Character varieties, earthquakes and essential surfaces

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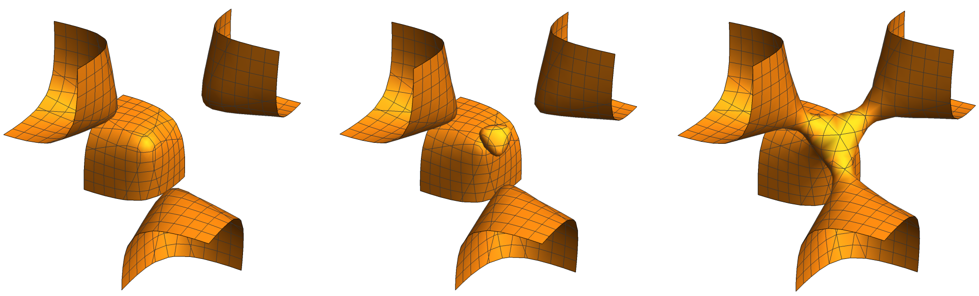

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Statement of originality and authorship

This is to certify that to the best of my knowledge, the content of this thesis is my own work. This thesis has not been submitted for any degree or other purposes. I certify that the intellectual content of this thesis is the product of my own work and that all the assistance received in preparing this thesis and sources have been acknowledged.

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Authorship attribution

The material in Sections 2.1 to 2.7 has been published in [Gar24]. Section 2.1 has been expanded on for unfamiliar readers, Sections 2.2 to 2.7 are mostly verbatim from the published paper.

The material in Sections 3.1 to 3.5 and Section 4.2 is in preparation as [GT] in collaboration with Stephan Tillmann. The material in Section 4.1 uses code originally written by Eric Chesebro and further developed by me. The material in Section 4.3 is in preparation as [GMT] in collaboration with Ben Martin and Stephan Tillmann. In all cases the research direction and mathematical analysis were developed jointly. I substantially contributed to the writing of the manuscript, the development and proof of key results, the computational analysis, and the creation of figures.

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As supervisor for the candidature upon which this thesis is based, I can confirm that the authorship attribution statements above are correct.

Stephan Tillmann   31 July 2024

Abstract

Character varieties of low-dimensional manifolds are a rich area of study that reflect geometric and topological information of the underlying manifold. We study character varieties from two perspectives, with a focus on hyperbolic manifolds. The first perspective uses Teichmüller space and earthquakes. Teichmüller space can be naturally identified with a component of the $\mathrm{SL}_2(\mathbb{R})$–character variety of a surface. We derive explicit forms of the earthquake deformations on Teichmüller space associated with simple closed curves of the once-punctured torus with both algebraic and geometric interpretations. Examining the limiting behaviour gives insight into earthquakes about measured geodesic laminations, of which simple closed curves are a special case. The second perspective uses essential surfaces and the theory of Culler and Shalen. In the seminal work of Culler and Shalen, essential surfaces in 3–manifolds are associated to ideal points of their $\mathrm{SL}_2(\mathbb{C})$–character varieties. Here, we lay a general foundation for this theory in arbitrary characteristic by using the same approach instead over the $\mathrm{SL}_2(\mathbb{F})$–variety of characters for $\mathbb{F}$ an arbitrary algebraically closed field. We provide several applications of the extended theory.

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[1] And Pauline, for feeding me.
[2] And Robbie, of course.
[3] And Tim! The list goes on and on.

Contents

Introduction

We study low-dimensional manifolds through their character variety. Given a manifold M and an algebraic group G, the G–character variety is loosely the space of conjugacy classes of representations from the fundamental group of M to G, with some conjugacy classes identified so that the resulting space is a variety. This has deep connections with the geometry and topology of the underlying manifold. We are particularly interested in the situation when M is a hyperbolic surface or 3–manifold and G is SL_2. This is because of the relationship between elements of the character variety and the orientation preserving isometry groups of the hyperbolic plane ($\mathrm{PSL}_2(\mathbb{R})$) and hyperbolic space ($\mathrm{PSL}_2(\mathbb{C})$).

In this thesis, we delve into two perspectives that illustrate this concept: studying geometric structures on surfaces via dynamics of character varieties and earthquakes; and, studying essential surfaces in 3–manifolds via ideal points of the related variety of characters and theory established by Culler and Shalen. We motivate and discuss these ideas, and then present the structure of the thesis with an emphasis on the main contributions.

On character varieties of surfaces and geometric structures

Thurston showed that certain elements of the character variety are associated with geometric structures of the manifolds via the holonomy map (see [Thu80]). The mapping class group is the group of isotopy classes of all orientation-preserving homeomorphisms on the manifold to itself. This acts naturally on the character variety and can be studied to understand the relationship between different geometric structures. This study is called the dynamics of character varieties and research in this way was spearheaded by Goldman [Gol97]. We especially care about the action of Dehn twists as the mapping class group is generated by finitely many Dehn twists. Defined intuitively, a Dehn twist about a simple closed curve is achieved by cutting the surface open along the curve, twisting around the curve, and then gluing the surface together again.

Of particular interest are hyperbolic structures, with the vast majority of compact, orientable surfaces being hyperbolic. Goldman [Gol88b] showed for every closed hyperbolic surface there is a component of the $\mathrm{PSL}_2(\mathbb{R})$–character variety, called the Hitchin component , that can be identified with the space of all finite-area complete hyperbolic structures of the surface. This space is called Teichmüller space and the corresponding study of the mapping class group action here began far earlier, with Fenchel-Nielsen deformations [FN] and then earthquakes [Thu86]. These generalise twisting about simple closed curves: Fenchel-Nielsen deformations allow for partial twists, and earthquakes extend to partial twists about measured geodesic laminations. In this way they can be seen as "fractional" Dehn twists, enabling continuous dynamics.

Earthquakes have proved to be powerful in the exploration of hyperbolic structures on surfaces. Most notably, a hyperbolic structure on a surface can be related to any other hyperbolic structure on a surface by a unique earthquake [Thu86]. This was used in the proof of the Nielsen realisation problem [Ker83]. Other flows have been related to the earthquakes. In particular, Mirzakhani showed a measurable conjugacy between the earthquake flow and Teichmüller horocycle flow [Mir08], which has been much studied since.

On character varieties of 3–manifolds and essential surfaces

The ground-breaking work of Culler and Shalen [CS83] detects essential surfaces in a 3–manifold by studying ideal points of their $\mathrm{SL}_2(\mathbb{C})$–character variety. Broadly, essential surfaces are orientable, incompressible, and boundary-incompressible surfaces in a 3–manifold M. These surfaces tell us about the inherent geometry and topology of M. For example, the prime decomposition and JSJ decomposition of 3–manifolds are obtained by cutting along essential spheres and essential tori, respectively. Thurston's hyperbolisation theorem implies every closed atoroidal 3–manifold with an essential surface admits a hyperbolic structure and thus the detection of essential surfaces is intimately connected with hyperbolic structures.

The method of Culler and Shalen relies on Bass-Serre theory (see [Ser77]) and the Stallings-Epstein-Waldhausen construction [Eps61, Sta59, Wal67]. Together, it underscores connections between the geometry of character varieties, group actions on trees, and the theory of incompressible surfaces in 3–manifolds. The techniques of Culler and Shalen's method have been incredibly influential in knot theory. They led to the definitions of the A–polynomial [CCG$^+$94], which encodes the boundary slopes of essential surfaces that are detected by ideal points of the character variety, and the Culler-Shalen norm [CGLS87], which measures the complexity of Dehn fillings of the manifold. They were also used in the proofs of numerous well-known theorems, including the weak Neuwirth conjecture [CS84], the Smith conjecture [Sha99], and the cyclic surgery theorem [CGLS87].

Not all essential surfaces can be detected in this manner and much work has been done to further characterise which surfaces are detected by which ideal points. The theory has been extended to $\mathrm{SL}_n(\mathbb{C})$-character varieties, which enables every essential surface to be detected for some n [FKN18]. My work extends the theory in another way, by adjusting the underlying field $\mathbb{C}$ to be any algebraically closed field $\mathbb{F}$ of arbitrary characteristic. Paoluzzi and Porti [PP20] have similarly studied the character varieties of knot complements in arbitrary characteristic and found discrepancies across characteristics (for example, in the number of components, dimension, intersection points).

A note on the two perspectives

The approaches outlined above for studying geometry and topology of surfaces and 3–manifolds through character varieties are not as disparate as they might seem. The first approach studies earthquakes on measured geodesic laminations of a surface as a natural way to understand Teichmüller space, which is a subvariety of the character variety. The second approach studies connections between the geometry of character varieties, splittings of groups, and incompressible surfaces in 3–manifolds. The techniques of Culler and Shalen's theory and Thurston's theory of measured geodesic laminations were generalised by

Morgan and Shalen in a series of papers, starting with [MS84]. Here they provide new proofs for results of Thurston regarding hyperbolic structures on manifolds.

Structure of the thesis and main results

Background material is covered in Chapter 1. We give concrete definitions for the character variety, the variety of characters, and related concepts (Section 1.1), with a focus on the varieties over SL_2 (Section 1.2). The variety of characters is the space of equivalence classes of representations under closure of their Zariski orbits. Over $\mathrm{SL}_2(\mathbb{C})$, this is the same as the classical definition of the character variety using the affine geometric invariant theory quotient; over $\mathrm{SL}_2(\mathbb{F})$, for a general algebraically closed field $\mathbb{F}$, they may not agree. We survey ways in which character varieties are studied and connected to geometry (Section 1.3) and we delve into the two perspectives that are explored further in this thesis with the relationship to Teichmüller space (Section 1.3.1) and essential surfaces (Section 1.3.2). This will set the scene for how we use and why we care about character varieties.

The main contributions of this thesis are contained in Chapters 2 to 4. In Chapter 2, we study earthquakes on the once-punctured torus, as a fundamental example of a hyperbolic surface. Here, we describe two methods to get an explicit form of earthquakes for all simple closed curves, giving new interpretations of earthquakes from both algebraic and geometric perspectives. Examining the limiting behaviour of each gives insight into earthquakes about measured geodesic laminations.

The first method (Section 2.2) follows Goldman [Gol03] in studying the dynamics of the $\mathrm{SL}_2(\mathbb{C})$–character variety of the once-punctured torus. The action of Dehn twists here gives an action of polynomial automorphisms on the $\mathrm{SL}_2(\mathbb{C})$–character variety that preserve level sets of the trace of a peripheral element, which relates to the geometry of the surface around the puncture. A specific level set within the $\mathrm{SL}_2(\mathbb{R})$–character variety is naturally identified with the Hitchin component. We use techniques from linear recurrence relations to explicitly extend the orbits of the Dehn twist about the meridian, longitude, and their product to a continuous flow, given in Theorem 2.12.

The second method (Section 2.3) examines the action directly on Teichmüller space. Cooper and Pfaff [CP24] introduce coordinates called triangle lengths, which measure the half lengths of the unique geodesics associated with the meridian, longitude, and their product. Using maximal collar neighbourhoods, results from [Bus78], and hyperbolic geometry returns an explicit expression for the earthquakes about these curves, given in Theorem 2.18.

It is straightforward to show the two independently derived expressions align on the level set of the character variety corresponding to Teichmüller space, proving that the continuous flow obtained from the character variety perspective is the earthquake in different coordinates. It remains an open problem to see if the flow from the first method has a geometric interpretation outside of Teichmüller space. We provide a change of coordinates algorithm to extend the expressions to all simple closed curves (Section 2.4). Example pictures of the earthquakes as paths in Teichmüller space using both methods are given in Figure 1.

