Asymptotic Analysis of Transcendental Solutions of Nonlinear Systems

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The results of Chapter 2 of this thesis are published in: J. Holroyd and N. Joshi. "On the perturbed second Painlevé equation". *Journal of Physics A: Mathematical and Theoretical* 56.1 (2023), p. 014002. I produced the mathematical results and wrote the initial drafts of the article.

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CHAPTER 1

Introduction

Asymptotic analysis is a cornerstone of mathematical physics and applied mathematics, offering insights into the behaviour of solutions of complex systems in limiting regimes. This thesis explores the asymptotic properties of transcendental solutions arising in nonlinear systems, focusing on Painlevé equations in both their continuous and q-difference forms.

The Painlevé equations are a class of nonlinear ordinary differential equations (ODEs) distinguished by the Painlevé property: their solutions possess no movable branch points or essential singularities in the complex plane. These equations have profound implications across diverse fields, from statistical mechanics and quantum field theory to special functions and integrable systems. Their transcendental solutions, which cannot be expressed in terms of elementary or classical special functions, exhibit complex asymptotic behaviours that demand careful study.

In addition to their classical (continuous) formulations, q-difference Painlevé equations have gained prominence for their role in discrete integrable systems and connections to orthogonal polynomials, lattice models, and combinatorics. The q-difference analogues extend the framework of Painlevé analysis to discrete systems that exhibit similar structural richness and integrable properties.

This thesis thoroughly examines the asymptotic properties of these systems' solutions. We illuminate the structures underlying exceptional transcendental solutions appearing in continuous and discrete settings by extending classical techniques such as divergent asymptotic expansions, exponential asymptotics, and isomonodromic deformation theory. Particular emphasis is placed on highlighting the interplay between the continuous and discrete realms, elucidating how q-deformations impact the asymptotic landscape.

Our findings contribute to a deeper understanding of nonlinear dynamics and integrable systems, demonstrating the universality of certain asymptotic behaviours of Painlevé transcendents and their relevance to modern mathematical and physical considerations.

1.1. Painlevé Equations

In the early 1900's, Painlevé [126, 127] and colleagues investigated second order ODEs [62] of the form

$$y'' = R\left(y, y', t\right), \tag{1.1}$$

where R is a rational function of $y = y(t)$ and $y' = dy/dt$, and is analytic with respect to t. Specifically, Painlevé sought to classify cases of Equation (1.1) satisfying what is now known as the Painlevé property [47, 50, 127]. Definition 1.2 below is the simplest way of defining the Painlevé property; we refer the reader to [97] for other definitions.

Definition 1.2 (Painlevé property)**.** *An equation of the form* (1.1) *satisfies the Painlevé property, given that all movable singularities of all solutions are poles. Here, the term "movable" refers to the location of the singularity depending on initial conditions.*

In this endeavour, Painlevé, Gambier [50], and Fuchs [48] proposed fifty canonical equations satisfying the Painlevé property. Of these, forty-four equations are exactly solvable by previously known elementary, elliptic, or classical special functions. The remaining six equations are nonlinear, second-order ODEs now known as the Painlevé equations. There exist solutions of the Painlevé equations which are rational [4], or given precisely in terms of classical special functions such as Airy or Bessel [116] (rational and special solutions only occur for specific values of parameters of the Painlevé equations). Beyond these particular cases, generic solutions of the Painlevé equations are new *transcendental* functions (not expressible in terms of known mathematical functions [152]), which are thus referred to as *Painlevé transcendents*.

Although arising via purely mathematical considerations, Painlevé equations are now known to be universal mathematical models, appearing extensively in the study of physical systems. Furthermore, Painlevé equations have attracted considerable attention due to their role as similarity reductions of infinite-dimensional, completely integrable systems, such as the Korteweg-de Vries equation [25, 41, 56]. All known reductions of such systems satisfy the Painlevé property [1, 2], which has subsequently become interchangeable with the concept of a system's *complete integrability*. Generally speaking, an integrable system allows for the global continuity of generic solutions and, therefore, is suitable for governing a physical phenomenon.

As indicated by the deep connection between Painlevé equations and integrability, it is unsurprising that Painlevé equations possess intriguingly rich mathematical properties. For instance, Painlevé equations also arise in the study of orthogonal polynomials [10, 12, 38, 40, 99], when considering the recurrence relations satisfied by orthogonal polynomials after choosing certain weight functions. Moreover, Painlevé equations are central in the study of monodromy preserving deformations of systems of linear differential equations [51].

Concerning mathematical physics, Painlevé transcendents arise as solutions of the nonlinear Schrödinger equation [14, 102] among many other soliton equations [1, 2]. In more recent developments, Painlevé transcendents appear in the study of quantum field theories [39, 130, 143, 144], and in the application of random matrix theory [31, 46, 150, 151]. Due to their prevalence as nonlinear mathematical models, Painlevé equations are commonly considered as defining new nonlinear special functions [13, 26, 64], analogous to the classical special functions defined as solutions of linear second order ODEs. It is no understatement to claim that Painlevé equations lie at the frontier of integrability and are pivotal in developing an understanding of nonlinear systems.

Due to their appearance in describing physical systems, the properties of solutions of Painlevé equations are an area of continual interest. Although generic solutions are transcendental, one may glean information regarding their behaviour under some limit of the independent variable via *asymptotic analysis* (see Section 1.4). Several key solutions of physical interest were originally identified via asymptotic analysis [111, 161] as the independent variable approaches a fixed singularity. For what remains of this section, we focus on the second Painlevé equation

$$\mathrm{P_{II}}: \qquad y'' = 2y^3 + ty + \alpha, \tag{1.3}$$

where differentiation is with respect to t and α is a constant parameter, as an example, as this is relevant to Chapter 2 where we consider a generalisation of $\mathrm{P_{II}}$ (not to mention, in Chapter 3 we consider a q-difference form of $\mathrm{P_{II}}$).

Boutroux initiated an asymptotic analysis of the Painlevé equations [16], considering generic behaviours of $\mathrm{P_I}$ and $\mathrm{P_{II}}$ as the independent variable approaches infinity, this being a fixed irregular singularity of the first five Painlevé equations. Considering $\mathrm{P_{II}}$, Boutroux transformed variables

$$y(t) = t^{1/2}u(z) \qquad \text{and} \qquad z = \frac{2}{3}t^{3/2},$$

which leads to a version of $\mathrm{P_{II}}$ that is more amenable to asymptotic analysis as the independent variable becomes large (now $z \to \infty$), that is

$$u'' = 2u^3 + u + \frac{2\alpha}{3z} - \frac{u'}{z} + \frac{u}{9z^2}, \tag{1.4}$$

where primes now denote differentiation with respect to z. The *Boutroux form* may be deduced similarly for any of the Painlevé equations. Crucially, we see that the above equation becomes autonomous in the limit $|z| \to \infty$; performing a first integration of the leading order autonomous equation yields the energy-like conserved (at leading order) quantity

$$E(u, u_z) = u_z^2 - u^4 - u^2.$$

Considering constant E above, the generic solution $u(z)$ is a (Jacobi) elliptic function, i.e., solving the ODE

$$w''(z) = 2w^3(z) + w(z).$$

In the case of the first and second Painlevé equations, these behaviours were first considered by Boutroux [15, 16]. Furthermore, for P_{I} and P_{II}, Joshi and Kruskal studied the modulation of these leading order elliptic functions (i.e., the slow variation of E above) as the angle of approach to infinity is varied within a bounded sector of the complex z plane [73, 75, 76]. An averaging method showed that the slow modulation of E is given at the leading order by complete elliptic integrals associated with the leading order elliptic function. Analogous results for the third, fourth, and fifth Painlevé equations are given in [77]. In Chapter 3, we extend these concepts to the q-difference setting, considering generic elliptic-type behaviours associated with the q-difference second Painlevé equation (discrete Painlevé equations are discussed in the following section).

We furthermore note that Joshi and Kruskal also developed a complex extension of the multiple-scales method [73, 75] in the study of global asymptotic properties of generic solutions of P_{I} and P_{II}. In comparison with the method of averaging, this encounters technical complications, requiring a non-conformal mapping of the complex plane, meaning that the equation and its solutions need to be embedded in a framework of non-analytic functions.

Moving on from this generic elliptic (doubly-periodic) behaviour, Boutroux discovered that as $t \to \infty$ on the boundaries of sectors given by (again in the case of P_{II})

$$S_n = \left\{ x \in \mathbb{C} \;\middle|\; (n-1)\pi/3 \leq \arg(x) \leq n\pi/3 \right\}, \quad n \in \mathbb{Z} \mod 6, \tag{1.5}$$

the asymptotic behaviour degenerates into singly periodic/trigonometric functions. The domains S_n are referred to as *Stokes sectors*. Generally, the asymptotic behaviour at $|t| = \infty$ of a given solution will fundamentally differ in different Stokes sectors; this is known as the nonlinear *Stokes phenomenon*, a core concept when considering an irregular singular point at infinity. Explicitly relating the asymptotic behaviours of a given solution as we encircle a point at infinity is known as a *connection problem* (connection problems also include relations between behaviours at other singular points of the equation, for instance, $t = 0$ and $t = \infty$ of P_{II}).

In the context of the celebrated Riemann-Hilbert approach (discussed in Section 1.3), the study of such connection formulae is termed *global asymptotic analysis* by Fokas et al. [45]. A Painlevé equation is said to be solved when the asymptotic behaviour of an arbitrary transcendent is entirely determined around every fixed singularity of the equation.

As one might expect based on the generic elliptic asymptotic behaviour, every non-rational solution of $\mathrm{P_{II}}$ is known to possess an infinite number of movable poles [60], which are typically distributed across the interior of each sector S_n as described in Equation (1.5). Boutroux described how lines of movable poles curve around to approach the boundaries of the sectors S_n. Solutions free of movable poles across one or more sectors S_n are exceptional, and those free of poles in a domain containing the real line are often of particular interest when describing physical phenomena. We highlight Boutroux's tritronquée solution, which is asymptotically free of poles in an extended sector of the complex plane, given by four contiguous sectors S_n, and the Hastings-McLeod solution, which is free of poles on the whole real-line (also it is real and positive here).

The Hastings-McLeod and tritronquée solutions are exceptional cases of transcendental solutions and are pivotal in the theory of the asymptotic behaviour of Painlevé transcendents. In Chapter 2, we consider a generalisation of the second Painlevé equation

$$\mathrm{P_{II}}(\mu): \qquad y''(t) = 2y(t)^3 + t^\mu y(t),$$

parameterised by $\mu \in \mathbb{Z}^+$, and primarily concern ourselves with the preservation of these important behaviours for $\mu > 1$. We note that for all $\mu \neq 1$, $\mathrm{P_{II}}(\mu)$ does not possess the Painlevé property. The family $\mathrm{P_{II}}(\mu)$ is a natural generalisation of the second Painlevé equation and was also briefly considered by Boutroux [15]. Regarding the linearisation of $\mathrm{P_{II}}(\mu)$ around $y = 0$, we have extended from Airy function solutions to a class of modified Bessel functions.

A similarly perturbed version of the first Painlevé equation has been previously considered [74, 121]. Tronquée and tritronquée solutions of perturbed $\mathrm{P_I}$ were shown to exist (and be unique in the tritronquée case) in [74]. Meanwhile, [121] studies the full exponentially-improved asymptotic expansion corresponding to a formal series solution of perturbed $\mathrm{P_I}$ (exponential asymptotics are discussed in Section 1.4). Tronquée solutions are uniquely characterised by the *Stokes multiplier*, an arbitrary constant multiplying the exponentially small terms. Asymptotic approximations for the locations of singularities along boundaries of validity (of singularity-free asymptotic behaviour) are deduced in terms of Stokes multipliers in [121].

We refer the reader to Boutroux [16] for detailed descriptions of the solutions he called *intégrales tronquées*, which are pole-free (for sufficiently large $|t|$) within two adjacent sectors of the form S_n defined in Equation (1.5). The region $S_n \cup S_{n+1}$ ($n \in \mathbb{Z}$ mod 6) described by such a pair of sectors has a bisector given by a ray $\arg t = n\pi/3$. The term "tronquée" arises from the fact that any line of poles that would usually lie on such a ray must be truncated as $|t|$ increases (due to the pole-free nature of the solution). There are three such rays in four contiguous sectors $S_n \cup S_{n+1} \cup S_{n+2} \cup S_{n+3}$ ($n \in \mathbb{Z}$ mod 6). Boutroux showed that, for each choice of such a region, there exists a unique solution he called *tritronquée* (triply-truncated), which is asymptotically pole-free in the region.

Meanwhile, the Hastings-McLeod solution of $\mathrm{P_{II}}$ is crucial in considering physical phenomena. The second Painlevé Equation has a rich history as a mathematical model, arising in an extensive range of applications (see, for example, [26, 46, 150, 151]). The case $\alpha = 0$ of $\mathrm{P_{II}}$ is relevant in many of these applications. The Hastings-McLeod solution corresponds to this case of $\mathrm{P_{II}}$ and now plays a critical role in modern random matrix theory due to its appearance in describing the Tracy-Widom distribution, which gives the distribution of the largest eigenvalue of Gaussian unitary ensembles of random matrices as the size of each matrix approaches infinity. The Hastings-McLeod solution also arises in the analogous distribution functions for the Gaussian orthogonal and symplectic ensembles [26, 35, 150].

There are many more physical models in which $\mathrm{P_{II}}$ arises. We mention fluid dynamics [3, 25, 26, 41, 56, 125], mathematical physics [46, 130, 143, 144, 151], and electrodynamics [7, 8]. Concerning plasma physics, de Boer and Ludford [34] posed an asymptotic boundary-value problem regarding a different generalisation of $\mathrm{P_{II}}$. Hastings and McLeod [58] not only showed the existence of a solution satisfying the required global asymptotic properties but also proved it to be unique, this being the now-famous Hastings-McLeod solution of $\mathrm{P_{II}}$. Moreover, the second Painlevé Equation arises in a model of steady, one-dimensional, two-ion electrodiffusion [7]; a review may be found in [28]. It is natural in many physical contexts to consider small perturbations of the model of interest. In Chapter 2, we describe such a perturbation of Bass' electrodiffusion model and show how $\mathrm{P_{II}}(\mu)$ arises naturally.

We also note that for particular values of the parameter α, $\mathrm{P_{II}}$ admits hierarchies of rational solutions [4, 5, 27, 49], and special solutions related to the classical Airy functions [50]. Beyond special solutions, transcendental solutions that vanish as $x \to +\infty$ satisfy the Airy equation to leading order, in the case $\alpha = 0$. The Hastings-McLeod solution first attracted attention in the asymptotic analysis of such solutions.

1.2. Discrete Painlevé Equations

In parallel with the classical (i.e., differential) Painlevé equations, there exist second-order, nonlinear difference equations known as *discrete Painlevé equations* [66, 71, 86, 140]. Each discrete Painlevé equation becomes a classical Painlevé equation in a certain *continuum limit* [134] and is denoted by the number of this corresponding Painlevé equation. For instance, consider the discrete equation

$$\frac{\eta_n}{y_{n+1}+y_n}+\frac{\eta_{n-1}}{y_n+y_{n-1}}+2y_n^2+t=0, \tag{1.6}$$

governing the sequence y_n for $n \in \mathbb{Z}$, as the independent variable η_n evolves additively, that is to say $\eta_n = n + c$ for arbitrary constant c, and t is a constant parameter. In this case, setting

$$y_n = \rho\left(1+\epsilon^2 u(x)\right), \quad \eta_n = \rho^3\left(4+\epsilon^4 x\right) \quad \text{and} \quad t=-6\rho^2,$$

with $\rho^3\epsilon^5 = 1$, yields exactly the first Painlevé equation ($\mathrm{P_I}$) $u'' = 6u^2 + x$ as $\epsilon \to 0$ [42, 53]. Therefore, Equation (1.6) is a discrete version of Painlevé I; this is an example of a continuum limit.

Continuing to use Equation (1.6) as an example, we discuss how discrete Painlevé equations may be obtained directly from classical Painlevé equations or, more specifically, from the recurrence relations satisfied by rational solutions of Painlevé equations. As previously mentioned, Painlevé equations can admit rational solutions; these occur for particular parameter values and are related by *auto-Bäcklund transformations*. Generally speaking, Bäcklund transformations are equations relating differential equations and their solutions, while auto-Bäcklund transformations relate distinct solutions of the same equation.

Rational solutions related by auto-Bäcklund transformations do not occur in the case of the first Painlevé equation since it does not involve a parameter. Considering the second Painlevé Equation (1.3), one may verify that $y(t) = 0$ when $\alpha = 0$, and $y(t) = -1/t$ when $\alpha = 1$, are rational solutions. These solutions form the seed of a hierarchy of infinitely many rational solutions, related by the auto-Bäcklund transformation [42, 53, 141]

$$\frac{\alpha+1/2}{y(t;\alpha+1)+y(t;\alpha)}+\frac{\alpha-1/2}{y(t;\alpha)+y(t;\alpha-1)}+2y(t;\alpha)^2+t=0.$$

The equation above is equivalent to the discrete Painlevé Equation (1.6), where α becomes the additive, discrete independent variable, and t is treated as a constant. We shall also note here that Equation (1.6) appeared first in the work of Jimbo and Miwa [65], although it was not connected with Painlevé equations at this time. In this fashion, many more discrete Painlevé equations arise via continuous Painlevé equations, for instance, [149].

Moving on from this direct connection between the discrete and continuous realms of Painlevé, it is true that discrete Painlevé equations, in their own right, are imbued with fascinating mathematical properties. Amazingly, discrete Painlevé equations appeared prior to their continuous counterpart. We mention the work of Laguerre [99] in 1885, who considered integrable, discrete, nonlinear (and nonautonomous) systems while studying orthogonal polynomials. In the context of recurrence relations satisfied by orthogonal polynomials, Laguerre obtained

$$x_{n+1} + x_n + x_{n-1} = \frac{1}{x_n}\left(n + \rho\Delta_n\right),$$

where $\Delta_n = \left(1 - (-1)^n\right)/2$ and $\rho > -1$. Then, in 1939, again in the study of orthogonal polynomials, Shohat [145] discovered the equation

$$x_{n+1} + x_n + x_{n-1} = \frac{z_n}{x_n} + 1, \tag{1.7}$$

where z_n is linear in $n \in \mathbb{Z}$. It was only in 1990, during work on a field-theoretic model of two-dimensional quantum gravity, that Brézin and Kazakov [17] discovered the continuum limit of Equation (1.7), under which it tends to the first Painlevé equation. From this point, Equation (1.7) was recognised as an additive, discrete version of $\mathrm{P_I}$, now termed d-$\mathrm{P_I}$. It was in the same year that Periwal and Shevitz [130] similarly computed a continuum limit of the difference equation

$$y_{n+1} + y_{n-1} = -\frac{2(n+1)y_n}{\lambda(1 - y_n^2)},$$

where y_n relates to the eigenvalues of a unitary matrix model of two-dimensional string theory [131]. Said continuum limit yields the $\alpha = 0$ case of $\mathrm{P_{II}}$ (1.3).

Unsurprisingly, these identifications in theoretical physics models spurred a growing interest in discrete analogues of Painlevé equations. It was furthermore quickly realised that discrete Panlevé equations appear as reductions of partial difference equations [114], analogous to what we see with the classical Painlevé equations and associated partial differential equations. However, the systematic construction of a discrete integrable version of a given Painlevé equation remained an open question of considerable interest.

A significant development in the area of discrete integrable mappings came in 1991 with the work of Grammaticos, Ramani, and Papageorgiou [54, 137] on the *singularity confinement property*. The singularity confinement property is a test of discrete integrability and uncovered many new discrete Painlevé equations when applied to a family of discrete equations known as QRT mappings [133], much in the same way that searching for ODEs with the Painlevé property lead to the classical Painlevé equations. This test considers the evolution of every possible singular behaviour of a given discrete mapping upon repeated iteration [53]. A singularity propagating indefinitely is said to be *essential*, while it is otherwise *confined*. It is natural to consider the singularity confinement property as a discrete manifestation of the Painlevé property [135, 138].

For example, one might consider a solution of d-P$_{\rm I}$ (1.7) where we set $z_n = \alpha n + \beta$. It is apparent that this mapping is singular with $x = 0$; suppose we choose some initial conditions $x_{n_0} = \gamma$ and $x_{n_0+1} = \epsilon$, with the intent of continuing this solution given that $\epsilon \to 0$. Here, we find that

$$x_{n_0+2} \sim \frac{z_{n_0+1}}{\epsilon}, \quad x_{n_0+3} \sim -\frac{z_{n_0+1}}{\epsilon},$$
$$x_{n_0+4} \sim -\frac{z_{n_0+4}\epsilon}{z_{n_0+1}}, \quad x_{n_0+5} \sim \frac{2\alpha + \gamma z_{n_0+1}}{z_{n_0+4}},$$

as $\epsilon \to 0$ (see Section 1.4 for asymptotic notation definitions). We see that x_{n_0+5} and subsequent iterates are of normal size and that the initial value γ is recovered; this implies that the singularity is confined, and the sequence has a unique global description.

As is already clear, there are not six canonical discrete Painlevé equations; instead, there are many discrete versions for each classical Painlevé equation. We have seen, for example, that difference Equations (1.6) and (1.7) are both discrete versions of Painlevé I. Sakai [140] classified the discrete Painlevé equations based on the resolutions of nine singularities on a complex projective space of dimension two, motivated by the work of Okamoto [117]. A comprehensive list of discrete Painlevé equations may be found in [140] and the review paper [86].

The list above includes two distinct varieties of difference equations: *additive* and *multiplicative*. Thus far, we have provided examples of additive difference equations where the independent variable appears in the form $z_n = \alpha n + \beta$, evolving arithmetically. In Chapters 3 and 4 of this thesis, we consider multiplicative discrete Painlevé equations, in which the independent variable is of the form $z_n = z_0 q^n$, evolving geometrically. Commonly, we refer to the multiplicative case as *q-difference*, where it is conventional to use q as the common ratio of the geometrically progressing independent variable.

Like continuous Painlevé equations, discrete Painlevé equations are prevalent in studying physical systems and are deeply intertwined with other mathematical objects. We have seen that these equations have a historical basis in the theory of orthogonal polynomials, where they continue to commonly appear [96, 99, 107, 108, 145, 154, 155]. Moreover, discrete Painlevé equations arise as similarity reductions of integrable lattice equations such as the differential-difference analogues of the KdV and modified KdV equations [113, 114]. Considering applications in mathematical physics, we have seen discrete Panlevé equations arising in two-dimensional field theoretic models [17, 44, 46, 63, 130, 131].

The generic solutions of discrete Painlevé equations are transcendental. Unsurprisingly, due to their role in mathematical physics, discrete Painlevé equations are considered as defining new nonlinear, discrete special functions analogous to their continuous counterparts. Further mirroring the differential Painlevé equations, for particular choices of parameters, there are solutions of discrete Painlevé equations which are rational or expressible in terms of (the discrete analogues of) classical special functions [57, 85, 87–89, 136].

Asymptotic analysis of discrete Painlevé equations, particularly in the q-difference cases, is less developed than that of the differential Painlevé equations. Analogous to classical Painlevé equations, it was shown in [68] that d-P_I admits divergent asymptotic series expansions as the independent variable moves to infinity, and that these expansions include exponentially small (beyond-all-orders) terms (divergent series expansions and exponential asymptotics are reviewed in Section 1.4). Building on this connection between the asymptotic properties of continuous and discrete Painlevé transcendents, Joshi and Lustri described how the nonlinear Stokes phenomenon manifests when considering d-P_I [69], and this concept is extended similarly to d-P_{II} in [79].

In the case of alternate d-P_I (which is Equation (1.6)), Joshi and Takei [84] give an explicit description of Stokes behaviour (i.e. providing connection formulae) using exact WKB analysis. In this vein, we further note that Xu and Zhao [162] considered discrete Painlevé V asymptotically via the Riemann-Hilbert approach, a method we further discuss in the following section.

By a similar method as in the additive discrete case, whereby a continuum limit is considered, we see a study of the nonlinear Stokes phenomenon as applied to the q-difference first Painlevé equation (q-P_I) in [78]. More pertinent to the content of this thesis are Joshi's aptly named *quicksilver* solutions [72] of q-P_I. These solutions are asymptotic to a divergent series expansion in a particular region of the complex plane as the independent variable becomes large. The term "quicksilver" is due to instability under perturbations in the initial value space of true solutions of this nature. In Chapter 3, we consider solutions of this kind arising in the case of q-P_{II}.

1.3. Isomonodromic Deformation

A widely used method of asymptotic analysis of Painlevé equations in the literature is the Riemann-Hilbert/isomonodromy approach [45], that is, the study of *monodromy-preserving deformations* of an associated linear system. Generally speaking, the *monodromy data* of a Fuchsian system encapsulates a description of behaviours of a fundamental solution around each fixed singularity of the system (and thus provides an explicit description of the linear Stokes phenomenon around a fixed singularity at infinity). Such monodromy data are confined to explicitly defined affine cubic surfaces [156], which act as moduli spaces of Painlevé equations [23, 139, 156]. Given a point on such a surface, the corresponding Painlevé solution is determined by a Riemann-Hilbert problem (RHP) with appropriate contours and jumps in terms of the monodromy data.

In this thesis, we shall refer to the *linear monodromy data* of a solution $y(t)$ of a Painlevé equation; this means the particular monodromy data of a linear system that has coefficients given appropriately in terms of t and $y(t)$ (i.e., such that the monodromy data is invariant under variation in $t \in \mathbb{C}$). We refer to such a linear system as the *linear monodromy problem* associated with a given Painlevé equation. Furthermore, we describe the solution $y(t)$ as being *monodromy solvable* when the associated linear monodromy data may be explicitly calculated. Monodromy solvable solutions occur due to the associated linear problem degenerating at a particular value of t so that it becomes solvable using classical special functions; this transpires, for example, in some instances of rational or special solutions of Painlevé equations. In Subsection 4.1.4 of Chapter 4, we show the explicit construction of linear monodromy data associated with solutions of the fourth Painlevé equation and derive the associated monodromy manifold.

A celebrated development in the theory of RHPs came with the method of steepest descent by Deift and Zhou [36]. In the context of monodromy-preserving deformations and the associated RHP, this method has been used to obtain global asymptotic results regarding the general solutions of Painlevé equations [43, 45]. A review of the application of this method to elliptic function behaviours of P_{I} and P_{II} may be found in [93]. Isomonodromy methods are a powerful tool in the search for connection formulae relating the behaviours of a given Painlevé transcendent at different singular points of the equation (i.e., at $t = 0$ and $t = \infty$ in the case of P_{II}).

The concept of an associated linear monodromy problem led to the discovery of the sixth Painlevé equation. Indeed, the initial classification of ODEs by Painlevé was incomplete. Fuchs uncovered the sixth Painlevé equation by considering generic, linear, second-order differential equations, say in z, with four regular singular points. He was interested in deforming the coefficients of such a system, in particular varying the location of one of the fixed singularities, say at $z = t$, such that the monodromy data is preserved. Fuchs found that this implies an additional equation involving differentiation with respect to t. Along with the original system in z, these are now referred to as a *Lax pair*. It is the compatibility of these equations, one being differential in z and the other in t, which leads to the coefficients of the original system being necessarily governed by a second order, nonlinear ODE in t: the sixth Painlevé equation. It is now recognised that the other Painlevé equations likewise represent monodromy preserving deformations of linear differential systems.

The core idea here is that a Lax pair implies a *monodromy mapping* between the solution space of a Painlevé equation and the monodromy space of an associated Fuchsian system (i.e., one with four regular singular points in the case of P_{VI}). In other words, the monodromy data represent integrals of motion of the corresponding Painlevé equation. Therefore, analysis of an associated linear system, for t approaching different singular points of the Painlevé equation, allows one to deduce connection results for a given Painlevé transcendent in terms of the conserved linear monodromy data.

However, we note that the monodromy data of a Painlevé transcendent cannot be explicitly calculated outside of exceptional cases. Considering the fourth Painlevé equation as an example (as this is an object of study in Chapter 4), Umemura [153] showed that generic solutions of this equation are new higher transcendental functions, but there are certain exceptional cases called "classical solutions" that arise for special values of parameters. These exceptional solutions correspond to explicitly solvable monodromy data of the associated Fuchsian system. Perhaps surprisingly, there furthermore exist transcendental Painlevé solutions with solvable monodromy. Outside of solutions given precisely by classical special functions [48], Fuchs first gave a monodromy solvable solution (in the case of P_{VI}); this is the linear monodromy data of Picard's solutions (rediscovered in [95, 110]), although, as is the case with rational and special solutions, these solutions only arise for particular parameter values of the Painlevé equation.

In Chapter 4, we consider a class of transcendental, monodromy solvable solutions that exist for general parameter values, known as *symmetric solutions*. A symmetric solution $y(t)$ requires particular initial values prescribed at the *reflection point*, say $t = 0$, and is then symmetric under specific rotations or reflections in the complex t plane. When evaluated at the reflection point $t = 0$, the associated linear system is exactly solvable in terms of classical special functions. These symmetric solutions are described for the first and second Painlevé equations by Kitaev, who calculates their linear monodromy data in [94]. For P_I and P_{II}, Kitaev showed that the associated linear systems degenerate exactly to Hankel and Whittaker equations, respectively. One may then calculate the linear monodromy data in these cases since asymptotic connection formulae regarding these classical special functions are well known; it is easiest to consult the Digital Library of Mathematical Functions (DLMF [37]) regarding these types of results. In Chapter 4, we consider the fourth Painlevé equation (P_{IV}) and its linear monodromy problem. Kaneko outlines symmetric solutions of the fourth Painlevé equation [90] and provides their linear monodromy data explicitly.

In recent developments, it has been recognised that the solutions of q-difference Painlevé equations may similarly be viewed as monodromy preserving deformations of associated q-difference linear systems, see for instance [82, 83]. General results regarding the q-difference monodromy problem are rooted in the classical studies of Birkhoff and Carmichael [11, 19] (we elaborate on the q-monodromy formulation of Birkhoff and Carmichael in Section 1.7), with more recent developments based on q-difference Galois theory by Ohyama, Ramis, Sauloy and Zhang; see for example [115, 142].

Mano [109] was the first to perform global asymptotic analysis of a discrete Painlevé equation from the isomonodromy point of view; this is regarding q-P_{VI}, serving as a q-analogue of Jimbo's classical work on P_{VI}. Joshi and Roffelsen have furthered work on connection formulae regarding q-P_{VI} in [80]. In [83] and [82], Joshi and Roffelsen consider RHPs associated with q-P_{VI} and q-P_{IV}, respectively, and construct bijective mappings between the solution spaces of the these q-difference Painlevé equations, and data on an associated q-monodromy surface.

Moreover, mirroring the differential setting, there exist q-difference (transcendental) symmetric solutions which correspond to explicitly solvable monodromy problems; we consider this in the case of q-P_{IV} [81] in Chapter 4.

1.4. Asymptotic Analysis

As it is core to this thesis, we shall precisely define the notation we use to express asymptotic analysis. Let $S \subseteq \mathbb{C}$ with limit point x_0, and suppose $f(x)$ and $g(x)$ are complex functions with S in their domain. We say that $f(x)$ is *of order* $g(x)$ as $x \to x_0$ in S, if there exists a constant $C \in \mathbb{R}^+$ and a punctured open domain $D \in \mathbb{C}$ of x_0, such that $|f(x)| \leq C|g(x)|$ for all $x \in D \cap S$. In this case we use the "big O" notation of Landau [100], writing $f(x) = \mathcal{O}\left(g(x)\right)$ as $x \to x_0$ in S.

Similarly, we say that $f(x)$ is *of order less than* $g(x)$ as $x \to x_0$ in S, if for every constant $C \in \mathbb{R}^+$, there exists a punctured open domain $D \in \mathbb{C}$ of x_0, such that again $|f(x)| \leq C|g(x)|$ holds for all $x \in D \cap S$. Here we use the "small o" notation, writing $f(x) = o\left(g(x)\right)$ as $x \to x_0$ in S. Finally, we say that $f(x)$ is *asymptotic* to $g(x)$ as $x \to x_0$ in S, if

$$\lim_{x \to x_0,\, x \in S} \frac{f(x)}{g(x)} = 1,$$

and we denote this by $f(x) \sim g(x)$ as $x \to x_0$ in S.

Furthermore, we shall discuss Poincaré's definition of an asymptotic power series and its limitations. By Poincaré [132], a function $f(x)$ is asymptotic to a power series as $x \to x_0$ in S, written

$$f(x) \sim \sum_{k=k_0}^{\infty} f_k (x - x_0)^k,$$

given that for every fixed $k_0 \leq N \in \mathbb{N}$ and sufficiently small $|x - x_0|$ we may write

$$\Big| \ f(x) - \sum_{k=k_0}^{N-1} f_k (x - x_0)^k \ \Big| = \mathcal{O}\left(|x - x_0|^N\right), \quad x \to x_0.$$

Note that this definition avoids being restricted to series that converge; throughout this thesis, we shall be primarily concerned with divergent series expansions.

We refer to this type of expansion as *algebraic asymptotic behaviour*. The above power series expansion fails to distinguish between functions that differ by exponentially small terms in the limit $x \to x_0$. Illustratively, one might consider that $\exp(-1/x) = o(x^n)$ for all $n \in \mathbb{N}$ as $x \to 0$ in $\mathbb{R}^+$. Equivalently, we notice that under Poincaré's definition, an exponentially small function may only be represented by the trivial series

$$e^{-1/x} \sim \sum_{k=0}^{\infty} 0 \cdot x, \quad x \to 0, \quad x \in \mathbb{R}^+.$$

We refer to such exponential effects as being hidden *beyond-all-orders*, leading to potential ambiguity in the distinction of functions defined by power series expansions. Moreover, as trivial as it is, the above power series expansion is only valid for $-\pi < \arg x < \pi$. Here, we see that exponential effects imply boundaries of validity for algebraic asymptotic behaviours, the boundaries being where exponential terms become oscillatory.

When considering the asymptotic behaviour of Painlevé transcendents as the independent variable approaches infinity, we deduce exponentially small terms, referred to as an "exponentially-improved" approximation by Olver [124]. The appearance of these exponential terms is emblematic of the nonlinear Stokes phenomenon as discussed in Section 1.1. We mention that the determination of exponentially small terms has been used to develop connection formulae (resolving the linear Stokes phenomenon) in the cases of such classical special functions as the gamma function [129, 159], generalised exponential functions [67, 124], Riccati equations [118, 119] and functions with integral representations [9, 122, 128, 160].

In Chapters 3 and 4, we apply the concepts of exponential asymptotics to q-difference Painlevé transcendents, seeing that exponentially small (beyond-all-orders) terms appear naturally in the form of classical q-special functions. Regarding solutions of a class of second order, linear difference equations, Olver [123] deduced exponentially improved asymptotic expansions in terms of inverse factorial series, with applications to Legendre functions. Olde Daalhuis [120] furthered these results to a larger class of equations with applications to hypergeometric functions.

In the development of exponential asymptotic methods for nonlinear problems, we direct the reader to the work of Chapman, Costin, King, and Kruskal [20, 21, 29, 32, 33, 92]. Uniform asymptotic expansions for solutions of nonlinear equations include Ricatti type equations [118, 119], Burgers equation [22], and regarding Painlevé equations [30, 31, 68, 84, 119]. Outside of purely mathematical considerations, it is worth noting that a myriad of physical phenomena occur on the exponentially small scale. Exponential asymptotics play a crucial role in physical applications such as crystal growth models [98] and fluid dynamics models [55, 103–106, 146]. The development of quantum field theories [6, 24, 39] relies on perturbation series, capturing effects that the Poincaré asymptotic power series cannot describe.

1.5. Solutions of Multiplicative Difference Equations

In this thesis, we consider the multiplicative variety of discrete equations. Given a constant $q \in \mathbb{C}^*$ with $|q| \neq 1$, a set $S \subset \mathbb{C}^*$ and a function $y : D \to \mathbb{C}$ where $D = \{q^k z \in \mathbb{C}^* \,|\, z \in S,\, k \in \mathbb{Z}\}$, a q-difference equation is a relation between values $y(q^k z)$ for k across some subset of $\mathbb{Z}$ and arbitrary $z \in S$. Generally speaking, a k^{th} order q-difference equation is of the form

$$F\left(y(q^k z), y(q^{k-1} z), \dots, y(qz), y(z), z\right) = 0, \quad z \in S. \tag{1.8}$$

The above q-difference equation is linear given that F is linear in $y(q^k z), \dots, y(z)$.

In the context of nonlinear q-difference equations such as discrete Painlevé equations, we consider the set S as containing a single point, say $S = \{z_0\}$ for some $z_0 \in \mathbb{C}^*$. Thus, a solution has the discrete domain $D = \{q^k z_0 \in \mathbb{C}^* \,|\, k \in \mathbb{Z}\}$ which from this point we will simply denote by $z_0 q^{\mathbb{Z}}$. The points $z_0 q^{\mathbb{Z}}$ lie along a curve in the complex plane known as a *q-spiral*, given by $z_0 q^t$ for $t \in \mathbb{R}$. Given parameter $q \in \mathbb{C}^*$ with $|q| \neq 1$, our choice of z_0 defines a distinct q-spiral along which the solution is iterated. Parameterising the independent variable by $z = z_n = z_0 q^n$ for $n \in \mathbb{Z}$, the solution is a sequence which we equivalently denote by $y = y_n = y(z)$.

Here we call n the *additive independent variable* while z is the *multiplicative independent variable*; note that under this parameterisation Equation (1.8) may be rewritten as the k^{th} order additive difference equation

$$F\left(y_{n+k}, y_{n+k-1}, \dots, y_{n+1}, y_n, z_0 q^n\right) = 0.$$

In the context of sequence solutions, we will also use bars to denote iterates where convenient: $n \to n+1$ implies $y \to \overline{y}$ and $n \to n-1$ implies $y \to \underline{y}$.

In the following section, we proceed to discuss the solutions of linear q-difference equations, this being of particular relevance to Chapter 4 where we consider solutions of q-$\mathrm{P_{IV}}$ as monodromy preserving deformations of a system of linear q-difference equations. In the linear case, we may instead consider $S = \mathbb{C}^*$ and construct meromorphic solutions $y(z)$ satisfying a given q-difference equation at every $z \in \mathbb{C}^*$.

1.6. Linear Multiplicative Difference Equations

This section examines first-order linear q-difference equations, whose solutions serve as fundamental building blocks in the linear q-difference equations theory. This foundational material traces back to the seminal work of Gasper and Rahman [52]. Building on this, in the following section, we delve into the classical contributions of Birkhoff and Carmichael [11, 19], focusing on linear q-difference systems and the associated monodromy data, which play a central role in understanding the global structure and behaviour of these systems.

Consider the generic, linear first-order q-difference equation

$$y(qz) = f(z)y(z), \tag{1.9}$$

where $f(z)$ is meromorphic on $\mathbb{C}^*$ and not identically zero. Contrary to the previous section, where we considered a discrete domain confined to a q-spiral, here we are interested in solutions $y(z)$ that are meromorphic on $\mathbb{C}^*$, satisfying Equation (1.9) on this domain.

First, considering the simplest case $f(z) \equiv 1$, we obtain solutions known as *q-elliptic functions*. In the general q-difference setting, any complex function satisfying $y(qz) = y(z)$ on its domain is referred to as *q-constant* or *q-periodic*, we call the function q-elliptic when it is meromorphic on the domain $\mathbb{C}^*$. A q-elliptic function is analogous to a constant in the theory of linear differential equations: the quotient of two solutions of Equation (1.9) is a q-elliptic function, and in this sense, Equation (1.9) is fully solved once one solution is found. We note that a q-elliptic function $y(z)$ is necessarily constant (in the usual sense) if $y(z)$ is analytic or has no zeros on $\mathbb{C}^*$.

We proceed to consider the case of Equation (1.9) given by $f(z) = 1/(1-z)$, and thus introduce the *q-Pochhammer* symbol, a cornerstone of q-calculus. For the remainder of this section, we shall assume $0 < |q| < 1$. The q-Pochhammer symbol is defined

$$(z;q)_n = \prod_{k=0}^{n-1}(1-q^k z) \quad \text{with} \quad (z;q)_0 = 1, \quad z\in\mathbb{C},$$

extending to the infinite product

$$(z;q)_\infty = \prod_{k=0}^{\infty}(1-q^k z),$$

which converges locally uniformly in z on $\mathbb{C}$. In particular, $(z;q)_\infty$ is an entire function satisfying $(qz;q)_\infty = (z;q)_\infty/(1-z)$, with $(0;q)_\infty = 1$ and simple zeros on the semi q-spiral $q^{-\mathbb{N}}$. Now, consider the product

$$\theta_q(z) = \theta(z;q) := (z;q)_\infty(q/z;q)_\infty, \quad z\in\mathbb{C}^*. \tag{1.10}$$

We see straightforwardly that this product of q-Pochhammer symbols satisfies the q-difference equation $\theta_q(qz) = -\theta_q(z)/z$ (now the case $f(z) = -1/z$ in Equation (1.9)), and furthermore $\theta_q(z) = \theta_q(q/z)$ by definition. We call (1.10) the *q-theta function*, this being due to its role as a q-analogue of the usual theta-functions in elliptic function theory. The q-theta function is holomorphic on $\mathbb{C}^*$, with essential singularities at $z=0$ and $z=\infty$ and simple zeros on the q-spiral $q^{\mathbb{Z}}$. Regarding products of q-Pochhammer symbols and q-theta functions, we also introduce the shorthand

$$\begin{aligned}\theta_q(z_1,\ldots,z_n) =& \theta_q(z_1)\times\cdots\times\theta_q(z_n),\\ (z_1,\ldots,z_n;q)_\infty =& (z_1;q)_\infty\times\cdots\times(z_n;q)_\infty.\end{aligned}$$

With the q-theta function at our disposal, we may construct a meromorphic solution of Equation (1.9) with $f(z) = \alpha\in\mathbb{C}^*$ being a constant, by setting for instance $y(z) = \theta_q(z/\alpha)/\theta_q(z)$. Such a ratio of q-theta functions is sometimes denoted $e_q(z;\alpha) := \theta_q(z/\alpha)/\theta_q(z)$ in the literature, this being considered a q-analogue of the exponential function.

In fact, by similarly utilising ratios of q-theta functions, we may introduce a generic description of q-elliptic functions. Firstly, we define the *fundumental q-annulus* by

$$D_q(r) = \{z\in\mathbb{C}^* \,|\, |q|r \le z < r\},$$

for some $r\in\mathbb{R}^+$. A q-elliptic function is uniquely characterised, up to multiplication by an arbitrary constant, by the locations (and orders) of its zeros and poles on any given fundamental q-annulus. Therefore, consider the solution of $y(qz) = y(z)$, meromorphic on $\mathbb{C}^*$, given by

$$y(z) = z^n\prod_{j=1}^{k}\frac{\theta_q(z/a_j)}{\theta_q(z/b_j)}, \quad \text{with} \quad \prod_{j=1}^{k}\frac{b_j}{a_j} = q^n, \quad n\in\mathbb{Z}.$$

The above solution has $k \in \mathbb{Z}^+$ zeros and poles (counting multiplicities), whose locations in a given fundamental q-annulus are straightforwardly determined by constants $a_1, \dots a_k$ and $b_1, \dots b_k$, respectively. We see that in order to satisfy $y(qz) = y(z)$, solutions have an equal number of zeros and poles (counting multiplicities) on a fundamental q-annulus, and the product of ratios b_j/a_j must belong to $q^{\mathbb{Z}}$.

