1. $D_L(w_1) = \{s_2\}$ (for if $s_0 \in D_L(w_1)$ then $s_0 \in D_L(w)$, a contradiction), thus $w = s_1s_2 \cdot w_2$ but $s_1s_2 \notin \Gamma$ so $w_2 \neq e$,
2. $D_L(w_2) = \{s_1\}$ (for if $s_0 \in D_L(w_2)$ then $w = s_1s_2s_0 \cdot w_3'$ with $\deg(\pi_{I,\mathsf{v}}(T_{w_3'})) = \mathbf{a}+2b-a \geq \mathbf{a} + b/2 > \mathbf{a}$, a contradiction), thus $w = s_1s_2s_1 \cdot w_3$ but $s_1s_2s_1 \notin \Gamma$ so $w_3 \neq e$,
3. $D_L(w_3) = \{s_2\}$ (for if $s_0 \in D_L(w_3)$ then $s_0 \in D_L(w_2)$, a contradiction), thus $w = s_1s_2s_1s_2 \cdot w_4$ but $s_1s_2s_1s_2 \notin \Gamma$ so $w_4 \neq e$.

From here we split into the two cases, $a/b < 3/2$ and $a/b = 3/2$. In the former case (case (iii)), if $s_0 \in D_L(w_4)$ then $w = s_1s_2s_1s_2s_0 \cdot w_5'$ with $\deg(\pi_{I,\mathsf{v}}(T_{w_4'}) = \mathbf{a}+3b-2a > \mathbf{a}$, a contradiction, so $D_L(w_4) = \{s_1\}$. Therefore, there exists w_5 such that $w = s_1s_2s_1s_2s_1 \cdot w_5$. If $w_5 \neq e$ then either $s_2 \in D_L(w_5)$, in which case $s_2 \in D_L(w)$, or $s_0 \in D_L(w_5)$, in which case $s_0 \in D_L(w_4)$. As both cases bring about a contradiction, we have that $w_5 = e$ and so $\mathbf{a} = 3a-2b$ and $\Gamma = \{12121\}$.

In the latter case (case (iv)), we have $s_1 \in D_L(w_4)$ or $s_0 \in D_L(w_4)$. If $s_0 \in D_L(w_4)$ then there exists w_5' such that $w = s_1s_2s_1s_2s_0 \cdot w_5'$ and $\deg(\pi_{I,\mathsf{v}}(T_{w_5'})) = \mathbf{a}$ (and so $w_5' \in \Gamma$). In addition, $s_1 \in D_L(w_5')$ and as s_1 and s_0 commute we have that $s_1 \in D_L(w_4)$. Thus, the final case to consider is $s_1 \in D_L(w_4)$. In this case there exists w_5 such that $w = s_1s_2s_1s_2s_1 \cdot w_5$. If $w_5 \neq e$ then $D_L(w_5) = \{s_0\}$ (for if $s_2 \in D_L(w_5)$ then $s_2 \in D_L(w)$) and similar arguments give $w = s_1s_2s_1s_2s_1s_0s_2s_1s_2s_1 \cdot v$. Iterating the argument shows that $\Gamma_{I,\mathsf{v}} = \{(12121)(02121)^k \mid k \geq 0\}$, as required.

We pose the following conjectural upper bound for $\mathbf{a}_{J,\mathsf{v}}$.

Conjecture 6.1.8. *If $(\pi_{J,\mathsf{v}}, M_{J,\mathsf{v}}, \mathsf{B}_{^JW})$ is bounded then the bound $\mathbf{a}_{J,\mathsf{v}}$ satisfies $\mathbf{a}_{J,\mathsf{v}} \leq L(\mathsf{w}_0)$ with equality if and only if $J = \emptyset$.*

When $\mathsf{v} = (\mathsf{v}_\alpha)_{\alpha \in \Phi_J}$ is such that $\mathsf{v}_{\alpha_j} = -\mathsf{q}^{-L(s_j)}$ for all $j \in J$ we have this upper bound. We denote this J-parameter system as $\hat{\mathsf{v}}$.

Theorem 6.1.9. *The bound of $(\pi_{J,\hat{\mathsf{v}}}, M_{J,\hat{\mathsf{v}}}, \mathsf{B}_{^JW})$ satisfies $\mathbf{a}_{J,\hat{\mathsf{v}}} \leq L(\mathsf{w}_0)$.*

Proof. By Theorem 5.3.3, for $w \in \widetilde{W}$ and $u, v \in \mathsf{B}_{^JW}$, we have

$$\deg[\pi_{J,\hat{\mathsf{v}}}(T_w)]_{u,v} \leq \max\{\deg \mathcal{Q}_{J,\hat{\mathsf{v}}}(p) \mid p \in \mathcal{P}_J(\vec{w}, u) \text{ with } \theta^J(p) = v\}.$$

By Proposition 4.3.6, for all J-folded paths p we have

$$\mathcal{Q}_{J,\hat{\mathsf{v}}}(p) = \mathcal{Q}(p_J) \prod_{\alpha - k\delta \in \widetilde{\Phi}^+} \mathsf{v}_{\alpha-k\delta}^{c_{\alpha,k}(p_J)}$$

where p_J is the J-straightening of p. By [29, Lemma 7.7] or [23, Lemma 6.2], for all positively folded paths p we have that $\deg \mathcal{Q}(p) \leq L(\mathsf{w}_0)$. Thus, as p_J is positively folded (see Proposition

4.2.5), $\deg \mathcal{Q}(p_J) \leq L(\mathsf{w}_0)$. The result follows as $\deg(\mathsf{v}_{\alpha-k\delta}^{c_{\alpha,k}(p_J)}) \leq 0$ by the definition of $\hat{\mathsf{v}}$ and by (4.3.1). □

We make the following conjecture connecting the representation bound and Lusztig's **a**-function, and connecting the set recognised by a representation and Kazhdan-Lusztig two-sided cells.

Conjecture 6.1.10. *Let* $(\pi_{J,\mathsf{v}}, M_{J,\mathsf{v}}, \mathsf{B}_{J_W})$ *be bounded. Then*

(1) *Lusztig's* **a***-function satisfies* $\mathbf{a}(w) = \mathbf{a}_{J,\mathsf{v}}$ *for all* $w \in \Gamma_{J,\mathsf{v}}$.

(2) *The set* $\Gamma_{J,\mathsf{v}}$ *is contained in a two sided Kazhdan-Lusztig cell of* $\widetilde{W}$.

Remark 6.1.11. It is not true in general that $\Gamma_{J,\mathsf{v}}$ equals a 2-sided cell. For example, in 6.1.7(*ii*) and (*iv*) the set $\Gamma_{I,\mathsf{v}}$ is strictly contained in a two sided cell (see [23, Figure 2] for the Kazhdan-Lusztig cell decomposition for W associated with $\Phi = G_2$).

6.2 Macdonalds c-function and further conjectures of $\mathbf{a}_{J,\mathsf{v}}$

In Section 3.4 we introduced the Plancherel formula, a spectral decomposition of the canonical trace function on $\widetilde{\mathcal{H}}$. The decomposition is an integral over the tempered representations of $\widetilde{\mathcal{H}}$ upon specialising variables. In this section we further explore this formula. In particular, we make a conjecture that draws a parallel between the affine situation and the finite (in which case the trace decomposes as a sum over characters of irreducible representations whose coefficients are connected to Lusztig's **a**-function).

The *Macdonald c-function* is as follows:

$$c(X) = \prod_{\alpha \in \Phi^+} \frac{1 - \mathsf{q}_\alpha^{-2} X^{-\alpha^\vee}}{1 - X^{-\alpha^\vee}}.$$

Recall that $\widetilde{\mathcal{H}}_{\mathbb{C}}$ is the algebra over $\mathbb{C}$ formed by extending the scalars of $\widetilde{\mathcal{H}}$ to $\mathbb{C}$ and taking q to $q \in \mathbb{R}$ with $q > 1$. Furthermore, $\overline{\mathcal{H}}_{\mathbb{C}}$ is the completion of $\widetilde{\mathcal{H}}_{\mathbb{C}}$ with respect to the operator norm defined from the trace function on $\widetilde{\mathcal{H}}$ in Section 3.4.

Recall the Plancherel formula from 3.4.3; the trace function on $\widetilde{\mathcal{H}}$ decomposes as

$$\mathrm{Tr}(h) = \int_{\mathrm{Irrep}(\overline{\mathcal{H}}_{\mathbb{C}})} \chi_\pi(h) d\mu(\pi)$$

where μ is the Plancherel measure and the irreducible representations of $\overline{\mathcal{H}}_{\mathbb{C}}$ are the tempered $\mathbb{C}$-extensions of the irreducible representations of $\widetilde{\mathcal{H}}$ taking $\mathsf{q} \mapsto q$.

Remark 6.2.1. Recall the representations of $\widetilde{\mathcal{H}}$ defined in Section 5.1, denoted $(\pi_{J,\mathsf{v}}, M_{J\mathsf{v}}, \mathsf{B}_{J_W})$. Extend the scalars of the representations to $\mathbb{C}$ and take $\mathsf{q} \mapsto q > 1$. Furthermore, take $\zeta_i \mapsto z_i$ for $i \in I\backslash J$ where $z_i \in \mathbb{C}$ with $|z_i| = 1$ (where $|\cdot|$ denotes complex modulus). By Theorem 6.1.4, $(\pi_{J,\mathsf{v}}, M_{J,\mathsf{v}})$ is bounded if and only if $\deg \mathsf{v}^\gamma \leq 0$ for all $\gamma \in P^+$. Equivalently this is when $|\varpi_{J,\mathsf{v}}(X^\gamma)| = |\mathsf{v}^\gamma| \leq 1$ for $\gamma \in P^+$, after specialising and extending scalars as $|z_i| = 1$. Thus, by Casselman's criterion for temperedness (see [40, Lemma 2.20]), the specialised form of the representation $(\pi_{J,\mathsf{v}}, M_{J,\mathsf{v}})$ is tempered if and only if it is bounded. Furthermore, by Corollary 5.4.7 the representation $(\pi_{J,\mathsf{v}}, M_{J,\mathsf{v}})$ is irreducible and thus will appear in the Plancherel formula decomposition.

To further understand the Plancherel formula we follow the setup of [41, §2.4] (which in turn follows [39]) and then complete a low rank example.

Let $z \in \mathrm{Hom}(P, \mathbb{C}^\times)$, writing $z(\gamma) = z^\gamma$ for $\gamma \in P$. Denote $z^{\omega_i} = z_i$ for $i \in I$. Define $G_z : \widetilde{\mathcal{H}}_\mathbb{C} \to \mathbb{C}$ by

$$G_z(h) = \sum_{\gamma \in P} z^{-\gamma} \mathrm{Tr}(X^\gamma h). \tag{6.2.1}$$

By [39, Corollary 3.2] this series is convergent for all $h \in \widetilde{\mathcal{H}}_\mathbb{C}$ whenever $|z^{\alpha_i^\vee}| < q_i^{-2}$. Furthermore, by [39, (3.9)], when the series is convergent we have that

$$G_z(h) = \frac{f_z(h)}{q_{\mathsf{w}_0}^2 c(z)c(z^{-1})} \tag{6.2.2}$$

where $c(z)$ is Macdonalds c-function and, for fixed $h \in \widetilde{\mathcal{H}}_\mathbb{C}$, the function $f_z(h)d(z)$ is a polynomial in $\{z^\gamma \mid \gamma \in P\}$ where $d(z) = \prod_{\alpha \in \Phi^+}(1 - z^{-\alpha^\vee})$. By [39, Theorem 3.7] or [41, (2.12)], combining (6.2.1) and (6.2.2) we have that

$$\mathrm{Tr}(h) = \frac{1}{q_{\mathsf{w}_0}^2} \int_{a_n\mathbb{T}} \cdots \int_{a_1\mathbb{T}} \frac{f_z(h)}{c(z)c(z^{-1})} dz_1 \cdots dz_n \tag{6.2.3}$$

where $\mathbb{T}$ is the group of complex numbers of modulus 1 and dz_i is the Haar measure on $\mathbb{T}$. Furthermore, $a_1, \ldots, a_n$ are defined so that $|z^{\alpha_i^\vee}| < q_i^{-2}$ when setting $|z_i| = a_i$. This is the starting point to calculating the Plancherel Theorem explicitly. To do this a number of contour shifts are completed to decompose the multi-integral into a sum of integrals over $\mathbb{T}^k$ for different $k \leq n$. See [41, §4] for the explicit Plancherel decomposition of all dimension 1 and 2 root systems. For understanding, we give the decomposition of $\Phi = A_1$ below.

Example 6.2.2. Let $\Phi = A_1$ and let $\mathsf{q} = \mathsf{q}_i$ for all $i \in I$. Taking $\mathsf{q} \mapsto q > 1$, the series $G_z(h)$ converges whenever $|z^{\alpha_1^\vee}| < q^{-2}$. Denote $z = z^{\omega_1}$. As $\alpha_1^\vee = 2\omega_1$, the series converges whenever $|z| < q^{-1}$. Fix $h \in \widetilde{\mathcal{H}}_\mathbb{C}$. By (6.2.3) we have

$$\mathrm{Tr}(h) = \frac{1}{q^2} \int_{q^{-1}a\mathbb{T}} \frac{f_z(h)}{c(z)c(z^{-1})} dz \tag{6.2.4}$$

where $0 < a < 1$ and

$$c(z)c(z^{-1}) = \frac{(1 - q^{-2}z^{-2})(1 - q^{-2}z^2)}{(1 - z^{-2})(1 - z^2)}.$$

Let $0 < b < 1$ be very close to 1. We want to extend the radius of the integral from $aq^{-1}\mathbb{T}$ to $b\mathbb{T}$. The z-poles between $aq^{-1}\mathbb{T}$ and $b\mathbb{T}$ are $z = \pm q^{-1}$. We compute the residues (where $dz = \frac{1}{2\pi i}\frac{dt}{t}$) as follows:

$$\begin{aligned} \mathrm{Res}_{z=\pm q^{-1}} \frac{f_z(h)}{q^2 c(z)c(z^{-1})} &= \lim_{z \to \pm q^{-1}} (1 - \pm q^{-1}z) \frac{f_z(h)(1 - z^{-2})(1 - z^2)}{q^2(1 - q^{-1}z^{-1})(1 + q^{-1}z^{-1})(1 - q^{-1}z)(1 + q^{-1}z)} \\ &= \begin{cases} -\frac{f_{q^{-1}}(h)(q^2-1)}{2(q^2+1)} & \text{if } z \to q^{-1} \\ -\frac{f_{-q^{-1}}(h)(q^2-1)}{2(q^2+1)} & \text{if } z \to -q^{-1} \end{cases} \end{aligned}$$

Consider the representations of $\widetilde{\mathcal{H}}$ formed from the set up in Chapter 5. We have the principal series representation $(\pi_{\emptyset,\hat{\mathsf{v}}}, M_{\emptyset,\hat{\mathsf{v}}})$ with $\psi_{\emptyset,\hat{\mathsf{v}}}(X^{\omega_1}) = \zeta \mapsto z$ the central character. We also have the one dimensional representation $(\pi_{I,\hat{\mathsf{v}}}, M_{I,\hat{\mathsf{v}}})$ with $\psi_{I,\hat{\mathsf{v}}}(T_1) = -\mathsf{q}^{-1}$ and $\psi_{I,\hat{\mathsf{v}}}(X^{\omega_1}) = -\mathsf{q}^{-1}\zeta_J^{\omega_1}$. We now specialise $\zeta_J^{\omega_1}$ to a complex number z_J with modulus 1. As $\zeta_J^{\alpha_1^\vee} = \zeta_J^{2\omega_1} = 1$ we have that $z_J^2 = 1$ and so $z_J = 1$ or -1. Thus, the two 1-dimensional representations are $\psi_{I,\hat{\mathsf{v}}}$ and $\psi'_{I,\hat{\mathsf{v}}}$ where $\psi_{I,\hat{\mathsf{v}}}(T_1) = -\mathsf{q}^{-1}$, $\psi_{I,\hat{\mathsf{v}}}(X^{\omega_1}) = -\mathsf{q}^{-1}$, $\psi'_{I,\hat{\mathsf{v}}}(T_1) = -\mathsf{q}^{-1}$ and $\psi'_{I,\hat{\mathsf{v}}}(X^{\omega_1}) = \mathsf{q}^{-1}$. Extending scalars and specialising q, let χ_z, χ_1 and χ_2 denote the associated characters of these three specialised representations, respectively.

By [41, Lemma 3.9] we have $\chi_z(h) = \sum_{w\in W_0} f_{wz}(h) = f_z(h) + f_{-z}(h)$ and by [41, Lemma 3.11], $\chi_1(h) = f_{-q^{-1}}(h)$ and $\chi_2(h) = f_{q^{-1}}(h)$. Thus, (6.2.4) becomes

$$\begin{aligned}\mathrm{Tr}(h) &= \frac{1}{q^2}\int_{b\mathbb{T}} \frac{f_z(h)}{c(z)c(z^{-1})} + \frac{(q^2-1)}{2(q^2+1)}(f_{q^{-1}}(h) + f_{-q^{-1}}(h)) \\ &= \frac{1}{2q^2}\int_{b\mathbb{T}} \frac{f_z(h)+f_{-z}(h)}{c(z)c(z^{-1})}dz + \frac{(q^2-1)}{2(q^2+1)}(f_{q^{-1}}(h) + f_{-q^{-1}}(h)) \\ &= \frac{1}{2q^2}\int_{\mathbb{T}} \frac{\chi_z(h)}{c(z)c(z^{-1})}dz + \frac{(q^2-1)}{2(q^2+1)}(\chi_1(h) + \chi_2(h))\end{aligned}$$

where we extend the integral from $b\mathbb{T}$ to $\mathbb{T}$ without encountering any z-poles.

Remark 6.2.3. The multi-integral in (6.2.3) will split, after completing contour shifts, into a sum of integrals of the form

$$\mathrm{Tr}(h) = \frac{1}{|W_0|q_{\mathsf{w}_0}^2}\int_{\mathbb{T}^n} \frac{\chi_z(h)}{c(z)c(z^{-1})}dz \ + \ \text{lower terms}$$

where 'lower terms' refers to integrals over lower dimensional tori that are linked with lower dimensional representations of $\overline{\mathcal{H}}_{\mathbb{C}}$. In Example 6.2.2 these lower terms are linked with 1-dimensional representations.

Let $\chi_{J,\mathsf{v}}$ denote the character of the representation $(\pi_{J,\mathsf{v}}, M_{J,\mathsf{v}})$. By [40, Theorem 3.25], the mass of the character $\chi_{J,\mathsf{v}}$ in the Plancherel formula is a constant multiple of the reciprocal of

$$\psi_{J,\mathsf{v}}(\mathsf{q}^{2L(\mathsf{w}_0)}c(X)c(X^{-1}))' \tag{6.2.5}$$

after specialising $\mathsf{q} \mapsto q$ and $\zeta_i \mapsto z_i$, where the prime indicates that any factors that are zero after applying $\psi_{J,\mathsf{v}}$ are removed. The following conjecture links the degree of this mass term with the bound of $\pi_{J,\mathsf{v}}$.

Conjecture 6.2.4. *If $(\pi_{J,\mathsf{v}}, M_{J,\mathsf{v}}, \mathsf{B}_{^J W})$ is bounded then the bound is given by*

$$\mathbf{a}_{J,\mathsf{v}} = \frac{1}{2}\deg\psi_{J,\mathsf{v}}(\mathsf{q}^{2L(\mathsf{w}_0)}c(X)c(X^{-1}))' = L(\mathsf{w}_0) - \frac{1}{2}\deg{\prod_{\alpha\in\Phi}}' \frac{1-\mathsf{v}^{\alpha^\vee}}{1-\mathsf{q}_\alpha^{-2}\mathsf{v}^{\alpha^\vee}}$$

where $\prod'$ indicates that any factors that are zero in the numerator or the denominator are omitted.

Conjecture 6.2.4 together with Conjecture 6.1.10 give a conjectural formula for Lusztig's **a**-function for the elements of $w \in \widetilde{W}$ that are recognised by some bounded representation $(\pi_{J,\mathsf{v}}, M_{J,\mathsf{v}}, \mathsf{B}_{^J W})$ (the elements in $\Gamma_{J,\mathsf{v}}$). Furthermore, Conjecture 6.2.4 gives an affine analogue to the link between generic degrees and Lusztig's **a**-function in the trace decomposition for $\mathcal{H}_0$ (see (3.4.1)).

Proposition 6.2.5. *Conjecture 6.2.4 implies Conjecture 6.1.8.*

Proof. We have

$$\prod_{\alpha\in\Phi}{}' \frac{1-\mathsf{v}^{\alpha^\vee}}{1-\mathsf{q}_\alpha^{-2}\mathsf{v}^{\alpha^\vee}} = \prod_{\alpha\in\Phi^+}{}' \frac{(1-\mathsf{v}^{\alpha^\vee})(1-\mathsf{v}^{-\alpha^\vee})}{(1-\mathsf{q}_\alpha^{-2}\mathsf{v}^{\alpha^\vee})(1-\mathsf{q}_\alpha^{-2}\mathsf{v}^{-\alpha^\vee})}.$$

Suppose that $\alpha\in\Phi^+$ and we have that $\mathsf{q}_\alpha=\mathsf{q}^a$ for some $a>0$. Furthermore, let $\mathsf{v}^{\alpha^\vee}=\mathsf{q}^k$ for some $k\in\mathbb{Z}$. Thus,

$$C_\alpha = \frac{(1-\mathsf{v}^{\alpha^\vee})(1-\mathsf{v}^{-\alpha^\vee})}{(1-\mathsf{q}_\alpha^{-2}\mathsf{v}^{\alpha^\vee})(1-\mathsf{q}_\alpha^{-2}\mathsf{v}^{-\alpha^\vee})} = \frac{(1-\mathsf{q}^k)(1-\mathsf{q}^{-k})}{(1-\mathsf{q}^{k-2a})(1-\mathsf{q}^{-k-2a})}.$$

Taking into account that factors that are zero are removed, it follows that

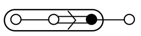
$$\deg \frac{(1-\mathsf{q}^k)(1-\mathsf{q}^{-k})}{(1-\mathsf{q}^{k-2a})(1-\mathsf{q}^{-k-2a})} = \begin{cases} |k| & \text{if } 0\le |k|\le 2a,\\ 2a & \text{if } 2a\le |k|. \end{cases}$$

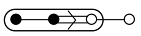
Hence, $\deg C_\alpha \ge 0$ for all $\alpha\in\Phi^+$ where equality occurs if and only if $\mathsf{v}^{\alpha^\vee}=1$. Hence,

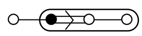
$$L(\mathsf{w}_0) - \frac{1}{2}\prod_{\alpha\in\Phi}{}' \frac{1-\mathsf{v}^{\alpha^\vee}}{1-\mathsf{q}_\alpha^{-2}\mathsf{v}^{\alpha^\vee}} = L(\mathsf{w}_0) - \frac{1}{2}\prod_{\alpha\in\Phi^+}{}' C_\alpha \le L(\mathsf{w}_0)$$

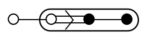
with equality if and only if $\mathsf{v}^{\alpha^\vee}=1$ for all $\alpha\in\Phi^+$ which occurs only in the case when $J=\emptyset$. □

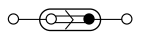
Example 6.2.6. Consider $\Phi=F_4$, with $L(s_1)=L(s_2)=b$ and $L(s_3)=L(s_4)=a$. Write $r=a/b$. The conjectural bounds $\mathbf{a}_\pi$ (from Conjecture 6.2.4) for a selection of the bounded representations of $\widetilde{\mathcal{H}}$ are as follows: (the notation is as in Example 6.1.6; J is circled, and the node associated with s_i is coloured black if $\mathsf{v}_{\alpha_i}=\mathsf{q}^{L(s_i)}$ and coloured white if $\mathsf{v}_{\alpha_i}=-\mathsf{q}^{-L(s_i)}$)

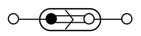
(1)  $\mathbf{a}_\pi = 2a+2b, 5a, 6a-b, 11a-7b$ for $r\in(0,2/3],[2/3,1],[1,6/5],[6/5,2]$.

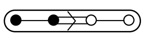
(2)  $\mathbf{a}_\pi = 4a+12b$ for $r\in[4,\infty)$.

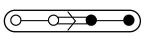
(3)  $\mathbf{a}_\pi = -2a+11b, a+6b, 2a+3b$ for $r\in[1,5/3],[5/3,3],[3,\infty)$.

(4)  $\mathbf{a}_\pi = 6a+4b, 9a+3b$ for $r\in(0,1/3],[1/3,1/2]$.

(5)  $\mathbf{a}_\pi = 4a+12b, 6a+4b$ for $r\in[2,4],[4,\infty)$.

(6)  $\mathbf{a}_\pi = 3a+6b, 11a+2b$ for $r\in(0,1/2],[1/2,1]$.

(7)  $\mathbf{a}_\pi = -2a+11b, -a+9b, 6b$ for $r\in[5/3,2],[2,3],[3,\infty)$.

(8)  $\mathbf{a}_\pi = 3a, 5a-b, 11a-7b$ for $r\in(0,1/2],[1/2,1],[1,6/5]$.

Thus Conjecture 6.1.10 predicts the existence of elements of $\widetilde{W}$ with the above $\mathbf{a}$-function values in the respective parameter ranges. See `https://github.com/ellielittle/ConjecturalBound` for MAGMA code that computes the conjectural bound for any Φ (reduced and irreducible), $J\subseteq I$ and J-parameter system v.

Theorem 6.2.7. *Conjectures 6.2.4 and 6.1.10 hold in the following cases.*
(1) *All extended affine Hecke algebras in the case* $J=\emptyset$.
(2) *All extended affine Hecke algebras of dimension* 1 *or* 2 *(rank* 2 *or* 3*).*

Proof. See [21, Theorem 5.20] for the non-reduced cases. Let Φ be reduced and $\widetilde{W}$ be its associated extended Weyl group.

(1) follows from Example 6.1.1 and [20, Theorem 4.6] which shows that $\Gamma_{\emptyset,\mathsf{v}}\cap W$ from Example 6.1.1 is equal to the lowest two-sided Kazhdan-Lusztig cell for W. Extending to $\widetilde{W}$,

as left cells (respectively right cells) are invariant under multiplication by $\sigma \in \Sigma$ on the right (respectively left), $\Gamma_{\emptyset,\mathsf{v}}$ is equal to the lowest Kazhdan-Lusztig cell for $\widetilde{W}$. Furthermore, it is well known that $\mathbf{a}(w) = L(\mathsf{w}_0)$ for elements of this cell.

(2) We have $\Phi \in \{A_1, A_2, G_2, C_2\}$, and consider the cases separately. First let $\Phi = A_1$. The case when $J = \emptyset$ is true by (1). Consider $J = \{1\}$. Letting $\mathsf{q}_1 = \mathsf{q}$, we have that $\mathsf{v}^{\alpha_1^\vee} = \mathsf{q}^{-2}$. Considering C_{α_1} (from the proof of Proposition 6.2.5) we have that $\deg C_{\alpha_1} = 2$ and thus

$$L(\mathsf{w}_0) - \frac{1}{2}\deg \prod_{\alpha\in\Phi^+}{}' C_\alpha = 0$$

In this case $\mathcal{A}_J = A_0$ and so the highest path bound is 0 (equivalently the matrix bound by Theorem 5.3.3). Furthermore, the only elements that reach this bound are $\Gamma_{I,\mathsf{v}} = \Sigma = \{e, \sigma\}$. By Example 3.2.5 when $a = b$ this is exactly the highest Kazhdan-Lusztig cell for $\widetilde{W}$. It is well known that $\mathbf{a}(w) = 0$ for the elements in this cell.

Now consider $\Phi = A_2$ and let $\mathsf{q}_1 = \mathsf{q}_2 = \mathsf{q}$. Again the case where $J = \emptyset$ follows from (1), and the case when $J = I$ is similar to that of $\Phi = A_1$ (the cell is $\Gamma_{I,\mathsf{v}} = \Sigma = \{e, \sigma, \sigma^2\}$). Let $J = \{1\}$. Then we have that $\deg C_{\alpha_1} = 2$, $\deg C_{\alpha_2} = 1$ and $\deg C_{\alpha_1+\alpha_2} = 1$ so

$$L(\mathsf{w}_0) - \frac{1}{2}\deg \prod_{\alpha\in\Phi^+}{}' C_\alpha = 1$$

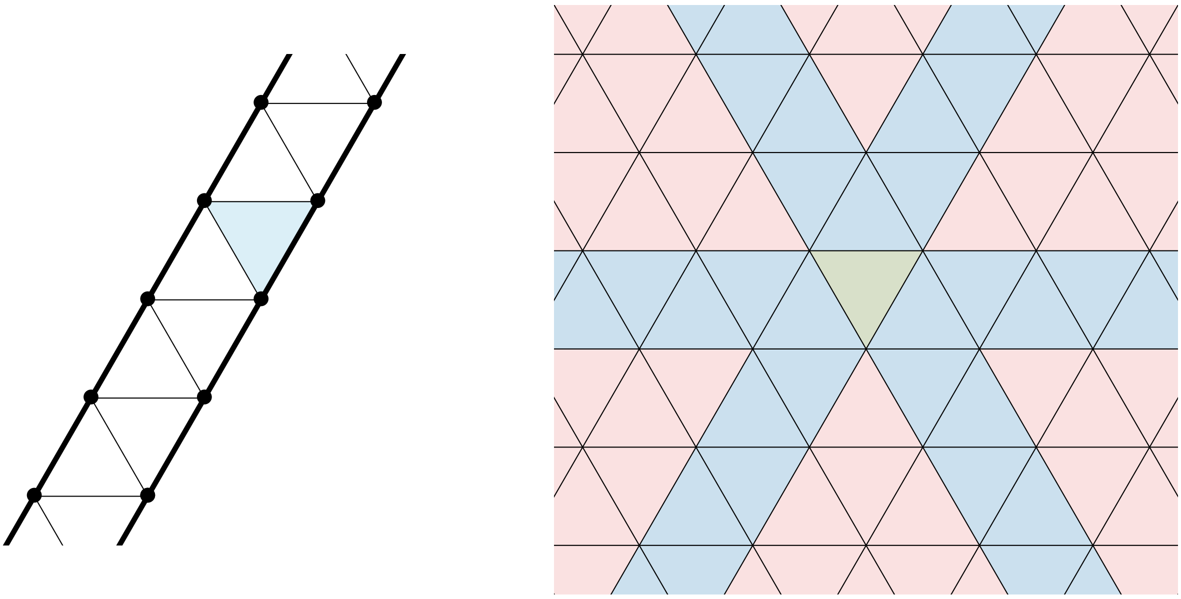
Consider $\mathcal{A}_J$, depicted in Figure 6.1(a), the paths with the highest bound are those that, beginning at uA_0 with $u \in {}^JW$, travel downwards, complete a fold and then travel upwards. If the path bounces as it travels downwards and then folds it must then travel upwards to remain a reduced expression. When travelling upwards paths cannot fold as every crossing is positive. If $\vec{w}$ is a path beginning at $u \in {}^JW$ that travels down $\mathcal{A}_J$, folds and then travels up, $u^{-1}w$ reaches the bound. Thus $\Gamma_{J,\mathsf{v}} \cap W$ is the set of the elements corresponding to the blue coloured alcoves in Figure 6.1(b) and $\Gamma_{J,\mathsf{v}} = (\Gamma_{J,\mathsf{v}} \cap W) \cup (\Gamma_{J,\mathsf{v}} \cap W)\sigma \cup (\Gamma_{J,\mathsf{v}} \cap W)\sigma^2$. Furthermore, $\mathbf{a}_{J,\mathsf{v}} = 1$. By [29, Figure 1], $\Gamma_{J,\mathsf{v}}$ is exactly a Kazhdan-Lusztig cell for $\widetilde{W}$ and it is well known that $\mathbf{a}(w) = 1$ for elements in this cell.

Let $\Phi = G_2$. Conjecture 6.1.10 is proved in [23], thus we only need to verify Conjecture 6.2.4. The case when $J = \emptyset$ is true by (1). Suppose that $J = \{1\}$ and let $\mathsf{q}_1 = \mathsf{q}^a$ and $\mathsf{q}_2 = \mathsf{q}^b$. As $\mathsf{v}^{\alpha_1^\vee} = \mathsf{q}^{-2a}$ and $\mathsf{v}^{\alpha_2^\vee} = -\mathsf{q}^a$ we have that

$\alpha \in \Phi^+$	$\deg C_\alpha$
α_1	$2a$
α_2	$2b$ if $a/b \geq 2$ a if $a/b < 2$
$\alpha_1 + \alpha_2$	a
$2\alpha_1 + \alpha_2$	a
$3\alpha_1 + \alpha_2$	$2b$ if $a/b \geq 2$ a if $a/b < 2$
$3\alpha_1 + 2\alpha_2$	0

Hence, as $L(\mathsf{w}_0) = 3a + 3b$ we have

$$L(\mathsf{w}_0) - \frac{1}{2}\deg \prod_{\alpha\in\Phi^+}{}' C_\alpha = \begin{cases} a+b & \text{if } a - 2b \geq 0 \\ 3b & \text{if } a - 2b < 0. \end{cases}$$



(a) $\mathcal{A}_J$ (b) The Kazhdan-Lusztig cells

Figure 6.1: Cells and fundamental J-alcove for $\Phi = A_2$

By [23, Theorem 7.10] these are equal to the bounds of the induced representation $\pi_{J,\mathsf{v}}$, completing this case.

The case when $J = \{2\}$ is similar, again using the results of [23, Theorem 7.10], and will be omitted.

Now consider $J = I$. There are 3 bounded I-parameter systems by Proposition 6.1.5. When $(\mathsf{v}_{\alpha_1}, \mathsf{v}_{\alpha_2}) = (-\mathsf{q}^{-a}, -\mathsf{q}^{-b})$ the bound is 0 and the representation recognises the trivial Kazhdan-Lusztig cell $\{e\}$. Consider $(\mathsf{v}_{\alpha_1}, \mathsf{v}_{\alpha_2}) = (\mathsf{q}^{a}, -\mathsf{q}^{-b})$ which is bounded only when $a/b \leq 3/2$. We have that $\mathsf{v}^{\alpha_1^\vee} = \mathsf{q}^{2a}$ and $\mathsf{v}^{\alpha_2^\vee} = \mathsf{q}^{-2b}$. We have the following degrees of C_α

$\alpha \in \Phi^+$	$\deg C_\alpha$
α_1	$2a$
α_2	$2b$
$\alpha_1 + \alpha_2$	$2a$
$2\alpha_1 + \alpha_2$	$-4a + 6b$ if $1 \leq a/b \leq 3/2$ $2a$ if $a/b \leq 1$
$3\alpha_1 + \alpha_2$	$2a - 2b$ if $1 \leq a/b \leq 3/2$ $2b - 2a$ if $a/b \leq 1$
$3\alpha_1 + 2\alpha_2$	$4b - 2a$ if $1 \leq a/b \leq 3/2$ $2b$ if $a/b \leq 1$

Thus,

$$L(\mathsf{w}_0) - \frac{1}{2} \deg \prod_{\alpha \in \Phi^+}{}' C_\alpha = \begin{cases} 3a - 2b & \text{if } 1 \leq a/b \leq 3/2 \\ a & \text{if } a/b \leq 1. \end{cases}$$

agreeing with the bounds given in Example 6.1.7. Furthermore, the cells recognised by this representation, given in Example 6.1.7, are contained in Kazhdan-Lusztig cells for the relevant parameter values by [23, Figure 2].

The case when $(\mathsf{v}_{\alpha_1}, \mathsf{v}_{\alpha_2}) = (-\mathsf{q}^{-a}, \mathsf{q}^{b})$ is similar and we omit the details. Furthermore, the analysis of the case when $\Phi = C_2$ is similar, using [22, Theorems 6.15,6.21,6.22], and we again omit the calculations. $\square$

Part II

The asymptotic Plancherel Theorem and Lusztig's asymptotic algebra in type A_n

Chapter 7

A_n background

In Chapters 1-6 our definitions were made for a general reduced irreducible root system Φ. We now set $\Phi = A_n$ and conduct a deeper analysis of this case, that is we will take the theory established in Chapters 1-5 and reduce to the case that $\Phi = A_n$. In this chapter we set the notation and conventions that will be used in the remainder of Part II and in Appendix A.

Section 7.1 gives an explicit description in type A_n of the definitions from Chapter 1. In particular, we now set a new basis for the underlying vector space V, the e-basis, and describe P and Q in terms of this basis. We also set up partition and tableau notation as the $J \subseteq I$ sets will now correspond to partitions of $n+1$ (and are thus denoted as J_λ for λ a partition of $n+1$).

Section 7.2 deals with the definitions established in Chapter 2. We explicitly describe the weights of the fundamental J_λ-alcove, using tableaux, and introduce a partial ordering on these weights. To explicitly describe the symmetries of the fundamental J_λ-alcove we define a particular group element u_λ and describe its properties. This element will be important throughout our analysis of $\Phi = A_n$.

Finally, Section 7.3 describes the definitions established in Chapter 3 and Chapter 5 in the $\Phi = A_n$ case. In addition, a λ-analogue to Schur functions is defined which will be used in Chapter 11.

7.1 The symmetric group and tableaux

Setting $\Phi = A_n$, we now define new notation and conventions to aid in our analysis in Chapters 8-11. In particular we set a new basis for our underlying vector space V, the e-basis. In this section, we will shift our definitions from Chapter 1 to this e-basis and introduce tableaux and window notation.

Let $V = \{v \in \mathbb{R}^{n+1} \mid v \cdot \mathbf{1} = 0\}$ where $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^{n+1}$. Denote

$$e_i = (0, \dots, 0, 1, 0, \dots, 0) - \frac{1}{n+1}\mathbf{1}$$

for $1 \leq i \leq n+1$ where the 1 is in the i-th place. We have that $e_i \in V$ and that $e_1 + \dots + e_{n+1} = 0$. Let $\langle \cdot, \cdot \rangle$ be the restriction of the standard inner product on $\mathbb{R}^{n+1}$ to V. Denote $e_{i,j} = e_i - e_j$.

Lemma 7.1.1. *Let $1 \le k \le l \le n+1$ and $1 \le i < j \le n+1$, we have*

$$\langle e_k + \cdots + e_l, e_{i,j} \rangle = \begin{cases} 0 & \text{if } k \le i < j \le l \text{ or } i < k \le l < j \text{ or } k \le l < i < j \text{ or } i < j < k \le l, \\ 1 & \text{if } k \le i \le l < j, \\ -1 & \text{if } i < k \le j \le l. \end{cases}$$

Proof. This is by direct calculation using the fact that $e_{i,j} = \frac{1}{n+1}(0, \ldots, 0, n+1, 0, \ldots, 0, -n-1, 0 \ldots, 0)$ (where $n+1$ is in the i-th position and $-n-1$ is in the j-th position) and $e_k + \cdots + e_l = \frac{1}{n+1}(-1-m, \ldots, -1-m, n-m, \ldots, n-m, -1-m, \ldots, -1-m)$ (where $n-m$ is the entry for the k-th to l-th positions and $m = l-k$). □

Let
$$\Phi^+ = \{e_{i,j} \mid 1 \le i < j \le n+1\}.$$

Thus $\Phi = \Phi^+ \cup (-\Phi^+)$ is a root system of type A_n. The simple roots are $\alpha_i = e_i - e_{i+1}$ for $1 \le i \le n$ and the highest root is $\varphi = \alpha_1 + \cdots + \alpha_n = e_{1,n+1}$.

The fundamental weights in this e-basis are $\omega_i = e_1 + \cdots + e_i$ for $1 \le i \le n$. By convention we set $\omega_0 = 0$ and $\omega_{n+1} = e_1 + \cdots + e_{n+1} = 0$. Hence, $\alpha_i = -\omega_{i-1} + 2\omega_i - \omega_{i+1}$ for all $1 \le i \le n$.

Recall from Chapter 1 that P is the $\mathbb{Z}$-span of the fundamental weights and that P^+ is the $\mathbb{N}$-span of the fundamental weights. Now considering the e-basis, P is the $\mathbb{Z}$-span of the e_i's for $1 \le i \le n+1$. A vector $a_1e_1 + \cdots + a_{n+1}e_{n+1} \in P$ is in P^+ if and only if $a_1 \ge a_2 \ge \cdots \ge a_{n+1}$.

From Chapter 1, Q is the $\mathbb{Z}$-span of the simple roots and Q^+ is the $\mathbb{N}$-span of the simple roots. Changing to the e-basis, a vector $a_1e_1 + \cdots + a_{n+1}e_{n+1} \in P$ is in Q if and only if $a_1 + \cdots + a_{n+1} = 0 \mod n+1$. Now consider $\gamma = a_1e_1 + \cdots + a_{n+1}e_{n+1} \in Q$ with $a_1 + \cdots + a_{n+1} = k(n+1)$ for some $k \in \mathbb{Z}$. As $e_1 + \cdots + e_{n+1} = 0$ we have that $\gamma = a_1'e_1 + \cdots + a_{n+1}'e_{n+1}$ where $a_i' = a_i - k$ and so $a_1' + \cdots + a_{n+1}' = 0$. Then $\gamma \in Q^+$ if and only if $a_1' + \cdots + a_i' \ge 0$ for all $1 \le i \le n+1$.

The Weyl group associated to Φ is denoted W_0 as it was in Chapter 1. In this case this group is isomorphic to the symmetric group on the set $\{1, \ldots, n+1\}$, denoted $\mathfrak{S}_{n+1}$. The action of $\mathfrak{S}_{n+1}$ on the e-basis is defined as $we_i = e_{w(i)}$ where $w \in \mathfrak{S}_{n+1}$ is a permutation of the numbers $\{1, \ldots, n+1\}$. Let $s_1, \ldots, s_n$ be the simple reflections of W_0 (associated with elementary transpositions in $\mathfrak{S}_{n+1}$).

Recall from Chapter 1 that the affine Weyl group associated to Φ is $W = Q \rtimes W_0$ and the extended affine Weyl group is $\widetilde{W} = P \rtimes W_0$. Furthermore, $\widetilde{W} = W \rtimes \Sigma$ where $\Sigma = P/Q$. In this case $\Sigma = P/Q \cong \mathbb{Z}/(n+1)\mathbb{Z}$. Setting $\sigma(i) = i+1 \mod n+1$ for $1 \le i \le n$, we have $\Sigma = \{e, \sigma, \sigma^2, \ldots, \sigma^n\}$. This set induces a permutation on the nodes of the Coxeter diagram of A_n, and thus on the simple reflections of W_0, with $\sigma^k s_i \sigma^{-k} = s_{\sigma^k(i)}$.

For the analysis of $\Phi = A_n$ we will only consider subsets J of I associated with a partition of $n+1$ (see [18, §2.3.7]). Therefore, many objects will be indexed by partitions of $n+1$ instead of J. We set up the following notation.

Let $\mathcal{P}(n+1)$ denote the set of partitions of $n+1$. For $\lambda \in \mathcal{P}(n+1)$ we write $\lambda \vdash n+1$. We will represent partitions by Young diagrams using English notation conventions (the number of boxes in each row decreases from top to bottom).

Definition 7.1.2. For $\lambda \vdash n+1$ we make the following definitions:
1. $r(\lambda)$ is the number of parts of λ, so $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{r(\lambda)})$.
2. $\lambda(0) = 0$ and $\lambda(i) = \lambda_1 + \cdots + \lambda_i$ for $1 \le i \le r(\lambda)$.
3. $\mathbf{t}_r(\lambda)$ is the standard tableau of shape λ filled by row.
4. $\mathbf{t}_c(\lambda)$ is the standard tableau of shape λ filled by column.

(5) $\lambda[i,j]$ is the element in the i-th row and j-th column of $\mathbf{t}_r(\lambda)$ (with $1 \le i \le r(\lambda)$ and $1 \le j \le \lambda_i$).
(6) $J_\lambda = \{1, 2, \ldots, n+1\} \backslash \{\lambda(1), \ldots, \lambda(r(\lambda))\}$.

Furthermore, let λ' denote the transposed partition of λ (obtained by reflecting λ in the main diagonal).

Example 7.1.3. Let $\lambda = (4,3,3,2,1,1) \vdash 14$ and so $r(\lambda) = 6$. Furthermore, $\lambda' = (6,4,3,1)$ and so $r(\lambda') = 4$. We have

$\mathbf{t}_r(\lambda) =$

1	2	3	4
5	6	7	
8	9	10	
11	12		
13			
14			

and $\mathbf{t}_c(\lambda) =$

1	7	11	14
2	8	12	
3	9	13	
4	10		
5			
6			

Then $\{\lambda(i) \mid 1 \le i \le r(\lambda)\} = \{4, 7, 10, 12, 13, 14\}$ (these are the elements in the final boxes of $\mathbf{t}_r(\lambda)$). Furthermore, $\{\lambda'(i) \mid 1 \le i \le r(\lambda')\} = \{6, 10, 13, 14\}$ (these are the elements in the final boxes of $\mathbf{t}_c(\lambda)$). Hence, $J_\lambda = \{1,2,3,5,6,8,9,11\}$ and $J_{\lambda'} = \{1,2,3,4,5,7,8,9,11,12\}$.

Note that, for $\lambda \vdash n+1$ $J_\lambda = I$ if and only if $\lambda = (n+1)$ and $J_\lambda = \emptyset$ if and only if $\lambda = (1^{n+1})$. From henceforth we will replace all J indexed notation by λ indexed notation. For example, ${}^\lambda W$, W_λ, Φ_λ, w_λ and Φ_λ will be used in place of ${}^{J_\lambda}W$, W_{J_λ}, Φ_{J_λ}, w_{J_λ} and Φ_{J_λ} (see Chapter 1 for the definitions of each of these symbols). Furthermore, recalling Definition 1.4.2, for $w \in \widetilde{W}$ we now notate $\theta_{J_\lambda}(w)$ as $\theta_\lambda(w)$ and $\theta^{J_\lambda}(w)$ as $\theta^\lambda(w)$.

For $w \in W_0$, the *one line expression* of w is the sequence $[w(1), \ldots, w(n+1)]$. For $1 \le i \le n$ we have $\ell(ws_i) = \ell(w) + 1$ if and only if $w(i+1) > w(i)$ and $\ell(s_i w) = \ell(w) + 1$ if and only if i and $i+1$ appear in ascending order in the one line notation of w. Let $A_L(w)$ denote the left ascent set of $w \in W_0$, that is

$$\begin{aligned} A_L(w) &= \{s_i \mid \ell(s_i w) > \ell(w)\} \\ &= \{s_i \mid i \text{ appears before } i+1 \text{ in the one line expression of } w\} \end{aligned} \tag{7.1.1}$$

and let $A_R(w)$ denote the right ascent set of $w \in W_0$, that is

$$A_R(w) = \{s_i \mid \ell(ws_i) > \ell(w)\} = \{s_i \mid w(i+1) > w(i)\}. \tag{7.1.2}$$

Lemma 7.1.4. *We have $w \in {}^\lambda W$ if and only if for each row of $\mathbf{t}_r(\lambda)$ the elements of the row appear in ascending order in the 1-line notation of w.*

Proof. By definition $w \in {}^\lambda W$ if and only if w is reduced on the left by all $j \in J_\lambda$. Equivalently, this is when $\ell(s_j w) > \ell(w)$ for all $j \in J_\lambda$. By (7.1.1) this occurs if and only if j appears before $j+1$ in the one line notation of w. The result follows as $j \in J_\lambda$ if and only if j and $j+1$ are in the same row of $\mathbf{t}_r(\lambda)$. ☐

Definition 7.1.5. The right λ-ascent set of $u \in {}^\lambda W$ is

$$A_\lambda(u) = \{s_i \mid \ell(us_i) > \ell(u) \text{ and } us_i \in {}^\lambda W\} = A_R(u) \cap \{s_i \mid us_i \in {}^\lambda W\}$$

Definition 7.1.6. Let $\lambda \vdash n+1$ and $w \in W_0$. Insert dividers into the one line notation $w = [w(1), w(2), \ldots, w(n+1)]$ forming "blocks" according to the following rules; for $1 \leq i \leq n$ insert a divider between $w(i)$ and $w(i+1)$ if either

1. $w(i+1) < w(i)$ or,
2. $w(i+1) > w(i)$ and the numbers $w(i)$ and $w(i+1)$ lie in a common row of $\mathbf{t}_r(\lambda)$

The resulting expression, with the one line notation split into blocks, is called the *λ-expression of w*. There is an associated composition to the λ-expression of w formed by the sequence of the lengths of the blocks (in the order they appear). Define $\boldsymbol{\mu}(w, \lambda) \in \mathcal{P}(n+1)$ to be the partition associated to this composition (formed by rearranging the sequence of lengths of the blocks into decreasing order).

Example 7.1.7. Let $\lambda = (4,3,3,2,1,1)$ and let $w = [1,3,7,2,6,11,12,5,4,14,9,10,13,8]$. Recalling $\mathbf{t}_r(\lambda)$ from Example 7.1.3 we have that the λ-expression of w is

$$[1 \mid 3,7 \mid 2,6,11 \mid 12 \mid 5 \mid 4,14 \mid 9 \mid 10,13 \mid 8].$$

The associated composition is $(1,2,3,1,1,2,1,2,1)$ and so $\boldsymbol{\mu}(w,\lambda) = (3,2,2,2,1,1,1,1,1)$.

Lemma 7.1.8. *For $u \in {}^{\lambda}W$ we have*

$$A_\lambda(u) = \{s_i \mid u(i) \text{ and } u(i+1) \text{ lie in a common block of the } \lambda\text{-expression of } u\}.$$

Proof. Let $u \in {}^{\lambda}W$ and suppose that $u(i)$ and $u(i+1)$ are in different blocks of the λ-expression of u. Then, by definition, either $u(i+1) < u(i)$, or $u(i+1) > u(i)$ and $u(i)$ and $u(i+1)$ are in the same row of $\mathbf{t}_r(\lambda)$. In the first case, we have that $\ell(us_i) < \ell(u)$ and thus $s_i \notin A_r(u)$ (see (7.1.2)). In the second case, as $s_i(i) = i+1$ and $s_i(i+1) = i$, we have that $us_i(i+1) = u(i)$ and $us_i(i) = u(i+1)$. As $u(i)$ and $u(i+1)$ lie in the same row of $\mathbf{t}_r(\lambda)$ we have that $us_i(i+1) < us_i(i)$. Therefore, the elements in the rows of $\mathbf{t}_r(\lambda)$ won't appear in ascending order in the one line notation of us_i and so, by Lemma 7.1.4, $us_i \notin {}^{\lambda}W$. Thus, in each case $s_i \notin A_\lambda(u)$.