A natural coordinate system for Teichmüller space is Fenchel-Nielsen length-twist coordinates. In these coordinates, we prove that the slope of an earthquake about a lamination will always converge to the inverse slope of the lamination (Section 2.5). We prove results relating the asymptotics of the earthquake

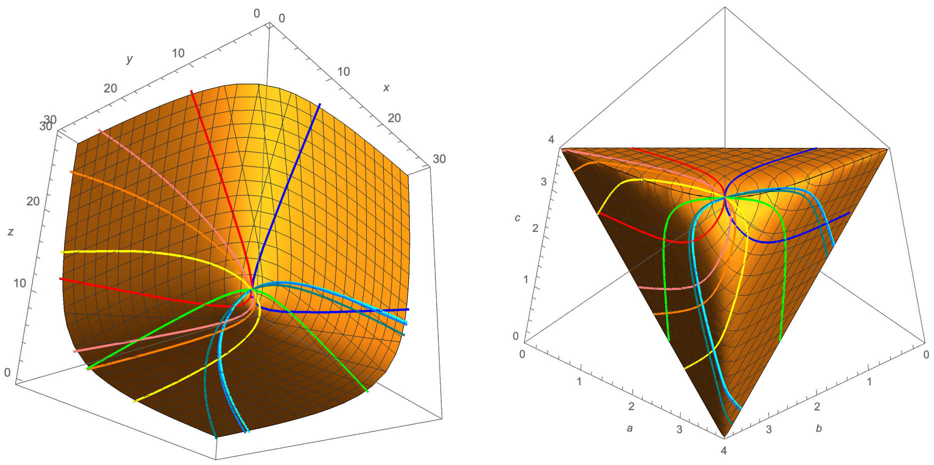

FIGURE 1: Example earthquakes on the once-punctured torus using the first method (left) and the second method (right). The earthquakes are shown as paths in Teichmüller space of the once-punctured torus using two different coordinates associated with the two methods; different colours represent earthquakes about different simple closed curves.

deformations and the initial lamination. A result on asymptotics of simple closed curves in Theorem 2.24 leads to the following corollary.

Corollary 2.25. *Let $\mathscr{L}$ be a measured geodesic lamination on the once-punctured torus $S_{1,1}$. Denote the associated earthquake in Fenchel-Nielsen length-twist coordinates as $E^{FN}_{\mathscr{L}}(\mathbf{u},s) = \left(\ell^{\mathscr{L}}_{\alpha}(\mathbf{u},s), \tau^{\mathscr{L}}_{\alpha}(\mathbf{u},s)\right) \subseteq \mathbb{R}_+ \times \mathbb{R}$.*

We find

$$\frac{\tau^{\mathscr{L}}_{\alpha}(\mathbf{u},s)}{\ell^{\mathscr{L}}_{\alpha}(\mathbf{u},s)} \to \frac{1}{slope(\mathscr{L})} \text{ as } s \to \infty \text{ for all } \mathbf{u} \in \mathscr{T}(S_{1,1}).$$

Theoretically, results for simple closed geodesics can be extended to measured geodesic laminations using a density argument. We discuss and plot several examples of earthquakes and review the practicalities of the extension to non-simple closed curve laminations, which is non-trivial (Sections 2.6 and 2.7). Calculations for examples of a family of earthquakes and their limiting behaviour are given in Appendix A. We finish the chapter by exploring possible extensions of the work to the hyperbolic plane (Section 2.8).

In Chapter 3, we study ideal points of character varieties and essential surfaces and lay the foundation to extend the theory of Culler and Shalen over any algebraically closed field $\mathbb{F}$ of arbitrary characteristic. The extension of the theory of Culler and Shalen is done over the $\mathrm{SL}_2(\mathbb{F})$–variety of characters, which a priori may not be the same as the $\mathrm{SL}_2(\mathbb{F})$–character variety. We apply the extended theory to a range of settings, including: the A–polynomial, detection of prime decomposition, and detection of the JSJ decomposition. Most of the approach follows aside from details in characteristic 2, where we require additional theory for character varieties and traces (Section 3.1). We can then prove similar results to Culler and Shalen regarding detecting essential surfaces over $\mathbb{F}$ an algebraically closed field of any characteristic p (Sections 3.2 and 3.3).

The A–polynomial arises from this theory and parametrises the boundary slopes detected using the Culler and Shalen machinery (Section 3.4). We show the definition of the A–polynomial still holds and prove a version of the boundary slopes theorem in characteristic p, which asserts that the boundary slopes encoded in the A–polynomial are precisely the boundary slopes of the Newton polygon of the A–polynomial. Comparing across characteristics, we prove:

Theorem 3.33. *Consider M with a single boundary torus $\partial M = T$. For all but finitely many primes p, the boundary curve of an essential surface in M is strongly detected by the A–polynomial over $\mathbb{C}$ if and only if it is strongly detected by the A–polynomial over $\mathbb{F}$ an algebraically closed field of characteristic p.*

We get similar results for the analogous definition of the A–polynomial over M with multiple boundary tori, such as Theorem 3.38.

We also give conditions under which the prime decomposition (Section 3.5) and JSJ decomposition (also known as toroidal decomposition) (Section 3.6) are detected, with examples. This is new work, as previous research typically focuses on detecting essential surfaces in irreducible manifolds. Using this theory, we construct examples where an essential surface is detected over $\mathbb{F}$ an algebraically closed field of positive characteristic p but is not detected over $\mathbb{C}$.

We continue to give illustrative examples for this theory in Chapter 4. In particular, we survey changes in dimension for closed Haken manifolds (Section 4.1), give detailed analysis of examples where the A–polynomial, dimension of the variety of characters, and number of components of the variety of characters change over varying characteristics (Section 4.2), and we provide a comprehensive study of a family of graph manifolds and Seifert fibered spaces, which gives infinitely many examples where an essential surface is never detected (Section 4.3). Calculations for curves in the variety of characters of the family of manifolds are given in Appendix B. Together the examples prove the following.

Theorem 4.1. *Let $\mathbb{F}$ be an algebraically closed field of arbitrary characteristic p.*

1. *There exists a manifold M with an essential surface $S \subset M$ that is not detected by the $\mathrm{SL}_2(\mathbb{C})$–variety of characters but is detected by the $\mathrm{SL}_2(\mathbb{F})$–variety of characters for some p.*
2. *There exists a manifold M with an essential surface $S \subset M$ that is not detected by the $\mathrm{SL}_2(\mathbb{F})$–variety of characters for all p.*

In particular, we will see examples where M is hyperbolic, M is a graph manifold, and where M is Seifert fibered. More work remains be done to understand why certain surfaces are detected in certain characteristics, and why some surfaces are not detected in any characteristic. We believe this could be connected to the underlying geometry and topology of both the surface and the 3–manifold it is embedded in.

Chapter 1

Character varieties and varieties of characters

In this chapter, we define the main mathematical objects studied in this thesis. We define the character variety, the variety of characters, and related concepts of the representation variety and character stack in Section 1.1. We then focus on the case of the SL_2–varieties in Section 1.2. Over SL_2, we can relate the variety of characters to trace functions and the trace ring. The main references for this content are [CS83, GM93]. The character variety can be studied in from many different perspectives. We highlight two perspectives related to geometry of the manifold in Section 1.3, the first on Teichmüller space and the second on essential surfaces. The main references for this content are [FM12, Gol22, Neu99]. We use this chapter to contextualise how we think about these objects for the purposes of this thesis. Most experts will be familiar with this material.

In particular: the sections on character varieties (Section 1.1 and Section 1.2) are relevant for Chapters 2 to 4; the section on Teichmüller space (Section 1.3.1) is relevant for Chapter 2; and the section on essential surfaces (Section 1.3.2) is relevant for Chapters 3 and 4.

1.1 Definitions

Let Γ be a finitely generated group and let G be an algebraic group over a field $\mathbb{F}$ with $G \subset \mathbb{F}^k$. Typically, G will be a Lie group.

Recall a *variety* (or *algebraic set*) in $\mathbb{F}^n$ is the set of zeroes over $\mathbb{F}$ of an ideal of $\mathbb{F}[x_1, \ldots, x_n]$. An *irreducible variety* is a variety whose defining ideal is prime. We can always equip the *Zariski topology* on an algebraic set, where the topology is defined by closed sets being the algebraic subsets of the variety.

Definition 1.1. *The* ***G–representation variety*** *of Γ is the space of representations $\rho\colon\ \Gamma \to G$,*

$$\mathrm{R}(\Gamma, G) = \mathrm{Hom}(\Gamma, G) = \{\rho\colon\ \Gamma \to G \ \ \textit{homomorphism}\}. \tag{1.1.1}$$

The representation variety is an algebraic set over $\mathbb{F}$, and is defined by polynomial equations over $\mathbb{F}$ from the relations of Γ and the embedding $G \subset \mathbb{F}^k$. Let $(\gamma_1, \ldots, \gamma_n)$ be the generators for Γ. Then

$$\begin{aligned} \varphi\colon\ \mathrm{R}(\Gamma, G) &\hookrightarrow G^n \subset \mathbb{F}^{kn} \\ \rho &\mapsto (\rho(\gamma_1), \ldots, \rho(\gamma_n)). \end{aligned}$$

The Zariski topology on $\mathbb{F}^{kn}$ induces a topology on $\mathrm{R}(\Gamma, G)$. Choosing a different ordered generating set $(\gamma_1', \ldots, \gamma_m')$ results in another embedding

$$\begin{aligned} \varphi'\colon\ \mathrm{R}(\Gamma, G) &\hookrightarrow G^m \subset \mathbb{F}^{km} \\ \rho &\mapsto (\rho(\gamma_1'), \ldots, \rho(\gamma_m')). \end{aligned}$$

There is a natural polynomial isomorphism between the images of φ and φ' obtained by expressing the generators of one generating set in terms of the other and vice versa. In particular, the topology on $\mathrm{R}(\Gamma, G)$ induced by the Zariski topology is independent of the choice of generating set.

The group G acts on $\mathrm{R}(\Gamma, G)$ by conjugation. For all $g \in G$ and $\rho\colon\ \Gamma \to G$, $g \cdot \rho$ gives a new representation defined by

$$\begin{aligned} g \cdot \rho\colon\ \Gamma &\to G \\ \gamma &\to g\rho(\gamma)g^{-1}. \end{aligned}$$

Given the embedding, we can also view this as the conjugation action on G^n. We say any two conjugate representations $\rho, \sigma \in \mathrm{R}(\Gamma, G)$ are **equivalent**, written $\rho \sim \sigma$. The action preserves the natural embedding of $\mathrm{R}(\Gamma, G) \subset \mathbb{F}^{kn}$ and is algebraic.

Definition 1.2. *The G–**character stack** of Γ is the space of conjugacy classes of representations,*

$$\mathrm{Hom}(\Gamma, G)/G. \tag{1.1.2}$$

The character stack is not necessarily an algebraic set. This is because the conjugation action is, in general, not proper. Recall, a group action of G on X is *proper* if the mapping $(g, x) \mapsto (g \cdot x, x)$ has the property that inverses of compact sets are compact for all $g \in G$, $x \in X$.

Example 1.3. *Let $G = \mathrm{SL}_2(\mathbb{C})$ and $\Gamma = \langle \gamma \rangle \cong \mathbb{Z}$. Take the representation ρ defined by*

$$\rho(\gamma) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and take σ to be the trivial representation. Then the closure of the orbit of ρ is

$$\overline{G}_\rho = \left\{ \tau\colon\ \mathbb{Z} \to \mathrm{SL}_2(\mathbb{C}) \;\middle|\; \tau(\gamma) = \begin{pmatrix} 1 - ac & a^2 \\ -c^2 & 1 + ac \end{pmatrix} \text{ with } a, c \in \mathbb{C} \right\}.$$

Then $\sigma \in \overline{G}_\rho$ and moreover $\sigma \in \overline{G}_\rho \setminus G_\rho$. Thus the conjugacy classes of ρ and σ cannot be separated by disjoint open sets in $\mathrm{Hom}(\mathbb{Z}, \mathrm{SL}_2(\mathbb{C}))/\,\mathrm{SL}_2(\mathbb{C})$. This means $\mathrm{Hom}(\mathbb{Z}, \mathrm{SL}_2(\mathbb{C}))/\,\mathrm{SL}_2(\mathbb{C})$ is not Hausdorff, and thus the action of $\mathrm{SL}_2(\mathbb{C})$ on $\mathrm{R}(\mathbb{Z}, \mathrm{SL}_2(\mathbb{C}))$ is not proper.