At this point, it is instructive to consider an arbitrary function $f(z)$ in Equation (1.9) that is meromorphic on $\mathbb{C}$. Here, one may determine $\alpha \in \mathbb{C}^*$ and $n \in \mathbb{Z}$ such that $f(z) = \alpha z^n g(z)$ where $g(z)$ is holomorphic at $z = 0$ with $g(0) = 1$. Therefore, by making the transformation

$$y(z) = e_q(z; \alpha)/\theta_q(-z)^n \psi(z), \tag{1.11}$$

we obtain the equation

$$\psi(qz) = g(z)\psi(z), \tag{1.12}$$

where $z = 0$ is now an *ordinary point* of the equation. There is a nonzero solution $\psi(z)$ holomorphic at $z = 0$, with a unique meromorphic continuation to $\mathbb{C}$. Indeed, suppose that such a solution exists and without loss of generality let $\psi(0) = 1$, then by Equation (1.12) we have evidently

$$\psi(q^{n+1}z) = \psi(z) \prod_{k=0}^{n} g(q^k z), \quad \forall n \in \mathbb{N},$$

and taking $n \to \infty$, we obtain the infinite product representation

$$\psi(z) = \prod_{k=0}^{\infty} 1/g(q^k z). \tag{1.13}$$

It is elementary to find then that the above infinite product converges uniformly in z on $\{z \in \mathbb{C} \,|\, |z| < r\}$, where we let $r \in \mathbb{R}^+$ be the radius of convergence of $1/g(z)$ about $z = 0$. Furthermore, we see that $\psi(0) = 1$. Therefore, (1.13) defines a solution $\psi(z)$ holomorphic around $z = 0$, and this infinite product representation also allows for meromorphic continuation into $\mathbb{C}$. One may equivalently construct solutions of Equation (1.9) about $z = \infty$ for any function $f(z)$ meromorphic on $\mathbb{C}^*$.

Before discussing the classical work of Birkhoff on systems of first order, linear q-difference equations, we shall further discuss q-theta functions, introducing a novel "standard" description of a holomorphic solution of $y(qz) = \alpha y(z)/z^n$ on $\mathbb{C}^*$ where $\alpha \in \mathbb{C}^*$ and $n \in \mathbb{Z}^+$. Following [82], for $\alpha \in \mathbb{C}^*$ and $n \in \mathbb{Z}^+$, we define $V_n(\alpha)$ as the set of all analytic q-theta functions $y(z)$ on $\mathbb{C}^*$ satisfying $y(qz) = \alpha y(z)/z^n$. Under the usual function addition and scalar multiplication, it is not difficult to show that $V_n(\alpha)$ forms a complex vector space of dimension n (see [82] for further details). As it shall prove pertinent to Chapter 4 of this thesis, we propose a useful set of basis functions that span the vector space $V_n(\alpha)$; these are the q-theta functions

$$\theta_{q^n}(-q^k z^n/\alpha) z^k \quad \text{for} \quad k \in \{0, 1, \ldots, n-1\}. \tag{1.14}$$

Firstly, we see that each of the n q-theta functions (1.14) are holomorphic on $\mathbb{C}^*$ satisfying $y(qz) = \alpha y(z)/z^n$. Building on this, we see that each function (1.14) introduces a distinct set of n zeros within a fundamental q-annulus: considering $D_q(|\alpha|^{1/n})$, the function $\theta_{q^n}(-q^k z^n/\alpha)z^k$ for $k \in \{0, \ldots, n-1\}$ has zeros distributed evenly on the circle given by $|z| = |\alpha|^{1/n}|q|^{1-k/n}$. It is not difficult to see that the functions are linearly independent elements of $V_n(\alpha)$. Take the inductive hypothesis that the functions $\theta_{q^n}(-q^k z^n/\alpha)z^k$ are linearly independent for $k \in \{0, 1, \ldots, n-2\}$, then consider the equation

$$\sum_{k=0}^{n-2} c_k \theta_{q^n}(-q^k z^n/\alpha) z^k = \theta_{q^n}(-q^{n-1} z^n/\alpha) z^{n-1}, \tag{1.15}$$

for some constants c_k. Evaluating Equation (1.15) at the n zeros (within a fundamental q-annulus) of $\theta_{q^n}(-q^{n-1}z^n/\alpha)$ yields an overdetermined, homogeneous linear system of equations for constants c_k, and it follows by the inductive hypothesis that the solution is unique and trivial, i.e., $c_k = 0$ for all $k \in \{0, \ldots, n-2\}$.

1.7. Birkhoff's Theory

In this section, we briefly summarise Birkhoff's approach to the global asymptotic analysis of systems of linear q-difference equations. Analogous to the prior section, we begin by considering the first-order, linear matrix q-difference equation

$$Y(qz) = A(z)Y(z), \tag{1.16}$$

where we restrict ourselves to the rank two case. Furthermore, suppose that

$$A(z) = \sum_{k=0}^{n} A_k z^k, \tag{1.17}$$

where $n \in \mathbb{N}$ is the degree of the equation. It is assumed that A_0, A_n are diagonalisable and we denote

$$A_0 = M_0 \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{pmatrix} M_0^{-1}, \quad A_n = M_\infty \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix} M_\infty^{-1}.$$

We assume that $\theta_1/\theta_2,\ \kappa_1/\kappa_2 \notin q^{\mathbb{Z}}$; these constraints are analogous to the *non-resonant condition* for differential equations. Now, we may define the *canonical solution at zero* and the *canonical solution at infinity*. In the prior section we showed that Equation (1.9), with any function $f(z)$ meromorphic on $\mathbb{C}$, is satisfied by (1.11) where $\psi(z)$ is a unique solution holomorphic around $z = 0$ with $\psi(0) = 1$. Equivalently, Carmichael [19] showed that by defining

$$Y_0(z) = M_0\Phi_0(z)E_0(z), \quad Y_\infty(z) = 1/\theta_q(-z)^n M_\infty \Phi_\infty(z)E_\infty(z),$$

where the matrices $E_0(z), E_\infty(z)$ with $|E_0(z)|, |E_\infty(z)| \neq 0$ satisfy

$$E_0(qz) = \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{pmatrix} E_0(z), \quad E_\infty(qz) = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix} E_\infty(z),$$

there exist unique fundamental solutions $\Phi_0(z)$ and $\Phi_\infty(z)$, holomorphic around $z = 0$ and $z = \infty$, respectively, satisfying $\Phi_0(0) = I$ and $\Phi_\infty(\infty) = I$.

Thus, we define the *connection matrix* $P(z)$ by $Y_\infty(z) = Y_0(z)P(z)$, which encapsulates q-monodromy data. Evidently, $P(z)$ is meromorphic on $\mathbb{C}^*$ and satisfies $P(qz) = P(z)$, i.e., it is comprised of q-elliptic functions. Considering this formulation, we notice that matrices E_0, E_∞ are defined somewhat ambiguously, only characterised up to q-elliptic functions. As we shall see in Chapter 4, it is perhaps more natural to directly consider the q-difference equations

$$\begin{aligned} M_0\Phi_0(qz) =& A(z)M_0\Phi_0(z) \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{pmatrix}^{-1}, \\ M_\infty\Phi_\infty(qz) =& 1/z^n A(z)M_\infty\Phi_\infty(z) \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}^{-1}, \end{aligned}$$

and define the corresponding connection matrix $C(z)$ by

$$M_\infty\Phi_\infty(z) = M_0\Phi_0(z)C(z).$$

The alternate portrayal of q-monodromy data, $C(z)$, is then meromorphic on $\mathbb{C}^*$ and satisfies

$$C(qz) = 1/z^n \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{pmatrix} C(z) \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}^{-1},$$

and therefore, the entries of $C(z)$ are q-theta functions belonging to the spaces $C_{jk}(z) \in V_n(\theta_j/\kappa_k)$ for $j, k \in \{1, 2\}$.

Consider $C(z)$ only up to multiplication from the left and right by invertible diagonal matrices, denoting the equivalence class $[C(z)]$. Then, given $\theta_1, \theta_2, \kappa_1, \kappa_2$, the celebrated *Riemann-Hilbert-Birkhoff correspondence* is the bijective mapping between matrix polynomials $A(z)$, up to conjugation by $GL_2(\mathbb{C})$, and corresponding $[C(z)]$. We mention that analogous to the discovery of $\mathrm{P_{VI}}$, Jimbo and Sakai [66] derived q-$\mathrm{P_{VI}}$ within Birkhoff's q-monodromy framework.

The general degree one case (that is, $n = 1$ in Equation (1.17)) was considered by Le Caine [101], who gave the fundamental solutions around $z = 0$ and $z = \infty$ in terms of ${}_2\phi_1$ q-hypergeometric series [52], and thus determined the corresponding connection matrix explicitly. We recall the basic hypergeometric series [52] with base q is

$$
\begin{aligned}
&{}_r\phi_s\left(a_1,\ldots,a_r;b_1,\ldots,b_s;q,z\right)\\
=&\sum_{k=0}^{\infty}\frac{(a_1,\ldots,a_r;q)_k}{(b_1,\ldots,b_s;q)_k(q;q)_k}\left\{(-1)^k q^{k(k-1)/2}\right\}^{1+s-r} z^k.
\end{aligned}
$$

In Appendix B, we consider a model q-linear system to demonstrate the explicit construction of a q-monodromy problem. We show the application of q-hypergeometric series [52] (analogous to the use of classical special functions) in giving an exact solution of this monodromy problem. The model linear matrix system $Y(qz) = A(z)Y(z)$ considered is similar to the linear system associated with q-$\mathrm{P_{VI}}$ [83], having a coefficient matrix $A(z)$ that is degree two.

We pay special attention to the q-Airy function, which serves a key role in Chapter 4 of this thesis when considering the linear q-monodromy problem associated with q-$\mathrm{P_{IV}}$. The q-Airy function arose in the study of special solutions of the second q-Painlevé equation [57, 89], and is the basic hypergeometric series

$$
\mathrm{Ai}_q(z)=\sum_{k=0}^{\infty}\frac{1}{(q^2;q^2)_k}q^{k(k-1)/2}z^k={}_1\phi_1\left(0;-q;q,-z\right).
$$

The q-Airy function satisfies the second-order linear q-difference equation

$$
\mathrm{Ai}_q(q^2z)+z\mathrm{Ai}_q(qz)-\mathrm{Ai}_q(z)=0,
$$

which may be shown to approach the classical Airy equation in a certain continuum limit. Utilising the q-Borel summation method introduced by Zhang [163], Morita [112] deduced a connection formula relating the q-Airy function $\mathrm{Ai}_q(z)$ to q-hypergeometric series around $z=\infty$, in particular

$$
\theta_q(z/q)\mathrm{Ai}_q(z)+\theta_q(-z/q)\mathrm{Ai}_q(-z)=(-1;q)_\infty\,\mathrm{A}_{q^2}(-q^3/z^2),\quad z\in\mathbb{C}^*, \tag{1.18}
$$

where $A_q(z)$ is known as the Ramanujan function, given by

$$
\mathrm{A}_q(z)=\sum_{k=0}^{\infty}\frac{1}{(q;q)_k}q^{k^2}(-z)^k={}_0\phi_1(-;0;q,-qz),
$$

satisfying

$$
qz\mathrm{A}_q(q^2z)-\mathrm{A}_q(qz)+\mathrm{A}_q(z)=0,\quad z\in\mathbb{C}.
$$

We find in Chapter 3 that the q-Airy equation arises when describing beyond-all-orders (exponentially small) perturbation terms corresponding to divergent, formal series solutions of q-$\mathrm{P_{II}}$ (and this concept similarly applies to other q-difference Painlevé equations).

1.8. Outline of Thesis

In Chapter 2, we study solutions of Equation $\mathrm{P_{II}}(\mu)$, which we refer to as *perturbed* Painlevé II, via asymptotic analysis. We first perform local asymptotic analysis of solutions $y(x)$ as the independent variable approaches infinity; in particular, we describe singularity-free behaviours in this limit. We consider a formal series solution around the fixed singularity at $x = \infty$, a divergent asymptotic series. Improving this asymptotic description with an exponentially small term, we see that this behaviour is characterised by one free parameter (Stokes multiplier). Moreover, we consider the corresponding behaviour on an *anti-Stokes line* (Stokes boundary), where exponential effects become oscillatory, and two free parameters characterise the behaviour. This analysis similarly applies to solutions that vanish as $|x| \to \infty$.

We proceed to prove the existence of solutions that are natural generalisations of the tronqueé and tritronqueé solutions of $\mathrm{P_{II}}$, asymptotic to the divergent power series expansion mentioned above in an extended open sector of the complex x plane as $|x| \to \infty$. Moreover, in the case of odd values of the integer parameter μ, we prove the unique existence of an appropriate generalisation of the Hastings-McLeod solution, satisfying certain asymptotic boundary conditions at $x = \pm\infty \in \mathbb{R}$. Finally, we consider a generalisation of Bass' model of electrodiffusion where $\mathrm{P_{II}}$ is known to arise; under modified conditions, we see that equations of the more general $\mathrm{P_{II}}(\mu)$ variety appear as mathematical models.

We move to the q-difference setting in Chapter 3, considering solutions $f(t)$ of q-$\mathrm{P_{II}}$ by asymptotic analysis. By the local asymptotic analysis of solutions, in both limits $t \to 0$ and $t \to \infty$, we provide leading order invariant quantities associated with the equation. We show that iterates of the solution are confined to certain algebraic curves in the phase space $(f(t), f(qt))$.

Outside of the generic behaviours, in the limit $|t| \to \infty$, we identify divergent power series expansions with no free parameters. We show that there exist true quicksilver solutions asymptotic to these series as $t \to \infty$ in a particular domain of the complex t plane; these solutions are characterised by a free parameter that multiplies an exponentially small term beyond all orders of the power series expansion. These solutions are analogous to tronqueé solutions in the q-difference setting, and following this analogy, we proceed to deduce an oscillatory-type asymptotic expansion on the Stokes-like boundaries.

Also, in Chapter 3, we consider the asymptotic limit $\epsilon \to 0$ where $t = t_0(1 + \epsilon)^n$ (i.e., $q \to 1$, a continuum limit), and thus analyse the full autonomous form of q-$\mathrm{P_{II}}$. Here, we show that generic solutions are parameterised in terms of elliptic functions, analogous to leading order autonomous (differential) Painlevé equations. Building on this analogy, we show that the slow modulation in this leading order elliptic behaviour, under repeated iteration in n, is described by complete elliptic integrals.

In Chapter 4, we endeavour to deduce asymptotic information regarding a symmetric solution $y(x)$ of $\mathrm{P_{IV}}$ as $|x| \to \infty$, via asymptotic analysis of the associated linear system in this limit. We then consider an analogous problem in the case of q-$\mathrm{P_{IV}}$. We apply a (q-)isomonodromic deformation approach to characterise symmetric solutions' asymptotic behaviour away from the reflection point. Furthermore, we find that this implies certain conditions on the parameters of the Painlevé equation leading to tronqueé/quicksilver-type symmetric solutions.

1.8.1. Original results. In this subsection, we summarise the original results reported in this thesis. In Chapter 2 the new results are described in Propositions 2.14 and 2.24, regarding the tritronqueé solution and Hastings-McLeod-type solution of $\mathrm{P_{II}}(\mu)$, respectively. The results of this chapter were published in [59].

In Section 3.2 of Chapter 3, for the first time we prove the existence of true "quicksilver" solutions of q-$\mathrm{P_{II}}$, and we proceed to describe a novel, local asymptotic expansion of solutions of q-$\mathrm{P_{II}}$, see Equation (3.25). Then, in Section 3.3 of Chapter 3, key new results are described in Theorems 3.40 and 3.47, concerning the generic elliptic asymptotic behaviour of q-$\mathrm{P_{II}}$ and its slow modulation under repeated iteration.

In Chapter 4, we provide detailed results concerning "anti-Stokes" local asymptotic expansions of solutions of $\mathrm{P_{IV}}$, see Equation (4.2) and subsequent exposition. In Equation (4.17), we provide an asymptotic representation of the linear monodromy data associated with a solution $y(x)$ of $\mathrm{P_{IV}}$, in the limit $x \to \infty$ on an anti-Stokes line. In Section 4.2 of Chapter 4, we similarly consider solutions of q-$\mathrm{P_{IV}}$, and the core result is given by Theorem 4.49.

CHAPTER 2

Solutions of Perturbed Painlevé II

In this chapter, we consider solutions of the perturbed second Painlevé equation

$$\mathrm{P}_{\mathrm{II}}(\mu): \qquad y''(x) = 2y(x)^3 + x^\mu y(x),$$

identified in an asymptotic limit, where we have introduced parameter $\mu \in \mathbb{Z}^+$, which is constant with respect to x. When $\mu = 1$, we have the $\alpha = 0$ case of the second Painlevé equation

$$\mathrm{P}_{\mathrm{II}}: \qquad y''(x) = 2y(x)^3 + xy(x) + \alpha.$$

As with P_{II}, the equations $\mathrm{P}_{\mathrm{II}}(\mu)$ are a family of second-order, nonlinear ODEs. A core distinction between P_{II} and $\mathrm{P}_{\mathrm{II}}(\mu)$ with $\mu \neq 1$ is that transcendental solutions of $\mathrm{P}_{\mathrm{II}}(\mu)$ are typically non-meromorphic in an arbitrary sector of the complex x-plane, featuring movable singularities which are not poles.

The overarching purpose of our analysis of $\mathrm{P}_{\mathrm{II}}(\mu)$ is to prove that it possesses solutions that are analogous to the Hastings-McLeod [58] and tritronqueé [16] solutions of P_{II}, respectively. The Hastings-McLeod and tritronqueé solutions are transcendental solutions of P_{II}, distinguished by their global asymptotic properties (see Section 1.1 of Chapter 1 for further background and details). Both solutions are pivotal in the theory of transcendental solutions of P_{II}.

The Hastings-McLeod solution is unique by virtue that it is real, positive, and holomorphic for all $x \in \mathbb{R}$, abiding by certain prescribed asymptotic behaviours as $x \to \pm\infty \in \mathbb{R}$. Furthermore, it is crucial in physical applications, random matrix theory, and more. The tritronqueé solution, uniquely, is holomorphic for all sufficiently large $|x|$ in four contiguous sectors of the complex x-plane given by

$$S_n = \left\{ x \in \mathbb{C} \;\middle|\; (n-1)\pi/3 \leq \arg(x) \leq n\pi/3 \right\}, \quad n \in \mathbb{Z} \mod 6.$$

We prove that natural generalisations of these solutions exist in the case of $\mathrm{P}_{\mathrm{II}}(\mu)$, which is surprising given that $\mathrm{P}_{\mathrm{II}}(\mu)$ with $\mu \neq 1$ does not possess additional core properties of P_{II}, such as the Painlevé property. Furthermore, we show that $\mu \neq 1$ cases of $\mathrm{P}_{\mathrm{II}}(\mu)$ are relevant as mathematical models in a physical application. We begin with a local asymptotic analysis of solutions satisfying $\mathrm{P}_{\mathrm{II}}(\mu)$ in the limit $|x| \to \infty$, laying foundational results concerning singularity-free asymptotic behaviours.

2.1. Local Asymptotic Analysis

We study the equation $\mathrm{P_{II}}(\mu)$, for $1 \neq \mu \in \mathbb{Z}^+$, as $|x| \to \infty$. This section provides an exhaustive list of singularity-free, exponentially-improved local asymptotic expansions around the fixed irregular singularity at $x = \infty$. Before considering the limit $|x| \to \infty$, we remark on the behaviour of solutions around a generic movable singularity, as this is a key distinguishing factor between $\mu = 1$ and $\mu \neq 1$ cases of $\mathrm{P_{II}}(\mu)$.

For $\mu \neq 1$, we find that the (infinitely many) movable singularities of a generic solution $y(x)$ satisfying $\mathrm{P_{II}}(\mu)$ are no longer poles. In other words, $\mathrm{P_{II}}(\mu)$ does not maintain the Painlevé property for $\mu \neq 1$; the solution $y(x)$ is multi-valued around an arbitrary movable singularity. We provide the asymptotic expansion of a solution as the independent variable approaches an arbitrary moveable singularity.

Corollary 2.1. *Let $y(x)$ be a solution of $P_{II}(\mu)$ and let $a \in \mathbb{C}^*$. The solution $y(x)$ is singular at $x = a$ if, and only if, either $+y(x)$ or $-y(x)$ admit the following asymptotic expansion*

$$\begin{aligned}\pm y(x) = &\frac{1}{x-a} - \frac{1}{6}a^{\mu}(x-a) - \frac{1}{4}a^{\mu-1}\mu(x-a)^2 + b(x-a)^3 \\ &+ \frac{1}{10}a^{\mu-2}\mu(\mu-1)\log(x-a)(x-a)^3 + \mathcal{O}\left(x-a\right)^4, \quad x \to a,\end{aligned}$$

where $b \in \mathbb{C}$ is an arbitrary constant.

Proof. The result is verified by substitution into $\mathrm{P_{II}}(\mu)$. In particular, we apply the standard method of dominant balance, formally satisfying $\mathrm{P_{II}}(\mu)$ by vanishing the coefficient of each power of $(x-a)$, respectively. □

The above expansion of $y(x)$ around a singularity contains free parameters a, b, which are representative of the freedom in two initial conditions, i.e., the singularity is movable. We note the presence of a term involving $\log(x-a)$, showing the multi-valued nature of the solution around the $x = a$ singularity; we also see that this term is removed solely when $\mu = 1$ (i.e., in the case of $\mathrm{P_{II}}$), given $\mu \in \mathbb{Z}^+$ and $a \neq 0$.

To begin the asymptotic analysis of $\mathrm{P_{II}}(\mu)$ as $|x| \to \infty$, it is natural first to establish a generalisation of Boutroux's transformation, prompting the following Lemma.

Lemma 2.2 (Boutroux form)**.** *Given $y(x)$ satisfying $P_{II}(\mu)$, we define $u(z)$ such that $y(x) = x^{\mu/2}u(z)$ where $z = 2/(\mu+2)x^{(\mu+2)/2}$, then $u(z)$ satisfies*

$$u''(z) = 2u(z)^3 + u(z) - \frac{3\mu}{\mu+2}\frac{u'(z)}{z} - \frac{\mu(\mu-2)}{(\mu+2)^2}\frac{u(z)}{z^2}. \tag{2.3}$$

Proof. Such a transformation arises by first considering the general transformation of variables $y(x) = f(x)u(z)$ and $z = g(x)$, for undetermined $f(x), g(x)$, to deduce an autonomous, *maximal dominant balance* of the equation. Then with $y(x)$ solving $\mathrm{P_{II}}(\mu)$ we obtain $u(z)$ solving

$$\begin{aligned} &f''(x)u(z) + \left(2f'(x)g'(x) + f(x)g''(x)\right) u'(z) + f(x)g'(x)^2 u''(z) \\ =&2f(x)^3u(z)^3 + x^\mu f(x)u(z). \end{aligned}$$

This prompts us to choose $f(x), g(x)$ such that $g'(x)^2 = f(x)^2 = x^\mu$, thus returning the autonomous version of $\mathrm{P_{II}}(\mu)$ at leading-order as $|z| \to \infty$, that is $w''(z) = 2w(z)^3 + w(z)$. These conditions immediately provide the transformation described in the Lemma and yield the given equation for $u(z)$. □

We now discern a power-series expansion around $z = \infty$ that satisfies the Boutroux form of $\mathrm{P_{II}}(\mu)$ seen in Equation (2.3).

Lemma 2.4. *Equation (2.3) is satisfied by the asymptotic series*

$$u_f(z) = \sum_{n=0}^{N} a_n/z^{2n} + \mathcal{O}\left(1/z^{2(N+1)}\right), \quad |z| \to \infty, \quad N \in \mathbb{Z}^+, \tag{2.5}$$

where the subscript f denotes a formal solution and the coefficients a_n, for $n \in \{0, 1, 2, \dots\}$, are given by

$$\begin{aligned} a_0 =& \frac{i}{\sqrt{2}}, \quad a_1 = -\frac{a_0}{2}\frac{\mu(\mu-2)}{(\mu+2)^2}, \\ a_n =& h_n a_{n-1} + 2a_0 \sum_{k=1}^{n-1} a_k a_{n-k} + \sum_{j=1}^{n-1}\sum_{k=1}^{n-j} a_j a_k a_{n-j-k}, \quad \forall n \geq 2, \end{aligned}$$

where

$$h_n = -(2n-1)(n-1) + \frac{3\mu}{\mu+2}(n-1) - \frac{\mu(\mu-2)}{2(\mu+2)^2}.$$

Proof. We substitute the form (2.5) into Equation (2.3) and balance powers of $1/z$. See Appendix A for a detailed analysis of series coefficients in a similar context. □

Corollary 2.6. *The series described in Lemma 2.4 diverges for $\mu \in \mathbb{Z}^+$ with $\mu \neq 2$. When $\mu = 2$, we have $a_n = 0$ for all $n \geq 1$, i.e. $u_f(z) \equiv i/\sqrt{2}$ is an exact solution of Equation (2.3). Furthermore, for $\mu \neq 2$, the coefficients a_n are given asymptotically by*

$$\begin{aligned} a_n =& c(\mu)(-2)^n \Gamma(n+1-\lambda_+)\Gamma(n+1-\lambda_-) \\ &\times \left(1 + \frac{3}{4}\frac{(\mu-2)\mu}{(2+\mu)^3}\frac{1}{n^3} + \mathcal{O}\left(1/n^4\right)\right), \end{aligned}$$

as $n \to \infty$ where

$$\lambda_{\pm} = \frac{3 + 3\mu \pm \sqrt{1 - 6\mu + 3\mu^2}}{2(2+\mu)},$$

and $c(\mu) \in \mathbb{C}^*$ is some constant which is independent of n and depends on μ.

Proof. See Appendix A for an analogous proof regarding the asymptotic behaviour of such series coefficients. □

2.1.1. Exponential asymptotics. We provide the first exponentially small correction term to the asymptotic series given in Lemma 2.4.

Corollary 2.7. *Equation (2.3) admits the following asymptotic expansion*

$$u(z) = u_f(z) + k \exp\left(S_{\pm}(z)\right)\left(1 + \mathcal{O}\left(1/z\right)\right) + \mathcal{O}\left(\exp\left(2S_{\pm}(z)\right)\right),$$

as $|z| \to \infty$ for $z \in \mathbb{C}^{\pm}$ where

$$S_{\pm}(z) = \pm i\sqrt{2}z - \frac{3\mu}{2(2+\mu)}\log(z), \quad \mathbb{C}^{\pm} := \left\{z \in \mathbb{C} \;\middle|\; \mathrm{Im}(\pm z) > 0\right\},$$

and $k \in \mathbb{C}$ is an arbitrary constant.

Proof. The expansion is a result of standard asymptotic methods; we substitute $u(z) = u_f(z) + v(z)$ into Equation (2.3), recalling u_f is a formal series solution. Then, by assuming $v(z)$ is exponentially small as $|z| \to \infty$, we consider terms that are linear in $v(z)$ and its derivatives, i.e., implying a second-order, homogeneous, linear ODE in $v(z)$. Furthermore, in a given sector of the complex z plane as $|z| \to \infty$, at most, one independent solution of this linear equation is valid (i.e., exponentially decaying), leading to a one-parameter family of behaviours.

So, assuming the form $v(z) = e^{S(z)}$ yields a first-order, nonlinear, nonhomogeneous ODE satisfied by $S'(z)$, and the result follows by the method of dominant balance as $|z| \to \infty$. We see that the leading exponential behaviour $e^{\pm i\sqrt{2}z}$ directly implies the given boundaries of validity, as this term must exponentially decay as $|z| \to \infty$ if we are to maintain $u(z) \sim u_f(z)$ in this limit. The expansion may be continued by adding doubly-exponentially small terms $e^{\pm 2S}$, triply exponentially small terms $e^{\pm 3S}$ and so forth, where for these parts of the expansions, we would proceed to consider terms which are quadratic in $v(z)$ and derivatives, and then terms which are cubic in $v(z)$ and its derivatives, and so on. □

In this case, the positive and negative real axes are anti-Stokes lines (boundaries of the valid domains in Corollary 2.7), where the exponential contribution becomes purely oscillatory. In the anti-Stokes directions, we obtain a two-parameter family of behaviours, as follows.

Corollary 2.8. *Let $\mu \in \mathbb{Z}^+$ with $\mu > 1$. Equation (2.3) admits the following asymptotic expansion*

$$\begin{aligned} u(z) =& u_f(z) + k_1\delta(z)e^{S(z)}\left(1 + \mathcal{O}\left(1/z, \delta^2\right)\right) \\ &+ k_2\delta(z)e^{-S(z)}\left(1 + \mathcal{O}\left(1/z, \delta^2\right)\right) + \mathcal{O}\left(\delta^2\right), \quad |z| \to \infty, \quad z \in \mathbb{R}, \end{aligned}$$

where

$$S(z) = i\sqrt{2}z - 3i\sqrt{2}\left(\frac{\mu+2}{\mu-1}\right)k_1k_2\delta(z)^2 z, \quad \delta(z) = z^{-3\mu/(4+2\mu)},$$

and $k_1, k_2 \in \mathbb{C}$ are arbitrary constants. Note that $\mu > 1$ implies $\delta \to 0$ with $|z|^{-3/2} \ll |\delta| \ll |z|^{-1/2}$.

Proof. Deducing an expansion in the anti-Stokes direction is more complex than the expansion deduced in Corollary 2.7 as we may no longer consider a linear ODE satisfied by a perturbation term $v(z)$. The function $v(z)$ may not be considered exponentially small where $u(z) = u_f(z)+v(z)$, so we may not simply disassociate terms of different degrees in $v(z)$ and its derivatives. To satisfy Equation (2.3) in this limit we consider a generalised Fourier series

$$u(z) = \sum_{k=-\infty}^{\infty} c_k(z)e^{kS(z)},$$

where we shall still find that $S(z) \sim i\sqrt{2}z$. The problem is to balance e^{kS} terms for each $k \in \mathbb{Z}$, respectively. This is made tractable by assuming that $c_{k_1}(z) \gg c_{k_2}(z) \gg c_{k_3}(z) \gg \dots$ as $|z| \to \infty$ given $|k_1| < |k_2| < |k_3| < \dots$. Indeed, substitution into Equation (2.3) corroborates this assumption and we see that $c_k(z) = \mathcal{O}\left(\delta(z)^{|k|}\right)$ as described in the Corollary. As the inherent symmetry in the proposed expansion shows, the coefficients of e^{kS} and e^{-kS} are equivalent. However, each side of this coin is characterised by an independent free parameter, hence the two-parameter family of behaviours. □

We turn our attention to solutions of $\mathrm{P_{II}}(\mu)$ that decay to zero as $|z| \to \infty$. Essentially, we consider an exponentially small perturbation about the identically zero $y(x) \equiv 0$ solution.

Remark 2.9. Assuming $y(x)$ exponentially decays to zero, the linearised $\mathrm{P_{II}}(\mu)$ is a generalised Airy equation $f''(x) = x^\mu f(x)$, whose solutions are in terms of the modified Bessel functions. The fundamental solution, which decays to zero as $x \to \infty$ on the positive real axis, is

$$f(x) = x^{1/2}K_{\frac{1}{2\alpha}}\left(x^\alpha/\alpha\right) = \left(\frac{\pi\alpha}{2}\right)^{1/2} x^{-\mu/4}\exp\left(-x^\alpha/\alpha\right)\left(1 + \mathcal{O}\left(1/x^\alpha\right)\right),$$

as $x \to \infty$, where $K_\nu(z)$ is the modified Bessel function of the second kind and $\alpha = (\mu+2)/2$. For known results concerning modified Bessel functions I_ν and

K_ν and their asymptotic expansions, see, for instance, [37, §10.40(i)]. We provide the following Lemma regarding the full Equation (2.3).

Lemma 2.10. *Equation (2.3) admits the following asymptotic expansion*

$$u(z) = k\exp\left(S_\pm(z)\right)\left(1+\mathcal{O}\left(1/z\right)\right)+\mathcal{O}\left(\exp\left(3S_\pm(z)\right)\right), \quad |z|\to\infty,$$

for $z\in\mathbb{C}^\mp$ *where*

$$S_\pm(z) = \pm z - \frac{3\mu}{2(2+\mu)}\log(z), \quad \mathbb{C}^\pm := \left\{z\in\mathbb{C}\;\middle|\;\mathrm{Re}(\pm z)>0\right\},$$

and $k\in\mathbb{C}$ *is an arbitrary constant.*

Proof. The proof is equivalent to the proof of Corollary 2.7. □

Again, we consider the corresponding anti-Stokes behaviour on the positive and negative imaginary axes in the z plane. We obtain the following.

Lemma 2.11. *Let* $\mu\in\mathbb{Z}^+$ *with* $\mu>1$. *Equation (2.3) admits the following asymptotic expansion*

$$\begin{aligned} u(z) =& k_1\delta(z)e^{S(z)}\left(1+\mathcal{O}\left(1/z,\delta^2\right)\right) \\ &+ k_2\delta(z)e^{-S(z)}\left(1+\mathcal{O}\left(1/z,\delta^2\right)\right)+\mathcal{O}\left(\delta^3\right), \quad |z|\to\infty, \quad z\in i\mathbb{R}, \end{aligned}$$

where

$$S(z) = z - \frac{3}{2}\left(\frac{\mu+2}{\mu-1}\right)k_1k_2\delta(z)^2 z, \quad \delta(z) = z^{-3\mu/(4+2\mu)},$$

and $k_1,k_2\in\mathbb{C}$ *are arbitrary constants. Note that* $\mu>1$ *implies* $\delta\to 0$ *with* $|z|^{-3/2}\ll|\delta|\ll|z|^{-1/2}$.

Proof. The proof is equivalent to the proof of Corollary 2.8. □

Remark 2.12. One might notice that the given anti-Stokes expansions require $\mu>1$, with the decaying term in $S(z)$ being multiplied by $1/(\mu-1)$ (also notice that this term is only vanishing as $|z|\to\infty$ given $\mu>1$). We point this out as it proves to be a fundamental distinction between P_{II} and $P_{II}(\mu)$ for any $\mu>1$. In the case of P_{II}, this is instead a term of order $\ln(z)$, that is, the exponential function has a prefactor, which is z to some power that depends on the free constant k_1k_2, furthermore implying a constraint on this constant to maintain asymptoticity of the expansion. This concept is apparent and further discussed in Chapters 3 and 4, where we deduce analogous expansions in the cases of q-P_{II} and P_{IV}, respectively.

2.2. Tritronqueé Solutions

This section's main object of study is the asymptotic series established in Lemma 2.4, which solves $\mathrm{P_{II}}(\mu)$. We prove the existence and uniqueness of a true solution asymptotic to this series for $|\arg(x)| \leq 2\pi/(\mu+2)$ as $|x| \to \infty$. This solution corresponds to Boutroux's tritronquée solution in the case $\mu = 1$. For $\mathrm{P_{II}}(\mu)$, the Stokes-boundaries are given by $\arg(x) = n\pi/(\mu+2)$ for $n \in \mathbb{Z}$; i.e., the generalised Stokes sectors are

$$S_n(\mu) = \left\{ x \in \mathbb{C} \;\middle|\; (n-1)\pi/(\mu+2) \leq \arg(x) \leq n\pi/(\mu+2) \right\}, \quad n \in \mathbb{Z}. \tag{2.13}$$

We set out to prove the following:

Proposition 2.14 (Tritronqueé solution). *Let $\mu \in \mathbb{Z}^+$. There exists a unique solution $Y_0(x)$ of $P_{II}(\mu)$ satisfying*

$$Y_0(x) \sim x^{\mu/2} u_f(z), \quad z = \frac{2}{\mu+2} x^{(\mu+2)/2} \quad \textit{as} \quad |x| \to \infty,$$

for $|\arg(x)| \leq 2\pi/(\mu+2)$ where $u_f(z)$ is the asymptotic series given in Lemma 2.4.

We first prove the existence of tronqueé solutions, described in the following proposition.

Proposition 2.15 (Tronqueé solutions). *Let $\mu \in \mathbb{Z}^+$ and let $\mathcal{S}$ be an open sector of the complex x-plane with vertex at the origin and angular width not exceeding $2\pi/(\mu+2)$. There exists a solution $y(x)$ of $P_{II}(\mu)$ satisfying*

$$y(x) \sim x^{\mu/2} u_f(z), \quad z = \frac{2}{\mu+2} x^{(\mu+2)/2} \quad \textit{as} \quad |x| \to \infty \quad \textit{for} \quad x \in \mathcal{S},$$

where $u_f(z)$ is the asymptotic series given in Lemma 2.4.

Proof. Let $y(x)$ be a solution of $\mathrm{P_{II}}(\mu)$. The corresponding Boutroux variables $u(z)$ and z are as described in Lemma 2.3. We further define $w(z) = u(z) - i/\sqrt{2}$ which satisfies

$$\begin{aligned} w''(z) =& 2w(z)^3 + 3i\sqrt{2}w(z)^2 - 2w(z) - \frac{3\mu}{\mu+2}\frac{w'(z)}{z} \\ &- \frac{\mu(\mu-2)}{(\mu+2)^2}\frac{w(z)+i/\sqrt{2}}{z^2}, \end{aligned}$$

and by Lemma 2.4 this is formally satisfied by the power series

$$w_f(z) = \sum_{n=1}^{\infty} a_n/z^{2n}.$$

Defining $w_1 = w$ and $w_2 = w'$, we convert this to the system

$$w_1' = w_2, \quad w_2' = 2w_1^3 + 3i\sqrt{2}w_1^2 - 2w_1 - \frac{3\mu}{\mu+2}\frac{w_2}{z} - \frac{\mu(\mu-2)}{(\mu+2)^2}\frac{w_1 + i/\sqrt{2}}{z^2}.$$

Denoting this system by

$$\begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} f_1(z, w_1, w_2) \\ f_2(z, w_1, w_2) \end{pmatrix},$$

we find that the corresponding Jacobian matrix, when evaluated at $(w_1, w_2) = (0, 0)$ and taking the limit $|z| \to \infty$, has nonzero eigenvalues $\lambda_\pm = \pm i\sqrt{2}$. We have now met the conditions required to apply W. Wasow's *Main Asymptotic Existence Theorem* [157, Theorem 12.1]; this shows that for any open sector Ω of the complex z plane with an angular width not exceeding π, there exists a solution satisfying $w(z) \sim w_f(z)$ as $|z| \to \infty$, in every proper subsector of Ω. Therefore, we have reached the desired result for the original variables $y(x)$. □

We now prove the main result of this section.

Proof of Proposition 2.14. Let $\epsilon \in \mathbb{R}^+$ and let y_1, y_2 be tronqueé solutions, having asymptotic behaviour described by Proposition 2.15 in the respective sectors

$$S_1 : -\epsilon < \arg(x) < \theta - \epsilon \quad \text{and} \quad S_2 : -\theta + \epsilon < \arg(x) < \epsilon,$$

where $\theta = 2\pi/(\mu+2)$. Firstly, we note that for some sufficiently large $x_0 \in \mathbb{R}^+$, the solutions $y_1(x)$ and $y_2(x)$ are holomorphic on the real interval $(x_0, \infty) \subset S_1 \cap S_2$. By Proposition 2.15, we may write

$$v(x) := y_1(x) - y_2(x) = o\left(x^{-n}\right), \quad |x| \to \infty, \quad x \in S_1 \cap S_2, \quad \forall n \in \mathbb{N}. \tag{2.16}$$

Furthermore, since $v''(x) = y_1''(x) - y_2''(x)$ and y_1, y_2 satisfy $\mathrm{P_{II}}(\mu)$, we obtain $v(x)$ satisfying the linear ODE

$$v''(x) = \left\{2y_1(x)^2 + 2y_1(x)y_2(x) + 2y_2(x)^2 + x^\mu\right\} v(x).$$

We proceed to write this in the form

$$v''(x) = \{\omega(x) - \psi(x)\}\, v(x), \tag{2.17}$$

where

$$\omega(x) = 2y_1(x)^2 + 2y_1(x)y_2(x) + 2y_2(x)^2 + 3x^\mu \quad \text{and} \quad \psi(x) = 2x^\mu.$$

It is evident that on (x_0, ∞): the functions ψ, ω are holomorphic, ψ is real and positive, and from the asymptotic expansion for y_1, y_2, we have

$$\omega(x) = \frac{3\mu(\mu-2)}{4x^2} + \mathcal{O}\left(\frac{1}{x^{\mu+4}}\right), \quad x \to \infty.$$

Taking arbitrary $a \in (x_0, \infty)$, we then deduce that the integral

$$F_a(x) = \int_a^x \left| \psi(t)^{-1/4} \frac{d^2}{dt^2} \left\{ \psi(t)^{-1/4} \right\} - \psi(t)^{-1/2} \omega(t) \right| dt, \tag{2.18}$$

taken along the real axis is finite for any $x \in (x_0, \infty)$, and in particular the integral converges as $x \to \infty$.

The study of solutions of Equation (2.17) in the limit $|x| \to \infty$ is the subject of theorems given by Olver [122] under the term *Liouville-Green* approximations. In this context, the integral (2.18) is called the *error-control function*. When this integral is bounded, Equation (2.17) satisfies the hypotheses of Olver's Theorems 2.1 and 2.2 (see [122, Chapter 6, §2]). Olver's theorem [122, Chapter 6, §2, Theorem 2.2] states that for $x \in (x_0, \infty)$, Equation (2.17) has general solution

$$\begin{aligned} v(x) =& c_1 \psi^{-1/4} \exp\left(i \int \psi^{1/2} dx \right) \big(1 + \delta_1(x)\big) \\ &+ c_2 \psi^{-1/4} \exp\left(-i \int \psi^{1/2} dx \right) \big(1 + \delta_2(x)\big), \end{aligned} \tag{2.19}$$

where c_1 and c_2 are some constants, and the functions δ_1, δ_2 satisfy

$$|\delta_j(x)| \le \exp\{F_a(x)\} - 1, \quad j \in \{1, 2\}.$$

Equation (2.19) contradicts the asymptotic behaviour (2.16) unless $c_1 = c_2 = 0$. It follows that $y_1(x) = y_2(x)$ for all $x \in S_1 \cup S_2$ with sufficiently large $|x|$. Since ϵ may be arbitrarily small, the proof is complete. □

Remark 2.20. In terms of the original variables $y(x)$, the asymptotic behaviour of the tronquée or tritronquée solution (in the appropriate sector of the complex plane) is given by

$$y_f(x) = x^{\mu/2} \sum_{n=0}^{N} \frac{(\mu+2)^{2n} a_n}{4^n x^{n(\mu+2)}} + \mathcal{O}\left(x^{\mu/2 - (N+1)(\mu+2)} \right), \quad |x| \to \infty,$$

for all $N \in \mathbb{Z}^+$, with coefficients a_n described in Lemma 2.4. Furthermore, in the same sector of the complex x-plane, the kth derivative $Y_0^{(k)}(x)$ is given asymptotically by the kth term-by-term differentiation of the above series; this is a consequence of Wasow's theorem [157, Theorem 8.8].