Conversely, suppose that $u(i)$ and $u(i+1)$ are in a common block of the λ-expression of u. Then $u(i+1) > u(i)$ and so, by (7.1.2), $s_i \in A_r(u)$. As $u \in {}^{\lambda}W$, by Lemma 7.1.4 the elements of each row of $\mathbf{t}_r(\lambda)$ appear in ascending order in the one line notation of u. Furthermore, as $u(i)$ and $u(i+1)$ lie in a common block of the λ-expression of u, $u(i)$ lies in a different row of $\mathbf{t}_r(\lambda)$ than $u(i+1)$. Thus, when swapping $u(i)$ and $u(i+1)$ in the one line expression of u to form the one line expression us_i the elements of the rows of $\mathbf{t}_r(\lambda)$ remain in ascending order. Hence, $us_i \in {}^{\lambda}W$ and so $s_i \in A_\lambda(u)$ as required. ☐

Let $\leq$ denote the dominance order on $\mathcal{P}(n+1)$. That is $\mu \leq \lambda$ if and only if $\mu(i) \leq \lambda(i)$ for all $i \geq 1$ where we set $\lambda_i = 0$ if $i > r(\lambda)$. For example, if $\lambda = (5,4,1,1)$ and $\mu = (5,3,1,1,1)$ then $\mu \leq \lambda$. Note that if $\mu \leq \lambda$ then $\lambda' \leq \mu'$.

Definition 7.1.9. For $\lambda \vdash n+1$ let

$$\mathbf{a}_\lambda = \sum_{i \geq 1} (i-1)\lambda_i.$$

Example 7.1.10. Take $\lambda = (4,3,3,2,1,1)$ as in Example 7.1.3. Then $\mathbf{a}_\lambda = 3 + 2(3) + 3(2) + 4 + 5 = 24$.

Lemma 7.1.11. *We have that*

$$\mathbf{a}_\lambda = \sum_{i \geq 1} (n+1-\lambda(i)) = \sum_{1 \leq i < j \leq r(\lambda)} \lambda_j = \frac{1}{2} \sum_{i \geq 1} \lambda'_i(\lambda'_i - 1) = \ell(\mathsf{w}_{\lambda'})$$

Proof. The first and second equality follows as

$$\sum_{i\geq 1}(n+1-\lambda(i)) = \sum_{1\leq i<j\leq r(\lambda)} \lambda_j = \sum_{i\geq 1} \text{the number of boxes below row } i$$

so the boxes of row 2 are counted once, the boxes of row 3 are counted twice and so on. The third equality follows as

$$\frac{1}{2}\sum_{i\geq 1}\lambda'_i(\lambda'_i-1) = \sum_{i\geq 1}\sum_{j=1}^{\lambda'_i-1} j$$

and so, thinking of each column, again the boxes of row 2 are counted once, row 3 twice and so on. It is well known that $\ell(\mathsf{w}_K) = \frac{1}{2}|K|(|K|+1)$ for $K \in \mathcal{K}(J_\lambda)$. Thus, the final equality follows by adding the lengths of the longest elements of each connected component of J_λ. □

Lemma 7.1.12. *Let $\lambda, \mu \in \mathcal{P}(n+1)$. If $\mu \leq \lambda$ then $\mathbf{a}_\mu \geq \mathbf{a}_\lambda$ with equality if and only if $\mu = \lambda$.*

Proof. As $\mathbf{a}_\lambda = \sum_{i\geq 1}(n+1-\lambda(i))$ by Lemma 7.1.11, this result follows directly by the definition of the dominance order. □

7.2 The λ-alcove and group G_λ

This section aims to explicitly describe the definitions set up in Chapter 2 for $\Phi = A_n$, now using the e-basis of V. In particular we will describe the weights and the symmetries of the fundamental J_λ-alcove. To describe the symmetry group we introduce an element $\mathsf{u}_\lambda \in {}^\lambda W$ that will play an important role throughout our analysis of $\Phi = A_n$. We will also define a partial ordering on the weights in the fundamental J_λ-alcove using the symmetry group.

Recall the definitions of $\mathcal{A}_J$, Q_J, $P^{(J)}$ and G_J from Chapter 2. As J will now be associated with a partition, we notate $\mathcal{A}_{J_\lambda}$, Q_{J_λ}, $P^{(J_\lambda)}$ and G_{J_λ} by $\mathcal{A}_\lambda$, Q_λ, $P^{(\lambda)}$ and G_λ.

Recall from Section 2.2 that P/Q_λ is isomorphic to the set of weights of the fundamental J_λ-alcove $\mathcal{A}_\lambda$. We now explicitly describe P/Q_λ for type A_n in terms of the e-basis.

If j, j' are in the same row of $\mathbf{t}_r(\lambda)$ then, as $e_j - e_{j'} \in \Phi_\lambda$, we have $e_j + Q_\lambda = e_{j'} + Q_\lambda$. Thus we can make the following definition; for $1 \leq i \leq r(\lambda)$ define

$$\tilde{e}_i = e_j + Q_\lambda \tag{7.2.1}$$

for any $\lambda(i-1)+1 \leq j \leq \lambda(i)$ (any j in row i of $\mathbf{t}_r(\lambda)$). Then, $P/Q_\lambda = \mathrm{span}_\mathbb{Z}\{\tilde{e}_1, \ldots, \tilde{e}_{r(\lambda)}\}$. Note that as $e_1 + \cdots + e_{n+1} = 0$ we have that

$$\lambda_1\tilde{e}_1 + \cdots + \lambda_{r(\lambda)}\tilde{e}_{r(\lambda)} = 0. \tag{7.2.2}$$

When $\lambda = (1^{n+1})$, $Q_\lambda = \{0\}$ and so $P/Q_\lambda = P$ and $\tilde{e}_i = e_i$. When $\lambda = (n+1)$, $Q_\lambda = Q$ and so $\tilde{e}_1 = e_j + Q$ for all $1 \leq j \leq n+1$. By (7.2.2) we have that $(n+1)\tilde{e}_1 = 0$, however $\tilde{e}_1 \neq 0$ as P/Q is nontrivial. Thus $P/Q = \mathbb{Z}\tilde{e}_1 = \{0, \tilde{e}_1, \ldots n\tilde{e}_1\}$.

Example 7.2.1. Let $\lambda = (2,2) \vdash 4$, thus

$$\mathbf{t}_r(\lambda) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}$$

We have that $P/Q_\lambda = \mathrm{span}_\mathbb{Z}\{\tilde{e}_1, \tilde{e}_2\}$ where $\tilde{e}_1 = e_1 + Q_\lambda = e_2 + Q_\lambda$ and $\tilde{e}_2 = e_3 + Q_\lambda = e_4 + Q_\lambda$. By (7.2.2), $2\tilde{e}_1 + 2\tilde{e}_2 = 0$. However, as $e_1 + e_3 \notin Q_\lambda$ we have that $\tilde{e}_1 + \tilde{e}_2 \neq 0$. Thus, $\tilde{e}_1 + \tilde{e}_2$ has order 2.

The natural quotient map $P \to P/Q_\lambda$ satisfies

$$d_1e_1 + \cdots + d_{n+1}e_{n+1} \mapsto a_1\tilde{e}_1 + \cdots a_{r(\lambda)}\tilde{e}_{r(\lambda)}, \text{ where } a_k = \sum_{i=\lambda(k-1)+1}^{\lambda(k)} d_i. \tag{7.2.3}$$

Example 7.2.2. Let $\lambda = (4,3,3) \vdash 10$ with

$$\mathbf{t}_r(\lambda) = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 \\ \cline{1-3} 8 & 9 & 10 \\ \cline{1-3}\end{array}$$

Let $\gamma = 2e_1 - 4e_3 + 6e_4 + e_6 + e_8 - 2e_9$. Via the map defined in (7.2.3) we have that $\gamma + Q_\lambda = 4\tilde{e}_1 + \tilde{e}_2 - \tilde{e}_3 \in P/Q_\lambda$.

Recall from Section 2.2 that the isomorphism $P^{(\lambda)} \xrightarrow{\sim} P/Q_\lambda$ is given by $\gamma \mapsto \gamma + Q_\lambda$. We now give the reverse isomorphism (for type A_n) in terms of the e-basis.

Proposition 7.2.3. *The isomorphism $P/Q_\lambda \xrightarrow{\sim} P^{(\lambda)}$ is given by*

$$a_1\tilde{e}_1 + \cdots + a_{r(\lambda)}\tilde{e}_{r(\lambda)} \mapsto \sum_{k=1}^{r(\lambda)} \left((b_k+1)(e_{\lambda[k,1]} + \cdots + e_{\lambda[k,c_k]}) + b_k(e_{\lambda[k,c_k+1]} + \cdots + e_{\lambda[k,\lambda_k]})\right)$$

where b_k and c_k are defined by $a_k = \lambda_k b_k + c_k$ with $0 \le c_k < \lambda_k$.

Proof. Let

$$\gamma = \sum_{k=1}^{r(\lambda)} \left((b_k+1)(e_{\lambda[k,1]} + \cdots + e_{\lambda[k,c_k]}) + b_k(e_{\lambda[k,c_k+1]} + \cdots + e_{\lambda[k,\lambda_k]})\right).$$

As $e_j + Q_\lambda = \tilde{e}_k$ for all j in the k-th row of $\mathbf{t}_r(\lambda)$, the k-th term of $\gamma + Q_\lambda$ (the component of $\gamma + Q_\lambda$ coming from the e_j's where j is in row k) is

$$b_k(e_{\lambda[k,1]} + \cdots + e_{\lambda[k,\lambda_k]}) + e_{\lambda[k,1]} + \cdots + e_{\lambda[k,c_k]} + Q_\lambda = \lambda_k b_k \tilde{e}_k + c_k\tilde{e}_k = a_k\tilde{e}_k$$

as required. Moreover, if $\alpha = e_{i,j} \in \Phi^+_\lambda$ then i, j lie in a common row of $\mathbf{t}_r(\lambda)$, say $i = \lambda[k,i']$ and $j = \lambda[k,j']$ where $1 \le i' < j' \le \lambda_k$. By Lemma 7.1.1, $\langle\gamma,\alpha\rangle \in \{0,1\}$ and so $\gamma \in P^{(\lambda)}$ (by the definition of $\mathcal{A}_\lambda$ and as $P^{(\lambda)} = \mathcal{A}_\lambda \cap P$). □

Remark 7.2.4. We can associate the elements of P with tableaux. Let $\lambda \vdash n+1$ and $\gamma = d_1e_1 + \cdots + d_{n+1}e_{n+1} \in P$. Identify γ with a tableau of shape λ by filling along the rows with $d_1, d_2, \ldots, d_{n+1}$. As $e_1 + \cdots + e_{n+1} = 0$, two filled tableaux are considered equal if they differ by a multiple of the constant tableau with every entry 1.

Proposition 7.2.3 gives a description of the elements of $P^{(\lambda)}$ in terms of the e-basis. Using this explicit description, the elements of $P^{(\lambda)}$ are precisely the elements whose associated tableau has k-th row of the form

b_k+1	b_k+1	$\cdots$	b_k+1	b_k	b_k	$\cdots$	b_k

where b_k+1 occurs c_k times (with $0 \le c_k < \lambda_k$), for all $1 \le k \le r(\lambda)$.

We can now explicitly describe the mapping $P \to P^{(\lambda)}$ in terms of these weight associated tableaux. That is, the weight $\gamma = d_1e_1 + \cdots d_{n+1}e_{n+1}$ is mapped to tableaux with rows of the above form where row k's data is formed by

$$\sum_{i=\lambda(k-1)+1}^{\lambda(k)} d_i = \lambda_k b_k + c_k.$$

For example, let $\lambda = (4, 3, 3)$ and $\gamma = 2e_1 - 4e_3 + 6e_4 + e_6 + e_8 - 2e_9$ as in Example 7.2.2. Then

$\gamma \mapsto$

1	1	1	1
1	0	0	
0	0	−1	

Another example, let $\lambda = (7, 3)$ then

$6e_1 + 4e_2 + 10e_3 + 11e_4 + 8e_7 + 6e_8 + 5e_{10} \mapsto$

6	6	6	6	5	5	5
4	4	3				

To explicitly describe the symmetries of the fundamental J-alcove for type $\Phi = A_n$ (the group G_λ) we introduce the following element of W_0. This element will be essential throughout our analysis of $\Phi = A_n$ and so we prove some useful properties here that will be used in later chapters.

Definition 7.2.5. Let u_λ be the element of W_0 given in one line notation by reading the columns of $\mathbf{t}_r(\lambda)$, top to bottom and left to right.

Recall the definition of the λ-expression of $w \in W_0$ and its corresponding partition $\boldsymbol{\mu}(w, \lambda)$ (Definition 7.1.6).

Lemma 7.2.6. *We have $\boldsymbol{\mu}(\mathsf{u}_\lambda, \lambda) = \lambda'$ and $A_\lambda(\mathsf{u}_\lambda) = J_{\lambda'}$. Moreover $\mathsf{u}_\lambda w \in {}^\lambda W$ for all $w \in W_{\lambda'}$.*

Proof. The blocks of the λ-expression of u_λ are the columns of $\mathbf{t}_r(\lambda)$ by the construction of u_λ. The first statement then follows by Lemma 7.1.8. Multiplying u_λ by $\mathsf{w}_{\lambda'}$ on the right permutes the elements within the blocks of the λ-expression of u_λ. Similarly, any element $w \in W_{\lambda'}$ will only shuffle elements within the blocks. Thus, by Lemma 7.1.4 and as the blocks cannot contain elements from the same row of $\mathbf{t}_r(\lambda)$, $\mathsf{u}_\lambda w \in {}^\lambda W$. □

Recall the Dominance Lemma below. We include the proof for completeness.

Lemma 7.2.7. *[43, Lemma 2.2.4] Let $\mathbf{t}$ be a tableaux of size λ and $\mathbf{s}$ be a tableaux of shape μ. If for all $1 \le i \le r(\mu)$, the elements of row i of $\mathbf{s}$ are all in different columns of $\mathbf{t}$ then $\lambda \ge \mu$.*

Proof. We can sort the entries of the columns of $\mathbf{t}$ so that the elements of rows 1 to i of $\mathbf{s}$ appear in rows 1 to i of $\mathbf{t}$. Thus,

$$\begin{aligned}\lambda_1 + \lambda_2 + \cdots + \lambda_i &= \text{ the number of elements in the first } i \text{ rows of } \mathbf{t} \\ &\ge \text{ the number of elements from the first } i \text{ rows of } \mathbf{s} \text{ in the first } i \text{ rows of } \mathbf{t} \\ &= \mu_1 + \cdots + \mu_i\end{aligned}$$

for all $1 \le i \le r(\mu)$, as required. □

Lemma 7.2.8. *For all $w \in W_0$ we have $\boldsymbol{\mu}(w, \lambda) \leq \lambda'$.*

Proof. Let $\mu = \boldsymbol{\mu}(w, \lambda)$. Let $\gamma_1, \ldots, \gamma_k$ be the blocks of the λ-expression of w arranged so that they are decreasing in length. Thus, $\mu = (|\gamma_1|, \ldots, |\gamma_k|)$. Let $\mathbf{t}$ be a (not necessarily column strict) tableau of shape μ given by entering the elements of γ_i into the i-th row. The elements of row i of $\mathbf{t}$ lie in different columns of $\mathbf{t}_c(\lambda')$ by the construction of the λ-expression ($\mathbf{t}_c(\lambda')$ is the transpose of $\mathbf{t}_r(\lambda)$). The result then follows from the Dominance Lemma, Lemma 7.2.7. □

Theorem 7.2.9. *Let $u \in {}^{\lambda}W$. We have*

(1) *$\ell(\mathsf{w}_{A_\lambda(u)}) \leq \ell(\mathsf{w}_{\lambda'})$ with equality if and only if $\boldsymbol{\mu}(u, \lambda) = \lambda'$, and*
(2) *$A_\lambda(u) = J_{\lambda'}$ if and only if $u = \mathsf{u}_\lambda$.*

Proof. Let $u \in {}^{\lambda}W$ and write $\mu = \boldsymbol{\mu}(u, \lambda) = (\mu_1, \ldots, \mu_m)$. By Lemma 7.1.8, the Young subgroup generated by $A_\lambda(u)$ is $\mathfrak{S}_{\mu_1} \times \cdots \times \mathfrak{S}_{\mu_m}$. Thus, $\ell(\mathsf{w}_{A_\lambda(u)}) = \ell(\mathsf{w}_\mu) = \mathbf{a}_{\mu'}$ (using Lemma 7.1.11). By Lemma 7.2.8 we have that $\mu \leq \lambda'$ and so $\mu' \geq \lambda$. Lemma 7.1.12 then gives that $\ell(\mathsf{w}_{A_\lambda(u)}) = \mathbf{a}_{\mu'} \leq \mathbf{a}_\lambda$ with equality if and only if $\mu = \lambda'$.

For the second statement, suppose that $u \in {}^{\lambda}W$ with $A_\lambda(u) = J_{\lambda'}$. Therefore, $\boldsymbol{\mu}(u, \lambda) = \lambda'$. If $u \neq \mathsf{u}_\lambda$ then the λ-expression of u is either a rearrangement of the blocks of the λ-expression of u_λ or there is a block of the λ-expression of u that is not a column of $\mathbf{t}_r(\lambda)$. In both cases the rows of $\mathbf{t}_r(\lambda)$ will not appear in ascending order in the one line notation of u. By Lemma 7.1.4 this implies that $u \notin {}^{\lambda}W$, a contradiction. The result follows as $A_\lambda(\mathsf{u}_\lambda) = J_{\lambda'}$ by Lemma 7.2.6. □

We now explicitly describe the finite symmetry group of $\mathcal{A}_\lambda$, denoted G_λ. Recall from Section 2.4 that

$$G_\lambda = \{g \in W_0 \mid g\mathcal{A}_\lambda = \mathcal{A}_\lambda\} = \{g \in W_0 \mid g\Phi_\lambda^+ = \Phi_\lambda^+\}.$$

Note that, as $\Phi = A_n$ is simply-laced, we have $g\rho_\lambda = \rho_\lambda$ for all $g \in G_\lambda$ where $\rho_\lambda = \rho_{J_\lambda}$.

Define

$$\mathbf{L}(\lambda) = \{\lambda_i \mid 1 \leq i \leq r(\lambda)\}$$

to be the set of lengths of the rows of λ. Then, for $l \in \mathbf{L}(\lambda)$ define

$$\kappa_\lambda(l) = \#\{1 \leq i \leq r(\lambda) \mid \lambda_i = l\}$$

to be the number of rows of length l. For example, if $\lambda = (4, 3, 3, 2, 1, 1)$ then $\mathbf{L}(\lambda) = \{1, 2, 3, 4\}$ and $\kappa_\lambda(4) = 1$, $\kappa_\lambda(3) = 2$, $\kappa_\lambda(2) = 1$ and $\kappa_\lambda(1) = 2$.

Proposition 7.2.10. *The group G_λ is the subgroup of $\mathfrak{S}_{n+1}$ stabilising each column of $\mathbf{t}_r(\lambda)$ and permuting the set of rows of $\mathbf{t}_r(\lambda)$. Thus G_λ is a Coxeter group of type $\prod_{l \in \mathbf{L}(\lambda)} A_{\kappa_\lambda(l)-1}$.*

Proof. By Proposition 2.4.3, $g \in G_\lambda$ maps connected components of J_λ to connected components of J_λ. As the connected components of J_λ correspond to rows of $\mathbf{t}_r(\lambda)$, g maps rows of $\mathbf{t}_r(\lambda)$ to rows of $\mathbf{t}_r(\lambda)$. As g maps Φ_λ^+ to Φ_λ^+ and as elements of Φ_λ^+ are of the form $e_{i,j}$ where $i < j$ and i and j are in the same row of $\mathbf{t}_r(\lambda)$, g preserves the columns of $\mathbf{t}_r(\lambda)$. Conversely, any $g \in W_0 = \mathfrak{S}_{n+1}$ that stabilises the columns and acts on the set of rows of $\mathbf{t}_r(\lambda)$ satisfies $g\Phi_\lambda^+ = \Phi_\lambda^+$ and so $g \in G_\lambda$. As g can only permute connected components of J_λ of the same size, we have that $G_\lambda \cong \prod_{l \in \mathbf{L}(\lambda)} \mathfrak{S}_{\kappa_\lambda(l)}$. □

Example 7.2.11. Let $\lambda = (5, 5, 3, 3, 3, 2, 1, 1, 1)$, and so

$$\mathbf{t}_r(\lambda) = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 7 & 8 & 9 & 10 \\ \hline 11 & 12 & 13 \\ \cline{1-3} 14 & 15 & 16 \\ \cline{1-3} 17 & 18 & 19 \\ \cline{1-3} 20 & 21 \\ \cline{1-2} 22 \\ \cline{1-1} 23 \\ \cline{1-1} 24 \\ \cline{1-1} \end{array}$$

Then the group $G_\lambda \cong \mathfrak{S}_2 \times \mathfrak{S}_3 \times \mathfrak{S}_3$ is generated by the involutions $(1, 6)(2, 7)(3, 8)(4, 9)(5, 10)$, $(11, 14)(12, 15)(13, 16)$, $(14, 17)(15, 18)(16, 19)$, $(22, 23)$, and $(23, 24)$.

Proposition 7.2.12. *We have* $G_\lambda = \{g \in \mathsf{u}_\lambda W_{\lambda'} \mathsf{u}_\lambda^{-1} \mid g\Phi_\lambda = \Phi_\lambda\}$. *In particular,* $\mathsf{u}_\lambda^{-1} G_\lambda \mathsf{u}_\lambda \leq W_{\lambda'}$.

Proof. By definition, the one line expression of u_λ is the columns of $\mathbf{t}_r(\lambda)$. It follows from this construction that $\mathsf{u}_\lambda^{-1} = \mathsf{u}_{\lambda'}$ (the element with one line expression formed from reading the rows of $\mathbf{t}_c(\lambda)$ from left to right and top to bottom or equivalently reading the columns of $\mathbf{t}_r(\lambda')$). It follows that $\mathsf{u}_\lambda^{-1} G_\lambda \mathsf{u}_\lambda$ is the subgroup of $\mathfrak{S}_{n+1}$ that stabilises each row and acts on the set of columns of $\mathbf{t}_r(\lambda')$. Hence, $\mathsf{u}_\lambda^{-1} G_\lambda \mathsf{u}_\lambda \leq W_{\lambda'}$ and so if $g \in G_\lambda$ then $g \in \mathsf{u}_\lambda W_{\lambda'} \mathsf{u}_\lambda^{-1}$ and $g\Phi_\lambda = \Phi_\lambda$ (as $g\Phi_\lambda^+ = \Phi_\lambda^+$).

On the other hand, suppose that $g = \mathsf{u}_\lambda w \mathsf{u}_\lambda^{-1}$ with $w \in W_{\lambda'}$ satisfying $g\Phi_\lambda = \Phi_\lambda$. If there is an $\alpha \in \Phi_\lambda^+$ such that $g\alpha = -\beta \in -\Phi_\lambda^+$ then $w\mathsf{u}_\lambda^{-1}\alpha = -\mathsf{u}_\lambda^{-1}\beta$. As $\mathsf{u}_\lambda w^{-1} \in {}^\lambda W$ by Lemma 7.2.6, we have that $w\mathsf{u}_\lambda^{-1}\alpha > 0$ which forces $\mathsf{u}_\lambda^{-1}\beta < 0$. However, as $\beta \in \Phi_\lambda^+$ and $\mathsf{u}_\lambda^{-1} \in {}^\lambda W$ (as $A_\lambda(\mathsf{u}_\lambda) = J_\lambda$) we have $\mathsf{u}_\lambda^{-1}\beta > 0$, a contradiction. □

Remark 7.2.13. It follows from the definition of u_λ that the group $\mathsf{u}_\lambda W_{\lambda'} \mathsf{u}_\lambda^{-1}$ is the subgroup of the symmetric group stabilising each column of $\mathbf{t}_r(\lambda)$.

Definition 7.2.14. The G_λ*-root system* is the subset $\Phi_{G_\lambda} = \Phi_{G_\lambda}^+ \cup (-\Phi_{G_\lambda}^+)$ of P/Q_λ with

$$\Phi_{G_\lambda}^+ = \{\tilde{e}_i - \tilde{e}_j \mid 1 \leq i < j \leq r(\lambda) \text{ with } \lambda_i = \lambda_j\}.$$

Note that Φ_{G_λ} is not a true root system as the group P/Q_λ can have torsion (see Example 7.2.1), however it will play an analogous role to a root system in setting up a dominance order on P/Q_λ.

As G_λ preserves Q_λ the equation $g(\gamma + Q_\lambda) = g\gamma + Q_\lambda$ defines an action on P/Q_λ. Explicitly, in terms of the e-basis, this action permutes the elements $\tilde{e}_1, \ldots, \tilde{e}_{r(\lambda)}$ with the constraint that if $g\tilde{e}_i = \tilde{e}_j$ then $\lambda_i = \lambda_j$ (as elements of G_λ permute rows of the same length). Set a fundamental domain for this action by

$$(P/Q_\lambda)_+ = \{a_1\tilde{e}_1 + \cdots + a_{r(\lambda)}\tilde{e}_{r(\lambda)} \mid a_i \geq a_j \text{ if } i < j \text{ with } \lambda_i = \lambda_j\}. \tag{7.2.4}$$

If $J_\lambda = \emptyset$ (and thus $\lambda = (1^{n+1})$), then $P/Q_\lambda = P$ and $P_+ = P^+$. The set $(P/Q_\lambda)_+$ gives a J_λ-analogue to dominant weights and are thus called *dominant* λ*-weights*. By definition G_λ also acts on $P^{(\lambda)}$. Let $P_+^{(\lambda)}$ be the fundamental domain of this action given by the image of $(P/Q_\lambda)_+$ under the isomorphism $P/Q_\lambda \to P^{(\lambda)}$ from Proposition 7.2.3. That is

$$P_+^{(\lambda)} = \{\gamma \in P^{(\lambda)} \mid \gamma + Q_\lambda \in (P/Q_\lambda)_+\}.$$

Example 7.2.15. Let $\lambda = (2,2)$ as in Example 7.2.1. The group G_λ is of type A_1 and generated by $(1,3)(2,4) = s_2s_1s_3s_2$. We have that $(P/Q_\lambda)_+ = \{a_1\tilde{e}_1 + a_2\tilde{e}_2 \mid a_1 \geq a_2\}$ and

$$P_+^{(\lambda)} = (\mathbb{Z}_{\geq 0}\omega_2) \cup (\omega_1 + \mathbb{Z}_{\geq 0}\omega_2) \cup (\omega_3 + \mathbb{Z}_{\geq 0}\omega_2) \cup (\omega_1 + \omega_3 + \mathbb{Z}_{\geq -1}\omega_2).$$

To see how to get $P_+^{(\lambda)}$ from $(P/Q_\lambda)_+$ consider the tableaux form of elements of $P_+^{(\lambda)}$ from Remark 7.2.4. There are four options for the tableaux corresponding to elements of $P_+^{(\lambda)}$:

b_1	b_1
b_2	b_2

b_1+1	b_1
b_2	b_2

b_1	b_1
b_2+1	b_2

b_1+1	b_1
b_2+1	b_2

with $b_1 \geq b_2$ (this forces $b_1 \geq 0$ as if b_1 is negative so is b_2 and we can add $-b_1$ to all entries, recalling that tableaux are equal if they differ by a constant tableau). This along with $\omega_4 = 0$ and the fact that

$$\omega_1 \leftrightarrow \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & 0 \\ \hline\end{array} \quad \text{and} \quad \omega_3 \leftrightarrow \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & 0 \\ \hline\end{array}$$

gives the result.

Example 7.2.16. Let $\lambda = (5,5,3,3,3,2,1,1,1)$ as in Example 7.2.11. Then

$$\Phi^+_{G_\lambda} = \{\tilde{e}_1 - \tilde{e}_2, \tilde{e}_3 - \tilde{e}_4, \tilde{e}_3 - \tilde{e}_5, \tilde{e}_4 - \tilde{e}_5, \tilde{e}_7 - \tilde{e}_8, \tilde{e}_7 - \tilde{e}_9, \tilde{e}_8 - \tilde{e}_9\},$$
$$(P/Q_\lambda)_+ = \{a_1\tilde{e}_1 + \cdots + a_9\tilde{e}_9 \mid a_1 \geq a_2, a_2 \geq a_4 \geq a_5 \text{ and } a_7 \geq a_8 \geq a_9\}.$$

Denote

$$Q^\lambda = \operatorname{span}_{\mathbb{Z}}(\Phi_{G_\lambda}) \quad \text{and} \quad Q^\lambda_+ = \operatorname{span}_{\mathbb{Z}_{\geq 0}}(\Phi^+_{G_\lambda}).$$

Define the λ-*dominance order* $\preccurlyeq_\lambda$ on P/Q_λ by

$$\gamma + Q_\lambda \preccurlyeq_\lambda \gamma' + Q_\lambda \text{ if and only if } \gamma' - \gamma + Q_\lambda \in Q^\lambda_+.$$

Via the isomorphism in Proposition 7.2.3, the λ-dominance order can also be considered a partial ordering on the elements of $P^{(\lambda)}$. The following lemmas give explicit conditions to be elements of Q_λ and Q^λ_+, aiding in understanding the partial order $\preccurlyeq_\lambda$. These lemmas give a J_λ-analogue to the e-basis interpretation of Q and Q^+ from Section 7.1.

Lemma 7.2.17. *Let $\gamma = \sum_{i=1}^{n+1} a_ie_i \in Q$, with the expression chosen so that $a_1 + \cdots + a_{n+1} = 0$. Then $\gamma \in Q_\lambda$ if and only if $\sum_{i=\lambda(k-1)+1}^{\lambda(k)} a_i = 0$ for each $1 \leq k \leq r(\lambda)$.*

Proof. This follows directly from the interpretation of Q in the e-basis from Section 7.1, now splitting into the different connected components of J_λ. The connected components of J_λ correspond to rows of $\mathbf{t}_r(\lambda)$. Each connected component has a corresponding root system, for example row k's root system is generated by $e_{i,i+1}$ for all $\lambda(k-1)+1 \leq i < \lambda(k)$. The weight γ is a root from the root system of row k if and only if $\gamma = \sum_{i=\lambda(k-1)+1}^{\lambda(k)} a_ie_i$ with $\sum_{i=\lambda(k-1)+1}^{\lambda(k)} a_i = 0$. The result follows as elements of Q_λ are combinations of roots from the root systems of each row. □

Lemma 7.2.18. *We have $\gamma + Q_\lambda \in Q^\lambda$ if and only if there is an expression $\gamma + Q_\lambda = \sum_{i=1}^{r(\lambda)} a_i \tilde{e}_i$ with*

$$\sum_{1 \le i \le r(\lambda),\, \lambda_i = l} a_i = 0 \quad \text{for all } l \in \mathbf{L}(\lambda),$$

and moreover $\gamma + Q^\lambda \in Q^\lambda_+$ if and only if $a_1 + \cdots + a_i \ge 0$ for all $1 \le i \le r(\lambda)$.

Proof. As with the proof of Lemma 7.2.17, this follows directly from the interpretation of Q and Q^+ in the e-basis. The set Q^λ is generated by $\tilde{e}_i - \tilde{e}_j$ for $1 \le i < j \le r(\lambda)$. Each row length $l \in \mathbf{L}(\lambda)$ has a corresponding 'root system'. Consider rows of length l, let $k = \min\{1 \le i \le r(\lambda) \mid \lambda_i = l\}$ and $m = \max\{1 \le i \le r(\lambda) \mid \lambda_i = l\}$. The 'root system' corresponding to length l is generated by $\tilde{e}_i - \tilde{e}_{i+1}$ for all $k \le i < m$. The element $\gamma + Q_\lambda = \sum_k^m a_i \tilde{e}_i$ is a linear combination of the elements of $\{\tilde{e}_i - \tilde{e}_{i+1} \mid k \le i < m\}$ (a member of this 'root system') if and only if

$$\sum_{1 \le i \le r(\lambda),\, \lambda_i = l} a_i = 0.$$

and is a positive linear combination of the elements of $\{\tilde{e}_i - \tilde{e}_{i+1} \mid k \le i < m\}$ if and only if $a_k + \cdots + a_i \ge 0$ for all $k \le i \le m$. The result follows as elements of Q^λ are combinations of members of the 'root systems' corresponding to each $l \in \mathbf{L}(\lambda)$. □

Note that, by the conditions for membership of Q^λ given in Lemma 7.2.18, if $\gamma + Q_\lambda \in Q^\lambda$ then $\gamma \in Q$. If $\lambda = (1^{n+1})$ then $Q_\lambda = \{0\}$ and $P/Q_\lambda = P$, so in this case $\preccurlyeq_\lambda$ is the usual dominance order on P given by $\gamma \preccurlyeq_\lambda \gamma'$ if and only if $\gamma' - \gamma \in Q^+$ (distinct from the dominance order on partitions defined in Section 7.1). Hence, λ-dominance on P/Q_λ is a J_λ-analogue of the usual dominance order on P. If $\lambda = (n+1)$ then $Q_\lambda = Q$, $P/Q_\lambda = \{0, \tilde{e}_1, 2\tilde{e}_1, \ldots, n\tilde{e}_1\}$ and $Q^\lambda = \{0\}$, so if $\gamma, \gamma' \in P/Q_\lambda$ we have $\gamma \preccurlyeq_\lambda \gamma'$ if and only if $\gamma = \gamma'$.

Example 7.2.19. Let $\gamma = (5,5,3,3,3,2,1,1,1)$ as in Example 7.2.11. Then Q^λ_+ consists of the elements $a_1\tilde{e}_1 + \cdots + a_9\tilde{e}_9$ with $a_i \in \mathbb{Z}$ satisfying $a_1 + a_2 = a_3 + a_4 + a_5 = a_6 = a_7 + a_8 + a_9 = 0$, $a_1 \ge 0$, $a_3 \ge 0$, $a_3 + a_4 \ge 0$, $a_7 \ge 0$ and $a_7 + a_8 \ge 0$.

7.3 The affine Hecke algebra of A_n, the modules M_λ and the ring $\mathbb{Z}[\zeta_\lambda]^{G_\lambda}$

We now explicitly describe the definitions set up in Chapter 3 and Chapter 5 for type $\Phi = A_n$. That is, the Hecke algebra relations, the ring $R[\zeta_{J_\lambda}]$ and the representations $(\pi_{J_\lambda,\mathsf{v}}, M_{J_\lambda,\mathsf{v}})$. As with Sections 7.1 and 7.2 we shift to the e-basis for these descriptions. We also define a λ-anaolgue to Schur functions that will be used in Chapter 11.

Let $\mathcal{H}$ denote the Hecke algebra associated with $\Phi = A_n$ and $\widetilde{\mathcal{H}}$ denote the corresponding extended Hecke algebra (see Section 3.1 for the definitions). Note that as Φ is simply-laced, $\mathsf{q}_{s_i} = \mathsf{q}_{s_j}$ for all $i, j \in I \cup \{0\}$. Hence, set $\mathsf{q} = \mathsf{q}_{s_i}$ for all $i \in I \cup \{0\}$ and the relations of $\widetilde{\mathcal{H}}$ from (3.1.1) become

$$\begin{aligned} T_w T_u &= T_{wu} && \text{if } \ell(wu) = \ell(w) + \ell(u) \\ T_w T_{s_i} &= T_{ws_i} + (\mathsf{q} - \mathsf{q}^{-1}) T_w && \text{if } \ell(ws_i) = \ell(w) - 1. \end{aligned}$$

Recall the definition of X_w for $w \in \widetilde{W}$ from Section 4.1 (see particularly (4.1.1)). Let $X_i = X^{e_i}$ for $1 \le i \le n+1$ (recall that $X^\gamma = X_{t_\gamma}$ for all $\gamma \in P$). As $e_1 + \cdots + e_{n+1} = 0$ we have that $X_1 \cdots X_{n+1} = 1$.

The Bernstein-Lusztig relation from (4.1.3) gives the following relations;

$$\begin{cases} T_jX_i = X_iT_j & \text{when } |i-j|>1, \\ T_i^{-1}X_iT_i^{-1} = X_{i+1} & \text{for } 1 \leq i \leq n. \end{cases}$$

In type A_n we have that $\mathsf{R} = \mathbb{Z}[q^{\pm 1}]$. Denote

$$\mathsf{R}[X] = \mathrm{span}_{\mathsf{R}}\{X^\gamma \mid \gamma \in P\}.$$

The group W_0 acts upon $\mathsf{R}[X]$ by linearly extending $w \cdot X^\gamma = X^{w\gamma}$. Let

$$\mathsf{R}[X]^{W_0} = \{f(X) \in \mathsf{R}[X] \mid w \cdot f(X) = f(X) \text{ for all } w \in W_0\}.$$

This is the ring of symmetric polynomials in $\mathsf{R}[X]$.

The Kazhdan-Lusztig cell decomposition of $\widetilde{\mathcal{H}}$ has been determined by Lusztig [30] and Shi [45]. The two-sided cells are indexed by partitions $\lambda \vdash n+1$. The following cells will be important for our analysis: for $\lambda \vdash n+1$ let

$$\Gamma_\lambda = \{w \in \widetilde{W} \mid w \sim_R \mathsf{w}_{\lambda'}\} \quad \text{and} \quad \Delta_\lambda = \{w \in \widetilde{W} \mid w \sim_{LR} \mathsf{w}_{\lambda'}\},$$

be the right cell containing $\mathsf{w}_{\lambda'}$ and the two sided cell containing $\mathsf{w}_{\lambda'}$ respectively. It is well known that for all $w \in \Delta_\lambda$ we have $\mathbf{a}(w) = \ell(\mathsf{w}_{\lambda'})$ (where $\mathbf{a}$ is Lusztig $\mathbf{a}$-function). Recall that the partial order $\leq_{LR}$ on $\widetilde{W}$ induces a partial order on the set of two sided cells from Section 3.2. For type A_n we have that $\Delta_\lambda \leq_{LR} \Delta_\mu$ if and only if $\lambda \leq \mu$ in the dominance order on partitions.

We will require the following result of Tanisaki and Xi.

Theorem 7.3.1 ([48, Theorem 4.3])**.** *Let $\lambda \vdash n+1$. The two-sided ideal*

$$\mathrm{span}_{\mathsf{R}}\{C_w \mid w \leq_{LR} \mathsf{w}_{\lambda'}\}/\mathrm{span}_{\mathsf{R}}\{C_w \mid w <_{LR} \mathsf{w}_{\lambda'}\}$$

is generated by the image of $C_{\mathsf{w}_{\lambda'}}$ in the quotient.

Lusztig proved that $P1-P15$ (stated in Section 3.2) are true for $\Phi = A_n$ in [35, §15] using the positivity properties of Elias and Williamson (see [14]). Thus we can define Lusztig's asymptotic algebra, $\mathcal{J}$, with generators $(\mathsf{t}_w)_{w\in\widetilde{W}}$ (see Section 3.2). Let $\mathcal{J}_\lambda$ denote the subring of $\mathcal{J}$ spanned by $\{\mathsf{t}_w \mid w \in \Delta_\lambda\}$.

We redefine $\mathsf{R}[\zeta_{J_\lambda}]$ from Section 5.1 so we are working in the e-basis of P instead of the ω-basis. Let $\zeta_1, \ldots, \zeta_{n+1}$ be commuting invertible indeterminants with the condition that $\zeta_1 \cdots \zeta_{n+1} = 1$. For $\gamma \in P$ denote $\zeta^\gamma = \zeta_1^{a_1} \cdots \zeta_{n+1}^{a_{n+1}}$ if $\gamma = \sum_{i=1}^{n+1} a_i e_i$. For $\lambda \vdash n+1$ let $\mathcal{I}_\lambda$ denote the ideal of the Laurent polynomial ring $\mathsf{R}[\zeta_1^{\pm1}, \ldots, \zeta_{n+1}^{\pm1}]$ generated by $\zeta^{\alpha_j} - 1$ for $j \in J_\lambda$. Let

$$\mathsf{R}[\zeta_\lambda] = \mathsf{R}[\zeta_1^{\pm1}, \ldots, \zeta_{n+1}^{\pm1}]/\mathcal{I}_\lambda \quad \text{and} \quad \mathbb{Z}[\zeta_\lambda] = \mathbb{Z}[\zeta_1^{\pm1}, \ldots, \zeta_{n+1}^{\pm1}]/\mathcal{I}_\lambda.$$

Remark 7.3.2. To shift between the e-basis setup above and the original setup of $\mathsf{R}[\zeta_J]$ from Section 5.1 (in the ω-basis) let $\zeta_i' = \zeta^{\omega_i}$. We have that $\zeta_1 = \zeta_1'$, $\zeta_{n+1} = \zeta_n'^{-1}$ and $\zeta_i = \zeta_i'\zeta_{i-1}'^{-1}$ for $1 < i < n+1$.

Write $\zeta_\lambda^\gamma = \zeta^\gamma + \mathcal{I}_\lambda$ and so $\zeta_\lambda^\gamma = 1$ for all $\gamma \in Q_\lambda$. As in Section 5.1 this implies that $\zeta_\lambda^\gamma = \zeta_\lambda^{\gamma^{(\lambda)}}$ for all $\gamma \in P$. We define $\zeta_\lambda^{\gamma+Q_\lambda} = \zeta_\lambda^\gamma$ for $\gamma + Q_\lambda \in P/Q_\lambda$ so that we can work in the quotient, noting that $\mathsf{R}[\zeta_\lambda] \cong \mathsf{R}[P/Q_\lambda]$.

Explicitly, since $\zeta_\lambda^\alpha = 1$ for all $\alpha \in \Phi_\lambda$ we have that $\zeta_j \zeta_{j'}^{-1} = 1$ for j and j' in the same row of $\mathbf{t}_r(\lambda)$. Thus, $\zeta_j + \mathcal{I}_\lambda = \zeta_{j'} + \mathcal{I}_\lambda$ for j, j' in the same row of $\mathbf{t}_r(\lambda)$. Therefore, we can define $z_1, \dots z_{r(\lambda)} \in \mathsf{R}[\zeta_\lambda]$ by

$$z_i = \zeta_j + \mathcal{I}_\lambda$$

for any j in the i-th row of $\mathbf{t}_r(\lambda)$ (that is, $z_i = \zeta_\lambda^{e_j}$ for all j in row i). As $e_1 + \cdots + e_{n+1} = 0$ we have that $z_1^{\lambda_1} \cdots z_{r(\lambda)}^{\lambda_{r(\lambda)}} = 1$ and

$$\mathsf{R}[\zeta_\lambda] = \mathsf{R}[z_1^{\pm 1}, \dots, z_{r(\lambda)}^{\pm 1}]/(z_1^{\lambda_1} \cdots z_{r(\lambda)}^{\lambda_{r(\lambda)}} - 1).$$

The group G_λ acts upon $\mathsf{R}[\zeta_\lambda]$ by $g(\zeta_\lambda^\gamma) = \zeta_\lambda^{g\gamma}$ for all $g \in G_\lambda$ and $\gamma \in P$. This action is well defined as if $\zeta_\lambda^\gamma = \zeta_\lambda^{\gamma'} \in \mathsf{R}[\zeta_\lambda]$ then $\gamma - \gamma' \in Q_\lambda$ and by the definition of G_λ this implies that $g\gamma - g\gamma' \in Q_\lambda$, thus $\zeta_\lambda^{g\gamma} = \zeta_\lambda^{g\gamma'}$. Explicitly, this action is given by permuting the variables $z_1, \dots, z_{r(\lambda)}$ under the condition that if $gz_i = z_j$ then $\lambda_i = \lambda_j$, mirroring the action of G_λ on the elements $\tilde{e}_i$ from Section 7.2.

Example 7.3.3. Let $\lambda = (5, 5, 3, 3, 3, 2, 1, 1, 1)$ as in Example 7.2.11. Then G_λ permutes the variables $z_1, \dots, z_9$ preserving the partition $\{z_1, z_2\} \cup \{z_3, z_4, z_5\} \cup \{z_6\} \cup \{z_7, z_8, z_9\}$.

Definition 7.3.4. Define the following two sets;

$$\mathsf{R}[\zeta_\lambda]^{G_\lambda} = \{p(\zeta_\lambda) \in \mathsf{R}[\zeta_\lambda] \mid g \cdot p(\zeta_\lambda) = p(\zeta_\lambda) \text{ for all } g \in G_\lambda\},$$
$$\mathbb{Z}[\zeta_\lambda]^{G_\lambda} = \{p(\zeta_\lambda) \in \mathbb{Z}[\zeta_\lambda] \mid g \cdot p(\zeta_\lambda) = p(\zeta_\lambda) \text{ for all } g \in G_\lambda\}.$$

Recall that $P_+^{(\lambda)}$ is a fundamental domain for the action of G_λ on $P^{(\lambda)}$ from Section 7.2 and so every element of $P^{(\lambda)}$ is of the form $g\gamma$ for some $g \in G_\lambda$ and $\gamma \in P_+^{(\lambda)}$. Thus, $\mathsf{R}[\zeta_\lambda]^{G_\lambda}$ has basis, as a free R-module, given by the monomials

$$\mathfrak{e}_\gamma(\zeta_\lambda) = \sum_{\gamma' \in G_\lambda \cdot \gamma} \zeta_\lambda^{\gamma'}, \quad \text{with } \gamma \in P_+^{(\lambda)}. \tag{7.3.1}$$

Similarly, $\mathbb{Z}[\zeta_\lambda]^{G_\lambda}$ has basis $\{\mathfrak{e}_\gamma(\zeta_\lambda) \mid \gamma \in P_+^{(\lambda)}\}$ as a free $\mathbb{Z}$-module.

Definition 7.3.5. For $\gamma \in P^{(\lambda)}$ (or $\gamma \in P/Q_\lambda$) let $\mathfrak{s}_\gamma(\zeta_\lambda)$ be the G_λ-*Schur function*

$$\mathfrak{s}_\gamma(\zeta_\lambda) = \sum_{g \in G_\lambda} \zeta_\lambda^{g\gamma} \prod_{\alpha \in \Phi_{G_\lambda}^+} \frac{1}{1 - \zeta_\lambda^{-g\alpha}}.$$

Proposition 7.3.6. *The elements $\mathfrak{s}_\gamma(\zeta_\lambda)$ are in $\mathbb{Z}[\zeta_\lambda]^{G_\lambda}$, and $\{\mathfrak{s}_\gamma(\zeta_\lambda) \mid \gamma \in P_+^{(\lambda)}\}$ is a basis of $\mathsf{R}[\zeta_\lambda]^{G_\lambda}$ (respectively $\mathbb{Z}[\zeta_\lambda]^{G_\lambda}$) as a free R-module (respectively free $\mathbb{Z}$-module).*

Proof. This follows from [38, (2.10) and Proposition 2.10], taking the theory for P^+ and W_0 and applying it to $P_+^{(\lambda)}$ and G_λ. □

For $\lambda = (n+1)$ we have $P^{(\lambda)} = \{0, \omega_1, \dots, \omega_n\}$, $\mathcal{A}_\lambda = A_0$ and $G_\lambda = \{0\}$. Thus, $\Phi_{G_\lambda} = \emptyset$ and so $\mathfrak{s}_{\omega_i}(\zeta_\lambda) = \zeta_\lambda^{\omega_i} = z_1^i$ where $z_1 = \zeta_\lambda^{e_1}$. For $\lambda = (1^{n+1})$ we have that $P^{(\lambda)} = P$ and $G_\lambda = W_0$. Hence, $\Phi_{G_\lambda} = \Phi$ and $s_\gamma(\zeta_\lambda)$ for $\gamma \in P^+$ is the classical Schur function given in [38, (2.10)].

Example 7.3.7. Let $\lambda = (2,2) \vdash 4$ as in Example 7.2.1 and Example 7.2.15. We have that $G_\lambda = \{e, s_2s_1s_3s_2\}$ and $\Phi^+_{G_\lambda} = \{\tilde{e}_1 - \tilde{e}_2\}$. The action of G_λ on P/Q_λ is given by $s_2s_1s_3s_2\tilde{e}_1 = \tilde{e}_2$. Let $\gamma = a\tilde{e}_1 + b\tilde{e}_2 \in P/Q_\lambda$. Then

$$\begin{aligned}\mathfrak{s}_\gamma(\zeta_\lambda) &= \zeta_\lambda^\gamma \frac{1}{1-\zeta_\lambda^{-\tilde{e}_1+\tilde{e}_2}} + \zeta_\lambda^{g\gamma}\frac{1}{1-\zeta_\lambda^{\tilde{e}_1-\tilde{e}_2}} \\ &= \frac{z_1^a z_2^b}{1-z_1^{-1}z_2} + \frac{z_1^b z_2^a}{1-z_1 z_2^{-1}} \\ &= \frac{z_1^{a+1}z_2^b - z_1^b z_2^{a+1}}{z_1 - z_2}\end{aligned}$$

as $\zeta_\lambda^{\tilde{e}_1} = z_1$ and $\zeta_\lambda^{\tilde{e}_2} = z_2$. In particular, as $z_1^2z_2^2 = 1$, we have that $\mathfrak{s}_{\tilde{e}_1+\tilde{e}_2}(\zeta_\lambda) = z_1z_2$ and so $\mathfrak{s}_{\tilde{e}_1+\tilde{e}_2}(\zeta_\lambda)^2 = 1$.

Let $(\pi_{J_\lambda,\mathsf{v}}, M_{J_\lambda,\mathsf{v}}, \mathsf{B}_{^J W})$ be the representation of $\widetilde{\mathcal{H}}$ defined in Chapter 5 with $\mathsf{B}_{^J W} = \{\mathbf{m}_u \mid u \in {}^\lambda W\}$. As Φ is simply-laced, $\mathsf{v}_\alpha = \mathsf{v}_\beta$ for all $\alpha, \beta \in \Phi_\lambda$. Furthermore, by Proposition 6.1.5 and Theorem 6.1.4 we have that $(\pi_{J_\lambda,\mathsf{v}}, M_{J_\lambda,\mathsf{v}})$ is bounded if and only if $\mathsf{v}_\alpha = -\mathsf{q}^{-1}$ for all $\alpha \in \Phi_\lambda$. We notate $(\pi_{J_\lambda,\mathsf{v}}, M_{J_\lambda,\mathsf{v}}, \mathsf{B}_{^J W})$ as $(\pi_\lambda, M_\lambda, \mathsf{B}_\lambda)$, setting v to be the J_λ-parameter system with $\mathsf{v}_\alpha = -\mathsf{q}^{-1}$ for all $\alpha \in \Phi_\lambda$. Similarly, we notate the 1-dimensional representation, $\psi_{J_\lambda,\mathsf{v}}$, of the Levi-subalgebra corresponding to J_λ (see Section 5.2) by ψ_λ and the function $\varpi_{J,\mathsf{v}}$ (see (5.1.1)) by ϖ_λ. By Proposition 5.2.1 we have that

$$\psi_\lambda(T_j) = -\mathsf{q}^{-1} \quad \text{and} \quad \psi_\lambda(X^\gamma) = (-\mathsf{q})^{-\langle\gamma, 2\rho_\lambda\rangle}\zeta_\lambda^\gamma$$

for all $j \in J_\lambda$ and $\gamma \in P$. Furthermore, we have that $\psi_\lambda(X^{e_i}) = (-\mathsf{q})^{-\langle e_i, 2\rho_\lambda\rangle}\zeta_\lambda^{e_i}$ for all $1 \le i \le n+1$. Let r and c be such that $i = \lambda[r,c]$ (so $1 \le r \le r(\lambda)$ and $1 \le c \le \lambda_r$). By Lemma 7.1.1

$$\langle e_i, 2\rho_\lambda\rangle = \sum_{j=i+1}^{\lambda(r)} 1 + \sum_{j=\lambda(r-1)+1}^{i-1} 1 = \lambda_r - 2c + 1,$$

hence,

$$\psi_\lambda(X^{e_i}) = (-\mathsf{q})^{2c-\lambda_r-1} z_r$$

for all $1 \le i \le n+1$. This implies that

$$\psi_\lambda(X^{\alpha_i}) = \begin{cases} \mathsf{q}^{-2} & \text{if } j \in J_\lambda, \\ (-\mathsf{q})^{\lambda_k+\lambda_{k+1}-2} z_k z_{k+1}^{-1} & \text{if } i = \lambda(k). \end{cases}$$

Using Lemma 7.1.4, the number of elements in ${}^\lambda W$, and the dimension of M_λ, is

$$N_\lambda = \frac{(n+1)!}{\lambda_1!\lambda_2!\cdots\lambda_{r(\lambda)}!}.$$

It is well known that the number of right (or left) cells within Δ_λ is equal to N_λ.