The action does not always behave this way.

Remark 1.4. *There are some examples where the conjugation action is proper, for example* $G = \mathrm{SU}(2)$ *on* $\mathrm{R}(\Gamma, \mathrm{SU}(2))$. *This can be verified directly looking at the conjugacy classes of* $\mathrm{SU}(2)$.

The notion of equivalence under conjugation is natural though. For example, Culler and Shalen prove the following result for $G = \mathrm{SL}_2(\mathbb{C})$. We state it for a more general class of groups.

Proposition 1.5 (Culler-Shalen)**.** *[CS83, Proposition 1.1.1] Let G be an irreducible algebraic group and let V be an irreducible component of* $\mathrm{R}(\Gamma, G)$. *Then any representation equivalent to a representation in V must itself belong to V.*

Proof. Take an irreducible algebraic group G. Previously we saw how $V \subseteq \mathrm{R}(\Gamma, G) \subseteq G^n$. The set $V \times G \subseteq G^{n+1}$ is the product of two affine varieties and is thus is an affine variety [Sha13, Theorem 1.6]. Define a map

$$\begin{aligned} f\colon\ V \times G &\to \mathrm{R}(\Gamma, G) \\ (x_1, \ldots, x_n, \gamma) &\mapsto \left(\gamma x_1 \gamma^{-1}, \ldots, \gamma x_n \gamma^{-1}\right). \end{aligned}$$

This is defined by polynomials in the coordinates and thus is regular.

We take $\overline{f(V \times G)} \subseteq \mathrm{R}(\Gamma, G)$ to be the closure of the image. This must also be a variety as the image of a variety under a regular map is a variety. Then $\overline{f(V \times G)}$ must be contained in a component V' of $\mathrm{R}(\Gamma, G)$. We have $V = f(V \times \{1\}) \subseteq V'$. Since V is an irreducible component of $\mathrm{R}(\Gamma, G)$ we must have $V = V'$. Thus $f(V \times G) \subseteq V$ and we are done. □

The issue with the character stack leads us to a stronger notion of equivalence to define the character variety.

Definition 1.6. *The* ***G–character variety*** *of* Γ *is space of equivalence classes of representations,*

$$\mathrm{C}(\Gamma, G) = \mathrm{Hom}(\Gamma, G) /\!\!/ G$$

where $/\!\!/$ *denotes the* affine geometric invariant theory quotient *of the representation variety.*

The geometric invariant theory quotient can be defined more explicitly when $\mathbb{F} = \mathbb{C}$. Denote the algebra of regular functions as $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]$ with the subalgebra of G-invariant functions $\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G$. Then the quotient is the spectrum of prime ideals $\mathrm{C}(\Gamma, G) = \mathrm{Spec}\left(\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G\right)$. This is the algebraic set such that $\mathbb{C}[\mathrm{Spec}\left(\mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G\right)] = \mathbb{C}[\mathrm{Hom}(\Gamma, G)]^G$. For example see [Sik12].

By definition, the character variety is an algebraic set.

Using this new definition, in Example 1.3 when we had $G = \mathrm{SL}_2(\mathbb{C})$, the two representations ρ and σ will now be in the same equivalence class. There are some cases where taking the geometric invariant theory quotient is the same as looking at conjugacy classes. For example when $G = \mathrm{SU}(2)$, $\mathrm{C}(\Gamma, \mathrm{SU}(2)) = \mathrm{Hom}(\Gamma, \mathrm{SU}(2)) / \mathrm{SU}(2)$.

A related concept is given by using the Zariski topology. We say that $\rho, \sigma \in \mathrm{R}(\Gamma, G)$ are **closure-equivalent**, written $\sim_c$, if the Zariski closures of their orbits intersect.

Definition 1.7. *The G–**variety of characters** of Γ is space of equivalence classes of representations,*

$$\mathrm{X}(\Gamma, G) = \mathrm{Hom}(\Gamma, G)/\sim_c .$$

We expand on this definition (and its name) in the next section and compare to $\mathrm{C}(\Gamma, G)$. In particular, by [Pro76] there is a natural identification $\mathrm{X}(\Gamma, \mathrm{SL}_2(\mathbb{C})) = \mathrm{C}(\Gamma, \mathrm{SL}_2(\mathbb{C}))$.

The formal definition and language used for the character variety is a matter of taste. Sometimes the variety of characters is referred to as the character variety; we could consider other concepts like the *Hausdorffization* of the character stack (as used in [Mon16], also sometimes referred to as Hausdorffication or Hausdorffification, if you wanted more syllables). See [Mar22] for a review of different definitions for the character variety. These are all equivalent definitions over $\mathbb{F} = \mathbb{C}$.

We denote the natural quotient map between the representation variety and the variety of characters

$$q\colon\ \mathrm{R}(\Gamma, G) \to \mathrm{X}(\Gamma, G). \tag{1.1.3}$$

For a compact, orientable manifold M with finitely generated fundamental group, take $\Gamma = \pi_1(M)$. The G–representation variety, character variety, and variety of characters are written $\mathrm{R}(M, G) = \mathrm{R}(\pi_1(M), G)$, $\mathrm{C}(M, G) = \mathrm{C}(\pi_1(M), G)$, $\mathrm{X}(M, G) = \mathrm{X}(\pi_1(M), G)$.

In this thesis, we will use the perspective of the variety of characters. In Chapter 2 we use the $\mathrm{SL}_2(\mathbb{C})$–character variety, which coincides with the $\mathrm{SL}_2(\mathbb{C})$–variety of characters; in Chapters 3 and 4 we use the $\mathrm{SL}_2(\mathbb{F})$–variety of characters, which may not coincide with the $\mathrm{SL}_2(\mathbb{F})$–character variety. Both the varieties have connections to the geometry and topology of M, which we expand on in Section 1.3.

1.2 The varieties over SL_2

Let Γ be a finitely generated group as before and $G = \mathrm{SL}_2(\mathbb{F})$ for $\mathbb{F}$ a field. Character varieties and varieties of characters over $\mathrm{SL}_2(\mathbb{F})$ are well-studied for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, in part due to connections with the isometry groups for the hyperbolic plane and hyperbolic 3–space.

We state some known definitions and results here first over $\mathbb{F}$ an algebraically closed field and then over $\mathbb{C}$, when the extension to general $\mathbb{F}$ is not already known. Many of these results are then extended to $\mathbb{F}$ in Chapter 3. We use E to denote the identity matrix in $\mathrm{SL}_2(\mathbb{F})$.

Definition 1.8. *The **character** of a representation $\rho\colon\ \Gamma \to \mathrm{SL}_2(\mathbb{F})$ is the trace function*

$$\begin{aligned} \tau_\rho\colon\ \Gamma &\to \mathbb{F} \\ \gamma &\mapsto \mathrm{tr}(\rho(\gamma)). \end{aligned} \tag{1.2.1}$$

Conjugate representations have the same trace function (see the trace identity in Equation (1.2.4)). Since the trace functions are algebraic, it follows that closure-equivalent representations have the same trace function.

Definition 1.9. *The* ***trace ring*** *of a representation* $\rho\colon\ \Gamma \to \mathrm{SL}_2(\mathbb{F})$ *is the ring of trace functions,*

$$\mathrm{T}(\Gamma,\mathbb{F}) = \langle \tau_\rho\colon\ \Gamma \to \mathbb{F} \mid \rho \in \mathrm{R}(\Gamma,\mathrm{SL}_2(\mathbb{F}))\rangle. \tag{1.2.2}$$

Culler and Shalen prove $\mathrm{X}(\Gamma,\mathrm{SL}_2(\mathbb{C})) \cong \mathrm{T}(\Gamma,\mathbb{C})$ [CS83].

In this section we review concepts and tools that help us to understand these objects. Section 1.2.1 reviews useful trace identities on $\mathrm{SL}_2(\mathbb{F})$; Section 1.2.2 demonstrates how they can be used to get finite coordinates for $\mathrm{T}(\Gamma,\mathbb{F})$; Section 1.2.3 introduces definitions for reducible and irreducible representations (and characters) with useful results over $\mathbb{F}$ and $\mathbb{C}$; and Section 1.2.4 gives an explicit example. The contents in Sections 1.2.1 to 1.2.3 are the main ingredients for proving the isomorphism $\mathrm{X}(\Gamma,\mathrm{SL}_2(\mathbb{C})) \cong \mathrm{T}(\Gamma,\mathbb{C})$.

We prove in Chapter 3 more generally that $\mathrm{X}(\Gamma,\mathrm{SL}_2(\mathbb{F})) \cong \mathrm{T}(\Gamma,\mathbb{F})$ for all algebraically closed fields $\mathbb{F}$ using a similar approach.

Remark 1.10. *We can define the trace ring for all matrix groups G and compare to the variety of characters and the character variety. They are not always the same. For example, Sikora proves in [Sik17] that* $\mathrm{C}(\Gamma,\mathrm{SO}(2n,\mathbb{C}))$ *is not generated by trace functions (nor is it generated by so-called "generalised trace functions") and is thus not the same as the variety of characters* $\mathrm{X}(\Gamma,\mathrm{SO}(2n,\mathbb{C}))$.

It is classical that the two constructions $\mathrm{C}(\Gamma,\mathrm{SL}_2(\mathbb{F}))$ and $\mathrm{X}(\Gamma,\mathrm{SL}_2(\mathbb{F}))$ define the same object for the complex numbers [Pro76]. This also holds for $\mathbb{F}$ an algebraically closed field of characteristic 0 or when Γ is the free group (see [Mar03]). We refer to the variety of characters as the character variety when it is clear they coincide.

For the rest of this section let $\mathbb{F}$ be an algebraically closed field.

1.2.1 Trace identities

We recall trace identities in $\mathrm{SL}_2(\mathbb{F})$ that will be used to describe the trace ring and the variety of characters. Most of this work is classical over $\mathbb{C}$ due to Vogt [Vog89] and Fricke [Fri96] and holds almost immediately over a general algebraically closed field $\mathbb{F}$. We follow the preliminary discussion in [GM93, Section 4], see also [Hor72] and [Mag80].

Let $A,B,C \in \mathrm{SL}_2(\mathbb{F})$. We have the following trace identities

$$\operatorname{tr}\left(A^{-1}\right) = \operatorname{tr}(A), \tag{1.2.3}$$
$$\operatorname{tr}\left(BAB^{-1}\right) = \operatorname{tr}(A), \tag{1.2.4}$$
$$\operatorname{tr}(A)\operatorname{tr}(B) = \operatorname{tr}(AB) + \operatorname{tr}\left(AB^{-1}\right). \tag{1.2.5}$$

Equation (1.2.4) is equivalent to $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. We also have

$$\operatorname{tr}(ACB) = \operatorname{tr}(A)\operatorname{tr}(BC) + \operatorname{tr}(B)\operatorname{tr}(AC) + \operatorname{tr}(C)\operatorname{tr}(AB) - \operatorname{tr}(A)\operatorname{tr}(B)\operatorname{tr}(C) - \operatorname{tr}(ABC). \tag{1.2.6}$$

These can all be derived from the Cayley-Hamilton theorem.