Corollary 2.21. *There exists a unique tritronqueé solution $Y_n(x)$ corresponding to each anti-Stokes ray, given by* $\arg(x) = 2\pi n/(\mu + 2)$ *for $n \in \mathbb{Z}$. Each $Y_n(x)$ is holomorphic for sufficiently large $|x|$ in an open sector comprised of four contiguous sectors described by Equation* (2.13) *and bisected by an anti-Stokes ray.*

Proof. Given tritronquée solution $Y_0(x)$ as described by Proposition 2.14, consider the discrete symmetry of $\mathrm{P_{II}}(\mu)$, given by the transformation

$$Y_n(x) = \exp\left(-\frac{2\pi\, i\, n}{\mu+2}\right) Y_0\left(\exp\left(-\frac{2\pi\, i\, n}{\mu+2}\right)x\right), \quad n \in \mathbb{Z}. \tag{2.22}$$

By substitution, it is straightforward to verify that Y_n for $n \in \mathbb{Z}$ also satisfies $\mathrm{P_{II}}(\mu)$. Given the properties of Y_0, it follows that Y_n is holomorphic in a rotated sector of the complex plane, as described. To first prove the existence and uniqueness of Y_0 as a "seed" tritronquée solution was convenient as we see that $\arg(x) = 0$ is an anti-Stokes ray for any $\mu \in \mathbb{Z}^+$. □

2.3. Hastings-McLeod Type Solution

In this section, we set out to prove the existence and uniqueness of a solution of $\mathrm{P_{II}}(\mu)$ mirroring the global asymptotic properties of the celebrated Hastings-McLeod solution of $\mathrm{P_{II}}$. Hastings and Mcleod studied a different generalisation of $\mathrm{P_{II}}$, that being $y''(x) = 2y^{2r+1} + xy$, which evidently coincides with $\mathrm{P_{II}}(\mu)$ when $r = \mu = 1$ (i.e. $\mathrm{P_{II}}$). The Hastings-Mcleod solution uniquely satisfies a certain asymptotic boundary-value problem on the real line (as $x \to \pm\infty$). In the following definition, we generalise these boundary conditions at $x = \pm\infty$ and thus define a Hastings-McLeod type solution of $\mathrm{P_{II}}(\mu)$.

Definition 2.23 (Hastings-McLeod type solution)**.** *Let $\mu \in \mathbb{Z}^+$. A solution $y(x)$ of $P_{II}(\mu)$ is called a Hastings-McLeod type solution if $y(x)$ is finite and positive for all $x \in \mathbb{R}$, having asymptotic behaviour $y(x) \sim y_f(x)$ as $x \to -\infty$ and $y(x) \sim kf(x)$ as $x \to +\infty$ for some $k \in \mathbb{R}^+$ where:*

a) *The asymptotic series $y_f(x)$ is described in Remark 2.20, obtained by expressing Lemma 2.4 in terms of the original (non-Boutroux) variables.*
b) *The function $f(x)$ satisfies $f''(x) = x^\mu f(x)$ and exponentially decays to zero as $x \to +\infty$, discussed in Remark 2.9.*

In this section, we set out to prove the following.

Proposition 2.24. *There is a unique Hastings-McLeod type solution of $P_{II}(\mu)$ for every odd value of $\mu \in \mathbb{Z}^+$, as described by Definition 2.23. The Hastings-McLeod type solution does not exist for even values of μ.*

2.3.1. Existence.

We first show a Hastings-McLeod type solution's existence (and non-existence in the even μ cases). The following Lemma serves as a cornerstone of the analysis in this section; we introduce an integral equation that is necessarily satisfied by a Hastings-McLeod-type solution.

Lemma 2.25. *Given $\mu \in \mathbb{Z}^+$ and $k \in \mathbb{R}^+$, there exists a unique solution $y_k(x)$ of $P_{II}(\mu)$ which is asymptotic to $kf(x)$ as $x \to +\infty$, $x \in \mathbb{R}$. The solution $y_k(x)$ may feature singularities at finite values of $x \in \mathbb{R}$; however, for all (x, k) in an open region of $\mathbb{R} \times \mathbb{R}^+$ where $y_k(x)$ is finite, the solution and its derivatives are continuous functions of k.*

Proof. Recall that $f(x)$ satisfies $f''(x) = x^\mu f(x)$. For some reference point $b \in \mathbb{R}$, we define the linearly independent solution of this equation by

$$g(x) = f(x) \int_b^x \frac{dt}{f(t)^2}.$$

Differentiating $g(x)$ twice, we see that $g(x)$ also satisfies $g''(x) = x^\mu g(x)$.

Using $f(x)$ as an integrating factor, we see that a solution $y(x)$ of $\mathrm{P_{II}}(\mu)$ satisfies

$$\frac{d}{dt}\left\{f(t)y'(t) - f'(t)y(t)\right\} = 2f(t)y(t)^3.$$

Then, integrating the above equation with respect to t, between arbitrary $a, s \in \mathbb{R}$ yields

$$\frac{d}{ds}\left\{\frac{y(s)}{f(s)}\right\} = \left\{f(a)y'(a) - f'(a)y(a)\right\}\frac{1}{f(s)^2} + \frac{2}{f(s)^2}\int_a^s f(t)y(t)^3 dt.$$

Performing a second integration, with respect to s, between $b, x \in \mathbb{R}$ yields

$$\begin{aligned} y(x) =& \frac{y(b)}{f(b)} f(x) + \left\{f(a)y'(a) - f'(a)y(a)\right\} f(x) \int_b^x \frac{ds}{f(s)^2} \\ &+ 2f(x)\int_b^x \frac{1}{f(s)^2}\int_a^s f(t)y(t)^3 dt ds. \end{aligned}$$

We denote $y_k(x)$, the solution satisfying the above equation with one degree of freedom fixed by taking the limit $a \to \infty$. By rearranging the order of integration of the double integral and using the definition of g, we find that $y_k(x)$ satisfies

$$\begin{aligned} y_k(x) =& kf(x) + 2\int_x^\infty \left\{f(x)g(t) - f(t)g(x)\right\} y_k(t)^3 dt, \qquad (2.26) \\ k =& \frac{y_k(b)}{f(b)} - 2\int_b^\infty g(t)y_k(t)^3 dt. \end{aligned}$$

By application of a standard contraction mapping argument, in an appropriate space of decaying, continuous functions equipped with the uniform norm, Equation (2.26) may be shown to have a unique solution, giving y_k and its continuous dependence on k. $\square$

Remark 2.27. From Equation (2.26), it is straightforward to obtain similarly

$$y_k'(x) = kf'(x) + 2\int_x^\infty \left\{f'(x)g(t) - f(t)g'(x)\right\} y_k(t)^3 dt,$$

where we may also note that

$$g'(x) = f'(x)\int_b^x \frac{dt}{f(t)^2} + \frac{1}{f(x)}.$$

We now provide sufficient conditions for $y_k(x)$ to become singular at some finite value of x.

Lemma 2.28. *Let $k \in \mathbb{R}^+$ and $x_0 \in \mathbb{R}$ be such that $y_k(x_0) > 0$ and $y_k'(x_0) < 0$ are finite. Then, given $y_k''(x) > y_k(x)^3$ on $(-\infty, x_0)$, the solution is singular at some finite value in this interval.*

Proof. For the sake of a contradiction, assume that $y_k(x)$ is analytic on $(-\infty, x_0)$. Since $y_k'(x_0) < 0$, there exists x_1 such that $y_k'(x) < 0$ on the open interval (x_1, x_0). Thus on (x_1, x_0) we have $y_k'(x)y_k''(x) < y_k(x)^3 y_k'(x)$ and integrating this autonomous inequality yields

$$y_k'(x_1)^2 > y_k'(x_0)^2 + \frac{1}{2}y_k(x_1)^4 - \frac{1}{2}y_k(x_0)^4. \tag{2.29}$$

Since $y_k'(x) < 0$ on (x_1, x_0), we have $y_k(x_1) > y_k(x_0)$, and thus Inequality (2.29) shows that $y_k'(x_1) < y_k'(x_0)$. Therefore, we deduce that $y_k(x)$ and $y_k'(x)$ are monotonically decreasing and increasing on $(-\infty, x_0)$, respectively, and Innequality (2.29) is valid for arbitrary $x_1 \in (-\infty, x_0)$. Thus, there exists $x_1 \in (-\infty, x_0)$ such that

$$\frac{1}{4}y_k(x)^4 + y_k'(x_0)^2 - \frac{1}{2}y_k(x_0)^4 > 0 \quad \text{for all} \quad x \in (-\infty, x_1),$$

and therefore Innequality (2.29) implies $y_k'(x)^2 > y_k(x)^4/4$, and thus $y_k'(x) < -y_k(x)^2/2$, for $x \in (-\infty, x_1)$. So, by another integration, we obtain the contradictory statement

$$\frac{1}{y_k(x_1)} - \frac{1}{y_k(x)} > \frac{1}{2}(x_1 - x) \quad \text{for all} \quad x \in (-\infty, x_1).$$

Indeed, the hypothesis may be relaxed to $y_k''(x) > y_k(x)^3$ on $(x_0 - a, x_0)$ for arbitrary $a \in \mathbb{R}^+$, and the solution will become singular in this interval given $y_k(x_0), |y_k'(x_0)| > K$ for some sufficiently large constant $K \in \mathbb{R}^+$ which is independent of x_0. $\square$

Corollary 2.30. *For even values of $\mu \in \mathbb{Z}^+$, there does not exist a Hastings-McLeod type solution of $P_{II}(\mu)$.*

Proof. Let $\mu \in \mathbb{Z}^+$ be even and say $y_k(x)$ is a Hastings-McLeod type solution of $\mathrm{P_{II}}(\mu)$. Then by the definitive asymptotic behaviour of $y_k(x)$, we have finite $y_k(x_0) > 0$ and $y_k'(x_0) < 0$ for some sufficiently large $x_0 \in \mathbb{R}$. Then since μ is even, we obtain $y_k''(x) = 2y_k(x)^3 + x^\mu y_k(x) > y_k(x)^3$ for all $x \in \mathbb{R}$. Thus, by

Lemma 2.28, the solution $y_k(x)$ becomes singular at some finite $(-\infty, x_0)$, which contradicts $y_k(x)$ being a Hastings-McLeod type solution. □

We now move on to the proof of the existence of a Hastings-McLeod type solution of $P_{II}(\mu)$ in the case of odd values of μ. We provide a topological argument, for which we first define the following sets:

Definition 2.31. *We define the following subsets of* $\mathbb{R}^+$.

a) *The set* $S_1 \subseteq \mathbb{R}^+$ *is the set of k values such that the solution* $y_k(x)$ *of* $P_{II}(\mu)$ *is singular at some* $x_s \in \mathbb{R}$*, and is finite and positive for* $x \in (x_s, +\infty)$.
b) *The set* $S_2 \subseteq \mathbb{R}^+$ *is the set of k values such that the solution* $y_k(x)$ *of* $P_{II}(\mu)$ *vanishes at some* $x_v \in \mathbb{R}$*, and is finite and positive for* $x \in (x_s, +\infty)$.

Lemma 2.32. *The set* S_1 *described in Definition 2.31 is non-empty.*

Proof. For sufficiently large x, i.e. where $f(x) > 0$ and $f'(x) < 0$, and considering $t \geq x$, we may write

$$f(x)g(t) - f(t)g(x) = f(t)f(x) \int_x^t \frac{ds}{f(s)^2} \geq 0.$$

Therefore, the integral Equation (2.26) implies that $y_k(x) > kf(x) > 0$ for all sufficiently large x. Similarly, after differentiating Equation (2.26) (see Remark 2.27), the integrand may be written

$$f'(x)g(t) - f(t)g'(x) = f'(x)f(t) \int_x^t \frac{ds}{f(s)^2} - \frac{f(t)}{f(x)} \leq 0,$$

and thus $y_k'(x) < kf'(x) < 0$ for all sufficiently large x. Furthermore, since $y_k''(x) > y_k(x)^3$ is certainly satisfied at least for positive $x \in \mathbb{R}$, we may choose sufficiently large $k \in \mathbb{R}^+$ so that $y_k(x)$ becomes singular at some finite x by the proof of Lemma 2.28. □

Lemma 2.33. *The set* S_1 *described in Definition 2.31 is open.*

Proof. Suppose we have $k_0 \in \mathbb{R}^+$ and $x_0 \in \mathbb{R}$ such that $y_{k_0}(x)$ is singular at $x = x_0$. Firstly note that we may choose $x_1 > x_0$ so that $|y_{k_0}(x_1)|$ and $|y_{k_0}'(x_1)|$ are arbitrarily large. Thus, say we choose x_1 satisfying

$$y_{k_0}(x_1)^2 > (|x_1| + 1)^\mu. \tag{2.34}$$

Since general $y_k(x)$ is continuous in k, the above inequality holds for all k in some neighbourhood of k_0. Inequality (2.34) implies that for all $x \in (x_1 - 1, x_1)$ we obtain

$$|x|^\mu < (|x_1| + 1)^\mu < y_k(x_1)^2 < y_k(x)^2.$$

Therefore, we may show that $y_k''(x) > y_k(x)^3$ on the interval $(x_1 - 1, x_1)$; this is evidently true for positive x, and for negative x we see that

$$\begin{aligned} y_k''(x) =& 2y_k(x)^3 + x^\mu y_k(x) = 2y_k(x)^3 - |x|^\mu y_k(x) \\ >& 2y_k(x)^3 - y_k(x)^3 = y_k(x)^3. \end{aligned} \tag{2.35}$$

Thus by the proof of Lemma 2.28, given sufficiently large $|y_k(x_1)|$ and $|y_k'(x_1)|$ the solution will become singular on $(x_1 - 1, x_1)$. □

Lemma 2.36. *The set S_2 described in Definition 2.31 is non-empty.*

Proof. For any odd integer $\mu > 1$, consider the graph of $p(x)$ given by the positive real root of $2p(x)^2 = -x^\mu$ on the interval $x \in (-\infty, 0]$. Let $x_0 \in (-\infty, 0]$, now note that $|y_k(x_0)| < p(x_0)$ implies $y_k''(x_0) < 0$, and thus $|y_k'(x_0)| < -p'(x_0)$ implies $y_k(x)$ will necessarily vanish at some $x \in (-\infty, x_0)$. Considering that $k = 0$ corresponds to the identically zero solution of $\mathrm{P_{II}}(\mu)$, by the continuity of $y_k(x)$ in k, we may choose x_0 and sufficiently small $k \in \mathbb{R}^+$ so that the discussed conditions are met, completing the proof. □

Lemma 2.37. *The set S_2 described in Definition 2.31 is open.*

Proof. Assuming the existence of an element $k_0 \in S_2$, i.e., $y_{k_0}(x)$ vanishes at some $x \in (-\infty, 0)$, it follows directly by the continuity of $y_k(x)$ in k that the argument in the proof of Lemma 2.36 applies for all k in some neighbourhood of k_0. □

With the above lemmas, we may conclude the proof of the existence of a Hastings-McLeod type solution of $\mathrm{P_{II}}(\mu)$ for odd values of $\mu > 1$. We have shown that $S_1, S_2 \subseteq \mathbb{R}^+$, as described in Definition 2.31, are non-empty, open intervals, and are disjoint by definition. Therefore, there exists at least one value $k \in \mathbb{R}^+$ which does not belong to $S_1 \cup S_2$, and thus there exists a solution $y_k(x)$ which is finite and positive for all $x \in \mathbb{R}$. This implies the desired asymptotic behaviour $y_k(x) \sim y_f(x)$ as $x \to -\infty$. Indeed, if we instead had $y_k(x)$ decaying to zero as $x \to -\infty$ then $y_k(x)$ would adopt oscillatory asymptotic behaviour (see Lemma 2.11), contradicting the fact that $k \notin S_2$.

2.3.2. Uniqueness. We now prove the uniqueness of the Hastings-McLeod type solution, completing the proof of Proposition 2.24. To this end, we first define the following function.

Definition 2.38 (Energy function)**.** *Letting $y_k(x)$ be a solution of $P_{II}(\mu)$, we define the corresponding* energy function *$V_k(x)$ by*

$$V_k(x) = y_k'(x)^2 - x^\mu y_k(x)^2 - y_k(x)^4.$$

Remark 2.39. Differentiating $V_k(x)$ and applying $\mathrm{P_{II}}(\mu)$ we obtain the equation $V_k'(x) = -\mu x^{\mu-1} y_k(x)^2$.

Lemma 2.40. *Let $\mu > 1$ be an odd integer and say $y_{k_1}(x), y_{k_2}(x)$ are Hastings-McLeod type solutions with $k_1 > k_2$, then $y_{k_1}(x) > y_{k_2}(x)$ for all $x \in \mathbb{R}$.*

Proof. Firstly note that asymptotic behaviour as $x \to +\infty$ implies $y_{k_1}(x) > y_{k_2}(x)$ for all sufficiently large x. For the sake of contradiction, say that there exists $x_0 \in \mathbb{R}$ such that $y_{k_1}(x_0) = y_{k_2}(x_0)$ and $y_{k_1}(x) > y_{k_2}(x)$ for all $x > x_0$. This further implies that $y_{k_1}'(x_0)^2 \leq y_{k_2}'(x_0)^2$. Considering the energy function $V_k(x)$, we see that

$$V_{k_1}(x_0) - V_{k_2}(x_0) = y_{k_1}'(x_0)^2 - y_{k_2}'(x_0)^2 \leq 0 \quad \Longrightarrow \quad V_{k_1}(x_0) \leq V_{k_2}(x_0). \tag{2.41}$$

Considering $V_k'(x) = -\mu x^{\mu-1} y_k(x)^2$, we see that $V_{k_1}'(x) < V_{k_2}'(x) < 0$ for all $x > x_0$, thus implying that

$$V_{k_1}(x) - V_{k_2}(x) < V_{k_1}(x_0) - V_{k_2}(x_0) \leq 0 \quad \Longrightarrow \quad V_{k_1}(x) < V_{k_2}(x),$$

for $x \in (x_0, +\infty)$.

However, directly from the definition of $V_k(x)$, and the asymptotic behaviour $y_{k_1}(x) \to k_1 f(x)$ and $y_{k_2}(x) \to k_2 f(x)$ as $x \to +\infty$, we calculate that $V_{k_1}(x) > V_{k_2}(x)$ as $x \to +\infty$ given $k_1 > k_2$. We have, therefore, reached the required contradiction. □

Remark 2.42. In terms of the energy function $V_k(x)$, Lemma 2.40 tells us that $V_{k_1}'(x) - V_{k_2}'(x) < 0$ for all $x \in \mathbb{R}$, given $k_1 > k_2$. Therefore, since $V_{k_1}(x) - V_{k_2}(x) > 0$ for sufficiently large x, we see that indeed $V_{k_1}(x) - V_{k_2}(x) > 0$ is true for all $x \in \mathbb{R}$.

We proceed to complete our proof of a unique Hastings-McLeod-type solution.

Theorem 2.43. *Let $\mu > 1$ be an odd integer and say $y_{k_1}(x), y_{k_2}(x)$ are Hastings-McLeod type solutions of $P_{II}(\mu)$, then $k_1 = k_2$.*

Proof. Suppose that $k_1 \neq k_2$ and without loss of generality, assume that $k_1 > k_2$. As in the proof of Proposition 2.14, we can appeal to another theorem of Olver [122, Theorem 2.1, p.193] to show that $p(x) = y_{k_1}(x) - y_{k_2}(x)$ satisfies

$$\begin{aligned} p(x) =& b_1 f^{-1/4} \exp\left(\int f^{1/2} dx\right) \big(1 + o(1)\big) \\ &+ b_2 f^{-1/4} \exp\left(-\int f^{1/2} dx\right) \big(1 + o(1)\big), \end{aligned} \tag{2.44}$$

as $x \to -\infty$ for some constants b_1 and b_2, where $f = -2\, x^{\mu}$. Olver's theorem also states that the behaviour (2.44) is twice differentiable. Equation (2.44) immediately contradicts the asymptotic behaviour $y_{k_j} \sim y_f$ for $k = 1, 2$ unless $b_1 = 0$. On the other hand, suppose $b_2 \neq 0$. Then we use the differentiability of the result (2.44) to obtain

$$\begin{aligned} V_{k_1}(x) - V_{k_2}(x) &= -2\left(2y_f(x)^3 + x^\mu\, y_f(x)\right) p(x)\big(1 + o(1)\big) \\ &= -2y_f''(x)p(x)\big(1 + o(1)\big). \end{aligned}$$

By Lemma 2.40 we have $p(x) > 0$ for all $x \in \mathbb{R}$, and the above statement thus implies $V_{k_1}(x) - V_{k_2}(x) < 0$ as $x \to -\infty$, this is a contradiction (see Remark 2.42) and so $b_2 = 0$. It follows that $y_{k_1}(x) = y_{k_2}(x)$ for sufficiently large negative $x \in \mathbb{R}$, contradicting the result of Lemma 2.40 unless $k_1 = k_2$. □

Remark 2.45. Consider the behaviours of $y_k(x)$ depending on the parameter $k \in \mathbb{R}^+$. For odd μ, suppose k^* is the unique value for which $y_{k*}(x)$ is the Hastings-Mcleod type solution. Using essentially the same argument as in Lemma 2.40, we may show that $0 < k < k^*$ implies $|y_k(x)| < y_{k^*}(x)$ for all $x \in \mathbb{R}$. So for $0 < k < k^*$, we have a family of bounded solutions on the real line; these solutions have oscillatory asymptotic behaviour given in Lemma (2.11). On the other hand, when $k^* < k$, we have positive and convex solutions for all $x \in \mathbb{R}$, becoming singular at some finite point as x decreases.

Remark 2.46. Figure 1 serves primarily as a visual demonstration of the solution types which are described in Definition 2.31, and the role of the Hastings-McLeod type solution as a distinct "interface" between solution families (i.e. corresponding to the sets S_1 and S_2 of k values). It is beyond the scope of this work to explicitly calculate $k^* = k^*(\mu)$ for odd $\mu \in \mathbb{Z}^+$, or provide reliable numerical estimates. However, we note that $k^*(1)$, corresponding to $\mathrm{P_{II}}$, was calculated explicitly by Hastings and McLeod [58]; in terms of the Airy function $\mathrm{Ai}(x)$ this result is given precisely by $y_{k^*}(x) \sim \mathrm{Ai}(x)$ as $x \to +\infty$.

Via rudimentary computational methods (in order to produce Figures 1 and 2), we determine estimates such as

$$k^*(1) \approx 0.184, \quad k^*(3) \approx 0.131, \quad k^*(5) \approx 0.107, \quad k^*(7) \approx 0.093,$$

noting here that the known value $k^*(1) = 1/(\pi\sqrt{3})$ was correctly determined up to four significant figures. The scaling of these values is with reference to the framework of modified Bessel functions; we write $y_k(x) \sim kf(x)$ as $x \to +\infty$ where $f(x)$ is exactly the function discussed in Remark 2.9.

2.4. Physical Application

In this section, we consider a generalisation of Bass' electrodiffusion model [7], which leads to $\mathrm{P_{II}}(\mu)$ arising naturally. We introduce a physical scenario involving the steady-state motion of charged chemical species in a fluid medium;

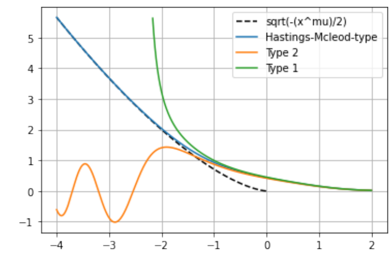

FIGURE 1. Solutions $y_k(x)$
Numerically computed solutions $y_k(x)$ of $\mathrm{P_{II}}(3)$. Orange is with $k < k^*$, green is with $k > k^*$, and blue is an approximation of the Hastings-McLeod type solution $k \approx k^*$ (see Remark 2.45).

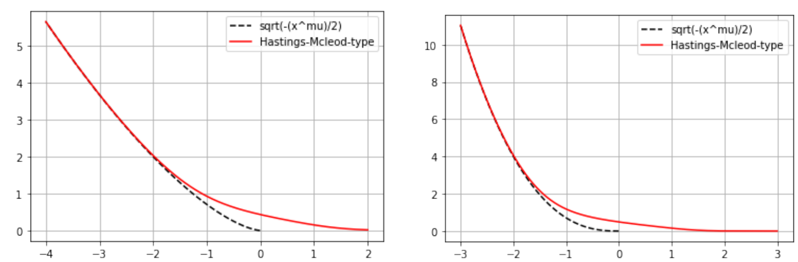

FIGURE 2. Hastings-McLeod type solutions
Numerically approximated Hastings-McLeod type solutions for $\mathrm{P_{II}}(3)$ (left) and $\mathrm{P_{II}}(5)$ (right).

specifically, we consider a strong, dilute, symmetric $\tilde{z}$-valent electrolyte. Denote the concentrations $c_+(\boldsymbol{x})$ and $c_-(\boldsymbol{x})$ of positively and negatively charged ions, respectively. The motion of ions is governed by the Nernst-Planck equation, which extends Fick's law of diffusion to include the effect of an electric field $\boldsymbol{E}(\boldsymbol{x})$ on charged particles, as follows:

$$\nabla \cdot \{\nabla c_\pm \mp \alpha c_\pm \boldsymbol{E}\} = 0, \quad \alpha = \frac{\tilde{z}e}{k_B T}, \tag{2.47}$$

where e is the elementary charge, k_B is the Boltzmann constant, and T is the absolute temperature. The above equation is due to the absence of volume sources of ions in strong electrolytes, i.e., the numbers of ions of each sort are conserved in each volume element.

This system is then augmented with Poisson's Equation, by which $\boldsymbol{E}$ satisfies

$$\nabla \cdot \boldsymbol{E} = \beta\,(c_+ - c_-)\,, \quad \beta = \frac{\tilde{z}e}{\epsilon},$$

where ϵ is the permittivity of the medium. Bass studies a strictly one-dimensional case of this model [7], where it is shown that $\boldsymbol{E}$ satisfies the second Painlevé equation (also see [8]). We consider a two-dimensional setting with $\boldsymbol{x} = (x, y)$ and $\boldsymbol{E} = (E_1, E_2)$. Thus, particle flux in the x-dimension is no longer necessarily constant, as the sources above assume.

Considering a one-dimensional domain $D = \{(x, y) \in \mathbb{R}^2 \,|\, x \in (0, \xi),\, y = \psi\}$ for some constants $\xi \in \mathbb{R}^+$ and $\psi \in \mathbb{R}$, we first impose the boundary conditions

$$\frac{\partial E_2}{\partial y} = 0, \quad \frac{\partial c_+}{\partial y} = -\frac{\partial c_-}{\partial y}, \quad \frac{\partial^2 c_+}{\partial y^2} = \frac{\partial^2 c_-}{\partial y^2}, \quad \boldsymbol{x} \in D,$$

thus implying certain symmetries regarding the movement of ions in the y dimension along this boundary. Then for $\boldsymbol{x} \in D$, and the arguments of functions referring to the value of $x \in (0, \xi)$, integration of the Nernst-Planck Equations (2.47) gives

$$\frac{\partial c_\pm}{\partial x} \mp \alpha c_\pm E_1 = \int_0^x g(t)dt + A_\pm(0), \tag{2.48}$$

where we have defined

$$g(x) = \alpha \frac{\partial c_+}{\partial y} E_2 - \frac{\partial^2 c_+}{\partial y^2}, \quad A_\pm(x) = \frac{\partial c_\pm}{\partial x} \mp \alpha c_\pm E_1,$$

and we may write Poisson's Equation

$$\frac{\partial E_1}{\partial x} = \beta \left(c_+ - c_-\right). \tag{2.49}$$

Thus, differentiating Poisson's Equation (2.49) and applying Equations (2.48) gives

$$\frac{\partial^2 E_1}{\partial x^2} = \alpha\beta(c_+ + c_-)E_1 + \beta(A_+(0) - A_-(0)). \tag{2.50}$$

Meanwhile, summing Equations (2.48) and applying Equation (2.49) yields

$$\frac{\partial c_+}{\partial x} + \frac{\partial c_-}{\partial x} = \frac{\alpha}{\beta} E_1 \frac{\partial E_1}{\partial x} + 2\int_0^x g(t)dt + A_+(0) + A_-(0),$$

so after an integration, we have

$$c_+ + c_- = \frac{\alpha}{2\beta} E_1^2 + 2\int_0^x (x - t) g(t)dt + (A_+(0) + A_-(0))x + B(0), \tag{2.51}$$

where

$$B(x) = c_+ + c_- - \frac{\alpha}{2\beta} E_1^2.$$

So now, combining Equations (2.50) and (2.51), we have

$$\begin{aligned}\frac{\partial^2 E_1}{\partial x^2} =& \frac{1}{2}\alpha^2 E_1^3 + \alpha\beta \left(2\int_0^x (x - t)g(t)dt + (A_+(0) + A_-(0))x + B(0)\right) E_1 \\ &+ \beta(A_+(0) - A_-(0)).\end{aligned}$$

If we furthermore consider the initial value space given by

$$\frac{\partial}{\partial x}(c_+ - c_-) = \alpha(c_+ + c_-)E_1 \quad \text{at} \quad x = 0,$$

then

$$\begin{aligned}\frac{\partial^2 E_1}{\partial x^2} =& \frac{1}{2}\alpha^2 E_1^3 + p(x)E_1,\\ p(x) =& \alpha\beta\left(2\int_0^x (x-t)g(t)dt + (A_+(0) + A_-(0))x + B(0)\right).\end{aligned}$$

In a one-dimensional model, we have $g = 0$, and thus, the coefficient function $p(x)$ is linear in x, resulting in the classical second Painlevé equation after an elementary transformation of variables. We see that here, the function $p(x)$ may be more complex, which essentially is due to a non-constant ion flux. We may assert that $p(x)$ is approximated by a μ-degree polynomial for some $\mu \geq 1$, thus giving a generalisation similar to our $\mathrm{P_{II}}(\mu)$. Obtaining $p(x)$ explicitly requires information regarding the boundary values

$$g = \alpha\frac{\partial c_+}{\partial y}E_2 - \frac{\partial^2 c_+}{\partial y^2} \quad \text{for} \quad \boldsymbol{x} \in D.$$

2.5. Summary

We investigated (via asymptotic analysis as the independent variable becomes large) solutions $y(x)$ of a family of nonlinear ODEs, parameterised by μ, which are perturbations of the second Painlevé equation; we also showed how such equations arise in an application, specifically, in a mathematical model of electrodiffusion. Our results show that the perturbed equation possesses natural generalisations of two exceptional transcendental solutions of $\mathrm{P_{II}}$: Boutroux's tritronqueé and the Hastings-McLeod solutions.

By asymptotic analysis of divergent power-series expansions in the limit $|x| \to \infty$, we found that the perturbed equation always (i.e., for any $\mu \in \mathbb{Z}^+$) has solutions which we call tritronqueé, first described by Boutroux for $\mathrm{P_{II}}$ (see Section 1.1 in Chapter 1 for a background). These solutions are asymptotic to a power-series expansion and, thus, singularity-free in a surprisingly broad annular region of the complex plane near infinity. We note that the angular width of this region of validity is monotonically decreasing towards zero as μ increases.

Furthermore, if our integer parameter μ is odd, $\mathrm{P_{II}}(\mu)$ has a solution analogous to the famous Hastings-McLeod solution of $\mathrm{P_{II}}$. The Hastings-McLeod type solution is holomorphic, real-valued, and positive on the entire real line, with known asymptotic behaviour in either direction. Fundamentally, this solution is defined by its global asymptotic properties, particularly the behaviours as the independent variable approaches positive and negative infinity on the real line. The Hastings-McLeod type solution is identified by considering a one-parameter family of solutions that uniquely satisfy a specific integral equation, one that implies well-defined exponentially decaying asymptotic behaviour in the positive infinity direction.

These results naturally raise similar questions regarding the perturbations of other Painlevé equations. They suggest that certain behaviours, important in both theory and application, are preserved by classes of perturbations. We extend the prevalence of exceptional, analytic, transcendental asymptotic behaviours to systems that are not strictly integrable and show the relevance of such models in a "near-Painlevé" physical application. The analysis of this chapter shows that celebrated asymptotic results speak to more profound phenomena beyond the realm of the Painlevé property.

CHAPTER 3

Asymptotic Analysis of q-Discrete Painlevé II

In this chapter, we focus on the asymptotic analysis of solutions of the q-difference second Painlevé equation

$$q\text{-P}_{\text{II}}: \qquad f(qt)f(t/q) = \frac{a(1+f(t)t)}{f(t)(f(t)+t)},$$

where f is a function of $t = t_0q^n$, $n \in \mathbb{Z}$, and quantities $a, q, t_0 \in \mathbb{C}^*$ are constant with respect to t. We note again here that being a q-difference equation, the independent variable t is iterated along a q-spiral $t_0q^{\mathbb{Z}} \subset \mathbb{C}$ for some initial point $t_0 \neq 0$ and constant $q \neq 0, 1$. The sequences t and f are equivalently described in terms of subscripts by $t = t_n = t_0q^n$ and $f = f_n = f(t_n)$, i.e., iterating the multiplicative variable $t \to qt$ is equivalent to iterating the additive variable $n \to n+1$. Bars may also be used to denote iterates, i.e. $\overline{f} = f_{n+1}$ and $\underline{f} = f_{n-1}$.

We first provide an exhaustive description of local asymptotic expansions of the solution $f(t)$, as $|t| \to \infty$ and $t \to 0$, respectively. Following this local asymptotic analysis of generic solutions, we prove the existence of quicksilver solutions [72] of q-P_{II}; these are identified via the asymptotic analysis of divergent power series expansions. Furthermore, we provide an oscillatory-type asymptotic expansion of f, which occurs on the boundaries of validity of these power-series behaviours, analogous to what we see in, for instance, Corollary 2.8 and Lemma 2.11 (anti-Stokes behaviour).

Finally, considering a continuum limit, we deduce that the full autonomous form of q-P_{II} is generically satisfied by elliptic-type behaviour at the leading order. Drawing further parallels with the differential scenario, we show that complete elliptic integrals describe the slow modulation in this elliptic behaviour.

3.1. Local Asymptotic Analysis

In our analysis of q-P_{II}, we seek to provide a complete picture of potential local asymptotic expansions of generic transcendental solutions, considering both limits $t \to 0$ and $|t| \to \infty$.

3.1.1. Limit t approaching infinity. Firstly, it is useful to write q-$\mathrm{P_{II}}$ in the expanded, additive form

$$f_{n+1}f_n^2f_{n-1}+t_nf_{n+1}f_nf_{n-1}=a+at_nf_n.$$

Naively assuming that $f=\mathcal{O}(1)$ as $|t|\to\infty$, we see that f_n is described to leading-order by the simple LHS of

$$f_{n+1}f_{n-1}-a=\frac{1}{t_nf_n}\left(a-f_{n+1}f_n^2f_{n-1}\right). \tag{3.1}$$

The equation $F_{n+1}F_{n-1}-a=0$ is solved rather trivially across the discrete $n\in\mathbb{Z}$ domain by setting $F_n=a^{1/2}\exp\{k_1i^n+k_2(-i)^n\}$, for arbitrary constants k_1,k_2. Here, F is q^4-*periodic*, a repeating sequence of four points in the n dimension.

Nevertheless, we will demonstrate the first summation of the autonomous LHS of Equation (3.1), yielding a leading-order conserved quantity; this provides further insight regarding the discussed sequence F_n, showing that iterates are confined to a specific algebraic curve in the phase space, which is parameterised in terms of an elliptic function. Furthermore, consideration of the conserved energy-like quantity reveals potential critical behaviours of solutions. It also provides a tool for directly analysing how this leading-order autonomous behaviour is slowly perturbed, adhering to the full non-autonomous equation q-$\mathrm{P_{II}}$.

Indeed, applying $(f_{n+1}-f_{n-1})/(f_{n+1}f_{n-1})$ as a summation factor, Equation (3.1) becomes

$$E_n-E_{n-1}=\frac{1}{t_nf_{n+1}f_nf_{n-1}}(f_{n+1}-f_{n-1})(a-f_{n+1}f_n^2f_{n-1}), \tag{3.2}$$

where we have defined E_n, which evidently is constant in the limit $t\to\infty$, by

$$E_n=E(f_{n+1},f_n)=\frac{1}{f_{n+1}f_n}\left(f_{n+1}^2f_n+f_{n+1}f_n^2+af_{n+1}+af_n\right).$$

If we take the convenient scaling

$$f_{n+1}=a^{1/2}x,\quad f_n=a^{1/2}y\quad\text{and}\quad E(f_{n+1},f_n)=-4a^{1/2}K(x,y),$$

then, we obtain the surface

$$x^2y+xy^2+4K(x,y)xy+x+y=0. \tag{3.3}$$

Furthermore, by making the following symmetric homographic transformation of variables

$$x=\frac{cX-1}{cX+1},\quad y=\frac{cY-1}{cY+1}\quad\text{such that}\quad c^4=\frac{K-1}{K+1},$$

we obtain the canonical biquadratic form

$$1+\gamma\left(X^2+Y^2\right)+X^2Y^2=0,\quad \gamma^2=\frac{K^2}{(K-1)(K+1)}.$$

For arbitrary $z \in \mathbb{C}$, this is now satisfied by $X = A\,\mathrm{sn}(z+p,k)$ and $Y = A\,\mathrm{sn}(z,k)$, where $\mathrm{sn}(z,k)$ is Jacobi's elliptic sine function of argument z and modulus k, and we require $k = -1/\gamma$, $A^2 = k$ and $\mathrm{sn}^2(p) = 1$. This result follows by substitution into Equation (3.3) and applying the addition theorem of $\mathrm{sn}(z+p,k)$.

As is hinted at by the machinery of the subsequent transformation, the surface (3.3) includes four critical points where the gradient of $K(x,y)$ vanishes, given by $K(1,1) = -1$, $K(-1,-1) = 1$, and $K(-1,1) = K(1,-1) = 0$. In neighbourhoods of these critical points lie *near-stationary* solutions of q-$\mathrm{P_{II}}$. For instance, corresponding to the point $K(1,1)$ we see that q-$\mathrm{P_{II}}$ is formally satisfied by a series solution

$$f_n = \sum_{k=0}^{\infty} d_k/t_n^k = \sqrt{a} + \frac{q}{1+q^2}(1-a)/t_n + \dots. \tag{3.4}$$

We considered solutions such that $f = \mathcal{O}(1)$ as $t \to \infty$. There are two other distinct balances of q-$\mathrm{P_{II}}$ as $t \to \infty$, these are $f = \mathcal{O}(t)$ and $f = \mathcal{O}(1/t)$, respectively. Indeed, setting $f_n = t_n g_n$ gives

$$g_n + 1 = \frac{a}{t_n^2 g_{n+1} g_{n-1}} + \frac{a}{t_n^4 g_{n+1} g_n g_{n-1}},$$

and setting $f_n = g_n/t_n$ we obtain

$$g_n + 1 = \frac{g_{n+1} g_n g_{n-1}}{a t_n^2} + \frac{g_{n+1} g_n^2 g_{n-1}}{a t_n^4}.$$

These are further examples of near-stationary solutions, where $f_n \sim -t_n$ or $f_n \sim -1/t_n$ as $t \to \infty$, each of these corresponding to a formal series solution of q-$\mathrm{P_{II}}$ in powers of $1/t_n^2$. In contrast with formal series (3.4), these solutions correspond asymptotically to base points of the surface (3.3), where K becomes indeterminate for $1/t = 0$. An analogous solution in the case of q-$\mathrm{P_I}$ is considered in [72], where it is termed a *quicksilver solution* due to its instability under perturbations in the initial-value space. These solutions are considered further in the subsequent section of this chapter.

As previously mentioned, we may consider the slow evolution of the discussed leading-order periodic behaviour by analysis of Equation (3.2), which provides the small difference $E_n - E_{n-1}$ for $t_n \gg 1$. We see directly from Equation (3.2) that

$$E_k - E_{k-1} = -\frac{1}{t_k}\left(L_k - L_{k-1}\right) \quad \text{for} \quad L_k := f_{k+1} f_k + \frac{a}{f_{k+1} f_k}, \quad \forall k \in \mathbb{Z}.$$

We may sum the above equation between arbitrary $k = m$ and $k = n$ to obtain

$$E_n - E_{m-1} = -\frac{L_n}{q t_n} + \frac{L_{m-1}}{t_m} + \frac{1}{q}(1-q)\sum_{k=m}^{n} \frac{L_k}{t_k}, \tag{3.5}$$

where we have applied summation by parts. Suppose that dependent on $m \in \mathbb{N}$, there exists some bound $\lambda_m \in \mathbb{R}^+$ with $|L_k| < \lambda_m$ for all $k \in \{m, m+1, \dots\}$. Then we may straightforwardly infer that in the limit $n \to \infty$, the conserved quantity E converges to some value E_∞ given by

$$E_\infty = E_m + \frac{L_m}{t_m} - (1 - 1/q) \sum_{k=m}^{\infty} \frac{L_k}{t_k},$$

and thus, we deduce a bound on the eventual variation in leading-order behaviour by noting that

$$|E_\infty - E_m| < \frac{\lambda_m}{|t_0|} \left(\frac{1 + |1/q| + |1/q|^m - |1/q|^{2m}}{1 - |1/q|^m} \right).$$

It is not unreasonable to assume the existence of such a λ_m since f assumes the discussed q^4-periodic behaviour for $|t_m| \gg 1$.

In this section, we have thus far considered a degenerate version of leading-order elliptic behaviour, recalling that the leading order behaviour is, in fact, q^4-periodic. The period in the discrete domain is reflected in the condition $\mathrm{sn}^2(p) = 1$, implying that the step in the argument of the elliptic sine function, that is, $z \to z + p$ as $n \to n + 1$, is simply a quarter-period. In Section 3.3 of this chapter, we consider generic elliptic-type behaviour at leading-order by instead taking the continuum limit $q \to 1$, which is analogous to the generic elliptic behaviour we see concerning transcendental solutions of the first five continuous Painlevé equations as the independent variable becomes large.

3.1.2. Limit t approaching zero. We may take the same approach in the case $t \to 0$, noticing that q-P$_{\mathrm{II}}$ becomes

$$E_n - E_{n-1} = \frac{t_n}{f_{n+1} f_{n-1}} \left(f_{n+1} - f_{n-1} \right) \left(a - f_{n+1} f_{n-1} \right),$$

where now we have defined

$$E_n = E(f_{n+1}, f_n) = \frac{1}{f_{n+1} f_n} \left(f_{n+1}^2 f_n^2 + a \right). \tag{3.6}$$

This leading-order invariant quantity is rather simple; here, we may solve a quadratic equation for $f_{n+1} f_n$ in terms of (leading-order) constant E; this implies that alternating iterates of f are constant (to leading order), including the unique (critical) case whereby said leading order constant is the same for all iterates, giving

$$f_n = \sum_{k=0}^{\infty} d_k t_n^k = a^{1/4} + \frac{q}{(1+q)^2} \left(a^{1/2} - 1 \right) t_n + \dots, \tag{3.7}$$

this being a formal series solution of q-P$_{\mathrm{II}}$ around $t = 0$.