The following proposition gives that the matrix entries of $\pi_\lambda(C_{\mathsf{w}_{\lambda'}})$ with respect to the basis B_λ are supported on the interval $[\mathsf{u}_\lambda, \mathsf{u}_\lambda\mathsf{w}_{\lambda'}]$ (note that by Lemma 7.2.6 we have that $[\mathsf{u}_\lambda, \mathsf{u}_\lambda\mathsf{w}_{\lambda'}] \subseteq {}^\lambda W$).

Proposition 7.3.8. *We have* $[\pi_\lambda(C_{\mathsf{w}_{\lambda'}})]_{u,v} = 0$ *unless* $u, v \in [\mathsf{u}_\lambda, \mathsf{u}_\lambda\mathsf{w}_{\lambda'}]$. *If* $x, y \in W_{\lambda'}$ *then*

$$[\pi_\lambda(C_{\mathsf{w}_{\lambda'}})]_{\mathsf{u}_\lambda x, \mathsf{u}_\lambda y} = \mathsf{q}^{\mathbf{a}_\lambda - \ell(x) - \ell(y)}.$$

Proof. Let $x \in W_{\lambda'}$. As $T_{x^{-1}}^{-1} C_{\mathsf{w}_{\lambda'}} = \mathsf{q}^{-\ell(x^{-1})} C_{\mathsf{w}_{\lambda'}} = \mathsf{q}^{-\ell(x)} C_{\mathsf{w}_{\lambda'}}$ (see Section 3.2) and by Proposition 5.1.3 we have

$$\mathbf{m}_{u_\lambda x} \cdot C_{\mathsf{w}_{\lambda'}} = \varpi_\lambda(X_{\mathsf{u}_\lambda}) \cdot T_{x^{-1}}^{-1} C_{\mathsf{w}_{\lambda'}} = \mathsf{q}^{-\ell(x)} \mathbf{m}_{\mathsf{u}_\lambda} \cdot C_{\mathsf{w}_{\lambda'}}.$$

By Proposition 3.2.2

$$\mathbf{m}_{\mathsf{u}_\lambda x} \cdot C_{\mathsf{w}_{\lambda'}} = \sum_{y \in W_{\lambda'}} \mathsf{q}^{\ell(\mathsf{w}_{\lambda'}) - \ell(x) - \ell(y)} \mathbf{m}_{\mathsf{u}_\lambda y}$$

and so the $\mathsf{u}_\lambda x$-th row of $\pi_\lambda(C_{\mathsf{w}_{\lambda'}})$ is as claimed (using Lemma 7.1.11).

It remains to prove that the other entries are zero, that is for $w \in {}^\lambda W$ such that $w \notin [\mathsf{u}_\lambda, \mathsf{u}_\lambda \mathsf{w}_\lambda]$ we have $\mathbf{m}_w \cdot C_{\mathsf{w}_{\lambda'}} = 0$. We write $w = w_1 w_2$ where w_1 is $J_{\lambda'}$-reduced on the right and $w_2 \in W_{\lambda'}$. Thus $J_{\lambda'} \subseteq A_R(w_1)$ but $A_R(w_1) \neq J_{\lambda'}$ as $w_1 \neq \mathsf{u}_\lambda$ (as $w \notin [\mathsf{u}_\lambda, \mathsf{u}_\lambda \mathsf{w}_\lambda]$). It follows that there exists $s' \in J_{\lambda'}$ such that $w_1 s' \notin {}^\lambda W$ and thus $w_1 s' = s w_1$ for some $s \in J_\lambda$ (see [1, p.79 (F)]).

Again by Proposition 3.2.2

$$\begin{aligned}
C_{\mathsf{w}_{\lambda'}} &= \sum_{y \in W_{\lambda'}} \mathsf{q}^{\ell(\mathsf{w}_{\lambda'}) - \ell(y)} X_y \\
&= \sum_{\substack{y \in W_{\lambda'} \\ ys' > y}} \mathsf{q}^{\ell(\mathsf{w}_{\lambda'}) - \ell(y)} X_y + \sum_{\substack{y \in W_{\lambda'} \\ ys' < y}} \mathsf{q}^{\ell(\mathsf{w}_{\lambda'}) - \ell(y)} X_y \\
&= \sum_{\substack{y \in W_{\lambda'} \\ ys' > y}} \mathsf{q}^{\ell(\mathsf{w}_{\lambda'}) - \ell(y)} X_y + \sum_{\substack{z \in W_{\lambda'} \\ zs' > z}} \mathsf{q}^{\ell(\mathsf{w}_{\lambda'}) - \ell(z) - \ell(s')} X_z T_{s'}^{-1} \\
&= (\mathsf{q}^{-1} T_{s'}^{-1} + 1) \sum_{\substack{y \in W_{\lambda'} \\ ys' > y}} \mathsf{q}^{\ell(\mathsf{w}_{\lambda'}) - \ell(y)} X_y.
\end{aligned}$$

Since $\mathbf{m}_w \cdot C_{\mathsf{w}_{\lambda'}} = \mathsf{q}^{-\ell(w_2)} \mathbf{m}_{w_1} \cdot C_{\mathsf{w}_{\lambda'}}$ and $X_{w_1}(\mathsf{q}^{-1} T_{s'}^{-1} + 1) = (\mathsf{q}^{-1} T_s^{-1} + 1) X_{w_1}$ it follows that

$$\begin{aligned}
\mathbf{m}_w \cdot C_{\mathsf{w}_{\lambda'}} &= \mathsf{q}^{-\ell(w_2)} \mathbf{m}_{w_1} \cdot (\mathsf{q}^{-1} T_{s'}^{-1} + 1) \sum_{\substack{y \in W_{\lambda'} \\ ys' > y}} \mathsf{q}^{\ell(\mathsf{w}_{\lambda'}) - \ell(y)} X_y \\
&= \mathsf{q}^{-\ell(w_2)} (\mathsf{q}^{-1} \psi_\lambda(T_s^{-1}) + 1) \varpi_\lambda \left(\sum_{\substack{y \in W_{\lambda'} \\ ys' > y}} \mathsf{q}^{\ell(\mathsf{w}_{\lambda'}) - \ell(y)} X_{w_1} X_y \right) = 0
\end{aligned}$$

with the final equality following from $\varpi_\lambda(T_s^{-1}) = -\mathsf{q}$. $\square$

Chapter 8

The killing property and boundedness

In Section 3.3 we introduced the notion of a balanced system of cell representations of $\widetilde{\mathcal{H}}$ as a family of matrix representations indexed by two-sided cells that satisfy 5 properties: $B1$ - $B5$. Recall the definition of $B1$, the killing property, and $B2$, boundedness, now restricting to the case $\Phi = A_n$:

$B1$ *The killing property*: If w is in a cell lower or incomparable to Δ_λ with respect to the left-right order then $\pi_\lambda(C_w) = 0$.

$B2$ *Boundedness*: The degree of q in the entries of the matrices $\pi_\lambda(T_w)$ for $w \in \widetilde{W}$ is bounded.

In this chapter we prove both properties and some consequences. In Section 8.1 we prove the killing property, Theorem 8.1.3. Boundedness is proved in Theorem 8.2.1 in Section 8.2, and the bound is shown to be $\ell(\mathsf{w}_{\lambda'})$. Furthermore, we prove that the elements whose matrices reach this bound are a subset of Δ_λ. In Chapter 10 we strengthen this result by showing that the elements that reach the bound are exactly Δ_λ, proving $B3$ for π_λ. In Section 8.3 we introduce the ring $\mathfrak{C}_\lambda$ of leading matrices (recalling the definition of leading matrices from Section 3.3) and define a map from Lusztig's asymptotic algebra to this ring which will be shown to be an isomorphism in Chapter 10.

8.1 Killing

In this section we prove the killing property for π_λ. We will need the following two lemmas.

Lemma 8.1.1. *Let $\lambda, \mu \vdash n+1$. If $\mu \not\geq \lambda$ then $u\Phi_{\mu'} \cap \{\alpha_j \mid j \in J_\lambda\} \neq \emptyset$ for all $u \in {}^\lambda W$.*

Proof. We will show the contrapositive: if $u \in {}^\lambda W$ with $u\Phi_{\mu'} \cap \{\alpha_j \mid j \in J_\lambda\} = \emptyset$ then $\mu \geq \lambda$.

Let $u \in {}^\lambda W$ such that $u\Phi_{\mu'} \cap \{\alpha_j \mid j \in J_\lambda\} = \emptyset$ and write $u = u_1 u_2$ where u_1 is $J_{\mu'}$-reduced on the right and $u_2 \in W_{\mu'}$. Thus $u_1 \in {}^\lambda W$ and $u\Phi_{\mu'} = u_1\Phi_{\mu'}$. As u_1 is $J_{\mu'}$ reduced on the right we have that $J_{\mu'} \subseteq A_R(u_1)$. If there exists some $s' \in J_{\mu'}$ such that $u_1 s' \notin {}^\lambda W$ then $u_1 s' = s u_1$ for some $s \in J_\lambda$. This implies that $u_1 \alpha_{s'} = \alpha_s$, contradicting the fact that $u_1\Phi_{\mu'} \cap \{\alpha_j \mid j \in J_\lambda\} = \emptyset$. Thus, no such s' exists and so $J_{\mu'} \subseteq A_\lambda(u_1)$. This implies that $\mu' \leq \boldsymbol{\mu}(u_1, \lambda)$ (as if $A_\lambda(u_1)\backslash J_{\mu'} \neq \emptyset$ then the λ-expression of u_1 is comprised of strictly less than $r(\mu')$ blocks, using Lemma 7.1.8). The result follows as $\boldsymbol{\mu}(u_1, \lambda) \leq \lambda'$ by Lemma 7.2.8. $\square$

Lemma 8.1.2. *If $\mu \not\geq \lambda$ then $\pi_\lambda(C_{\mathsf{w}_{\mu'}}) = 0$.*

Proof. Applying π_λ to the formula in Theorem 5.4.2 we have that

$$\pi_\lambda(C_{w'_\mu}) = \mathsf{q}^{\ell(\mathsf{w}_{\mu'})} \sum_{w \in W_{\mu'}} \mathsf{q}^{-\ell(w)} \pi_\lambda(c_w^{\mu'}(X) U_w) \quad \text{where} \quad c_w^{\mu'}(X) = \prod_{\beta \in \Phi^+_{\mu'} \setminus \Phi(w)} \frac{1 - \mathsf{q}^{-2} X^{-\beta}}{1 - X^{-\beta}}$$

as $\beta^\vee = \beta$ for all $\beta \in \Phi^+$. Let $B' = \{\varpi_\lambda(U_u) \mid u \in {}^\lambda W\}$. By Remark 5.4.5, for $w \in W_{\mu'}$, $\pi_\lambda(c_w^{\mu'}(X); B')$ is diagonal and $\pi_\lambda(U_w; B')$ has distinct support (the places of the non-zero entries; row u of $\pi_\lambda(U_w; B')$ has a unique entry in the uw-th position if $uw \in {}^\lambda W$ and is zero otherwise). Thus $\pi_\lambda(C_{\mathsf{w}_{\mu'}}; B') = 0$ if and only if $\pi_\lambda(c_w^{\mu'}(X) U_w; B') = 0$ for all $w \in W_{\mu'}$. This is equivalent to the statement

$$\varpi_\lambda(U_u) \cdot c_w^{\mu'}(X) U_w = 0 \text{ for all } u \in {}^\lambda W \text{ and } w \in W_{\mu'}.$$

By Proposition 5.1.3 and Proposition 5.4.1(2) we have

$$\begin{aligned} \varpi_\lambda(U_u) \cdot c_w^{\mu'}(X) U_w &= \varpi_\lambda \left(\prod_{\beta \in \Phi^+_{\mu'} \setminus \Phi(w)} \frac{1 - \mathsf{q}^{-2} X^{-u\beta}}{1 - X^{-u\beta}} U_u U_w \right) \\ &= \left(\prod_{\beta \in \Phi^+_{\mu'} \setminus \Phi(w)} \frac{1 - \mathsf{q}^{-2} \psi_\lambda(X^{-u\beta})}{1 - \psi_\lambda(X^{-u\beta})} \right) \varpi_\lambda(U_u U_w) \\ &= \left(\prod_{\beta \in \Phi^+_{\mu'} \setminus \Phi(w)} \frac{1 - \mathsf{q}^{-2} \psi_\lambda(X^{-u\beta})}{1 - \psi_\lambda(X^{-u\beta})} \right) \varpi_\lambda(U_u) \cdot U_w \end{aligned}$$

As row u of $\pi_\lambda(U_w; B')$ is only nonzero if $uw \in {}^\lambda W$, when $uw \notin {}^\lambda W$ we have $\varpi_\lambda(U_u) \cdot U_w = 0$ and thus $\varpi_\lambda(U_u) \cdot c_w^{\mu'}(X) U_w = 0$. So assume that $uw \in {}^\lambda W$.

As $\mu \not\geq \lambda$, by Lemma 8.1.1 we have that $u\Phi_{\mu'} \cap \{\alpha_j \mid j \in J_\lambda\} \neq \emptyset$ for all $u \in {}^\lambda W$. Thus there exists $j \in J_\lambda$ such that $u^{-1}\alpha_j = \beta \in \Phi_{\mu'}$. Since $u \in {}^\lambda W$ we have that $\beta > 0$, so $\beta \in \Phi^+_{\mu'}$. Moreover, $w^{-1}\beta = (uw)^{-1}\alpha_j > 0$ as $uw \in {}^\lambda W$ by assumption. Hence, $\beta \in \Phi^+_{\mu'} \setminus \Phi(w)$ and so β appears in the product of $\varpi_\lambda(U_u) \cdot c_w^{\mu'}(X) U_w$ with corresponding factor

$$\frac{1 - \mathsf{q}^{-2} \psi_\lambda(X^{-u\beta})}{1 - \psi_\lambda(X^{-u\beta})} = \frac{1 - \mathsf{q}^{-2} \psi_\lambda(X^{-\alpha_j})}{1 - \psi_\lambda(X^{-\alpha_j})} = 0$$

as $\psi_\lambda(X^{\alpha_j}) = \mathsf{q}^{-2}$ for all $j \in J_\lambda$ (see Section 7.3). Thus $\pi_\lambda(C_{\mathsf{w}_{\mu'}}) = 0$. □

Recall from Section 7.3 that $\Delta_\mu \leq_{LR} \Delta_\lambda$ if and only if $\mu \leq \lambda$. We now prove the killing property.

Theorem 8.1.3. *Let $\lambda, \mu \vdash n+1$. If $w \in \Delta_\mu$ with $\mu \not\geq \lambda$ then $\pi_\lambda(C_w) = 0$.*

Proof. Let $w \in \Delta_\mu$. By Theorem 7.3.1, there exist $h, h' \in \widetilde{\mathcal{H}}$ such that

$$hC_{\mathsf{w}_{\mu'}}h' = C_w + \sum_{z \in \widetilde{W}, z <_{LR} \mathsf{w}_{\mu'}} a_z C_z \quad \text{with } a_z \in \mathsf{R}. \tag{8.1.1}$$

We proceed by induction on μ with respect to the dominance order. If $\mu \geq \lambda$ then there is nothing to prove, so assume that $\mu \not\geq \lambda$. For the base case, let $\mu = 1^{n+1}$. As this is the unique lowest partition in the dominance order (8.1.1) becomes $hC_{\mathsf{w}_{\mu'}}h' = C_w$ for all $w \in \Delta_\mu$ and the base case holds by Lemma 8.1.2.

Assume that for all $\nu < \mu$, if $w \in \Delta_\nu$ then $\nu \ngeq \lambda$ implies that $\pi_\lambda(C_w) = 0$. In the sum in (8.1.1) for $z <_{LR} \mathsf{w}_{\mu'}$ if $a_z \neq 0$ then $z \in \Delta_\nu$ for some $\nu < \mu$. As $\nu < \mu$ and $\mu \ngeq \lambda$ it follows that $\nu \ngeq \lambda$. Thus applying the inductive hypothesis we have $\pi_\lambda(C_z) = 0$. The equation (8.1.1) then becomes $hC_{\mathsf{w}_{\mu'}}h' = C_w$ and $\pi_\lambda(C_w) = 0$ for all $w \in \Delta_\mu$ follows by Lemma 8.1.2. □

8.2 Boundedness

Using the killing property from Section 8.1, we now show that π_λ is bounded with bound $\ell(\mathsf{w}_{\lambda'})$ and that the elements that reach this bound are a subset of Δ_λ.

Theorem 8.2.1. *The degree in* q *of the entries of the matrices* $\pi_\lambda(T_w)$ *for* $w \in \widetilde{W}$ *is bounded by* $\ell(\mathsf{w}_{\lambda'})$. *Moreover, if* $\deg \pi_\lambda(T_w) = \ell(\mathsf{w}_{\lambda'})$ *then* $w \in \Delta_\lambda$.

Proof. Let $N = \max\{\deg[\pi_\lambda(T_w)]_{u,v} \mid w \in \widetilde{W},\ u, v \in {}^\lambda W\}$ which is well defined by Theorem 6.1.9 (as $\mathsf{v}_\alpha = -\mathsf{q}^{-1}$ for all $\alpha \in \Phi_J$). Let $w \in \widetilde{W}$ be such that $\pi_\lambda(T_w)$ attains the degree N, and suppose that this degree is attained on the u-th row of $\pi_\lambda(T_w)$. Let $v_1, \ldots, v_k \in {}^\lambda W$ be the columns of $\pi_\lambda(T_w)$ where $\deg[\pi_\lambda(T_w)]_{u,v_j} = N$ for $1 \leq j \leq k$. Thus we can write

$$[\pi_\lambda(T_w)]_{u,v_j} = a_j(\zeta_\lambda)\mathsf{q}^N + (\text{terms of strictly lower } \mathsf{q} \text{ degree}),$$

with $a_j(\zeta_\lambda) \in \mathsf{R}[\zeta_\lambda]$. By Lemma 5.3.7 we have that $\pi_\lambda(h^*) = \pi_\lambda(h)^*$ and so

$$[\pi_\lambda(T_{w^{-1}})]_{v_j,u} = a_j(\zeta_\lambda^{-1})\mathsf{q}^N + (\text{terms of strictly lower } \mathsf{q} \text{ degree}).$$

By the triangularity between C_w and T_w

$$\begin{aligned}[\pi_\lambda(C_w)]_{u,v_j} &= a_j(\zeta)\mathsf{q}^N + (\text{terms of strictly lower } \mathsf{q} \text{ degree}), \text{ and}\\ [\pi_\lambda(C_{w^{-1}})]_{v_j,u} &= a_j(\zeta_\lambda^{-1})\mathsf{q}^N + (\text{terms of strictly lower } \mathsf{q} \text{ degree}).\end{aligned}$$

Thus,

$$\begin{aligned}[\pi_\lambda(C_w)\pi_\lambda(C_{w^{-1}})]_{u,u} = [a_1(\zeta_\lambda)a_1(\zeta_\lambda^{-1}) + \cdots + a_k(\zeta_\lambda)a_k(\zeta_\lambda^{-1})]\mathsf{q}^{2N}&\\ + (\text{terms of strictly lower } \mathsf{q} \text{ degree}),&\end{aligned}$$

We claim that the coefficient of q^{2N} does not vanish. Let $a(\zeta_\lambda) = \sum_{\gamma \in P^{(\lambda)}} a_\gamma \zeta_\lambda^\gamma \in \mathsf{R}[\zeta_\lambda]$ with $a_\gamma \in \mathbb{Z}$, then

$$a(\zeta_\lambda)a(\zeta_\lambda^{-1}) = \sum_{\gamma_1,\gamma_2 \in P^{(\lambda)}} a_{\gamma_1}a_{\gamma_2}\zeta_\lambda^{\gamma_1-\gamma_2} = \sum_{\gamma \in P^{(\lambda)}} a_\gamma^2 + \text{ terms involving } \zeta_\lambda,$$

and the constant term, $\sum_{\gamma \in P^{(\lambda)}} a_\gamma^2$, is strictly positive. Since $[a_1(\zeta_\lambda)a_1(\zeta_\lambda^{-1}) + \cdots + a_k(\zeta_\lambda)a_k(\zeta_\lambda^{-1})]$ is a sum of terms of the form of $\sum_{\gamma \in P^{(\lambda)}} a_\gamma \zeta_\lambda^\gamma$ with $a_\gamma \in \mathbb{Z}$, it cannot vanish as claimed. Thus, $\pi_\lambda(C_w)\pi_\lambda(C_{w^{-1}})$ attains a degree of q^{2N} in the (u, u)-th entry.

On the other hand,

$$\pi_\lambda(C_w)\pi_\lambda(C_{w^{-1}}) = \sum_z h_{w,w^{-1},z}\pi_\lambda(C_z). \tag{8.2.1}$$

where $h_{w,w^{-1},z}$ are the structure constants with respect to w, w^{-1} and z (see Section 3.2). By Theorem 8.1.3 the sum in (8.2.1) is over z in two-sided cells higher than or equal to Δ_λ with respect to $\leq_{LR}$. Consider $z \in \Delta_\mu$ with $\mu \geq \lambda$. As $\mathbf{a}(z) = \ell(\mathsf{w}_{\mu'})$ we have that $\mathbf{a}(z) \leq \ell(\mathsf{w}_{\lambda'})$ for all z in the sum of (8.2.1) by Lemma 7.1.12. Thus, $\deg(h_{w,w^{-1},z}) \leq \mathbf{a}(z) \leq \ell(\mathsf{w}_{\lambda'})$. Furthermore, $\deg(\pi_\lambda(C_z)) \leq N$ and so the right hand side of (8.2.1) is bounded by $N + \ell(\mathsf{w}_{\lambda'})$. Thus, $2N \leq N + \ell(\mathsf{w}_{\lambda'})$ and so the representation is bounded by $\ell(\mathsf{w}_{\lambda'})$.

Suppose that $\pi_\lambda(T_w)$ attains this optimal degree $\ell(\mathsf{w}_{\lambda'})$ in the (u, v)-th entry. By the arguments above, $\pi_\lambda(C_w)\pi_\lambda(C_{w^{-1}})$ attains a bound of $\mathsf{q}^{2\ell(\mathsf{w}_{\lambda'})}$ in the (u, u)-th position. Equation (8.2.1) implies that there exists $z \geq_{LR} \mathsf{w}_{\lambda'}$ such that $\deg h_{w,w^{-1},z} = \ell(\mathsf{w}_{\lambda'})$ and $\deg[\pi_\lambda(C_z)]_{u,u} = \ell(\mathsf{w}_{\lambda'})$. As $\mathbf{a}(z) \leq \ell(\mathsf{w}_{\lambda'})$ this forces $\mathbf{a}(z) = \ell(\mathsf{w}_{\lambda'})$ and hence $z \sim_{LR} \mathsf{w}_{\lambda'}$ by $P11$ (see Section 3.2). As $\deg h_{w,w^{-1},z} = \ell(\mathsf{w}_{\lambda'})$ we have that $\gamma_{w,w^{-1},z^{-1}} \neq 0$ (recall that $\gamma_{w,w^{-1},z^{-1}}$ is the constant term of $\mathsf{q}^{-\ell(\mathsf{w}_{\lambda'})}h_{w,w^{-1},z}$). Thus by $P8$, $w \sim_R z$ and so $w \in \Delta_\lambda$ as required. □

Corollary 8.2.2. *The set of elements $w \in \widetilde{W}$ such that the matrix $\pi_\lambda(T_w)$ attains the bound $\ell(\mathsf{w}_{\lambda'})$ is a subset of the two-sided cell Δ_λ.*

Proof. This is immediate from Theorem 8.2.1. □

In Chapter 10 we will improve upon Corollary 8.2.2 to show that the elements that reach the bound are precisely the elements of Δ_λ. That is that π_λ recognises the cell Δ_λ in the sense that $\Gamma_{\pi_\lambda, M_\lambda, \mathsf{B}_\lambda} = \Delta_\lambda$ (see Section 3.3).

8.3 Leading matrices

Recall the definition of leading matrices from Section 3.3. For simplicity we will notate $\mathfrak{c}_{\pi_\lambda, M_\lambda, \mathsf{B}_\lambda}(w)$ for $w \in \widetilde{W}$ by $\mathfrak{c}_\lambda(w)$. By Theorem 8.2.1 the definition of $\mathfrak{c}_\lambda(w)$ is as follows: for $w \in \widetilde{W}$, its λ-leading matrix is given by

$$\mathfrak{c}_\lambda(w) = \mathsf{q}^{-\ell(\mathsf{w}_{\lambda'})}\pi_\lambda(C_w)|_{\mathsf{q}^{-1}=0} \in \mathrm{Mat}_{N_\lambda}(\mathbb{Z}[\zeta_\lambda]).$$

Note that by Lemma 3.3.3 we also have $\mathfrak{c}_\lambda(w) = \mathsf{q}^{-\ell(\mathsf{w}_{\lambda'})}\pi_\lambda(T_w)|_{\mathsf{q}^{-1}=0}$. In this section we define the ring of these leading matrices and prove that the map from $\mathscr{J}_\lambda$ to this ring, defined by $\mathsf{t}_w \mapsto \mathfrak{c}_\lambda(w)$, is a surjective homomorphism of unital rings.

By Corollary 8.2.2 if $\mathfrak{c}_\lambda(w) \neq 0$ then $w \in \Delta_\lambda$ and by Proposition 7.3.8 we have that $\mathfrak{c}_\lambda(\mathsf{w}_{\lambda'}) = E_{\mathsf{u}_\lambda, \mathsf{u}_\lambda}$ (where $E_{u,v}$ is the matrix with a 1 in the (u, v)-th position and 0 elsewhere).

Definition 8.3.1. For $\lambda \vdash n + 1$, define

$$\mathfrak{C}_\lambda = \mathrm{span}_\mathbb{Z}\{\mathfrak{c}_\lambda(w) \mid w \in \Delta_\lambda\} \quad \text{and} \quad \mathfrak{C} = \bigoplus_{\lambda \vdash n+1} \mathfrak{C}_\lambda.$$

Recall the definition of $\gamma_{x,y,z^{-1}} \in \mathbb{Z}$ from Section 3.2 as the constant term of $\mathsf{q}^{-\ell(\mathsf{w}_{\lambda'})}h_{x,y,z}$ for $x, y, z \in \widetilde{W}$.

Proposition 8.3.2. *The $\mathbb{Z}$-module $\mathfrak{C}_\lambda$ is an associative $\mathbb{Z}$-algebra under matrix multiplication. Moreover, for $x, y \in \Delta_\lambda$ we have*

$$\mathfrak{c}_\lambda(x)\mathfrak{c}_\lambda(y) = \sum_{z \in \Delta_\lambda} \gamma_{x,y,z^{-1}}\mathfrak{c}_\lambda(z).$$

Thus the linear map $\mathscr{J}_\lambda \to \mathfrak{C}_\lambda$, $\mathsf{t}_w \mapsto \mathfrak{c}_\lambda(w)$, is a surjective homomorphism of unital rings.

Proof. For $x, y \in \Delta_\lambda$ we have

$$[\mathsf{q}^{-\ell(\mathsf{w}_{\lambda'})}\pi_\lambda(C_x)][\mathsf{q}^{-\ell(\mathsf{w}_{\lambda'})}\pi_\lambda(C_y)] = \sum_{z\in\widetilde{W}}[\mathsf{q}^{-\ell(\mathsf{w}_{\lambda'})}h_{x,y,z}][\mathsf{q}^{-\ell(\mathsf{w}_{\lambda'})}\pi_\lambda(C_z)].$$

Specialising $\mathsf{q}^{-1} = 0$ on the left hand side gives $\mathfrak{c}_\lambda(x)\mathfrak{c}_\lambda(y)$. By Theorem 8.1.3 the sum on the right hand side is over $z \in \Delta_\mu$ with $\mu \geq \lambda$ and all other terms vanish. If $\mu \geq \lambda$ then $\ell(\mathsf{w}_{\mu'}) \leq \ell(\mathsf{w}_{\lambda'})$ so $\deg(\mathsf{q}^{-\ell(\mathsf{w}_{\lambda'})}h_{x,y,z}) \leq 0$ for the z terms in the sum (as $h_{x,y,z} \leq \ell(\mathsf{w}_{\mu'})$ for $z \in \Delta_\mu$). Thus, each term $[\mathsf{q}^{-\ell(\mathsf{w}_{\lambda'})}h_{x,y,z}][\mathsf{q}^{-\ell(\mathsf{w}_{\lambda'})}\pi_\lambda(C_z)]$ can be specialised at $\mathsf{q}^{-1} = 0$. If this specialisation is non-zero then $\deg \pi_\lambda(C_z) = \ell(\mathsf{w}_{\lambda'})$ and so $z \in \Delta_\lambda$ by Corollary 8.2.2. Upon specialisation, for such $z \in \Delta_\lambda$ we have $\mathsf{q}^{-\ell(\mathsf{w}_{\lambda'})}h_{x,y,z} \mapsto \gamma_{x,y,z^{-1}}$ and the result follows. □

Chapter 9

λ-Satake isomorphism

In this chapter we will give a λ-relative version of the Satake isomorphism using π_λ. The results of this chapter, leading up to the λ-relative Satake isomorphism, will be used in Chapter 11 to describe Lusztig's asymptotic algebra. First recall the classical Satake isomorphism. Let

$$\mathbf{1}_0 = \frac{\mathsf{q}^{\ell(\mathsf{w}_0)}}{W(\mathsf{q}^2)} C_{\mathsf{w}_0}$$

where $W(\mathsf{q}^2) = \sum_{w \in W_0} \mathsf{q}^{2\ell(w)}$. This normalisation of C_{w_0} is such that $\mathbf{1}_0^2 = \mathbf{1}_0$ and $\mathbf{1}_0 \widetilde{\mathcal{H}} \mathbf{1}_0$ is a unital algebra with identity $\mathbf{1}_0$. Recall that $\mathsf{R}[X]^{W_0}$ is the ring of symmetric functions in X. It is well know that $\mathsf{R}[X]^{W_0}$ is the centre of $\widetilde{\mathcal{H}}$ (see for example [38, Theorem 1.4]). By [38, Proposition 2.3(b)], the set $\{s_\gamma \mid \gamma \in P^+\}$ is a basis of $\mathsf{R}[X]^{W_0}$ where the s_γ's are the classical Schur functions from [38, (2.10)]. The classical Satake isomorphism is

$$Z(\widetilde{\mathcal{H}}) = \mathsf{R}[X]^{W_0} \cong \mathbf{1}_0 \widetilde{\mathcal{H}} \mathbf{1}_0$$

where the isomorphism is such that $s_\gamma \mapsto s_\gamma \mathbf{1}_0 = \mathbf{1}_0 s_\gamma \mathbf{1}_0$.

The λ-analogue to this isomorphism proved in this chapter is $\pi_\lambda(\mathbf{1}_{\lambda'} \widetilde{\mathcal{H}}_{\mathsf{R}'} \mathbf{1}_{\lambda'}) \cong \mathsf{R}'[\zeta_\lambda]^{G_\lambda}$ where $\mathbf{1}_{\lambda'}$ is a normalisation of $C_{\mathsf{w}_{\lambda'}}$ and R' is the extension of R so that the associated algebra extension $\widetilde{\mathcal{H}}_{\mathsf{R}'}$ contains $\mathbf{1}_{\lambda'}$.

To prove this isomorphism we first define a λ-analogue to $\mathbf{1}_0$. Define

$$\mathbf{1}_{\lambda'} = \frac{\mathsf{q}^{\ell(\mathsf{w}_{\lambda'})}}{W_{\lambda'}(\mathsf{q}^2)} C_{\mathsf{w}_{\lambda'}} \quad \text{where} \quad W_{\lambda'}(\mathsf{q}^2) = \sum_{w \in W_{\lambda'}} \mathsf{q}^{2\ell(w)}.$$

To determine properties of $\pi_\lambda(\mathbf{1}_{\lambda'} \widetilde{\mathcal{H}}_{\mathsf{R}'} \mathbf{1}_{\lambda'})$ we will explore the properties of $\pi_\lambda(C_{\mathsf{w}_{\lambda'}} \widetilde{\mathcal{H}} C_{\mathsf{w}_{\lambda'}})$.

Definition 9.0.1. Let $f_\lambda : \widetilde{\mathcal{H}} \to \mathsf{R}[\zeta_\lambda]$ be defined by $f_\lambda(h) = \chi_\lambda(h C_{\mathsf{w}_{\lambda'}})$ where χ_λ is the character of (π_λ, M_λ). We extend the definition of f_λ to operators U acting on M_λ on the right (as we did in Remark 5.4.5).

Theorem 9.0.2. *We have $\pi_\lambda(C_{\mathsf{w}_{\lambda'}} h C_{\mathsf{w}_{\lambda'}}) = f_\lambda(h) \pi_\lambda(C_{\mathsf{w}_{\lambda'}})$ for all $h \in \widetilde{\mathcal{H}}$.*

Proof. By the definition of matrix multiplication we have

$$[\pi_\lambda(C_{\mathsf{w}_{\lambda'}} h C_{\mathsf{w}_{\lambda'}})]_{u,v} = \sum_{u_1, u_2 \in {}^\lambda W} [\pi_\lambda(C_{\mathsf{w}_{\lambda'}})]_{u,u_1} [\pi_\lambda(h)]_{u_1,u_2} [\pi_\lambda(C_{\mathsf{w}_{\lambda'}})]_{u_2,v}.$$

By Proposition 7.3.8 this entry is zero unless $u = \mathsf{u}_\lambda x$ and $v = \mathsf{u}_\lambda y$ for some $x, y \in W_{\lambda'}$. Furthermore, in the sum we have that $u_1 = \mathsf{u}_\lambda x'$ and $u_2 = \mathsf{u}_\lambda y'$ for some $x', y' \in W_{\lambda'}$. Thus,

$$\begin{aligned}[\pi_\lambda(C_{\mathsf{w}_{\lambda'}} h C_{\mathsf{w}_{\lambda'}})]_{\mathsf{u}_\lambda x, \mathsf{u}_\lambda y} &= \sum_{x', y' \in W_{\lambda'}} \mathsf{q}^{\mathbf{a}_\lambda - \ell(x) - \ell(x')} \mathsf{q}^{\mathbf{a}_\lambda - \ell(y) - \ell(y')} [\pi_\lambda(h)]_{\mathsf{u}_\lambda x', \mathsf{u}_\lambda y'} \\ &= \mathsf{q}^{\mathbf{a}_\lambda - \ell(x) - \ell(y)} \sum_{x', y' \in W_{\lambda'}} \mathsf{q}^{\mathbf{a}_\lambda - \ell(x') - \ell(y')} [\pi_\lambda(h)]_{\mathsf{u}_\lambda x', \mathsf{u}_\lambda y'}\end{aligned}$$

and so $\pi_\lambda(C_{\mathsf{w}_{\lambda'}} h C_{\mathsf{w}_{\lambda'}}) = f'_\lambda(h) \pi_\lambda(C_{\mathsf{w}_{\lambda'}})$ where

$$f'_\lambda(h) = \sum_{x, y \in W_{\lambda'}} \mathsf{q}^{\mathbf{a}_\lambda - \ell(x) - \ell(y)} [\pi_\lambda(h)]_{\mathsf{u}_\lambda x, \mathsf{u}_\lambda y}.$$

This implies that $\chi_\lambda(C_{\mathsf{w}_{\lambda'}} h C_{\mathsf{w}_{\lambda'}}) = f'_\lambda(h) \chi_\lambda(C_{\mathsf{w}_{\lambda'}})$, and Proposition 7.3.8 gives that

$$\chi_\lambda(C_{\mathsf{w}_{\lambda'}}) = \sum_{x \in W_{\lambda'}} \mathsf{q}^{\mathbf{a}_\lambda - 2\ell(x)} = \mathsf{q}^{\mathbf{a}_\lambda} W_{\lambda'}(\mathsf{q}^{-2}) = \mathsf{q}^{\ell(\mathsf{w}_{\lambda'})} W_{\lambda'}(\mathsf{q}^{-2})$$

as $\mathbf{a}_\lambda = \ell(\mathsf{w}_{\lambda'})$ by Lemma 7.1.11.

On the other hand, we have

$$\chi_\lambda(C_{\mathsf{w}_{\lambda'}} h C_{\mathsf{w}_{\lambda'}}) = \chi_\lambda(h C^2_{\mathsf{w}_{\lambda'}}) = \mathsf{q}^{\ell(\mathsf{w}_{\lambda'})} W_{\lambda'}(\mathsf{q}^{-2}) \chi_\lambda(h C_{\mathsf{w}_{\lambda'}})$$

as $C^2_{\mathsf{w}_{\lambda'}} = \mathsf{q}^{\ell(\mathsf{w}_{\lambda'})} W_{\lambda'}(\mathsf{q}^{-2}) C_{\mathsf{w}_{\lambda'}}$ (See Section 3.2). Therefore, $f'_\lambda(h) = \chi_\lambda(h C_{\mathsf{w}_{\lambda'}}) = f_\lambda(h)$ as required. □

Corollary 9.0.3. *The algebra $\pi_\lambda(C_{\mathsf{w}_{\lambda'}} \widetilde{\mathcal{H}} C_{\mathsf{w}_{\lambda'}})$ is commutative.*

Proof. The result follows directly from Theorem 9.0.2: for $h_1, h_2 \in \widetilde{\mathcal{H}}$ we have

$$\pi_\lambda(C_{\mathsf{w}_{\lambda'}} h_1 C_{\mathsf{w}_{\lambda'}}) \pi_\lambda(C_{\mathsf{w}_{\lambda'}} h_2 C_{\mathsf{w}_{\lambda'}}) = f_\lambda(h_1) f_\lambda(h_2) \pi_\lambda(C^2_{\mathsf{w}_{\lambda'}}) = \pi_\lambda(C_{\mathsf{w}_{\lambda'}} h_2 C_{\mathsf{w}_{\lambda'}}) \pi_\lambda(C_{\mathsf{w}_{\lambda'}} h_1 C_{\mathsf{w}_{\lambda'}}).$$

□

Recall from Section 3.2 that the bar involution on R is defined by $\overline{\mathsf{q}} \mapsto \mathsf{q}^{-1}$. We extend this definition from $\mathsf{R} \to \mathsf{R}[\zeta_\lambda]$ by setting $\overline{\zeta^\gamma_\lambda} = \zeta^\gamma_\lambda$. The following property of $f_\lambda(h)$ will be required in the proof of Proposition 11.2.8.

Corollary 9.0.4. *We have $f_\lambda(\overline{h}) = \overline{f_\lambda(h)}$ for all $h \in \widetilde{\mathcal{H}}$.*

Proof. We claim that $\chi_\lambda(\overline{h}) = \overline{\chi_\lambda(h)}$ for all $f \in \widetilde{\mathcal{H}}$. The result follows directly from this claim and the fact that $f_\lambda(h) = \chi_\lambda(h C_{\mathsf{w}_{\lambda'}})$.

It is sufficient to prove the claim for $h = T_w$ by linearity. Let $(\pi'_\lambda, M'_\lambda)$ denote the representation $(\pi'_{J_\lambda, \mathsf{v}}, M'_{J_\lambda, \mathsf{v}})$ of $\widetilde{\mathcal{H}}$ induced from $\mathcal{L}_{J_\lambda}$ with basis $\{\xi_\lambda \otimes X_u \mid u \in {}^\lambda W\}$ (see Section 5.2). Let $\mathsf{B}'_\lambda = \{\xi_\lambda \otimes T_u \mid u \in {}^\lambda W\}$, which is clearly a basis of M'_λ by Proposition 5.2.4 and the triangularity between X_u and T_u.

Let $u \in {}^\lambda W$, then

$$(\xi_\lambda \otimes T_u) \cdot \overline{T_w} = \xi_\lambda \otimes \overline{X_u T_w} = \sum_{v \in {}^\lambda W} (\xi_\lambda \otimes T_v) \overline{[\pi'_\lambda(T_w)]_{u,v}}$$

as $\overline{T^{-1}_{u^{-1}}} = \overline{X_u} = T_u$, where $\pi'_\lambda(T_w)$ is with respect to the basis $\{\xi_\lambda \otimes X_u \mid u \in {}^\lambda W\}$. Thus, $[\pi_\lambda(\overline{T_w}; \mathsf{B}'_\lambda)]_{u,v} = \overline{[\pi_\lambda(T_w)]_{u,v}}$. The claim follows by Theorem 5.2.5 and the fact that the trace of a matrix is basis independent. □

Recall the Macdonald c-function from Section 6.2. We define a λ-analogue to this function by

$$c_\lambda(X) = \prod_{\alpha\in\Phi_\lambda^+} \frac{1-\mathsf{q}^{-2}X^{-\alpha}}{1-X^{-\alpha}}.$$

Also, recall the definition of ψ_λ from Section 5.2 and the explicit description of ψ_λ in type A_n from Section 7.3. As in Remark 5.4.5 we extend the definition of ψ_λ to rational functions of X whose denominators do not vanish upon application of ψ_λ.

Lemma 9.0.5. *If $u \in {}^\lambda W$ then $\psi_\lambda(u \cdot c_{\lambda'}(X)) = 0$ unless $u \in \mathsf{u}_\lambda W_{\lambda'}$.*

Proof. As $w \in W_0$ acts on $\mathsf{R}[X]$ by $w \cdot X^\gamma = X^{w\gamma}$ and as $\psi_\lambda(X^\gamma) = \mathsf{v}^{\langle\gamma,2\rho_\lambda\rangle}\zeta_\lambda^\gamma$, we have

$$\psi_\lambda(u\cdot c_{\lambda'}(X)) = \psi_\lambda\left(\prod_{\alpha\in\Phi_{\lambda'}^+}\frac{1-\mathsf{q}^{-2}X^{-u\alpha}}{1-X^{-u\alpha}}\right) = \prod_{\alpha\in\Phi_{\lambda'}^+}\frac{1-\mathsf{q}^{-2}\mathsf{v}^{-\langle u\alpha,2\rho_\lambda\rangle}\zeta_\lambda^{-u\alpha}}{1-\mathsf{v}^{-\langle u\alpha,2\rho_\lambda\rangle}\zeta_\lambda^{-u\alpha}}.$$

Note that for $\alpha \in \Phi_{\lambda'}^+$ if $u\alpha = \alpha_s$ for some $s \in J_\lambda$ then the numerator of the corresponding term in the product is zero as $\zeta_\lambda^{\alpha_s} = 1$ and $\mathsf{v}^{-\langle\alpha_s,2\rho_\lambda\rangle} = \mathsf{q}^2$. Thus, proving the result is equivalent to proving that if $u \in {}^\lambda W$ with $u \notin \mathsf{u}_\lambda W_{\lambda'}$ then there exists some $\alpha \in \Phi_{\lambda'}^+$ and $s \in J_\lambda$ such that $u\alpha = \alpha_s$.

Let $u \in {}^\lambda W$ with $u \notin \mathsf{u}_\lambda W_{\lambda'}$ and write $u = u_1u_2$ where u_1 is $J_{\lambda'}$-reduced on the right and $u_2 \in W_{\lambda'}$. Hence, $J_{\lambda'} \subseteq A_R(u_1)$. However, by Theorem 7.2.9 we have $J_{\lambda'} \neq A_\lambda(u_1)$ as $u \notin \mathsf{u}_\lambda W_{\lambda'}$ implies that $u_1 \neq \mathsf{u}_\lambda$. Hence, there exists some $s' \in J_{\lambda'}$ with $\ell(u_1s') = \ell(u_1)+1$ and $u_1s' = su_1$ for some $s \in J_\lambda$ (see [1, p.79 (F)]). Thus, $u_1\alpha_{s'} = \alpha_s$.

Let $\alpha = u_2^{-1}\alpha_{s'}$. Then $\alpha \in \Phi_{\lambda'}$ as $u_2 \in W_{\lambda'}$. Furthermore, $\alpha > 0$ as otherwise $u_2^{-1}u_1^{-1}\alpha_s = u_2^{-1}\alpha_{s'} = \alpha < 0$ which contradicts $u \in {}^\lambda W$. Thus $\alpha \in \Phi_{\lambda'}^+$ and $u\alpha = u_1u_2u_2^{-1}\alpha_{s'} = \alpha_s$ as required. ☐

Lemma 9.0.6. *We have $\psi_\lambda(\mathsf{u}_\lambda \cdot p(X)) \in \mathsf{R}[\zeta_\lambda]^{G_\lambda}$ for all $p(X) \in \mathsf{R}[X]^{W_{\lambda'}}$ where $\mathsf{R}[X]^{W_{\lambda'}} = \{f(X) \in \mathsf{R}[X] \mid w\cdot f(X) = f(X) \text{ for all } w \in W_{\lambda'}\}$.*

Proof. The set of monomials of type $p(X) = \sum_{\gamma'\in W_{\lambda'}\cdot\gamma} X^{\gamma'}$ for $\gamma \in P$ span $\mathsf{R}[X]^{W_{\lambda'}}$, thus it is sufficient to prove the claim for monomials of this type. Set $p(X) = \sum_{\gamma'\in W_{\lambda'}\cdot\gamma} X^{\gamma'}$. If $g \in G_\lambda$ then

$$g\cdot\psi_\lambda(\mathsf{u}_\lambda\cdot p(X)) = g\cdot\psi_\lambda\left(\sum_{\gamma'\in W_{\lambda'}\cdot\gamma} X^{\mathsf{u}_\lambda\gamma'}\right) = \sum_{\gamma'\in W_{\lambda'}\cdot\gamma}\mathsf{v}^{\langle\mathsf{u}_\lambda\gamma',2\rho_\lambda\rangle}\zeta_\lambda^{g\mathsf{u}_\lambda\gamma'}. \tag{9.0.1}$$

Write $g\mathsf{u}_\lambda\gamma' = \mathsf{u}_\lambda(\mathsf{u}_\lambda^{-1}g\mathsf{u}_\lambda)\gamma'$. By Proposition 7.2.12 we have that $\mathsf{u}_\lambda^{-1}g\mathsf{u}_\lambda \in W_{\lambda'}$ and thus $\mathsf{u}_\lambda^{-1}g\mathsf{u}_\lambda\gamma' \in W_{\lambda'}\cdot\gamma'$ and so $\mathsf{u}_\lambda^{-1}g\mathsf{u}_\lambda\gamma' \in W_{\lambda'}\cdot\gamma$. Consequently, we can change the variable in the sum of (9.0.1) as follows: letting $\gamma'' = \mathsf{u}_\lambda^{-1}g\mathsf{u}_\lambda\gamma'$ we have

$$g\cdot\psi_\lambda(\mathsf{u}_\lambda\cdot p(X)) = \sum_{\gamma''\in W_{\lambda'}\cdot\gamma}\mathsf{v}^{\langle g^{-1}\mathsf{u}_\lambda\gamma'',2\rho_\lambda\rangle}\zeta_\gamma^{\mathsf{u}_\lambda\gamma''}.$$

As $g\rho_\lambda = \rho_\lambda$ for all $g \in G_\lambda$ (see Section 7.2) we have that $\langle g^{-1}\mathsf{u}_\lambda\gamma'',2\rho_\lambda\rangle = \langle\mathsf{u}_\lambda\gamma'',2\rho_\lambda\rangle$. Thus, $g\cdot\psi_\lambda(\mathsf{u}_\lambda\cdot p(X)) = \psi_\lambda(\mathsf{u}_\lambda\cdot p(X))$ as required. ☐

Theorem 9.0.7. *We have $f_\lambda(h) \in \mathsf{R}[\zeta_\lambda]^{G_\lambda}$ for all $h \in \widetilde{\mathcal{H}}$.*

Proof. By the triangulaity between T_w and U_w for $w \in W_0$ (see the proof of Proposition 5.4.4) all elements of $\widetilde{\mathcal{H}}$ are a linear combination of $p(X)U_v$ where $v \in W_0$ and $p(X)$ is a rational function in X^γ with non-vanishing denominator upon applying ψ_λ (see Remark 5.4.5). By the linearity of f_λ it is sufficient to prove the property for $h = p(X)U_v$ where $v \in W_0$ and $p(X)$ is a rational function with non-vanishing denominator. By Definition 9.0.1 we have $f_\lambda(p(X)U_v) = \chi_\lambda(p(X)U_vC_{\mathsf{w}_{\lambda'}})$. By Theorem 5.4.2

$$C_{\mathsf{w}_{\lambda'}} = \sum_{w \in W_{\lambda'}} \mathsf{q}^{\ell(\mathsf{w}_{\lambda'})-\ell(w)} c_w^{\lambda'}(X) U_w \quad \text{where} \quad c_w^{\lambda'}(X) = \prod_{\beta \in \Phi^+_{\lambda'} \setminus \Phi(w)} \frac{1-\mathsf{q}^{-2}X^{-\beta}}{1-X^{-\beta}}.$$

Thus, by Proposition 5.4.1(2)

$$\begin{aligned}\chi_\lambda(p(X)U_vC_{\mathsf{w}_{\lambda'}}) &= \chi_\lambda\left(p(X)\sum_{w\in W_{\lambda'}} \mathsf{q}^{\ell(\mathsf{w}_{\lambda'})-\ell(w)}(v\cdot c_w^{\lambda'}(X))U_vU_w\right)\\ &= \sum_{w\in W_{\lambda'}} \mathsf{q}^{\ell(\mathsf{w}_{\lambda'})-\ell(w)}\chi_\lambda(p(X)(v\cdot c_w^{\lambda'}(X))U_vU_w).\end{aligned}$$

By Remark 5.4.5 we have that $\chi_\lambda(U_w) = 0$ unless $w = e$ and that $\pi_\lambda(r(X); B')$ (with $B' = \{\varpi_\lambda(U_u) \mid u \in {}^\lambda W\}$) is a diagonal matrix for all rational functions $r(X)$ (with non-vanishing denominator). Furthermore, by Proposition 5.4.1(4) U_vU_w is a rational function multiple of U_{vw}. Thus,

$$\chi_\lambda(p(X)U_vC_{\mathsf{w}_{\lambda'}}) = \begin{cases} 0 & \text{if } v \notin W_{\lambda'},\\ \mathsf{q}^{\ell(\mathsf{w}_{\lambda'})-\ell(v)}\chi_\lambda(p(X)(v\cdot c_{v^{-1}}^{\lambda'}(X))U_vU_{v^{-1}}) & \text{if } v \in W_{\lambda'}.\end{cases}$$

For $v \in W_{\lambda'}$ we have

$$v\cdot c_{v^{-1}}^{\lambda'}(X) = \prod_{\alpha\in\Phi^+_{\lambda'}\setminus\Phi(v^{-1})} \frac{1-\mathsf{q}^{-2}X^{-v\alpha}}{1-X^{-v\alpha}} = \prod_{\beta\in\Phi^+_{\lambda'}\setminus\Phi(v)} \frac{1-\mathsf{q}^{-2}X^{-\beta}}{1-X^{-\beta}}$$

taking $\beta = v\alpha$ as $\Phi^+_{\lambda'}\setminus\Phi(v^{-1}) = \{\alpha \in \Phi^+_{\lambda'} \mid v\alpha > 0\} = \{v\beta \in \Phi^+_{\lambda'} \mid v^{-1}v\beta > 0\} = \Phi^+_{\lambda'}\setminus\Phi(v)$.