Take the commutator to be $[A, B] = ABA^{-1}B^{-1}$. Applying Equation (1.2.6) we have

$$\operatorname{tr}[A, B] = (\operatorname{tr}(A))^2 + (\operatorname{tr}(B))^2 + (\operatorname{tr}(AB))^2 - \operatorname{tr}(A)\operatorname{tr}(B)\operatorname{tr}(AB) - 2 \tag{1.2.7}$$

and

$$\operatorname{tr}[A, BC] = (\operatorname{tr}(A))^2 + (\operatorname{tr}(BC))^2 + (\operatorname{tr}(ABC))^2 - \operatorname{tr}(A)\operatorname{tr}(BC)\operatorname{tr}(ABC) - 2. \tag{1.2.8}$$

Suppose we are given a word W of length L in the matrices $A_1, \dots, A_n \in \mathrm{SL}_2(\mathbb{F})$ and their inverses. We regard the identity element as a word of length zero. The trace identities can be used to write $\operatorname{tr}(W)$ as a polynomial with integral coefficients in the traces of words of length at most L with the property that each word contains each letter A_i at most once and the A_i respect the natural cyclic order of the index set. This is recorded in the below result.

Lemma 1.11 (Vogt, Fricke)**.** *Let $\mathbb{F}$ be an algebraically closed field. The trace of any word in the matrices $A_1, \dots, A_n \in \mathrm{SL}_2(\mathbb{F})$ and their inverses is a polynomial with integral coefficients in the $2^n - 1$ traces*

$$\operatorname{tr}(A_{j_1}A_{j_2}\dots A_{j_m}), \quad 1 \le j_1 < j_2 < \dots < j_m \le n, \quad m \le n.$$

Moreover, the constant term of the polynomial is an integral multiple of the trace of the identity matrix.

Proof. A method to determine the polynomial claimed in the statement is as follows. Using Equation (1.2.4) in the equivalent form $\operatorname{tr}(BA) = \operatorname{tr}(AB)$ and induction, we see that we may cyclically permute the argument in the trace function.

Using Equation (1.2.6), the previous observation regarding cyclic permutation, and induction, we may write the trace of W as a polynomial with integral coefficients in traces of words of length at most L that have the property that they are of the form $A_1^{m_1} \cdots A_n^{m_n}$ with $m_i \in \mathbb{Z}$. Note that $\sum |m_i| \le L$.

Using Equation (1.2.5) in the equivalent form $\operatorname{tr}\left(AB^{-1}\right) = \operatorname{tr}(A)\operatorname{tr}(B) - \operatorname{tr}(AB)$ and cyclic permutation, we may now arrange for all exponents to be positive, at the expense of increasing the number of terms in the polynomial expression for $\operatorname{tr}(W)$ but maintaining the property that each term has the required cyclic order.

Now Equation (1.2.5) also implies

$$\operatorname{tr}\left(A^{m+1}C\right) = \operatorname{tr}(A)\operatorname{tr}\left(A^mC\right) - \operatorname{tr}\left(A^{m-1}C\right), \quad m \ge 1 \tag{1.2.9}$$

and hence we may inductively transform the words in the terms of the polynomial to contain each letter at most once, again maintaining the order property. This gives the desired result. □

Applying Equation (1.2.6) three times to different partitions of the product $ABCD$ as in [GM93, Lemma 4.1.1] implies the following:

$$\begin{aligned} 2\operatorname{tr}(ABCD) = &\operatorname{tr}(AC)\operatorname{tr}(B)\operatorname{tr}(D) - \operatorname{tr}(A)\operatorname{tr}(B)\operatorname{tr}(CD) - \operatorname{tr}(B)\operatorname{tr}(C)\operatorname{tr}(DA) \\ &- \operatorname{tr}(AC)\operatorname{tr}(BD) + \operatorname{tr}(AB)\operatorname{tr}(CD) + \operatorname{tr}(BC)\operatorname{tr}(DA) \\ &- \operatorname{tr}(ACB)\operatorname{tr}(D) + \operatorname{tr}(A)\operatorname{tr}(BCD) + \operatorname{tr}(B)\operatorname{tr}(CDA) + \operatorname{tr}(C)\operatorname{tr}(DAB). \end{aligned} \tag{1.2.10}$$

Equation (1.2.10) involves the term $\operatorname{tr}(ACB)$, which does not respect the natural cyclic order but can be substituted using Equation (1.2.6). Then over an algebraically closed field $\mathbb{F}$ of characteristic $p \neq 2$ we can use induction on Equation (1.2.10) and the above observations to write the trace of any word W of length $L \geq 4$ in the matrices $A_1, \ldots, A_n$ and their inverses as a polynomial in the traces of the A_i and their cyclically ordered double and triple products. Moreover, the constant term of the polynomial is an integral multiple of 2.

Corollary 1.12. *Let $\mathbb{F}$ be an algebraically closed field of characteristic $p \neq 2$. The trace of any word in the matrices $A_1, \ldots, A_n \in \mathrm{SL}_2(\mathbb{F})$ and their inverses is a polynomial in the $d = n + \binom{n}{2} + \binom{n}{3}$ traces*

$$\operatorname{tr}(A_{j_1} \ldots A_{j_m}), \quad 1 \leq j_1 < j_2 < \ldots < j_m \leq n, \quad m \leq 3.$$

1.2.2 Coordinates for the trace ring

The trace identities give us a way to define *trace coordinates* for the trace ring $\mathrm{T}(\Gamma, \mathbb{F})$. There is a natural map $\mathrm{R}(\Gamma, \mathrm{SL}_2(\mathbb{F})) \to \mathrm{T}(\Gamma, \mathbb{F})$ taking representations to their trace function.

Given the traces from Lemma 1.11, the embedding $\varphi\colon\ \mathrm{R}(\Gamma, \mathrm{SL}_2(\mathbb{F})) \hookrightarrow \mathrm{SL}_2(\mathbb{F})^n \subset \mathbb{F}^{4n}$ induces a natural embedding $\varphi_n\colon\ \mathrm{T}(\Gamma, \mathbb{F}) \hookrightarrow \mathbb{F}^{2^n-1}$ where the image is the traces of ordered products of generators. We call these the **long trace coordinates**,

$$\mathrm{Tr_n}\colon\ \varphi(\mathrm{R}(\Gamma, \mathrm{SL}_2(\mathbb{F}))) \to \varphi_n(\mathrm{T}(\Gamma, \mathbb{F})).$$

Given the traces from Corollary 1.12, over a field $\mathbb{F}$ of characteristic $p \neq 2$ we can push this further by composing $\varphi_n\colon\ \mathrm{T}(\Gamma, \mathbb{F}) \hookrightarrow \mathbb{F}^{2^n-1}$ with the projection r onto the $d = n + \binom{n}{2} + \binom{n}{3}$ traces of ordered single, double, and triple products of generators. We call these the **short trace coordinates**,

$$\mathrm{tr_n} = r \circ \mathrm{Tr_n}\colon\ \varphi(\mathrm{R}(\Gamma, \mathrm{SL}_2(\mathbb{F}))) \to \varphi_d(\mathrm{T}(\Gamma, \mathbb{F})).$$

Choosing a different ordered generating set for Γ would give another set of long and short trace coordinates. We can use the trace identities Equations (1.2.3) to (1.2.6) to get a polynomial automorphism between the two sets of coordinates.

We get the following commutative diagram:

$$\begin{tikzcd}
\mathrm{R}(\Gamma, \mathrm{SL}_2(\mathbb{F})) \arrow[r] \arrow[d, "\varphi"'] & \mathrm{T}(\Gamma, \mathbb{F}) \arrow[d, "\varphi_n"'] \arrow[dr, "\varphi_d"] & \\
\mathrm{SL}_2(\mathbb{F})^n \arrow[r, "\mathrm{Tr_n}"] & \mathbb{F}^{2^n-1} \arrow[r, "r"] & \mathbb{F}^d
\end{tikzcd}$$

In particular the restriction of the projection map $\varphi_n(\mathrm{T}(\Gamma, \mathbb{F})) \to \varphi_d(\mathrm{T}(\Gamma, \mathbb{F}))$ has a polynomial inverse arising from the trace identities over a field $\mathbb{F}$ of characteristic $p \neq 2$.

1.2.3 Properties of representations and characters

From now on, identify $\mathrm{R}(\Gamma, \mathrm{SL}_2(\mathbb{F}))$ with its image in $\mathrm{SL}_2(\mathbb{F})^n$.

Definition 1.13. *We say that a subgroup of* $\mathrm{SL}_2(\mathbb{F})$ *is* ***reducible*** *if its action on* $\mathbb{F}^2$ *has an invariant 1–dimensional subspace. Otherwise it is* ***irreducible****.*

We call a representation *(or equivalently an n–tuple of matrices)* ***irreducible*** *(resp.* ***reducible****) if it generates an irreducible (resp. reducible) subgroup of* $\mathrm{SL}_2(\mathbb{F})$*. This property is invariant under conjugation, and hence inherited by conjugacy classes.*

In particular, all abelian subgroups of $\mathrm{SL}_2(\mathbb{F})$ are reducible. We call a representation (or equivalently an n–tuple of matrices) **abelian** if it generates an abelian subgroup of $\mathrm{SL}_2(\mathbb{F})$.

We collect some useful results here about irreducible and reducible representations and characters. Most of the results are classical or are due to [CS83].

Remark 1.14. *Since we are interested in representations up to conjugacy, we can often conjugate pairs of matrices to be in their simplest form.*

If $A,B \in \mathrm{SL}_2(\mathbb{F})$ *form a reducible pair, then we may conjugate so that*

$$A = \begin{pmatrix} s & * \\ 0 & s^{-1} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} t & * \\ 0 & t^{-1} \end{pmatrix}, \quad s,t \in \mathbb{F} \setminus \{0\}. \tag{1.2.11}$$

If $A,B \in \mathrm{SL}_2(\mathbb{F})$ *form an irreducible pair, then we may conjugate so that*

$$A = \begin{pmatrix} s & 1 \\ 0 & s^{-1} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} t & 0 \\ u & t^{-1} \end{pmatrix}, \quad s,t,u \in \mathbb{F} \setminus \{0\} \tag{1.2.12}$$

where $u \neq -\left(s-s^{-1}\right)\left(t-t^{-1}\right)$*. This is useful for calculations.*

Note the requirement $u \neq -\left(s-s^{-1}\right)\left(t-t^{-1}\right)$ *in Equation (1.2.12) is because the invariant subspaces of* A *are spanned by* $(1,0)^\intercal$ *and* $\left(1,s^{-1}-s\right)^\intercal$ *and those of* B *by* $(0,1)^\intercal$ *and* $\left(t-t^{-1},u\right)^\intercal$*. Then* $u = -\left(s-s^{-1}\right)\left(t-t^{-1}\right)$ *gives* $\left(t-t^{-1}\right)\left(1,s^{-1}-s\right)^\intercal = \left(t-t^{-1},u\right)^\intercal$*, a contradiction to irreducibility.*

Lemma 1.15. $A,B \in \mathrm{SL}_2(\mathbb{F})$ *have a common invariant 1–dimensional subspace if and only if*

$$\operatorname{tr}\left(ABA^{-1}B^{-1}\right) = 2. \tag{1.2.13}$$

Proof. If $A,B \in \mathrm{SL}_2(\mathbb{F})$ have a common invariant 1–dimensional subspace, then we may conjugate them to the form given in Equation (1.2.11). Then $\operatorname{tr}\left(ABA^{-1}B^{-1}\right) = 2$.

Now suppose $A,B \in \mathrm{SL}_2(\mathbb{F})$ have no common 1–dimensional invariant subspace. We may conjugate so they are in the form given in Equation (1.2.12). Then

$$\operatorname{tr}\left(ABA^{-1}B^{-1}\right) = 2 + u^2 + u\left(s-s^{-1}\right)\left(t-t^{-1}\right).$$

If $\operatorname{tr}\left(ABA^{-1}B^{-1}\right) = 2$, then $u \neq 0$ implies $0 \neq u = -\left(s-s^{-1}\right)\left(t-t^{-1}\right)$. This gives a contradiction to Equation (1.2.12). Hence $\operatorname{tr}\left(ABA^{-1}B^{-1}\right) \neq 2$. □

We can extend this idea to a triple of matrices.