The above behaviour indicates that one should consider alternating iterates of f_n having different orders of magnitude as $t_n \to 0$. We decompose q-$\mathrm{P_{II}}$ into the system

$$\begin{aligned} v_k u_k^2 v_{k-1} + z_k v_k u_k v_{k-1} =& a + a z_k u_k, \\ u_{k+1} v_k^2 u_k + q z_k u_{k+1} v_k u_k =& a + a q z_k v_k, \end{aligned}$$

by setting $u_k = f_{2k}$, $v_k = f_{2k+1}$ and $z_k = t_{2k}$, for $k \in \mathbb{Z}$. The limit $t \to 0$ is now $z \to 0$, and we obtain a maximal dominant balance of the system by considering $u = \mathcal{O}(z)$ and $v = \mathcal{O}(1/z)$ (or vice-versa). Indeed, making the transformation $u_k = a z_k g_k$ and $v_k = h_k/(q z_k)$ gives the system

$$\begin{aligned} a h_k g_k^2 h_{k-1} + h_k g_k h_{k-1} - 1 =& a z_k^2 g_k, \\ a g_{k+1} h_k^2 g_k - h_k - 1 =& - a q^2 z_k^2 g_{k+1} h_k g_k. \end{aligned} \tag{3.8}$$

The leading-order invariant quantity is less obvious here. Taking a linear combination of the coupled difference equations above (after applying a summation factor to each), we deduce

$$E_k - E_{k-1} = - a z_k^2 \left(q^2 g_{k+1} - q^2 g_k + 1/h_k - 1/h_{k-1} \right),$$

where the (leading-order) invariant quantity is

$$E_k = E(g_{k+1}, h_k) = \frac{1}{g_{k+1} h_k} \left(a g_{k+1}^2 h_k^2 + g_{k+1} h_k^2 + h_k + 1 \right). \tag{3.9}$$

So, here we consider the surface

$$0 = a \overline{x}^2 y^2 + \overline{x} y^2 - E(\overline{x}, y) \overline{x} y + y + 1. \tag{3.10}$$

The above equation represents genus-zero algebraic level curves, which may be parameterised via trigonometric functions. Taking the asymmetric, homographic transformation of variables $\overline{x} = \alpha \overline{X} + \beta$ and $y = 1/(\gamma Y + \delta)$, for these constants satisfying

$$\alpha^2 = -\frac{1 + a + E}{a\left(4a - E^2\right)}, \quad \beta = -\frac{2 + E}{4a - E^2},$$
$$\gamma^2 = -\frac{1 + a + E}{4a - E^2}, \quad \delta = -\frac{2a + E}{4a - E^2},$$

we obtain the canonical biquadratic equation

$$0 = 1 + \frac{E}{a^{1/2}} \overline{X} Y + \overline{X}^2 + Y^2.$$

This is satisfied for any $z \in \mathbb{C}$ simply by $\overline{X} = A \sin(z + p)$ and $Y = A \sin(z)$, given that $A^2 = -4a/\left(4a - E^2\right)$ and $A^2 \sin^2(p) = -1$, by the addition theorem for $\sin(z + p)$.

The surface (3.10) features a single critical point given by $E(1/(1 - a), a - 1) = -a - 1$, once again corresponding to near-stationary solutions for g and

h. In a neighbourhood of this point we have system (3.8) formally satisfied by series solutions $G_k = \sum_{j=0} c_j z_k^{2j}$ and $H_k = \sum_{j=0} d_j z_k^{2j}$ where of course $c_0 = -1/(a-1)$ and $d_0 = a-1$.

Outside of this critical behaviour, there is an apparent issue with points in the initial value space satisfying $E^2 = 4a$. These points in the initial value space correspond to u_k and v_k remaining of constant order as $z_k \to 0$. Indeed, consider the sub-maximal dominant balance of the system form of q-$\mathrm{P_{II}}$ given by

$$v_k u_k^2 v_{k-1} \sim a \quad \text{and} \quad u_{k+1} v_k^2 u_k \sim a.$$

We may explicitly solve this simple system for u and v by setting $u_k = u_0 r^k$ and $v_k = (a/r)^{1/2}/u_k$, with initial conditions represented by constants u_0 and r. To remain consistent, we assert that $|r| = 1$, i.e., if not, we should consider the general system (3.8). The case $r = 1$ returns exactly what has already been discussed concerning (3.6). However, the general restraint $|r| = 1$ implies that u_k and v_k are parameterised trigonometrically, evolving on circles in the complex plane centred at the origin.

Analogous to Equation (3.5), we may deduce that the leading-order invariant quantity (3.9) evolves according to

$$\begin{aligned} E_k - E_0 =& - aq^4 z_k^2 L_k + aq^4 z_0^2 L_0 + a\left(q^4 - 1\right)\sum_{j=1}^{k} z_j^2 L_j, \\ L_k :=& q^2 g_{k+1} + 1/h_k. \end{aligned}$$

Remark 3.11 (Rational solutions)**.** We notice that when $a = 1$, an exact solution is $f(t) \equiv 1$. This is the seed of a hierarchy of rational solutions $f^{(k)}$, generated by the auto-Bäcklund transformation [70]

$$f^{(k+1)} = \frac{qf^{(k)}\underline{f}^{(k)} + a^{(k)}q^2(1 + tf^{(k)})}{f^{(k)}(qf^{(k)}\underline{f}^{(k)} + tf^{(k)} + 1)}, \quad a^{(k+1)} = q^4 a^{(k)}, \quad f^{(0)} = a^{(0)} = 1.$$

It is then straightforward to see that these solutions are asymptotically given by series (3.4) and (3.7) in the limits $|t| \to \infty$ and $|t| \to 0$, respectively, corresponding to critical points on surfaces (3.3) and (3.6).

3.1.3. Special solutions. Here we give a quick review of how one may generate solutions of q-$\mathrm{P_{II}}$ for certain choices of parameter a, which are given exactly in terms of ratios of iterates of the q-Airy function, mirroring the theory of special solutions of the continuous second Painlevé equation. We proceed to discuss the asymptotic behaviour of these solutions in relation to this section's previous local asymptotic analysis.

In [70], we see the systematic construction of *Miura transformations* of discrete Painlevé equations. We consider transforming q-$\mathrm{P_{II}}$ into the system

$$y = \frac{\overline{f}}{qt^2}(t+f), \quad f = \frac{t^4 y\underline{y} - a}{t(a+t^2 y)}. \tag{3.12}$$

Eliminating y returns f satisfying q-$\mathrm{P_{II}}$, while eliminating f we obtain a form of the equation known as q-P_{34}, this being

$$\left(q^4 t^4 \overline{y} y - a\right)\left(t^4 y\underline{y} - a\right) = at^2\left(q^2 t^2 y + 1\right)\left(t^2 y + a\right).$$

Therefore, choosing the parameter $a = 1/q^2$, we see that $y = -1/\left(q^2 t^2\right)$ exactly satisfies the above equation. Looking back to the system (3.12), we see that in this case, f satisfies $1 + qt\overline{f} + q\overline{f}f = 0$. We are now in a position to linearise this first-order equation by setting $f = \underline{R}/R$, which returns the q-Airy equation

$$\overline{R} + qtR + q\underline{R} = 0. \tag{3.13}$$

Recalling $t = t_n = t_0 q^n$, a q-Airy solution may be expressed asymptotically as $|t| \to \infty$ by a linear combination of the expansions

$$R^+ = (-t_0)^n q^{n(n+1)/2}\left(1 - \frac{q^2}{(1-q)(1+q)}\frac{1}{t^2} + \mathcal{O}\left(1/t^4\right)\right), \tag{3.14}$$

$$R^- = (-1/t_0)^n q^{-n(n+1)/2}\left(1 + \frac{1}{q^2(1-q)(1+q)}\frac{1}{t^2} + \mathcal{O}\left(1/t^4\right)\right). \tag{3.15}$$

Taking ratios of iterates of these expansions leads directly to the aforementioned quicksilver-type series solutions $f \sim -t$ or $f \sim -1/t$ as $|t| \to \infty$.

Meanwhile, as $t \to 0$, we have singly periodic behaviour given by the leading order $\overline{R} \sim -q\underline{R}$. In terms of f, this corresponds to the degenerate behaviour associated with the surface (3.6), which may be explicitly parameterised by $f_n \sim -\cot\left(\theta_0 + \pi n/2\right)/\sqrt{q}$. However, suppose we allow alternating iterates of f_n to be of different orders of magnitude as $t \to 0$. In that case, we arrive at a case of the series solutions arising in a neighbourhood of the critical point of surface (3.10).

As noted in Remark 3.11, one may construct rational solutions corresponding to parameters $a = q^{4k}$, for natural numbers k; we could similarly use the auto-Bäcklund transformation to construct solutions which are rational with respect to the q-Airy function, for parameters $a = q^{2(2k-1)}$.

3.2. Quicksilver Solutions

In this section, we pay closer attention to the previously mentioned formal power series expansions around $t = \infty$, given to leading order by $f \sim -t$ and $f \sim -1/t$ as $t \to \infty$. We analyse the coefficients in these power series and seek to show that, although these series are divergent, true analytic solutions asymptotic to such series exist in certain q-domains. These solutions are termed "quicksilver", following [72].

Quicksilver solutions are in parallel with the study of *tronqueé* solutions of the continuous second Painlevé equation [16], which (atypically for transcendental solutions of P_{II}) are free of poles for sufficiently large independent variable in a particular open sector of the complex plane, similarly being asymptotic to a divergent power series expansion in such a region. Here, we expect regions of validity to be bounded by the arcs of q-spirals since solutions are iterated along q-spirals in the t-plane, contrasting the traditional sectorial regions in the case of continuous Painlevé equations.

3.2.1. Power series expansions. We begin by defining the following power series expansions

$$G_n = -t_n \sum_{k=0}^{\infty} g_k/t_n^{2k}, \quad D_n = -\frac{1}{t_n}\sum_{k=0}^{\infty} d_k/t_n^{2k}, \quad g_0 = d_0 = 1.$$

Considering the growing solution G_n, we find that $g_1 = -a$, $g_2 = a - a^2q^2 - a^2/q^2$, and for all $n \geq 3$ coefficients g_n satisfy the nonlinear difference equation

$$a g_{n-1} + \sum_{j=0}^{n-1}\sum_{k=0}^{n-j-1}\sum_{l=1}^{n-j-k} g_j g_k g_l g_{n-j-k-l} q^{2(n-2j-k-l)} = 0,$$

where rearranging for g_n yields

$$\begin{aligned} g_n = &- a g_{n-1} - \sum_{j=1}^{n-1}\sum_{k=0}^{n-j-1}\sum_{l=1}^{n-j-k} g_j g_k g_l g_{n-j-k-l} q^{2(n-2j-k-l)} \\ &- \sum_{k=1}^{n-1}\sum_{l=0}^{n-k} g_k g_l g_{n-k-l} q^{2(n-k-l)}. \end{aligned}$$

Meanwhile, considering D_n we find $d_1 = 1/a$ and for all $n \geq 0$ that d_{n+2} is given by

$$\begin{aligned} d_{n+2} = &\frac{q^2}{a}\sum_{j=0}^{n+1}\sum_{k=0}^{n-j+1} d_j d_k d_{n-j-k+1} q^{2(n-2j-k)} \\ &- \frac{1}{a}\sum_{j=0}^{n}\sum_{k=0}^{n-j}\sum_{l=0}^{n-j-k} d_j d_k d_l d_{n-j-k-l} q^{2(n-2j-k-l)}. \end{aligned}$$

(Further details are provided in Appendix A.)

The above G_n and D_n are both divergent asymptotic series. Without loss of generality taking the case $|q| > 1$, the growth of coefficients is given by, respectively

$$g_n = k_g a^n q^{n(n-1)} \left(1 - \frac{a - q^2 + 2aq^4}{a(q-1)(q+1)} q^{-2n} + \mathcal{O}\left(1/q^{4n}\right)\right), \tag{3.16}$$

$$d_n = k_d a^{-n} q^{n(n-1)} \left(1 - \frac{1 + 2q^2 - aq^2}{(q-1)(q+1)} q^{-2n} + \mathcal{O}\left(1/q^{4n}\right)\right), \tag{3.17}$$

for some constants k_g and k_d, as $n \to \infty$. This behaviour may be proved directly from the nonlinear difference equations governing sequences g_n and d_n. In Appendix A, we provide this proof for a case of the coefficients d_n. Indeed, consider the transformation

$$d_n = a^{-n} q^{n(n-1)} \frac{(p_2; p_2)_{n-1}}{(p_1; p_1)_{n-1}} c_n, \quad n > 0,$$

with $p_1 = 1/q^2, p_2 = 1/q^6$ and $c_0 = 3$ (here we utilise the q-Pochhammer symbol, see Section 1.6 for details). Taking the case $0 > a \in \mathbb{R}$, and assuming $e^2 < q \in \mathbb{R}$ to obtain simpler explicit estimates (although this may improved to any fixed $q > 1$), we show inductively that the coefficients c_n are monotonically increasing and bounded above for all $n \geq 1$, i.e. converging as $n \to \infty$. For instance, taking the value $|a| = 1$ we obtain

$$d_n \to c_\infty a^{-n} q^{n(n-1)} \frac{(p_2; p_2)_{n-1}}{(p_1; p_1)_{n-1}}, \quad n \to \infty, \tag{3.18}$$

where c_∞ is some constant with explicit estimate $1 < c_\infty < \Phi \approx 3.20474$.

Let S_0 be a disk in $\mathbb{C}$ whose points are at least $r_0 > 0$ away from the origin. Define S to be a union of open sets $S_n = q^n S_0$. There is a finite collection of sectors $\Omega_k = \{t \in \mathbb{C} \,|\, r_0 < 1/|t| \leq r_k, \alpha_k \leq \arg t \leq \beta_k\}$ in $\mathbb{C}^*$ such that each S_n is contained in one of the Ω_k. Let $\Omega = \cup \Omega_k$. A standard use of the Borel-Ritt theorem [61] provides a function $F(t)$ which is analytic in Ω and $F \sim -\frac{1}{t_n}\sum_{k=0}^{\infty} d_k/t_n^{2k}$ as $t \to \infty$ in Ω.

3.2.2. True solutions. Analogous to the asymptotic theory regarding solutions of continuous Painlevé equations, we consider a perturbation that appears beyond all orders of the proposed power series expansion. If we substitute $f_n = D_n + v_n$ into q-$\mathrm{P_{II}}$ then we obtain

$$\frac{D_{n+1}}{D_{n-1}} v_{n-1} + \frac{2D_{n-1}D_nD_{n+1} - at_n + D_{n-1}D_{n+1}t_n}{D_{n-1}D_n(D_n + t_n)} v_n + v_{n+1} = \mathcal{Q}, \tag{3.19}$$

where $\mathcal{Q}$ is a quartic polynomial in v_{n-1}, v_n, v_{n+1}, but crucially includes no terms of degree less than two. Consider P_n satisfying the above linear homogeneous equation (i.e., neglecting $\mathcal{Q}$). To leading order as $t \to \infty$, this is a q-Airy equation

$$\left(1/q^2 + \mathcal{O}\left(1/t_n^2\right)\right) P_{n-1} + \left(-at_n^2/q + \mathcal{O}(1)\right) P_n + P_{n+1} = 0. \tag{3.20}$$

Remark 3.21. The leading-order equation $\underline{P} - aqt^2 P + q^2 \overline{P}$ becomes the classical Airy equation in the continuum limit $P_n = u(s)/q^n$, $n = s/\epsilon$, $q = 1 + \epsilon^2/4$ and $t_0 = \sqrt{2/a}$, as $\epsilon \to 0$.

Similar to expansions (3.14), Equation (3.20) has fundamental asymptotic expansions as $t \to \infty$ given by

$$\begin{aligned} P_n^- &= a^{-n} t_0^{-2n} q^{-n^2-2n} \left(1 + \mathcal{O}\left(1/t_n^2\right)\right), \\ P_n^+ &= a^n t_0^{2n} q^{n^2-2n} \left(1 + \mathcal{O}\left(1/t_n^2\right)\right). \end{aligned} \tag{3.22}$$

We proceed to work in the region of Ω where $|P^+| \gg |P^-|$, which we continue to denote by Ω. So, our assumptions are corroborated by setting $v_n \sim kP_n^-$ for arbitrary constant k, in an open region of the t-plane where $P^-(t)$ and $P^+(t)$ are exponentially decaying and growing, respectively, as $|t| \to \infty$.

Now considering the full Equation (3.19) for v, we apply $P_n/(D_{n+1}D_n)$ as a summation factor and then notice that

$$\left(\frac{P_k v_{k+1}}{D_{k+1}D_k} - \frac{P_{k-1} v_k}{D_k D_{k-1}}\right) - \left(\frac{P_{k+1} v_k}{D_{k+1}D_k} - \frac{P_k v_{k-1}}{D_k D_{k-1}}\right) = \frac{P_k \mathcal{Q}_k}{D_{k+1}D_k}.$$

Thus, after performing a first summation between $k = j$ and $k = m$, we obtain

$$\frac{v_j}{P_j} - \frac{v_{j-1}}{P_{j-1}} = \frac{D_j D_{j-1}}{P_j P_{j-1}} \left(\frac{P_m v_{m+1}}{D_{m+1}D_m} - \frac{P_{m+1} v_m}{D_{m+1}D_m}\right) - \frac{D_j D_{j-1}}{P_j P_{j-1}} \sum_{k=j}^{m} \frac{P_k \mathcal{Q}_k}{D_{k+1}D_k}.$$

Here, we take the limit $m \to \infty$ and let our summation factor $P_n = P_n^-$ be the solution which exponentially decays as $n \to \infty$ (in some region of the t-plane). We thus remove one degree of freedom, obtaining the summation equation satisfied by a family of solutions $v_n^{(k)}$ with the property $v_n^{(k)} \sim kP_n^-$ for the remaining arbitrary constant k. After taking $m \to \infty$ we have

$$\frac{v_j}{P_j} - \frac{v_{j-1}}{P_{j-1}} = -\frac{D_j D_{j-1}}{P_j P_{j-1}} \sum_{k=j}^{\infty} \frac{P_k \mathcal{Q}_k}{D_{k+1}D_k}.$$

and summing again between $j = r$ and $j = n$ yields

$$v_n = \frac{v_{r-1}}{P_{r-1}} P_n - P_n \sum_{j=r}^{n} \frac{D_j D_{j-1}}{P_j P_{j-1}} \sum_{k=j}^{\infty} \frac{P_k \mathcal{Q}_k}{D_{k+1}D_k}.$$

In fact, after re-ordering the double sum, we may write

$$v_n^{(k)} = kP_n + \sum_{k=n}^{\infty} \left\{P_n Q_k - P_k Q_n\right\} \frac{\mathcal{Q}_k}{D_{k+1}D_k}, \quad k = \frac{v_{r-1}}{P_{r-1}} - \sum_{k=r}^{\infty} \frac{Q_k \mathcal{Q}_k}{D_{k+1}D_k}, \tag{3.23}$$

where we have defined Q_n, a linearly independent solution of Equation (3.20), given by

$$Q_n = P_n \sum_{j=r}^{n} \frac{D_j D_{j-1}}{P_j P_{j-1}}.$$

Now consider the function $\mathcal{Q} = \mathcal{Q}(v_n, v_{n-1}, t_n)$. Upon closer inspection, we find that

$$\mathcal{Q}(v_n, v_{n-1}, t_n) = \frac{\mathcal{R}(v_n, v_{n-1}, t_n)}{D_{n-1}D_n(D_n+t_n)(D_{n-1}+v_{n-1})(D_n+v_n)(D_n+t_n+v_n)},$$

where $\mathcal{R}(v_n, v_{n-1}, t_n)$ is simply a polynomial in v_n, v_{n-1}, t_n and iterates of D_n. It follows that $\mathcal{Q}(x, y, t)$ is holomorphic in a domain

$$\mathcal{D} = \left\{ (x, y, t) \in \mathbb{C}^3 \;\middle|\; |1/t| < \epsilon,\ \|(x, y)\| \leq \delta \right\},$$

for some constants $\epsilon, \delta \in \mathbb{R}^+$. Moreover, due to D_n being a formal series solution of q-$\mathrm{P_{II}}$, at $(x, y) = (0, 0)$ we see that

$$\mathcal{Q}(0,0,t),\ \partial_x\mathcal{Q}(0,0,t),\ \partial_y\mathcal{Q}(0,0,t) = \mathcal{O}\left(1/|t|^N\right), \quad \forall N \in \mathbb{N}.$$

In other words, for $(v_n, v_{n-1}, t) \in \mathcal{D}$, the Taylor expansion for $\mathcal{Q}(v_n, v_{n-1}, t_n)$ has terms constant and linear in v_n, v_{n-1} which are arbitrarily small (as $|t|$ becomes arbitrarily large), followed by terms quadratic in v_n, v_{n-1}. It follows that we may assume estimates

$$|\mathcal{Q}(x,y,t)| \leq C_1\|(x,y)\|^2 + C_2/|t|, \quad \|(\partial_x\mathcal{Q}, \partial_y\mathcal{Q})\| \leq C_3\|(x,y)\| + C_4/|t|,$$

for some constants $C_1, C_2, C_3, C_4 \in \mathbb{R}^+$, where $(x, y, t) \in \mathcal{D}$ with some constants $\epsilon, \delta \in \mathbb{R}^+$.

Appropriate application of the contraction mapping theorem to summation Equation (3.23) then gives the desired true solution $v_n^{(k)} \sim kP_n^-$ in Ω defined on a semi-infinite q-spiral, see [72] for further details of the rest of this argument.

Remark 3.24. One may generate analogous results concerning the growing series solution G_n. Setting $f_n = G_n + u_n$, Equation (3.20) becomes

$$\left(q^2 + \mathcal{O}\left(1/t_n^2\right)\right) P_{n-1} + \left(-qt_n^2/a + \mathcal{O}(1)\right) P_n + P_{n+1} = 0.$$

So the perturbation u_n, while assumed to be exponentially small, also satisfies a q-Airy type equation to leading order, with fundamental asymptotic expansions as $t \to \infty$ given by

$$P_n^- = a^n t_0^{-2n} q^{-n^2}\left(1 + \mathcal{O}\left(1/t_n^2\right)\right), \quad P_n^+ = a^{-n} t_0^{2n} q^{n^2}\left(1 + \mathcal{O}\left(1/t_n^2\right)\right).$$

3.2.3. Anti-Stokes behaviour. Considering expansions (3.22), the perturbation term $v_n \sim kP_n^-$ as $|t| \to \infty$ is valid for t in an open region of the complex plane bounded by arcs where $|P^+/P^-|$ no longer grows or decays exponentially. Equivalently, Stokes boundaries are where q^{n^2}, q^{-n^2} are no longer exponentially large for $|t_n| = |t_0 q^n| \gg 1$. Say that $t/t_0 = r_t e^{i\theta_t}$ and $q = r_q e^{i\theta_q}$; we have $n = \ln(t/t_0)/\ln(q)$ and therefore

$$q^{n^2} = (t/t_0)^{\ln(t/t_0)/\ln(q)} = r_t^{\ln(t/t_0)/\ln(q)} e^{i\theta_t \ln(t/t_0)/\ln(q)},$$

where

$$\ln(t/t_0)/\ln(q) = \frac{\ln(r_t) + i\theta_t}{\ln(r_q) + i\theta_q} = \frac{\ln(r_q)\ln(r_t) + \theta_q\theta_t}{\ln(r_q)^2 + \theta_q^2} + i\frac{\ln(r_q)\theta_t - \theta_q \ln(r_t)}{\ln(r_q)^2 + \theta_q^2}.$$

Therefore, to avoid exponential growth of q^{n^2} or q^{-n^2} in r_t as $r_t \to \infty$ (interpreting this to mean simply any growth that exceeds that of any bounded power of r_t), we require $\ln(r_q)\ln(r_t) = -\theta_q\theta_t$, giving spirals in the complex t-plane.

Analogous to the asymptotic study of solutions of continuous Painlevé equations, we expect that on the Stokes boundary, v has an asymptotic expansion with two free constants. Mirroring the form of (4.2), which we use in the corresponding scenario for $\mathrm{P_{IV}}$, we find that v_n admits the expansion

$$v_n = \tilde{D}_n + t_n \sum_{k=1}^{\infty} \frac{1}{t_n^{3k}} \left(R_n^{(k)} S_n^k + T_n^{(k)}/S_n^k \right). \tag{3.25}$$

Here each $R_n^{(k)}$ and $T_n^{(k)}$ for $k \in \mathbb{Z}^+$ are series in powers of $1/t_n^2$ and (at most) constant as $|t| \to \infty$. The two free parameters in this expansion are given by k_1, k_2 where $R_n^{(1)} \to k_1$ and $T_n^{(1)} \to k_2$ as $|t| \to \infty$. Furthermore, we have $S_n = s^n q^{n^2}$, where analogous to the differential case, the constant s depends on the product of parameters $k_1 k_2$. However, here we arrive at a more complex scenario, whereby to obtain s, we must satisfy the sixth-degree polynomial

$$k_1^3 k_2^3 s^6 - 3ak_1^2 k_2^2 t_0^4 s^4 + 3a^2 k_1 k_2 t_0^8 s^2 + a^2 t_0^{10} s - a^3 t_0^{12} = 0.$$

As we see in the equivalent case with vanishing series solutions of $\mathrm{P_{IV}}$ on an anti-Stokes ray, the leading algebraic behaviour D_n is not preserved, rather now depending on the product $k_1 k_2$, since we calculate

$$\tilde{D}_n = \frac{k_1 k_2 s^2}{k_1 k_2 s^2 - a t_0^4} \times \frac{1}{t_n} + \mathcal{O}\left(1/t_n^3\right).$$

Moreover, again equivalent to the differential case, to maintain asymptoticity of expansion (3.25), we require a condition on $|s|$. Assuming $q^{n^2} = \mathcal{O}(1)$, we see that subsequent terms in the sum of (3.25) decay in powers of $s^{\pm n}/t_n^3$, so we obtain the constraint $|s|, 1/|s| < |q|^3$. It is encouraging to note that this expansion degenerates to exactly one of (3.22) by taking either $k_1 = 0$ or $k_2 = 0$, i.e. the quicksilver (tronqueé-like) solutions.

3.3. Generic Elliptic Behaviour

In this section, we progress the concept of generic elliptic behaviour (see Section 1.1 of Chapter 1 for background) to the q-difference setting. We consider q-P$_{\mathrm{II}}$ with $q = 1 + \epsilon$ where $0 < |\epsilon| \ll 1$ is a small parameter; that is, by decreasing the step size $|t_{n+1} - t_n|$, we consider a generic solution of q-P$_{\mathrm{II}}$ while staying within a bounded region near an arbitrary point $t_0 \in \mathbb{C}^*$. In this asymptotic context, we show that the leading-order autonomous form of q-P$_{\mathrm{II}}$ is solved in terms of Jacobi elliptic functions. Furthermore, we extend the averaging method and describe the slow evolution of this leading-order elliptic behaviour in terms of complete elliptic integrals. Finally, we consider critical points in the initial value space where solutions degenerate to being singly periodic, as is the case for continuous Painlevé transcendents along certain rays in the complex plane (Stokes boundaries).

This investigation provides a new fundamental link between discrete and continuous Painlevé equations and develops a new understanding of the generic behaviour of a q-difference Painlevé equation.

3.3.1. Elliptic behaviour. In this section, we consider the autonomous version of q-P$_{\mathrm{II}}$, obtained at leading order by setting $q = 1 + \epsilon$, where $\epsilon \in \mathbb{C}^*$ is a small parameter. After providing a bound on the error associated with the leading-order equation, we give the generic solution in terms of Jacobi elliptic functions in the limit $\epsilon \to 0$.

We begin by formally defining a solution of q-P$_{\mathrm{II}}$, the quantity E which is conserved at leading order, and a quantity L which arises naturally when considering the slight difference in E as we iterate $n \to n + 1$.

Definition 3.26. *Let $a, f_0, f_1, t_0, \epsilon \in \mathbb{C}^*$, with $|\epsilon| < 1$. We define the sequence $t : \mathbb{N} \to \mathbb{C}$ by $t_n = t_0(1 + \epsilon)^n$, and the corresponding sequence $f : \mathbb{N} \to \mathbb{C}$ by*

$$f_{n+1} f_{n-1} = \frac{a(1 + f_n t_n)}{f_n(f_n + t_n)}. \tag{3.27}$$

Definition 3.28. *We define the sequences $E : \mathbb{N} \to \mathbb{C}$ and $L : \mathbb{N} \to \mathbb{C}$ by*

$$E_n = \frac{f_{n+1}^2 f_n^2 + t_0 f_{n+1} f_n (f_{n+1} + f_n) + a t_0 (f_{n+1} + f_n) + a}{f_{n+1} f_n}, \tag{3.29}$$

$$L_n = \frac{(a + f_{n+1} f_n)(f_{n+1} + f_n)}{f_{n+1} f_n}. \tag{3.30}$$

We study the sequence f_n in a domain where it remains bounded and sufficiently away from zero, which is always possible due to the meromorphic nature of solutions of discrete Painlevé equations. This sentiment is expressed explicitly in the following definition.

Definition 3.31. *Given some $n \in \mathbb{N}$, we denote finite value $J_n \in \mathbb{R}^+$ such that for all $k \in \mathbb{N}$, with $0 \leq k \leq n$, the sequence f_k satisfies*

$$|L_k| = \left| \frac{a}{f_{k+1}} + \frac{a}{f_k} + f_{k+1} + f_k \right| \leq J_n.$$

Across an arbitrary $n \in \mathbb{N}$ number of steps, we proceed to give a bound on the difference $|E_n - E_0|$, and thus provide sufficient conditions for this difference to remain small as $\epsilon \to 0$.

Lemma 3.32. *Let $n \in \mathbb{N}$ and assume the existence of finite value J_n as described in Definition 3.31. Then we obtain*

$$|E_n - E_0| < 2\,|t_0|\,J_n \left(e^{|\epsilon|n} - 1 \right),$$

and thus $|E_n - E_0| \ll 1$ as $\epsilon \to 0$ while $|\epsilon| n \ll 1$.

Proof. Let $n \in \mathbb{N}$. From definitions 3.26 and 3.28 we directly compute

$$\begin{aligned} E_n - E_{n-1} =& \frac{f_{n+1} - f_{n-1}}{f_{n+1} f_n f_{n-1}} \left(f_{n+1} f_n^2 f_{n-1} + t_0 f_{n+1} f_n f_{n-1} - a t_0 f_n - a \right) \\ =& -(t_n - t_0) \frac{(f_{n+1} - f_{n-1})(f_{n+1} f_{n-1} - a)}{f_{n+1} f_{n-1}} \\ =& -P_n (L_n - L_{n-1}), \end{aligned} \tag{3.33}$$

where we denote P_n, for $n \in \mathbb{N}$, the binomial expansion

$$P_n = t_n - t_0 = t_0 \sum_{k=1}^{n} \binom{n}{k} \epsilon^k \quad \text{with} \quad P_0 = 0.$$

Then, it is straightforward to see that

$$|P_n| \leq |t_0| \sum_{k=1}^{n} \frac{(n\,|\epsilon|)^k}{k!} < |t_0| \left(-1 + \sum_{k=0}^{\infty} \frac{(n\,|\epsilon|)^k}{k!} \right) = |t_0| \left(e^{n|\epsilon|} - 1 \right),$$

which immediately yields

$$|E_n - E_{n-1}| < |t_0|\,|L_n - L_{n-1}| \left(e^{|\epsilon|n} - 1 \right). \tag{3.34}$$

Summing the equation $E_k - E_{k-1} = -P_k(L_k - L_{k-1})$ across $k = 1$ to $k = n$ and applying summation by parts (analogous to integration by parts), we obtain

$$E_n - E_0 = -L_n P_n + t_0 \epsilon \sum_{k=0}^{n-1} L_k \left(\frac{P_k}{t_0} + 1 \right). \tag{3.35}$$

Using the fact that $|P_k| < |t_0|(e^{k|\epsilon|} - 1)$ for any $k \in \mathbb{N}$, it is straightforward to furthermore see that

$$\left|\sum_{k=0}^{n-1} L_k \left(\frac{P_k}{t_0} + 1\right)\right| < \sum_{k=0}^{n-1} |L_k|\, e^{|\epsilon|k} \leq J_n \sum_{k=0}^{n-1} e^{|\epsilon|k} = J_n \frac{e^{|\epsilon|n} - 1}{e^{|\epsilon|} - 1}.$$

Thus from Equation (3.35) we obtain

$$|E_n - E_0| < |L_n P_n| + |t_0 \epsilon| J_n \frac{e^{|\epsilon|n} - 1}{e^{|\epsilon|} - 1} < 2\,|t_0|\, J_n \left(e^{|\epsilon|n} - 1\right),$$

as required. □

Having established an error-bound, we consider the leading-order, autonomous equation

$$f_{n+1}^2 f_n^2 + t_0 f_{n+1} f_n (f_{n+1} + f_n) - E_0 f_{n+1} f_n + a t_0 (f_{n+1} + f_n) + a = H_n, \tag{3.36}$$

for arbitrary $n \in \mathbb{N}$, where by Lemma 3.32

$$|H_n| = |f_{n+1} f_n (E_n - E_0)| < 2\,|t_0 f_{n+1} f_n|\, J_n \left(e^{|\epsilon|n} - 1\right), \tag{3.37}$$

so that $|H_n| \ll 1$ as $\epsilon \to 0$ while $|\epsilon| n \ll 1$, given the existence of finite value $J_n \in \mathbb{R}^+$.

We proceed to define a linear fractional transformation of the function f_n, thus converting the LHS of Equation (3.36) to the canonical biquadratic form $x^2y^2 + \gamma(x^2 + y^2) + \zeta xy + 1$ for some $\gamma, \zeta \in \mathbb{C}$. These constants are provided in the following lemma, where we take the case $a = 1$. Setting $a = 1$ provides a more tractable illustrative example while not being a meaningful degeneration of the problem.

Lemma 3.38. *With conditions $E_0 \neq 2 \pm 4t_0$, define the sequence $F : \mathbb{N} \to \mathbb{C}$ by*

$$f_n = \frac{c + F_n}{c - F_n} \quad \text{with } c \in \mathbb{C}^* \text{ such that} \quad c^4 = \frac{2 - E_0 - 4t_0}{2 - E_0 + 4t_0}.$$

Then considering bounded $n \in \mathbb{N}$ and taking the case $a = 1$, Equation (3.36) becomes

$$F_{n+1}^2 F_n^2 + \gamma \left(F_{n+1}^2 + F_n^2\right) + \zeta F_{n+1} F_n + 1 = K_n \to 0, \quad \epsilon \to 0,$$

where

$$\gamma = \frac{(2 + E_0)c^2}{2 - E_0 - 4t_0}, \quad \zeta = \frac{8c^2}{2 - E_0 - 4t_0}, \quad K_n = \frac{(c - F_n)^2 (c - F_{n+1})^2}{2 - E_0 - 4t_0} H_n.$$

Proof. The lemma may be verified straightforwardly via substitution. Considering Equation (3.37), we see that $K_n \to 0$ as $\epsilon \to 0$ given n is bounded (or $|\epsilon| n \ll 1$), F_n is finite and $E_0 \neq 2 \pm 4t_0$. The condition $E_0 \neq 2 \pm 4t_0$ on the initial value space of f is discussed in the subsequent remark. □

Remark 3.39. The transformation described in Lemma 3.38, applied to Equation (3.36), breaks down with initial conditions causing $E_0 = 2 \pm 4t_0$. By observing Equation (3.27), we see that these conditions correspond to stationary solutions $f_n = \pm 1$ for all $n \in \mathbb{N}$ (in the $a = 1$ case). Expansions of the solution f near critical points in the initial value space are given in Subsection 3.3.3.

This subsection's core result is that F_n described in Lemma 3.38 is given by the Jacobi elliptic sine function in the limit $\epsilon \to 0$.

Theorem 3.40. *Consider bounded $n \in \mathbb{N}$ and $a = 1$. Regarding the sequence F_n described in Lemma 3.38 we obtain*

$$F_n \to A\,sn(z_0 + pn; k), \quad \epsilon \to 0.$$

Here, $sn(z; k)$ is Jacobi's elliptic sine function of argument z and modulus k. Furthermore, $A^2 = k$ and k satisfying $0 < |k| < 1$ is uniquely determined by

$$k^2 + \beta k + 1 = 0, \quad \beta := \frac{4 + 4\gamma^2 - \zeta^2}{4\gamma}.$$

Finally, p is chosen such that $ksn^2(p; k) = -1/\gamma$ and z_0 is representative of the initial condition f_0, satisfying $Asn(z_0; k) = F_0$.

Proof. Let $A, k, p, z \in \mathbb{C}$ and define $G(z) := A\,\mathrm{sn}(z; k)$. The result follows by applying the well-known addition theorem regarding the Jacobi elliptic sine function:

$$\mathrm{sn}(x + y) = \frac{\mathrm{sn}(x)\mathrm{cn}(y)\mathrm{dn}(y) + \mathrm{sn}(y)\mathrm{cn}(x)\mathrm{dn}(x)}{1 - k^2\mathrm{sn}^2(x)\mathrm{sn}^2(y)}, \quad \forall x, y \in \mathbb{C},$$

where we have suppressed the modulus k for simplicity, and the elliptic functions sn, cn, and dn are related by

$$\mathrm{cn}^2(x) + \mathrm{sn}^2(x) = 1, \quad \mathrm{dn}^2(x) + k^2\mathrm{sn}^2(x) = 1, \quad \forall x \in \mathbb{C}.$$

Applying the above formulas, we find that $G(z)$ satisfies

$$G(z)^2G(z+p)^2 + \gamma\left(G(z)^2 + G(z+p)^2\right) + \zeta G(z)G(z+p) + 1 = 0,$$

for all $z \in \mathbb{C}$, given we choose A satisfying $A^2 = k$, p satisfying $G(p)^2 = -1/\gamma$, and k satisfying

$$k^2 + \beta k + 1 = 0, \quad \beta := \frac{4 + 4\gamma^2 - \zeta^2}{4\gamma}.$$

Without loss of generality, we take the solution of the above quadratic satisfying $|k| < 1$, noting that this equation is invariant under $k \to 1/k$. Considering Lemma 3.38, it now follows that $F_n \to G(z_0 + pn)$ as $\epsilon \to 0$, given we choose appropriate initial condition $z_0 \in \mathbb{C}$ satisfying $G(z_0) = F_0$. ∎

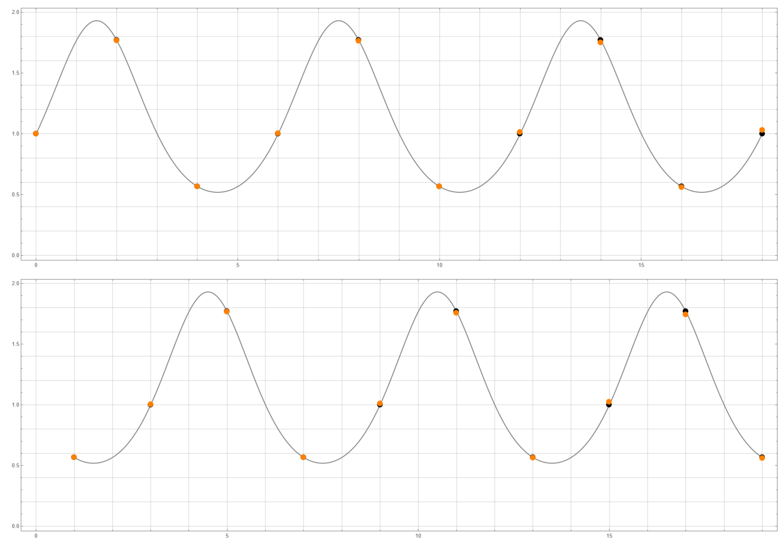

FIGURE 1. Elliptic behaviour example A
Parameters and initial conditions are $a = f_0 = t_0 = 1$, $E_0 = 20/3$ and $\epsilon = 1/1000$. The top and bottom plots distinguish between even and odd iterates of f_n. The orange points represent the numerically computed sequence f_n solving Equation (3.27), and the black points are given by Theorem 3.40.

Utilising Theorem 3.40, we proceed to define appropriate continuous functions $f(x)$, $E(x)$, $L(x)$, which will be important to Subsection 3.3.2.

Definition 3.41. *Following Lemma 3.38 and Theorem 3.40, we define the continuous function $f : \mathbb{R} \to \mathbb{C}$ so that $f(n) = f_n$ for bounded $n \in \mathbb{N}$, and*

$$f(x) \to \frac{c + A\,sn(z_0 + px)}{c - A\,sn(z_0 + px)}, \quad \epsilon \to 0,$$

for bounded $x \in \mathbb{R}$. We likewise define corresponding continuous functions $E(x)$ and $L(x)$ (see Definition 3.28).

In Figures 1 and 2, we show examples of numerically computed sequences f_n satisfying Equation (3.27) and the corresponding approximations given by Theorem 3.40.

We proceed to discuss the periodicity of $G(z) := A\,\mathrm{sn}(z;k)$, this having two fundamental periods which are linearly independent in the complex z-plane, given $k \neq 0, \pm 1$.

Definition 3.42. *Given the function $G(z) := A\,sn(z;k)$, where $A^2 = k$ and $0 < |k| < 1$, we define constants $\Omega_1, \Omega_2 \in \mathbb{C}^*$ such that $G(z + \Omega_j) = G(z)$ for all $z \in \mathbb{C}$ and $j \in \{1, 2\}$. Furthermore, $\Omega_1 \neq r\Omega_2$ for all $r \in \mathbb{R}^*$.*

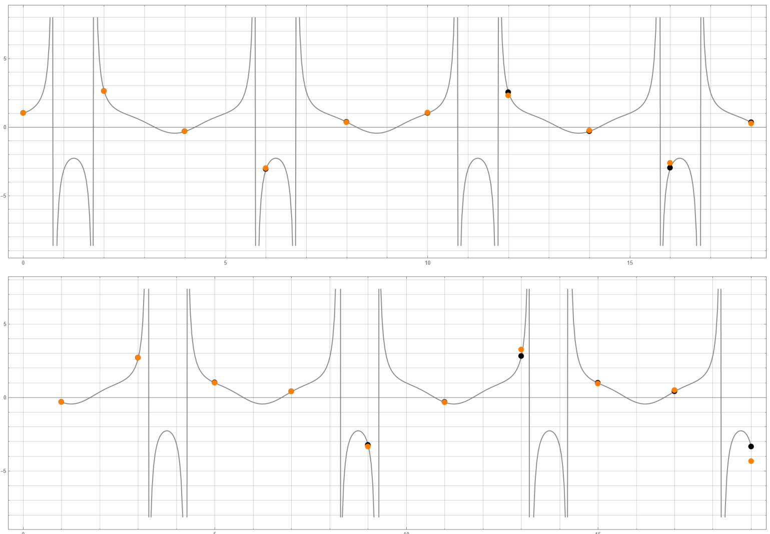

FIGURE 2. Elliptic behaviour example B
Parameters and initial conditions are $a = f_0 = 1$, $t_0 = -\sqrt{2}$, $E_0 = -1.4$ and $\epsilon = 1/1000$. The top and bottom plots distinguish between even and odd iterates of f_n. The orange points represent the numerically computed sequence f_n solving Equation (3.27), and the black points are given by Theorem 3.40.