Applying Proposition 5.4.1(3) $\ell(v)$ times gives

$$U_vU_{v^{-1}} = \mathsf{q}^{2\ell(v)} \prod_{\beta\in\Phi(v)} \frac{(1-\mathsf{q}^{-2}X^{-\beta})(1-\mathsf{q}^{-2}X^{\beta})}{(1-X^{-\beta})(1-X^{\beta})}.$$

Thus, for $v \in W_{\lambda'}$

$$\begin{aligned}(v\cdot c_{v^{-1}}^{\lambda'})U_vU_{v^{-1}} &= \mathsf{q}^{2\ell(v)} \prod_{\beta\in\Phi^+_{\lambda'}\setminus\Phi(v)} \frac{1-\mathsf{q}^{-2}X^{-\beta}}{1-X^{-\beta}} \prod_{\alpha\in\Phi(v)} \frac{(1-\mathsf{q}^{-2}X^{-\alpha})(1-\mathsf{q}^{-2}X^{\alpha})}{(1-X^{-\alpha})(1-X^{\alpha})}\\ &= \mathsf{q}^{2\ell(v)} c_{\lambda'}(X) \prod_{\beta\in\Phi(v)} \frac{1-\mathsf{q}^{-2}X^{\beta}}{1-X^{\beta}}\end{aligned}$$

where $c_{\lambda'}(X)$ is the λ'-analogue of the Macdonald c-function. Computing the trace using the intertwiner basis of π_λ, $B' = \{\varpi_\lambda(U_u) \mid u \in {}^\lambda W\}$, we have

$$\begin{aligned}\chi_\lambda(p(X)U_vC_{\mathsf{w}_{\lambda'}}) &= \mathsf{q}^{\ell(\mathsf{w}_{\lambda'})-\ell(v)}\chi_\lambda\left(p(X)\mathsf{q}^{2\ell(v)}c_{\lambda'}(X)\prod_{\beta\in\Phi(v)}\frac{1-\mathsf{q}^{-2}X^\beta}{1-X^{-\beta}}\right)\\ &= \mathsf{q}^{\ell(\mathsf{w}_{\lambda'})+\ell(v)}\sum_{u\in{}^\lambda W}\psi_\lambda\left(u\cdot\left(p(X)c_{\lambda'}(X)\prod_{\beta\in\Phi(v)}\frac{1-\mathsf{q}^{-2}X^\beta}{1-X^\beta}\right)\right)\end{aligned}$$

as $\pi_\lambda(X^\gamma; B')$ is diagonal with entries $\psi_\lambda(X^{u\gamma})$ for all $u \in {}^\lambda W$ (see Remark 5.4.5). By Lemma 9.0.5 we have that $\psi_\lambda(u \cdot c_{\lambda'}(X)) = 0$ unless $u \in \mathsf{u}_\lambda W_{\lambda'}$. Thus the sum over ${}^\lambda W$ above becomes a sum over $W_{\lambda'}$ as follows:

$$\begin{aligned}\chi_\lambda(p(X)U_vC_{\mathsf{w}_{\lambda'}}) &= \mathsf{q}^{\ell(\mathsf{w}_{\lambda'})\ell(v)}\sum_{y\in W_{\lambda'}}\psi_\lambda\left(\mathsf{u}_\lambda y\cdot\left(p(X)c_{\lambda'}(X)\prod_{\beta\in\Phi(v)}\frac{1-\mathsf{q}^{-2}X^\beta}{1-X^\beta}\right)\right)\\ &= \mathsf{q}^{\ell(\mathsf{w}_{\lambda'})+\ell(v)}\psi_\lambda\left(\mathsf{u}_\lambda\cdot\left(\sum_{y\in W_{\lambda'}}p(yX)c_{\lambda'}(yX)\prod_{\beta\in\Phi(v)}\frac{1-\mathsf{q}^{-2}X^{y\beta}}{1-X^{y\beta}}\right)\right). \quad (9.0.2)\end{aligned}$$

As the sum is $W_{\lambda'}$-invariant the result follows by Lemma 9.0.6. □

Remark 9.0.8. The proof of Theorem 9.0.7 shows that $f_\lambda(p(X)U_v) = 0$ if $v \notin W_{\lambda'}$, and (9.0.2) computes $f_\lambda(h)$ explicitly when h is written in the form $h = \sum_{v\in W_0} p_v(X)U_v$. That is $f_\lambda(h) = \sum_{v\in W_0} f_\lambda(p_v(X)U_v)$ with

$$f_\lambda(p_v(X)U_v) = \mathsf{q}^{\ell(\mathsf{w}_{\lambda'})+\ell(v)}\psi_\lambda\left(\mathsf{u}_\lambda\cdot\left(\sum_{y\in W_{\lambda'}}p_v(yX)c_{\lambda'}(yX)\prod_{\beta\in\Phi(v)}\frac{1-\mathsf{q}^{-2}X^{y\beta}}{1-X^{y\beta}}\right)\right).$$

Corollary 9.0.9. *For $\gamma \in P$ we have*

$$f_\lambda(X^\gamma) = \chi_\lambda(X^\gamma C_{\mathsf{w}_{\lambda'}}) = \mathsf{q}^{\ell(\mathsf{w}_{\lambda'})}\psi_\lambda(\mathsf{u}_\lambda\cdot P_\gamma^{\lambda'}(X))$$

where $P_\gamma^{\lambda'}(X)$ is the λ'-relative Macdonald spherical function

$$P_\gamma^{\lambda'}(X) = \sum_{y\in W_{\lambda'}} X^{y\gamma}\prod_{\alpha\in\Phi^+_{\lambda'}}\frac{1-\mathsf{q}^{-2}X^{-y\alpha}}{1-X^{-y\alpha}} = \sum_{y\in W_{\lambda'}} y\cdot(X^\gamma c_{\lambda'}(X)).$$

Proof. By the definition of f_λ and (9.0.2) in Theorem 9.0.7 we have

$$f_\lambda(X^\gamma) = \chi_\lambda(X^\gamma C_{\mathsf{w}_{\lambda'}}) = \mathsf{q}^{\ell(\mathsf{w}_{\lambda'})}\psi_\lambda\left(\mathsf{u}_\lambda\cdot\sum_{y\in W_{\lambda'}}X^{y\gamma}c_{\lambda'}(yX)\right)$$

as required. □

Let R' be the ring formed by adjoining R with the inverse of $W_{\lambda'}(\mathsf{q}^2) = \sum_{w\in W_{\lambda'}}\mathsf{q}^{2\ell(w)}$. Denote $\widetilde{\mathcal{H}}_{\mathsf{R}'}$ to be the extension of $\widetilde{\mathcal{H}}$ to include scalars in R'. We also extend the scalars of the representation (π_λ, M_λ) so that the representation is now defined over the ring $\mathsf{R}'[\zeta_\lambda]$. Recall that

$$\mathbf{1}_{\lambda'} = \frac{\mathsf{q}^{\ell(\mathsf{w}_{\lambda'})}}{W_{\lambda'}(\mathsf{q}^2)} C_{\mathsf{w}_{\lambda'}}.$$

As $C_{\mathsf{w}_{\lambda'}}^2 = \mathsf{q}^{-\ell(\mathsf{w}_{\lambda'})} W_{\lambda'}(\mathsf{q}^2) C_{\mathsf{w}_{\lambda'}}$ (see Section 3.2) we have that $\mathbf{1}_{\lambda'}^2 = \mathbf{1}_{\lambda'}$. This then implies that the algebra $\pi_\lambda(\mathbf{1}_{\lambda'} \widetilde{\mathcal{H}}_{\mathsf{R}'} \mathbf{1}_{\lambda'})$ is unital with identity $\mathbf{1}_{\lambda'}$. Furthermore, this algebra is commutative by Corollary 9.0.3.

We normalise f_λ as follows. Let

$$\tilde{f}_\lambda(h) = \frac{\ell(\mathsf{w}_{\lambda'})}{W_{\lambda'}(\mathsf{q}^2)} f_\lambda(h).$$

Then by Theorem 9.0.2 we have that

$$\pi_\lambda(\mathbf{1}_{\lambda'} h \mathbf{1}_{\lambda'}) = \frac{\mathsf{q}^{2\ell(\mathsf{w}_{\lambda'})}}{W_{\lambda'}(\mathsf{q}^2)^2} f_\lambda(h) \pi_\lambda(C_{\mathsf{w}_{\lambda'}}) = \frac{\mathsf{q}^{\ell(\mathsf{w}_{\lambda'})}}{W_{\lambda'}(\mathsf{q}^2)} f_\lambda(h) \pi_\lambda(\mathbf{1}_{\lambda'}) = \tilde{f}_\lambda(h) \pi_\lambda(\mathbf{1}_{\lambda'}).$$

We now prove the λ-relative Satake isomorphism. In the proof we use Corollary 11.2.9 which will be proved in Section 11.2 (note that there is no circular argument as Corollary 11.2.9 follows from Proposition 11.2.8 whose proof is not reliant on the Satake isomorphism).

Theorem 9.0.10. *We have* $\pi_\lambda(\mathbf{1}_{\lambda'} \widetilde{\mathcal{H}}_{\mathsf{R}'} \mathbf{1}_{\lambda'}) \cong \mathsf{R}'[\zeta_\lambda]^{G_\lambda}$*, with the isomorphism given by*

$$\pi_\lambda(\mathbf{1}_{\lambda'} h \mathbf{1}_{\lambda'}) \leftrightarrow \tilde{f}_\lambda(h).$$

Proof. Let $\Theta : \pi_\lambda(\mathbf{1}_{\lambda'} \widetilde{\mathcal{H}}_{\mathsf{R}'} \mathbf{1}_{\lambda'}) \to \mathsf{R}'[\zeta_\lambda]^{G_\lambda}$ be defined by $\Theta(\pi_\lambda(\mathbf{1}_{\lambda'} h \mathbf{1}_{\lambda'})) = \tilde{f}_\lambda(h)$. By Theorem 9.0.2, for $h_1, h_2 \in \widetilde{\mathcal{H}}_{\mathsf{R}'}$ we have that if $\pi_\lambda(\mathbf{1}_{\lambda'} h_1 \mathbf{1}_{\lambda'}) = \pi_\lambda(\mathbf{1}_{\lambda'} h_2 \mathbf{1}_{\lambda'})$ then

$$\tilde{f}_\lambda(h_1) \pi_\lambda(\mathbf{1}_{\lambda'}) = \pi_\lambda(\mathbf{1}_{\lambda'} h_1 \mathbf{1}_{\lambda'}) = \pi_\lambda(\mathbf{1}_{\lambda'} h_2 \mathbf{1}_{\lambda'}) = \tilde{f}_\lambda(h_2) \pi_\lambda(\mathbf{1}_{\lambda'}).$$

In addition, Theorem 9.0.7 gives that $\tilde{f}_\lambda(h) \in \mathsf{R}'[\zeta_\lambda]^{G_\lambda}$ for all $h \in \widetilde{\mathcal{H}}_{\mathsf{R}'}$. Therefore, Θ is well defined. The function is surjective by Corollary 11.2.9. For injectivity if $D_1 = \pi_\lambda(\mathbf{1}_{\lambda'} h_1 \mathbf{1}_{\lambda'})$ and $D_2 = \pi_\lambda(\mathbf{1}_{\lambda'} h_2 \mathbf{1}_{\lambda'})$ with $\Theta(D_1) = \Theta(D_2)$ then

$$D_1 = \tilde{f}_\lambda(h_1) \pi_\lambda(\mathbf{1}_{\lambda'}) = \Theta(D_1) \pi_\lambda(\mathbf{1}_{\lambda'}) = \Theta(D_2) \pi_\lambda(\mathbf{1}_{\lambda'}) = \tilde{f}_\lambda(h_2) \pi_\lambda(\mathbf{1}_{\lambda'}) = D_2.$$

All that remains to prove is that Θ is a homomorphism. Let $D_1 = \pi_\lambda(\mathbf{1}_{\lambda'} h_1 \mathbf{1}_{\lambda'})$ and $D_2 = \pi_\lambda(\mathbf{1}_{\lambda'} h_2 \mathbf{1}_{\lambda'})$, then

$$\Theta(D_1 D_2) = \Theta(\tilde{f}_\lambda(h_1) \pi_\lambda(\mathbf{1}_{\lambda'}) \tilde{f}_\lambda(h_2) \pi_\lambda(\mathbf{1}_{\lambda'})) = \tilde{f}_\lambda(h_1) \tilde{f}_\lambda(h_2) \Theta(\pi_\lambda(\mathbf{1}_{\lambda'}^2)) = \tilde{f}_\lambda(h_1) \tilde{f}_\lambda(h_2) \tilde{f}_\lambda(1).$$

By Proposition 7.3.8 we have that $f_\lambda(1) = \chi_\lambda(C_{\mathsf{w}_{\lambda'}}) = \sum_{u \in W_{\lambda'}} \mathsf{q}^{\ell(\mathsf{w}_{\lambda'}) - 2\ell(x)} = \mathsf{q}^{\ell(\mathsf{w}_{\lambda'})} W_{\lambda'}(\mathsf{q}^{-2})$. Moreover, as $C_{\mathsf{w}_{\lambda'}}^2 = \mathsf{q}^{\ell(\mathsf{w}_{\lambda'})} W_{\lambda'}(\mathsf{q}^{-2}) C_{\mathsf{w}_{\lambda'}} = \mathsf{q}^{-\ell(\mathsf{w}_{\lambda'})} W_{\lambda'}(\mathsf{q}^2) C_{\mathsf{w}_{\lambda'}}$ (see Section 3.2) we have that $f_\lambda(1) = \mathsf{q}^{-\ell(\mathsf{w}_{\lambda'})} W_{\lambda'}(\mathsf{q}^2)$. Thus, $\tilde{f}_\lambda(1) = 1$ and so Θ is a homomorphism. □

Remark 9.0.11. To recover the classical Satake isomorphism, take $\lambda = (1^{n+1})$. Then π_λ is the principal series representation. It is well known that this representation is faithful, we will give a brief description of this fact using the intertwiner basis from Remark 5.4.5.

Let $h \in \widetilde{\mathcal{H}}$ such that $\pi_\lambda(h) = 0$. Recall that all elements of $\widetilde{\mathcal{H}}$ can be written as a linear combination of $p(X) U_v$ where $v \in W_0$ and $p(X)$ is a rational function in X with coefficients in R and a non-vanishing denominator upon application of ψ_λ. Thus, write $h = \sum_{v \in W_0} p_v(X) U_v$. Let

$B' = \{\varpi_\lambda(U_u) \mid u \in {}^\lambda W\}$. By Remark 5.4.5 we have that $\pi_\lambda(U_v; B')$ is the matrix which has at most one non-zero entry in each row and column where the u-th row is non-zero only if $uv \in {}^\lambda W$ with an entry in the uv-th column. In the case when $\lambda = (1^{n+1})$, ${}^\lambda W = W_0$ and so $uv \in {}^\lambda W$ for all $u, v \in W_0$. Consider $u = e$, then there is an entry in the (u, v)-th position of $\pi(U_v; B')$. Thus the matrices $(\pi(U_v; B'))_{v \in W_0}$ are free over R. In addition, by Remark 5.4.5 $\pi_\lambda(r(X); B')$ is a diagonal matrix for all rational functions $r(X)$ (with non-vanishing denominator) and so $\pi_\lambda(h) = 0$ if and only if $\pi(p_v(X)U_v) = 0$ for all $v \in W_0$.

Let $v \in W_0$. Remark 5.4.5 states that the diagonal entries of $\pi_\lambda(X^\gamma; B')$ are $\psi_\lambda(X^\gamma) = \mathsf{v}^{u\gamma}\zeta_\lambda^{u\gamma} = \zeta^{u\gamma}$ with $u \in W_0$. Thus, $\pi_\lambda(p_v(X); B')$ is a diagonal matrix with entries rational functions in ζ and each diagonal entry is non-zero unless $p_v(X) = 0$. Therefore, $\pi_\lambda(p_v(X)U_v; B')$ is the matrix formed by taking $\pi_\lambda(U_v; B')$ and multiplying row u by $\psi_\lambda(p_v(u \cdot X))$ for all $u \in W_0$. This matrix is never zero, so the kernel of π_λ is trivial and π_λ is a faithful representation.

Therefore, $\pi_\lambda(\widetilde{\mathcal{H}}) \cong \widetilde{\mathcal{H}}$ and so with Theorem 9.0.10 we recover the classical Satake isomorphism $\mathbf{1}_0\widetilde{\mathcal{H}}\mathbf{1}_0 \cong \mathsf{R}[X]^{W_0}$.

Chapter 10

The asymptotic Plancherel formula

In Section 3.4 we defined a trace function on $\widetilde{\mathcal{H}}$ and described its spectral decomposition, the Plancherel Theorem. We also defined a trace function on Lusztig's asymptotic algebra $\mathcal{J}$. In this chapter, specialising to type A_n, we give an explicit description of the Plancherel Theorem and then prove a spectral decomposition for the trace on $\mathcal{J}$, called the asymptotic Plancherel Theorem. The decompositions are given as non-analytic formulas with the link to the analytic form described for both the regular and asymptotic cases. Section 10.1 gives the Plancherel Theorem description while Section 10.2 deals with the asymptotic case. Using the asymptotic Plancherel formula we prove that the elements $w \in \widetilde{W}$ such that $\pi_\lambda(T_w)$ reaches the q degree bound $\ell(\mathsf{w}_{\lambda'})$ are precisely the elements Δ_λ and we prove that $\mathcal{J}_\lambda$ is isomorphic to the $\mathbb{Z}$-span of the leading matrices of π_λ.

10.1 The Plancherel Theorem for $\widetilde{A}_n$

In this section we give an explicit description of the Plancherel formula (see Section 3.4) for type A_n. Let $\mathsf{v} = -\mathsf{q}^{-1}$ (recall that $\mathsf{v}_\alpha = -\mathsf{q}^{-1}$ for all $\alpha \in \Phi_\lambda$). Let

$$C_\lambda(\mathsf{q}) = \mathsf{q}^{-n(n+1)} \prod_{i=1}^{r(\lambda)} \frac{\mathsf{q}^{\lambda_i^2-\lambda_i}(1-\mathsf{q}^{-2})^{\lambda_i}}{1-\mathsf{q}^{-2\lambda_i}} \quad \text{and} \quad c^\lambda(\zeta_\lambda) = \prod_{\substack{1\le i<j\le r(\lambda)\\ 1\le k\le \lambda_j}} \frac{1-\mathsf{v}^{\lambda_i-\lambda_j+2k} z_i^{-1} z_j}{1-\mathsf{v}^{-\lambda_i-\lambda_j+2k} z_i^{-1} z_j}.$$

Recall the inner product on $\widetilde{\mathcal{H}}$ defined in Section 3.4 and its analytic decomposition from (3.4.3) and (6.2.3). As in Remark 6.2.3 we specialise $\mathsf{q} \mapsto q > 1$ real and z_i to a complex number with modulus 1 for $1 \le i \le r(\lambda)$. Let $\mathbb{T}^\lambda = \{(z_1, z_2, \ldots, z_{r(\lambda)}) \in \mathbb{T}^{r(\lambda)} \mid z_1^{\lambda_1} z_2^{\lambda_2} \cdots z_{r(\lambda)}^{\lambda_{r(\lambda)}} = 1\}$ where $\mathbb{T}$ is the group of complex numbers with modulus 1. Then, by [2, Remark 5.6] and [40] the explicit analytic decomposition when restricting to type A_n is

$$\langle h_1, h_2 \rangle = \sum_{\lambda \vdash n+1} \frac{C_\lambda(q)}{|G_\lambda|} \int_{\mathbb{T}^\lambda} \chi_\lambda(h_1 h_2^*) d\mu_\lambda(z_\lambda) \quad \text{where} \quad d\mu_\lambda(z_\lambda) = \frac{dz_\lambda}{|c^\lambda(z_\lambda)|^2}. \tag{10.1.1}$$

Note that in [2, Remark 5.6] Aubert and Plymen use compositions instead of partitions. Combining composition contributions and changing to the representation defined in Chapter 5 recovers (10.1.1) from [2, Remark 5.6]. In particular, the numerical constant becomes $1/|G_\lambda|$. This constant is also forced by the results of Chapter 11 (see Remark 11.2.11).

We now construct a combinatorial version of the Plancherel Theorem (10.1.1) which is equivalent to the analytic version upon specialisation of q and z_i. For $p(\zeta_\lambda) \in \mathsf{R}[\zeta_\lambda]$ we can expand $p(\zeta_\lambda)/(c^\lambda(\zeta_\lambda)(c^\lambda(\zeta_\lambda^{-1}))$ into a series using

$$\frac{1}{1-\mathsf{v}^{\lambda_i-\lambda_j+2k}z_i^{-1}z_j} = \sum_{r\geq 0} \mathsf{v}^{(\lambda_i-\lambda_j+2k)r} z_i^{-r} z_j^r \tag{10.1.2}$$

noting that this choice of expansion keeps q bounded from above as $\lambda_i - \lambda_j + 2k > 0$. Write

$$\left[\frac{p(\zeta_\lambda)}{c^\lambda(\zeta_\lambda)c^\lambda(\zeta_\lambda^{-1})}\right]_{\mathrm{ct}}$$

for the constant of ζ_λ^0 in this expansion.

Definition 10.1.1. For $h_1, h_2 \in \widetilde{\mathcal{H}}$ define

$$\langle h_1, h_2\rangle_\lambda = \frac{C_\lambda(\mathsf{q})}{|G_\lambda|}\left[\frac{\chi_\lambda(h_1h_2^*)}{c^\lambda(\zeta_\lambda)c^\lambda(\zeta_\lambda^{-1})}\right]_{\mathrm{ct}}.$$

To connect this definition to the terms in the analytic expression of the Plancherel formula, consider a term of 10.1.1

$$\frac{C_\lambda(q)}{|G_\lambda|}\int_{\mathbb{T}^\lambda} \frac{\chi_\lambda(h_1h_2^*)}{|c_\lambda(z_\lambda)|^2}dz_\lambda. \tag{10.1.3}$$

As $|z_i| = 1$ (for $1 \leq i \leq r(\lambda)$) we have that $c^\lambda(z_\lambda^{-1}) = \overline{c^\lambda(z_\lambda)}$, so $c^\lambda(z_\lambda)c^\lambda(z_\lambda^{-1}) = |c^\lambda(z_\lambda)|^2$. Using (10.1.2) we can expand the integrand of (10.1.3) into a series in $z_1, \ldots, z_{r(\lambda)}$ noting that this choice of expansion gives an absolutely convergent series as $q > 1$. We have that $\int_{\mathbb{T}^\lambda} z_\lambda^\gamma dz_\lambda = \delta_{0,\gamma}$ as dz_λ is the Haar measure. Thus upon integrating the series term by term

$$\frac{C_\lambda(q)}{|G_\lambda|}\int_{\mathbb{T}^\lambda} \frac{\chi_\lambda(h_1h_2^*)}{|c_\lambda(z_\lambda)|^2}dz_\lambda = \langle h_1, h_2\rangle_\lambda \tag{10.1.4}$$

where we specialise $\mathsf{q} \mapsto q > 1$ and z_i $(1 \leq i \leq r(\lambda))$ to a complex number of modulus 1 in $\langle h_1, h_2\rangle_\lambda$. Combining (10.1.4) and (10.1.1), we have the following non-analytic version of the Plancherel Theorem.

Theorem 10.1.2. *[40],[2, Remark 5.6] For all $h_1, h_2 \in \widetilde{\mathcal{H}}$ we have*

$$\langle h_1, h_2\rangle = \sum_{\lambda\vdash n+1} \langle h_1, h_2\rangle_\lambda.$$

Example 10.1.3. Consider $n = 1$. By direct calculation

$$C_{(1,1)}(\mathsf{q}) = \mathsf{q}^{-2} \qquad c^{(1,1)}(\zeta_{(1,1)}) = \frac{1-\mathsf{q}^{-2}z_1^{-1}z_2}{1-z_1^{-1}z_2} \quad \text{where} \quad z_1z_2 = 1$$

$$C_{(2)}(\mathsf{q}) = \frac{1-\mathsf{q}^{-2}}{1+\mathsf{q}^{-2}} \qquad c^{(2)}(\zeta_{(2)}) = 1.$$

Furthermore, $|G_{(2)}| = 1$ and $|G_{(1,1)}| = 2$. Specialising variables we recover the analytic Plancherel formula from Example 6.2.2, noting that $\mathbb{T}^{(2)} = \{z \in \mathbb{C} \mid z^2 = 1\} = \{1, -1\}$ so the normalised Haar measure on $\mathbb{T}^{(2)}$ is the discrete measure assigning mass 1/2 to each atom.

Now consider $n = 2$. Again, by direct calculation we have

$$C_{(1,1,1)}(\mathsf{q}) = \mathsf{q}^{-6} \qquad C_{(2,1)}(\mathsf{q}) = \mathsf{q}^{-4}\frac{1-\mathsf{q}^{-2}}{1+\mathsf{q}^{-2}} \qquad C_{(3)}(\mathsf{q}) = \frac{(1-\mathsf{q}^{-2})^3}{1-\mathsf{q}^{-6}}$$

$$c^{(1,1,1)}(\zeta_{(1,1,1)}) = \frac{(1-\mathsf{q}^{-2}z_1^{-1}z_2)(1-\mathsf{q}^{-2}z_2^{-1}z_3)(1-\mathsf{q}^{-2}z_1^{-1}z_3)}{(1-z_1^{-1}z_2)(1-z_2^{-1}z_3)(1-z_1^{-1}z_3)} \quad \text{where } z_1z_2z_3 = 1$$

$$c^{(2,1)}(\zeta_{(2,1)}) = \frac{1+\mathsf{q}^{-3}z_1^{-1}z_2}{1+\mathsf{q}z_1^{-1}z_2} \quad \text{where } z_1^2z_2 = 1$$

$$c^{(3)}(\zeta_{(3)}) = 1.$$

We also have $|G_{(1,1,1)}| = 6$ and $|G_{(2,1)}| = |G_{(3)}| = 1$. The analytic decomposition for this case is explicitly calculated in [41, Theorem 4.3]. To recover this decomposition note that $\mathbb{T}^{(3)} = \{z \in \mathbb{C} \mid z^3 = 1\} = \{1, e^{2\pi i/3}, e^{4\pi i/3}\}$. Thus, the Haar measure on $\mathbb{T}^{(3)}$ is the discrete measure assigning mass $1/3$ to each atom. With this we recover the analytic expression [41, Theorem 4.3] upon specializing variables.

Proposition 10.1.4. *For $\lambda \vdash n+1$ we have*

$$\deg \frac{C_\lambda(\mathsf{q})}{c^\lambda(\zeta_\lambda)c^\lambda(\zeta_\lambda^{-1})} = -2\ell(\mathsf{w}_{\lambda'}).$$

Proof. By definition, and as $\lambda_i > 0$ for all $1 \le i \le r(\lambda)$,

$$\deg C_\lambda(\mathsf{q}) = -n(n+1) + \sum_{i=1}^{r(\lambda)} \lambda_i^2 - \lambda_i = -n^2 - n - n - 1 + \sum_{i=1}^{r(\lambda)} \lambda_i^2$$

using the fact that $\sum_{i=1}^{r(\lambda)} \lambda_i = n+1$. In addition, as $\lambda_i - \lambda_j + 2k > 0$ and $-\lambda_i - \lambda_j + 2k \le 0$ for all $1 \le i < j \le r(\lambda)$ and $1 \le k \le \lambda_j$ we have

$$\begin{aligned}\deg\left(\frac{1}{c^\lambda(\zeta_\lambda)c^\lambda(\zeta_\lambda^{-1})}\right) &= \sum_{\substack{1\le i<j\le r(\lambda)\\ 1\le k\le\lambda_j}} 2(\lambda_i - \lambda_j - 2k)\\ &= \sum_{1\le i<j\le r(\lambda)} 2(\lambda_i\lambda_j + \lambda_j^2 - \lambda_j(\lambda_j+1))\\ &= \sum_{1\le i<j\le r(\lambda)} 2\lambda_j(\lambda_i - 1).\end{aligned}$$

Thus,

$$\deg \frac{C_\lambda(\mathsf{q})}{c^\lambda(\zeta_\lambda)c^\lambda(\zeta_\lambda^{-1})} = -(n+1)^2 + \sum_{i=1}^{r(\lambda)} \lambda_i^2 + \sum_{1\le i<j\le r(\lambda)} 2\lambda_i\lambda_j - \sum_{1\le i<j\le r(\lambda)} 2\lambda_j.$$

By Lemma 7.1.11 we have that $\sum_{1\le i<j\le r(\lambda)} 2\lambda_j = 2\ell(\mathsf{w}_{\lambda'})$, thus all there remains to prove is that $(n+1)^2 = \sum_{i=1}^{r(\lambda)} \lambda_i^2 + \sum_{1\le i<j\le r(\lambda)} 2\lambda_i\lambda_j$. This follows from the fact that

$$\sum_{i=1}^{r(\lambda)} \lambda_i \sum_{j\ne i} \lambda_j = \sum_{i=1}^{r(\lambda)} \sum_{1\le j<i\le r(\lambda)} \lambda_i\lambda_j + \sum_{i=1}^{r(\lambda)} \sum_{1\le i<j\le r(\lambda)} \lambda_i\lambda_j = 2\sum_{1\le i<j\le r(\lambda)} \lambda_i\lambda_j$$

with

$$\sum_{i=1}^{r(\lambda)} \lambda_i \left(\lambda_i + \sum_{j \neq i} \lambda_j \right) = \sum_{i=1}^{r(\lambda)} \lambda_i (n+1) = (n+1)^2$$

□

Remark 10.1.5. By Remark 6.2.3, (10.1.1) and Theorem 10.1.2, $\psi_\lambda(\mathsf{q}^{2\ell(\mathsf{w}_0)} c(X)c(X^{-1}))'$ is a constant multiple of $C_\lambda(\mathsf{q})/(c^\lambda(\zeta_\lambda)c^\lambda(\zeta_\lambda^{-1}))$ where $c(X)$ is Macdonalds c-function (see Section 6.2) and the $'$ notation denotes that all factors that are zero in the numerator or denominator are omitted. Thus, Proposition 10.1.4 with Theorem 8.2.1 confirms Conjecture 6.2.4 for type A_n.

Corollary 10.1.6. *For $\lambda \vdash n+1$ and $u, v \in \widetilde{W}$ we have $\deg\langle T_u, T_v\rangle_\lambda \leq 0$, and if equality holds then $\deg \pi_\lambda(T_u) = \deg \pi_\lambda(T_v) = \ell(\mathsf{w}_{\lambda'})$, and hence $u, v \in \Delta_\lambda$.*

Proof. By definition, we have

$$\langle T_u, T_v \rangle_\lambda = \frac{C_\lambda(\mathsf{q})}{|G_\lambda|} \left[\frac{\chi_\lambda(T_u T_{v^{-1}})}{c^\lambda(\zeta_\lambda)c^\lambda(\zeta_\lambda^{-1})} \right]_{\mathrm{ct}}.$$

By Theorem 8.2.1, $\deg \pi_\lambda(T_u) \leq \ell(\mathsf{w}_{\lambda'})$ and $\deg \pi_\lambda(T_v) \leq \ell(\mathsf{w}_{\lambda'})$. Thus, $\chi_\lambda(T_u T_v^{-1}) \leq 2\ell(\mathsf{w}_{\lambda'})$. If equality holds then $\deg \pi_\lambda(T_u) = \deg \pi_\lambda(T_v) = \ell(\mathsf{w}_{\lambda'})$ and so, by Theorem 8.2.1, $u, v \in \Delta_\lambda$. The result follows from Proposition 10.1.4. □

10.2 The asymptotic Plancherel Theorem

In this section, using the Plancherel formula, we give a spectral decomposition, called the asymptotic Plancherel formula, for the trace function defined on $\mathcal{J}$ in Section 3.4. We also prove that the elements of $\widetilde{W}$ whose image in π_λ reaches the maximal q degree are precisely the elements in Δ_λ (improving on the result of Corollary 8.2.2) and that $\mathcal{J} \cong \mathfrak{C}$.

Proposition 10.2.1. *For $\lambda \vdash n+1$ we have*

$$\left(\frac{\mathsf{q}^{2\ell(\mathsf{w}_{\lambda'})} C_\lambda(\mathsf{q})}{c^\lambda(\zeta_\lambda)c^\lambda(\zeta_\lambda^{-1})} \right)\Bigg|_{\mathsf{q}^{-1}=0} = \prod_{\alpha \in \Phi_{G_\lambda}} (1 - \zeta_\lambda^\alpha).$$

Proof. By Proposition 10.1.4, the specialisation of $\mathsf{q}^{2\ell(\mathsf{w}_{\lambda'})} C_\lambda(\mathsf{q})/(c^\lambda(\zeta_\lambda)c^\lambda(\zeta_\lambda^{-1}))$ at $\mathsf{q}^{-1} = 0$ exists and is a nonzero rational function in ζ_λ. Consider a typical term of $1/(c^\lambda(\zeta_\lambda)c^\lambda(\zeta_\lambda^{-1}))$:

$$\begin{aligned} &\frac{(1 - \mathsf{v}^{-\lambda_i - \lambda_j + 2k} z_i^{-1} z_j)(1 - \mathsf{v}^{-\lambda_i - \lambda_j + 2k} z_i z_j^{-1})}{(1 - \mathsf{v}^{\lambda_i - \lambda_j + 2k} z_i^{-1} z_j)(1 - \mathsf{v}^{\lambda_i - \lambda_j + 2k} z_i z_j^{-1})} \\ &\qquad = (-\mathsf{q})^{2(\lambda_i + \lambda_j - 2k)} \frac{((-\mathsf{q})^{-\lambda_i - \lambda_j + 2k} - z_i^{-1} z_j)((-\mathsf{q})^{-\lambda_i - \lambda_j + 2k} - z_i z_j^{-1})}{(1 - (-\mathsf{q})^{-\lambda_i + \lambda_j - 2k} z_i^{-1} z_j)(1 - (-\mathsf{q})^{-\lambda_i + \lambda_j - 2k} z_i z_j^{-1})} \end{aligned}$$

(noting that $\lambda_i - \lambda_j + 2k > 0$ and $-\lambda_i - \lambda_j + 2k \leq 0$). The q term out the front will be absorbed into the overall degree. If $\lambda_i + \lambda_j - 2k > 0$ then, upon specialising $\mathsf{q}^{-1} = 0$, the above term will contribute $(-z_i^{-1}z_j)(-z_iz_j^{-1}) = 1$. If $\lambda_i + \lambda_j - 2k = 0$, which forces $\lambda_i = \lambda_j = k$, then the term will contribute $(1 - z_i^{-1}z_j)(1 - z_iz_j^{-1})$. Thus,

$$\left(\frac{\mathsf{q}^{2\ell(\mathsf{w}_{\lambda'})}C_\lambda(\mathsf{q})}{c^\lambda(\zeta_\lambda)c^\lambda(\zeta_\lambda^{-1})}\right)\Bigg|_{\mathsf{q}^{-1}=0} = \prod_{\substack{1\leq i<j\leq r(\lambda)\\ \lambda_i=\lambda_j}} (1 - z_i^{-1}z_j)(1 - z_iz_j^{-1}) = \prod_{\alpha\in\Phi_{G_\lambda}} (1 - \zeta_\lambda^\alpha)$$

as required. □

Recall the definition of leading matrices for type A_n:

$$\mathfrak{c}_\lambda(w) = \mathsf{q}^{-\ell(\mathsf{w}_{\lambda'})}\pi_\lambda(C_w)|_{\mathsf{q}^{-1}=0} = \mathsf{q}^{-\ell(\mathsf{w}_{\lambda'})}\pi_\lambda(T_w)|_{\mathsf{q}^{-1}=0}$$

(where the second equality follows from Lemma 3.3.3). By Lemma 5.3.7 we have that

$$\mathfrak{c}_\lambda(w^{-1}) = \mathfrak{c}_\lambda(w)^*$$

where $\mathfrak{c}_\lambda(w)^*$ denotes the conjugate transpose of $\mathfrak{c}_\lambda(w)$ as defined in Section 5.3. For $\lambda \vdash n+1$ define $\pi_\lambda^\infty : \mathcal{J} \to \mathrm{Mat}_{N_\lambda}(\mathbb{Z}[\zeta_\lambda])$ by

$$\pi_\lambda^\infty\left(\sum_{w\in\widetilde{W}} a_w\mathsf{t}_w\right) = \sum_{w\in\Delta_\lambda} a_w\mathfrak{c}_\lambda(w).$$

In particular, note that $\pi_\lambda^\infty(\mathsf{t}_w) = 0$ if $w \notin \Delta_\lambda$. By Proposition 8.3.2, π_λ^∞ is a matrix representation of $\mathcal{J}$. Denote the character of π_λ^∞ by χ_λ^∞. Thus,

$$\chi_\lambda^\infty(A) = \mathsf{tr}(\pi_\lambda^\infty(A)), \quad \text{for } A \in \mathcal{J}$$

where tr denotes matrix trace.

Recall the definition of Tr^∞ and the corresponding inner product $\langle\cdot,\cdot\rangle^\infty$ from Section 3.4. The following *asymptotic Plancherel Theorem* gives a spectral decomposition of this inner product, mirroring the Plancherel Theorem (Theorem 10.1.2) at the asymptotic level.

Theorem 10.2.2. *For $A, B \in \mathcal{J}$ we have*

$$\langle A, B\rangle^\infty = \sum_{\lambda\vdash n+1} \langle A, B\rangle_\lambda^\infty \quad \text{where} \quad \langle A, B\rangle_\lambda^\infty = \frac{1}{|G_\lambda|}\left[\chi_\lambda^\infty(AB^*)\prod_{\alpha\in\Phi_{G_\lambda}}(1 - \zeta_\lambda^\alpha)\right]_{\mathrm{ct}}.$$

Moreover, for each $\lambda \vdash n+1$ the bilinear form $\langle\cdot,\cdot\rangle_\lambda^\infty$ is an inner product on the $\mathbb{Z}$-module $\mathcal{J}_\lambda$, and the elements t_w, $w \in \Delta_\lambda$, form an orthonormal basis.

Proof. By the linearity of the matrix trace function, it is sufficient to prove that

$$\sum_{\lambda\vdash n+1} \langle \mathsf{t}_u, \mathsf{t}_v\rangle_\lambda^\infty = \mathrm{Tr}^\infty(\mathsf{t}_u\mathsf{t}_v^*) = \delta_{u,v} \quad \text{for all } u, v \in \widetilde{W}$$

by Theorem 3.4.4. Let $u \nsim_{LR} v$. Recall that

$$\mathsf{t}_u\mathsf{t}_{v^{-1}} = \sum_{y\in\widetilde{W}} \gamma_{u,v^{-1},y^{-1}}\mathsf{t}_y.$$

In the sum, if $\gamma_{u,v^{-1},y^{-1}} \neq 0$ we have that $u \sim_L v$ by $P8$. Thus, as $u \nsim_{LR} v$, we have $\mathsf{t}_u\mathsf{t}_{v^{-1}} = 0$. It then follows that $\chi_\lambda^\infty(\mathsf{t}_u\mathsf{t}_v^{-1}) = 0$ for all $\lambda \vdash n+1$ and so $\sum_{\lambda \vdash n+1}\langle \mathsf{t}_u, \mathsf{t}_v\rangle_\lambda^\infty = 0$.

Thus, it remains to consider $u, v \in \Delta_\mu$ for some $\mu \vdash n+1$. By Theorem 10.1.2 and Lemma 3.4.1

$$\sum_{\lambda \vdash n+1} \langle T_u, T_v\rangle_\lambda = \delta_{u,v}.$$

By Corollary 10.1.6 the specialisation $(\langle T_u, T_v\rangle_\lambda)|_{\mathsf{q}^{-1}=0}$ exists and is zero unless $\mu = \lambda$. Therefore,

$$\left(\sum_{\lambda \vdash n+1} \langle T_u, T_v\rangle_\lambda\right)\Bigg|_{\mathsf{q}^{-1}=0} = \langle T_u, T_v\rangle_\mu|_{\mathsf{q}^{-1}=0} = \delta_{u,v}.$$

By Theorem 8.2.1 and the definition of $\mathfrak{c}_\lambda(w)$ from Lemma 3.3.3

$$\chi_\mu(T_uT_{v^{-1}}) = \mathsf{q}^{2\ell(\mathsf{w}_{\mu'})}\mathsf{tr}(\mathfrak{c}_\mu(u)\mathfrak{c}_\mu(v)^*) + (\text{terms of strictly lower } \mathsf{q} \text{ degree}),$$

with χ_μ the character of (π_μ, M_μ). Then, by Proposition 10.2.1 we have

$$C_\mu(\mathsf{q})\frac{\chi_\mu(T_uT_{v^{-1}})}{c^\mu(\zeta_\mu)c^\mu(\zeta_\mu^{-1})} = \chi_\mu^\infty(\mathsf{t}_u\mathsf{t}_v^*)\prod_{\alpha\in\Phi_{G_\mu}}(1-\zeta_\mu^\alpha) + (\text{terms of } \mathsf{q} \text{ degree} < 0)$$

as $\mathsf{tr}(\mathfrak{c}_\mu(u)\mathfrak{c}_\mu(v)^*) = \mathsf{tr}(\pi_\mu^\infty(\mathsf{t}_u)\pi_\mu^\infty(\mathsf{t}_v^*)) = \chi_\mu^\infty(\mathsf{t}_u\mathsf{t}_v^*)$. Upon specialising we have

$$\begin{aligned}\langle T_u, T_v\rangle_\mu|_{\mathsf{q}^{-1}=0} &= \left(\frac{1}{|G_\mu|}\left[\frac{C_\mu(\mathsf{q})\chi_\mu(T_uT_v^*)}{c^\mu(\zeta_\mu)c^\mu(\zeta_\mu^{-1})}\right]_{\mathrm{ct}}\right)\Bigg|_{\mathsf{q}^{-1}=0}\\ &= \frac{1}{|G_\mu|}\left[\chi_\mu^\infty(\mathsf{t}_u\mathsf{t}_v^*)\prod_{\alpha\in\Phi_{G_\mu}}(1-\zeta_\mu^\alpha)\right]_{\mathrm{ct}} = \langle \mathsf{t}_u, \mathsf{t}_v\rangle_\mu^\infty\end{aligned}$$

and the result follows as $\delta_{u,v} = \langle T_u, T_v\rangle_\mu|_{\mathsf{q}^{-1}=0}$.

It is clear that $\langle\cdot,\cdot\rangle_\lambda^\infty$ is linear and symmetry follows as the matrix trace is symmetric. For $A = \sum_{w\in\widetilde{W}} a_w\mathsf{t}_w \in \mathcal{J}$ we have

$$\langle A, A\rangle_\lambda^\infty = \sum_{w,u\in\widetilde{W}} a_wa_u\langle \mathsf{t}_w\mathsf{t}_u\rangle_\lambda^\infty = \sum_{w,u\in\widetilde{W}} a_wa_u\delta_{w,u} = \sum_{w\in\widetilde{W}} a_w^2$$

and so $\langle\cdot,\cdot\rangle_\lambda^\infty$ is an inner product as required. □

The following two theorems are consequences of the asymptotic Placherel theorem. In particular, note that Theorem 10.2.3 improves on Corollary 8.2.2 to say that the set of elements $w \in \widetilde{W}$ such that $\pi_\lambda(T_w)$ attains the bound $\ell(\mathsf{w}_{\lambda'})$ is precisely the two sided cell Δ_λ.

Theorem 10.2.3. *We have $\mathfrak{c}_\lambda(w) \neq 0$ if and only if $w \in \Delta_\lambda$.*

Proof. The forwards implication is true by Corollary 8.2.2, so it remains to prove that if $w \in \Delta_\lambda$ then $\deg \pi_\lambda(T_w) = \ell(\mathsf{w}_{\lambda'})$. If $w \in \Delta_\lambda$ then by Theorem 10.2.2, $\langle \mathsf{t}_w, \mathsf{t}_w\rangle_\lambda^\infty = 1$. Then, by the proof of Theorem 10.2.2 we have $\langle \mathsf{t}_w, \mathsf{t}_w\rangle_\lambda^\infty = \langle T_w, T_w\rangle_\lambda|_{\mathsf{q}^{-1}=0} = \delta_{w,w} = 1$, so $\deg\langle T_w, T_w\rangle_\lambda = 0$. The result then follows from Corollary 10.1.6. □

Theorem 10.2.4. *We have $\mathcal{J}_\lambda \cong \mathfrak{C}_\lambda$ as $\mathbb{Z}$-algebras, with $\mathsf{t}_w \mapsto \mathfrak{c}_\lambda(w)$. Thus $\mathcal{J} \cong \mathfrak{C}$.*

Proof. From Proposition 8.3.2 the map $\psi : \mathcal{J}_\lambda \to \mathfrak{C}_\lambda$, with $\mathsf{t}_w \mapsto \mathfrak{c}_\lambda(w)$, is a surjective ring homomorphism. It remains to prove that the map is injective. Let $a = \sum_{w\in\Delta_\lambda} a_w \mathsf{t}_w \in \mathcal{J}_\lambda$ with $\psi(a) = 0$. Thus,

$$\psi\left(\sum_{w\in\Delta_\lambda} a_w \mathsf{t}_w\right) = \sum_{w\in\Delta_\lambda} a_w \mathfrak{c}_\lambda(w) = 0,$$

and so for each $v \in \Delta_\lambda$ we have $\sum_{w\in\Delta_\lambda} a_w \mathfrak{c}_\lambda(w)\mathfrak{c}_\lambda(v)^* = 0$. Taking traces and multiplying by $(1/|G_\lambda|)\prod_{\alpha\in\Phi_{G_\lambda}}(1-\zeta_\lambda^\alpha)$ we have

$$\frac{1}{|G_\lambda|}\prod_{\alpha\in\Phi_{G_\lambda}}(1-\zeta_\lambda^\alpha)\mathsf{tr}\left(\sum_{w\in\Delta_\lambda} a_w \mathfrak{c}_\lambda(w)\mathfrak{c}_\lambda(v)^*\right) = \frac{1}{|G_\lambda|}\prod_{\alpha\in\Phi_{G_\lambda}}(1-\zeta_\lambda^\alpha)\sum_{w\in\Delta_\lambda} a_w \chi_\lambda^\infty(\mathsf{t}_w\mathsf{t}_v^*) = 0.$$

Taking the constant term of ζ_λ^0 on each side we have that $\sum_{w\in\Delta_\lambda} a_w\langle \mathsf{t}_w, \mathsf{t}_v\rangle_\lambda^\infty = 0$ and so $a_v = 0$ for all $v \in \Delta_\lambda$ as $\langle \mathsf{t}_w, \mathsf{t}_v\rangle_\lambda^\infty = \delta_{w,v}$ by Theorem 10.2.2. □

Remark 10.2.5. As for the Plancherel theorem (Theorem 10.1.2) we have an analytic version of the asymptotic Plancherel theorem. Specialise $\mathsf{q} \mapsto q > 1$ real and z_i to a complex number of modulus 1, for all $1 \le i \le r(\lambda)$. As $\int_{\mathbb{T}^\lambda} z_\lambda^\gamma dz_\lambda = \delta_{\gamma,0}$, integrating term by term we have

$$\frac{1}{|G_\lambda|}\int_{\mathbb{T}^\lambda}\chi_\lambda^\infty(AB^*)\prod_{\alpha\in\Phi_{G_\lambda}}(1-z_\lambda^{-\alpha})dz_\lambda = \langle A, B\rangle_\lambda^\infty$$

for $A, B \in \mathcal{J}$. As z_i has modulus 1, $\overline{(1-z_\lambda^{-\alpha})} = (1-z_\lambda^\alpha)$. Hence, the analytic version of the asymptotic Plancherel theorem is

$$\langle A, B\rangle^\infty = \sum_{\lambda\vdash n+1}\int_{\mathbb{T}^\lambda}\chi_\lambda^\infty(AB^*)d\mu_\lambda^\infty(z_\lambda) \quad \text{where} \quad d\mu_\lambda^\infty(z_\lambda) = \left|\prod_{\alpha\in\Phi_{G_\lambda}^+}(1-z_\lambda^{-\alpha})\right|^2 dz_\lambda.$$

Chapter 11

Lusztig's asymptotic algebra

In this chapter we give an explicit description of Lusztig's asymptotic algebra for type A_n, a culmination of the results of Chapters 7-10. To do so we will describe a subring of $\mathcal{J}_\lambda$ and use a result of Xi from [49]. Let

$$\mathcal{J}_{\Gamma_\lambda \cap \Gamma_\lambda^{-1}} = \mathrm{span}_{\mathbb{Z}}\{\mathsf{t}_w \mid w \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}\} \quad \text{and} \quad \mathfrak{C}_{\Gamma_\lambda \cap \Gamma_\lambda^{-1}} = \mathrm{span}_{\mathbb{Z}}\{\mathfrak{c}_\lambda(w) \mid w \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}\}.$$

As Γ_λ^{-1} is a left cell, $\mathcal{J}_{\Gamma_\lambda \cap \Gamma_\lambda^{-1}}$ is a ring (for example, this can be seen using $P8$). By Theorem 10.2.4 we then have that $\mathcal{J}_{\Gamma_\lambda \cap \Gamma_\lambda^{-1}} \cong \mathfrak{C}_{\Gamma_\lambda \cap \Gamma_\lambda^{-1}}$.

We state the following result of Xi that will be instrumental in the results of this chapter. The result reduces the understanding of $\mathcal{J}_\lambda$ to $\mathcal{J}_{\Gamma_\lambda \cap \Gamma_\lambda^{-1}}$. Fix bijections $\phi_{ij} : \Gamma_i \cap \Gamma_j^{-1} \to \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ as in [49, §2.3], and write $E_{ij}(a)$ for the matrix with a in the (i,j)-th position and zero elsewhere.