Lemma 1.16. *Suppose $(A,B,C) \in \mathrm{SL}_2(\mathbb{F})^3$ is irreducible, but any two of the matrices have a common invariant 1–dimensional subspace. Then* $\mathrm{tr}[\,A,\ BC\,] \neq 2$.

Proof. Let e_1 be a common eigenvector of A and B, and e_2 be a common eigenvector of A and C. Since $(A,B,C) \in \mathrm{SL}_2(\mathbb{F})^3$ is irreducible, (e_1,e_2) is a basis of $\mathbb{F}^2$, and hence up to conjugation we may assume

$$A = \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix}, \quad B = \begin{pmatrix} s & 1 \\ 0 & s^{-1} \end{pmatrix}, \quad C = \begin{pmatrix} t & 0 \\ u & t^{-1} \end{pmatrix}, \quad s,t,u,v \in \mathbb{F} \setminus \{0\}.$$

Now $\mathrm{tr}[\,B,\ C\,] = 2$ and $u \neq 0$ implies $0 \neq u = -\left(s - s^{-1}\right)\left(t - t^{-1}\right)$. Then $\mathrm{tr}[\,A,\ BC\,] = 2$ if and only if $\left(v^2-1\right)\left(s^2-1\right)\left(t^2-1\right) = 0$. Since $(A,B,C) \in \mathrm{SL}_2(\mathbb{F})^3$ is irreducible, we have $v^2 \neq 1$ and thus $\left(v^2-1\right)\left(s^2-1\right)\left(t^2-1\right) \neq 0$. This gives $\mathrm{tr}[\,A,\ BC\,] \neq 2$ as required. ☐

Another way of formulating the statement in Lemma 1.16 is: if $(A,B,C) \in \mathrm{SL}_2(\mathbb{F})^3$ is irreducible, but any two of the matrices have a common invariant 1–dimensional subspace, then A and BC do not share a common invariant 1–dimensional subspace.

We can understand reducible representations through traces of commutators.

Lemma 1.17 (Culler-Shalen)**.** *[CS83, Lemma 1.2.1] Let $\rho\colon\ \Gamma \to \mathrm{SL}_2(\mathbb{F})$ be a representation with non-abelian image. The following are equivalent:*

(i) ρ is reducible.

(ii) $\tau_\rho(c) = 2$ for every element c in the commutator subgroup of Γ.

Proof. Suppose that ρ is reducible over $\mathbb{F}$. Then ρ is equivalent to a representation by upper triangular matrices. The commutator of two upper triangular matrices is upper triangular with diagonal elements equal to 1. It follows that $\tau_\rho(c) = 2$ for all commutators $c \in [\,\Gamma\,,\ \Gamma\,]$ thus Item (i) implies Item (ii).

Suppose now $\tau_\rho(c) = 2$ for all $c \in [\,\Gamma\,,\ \Gamma\,]$. Since $\rho(\Gamma)$ is non-abelian there exists $c \in [\,\Gamma\,,\ \Gamma\,]$ such that $\rho(c) \neq E$. Since $\tau_\rho(c) = 2$, $\rho(c)$ must have a unique 1–dimensional invariant subspace. Label this subspace L. Assume there exists a $c' \in [\,\Gamma\,,\ \Gamma\,]$ such that L is not invariant under $\rho(c')$. Then $\rho(c') \neq E$ and $\rho(c')$ has a different 1–dimensional invariant subspace, say $L' \neq L$. Up to conjugation, we can then write the image of ρ on c and c' as

$$\rho(c) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad \rho(c') = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}, \quad a,b \in \mathbb{F} \setminus \{0\}.$$

Then $\mathrm{tr}(\rho(cc')) = 2 + ab \neq 2$ which contradicts our assumption. Therefore all elements $c \in [\,\Gamma\,,\ \Gamma\,]$ must share a 1–dimensional invariant subspace L. We have $\rho([\,\Gamma\,,\ \Gamma\,])$ is normal in $\rho(\Gamma)$ and thus L will also be invariant under ρ, which implies ρ is reducible. ☐

Remark 1.18. *Abelian representations $\rho\colon\ \Gamma \to \mathrm{SL}_2(\mathbb{F})$ also satisfy Items (i) and (ii). Hence reducible representations form a Zariski closed set by Hilbert's basis theorem.*

There are no comparable polynomial equations that define irreducible representations or characters, but we do have the following over $\mathbb{C}$. We later prove a similar result over $\mathbb{F}$ in Chapter 3.

Proposition 1.19 (Culler-Shalen). *[CS83, Proposition 1.5.2] If $\rho, \sigma \in \mathrm{R}(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ with $\tau_\rho = \tau_\sigma$ and ρ is irreducible then ρ and σ are equivalent.*

Proof. First note the following claim (which is recorded in [CS83, Lemma 1.5.1]):

Claim 1.20. *If $\rho\colon \Gamma \to \mathrm{SL}_2(\mathbb{C})$ is an irreducible representation and $\rho(g) \neq \pm E$ for a given $g \in \Gamma$, then there exists $h \in \Gamma$ such that the restriction of ρ to the subgroup generated by g and h is irreducible and $\tau_\rho(h) \neq \pm 2$.*

This lets us find an element h that is irreducible with g *and* cannot have the same trace as plus or minus the identity.

Proof of Claim. By irreducibility of Γ and the fact that $\rho(g) \neq \pm E$, we must have an element $h_0 \in \Gamma$ such that the restriction of ρ to the subgroup generated by g and h_0 is irreducible. If $\tau_\rho(h_0) \neq \pm 2$ then take $h = h_0$.

Assume we are in the case where $\tau_\rho(h_0) = \pm 2$. Then up to conjugation we can write

$$\rho(h_0) = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \rho(g) = \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix}, \quad a, c \in \mathbb{C} \setminus \{0\}.$$

For $n \in \mathbb{Z}$ we get $\tau_\rho(gh_0^n) = (\pm 1)^n (a + a^{-1} + cn)$. Then there exists an $n \neq 0$ such that $a + a^{-1} + cn \neq 2$ and thus $\tau_\rho(gh_0^n) \neq \pm 2$. Take $h = gh_0^n$. Note $\tau_\rho([\,g,\,h\,]) = 2 + c^2 n^2 \neq 2$ so by Lemma 1.17 the restriction of ρ to the subgroup generated by g and h is irreducible. This finishes the proof of the claim. □

Given the claim and that ρ is irreducible we can find $g, h \in \Gamma$ such that the restriction of ρ to the subgroup generated by g and h is irreducible, $\rho(g) \neq \pm E$ and $\tau_\rho(h) \neq \pm 2$. Let G the subgroup generated by $g, h \in \Gamma$. Since $\tau_\rho = \tau_\sigma$ it follows that σ restricted to G is also irreducible. Up to conjugation we can assume

$$\rho(h) = \sigma(h) = \begin{pmatrix} e & 0 \\ 0 & e^{-1} \end{pmatrix}, \quad e \in \mathbb{C} \setminus \{0, 1, -1\}$$

and

$$\rho(g) = \begin{pmatrix} a & 1 \\ c & d \end{pmatrix}, \quad \sigma(g) = \begin{pmatrix} a' & 1 \\ c' & d' \end{pmatrix}, \quad a, a', c, c', d, d' \in \mathbb{C}, \text{ with } c, c' \neq 0.$$

Take $\gamma \in \Gamma$ and let

$$\rho(\gamma) = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \quad \sigma(\gamma) = \begin{pmatrix} p' & q' \\ r' & s' \end{pmatrix}.$$

Analysing the trace function on combinations of g, h, γ we have

$$\begin{aligned} ep + e^{-1}s &= \tau_\rho(h\gamma) = \tau_\sigma(h\gamma) = ep' + e^{-1}s', \\ ae + de^{-1} &= \tau_\rho(hg) = \tau_\sigma(hg) = a'e + d'e^{-1}. \end{aligned}$$

Given $e \neq \pm 1$ we get $p = p'$, $s = s'$, $a = a'$, $d = d'$. The determinant condition on $\rho(g)$ and $\sigma(g)$ forces $c = c'$. We then have

$$(ap + r)e + (cq + ds)e^{-1} = \tau_\rho(hg\gamma) = \tau_\sigma(hg\gamma) = (ap + r')e + (cq' + ds)e^{-1},$$

with $c \neq 0$. This forces $r = r'$, $q = q'$. Therefore $\rho(\gamma) = \sigma(\gamma)$ and, since γ was arbitrary, $\rho = \sigma$. This completes the proof. $\square$

From these results we can start to see how the image of reducible and irreducible representations in $q\colon\ \mathrm{R}(\Gamma, \mathrm{SL}_2(\mathbb{C})) \to \mathrm{X}(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ are defined by trace, which is how the isomorphism between the variety of characters and the trace ring is established. With this isomorphism, the trace coordinates from Section 1.2.2 give coordinates for the $\mathrm{SL}_2(\mathbb{C})$–variety of characters.

Definition 1.21. *We say that an element of the* variety of characters *is* ***irreducible*** *(resp.* ***reducible****) if it is in the image of an irreducible (resp. reducible) representation under the quotient map q.*

1.2.4 Example: the free group on two generators

Consider $\Gamma = \langle \alpha, \beta \rangle$ the free group on two elements. We can fully describe $\mathrm{X}(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ following well-known classical arguments that originate from [Fri96, FK97, FK12]. In terms of manifolds, this is the $\mathrm{SL}_2(\mathbb{C})$–character variety of the once-punctured torus $S_{1,1}$ with $\pi_1(S_{1,1}) \cong \Gamma$.

Lemma 1.22. *The embedding $\varphi_2\colon\ \mathrm{X}(\Gamma, \mathrm{SL}_2(\mathbb{C})) \hookrightarrow \mathbb{C}^{2^2-1} = \mathbb{C}^3$ is an isomorphism. That is, for all $(x, y, z) \in \mathbb{C}^3$ there exists some $(A, B) \in \mathrm{SL}_2(\mathbb{C})^2$ such that* $\mathrm{tr}(A) = x$, $\mathrm{tr}(B) = y$, *and* $\mathrm{tr}(AB) = z$. *Moreover, (A, B) is reducible if and only if $x^2 + y^2 + z^2 - xyz = 4$.*

Proof. Consider the representation variety $\mathrm{R}(\Gamma, \mathrm{SL}_2(\mathbb{C})) = \mathrm{SL}_2(\mathbb{C})^2$. It follows from Lemma 1.11 and the trace coordinates that

$$\mathrm{X}(\Gamma, \mathrm{SL}_2(\mathbb{C})) = \{(\mathrm{tr}(A), \mathrm{tr}(B), \mathrm{tr}(AB)) \in \mathbb{C}^3 \mid (A, B) \in \mathrm{R}(\Gamma, \mathrm{SL}_2(\mathbb{C}))\}.$$

We want to show $\mathrm{X}(\Gamma, \mathrm{SL}_2(\mathbb{C})) \cong \mathbb{C}^3$.

Let $\chi = (x, y, z) \in \mathbb{C}^3$. We need to show there is a pair of matrices A, B such that $(\mathrm{tr}(A), \mathrm{tr}(B), \mathrm{tr}(AB)) = \chi$.