Corollary 3.43. *The constants Ω_1, Ω_2 described in Definition 3.42 are given by the complete elliptic integral of the first kind*

$$\Omega(k) := \oint \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}}. \tag{3.44}$$

These may be expanded in powers of $m := k^2$ as follows

$$\Omega_1(m) = 2h(m), \quad \Omega_2(m) = ih(1-m), \quad h(m) := \sum_{j=0}^{\infty} \frac{\Gamma(j+1/2)^2}{\Gamma(j+1)^2} m^j.$$

Proof. Let $G(z) := A\,\mathrm{sn}(z;k)$, where $A^2 = k$ and $0 < |k| < 1$. For all $z \in \mathbb{C}$, it is well known that the function $G(z)$ satisfies

$$\left(\frac{dG}{dz}\right)^2 = \left(k - G^2\right)\left(1 - kG^2\right).$$

The two independent periods of $G(z)$ correspond to the two independent closed contours associated with the elliptic integral of the first kind

$$\Omega(k) := \oint \frac{\mathrm{d}G}{\sqrt{(k-G^2)(1-kG^2)}} = \oint \frac{\mathrm{d}u}{\sqrt{(1-u^2)(1-k^2u^2)}}.$$

Letting prime denote differentiation with respect to k, we find that the integral Ω satisfies the ODE

$$k(1-k^2)\Omega'' + (1-3k^2)\Omega' - k\Omega = 0.$$

Converting to the parameter $m = k^2$ (and prime now denotes differentiation with respect to m) gives the canonical hypergeometric differential equation

$$m(1-m)\Omega'' + (1-2m)\Omega' - \frac{1}{4}\Omega = 0.$$

This ODE has regular singular points at $m \in \{0,1\}$. Furthermore, it is invariant under the transformation $m \to 1-m$. Linearly independent solutions are given by

$$\begin{aligned} h_1(m) =& \sum_{j=0}^{\infty} \frac{\Gamma(j+1/2)^2}{\Gamma(j+1)^2} m^j, \\ h_2(m) =& \log(m) \sum_{j=0}^{\infty} \frac{\Gamma(j+1/2)^2}{\Gamma(j+1)^2} m^j \\ &+ 2\sum_{j=1}^{\infty} \frac{\Gamma(j+1/2)^2}{\Gamma(j+1)^2} \left(\psi(j+1/2) - \psi(j+1) + 2\log(2)\right) m^j, \end{aligned}$$

where we denote the usual gamma function $\Gamma(z)$ and the digamma function $\psi(z) := \Gamma'(z)/\Gamma(z)$. Expansions around $m = 1$ are obtained by simply transforming $m \to 1-m$. □

3.3.2. Averaging method. This section considers the order ϵ perturbation associated with the leading-order autonomous equation. We approximate the variation in E_n over a fundamental period of the leading-order elliptic behaviour in terms of complete elliptic integrals.

We first give an expression for the $\mathcal{O}(\epsilon)$ part of $E_n - E_0$ for arbitrary n, and provide a bound on smaller order terms which we denote by M_n.

Lemma 3.45. *For any $n \in \mathbb{N}$ we have*

$$E_n - E_0 = -\left(nL_n - \sum_{k=0}^{n-1} L_k\right) t_0\epsilon + M_n, \tag{3.46}$$

where

$$|M_n| < 2|t_0|J_n\left(e^{|\epsilon|n} - 1 - |\epsilon|n\right) = \mathcal{O}\left(|\epsilon|^2 n^2\right), \quad |\epsilon|n \ll 1.$$

Proof. Defining $\tilde{P}_n = P_n - t_0\epsilon n$, following Equation (3.33) we have

$$E_k - E_{k-1} = -t_0\epsilon(L_k - L_{k-1})k - (L_k - L_{k-1})\tilde{P}_k, \quad \forall k \in \mathbb{N}.$$

Therefore, for arbitrary $n \in \mathbb{N}$, we sum the above equations to obtain

$$E_n - E_0 = -\left(nL_n - \sum_{k=0}^{n-1} L_k\right) t_0\epsilon + M_n,$$

where, similar to Equation (3.35), we express M in the form

$$M_n = -\sum_{k=1}^{n}(L_k - L_{k-1})\tilde{P}_k = -\tilde{P}_n L_n + t_0\epsilon \sum_{k=0}^{n-1} L_k \left(\frac{\tilde{P}_k}{t_0} + \epsilon k\right).$$

Then using the fact that $|\tilde{P}_n| < |t_0|(e^{|\epsilon|n} - 1 - |\epsilon|n)$ for all $n \in \mathbb{N}$, we see that

$$\begin{aligned} |M_n| <& |t_0| J_n \left\{ \left(e^{|\epsilon|n} - 1 - |\epsilon|n\right) + |\epsilon| \sum_{k=0}^{n-1} \left(e^{|\epsilon|k} - 1\right) \right\} \\ <& 2|t_0| J_n \left(e^{|\epsilon|n} - 1 - |\epsilon|n\right), \end{aligned}$$

as required. □

We shall apply the averaging method, deducing an approximation for the variation in E over a fundamental period of the leading-order elliptic behaviour. For $a = 1$ and all $0 \le n \ll 1/|\epsilon|$, as $\epsilon \to 0$ we have coordinates (f_{n+1}, f_n) confined to a level curve of the surface

$$E_0(x, y) = \frac{x^2y^2 + t_0xy(x + y) + t_0(x + y) + 1}{xy}.$$

In what proceeds, we suppose that there is some $\eta \in \mathbb{N}$ which is a *near-period* of the discrete function f_n. That is, η is close to a finite period of the leading order elliptic behaviour, given by Integral (3.44) with some non-trivial closed contour.

Theorem 3.47. *Let $a = 1$ and assume $|L^{(k)}(0)| \le R$ for all $k \in \mathbb{N}$ and some finite $R \in \mathbb{R}^+$. Furthermore, suppose we have $\eta \in \mathbb{N}$ such that $|\eta - \Omega/p| \ll 1$, where $\Omega(k)$ is a period of the leading-order elliptic function given by Integral (3.44). Then, regarding the difference $E_\eta - E_0$ we obtain*

$$E_\eta - E_0 \sim -\frac{t_0\epsilon}{p}\left\{(4 + L_0)\Omega(k) - 8\Pi(k/c^2, k)\right\} + S_\eta, \quad \epsilon \to 0,$$

where $\Pi(\alpha^2, k)$ is the complete elliptic integral of the third kind with modulus k and parameter α^2, and

$$\begin{aligned} |S_\eta| <& R|t_0\epsilon| \left\{ |\eta - \Omega/p| e^{|\eta - \Omega/p|} + (|\Omega/p| + 1)\left(e^{|\eta-\Omega/p|} - 1\right)\right\} \\ =& \mathcal{O}\left(|\epsilon||\eta - \Omega/p|\right). \end{aligned}$$

Proof. For finite $\eta \in \mathbb{N}$, in Lemma 3.45 we showed that

$$E_\eta - E_0 \sim -\left((\eta + 1)L_\eta - L_0 - \sum_{k=1}^{\eta} L_k\right) t_0\epsilon, \quad \epsilon \to 0.$$

We apply the Euler-Maclaurin formula to the sum in the above equation, seeing that

$$\sum_{k=1}^{\eta} L(k) = \int_0^{\Omega/p} L(x)dx + \int_{\Omega/p}^{\eta} L(x)dx + \sum_{k=1}^{\infty} \frac{B_k}{k!}\left(L^{(k-1)}(\eta) - L^{(k-1)}(0)\right),$$

where B_k, for $k \in \mathbb{N}$, are the Bernoulli numbers. Now, we utilise the expansion

$$L^{(k)}(\eta) = \sum_{j=k}^{\infty} \frac{L^{(j)}(\Omega/p)}{(j-k)!}(\eta - \Omega/p)^{j-k}, \quad k \in \mathbb{N},$$

also noting that $L^{(j)}(\Omega/p) \to L^{(j)}(0)$ as $\epsilon \to 0$ for all $j \in \mathbb{N}$. Applying these concepts, we obtain

$$E_\eta - E_0 \sim -t_0\epsilon\left(\frac{\Omega}{p}L_0 - \int_0^{\Omega/p} L(x)dx\right) + S_\eta, \tag{3.48}$$

where

$$S_\eta = -\, t_0\epsilon \sum_{k=1}^{\infty} \frac{\theta_k}{k!}(\eta - \Omega/p)^k,$$

and

$$\theta_k = (k-1)L^{(k-1)}(0) + (\Omega/p + 1)\, L^{(k)}(0) - \sum_{j=1}^{\infty} \frac{B_j}{j!} L^{(j+k-1)}(0).$$

Using the assumption that $|L^{(k)}(0)| \le R$ for all $k \in \mathbb{N}$, and also noting

$$\sum_{j=1}^{\infty} |B_j/j!| < 1,$$

we may verify that

$$|S_\eta| < R|t_0\epsilon| \left\{ |\eta - \Omega/p| e^{|\eta - \Omega/p|} + (|\Omega/p| + 1)\left(e^{|\eta-\Omega/p|} - 1\right)\right\} \ll |\epsilon|,$$

for $|\eta - \Omega/p| \ll 1$. Regarding the integral in Equation (3.48), recall that in the $a = 1$ case we have simply

$$L(x) = f(x) + \frac{1}{f(x)} + f(x+1) + \frac{1}{f(x+1)},$$

and the continuous behaviour of $f(x)$ as $\epsilon \to 0$ is stated in Definition 3.41. Then, since the integral is across a period of the leading-order elliptic behaviour, we find that

$$\begin{aligned}\int_0^{\Omega/p} L(x)dx =& \frac{4}{p}\oint \left(\frac{1 + ku^2/c^2}{1 - ku^2/c^2}\right) \frac{\mathrm{d}u}{\sqrt{(1-u^2)(1-k^2u^2)}} \\ =& \frac{4}{p}\left(2\Pi(k/c^2, k) - \Omega(k)\right),\end{aligned}$$

where $\Pi(k/c^2, k)$ is a standard notation for the complete elliptic integral of the third kind, with parameter k/c^2 and modulus k (see [18] for a useful text on elliptic integrals and their identities). Applying this to Equation (3.48) completes the proof. □

3.3.3. Critical points. In the $a = 1$ case, we find that critical points $k = \pm 1$ are given by two potential conditions in terms of E_0: (a) $2 + E_0 + t_0^2 = 0$ and (b) $|E_0| \to \infty$. On the other hand, we find that the three conditions give $k = 0$: (a) $2 + E_0 = 0$, (b) $2 - E_0 + 4t_0 = 0$ and (c) $2 - E_0 - 4t_0 = 0$. We proceed to give a summary of expansions for f_n as E_0 approaches each of these critical points, whereby f is now degenerating from elliptic to constant or singly-periodic in the limit $\epsilon \to 0$. Here we may apply known results regarding the elliptic sine $\mathrm{sn}(z; k)$ as the modulus k approaches $0, \pm 1$, recalling that for bounded $n \in \mathbb{N}$ we have

$$f_n \sim g(z) := \frac{c + A\,\mathrm{sn}(z;k)}{c - A\,\mathrm{sn}(z;k)}, \quad z := z_n = z_0 + pn, \quad \epsilon \to 0.$$

These degenerate behaviours correspond to certain algebraic curves in the initial value space, those being level curves of the surface given by

$$E_0(f_0, f_1) = \frac{f_1^2 f_0^2 + t_0 f_1 f_0 (f_1 + f_0) + t_0(f_1 + f_0) + 1}{f_1 f_0},$$

where the specific level curves in question include (or are solely) a critical point of the surface.

In the $k^2 \to 1$ cases, we obtain critical behaviours:

$$g(z) = \frac{1 + \tanh(z)}{1 - \tanh(z)} - \frac{4t_0 \tanh(z)}{(\tanh(z) - 1)^2} E_0^{-1} + \mathcal{O}\left(E_0^{-3/2}\right), \quad |E_0| \to \infty,$$

and

$$g(z) = \frac{c_0 + \tanh(z)}{c_0 - \tanh(z)} + \mathcal{O}\left(\delta^{1/2}\right), \quad \delta \to 0,$$

$$\text{where} \quad \delta = E_0 + 2 + t_0^2, \quad c_0^4 = \frac{(t_0 - 2)^2}{(t_0 + 2)^2}.$$

In the $k^2 \to 0$ cases, we obtain critical behaviours:

$$g(z) = 1 + \frac{2A_0 \sin(z)}{c_0} \delta^{1/2} + \frac{2A_0^2 \sin^2(z)}{c_0^2} \delta + \mathcal{O}\left(\delta^{3/2}\right), \quad \delta \to 0,$$

$$\text{where} \quad \delta = E_0 + 2, \quad c_0^4 = \frac{1 - t_0}{1 + t_0}, \quad A_0^4 = -\frac{(t_0 + 1)(t_0 - 1)}{16 t_0^4},$$

and

$$g(z) = 1 + \frac{2A_0\sin(z)}{c_0}\delta^{1/2} + \frac{2A_0^2\sin^2(z)}{c_0^2}\delta + \mathcal{O}\left(\delta^{3/2}\right), \quad \delta \to 0,$$

$$\text{where} \quad \delta = E_0 - 2(1+2t_0), \quad c_0^4 = 8t_0, \quad A_0^4 = \frac{(t_0+1)^2}{2t_0(t_0+2)^2},$$

and

$$g(z) = \frac{c_0 + A_0\sin(z)}{c_0 - A_0\sin(z)} + \mathcal{O}(\delta), \quad \delta \to 0,$$

$$\text{where} \quad \delta = E_0 - 2(1-2t_0), \quad c_0^4 = -\frac{1}{8t_0}, \quad A_0^4 = -\frac{(t_0-1)^2}{2t_0(t_0-2)^2}.$$

Remark 3.49. We note that the above cases with $g \to 1$ correspond to perturbations about the exact solution $f(t) \equiv 1$ when $a = 1$, which is the seed of a known hierarchy of rational solutions which exist for parameter values $a = q^{4k}$, $k \in \{0, 1, 2, \dots\}$.

3.4. Summary

We began this chapter by performing local asymptotic analysis of generic solutions $f(t)$ of q-$\mathrm{P_{II}}$ in both limits $|t| \to \infty$ and $t \to 0$. We saw that these asymptotic behaviours are expressible in terms of leading-order invariant quantities. Consequently, solutions are restricted to certain algebraic curves in the phase plane of $(f(t), f(qt))$ for t in these limits. Parameterised by leading-order conserved quantity (summation constant) E, these are level curves of a surface, the critical points of which correspond to "near-stationary" solutions, where $f(t)$ is given asymptotically by a power-series expansion. We also deduce expressions describing the slow evolution of quantity E under repeated iteration $t \to qt$.

In particular, in the limit $|t| \to \infty$, we found that iterates of generic $f(t)$ belong to genus-one (elliptic-type) algebraic curves. Meanwhile, in the limit $t \to 0$, we deduced generic behaviour by decomposing q-$\mathrm{P_{II}}$ into a system of even and odd iterates of $f(t)$, finding then that iterates belong asymptotically to genus-zero algebraic curves (parameterised trigonometrically).

Beyond these generic behaviours, we found in the limit $|t| \to \infty$ that q-$\mathrm{P_{II}}$ is satisfied by other distinct dominant balances; these corresponding to base-points of the surface mentioned above, and q-$\mathrm{P_{II}}$ is formally satisfied here by a divergent asymptotic series expansion (with no free parameters). Indeed, mirroring the differential case, these expansions are characterised by beyond-all-orders perturbation terms multiplied by a free constant, a discrete version of the Stokes multiplier. In Section 3.2, we studied these divergent expansions and discerned the asymptotic behaviour of coefficients in these series. We proceed to prove there exist true solutions asymptotic to these series in specific domains of the t-plane; these are termed quicksilver solutions following the work of Joshi in [72] and are considered the q-difference analogues of Boutroux's tronqueé solutions.

Further following a path that parallels asymptotic analysis of the continuous Painlevé equations, we consider an asymptotic expansion of solution $f(t)$ on the boundary of validity of the discussed power-series behaviour (which in this discrete case corresponds to q-spirals in the t-plane). To the best of our knowledge, this is the first time an expansion of this kind (see Equation (3.25)) has been attributed to the solution of a q-difference Painlevé equation. This oscillatory-type expansion is, in principle, a generalised Fourier series, mirroring the form we see in the continuous case (for example, in the case of $\mathrm{P_{IV}}$ see Equation (4.2)).

In Section 3.3, we gave a generic transcendental solution $f(t)$ of the q-difference second Painlevé equation to leading order by elliptic functions in a bounded region near an arbitrary point in the t-plane, achieved by considering the asymptotic limit $\epsilon \to 0$ where the independent variable is $t = t_0(1+\epsilon)^n$. Furthermore, by a method of averaging analogous to what we see for continuous Painlevé equations in, for instance, [77], we studied the slow evolution of this leading-order elliptic behaviour as we traverse an edge of the associated period parallelogram; this slow modulation is given in terms of complete elliptic integrals.

Future questions include how this analysis applies to other q-difference Painlevé equations, particularly considering that analogous results between $\mathrm{P_I}$ to $\mathrm{P_V}$ share a remarkable similarity when considering their generic elliptic behaviours [77]. Further, little exploration in the literature currently exists concerning elliptic-type behaviours associated with more complex q-difference Painlevé equations, potentially in a different asymptotic limit to what has been considered here. We deduced novel oscillatory-type (anti-Stokes-like) expansions in the case of q-$\mathrm{P_{II}}$, strengthening the deep connection between asymptotic behaviours of continuous and q-difference Painlevé transcendents.

CHAPTER 4

Symmetric Solutions and Isomonodromic Deformation

This chapter considers the linear monodromy problem associated with solutions of the continuous and q-difference fourth Painlevé equations, respectively. In particular, we focus on symmetric solutions of these equations, where the corresponding monodromy data may be explicitly calculated. Symmetric solutions exist for general parameter values of the Painlevé equation, occurring for particular initial values at a reflection point, this being $x = 0$ in the case of P_{IV} and $t = i$ in the case of q-P_{IV}.

With explicitly known, conserved monodromy data in mind, we consider the deformation of these associated linear systems: taking the asymptotic limit $x \to \infty$ in the case of P_{IV}, and $t \to 0$ in the case of q-P_{IV}. We note here that for the q-difference equation, we could equivalently consider $t \to \infty$; the symmetry condition relates behaviours as $t \to 0$ and $t \to \infty$, both of these limits are analogous to $|x| \to \infty$ in the continuous case. Indeed, from the perspective of the additive discrete variable n, where $t = iq^n$ and $0 < |q| < 1$, we have $t \to 0$ corresponding to $n \to +\infty$ while $t \to \infty$ corresponds to $n \to -\infty$.

By deforming the linear systems in this way, we may express the asymptotic behaviour of symmetric Painlevé transcendents (as $|x| \to \infty$ and $t \to 0$) in terms of the conserved monodromy data. In doing this, we obtain certain conditions in terms of the parameters of the Painlevé equations, these determining the asymptotic behaviour of the symmetric transcendents away from the reflection points, also including cases where the symmetric solution is tronquée (or analogously a quicksilver solution in the discrete case). This isomonodromic approach to connection formulae is novel in the q-difference case.

4.1. Symmetric Solutions of Painlevé IV

Here, we consider the fourth Painlevé transcendent $y(x)$ as a monodromy preserving deformation of a system of linear differential equations [65]. In the case of a symmetric solution of P_{IV}, introduced in [90], such preserved monodromy data (i.e., constant in x) is known explicitly. Thus, we consider the associated linear problem in the asymptotic limit $|x| \to \infty$ to discern information regarding the behaviour of the symmetric Painlevé transcendent in this limit. Such methods are applied to the first and second Painlevé equations in, for instance, [91, 94].

4.1.1. Local asymptotic analysis of P_{IV}. We shall inform our approach to the isomonodromic deformation problem by first carrying out local asymptotic analysis of the scalar fourth Painlevé equation

$$P_{IV}: \qquad 2y\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2 + 3y^4 + 8xy^3 + 4(x^2 - 2b + 1)y^2 - 16a^2,$$

as $|x| \to \infty$. Specifically, we consider *anti-Stokes behaviours*. In this endeavour, we should consider leading order algebraic behaviours as $|x| \to \infty$. The equation P_{IV} is formally satisfied by four distinct power-series expansions around $x = \infty$, each being a divergent asymptotic series in descending powers of x^2. Giving a few terms in these series:

$$\begin{aligned}
&1. \qquad Y(x) = -2x + \frac{1-2b}{x} + \frac{1-a^2-3b+3b^2}{x^3} + \mathcal{O}\left(1/x^5\right),\\
&2. \qquad Y(x) = -\frac{2x}{3} - \frac{1-2b}{x} - \frac{1-9a^2-3b+3b^2}{x^3} + \mathcal{O}\left(1/x^5\right),\\
&3. \qquad Y(x) = \frac{2a}{x} - \frac{a+4a^2-2ab}{x^3} + \mathcal{O}\left(1/x^5\right),\\
&4. \qquad Y(x) = -\frac{2a}{x} + \frac{a-4a^2-2ab}{x^3} + \mathcal{O}\left(1/x^5\right).
\end{aligned} \tag{4.1}$$

We seek to generalise these behaviours to the anti-Stokes rays in the complex x-plane, where oscillatory exponential effects are incorporated. Due to their purely oscillatory nature, the exponential terms interact with the above series; we will see in the vanishing cases (3 and 4) that not even the leading order term above is preserved but depends on a product of Stokes multipliers.

Essentially, we consider the generalised Fourier series

$$y(x) = x^\alpha Y(x) + x^\beta \sum_{k=1}^{\infty} x^{\gamma k}\left(P_k(x)e^{S(x)k} + Q_k(x)e^{-S(x)k}\right). \tag{4.2}$$

Exponents α, β, γ are some constants to be determined where we assume $x^\gamma \to 0$ as $|x| \to \infty$. The functions $Y(x)$, and $P_k(x), Q_k(x)$ for $k \in \{0, 1, 2, \dots\}$, are once again power-series in descending powers of x^2, and are constant as $|x| \to \infty$. We denote $P_1(x) \to p$ and $Q_1(x) \to q$ as $|x| \to \infty$, with p and q being the two free constants (Stokes multipliers) in these general behaviours. This line of investigation leads to the following two-parameter families of solutions.

In Case 1 of Equations (4.1), the constants in Equation (4.2) are $\alpha = \beta = 1$ and $\gamma = -1$, and $S(x) = x^2 - (2-4b+3pq)\ln x$. We thus see that the anti-Stokes lines, on which we consider this behaviour, are given by $\arg x = \pi(1+2n)/4$ for

$n \in \mathbb{Z}$. The assumption that $x^\gamma \to 0$ has been corroborated. However, the $\ln x$ term within the function $S(x)$ implies that $|\Re(2-4b+3pq)| < 1$ in order to maintain asymptoticity. We denote $\delta := x^{|\Re(2-4b+3pq)|-1}$ to conveniently describe the orders of neglected terms. The aforementioned condition implies $\mathcal{O}(1/x) \leq \delta < \mathcal{O}(1)$ as $|x| \to \infty$, and then we may express the solution

$$y(x) = -2x + pe^{S(x)} + qe^{-S(x)} + \mathcal{O}\left(1/x, x\delta^2\right). \tag{4.3}$$

We would expect that Case 2 has a similar structure to Case 1, and indeed, we still obtain $\alpha = \beta = 1$ and $\gamma = -1$. Although, we now have $S(x) = i(x^2 - 3pq\ln x)/\sqrt{3}$, and thus the anti-Stokes rays are given by $\arg x = n\pi/2$ for $n \in \mathbb{Z}$. From $S(x)$ we again have a corresponding constraint $\sqrt{3}|\Im(pq)| < 1$. Now defining $\delta := x^{\sqrt{3}|\Im(pq)|-1}$ with $\mathcal{O}(1/x) \leq \delta < \mathcal{O}(1)$, we similarly obtain the leading order

$$y(x) = -\frac{2x}{3} + pe^{S(x)} + qe^{-S(x)} + \mathcal{O}\left(1/x, x\delta^2\right). \tag{4.4}$$

Cases 3 and 4 are very much distinct from Cases 1 and 2. Interestingly, the (algebraic) leading order term in the expansion depends on the product of Stokes multipliers pq. Here, the constants in Equation (4.2) are $\alpha = -1$, $\beta = 1$ and $\gamma = -2$. Denoting the constant $r := 2\sqrt{a^2 + pq}$, we have $S(x) = x^2 + (1 - 2b + 3r)\ln x$. So, as in Case 1, the anti-Stokes rays are given by $\arg x = \pi(1+2n)/4$ for $n \in \mathbb{Z}$, and now we have the constraint $|\Re(1-2b+3r)| < 2$. Thus, defining $\delta = x^{|\Re(1-2b+3r)|-2}$ where $\mathcal{O}(1/x^2) \leq \delta < \mathcal{O}(1)$, we obtain

$$y(x) = \frac{r}{x} + \frac{p}{x}e^{S(x)} + \frac{q}{x}e^{-S(x)} + \mathcal{O}\left(1/x^3, \delta/x, x\delta^2\right). \tag{4.5}$$

Case 4 is given by simply transforming $r \to -r$ in Case 3.

Taking the case of p or q equal to zero in these expansions corresponds to a *tronqueé solution*, with the expansion becoming valid on one side of the anti-Stokes line (where the remaining exponential term decays), in a sector extending to the following anti-Stokes line. When $p = q = 0$, we obtain the unique *tritronqueé solution*, with the expansion valid on both sides of the anti-Stokes line, in a sector extending to the following anti-Stokes line in both directions. As one would hope, we see that when p or q equals zero, we recover the power series in Equation (4.1), along with an exponentially small perturbation.

4.1.2. Symmetric solutions of $\mathrm{P_{IV}}$. The symmetric solutions of $\mathrm{P_{IV}}$, outlined in [90], stem from the fact $\mathrm{P_{IV}}$ is invariant under the transformation $x \to -x$ and $y \to -y$. Indeed, one may construct a solution $y(x)$ of $\mathrm{P_{IV}}$ which is odd, that is, $y(-x) = -y(x)$. These solutions are holomorphic at the origin (the reflection point), determined here by their Taylor expansion; we have within some radius of convergence

$$y(x) = \sum_{n=0}^{\infty} c_n x^{2n+1}, \quad c_0 = 4a, \quad c_1 = \frac{8}{3}a(1-2b).$$

For for all $n > 1$, the coefficient c_n is determined by

$$\begin{aligned} &8a(2n+1)(2n-1)c_n \\ &= -4\sum_{j=1}^{n-1} c_j c_{n-j}(2n-2j+1)(n-j) + \sum_{j=1}^{n-1} c_j c_{n-j}(2j+1)(2n-2j+1) \\ &\quad + 3\sum_{j=0}^{n-2}\sum_{k=0}^{n-j-2}\sum_{l=0}^{n-j-k-2} c_j c_k c_l c_{n-j-k-l-2} + 8\sum_{j=0}^{n-2}\sum_{k=0}^{n-j-2} c_j c_k c_{n-j-k-2} \\ &\quad + 4\sum_{j=0}^{n-2} c_j c_{n-j-2} + 4(1-2b)\sum_{j=0}^{n-1} c_j c_{n-j-1}. \end{aligned}$$

4.1.3. Direct monodromy problem. In accordance with [65], a solution $y(x)$ of P_{IV} represents a monodromy preserving deformation of the associated (2×2) linear system

$$\Psi'(\xi) = A(\xi)\Psi(\xi), \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{4.6}$$

where $A_{11} = -A_{22}$ and

$$A_{11} = \xi + x + \frac{a-z}{\xi}, \quad \frac{2}{f}A_{12} = 2 - \frac{y}{\xi}, \quad \frac{f}{2}A_{21} = z - a - b + \frac{z(z-2a)}{y\xi},$$

for functions (of the Painlevé independent variable) $z(x)$ and $f(x)$ satisfying

$$4z = y^2 + 2xy - \frac{dy}{dx} + 4a \quad \text{and} \quad \frac{df}{dx} = -(y+2x)f. \tag{4.7}$$

Taking an entry $\psi(\xi)$ of the solution of this system, and making the convenient transformation $\psi(\xi) = \phi(\xi)\sqrt{(y-2\xi)/(-2\xi)}$, leads to a scalar, second-order ODE of the form $\phi''(\xi) = P(\xi)\phi(\xi)$. Motivated by expansions (4.3), (4.4) and (4.5), we should specify the sizes of y and z as $|x| \to \infty$ according to the following cases:

$$\begin{aligned} &1. \quad y = -2x + u, \quad z = xv, \\ &2. \quad y = -2x/3 + u, \quad z = -2x^2/9 + xv, \\ &3. \quad y = u/x, \quad z = v. \end{aligned} \tag{4.8}$$

After scaling the independent variable of the linear problem by $\xi = -x\lambda$, we obtain the modified ODE $\epsilon^2\phi''(\lambda) = Q(\lambda)\phi(\lambda)$, where we denote small parameter $\epsilon := 1/x^2$ and $Q(\lambda)$ may be expanded in the form $Q(\lambda) = \sum_{k=0} Q_k(\lambda)\epsilon^k$. Each

Q_k is constant in the limit $\epsilon \to 0$, and Q_0 is non-zero. Here, we see a generalisation of the classical WKB problem; one might review, for instance, [9].

Consider the transformation $\phi(\lambda) = g(\lambda)\omega(h(\lambda))$ where

$$g(\lambda) = R(\lambda)^{-1/4}, \quad h'(\lambda) = R(\lambda)^{1/2}, \quad R(\lambda) = Q_0(\lambda) + Q_1(\lambda)\epsilon.$$

We thus obtain the equation

$$\epsilon^2\omega_{hh} = \left(1 + \mu\epsilon^2\right)\omega, \quad \mu = \frac{1}{4}\frac{R''}{R^2} - \frac{5}{16}\frac{(R')^2}{R^3} + \frac{1}{R}\sum_{k=0} Q_{k+2}\epsilon^k, \tag{4.9}$$

which is a slight generalisation (since μ is given in powers of ϵ) of equations studied extensively by Olver [122, Chapter 10]. It is then straightforward to discern that Equation (4.9) is formally satisfied by the two series $\omega_\pm = e^{\pm h/\epsilon}\sum_{k=0} A_k\epsilon^k$ with $A_0 = 1$, and for all $k \geq 0$ the functions A_k satisfy

$$\mu A_k = \pm 2\frac{d}{dh}A_{k+1} + \frac{d^2}{dh^2}A_k.$$

Since we will only be interested in these expansions' leading order, consider the solutions $\omega_\pm = e^{\pm h/\epsilon} + \delta_\pm$. Then we obtain $\delta_\pm$ satisfying the nonhomogeneous equation

$$\frac{d^2\delta_\pm}{dh^2} - \left(\frac{1}{\epsilon^2} + \mu\right)\delta_\pm = \pm 2e^{\pm h/\epsilon}\frac{dA_1}{dh},$$

and it follows by Olver's Theorem 3.1 [122, Chapter 10, §3] that $\delta_\pm$ satisfies

$$|\delta_\pm| \leq \left|e^{\pm h/\epsilon}\right| \exp\left\{V\right\}V, \quad V = |\epsilon|\int_\sigma^h |\mu(v)dv|, \tag{4.10}$$

for reference point σ, given that the real part of $\pm v/\epsilon$ is non-decreasing as v travels along a contour from σ to h.

Considering Cases 1 and 3 of (4.8), we find that Q_0 and Q_1 have a relatively simple common form, this being

$$Q_0(\lambda) = (\lambda - 1)^2 \quad \text{and} \quad Q_1(\lambda) = 1 - 2b + \alpha/\lambda,$$

where α is constant with respect to λ, depending on potentially a, b, u, v. We therefore consider the function

$$H(\lambda) = \exp\left\{\frac{1}{\epsilon}\left(\lambda^2/2 - \lambda\right) + \frac{1}{2}(1 - 2b + \alpha)\log(\lambda - 1) - \frac{1}{2}\alpha\log(\lambda)\right\}. \tag{4.11}$$

The argument of the above exponential is an anti-derivative of $h'(\lambda)/\epsilon$ to leading order, where we have expanded $R^{1/2}$ in powers of ϵ, i.e. $\omega = H^{\pm 1}(1 + o(1))$ is valid given $|\lambda - 1| \gg 1/|x|$ as $|x| \to \infty$ (this condition also agrees with the error control described in (4.10)).

By Olver's theorem (recalling (4.10)), we will consider solutions along paths in the complex λ-plane where $\Re(H)$ is constant. Since we are considering Case 1 or 3 of (4.8), we set $\arg(x) = \pi/4$, and thus $\arg(\epsilon) = -\pi/2$. Then for $|\lambda| \gg 1$, these desirable contours converge to the rays given by $\arg\lambda = n\pi/2$ for $n \in \mathbb{Z}$. On the other hand, consider an annular region close to the turning point at $\lambda = 1$, making the transformation of independent variable $\zeta := x\sqrt{2}(\lambda - 1)$ where $1 \ll |\zeta| \ll |x|$. In this region of the ζ-plane, the aforementioned valid contours of integration emanate from the turning point along rays asymptotically given by $\arg\zeta = (2n + 1)\pi/4$ for $n \in \mathbb{Z}$.

Getting back to Equation (4.11), we see that $H \to e^c e^{\theta(\xi)}$ as $|\lambda| \to \infty$ and $H \to e^d e^{\tau(\zeta)}$ for $1 \ll |\zeta| \ll |x|$, where we denote the following useful functions and constants

$$\theta(\xi) = \xi^2/2 + x\xi + (1 - 2b)\log(\xi)/2, \quad \tau(\zeta) = \zeta^2/4 + \gamma\log(\zeta),$$
$$\gamma = \frac{1 - 2b + \alpha}{2}, \quad c = -(1 - 2b)\log(-x)/2,$$
$$\text{and} \quad d = -x^2/2 - \gamma\log(x) - \gamma\log(2)/2.$$

Indeed, the variable ζ was specifically chosen to give the canonical form of the parabolic cylinder function near the turning point; one may verify that at leading order in this region ϕ satisfies the parabolic cylinder equation

$$\frac{d^2\phi}{d\zeta^2} = \left(\frac{1}{4}\zeta^2 + \gamma + o(1)\right)\phi.$$

Solutions of the parabolic cylinder equation are well-understood, classical special functions, view for instance [37, §12.9(i)]. We recall the fundamental pair of solutions $U(\gamma, \zeta)$ and $U(-\gamma, -i\zeta)$; these are entire functions of both argument ζ and parameter γ, where $U(\gamma, \zeta)$ for $1 \ll |\zeta|$ satisfies

$$\zeta^{1/2}U(\gamma, \zeta) \to e^{-\tau(\zeta)}, \quad |\arg\zeta| < 3\pi/4,$$
$$\zeta^{1/2}U(\gamma, \zeta) \to e^{-\tau(\zeta)} \pm \frac{i\sqrt{2\pi}}{\Gamma\left(\frac{1}{2} + \gamma\right)} e^{\mp i\pi\gamma} e^{\tau(\zeta)}, \quad \pi/4 < \pm\arg\zeta < 5\pi/4.$$

Definition 4.12. *Recall the changes of variables $\xi = -x\lambda$ and $\zeta = x\sqrt{2}(\lambda - 1)$, where furthermore $|x| \gg 1$ with $\arg x = \pi/4$. We define arcs $\mathcal{T}_j$, $j \in \{2, 4\}$, in the complex λ-plane where $|\lambda - 1| \gg 1/|x|$ and $\Re(H)$ is constant as discussed above. Moreover, $\mathcal{T}_2$ is a domain asymptotically converging to the ray $\arg\xi = 3\pi/4$ as $|\lambda| \to \infty$, and the ray $\arg\zeta = -\pi/4$ as $\lambda \to 1$. Similarly, $\mathcal{T}_4$ converges to the ray $\arg\xi = 7\pi/4$ as $|\lambda| \to \infty$ and $\arg\zeta = 3\pi/4$ as $\lambda \to 1$.*

Definition 4.13 (Fundamental solutions at infinity)**.** *Suppose we define solutions $\omega_\pm^{(j)}$, $j \in \{2, 4\}$, of Equation* (4.9) *such that $\omega_\pm^{(j)}(\lambda) \sim H^{\pm 1}(\lambda)$ for $\lambda \in \mathcal{T}_j$.*

Using the fact that solutions are given asymptotically by parabolic cylinder functions in an annular region near the turning point, we may determine explicit relationships between fundamental solutions, which were defined via their expansions in $\mathcal{T}_2$ and $\mathcal{T}_4$, respectively.

Lemma 4.14. *Denoting the two-by-two connection matrix* Θ *such that*

$$(\omega_+, \omega_-)^{(2)} = (\omega_+, \omega_-)^{(4)}\,\Theta, \quad \Theta := \begin{pmatrix} \theta_1 & \theta_2 \\ \theta_3 & \theta_4 \end{pmatrix}, \quad |\Theta| = 1,$$

we find that $\theta_4 = 1$ *and*

$$\theta_1 = 1 - \frac{2\pi e^{-i\pi\gamma}}{\Gamma\left(\frac{1}{2}-\gamma\right)\Gamma\left(\frac{1}{2}+\gamma\right)},$$
$$\theta_2 = \frac{i\sqrt{2\pi}}{\Gamma\left(\frac{1}{2}+\gamma\right)} e^{-i\pi\gamma} e^{-2d}, \quad \theta_3 = \frac{i\sqrt{2\pi}}{\Gamma\left(\frac{1}{2}-\gamma\right)} e^{2d}.$$

Proof. The existence of such a matrix Θ follows immediately from the fact that $\omega_\pm^{(j)}$ for $j \in \{2,4\}$ are all solutions of the same linear, second-order ODE. Let $\lambda \in \mathcal{T}_2$. Now set $\lambda = 1 + \zeta/\left(x\sqrt{2}\right)$ and $\tilde{\lambda} = 1 - \zeta/\left(x\sqrt{2}\right)$, then as $\lambda \to 1$ we have by definition

$$\omega_+^{(2)}(\lambda) \to e^{-i\pi/4} e^{i\pi\gamma/2} e^d \zeta^{1/2} U(-\gamma, -i\zeta) + \frac{i\sqrt{2\pi}}{\Gamma\left(\frac{1}{2}-\gamma\right)} e^d \zeta^{1/2} U(\gamma, \zeta),$$
$$\omega_-^{(2)}(\lambda) \to e^{-d} \zeta^{1/2} U(\gamma, \zeta).$$

Then, since these solutions are given asymptotically by the above parabolic cylinder functions in an annular region around the turning point, we may further obtain

$$\begin{aligned}
\omega_+^{(2)}(\tilde{\lambda}) &\sim e^{i\pi/4} e^{i\pi\gamma/2} e^d \zeta^{1/2} U(-\gamma, i\zeta) - \frac{\sqrt{2\pi}}{\Gamma\left(\frac{1}{2}-\gamma\right)} e^d \zeta^{1/2} U(\gamma, -\zeta) \\
&\sim \frac{i\sqrt{2\pi}}{\Gamma\left(\frac{1}{2}-\gamma\right)} e^{-i\pi\gamma} e^d e^{-\tau(\zeta)} + \left(e^{i\pi\gamma} - \frac{2\pi}{\Gamma\left(\frac{1}{2}-\gamma\right)\Gamma\left(\frac{1}{2}+\gamma\right)}\right) e^d e^{\tau(\zeta)} \\
&\sim \frac{i\sqrt{2\pi}}{\Gamma\left(\frac{1}{2}-\gamma\right)} e^{2d} \omega_-^{(4)}(\tilde{\lambda}) + \left(1 - \frac{2\pi e^{-i\pi\gamma}}{\Gamma\left(\frac{1}{2}-\gamma\right)\Gamma\left(\frac{1}{2}+\gamma\right)}\right) \omega_+^{(4)}(\tilde{\lambda}),
\end{aligned}$$

and

$$\begin{aligned}
\omega_-^{(2)}(\tilde{\lambda}) &\sim i e^{-d} \zeta^{1/2} U(\gamma, -\zeta) \sim e^{-i\pi\gamma} e^{-d} e^{-\tau(\zeta)} + \frac{i\sqrt{2\pi}}{\Gamma\left(\frac{1}{2}+\gamma\right)} e^{-d} e^{\tau(\zeta)} \\
&\sim \omega_-^{(4)}(\tilde{\lambda}) + \frac{i\sqrt{2\pi}}{\Gamma\left(\frac{1}{2}+\gamma\right)} e^{-i\pi\gamma} e^{-2d} \omega_+^{(4)}(\tilde{\lambda}),
\end{aligned}$$

so we have the result. □

Remark 4.15. Applying "Euler's reflection formula" regarding the gamma function, one may note for $1/2+\gamma \notin \mathbb{Z}$ that

$$\theta_1 = 1 - \frac{2\pi e^{-i\pi\gamma}}{\Gamma\left(\frac{1}{2}-\gamma\right)\Gamma\left(\frac{1}{2}+\gamma\right)} = 1 - 2e^{-i\pi\gamma}\sin(\pi(1/2+\gamma)) = -e^{-2i\pi\gamma}.$$

4.1.4. Monodromy data. The *monodromy data* of the linear system is invariant under deformation of the parameter x, given the parameter $y = y(x)$ is a solution of P_{IV}. We express the monodromy data of the system via *Stokes matrices*, which relate *canonical solutions at infinity*. These linear system solutions are defined via asymptotic expansions as $\xi \to \infty$ in a given sector of the complex ξ plane. Firstly, we define sectors:

$$\Omega_n = \left\{\xi \in \mathbb{C} : \frac{\pi}{2}(n-1) < \arg\xi < \frac{\pi}{2}n\right\}.$$

Then the canonical solutions at infinity $\Psi^{(n)}(\xi)$ for $n \in \mathbb{Z}$ are defined as follows.

Definition 4.16. *For $n \in \mathbb{Z}$, the solution $\Psi^{(n)}$ is uniquely determined by the behaviour*

$$\Psi^{(n)}(\xi) = \begin{pmatrix} e^{\sigma(\xi)} & -\frac{f}{2\xi}e^{-\sigma(\xi)} \\ \frac{z-a-b}{f\xi}e^{\sigma(\xi)} & e^{-\sigma(\xi)} \end{pmatrix}(I + \mathcal{O}(1/\xi)), \quad |\xi| \to \infty, \quad \xi \in \Omega_n,$$

where $\sigma(\xi) = \xi^2/2 + x\xi - b\log(\xi)$.