Theorem ([49, Theorems 2.3.2 and 8.4.2])**.** We have:

(1) The map $\mathsf{t}_w \mapsto \mathsf{t}_{\phi_{ii}(w)}$ induces a ring isomorphism $\mathcal{J}_{\Gamma_i \cap \Gamma_i^{-1}} \to \mathcal{J}_{\Gamma_\lambda \cap \Gamma_\lambda^{-1}}$.

(2) The map $\mathsf{t}_w \mapsto E_{ij}(\mathsf{t}_{\phi_{ij}(w)})$, for $w \in \Gamma_i \cap \Gamma_j^{-1}$, defines an isomorphism from $\mathcal{J}_\lambda$ to $\mathrm{Mat}_{N_\lambda}(\mathcal{J}_{\Gamma_\lambda \cap \Gamma_\lambda^{-1}})$.

In Section 11.1 we give a preliminary result about matrices and then prove that for $w \in \Gamma_i \cap \Gamma_j^{-1}$ the matrix $\mathfrak{c}_\lambda(w)$ has a unique non-zero entry. The placement of this entry is determined, and an ordering on ${}^\lambda W$ is set. In particular, it is shown that $w \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ if and only if the matrix $\mathfrak{c}_\lambda(w)$ has a nonzero entry in the $(\mathsf{u}_\lambda, \mathsf{u}_\lambda)$-th position. In Section 11.2 we introduce an inner product on $\mathbb{Z}[\zeta_\lambda]$ that reflects the inner product defined in Theorem 10.2.2 entry-wise (in terms of the entries of the matrices $\pi_\lambda^\infty(\cdot)$) and show that the G_λ-Schur functions are orthonormal with respect to this inner product. The main result of the section, and the chapter itself, is an explicit description of $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$. The set is shown to be equal to the set of maximal double coset representatives for the cosets $W_{\lambda'} \mathsf{u}_\lambda^{-1} \tau_\gamma \mathsf{u}_\lambda W_{\lambda'}$ (with $\gamma \in P_+^{(\lambda)}$) and it is shown that the non-zero entry of the leading matrices of these elements is a G_λ-Schur function. It follows from this description that $\mathcal{J}_{\Gamma_\lambda \cap \Gamma_\lambda^{-1}}$ is isomorphic to the ring $\mathbb{Z}[\zeta_\lambda]^{G_\lambda}$, giving a λ-relative version of [27, Proposition 8.6] (see also [38, Theorem 2.22(b)]) and recovering the main result of [49] (see Remark 11.2.13 for the difference between their result and the one given in Section 11.2).

Finally, in Section 11.3 we give the description of $\mathcal{J}_\lambda$ by explicitly describing $\mathfrak{C}_\lambda$ (as $\mathcal{J}_\lambda \cong \mathfrak{C}_\lambda$ by Theorem 10.2.4). It is shown that, from this description and from the results of Part II, that the matrix representations $(\pi_\lambda)_{\lambda \vdash n+1}$ form a balanced system of cell representations (in terms of the criteria listed in Section 3.3).

11.1 The set $\Gamma_i \cap \Gamma_j^{-1}$

In preparation for explicitly describing $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$, in this section we show that the matrix $\mathfrak{c}_\lambda(w)$ has a unique non-zero term when $w \in \Gamma_i \cap \Gamma_j^{-1}$, we prove the placement of this non-zero term, and we choose an ordering on ${}^\lambda W$.

Let $\mathrm{Mat}_N(\mathbb{Z}[\zeta_\lambda])$ be the algebra of $N \times N$ matrices with entries in $\mathbb{Z}[\zeta_\lambda]$ and for $A \in \mathrm{Mat}_N(\mathbb{Z}[\zeta_\lambda])$ let A^* denote the matrix formed by transposing A and performing the conjugation $\zeta_\lambda^\gamma \mapsto \zeta_\lambda^{-\gamma}$ entry-wise.

Lemma 11.1.1. *Let $A \in \mathrm{Mat}_N(\mathbb{Z}[\zeta_\lambda])$ be an idempotent matrix of rank 1 with $A^* = A$. Then $A = E_{k,k}$ for some $1 \leq k \leq N$.*

Proof. Think of A as an operator on the module $M = \mathbb{Z}[\zeta_\lambda]^N$ where elements of M are column vectors with entries in $\mathbb{Z}[\zeta_\lambda]$. Let $M_0 = \ker(A)$ and $M_1 = \{x \in M \mid Ax = x\}$ denote the 0-eigenspace and 1-eigenspace of A respectively. Let $x \in M$. We can write $x = (x - Ax) + Ax$. As A is idempotent we have that $A(x - Ax) = Ax - Ax = 0$ and so $x - Ax \in M_0$. In addition, we have that $A(Ax) = Ax$ and so $Ax \in M_1$. Thus, $M = M_0 \oplus M_1$. It follows that $\dim(M_1) = 1$ as $\mathrm{rank}(A) = 1$. Denote x_0 to be the generator of M_1. By definition $Ax_0 = x_0$ and thus $(Ax_0)^* = x_0^* A = x_0^*$ as $A^* = A$.

Let $y \in M = M_0 \oplus M_1$. We have $Ay = \mu_y x_0$ for some $\mu_y \in \mathbb{Z}[\zeta_\lambda]$ (with $\mu_y = 0$ if $y \in M_0$). Then, denoting $\nu = x_0^* x_0 \in \mathbb{Z}[\zeta_\lambda]$ we have

$$\mu_y \nu = \mu_y x_0^* x_0 = x_0^*(\mu_y x_0) = x_0^*(Ay) = x_0^* y.$$

Thus,

$$\nu A y = \nu \mu_y x_0 = x_0(\nu \mu_y) = x_0 x_0^* y$$

for all $y \in M$ and so $\nu A = x_0 x_0^*$. Write $x_0 = (x_1, \ldots, x_N)^T$ (where T denotes matrix transposition). As $x_0 \neq 0$ there is some $1 \leq k \leq N$ such that $x_k \neq 0$. Furthermore, we have that

$$\nu = x_0^* x_0 = x_1 \mathrm{conj}(x_1) + \cdots + x_N \mathrm{conj}(x_N)$$

where $\mathrm{conj}(x_i)$ is the conjugation of $x_i \in \mathbb{Z}[\zeta_\lambda]$ (as in Section 5.3). Thus, $\nu A = x_0 x_0^*$ implies that $\nu a_{kk} = x_k \mathrm{conj}(x_k)$ where $A = (a_{ij})$. For $f(\zeta_\lambda) \in \mathbb{Z}[\zeta_\lambda]$, writing $f(\zeta_\lambda) = \sum_\gamma c_\gamma \zeta_\lambda^\gamma$ with $c_\gamma \in \mathbb{Z}$, we have that $[f(\zeta_\lambda)\mathrm{conj}(f(\zeta_\lambda))]_{\mathrm{ct}} = \sum_\gamma c_\gamma^2$ (where ct is as in Section 10.1) and so $[f(\zeta_\lambda)\mathrm{conj}(f(\zeta_\lambda))]_{\mathrm{ct}}$ is strictly positive when $f(\zeta_\lambda) \neq 0$. Thus, for $1 \leq k \leq N$ such that $x_k \neq 0$ we have $[x_k \mathrm{conj}(x_k)]_{\mathrm{ct}} > 0$ and so $[\nu]_{\mathrm{ct}} \geq [x_k \mathrm{conj}(x_k)]_{\mathrm{ct}}$ with equality if and only if $x_i = 0$ for all $i \neq k$. Thus, as $a_{kk} \in \mathbb{Z}$ (because $A^* = A$) and as $\nu a_{kk} = x_k \mathrm{conj}(x_k)$ we have that $[\nu]_{\mathrm{ct}} \leq [a_{kk}\nu]_{\mathrm{ct}} = [x_k \mathrm{conj}(c_k)]_{\mathrm{ct}}$ and so $[\nu]_{\mathrm{ct}} = [x_k \mathrm{conj}(x_k)]_{\mathrm{ct}}$. Therefore, $x_i = 0$ for all $i \neq k$ and so $\nu = x_k \mathrm{conj}(x_k)$. On the other hand $x_0 x_0^* = x_k \mathrm{conj}(x_k) E_{k,k}$ and so the equation $\nu A = x_0 x_0^*$ gives that $A = E_{k,k}$ as required. □

Recall from Chapter 7 that $N_\lambda = \dim(\pi_\lambda)$ is the number of right cells within Δ_λ. Let $\Gamma_1, \ldots, \Gamma_{N_\lambda}$ denote these right cells and let $d_i \in \mathcal{D}$ be the unique distinguished involution such that $d_i \in \Gamma_i$ (see $P13$ in Section 3.2). We establish a specific ordering on the elements of ${}^\lambda W$, and thus the rows and columns of $\pi_\lambda(\widetilde{\mathcal{H}})$, as follows.

Theorem 11.1.2. *Let $w \in \Gamma_i \cap \Gamma_j^{-1}$. The elements of ${}^\lambda W$ can be ordered such that the matrix $\mathfrak{c}_\lambda(w)$ has a unique non-zero term, in position (i, j). With this ordering $\mathfrak{c}_\lambda(d_i) = E_{i,i}$.*

Proof. Let $A_i = \mathfrak{c}_\lambda(d_i)$ for $1 \le i \le N_\lambda$. By definition

$$\mathsf{t}_{d_i}\mathsf{t}_{d_j} = \sum_{z\in\widetilde{W}} \gamma_{d_i,d_j,z^{-1}}\mathsf{t}_z.$$

For $z \in \widetilde{W}$ such that $\gamma_{d_i,d_j,z^{-1}} \neq 0$ by $P7$ and $P2$ we have that $d_i = d_j = z$. Thus, if $i \neq j$ then $\mathsf{t}_{d_i}\mathsf{t}_{d_j} = 0$. In addition, $\mathsf{t}_{d_i}^2 = \gamma_{d_i,d_i,d_i}\mathsf{t}_{d_i} = \pm\mathsf{t}_{d_i}$ by $P5$ and positivity of the structure constants forces $\mathsf{t}_{d_i}^2 = \mathsf{t}_{d_i}$. It follows by $\mathfrak{C}_\lambda \cong \mathcal{J}_\lambda$ (Theorem 10.2.4) that the matrices $A_1, \dots, A_{N_\lambda} \in \mathrm{Mat}_{N_\lambda}(\mathbb{Z}[\zeta_\lambda])$ satisfy $A_iA_j = 0$ if $i \neq j$ and $A_i^2 = A_i$, and hence are pairwise commuting idempotent matrices. As A_i is idempotent the argument at the beginning of the proof of Lemma 11.1.1 gives that A_i is diagonalisable and has eigenvalues 1 and 0, and so as the A_i's are pairwise commuting matrices they are simultaneously diagonalisable. Hence, there exists $P \in \mathrm{Mat}_{N_\lambda}(\mathbb{Z}[\zeta_\lambda])$ and $D_1, \dots, D_{N_\lambda} \in \mathrm{Mat}_{N_\lambda}(\mathbb{Z}[\zeta_\lambda])$ such that $A_i = PD_iP^{-1}$ for all $1 \le i \le N_\lambda$.

The diagonal entries of D_i are 0 and 1 (as $A^2 = A$ implies that $D^2 = D$ or as 0 and 1 are the eigenvalues of A_i). Furthermore, as the A_i's are pairwise commuting idempotent matrices, $(\sum_{i=1}^{N_\lambda} A_i)^2 = \sum_{i=1}^{N_\lambda} A_i$ implies that $(\sum_{i=1}^{N_\lambda} D_i)^2 = \sum_{i=1}^{N_\lambda} D_i$. This forces the D_i's to have their 1 entries in different places and to only have one nonzero entry each, otherwise there exists some i, j such that $D_iD_j \neq 0$. Therefore, $D_i = E_{j,j}$ for some $1 \le j \le N_\lambda$ and so A_i has rank 1. By Lemma 5.3.7 and as $d_i^{-1} = d_i$ we have that $A_i^* = A_i$. Thus, A_i satisfies the hypothesis in Lemma 11.1.1 and so $A_i = E_{k_i,k_i}$ for some $1 \le k_i \le N_\lambda$. By the isomorphism in Theorem 10.2.4, the A_i matrices are all distinct and so the mapping $\pi : (1, \dots, N_\lambda) \mapsto (k_1, \dots, k_{N_\lambda})$ is bijective. Thus, we may order ${}^\lambda W$ so that $A_i = E_{i,i}$.

Suppose that $w \in \Gamma_i \cap \Gamma_j^{-1}$. Using $P2$ and $P7$ as we did above, we have that $\mathsf{t}_{d_i}\mathsf{t}_w = \sum_{z\in\widetilde{W}} \gamma_{d_i,w,z^{-1}}\mathsf{t}_z$ and if $\gamma_{d_i,w,z^{-1}} \neq 0$ then $w = z$. Thus, $\mathsf{t}_{d_i}\mathsf{t}_w = \mathsf{t}_w$, using the positivity of structure constant for equal parameters, $P5$ and the fact that $w \in \Gamma_i$. By Theorem 10.2.4 this then implies that $\mathfrak{c}_\lambda(d_i)\mathfrak{c}_\lambda(w) = \mathfrak{c}_\lambda(w)$ and so the non-zero elements of $\mathfrak{c}_\lambda(w)$ are forced to lie in the i-th row. Similarly, using $P2$, $P7$, $P5$ and the fact that $w \in \Gamma_j^{-1}$ we have that $\mathfrak{c}_\lambda(w)\mathfrak{c}_\lambda(d_j) = \mathfrak{c}_\lambda(w)$ which forces the non-zero elements of $\mathfrak{c}_\lambda(w)$ to be in the j-th row. Hence, $\mathfrak{c}_\lambda(w)$ has a unique non-zero entry in position (i, j). □

Corollary 11.1.3. *We have $w \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ if and only if the matrix $\mathfrak{c}_\lambda(w)$ has a non-zero entry in the $(\mathsf{u}_\lambda, \mathsf{u}_\lambda)$-position.*

Proof. The forward implication follows immediately from Theorem 11.1.2 and the fact that $\mathfrak{c}_\lambda(\mathsf{w}_{\lambda'}) = E_{\mathsf{u}_\lambda,\mathsf{u}_\lambda}$ (see Proposition 7.3.8). For the reverse implication, let $\mathfrak{c}_\lambda(w)$ have a non-zero entry in the $(\mathsf{u}_\lambda, \mathsf{u}_\lambda)$-position. As $\mathfrak{c}_\lambda(\mathsf{w}_{\lambda'}) = E_{\mathsf{u}_\lambda,\mathsf{u}_\lambda}$ it follows that $\mathfrak{c}_\lambda(w)\mathfrak{c}_\lambda(\mathsf{w}_{\lambda'}) \neq 0$ and $\mathfrak{c}_\lambda(\mathsf{w}_{\lambda'})\mathfrak{c}_\lambda(w) \neq 0$. By Theorem 10.2.4 this implies that there exists $z \in \widetilde{W}$ such that $\gamma_{\mathsf{w}_{\lambda'},w,z^{-1}} \neq 0$ and $\gamma_{w,\mathsf{w}_{\lambda'},z^{-1}} \neq 0$. By $P7$ we then have $\gamma_{w,z^{-1},\mathsf{w}_{\lambda'}} \neq 0$ and $\gamma_{z^{-1},w,\mathsf{w}_{\lambda'}} \neq 0$, which in turn imply that $\mathsf{w}_{\lambda'} \le_R w$ and $\mathsf{w}_{\lambda'} \le_L w$. Thus, with $P9$ and $P10$ and as $\mathbf{a}(w) = \mathsf{w}_{\lambda'} = \ell(\mathsf{w}_{\lambda'})$ we have $w \sim_L \mathsf{w}_{\lambda'}$, which implies $w^{-1} \sim_R \mathsf{w}_{\lambda'}$, and $w \sim_R \mathsf{w}_{\lambda'}$ as required. □

11.2 The set $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$ and the ring $\mathcal{J}_{\Gamma_\lambda \cap \Gamma_\lambda^{-1}}$

Recall the definition of the G_λ-Schur functions $\mathfrak{s}_\gamma(\zeta_\lambda)$ from Definition 7.3.5, the monomials $\mathfrak{e}_\gamma(\zeta_\lambda)$ from (7.3.1) and the λ-dominance order on P/Q_λ (and consequently $P^{(\lambda)}$) from Section 7.2. In this section we define an inner product on elements of $\mathbb{Z}[\zeta_\lambda]$ connected to the inner product on $\mathcal{J}$ defined in Theorem 10.2.2 and show that the G_λ-Schur functions are orthonormal with respect to this inner product. We then prove that these Schur functions are the contents of the leading matrices of particular double coset representatives. Finally we prove that $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$ is exactly the set of these double coset representatives and that $\mathcal{J}_{\Gamma_\lambda \cap \Gamma_\lambda^{-1}}$ is isomorphic to $\mathbb{Z}[\zeta_\lambda]^{G_\lambda}$.

Definition 11.2.1. For $f(\zeta_\lambda), g(\zeta_\lambda) \in \mathbb{Z}[\zeta_\lambda]$ define

$$\langle f(\zeta_\lambda), g(\zeta_\lambda)\rangle_\lambda^\infty = \frac{1}{|G_\lambda|}\left[f(\zeta_\lambda)\cdot \mathrm{conj}(g(\zeta_\lambda))\cdot \prod_{\alpha\in\Phi_{G_\lambda}}(1-\zeta_\lambda^\alpha)\right]_{\mathrm{ct}}.$$

By Corollary 11.1.3 if $A, B \in \mathcal{J}_{\Gamma_\lambda\cap\Gamma_\lambda^{-1}}$ then

$$\pi_\lambda^\infty(A) = a(\zeta_\lambda)E_{\mathsf{u}_\lambda,\mathsf{u}_\lambda} \quad \text{and} \quad \pi_\lambda^\infty(B) = b(\zeta_\lambda)E_{\mathsf{u}_\lambda,\mathsf{u}_\lambda}$$

for some $a(\zeta_\lambda), b(\zeta_\lambda) \in \mathbb{Z}[\zeta_\lambda]$ where π_λ^∞ is the $\mathbb{Z}$-linear extension of $\pi_\lambda^\infty(\mathsf{t}_w) = \mathfrak{c}_\lambda(w)$, as in Section 10.2. Thus,

$$\langle A, B\rangle_\lambda^\infty = \langle a(\zeta_\lambda), b(\zeta_\lambda)\rangle_\lambda^\infty$$

where $\langle\cdot,\cdot\rangle_\lambda^\infty : \mathcal{J}\times\mathcal{J}\to\mathbb{Z}$ is as in Definition 3.4.3.

Lemma 11.2.2. *The Schur functions $\mathfrak{s}_\gamma(\zeta_\lambda)$ with $\gamma \in (P/Q_\lambda)_+$, are the unique elements of $\mathbb{Z}[\zeta_\lambda]^{G_\lambda}$ satisfying:*
(1) $\mathfrak{s}_\gamma(\zeta_\lambda) = \mathfrak{e}_\gamma(\zeta_\lambda) + \sum_{\gamma'\prec_\lambda\gamma} a_{\gamma,\gamma'}\mathfrak{e}_{\gamma'}(\zeta_\lambda)$ *with* $a_{\gamma,\gamma'}\in\mathbb{Z}$, *and*
(2) $\langle \mathfrak{s}_\gamma(\zeta_\lambda), \mathfrak{s}_{\gamma'}(\zeta_\lambda)\rangle_\lambda^\infty = \delta_{\gamma,\gamma'}$.
Moreover, we have $a_{\gamma,\gamma'}\geq 0$.

Proof. The fact that the Schur functions satisfy (1) and (2) is classical. For example taking P and W_0 to be $P_+^{(\lambda)}$ and G_λ in [38, (3.14)] we have

$$\mathfrak{s}_\gamma(\zeta_\lambda) = \sum_{\gamma'\preccurlyeq_\lambda\gamma} K_{\gamma,\gamma'}\mathfrak{e}_{\gamma'}(\zeta_\lambda)$$

where $K_{\gamma,\gamma'}$ are the Kostka numbers. It is well known that $K_{\gamma,\gamma'}\geq 0$ and $K_{\gamma,\gamma}=1$, so $\mathfrak{s}_\lambda(\zeta_\lambda)$ satisfies (1) taking $a_{\gamma,\gamma'} = K_{\gamma,\gamma'}$. Furthermore, $\mathfrak{s}_\lambda(\zeta_\lambda)$ satisfies (2) by [38, Proposition 3.4].

To prove uniqueness, suppose that $\mathfrak{s}'_\gamma(\zeta_\lambda)$ satisfies (1) and (2) and that the $\mathfrak{s}'_{\gamma'}(\zeta_\lambda)$ are determined for all $\gamma'\prec_\lambda\gamma$. By (1) and (2) it follows that $\{\mathfrak{s}'_{\gamma'}(\zeta_\lambda) \mid \gamma'\prec_\lambda\gamma\}$ is an orthonormal basis of the G_λ invariant functions spanned by $\{\mathfrak{e}_\lambda(\zeta_\lambda)\mid\gamma'\prec_\lambda\gamma\}$. As $\mathfrak{s}'_\gamma(\zeta_\lambda)$ satisfies (1) we have that

$$\mathfrak{s}'_\gamma(\zeta_\lambda) = \mathfrak{e}_\gamma(\zeta_\lambda) + \sum_{\gamma'\prec_\lambda\gamma} b_{\gamma,\gamma'}\mathfrak{s}'_{\gamma'}(\zeta_\lambda).$$

Applying $\langle\cdot,\mathfrak{s}'_{\gamma'}(\zeta_\lambda)\rangle_\lambda^\infty$ for $\gamma'\prec_\lambda\gamma$ on either side and applying (2) gives that the integers $b_{\gamma,\gamma'}$ are uniquely determined by $b_{\gamma,\gamma'} = -\langle\mathfrak{e}_\gamma(\zeta_\lambda),\mathfrak{s}'_{\gamma'}(\zeta_\lambda)\rangle_\lambda^\infty$. As $a_{\gamma,\gamma'} = -\langle\mathfrak{e}_\gamma(\zeta_\lambda),\mathfrak{s}_{\gamma'}(\zeta_\lambda)\rangle_\lambda^\infty$ we have $\mathfrak{s}_\gamma(\zeta_\lambda) = \mathfrak{s}'_\gamma(\zeta_\lambda)$ as required. □

We introduce the following maximal length coset representatives, which will be shown to explicitly describe $\Gamma_\lambda\cap\Gamma_\lambda^{-1}$ in Theorem 11.2.10.

Definition 11.2.3. For $\gamma\in P^{(\lambda)}$ let m_γ be the longest element of the double coset $W_{\lambda'}\mathsf{u}_\lambda^{-1}\tau_\gamma\mathsf{u}_\lambda W_{\lambda'}$.

Proposition 11.2.4. *If $\gamma\in P^{(\lambda)}$ and $g\in G_\lambda$ then $\mathsf{m}_{g\gamma} = \mathsf{m}_\gamma$.*

Proof. By Lemma 2.4.5 we have that $g\tau_\gamma g^{-1} = \tau_{g\gamma}$ for all $g\in G_\lambda$, and so

$$\mathsf{u}_\lambda^{-1}\tau_{g\gamma}\mathsf{u}_\lambda = (\mathsf{u}_\lambda^{-1}g\mathsf{u}_\lambda)(\mathsf{u}_\lambda^{-1}\tau_\gamma\mathsf{u}_\lambda)(\mathsf{u}_\lambda^{-1}g^{-1}\mathsf{u}_\lambda).$$

By Proposition 7.2.12 $\mathsf{u}_\lambda^{-1}g\mathsf{u}_\lambda\in W_{\lambda'}$, thus $\mathsf{u}_\lambda^{-1}\tau_{g\gamma}\mathsf{u}_\lambda\in W_{\lambda'}\mathsf{u}_\lambda^{-1}\tau_\gamma\mathsf{u}_\lambda W_{\lambda'}$ and hence the result. □

Theorem 11.2.5. *Let $\gamma \in P^{(\lambda)}$. There exist $x, y \in W_{\lambda'}$ such that $\mathsf{m}_\gamma = x\mathsf{u}_\lambda^{-1}\tau_\gamma\mathsf{u}_\lambda y$ with*

$$\ell(\mathsf{m}_\gamma) = \ell(x) + \ell(\mathsf{u}_\lambda^{-1}\tau_\gamma\mathsf{u}_\lambda) + \ell(y) \quad \text{and} \quad \ell(x) + \ell(y) = \ell(\mathsf{w}_{\lambda'}).$$

If $\gamma, \gamma' \in P_+^{(\lambda)}$ with $\gamma + Q_\lambda \preccurlyeq_\lambda \gamma' + Q_\lambda$ then $\ell(\mathsf{m}_\gamma) \leq \ell(\mathsf{m}_{\gamma'})$ with equality if and only if $\gamma = \gamma'$.

Proof. See Appendix A. □

Proposition 11.2.6. *If $\gamma \in P_+^{(\lambda)}$ then there is an integer $c > 0$ such that*

$$[\mathfrak{c}_\lambda(\mathsf{m}_\gamma)]_{\mathsf{u}_\lambda,\mathsf{u}_\lambda} = c\zeta_\lambda^\gamma + (\mathbb{Z}\text{-linear combination of terms } \zeta_\lambda^{\gamma'} \text{ with } \gamma \not\preccurlyeq_\lambda \gamma').$$

Thus, $\pi_\lambda(T_{\mathsf{m}_\gamma})$ attains the bound $\ell(\mathsf{w}_{\lambda'})$.

Proof. By Theorem 11.2.5 $\mathsf{m}_\gamma = x\mathsf{u}_\lambda^{-1}\tau_\gamma\mathsf{u}_\lambda y$ for some $x, y \in W_{\lambda'}$ with $\ell(\mathsf{m}_\gamma) = \ell(x)+\ell(\mathsf{u}_\lambda^{-1}\tau_\gamma\mathsf{u}_\lambda)+\ell(y)$ and $\ell(x) + \ell(y) = \ell(\mathsf{w}_{\lambda'})$. Let p_0 be the path starting at u_λ of type $\vec{\mathsf{m}}_\gamma = x \cdot (\mathsf{u}_\lambda^{-1}\tau_\gamma\mathsf{u}_\lambda) \cdot y$, (choosing any reduced expressions for x, y and $\mathsf{u}_\lambda^{-1}\tau_\gamma\mathsf{u}_\lambda$) such that the first $\ell(x)$ steps are folds, the next $\ell(\mathsf{u}_\lambda^{-1}\tau_\gamma\mathsf{u}_\lambda)$ steps are crossings and the final $\ell(y)$ steps are folds. We claim that p_0 is a J_λ-folded alcove path.

First, as $\ell(\mathsf{u}_\lambda s_j) = \ell(\mathsf{u}_\lambda) + 1$ and $\mathsf{u}_\lambda s_j \in {}^\lambda W$ for all $j \in J_{\lambda'}$ (see Theorem 7.2.9(2)), the first $\ell(x)$ steps are positive folds (and not bounces) given that $x \in W_{\lambda'}$. The next $\ell(\mathsf{u}_\lambda^{-1}\tau_\gamma\mathsf{u}_\lambda)$ steps must remain in $\mathcal{A}_\lambda$ to avoid forced bounces. First note that the starting alcove $\mathsf{u}_\lambda A_0 \subseteq \mathcal{A}_\lambda$ as $\mathsf{u}_\lambda \in {}^\lambda W$. By Theorem 2.3.8 the end alcove $\mathsf{u}_\lambda(\mathsf{u}_\lambda^{-1}\tau_\gamma\mathsf{u}_\lambda)A_0 = \tau_\gamma\mathsf{u}_\lambda A_0 \subseteq \mathcal{A}_\lambda$ as well. By [1, Proposition 3.94], $\mathcal{A}_\lambda$ is convex as it is an intersection of half-spaces. Thus, reduced paths beginning and ending in $\mathcal{A}_\lambda$ remain in $\mathcal{A}_\lambda$, so the path from $\mathsf{u}_\lambda A_0$ to $\tau_\gamma\mathsf{u}_\lambda A_0$ of reduced type $\mathsf{u}_\lambda^{-1}\tau_\gamma\mathsf{u}_\lambda$ remains within $\mathcal{A}_\lambda$. Finally, as $\mathsf{u}_\lambda s_j \in {}^\lambda W$ for all $j \in J_{\lambda'}$ (by Theorem 7.2.9(2)), $\tau_\gamma\mathsf{u}_\lambda s_j A_0 \subseteq \mathcal{A}_\lambda$ by Theorem 2.3.8 and so the last $\ell(y)$ folds are not forced bounces. It remains to show that these folds are positively oriented. By Definition 2.3.1 we have $\tau_\gamma\mathsf{u}_\lambda = t_\gamma\mathsf{y}_\gamma\mathsf{u}_\lambda$. In addition, $\ell(\mathsf{y}_\gamma\mathsf{u}_\lambda s_j) = \ell(\mathsf{y}_\gamma\mathsf{u}_\lambda) + 1$ for all $j \in J_{\lambda'}$ as otherwise we have a contradiction with $\mathsf{u}_\lambda \in {}^\lambda W$ (using [1, pg.79 (F)]). So $t_\gamma\mathsf{y}_\gamma\mathsf{u}_\lambda\alpha_{s_j} \in \Phi^+ + \mathbb{Z}\delta$ and thus, by (1.3.1) and the fact that $y \in W_{\lambda'}$, this implies that the folds are positively oriented.

Since $\ell(x) + \ell(y) = \ell(\mathsf{w}_{\lambda'})$ we have that $\mathcal{Q}_{J_\lambda}(p_0) = (\mathsf{q} - \mathsf{q}^{-1})^{\ell(\mathsf{w}_{\lambda'})}$. In addition, as $\mathrm{end}(p_0) = \tau_\gamma\mathsf{u}_\lambda$ we have $\mathrm{wt}(p_0) = \gamma$ and $\theta^\lambda(p_0) = \mathsf{u}_\lambda$. Hence, by Theorem 5.3.3

$$[\pi_\lambda(T_{\mathsf{m}_\gamma})]_{\mathsf{u}_\lambda,\mathsf{u}_\lambda} = (\mathsf{q} - \mathsf{q}^{-1})^{\ell(\mathsf{w}_{\lambda'})}\zeta_\lambda^\gamma + \sum_{p \in \mathcal{P}_{J_\lambda}(\vec{\mathsf{m}}_\gamma, \mathsf{u}_\lambda)_{\mathsf{u}_\lambda} \setminus \{p_0\}} \mathcal{Q}_{J_\lambda}(p)\zeta_\lambda^{\mathrm{wt}(p)}.$$

By Theorem 8.2.1 the bound of π_λ is $\ell(\mathsf{w}_{\lambda'})$ and so multiplying the above equation by $\mathsf{q}^{-\ell(\mathsf{w}_{\lambda'})}$ gives

$$\mathsf{q}^{-\ell(\mathsf{w}_{\lambda'})}[\pi_\lambda(T_{\mathsf{m}_\gamma})]_{\mathsf{u}_\lambda,\mathsf{u}_\lambda} = \zeta_\lambda^\gamma + (\mathbb{Z}[\mathsf{q}^{-1}]\text{-linear combination of terms } \zeta_\lambda^{\gamma'} \text{ with } \gamma' \in P).$$

Thus, it remains to prove that if $p \in \mathcal{P}_{J_\lambda}(\vec{\mathsf{m}}_\gamma, \mathsf{u}_\lambda)_{\mathsf{u}_\lambda}$ with $\mathrm{wt}(p) = \gamma'$ and $\gamma \preccurlyeq_\lambda \gamma'$ then $\deg \mathcal{Q}_{J_\lambda}(p) \leq \ell(\mathsf{w}_{\lambda'})$ and if equality holds then $\gamma = \gamma'$ and $\mathcal{Q}_{J_\lambda}(p)$ has a positive leading coefficient.

Let $p \in \mathcal{P}_{J_\lambda}(\vec{\mathsf{m}}_\gamma, \mathsf{u}_\lambda)_{\mathsf{u}_\lambda}$ with $\mathrm{wt}(p) = \gamma'$. By Corollary 2.3.10 $\theta^\lambda(p) = \mathsf{u}_\lambda$ and so $\mathrm{end}(p) = \tau_{\gamma'}\mathsf{u}_\lambda$. Let $N = f(p) + b(p)$ be the total number of folds and bounces in p. Writing $\mathsf{m}_\gamma = s_{i_1}s_{i_2}\cdots s_{i_l}\sigma^k$ $(0 \leq k \leq n)$ we have $\mathrm{end}(p) = \tau_{\gamma'}\mathsf{u}_\lambda = \mathsf{u}_\lambda s_{i_1}s_{i_2}\cdots\hat{s}_{i_{j_1}}\cdots\hat{s}_{i_{j_N}}\cdots s_{i_l}\sigma^k$ where the folds and bounces of p occur at the indices $1 \leq j_1 < \cdots < j_N \leq l$ and $\hat{s}_{i_{j_k}}$ indicates the omission of the generator in the expression. Thus, $\ell(\mathsf{u}_\lambda^{-1}\tau_{\gamma'}\mathsf{u}_\lambda) \leq \ell(\mathsf{m}_\gamma) - N$, and so $N \leq \ell(\mathsf{m}_\gamma) - \ell(\mathsf{u}_\lambda^{-1}\tau_{\gamma'}\mathsf{u}_\lambda)$. Furthermore, as $\ell(\mathsf{m}_{\gamma'}) = \ell(\mathsf{u}_\lambda^{-1}\tau_{\gamma'}\mathsf{u}_\lambda) + \ell(\mathsf{w}_{\lambda'})$ we have

$$N \leq \ell(\mathsf{w}_{\lambda'}) + \ell(\mathsf{m}_\gamma) - \ell(\mathsf{m}_{\gamma'}).$$

By Definition 4.3.3 $\mathcal{Q}_{J_\lambda}(p) = (-\mathsf{q}^{-1})^{-b(p)}(\mathsf{q} - \mathsf{q}^{-1})^{f(p)}$ and so

$$\deg \mathcal{Q}_{J_\lambda}(p) = f(p) - b(p) < N \leq \ell(\mathsf{w}_{\lambda'}) + \ell(\mathsf{m}_\gamma) - \ell(\mathsf{m}_{\gamma'}),$$

with equality if and only if $f(p) = \ell(\mathsf{w}_{\lambda'}) + \ell(\mathsf{m}_\gamma) - \ell(\mathsf{m}_{\gamma'})$ and $b(p) = 0$. The result follows from Theorem 11.2.5 as when $\gamma \preccurlyeq_\lambda \gamma'$ we have $\ell(\mathsf{m}_\gamma) \leq \ell(\mathsf{m}_{\gamma'})$ with equality if and only if $\gamma = \gamma'$. □

Definition 11.2.7. For $w \in \widetilde{W}$ let $\mathfrak{f}_w(\zeta_\lambda) = \mathsf{q}^{-\ell(\mathsf{w}_{\lambda'})}[\pi_\lambda(C_w)]_{\mathsf{u}_\lambda,\mathsf{u}_\lambda}$.

Note that by Theorem 8.2.1 we have that $\mathfrak{f}_w(\zeta_\lambda) \in (\mathbb{Z}[\mathsf{q}^{-1}])[\zeta_\lambda]$ and by Corollary 11.1.3 if $\mathfrak{f}_w(\zeta_\lambda) \neq 0$ then $w \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$.

Recall the definition of $f_\lambda(h)$ from Definition 9.0.1.

Proposition 11.2.8. *If $w \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ we have*

$$\pi_\lambda(C_w) = \mathfrak{f}_w(\zeta_\lambda)\pi_\lambda(C_{\mathsf{w}_{\lambda'}}) \quad \textit{with} \quad \mathfrak{f}_w(\zeta_\lambda) = \frac{\mathsf{q}^{2\ell(\mathsf{w}_{\lambda'})}}{W_{\lambda'}(\mathsf{q}^2)^2} f_\lambda(C_w)$$

and $\mathfrak{f}_w(\zeta_\lambda) \in \mathbb{Z}[\zeta_\lambda]^{G_\lambda}$. In particular, $\mathfrak{f}_{\mathsf{m}_\gamma}(\zeta_\lambda) = \mathfrak{s}_\gamma(\zeta_\lambda)$ for $\gamma \in P^{(\lambda)}$ and so $\mathfrak{c}_\lambda(\mathsf{m}_\gamma) = \mathfrak{s}_\gamma(\zeta_\lambda)E_{\mathsf{u}_\lambda,\mathsf{u}_\lambda}$.

Proof. If $w \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ then by Theorem 11.1.2 the matrix $\pi_\lambda(C_w)$ attains the bound $\ell(\mathsf{w}_{\lambda'})$ in the $(\mathsf{u}_\lambda, \mathsf{u}_\lambda)$-position and only in this position. As $w \in \Gamma_\lambda$, by [25, Proposition 2.4] we have that $D_L(w) = J_{\lambda'}$ and so $w = \mathsf{w}_{\lambda'}u$ with u reduced on the left by $J_{\lambda'}$. This implies that $\ell(\mathsf{w}_{\lambda'}w) = \ell(w) - \ell(\mathsf{w}_{\lambda'})$. By a similar argument, as $w \in \Gamma_\lambda^{-1}$, we also have that $\ell(w\mathsf{w}_{\lambda'}) = \ell(w) - \ell(\mathsf{w}_{\lambda'})$. Thus, by Corollary 3.2.3 we have

$$C_{\mathsf{w}_{\lambda'}}C_wC_{\mathsf{w}_{\lambda'}} = \mathsf{q}^{-2\ell(\mathsf{w}_{\lambda'})}W_{\lambda'}(\mathsf{q}^2)^2C_w.$$

With Theorem 9.0.2 this gives that

$$\pi_\lambda(C_{\mathsf{w}_{\lambda'}}C_wC_{\mathsf{w}_{\lambda'}}) = \mathsf{q}^{-2\ell(\mathsf{w}_{\lambda'})}W_{\lambda'}(\mathsf{q}^2)^2\pi_\lambda(C_w) = f_\lambda(C_w)\pi_\lambda(C_{\mathsf{w}_{\lambda'}}). \tag{11.2.1}$$

Reading the $(\mathsf{u}_\lambda, \mathsf{u}_\lambda)$-entry, using Proposition 7.3.8 for $\pi_\lambda(C_{\mathsf{w}_{\lambda'}})$, we have $\mathsf{q}^{-\ell(\mathsf{w}_{\lambda'})}W_{\lambda'}(\mathsf{q}^2)^2\mathfrak{f}_w(\zeta_\lambda) = \mathsf{q}^{\ell(\mathsf{w}_{\lambda'})}f_\lambda(C_w)$. Thus, $f_\lambda(C_w)$ is divisible (in $\mathsf{R}[\zeta_\lambda]$) by $W_{\lambda'}(\mathsf{q}^2)^2$, we have

$$\mathfrak{f}_w(\zeta_\lambda) = \frac{\mathsf{q}^{2\ell(\mathsf{w}_{\lambda'})}}{W_{\lambda'}(\mathsf{q}^2)^2} f_\lambda(C_w)$$

and (11.2.1) becomes $\pi_\lambda(C_w) = \mathfrak{f}_w(\zeta_\lambda)\pi_\lambda(C_{\mathsf{w}_{\lambda'}})$. As $\mathsf{q}^{\ell(\mathsf{w}_{\lambda'})}W_{\lambda'}(\mathsf{q}^{-2})C_{\mathsf{w}_{\lambda'}} = \mathsf{q}^{-\ell(\mathsf{w}_{\lambda'})}W_{\lambda'}(\mathsf{q}^2)C_{\mathsf{w}_{\lambda'}}$ (see Section 3.2), by Corollary 9.0.4 we have $\overline{\mathfrak{f}_w(\zeta_\lambda)} = \mathfrak{f}_w(\zeta_\lambda)$. Thus, since $\mathfrak{f}_w(\zeta_\lambda) \in (\mathbb{Z}[\mathsf{q}^{-1}])[\zeta_\lambda]$ this forces $\mathfrak{f}_w(\zeta_\lambda) \in \mathbb{Z}[\zeta_\lambda]$ and by Theorem 9.0.7 $\mathfrak{f}_w(\zeta_\lambda) \in \mathbb{Z}[\zeta_\lambda]^{G_\lambda}$.

Let $\gamma \in P^{(\lambda)}$. It remains to show that $\mathfrak{f}_{\mathsf{m}_\gamma}(\zeta_\lambda) = \mathfrak{s}_\gamma(\zeta_\lambda)$. By Proposition 11.2.4 $\mathsf{m}_{g\gamma} = \mathsf{m}_\gamma$ for all $g \in G_\lambda$ so we may assume that $\gamma \in P_+^{(\lambda)}$ as $P_+^{(\lambda)}$ is a fundamental domain for the action of G_λ on $P^{(\lambda)}$ (see Section 7.2). By Proposition 11.2.6 we have

$$\mathfrak{f}_{\mathsf{m}_\gamma}(\zeta_\lambda) = c\zeta_\lambda^\gamma + (\mathbb{Z}\text{-linear combination of terms } \zeta_\lambda^{\gamma'} \text{ with } \gamma \not\preccurlyeq_\lambda \gamma'), \tag{11.2.2}$$

where $c > 0$ is an integer. As $\deg[\pi_\lambda(T_{\mathsf{m}_\gamma})]_{\mathsf{u}_\lambda,\mathsf{u}_\lambda} = \ell(\mathsf{w}_{\lambda'})$ (by Proposition 11.2.6) we have that $\mathsf{m}_\gamma \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ by Corollary 11.1.3 and so $\pi_\lambda^\infty(\mathsf{t}_{\mathsf{m}_\gamma}) = c_\lambda(\mathsf{m}_\gamma) = \mathfrak{f}_{\mathsf{m}_\gamma}(\zeta_\lambda)E_{\mathsf{u}_\lambda,\mathsf{u}_\lambda}$ by Theorem 11.1.2. Thus,

$$\langle \mathfrak{f}_{\mathsf{m}_\gamma}(\zeta_\lambda), \mathfrak{f}_{\mathsf{m}_\gamma}(\zeta_\lambda)\rangle_\lambda^\infty = \langle \mathsf{t}_{\mathsf{m}_\gamma}, \mathsf{t}_{\mathsf{m}_\gamma}\rangle_\lambda^\infty = 1$$

by Theorem 10.2.2. As $\mathfrak{f}_\gamma(\zeta_\lambda) \in \mathbb{Z}[\zeta_\lambda]^{G_\lambda}$ and as the Schur functions $(\mathfrak{s}_\gamma(\zeta_\lambda))_{\gamma \in P_+^{(\lambda)}}$ are a $\mathbb{Z}$-basis of this ring we have that

$$\mathfrak{f}_{\mathsf{m}_\gamma}(\zeta_\lambda) = \sum_{\gamma' \in (P/Q_\lambda)_+} a_{\gamma'}\mathfrak{s}_{\gamma'}(\zeta_\lambda)$$

for some integers $a_{\gamma'}$. As $\langle \mathfrak{f}_{\mathsf{m}_\gamma}(\zeta_\lambda), \mathfrak{f}_{\mathsf{m}_\gamma}(\zeta_\lambda)\rangle_\lambda^\infty = 1$ we have $\sum_{\gamma'} a_{\gamma'}^2 = 1$ by Lemma 11.2.2(2) and as the $a_{\gamma'}$'s are integers this forces $\mathfrak{f}_{\mathsf{m}_\gamma}(\zeta_\lambda) = \pm\mathfrak{s}_{\gamma_1}(\zeta_\lambda)$ for some $\gamma_1 \in P_+^{(\lambda)}$. By Lemma 11.2.2(1) we have that

$$\mathfrak{s}_{\gamma_1}(\zeta_\lambda) = \mathfrak{e}_{\gamma_1}(\zeta_\lambda) + \sum_{\gamma' \prec_\lambda \gamma_1} a_{\gamma_1,\gamma'}\mathfrak{e}_{\gamma'}(\zeta_\lambda) = (\text{a } \mathbb{Z}_{\geq 0}\text{-linear combinations of } \zeta_\lambda^{\gamma'} \text{ with } \gamma' \in P^{(\lambda)}),$$

and as $c > 0$ in (11.2.2) it follows that $\mathfrak{f}_{\mathsf{m}_\gamma}(\zeta_\lambda) = \mathfrak{s}_{\gamma_1}(\zeta_\lambda)$. Furthermore, Lemma 11.2.2(1) gives that $\gamma \preccurlyeq_\lambda \gamma_1$ which, with (11.2.2), forces $\gamma = \gamma_1$ (as $\gamma \in P_+^{(\lambda)}$). □

The following Corollary is used to prove Theorem 9.0.10.

Corollary 11.2.9. *The map $f_\lambda : \widetilde{\mathcal{H}} \to \mathsf{R}[\zeta_\lambda]^{G_\lambda}$ is surjective.*

Proof. By Proposition 11.2.8 we have

$$\mathfrak{f}_{\mathsf{m}_\gamma}(\zeta_\lambda) = \mathfrak{s}_\gamma(\zeta_\lambda) = \frac{\mathsf{q}^{2\ell(\mathsf{w}_{\lambda'})}}{W_{\lambda'}(\mathsf{q}^2)^2} f_\lambda(C_{\mathsf{m}_\gamma}).$$

Therefore, $f_\lambda(C_{\mathsf{m}_\gamma}) = \mathsf{q}^{-2\ell(\mathsf{w}_{\lambda'})}W_{\lambda'}(\mathsf{q}^2)^2\mathfrak{s}_\gamma(\zeta_\lambda)$ and the result follows as the Schur functions form a basis of $\mathsf{R}[\zeta_\lambda]^{G_\lambda}$. □

Theorem 11.2.10. *We have $\Gamma_\lambda \cap \Gamma_\lambda^{-1} = \{\mathsf{m}_\gamma \mid \gamma \in P_+^{(\lambda)}\}$, and the linear map*

$$\mathcal{J}_{\Gamma_\lambda \cap \Gamma_\lambda^{-1}} \to \mathbb{Z}[\zeta_\lambda]^{G_\lambda} \quad \textit{with} \quad \mathsf{t}_{\mathsf{m}_\gamma} \mapsto \mathfrak{s}_\gamma(\zeta_\lambda)$$

is an isomorphism of unital rings.

Proof. By Proposition 11.2.6 $\deg[\pi_\lambda(T_{\mathsf{m}_\gamma})]_{\mathsf{u}_\lambda,\mathsf{u}_\lambda} = \ell(\mathsf{w}_{\lambda'})$ and so, by Corollary 11.1.3, we have that $\mathsf{m}_\gamma \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ for all $\gamma \in P_+^{(\lambda)}$. Thus, $\{\mathsf{m}_\gamma \mid \gamma \in P_+^{(\lambda)}\} \subseteq \Gamma_\lambda \cap \Gamma_\lambda^{-1}$.

To prove the reverse containment, let $w \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$. By Theorem 11.1.2 and Proposition 11.2.8 we have that $\pi_\lambda^\infty(\mathsf{t}_w) = \mathfrak{c}_\lambda(w) = \mathfrak{f}_w(\zeta_\lambda)E_{\mathsf{u}_\lambda,\mathsf{u}_\lambda}$ and that $\mathfrak{f}_w(\zeta_\lambda) \in \mathbb{Z}[\zeta_\lambda]^{G_\lambda}$. If $w \notin \{\mathsf{m}_\gamma \mid \gamma \in P_+^{(\lambda)}\}$ then by Theorem 10.2.2 and as $\pi_\lambda^\infty(\mathsf{t}_{\mathsf{m}_\gamma}) = \mathfrak{s}_\gamma(\zeta_\lambda)E_{\mathsf{u}_\lambda,\mathsf{u}_\lambda}$ by Proposition 11.2.8 we have

$$\langle \mathfrak{f}_w(\zeta_\lambda), \mathfrak{s}_\gamma(\zeta_\lambda)\rangle_\lambda^\infty = \langle \mathsf{t}_w, \mathsf{t}_{\mathsf{m}_\gamma}\rangle_\lambda^\infty = 0$$

for all $\gamma \in P_+^{(\lambda)}$. This contradicts the fact that $(\mathfrak{s}_\gamma(\zeta_\lambda))_{\gamma \in P_+^{(\lambda)}}$ forms a basis of $\mathbb{Z}[\zeta_\lambda]^{G_\lambda}$ and so $w = \mathsf{m}_\gamma$ for some $\gamma \in P_+^{(\lambda)}$. Thus, $\{\mathsf{m}_\gamma \mid \gamma \in P_+^{(\lambda)}\} = \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ as required.

Hence, the mapping $\mathfrak{C}_{\Gamma_\lambda \cap \Gamma_\lambda^{-1}} \to \mathbb{Z}[\zeta_\lambda]^{G_\lambda}$ with $\mathfrak{c}_\lambda(\mathsf{m}_\gamma) \mapsto \mathfrak{s}_\gamma(\zeta_\lambda)$ is a ring isomorphism and the result follows from Theorem 10.2.4. □

The following remarks can be made about Proposition 11.2.8 and Theorem 11.2.10.

Remark 11.2.11. Recall the analytic description of the asymptotic Plancherel formula given in (10.1.1). The constant of each term in the sum of $\langle h_1, h_2\rangle$ from (10.1.1) is forced to be $\frac{1}{|G_\lambda|}$ by Proposition 11.2.8 and Lemma 11.2.2. To see this let $\langle f(\zeta_\lambda), g(\zeta_\lambda)\rangle_\lambda^\infty$ be as defined in Definition 11.2.1 and define

$$(f(\zeta_\lambda), g(\zeta_\lambda))_\lambda^\infty = (A,B)_\lambda^\infty = \frac{K}{|G_\lambda|}\left[\chi_\lambda^\infty(AB^*)\prod_{\alpha\in\Phi_{G_\lambda}}(1-\zeta_\lambda^\alpha)\right]_{\mathrm{ct}}$$

with $\pi_\lambda^\infty(A) = f(\zeta_\lambda)E_{\mathsf{u}_\lambda,\mathsf{u}_\lambda}$ and $\pi_\lambda^\infty(B) = g(\zeta_\lambda)E_{\mathsf{u}_\lambda,\mathsf{u}_\lambda}$. Thus, $(f(\zeta_\lambda), g(\zeta_\lambda))_\lambda^\infty = K\langle f(\zeta_\lambda), g(\zeta_\lambda)\rangle_\lambda^\infty$. We have that $(\mathsf{t}_w, \mathsf{t}_v)_\lambda^\infty = \delta_{w,v}$ for all $w, v \in \widetilde{W}$ as the proof of Theorem 10.2.2 did not rely on $K = 1$. By Proposition 11.2.8 we have that $\pi_\lambda^\infty(\mathsf{t}_{\mathsf{m}_\gamma}) = \mathfrak{c}_\lambda(\mathsf{m}_\gamma) = \mathfrak{s}_\gamma(\zeta_\lambda)E_{\mathsf{u}_\lambda,\mathsf{u}_\lambda}$, and so $(\mathfrak{s}_\gamma(\zeta_\lambda), \mathfrak{s}_\gamma(\zeta_\lambda))_\lambda^\infty = (\mathsf{t}_{\mathsf{m}_\gamma}, \mathsf{t}_{\mathsf{m}_\gamma})_\lambda^\infty = 1$. In addition, by Lemma 11.2.2 we have that $\langle\mathfrak{s}_\gamma(\zeta_\lambda), \mathfrak{s}_\gamma(\zeta_\lambda)\rangle_\lambda^\infty = 1$. Thus,

$$1 = (\mathfrak{s}_\gamma(\zeta_\lambda), \mathfrak{s}_\gamma(\zeta_\lambda))_\lambda^\infty = K\langle\mathfrak{s}_\gamma(\zeta_\lambda), \mathfrak{s}_\gamma(\zeta_\lambda)\rangle_\lambda^\infty = K$$

as expected.