We may choose $d_x, d_y \in \mathbb{C} \setminus \{0\}$ such that $x = d_x + d_x^{-1}$ and $y = d_y + d_y^{-1}$. Let

$$A = \begin{pmatrix} d_x & 1 \\ 0 & d_x^{-1} \end{pmatrix}, \quad B = \begin{pmatrix} d_u & 0 \\ u & d_y^{-1} \end{pmatrix}$$

with $u \in \mathbb{C}$ to be determined. Then

$$AB = \begin{pmatrix} d_x d_y + u & d_y^{-1} \\ d_x^{-1} u & d_x^{-1} d_y^{-1} \end{pmatrix}.$$

We can choose $u = z - (d_x d_y + d_x^{-1} d_y^{-1})$. Then $\mathrm{tr}(AB) = (d_x d_y + d_x^{-1} d_y^{-1}) + u = z$ and $(\mathrm{tr} A, \mathrm{tr} B, \mathrm{tr}(AB)) = (x, y, z)$. This completes the first part of the proof.

Consider now the reducible representations. By Lemma 1.15, (A, B) is reducible if and only if $\mathrm{tr}[A,\ B] = 2$. The trace identity Equation (1.2.7) gives $\mathrm{tr}[\ A,\ B\ ] = x^2 + y^2 + z^2 - xyz - 2$. Then the reducible representations are precisely $x^2 + y^2 + z^2 - xyz = 4$ for $(x, y, z) = (\mathrm{tr}(A), \mathrm{tr}(B), \mathrm{tr}(AB))$. This finishes the proof. $\square$

We will see the same result follows for $\mathbb{F}$ an algebraically closed field using results from Chapter 3.

1.3 Connections to geometry

Representations and characters of manifolds are naturally connected to geometry. A **geometric structure** (G,X) is a homogeneous space X together with a group G of transformations of X. A geometric structure can be placed on a manifold using a system of coordinate charts. A (G,X)**–manifold** is a manifold M endowed with a (G,X)–structure in the sense that there is an atlas of charts on M over X whose transition functions are restrictions of elements of G. The formalisation and study of the geometric structures in this way was started by Ehresmann in [Ehr36]. If X is a simply connected Riemannian manifold of constant sectional curvature and G is its group of isometries then a (G,X)–structure on M is equivalent to giving M a Riemannian metric of the same curvature.

Let $\widetilde{M}$ denote the universal cover of M. Associated with every (G,X)–structure on M is a **developing map** and a representation called the **holonomy**, written

$$\mathrm{dev}\colon\ \widetilde{M} \to X, \quad \rho\colon\ \pi_1(M) \to G$$

such that

$$\mathrm{dev}(\gamma \circ x) = \rho(\gamma) \circ \mathrm{dev}(x), \quad \text{for all } \gamma \in \pi_1(m),\ x \in \widetilde{M}.$$

The (G,X)–manifold M is complete if and only if dev is a covering map and, if X is simply connected, M is isometric to X/Γ for $\Gamma = \rho(\pi_1(M))$ a discrete subgroup of G. The pair (dev,ρ) is called the **development pair**. If the action by G is analytic then it is uniquely determined up to conjugation by an element of G.

We can define a topology on (G,X)–manifolds in terms of atlases of coordinate charts (see [Gol22]). Consider the Euclidean topology on $\mathrm{R}(M,G)$. Thurston proved the following connection between geometric structures and representations using the holonomy.

Theorem 1.23 (Deformation theorem, [Thu80])**.** *Let G be a Lie group that acts transitively on manifold X. Let M be a compact (G,X)–manifold with holonomy $\rho\colon\ \pi_1(M) \to G$.*

- *Suppose ρ' is sufficiently close to ρ in the representation variety $\mathrm{R}(M,G)$. Then there exists a nearby (G,X)–structure on M with holonomy representation ρ'.*
- *If M' is another manifold with a (G,X)–structure near M with the same holonomy ρ then M' is isomorphic to M by an isomorphism that is isotopic to the identity.*

This result and its consequences are discussed in more detail in [Gol88a, Gol22]. In particular, this implies the set of holonomy representations of (G,X)–structures on M is open in $\mathrm{R}(M,G)$. Goldman [Gol22] elaborates on how these ideas can be used to see the *deformation space* $\mathrm{Def}_{(G,X)}(M)$ of equivalence classes of (marked) geometric structures is "locally modelled" on the character stack $\mathrm{Hom}(\pi_1(M),G)/G$.

We focus on two approaches for how we can study geometry of low-dimensional manifolds through character varieties and varieties of characters. Section 1.3.1 defines Teichmüller space on a surface and then illustrates connections between Teichmüller space and character varieties. This motivates Chapter 2, where we study the action of Dehn twists on the character variety and earthquakes on Teichmüller space. Section 1.3.2 defines essential surfaces in 3–manifolds and outlines connections between essential surfaces and geometry and topology. We then discuss the relation with character varieties using the theory of Culler and Shalen. This motivates Chapter 3, where we extend the theory of Culler and Shalen from character varieties over $\mathbb{C}$ to varieties of characters over a general algebraically closed field $\mathbb{F}$. We continue to use E to denote the identity matrix in $\mathrm{SL}_2(\mathbb{F})$.

1.3.1 Teichmüller space

Let S be a topologically finite orientable surface of negative Euler characteristic. We introduce the space of hyperbolic structures on S, called Teichmüller space, from two angles: first from the perspective of metrics and lengths, and second from the perspective of representations. We use the latter perspective to see the relationship between Teichmüller space and the character variety of S. A repeating example throughout will be the once-punctured torus $S_{1,1}$.

A *hyperbolic metric* on S is a complete Riemannian metric on S of constant sectional curvature -1. A *hyperbolic structure* (X,ϕ) on S is a surface X with a hyperbolic metric and a diffeomorphism $\phi : S \to X$. The diffeomorphism ϕ is called the *marking*.

Definition 1.24. *The **Teichmüller space** of S is the space of all homotopy classes of hyperbolic structures of finite area*

$$\mathscr{T}(S) = \{(X,\phi)\}/\sim,$$

where two hyperbolic structures (X_1,ϕ_1) and (X_2,ϕ_2) are homotopic if there is an isometry $I\colon X_1 \to X_2$ such that $I \circ \phi_1$ is homotopic to ϕ_2.

There are many equivalent ways to define Teichmüller space. For example, we could define Teichmüller space of S as the space of homotopy classes of complex structures on S. If $\chi(S) < 0$ then each class of complex structures is associated with a unique hyperbolic metric using the uniformization theorem. We present an example and then elaborate on two viewpoints of Teichmüller space.

Example 1.25 (Once-punctured torus)**.** *Consider $\mathscr{T}(S_{1,1})$. We can picture $S_{1,1}$ as a quadrilateral (or "lattice") in $\mathbb{R}^2$ with opposite edges identified and a point removed (the puncture). This is similar to the torus $\mathbb{T}$.*

The following bijection is well-known:

$$\mathscr{T}(S_{1,1}) \cong \mathbb{H}^2.$$

This can be proved using the forgetful *map defined by taking a class of hyperbolic structures on $S_{1,1}$ to the associated class of complex structures on $\mathbb{T}$ and forgetting about the puncture. See for example [Pet24, Proposition 4.1.5]. We get the result by then seeing that the space of classes of complex structures on the torus is in bijection with $\mathbb{H}^2$. See for example [FM12, Proposition 10.1]; compare with [IT92, Pet24].*

1.3.1.1 Teichmüller space through Fenchel-Nielsen length-twist coordinates

A useful tool in studying Teichmüller space is the length of curves. Given a metric, all equivalence classes of simple closed curves $[\gamma] \in \pi_1(S)$ that are not homotopic into a neighbourhood of a puncture must contain a unique geodesic (see [FM12, Proposition 1.3]).

A natural coordinate system on $\mathscr{T}(S)$ is *Fenchel-Nielsen length-twist coordinates*, which were introduced in [FN] (see also [FN02]). A closed hyperbolic surface S_g with genus g has a decomposition into $2g-2$ pairs of pants with $3g-3$ distinct curves that are the boundary components for a pair of pants in the decomposition. Let the boundary components of the pairs of pants be labelled $\gamma_1, \dots, \gamma_{3g-3}$. Then for a hyperbolic structure $x \in \mathscr{T}(S_g)$, the **length parameter** associated with γ_i from the pair of pants decomposition is

$$\ell_i(x) = \ell_x(\gamma_i),$$

the length of the geodesic associated with γ_i under the metric from x. There will be $3g-3$ length parameters in total, one for each curve γ_i.

In order to get coordinates for S_g, we need to be able to measure how the pairs of pants are glued together. We can also choose geodesics on the interior of each pair of pants, called *seams*, which are the three unique disjoint geodesics connecting pairs of boundary components. Take a seam δ_j that touches geodesic γ_i. The twisting numbers t_L and t_R of δ_j at γ_i measure how much δ_j twists to the left or right respectively of γ_i. Then for a hyperbolic structure $x \in \mathscr{T}(S_g)$, the **twist parameter** associated with γ_i is

$$\theta_i(x) = 2\pi \frac{t_L - t_R}{\ell_x(\gamma_i)}.$$

The twist parameter is well-defined in that, if we had taken δ_k the other seam that touches γ_i, we would get the same result. There will be $3g-3$ twist parameters in total, one for each curve γ_i.

Definition 1.26. *Let S_g be a closed hyperbolic surface of genus g with a given pair of pants decomposition. The **Fenchel-Nielsen length-twist coordinates** of $\mathscr{T}(S_g)$ are the $3g-3$ length and twist parameters of $\mathscr{T}(S_g)$ under the decomposition, given by*

$$\begin{aligned} \mathrm{FN}\colon\ \mathscr{T}(S_g) &\to \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3} \\ x &\mapsto (\ell_1(x), \dots, \ell_{3g-3}(x), \theta_1(x), \dots, \theta_{3g-3}(x)). \end{aligned} \tag{1.3.1}$$

The coordinates allow us to better understand the topology of $\mathscr{T}(S_g)$. In fact, the map FN is a homeomorphism and, for a closed hyperbolic surface,

$$\mathscr{T}(S_g) \cong \mathbb{R}^{6g-6}.$$

This result was first proved independently of these coordinates in [FK97, FK12]. See [FM12, Theorem 10.6] for a proof using the map FN.

Similar results can be derived for surfaces with boundary and punctures. For example, a compact hyperbolic surface $S_{g,n}$ with genus g and n punctures we get a homeomorphism

$$\mathscr{T}(S_{g,n}) \cong \mathbb{R}^{6g-6+2n}.$$

See [FM12, Section 10.6.3].

Example 1.27 (Once-punctured torus)**.** *This result gives $\mathscr{T}(S_{1,1}) \cong \mathbb{R}^2$ with only one length and one twist coordinate. This is consistent with our previous claim in Example 1.25 that $\mathscr{T}(S_{1,1}) \cong \mathbb{H}^2$.*

1.3.1.2 Teichmüller space through representations

In the language of (G,X)–structures, a hyperbolic structure is framed as $X = \mathbb{H}^2$, $G \leq \mathrm{Isom}^+(\mathbb{H}^2)$ the orientation-preserving isometry group of $\mathbb{H}^2$. Isometries $f \in \mathrm{Isom}^+(\mathbb{H}^2)$ are in bijection with *Möbius transformations* given by

$$F = \left\{ \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\} \in \mathrm{PSL}_2(\mathbb{R}), \quad F \cdot z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

for all $z \in \mathbb{H}^2$. Then $\mathrm{Isom}^+(\mathbb{H}^2) \cong \mathrm{PSL}_2(\mathbb{R})$. We also get the full isometry group $\mathrm{Isom}(\mathbb{H}^2) \cong \mathrm{PGL}_2(\mathbb{R})$.

Each hyperbolic structure on S has an associated holonomy map $\rho \colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{R})$ defined up to conjugacy. In fact, for all surfaces S with a hyperbolic structure there is a natural bijection between Teichmüller space and the $\mathrm{PGL}_2(\mathbb{R})$ conjugacy classes of discrete, faithful representations $\rho \colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{R})$ (see for example [FM12, Proposition 10.2]). Recall such a representation ρ is called *discrete* if $\rho(\pi_1(S))$ is discrete in $\mathrm{PSL}_2(\mathbb{R})$ and *faithful* if ρ is injective.