Then, Stokes matrices S_n for $n \in \mathbb{Z}$ are such that $\Psi^{(n+1)} = \Psi^{(n)}S_n$, and have structure

$$S_{2k+1} = \begin{pmatrix} 1 & s_{2k+1} \\ 0 & 1 \end{pmatrix}, \quad S_{2k} = \begin{pmatrix} 1 & 0 \\ s_{2k} & 1 \end{pmatrix}, \quad k \in \mathbb{Z},$$

where constants s_n for $n \in \mathbb{Z}$ are *Stokes multipliers*, four of which are non-trivial as these are constrained by

$$\Psi^{(5)}\left(e^{2i\pi}\xi\right) = \Psi^{(1)}(\xi)M_\infty, \quad M_\infty = \begin{pmatrix} e^{-2i\pi b} & 0 \\ 0 & e^{2i\pi b} \end{pmatrix}.$$

We may furthermore introduce the *canonical solution at zero* by

$$\Psi_0(\xi) = (\mathcal{O}(1) + \mathcal{O}(\xi))\begin{pmatrix} \xi^a & 0 \\ 0 & \xi^{-a} \end{pmatrix},$$

with local monodromy around the origin given by

$$\Psi_0\left(e^{2i\pi}\xi\right) = \Psi_0(\xi)M_0, \quad M_0 = \begin{pmatrix} e^{2i\pi a} & 0 \\ 0 & e^{-2i\pi a} \end{pmatrix}.$$

We complete our introduction of non-trivial monodromy data by introducing a matrix J such that

$$\Psi^{(1)} = \Psi_0 J, \quad J := \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where one of the constants $\alpha, \beta, \gamma, \delta$ is determined by the others, since without loss of generality, we may assume that Ψ_0 is normalised such that $|J| = 1$.

One may show that only two free parameters fully determine the system's monodromy data. Indeed, consider that

$$\begin{aligned} \Psi^{(5)}\left(e^{2i\pi}\xi\right) &= \Psi^{(1)}(\xi) M_\infty, \\ \implies \Psi^{(1)}\left(e^{2i\pi}\xi\right) S_1 S_2 S_3 S_4 &= \Psi^{(1)}(\xi) M_\infty, \\ \implies \Psi_0\left(e^{2i\pi}\xi\right) J S_1 S_2 S_3 S_4 &= \Psi_0(\xi) J M_\infty, \\ \implies \Psi_0\left(\xi\right) M_0 J S_1 S_2 S_3 S_4 &= \Psi_0(\xi) J M_\infty. \end{aligned}$$

Thus, we have the cyclic relationship $M_0 J S_1 S_2 S_3 S_4 = J M_\infty$. Eliminating α, β, γ and δ yields a relationship between s_1, s_2, s_3, s_4, i.e., any three Stokes multipliers determine the final one. In particular, the monodromy manifold is given by a hyper-surface satisfying

$$\begin{aligned} s_1(s_2 + s_4 + s_2 s_3 s_4) = &- 1 + e^{2i(a-b)\pi} - e^{-4ib\pi} + e^{-2i(a+b)\pi} \\ &- e^{-4ib\pi} s_2 s_3 - s_3 s_4. \end{aligned}$$

Furthermore, we note that parameters s_j and s_{j+2}, for any $j \in \mathbb{Z}$, are related due to a symmetry of the linear system (4.6) as $\xi \to -\xi$, thus reducing the number of free parameters to two.

The system (4.6) is invariant under $\xi \to -\xi$ given we also transform the parameters (Painlevé variables) by $x \to -x$, $y \to -y$ and $f \to -f$. Considering these parameters as corresponding to the reflection point of a symmetric solution of $\mathrm{P_{IV}}$, the constraint on monodromy data manifests as $s_{j+2} = -s_j e^{2i\pi b}$ for odd values of $j \in \mathbb{Z}$ and $s_{j+2} = -s_j e^{-2i\pi b}$ for even values of $j \in \mathbb{Z}$; we may note that simply $s_j s_{j+1} = s_{j+2} s_{j+3}$ for all $j \in \mathbb{Z}$.

We consider symmetric solutions of $\mathrm{P_{IV}}$, for which the monodromy data has been explicitly calculated; see [90] for details. For further exposition regarding the generic linear monodromy data and its correspondence with the solution space of $\mathrm{P_{IV}}$, we refer the reader to, for instance, [43].

4.1.5. Isomonodromic deformation. By virtue of our earlier transformations, and Definition 4.13, we may express the canonical solutions $\Psi^{(2)}$ and $\Psi^{(4)}$

by

$$\Psi^{(j)} = r \begin{pmatrix} A\omega_{+}^{(j)} & B\omega_{-}^{(j)} \\ \cdots & \cdots \end{pmatrix}, \quad r = R^{-1/4}\sqrt{\frac{y-2\xi}{-2\xi}}, \quad j \in \{2,4\},$$

for some constants A, B. Then considering Definition 4.16, using the fact that $r \sim (-x)^{1/2}/\xi^{1/2}$ as $|\xi| \to \infty$, we see that these constants are necessarily

$$A = e^{-b\log(-x)} \quad \text{and} \quad B = -(f/2)e^{(b-1)\log(-x)}.$$

By definition it is true that $\Psi^{(2)} = \Psi^{(4)}(S_2S_3)^{-1}$, so Lemma 4.14 is equivalent to the connection result

$$1 + s_2s_3 = -e^{-2i\pi\gamma}, \quad s_2 = -\frac{A}{B}\frac{i\sqrt{2\pi}}{\Gamma\left(\frac{1}{2}-\gamma\right)}e^{2d}, \quad s_3 = -\frac{B}{A}\frac{i\sqrt{2\pi}}{\Gamma\left(\frac{1}{2}+\gamma\right)}e^{-2d-i\pi\gamma}. \tag{4.17}$$

Setting $\Delta := 1/2 - \gamma$, these equations are equivalent to $\Delta = \ln(1+s_2s_3)/(2i\pi)$, and

$$f(x) = \frac{\sqrt{2\pi}}{s_2\Gamma(\Delta)}e^{i\pi\Delta/2+3i\pi/4}e^{-T(x)}, \quad \Delta = f(x)\frac{\sqrt{2\pi}}{s_3\Gamma(-\Delta)}e^{i\pi\Delta/2+3i\pi/4}e^{T(x)},$$

$$T(x) := x^2 + 2(b-\Delta)\ln x - \frac{1}{2}(1+2\Delta)\ln 2 + \frac{i\pi}{4}(2\Delta+8b-3). \tag{4.18}$$

In particular, in terms of functions $u(x), v(x)$ we obtain

$$\text{Case 1}: \quad \Delta = \left(1 - uv - v^2\right)/2,$$
$$\text{Case 3}: \quad \Delta = \left(au + bu - 2av - uv + v^2\right)/u.$$

Considering Case 3, combining Equations (4.18) with expansion (4.5) leads us to

$$u(x) \sim \frac{1}{\Delta}(a+b)pe^{S(x)} - \frac{1}{\Delta}(a-b)qe^{-S(x)}, \tag{4.19}$$

which is contradictory to the assumed asymptotic expansion (4.5) for general parameters a, b, when $p, q \neq 0$. In fact, for general p, q, this is consistent only when $a = 0$, but this symmetric solution of P_{IV} is simply the identically zero solution $y(x) \equiv 0$. In other words, we have obtained the trivial case whereby the symmetric solution is given by expansion (4.5) with $p = q = 0$ when $a = 0$.

More generally, behaviour (4.19) becomes consistent with expansion (4.5) given just one of p or q is zero, hence giving vanishing, tronqueé-symmetric solutions. However, these solutions exist for specific values of a and b, assuming there are solutions (a, b) of

$$p = 0 \implies b - a = -\frac{i}{2\pi}\ln(1+s_2s_3) \quad \text{or}$$
$$q = 0 \implies a + b = -\frac{i}{2\pi}\ln(1+s_2s_3).$$

Considering Case 1, on the other hand, we find that expansion (4.3) is consistent in general, with the condition $pq = 2\Delta - 1$. This further implies that (tri-)tronqueé solutions occur when (a, b) satisfies $i\pi = \ln(1 + s_2 s_3)$.

Assuming that $y(x)$ is a symmetric solution of $\mathrm{P_{IV}}$, one may calculate the linear monodromy data in exact and explicit terms; this is due to the linear problem being solvable exactly by Whittaker functions (for instance, see [37, §13.19(i)] regarding connection results) at the *reflection point* $x = 0$; these results are given in [90]. We see that the product of Stokes multipliers $s_2 s_3$ is given by

$$s_2 s_3 = \frac{4i\pi^2 e^{i\pi b}}{\Gamma\left(\frac{1}{2}(a+b)\right)\Gamma\left(\frac{1}{2}(1-a+b)\right)\Gamma\left(\frac{1}{2}(1+a-b)\right)\Gamma\left(\frac{1}{2}(2-a-b)\right)}. \tag{4.20}$$

Given $a + b$ is not an even integer, and $a - b$ is not an odd integer, the above expression for $s_2 s_3$ is equivalent to

$$1 + s_2 s_3 = e^{2i\pi b} + e^{i\pi(a+b)} - e^{i\pi(b-a)}.$$

Consistent with taking a limit in (4.20), the excluded cases, where the Gamma functions have poles, correspond to $s_2 s_3 = 0$.

To summarise what the associated linear monodromy problem has revealed regarding a symmetric solution of $\mathrm{P_{IV}}$, consider the asymptotic limit $|x| \to \infty$ with $\arg x = \pi/4$. In general, a symmetric solution $y(x)$ of $\mathrm{P_{IV}}$ adopts the asymptotic behaviour

$$\begin{aligned} y(x) &= -2x + pe^{S(x)} + q/e^{S(x)} + \mathcal{O}\left(1/x, x\delta^2\right), \\ S(x) &= x^2 - (2 - 4b + 3pq)\ln x, \end{aligned}$$

where the product of Stokes multipliers, pq, satisfies

$$e^{i\pi pq} = e^{-i\pi(a-b)} - e^{i\pi(a+b)} - e^{2i\pi b}. \tag{4.21}$$

Moreover, the expansion is valid in some region of the parameter space $(a, b) \in \mathcal{D} \subset \mathbb{C}^2$ such that $|\Re(2 - 4b + 3pq)| < 1$. Here, we have defined $\delta := x^{|\Re(2-4b+3pq)|-1}$ to describe orders of neglected terms.

A degenerate case of the associated linear monodromy data is when $s_2 s_3 = 0$ (Stokes multipliers of the linear system vanishing), occurring when either $a + b$ is an even integer or $a - b$ is an odd integer. Moreover, this scenario is consistent with the above behaviour, corresponding to the case of pq being an odd integer.

Furthermore, there exists a non-empty set of parameter values $(a, b) \in \mathbb{C}^2$ such that the symmetric solution is tronqueé, i.e. $pq = 0$. This occurs when parameters (a, b) satisfy $1 = e^{-i\pi(a-b)} - e^{i\pi(a+b)} - e^{2i\pi b}$. If we consider, for instance, real parameters $a, b \in \mathbb{R}$, then this condition may be simplified to $\cos(\pi b) = 0$ and

$\sin(\pi a) = 0$. Therefore, in the case of $a, b \in \mathbb{R}$, we propose that symmetric solutions are tronqueé when $a = m$ and $b = n + 1/2$, for any $m, n \in \mathbb{Z}$.

It is also possible that a symmetric solution has one of the expansions

$$y(x) = \pm\frac{2a}{x} + \frac{p}{x}e^{S(x)} + \frac{q}{x}e^{-S(x)} + \mathcal{O}\left(1/x^3, \delta/x, x\delta^2\right),$$

where $S(x) = x^2 + (1 - 2b \pm 6a)\ln x$ and one of p or q is equal to zero (tronqueé solutions). We have set $\delta = x^{|\Re(1-2b\pm6a)|-2}$ and require $|\Re(1 - 2b \pm 6a)| < 2$. To abide by the constraint $pq = 0$, this is only valid for parameters (a, b) satisfying either

$$e^{2i\pi b} + e^{i\pi(a+b)} = e^{-i\pi(a-b)} + e^{-2i\pi(a-b)} \qquad \Longrightarrow \qquad p = 0,$$
$$\text{or} \quad e^{2i\pi b} + e^{i\pi(a+b)} = e^{-i\pi(a-b)} + e^{2i\pi(a+b)} \qquad \Longrightarrow \qquad q = 0.$$

A trivial case of this is when $a = p = q = 0$, and then the symmetric solution is identically zero. We may further note that the $p = 0$ condition is satisfied by, for example, the degenerate case whereby $a - b$ is an odd integer (and $s_2s_3 = 0$). Similarly, a solution to the $q = 0$ condition is to let $a + b$ be an even integer (also giving $s_2s_3 = 0$).

Remark 4.22. When $a + b = 0$, i.e., corresponding to $s_2s_3 = 0$, the symmetric solution $y(x)$ is given exactly in terms of classical special functions. In this degenerate case, the function $z(x)$ is identically zero, and then $f(x)$ satisfies the second-order ODE

$$f_{xx} + 2xf_x + 2(1 + 2a)f = 0.$$

Thus making the transformation $f(x) = e^{-x^2/2}g(x)$ gives $g(x)$ satisfying the parabolic cylinder equation $g_{xx} = \left(x^2 - 1 - 4a\right) g$. Then by definition

$$y = -2x - f_x/f = -x - g_x/g.$$

Rational and special solution cases are further discussed in [90].

4.2. Symmetric Solutions of q-Painlevé IV

Here, we seek to develop a q-isomonodromic deformation method to determine global asymptotic properties of symmetric solutions of the q-Painlevé IV system by considering the associated q-linear system in the limit $t \to 0$.

4.2.1. Local asymptotic analysis of q-P$_{\text{IV}}$. In this subsection we introduce the system known as q-P$_{\text{IV}}$ and describe symmetric solutions. We proceed to give a local asymptotic expansion for these solutions as $t \to 0$, which, in general, is determined up to two free constants.

The q-Painlevé IV system, denoted q-$\mathrm{P_{IV}}$, is

$$\begin{aligned}\frac{\bar{f}_0}{a_0a_1f_1} &= \frac{1+a_2f_2(1+a_0f_0)}{1+a_0f_0(1+a_1f_1)},\\ \frac{\bar{f}_1}{a_1a_2f_2} &= \frac{1+a_0f_0(1+a_1f_1)}{1+a_1f_1(1+a_2f_2)},\\ \frac{\bar{f}_2}{a_2a_0f_0} &= \frac{1+a_1f_1(1+a_2f_2)}{1+a_2f_2(1+a_0f_0)},\end{aligned}$$

where $q \in \mathbb{C}^*$ satisfies $0 < |q| < 1$, each f_k is a function of $t \in \mathbb{C}^*$ and each a_k is a constant parameter. The bars refer to iteration in t, that is, $t \to qt$. The system is further constrained by $f_0f_1f_2 = t^2$ and $a_0a_1a_2 = q$. We consider the symmetric solutions of q-$\mathrm{P_{IV}}$, introduced in [81], which satisfy

$$f_k(iq^m) = \frac{1}{f_k(iq^{-m})}, \quad m \in \mathbb{Z}, \quad k \in \{0,1,2\}.$$

There exist precisely four distinct symmetric solutions (see [81] for details), determined by the initial conditions

$$(f_0(i), f_1(i), f_2(i)) \in \{(-1,1,1), (1,-1,1), (1,1,-1), (-1,-1,-1)\}.$$

Now, we consider a transformation of variables amenable to asymptotic analysis as $t \to 0$.

Definition 4.23. *We define $\epsilon(t)$ such that $\epsilon^3 = t$ with $\epsilon(i) = -i$, and define $u(t), v(t)$ such that $f_0 = \epsilon^2 u$ and $f_1 = \epsilon^2 v$.*

Considering the limit $t \to 0$ (and thus $\epsilon \to 0$), we obtain:

Lemma 4.24. *Let $t = iq^m$ for $m \in \mathbb{Z}$. The system q-P_{IV} admits the local asymptotic expansion*

$$\begin{aligned}u(t) &= (a_1/a_2)^{1/3} \exp\left(C_1 e^{2i\pi m/3} + C_2 e^{-2i\pi m/3}\right)\left(1 + \mathcal{O}\left(\epsilon^2\right)\right),\\ v(t) &= (a_2/a_0)^{1/3} \exp\left(B_1 e^{2i\pi m/3} + B_2 e^{-2i\pi m/3}\right)\left(1 + \mathcal{O}\left(\epsilon^2\right)\right),\end{aligned}$$

as $\epsilon \to 0$, where $B_1 = C_1 e^{2i\pi/3}$ and $B_2 = C_2 e^{-2i\pi/3}$ for arbitrary constants $C_1, C_2 \in \mathbb{C}$.

Proof. The lemma is straightforwardly verified via substitution into the q-$\mathrm{P_{IV}}$ system.

□

Remark 4.25. On the domain $t \in iq^{\mathbb{Z}}$ we say that u, v are q^3-periodic in the limit $t \to 0$. To this end, we define constants $u_k, v_k \in \mathbb{C}$, for $k \in \{0,1,2\}$, such that $u(q^k t) \to u_k$ and $v(q^k t) \to v_k$ as $t \to 0$, where $t = iq^m$ with $m \in \mathbb{Z}$ and we fix

$m = 0 \mod 3$. The problem is then to discern two of these constants, from which B_1, B_2, C_1, C_2 follow trivially; for instance, given v_0, v_1, we calculate

$$B_1 = -\frac{i}{\sqrt{3}}\left\{\ln\left(\frac{v_1}{A}\right) + \ln\left(\frac{v_0}{A}\right)e^{i\pi/3}\right\},$$
$$B_2 = \frac{i}{\sqrt{3}}\left\{\ln\left(\frac{v_1}{A}\right) - \ln\left(\frac{v_0}{A}\right)e^{2i\pi/3}\right\},$$

where $A = (a_2/a_0)^{1/3}$.

Remark 4.26 (Invariant quantity)**.** After scaling $v = (a_2/a_0)^{1/3}y$, we find that the behaviour described in Lemma 4.24 is associated with the leading-order conserved quantity $K(\overline{y}, y)$, where

$$K(x, y) = \frac{1}{xy}\left(x^2y^2 + x + y\right).$$

Moreover, the above surface features three critical points given by $x^3 = 1$ with $y = 1/x^2$. These points in the phase-space $(\overline{y}, y)$ correspond to near-stationary solutions. Indeed, see that the q-$\mathrm{P_{IV}}$ is satisfied by a formal series of the form

$$f_k = \sum_{n=1}^{\infty} c_n^{(k)} t^{2n/3}, \quad k \in \{0, 1, 2\}, \quad t \to 0,$$

where in particular $c_1^{(k)} = (a_{k+1}/a_{k+2})^{1/3}$ taking mod 3 of the parameter indices.

Remark 4.27 (Quicksilver solutions)**.** Equivalent to what was determined in the study of q-$\mathrm{P_{II}}$, beyond the general behaviour given in Lemma 4.24, we see that q-$\mathrm{P_{IV}}$ is satisfied by quicksilver solutions, given by a balance of the system whereby one solution f_j for $j \in \{0, 1, 2\}$ vanishes with order t^2 as $t \to 0$. See that q-$\mathrm{P_{IV}}$ is satisfied by formal series solutions

$$f_k = \sum_{n=0}^{\infty} c_n^{(k)} t^{2n}, \quad k \in \{0, 1\} \quad \text{and} \quad f_2 = \sum_{n=1}^{\infty} c_n^{(2)} t^{2n}, \quad t \to 0,$$

where the leading coefficients are $c_0^{(0)} = -a_0$, $c_0^{(1)} = -1/a_1$, and $c_1^{(2)} = a_1/a_0$.

Furthermore, these series solutions lead to exponentially small perturbation terms given by q-Airy functions, as seen in the case of quicksilver solutions of q-$\mathrm{P_{II}}$. For instance, setting $f_1 = F_1 + g_1$ where F_1 is the above formal series solution, we obtain g_1 satisfying the linearised, leading-order equation

$$\overline{\overline{g}}_1 + \frac{a_0^2}{a_1^2 q^3 t^2}\overline{g}_1 + g_1 \to 0, \quad t \to 0.$$

Following the analysis shown in Chapter 3, regarding quicksilver solutions of q-$\mathrm{P_{II}}$, one may, therefore, deduce the existence of true solutions that are asymptotic to the proposed quicksilver series in some open (Stokes-like) region of the complex t-plane as $t \to 0$.

4.2.2. Direct monodromy problem. Here, we introduce the Lax pair associated with q-P_{IV} and deduce asymptotic representations for the canonical solutions at $z = 0$ and $z = \infty$, as $t \to 0$. Specifically, we consider an annular region in the z-plane where $1 \gg |z| \gg |\epsilon|$, and we find that q-Airy equations appear in the leading order when considering the spectral equation.

We recall the Lax pair of q-P_{IV}, given in [83], the coupled matrix equations $Y(qz,t) = A(z,t)Y(z,t)$ and $Y(z,qt) = B(z,t)Y(z,t)$ where

$$\begin{aligned} A(z,t) &= \begin{pmatrix} \rho & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -iq\frac{t}{f_2}z & 1 \\ -1 & -iq\frac{f_2}{t}z \end{pmatrix} \begin{pmatrix} -ia_0a_2\frac{t}{f_0}z & 1 \\ -1 & -ia_0a_2\frac{f_0}{t}z \end{pmatrix} \\ &\quad \times \begin{pmatrix} -ia_0\frac{t}{f_1}z & 1 \\ -1 & -ia_0\frac{f_1}{t}z \end{pmatrix} \begin{pmatrix} \rho^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \\ B(z,t) &= \begin{pmatrix} 0 & -b\rho \\ 1/(b\rho) & 0 \end{pmatrix} + \begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

with

$$b = \frac{t(1 + a_1f_1(1 + a_2f_2))}{i(qt^2 - 1)f_2}.$$

We refer to $Y(qz,t) = A(z,t)Y(z,t)$ as the spectral equation. Compatibility of the Lax pair, that is $A(z,qt)B(z,t) = B(qz,t)A(z,t)$, is equivalent to (f_0, f_1, f_2) satisfying q-P_{IV} and ρ satisfying the auxiliary equation $\bar{\rho} = b^2\rho$. Associated with this Lax pair, we now introduce canonical solutions at $z = 0$ and $z = \infty$, and the connection matrix, as seen in [81, 83] (we refer back to the introduction of these concepts in Section 1.7).

Definition 4.28 (Canonical solution at $z = 0$). *For any fixed $d, t \in \mathbb{C}^*$, there exists a unique matrix solution $\Phi_0(z,t)$, meromorphic on $z \in \mathbb{C}^*$, such that*

$$\Phi_0(qz,t) = A(z,t)\Phi_0(z,t)\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad \Phi_0(z,t) = M_0 + \mathcal{O}(z), \quad z \to 0,$$

where

$$M_0 = d\begin{pmatrix} \rho & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}.$$

Definition 4.29 (Canonical solution at $z = \infty$). *For any fixed $t \in \mathbb{C}^*$, there exists a unique matrix solution $\Phi_\infty(z,t)$, meromorphic on $z \in \mathbb{C}^*$, such that as $z \to \infty$*

$$\Phi_\infty(qz,t) = \frac{1}{ia_0^2a_2qz^3}A(z,t)\Phi_\infty(z,t)\begin{pmatrix} 1/t & 0 \\ 0 & t \end{pmatrix}, \quad \Phi_\infty(z,t) = I + \mathcal{O}(1/z).$$

We define the corresponding connection matrix by $\Phi_\infty(z,t) = \Phi_0(z,t)C(z,t)$. Furthermore, d satisfies the time evolution $\bar{d} = (i/b)d$ and the connection matrix satisfies $C(z,qt) = \sigma_3 C(z,t) z^{-\sigma_3}$; these are direct results of the compatibility condition as pointed out in [81]. It is shown in [83] that $C(z,t)$ is of the form

$$C(z,t) = \begin{pmatrix} C_1(z,t) & C_2(z,t) \\ -C_1(-z,t) & C_2(-z,t) \end{pmatrix},$$

where C_1, C_2 belong to the respective spaces

$$C_1(z,t) \in V_3\left(\frac{a_1}{a_0 q^2 t}\right), \quad C_2(z,t) \in V_3\left(\frac{a_1 t}{a_0 q^2}\right). \tag{4.30}$$

These basic properties follow somewhat straightforwardly from the definitions of $\Phi_0(z,t)$, $\Phi_\infty(z,t)$ and $C(z,t)$.

We give an asymptotic representation of $\Phi_0(z,t)$ in an annular region where $1 \gg |z| \gg |\epsilon|$.

Lemma 4.31. *Let $k \in \mathbb{Z}$ and let $z, \epsilon \in \mathbb{C}^*$ such that $1 \gg |z| \gg |\epsilon|$. There exist unique functions $F_k(z,t)$ and $G_k(z,t)$, which are rational with respect to z, such that*

$$\phi_0^{(4)}(z,t) \sim F_k(z,t) Ai_p\left(q^k \zeta\right) + G_k(z,t) Ai_p\left(p q^k \zeta\right),$$

where $p = q^3$, $\zeta = a_0 q^2 z^3 / (a_1 t)$, and here $\phi_0^{(4)}(z,t)$ denotes the bottom-right entry of the 2×2 matrix solution $\Phi_0(z,t)$.

Proof. For simplicity we abbreviate $y(z,t) := \phi_0^{(4)}(z,t)$. Via consideration of the spectral equation, we find that $y(z,t)$ satisfies the second order, linear q-difference equation

$$\begin{aligned} 0 = &\left(1 + \mathcal{O}\left(\epsilon^2/z^2\right)\right) y(q^2 z, t) + \left(\frac{a_0 q^5 z^3}{a_1 \epsilon^3} + \mathcal{O}\left(z/\epsilon\right)\right) y(qz,t) \\ &+ \left(-q^2 + \mathcal{O}\left(\epsilon^2/z^2\right)\right) y(z,t). \end{aligned}$$

The above equation is satisfied at leading order by

$$y(z,t) \sim f(z,t) \mathrm{Ai}_{q^3}\left(\frac{a_0 q^4 z^3}{a_1 \epsilon^3}\right), \tag{4.32}$$

where $f(z,t)$ is some function satisfying $f(qz,t) = q f(z,t)$ and $\mathrm{Ai}_q(x)$ is the q-Airy function (see Section 1.6).

We fix $f(z,t)$ by considering the condition on the determinant of $\Phi_0(z,t)$, which is $|\Phi_0(z,t)| \to |M_0| = 2id^2\rho$ as $z \to 0$. Also computing the determinant from (4.32) we deduce the condition

$$\frac{i a_1 v \rho}{q} \frac{\epsilon^2}{z^2} f(-z,t) f(z,t) \left\{ \mathrm{Ai}_p(\xi) \mathrm{Ai}_p(-p\xi) + \mathrm{Ai}_p(-\xi) \mathrm{Ai}_p(p\xi) \right\} = 2id^2\rho,$$

where $p = q^3$ and $\xi = a_0 q^4 z^3/(a_1 t)$. To proceed, we require the identity

$$\mathrm{Ai}_p(\xi)\mathrm{Ai}_p(-p\xi) + \mathrm{Ai}_p(-\xi)\mathrm{Ai}_p(p\xi) = 2,$$

for all $\xi \in \mathbb{C}^*$, this follows from the fact that $\mathrm{Ai}_p(\xi)\mathrm{Ai}_p(-p\xi)-1$ is an odd function in ξ which may be proved by writing $\mathrm{Ai}_p(\xi)$ as its convergent power series in ξ. With this fact, we see that

$$f(z,t) = \frac{idz}{\epsilon}\sqrt{\frac{q}{a_1 v}},$$

noting that this does indeed satisfy the original condition $f(qz,t) = qf(z,t)$. Concerning the functions F_k, G_k, we have thus far determined a representation whereby

$$F_2(z,t) = f(z,t)\left(1 + \mathcal{O}\left(\epsilon^2/z^2\right)\right), \quad G_2(z,t) = f(z,t)\left(\mathcal{O}\left(\epsilon^5/z^5\right)\right).$$

Then, it follows from the compatibility of the Lax pair that

$$y(z,qt) = \frac{1}{b\rho}\frac{A_4(z,t)}{A_3(z,t)}y(z,t) + \frac{i}{b\rho}\frac{1}{A_3(z,t)}y(qz,t),$$

where A_3, A_4 are entries of the coefficient matrix $A(z,t)$, in particular

$$\begin{aligned} A_3(z,t) =& \frac{q^2}{a_1 v\rho}\frac{z^2}{\epsilon^2} + \frac{1}{\rho} + \mathcal{O}\left(z^2\right), \\ A_4(z,t) =& \frac{ia_0 q^2}{a_1}\frac{z^3}{\epsilon^3} + \frac{i(q + a_0 uv^2)}{uv}\frac{z}{\epsilon} + \mathcal{O}(z\epsilon). \end{aligned}$$

Therefore, we obtain the recurrence system

$$\begin{aligned} F_{k+1}(z,qt) =& \frac{1}{b\rho}\frac{1}{A_3(z,t)}\left\{A_4(z,t)F_k(z,t) + iG_k(qz,t)\right\}, \\ G_{k+1}(z,qt) =& \frac{1}{b\rho}\frac{1}{A_3(z,t)} \\ & \times\left\{A_4(z,t)G_k(z,t) + iF_k(qz,t) - \frac{ia_0 q^2 z^3}{a_1 t}q^k G_k(qz,t)\right\}, \end{aligned}$$

and the lemma is proved inductively. □

4.2.3. Isomonodromic deformation. This section deduces an asymptotic representation for the connection matrix as $t \to 0$. As in the classical isomonodromic deformation method, we use known connection results regarding leading order special functions (in this case, q-Airy). We then relate $C(z, t \to 0)$ and $C(z,i)$ via the simple time evolution $C(z,q^m i) = \sigma_3^m C(z,i)z^{-m\sigma_3}$, thus constraining the value of a symmetric solution (f_0, f_1, f_2) of q-$\mathrm{P_{IV}}$ at a fixed $|t| \ll 1$. For simplicity, we take the specific case of symmetric solution given by $(f_0, f_1, f_2) = (-1,-1,-1)$ when $t = i$.

Firstly, we give the connection matrix at the reflection point $t = i$, where the direct monodromy problem is exactly solvable in terms of basic hypergeometric series. It is shown in [81] that $C_2(z,i) = 2ic_0^3 g(z)$ where $c_0 = \theta_q(i)/\theta_q(-1)\sqrt{i/2}$ and

$$\begin{aligned} g(z) = +\,\theta_q\left(+a_0 iz, -\frac{q}{a_1}iz, +qiz\right) - \theta_q\left(+a_0 iz, +\frac{q}{a_1}iz, -qiz\right) \\ -\,\theta_q\left(-a_0 iz, -\frac{q}{a_1}iz, -qiz\right) - \theta_q\left(-a_0 iz, +\frac{q}{a_1}iz, +qiz\right). \end{aligned} \tag{4.33}$$

Remark 4.34. Regarding the other entry of the connection matrix, $C_1(z,t)$, in the case of symmetric solutions, we have simply $C_1(z,i) = -iC_2(-z,i)$. Considering the time evolution of the connection matrix, this leads to a more general relation whereby $C_1(z,t) = -i(-iz)^{-2m}C_2(-z,t)$ for $t = iq^m$ and $m \in \mathbb{Z}$.

The following lemma allows us to convert from the triple products of q-theta functions seen in Equation (4.33) to q^3-theta functions as seen in the $n = 3$ case of (1.14).

Lemma 4.35. *Let $a, b, c, z \in \mathbb{C}^*$. We give the following identity concerning products of q-theta functions*

$$\delta\theta_q(az, bz, cz) = \tau_0\theta_{q^3}\left(abcz^3\right) - c\tau_1\theta_{q^3}\left(qabcz^3\right)z + qc^2\tau_2\theta_{q^3}\left(q^2abcz^3\right)z^2,$$

where the constant coefficients δ and τ_k, for $k \in \{0,1,2\}$, are

$$\begin{aligned} \delta &= \frac{(q;q)_\infty^3}{(q^2;q^2)_\infty (q^3;q^3)_\infty (q^6;q^6)_\infty}, \\ \tau_k &= \theta_{q^2}\left(-\frac{aq}{b}\right)\theta_{q^6}\left(-\frac{abq^3}{c^2q^{2k}}\right) + \frac{bq}{cq^k}\theta_{q^2}\left(-\frac{a}{b}\right)\theta_{q^6}\left(-\frac{abq^6}{c^2q^{2k}}\right). \end{aligned}$$

Proof. Firstly we note that $\theta_q(z)$, with $q, z \in \mathbb{C}^*$ and $|q| \neq 1$, may be expressed by the convergent sum

$$\theta_q(z) = \frac{1}{(q;q)_\infty}\sum_{k\in\mathbb{Z}} q^{k(k-1)/2}(-z)^k.$$

The lemma follows by a rather lengthy manipulation, beginning with

$$\theta_q(az)\theta_q(bz)\theta_q(cz) = \frac{1}{(q;q)_\infty^3}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} q_j q_k q_l a^j b^k c^l (-z)^{j+k+l},$$

of which further details are omitted here for conciseness. □

Remark 4.36. Although less relevant to this section as we consider a connection matrix comprised of triple q-theta products, we nonetheless supply an analogous identity concerning double q-theta products, this being more simply

$$\frac{(q;q)_\infty^2}{(q^2;q^2)_\infty^2}\theta_q(az, bz) = \theta_{q^2}(-qa/b)\theta_{q^2}\left(-abz^2\right) - b\theta_{q^2}(-a/b)\theta_{q^2}\left(-qabz^2\right)z,$$

for arbitrary $a, b, z \in \mathbb{C}^*$.

In the following lemmas, we describe $C_2(z,t)$ at both $t = i$ and as $t \to 0$.

Lemma 4.37. *For $z, t \in \mathbb{C}^*$ the connection matrix entry $C_2(z,t)$ is of the form*

$$C_2(z,t) = R(q^2t)\theta_p(-\zeta) + R(qt)\theta_p(-q\zeta)z + R(t)\theta_p(-q^2\zeta)z^2,$$

for some function $R(t)$ which is constant with respect to z, and once again $p = q^3$, $\zeta = a_0q^2z^3/(a_1t)$.

Proof. We first recall that $C_2(z,t) \in V_3\left(a_1t/a_0q^2\right)$, and thus consider the basis functions (1.14) introduced in Section 1.6

$$\left\{\theta_p(-\zeta), \theta_p(-q\zeta)z, \theta_p\left(-q^2\zeta\right)z^2\right\},$$

in spanning this space. Therefore, we may write $C_2(z,t)$ in the form

$$C_2(z,t) = R_0(t)\theta_p(-\zeta) + R_1(t)\theta_p(-q\zeta)z + R_2(t)\theta_p(-q^2\zeta)z^2, \tag{4.38}$$

for some R_0, R_1, R_2 which may depend on t but not z. Now, we consider the time evolution $C(z,qt) = \sigma_3 C(z,t)z^{-\sigma_3}$, in particular giving simply

$$C_2(z,qt) = C_2(z,t)z. \tag{4.39}$$

Substituting the generic form (4.38) into the time evolution equation (4.39) we see that

$$\begin{aligned} &R_0(qt)\theta_p(-\zeta/q) + R_1(qt)\theta_p(-\zeta)z + R_2(qt)\theta_p(-q\zeta)z^2 \\ =&R_0(t)\theta_p(-\zeta)z + R_1(t)\theta_p(-q\zeta)z^2 + R_2(t)\theta_p(-q^2\zeta)z^3. \end{aligned} \tag{4.40}$$

Then, applying the fundamental property of q-theta functions, that is $\theta_q(qz) = -(1/z)\theta_q(z)$, Equation (4.40) becomes

$$R_0(qt)\theta_p(-\zeta/q) + R_1(qt)\theta_p(-\zeta)z + R_2(qt)\theta_p(-q\zeta)z^2 \tag{4.41}$$

$$= \frac{a_1t}{a_0q}R_2(t)\theta_p(-\zeta/q) + R_0(t)\theta_p(-\zeta)z + R_1(t)\theta_p(-q\zeta)z^2. \tag{4.42}$$

Therefore, we have shown that $R_0(qt) = a_1t/(a_0q)R_2(t)$, $R_1(qt) = R_0(t)$ and $R_2(qt) = R_1(t)$. Considering another iteration $t \to qt$ of Equation (4.41) completes the result. □

Lemma 4.43. *Concerning the function $R(t)$ described in Lemma 4.37, we obtain values*

$$R(q^2i) = -\frac{4ic_0^3}{a_1\delta}\alpha, \quad R(qi) = \frac{4c_0^3q}{a_1\delta}\beta, \quad R(i) = \frac{4ic_0^3q^2}{a_1\delta}\gamma,$$

where the constants $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ are given by

$$\begin{aligned}
\alpha &= a_1\theta_{q^2}(-a_0a_1)\theta_{q^6}(-a_0q^2/a_1) + q\theta_{q^2}(a_0a_1/q)\theta_{q^6}(a_0q^5/a_1),\\
\beta &= a_1\theta_{q^2}(-a_0a_1)\theta_{q^6}(-a_0/a_1) - \theta_{q^2}(a_0a_1/q)\theta_{q^6}(a_0q^3/a_1),\\
\gamma &= a_1q\theta_{q^2}(-a_0a_1)\theta_{q^6}(-a_0/a_1q^2) + \theta_{q^2}(a_0a_1/q)\theta_{q^6}(a_0q/a_1),\\
\delta &= \frac{(q;q)_\infty^3}{(q^2;q^2)_\infty (q^3;q^3)_\infty (q^6;q^6)_\infty}. \qquad (4.44)
\end{aligned}$$

Proof. The lemma follows by the explicit solution of the linear system at $t = i$, see (4.33), and then applying Lemma 4.35. □

Lemma 4.45. *Let $t = iq^m$ for $m \in \mathbb{Z}$. The function $R(t)$ described in Lemma 4.37 is given asymptotically by*

$$R(t) \sim \frac{i}{(-1;q^3)_\infty}\frac{a_0}{d\epsilon^2}\sqrt{\frac{qv}{a_1}}y_m, \quad t \to 0,$$

where

$$y_m = b_1e^{2i\pi m/3} + b_2e^{-2i\pi m/3} + 1/3,$$

for some $b_1, b_2 \in \mathbb{C}$ which are constant with respect to t.

Proof. Consider the asymptotic region given by $1 \gg |z| \gg |\epsilon|$. We recall the fundamental definition of the connection matrix $\Phi_\infty(z,t) = \Phi_0(z,t)C(z,t)$; following Definition 4.29, the bottom-right entry of this equation implies that

$$C_2(-z,t)\phi_0^{(4)}(z,t) + C_2(z,t)\phi_0^{(4)}(-z,t) \to 1, \quad t \to 0. \qquad (4.46)$$

So, here we apply Lemma 4.31, calculating

$$\begin{aligned}
&C_2(-z,t)\phi_0^{(4)}(z,t)\\
=&\left(R(q^2t)\theta_{q^3}(\zeta) - R(qt)\theta_{q^3}(q\zeta)z + R(t)\theta_{q^3}\left(q^2\zeta\right)z^2\right)\\
&\times\left(F_k(z,t)\mathrm{Ai}_p(q^k\zeta) + G_k(z,t)\mathrm{Ai}_p(pq^k\zeta)\right)\\
=&R(q^2t)F_3(z,t)\theta_p(\zeta)\mathrm{Ai}_p(p\zeta) - \zeta R(q^2t)G_3(z,t)\theta_p(p\zeta)\mathrm{Ai}_p\left(p^2\zeta\right)\\
&- zR(qt)F_4(z,t)\theta_p(q\zeta)\mathrm{Ai}_p(pq\zeta) + qz\zeta R(qt)G_4(z,t)\theta_p(pq\zeta)\mathrm{Ai}_p\left(p^2q\zeta\right)\\
&- \left(qz^2/\zeta\right)R(t)F_2(z,t)\theta_p\left(q^2\zeta/p\right)\mathrm{Ai}_p\left(q^2\zeta\right)\\
&+ z^2R(t)G_2(z,t)\theta_p\left(q^2\zeta\right)\mathrm{Ai}_p\left(pq^2\zeta\right).
\end{aligned}$$

Then applying the connection formula (1.18), we obtain

$$\begin{aligned}
&C_2(-z,t)\phi_0^{(4)}(z,t)/(-1;p)_\infty + C_2(z,t)\phi_0^{(4)}(-z,t)/(-1;p)_\infty\\
\to&\left(F_3(z,t) - \zeta G_3(z,t)\right)R(q^2t) - z\left(F_4(z,t) - q\zeta G_4(z,t)\right)R(qt)\\
&- \frac{qz^2}{\zeta}F_2(z,t)R(t).
\end{aligned}$$

Therefore, substituting in F_k and G_k for $k \in \{2,3,4\}$ in accordance with Lemma 4.31, and considering (4.46), we see that the function $R(t)$ must satisfy the linear, second order q-difference equation

$$q^{1/3}\sqrt{\frac{\bar{v}}{v}}R(q^2t) + \frac{a_1\bar{v}\epsilon}{q^{2/3}}\sqrt{\frac{q}{a_1\bar{v}}}R(qt) + \frac{\epsilon^2}{a_0}\sqrt{\frac{a_1}{qv}}R(t) = \frac{i}{d}\frac{1}{(-1;p)_\infty}.$$

Then, making the transformation

$$R(t) = \frac{i}{(-1;p)_\infty}\frac{a_0}{d\epsilon^2}\sqrt{\frac{qv}{a_1}}y(t),$$

results in $y(q^2t) + y(qt) + y(t) = 1$, so with $t = t_m = iq^m$, $m \in \mathbb{Z}$, we obtain general solution

$$y(t_m) = y_m = b_1e^{2i\pi m/3} + b_2e^{-2i\pi m/3} + 1/3,$$

completing the proof. □

We now use the relation $C(z, q^m i) = \sigma_3^m C(z,i)z^{-m\sigma_3}$ to provide information regarding the symmetric solution of q-P$_{\mathrm{IV}}$ as $t \to 0$.

Lemma 4.47. *Setting $t = iq^m$, let $m = 3n$ for $n \in \mathbb{Z}$. The function $R(t)$ described in Lemma 4.37 satisfies*

$$R(q^2t)\mu_n = R(q^2i), \quad iR(qt)\epsilon\mu_n = R(qi), \quad -R(t)\epsilon^2\mu_n = R(i), \tag{4.48}$$

where

$$\mu_n = \left(-ia_0q^2/a_1\right)^n q^{-3n(n+1)/2}.$$

Proof. With $t = iq^m$, $m \in \mathbb{Z}$, the proof of this lemma is essentially an m-times iteration of what is seen in the proof of Lemma 4.37, i.e. utilising the more general time evolution $C(z, q^m i) = \sigma_3^m C(z,i)z^{-m\sigma_3}$, and noting the general identity

$$\theta_q(q^n z) = (-1/z)^n q^{-n(n-1)/2}\theta_q(z), \quad \forall n \in \mathbb{Z}.$$

□

Theorem 4.49. *Letting $t = iq^m$, with $m = 3n$ for $n \in \mathbb{Z}$, we obtain*

$$v(t) \sim -\frac{q}{a_1}\frac{R(q^2i)^2}{R(qi)^2}\frac{y_1^2}{y_2^2}, \quad v(qt) \sim \frac{a_1}{a_0^2q^{5/3}}\frac{R(i)^2}{R(q^2i)^2}\frac{y_2^2}{y_0^2}, \quad t \to 0,$$

where y_k, for $k \in \{0,1,2\}$, is as described in Lemma 4.45. Furthermore

$$d(t) \sim \frac{1}{(-1;q^3)_\infty}\frac{a_0q}{a_1}\frac{R(q^2i)}{R(i)R(qi)}\frac{y_0y_1}{y_2}\mu_n, \quad t \to 0,$$

for the sequence μ_n described in Lemma 4.47.