Remark 11.2.12. Recall that $\mathsf{m}_{g\gamma} = \mathsf{m}_\gamma$ for all $\gamma \in P^{(\lambda)}$ and $g \in G_\lambda$ by Proposition 11.2.4. This can be strengthened by Proposition 11.2.8 to $\mathsf{m}_{\gamma_1} = \mathsf{m}_{\gamma_2}$ if and only if $\gamma_2 \in G_\lambda\gamma_1$. To see this let $\gamma_1, \gamma_2 \in P_+^{(\lambda)}$ with $\gamma_1 \neq \gamma_2$. By Proposition 11.2.8 we have that

$$\pi_\lambda(C_{\mathsf{m}_{\gamma_1}}) = \mathfrak{s}_{\gamma_1}(\zeta_\lambda)\pi_\lambda(C_{\mathsf{w}_{\lambda'}}) \neq \mathfrak{s}_{\gamma_2}(\zeta_\lambda)\pi_\lambda(C_{\mathsf{w}_{\lambda'}}) = \pi_\lambda(C_{\mathsf{m}_{\gamma_2}})$$

and so $\mathsf{m}_{\gamma_1} \neq \mathsf{m}_{\gamma_2}$.

Remark 11.2.13. The description of $\mathcal{J}_{\Gamma_\lambda\cap\Gamma_\lambda^{-1}}$ in Theorem 11.2.10 is in terms of the G_λ-symmetric functions of ζ_λ whereas the description of $\mathcal{J}_{\Gamma_\lambda\cap\Gamma_\lambda^{-1}}$ in [49, Theorem 8.4.5] (and the description in [26]) is in terms of representation rings. To translate between these definitions, for each $\lambda \vdash n+1$ let u_λ be the unipotent element of $\mathsf{SL}_{n+1}(\mathbb{C})$ with Jordan blocks given by the partition λ. Let F_λ be the maximal reductive subgroup of the centraliser in $\mathsf{SL}_{n+1}(\mathbb{C})$ of u_λ. Then $\mathbb{Z}[\zeta_\lambda]^{G_\lambda}$ is isomorphic to the representation ring of F_λ. Note that in [49] Xi works with the group $W \rtimes \Omega$, with $\Omega \cong \mathbb{Z}$, instead of $W \rtimes \Sigma$. Elements of $W \rtimes \Omega$ are of the form $s_{i_1}\cdots s_{i_l}\sigma^k$ with no restriction on $k \in \mathbb{Z}$ (and σ the generator of Ω), differing from elements of $W \rtimes \Sigma$ which are of the form $s_{i_1}\cdots s_{i_l}\sigma^k$ with $0 \leq k \leq n$ (and σ as in Section 7.1). Thus, Xi is working in $\mathsf{GL}_{n+1}(\mathbb{C})$ instead of $\mathsf{SL}_{n+1}(\mathbb{C})$.

11.3 Lusztig's asymptotic algebra

In this final section we give an explicit description of $\mathfrak{C}_\lambda$. By Theorem 10.2.4 this gives an explicit description of Lusztig's asymptotic algebra $\mathcal{J} = \bigoplus_{\lambda\vdash n+1}\mathcal{J}_\lambda$. Using this description we then prove that the family of matrix representations $(\pi_\lambda)_{\lambda\vdash n+1}$ forms a balanced system of cell representatives (see Section 3.3).

As in Section 11.1 let $d_1, \ldots d_{N_\lambda}$ denote the distinguished involutions in Δ_λ and $\Gamma_1, \ldots, \Gamma_{N_\lambda}$ be the right cells in Δ_λ such that $d_i \in \Gamma_i$. Now fix the ordering so that $d_1 = \mathsf{w}_{\lambda'}$ (so $\Gamma_1 = \Gamma_\lambda$). We order ${}^\lambda W$ as in Theorem 11.1.2 so that $\mathfrak{c}_\lambda(d_i) = E_{i,i}$. Recall that there exists bijections $\phi_{ij} : \Gamma_i \cap \Gamma_j^{-1} \to \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ (from [49, §2.3]). Throughout this section we will use the Theorems of Xi [49, Theorems 2.3.2 and 8.4.2] often so will recall them here (let $E_{ij}(a)$ denote the matrix with a in the (i,j)-th position).

Theorem 11.3.1 ([49, Theorems 2.3.2 and 8.4.2])**.** *We have:*

(1) *The map* $\mathsf{t}_w \mapsto \mathsf{t}_{\phi_{ii}(w)}$ *induces a ring isomorphism* $\mathcal{J}_{\Gamma_i\cap\Gamma_i^{-1}} \to \mathcal{J}_{\Gamma_\lambda\cap\Gamma_\lambda^{-1}}$.

(2) *The map* $\mathsf{t}_w \mapsto E_{ij}(\mathsf{t}_{\phi_{ij}(w)})$, *for* $w \in \Gamma_i \cap \Gamma_j^{-1}$, *defines an isomorphism from* $\mathcal{J}_\lambda$ *to* $\mathrm{Mat}_{N_\lambda}(\mathcal{J}_{\Gamma_\lambda\cap\Gamma_\lambda^{-1}})$.

Theorem 11.3.2. *There are weights* $\gamma_1, \dots, \gamma_{N_\lambda} \in P^{(\lambda)}$ *and* $\epsilon_1, \dots, \epsilon_{N_\lambda} \in \{-1, 1\}$ *such that*

$$D^{-1}\mathfrak{C}_\lambda D = \mathrm{Mat}_{N_\lambda}(\mathbb{Z}[\zeta_\lambda]^{G_\lambda}) \quad \textit{where} \quad D = \mathrm{diag}(\epsilon_1\zeta_\lambda^{\gamma_1}, \dots, \epsilon_{N_\lambda}\zeta_\lambda^{\gamma_{N_\lambda}}).$$

Moreover there is a function $h_\lambda : \Delta_\lambda \to P_+^{(\lambda)}$ *such that:*

(*a*) *for each* $1 \le i, j \le N_\lambda$ *the map* $h_\lambda : \Gamma_i \cap \Gamma_j^{-1} \to P_+^{(\lambda)}$ *is bijective, and*

(*b*) *if* $w \in \Gamma_i \cap \Gamma_j^{-1}$ *then* $D^{-1}\mathfrak{c}_\lambda(w)D = \mathfrak{s}_{h_\lambda(w)}(\zeta_\lambda)E_{ij}$.

Proof. By Theorem 11.1.2 we have that $\mathfrak{c}_\lambda(d_i) = E_{i,i}$. Let $w_{ij} = \phi_{ij}^{-1}(\mathsf{w}_{\lambda'}) \in \Gamma_i \cap \Gamma_j^{-1}$. By [49, Lemma 2.3.1] $\phi_{ii}(d_i) = \mathsf{w}_{\lambda'}$ and $\phi_{ij}(w_{ji}^{-1}) = (\phi_{ji}(w_{ji}))^{-1}$, which implies that $\phi_{ji}^{-1}(\mathsf{w}_{\lambda'}) = (\phi_{ij}^{-1}(\mathsf{w}_{\lambda'}))^{-1}$. Thus, $w_{ii} = d_i$ and $w_{ji} = w_{ij}^{-1}$ for all $1 \le i, j \le N_\lambda$.

We have that $\mathsf{t}_{\mathsf{w}_{\lambda'}}^2 = \mathsf{t}_{\mathsf{w}_{\lambda'}}$ by the proof of Theorem 11.1.2. Thus, $E_{ij}(\mathsf{t}_{\mathsf{w}_{\lambda'}})E_{ji}(\mathsf{t}_{\mathsf{w}_{\lambda'}}) = E_{ii}(\mathsf{t}_{\mathsf{w}_{\lambda'}}^2) = E_{ii}(\mathsf{t}_{\mathsf{w}_{\lambda'}})$. Therefore, under the isomorphism in Theorem 11.3.1(2) we have $\mathsf{t}_{w_{ij}}\mathsf{t}_{w_{ji}} = \mathsf{t}_{w_{ij}}\mathsf{t}_{w_{ij}^{-1}} = \mathsf{t}_{d_i}$. By Theorem 11.1.2 $\mathfrak{c}_\lambda(w_{ij}) = a(\zeta_\lambda)E_{ij}$ for some $a(\zeta_\lambda) \in \mathbb{Z}[\zeta_\lambda]$. Thus, applying Theorem 10.2.4 and the fact that $\mathfrak{c}_\lambda(w_{ij}^{-1}) = \mathfrak{c}_\lambda(w_{ij})^*$ (see Lemma 5.3.7), we have

$$E_{ii} = \mathfrak{c}_\lambda(d_i) = \mathfrak{c}_\lambda(w_{ij})\mathfrak{c}_\lambda(w_{ij}^{-1}) = a(\zeta_\lambda)a(\zeta_\lambda^{-1})E_{ii}$$

and so $a(\zeta_\lambda)a(\zeta_\lambda^{-1}) = 1$. Writing $a(\zeta_\lambda) = \sum_{\gamma\in P^{(\lambda)}} c_\gamma\zeta_\lambda^\gamma$ with $c_\gamma \in \mathbb{Z}$ (as was done in the proof of Theorem 8.2.1) it follows that

$$a(\zeta_\lambda)a(\zeta_\lambda^{-1}) = \sum_{\gamma,\gamma'\in P^{(\lambda)}} c_\gamma c_{\gamma'}\zeta_\lambda^{\gamma-\gamma'} = \sum_{\gamma\in P^{(\lambda)}} c_\gamma^2 + (\text{terms involving } \zeta_\lambda).$$

It follows that $\sum_{\gamma\in P^{(\lambda)}} c_\gamma^2 = 1$ and, as $c_\gamma \in \mathbb{Z}$, this forces $a(\zeta_\lambda) = \epsilon_{ij}\zeta_\lambda^{\gamma_{ij}}$ for some $\gamma_{ij} \in P^{(\lambda)}$ and $\epsilon_{ij} = \pm 1$. Thus, $\mathfrak{c}(w_{ij}) = \epsilon_{ij}\zeta_\lambda^{\gamma_{ij}}E_{ij}$ and, as $w_{ji} = w_{ij}^{-1}$ and $w_{ii} = d_i$, we have $\epsilon_{ij} = \epsilon_{ji}$, $\epsilon_{ii} = 1$ and $\gamma_{ij} = -\gamma_{ji}$.

As $E_{ij}(\mathsf{t}_{\mathsf{w}_{\lambda'}})E_{jk}(\mathsf{t}_{\mathsf{w}_{\lambda'}})E_{ki}(\mathsf{t}_{\mathsf{w}_{\lambda'}}) = E_{ii}(\mathsf{t}_{\mathsf{w}_{\lambda'}})$, by Theorem 11.3.1(2), we have $\mathsf{t}_{w_{ij}}\mathsf{t}_{w_{jk}}\mathsf{t}_{w_{ki}} = \mathsf{t}_{d_i}$ and so by Theorem 10.2.4 $\mathfrak{c}_\lambda(w_{ij})\mathfrak{c}_\lambda(w_{jk})\mathfrak{c}_\lambda(w_{ki}) = \mathfrak{c}_\lambda(d_i) = E_{ii}$. On the other hand, as $\mathfrak{c}(w_{ij}) = \epsilon_{ij}\zeta_\lambda^{\gamma_{ij}}E_{ij}$ we have

$$\mathfrak{c}_\lambda(w_{ij})\mathfrak{c}_\lambda(w_{jk})\mathfrak{c}_\lambda(w_{ki}) = \epsilon_{ij}\epsilon_{jk}\epsilon_{ki}\zeta_\lambda^{\gamma_{ij}+\gamma_{jk}+\gamma_{ki}}E_{ii}$$

and so $\gamma_{ij} + \gamma_{jk} + \gamma_{ki} = 0$ and $\epsilon_{ij}\epsilon_{jk}\epsilon_{ki} = 1$.

We claim that there exists weights $\gamma_1, \dots, \gamma_{N_\lambda} \in P^{(\lambda)}$and signs $\epsilon_1, \dots, \epsilon_{N_\lambda} \in \{-1, 1\}$ such that $\zeta_\lambda^{\gamma_i-\gamma_j} = \zeta_\lambda^{\gamma_{ij}}$ and $\epsilon_{ij} = \epsilon_i\epsilon_j$ for $1 \le i, j \le N_\lambda$. We proceed by induction using the fact that $\epsilon_{ij}\epsilon_{jk} = \epsilon_{ik}$ and $\zeta_\lambda^{\gamma_{ij}}\zeta_\lambda^{\gamma_{jk}} = \zeta_\lambda^{-\gamma_{ki}} = \zeta_\lambda^{\gamma_{ik}}$, as $\gamma_{ij} + \gamma_{jk} + \gamma_{ki} = 0$, for all $1 \le i, j, k \le N_\lambda$. If $i = j$ then $\gamma_{ii} = 0$ so $\zeta_\lambda^{\gamma_i-\gamma_i} = \zeta_\lambda^{\gamma_{ii}}$ for any $\gamma_i \in P^{(\lambda)}$ and $\epsilon_{ii} = 1$ so $\epsilon_{ii} = \epsilon_i\epsilon_i$ for $\epsilon_i = \pm 1$. In particular, if $1 \le i, j \le 1$, that is $i = j = 1$, the claim holds. Now, assume that there exists some $\gamma_1, \dots, \gamma_n \in P^{(\lambda)}$ and $\epsilon_1, \dots, \epsilon_n \in \{1, -1\}$ such that $\zeta_\lambda^{\gamma_i-\gamma_j} = \zeta_\lambda^{\gamma_{ij}}$ and $\epsilon_i\epsilon_j = \epsilon_{ij}$ for $1 \le i, j \le n < N_\lambda$. We only need to consider the case when $i \ne j$ as the case when $i = j$ was completed above. If $i = n+1$ and $j < n+1$ then setting $\zeta_\lambda^{\gamma_{n+1}} = \zeta_\lambda^{\gamma_{(n+1)1}+\gamma_1}$ and $\epsilon_{n+1} = \epsilon_{(n+1)1}\epsilon_1$ we have

$$\zeta_\lambda^{\gamma_{n+1}-\gamma_j} = \zeta_\lambda^{\gamma_{(n+1)1}+\gamma_1-\gamma_j} = \zeta_\lambda^{\gamma_{(n+1)1}+\gamma_{1j}} = \zeta_\lambda^{\gamma_{(n+1)j}} \text{ and } \epsilon_{n+1}\epsilon_j = \epsilon_{(n+1)1}\epsilon_1\epsilon_j = \epsilon_{(n+1)1}\epsilon_{1j} = \epsilon_{(n+1)j}.$$

Similarly, if $i < n+1$ and $j = n+1$ then again setting $\zeta_\lambda^{\gamma_{n+1}} = \zeta_\lambda^{\gamma_{(n+1)1}+\gamma_1} = \zeta_\lambda^{-\gamma_{1(n+1)}+\gamma_1}$ and $\epsilon_{n+1} = \epsilon_{(n+1)1}\epsilon_1$ gives that $\zeta_\lambda^{\gamma_i-\gamma_{n+1}} = \zeta_\lambda^{\gamma_{i(n+1)}}$ and $\epsilon_i\epsilon_{n+1} = \epsilon_{i(n+1)}$ as required. The result follows by induction. It follows that

$$\mathfrak{c}_\lambda(w_{ij}) = DE_{ij}D^{-1} \quad \text{with} \quad D = \operatorname{diag}(\epsilon_1\zeta_\lambda^{\gamma_1}, \ldots, \epsilon_{N_\lambda}\zeta_\lambda^{\gamma_{N_\lambda}}).$$

Let $\mathcal{R}_{ij} = \{[D^{-1}AD]_{ij} \mid A \in \mathfrak{C}_\lambda\}$. We claim that $\mathcal{R}_{ij} = \mathbb{Z}[\zeta_\lambda]^{G_\lambda}$. Suppose that $a(\zeta_\lambda), b(\zeta_\lambda) \in \mathcal{R}_{ij}$, thus there exists $A, B \in \mathfrak{C}_\lambda$ such that $[D^{-1}AD]_{ij} = a(\zeta_\lambda)$ and $[D^{-1}BD]_{ij} = b(\zeta_\lambda)$. Then

$$\begin{aligned} D^{-1}\mathfrak{c}_\lambda(d_i)A\mathfrak{c}_\lambda(w_{ji})B\mathfrak{c}_\lambda(d_j)D &= D^{-1}E_{ii}ADD^{-1}\mathfrak{c}_\lambda(w_{ji})DD^{-1}BE_{jj}D \\ &= E_{ii}D^{-1}ADE_{ji}D^{-1}BDE_{jj} \\ &= a(\zeta_\lambda)b(\zeta_\lambda)E_{ij} \end{aligned}$$

and so $\mathcal{R}_{ij}$ is a ring. Moreover,

$$D^{-1}\mathfrak{c}_\lambda(w_1i)A\mathfrak{c}_\lambda(w_{j1})D = D^{-1}\mathfrak{c}_\lambda(w_{1i})DD^{-1}ADD^{-1}\mathfrak{c}_\lambda(w_{j1})D = E_{1i}D^{-1}ADE_{j1} = a(\zeta_\lambda)E_{11}$$

so $\mathfrak{c}_\lambda(w_{1i})A\mathfrak{c}_\lambda(w_{j1}) = a(\zeta_\lambda)E_{11} = a(\zeta_\lambda)\mathfrak{c}_\lambda(\mathsf{w}_{\lambda'}) \in \mathfrak{C}_{\Gamma_\lambda\cap\Gamma_\lambda^{-1}}$ (by Theorem 11.1.2). Thus, as $\mathfrak{c}_\lambda(w_{1,i}), A, \mathfrak{c}_\lambda(w_{j1}) \in \mathfrak{C}_\lambda$ and by Theorem 11.2.10 we have that $a(\zeta_\lambda) \in \mathbb{Z}[\zeta_\lambda]^{G_\lambda}$, so $\mathcal{R}_{ij} \subseteq \mathbb{Z}[\zeta_\lambda]^{G_\lambda}$.

For the reverse inclusion, note that

$$\begin{aligned} D^{-1}\mathfrak{c}_\lambda(w_{i1})\mathfrak{c}_\lambda(\mathsf{m}_\gamma)\mathfrak{c}_\lambda(w_{1j})D &= \mathfrak{s}_\gamma(\zeta_\lambda)D^{-1}\mathfrak{c}_\lambda(w_{i1})DD^{-1}E_{11}DD^{-1}\mathfrak{c}_\lambda(w_{1j})D \\ &= \mathfrak{s}_\gamma(\zeta_\lambda)D^{-1}\mathfrak{c}_\lambda(w_{i1})DE_{11}D^{-1}\mathfrak{c}_\lambda(w_{1j})D = \mathfrak{s}_\gamma(\zeta_\lambda)E_{ij} \end{aligned}$$

and so $\mathbb{Z}[\zeta_\lambda]^{G_\lambda} \subseteq \mathcal{R}_{ij}$ by Proposition 7.3.6. Thus, $\mathcal{R}_{ij} = \mathbb{Z}[\zeta_\lambda]^{G_\lambda}$, so $D^{-1}\mathfrak{C}_\lambda D = \operatorname{Mat}_{N_\lambda}(\mathbb{Z}[\zeta_\lambda]^{G_\lambda})$.

Let $w \in \Gamma_i \cap \Gamma_j^{-1}$, and write $D^{-1}\mathfrak{c}_\lambda(w)D = a(\zeta_\lambda)E_{ij}$ for some $a(\zeta_\lambda) \in \mathbb{Z}[\zeta_\lambda]^{G_\lambda}$. We have that $\pi_\lambda^\infty(\mathsf{t}_w) = \mathfrak{c}_\lambda(w) = a(\zeta_\lambda)DE_{ij}D^{-1}$, so

$$\begin{aligned} \chi_\lambda^\infty(\mathsf{t}_w\mathsf{t}_w^*) &= \mathsf{tr}(a(\zeta_\lambda)a(\zeta_\lambda^{-1})DE_{ij}D^{-1}(DE_{ij}D^{-1})^*) \\ &= \mathsf{tr}(a(\zeta_\lambda)a(\zeta_\lambda^{-1})DE_{ij}D^{-1}DE_{ji}D^{-1}) \\ &= \mathsf{tr}(a(\zeta_\lambda)a(\zeta_\lambda^{-1})E_{ii}). \end{aligned}$$

Thus, by Theorem 10.2.2 we have that $\langle \mathsf{t}_w, \mathsf{t}_w\rangle_\lambda^\infty = \langle a(\zeta_\lambda), a(\zeta_\lambda)\rangle_\lambda^\infty = 1$. Writing $a(\zeta_\lambda) = \sum_{\gamma\in(P/Q_\lambda)_+} a_\gamma\mathfrak{s}_\gamma(\zeta_\lambda)$, with a_γ integers (as the Schur functions are a basis of $\mathbb{Z}[\zeta_\lambda]^{G_\lambda}$), it follows that $\sum_\gamma a_\gamma^2 = 1$ by Lemma 11.2.2. Thus, $a(\zeta_\lambda) = \epsilon_w\mathfrak{s}_{h_\lambda(w)}(\zeta_\lambda)$ for some $h_\lambda(w) \in P_+^{(\lambda)}$ and $\epsilon_w = \pm 1$. Consider this map $h_\lambda : \Gamma_i \cap \Gamma_j^{-1} \to P_+^{(\lambda)}$. If $w \neq v$ with $w, v \in \Gamma_i \cap \Gamma_j^{-1}$ then $h_\lambda(w) \neq h_\lambda(v)$ as otherwise $\langle \mathsf{t}_w, \mathsf{t}_v\rangle_\lambda^\infty = \langle \epsilon_w\mathfrak{s}_{h_\lambda(w)}(\zeta_\lambda), \epsilon_v\mathfrak{s}_{h_\lambda(v)}(\zeta_\lambda)\rangle_\lambda^\infty \neq 0$, a contradiction by Theorem 10.2.2. Furthermore, the map is surjective by $\mathcal{R}_{ij} = \mathbb{Z}[\zeta_\lambda]^{G_\lambda}$ with the fact that $\{\mathfrak{s}_\gamma(\zeta_\lambda) \mid \gamma \in P_+^{(\lambda)}\}$ is a basis of $\mathbb{Z}[\zeta_\lambda]^{G_\lambda}$. Thus, h_λ is bijective as required.

Finally, we claim that $\epsilon_w = 1$ for all $w \in \Delta_\lambda$. By Proposition 11.2.8 we have

$$\begin{aligned} D^{-1}\mathfrak{c}_\lambda(w_{1i})\mathfrak{c}_\lambda(w)\mathfrak{c}_\lambda(w_{j1})D &= D^{-1}\mathfrak{c}_\lambda(w_{1i})DD^{-1}\mathfrak{c}_\lambda(w)DD^{-1}\mathfrak{c}_\lambda(w_{j1})D \\ &= \epsilon_w\mathfrak{s}_{h_\lambda(w)}(\zeta_\lambda)E_{11} \\ &= \epsilon_w\mathfrak{s}_{h_\lambda(w)}(\zeta_\lambda)D^{-1}E_{11}D \\ &= \epsilon_w(\zeta_\lambda)D^{-1}\mathfrak{c}_\lambda(\mathsf{m}_{h_\lambda(w)})D \end{aligned}$$

so $\mathfrak{c}_\lambda(w_{1i})\mathfrak{c}_\lambda(w)\mathfrak{c}_\lambda(w_{j1}) = \epsilon_w\mathfrak{c}_\lambda(\mathsf{m}_{h_\lambda(w)})$. By Theorem 10.2.4 this implies that $\mathsf{t}_{w_{1i}}\mathsf{t}_w\mathsf{t}_{w_{j1}} = \epsilon_w\mathsf{t}_{\mathsf{m}_{h_\lambda(w)}}$ and by the positivity of structure constants (of $\widetilde{\mathcal{H}}$ and thus $\mathcal{J}_\lambda$) this forces $\epsilon_w = 1$. $\square$

We make the following conjecture.

Conjecture 11.3.3. *In Theorem 11.3.2 we have* $\epsilon_i = 1$ *for all* $1 \leq i \leq N_\lambda$.

Recall the definition of a balanced system of cell representations from Section 3.3, in particular recall the descriptions of the conditions $B1$-$B5$ and $\tilde{B}$. The following corollary shows that Theorem 11.3.2, along with other result from Part II, give that $(\pi_\lambda)_{\lambda\vdash n+1}$ is a balanced system of cell representations of $\widetilde{\mathcal{H}}$.

Corollary 11.3.4. *The matrix representations* $\pi_\lambda(\cdot)$, *with* $\lambda \vdash n+1$, *satisfy the conditions* $B1$ - $B5$ *and* $\tilde{B}$. *Thus they form a balanced system of cell representations*

Proof. Theorem 8.1.3 gives that π_λ satisfies $B1$ and Theorem 8.2.1 gives that π_λ satisfies $B2$, with the bound being $\ell(\mathsf{w}_{\lambda'})$. Theorem 10.2.3 gives that $\mathfrak{c}_\lambda(w) \neq 0$ if and only if $w \in \Delta_\lambda$ and so $B3$ is also satisfied. Condition $B4$, that the leading matrices are free over $\mathbb{Z}$, follows from the freeness of the Schur functions and the fact that $D^{-1}\mathfrak{c}_\lambda(w)D = \mathfrak{s}_{h_\lambda(w)}(\zeta_\lambda)E_{ij}$ by Theorem 11.3.2. Condition $B5$ follows from Lemma 7.1.12 and Theorem 8.2.1. Finally, to show that $\tilde{B}$ is satisfied consider $z \in \Delta_\lambda$. Let $1 \leq i, j \leq N_\lambda$ such that $z \in \Gamma_i \cap \Gamma_j^{-1}$. By Theorem 11.3.2 we have

$$D^{-1}\mathfrak{c}_\lambda(z)\mathfrak{c}_\lambda(d_j)D = D^{-1}\mathfrak{s}_{h_\lambda(z)}(\zeta_\lambda)E_{ij}E_{jj}D = D^{-1}\mathfrak{s}_{h_\lambda(z)}(\zeta_\lambda)E_{ij}D$$

and so $\mathfrak{c}_\lambda(z)\mathfrak{c}_\lambda(d_j) = \mathfrak{c}_\lambda(z)$. Therefore, $\tilde{\gamma}_{z,d_j,z^{-1}} \neq 0$ as required (an analogous argument can be made instead using d_i). □

Remark 11.3.5. Conjugation by D in Theorem 11.3.2 amounts to choosing a (signed) basis associated to a fundamental domain F for the action of $\mathbb{T}_{J_\lambda}$ on $\mathbb{W}^{J_\lambda}$. Proposition 5.3.4 gives the description of the basis and Theorem 5.3.5 gives the path formula with respect to this basis. Specifically, the fundamental domain chosen is $\mathsf{F} = \{\tau_{\gamma_i}^{-1}u_i \mid 1 \leq i \leq N_\lambda\}$ where ${}^\lambda W = \{u_i \mid 1 \leq i \leq N_\lambda\}$ is ordered as in Theorem 11.1.2, the associated (signed) basis is then $\{\epsilon_i\varpi_\lambda(X_u) \mid u \in \mathsf{F}\}$.

Example 11.3.6. Consider A_3 with $\lambda = (2,2)$ as in Examples 7.2.1, 7.2.15 and 7.3.7. We have that $\mathsf{u}_\lambda = s_2$ and so we order ${}^\lambda W$ as follows:

$${}^\lambda W = \{s_2, e, s_2s_1, s_2s_3, s_2s_1s_3, s_2s_1s_3s_2\}.$$

As in Example 7.3.7 we have $\zeta_\lambda^{\tilde{e}_1} = z_1$ and $\zeta_\lambda^{\tilde{e}_2} = z_2$ with $z_1^2z_2^2 = 1$. Using Theorem 5.3.3 we have the following matrices (with the basis ordered with respect to the above ordering of ${}^\lambda W$):

$$\pi_\lambda(T_1) = \begin{bmatrix} \mathsf{Q} & 0 & 1 & 0 & 0 & 0 \\ 0 & -\mathsf{q}^{-1} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathsf{Q} & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mathsf{q}^{-1} \end{bmatrix} \quad \pi_\lambda(T_2) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & \mathsf{Q} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathsf{q}^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathsf{q}^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathsf{Q} & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\pi_\lambda(T_3) = \begin{bmatrix} \mathsf{Q} & 0 & 0 & 1 & 0 & 0 \\ 0 & -\mathsf{q}^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathsf{Q} & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mathsf{q}^{-1} \end{bmatrix} \quad \pi_\lambda(T_0) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & z_1/z_2 \\ 0 & 0 & 0 & 0 & z_1/z_2 & 0 \\ 0 & 0 & -\mathsf{q}^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathsf{q}^{-1} & 0 & 0 \\ 0 & z_2/z_1 & 0 & 0 & \mathsf{Q} & 0 \\ z_2/z_1 & 0 & 0 & 0 & 0 & \mathsf{Q} \end{bmatrix}$$

$$\pi_\lambda(T_\sigma) = \begin{bmatrix} 0 & 0 & 0 & 0 & z_1 & 0 \\ 0 & 0 & 0 & z_1 & 0 & 0 \\ 0 & z_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & z_1 \\ z_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & z_2 & 0 & 0 & 0 \end{bmatrix}$$

where $\mathsf{Q} = \mathsf{q} - \mathsf{q}^{-1}$. MAGMA code calculating these matrices can be found at `https://github.com/ellielittle/A3Matrices`. The distinguished involutions are

$$d_1 = s_1s_3 \quad d_2 = s_2s_1s_3s_2 \quad d_3 = s_3s_2s_0s_3 \quad d_4 = s_1s_2s_0s_1 \quad d_5 = s_0s_2 \quad d_6 = s_0s_1s_3s_0.$$

We have the following bijections (these bijections are not unique)

$$\begin{aligned}
&\Gamma_2 \to \Gamma_1,\ 2w \mapsto w \\
&\Gamma_3 \to \Gamma_1,\ 3\sigma^3 w \mapsto w \\
&\Gamma_4 \to \Gamma_1,\ 1\sigma^3 w \mapsto w \\
&\Gamma_5 \to \Gamma_1,\ \sigma^3 w \mapsto w \\
&\Gamma_6 \to \Gamma_1,\ 0\sigma^2 w \mapsto w.
\end{aligned}$$

Thus, we can find bijections $\phi_{ij} : \Gamma_i \cap \Gamma_j^{-1} \to \Gamma_1 \cap \Gamma_1^{-1}$ (see the proof of Theorem 11.3.2 and [49, §2.3]). The corresponding elements $w_{ij} = \phi_{ij}^{-1}(\mathsf{w}_{\lambda'})$ are

$$\begin{array}{lllll}
w_{12} = s_1s_3s_2 & w_{13} = s_1s_3s_0\sigma & w_{14} = s_1s_3s_2\sigma & w_{15} = s_1s_3\sigma & w_{16} = s_1s_3s_2\sigma^2 \\
w_{23} = s_2s_1s_3s_0\sigma & w_{24} = s_2s_1s_3s_2\sigma & w_{25} = s_2s_1s_3\sigma & w_{26} = s_2s_1s_3s_2\sigma^2 & w_{34} = s_3s_2s_0s_1 \\
w_{35} = s_3s_2s_0 & w_{36} = s_3s_2s_0s_1\sigma & w_{45} = s_1s_2s_0 & w_{46} = s_1s_2s_0s_1\sigma & w_{56} = s_2s_0s_1\sigma
\end{array}$$

(with $w_{ii} = d_i$ and $w_{ji} = w_{ij}^{-1}$). For example, ϕ_{34}^{-1} maps 13 to $3\sigma^3 13\sigma 1 = 3201$ as required. Using the MAGMA code we calculate

$$\mathfrak{c}_\lambda(w_{12}) = E_{12}, \quad \mathfrak{c}_\lambda(w_{13}) = z_1E_{13}, \quad \mathfrak{c}_\lambda(w_{14}) = z_1E_{14}, \quad \mathfrak{c}_\lambda(w_{15}) = z_1E_{15}, \quad \mathfrak{c}_\lambda(w_{16}) = z_1^2E_{16}.$$

Thus, the conjugating matrix (from Theorem 11.3.2) can be taken as

$$D = \mathrm{diag}(1, 1, z_1^{-1}, z_1^{-1}, z_1^{-1}, z_1^{-2}).$$

Then $D^{-1}\mathfrak{c}_\lambda(w_{ij})D = E_{ij}$ for all $1 \leq i, j \leq 6$ (noting that Conjecture 11.3.3 holds in this case). By Theorem 11.3.2 it then follows that $D^{-1}\mathfrak{C}_\lambda D = \mathrm{Mat}_6(\mathbb{Z}[\zeta_\lambda]^{G_\lambda})$. For example, take $w = s_2s_0s_1s_3s_2s_0s_1s_3s_2s_0\sigma^2$. Using the MAGMA code we have

$$D^{-1}\mathfrak{c}_\lambda(w)D = (z_1^5z_2 + z_1z_2^5 + z_1^2 + z_2^2 + z_1z_2)E_{56} = \frac{z_1^6z_2 - z_1z_2^6}{z_1 - z_2}E_{56} = \mathfrak{s}_{5\tilde{e}_1+\tilde{e}_2}(\zeta_\lambda)E_{56},$$

by Example 7.3.7. Thus $w \in \Gamma_5 \cap \Gamma_6^{-1}$ and $h_\lambda(w) = 3e_1 + 2e_2 + e_3 \in P_+^{(\lambda)}$.

Part III

Appendices

Appendix A

The length of m_γ

Recall from Section 11.2 that m_γ is the maximal length representative of the double coset $W_{\lambda'}\mathsf{u}_\lambda^{-1}\tau_\gamma\mathsf{u}_\lambda W_{\lambda'}$ for $\gamma \in P^{(\lambda)}$. In this appendix we will prove the important property of these elements stated in Theorem 11.2.5. The results of Chapter 11 are reliant on this theorem. We recall the theorem here:

Theorem. Let $\gamma \in P^{(\lambda)}$. There exist $x, y \in W_{\lambda'}$ such that $\mathsf{m}_\gamma = x\mathsf{u}_\lambda^{-1}\tau_\gamma\mathsf{u}_\lambda y$ with

$$\ell(\mathsf{m}_\gamma) = \ell(x) + \ell(\mathsf{u}_\lambda^{-1}\tau_\gamma\mathsf{u}_\lambda) + \ell(y) \quad \text{and} \quad \ell(x) + \ell(y) = \ell(\mathsf{w}_{\lambda'}).$$

If $\gamma, \gamma' \in P_+^{(\lambda)}$ with $\gamma + Q_\lambda \preccurlyeq_\lambda \gamma' + Q_\lambda$ then $\ell(\mathsf{m}_\gamma) \leq \ell(\mathsf{m}_{\gamma'})$ with equality if and only if $\gamma = \gamma'$.

In Section A.1 we give some general results of maximal length double coset representatives that will be used in the remaining sections. Section A.2 gives the proof of the first half of Theorem 11.2.5, describing the length of $\ell(\mathsf{m}_\gamma)$. In Section A.3 we compute the length of $\mathsf{u}_\lambda^{-1}\tau_\gamma\mathsf{u}_\lambda$ and, finally, in Section A.4 we use this length description to prove the second half of Theorem 11.2.5, the monotonicity of $\ell(\mathsf{m}_\gamma)$ with respect to $\preccurlyeq_\lambda$.

A.1 Maximal length double coset representatives

In this section we prove some preliminary results about maximal length double coset representatives that will be used in the subsequent sections. The results of this section are for general affine Weyl groups, so the notation $J \subseteq I$ is used (instead of $\lambda \vdash n+1$).

Lemma A.1.1. *Let $\gamma \in P$ and $u \in W_0$. For $1 \leq j \leq n$ we have:*

(1) $\ell(s_j t_\gamma u) = \ell(t_\gamma u) - 1$ *if and only if* $\langle \gamma, \alpha_j \rangle \leq 0$ *and if* $\langle \gamma, \alpha_j \rangle = 0$ *then* $u^{-1}\alpha_j < 0$.

(2) $\ell(t_\gamma u s_j) = \ell(t_\gamma u) - 1$ *if and only if* $\langle u^{-1}\gamma, \alpha_j \rangle \geq 0$ *and if* $\langle u^{-1}\gamma, \alpha_j \rangle = 0$ *then* $u\alpha_j < 0$.

Proof. It is sufficient to prove (1) as (2) follows from (1) noting that $(t_\gamma u)^{-1} = t_{-u^{-1}\gamma}u^{-1}$ and $\ell(t_\gamma u s_j) = \ell(s_j(t_\gamma u)^{-1})$. By [37, (2.4.1)] we have that

$$\ell(t_\gamma u) = \sum_{\alpha \in \Phi^+} |\langle \gamma, \alpha \rangle - \chi^-(u^{-1}\alpha)|,$$

where $\chi^-(\alpha) = 1$ if $\alpha \in \Phi^-$ and 0 otherwise (this formula is counting the hyperplanes separating A_0 from $t_\gamma u A_0$). As the simple reflection s_j permutes $\Phi^+\backslash\{\alpha_j\}$, and as $s_j t_\gamma u = t_{s_j\gamma}u$ we have that

$$\ell(s_j t_\gamma u) = |\langle \gamma, \alpha_j \rangle + \chi^-(-u^{-1}\alpha_j)| + \sum_{\alpha \in \Phi^+\backslash\{\alpha_j\}} |\langle \gamma, \alpha \rangle - \chi^-(u^{-1}\alpha)|,$$

and so $\ell(t_\gamma u) - \ell(s_j t_\gamma u) = |\langle\gamma,\alpha_j\rangle - \chi^-(u^{-1}\alpha_j)| - |\langle\gamma,\alpha_j\rangle + \chi^-(-u^{-1}\alpha_j)|$.

If $\langle\gamma,\alpha_j\rangle < 0$ there are two cases: if $u^{-1}\alpha_j < 0$ then $\ell(t_\gamma u) - \ell(s_j t_\gamma u) = 1$ and if $u^{-1}\alpha_j > 0$ then $\ell(t_\gamma u) - \ell(s_j t_\gamma u) = 1$. Whereas if $\langle\gamma,\alpha_j\rangle > 0$ then: if $u^{-1}\alpha_j < 0$ then $\ell(t_\gamma u) - \ell(s_j t_\gamma u) = -1$ and if $u^{-1}\alpha_j > 0$ then $\ell(t_\gamma u) - \ell(s_j t_\gamma u) = -1$. Finally, if $\langle\gamma,\alpha_j\rangle = 0$ we have that if $u^{-1}\alpha_j < 0$ then $\ell(t_\gamma u) - \ell(s_j t_\gamma u) = 1$ and if $u^{-1}\alpha_j > 0$ then $\ell(t_\gamma u) - \ell(s_j t_\gamma u) = -1$. □

Definition A.1.2. For $\gamma \in P$, $u \in W_0$ and $J \subseteq I$ let

$$L_J(\gamma,u) = \{\alpha \in \Phi_J^+ \mid \langle\gamma,\alpha\rangle > 0 \text{ or } \langle\gamma,\alpha\rangle = 0 \text{ and } u^{-1}\alpha > 0\}$$
$$R_J(\gamma,u) = \{\alpha \in \Phi_J^+ \mid \langle u^{-1}\gamma,\alpha\rangle < 0 \text{ or } \langle u^{-1}\gamma,\alpha\rangle = 0 \text{ and } u\alpha > 0\}.$$

Lemma A.1.3. *For $\gamma \in P$, $u \in W_0$, and $J \subseteq I$ there exist (unique) elements $x, x' \in W_J$ with $\Phi(x) = L_J(\gamma,u)$ and $\Phi(x') = R_J(\gamma,u)$. Moreover $x^{-1}t_\gamma u$ is of maximal length in $W_J t_\gamma u$ and so $\ell(x^{-1}t_\gamma u) = \ell(x) + \ell(t_\gamma u)$, and $t_\gamma u x'$ is of maximal length in $t_\gamma u W_J$ and so $\ell(t_\gamma u x') = \ell(t_\gamma u) + \ell(x')$.*

Proof. It is sufficient to consider the coset $W_J t_\gamma u$ (as the other coset result follows by considering $(t_\gamma u)^{-1} = t_{-u^{-1}\gamma}u^{-1}$). We argue by induction on $|L_J(\gamma,u)|$. If $|L_J(\gamma,u)| = 0$ then $\ell(s_j t_\gamma u) = \ell(t_\gamma u) - 1$ for all $j \in J$ (by Lemma A.1.1) and so $t_\gamma u$ is of maximal length in $W_J t_\gamma u$ and $x = e$. Suppose that $|L_J(\gamma,u)| > 0$. Then there exists $j \in J$ such that $\alpha_j \in L_J(\gamma,u)$ and $\ell(s_j t_\gamma u) = \ell(t_\gamma u) + 1$ (as otherwise $L_J(\gamma,u) = \emptyset$ by Lemma A.1.1).

We claim that $L_J(s_j\gamma, s_j u) = s_j(L_J(\gamma,u)\backslash\{\alpha_j\})$ (noting that $s_j t_\gamma u = t_{s_j\gamma}s_j u$). By Lemma A.1.1 we have that $\alpha_j \notin L_J(s_j\gamma, s_j u)$ as otherwise $\langle s_j\gamma,\alpha_j\rangle > 0$, in which case $\langle\gamma,\alpha_j\rangle < 0$ contradicting $\alpha_j \in L_J(\gamma,u)$, or $\langle s_j\gamma,\alpha_j\rangle = 0$ with $u^{-1}s_j\alpha_j > 0$, in which case $\langle\gamma,\alpha\rangle = 0$ with $u^{-1}\alpha_j < 0$ again contradicting $\alpha_j \in L_J(\gamma,u)$. Suppose that $\alpha \in L_J(s_j\gamma, s_j u)$. Then either $\langle\gamma, s_j\alpha\rangle > 0$ or $\langle\gamma, s_j\alpha\rangle = 0$ with $u^{-1}s_j\alpha > 0$. As $\alpha \in \Phi^+\backslash\{\alpha_j\}$ and since s_j permutes $\Phi^+\backslash\{\alpha_j\}$, it follows that $s_j\alpha \in L_J(\gamma,u)$ and so $\alpha \in s_j(L_J(\gamma,u)\backslash\{\alpha_j\})$. Thus, $L_J(s_j\gamma, s_j u) \subseteq s_j(L_J(\gamma,u)\backslash\{\alpha_j\})$. For the converse suppose that $\alpha \in s_j(L_J(\gamma,u)\backslash\{\alpha_j\})$ then there exists $\beta \in L_J(\gamma,u)\backslash\{\alpha_j\}$ such that $\alpha = s_j\beta$ and $\langle\gamma,\beta\rangle > 0$ or $\langle\gamma,\beta\rangle = 0$ with $u^{-1}\beta > 0$. Thus, $\alpha \in L_J(s_j\gamma, s_j u)$ and so the claim holds.

With $L_J(s_j\gamma, s_j u) = s_j(L_J(\gamma,u)\backslash\{\alpha_j\})$, it follows by induction that there exists $j_1, \ldots j_k \in J$ such that, writing $x^{-1} = s_{j_1}\cdots s_{j_k}$, we have $\ell(x^{-1}t_\gamma u) = k + \ell(t_\gamma u)$ and $L_J(x\gamma, xu) = \emptyset$. Thus, $x^{-1}t_\gamma u$ is of maximal length in $W_J t_\gamma u$ and $\Phi(x) = L_J(\gamma,u)$, completing the proof. □

Corollary A.1.4. *Let $\gamma \in P$, $u \in W_0$, and $J \subseteq I$. Let m be the longest element of $W_J t_\gamma u W_J$. Then*

$$m = x^{-1}t_\gamma u y \quad \textit{with} \quad \ell(m) = \ell(x) + \ell(t_\gamma u) + \ell(y),$$

where $\Phi(x) = L_J(\gamma,u)$ and $\Phi(y) = R_J(x^{-1}\gamma, x^{-1}u)$. In particular,

$$\ell(m) - \ell(t_\gamma u) = |L_J(\gamma,u)| + |R_J(x^{-1}\gamma, x^{-1}u)|.$$

Proof. By Lemma A.1.3 we have that $x^{-1}t_\gamma u = t_{x^{-1}\gamma}x^{-1}u$ is of maximal length in $W_J t_\gamma u$ and $\ell(x^{-1}t_\gamma u) = \ell(x) + \ell(t_\gamma u)$, where x is such that $\Phi(x) = L_J(\gamma,u)$. Then, again using Lemma A.1.3, we have that $(x^{-1}t_\gamma u)y$ is of maximal length in $x^{-1}t_\gamma u W_J$ and

$$\ell(x^{-1}t_\gamma u y) = \ell(x^{-1}t_\gamma u) + \ell(y) = \ell(x) + \ell(t_\gamma u) + \ell(y),$$

where y is such that $\Phi(y) = R_J(x^{-1}\gamma, x^{-1}u)$. So $x^{-1}t_\gamma u y$ is of maximal length in $W_J t_\gamma u W_J$, and the result follows. □

A.2 The elements m_γ

In this section we will prove that $\ell(\mathsf{m}_\gamma) = \ell(\mathsf{u}_\lambda^{-1}\tau_\gamma\mathsf{u}_\lambda) + \ell(\mathsf{w}_{\lambda'})$, thus completing the proof of the first part of Theorem 11.2.5.

Definition A.2.1. For $\gamma \in P^{(\lambda)}$ let

$$L(\gamma) = \{\beta \in \mathsf{u}_\lambda\Phi^+_{\lambda'} \mid \langle\gamma,\beta\rangle > 0 \text{ or } \langle\gamma,\beta\rangle = 0 \text{ and } \mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma^{-1}\beta > 0\},$$
$$R(\gamma) = \{\beta \in \mathsf{u}_\lambda\Phi^+_{\lambda'} \mid \langle\mathsf{w}_\lambda\gamma,\beta\rangle < 0 \text{ or } \langle\mathsf{w}_\lambda\gamma,\beta\rangle = 0 \text{ and } \mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma\beta > 0 \text{ and } \mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma\beta \notin \Phi^+_{\lambda'}\}.$$

Proposition A.2.2. *We have* $\ell(\mathsf{m}_\gamma) - \ell(\mathsf{u}_\lambda^{-1}\tau_\gamma\mathsf{u}_\lambda) = |L(\gamma)| + |R(\gamma)|$.

Proof. Note that $L(\gamma) = \mathsf{u}_\lambda L_{J_{\lambda'}}(\mathsf{u}_\lambda^{-1}\gamma, \mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma\mathsf{u}_\lambda)$ (using Definition A.1.2). Then Corollary A.1.4 gives that $\ell(\mathsf{m}_\gamma) - \ell(\mathsf{u}_\lambda^{-1}\tau_\gamma\mathsf{u}_\lambda) = |L(\gamma)| + |R'(\gamma)|$ where $R'(\gamma) = R_{J_{\lambda'}}(x^{-1}\mathsf{u}_\lambda^{-1}\gamma, x^{-1}\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma\mathsf{u}_\lambda)$ and $\Phi(x) = L_{J_{\lambda'}}(\mathsf{u}_\lambda^{-1}\gamma, \mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma\mathsf{u}_\lambda) = \mathsf{u}_\lambda^{-1}L(\gamma)$. Thus, all that is required to prove is that $R(\gamma) = R'(\gamma)$.

Recall that by Definition 2.3.1 we have $\mathsf{y}_\gamma^{-1} = \mathsf{w}_\lambda\mathsf{w}_{J_\lambda\backslash J_\lambda(\gamma)}$ with $J_\lambda(\gamma) = \{j \in J_\lambda \mid \langle\gamma,\alpha_j\rangle = 1\}$. Thus, $\langle\gamma,\alpha_j\rangle = 0$ for all $j \in J_\lambda\backslash J_\lambda(\gamma)$ and so $s_j(\gamma) = \gamma$ for all $j \in J_\lambda\backslash J_\lambda(\gamma)$. Therefore, $\mathsf{y}_\gamma^{-1}\gamma = \mathsf{w}_\lambda\mathsf{w}_{J_\lambda\backslash J_\lambda(\gamma)}\gamma = \mathsf{w}_\lambda\gamma$ and it remains to prove that for $\beta \in \mathsf{u}_\lambda\Phi^+_{\lambda'}$ if $\langle\mathsf{w}_\lambda\gamma,\beta\rangle = 0$ then $x^{-1}\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma\beta > 0$ if and only if $\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma\beta > 0$ and $\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma\beta \notin \Phi^+_{\lambda'}$. Suppose that $x^{-1}\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma\beta > 0$. If $\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma\beta < 0$ then $-\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma\beta \in \Phi(x) = \mathsf{u}_\lambda^{-1}L(\gamma)$ and so $-\mathsf{y}_\gamma\beta \in L(\gamma)$. Thus, since $\langle\gamma, -\mathsf{y}_\gamma\beta\rangle = -\langle\mathsf{w}_\lambda\gamma,\beta\rangle = 0$ we have that $-\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma^{-1}\mathsf{y}_\gamma\beta > 0$. However, $-\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma^{-1}\mathsf{y}_\gamma\beta = -\mathsf{u}_\lambda^{-1}\beta \in -\Phi^+_{\lambda'}$ (as $\beta \in \mathsf{u}_\lambda\Phi^+_{\lambda'}$), a contradiction. Thus, $\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma\beta > 0$. This implies that $\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma\beta \notin \Phi(x)$ and so $\mathsf{y}_\gamma\beta \notin L(\gamma)$. If $\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma\beta \in \Phi^+_{\lambda'}$ then $\mathsf{y}_\gamma\beta \in \mathsf{u}_\lambda\Phi^+_{\lambda'}$. Thus $\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma^{-1}\mathsf{y}_\gamma\beta = \mathsf{u}_\lambda^{-1}\beta \in \Phi^+_{\lambda'}$ and as $\langle\gamma,\mathsf{y}_\gamma\beta\rangle = 0$ we have $\mathsf{y}_\gamma\beta \in L(\gamma)$, a contradiction.