Goldman [Gol88b] showed for every closed hyperbolic surface, this space of conjugacy classes of discrete and faithful representations forms two components of the $\mathrm{PSL}_2(\mathbb{R})$–character variety, which can themselves be identified. This is called the *Hitchin component* after Hitchin extended this idea to representations to $\mathrm{PSL}_n(\mathbb{R})$ in [Hit92].

Then we can study the hyperbolic structures (and the associated hyperbolic metrics) via representations. Let us see more explicitly how this relates to the geometry of a hyperbolic surface. It is known that each $f \in \mathrm{Isom}^+(\mathbb{H}^2)$ extends uniquely to a map $\overline{f}$ on $\overline{\mathbb{H}^2} = \mathbb{H}^2 \cup \partial\mathbb{H}^2$, for $\partial\mathbb{H}^2 = \mathbb{R} \cup \{\infty\}$ the boundary and that this can be used to classify the non-trivial isometries [Thu80]. Let $F \in \mathrm{PSL}_2(\mathbb{R})$ be the matrix associated with f.

- If $\overline{f}$ fixes a point $p \in \mathbb{H}^2$, then f is a rotation about p and $|\mathrm{tr}(F)| < 2$. We call f **elliptic**. In this case, F will be conjugate to the form

$$\left\{ \pm \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \right\} \in \mathrm{PSL}_2(\mathbb{R}).$$

- If $\overline{f}$ fixes a exactly one point in $\partial\mathbb{H}^2$, then $F \neq \{\pm E\}$ and $|\mathrm{tr}(F)| = 2$. We call f **parabolic**. In this case, F will be conjugate to the form

$$\left\{ \pm \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix} \right\} \in \mathrm{PSL}_2(\mathbb{R}).$$

- If $\overline{f}$ fixes exactly two points in $\partial\mathbb{H}^2$, then $|\mathrm{tr}(F)| > 2$. We call f **hyperbolic** (or **loxodromic**). In this case, F will be conjugate to the form

$$\left\{ \pm \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \right\} \in \mathrm{PSL}_2(\mathbb{R}), \quad \lambda > 1.$$

Examples are shown in Figure 1.1. If $\overline{f}$ fixes at least three points in $\overline{\mathbb{H}^2}$, then f is the identity.

The geometry of $S \cong \mathbb{H}^2/f$ varies depending on the form of f under this classification. If f is elliptic and $\langle f \rangle \leq \mathrm{PSL}_2(\mathbb{R})$ is discrete then S will have a conic singularity; if f is parabolic then S will have a cusp; if f is loxodromic then S will have a funnel. These geometries and how they relate to the isometry are shown in Figure 1.1. Considering again an equivalence class of representations in the character variety, we can see how the associated geometric structure will depend on the image of the simple closed curves in $\mathrm{Isom}^+(\mathbb{H}^2)$.

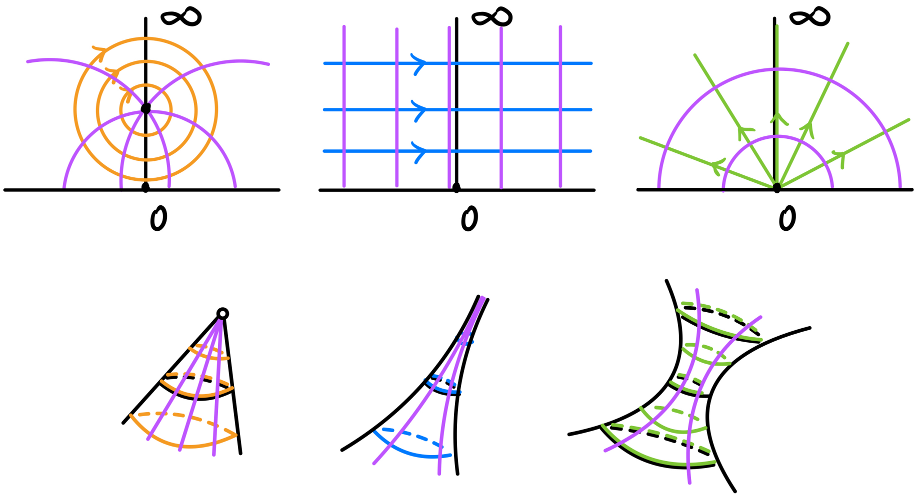

FIGURE 1.1: Example isometries of the hyperbolic plane (top) and example quotients $\mathbb{H}^2/f$ (bottom) for elliptic (left), parabolic (middle), and loxodromic (right) isometry f.

Example 1.28 (Once-punctured torus)**.** *The character variety of $S_{1,1}$ was touched on in Section 1.2.4. We have $\pi_1(S_{1,1}) = \langle \alpha, \beta \rangle$ for generators α, β the meridian and longitude. The choice of generators is called the framing. Then we saw $\mathrm{X}(\Gamma, \mathrm{SL}_2(\mathbb{C})) \cong \mathbb{C}^3$ via the map*

$$\begin{aligned} \mathrm{X}(\Gamma, \mathrm{SL}_2(\mathbb{C})) &\to \mathbb{C}^3 \\ [\rho] &\mapsto \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \mathrm{tr}(\rho(\alpha)) \\ \mathrm{tr}(\rho(\beta)) \\ \mathrm{tr}(\rho(\alpha\beta)) \end{pmatrix}. \end{aligned}$$

The (x, y, z) are called the trace coordinates. We also saw the reducible representations are defined by an equation in terms of the trace of the image of the commutator $K = [\,\alpha,\,\beta\,]$.

Look at the representations restricted to $\mathrm{SL}_2(\mathbb{R})$. The geometry around the puncture on $S_{1,1}$ can be determined by the image of the representation on the peripheral element K. Using the classification of isometries we get that:

- *If* $|\mathrm{tr}(K)| < 2$ *or* $\mathrm{tr}(K) = 2$*,* $S_{1,1}$ *has a conic singularity (incomplete);*
- *If* $\mathrm{tr}(K) = -2$*,* $S_{1,1}$ *has a cusp (complete with finite volume);*
- *If* $|\mathrm{tr}(K)| > 2$*,* $S_{1,1}$ *has a funnel (complete with geodesic waist and infinite volume).*

The options are shown in Figure 1.2.

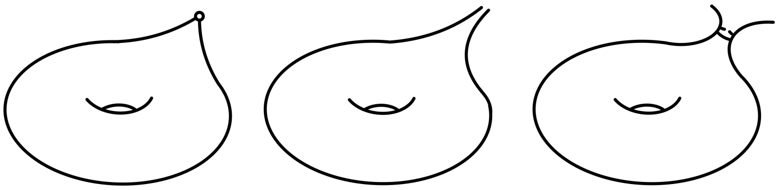

FIGURE 1.2: Example once-punctured torus with a conic singularity point (left), cusp (middle), and funnel (right).

The complete, finite-area hyperbolic structures occur precisely when $\mathrm{tr}(K) = -2$ *and* $S_{1,1}$ *has a cusp. We see* $\mathscr{T}(S_{1,1})$ *can be identified with a component of the subvariety defined by*

$$x^2 + y^2 + z^2 - xyz = 0.$$

See [Kee74] for a comprehensive proof of this separate from the character variety perspective.

The character variety under the different cases for $\mathrm{tr}(K)$ *is shown in Figure 1.3. When* $\mathrm{tr}(K) = -2$ *there are four components and* $\mathscr{T}(S_{1,1})$ *can be identified with any of them. See also [Gol03] for a discussion.*

1.3.2 Essential surfaces

We now switch gears to understand how we can use varieties of characters to study 3–manifolds. Let M be a compact, orientable 3–manifold. A key idea is decompositions of the manifold and embedded surfaces within them. We start by defining what an essential surface is. We will then discuss how they are related to geometry and topology via 3–manifold decompositions and how they can be related to varieties of characters via splittings of groups.

Intuitively, an essential surface holds intrinsic information about the topology of M (as the name suggests). We make this more precise in the following paragraphs using standard definitions from [Jac80].

An (embedded) surface S in M will always mean a 2–dimensional piecewise linear submanifold *properly embedded* in M. That is, a closed subset of M with $\partial S = S \cap \partial M$. If the 3–manifold is not compact, we replace it by a compact core.

We separate the cases for when S has a component that is a 2–sphere or not separately. An embedded sphere $\mathbb{S}^2$ in M is called **incompressible** if it does not bound an embedded ball in M. An embedded

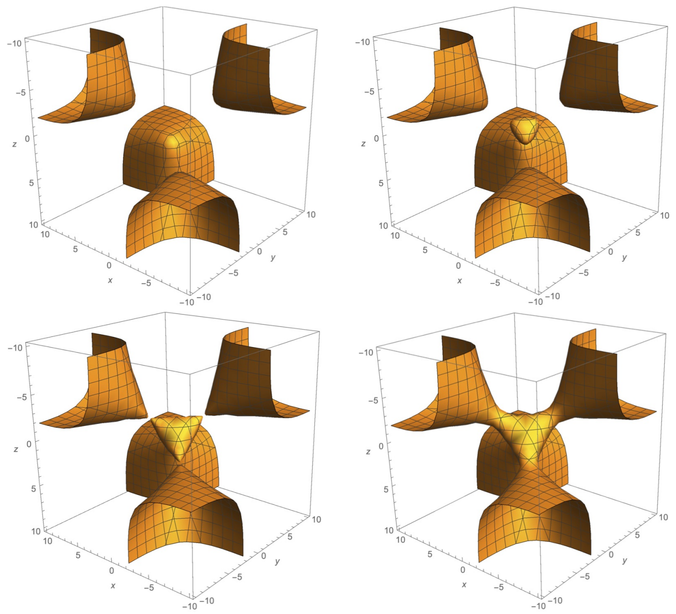

FIGURE 1.3: The level sets $\mathrm{tr}(K) = -2$ (top left), $\mathrm{tr}(K) = 0$ (top right), $\mathrm{tr}(K) = 2$ (bottom left), $\mathrm{tr}(K) = 4$ (bottom right) from Example 1.28.

sphere $\mathbb{S}^2$ in M is called **essential** if it is incompressible. We say M is **irreducible** if it contains no incompressible 2–spheres.

A surface S without 2–sphere components in M is called **incompressible** if for each disc $D \subset M$ with $D \cap S = \partial D$ there is a disc $D' \subset S$ with $\partial D' = \partial D$. Note that *every* properly embedded disc in M is incompressible.

A surface S in M is **∂–compressible** if either

1. S is a disc and S is parallel to a disc in ∂M, or
2. S is not a disc and there exists a disc $D \subset M$ such that: (i) $D \cap S = \alpha$ is an arc in ∂D, (ii) $D \cap \partial M = \beta$ is an arc in ∂D, (iii) $\alpha \cap \beta = \partial\alpha = \partial\beta$, (iv) $\alpha \cup \beta = \partial D$, and (v) either α does not separate S or α separates S into two components and the closure of neither is a disc.

Otherwise S is **∂–incompressible**.

We define essential surfaces more broadly using an approach from Shalen [Sha02].

Definition 1.29. *[Sha02] A surface S in a compact, irreducible, orientable 3–manifold M is **essential** if it has the following properties:*

1. *S is bicollared;*
2. *the inclusion homomorphism $\pi_1(S_i) \to \pi_1(M)$ is injective for every component S_i of S;*
3. *no component of S is a 2–sphere;*
4. *no component of S is boundary parallel;*

5. *S is non-empty.*

In the first condition, bicollared means S admits a map $h\colon\ S\times[-1,1]\to M$ that is a homeomorphism onto a neighbourhood of S in M such that $h(x,0)=x$ for every $x\in S$ and $h(S\times[-1,1])\cap\partial M=h(\partial S\times[-1,1])$. The surface S being bicollared in orientable M implies S is orientable. The second condition is equivalent to saying that there are no compression discs for the surface (see [Hem76, Lemma 6.1]) and thus S is incompressible.