Proof. Here, we have applied the result of Lemma 4.45 to Equations (4.48) in Lemma 4.47 and performed an inversion for the Painlevé functions v, $\overline{v}$, and d. The result follows by furthermore recalling that $\bar{d} = (i/b)d$ and $v\bar{v}\bar{\bar{v}} \sim a_2/a_0$ as $t \to 0$. ☐

Concerning real symmetric solutions, we deduce the following:

Remark 4.50. In the case of real parameters $a_0, a_1, q \in \mathbb{R}$, and a corresponding real symmetric solution $v(t) \in \mathbb{R}$ for $t = iq^m$, $m \in \mathbb{Z}$, the results of Theorem 4.49 are in terms of one free parameter and its complex conjugate, that is to say

$$y_k = b_1 e^{2i\pi k/3} + \overline{b}_1 e^{-2i\pi k/3} + 1/3.$$

Remark 4.51. For the special choice of parameters $a_0 = a_1 = q^{1/3}$ we obtain the rational symmetric solution $u = v = 1$ for all $t \in iq^{\mathbb{Z}}$ (pointed out in [81]); this is consistent with the $b_1 = b_2 = 0$ case of Theorem 4.49.

The result obtained in Theorem 4.49 corroborates the proposed general asymptotic behaviour of solutions of q-P_{IV} as $t \to 0$, described in Lemma 4.24, and explicitly relates this to the connection matrix of the linear system as $t \to 0$. However, our assumptions regarding the asymptotic behaviour of $v(t)$ are apparently contradicted in the case of $R\left(q^k i\right) = 0$, for any $k \in \{0, 1, 2\}$, as seen by Theorem 4.49.

Eliminating our assumed dominant balance of q-P_{IV} as $t \to 0$, we must assume that these cases correspond to quicksilver solutions of q-P_{IV}, discussed in Remark 4.27. Recall from Chapter 3 (discussed in Section 3.2) that quicksilver solutions are analogous to tronqueé solutions in the case of continuous Painlevé equations, these being asymptotic to a formal series solutions (with no free parameters) in an open, bounded (Stokes-like) region of the complex t-plane, featuring a beyond-all-orders perturbation term which is multiplied by a free parameter (Stokes multiplier).

With this in mind, we arrive at a scenario equivalent to what has been seen in the study of symmetric solutions of continuous P_{IV} via the isomonodromic deformation, whereby specific parameter values lead to the symmetric solutions becoming tronqueé/quicksilver. In this case, we have nonlinear constraints in a_0, a_1, q, where these parameters appear in the arguments of q-theta functions, as opposed to appearing in the arguments of classical exponential functions as seen in the continuous P_{IV} case.

These constraints correspond to vanishing any constants α, β, γ, which were given in (4.44). For instance, $R(q^2 i)$ vanishes, and v becomes singular in Theorem 4.49, when parameters (a_0, a_1, q) satisfy

$$0 = a_1\theta_{q^2}(-a_0a_1)\theta_{q^6}(-a_0q^2/a_1) + q\theta_{q^2}(a_0a_1/q)\theta_{q^6}(a_0q^5/a_1),$$

which we propose is analogous to such conditions as $1 = e^{-i\pi(a-b)} - e^{i\pi(a+b)} - e^{2i\pi b}$, leading to tronqueé-symmetric solutions of (continuous) $\mathrm{P_{IV}}$.

Remark 4.52. Recall that the connection matrix expresses q-monodromy data

$$C(z,t) = \begin{pmatrix} C_1(z,t) & C_2(z,t) \\ -C_1(-z,t) & C_2(-z,t) \end{pmatrix},$$

where in the case of symmetric solutions $C_1(z,t) = -i(-iz)^{-2m}C_2(-z,t)$ for $t = iq^m$ and $m \in \mathbb{Z}$, and we write

$$C_2(z,t) = R(q^2t)\theta_p(-\zeta) + R(qt)\theta_p(-q\zeta)z + R(t)\theta_p(-q^2\zeta)z^2,$$

having set $p = q^3$ and $\zeta = a_0q^2z^3/(a_1t)$. Here, we utilise the basis-functions

$$\left\{\theta_p(-\zeta), \, \theta_p(-q\zeta)\,z, \, \theta_p\left(-q^2\zeta\right)z^2\right\} \tag{4.53}$$

to span the three dimensional vector space $C_2(z,t) \in V_3\left(a_1t/a_0q^2\right)$. Then, taking Lemma 4.47 into account, we see in the case of quicksilver solutions that $C_2(z,t)$ is confined to a hyperplane spanned by (at most) two elements of (4.53).

4.3. Summary

In Section 4.1, we consider symmetric solutions of $\mathrm{P_{IV}}$ and use the method of isomonodromic deformation to deduce asymptotic information regarding these solutions in the limit $|x| \to \infty$. In particular, we consider the ray given by $\arg(x) = \pi/4$, an anti-Stokes ray where exponential effects become oscillatory, and generic solutions depend upon two free parameters (Stokes multipliers). We deduce that in the case of general parameters (a, b) of $\mathrm{P_{IV}}$, this anti-Stokes behaviour is given by Expansion (4.3), and the product of Stokes multipliers is exactly determined by these parameters as described in Section 4.1.5, specifically Equation (4.21). Furthermore, having explicitly related the Stokes multipliers to the linear monodromy data, and thus the parameters of $\mathrm{P_{IV}}$ in the case of symmetric solutions, we deduce conditions that lead to symmetric-tronqueé solutions.

Section 4.2 tackles these concepts in the q-difference setting. We give an explicit relation between the connection matrix of the q-linear problem at the reflection point $t = i$ (which encapsulates monodromy data) and symmetric solutions of q-$\mathrm{P_{IV}}$ in the limit $t \to 0$. We obtain analogous conditions on the parameters of q-$\mathrm{P_{IV}}$, leading to a breakdown in the generic asymptotic behaviour of solutions, and we instead have quicksilver-symmetric solutions. As has also been discussed in Chapter 3 when considering quicksilver solutions of q-$\mathrm{P_{II}}$, these solutions are asymptotic to divergent power series expansions with no free parameters, as is the case with tronqueé solutions in the continuous setting.

We also notice that these exceptional cases correspond to entries of the connection matrix lying within certain subspaces of the usual three-dimensional vector space of q-theta functions (see Remark 4.52). This scenario is analogous to the differential setting, recalling that linear monodromy data are constrained to particular cubic surfaces, further representing moduli spaces of the corresponding Painlevé equations. In this context, tronqueé solutions correspond to lower-dimensional geometric objects (in this case, lines) embedded in the more general manifold.

The analysis of this chapter represents a key step in the development of isomonodromic deformation theory as applied to the q-difference setting.

CHAPTER 5

Concluding Remarks

This thesis delves into the asymptotic analysis of transcendental solutions of nonlinear systems, focusing on continuous and discrete Painlevé equations. We illuminate deep structural connections by investigating the interplay between continuous and discrete regimes and extend our understanding of these transcendental phenomena.

In Chapter 2, we introduced perturbations of the second Painlevé equation, called $\mathrm{P_{II}}(\mu)$. Through asymptotic analysis as the independent variable approaches infinity, we identified exceptional solutions that generalise the classical tronquée, tritronquée, and Hastings-McLeod solutions of $\mathrm{P_{II}}$. These solutions hold a distinguished position in the asymptotic theory of Painlevé equations, and preserving these critical behaviours in the case of $\mathrm{P_{II}}(\mu)$ (where the Painlevé property no longer holds) is encouraging and intriguing.

Through the asymptotic analysis of divergent power series at infinity, we showed that for every $\mu \in \mathbb{Z}^+$, there exists, uniquely, a true solution $y(x)$ of $\mathrm{P_{II}}(\mu)$ which is holomorphic for sufficiently large $|x|$ in a given sector of angular width $4\pi/(\mu+2)$; this is a natural generalisation of Boutroux's tritronqueé solution of $\mathrm{P_{II}}$. This exceptional transcendental solution speaks to an intriguing underlying phenomenon that persists beyond strict integrability, raising questions concerning how such behaviours manifest in similar perturbations of Painlevé equations.

The Hastings-McLeod solution of $\mathrm{P_{II}}$ is also characterised by its unique asymptotic properties. These asymptotic behaviours pertain to the real line, making this solution of grand importance across applications in physics and random matrix theory, as well as intriguing from a purely mathematical point of view. A natural generalisation of this pivotal solution uniquely exists for odd values of $\mu \in \mathbb{Z}^+$ in the $\mathrm{P_{II}}(\mu)$ case. These theoretical insights are amplified as we demonstrated the relevance of $\mathrm{P_{II}}(\mu)$ in a mathematical model of electrodiffusion, suggesting broader applicability for perturbed Painlevé systems. Generally speaking, $\mathrm{P_{II}}(\mu)$ may become relevant when considering a higher-dimensional extension of a classical model in which $\mathrm{P_{II}}$ arises, leading to questions regarding similar perturbations of other Painlevé-type mathematical models.

Chapter 3 transitions to the q-difference setting, examining the q-difference second Painlevé equation in various asymptotic limits. Generic solutions were shown to associate with particular algebraic curves in phase space as $t \to 0$ or $|t| \to \infty$, reflecting genus-zero and genus-one structures, respectively. Exceptional solutions, termed quicksilver solutions, were identified as discrete analogues of tronquée solutions, asymptotic to divergent power series within specific t-plane regions. On Stokes-like boundaries of such power series behaviour given by q-spirals, we deepen the connection between continuous and discrete Painlevé transcendents by identifying novel oscillatory-type expansions, offering the first such characterisation for a q-difference system.

We further endeavoured to extend an understanding of the role of elliptic functions in the generic asymptotic behaviour of a q-Painlevé transcendent. By studying a continuum limit where $q \to 1$, we established that generic solutions are parameterised by elliptic functions to leading-order, with slow modulation in this behaviour described by complete elliptic integrals under repeated iteration in the discrete domain. This work on the elliptic behaviour of q-$\mathrm{P_{II}}$ offers new insight into the role of elliptic functions in discrete nonlinear systems. It raises questions regarding the application of these concepts to other discrete Painlevé equations.

Chapter 4 explored symmetric solutions of the fourth Painlevé equation and its q-difference counterpart. For $\mathrm{P_{IV}}$, we used the method of isomonodromic deformations to characterise asymptotic behaviours of symmetric solutions on an anti-Stokes ray as the independent variable becomes large. This analysis leads to valuable new insight regarding necessary conditions on the parameters of $\mathrm{P_{IV}}$, leading to symmetric solutions that are (tri-)tronquée. Here, we intertwine the exceptional asymptotic properties of Boutroux's tronqueé solution with yet another area of Painlevé analysis: the symmetric solution characterised by distinct analytic behaviour at the origin, leading to a reflective symmetry in the complex plane.

We sought to apply similar ideas regarding symmetric solutions of q-$\mathrm{P_{IV}}$; this revealed conditions on the parameters of q-$\mathrm{P_{IV}}$, leading to the breakdown of generic asymptotic behaviour away from the reflection point, implying quicksilver-symmetric solutions described by divergent asymptotic series with no free parameters. We furthermore described the implication of these conditions on the preserved q-monodromy data, where reflecting the continuous case, we see that this monodromy data is represented by certain subspaces of the usual three-dimensional vector space of q-theta functions. This work offers a new development in the continually evolving connection between the concept of isomonodromic deformation applied to continuous and q-difference settings.

These results have advanced our understanding of the asymptotic structures underpinning Painlevé equations and their discrete analogues. By introducing "perturbations" of the second Painlevé equation and demonstrating the persistence of exceptional transcendental solutions beyond strict integrability, we have provided new insight regarding the robustness of celebrated Painlevé-type asymptotic behaviours. Several promising directions emerge for future research. A natural extension would be to investigate whether perturbations of other Painlevé equations are applicable as physical models and whether important transcendental behaviours are preserved/generalised.

The analysis of quicksilver solutions in the q-difference setting further bridges the gap between continuous and discrete Painlevé systems, revealing that key asymptotic phenomena, such as divergent series expansions and Stokes structures, manifest in both realms, albeit in distinct ways. The novel identification of an oscillatory asymptotic expansion appearing on Stokes-like boundaries in the q-difference setting provides new questions for future study. These bounded/oscillatory behaviours may be important when considering an application of a q-difference Painlevé equation; it is therefore important to explore a comprehensive classification of such asymptotic expansions (as has been achieved in the differential setting).

Moreover, our characterisation of elliptic behaviour in the q-Painlevé setting suggests a more profound role for elliptic functions in studying discrete nonlinear systems, hinting at broader connections between Painlevé analysis, algebraic geometry, and special functions. Another avenue of research lies in further exploring the role of elliptic functions in discrete nonlinear systems by applying the local asymptotic methods of this thesis to other Painlevé difference equations. One might also consider the implication of such behaviours representing isomonodromic deformations of a corresponding discrete linear system.

While our study of symmetric solutions of P_{IV} and q-P_{IV} as isomonodromic deformations has uncovered new thematic links between continuous and discrete regimes, further classification of such solutions across the discrete Painlevé equations may reveal more profound underlying concepts. The q-analogue of an isomonodromic deformation method applied to the global asymptotic analysis of generic q-Painlevé transcendents remains an important open area of research. Considering exceptional transcendental solutions (for instance, solutions we determined to be both symmetric and tronqueé) in the context of physical models, such as those in fluid dynamics, nonlinear optics, or statistical physics, may lead to novel interpretations of Painlevé phenomena in real-world systems, reinforcing the significance of these equations in both pure and applied mathematics.

Bibliography

[1] M. J. Ablowitz, A. Ramani, and H. Segur. "Nonlinear evolution equations and ordinary differential equations of Painlevé type". *Lettere al Nuovo Cimento* 23.9 (1978), pp. 333–338.

[2] M. J. Ablowitz, A. Ramani, and H. Segur. "A connection between nonlinear evolution equations and ordinary differential equations of P-type. II". *Journal of Mathematical Physics* 21.5 (1980), pp. 1006–1015.

[3] M. J. Ablowitz and H. Segur. "Exact linearization of a Painlevé transcendent". *Physical Review Letters* 38.20 (1977), pp. 1103–1106.

[4] H. Airault. "Rational solutions of Painlevé equations". *Studies in Applied Mathematics* 61.1 (1979), pp. 31–53.

[5] D. W. Albrecht, E. L. Mansfield, and A. E. Milne. "Algorithms for special integrals of ordinary differential equations". *Journal of Physics A: Mathematical and General* 29.5 (1996), pp. 973–991.

[6] I. Aniceto, R. Schiappa, and M. Vonk. "The resurgence of instantons in string theory". *Communications in Number Theory and Physics* 6.2 (2012), pp. 339–496.

[7] L. Bass. "Irreversible interactions between metals and electrolytes". *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*. Vol. 277. 1368. The Royal Society London, 1964, pp. 125–136.

[8] L. Bass et al. "Electrical structures of interfaces: a Painlevé II model". *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*. Vol. 466. 2119. The Royal Society Publishing, 2010, pp. 2117–2136.

[9] C. M. Bender and S. A. Orszag. *Advanced mathematical methods for scientists and engineers I: Asymptotic methods and perturbation theory*. Springer Science & Business Media, 2013.

[10] M. Bertola and A. Tovbis. "Asymptotics of orthogonal polynomials with complex varying quartic weight: global structure, critical point behavior and the first Painlevé equation". *Constructive Approximation* 41.3 (2015), pp. 529–587.

[11] G. D. Birkhoff. "The generalized Riemann problem for linear differential equations and the allied problems for linear difference and q-difference

equations". *Proceedings of the American Academy of Arts and Sciences*. Vol. 49. 9. 1913, pp. 521–568.

[12] L. Boelen, G. Filipuk, and W. Van Assche. "Recurrence coefficients of generalized Meixner polynomials and Painlevé equations". *Journal of Physics A: Mathematical and Theoretical* 44.3 (2010), p. 035202.

[13] F. Bornemann et al. "A request: the Painlevé project". *AMSTAT News* 57.11 (2010), p. 1389.

[14] S. Boscolo et al. "Self-similar parabolic optical solitary waves". *Theoretical and Mathematical Physics* 133.3 (2002), pp. 1647–1656.

[15] P. Boutroux. "Recherches sur les transcendantes de M. Painlevé et l'étude asymptotique des équations différentielles du second ordre (suite)". *Annales Scientifiques de l'École Normale Supérieure*. Vol. 31. 1914, pp. 99–159.

[16] P. Boutroux. "Recherches sur les transcendantes de M. Painlevé et l'étude asymptotique des équations différentielles du second ordre". *Annales Scientifiques de l'École Normale Supérieure*. Vol. 30. 1913, pp. 255–375.

[17] E. Brezin and V. Kazakov. "Exactly solvable field theories of closed strings". *Physics Letters B* 236.2 (1990), pp. 144–150.

[18] P. F. Byrd and M. D. Friedman. *Handbook of elliptic integrals for engineers and physicists*. Vol. 67. Springer, 2013.

[19] R. D. Carmichael. "The general theory of linear q-difference equations". *American Journal of Mathematics* 34.2 (1912), pp. 147–168.

[20] S. J. Chapman, J. King, and K. Adams. "Exponential asymptotics and Stokes lines in nonlinear ordinary differential equations". *Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences*. Vol. 454. 1978. The Royal Society, 1998, pp. 2733–2755.

[21] S. J. Chapman and D. B. Mortimer. "Exponential asymptotics and Stokes lines in a partial differential equation". *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*. Vol. 461. 2060. The Royal Society London, 2005, pp. 2385–2421.

[22] S. J. Chapman et al. "Why is a shock not a caustic? The higher-order Stokes phenomenon and smoothed shock formation". *Nonlinearity* 20.10 (2007), pp. 2425–2452.

[23] L. O. Chekhov, M. Mazzocco, and V. N. Rubtsov. "Painlevé monodromy manifolds, decorated character varieties, and cluster algebras". *International Mathematics Research Notices* 2017.24 (2017), pp. 7639–7691.

[24] A. Cherman et al. "Resurgence in quantum field theory: nonperturbative effects in the principal chiral model". *Physical Review Letters* 112.2 (2014), p. 021601.

[25] T. Claeys and T. Grava. "Painlevé II asymptotics near the leading edge of the oscillatory zone for the Korteweg-de Vries equation in the small-dispersion limit". *Communications on Pure and Applied Mathematics* 63.2 (2010), pp. 203–232.

[26] P. A. Clarkson. "Painlevé equations-nonlinear special functions". *Orthogonal Polynomials and Special Functions*. Springer, 2006, pp. 331–411.

[27] P. A. Clarkson and E. L. Mansfield. "The second Painlevé equation, its hierarchy and associated special polynomials". *Nonlinearity* 16.3 (2003), R1–R26.

[28] R. Conte. *The Painlevé property: one century later*. Springer Science & Business Media, 2012.

[29] O. Costin and R. Costin. "On the formation of singularities of solutions of nonlinear differential systems in antistokes directions". *Inventiones Mathematicae* 145.3 (2001), pp. 425–485.

[30] O. Costin, R. Costin, and M. Huang. "Tronquée solutions of the Painlevé equation P_I". *Constructive Approximation* 41.3 (2015), pp. 467–494.

[31] O. Costin and R. Costin. "Asymptotic properties of a family of solutions of the Painlevé equation P_{VI}". *International Mathematics Research Notices* 2002.22 (2002), pp. 1167–1182.

[32] O. Costin and M. D. Kruskal. "On optimal truncation of divergent series solutions of nonlinear differential systems". *Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences*. Vol. 455. 1985. The Royal Society, 1999, pp. 1931–1956.

[33] O. Costin and M. D. Kruskal. "Optical uniform estimates and rigorous asymptotics beyond all orders for a class of ordinary differential equations". *Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences*. Vol. 452. 1948. The Royal Society London, 1996, pp. 1057–1085.

[34] P. De Boer and G. Ludford. "Spherical electric probe in a continuum gas". *Plasma Physics* 17.1 (1975), pp. 29–43.

[35] P. Deift. "Universality for mathematical and physical systems". *International Congress of Mathematicians.* Vol. I. European Mathematical Society, Zürich, 2007, pp. 125–152.

[36] P. Deift and X. Zhou. "A steepest descent method for oscillatory Riemann–Hilbert problems. Asymptotics for the MKdV equation". *Annals of Mathematics* 137.2 (1993), pp. 295–368.

[37] *NIST Digital Library of Mathematical Functions*. http://dlmf.nist.gov/, Release 1.1.7 of 2022-10-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.

[38] M. Duits and A. Kuijlaars. "Painlevé I asymptotics for orthogonal polynomials with respect to a varying quartic weight". *Nonlinearity* 19.10 (2006), pp. 2211–2245.

[39] G. V. Dunne and M. Ünsal. "Resurgence and trans-series in quantum field theory: the $\mathbb{CP}^{N-1}$ model". *Journal of High Energy Physics* 2012.11 (2012), pp. 1–86.

[40] G. Filipuk, W. Van Assche, and L. Zhang. "The recurrence coefficients of semi-classical Laguerre polynomials and the fourth Painlevé equation". *Journal of Physics A: Mathematical and Theoretical* 45.20 (2012), p. 205201.

[41] A. Fokas and M. Ablowitz. "Linearization of the Korteweg-de Vries and Painlevé II equations". *Physical Review Letters* 47.16 (1981), pp. 1096–1100.

[42] A. Fokas, B. Grammaticos, and A. Ramani. "From continuous to discrete Painlevé equations". *Journal of Mathematical Analysis and Applications* 180.2 (1993), pp. 342–360.

[43] A. Fokas and X. Zhou. "On the solvability of Painlevé II and IV". *Communications in Mathematical Physics* 143.2 (1992), pp. 601–622.

[44] A. Fokas, A. Its, and A. Kitaev. "Discrete Painlevé equations and their appearance in quantum gravity". *Communications in Mathematical Physics* 142.2 (1991), pp. 313–344.

[45] A. Fokas et al. *Painlevé transcendents: the Riemann-Hilbert approach*. Vol. 128. American Mathematical Society, 2023.

[46] P. J. Forrester and N. S. Witte. "Painlevé II in random matrix theory and related fields". *Constructive Approximation* 41.3 (2015), pp. 589–613.

[47] R. Fuchs. "Sur quelque équations différentielles linéaires du second ordre". *Comptes Rendus des Séances de l'Académie des Sciences* 141 (1905), pp. 555–558.

[48] R. Fuchs. "Über lineare homogene differentialgleichungen zweiter ordnung mit drei im Endlichen gelegenen wesentlich singulären stellen". *Mathematische Annalen* 63.3 (1907), pp. 301–321.

[49] S. Fukutani, K. Okamoto, and H. Umemura. "Special polynomials and the Hirota bilinear relations of the second and the fourth Painlevé equations". *Nagoya Mathematical Journal* 159 (2000), pp. 179–200.

[50] B. Gambier. "Sur les équations différentielles du second ordre et du premier degré dont l'intégrale générale est à points critiques fixes". *Acta Mathematica* 33.1 (1910), pp. 1–55.

[51] R. Garnier. "Sur des équations différentielles du troisième ordre dont l'intégrale générale est uniforme et sur une classe d'équations nouvelles d'ordre supérieur dont l'intégrale générale a ses points critiques fixes".

Annales Scientifiques de l'École Normale Supérieure. Vol. 29. 1912, pp. 1–126.

[52] G. Gasper and M. Rahman. *Basic hypergeometric series*. Vol. 96. Cambridge University Press, 2011.

[53] B. Grammaticos and A. Ramani. "Discrete Painlevé equations: a review". *Discrete Integrable Systems* 644 (2004), pp. 245–321.

[54] B. Grammaticos, A. Ramani, and V. Papageorgiou. "Do integrable mappings have the Painlevé property?" *Physical Review Letters* 67.14 (1991), pp. 1825–1828.

[55] C. C. Green, C. J. Lustri, and S. W. McCue. "The effect of surface tension on steadily translating bubbles in an unbounded Hele-Shaw cell". *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*. Vol. 473. 2201. The Royal Society Publishing, 2017, pp. 1–20.

[56] V. I. Gromak, I. Laine, and S. Shimomura. *Painlevé differential equations in the complex plane*. Vol. 28. Walter de Gruyter, 2008.

[57] T. Hamamoto, K. Kajiwara, and N. S. Witte. "Hypergeometric solutions to the q-Painlevé equation of type $(A_1 + A_1')^{(1)}$". *International Mathematics Research Notices* 2006.9 (2006), p. 84619.

[58] S. P. Hastings and J. B. Mcleod. "A boundary value problem associated with the second Painlevé transcendent and the Korteweg-de Vries equation". *Archive for Rational Mechanics and Analysis* 73.1 (1980), pp. 31–51.

[59] J. Holroyd and N. Joshi. "On the perturbed second Painlevé equation". *Journal of Physics A: Mathematical and Theoretical* 56.1 (2023), p. 014002.

[60] P. Howes and N. Joshi. "Global asymptotics of the second Painlevé equation in Okamoto's space". *Constructive Approximation* 39.1 (2014), pp. 11–41.

[61] P.-F. Hsieh and Y. Sibuya. *Basic theory of ordinary differential equations*. Springer Science & Business Media, 2012.

[62] E. L. Ince. *Ordinary differential equations*. Dover Books on Mathematics. Dover Publications, New York, 1956.

[63] A. Its, A. Kitaev, and A. Fokas. "The isomonodromy approach in the theory of two-dimensional quantum gravitation". *Russian Mathematical Surveys* 45.6 (1990), pp. 155–157.

[64] A. Its. "The Painlevé transcendents as nonlinear special functions". *Painlevé Transcendents: Their Asymptotics and Physical Applications*. Vol. 278. Springer, 1992, pp. 49–59.

[65] M. Jimbo and T. Miwa. "Monodromy perserving deformation of linear ordinary differential equations with rational coefficients. II". *Physica D: Nonlinear Phenomena* 2.3 (1981), pp. 407–448.

[66] M. Jimbo and H. Sakai. “A q-analog of the sixth Painlevé equation”. *Letters in Mathematical Physics* 38.2 (1996), pp. 145–154.

[67] D. Jones. “Asymptotic behavior of integrals”. *SIAM Review* 14.2 (1972), pp. 286–317.

[68] N. Joshi. “Irregular singular behaviour in the first discrete Painlevé equation”. *Symmetries and Integrability of Difference Equations III* (2000), pp. 237–243.

[69] N. Joshi and C. J. Lustri. “Stokes phenomena in discrete Painlevé I”. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*. Vol. 471. 2177. The Royal Society Publishing, 2015, p. 20140874.

[70] N. Joshi, A. Ramani, and B. Grammaticos. “A bilinear approach to discrete Miura transformations”. *Physics Letters A* 249.1-2 (1998), pp. 59–62.

[71] N. Joshi. *Discrete Painlevé equations*. Vol. 131. American Mathematical Society, 2019.

[72] N. Joshi. “Quicksilver solutions of a q-difference first Painlevé equation”. *Studies in Applied Mathematics* 134.2 (2015), pp. 233–251.

[73] N. Joshi. “The connection problem for the first and the second Painlevé transcendents”. PhD thesis. Princeton University, 1987.

[74] N. Joshi. “Tritronquée solutions of perturbed first Painlevé equations”. *Theoretical and Mathematical Physics* 137.2 (2003), pp. 1515–1519.

[75] N. Joshi and M. D. Kruskal. “An asymptotic approach to the connection problem for the first and the second Painlevé equations”. *Physics Letters A* 130.3 (1988), pp. 129–137.

[76] N. Joshi and M. D. Kruskal. “The Painlevé connection problem: an asymptotic approach. I”. *Studies in Applied Mathematics* 86.4 (1992), pp. 315–376.

[77] N. Joshi and E. Liu. “Asymptotic behaviours given by elliptic functions in $\mathrm{P_I}$-$\mathrm{P_V}$”. *Nonlinearity* 31.8 (2018), pp. 3726–3747.

[78] N. Joshi, C. J. Lustri, and S. Luu. “Nonlinear q-Stokes phenomena for q-Painlevé I”. *Journal of Physics A: Mathematical and Theoretical* 52.6 (2019), p. 065204.

[79] N. Joshi, C. J. Lustri, and S. Luu. “Stokes phenomena in discrete Painlevé II”. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*. Vol. 473. 2198. The Royal Society Publishing, 2017, p. 20160539.

[80] N. Joshi and P. Roffelsen. “Analytic solutions of qP (A1) near its critical points”. *Nonlinearity* 29.12 (2016), p. 3696.

[81] N. Joshi and P. Roffelsen. “On symmetric solutions of the fourth q-Painlevé equation”. *Journal of Physics A: Mathematical and Theoretical* 56.18 (2023), p. 185201.

[82] N. Joshi and P. Roffelsen. “On the monodromy manifold of q-Painlevé VI and its Riemann–Hilbert problem”. *Communications in Mathematical Physics* 404.1 (2023), pp. 97–149.

[83] N. Joshi and P. Roffelsen. “On the Riemann-Hilbert problem for a q-difference Painlevé equation”. *Communications in Mathematical Physics* 384.1 (2021), pp. 549–585.

[84] N. Joshi and Y. Takei. “On Stokes phenomena for the alternate discrete P_I equation”. *Analytic, Algebraic and Geometric Aspects of Differential Equations*. Springer. 2017, pp. 369–381.

[85] K. Kajiwara. “The discrete Painlevé II equation and the classical special functions”. *Symmetries and Integrability of Difference Equations*. Cambridge University Press, 1999, pp. 217–227.

[86] K. Kajiwara, M. Noumi, and Y. Yamada. “Geometric aspects of Painlevé equations”. *Journal of Physics A: Mathematical and Theoretical* 50.7 (2017), p. 073001.

[87] K. Kajiwara, K. Yamamoto, and Y. Ohta. “Rational solutions for the discrete Painlevé II equation”. *Physics Letters A* 232.3-4 (1997), pp. 189–199.

[88] K. Kajiwara et al. “Casorati determinant solutions for the discrete Painlevé II equation”. *Journal of Physics A: Mathematical and General* 27.3 (1994), pp. 915–922.

[89] K. Kajiwara et al. “Hypergeometric solutions to the q-Painlevé equations”. *International Mathematics Research Notices* 2004.47 (2004), pp. 2497–2521.

[90] K. Kaneko. “A new solution of the fourth Painleve equation with a solvable monodromy”. *Proceedings of the Japan Academy. Series. A, Mathematical Sciences* 81.5 (2005), pp. 75–79.

[91] A. Kapaev. “Asymptotics of solutions of the Painlevé equation of the first kind”. *Nonlinear Studies Preprint* (1988).

[92] J. King and S. J. Chapman. “Asymptotics beyond all orders and Stokes lines in nonlinear differential-difference equations”. *European Journal of Applied Mathematics* 12.4 (2001), pp. 433–463.

[93] A. Kitaev. “Elliptic asymptotics of the first and the second Painlevé transcendents”. *Russian Mathematical Surveys* 49.1 (1994), pp. 81–150.

[94] A. Kitaev. “Symmetric solutions for the first and second Painlevé equations”. *Journal of Mathematical Sciences* 73.4 (1995), pp. 494–499.

[95] A. Kitaev and D. Korotkin. “On solutions of the Schlesinger equations in terms of ζ-functions”. *International Mathematics Research Notices* 1998.17 (1998), pp. 877–905.

[96] A. Knizel. “Moduli spaces of q-connections and gap probabilities”. *International Mathematics Research Notices* 2016.22 (2016), pp. 6921–6954.

[97] M. D. Kruskal and P. A. Clarkson. “The Painlevé-Kowalevski and poly-Painlevé tests for integrability”. *Studies in Applied Mathematics* 86.2 (1992), pp. 87–165.

[98] M. D. Kruskal and H. Segur. “Asymptotics beyond all orders in a model of crystal growth”. *Studies in Applied Mathematics* 85.2 (1991), pp. 129–181.

[99] E. Laguerre. “Sur la réduction en fractions continues d’une fraction qui satisfait à une équation différentielle linéaire du premier ordre dont les coefficients sont rationnels”. *Journal de Mathématiques Pures et Appliquées* 1 (1885), pp. 135–165.

[100] E. Landau. *Handbuch der lehre von der verteilung der primazahlen*. Vol. 1. BG Teubner, 1909.

[101] J. Le Caine. “The linear q-difference equation of the second order”. *American Journal of Mathematics* 65.4 (1943), pp. 585–600.

[102] X. Lü and M. Peng. “Painlevé-integrability and explicit solutions of the general two-coupled nonlinear Schrödinger system in the optical fiber communications”. *Nonlinear Dynamics* 73.1-2 (2013), pp. 405–410.

[103] C. J. Lustri and S. J. Chapman. “Steady gravity waves due to a submerged source”. *Journal of Fluid Mechanics* 732 (2013), pp. 660–686.

[104] C. J. Lustri and S. J. Chapman. “Unsteady flow over a submerged source with low Froude number”. *European Journal of Applied Mathematics* 25.5 (2014), pp. 655–680.

[105] C. J. Lustri, S. W. McCue, and B. J. Binder. “Free surface flow past topography: a beyond-all-orders approach”. *European Journal of Applied Mathematics* 23.4 (2012), pp. 441–467.

[106] C. J. Lustri, S. W. McCue, and S. J. Chapman. “Exponential asymptotics of free surface flow due to a line source”. *The IMA Journal of Applied Mathematics* 78.4 (2013), pp. 697–713.

[107] A. Magnus. “Freud’s equations for orthogonal polynomials as discrete Painlevé equations”. *Symmetries and Integrability of Difference Equations*. Cambridge University Press, 1999, pp. 228–244.

[108] A. Magnus. “Painlevé-type differential equations for the recurrence coefficients of semi-classical orthogonal polynomials”. *Journal of Computational and Applied Mathematics* 57.1-2 (1995), pp. 215–237.

[109] T. Mano. “Asymptotic behaviour around a boundary point of the q-Painlevé VI equation and its connection problem”. *Nonlinearity* 23.7 (2010), pp. 1585–1608.

[110] M. Mazzocco. “Picard and Chazy solutions to the Painlevé VI equation”. *Mathematische Annalen* 321.1 (2001), pp. 157–195.

[111] B. M. McCoy, C. A. Tracy, and T. T. Wu. “Painlevé functions of the third kind”. *Journal of Mathematical Physics* 18.5 (1977), pp. 1058–1092.

[112] T. Morita. “The Stokes phenomenon for the q-difference equation satisfied by the Ramanujan entire function”. *The Ramanujan Journal* 34.3 (2014), pp. 329–346.

[113] F. Nijhoff. “Discrete Painlevé equations and symmetry reduction on the lattice”. *Oxford Lecture Series in Mathematics and its Applications* 16 (1999), pp. 209–234.

[114] F. Nijhoff and V. Papageorgiou. “Similarity reductions of integrable lattices and discrete analogues of the Painlevé II equation”. *Physics Letters A* 153.6-7 (1991), pp. 337–344.

[115] Y. Ohyama, J.-P. Ramis, and J. Sauloy. “The space of monodromy data for the Jimbo–Sakai family of q-difference equations”. *Annales de la Faculté des Sciences de Toulouse: Mathématiques*. Vol. 29. 5. 2020, pp. 1119–1250.

[116] K. Okamoto. “Studies on the Painlevé equations: III. Second and fourth painlevé equations, P_{II} and P_{IV}”. *Mathematische Annalen* 275.2 (1986), pp. 221–255.

[117] K. Okamoto. “Sur les feuilletages associés aux équation du second ordre à points critiques fixes de P. Painlevé spaces des conditions initiales”. *Japanese Journal of Mathematics. New Series* 5.1 (1979), pp. 1–79.

[118] A. Olde Daalhuis. “Hyperasymptotics for nonlinear ODEs I. A Riccati equation”. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*. Vol. 461. 2060. The Royal Society London, 2005, pp. 2503–2520.

[119] A. Olde Daalhuis. “Hyperasymptotics for nonlinear ODEs II. The first Painlevé equation and a second-order Riccati equation”. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*. Vol. 461. 2062. The Royal Society London, 2005, pp. 3005–3021.

[120] A. Olde Daalhuis. “Inverse factorial-series solutions of difference equations”. *Proceedings of the Edinburgh Mathematical Society*. Vol. 47. 2. Cambridge University Press, 2004, pp. 421–448.

[121] A. Olde Daalhuis. “Exponentially-improved asymptotics and numerics for the (un) perturbed first Painlevé equation”. *Journal of Physics A Mathematical General* 55.30 (2022), p. 304004.

[122] F. Olver. *Asymptotics and special functions*. CRC Press, 1997.

[123] F. Olver. “Resurgence in difference equations, with an application to Legendre functions”. *Special Functions*. World Scientific, 2000, pp. 221–235.

[124] F. Olver. “Uniform, exponentially improved, asymptotic expansions for the generalized exponential integral”. *Journal on Mathematical Analysis* 22.5 (1991), pp. 1460–1474.

[125] S. Olver. “Numerical solution of Riemann–Hilbert problems: Painlevé II”. *Foundations of Computational Mathematics* 11.2 (2011), pp. 153–179.

[126] P. Painlevé. "Mémoire sur les équations différentielles dont l'intégrale générale est uniforme". *Bulletin de la Société Mathématique de France* 28 (1900), pp. 201–261.

[127] P. Painlevé. "Sur les équations différentielles du second ordre et d'ordre supérieur dont l'intégrale générale est uniforme". *Acta Mathematica* 25.1 (1902), pp. 1–85.

[128] R. Paris. "Smoothing of the Stokes phenomenon using Mellin-Barnes integrals". *Journal of Computational and Applied Mathematics* 41.1-2 (1992), pp. 117–133.

[129] R. Paris and A. D. Wood. "Exponentially-improved asymptotics for the gamma function". *Journal of Computational and Applied Mathematics* 41.1-2 (1992), pp. 135–143.

[130] V. Periwal and D. Shevitz. "Exactly solvable unitary matrix models: multicritical potentials and correlations". *Nuclear Physics B* 344.3 (1990), pp. 731–746.

[131] V. Periwal and D. Shevitz. "Unitary-matrix models as exactly solvable string theories". *Physical Review Letters* 64.12 (1990), pp. 1326–1329.

[132] H. Poincaré. "Sur les intégrales irrégulières: Des équations linéaires". *Acta Mathematica* 8.1 (1886), pp. 295–344.

[133] G. R. W. Quispel, J. A. Roberts, and C. J. Thompson. "Integrable mappings and soliton equations". *Physics Letters A* 126.7 (1988), pp. 419–421.

[134] A. Ramani and B. Grammaticos. "Discrete Painlevé equations: coalescences, limits and degeneracies". *Physica A: Statistical Mechanics and its Applications* 228.1-4 (1996), pp. 160–171.

[135] A. Ramani, B. Grammaticos, and K. Tamizhmani. "Painlevé analysis and singularity confinement: the ultimate conjecture". *Journal of Physics A: Mathematical and General* 26.2 (1993), pp. L53–L58.

[136] A. Ramani et al. "Special function solutions of the discrete Painlevé equations". *Computers & Mathematics with Applications* 42.3-5 (2001), pp. 603–614.

[137] A. Ramani, B. Grammaticos, and J. Hietarinta. "Discrete versions of the Painlevé equations". *Physical Review Letters* 67.14 (1991), pp. 1829–1832.

[138] A. Ramani et al. "The redemption of singularity confinement". *Journal of Physics A: Mathematical and Theoretical* 48.11 (2015), 11FT02.

[139] P. Roffelsen. "On the global asymptotic analysis of a q-discrete Painlevé equation". PhD thesis. University of Sydney, 2017.

[140] H. Sakai. "Rational surfaces associated with affine root systems and geometry of the Painlevé equations". *Communications in Mathematical Physics* 220.1 (2001), pp. 165–229.

[141] J. Satsuma et al. “Bilinear discrete Painlevé II and its particular solutions”. *Journal of Physics A: Mathematical and General* 28.12 (1995), pp. 3541–3548.

[142] J. Sauloy. “Galois theory of q-difference equations: The "analytical" approach”. *Differential Equations and the Stokes Phenomenon*. World Scientific, 2002, pp. 277–292.

[143] R. Schiappa and R. Vaz. “The resurgence of instantons: multi-cut Stokes phases and the Painlevé II equation”. *Communications in Mathematical Physics* 330.2 (2014), pp. 655–721.

[144] N. Seiberg and D. Shih. “Flux vacua and branes of the minimal superstring”. *Journal of High Energy Physics* 2005.01 (2005), pp. 1259–1296.

[145] J. Shohat. “A differential equation for orthogonal polynomials”. *Duke Mathematical Journal* 5.2 (1939), pp. 401–417.

[146] S. Tanveer. “Analytic theory for the determination of velocity and stability of bubbles in a Hele-Shaw cell: Part I: Velocity selection”. *Theoretical and Computational Fluid Dynamics* 1.3 (1989), pp. 135–163.

[147] J. Thomae. “Beiträge zur theorie der durch die Heinesche reihe: darstellbaren functionen.” *Journal für die Reine und Angewandte Mathematik* 70 (1869), pp. 258–281.

[148] J. Thomae. “Les séries Heinéennes supérieures, ou les séries de la forme”. *Annali di Matematica Pura ed Applicata (1867-1897)* 4.1 (1870), pp. 105–138.

[149] T. Tokihiro, B. Grammaticos, and A. Ramani. “From the continuous $\mathrm{P_V}$ to discrete Painlevé equations”. *Journal of Physics A: Mathematical and General* 35.28 (2002), pp. 5943–5950.

[150] C. A. Tracy and H. Widom. “Level-spacing distributions and the Airy kernel”. *Communications in Mathematical Physics* 159.1 (1994), pp. 151–174.

[151] C. A. Tracy and H. Widom. “Random unitary matrices, permutations and Painlevé”. *Communications in Mathematical Physics* 207.3 (1999), pp. 665–685.

[152] H. Umemura. “Invitation to Galois theory”. *Differential Equations and Quantum Groups*. Vol. 9. Irma Lectures in Mathematics and Theoretical Physics. European Mathematical Society, Zürich, 2007, pp. 269–289.

[153] H. Umemura and H. Watanabe. “Solutions of the second and fourth Painlevé equations, I”. *Nagoya Mathematical Journal* 148 (1997), pp. 151–198.

[154] W. Van Assche. “Discrete Painlevé equations for recurrence coefficients of orthogonal polynomials”. *Difference Equations, Special Functions and Orthogonal Polynomials*. World Scientific, 2007, pp. 687–725.

[155] W. Van Assche, G. Filipuk, and L. Zhang. “Multiple orthogonal polynomials associated with an exponential cubic weight”. *Journal of Approximation Theory* 190.1 (2015), pp. 1–25.

[156] M. Van der Put and M.-H. Saito. “Moduli spaces for linear differential equations and the Painlevé equations”. *Annales de l’Institut Fourier*. Vol. 59. 7. 2009, pp. 2611–2667.

[157] W. Wasow. *Asymptotic expansions for ordinary differential equations*. Courier Dover Publications, 2018.

[158] G. N. Watson. “The continuation of functions defined by generalized hypergeometric series”. *Transactions of the Cambridge Philosophical Society* 21 (1910), pp. 281–299.

[159] E. T. Whittaker and G. N. Watson. *A course of modern analysis: an introduction to the general theory of infinite processes and of analytic functions; with an account of the principal transcendental functions*. University Press, 1920.