For the converse suppose that $\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma\beta > 0$ and $\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma\beta \notin \Phi^+_{\lambda'}$. If $x^{-1}\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma\beta < 0$ then $\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma\beta \in \Phi(x) = \mathsf{u}_\lambda^{-1}L(\gamma)$. Thus, $\mathsf{y}_\gamma\beta \in L(\gamma) \subseteq \mathsf{u}_\lambda\Phi^+_{\lambda'}$ which contradicts the fact that $\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma\beta \notin \Phi^+_{\lambda'}$, completing the proof. □

Recall that $\lambda[k,i]$ is the entry in the k-th row and i-th column of $\mathbf{t}_r(\lambda)$ from Definition 7.1.2.

Lemma A.2.3. *We have*

$$\mathsf{u}_\lambda\Phi^+ = \{e_{\lambda[k,i]} - e_{\lambda[l,j]} \mid 0 \le k,l \le r(\lambda),\ 1 \le i \le j \le \lambda_l,\ \textit{and if } i = j \textit{ then } k < l\},\ \textit{and}$$
$$\mathsf{u}_\lambda\Phi^+_{\lambda'} = \{e_{\lambda[k,i]} - e_{\lambda[l,i]} \mid 1 \le k < l \le r(\lambda),\ 1 \le i \le \lambda_l\}.$$

Proof. Recall that the one-line notation of u_λ is the column reading of $\mathbf{t}_r(\lambda)$. Thus, $\mathsf{u}_\lambda\Phi^+$ consists of the roots $e_{i_1} - e_{i_2}$ where i_1 occurs before i_2 in the one-line notation of u_λ. As $\Phi^+_{\lambda'}$ consists of the roots $e_{i'_1} - e_{i'_2}$ with i'_1, i'_2 in the same column of $\mathbf{t}_c(\lambda)$ and $i'_1 < i'_2$, we have that $\mathsf{u}_\lambda\Phi^+_{\lambda'}$ consists of the roots $e_{i_1} - e_{i_2}$ with i_1 and i_2 in the same column of $\mathbf{t}_r(\lambda)$ and i_1 in a higher row than i_2. □

Lemma A.2.4. *Let* $\beta = e_{\lambda[k,i]} - e_{\lambda[l,j]}$ *with* $1 \le k < l \le r(\lambda)$, $1 \le i \le \lambda_k$, *and* $1 \le j \le \lambda_l$. *Then*

(1) $\mathsf{u}_\lambda^{-1}\beta > 0$ *if and only if* $i \le j$.
(2) $\mathsf{u}_\lambda^{-1}\beta \in \Phi^+_{\lambda'}$ *if and only if* $i = j$.

Proof. Again this follows directly from the definition of u_λ (it can help to note that the one-line notation of u_λ^{-1} is the row reading word of $\mathbf{t}_c(\lambda)$). □

For $1 \le k < l \le r(\lambda)$ define

$$A_{k,l} = \{e_{\lambda(k-1)+i} - e_{\lambda(l-1)+i} \mid 1 \le i \le \lambda_l\}.$$

These sets decompose $\mathsf{u}_\lambda \Phi^+_{\lambda'}$ as follows:

$$\mathsf{u}_\lambda \Phi^+_{\lambda'} = \bigsqcup_{1 \le k < l \le r(\lambda)} A_{k,l}. \tag{A.2.1}$$

For the next three results (Lemma A.2.5, Lemma A.2.6 and Proposition A.2.8) we will set the following notation: let

$$\begin{aligned}&(1)\ \beta = e_{\lambda[k,i]} - e_{\lambda[l,i]} \text{ for } 1 \le k < l \le r(\lambda) \text{ and } 1 \le i \le \lambda_l, \text{ and}\\ &(2)\ \gamma \in P^{(\lambda)} \text{ with } \gamma + Q_\lambda = \sum_{j=1}^{r(\lambda)} a_j \tilde{e}_j \text{ with } a_j = \lambda_j b_j + c_j,\end{aligned} \tag{A.2.2}$$

where, in (2), we have $0 \le c_j < \lambda_j$ for all $0 \le j \le r(\lambda)$ as we did in Proposition 7.2.3. For $1 \le j \le r(\lambda)$ set $c^*_j = \lambda_j - c_j$.

The following two lemmas will be used in Proposition A.2.8 where we explicitly describe $L(\gamma)$ and $R(\gamma)$.

Lemma A.2.5. *Let* β *and* γ *be as in (A.2.2). We have* $\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma^{-1}\beta > 0$ *if and only if either:*
(*a*) $i \le c_k$ *and* $i \le c_l$ *with* $c^*_k \le c^*_l$, *or*
(*b*) $i > c_k$ *and* $i \le c_l$, *or*
(*c*) $i > c_k$ *and* $i > c_l$ *with* $c_l \le c_k$.

Proof. As $\mathsf{y}_\gamma = \mathsf{w}_{J_\lambda \setminus J_\lambda(\gamma)}\mathsf{w}_\lambda$, for $\beta = e_{\lambda[k,i]} - e_{\lambda[l,i]}$ we have $\mathsf{y}_\gamma^{-1}\beta = e_{\lambda[k,i']} - e_{\lambda[l,j']}$ with

$$i' = \begin{cases} c^*_k + i & \text{if } i \le c_k \\ i - c_k & \text{if } i > c_k \end{cases} \qquad j' = \begin{cases} c^*_l + i & \text{if } i \le c_l \\ i - c_l & \text{if } i > c_l. \end{cases}$$

The result follows by Lemma A.2.4 which states that $\mathsf{u}_\lambda^{-1}\beta \ge 0$ if and only if $i' \le j'$ (noting that if $i \le c_k$ and $i > c_l$ then $\mathsf{u}_\lambda^{-1}\beta$ is never positive as $c^*_k \le -c_l$ never occurs, and if $i > c_k$ and $i \le c_l$ then $\mathsf{u}_\lambda^{-1}\beta$ is always positive as $-c_k \le c^*_l$ always holds). □

Lemma A.2.6. *Let* β *and* γ *be as in (A.2.2). We have* $\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma\beta > 0$ *with* $\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma\beta \notin \Phi^+_{\lambda'}$ *if and only if either:*
(*a*) $i \le c^*_k$ *and* $i \le c^*_l$ *with* $c_k < c_l$, *or*
(*b*) $i > c^*_k$ *and* $i \le c^*_l$, *or*
(*c*) $i > c^*_k$ *and* $i > c^*_l$ *with* $c^*_l < c^*_k$.

Proof. Similarly to Lemma A.2.5, for $\beta = e_{\lambda[k,i]} - e_{\lambda[l,i]}$ we have $\mathsf{y}_\gamma\beta = e_{\lambda[k,i']} - e_{\lambda[l,j']}$ with

$$i' = \begin{cases} i - c^*_k & \text{if } i > c^*_k \\ c_k + i & \text{if } i \le c^*_k \end{cases} \qquad j' = \begin{cases} i - c^*_l & \text{if } i > c^*_l \\ c_l + i & \text{if } i \le c^*_l. \end{cases}$$

The result follows from Lemma A.2.4 which states that $\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma\beta > 0$ if and only if $i' \le j'$ and if $i' = j'$ then $\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma\beta \in \Phi^+_{\lambda'}$. □

Remark A.2.7. The results and proofs of Lemma A.2.5 and Lemma A.2.6 can be thought of within tableaux. By Lemma A.2.3 we know that the roots of $\mathsf{u}_\lambda\Phi^+_{\lambda'}$ are of the form $e_{i_1} - e_{i_2}$ where i_1, i_2 are in the same column of $\mathbf{t}_r(\lambda)$ and i_1 is above i_2. As $\mathsf{y}_\gamma = \mathsf{w}_{J_\lambda \setminus J_\lambda(\gamma)}\mathsf{w}_\lambda$, the roots $\mathsf{y}_\gamma\mathsf{u}_\lambda\Phi^+_{\lambda'}$ are of the form $e_{i'_1} - e_{i'_2}$ where i'_1, i'_2 are in the same column (with i'_1 above i'_2) of the following modification of $\mathbf{t}_r(\lambda)$: for each row, $1 \le j \le r(\lambda)$, swap the first c_j elements with the last c^*_j elements. For example, for $\lambda = (4,3,3,2)$ with $c_1 = 2$, $c_2 = 2$, $c_3 = 1$ and $c_4 = 0$, the modified tableaux is

3	4	1	2
7	5	6	
9	10	8	
11	12		

so for $\beta = e_1 - e_5 \in \mathsf{u}_\lambda\Phi^+_{\lambda'}$ we have $\mathsf{y}_\gamma\beta = e_3 - e_7$. Similarly, the roots $\mathsf{y}_\gamma^{-1}\mathsf{u}_\lambda\Phi^+_{\lambda'}$ are of the form $e_{i'_1} - e_{i'_2}$ where i'_1, i'_2 are in the same column (with i'_1 above i'_2) of the following modification of $\mathbf{t}_r(\lambda)$: for each row, $1 \le j \le r(\lambda)$, swap the first c^*_j elements with the last c_j elements. The results of Lemma A.2.5 and Lemma A.2.6 then fall out by considering the relative positions of c_k and c_l in the modified tableaux for all $1 \le k < l \le r(\lambda)$.

Proposition A.2.8. *Let β and γ be as in (A.2.2). We have $\beta \in L(\gamma)$ if and only if either $b_k - b_l \ge 1$, or $b_k - b_l = 0$ with $c_k \ge c_l$ and either:*
(a) $c^*_k > c^*_l$ with $c_l < i \le \lambda_l$, or
(b) $c^*_k \le c^*_l$ with $1 \le i \le \lambda_l$.
We have $\beta \in R(\gamma)$ if and only if either $b_k - b_l \le -1$, or $b_k - b_l = 0$ with $c^*_k > c^*_l$ and either:
(a) $c_k \ge c_l$ with $c^*_l < i \le \lambda_l$, or
(b) $c_k < c_l$ with $1 \le i \le \lambda_l$.

Proof. By Lemma 7.1.1 and Proposition 7.2.3 we have

$$\langle \gamma, \beta \rangle = \begin{cases} b_k - b_l & \text{if } i \le c_k \text{ and } i \le c_l, \text{ or } i > c_k \text{ and } i > c_l \\ b_k - b_l + 1 & \text{if } i \le c_k \text{ and } i > c_l \\ b_k - b_l - 1 & \text{if } i > c_k \text{ and } i \le c_l. \end{cases}$$

We have the following cases:

- If $b_k - b_l \le 2$ then $\langle \gamma, \beta \rangle < 0$ and so $\beta \in L(\gamma)$.
- If $b_k - b_l = -1$ then $\beta \notin L(\gamma)$ unless $i \le c_k$, $i > c_l$ and $\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma^{-1}\beta > 0$. By Lemma A.2.5 $\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma^{-1}\beta < 0$ when $i \le c_k$ and $i > c_l$ so $\beta \notin L(\gamma)$.
- If $b_k - b_l = 0$ we have the following subcases:
  - (a) If $i \le c_k$ and $i > c_l$ then $\beta \in L(\gamma)$.
  - (b) If $i > c_k$ and $i \le c_l$ then $\beta \notin L(\gamma)$.
  - (c) If $i \le c_k$ and $i \le c_l$ then, as $\langle \gamma, \beta \rangle = 0$, $\beta \in L(\gamma)$ if and only if $c^*_k \le c^*_l$ (by Lemma A.2.5).
  - (d) Similarly, if $i > c_k$ and $i > c_l$ then $\beta \in L(\gamma)$ if and only if $c_l \le c_k$.
- If $b_k - b_l = 1$ then $\beta \in L(\gamma)$ unless $i > c_k$ and $i \le c_l$ with $\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma^{-1}\beta < 0$. However, by Lemma A.2.5, when $i > c_k$ and $i \le c_l$ we have $\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma^{-1}\beta > 0$ and so $\beta \in L(\gamma)$.
- If $b_k - b_l \ge 2$ then $\beta \in L(\gamma)$.

Thus, when $b_k - b_l = 0$, $\beta \in L(\gamma)$ if and only if $c_l < i \le c_k$, or $i \le c_k$ and $i \le c_k$ with $c^*_k \le c^*_l$, or $i > c_k$ and $i > c_l$ with $c_l \le c_k$. These cases become the required inequalities noting that the situation when $c_k < c_l$ and $c^*_k \le c^*_l$ is impossible (as it implies $\lambda_k \le \lambda_l - (c_l - c_k) < \lambda_l$) and noting that if $c_l \ge c_k$ then there is no i such that $c_l < i \le c_k$.

For $R(\gamma)$, noting that $\mathsf{w}_\lambda\beta = e_{\lambda[k,\lambda_k-i+1]} - e_{\lambda[l,\lambda_l-i+1]}$ and by Lemma 7.1.1 and Proposition 7.2.3, we have

$$\langle \mathsf{w}_\lambda\gamma, \beta\rangle = \begin{cases} b_k - b_l & \text{if } i \le c_k^* \text{ and } i \le c_l^*, \text{ or } i > c_k^* \text{ and } i > c_l^* \\ b_k - b_l + 1 & \text{if } i > c_k^* \text{ and } i \le c_l^* \\ b_k - b_l - 1 & \text{if } i \le c_k^* \text{ and } i > c_l^*. \end{cases}$$

We have the following cases:

- If $b_k - b_l \le -2$ then $\beta \in R(\gamma)$.
- If $b_k - b_l = -1$ then $\beta \in R(\gamma)$ unless $i > c_k^*$ and $i \le c_l^*$ and $\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma\beta < 0$ or $\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma\beta \in \Phi_{\lambda'}^+$. By Lemma A.2.6 when $i > c_k^*$ and $i \le c_l^*$ we have $\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma\beta > 0$ and $\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma\beta \notin \Phi_{\lambda'}^+$ and so $\beta \in R(\gamma)$.
- If $b_k - b_l = 0$ we have the following subcases:
  - (a) If $i > c_k^*$ and $i \le c_k^*$ then $\beta \notin R(\gamma)$.
  - (b) If $i \le c_k^*$ and $i > c_l^*$ then $\beta \in R(\gamma)$.
  - (c) If $i \le c_k^*$ and $i \le c_l^*$ then $\beta \in R(\gamma)$ if and only if $c_k < c_l$ (by Lemma A.2.6).
  - (d) Similarly, if $i > c_k^*$ and $i > c_l^*$ then $\beta \in R(\gamma)$ if and only if $c_l^* < c_k^*$.
- If $b_k - b_l = 1$ then $\beta \notin R(\gamma)$ unless $i \le c_k^*$ and $i > c_l^*$ with $\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma\beta > 0$ and $\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma\beta \notin \Phi_{\lambda'}^+$. However, by Lemma A.2.6, when $i \le c_k^*$ and $i > c_l^*$ then $\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma\beta < 0$ or $\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma\beta \in \Phi_{\lambda'}^+$ and so $\beta \notin R(\gamma)$.
- If $b_k - b_l \ge 2$ then $\beta \notin R(\gamma)$.

Thus, when $b_k - b_l = 0$ we have that $\beta \in R(\gamma)$ if and only if $c_l^* < i \le c_k^*$, or $i \le c_k^*$ and $i \le c_l^*$ with $c_k < c_l$, or $i > c_k^*$ and $i > c_l^*$ with $c_l^* < c_k^*$. These cases become the inequalities stated in the lemma, completing the proof. □

Corollary A.2.9. *Let* $\gamma \in P^{(\lambda)}$. *For* $1 \le k < l \le r(\lambda)$ *we have*

$$|L(\gamma) \cap A_{k,l}| + |R(\gamma) \cap A_{k,l}| = \lambda_l.$$

Proof. Write $\gamma + Q_\lambda = \sum_{j=1}^{r(\lambda)} a_j\tilde{e}_j$ with $a_j = \lambda_j b_j + c_j$ as in (A.2.2)(2). By Lemma A.2.8 if $|b_k - b_l| \ge 1$ then $|L(\gamma) \cap A_{k,l}| + |R(\gamma) \cap A_{k,l}| = \lambda_l$. If $b_k - b_l = 0$ then

$$|L(\gamma) \cap A_{k,l}| = \begin{cases} \lambda_l - c_l & c_k \ge c_l \text{ and } c_k^* > c_l^* \\ \lambda_l & c_k \ge c_l \text{ and } c_k^* \le c_l^* \\ 0 & \text{otherwise,} \end{cases}$$

and

$$|R(\gamma) \cap A_{k,l}| = \begin{cases} c_l & c_k \ge c_l \text{ and } c_k^* > c_l^* \\ \lambda_l & c_k < c_l \text{ and } c_k^* > c_l^* \\ 0 & \text{otherwise.} \end{cases}$$

The result then follows acknowledging that the case when $c_k < c_l$ and $c_k^* \le c_l^*$ is not possible. □

Theorem A.2.10. *Let* $\gamma \in P^{(\lambda)}$. *There exist* $x, y \in W_{\lambda'}$ *such that* $\mathsf{m}_\gamma = x\mathsf{u}_\lambda^{-1}\tau_\gamma\mathsf{u}_\lambda y$ *with*

$$\ell(\mathsf{m}_\gamma) = \ell(x) + \ell(\mathsf{u}_\lambda^{-1}\tau_\gamma\mathsf{u}_\lambda) + \ell(y) \quad \textit{and} \quad \ell(x) + \ell(y) = \ell(\mathsf{w}_{\lambda'}).$$

Proof. By (A.2.1) and Corollary A.2.9 we have that

$$|L(\gamma)| + |R(\gamma)| = \sum_{1\le k<l\le r(\lambda)} (|L(\gamma) \cap A_{k,l}| + |R(\gamma) \cap A_{k,l}|) = \sum_{1\le k<l\le r(\lambda)} \lambda_l = \ell(\mathsf{w}_{\lambda'})$$

where the last equality follows from Lemma 7.1.11. The result then follows from Proposition A.2.2. □

A.3 The length of $\mathsf{u}_\lambda^{-1}\tau_\gamma\mathsf{u}_\lambda$

In this section we calculate $\ell(\mathsf{u}_\lambda^{-1}\tau_\gamma\mathsf{u}_\lambda)$ which will be used, with the results of the last section, to prove the second part of Theorem 11.2.5 in the next section. To begin we make some notation definitions.

For $1 \le k < l \le r(\lambda)$ let

$$\beta^+(k,l;i,j) = e_{\lambda[k,i]} - e_{\lambda[l,j]} \qquad \text{for } 1 \le i \le j \le \lambda_l$$
$$\beta^-(k,l;i,j) = e_{\lambda[l,j]} - e_{\lambda[k,i]} \qquad \text{for } 1 \le j \le \lambda_l \text{ and } j < i \le \lambda_k.$$

Furthermore, define the sets of these roots as follows:

$$B^+_{k,l} = \{\beta^+(k,l;i,j) \mid 1 \le i \le j \le \lambda_l\} \quad \text{and} \quad B^-_{k,l} = \{\beta^-(k,l;i,j) \mid 1 \le j \le \lambda_l \text{ and } j < i \le \lambda_k\}.$$

Let $B_{k,l} = B^+_{k,l} \cup B^-_{k,l}$. By Lemma A.2.3 we have

$$\mathsf{u}_\lambda\Phi^+\backslash\Phi_\lambda = \bigsqcup_{1\le k<l\le r(\lambda)} B_{k,l}.$$

Let $\Phi^+_{k,l} = B^+_{k,l} \cup (-B^-_{k,l})$. Thus, $\Phi^+_{k,l}$ consists precisely of the roots $e_p - e_q \in \Phi^+$ with p in row k and q in row l of $\mathbf{t}_r(\lambda)$. Furthermore, $\Phi^+\backslash\Phi_\lambda = \bigsqcup_{1\le k<l\le r(\lambda)} \Phi^+_{k,l}$.

Example A.3.1. Let $\lambda = (6,4)$, so

$$\mathbf{t}_r(\lambda) = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 7 & 8 & 9 & 10 \\ \cline{1-4}\end{array}$$

and we have

$$B^+_{k,l} = \{e_{1,7}, e_{1,8}, e_{1,9}, e_{1,10}, e_{2,8}, e_{2,9}, e_{2,10}, e_{3,9}, e_{3,10}, e_{4,10}\} \text{ and}$$
$$B^-_{k,l} = \{e_{7,2}, e_{7,3}, e_{7,4}, e_{7,5}, e_{7,6}, e_{8,3}, e_{8,4}, e_{8,5}, e_{8,6}, e_{9,4}, e_{9,5}, e_{9,6}, e_{10,5}, e_{10,6}\},$$

where $e_{p,q} = e_p - e_q$.

As in the previous section, write $\gamma + Q_\lambda = \sum_{m=1}^{r(\lambda)} a_m\tilde{e}_m$, where $a_m = \lambda_m b_m + c_m$ with $0 \le c_m < \lambda_m$ for $1 \le m \le r(\lambda)$, and let $c^*_m = \lambda_m - c_m$.

Lemma A.3.2. *Let $\gamma \in P^{(\lambda)}$ and let $\beta^+ = \beta^+(k,l;i,j)$ (respectively $\beta^- = \beta^-(k,l;i,j)$). We have $\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma^{-1}\beta^+ > 0$ (respectively $\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma^{-1}\beta^- > 0$) if and only if either:*
(a) $i \le c_k$, $j \le c_l$ with $i + c^*_k \le j + c^*_l$ (respectively $i \le c_k$, $j \le c_l$ with $j + c^*_l < i + c^*_k$), or
(b) $i \le c_k$, $j > c_l$ with $i + c^*_k \le j - c_l$ (respectively $i \le c_k$, $j > c_l$), or
(c) $i > c_k$, $j \le c_l$ (respectively $i > c_k$, $j \le c_l$ with $j + c^*_l < i - c_k$), or
(d) $i > c_k$, $j > c_l$ with $i - c_k \le j - c_l$ (respectively $i > c_k$, $j > c_l$ with $j - c_l < i - c_k$).

Proof. As in Lemma A.2.5, by the definition of y_γ we have that $\mathsf{y}_\gamma^{-1}\beta^+ = e_{\lambda[k,i']} - e_{\lambda[l,j']}$ and $\mathsf{y}_\gamma^{-1}\beta^- = e_{\lambda[l,j']} - e_{\lambda[k,i']}$ where

$$i' = \begin{cases} c^*_k + i & \text{if } i \le c_k \\ i - c_k & \text{if } i > c_k \end{cases} \qquad j' = \begin{cases} c^*_l + j & \text{if } j \le c_l \\ j - c_l & \text{if } j > c_l. \end{cases} \tag{A.3.1}$$

By Lemma A.2.4 we have $\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma^{-1}\beta^+ > 0$ if and only if $i' \le j'$ and $\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma^{-1}\beta^- > 0$ if and only if $j' < i'$, and the result follows. $\square$

Lemma A.3.3. *If* $\gamma \in P^{(\lambda)}$ *and* $\alpha \in \Phi^+$ *with* $\mathsf{u}_\lambda\alpha \in \Phi_\lambda$ *then* $\langle \gamma, \mathsf{u}_\lambda\alpha\rangle = \chi^-(\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma^{-1}\mathsf{u}_\lambda\alpha)$.

Proof. Let $\beta = \mathsf{u}_\lambda\alpha \in \Phi_\lambda$ with $\alpha \in \Phi^+$. Thus, by Lemma A.2.3, $\beta = e_{\lambda[k,i]} - e_{\lambda[k,j]}$ with $1 \le k \le r(\lambda)$ and $1 \le i < j \le \lambda_k$. Write $\gamma + Q_\lambda = \sum_{m=1}^{r(\lambda)} a_m\tilde{e}_m$ as in (A.2.2)(2) and so $a_k = \lambda_k b_k + c_k$ with $0 \le c_k < \lambda_k$. By Lemma 7.1.1 and Proposition 7.2.3 we have

$$\langle \gamma, \beta\rangle = \begin{cases} 0 & \text{if either } i, j \le c_k \text{ or } i, j > c_k \\ 1 & \text{if } i \le c_k < j. \end{cases}$$

We claim that in the first case $\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma^{-1}\beta > 0$ while in the second case $\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma^{-1}\beta < 0$. If $i, j \le c_k$ then $\mathsf{y}_\gamma^{-1}\beta = e_{\lambda[k,i+c_k^*]} - e_{\lambda[k,j+c_k^*]}$ by (A.3.1) and so $\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma^{-1}\beta > 0$ as $c_k^* + i < j + c_k^*$ (by Lemma A.2.4). Similarly, if $i, j > c_k$ then using (A.3.1) and Lemma A.2.4 we have that $\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma^{-1}\beta > 0$ as $i - c_k < j - c_k$.

On the other hand, if $i \le c_k < j$ then $\mathsf{y}_\gamma^{-1}\beta = e_{\lambda[k,c_k^*+1]} - e_{\lambda[k,j-c_k]}$ by (A.3.1). As $i + \lambda_k > j$ we have $i + c_k^* = i + \lambda_k - c_k > j - c_k$ and so by Lemma A.2.4, $\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma^{-1}\beta < 0$ as required. □

Theorem A.3.4. *Let* $\gamma \in P^{(\lambda)}$ *with* $\gamma + Q_\lambda = \sum_{k=1}^{r(\lambda)} a_k\tilde{e}_k$ *where* $a_k = \lambda_k b_k + c_k$ *with* $0 \le c_k < \lambda_k$, *and let* $c_k^* = \lambda_k - c_k$. *Then*

$$\ell(\mathsf{u}_\lambda^{-1}\tau_\gamma\mathsf{u}_\lambda) = \sum_{1\le k<l\le r(\lambda)} z(\gamma,k,l) \quad \textit{where} \quad z(\gamma,k,l) = \begin{cases} |\lambda_l a_k - \lambda_k a_l| & \textit{if } b_k \ne b_l \\ c_l^*|c_k - c_l| + c_l|c_k^* - c_l^*| & \textit{if } b_k = b_l. \end{cases}$$

Proof. For $\beta \in \Phi$ let $h(\gamma,\beta) = \langle\gamma,\beta\rangle - \chi^-(\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma^{-1}\beta)$. Since $\mathsf{u}_\lambda^{-1}\tau_\gamma\mathsf{u}_\lambda = t_{\mathsf{u}_\lambda^{-1}\gamma}\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma\mathsf{u}_\lambda$, by [37, (2.4.1)] we have

$$\ell(\mathsf{u}_\lambda^{-1}\tau_\gamma\mathsf{u}_\lambda) = \sum_{\alpha\in\Phi^+} |\langle \mathsf{u}_\lambda^{-1}\gamma, \alpha\rangle - \chi^-(\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma^{-1}\mathsf{u}_\lambda\alpha)| = \sum_{\beta\in\mathsf{u}_\lambda\Phi^+} |h(\gamma,\beta)|.$$

By Lemma A.3.3, $h(\gamma,\beta) = 0$ if $\beta \in \Phi_\lambda$, so we can eliminate these roots from the sum. Recall that $\mathsf{u}_\lambda\Phi^+\backslash\Phi_\lambda = \bigsqcup_{1\le k<l\le r(\lambda)} B_{k,l}$, thus we have

$$\ell(\mathsf{u}_\lambda^{-1}\tau_\gamma\mathsf{u}_\lambda) = \sum_{1\le k<l\le r(\lambda)} \sum_{\beta\in B_{k,l}} |h(\gamma,\beta)|.$$

Therefore, to prove the theorem it remains to show that

$$\sum_{\beta\in B_{k,l}} |h(\gamma,\beta)| = z(\gamma,k,l). \tag{A.3.2}$$

Let $\beta^+ = \beta^+(k,l;i,j) \in B_{k,l}^+$ and $\beta^- = \beta^-(k,l;i,j) \in B_{k,l}^-$.

By definition $\beta^+ = e_{\lambda[k,i]} - e_{\lambda[l,j]}$ where $1 \le i < j \le \lambda_l$ (and $k < l$) and by Lemma 7.1.1 and Proposition 7.2.3 we have

$$\langle\gamma,\beta^+\rangle = \begin{cases} b_k - b_l & \text{if } i \le c_k,\ j \le c_l \text{ or } i > c_k,\ j > c_l \\ b_k - b_l + 1 & \text{if } i \le c_k,\ j > c_l \\ b_k - b_l - 1 & \text{if } i > c_k,\ j \le c_l. \end{cases} \tag{A.3.3}$$

Thus, it follows that if $b_k - b_l \le -1$ then $h(\gamma,\beta^+) \le 0$ and if $b_k - b_l \ge 1$ then $h(\gamma,\beta^+) \ge 0$ (noting that if $b_k - b_l = 1$ with $i > c_k$ and $j \le c_l$ then by Lemma A.3.2 $\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma^{-1}\beta^+ > 0$ and so $h(\gamma,\beta^+) = -\chi^-(\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma^{-1}\beta^+) = 0$).

Similarly, by definition $\beta^- = e_{\lambda[l,j]} - e_{\lambda[k,i]}$ with $1 \leq j \leq \lambda_l$ and $j < i \leq \lambda_k$ (and $k < l$) so by Lemma 7.1.1 and Proposition 7.2.3 we have

$$\langle \gamma, \beta^- \rangle = \begin{cases} b_l - b_k & \text{if } i \leq c_k,\ j \leq c_l \text{ or } i > c_k,\ j > c_l \\ b_l - b_k + 1 & \text{if } i > c_k,\ j \leq c_l \\ b_l - b_k - 1 & \text{if } i \leq c_k,\ j > c_l \end{cases} \tag{A.3.4}$$

Thus, if $b_k - b_l \geq 1$ then $h(\gamma, \beta^-) \leq 0$ and if $b_k - b_l \leq -1$ then $h(\gamma, \beta^-) \geq 0$ (noting that if $b_k - b_l = -1$ with $i \leq c_k$ and $j > c_l$ then by Lemma A.3.2 we have $\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma^{-1}\beta^- > 0$ and so $h(\gamma, \beta^-) = 0$).

Note that;

$$\begin{aligned} &\text{if } b_k > b_l \text{ then } \lambda_l a_k - \lambda_k a_l = \lambda_k\lambda_l(b_k - b_l) + (\lambda_l c_k - \lambda_k c_l) > \lambda_k\lambda_l - \lambda_k\lambda_l = 0, \\ &\text{if } b_k < b_l \text{ then } \lambda_l a_k - \lambda_k a_l = \lambda_k\lambda_l(b_k - b_l) + (\lambda_l c_k - \lambda_k c_l) < \lambda_k\lambda_l - \lambda_k\lambda_l = 0, \end{aligned} \tag{A.3.5}$$

as $0 \leq c_k < \lambda_k$ and $0 \leq c_l < \lambda_l$.

We split into the following cases to prove (A.3.2):

Case 1: Suppose that $b_k - b_l \geq 1$. From (A.3.3) and (A.3.4) we have $h(\gamma, \beta^+) \geq 0$ and $h(\gamma, \beta^-) \leq 0$. Thus,

$$\begin{aligned} \sum_{\beta \in B_{k,l}} |h(\gamma, \beta)| &= \sum_{\beta \in B_{k,l}^+} h(\gamma, \beta) - \sum_{\beta \in B_{k,l}^-} h(\gamma, \beta) \\ &= \sum_{\beta \in B_{k,l}^+} \langle \gamma, \beta \rangle - \sum_{\beta \in B_{k,l}^-} \langle \gamma, \beta \rangle - \sum_{\beta \in B_{k,l}^+} \chi^-(\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma^{-1}\beta) + \sum_{\beta \in B_{k,l}^-} \chi^-(\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma^{-1}\beta) \\ &= \langle \gamma, 2\rho_{k,l} \rangle - \sum_{\beta \in B_{k,l}^+} \chi^-(\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma^{-1}\beta) + \sum_{\beta \in B_{k,l}^-} \chi^-(\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma^{-1}\beta). \end{aligned}$$

where $2\rho_{k,l} = \sum_{\beta \in B_{k,l} \cup (-B_{k,l}^-)} \beta = \sum_{\beta \in \Phi_{k,l}^+} \beta$. Let $S = -\sum_{\beta \in B_{k,l}} |h(\gamma, \beta)| + \langle \gamma, 2\rho_{k,l} \rangle = \sum_{\beta \in B_{k,l}^+} \chi^-(\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma^{-1}\beta) - \sum_{\beta \in B_{k,l}^-} \chi^-(\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma^{-1}\beta)$.

We claim that $B_{k,l}^+ = \Phi_{k,l}^+ \backslash \Phi(\mathsf{u}_\lambda)$ and $-B_{k,l}^- = \Phi(\mathsf{u}_\lambda) \cap \Phi_{k,l}$. We have that $\mathsf{u}_\lambda \Phi^+ \backslash \Phi_\lambda = \sqcup_{1 \leq k < l \leq r(\lambda)} (B_{k,l}^+ \cup B_{k,l}^-)$ where $B_{k,l}^+ \subseteq \Phi^+$ and $B_{k,l}^- \subseteq \Phi^-$. Thus, if $\beta \in B_{k,l}^+$ then $\beta > 0$ and there exists $\alpha \in \Phi^+$ such that $\beta = \mathsf{u}_\lambda \alpha$ so $\beta \notin \Phi(\mathsf{u}_\lambda)$. On the other hand, if $\beta \in B_{k,l}^-$ then $\beta < 0$ and there exists $\alpha \in \Phi^+$ such that $\beta = \mathsf{u}_\lambda \alpha$ and so $-\beta \in \Phi(\mathsf{u}_\lambda)$. The proof of the reverse containment is similar, proving the claim.

Therefore,

$$\begin{aligned} S &= \sum_{\beta \in \Phi_{k,l}^+ \backslash \Phi(\mathsf{u}_\lambda)} \chi^-(\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma^{-1}\beta) - \sum_{\beta \in \Phi(\mathsf{u}_\lambda) \cap \Phi_{k,l}} \chi^-(-\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma^{-1}\beta) \\ &= \sum_{\beta \in \Phi_{k,l}^+} \chi^-(\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma^{-1}\beta) - \sum_{\beta \in \Phi(\mathsf{u}_\lambda) \cap \Phi_{k,l}} [\chi^-(-\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma^{-1}\beta) + \chi^-(\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma^{-1}\beta)] \\ &= |\Phi(\mathsf{y}_\gamma \mathsf{u}_\lambda) \cap \Phi_{k,l}| - |\Phi(\mathsf{u}_\lambda) \cap \Phi_{k,l}| \end{aligned}$$

as every term in the second sum is 1. Since $\ell(\mathsf{y}_\gamma \mathsf{u}_\lambda) = \ell(\mathsf{u}_\lambda) + \ell(\mathsf{y}_\gamma)$ (as u_λ is J_λ reduced on the left) we have that $\Phi(\mathsf{y}_\gamma \mathsf{u}_\lambda) = \Phi(\mathsf{y}_\gamma) \sqcup \mathsf{y}_\gamma \Phi(\mathsf{u}_\lambda)$ (by (1.1.1)). Thus, $|\Phi(\mathsf{y}_\gamma \mathsf{u}_\lambda) \cap \Phi_{k,l}| =$

$|\Phi(\mathsf{y}_\gamma) \cap \Phi_{k,l}| + |\Phi(\mathsf{u}_\lambda) \cap \mathsf{y}_\gamma^{-1}\Phi_{k,l}|$. However, as $\Phi(\mathsf{y}_\gamma) \subseteq \Phi_\lambda^+$ we have $\Phi(\mathsf{y}_\gamma) \cap \Phi_{k,l} = \emptyset$ and $\mathsf{y}_\gamma^{-1}\Phi_{k,l} = \Phi_{k,l}$ which reduces $S = 0$. Hence,

$$\sum_{\beta \in B_{k,l}} |h(\gamma, \beta)| = \langle \gamma, 2\rho_{k,l} \rangle.$$

Then, as

$$2\rho_{k,l} = \sum_{\beta \in \Phi_{k,l}^+} \beta = \sum_{i,j} e_{\lambda[k,i]} - e_{\lambda[l,j]} = \lambda_l \sum_{i=1}^{\lambda_k} e_{\lambda[k,i]} + \lambda_k \sum_{j=1}^{\lambda_l} e_{\lambda[l,j]}$$

and by Lemma 7.1.1 and Proposition 7.2.3 we have $\langle \gamma, 2\rho_{k,l} \rangle = \lambda_l a_k - \lambda_k a_l$ as required (noting (A.3.5)).

Case 2: Suppose that $b_k - b_l \leq -1$. From (A.3.3) and (A.3.4) we have $h(\gamma, \beta^+) \leq 0$ and $h(\gamma, \beta^-) \geq 0$, and so

$$\sum_{\beta \in B_{k,l}} |h(\gamma, \beta)| = -\sum_{\beta \in B_{k,l}^+} h(\gamma, \beta) + \sum_{\beta \in B-k,l^-} h(\gamma, \beta) = -\langle \gamma, 2\rho_{k,l} \rangle = \lambda_k a_l - \lambda_l a_k$$

so once again (A.3.2) holds (noting (A.3.5)).

Case 3: Suppose that $b_k - b_l = 0$.

In this case

$$z(\gamma, k, l) = \begin{cases} \lambda_k c_l - \lambda_l c_k & \text{if } c_k \leq c_l \text{ (and hence } c_k^* \geq c_l^*) \\ \lambda_l c_k - \lambda_k c_l & \text{if } c_k > c_l \text{ and } c_k^* \leq c_l^* \\ c_l^*(c_k - c_l) + c_l(c_k^* - c_l^*) & \text{if } c_k > c_l \text{ and } c_k^* > c_l^*. \end{cases} \tag{A.3.6}$$

Let $\chi^+(\cdot)$ be the characteristic function of Φ^+, thus $1 - \chi^-(\cdot) = \chi^+(\cdot)$. By (A.3.3) and (A.3.4) we have the following:

- If $i \leq c_k$ and $j \leq c_l$, or $i > c_k$ and $j > c_l$ then $|h(\gamma, \beta^+)| = \chi^-(\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma^{-1}\beta^+)$ and $|h(\gamma, \beta^-)| = \chi^-(\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma^{-1}\beta^-)$.
- If $i \leq c_k$ and $j > c_l$ then $|h(\gamma, \beta^+)| = \chi^+(\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma^{-1}\beta^+)$ and $|h(\gamma, \beta^-)| = 1$ (by Lemma A.3.2).
- If $i > c_k$ and $j \leq c_l$ then $|h(\gamma, \beta^+)| = 1$ (by Lemma A.3.2) and $|h(\gamma, \beta^-)| = \chi^+(\mathsf{u}_\lambda^{-1}\mathsf{y}_\gamma^{-1}\beta^-)$.

Thus, using Lemma A.3.2, we have that $\sum_{\beta \in B_{k,l}} |h(\gamma, \beta)| = \sum_{i=1}^{8} |X_i|$, where

$$\begin{aligned}
X_1 &= \{(i,j) \mid 1 \leq i \leq j \leq \lambda_l,\ i \leq c_k,\ j \leq c_l,\ i + c_k^* > j + c_l^*\} \\
X_2 &= \{(i,j) \mid 1 \leq i \leq j \leq \lambda_l,\ i > c_k,\ j > c_l,\ i - c_k > j - c_l\} \\
X_3 &= \{(i,j) \mid 1 \leq i \leq j \leq \lambda_l,\ i \leq c_k,\ j > c_l,\ i + c_k^* \leq j - c_l\} \\
X_4 &= \{(i,j) \mid 1 \leq i \leq j \leq \lambda_l,\ i > c_k,\ j \leq c_l\} \\
X_5 &= \{(i,j) \mid 1 \leq j \leq \lambda_l,\ j < i \leq \lambda_k,\ i \leq c_k,\ j \leq c_l,\ i + c_k^* \leq j + c_l^*\} \\
X_6 &= \{(i,j) \mid 1 \leq j \leq \lambda_l,\ j < i \leq \lambda_k,\ i > c_k,\ j > c_l,\ i - c_k \leq j - c_l\} \\
X_7 &= \{(i,j) \mid 1 \leq j \leq \lambda_l,\ j < i \leq \lambda_k,\ i \leq c_k,\ j > c_l\} \\
X_8 &= \{(i,j) \mid 1 \leq j \leq \lambda_l,\ j < i \leq \lambda_k,\ i > c_k,\ j \leq c_l,\ i - c_k > j + c_l^*\}.
\end{aligned}$$

Thus, it remains to prove the cardinalities of each of these sets. There are three cases. We will prove the first case in full, the remaining two cases follow similarly.

Let $c_k \leq c_l$. As $\lambda_k \geq \lambda_l$ this forces $c_k^* \geq c_l^*$. Figure A.1 illustrates the possible regions in which $X_1, \ldots, X_8$ occur in the (i,j)-plane; X_1 is a subset of region (a), X_2 is a subset of region (b), X_3 is a subset of region (c), X_4 is equal to region (d) (including the boundaries except the line $i = c_k$), X_5 is a subset of region (e), X_6 is a subset of region (f) and X_8 is a subset of region (g).

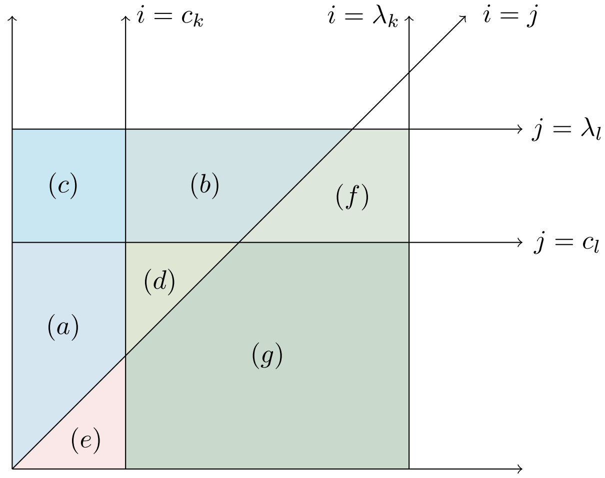
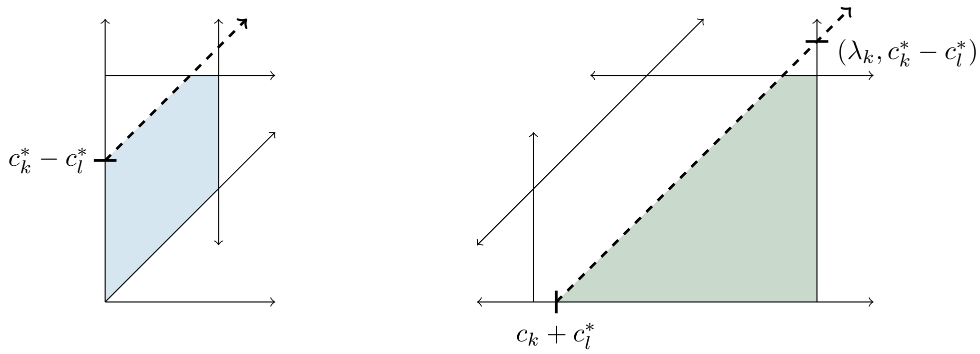


Figure A.1: The regions of the (i,j)-plane when $c_k \leq c_l$.



(i) X_1 in region (a) (ii) X_8 in region (g)

Figure A.2: The points of X_1 and X_8.

Consider X_1. If $c_l < c_k^* - c_l^*$ then X_1 is equal to region (a) from Figure A.1 (including all boundaries) as the dotted line ($j = i + c_k^* - c_l^*$) in Figure A.2(i) lies completely above the region. Counting the integer points gives

$$|X_1| = \sum_{i=1}^{c_k} \sum_{j=i}^{c_l} 1 = c_k(c_l+1) - \frac{1}{2}c_k(c_k+1).$$

If $c_l \geq c_k^* - c_l^*$ then the dotted line divides the region and so X_1 is the points in the shaded section of Figure A.2(i) (not including the points on the dashed line). Counting the integer points we have

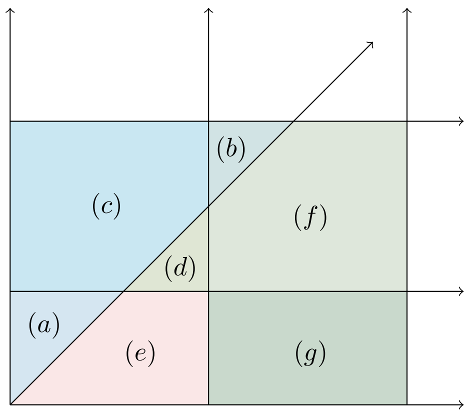
$$|X_1| = \sum_{i=1}^{c_k} \sum_{j=i}^{c_k^*-c_l^*} 1 + \sum_{j=c_k^*-c_l^*+1}^{c_l} \sum_{i=j+c_l^*-c_k^*+1}^{c_k} 1 = c_k(c_k^* - c_l^*) - \frac{1}{2}(\lambda_k - \lambda_l)(\lambda_k - \lambda_l - 1).$$

A similar process, using Figure A.2(ii) and adjusting the dotted line depending on whether $c_k^* - c_l^* \leq c_l$, gives that

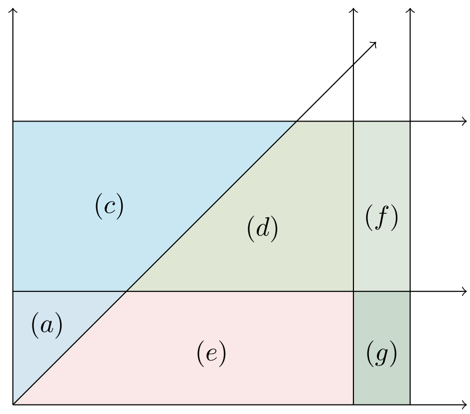
$$|X_8| = \begin{cases} \frac{1}{2}(c_k^* - c_l^* - 1)(c_k^* - c_l^*) & \text{if } c_l \geq c_k^* - c_l^* \\ c_l(c_k^* - c_l^*) - \frac{1}{2}c_l(c_l + 1) & \text{if } c_l < c_k^* - c_l^*. \end{cases}$$

A similar analysis shows that $|X_2| = (c_l - c_k)(\lambda_l - c_l)$ (using Picks theorem can help here), $|X_4| = \frac{1}{2}(c_l - c_k)(c_l - c_k)(c_l - c_k + 1)$ and $|X_3| = |X_5| = |X_6| = |X_7| = 0$. Summing gives $\sum_{\beta \in B_{k,l}} |h(\gamma, \beta)| = \lambda_k c_l - \lambda_l c_k$ as required (see (A.3.6)).

When $c_k > c_l$ the possible regions split into two cases; $c_k \leq \lambda_l$ and $c_k > \lambda_l$. Figure A.3 gives the possible regions in which the sets occur; X_1 is a subset of (a), X_2 is a subset of (b), X_3 is a subset of (c), X_5 is a subset of (e), X_6 is a subset of (f), X_7 is equal to region (d) when $c_k \leq \lambda_l$ (excluding the boundary points on line $j = i$ and $j = c_l$) and X_8 is a subset of region (g).



(i) When $c_k \leq \lambda_l$



(i) When $c_k > \lambda_l$

Figure A.3: The regions of the (i, j)-plane when $c_k > c_l$.

Similar calculations to the ones for when $c_k \leq c_l$ show that the sets have the following cardinalities.

When $c_k^* \leq c_l^*$: $|X_1| = |X_2| = |X_4| = |X_8| = 0$, $|X_3| = \frac{1}{2}(c_l^* - c_k^*)(c_l^* - c_k^* + 1)$, $|X_5| = c_l(c_l^* - c_k^*)$, and

$$|X_6| = \begin{cases} (\lambda_l - c_k)(c_k - c_l) + \frac{1}{2}(c_k - c_l)(c_k - c_l + 1) - \frac{1}{2}(c_l^* - c_k^*)(c_l^* - c_k^* + 1) & \text{if } c_k \leq \lambda_l \\ \frac{1}{2}c_l^*(c_l^* + 1) - \frac{1}{2}(c_l^* - c_k^*)(c_l^* - c_k^* + 1) & \text{if } c_k > \lambda_l \end{cases}$$

$$|X_7| = \begin{cases} \frac{1}{2}(c_k - c_l - 1)(c_k - c_l) & \text{if } c_k \leq \lambda_l \\ \frac{1}{2}(c_k - c_l - 1)(c_k - c_l) - \frac{1}{2}(c_k - \lambda_l - 1)(c_k - \lambda_l) & \text{if } c_k > \lambda_l. \end{cases}$$

Thus $\sum_{\beta \in B_{k,l}} |h(\gamma, \beta)| = \lambda_l c_k - \lambda_k c_l$ as required.

When $c_k^* > c_l^*$*:* $|X_2| = |X_3| = |X_4| = |X_5| = 0$, and

$$|X_1| = \begin{cases} \frac{1}{2}c_l(c_l+1) & \text{if } c_l < c_k^* - c_l^* \\ (c_l - (c_k^* - c_l^*))(c_k^* - c_l^*) + \frac{1}{2}(c_k^* - c_l^*)(c_k^* - c_l^* + 1) & \text{if } c_l \geq c_k^* - c_l^* \end{cases}$$

$$|X_6| = \begin{cases} (\lambda_l - c_k)(c_k - c_l) + \frac{1}{2}(c_k - c_l)(c_k - c_l + 1) & \text{if } c_k \leq \lambda_l \\ \frac{1}{2}c_l^*(c_l^* + 1) & \text{if } c_k > \lambda_l \end{cases}$$

$$|X_7| = \begin{cases} \frac{1}{2}(c_k - c_l - 1)(c_k - c_l) & \text{if } c_k \leq \lambda_l \\ \frac{1}{2}(c_k - c_l - 1)(c_k - c_l) - \frac{1}{2}(c_k - \lambda_l - 1)(c_k - \lambda_l) & \text{if } c_k > \lambda_l \end{cases}$$

$$|X_8| = \begin{cases} \frac{1}{2}c_l(c_l - 1) + c_l(c_k^* - \lambda_l) & \text{if } c_l < c_k^* - c_l^* \\ \frac{1}{2}(c_k^* - c_l^* - 1)(c_k^* - c_l^*) & \text{if } c_l \geq c_k^* - c_l^*. \end{cases}$$

Thus $|X_1| + |X_8| = c_l(c_k^* - c_l^*)$ (in both cases) and $|X_6| + |X_7| = c_l^*(c_k - c_l)$ (in both cases), and hence $\sum_{\beta \in B_{k,l}} |h(\gamma, \beta)| = c_l^*(c_k - c_l) + c_l(c_k^* - c_l^*)$ as required, completing the proof. □

Remark A.3.5. By (A.3.6) we have that $z(\gamma, k, l) = |\lambda_l a_k - \lambda_k a_l|$ in all cases except for $b_k = b_l$ with $c_k > c_l$ and $c_k^* > c_l^*$. If $\lambda = (d^r)$ (that is λ is rectangular with r rows of length d) then the case when $c_k > c_l$ and $c_k^* > c_l^*$ cannot happen as $\lambda_k = \lambda_l = d$ implies that $\lambda_k - c_k < \lambda_l - c_l$. Thus, in this case $z(\gamma, k, l) = d|a_k - a_l|$ for all $k < l$ and so

$$\ell(\mathsf{u}_\lambda^{-1}\tau_\gamma \mathsf{u}_\lambda) = d \sum_{1 \leq k < l \leq r(\lambda)} |a_k - a_l|.$$

Example A.3.6. Let $\lambda = (4, 2)$ with $\gamma = 2\tilde{e}_1 + \tilde{e}_2$. Recall that from Remark 7.2.4 we can write γ in a tableaux as follows:

$$\gamma \leftrightarrow \begin{array}{|c|c|c|c|} \hline 1 & 1 & 0 & 0 \\ \hline 1 & 0 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}$$

We have $b_1 = b_2 = 0$ and $c_1 = 2$, $c_2 = 1$, $c_1^* = 2$ and $c_2^* = 1$ and so are in the case where $c_1 > c_2$ and $c_1^* > c_2^*$. Therefore, by Theorem A.3.4 we have that

$$\ell(\mathsf{u}_\lambda^{-1}\tau_\gamma \mathsf{u}_\lambda) = z(\gamma, 1, 2) = c_2^*(c_1 - c_2) + c_2(c_1^* - c_2^*) = 2.$$

Indeed by direct calculation $\mathsf{u}_\lambda^{-1}\tau_\gamma \mathsf{u}_\lambda = (s_4 s_5 s_2 s_3 s_4)(s_4 s_3 s_0 s_1 \sigma^3)(s_4 s_3 s_2 s_5 s_4) = s_1 s_4 \sigma^3$.