[160] R. Wong. *Asymptotic approximations of integrals*. SIAM, 2001.

[161] T. T. Wu et al. “Spin-spin correlation functions for the two-dimensional Ising model: Exact theory in the scaling region”. *Physical Review B* 13.1 (1976), pp. 316–374.

[162] S. Xu and Y. Zhao. “Asymptotics of discrete Painlevé V transcendents via the Riemann–Hilbert approach”. *Studies in Applied Mathematics* 130.3 (2013), pp. 201–231.

[163] C. Zhang. “Une sommation discrète pour des équations aux q-différences linéaires et à coefficients analytiques: théorie générale et exemples”. *Differential Equations and the Stokes Phenomenon*. World Scientific, 2002, pp. 309–329.

APPENDIX A

Analysis of Series Coefficients

Primarily, this appendix supplies a formal proof of the asymptotic behaviour (3.16) of coefficients d_n (of the vanishing q-$\mathrm{P_{II}}$ quicksilver solution formal series) as $n \to \infty$ (specifically, in Section A3). We remove ambiguity by confining our argument to the case $\arg a = \pi$, i.e., $0 > a \in \mathbb{R}$. We show an explicit estimate for the constant c_∞ in (3.18), arbitrarily taking $|a| = 1$ and assuming bound $q > e^2$.

We first demonstrate the development of the full nonlinear difference equations governing the coefficients in the growing and decaying divergent power-series expansions of q-$\mathrm{P_{II}}$. To this end, we remark on an identity that proves a core starting point for analysing such series coefficients.

Remark A.1. The following identity for reordering an arbitrary n number of sums is true.

$$\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_n=0}^{\infty} = \sum_{r=0}^{\infty}\sum_{k_1=0}^{r}\sum_{k_2=0}^{r-k_1}\cdots\sum_{k_{n-1}=0}^{r-k_1-\cdots-k_{n-2}},$$

where $r = k_1 + k_2 + \cdots + k_n$.

A1. Growing Quicksilver Solution of q-$\mathrm{P_{II}}$

Substituting the formal series expansion G_n into q-$\mathrm{P_{II}}$ gives

$$-\frac{a}{t_n^4} + \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} g_j g_k g_l g_m q^{2(m-j)} t_n^{-2(j+k+l+m)}$$
$$= -\frac{a}{t_n^2}\sum_{k=0}^{\infty} g_k t_n^{-2k} + \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} g_j g_k g_l q^{2(l-j)} t_n^{-2(j+k+l)}.$$

Reordering the sums we obtain

$$-\frac{a}{t_n^4} + \sum_{r=0}^{\infty}\sum_{j=0}^{r}\sum_{k=0}^{r-j}\sum_{l=0}^{r-j-k} g_j g_k g_l g_{r-j-k-l} q^{2(r-2j-k-l)} t_n^{-2r}$$
$$= -\frac{a}{t_n^2}\sum_{r=0}^{\infty} g_r t_n^{-2r} + \sum_{r=0}^{\infty}\sum_{j=0}^{r}\sum_{k=0}^{r-j} g_j g_k g_{r-j-k} q^{2(r-2j-k)} t_n^{-2r}.$$

Since $g_0 = 1$ this becomes

$$-\frac{a}{t_n^4} + \sum_{r=1}^{\infty}\sum_{j=0}^{r}\sum_{k=0}^{r-j}\sum_{l=0}^{r-j-k} g_j g_k g_l g_{r-j-k-l} q^{2(r-2j-k-l)} t_n^{-2r}$$
$$= -a\sum_{r=1}^{\infty} g_{r-1} t_n^{-2r} + \sum_{r=1}^{\infty}\sum_{j=0}^{r}\sum_{k=0}^{r-j} g_j g_k g_{r-j-k} q^{2(r-2j-k)} t_n^{-2r},$$

and we eliminate $\mathcal{O}(1/t^2)$ terms by setting $g_1 = -a$, giving

$$-\frac{a}{t_n^4} + \sum_{r=2}^{\infty}\sum_{j=0}^{r}\sum_{k=0}^{r-j}\sum_{l=0}^{r-j-k} g_j g_k g_l g_{r-j-k-l} q^{2(r-2j-k-l)} t_n^{-2r}$$
$$= -a\sum_{r=2}^{\infty} g_{r-1} t_n^{-2r} + \sum_{r=2}^{\infty}\sum_{j=0}^{r}\sum_{k=0}^{r-j} g_j g_k g_{r-j-k} q^{2(r-2j-k)} t_n^{-2r}.$$

The $\mathcal{O}(1/t^4)$ terms are eliminated by setting $g_2 = a - a^2q^2 - a^2/q^2$, giving

$$\sum_{r=3}^{\infty}\sum_{j=0}^{r}\sum_{k=0}^{r-j}\sum_{l=0}^{r-j-k} g_j g_k g_l g_{r-j-k-l} q^{2(r-2j-k-l)} t_n^{-2r}$$
$$= -a\sum_{r=3}^{\infty} g_{r-1} t_n^{-2r} + \sum_{r=3}^{\infty}\sum_{j=0}^{r}\sum_{k=0}^{r-j} g_j g_k g_{r-j-k} q^{2(r-2j-k)} t_n^{-2r}.$$

Hence for $r \geq 3$ we have

$$a g_{r-1} + \sum_{j=0}^{r}\sum_{k=0}^{r-j}\sum_{l=0}^{r-j-k} g_j g_k g_l g_{r-j-k-l} q^{2(r-2j-k-l)}$$
$$= \sum_{j=0}^{r}\sum_{k=0}^{r-j} g_j g_k g_{r-j-k} q^{2(r-2j-k)},$$

which, more concisely, is equivalent to

$$a g_{r-1} + \sum_{j=0}^{r-1}\sum_{k=0}^{r-j-1}\sum_{l=1}^{r-j-k} g_j g_k g_l g_{r-j-k-l} q^{2(r-2j-k-l)} = 0.$$

Rearranging for g_r gives

$$g_r = -a g_{r-1} - \sum_{j=1}^{r-1}\sum_{k=0}^{r-j-1}\sum_{l=1}^{r-j-k} g_j g_k g_l g_{r-j-k-l} q^{2(r-2j-k-l)}$$
$$- \sum_{k=1}^{r-1}\sum_{l=0}^{r-k} g_k g_l g_{r-k-l} q^{2(r-k-l)}.$$

Similarly isolating g_{r-1} we obtain

$$\begin{aligned} g_r =& a\left(q^{2r-2}+q^2+1+q^{-2}+q^{2-2r}\right)g_{r-1} \\ &-\sum_{j=2}^{r-2}\sum_{k=0}^{r-j-1}\sum_{l=1}^{r-j-k} g_j g_k g_l g_{r-j-k-l} q^{2(r-2j-k-l)} + \frac{a}{q^2}\sum_{k=1}^{r-2} g_k g_{r-k-1} q^{2(r-k)} \\ &+\frac{a}{q^4}\sum_{k=1}^{r-2}\sum_{l=0}^{r-k-1} g_k g_l g_{r-k-l-1} q^{2(r-k-l)} - \sum_{k=2}^{r-2}\sum_{l=0}^{r-k} g_k g_l g_{r-k-l} q^{2(r-k-l)}. \end{aligned}$$

If we transform the sequence by writing $g_r = a^r h_r$, then we entirely remove the dependence of this equation on a, obtaining

$$\begin{aligned} h_r =& \left(q^{2r-2}+q^2+1+q^{-2}+q^{2-2r}\right)h_{r-1} \\ &-\sum_{j=2}^{r-2}\sum_{k=0}^{r-j-1}\sum_{l=1}^{r-j-k} h_j h_k h_l h_{r-j-k-l} q^{2(r-2j-k-l)} + \frac{1}{q^2}\sum_{k=1}^{r-2} h_k h_{r-k-1} q^{2(r-k)} \\ &+\frac{1}{q^4}\sum_{k=1}^{r-2}\sum_{l=0}^{r-k-1} h_k h_l h_{r-k-l-1} q^{2(r-k-l)} - \sum_{k=2}^{r-2}\sum_{l=0}^{r-k} h_k h_l h_{r-k-l} q^{2(r-k-l)}. \end{aligned}$$

Without loss of generality, take the case $|q| > 1$. Let's assume that for $r \gg 1$, the coefficients h_r are given to leading order by $h_r = \left(q^{2r}/q^2\right)h_{r-1}$. This implies $h_r \sim Kq^{r(r-1)}$ for some constant K. Continuing this train of thought, we obtain

$$h_r = Kq^{r(r-1)}\left(1 - \frac{a-q^2+2aq^4}{a(q-1)(q+1)}q^{-2r} + \mathcal{O}\left(q^{-4r}\right)\right), \quad r \gg 1.$$

A2. Vanishing Quicksilver Solution of q-P$_{\text{II}}$

Substituting formal series D_n into q-P$_{\text{II}}$ gives

$$\begin{aligned} &\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} d_j d_k d_l d_m q^{2(m-j)} t_n^{-2(j+k+l+m)} - at_n^4 \\ =& -at_n^4\sum_{r=0}^{\infty} d_r t_n^{-2r} + t_n^2\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} d_j d_k d_l q^{2(l-j)} t_n^{-2(j+k+l)}, \end{aligned}$$

and rearranging the orders of summation yields

$$\begin{aligned} &\sum_{r=0}^{\infty}\sum_{j=0}^{r}\sum_{k=0}^{r-j}\sum_{l=0}^{r-j-k} d_j d_k d_l d_{r-j-k-l} q^{2(r-2j-k-l)} t_n^{-2r} - at_n^4 \\ =& -at_n^4\sum_{r=0}^{\infty} d_r t_n^{-2r} + t_n^2\sum_{r=0}^{\infty}\sum_{j=0}^{r}\sum_{k=0}^{r-j} d_j d_k d_{r-j-k} q^{2(r-2j-k)} t_n^{-2r}. \end{aligned}$$

Thus using $d_0 = 1$, we see that

$$\sum_{r=0}^{\infty}\sum_{j=0}^{r}\sum_{k=0}^{r-j}\sum_{l=0}^{r-j-k} d_j d_k d_l d_{r-j-k-l} q^{2(r-2j-k-l)} t_n^{-2r}$$
$$= -\, a t_n^4 \sum_{r=1}^{\infty} d_r t_n^{-2r} + t_n^2 \sum_{r=0}^{\infty}\sum_{j=0}^{r}\sum_{k=0}^{r-j} d_j d_k d_{r-j-k} q^{2(r-2j-k)} t_n^{-2r}.$$

Furthermore, by setting $d_1 = 1/a$ we have

$$\sum_{r=0}^{\infty}\sum_{j=0}^{r}\sum_{k=0}^{r-j}\sum_{l=0}^{r-j-k} d_j d_k d_l d_{r-j-k-l} q^{2(r-2j-k-l)} t_n^{-2r}$$
$$= -\, a \sum_{r=2}^{\infty} d_r t_n^{-2r+4} + \sum_{r=1}^{\infty}\sum_{j=0}^{r}\sum_{k=0}^{r-j} d_j d_k d_{r-j-k} q^{2(r-2j-k)} t_n^{-2r+2}.$$

Therefore, for $r \geq 0$ it is true that

$$a d_{r+2} = q^2 \sum_{j=0}^{r+1}\sum_{k=0}^{r-j+1} d_j d_k d_{r-j-k+1} q^{2(r-2j-k)}$$
$$- \sum_{j=0}^{r}\sum_{k=0}^{r-j}\sum_{l=0}^{r-j-k} d_j d_k d_l d_{r-j-k-l} q^{2(r-2j-k-l)}.$$

If we isolate the d_{r+1} term, we obtain

$$a d_{r+2} = \left(q^{2r+2} + 1 + q^{-2r-2}\right) d_{r+1}$$
$$+ q^2 \sum_{j=1}^{r}\sum_{k=0}^{r-j+1} d_j d_k d_{r-j-k+1} q^{2(r-2j-k)} + q^2 \sum_{j=1}^{r} d_j d_{r-j+1} q^{2(r-j)}$$
$$- \sum_{j=0}^{r}\sum_{k=0}^{r-j}\sum_{l=0}^{r-j-k} d_j d_k d_l d_{r-j-k-l} q^{2(r-2j-k-l)}.$$

For these coefficients, assuming $d_{r+2} \gg d_{r+1} \gg \ldots$, we similarly obtain an expansion

$$d_r = K a^{-r} q^{r(r-1)} \left(1 - \frac{1 + 2q^2 - aq^2}{(q-1)(q+1)} q^{-2r} + \mathcal{O}\left(q^{-4r}\right)\right), \quad r \gg 1.$$

A3. Proof of Coefficient Growth

We now give explicit proof of the above divergent asymptotic behaviour. We restrict ourselves to the case $1 < q \in \mathbb{R}$ and $0 > a \in \mathbb{R}$, also eventually taking $q > e^2$ to obtain explicit estimates, although these bounds may be generalised.

First, make the transformation $d_r = a^{-r}q^{r(r-1)}b_r$, we have $b_0 = b_1 = 1$ and

$$b_{r+2} = \left(1 + q^{-2r-2} + q^{-4r-4}\right) b_{r+1} + \sum_{j=1}^{r} \sum_{k=0}^{r-j+1} q^{G(r,j,k)} b_j b_k b_{r-j-k+1} + \sum_{j=1}^{r} q^{F(r,j)} b_j b_{r-j+1} - a \sum_{j=0}^{r} \sum_{k=0}^{r-j} \sum_{l=0}^{r-j-k} q^{H(r,j,k,l)} b_j b_k b_l b_{r-j-k-l},$$

where

$$\begin{aligned} &F(r,j) = -2j(2-j+r),\\ &G(r,j,k) = 2\left(-3j + j^2 - 2k + jk + k^2 - jr - kr\right),\\ &H(r,j,k,l)\\ =&2\left(-1 - 2j + j^2 - k + jk + k^2 - l + jl + kl + l^2 - r - jr - kr - lr\right). \end{aligned}$$

As mentioned, we shall remove some ambiguity by taking the example $1 < q \in \mathbb{R}$ and $0 > a \in \mathbb{R}$. Considering the part of the equation involving b_{r+2} and b_{r+1} (the leading coefficients), we see that

$$b_{r+1} = \left(1 + q^{-2r} + q^{-4r}\right) b_r = \left(\frac{1 - p_2^r}{1 - p_1^r}\right) b_r, \quad r > 0,$$

where $p_1 = 1/q^2$ and $p_2 = 1/q^6$, implying that we should make the transformation

$$b_r = \frac{(p_2; p_2)_{r-1}}{(p_1; p_1)_{r-1}} c_r, \quad r > 0 \quad \text{with} \quad c_0 = 3.$$

Thus we obtain

$$\begin{aligned} c_{r+2} - c_{r+1} =& \frac{(p_1; p_1)_{r+1}}{(p_2; p_2)_{r+1}} \sum_{j=1}^{r} q^{F(r,j)} b_j b_{r-j+1}\\ &+ \frac{(p_1; p_1)_{r+1}}{(p_2; p_2)_{r+1}} \sum_{j=1}^{r} \sum_{k=0}^{r-j+1} q^{G(r,j,k)} b_j b_k b_{r-j-k+1}\\ &+ |a| \frac{(p_1; p_1)_{r+1}}{(p_2; p_2)_{r+1}} \sum_{j=0}^{r} \sum_{k=0}^{r-j} \sum_{l=0}^{r-j-k} q^{H(r,j,k,l)} b_j b_k b_l b_{r-j-k-l}, \end{aligned}$$

Now, we define a couple of useful functions

$$\begin{aligned} \tau_q(n) =& \frac{1 - p_1^n}{1 - p_2^n}, \quad n > 0 \quad \text{with} \quad \tau(0) = 1/3,\\ \psi_q(m,n) =& \begin{bmatrix} m \\ n \end{bmatrix}_{p_1} \begin{bmatrix} m \\ n \end{bmatrix}_{p_2}^{-1}, \quad m \geq n \geq 0, \end{aligned}$$

where here we are referring to the q-binomial coefficients (also known as Gaussian binomial coefficients), and we introduce the shorthand $\tau(n_1)\tau(n_2)\cdots = \tau(n_1, n_2, \ldots)$.

Using these notations, with $c_0 = 3$ and $c_1 = 1$, for any $r \geq 0$ the coefficients c_r are given by the following equation:

$$\begin{aligned} c_{r+2} - c_{r+1} =& \sum_{j=1}^{r} \left\{ q^{F(r,j)} + q^{G(r,j,r-j+1)} + q^{G(r,j,0)} \right\} A(r,j) c_j c_{r-j+1} \\ &+ \sum_{j=1}^{r-1} \sum_{k=1}^{r-j} q^{G(r,j,k)} B(r,j,k) c_j c_k c_{r-j-k+1} \\ &+ |a| \sum_{j=0}^{r} \sum_{k=0}^{r-j} \sum_{l=0}^{r-j-k} q^{H(r,j,k,l)} C(r,j,k,l) c_j c_k c_l c_{r-j-k-l}, \end{aligned}$$

where

$$\begin{aligned} A(r,j) =& \tau_q(r+1,j)\psi_q(r,j), \\ B(r,j,k) =& \tau_q(r+1,j,k)\psi_q(r,j+k)\psi_q(j+k,j), \\ C(r,j,k) =& \tau_q(r+1, r-j-k-l, j, k, l)\psi_q(r, j+k+l) \\ & \times \psi_q(j+k+l, j+k)\psi_q(j+k,j). \end{aligned}$$

Considering the functions F, G, H, we find that for indices j, k, r in the appropriate spaces given the above sums, it is true that

$$\begin{aligned} &F(r,j) \leq -2(1+r), \quad G(r,j,k) \leq -4(1+r), \quad H(r,j,k,l) \leq -2(1+r), \\ &G(r,j,r-j+1) \leq -4(1+r), \quad G(r,j,0) \leq -2(2+r). \end{aligned}$$

Now, we note that the quantity $\psi_q(m,n)$, for all $m, n \in \mathbb{N}$ with $0 \leq n \leq m$, is bounded by some finite value which depends on q but not on m, n. This constant may be expressed by $\Psi_q = \lim_{m\to\infty} \psi_q(2m, m)$. We will assume $q > e^2$ to simplify explicit estimates, though this may be improved to any fixed $q > 1$. In this example we have $\Psi_q < \Psi_{e^2} \approx 1.019$ (this number approaches one as q becomes large). Also note that evidently $\tau_q(n) < 1$ for all $n \in \mathbb{N}$ and $q > 1$.

Therefore, the coefficients c_r are monotonically increasing for $r \geq 1$, and for all $r \geq 0$ we have

$$\begin{aligned} c_{r+2} - c_{r+1} <& \Psi_{e^2} \sum_{j=1}^{r} q^{-4-4r} \left(1 + q^{2r} + q^{2+2r}\right) c_j c_{r-j+1} \\ &+ \Psi_{e^2}^2 \sum_{j=1}^{r-1} \sum_{k=1}^{r-j} q^{-4(1+r)} c_j c_k c_{r-j-k+1} \\ &+ |a| \Psi_{e^2}^3 \sum_{j=0}^{r} \sum_{k=0}^{r-j} \sum_{l=0}^{r-j-k} q^{-2(1+r)} c_j c_k c_l c_{r-j-k-l}, \end{aligned}$$

In the interest of an inductive argument, assume that the coefficients c_k are bounded above by some constant Φ for all $k \in \{0, 1, \dots, n\}$. Therefore, we see for all $0 \leq r \leq n$ that

$$\begin{aligned} c_{r+2} - c_{r+1} <& \Psi_{e^2} \Phi^2 q^{-4-4r} \left(1 + q^{2r} + q^{2+2r}\right) r + \frac{1}{2} \Psi_{e^2}^2 \Phi^3 q^{-4(1+r)} r(r-1) \\ &+ \frac{1}{6} |a| \Psi_{e^2}^3 \Phi^4 q^{-2(1+r)} (1+r)(2+r)(3+r), \end{aligned}$$

Therefore, we sum the above inequality between $r = 0$ and $r = n - 1$, for $n \geq 1$, to obtain

$$\begin{aligned} & c_{n+1} - c_1 \\ <& \Psi_{e^2} \Phi^2 \sum_{r=0}^{n-1} q^{-4-4r} \left(1 + q^{2r} + q^{2+2r}\right) r + \frac{1}{2} \Psi_{e^2}^2 \Phi^3 \sum_{r=0}^{n-1} q^{-4(1+r)} r(r-1) \\ &+ \frac{1}{6} |a| \Psi_{e^2}^3 \Phi^4 \sum_{r=0}^{n-1} q^{-2(1+r)} (1+r)(2+r)(3+r) \\ <& \Psi_{e^2} \Phi^2 \sum_{r=0}^{\infty} \left(e^{-8-8r} + e^{-8-4r} + e^{-4-4r}\right) r + \frac{1}{2} \Psi_{e^2}^2 \Phi^3 \sum_{r=0}^{\infty} e^{-8(1+r)} r(r-1) \\ &+ \frac{1}{6} |a| \Psi_{e^2}^3 \Phi^4 \sum_{r=0}^{\infty} e^{-4(1+r)} (1+r)(2+r)(3+r) \\ =& \Psi_{e^2} \Phi^2 \frac{1 + 4e^4 + 3e^8 + e^{12}}{e^4 (e^8 - 1)^2} + \Psi_{e^2}^2 \Phi^3 \frac{1}{(e^8 - 1)^3} + |a| \Psi_{e^2}^3 \Phi^4 \frac{e^{12}}{(e^4 - 1)^4}, \end{aligned}$$

So the statement

$$c_{n+1} < \frac{1 + 4e^4 + 3e^8 + e^{12}}{e^4 (e^8 - 1)^2} \Psi_{e^2} \Phi^2 + \frac{\Psi_{e^2}^2}{(e^8 - 1)^3} \Phi^3 + \frac{e^{12} |a| \Psi_{e^2}^3}{(e^4 - 1)^4} \Phi^4 + 1 = \Phi,$$

for $n \geq 1$ completes the proof, given the existence of positive real solution Φ of the above quartic with $\Phi > 3$ (satisfying the base cases). For instance, in the case $|a| = 1$, we obtain a solution $\Phi \approx 3.20474$.

APPENDIX B

Model q-Monodromy Problem

Here, we consider a q-difference, linear (two-by-two) matrix system $Y(qz) = A(z)Y(z)$. In this case, the coefficient matrix $A(z)$ is quadratic in z and, therefore, is a degenerate (factorisable) case of the q-linear system associated with q-P_{VI} [83]. Furthermore, the factorised form of $A(z)$ is similar to what is seen in the linear problem associated with q-P_{IV} [82]. However, that case of the coefficient matrix is cubic in z, involving an extra matrix factor of the same form.

We provide this example to showcase the construction of a *direct monodromy problem* for a q-difference linear system. Moreover, said monodromy problem is precisely solvable in this case; we demonstrate the application of q-hypergeometric series [52] (also known as basic hypergeometric series) in achieving this.

We introduce the coefficient matrix

$$A(z) = \begin{pmatrix} z & 1 \\ b^2 & z \end{pmatrix} \begin{pmatrix} z & 1 \\ c^2 & z \end{pmatrix},$$

with two parameters $b, c \in \mathbb{C}^*$. The entries of the top row of $Y(z)$ satisfy the linear, second-order q-difference equation

$$q\left(b^2 - z^2\right)\left(c^2 - z^2\right) y(z) - \left(c^2 + b^2 q + q(1+q)z^2\right) y(qz) + y(q^2 z) = 0.$$

The system degenerates at turning points $z \in \{\pm b, \pm c\}$, where $\det A$ vanishes. We shall proceed to construct power series solutions of this system around $z = 0$.

B1. Series Solutions Around the Origin

We make the change of variables $t = z^2$, $v(t) = y(z)$ and $p = q^2$, and the system is recast as

$$q\left(b^2 - t\right)\left(c^2 - t\right) v(t) - \left(c^2 + b^2 q + q(1+q)t\right) v(pt) + v(p^2 t) = 0.$$

Therefore, transforming the solution by $v(t) = u(t)/(t/t_0; p)_\infty$ for some $t_0 \in \mathbb{C}^*$, which further implies

$$v(pt) = \frac{t_0 - t}{t_0 (t/t_0; p)_\infty} u(pt), \quad v(p^2 t) = \frac{(t_0 - pt)(t_0 - t)}{t_0^2 (t/t_0; p)_\infty} u(p^2 t),$$

we obtain a difference equation

$$0 = qt_0^2\left(b^2-t\right)\left(c^2-t\right)u(t) - t_0\left(c^2+b^2q+q(1+q)t\right)(t_0-t)u(pt) \\ + (t_0-pt)(t_0-t)u(p^2t).$$

Thus, by choosing $t_0 = b^2$ we have

$$qb^4\left(c^2-t\right)u(t) - b^2\left(c^2+b^2q+q(1+q)t\right)u(pt) + (b^2-pt)u(p^2t) = 0,$$

and choosing $t_0 = c^2$ gives

$$qc^4\left(b^2-t\right)u(t) - c^2\left(c^2+b^2q+q(1+q)t\right)u(pt) + (c^2-pt)u(p^2t) = 0,$$

which are both of the general form

$$(\alpha+\beta t)\,u(t) + (\gamma+\delta t)\,u(pt) + (\epsilon+\zeta t)\,u(p^2t) = 0.$$

Now, we show that solutions of this general system are given in terms of standard q-hypergeometric functions. To this end, consider a series solution of the form

$$u(t) = f(t)\sum_{k=0}^{\infty} a_k t^k,$$

where $a_0 = 1$ and $f(t)$ satisfies $f(pt) = \lambda f(t)$ for some constant $\lambda \in \mathbb{C}^*$. This straightforwardly implies characteristic equation $\alpha+\gamma\lambda+\epsilon\lambda^2 = 0$, in fact, regardless of the choice of $t_0 \in \{b^2, c^2\}$, we find solutions $\lambda_0 = b^2q$ and $\lambda_1 = c^2$. For $k \geq 0$ the coefficients a_k must satisfy

$$a_{k+1} = -\frac{\beta+\delta\lambda p^k+\zeta\lambda^2p^{2k}}{\alpha+\gamma\lambda p^{k+1}+\epsilon\lambda^2p^{2(k+1)}}a_k.$$

When $t_0 = b^2$ the above statement becomes

$$a_{k+1} = \frac{1}{c^2}\frac{\left(1+p^k\lambda/b^2\right)\left(1+p^kq\lambda/b^2\right)}{\left(1-p^kp\lambda/c^2\right)\left(1-p^kq\lambda/b^2\right)}a_k,$$

and thus

$$\lambda = b^2q \implies a_k = \frac{1}{c^{2k}}\frac{(-q;p)_k\,(-p;p)_k}{(q^3b^2/c^2;p)_k\,(p;p)_k},$$

$$\lambda = c^2 \implies a_k = \frac{1}{c^{2k}}\frac{(-c^2/b^2;p)_k(-qc^2/b^2;p)_k}{(qc^2/b^2;p)_k\,(p;p)_k}.$$

Meanwhile when $t_0 = c^2$ we have

$$a_{k+1} = \frac{1}{b^2}\frac{\left(1+p^k\lambda/c^2\right)\left(1+p^kq\lambda/c^2\right)}{\left(1-p^kp\lambda/c^2\right)\left(1-p^kq\lambda/b^2\right)}a_k,$$

and thus

$$\lambda = b^2 q \implies a_k = \frac{1}{b^{2k}} \frac{\left(-qb^2/c^2;p\right)_k \left(-pb^2/c^2;p\right)_k}{\left(q^3b^2/c^2;p\right)_k (p;p)_k}$$
$$\lambda = c^2 \implies a_k = \frac{1}{b^{2k}} \frac{(-1;p)_k (-q;p)_k}{\left(qc^2/b^2;p\right)_k (p;p)_k}.$$

At this point, we have reached four potential solutions of the second-order system, between which we must expect linear dependencies. We deduced two series solutions around $z = 0$ being

$$\begin{aligned}
Y_1(z) &= \frac{1}{(z^2/b^2;q^2)_\infty} \frac{\theta_{q^2}(z^2)}{\theta_{q^2}(b^2qz^2)} \sum_{k=0}^{\infty} \frac{\left(-q;q^2\right)_k \left(-q^2;q^2\right)_k}{(q^3b^2/c^2;q^2)_k (q^2;q^2)_k} (z/c)^{2k} \\
&= \frac{1}{(z^2/b^2;q^2)_\infty} \frac{\theta_{q^2}(z^2)}{\theta_{q^2}(b^2qz^2)} {}_2\phi_1\left(-q,-q^2;q^3b^2/c^2;q^2,z^2/c^2\right), \\
Y_2(z) &= \frac{1}{(z^2/b^2;q^2)_\infty} \frac{\theta_{q^2}(z^2)}{\theta_{q^2}(c^2z^2)} \sum_{k=0}^{\infty} \frac{\left(-c^2/b^2;q^2\right)_k \left(-qc^2/b^2;q^2\right)_k}{(qc^2/b^2;q^2)_k (q^2;q^2)_k} (z/c)^{2k} \\
&= \frac{1}{(z^2/b^2;q^2)_\infty} \frac{\theta_{q^2}(z^2)}{\theta_{q^2}(c^2z^2)} {}_2\phi_1\left(-c^2/b^2,-qc^2/b^2;qc^2/b^2;q^2,z^2/c^2\right),
\end{aligned}$$

which are defined in this way for $|z| < |c|$, and we see the solutions feature poles for $z \in \pm\{b, b/q, b/q^2, \dots\}$. Alternatively, we deduced series

$$\begin{aligned}
Y_3(z) &= \frac{1}{(z^2/c^2;q^2)_\infty} \frac{\theta_{q^2}(z^2)}{\theta_{q^2}(c^2z^2)} \sum_{k=0}^{\infty} \frac{\left(-1;q^2\right)_k \left(-q;q^2\right)_k}{(qc^2/b^2;q^2)_k (q^2;q^2)_k} (z/b)^{2k} \\
&= \frac{1}{(z^2/c^2;q^2)_\infty} \frac{\theta_{q^2}(z^2)}{\theta_{q^2}(c^2z^2)} {}_2\phi_1\left(-1,-q;qc^2/b^2;q^2,z^2/b^2\right), \\
Y_4(z) &= \frac{1}{(z^2/c^2;q^2)_\infty} \frac{\theta_{q^2}(z^2)}{\theta_{q^2}(b^2qz^2)} \sum_{k=0}^{\infty} \frac{\left(-qb^2/c^2;q^2\right)_k \left(-q^2b^2/c^2;q^2\right)_k}{(q^3b^2/c^2;q^2)_k (q^2;q^2)_k} (z/b)^{2k} \\
&= \frac{1}{(z^2/c^2;q^2)_\infty} \frac{\theta_{q^2}(z^2)}{\theta_{q^2}(b^2qz^2)} {}_2\phi_1\left(-qb^2/c^2,-q^2b^2/c^2;q^3b^2/c^2;q^2,z^2/b^2\right),
\end{aligned}$$

defined this way for $|z| < |b|$ and admitting poles for $z \in \pm\{c, c/q, c/q^2, \dots\}$.

By considering expansions around $z = 0$, we find that, in fact, $Y_1 = Y_4$ and $Y_2 = Y_3$. For example, see that

$$
\begin{aligned}
&\left\{\frac{\theta_{q^2}(z^2)}{\theta_{q^2}(b^2qz^2)}\right\}^{-1} Y_1(z)\\
=&\left(1+\frac{z^2}{b^2(1-q^2)}+\mathcal{O}\left(z^4\right)\right)\sum_{k=0}^{\infty}\frac{\left(-q;q^2\right)_k\left(-q^2;q^2\right)_k}{(q^3b^2/c^2;q^2)_k\,(q^2;q^2)_k}(z/c)^{2k}\\
=&\left(1+\frac{z^2}{b^2(1-q^2)}+\mathcal{O}\left(z^4\right)\right)\left(1+\frac{(1+q)(1+q^2)}{(1-q^3b^2/c^2)(1-q^2)}\frac{z^2}{c^2}+\mathcal{O}\left(z^4\right)\right)\\
=&1+\left(\frac{c^2+b^2+b^2q+b^2q^2}{b^2(c^2-q^3b^2)}\right)\frac{z^2}{1-q^2}+\mathcal{O}\left(z^4\right), \quad z\to 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\{\frac{\theta_{q^2}(z^2)}{\theta_{q^2}(b^2qz^2)}\right\}^{-1} Y_4(z)\\
=&\left(1+\frac{z^2}{c^2(1-q^2)}+\mathcal{O}\left(z^4\right)\right)\sum_{k=0}^{\infty}\frac{\left(-qb^2/c^2;q^2\right)_k\left(-q^2b^2/c^2;q^2\right)_k}{(q^3b^2/c^2;q^2)_k\,(q^2;q^2)_k}(z/b)^{2k}\\
=&\left(1+\frac{z^2}{c^2(1-q^2)}+\mathcal{O}\left(z^4\right)\right)\left(1+\frac{(1+qb^2/c^2)(1+q^2b^2/c^2)}{(1-q^3b^2/c^2)(1-q^2)}\frac{z^2}{b^2}+\mathcal{O}\left(z^4\right)\right)\\
=&1+\frac{b^2+c^2+qb^2+q^2b^2}{b^2(c^2-q^3b^2)}\frac{z^2}{1-q^2}+\mathcal{O}\left(z^4\right), \quad z\to 0.
\end{aligned}
$$

Inadvertently, we have proved the identity

$$
\begin{aligned}
&{}_2\phi_1\left(-q,-q^2;q^3b^2/c^2;q^2,z^2/c^2\right)\\
=&\frac{(z^2/b^2;q^2)_\infty}{(z^2/c^2;q^2)_\infty}{}_2\phi_1\left(-qb^2/c^2,-q^2b^2/c^2;q^3b^2/c^2;q^2,z^2/b^2\right),
\end{aligned}
$$

and a similar identity considering $Y_2 = Y_3$.

B2. Connection Formulae

Having made this clarification, suppose we define the independent solutions

$$Y_1 = \frac{f}{(\alpha\zeta;p)_\infty}u_1, \quad Y_2 = \frac{f}{(\alpha\zeta;p)_\infty}u_2,$$

where we have defined $f(z) = \theta_{q^2}(\lambda_1 z^2)/\theta_{q^2}(b^2q\lambda_1 z^2)$ for some constant $\lambda_1 \in \mathbb{C}^*$, $z^2 = c^2\zeta$, $\alpha = c^2/b^2$, and

$$u_1 = {}_2\phi_1\left(-q,-q^2;q^3/\alpha;p,\zeta\right), \quad u_2 = \frac{\theta_p(\zeta)}{\theta_p(\alpha\zeta/q)}{}_2\phi_1\left(-\alpha,-q\alpha;q\alpha;p,\zeta\right).$$

We then may utilise known connection formulae regarding such solutions defined by basic hypergeometric series (see, for instance, the work of Watson [158], and the work of Thomae [147, 148]). In particular, we have

$$\begin{aligned}
u_1 =& \frac{(-p,-p/\alpha;p)_\infty}{(q^3/\alpha,q;p)_\infty} v_1 + \frac{(-q,-q/\alpha;p)_\infty}{(q^3/\alpha,1/q;p)_\infty} v_2,\\
u_2 =& \frac{(-q\alpha,-q;p)_\infty}{(q\alpha,q;p)_\infty}\frac{\theta_p(-\alpha\zeta)\theta_p(\zeta)}{\theta_p(\alpha\zeta/q)\theta_p(-q\zeta)} v_1 + \frac{(-\alpha,-1;p)_\infty}{(q\alpha,1/q;p)_\infty}\frac{\theta_p(-q\alpha\zeta)\theta_p(\zeta)}{\theta_p(\alpha\zeta/q)\theta_p(-p\zeta)} v_2,
\end{aligned}$$

where v_1, v_2 are in terms of series around $z=\infty$, these being

$$\begin{aligned}
v_1 &= \frac{\theta_p(-q\zeta)}{\theta_p(\zeta)} {}_2\phi_1\left(-q,-\alpha;q;p,p/(\alpha\zeta)\right),\\
v_2 &= \frac{\theta_p(-p\zeta)}{\theta_p(\zeta)} {}_2\phi_1\left(-q^2,-q\alpha;q^3;p,p/(\alpha\zeta)\right).
\end{aligned}$$

B2.1. Global behaviour of fundamental solutions. Let $\lambda_1 = 1/(b^2q)$. Then for $|\zeta|\geq 1$ we may write independent solutions Y_1, Y_2 in terms of convergent series around $z=\infty$ as follows

$$\begin{aligned}
Y_1 =& \frac{(-p,-p/\alpha;p)_\infty}{(q^3/\alpha,q;p)_\infty}\frac{1}{(\alpha\zeta;p)_\infty}\frac{\theta_p(\alpha\zeta/q)}{\theta_p(c^2\zeta)}\frac{\theta_p(-q\zeta)}{\theta_p(\zeta)} {}_2\phi_1\left(-q,-\alpha;q;p,p/(\alpha\zeta)\right)\\
&+ \frac{(-q,-q/\alpha;p)_\infty}{(q^3/\alpha,1/q;p)_\infty}\frac{1}{(\alpha\zeta;p)_\infty}\frac{\theta_p(\alpha\zeta/q)}{\theta_p(c^2\zeta)}\\
&\times \frac{\theta_p(-p\zeta)}{\theta_p(\zeta)} {}_2\phi_1\left(-q^2,-q\alpha;q^3;p,p/(\alpha\zeta)\right),\\
Y_2 =& \frac{(-q\alpha,-q;p)_\infty}{(q\alpha,q;p)_\infty}\frac{1}{(\alpha\zeta;p)_\infty}\frac{\theta_p(-\alpha\zeta)}{\theta_p(c^2\zeta)} {}_2\phi_1\left(-q,-\alpha;q;p,p/(\alpha\zeta)\right)\\
&+ \frac{(-\alpha,-1;p)_\infty}{(q\alpha,1/q;p)_\infty}\frac{1}{(\alpha\zeta;p)_\infty}\frac{\theta_p(-q\alpha\zeta)}{\theta_p(c^2\zeta)} {}_2\phi_1\left(-q^2,-q\alpha;q^3;p,p/(\alpha\zeta)\right),
\end{aligned}$$

whereas on the punctured unit disc $|\zeta|<1$ we have these solutions described by convergent series around $z=0$ as follows

$$\begin{aligned}
Y_1 &= \frac{1}{(\alpha\zeta;p)_\infty}\frac{\theta_p(\alpha\zeta/q)}{\theta_p(c^2\zeta)} {}_2\phi_1\left(-q,-q^2;q^3/\alpha;p,\zeta\right),\\
Y_2 &= \frac{1}{(\alpha\zeta;p)_\infty}\frac{\theta_p(\zeta)}{\theta_p(c^2\zeta)} {}_2\phi_1\left(-\alpha,-q\alpha;q\alpha;p,\zeta\right),
\end{aligned}$$

where $p=q^2$, $z^2=c^2\zeta$, and $\alpha = c^2/b^2$. The non-resonance conditions following our choice of λ_1 imply that this is valid for $z\notin \pm q^{\mathbb{Z}}$.

B3. Monodromy Problem

Recall that $Y(qz) = A(z)Y(z)$ where

$$A(z) = \begin{pmatrix} z & 1 \\ b^2 & z \end{pmatrix} \begin{pmatrix} z & 1 \\ c^2 & z \end{pmatrix} = \begin{pmatrix} c^2 + z^2 & 2z \\ (b^2 + c^2)z & b^2 + z^2 \end{pmatrix},$$
$$Y(z) = \begin{pmatrix} y_1(z) & y_2(z) \\ y_3(z) & y_4(z) \end{pmatrix}.$$

Let $Y_0(z)$ be a solution of this system and now define the function $\Phi_0(z)$ by asserting that $Y_0(z) = \Phi_0(z)E_0(z)$ where

$$E_0(qz) = \begin{pmatrix} c^2 & 0 \\ 0 & b^2 \end{pmatrix} E_0(z), \quad |E_0(z)| \neq 0.$$

Therefore, we see that $\Phi_0(z)$ satisfies

$$\Phi_0(qz) = A(z)\Phi_0(z) \begin{pmatrix} 1/c^2 & 0 \\ 0 & 1/b^2 \end{pmatrix}.$$

We now fix the solution Φ_0, which we shall refer to as the canonical solution at $z = 0$, by writing $\Phi_0(z) = I + \mathcal{O}(z)$ as $z \to 0$. Similarly we define a solution $Y_\infty(z)$ by $Y_\infty(z) = \Phi_\infty(z)E_\infty(z)$ where $E_\infty(qz) = z^2 E_\infty(z)$, i.e., $\Phi_\infty(qz) = A(z)\Phi_\infty(z)/z^2$. Then, we may similarly fix Φ_∞, the canonical solution at $z = \infty$, by writing $\Phi_\infty(z) = I + \mathcal{O}(1/z)$ as $z \to \infty$. The monodromy data of the system is then encapsulated by the connection matrix defined by $\Phi_\infty(z) = \Phi_0(z)C(z)$, which furthermore has the property

$$C(qz) = \frac{1}{z^2} \begin{pmatrix} c^2 & 0 \\ 0 & b^2 \end{pmatrix} C(z).$$

From our earlier analysis of series solutions around $z = 0$, we see that these definitions imply

$$(\alpha\zeta; p)_\infty \Phi_0 = \begin{pmatrix} {}_2\phi_1(-\alpha, -q\alpha; q\alpha; p, \zeta) & Bz\,{}_2\phi_1(-q, -q^2; q^3/\alpha; p, \zeta) \\ \cdots & \cdots \end{pmatrix},$$

where $B := 2/(b^2 q - c^2)$. Similarly, considering Φ_∞ we find that

$$\frac{\Phi_\infty}{(p/\alpha\zeta; p)_\infty} = \begin{pmatrix} {}_2\phi_1\left(-q, -\alpha; q; p, p/\alpha\zeta\right) & D\,{}_2\phi_1\left(-q^2, -q\alpha; q^3; p, p/\alpha\zeta\right)/z \\ \cdots & \cdots \end{pmatrix},$$

where $D = 2q/(1 - q)$. So now, considering the earlier discussed connection results regarding these basic hypergeometric series, we obtain the connection matrix explicitly by seeing that

$$\Phi_0 = \Phi_\infty \begin{pmatrix} \frac{(-q\alpha,-q;p)_\infty}{(q\alpha,q;p)_\infty} \frac{\theta_p(-\alpha\zeta)}{\theta_p(\zeta)\theta_p(\alpha\zeta)} & B\frac{(-p,-p/\alpha;p)_\infty}{(q^3/\alpha,q;p)_\infty} \frac{\theta_p(-q\zeta)z}{\theta_p(\zeta)\theta_p(\alpha\zeta)} \\ \frac{1}{D} \frac{(-\alpha,-1;p)_\infty}{(q\alpha,1/q;p)_\infty} \frac{\theta_p(-q\alpha\zeta)z}{\theta_p(\zeta)\theta_p(\alpha\zeta)} & \frac{B}{D} \frac{(-q,-q/\alpha;p)_\infty}{(q^3/\alpha,1/q;p)_\infty} \frac{c^2\theta_p(-\zeta)}{\theta_p(\zeta)\theta_p(\alpha\zeta)} \end{pmatrix}.$$