A.4 Monotonicity of $\ell(\mathsf{m}_\gamma)$ with respect to $\preccurlyeq_\lambda$

This section completes the proof of Theorem 11.2.5, by proving the monotonicity of the length of m_γ with respect to $\preccurlyeq_\lambda$.

Theorem A.4.1. *If* $\gamma, \gamma' \in P_+^{(\lambda)}$ *with* $\gamma + Q_\lambda \preccurlyeq_\lambda \gamma' + Q_\lambda$ *then* $\ell(\mathsf{m}_\gamma) \leq \ell(\mathsf{m}_{\gamma'})$ *with equality if and only if* $\gamma = \gamma'$*.*

Proof. Let $B_1, \ldots, B_t \subseteq \{1, 2, \ldots, r(\lambda)\}$ be the 'blocks' of $\mathbf{t}_r(\lambda)$ (note $t = |L(\lambda)|$). That is the sets of rows with the same length: if $i, j \in B_p$ then $\lambda_i = \lambda_j$ (for example, if $\lambda = (5, 5, 3, 3, 3, 2, 1, 1, 1)$ as in Example 7.2.11 then $B_1 = \{1, 2\}$, $B_2 = \{3, 4, 5\}$, $B_3 = \{6\}$ and $B_4 = \{7, 8, 9\}$).

By Theorem A.2.10 it is sufficient to prove that $\ell(\mathsf{u}_\lambda^{-1}\tau_\gamma \mathsf{u}_\lambda) \leq \ell(\mathsf{u}_\lambda^{-1}\tau_{\gamma'} \mathsf{u}_\lambda)$ with equality if and only if $\gamma = \gamma'$. We can make the following further simplifications. As $\gamma + Q_\lambda \preccurlyeq_\lambda \gamma' + Q_\lambda$

we can write $\gamma' + Q_\lambda = \gamma + \epsilon + Q_\lambda$ with $\epsilon \in Q_+^\lambda$. By Lemma 7.2.18 when writing $\epsilon = \sum_{i=1}^{r(\lambda)} d_i \tilde{e}_i$ we have $\sum_{i\in B_p} d_i = 0$ for all $1 \leq p \leq t$ and $d_1 + d_2 + \cdots + d_i \geq 0$ for all $1 \leq i \leq r(\lambda)$. Let $\epsilon = \epsilon_1 + \cdots + \epsilon_t$ with $\epsilon_p = \sum_{i\in B_p} d_i \tilde{e}_i$. As $\gamma' \in P_+^{(\lambda)}$ we have that $\gamma + \epsilon_p \in (P/Q_\lambda)_+$ (as dominance is defined within blocks, see (7.2.4)). Furthermore, $\gamma + Q_\lambda \preccurlyeq_\lambda \gamma + \epsilon_p + Q_\lambda \preccurlyeq_\lambda \gamma' + Q_\lambda$ so it is equivalent to prove the result for $\gamma' + Q_\lambda = \gamma + \epsilon_p + Q_\lambda$ (that is, $\gamma + Q_\lambda$ and $\gamma' + Q_\lambda$ only differ in one block). By a well known property of the dominance order (see for example [47, Corollary 2.7]) we have that $\epsilon_p \in \Phi_{G_\lambda}^+$ and so $\epsilon_p = \tilde{e}_i - \tilde{e}_j$ for some $i, j \in B_p$ with $i < j$. Thus we henceforth assume that $\gamma, \gamma' \in P_+^{(\lambda)}$ with $\gamma' + Q_\lambda = \gamma + \tilde{e}_i - \tilde{e}_j + Q_\lambda$ for some $i, j \in B_p$ with $i < j$.

Let $M(\gamma) = \ell(\mathsf{u}_\lambda^{-1} \tau_\gamma \mathsf{u}_\lambda)$. By Theorem A.3.4 we have

$$M(\gamma) = \sum_{1\leq k<l\leq r(\lambda)} z(\gamma,k,l) = \sum_{p=1}^{t} \sum_{\substack{k,l\in B_p,\\ k<l}} z(\gamma,k,l) + \sum_{1\leq p<q\leq t} \sum_{\substack{k\in B_p,\\ l\in B_q}} z(\gamma,k,l)$$

and similarly for $M(\gamma')$.

Write $\gamma + Q_\lambda = \sum_{m=1}^{r(\lambda)} a_m \tilde{e}_m$ and $\gamma' + Q_\lambda = \sum_{m=1}^{r(\lambda)} a'_m \tilde{e}_m$ as in (A.2.2)(2). As $\gamma' - \gamma + Q_\lambda = \tilde{e}_i - \tilde{e}_j + Q_\lambda$ we have that $a'_m = a_m$ unless $m \in \{i, j\}$, in which case $a'_i = a_i + 1$ and $a'_j = a_j - 1$. Let $B_{<p} = B_1 \cup \cdots \cup B_{p-1}$ and $B_{>p} = B_{p+1} \cup \cdots \cup B_t$. As $z(\gamma', k, l) = z(\gamma, k, l)$ whenever $k, l \notin \{i, j\}$ we have $M(\gamma') - M(\gamma) = A + B + C$ where

$$\begin{aligned}
A &= \sum_{\substack{k,l\in B_p,\\ k<l}} (z(\gamma',k,l) - z(\gamma,k,l)),\\
B &= \sum_{k\in B_{<p}} (z(\gamma',k,i) - z(\gamma,k,i) + z(\gamma',k,j) - z(\gamma,k,j)),\\
C &= \sum_{l\in B_{>p}} (z(\gamma',i,l) - z(\gamma,i,l) + z(\gamma',j,l) - z(\gamma,j,l)).
\end{aligned}$$

It is required to show that $M(\gamma') - M(\gamma) > 0$. Let $\lambda_0 = \lambda_k$ for all $k \in B_p$ (so λ_0 is the length of the rows in block B_p). We will consider sum A first. For $k, l \in B_p$ with $k < l$ we have $a_k \geq a_l$ as $\gamma \in P_+^{(\lambda)}$ (by the dominance in blocks, see (7.2.4)). Thus $b_k \geq b_l$ and if $b_k = b_l$ then $c_k \geq c_l$ (which forces $\lambda_k - c_k \leq \lambda_l - c_l$) and so $z(\gamma, k, l) = |\lambda_0 a_k - \lambda_0 a_l|$ and $z(\gamma', k, l) = |\lambda_0(a'_k - a'_l)|$ by (A.3.6). Thus, as $a_k \geq a_l$ we have $z(\gamma, k, l) = \lambda_0(a_k - a_l)$ and $z(\gamma', k, l) = \lambda_0(a'_k - a'_l)$. Then, as $a'_k = a_k$ and $a'_l = a_l$ when $k, l \notin \{i, j\}$ and $a'_i = a_i + 1$ and $a'_j = a_j - 1$ we have that

$$\begin{aligned}
A = \sum_{\substack{k,l\in B_p,\\ k<l}} (z(\gamma',k,l) - z(\gamma,k,l)) &= \lambda_0 \sum_{\substack{l\in B_p,\\ i<l}} 1 - \lambda_0 \sum_{\substack{l\in B_p,\\ j<l}} 1 - \lambda_0 \sum_{\substack{k\in B_p,\\ i>k}} 1 + \lambda_0 \sum_{\substack{k\in B_p,\\ j>k}} 1\\
&= 2\lambda_0(j - i) > 0.
\end{aligned}$$

As this is a strict inequality, to prove that $M(\gamma') - M(\gamma) > 0$ it remains to prove that $B \geq 0$ and $C \geq 0$. We have that $a'_i = a_i + 1$ and $a'_j = a_j - 1$, and

$$b'_i = \begin{cases} b_i & \text{if } c_i < \lambda_i - 1\\ b_i + 1 & \text{if } c_i = \lambda_i - 1 \end{cases} \qquad c'_i = \begin{cases} c_i + 1 & \text{if } c_i < \lambda_i - 1\\ 0 & \text{if } c_i = \lambda_i - 1 \end{cases}$$

$$b'_j = \begin{cases} b_j & \text{if } c_j > 0\\ b_j - 1 & \text{if } c_j = 0 \end{cases} \qquad c'_j = \begin{cases} c_j - 1 & \text{if } c_j > 0\\ \lambda_j - 1 & \text{if } c_j = 0. \end{cases}$$

For sum B, let

$$x_1(k) = z(\gamma', k, i) - z(\gamma, k, i) \text{ and } x_2(k) = z(\gamma', k, j) - z(\gamma, j, k),$$

and set $x(k) = x_1(k) + x_2(k)$ for $k \in B_{<p}$. We claim that $x(k) \geq 0$ for all $k \in B_{<p}$. First consider $x_1(k)$. As before, let $c_m^* = \lambda_m - c_m$ for $1 \leq m \leq r(\lambda)$ and note $0 \leq c_m < \lambda_m$. We claim that

$$x_1(k) = \begin{cases} -\lambda_k & \text{if } b_k > b_i \\ -\lambda_k & \text{if } b_k = b_i = b_i' \text{ with } c_k > c_i \text{ and } c_k^* < c_i^* \\ \lambda_k - 2(c_i^* - c_i - 1 + c_k) & \text{if } b_k = b_i = b_i' \text{ with } c_k > c_i \text{ and } c_k^* \geq c_i^* \\ \lambda_k & \text{if } b_k = b_i \text{ with } c_k \leq c_i \\ \lambda_k - 2(c_k - c_i) & \text{if } b_k = b_i \text{ with } b_i' = b_i + 1 \text{ and } c_k > c_i \\ \lambda_k & \text{if } b_k < b_i. \end{cases} \tag{A.4.1}$$

We consider the following cases:

- If $b_k > b_i$ then either $b_k \geq b_i + 2$ or $b_k = b_i + 1$. In the first case $b_k > b_i, b_i'$ and so

$$x_1(k) = (\lambda_0 a_k - \lambda_k(a_i + 1)) - \lambda_0 a_k + \lambda_k a_i = -\lambda_k$$

by (A.3.5). If $b_k = b_i + 1$ with $b_i = b_i'$ we also have $b_k > b_i, b_i'$ and so $x_1(k) = -\lambda_k$ (by the calculation above). In the final case $b_k = b_i + 1 = b_i'$ and so $c_i' = 0$, $c_i'^* = \lambda_0$ and $c_i = \lambda_0 - 1$. As $b_k = b_i + 1 > b_i$ we have $z(\gamma', k, i) = c_i'^*|c_k - c_i'| + c_i'|c_k^* - c_i'^*| = \lambda_0 c_k$ and $z(\gamma, k, i) = \lambda_0 a_l - \lambda_k a_i$ (using (A.3.5)) and so

$$x_1(k) = \lambda_0 c_k - (\lambda_0 a_k - \lambda_k a_i) = \lambda_0 \lambda_k (b_i - b_k) + \lambda_k c_i = -\lambda_k.$$

- If $b_k = b_i = b_i'$ then $c_i < \lambda_i - 1$, $c_i' = c_i + 1$ and $c_i'^* = c_i^* - 1$. Thus,

$$x_1(k) = (c_i^* - 1)|c_k - c_i - 1| + (c_i + 1)|c_k^* - c_i^* + 1| - c_i^*|c_k - c_i| - c_i|c_k^* - c_i^*|.$$

  If $c_k > c_i$ and $c_k^* < c_i^*$ then

$$x_1(k) = (c_i^* - 1)(c_k - c_i - 1) + (c_i + 1)(-c_k^* + c_i^* - 1) - c_i^*(c_k - c_i) - c_i(-c_k^* + c_i^*) = -\lambda_k.$$

  If $c_k > c_i$ and $c_k^* \geq c_i^*$ then

$$\begin{aligned} x_1(k) &= (c_i^* - 1)(c_k - c_i - 1) + (c_i + 1)(c_k^* - c_i^* + 1) - c_i^*(c_k - c_i) - c_i(c_k^* - c_i^*) \\ &= \lambda_k - 2(c_i^* - c_i - 1 + c_k). \end{aligned}$$

  If $c_k \leq c_i$ (and thus $c_k^* = \lambda_k - c_k \geq \lambda_0 - c_i = c_i^*$ as $\lambda_k > \lambda_0$) we have

$$x_1(k) = (c_i^* - 1)(-c_k + c_i + 1) + (c_i + 1)(c_k^* - c_i^* + 1) - c_i^*(-c_k + c_i) - c_i(c_k^* - c_i^*) = \lambda_k.$$

- If $b_k = b_i = b_i' - 1$ then $c_i = \lambda_0 - 1$, $c_i^* = 1$ and $c_i' = 0$. As $b_k = b_i < b_i'$ we have $z(\gamma', k, i) = \lambda_k(a_i + 1) - \lambda_0 a_k = \lambda_0 c_k^*$ (by (A.3.5)) and $z(\gamma, k, i) = |c_k - c_i| + c_i(c_k^* - 1)$, so

$$x_1(k) = \lambda_0 c_k^* - |c_k - c_i| - c_i(c_k^* - 1).$$

  If $c_k > c_i$ then $x_1(k) = \lambda_k + 2(c_i - c_k)$ and if $c_k \leq c_i$ then $x_1(k) = \lambda_k$, as required.

- Finally, if $b_k \le b_i - 1$ then $b_k < b_i, b'_i$ and so by (A.3.5) we have

$$x_1(k) = -\lambda_0 a_k + \lambda_k a'_i + \lambda_0 a_k - \lambda_k a_i = \lambda_k.$$

Thus, the claim in (A.4.1) is proved. We now consider $x_2(k)$. We claim that

$$x_2(k) = \begin{cases} \lambda_k & \text{if } b_k > b'_j \\ \lambda_k & \text{if } b_k = b'_j = b_j \text{ with } c_k > c'_j \text{ and } c^*_k < c'^*_j \\ -\lambda_k + 2(c'^*_j - c'_j - 1 + c_k) & \text{if } b_k = b'_j = b_j \text{ with } c_k > c'_j \text{ and } c^*_k \ge c'^*_j \\ -\lambda_k & \text{if } b_k = b'_j \text{ with } c_k \le c'_j \\ -\lambda_k + 2(c_k - c'_j) & \text{if } b_k = b'_j \text{ with } b_j = b'_j + 1 \text{ and } c_k > c'_j \\ -\lambda_k & \text{if } b_k < b'_j. \end{cases} \tag{A.4.2}$$

We have the following cases:

- Suppose $b_k > b'_j$. If $b_k > b_j \ge b'_j$ then

$$x_2(k) = \lambda_0 a_k - \lambda_k a'_j - \lambda_0 a_k + \lambda_k a_j = \lambda_k.$$

  If $b_k \le b_j$ then, as $b_k > b'_j$ we have that $b_k = b_j = b'_j + 1$ and so $c_j = 0$, $c^*_j = \lambda_0$ and $c'_j = \lambda_j - 1$. Thus, noting (A.3.5) we have $x_2(k) = \lambda_0 a_k - \lambda_k a'_j - c^*_j|c_k - c_j| - c_j|c^*_k - c^*_j| = \lambda_0 c_k + \lambda_k - \lambda_0 c_k = \lambda_k$.
- If $b_k = b'_j = b_j$ then $c'_j = c_j - 1$ and ${c'_j}^* = c^*_j + 1$ thus

$$x_2(k) = {c'_j}^*|c_k - c'_j| + c'_j|c^*_k - {c'_j}^*| - ({c'_j}^* - 1)|c_k - c'_j - 1| - (c'_j + 1)|c^*_k - {c'_j}^* + 1|.$$

  If $c_k > c'_j$ and $c^*_k < {c'_j}^*$ then

$$x_2(k) = {c'_j}^*(c_k - c'_j) + c'_j(-c^*_k + {c'_j}^*) - ({c'_j}^* - 1)(c_k - c'_j - 1) - (c'_j + 1)(-c^*_k + {c'_j}^* - 1) = \lambda_k.$$

  If $c_k > c'_j$ and $c^*_k \ge {c'_j}^*$ then

$$\begin{aligned} x_2(k) &= {c'_j}^*(c_k - c'_j) + c'_j(c^*_k - {c'_j}^*) - ({c'_j}^* - 1)(c_k - c'_j - 1) - (c'_j + 1)(c^*_k - {c'_j}^* + 1) \\ &= -\lambda_k + 2({c'_j}^* - c'_j + c_k - 1). \end{aligned}$$

  If $c_k \le c'_j$ (and so $c^*_k = \lambda_k - c_k \ge \lambda_0 - c'_j = {c'_j}^*$ as $\lambda_k > \lambda_0$) then

$$x_2(k) = {c'_j}^*(-c_k + c'_j) + c'_j(c^*_k - {c'_j}^*) - ({c'_j}^* - 1)(-c_k + c'_j + 1) - (c'_j + 1)(c^*_k - {c'_j}^* + 1) = -\lambda_k.$$

- If $b_k = b'_j = b_j - 1$ then $c_j = 0$, $c'_j = \lambda_0 - 1$ and ${c'_j}^* = 1$ and so

$$x_2(k) = |c_k - c'_j| + c'_j(c^*_k - 1) - (\lambda_k a_j - \lambda_0 a_k) = |c_k - c'_j| - c^*_k - c'_j.$$

  Thus, if $c_k > c'_j$ we have $x_2(k) = -\lambda_k - 2(c'_j - c_k)$ and if $c_k \le c'_j$ then $x_2(k) = -\lambda_k$.
- For the final case, let $b_k < b'_j \le b_j$. Using (A.3.5), we have that

$$x_2(k) = \lambda_k a'_j - \lambda_0 a_k - \lambda_k a_j + \lambda_0 a_k = -\lambda_k.$$

We now consider the different possible values of $x_1(k)$ and $x_2(k)$ from (A.4.1) and (A.4.2) to show that in all cases $x(k) \ge 0$. Recall from (7.2.4), and the fact that $\gamma + Q_\lambda \in (P/Q_\lambda)_+$, that $a_i \ge a_j$ and so $b_i \ge b_j$ and if $b_i = b_j$ then $c_i \ge c_j$.

(1) Suppose that $x_1(k) = -\lambda_k$. By (A.4.1) there are two cases; $b_k > b_i$ or $b_k = b_i = b'_i$ with $c_k > c_i$ and $c^*_k < c^*_i$.
   - In the first case we have that $b_k > b'_j$ as otherwise $b_i < b_k \le b'_j \le b_j$ which contradicts dominance. Thus, $x_2(k) = \lambda_k$ and so $x(k) = 0$.
   - In the second case if $b_k > b'_j$ then $x(k) = 0$. If $b_k \le b'_j$ then we have $b_j \ge b'_j \ge b_k = b_i$ which forces $b_i = b_j = b'_j = b_k$ by dominance. Thus, $c_i \ge c_j$ and $c^*_i \le c^*_j$. Furthermore, as $c'_j = c_j - 1$ and ${c'_j}^* = c^*_j + 1$ we have that $c'_j < c'_j + 1 = c_j \le c_i < c_k$ and $c^*_k < c^*_i = \lambda_0 - c_i \le c^*_j = {c'_j}^* - 1 < {c'_j}^*$ and so $x_2(k) = \lambda_k$ by (A.4.2). So again $x(k) = 0$.

(2) Suppose that $x_1(k) = \lambda_k$. By (A.4.2) we have that $x(k) \ge 0$ for all cases, for example if $x_2(k) = -\lambda_k + 2({c'_j}^* - c'_j - 1 + c_k)$ then $c_k > c'_j$ and so $x(k) = 2(c_k - c'_j) + 2({c'_j}^* - 1) > 0$.

(3) Suppose that $x_1(k) = \lambda_k - 2(c^*_i - c_i - 1 + c_k)$ and so $b_k = b_i = b'_i$, $c_k > c_i$ and $c^*_k \ge c^*_i$. We consider two cases:
   - If $b_k > b'_j$ then
$$x(k) = 2(\lambda_k - c^*_i + c_i + 1 - c_k) = 2(c^*_k - c^*_i + c_i + 1) \ge 0.$$
   - If $b_k \le b'_j$ then $b_j \ge b'_j \ge b_k = b_i$ which forces $b_i = b_j = b'_j = b_k$, $c_i \ge c_j$ and $c^*_i \le c^*_j$ by dominance. Thus we have that $c'_j = c_j - 1 \le c_i - 1 < c_i < c_k$. If $c^*_k < {c'_j}^*$ then $x_2(k) = \lambda_k$ and so $x(k) \ge 0$ as in the above case (when $b_k > b'_j$). If $c^*_k \ge {c'_j}^*$ then
$$x(k) = \lambda_k - 2(c^*_i - c_i - 1 + c_k) - \lambda_k + 2({c'_j}^* - c'_j - 1 + c_k) = 2(c_i - c'_j) = 2(c_i - c_j + 1) \ge 0.$$

(4) Finally, suppose that $x_1(k) = \lambda_k - 2(c_k - c_i)$ and so $b_k = b_i = b'_i - 1$ and $c_k > c_i$. First we note that $x_2(k) = -\lambda_k$ is not possible in this case, there are two cases to consider: $b_k = b'_j$ with $c_k \le c'_j$, and $b_k < b'_j$. If $b_k = b'_j$ with $c_k \le c'_j$ then $b'_j = b_j$, as otherwise $b_j = b'_j + 1 = b_k + 1 = b_i + 1 > b_i$ contradicting dominance, and so $b_i = b_j$ and $c_i \ge c_j$. However, in this case we have $c_i < c_k \le c'_j = c_j - 1 \le c_j$, a contradiction. On the other hand, if $b_k < b'_j$ we have that $b_i = b_k < b'_j \le b_j$ which contradicts dominance. We will now consider the 3 other options for the value of $x_2(k)$:
   - If $x_2(k) = \lambda_k$ then $x(k) = 2(c^*_k + c_i) > 0$.
   - If $x_2(k) = -\lambda_k + 2(c_k - c'_j)$ then $b_j = b'_j + 1$ and so $c'_j = \lambda_0 - 1$. As $b'_i = b_i + 1$ we also have $c_i = \lambda_0 - 1$ so
$$x(k) = 2(c_k - c'_j) - 2(c_k - c_i) = 0.$$
   - If $x_2(k) = -\lambda_k + 2({c'_j}^* - c'_j - 1 + c_k)$ then $b_k = b'_j = b_j$, $c_k > c_j$, $c^*_k \ge c^*_j$ and $c'_j = c_j - 1$. Thus, $b_i = b_j$ and so $c_i \ge c_j = c'_j + 1 > c'_j$. Therefore,
$$x(k) = 2({c'_j}^* - c'_j - 1 + c_k) - 2(c_k - c_i) = 2{c'_j}^* + 2c_i - 2c_j \ge 0$$

Thus, $x(k) \ge 0$ for all $k \in B_{<p}$. A similar analysis shows that $C \ge 0$: letting $l \in B_{>p}$, $y_1(l) = z(\gamma', i, l) - z(\gamma, i, l)$, $y_2(l) = z(\gamma', j, l) - z(\gamma, j, l)$ and $y(l) = y_1(l) + y_2(l)$ and then showing that $y(l) \ge 0$ by considering the different possible values of $y_1(l)$ and $y_2(l)$. We omit the details. Thus, $M(\gamma') - M(\gamma) = A + B + C > 0$ as required. □

Appendix B

Folding tables

In this appendix we introduce the concept of folding tables, and give examples of such tables using MAGMA code. Folding tables were introduced by Guilhot and Parkinson in [23], [22] to find the bounding degree of the v-mass of J-folded alcove walks for the affine Coxeter groups of type G_2 and C_2. They are a tabular display of data, and depend on an element $\vec{w} \in W$, that allows one to read off J-folded alcove paths of type $\vec{w}$.

Recall the definition of v-mass from Definition 4.3.3. The v-mass of a path is encoded into its folding table. Using these tables Parkinson and Guilhot found the degree bound of the v-mass of J-folded paths of $\widetilde{G}_2$ and $\widetilde{C}_2$ and thus, using a version of Theorem 5.3.3, determined the bound of $\widetilde{\mathcal{H}}$ representations (equivalent to our representations $\pi_{\mathsf{v},J}$). To do this they noted that one can decompose $w \in W$ into $w = v \cdot t_\gamma \cdot \mathsf{b}$ where $v \in W_0$, $\gamma \in P^+$ and $\mathsf{b} \in B_0$ with $B_0 = \{w \in \widetilde{W} \mid wA_0 \subseteq \{x \in V \mid 0 \leq \langle x, \alpha_i \rangle \leq 1 \text{ for all } i \in I\}\}$, the alcoves in the area bounded by $H_{\alpha_i,0}$ and $H_{\alpha_i,1}$ for all $i \in I$ (see [4, §3] or [23, §4.2]). Choosing a reduced expression $\vec{w} = \vec{v} \cdot \vec{t}_\gamma \cdot \vec{\mathsf{b}}$ they then used folding tables for $\vec{v}$, $\vec{t}_\gamma$ and $\vec{\mathsf{b}}$ to find the possible highest values of the v-mass of a J-folded path of type $\vec{w}$. In other types the number of paths and the cancellations between paths makes this method unrealistic. Recall from Remark 6.1.3 that the bound of the paths can be higher than the bound of the matrix representation but will be cancelled out by other paths in Theorem 5.3.3. In low rank cases these cancellations can be managed but in higher rank this becomes increasingly difficult. Thus, the theory of folding tables was not used in the results of this thesis. However, there is still merit in having a simple way to find J-folded alcove paths to complete examples and test conjectures.

The author created MAGMA code to calculate folding tables for any element of any affine Coxeter group (irreducible and reduced). The code can be found at `https://github.com/ellielittle/FoldingTables`. This code was used throughout the creation of the thesis to find examples, calculate matrices, learn about J-folded paths and test conjectures. In this appendix folding tables are defined and then two examples are calculated using the MAGMA code.

Definition B.0.1. ([23, Definition 7.4]) Let $\vec{x}$ be a reduced expression of $x \in W$. Let $u_1, \ldots, u_r$ be the elements of ${}^JW$ ordered so that $\ell(u_i) \leq \ell(u_{i+1})$. Denote p^i to be the straight path in $\mathcal{P}_J(\vec{x}, u_i)$. If the k-th step of p^i is a negative crossing, let p^i_k denote the path in $\mathcal{P}_J(\vec{x}, u_i)$ formed from p^i by folding at step k.

For each $1 \leq i \leq r$ and $1 \leq k \leq \ell(x)$ define $\mathsf{ft}_{i,k}(\vec{x}) \in \{-, *, 1, 2, 3, \ldots, r\}$ as follows:

- If p^i has a positive crossing at the k-th step, then $\mathsf{ft}_{i,k} = -$,
- If p^i has a bounce at the k-th step, then $\mathrm{ft}_{i,k} = *$, and

- If p^i has a negative crossing at the k-th step, then $\mathsf{ft}_{i,k} = j$, where $u_j = \theta^J(p')$ and p' is the straight path in $\mathcal{P}_J(\mathrm{rev}(\vec{x}), \theta^J(p^i_k))$

The *J-folding table of type $\vec{x}$* is then the $r \times \ell(x)$ array where the (i,k)-th entry is $\mathsf{ft}_{i,k}$.

To motivate the final point in the definition: j is chosen so that the paths p^i_k and $\tau_{(\mathrm{wt}(p^i_k)-\mathrm{wt}(p^j))^J}p^j$ agree after the k-th step.

The folding table is read left to right to find J-folded alcoves paths of type $\vec{x}$. To find a path $p \in \mathcal{P}_J(\vec{x}, u_i)$ begin in position $(i, 1)$ and then move through the row from left to right. The symbol $-$ denotes a positive crossing, the symbol $*$ denotes a bounce and a number $1 \leq j \leq r$ is a possible fold or negative crossing. That is, at $\mathsf{ft}_{i,k} = j$ one can choose to fold or cross. If one chooses to take a fold at step k then they move to position $(j, k+1)$ in the folding table and continue to build the path moving left to right in row j. If they choose to cross they remain in row i (as is done with the symbol $-$).

By [23, Proposition 7.8] all positively folded paths of type $\vec{x} \in W$ can be found in its folding table. Note that if $p = (v_0, v_1, \ldots, v_l) \in \mathcal{P}_J(\vec{x}, u_i)$ then $p' = (v_0, v_1, \ldots, v_l, v_l\sigma) \in \mathcal{P}_J(\vec{x}\sigma, u_i)$ for any $\sigma \in \Sigma$, so the folding tables can also be used to find all J-folded paths of type $\vec{x} = s_{i_1}s_{i_2}\cdots s_{i_l}\sigma \in \widetilde{W}$.

Example B.0.2. Let $\Phi = F_4$ and $J = \{1, 2, 3\}$. Let ${}^JW$ be ordered as follows:

$$\{e, s_4, s_4s_3, s_4s_3s_2, s_4s_3s_2s_1, s_4s_3s_2s_3, s_4s_3s_2s_1s_3, s_4s_3s_2s_3s_4, s_4s_3s_2s_1s_3s_2, s_4s_3s_2s_1s_3s_4,$$
$$s_4s_3s_2s_1s_3s_2s_3, s_4s_3s_2s_1s_3s_2s_4, s_4s_3s_2s_1s_3s_2s_3s_4, s_4s_3s_2s_1s_3s_2s_4s_3, s_4s_3s_2s_1s_3s_2s_3s_4s_3,$$
$$s_4s_3s_2s_1s_3s_2s_4s_3s_2, s_4s_3s_2s_1s_3s_2s_3s_4s_3s_2, s_4s_3s_2s_1s_3s_2s_4s_3s_2s_1, s_4s_3s_2s_1s_3s_2s_3s_4s_3s_2s_1,$$
$$s_4s_3s_2s_1s_3s_2s_3s_4s_3s_2s_3, s_4s_3s_2s_1s_3s_2s_3s_4s_3s_2s_1s_3, s_4s_3s_2s_1s_3s_2s_3s_4s_3s_2s_1s_3s_2,$$
$$(s_4s_3s_2s_1s_3s_2s_3)^2, s_4s_3s_2s_1s_3s_2s_3s_4s_3s_2s_1s_3s_2s_3s_4\}.$$

By [28] we set $\vec{t}_{\omega_2} = s_{i_1}s_{i_2}\cdots s_{i_l} = s_0s_2s_4s_3s_1s_4s_2s_3s_4s_3s_2s_4s_1s_3s_4s_2$. The folding table of $\vec{t}_{\omega_2}$ is displayed in Figure B.1. Let $L(s_1) = L(s_2) = b$ and $L(s_3) = L(s_4) = a$ as in Example 6.1.6. We can read the v-mass of a path from the folding table while finding the path, when a fold is taken at step k it contributes $(\mathsf{q}_{i_k} - \mathsf{q}_{i_k}^{-1})$ to the v-mass and when a bounce occurs at step k it contributes $\mathsf{v}_{\alpha_{i_k}}$ if $i_k \in \{1, 2, 3\}$ (with $\mathsf{v}_{\alpha_1} = \mathsf{v}_{\alpha_2}$), v_{α_3} if $i_k = 4$ and $\mathsf{v}_{\alpha_1} = \mathsf{v}_{\alpha_2}$ if $i_k = 0$. Using the table some J-folded paths of type $\vec{t}_{\omega_2}$, and their v-masses, are as follows: (we write i_k instead of s_{i_k} and $\hat{i}_k$ to denote a fold and $\check{i}_k$ to denote a bounce)

$$p_1 = \hat{0}2\check{4}\hat{3}\check{1}\check{4}234\check{3}\check{2}\hat{4}1\check{3}\hat{4}2 \in \mathcal{P}_J(\vec{t}_{\omega_2}u_{21}), \quad \mathcal{Q}_J(p_1) = (\mathsf{q}_5 - \mathsf{q}_5^{-1})(\mathsf{q}_3 - \mathsf{q}_3^{-1})(\mathsf{q}_4 - \mathsf{q}_4^{-1})^2\mathsf{v}_{\alpha_1}^2\mathsf{v}_{\alpha_3}^4,$$
$$p_2 = \check{0}2\hat{4}3\check{1}423\check{4}324\check{1}\hat{3}42 \in \mathcal{P}_J(\vec{t}_{\omega_2}, u_{16}), \quad \mathcal{Q}_J(p_2) = (\mathsf{q}_3 - \mathsf{q}_3^{-1})(\mathsf{q}_4 - \mathsf{q}_4^{-1})\mathsf{v}_{\alpha_1}^3\mathsf{v}_{\alpha_3},$$
$$p_3 = 0\check{2}4\hat{3}142343\check{2}\hat{4}\check{1}342 \in \mathcal{P}_J(\vec{t}_{\omega_2}, u_1), \quad \mathcal{Q}_J(p_3) = (\mathsf{q}_3 - \mathsf{q}_3^{-1})(\mathsf{q}_4 - \mathsf{q}_4^{-1})\mathsf{v}_{\alpha_1}^3.$$

Example B.0.3. Let $\Phi = A_6$ and $\lambda = (4, 3)$ so $J_\lambda = \{1, 2, 3, 5, 6\}$. Let ${}^\lambda W$ be ordered as follows:

$$\{e, s_4, s_4s_3, s_4s_5, s_4s_3s_2, s_4s_3s_5, s_4s_5s_6, s_4s_3s_2s_1, s_4s_3s_2s_5, s_4s_3s_5s_4, s_4s_3s_5s_6, s_4s_3s_2s_1s_5,$$
$$s_4s_3s_2s_5s_4, s_4s_3s_2s_5s_6, s_4s_3s_5s_4s_6, s_4s_3s_2s_1s_5s_4, s_4s_3s_2s_1s_5s_6, s_4s_3s_2s_5s_4s_3, s_4s_3s_2s_5s_4s_6,$$
$$s_4s_3s_5s_4s_6s_5, s_4s_3s_2s_1s_5s_4s_3, s_4s_3s_2s_1s_5s_4s_6, s_4s_3s_2s_5s_4s_3s_6, s_4s_3s_2s_5s_4s_6s_5, s_4s_3s_2s_1s_5s_4s_3s_2,$$
$$s_4s_3s_2s_1s_5s_4s_3s_6, s_4s_3s_2s_1s_5s_4s_6s_5, s_4s_3s_2s_5s_4s_3s_6s_5, s_4s_3s_2s_1s_5s_4s_3s_2s_6, s_4s_3s_2s_1s_5s_4s_3s_6s_5,$$
$$s_4s_3s_2s_5s_4s_3s_6s_5s_4, s_4s_3s_2s_1s_5s_4s_3s_2s_6s_5, s_4s_3s_2s_1s_5s_4s_3s_6s_5s_4, s_4s_3s_2s_1s_5s_4s_3s_2s_6s_5s_4,$$
$$s_4s_3s_2s_1s_5s_4s_3s_2s_6s_5s_4s_3\}$$

By [28] we have that $\vec{t}_{\omega_4} = s_0 s_6 s_5 s_1 s_0 s_6 s_2 s_1 s_0 s_3 s_2 s_1 \sigma^4$ where $\sigma(i) = i+1 \mod 7$. The folding table of $\vec{t}_\omega \sigma^{-4}$ is given in Figure B.2. Recall that $\mathsf{v}_\alpha = -\mathsf{q}^{-1}$ for all $\alpha \in \Phi_J$ where $\mathsf{q} = \mathsf{q}_1 = \mathsf{q}_2 = \mathsf{q}_3 = \mathsf{q}_4 = \mathsf{q}_5 = \mathsf{q}_6 = \mathsf{q}_0$. The v-mass of a path can be read from the folding table as follows: a fold contributes $(\mathsf{q} - \mathsf{q}^{-1})$ to the v-mass and a bounce contributes $-\mathsf{q}^{-1}$ to the v-mass. Using the table, the below are some J-folded paths of type $\vec{t}_{\omega_4}$ and their v-masses (as in Example B.0.2 we write i_k instead of s_{i_k} and $\hat{i}_k$ for a fold and $\check{i}_k$ for a bounce).

$$p_1 = 0\check{6}\hat{5}\check{1}06\check{2}10\hat{3}21\sigma^4 \in \mathcal{P}_\lambda(\vec{t}_{\omega_4}, u_{22}), \qquad \mathcal{Q}_\lambda(p_1) = (\mathsf{q} - \mathsf{q}^{-1})^2(-\mathsf{q}^{-1})^3,$$
$$p_2 = 06\hat{5}1\hat{0}6\hat{2}10321\sigma^4 \in \mathcal{P}_\lambda(\vec{t}_{\omega_4}, u_{35}), \qquad \mathcal{Q}_\lambda(p_2) = (\mathsf{q} - \mathsf{q}^{-1})^3,$$
$$p_3 = 06\hat{5}\check{1}\check{0}6\hat{2}\check{1}0\check{3}\hat{2}1\sigma^4 \in \mathcal{P}_\lambda(\vec{t}_{\omega_4}, u_{33}), \qquad \mathcal{Q}_\lambda(p_3) = (\mathsf{q} - \mathsf{q}^{-1})^3(-\mathsf{q}^{-1})^4.$$

	s_0	s_2	s_4	s_3	s_1	s_4	s_2	s_3	s_4	s_3	s_2	s_4	s_1	s_3	s_4	s_2
1	–	∗	2	4	–	∗	∗	–	–	∗	–	7	∗	–	–	∗
2	–	∗	–	∗	–	4	–	–	–	–	–	10	–	∗	–	∗
3	–	4	∗	8	∗	6	∗	∗	–	–	–	∗	–	–	–	–
4	–	–	∗	–	–	–	–	–	–	2	∗	21	∗	1	–	3
5	∗	∗	∗	12	–	9	∗	∗	–	–	–	∗	∗	–	–	∗
6	–	∗	8	∗	∗	–	∗	–	∗	3	∗	14	∗	∗	–	∗
7	∗	9	10	17	∗	11	2	1	∗	–	–	–	∗	–	–	–
8	–	∗	–	–	∗	∗	–	–	–	∗	∗	15	–	3	∗	∗
9	∗	–	12	∗	–	–	∗	–	∗	5	∗	18	2	∗	–	7
10	∗	12	–	13	∗	16	∗	2	–	–	–	–	–	–	∗	–
11	∗	∗	13	16	∗	–	4	∗	1	7	∗	∗	3	–	–	∗
12	∗	–	–	–	–	∗	–	–	–	∗	∗	19	∗	5	∗	10
13	∗	∗	–	–	∗	17	–	∗	–	–	∗	∗	–	10	2	∗
14	∗	16	15	20	1	∗	8	6	3	∗	∗	–	∗	–	∗	–
15	∗	17	–	∗	2	20	∗	8	∗	–	∗	–	–	∗	3	–
16	∗	–	17	–	∗	–	∗	–	4	10	1	23	8	11	∗	14
17	∗	–	–	–	∗	–	–	–	∗	13	2	24	∗	7	4	15
18	1	∗	19	22	∗	∗	12	9	5	∗	∗	–	10	–	∗	∗
19	2	∗	–	∗	∗	22	∗	12	∗	–	∗	–	∗	∗	5	∗
20	∗	∗	∗	–	4	–	∗	∗	8	15	3	∗	∗	14	6	∗
21	3	22	∗	24	5	23	13	4	10	–	∗	–	∗	–	7	–
22	4	–	∗	–	∗	–	∗	∗	12	19	5	∗	13	18	9	21
23	6	∗	24	∗	9	–	17	16	2	21	7	–	15	∗	11	∗
24	8	∗	–	–	12	∗	∗	17	13	∗	10	–	∗	21	1	∗

Figure B.1: The folding table of t_{ω_2} for $\Phi = F_4$ and $J = \{1, 2, 3\}$.

	s_0	s_6	s_5	s_1	s_0	s_6	s_2	s_1	s_0	s_3	s_2	s_1
1	–	–	–	–	–	–	–	–	–	–	–	–
2	–	–	*	–	–	*	–	–	*	*	*	1
3	–	–	*	–	–	*	*	*	1	–	–	*
4	–	*	–	–	*	–	–	*	–	*	1	*
5	–	–	*	*	*	1	–	–	*	–	–	*
6	–	*	–	–	*	–	*	1	*	–	*	–
7	*	–	–	*	–	–	*	–	–	1	*	*
8	*	*	1	–	–	*	–	–	*	–	–	*
9	–	*	–	*	1	*	–	*	–	–	*	–
10	–	*	*	–	*	*	*	2	4	*	3	6
11	*	–	–	*	–	–	1	*	*	*	–	–
12	*	1	*	–	*	–	–	*	–	–	*	–
13	–	*	*	*	2	4	–	*	*	*	5	9
14	*	–	–	1	*	*	*	–	–	*	–	–
15	*	–	*	*	–	*	2	*	7	3	*	11
16	*	2	4	–	*	*	–	*	*	*	8	12
17	1	*	*	*	–	–	*	–	–	*	–	–
18	–	*	*	*	3	6	*	5	9	–	*	*
19	*	–	*	2	*	7	*	–	*	5	*	14
20	*	*	–	*	*	–	4	7	*	6	11	*
21	*	3	6	–	*	*	*	8	12	–	*	*
22	2	*	7	*	–	*	*	–	*	8	*	17
23	*	–	*	3	*	11	5	*	14	*	–	*
24	*	*	–	4	7	*	*	*	–	9	14	*
25	*	5	9	*	8	12	–	*	*	–	*	*
26	3	*	11	*	–	*	8	*	17	*	–	*
27	4	7	*	*	*	–	*	*	–	12	17	*
28	*	*	–	6	11	*	9	14	*	*	*	–
29	5	*	14	8	*	17	*	–	*	*	–	*
30	6	11	*	*	*	–	12	17	*	*	*	–
31	*	*	*	10	15	20	13	19	24	18	23	28
32	9	14	*	12	17	*	*	*	–	*	*	–
33	10	15	20	*	*	*	16	22	27	21	26	30
34	13	19	24	16	22	27	*	*	*	25	29	32
35	18	23	28	21	26	30	25	29	32	*	*	*

Figure B.2: The folding table of $t_{\omega_4}\sigma^{-4}$ for $\Phi = A_6$ and $J = \{1, 2, 3, 5, 6\}$.

Bibliography

[1] P. Abramenko and K. Brown. *Buildings: Theory and Applications*, volume 248 of *Graduate texts in mathematics*. Springer, 2008.

[2] A.-M. Aubert and R. Plymen. Plancherel measure for $GL(n, f)$ and $GL(m, d)$: explicit formulas and Bernstein decomposition. *Journal of Number Theory*, 112(1):26–66, 2005.

[3] A. Björner and F. Brenti. *Combinatorics of Coxeter groups*, volume 231. Springer, 2005.

[4] J. Blasiak. A factorization theorem for affine Kazhdan-Lusztig basis elements. *Preprint*, 2009.

[5] C. Bonnafé. *Kazhdan-Lusztig cells with unequal parameters*, volume 24 of *Algebra and Applications*. Springer, 2017.

[6] C. Bonnafé and L. Iancu. Left cells in type B_n with unequal parameters. *Representation Theory of the American Mathematical Society*, 7(22):587–609, 2003.

[7] N. Bourbaki. *Lie groups and Lie algebras: Chapters 4-6*. Elements of Mathematics. Springer-Verlag, 2002.

[8] N. Chapelier-Laget, J. Guilhot, E. Little, and J. Parkinson. The asymptotic Plancherel formula and Lusztig's asymptotic algebra for $\tilde{A}_n$. *Preprint*, 2024.

[9] V. Deodhar. On some geometric aspects of Bruhat orderings ii. the parabolic analogue of Kazhdan-Lusztig polynomials. *Journal of Algebra*, 111(2):483–506, 1987.

[10] V. Deodhar. A combinatorial setting for questions in Kazhdan-Lusztig theory. *Geometriae Dedicata*, 36(1):95–119, 1990.

[11] V. Deodhar. Duality in parabolic set up for questions in Kazhdan-Lusztig theory. *Journal of Algebra*, 142(1):201–209, 1991.

[12] V. Deodhar. J-chains and multichains, duality of Hecke modules, and formulas for parabolic Kazhdan–Lusztig polynomials. *Journal of Algebra*, 190(1):214–225, 1997.

[13] J. Dixmier. C^*-algebras. *North-Holland Mathematical Library*, 15, 1977.

[14] B. Elias and G. Williamson. The Hodge theory of Soergel bimodules. *Annuals of Mathematics*, 180:1089–1136, 2014.

[15] J. Gao and X. Xie. Conjectures P1-P15 for hyperbolic Coxeter groups of rank 3. *Journal of Algebra*, 567:139–195, 2021.

[16] S. Gaussent and P. Littelmann. LS galleries, the path model, and MV cycles. *Duke Mathematical Journal*, 127:35–88, 2005.

[17] M. Geck. On Iwahori-Hecke algebras with unequal parameters and Lusztig's isomorphism theorem. *Pure and Applied Mathematics*, 7:587–620, 2011.

[18] M. Geck and G. Pfeiffer. *Characters of Coxeter groups and Iwahori-Hecke algebras*, volume 21 of *London Mathematical Society Monographs*. Oxford University Press, 2000.

[19] U. Görtz. Alcove walks and nearby cycles on affine flag manifolds. *Journal of Algebraic Combinatorics*, 26:415–430, 2007.

[20] J. Guilhot. On the lowest two-sided cell in affine Weyl groups. *Representation Theory of the American Mathematical Society*, 12(13):327–345, 2008.

[21] J. Guilhot, E. Little, and J. Parkinson. On J-folded alcove paths and combinatorial representations of affine Hecke algebras. *Algebras and Representation Theory*, 27(6):2131–2185, 2024.

[22] J. Guilhot and J. Parkinson. Balanced representations, the asymptotic Plancherel formula, and Lusztig's conjectures for $\tilde{C}_2$. *Algebraic Combinatorics*, 2(5):969–1031, 2019.

[23] J. Guilhot and J. Parkinson. A proof of Lusztig's conjectures for affine type $\widetilde{G_2}$ with arbitrary parameters. *Proceedings of the London Mathematical Society*, 118(5):1188–1244, 2019.

[24] C. Hohlweg, P. Nadeau, and N. Williams. Automata, reduced words and Garside shadows in Coxeter groups. *Journal of Algebra*, 457:431–456, 2016.

[25] D. Kazhdan and G. Lusztig. Representations of Coxeter groups and Hecke algebras. *Inventiones mathematicae*, 53(2):165–184, 1979.

[26] D. Kim and P. Pylyavskyy. Asymptotic Hecke algebras and Lusztig-Vogan bijection via affine matrix-ball construction. *International Mathematics Research Notices*, (18):16051–16103, 2023.

[27] G. Lusztig. Singularities, character formulas, and a q-analog of weight multiplicities. *Astérisque*, 101(102):208–229, 1983.

[28] G. Lusztig. Some examples of square integrable representations of semisimple p-adic groups. *Transactions of the American Mathematical Society*, 277(2):623–653, 1983.

[29] G. Lusztig. Cells in affine Weyl groups. In *Algebraic groups and related topics*, volume 6 of *Advanced Studies in Pure Mathematics*, pages 255–287, 1985.

[30] G. Lusztig. The two-sided cells of the affine Weyl group of type a_n. In *Infinite dimensional groups with applications*, pages 275–283. Springer-Verlag, 1985.

[31] G. Lusztig. Cells in affine Weyl groups, II. *Journal of algebra*, 109(2):536–548, 1987.

[32] G. Lusztig. Affine Hecke algebras and their graded version. *Journal of the American Mathematical Society*, 2(3):599–635, 1989.

[33] G. Lusztig. Cells in affine Weyl groups, IV. *Journal of the Faculty of Science, University of Tokyo*, 36(2):297–328, 1989.

[34] G. Lusztig. Periodic W-graphs. *Representation Theory of the American Mathematical Society*, 1(11):207–279, 1997.

[35] G. Lusztig. *Hecke algebras with unequal parameters*, volume 18 of *CRM Monograph Series*. American Mathematical Society, 2003.

[36] I. G. Macdonald. *Spherical functions on a group of p-adic type*, volume 2 of *Publications of the Ramanujan Institute*. Ramanujan Institute, 1971.

[37] I. G. Macdonald. *Affine Hecke algebras and orthogonal polynomials*, volume 157. Cambridge University Press, 2003.

[38] K. Nelsen and A. Ram. Kostka-Foulkes polynomials and Macdonald spherical functions. In *Surveys in combinatorics*, volume 307 of *London Mathematical Society Lecture Notes*, pages 325–370. Cambridge Univeristy Press, 2003.

[39] E. Opdam. A generating function for the trace of the Iwahori-Hecke algebra. In *Studies in memory of Issai Schur*, pages 301–323. Springer, 2003.

[40] E. Opdam. On the spectral decomposition of affine Hecke algebras. *Journal of the Institute of Mathematics of Jussieu*, 3(4):531–648, 2004.

[41] J. Parkinson. On calibrated representations and the Plancherel Theorem for affine Hecke algebras. *Journal of Algebraic Combinatorics*, 40:331–371, 2014.

[42] A. Ram. Alcove walks, Hecke algebras, spherical functions, crystals and column strict tableaux. *Pure and Applied Mathematics Quarterly (special issue in honor of Robert MacPherson)*, 2(4):963–1013, 2006.

[43] B. Sagan. *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions*, volume 203 of *Graduate Texts in Mathematics*. Springer, 2 edition, 2001.

[44] C. Schwer. Galleries, Hall-Littlewood polynomials, and structure constants of the spherical Hecke algebra. *International Mathematics Research Notices*, 2006.

[45] J.-y. Shi. *The Kazhdan-Lusztig cells in certain affine Weyl groups*, volume 1179 of *Lecture Notes in Mathematics*. Springer-Verlag, 1986.

[46] J.-y. Shi and G. Yang. The weighted universal Coxeter group and some related conjectures of Lusztig. *Journal of Algebra*, 441:678–694, 2015.

[47] J. R. Stembridge. The partial order of dominant weights. *Advances in Mathematics*, 136(2):340–364, 1998.

[48] T. Tanisaki and N. Xi. Kazhdan-Lusztig basis and a geometric filtration of an affine Hecke algebra. *Nagoya Mathematical Journal*, 182:285–311, 2006.

[49] N. Xi. The based ring of two-sided cells of affine Weyl groups of type $\tilde{A}_{n-1}$. *Memoirs of the American Mathematical Society*, 157(749), 2002.

[50] X. Xie. Conjectures P1-P15 for Coxeter groups with complete graph. *Advances in Mathematics*, 379:107565, 2021.