See [Sha02] for a further discussion of the definition and its ramifications. Of particular note, Hatcher [Hat07, Lemma 1.10] implies that if each boundary component of the essential surface S lies on a torus boundary component of M, then S is both incompressible and ∂–incompressible. A compact, irreducible 3–manifold that contains an essential surface is called **Haken**.

Of particular interest are knot (or link) complements. Given a knot $K\colon\ \mathbb{S}^1\hookrightarrow\mathbb{S}^3$, let $N(K)$ be a neighbourhood of the knot. The *knot complement* is the complement of $N(K)$ in the 3–sphere $\mathbb{S}^3$, written $\mathbb{S}^3\setminus N(K)$. The concept for a link is the same with now possibly multiple circles $\mathbb{S}^1$ embedded in $\mathbb{S}^3$. All 3–manifolds can be obtained through a process called Dehn surgery on a link complement [Lic62, Wal60]. In knot and link complements, it is known there is always at least one essential surface, the least genus Seifert surface (due to [FP30, Sei35]).

Definition 1.30. *A **Seifert surface** for the knot or link $K\subset\mathbb{S}^2$ is a compact, connected, orientable surface S embedded in $\mathbb{S}^3$ whose boundary is K.*

1.3.2.1 3–manifold decompositions

In the language of (G,X)–structures, a hyperbolic structure on a 3–manifold M is framed as $X=\mathbb{H}^3$, $G\subset\mathrm{Isom}^+(\mathbb{H}^3)$ the orientation-preserving isometry group of $\mathbb{H}^3$.

It is known $\mathrm{Isom}^+(\mathbb{H}^3)\cong\mathrm{PSL}_2(\mathbb{C})$. An isometry $f\in\mathrm{Isom}^+(\mathbb{H}^3)$ is determined by its action in $\partial\mathbb{H}^3=\mathbb{C}^2\cup\{\infty\}$, which is in turn defined by a Mobius tranformation on the boundary given by

$$F=\left\{\pm\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right\}\in\mathrm{PSL}_2(\mathbb{C}),\quad F\circ z=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\circ z=\frac{az+b}{cz+d}$$

for all $z\in\partial\mathbb{H}^3$.

Using the holonomy, we can think of hyperbolic structures on M as inducing a representation $\rho\colon\ \pi_1(M)\to\mathrm{PSL}_2(\mathbb{C})$. In dimension 2 we could study the hyperbolic structures using Teichmüller space. In higher dimensions, Mostow-Prasad rigidity says that the geometry of a complete, finite-volume hyperbolic n–manifold, for $n>2$, is determined by the fundamental group and hence unique [Mos68, Pra73]. We can, however, use essential surfaces in M to better understand the topology.

To do this we highlight three results about 3–manifold decompositions that relate to essential surfaces. The results are the prime decomposition, the JSJ decomposition, and the geometrization theorem.

We say M is **prime** if it is not homeomorphic to a connected sum of manifolds, aside from the trivial connected sum $M\cong M\#\mathbb{S}^3$.

Theorem 1.31 (Prime decomposition, [Kne29, Mil62]). *Every compact, orientable 3–manifold is the connected sum of a unique (up to homeomorphism) finite collection of prime 3–manifolds.*

We say M is **atoroidal** if there are no essential tori in M and **Seifert-fibered** if M is a circle bundle over a 2–dimensional orbifold (a surface with marked points where there are singular fibres).

Theorem 1.32 (JSJ decomposition, [JS79, Joh79]). *Every irreducible, orientable closed 3–manifold has a unique (up to isotopy) minimal collection of disjointly embedded incompressible tori such that each component of the 3–manifold obtained by cutting along the tori is either atoroidal or Seifert-fibered.*

Along the prime decomposition will be a finite collection of essential 2–spheres. Along the JSJ decomposition will be a finite collection of essential tori.

The geometrization theorem uses these ideas to give an analogue to the uniformization theorem for 3–manifolds. Originally conjectured by Thurston [Thu82], the theorem was proved by Perelman [Per02, Per03a, Per03b] following a program started by Hamilton [Ham82].

Theorem 1.33 (Geometrization theorem). *The interior of every compact, orientable 3–manifold can be decomposed into pieces that have uniform geometry. More specifically, if a 3–manifold cannot be decomposed along essential spheres or tori, then it has a uniform geometry.*

There are eight options for the uniform geometry: $\mathbb{S}^3$, $\mathbb{E}^3$, $\mathbb{H}^3$, $\mathbb{S}^2\times\mathbb{E}$, $\mathbb{H}^2\times\mathbb{E}$, Nil, Sol, PSL. We focus on the case when the 3–manifold is hyperbolic, which can only occur if it is atoroidal (see [BBB$^+$10, Neu99]).

Thurston initially proved the geometrisation conjecture for Haken manifolds, which gave the following.

Theorem 1.34 (Thurston, [Thu82]). *Let M be a compact, orientable, irreducible, atoroidal, Haken 3–manifold with $\chi(\partial M) = 0$. Then the interior of M has a complete hyperbolic structure of finite volume.*

Note $\chi(\partial M) = 0$ for orientable M implies the boundary of M consists of tori.

In fact, the requirement for M to be Haken is unnecessarily strong. The proof of the geometrization theorem by Perelman shows that for M a closed 3–manifold it is sufficient to assume M is orientable, irreducible, and atoroidal with infinite fundamental group instead (see [BBB$^+$10]).

A related result also due to Thurston is:

Theorem 1.35 (Thurston, [Thu82]). *Let M be a compact, orientable 3–manifold with $\partial M \neq \emptyset$ consisting of tori. Then the interior of M has a complete hyperbolic structure of finite volume if and only if M contains no essential surfaces that are spheres, projective planes, tori, discs, annuli, Mobius bands, or Klein bottles.*

Using these results we can start to get a sense of how essential surfaces are related to the geometry and topology of M.

1.3.2.2 Splittings of groups

Culler and Shalen devised a way to detect essential surfaces in 3–manifolds by studying *ideal points* of their $\mathrm{SL}_2(\mathbb{C})$–character varieties. This is done by finding ideal points of curves in the character variety and applying Bass-Serre theory [Bas93, Ser03]. This will give a splitting of the fundamental group $\pi_1(M)$ (formally, an isomorphism of $\pi_1(M)$ with the fundamental group of a graph of groups). The Stallings-Epstein-Waldhausen construction [Eps61, Sta59, Wal67] shows how to associate an essential surface S in M with any such splitting.

We outline this theory in more detail in Chapter 3. The same surface might be detected by multiple ideal points, and there are some surfaces that are never detected by this machinery.

Given the connections between $\mathrm{SL}_2(\mathbb{C})$ and $\mathrm{Isom}^+(\mathbb{H}^3)$, explicit examples can be constructed and studied further. Based on Thurston [Thu80, Theorem 5.6], Culler and Shalen [CS83, Proposition 3.2.1] provide a bound on the dimension of the $\mathrm{SL}_2(\mathbb{C})$–character variety.

Proposition 1.36 (Thurston)**.** *Let M be a compact orientable 3–manifold. Let $\rho_0\colon\ \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ be an irreducible representation such that for each torus component T of ∂M, $\rho_0(\mathrm{im}(\pi_1(T) \to \pi_1(M))) \not\subseteq \{\pm E\}$.*

Let R_0 be an irreducible component of $\mathrm{R}(M,\mathbb{C})$ containing ρ_0. Then $X_0 = \mathrm{tr_n}(\mathrm{R_0})$ has dimension $\geq s - 3\chi(M)$, where s is the number of torus components of ∂M.

We know a hyperbolic manifold M must admit a (discrete and faithful) representation $\pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$. Then there is a component of the character variety X_0 that contains the character of this representation and the result says if $s > 0$ then X_0 must have dimension > 0. Then there will be a curve in X_0, and we can apply the theory of Culler and Shalen to detect at least one essential surface.

Chapter 2

Earthquakes and character varieties

In this chapter, we study earthquakes on the example of the once-punctured torus $S_{1,1}$ as a fundamental example of a hyperbolic surface. This is research that has been published in [Gar24]. Earthquake deformations about simple closed curves can be viewed as "fractional" Dehn twists. Consider a hyperbolic orientable surface S with a homotopy class of simple closed curves $\gamma \in \pi_1(S)$. Defined intuitively, a Dehn twist about γ is achieved by cutting the surface open along γ, twisting around γ fully, and then gluing the surface together again. An earthquake deformation about γ is achieved by cutting the surface open along γ, twisting around γ by some distance s, and then gluing the surface together again. Earthquakes can be done over the broader space of measured geodesic laminations, within which simple closed curves form a dense subset.

Earthquakes were first introduced by Thurston as a tool to study hyperbolic structures (see [CEM06] for an exposition). Most notably Thurston proved the earthquake theorem, which says the following:

Theorem 2.1 (Thurston's earthquake theorem)**.** *For any two points* $\mathbf{p}, \mathbf{q} \in \mathscr{T}(S)$*, there exists a unique earthquake that connects* $\mathbf{p}$ *to* $\mathbf{q}$.

A written proof of the earthquake theorem was first given in [Ker83], where certain convexity properties of earthquakes on length functions are established and used.

Earthquakes were famously used to answer the Nielsen realisation problem in the affirmative in [Ker83].

Theorem 2.2 (Nielsen realisation)**.** *Every finite subgroup of the mapping class group fixes a point of Teichmüller space.*

Previous research from Waterman and Wolpert [WW83, Wol83], Kerckhoff [Ker85], and Weiss [Wei89] focuses on earthquakes as specific integral curves. We study earthquakes from an alternative perspective.

The chapter proceeds as follows. Section 2.1 describes preliminary concepts and results, recalls trace coordinates, and describes the triangle length coordinates from [CP24]. We start our research studying the action of Dehn twists on the character variety using trace coordinates and employing the connection between Teichmüller space and the character variety in Section 2.2. We use linear recurrence relations to extend orbits of Dehn twists to a continuous flow, given in Theorem 2.12. This relates to the broader study of dynamics of character varieties, where the action of the mapping class group on the character variety is explored.

We then study earthquakes more explicitly on Teichmüller space using the triangle length coordinates based on the length of certain curves in Section 2.3. Through this we obtain a formula for the earthquake deformation in Theorem 2.18. The two approaches give results that agree, proving the continuous flow in the first method is indeed the earthquake deformation and giving both algebraic and geometric interpretations, respectively, of earthquakes. Section 2.4 extends the results from all simple closed curves using a change of coordinates algorithm.

We then focus on earthquake deformations in Fenchel-Nielsen length-twist coordinates, where we get more results for the behaviour and asymptotics in Section 2.5. In particular, see Corollary 2.25. We then explore how results for simple closed geodesics can be extended to measured geodesic laminations using a density argument from Thurston [Thu80] in Section 2.6 and how the expressions for earthquakes can be translated to other coordinate systems in Section 2.7. Section 2.8 reflects on initial results for how this could be explicitly transferred to an action on the hyperbolic plane.

Section 2.1 contains information from [Gar24] that has been expanded on for unfamiliar readers; Sections 2.2 to 2.7 are almost verbatim from [Gar24], with adjustments for consistency; and Section 2.8 contains remarks and calculations that are previously unpublished.

2.1 Preliminaries

We formalise the definitions and basic results for the mapping class group, Dehn twists, earthquakes, and measured geodesic laminations. The fundamental group of $S_{1,1}$ is the free group of rank two and the universal cover of $S_{1,1}$ is the hyperbolic plane $\mathbb{H}^2$. The Teichmüller space of $S_{1,1}$, $\mathscr{T}(S_{1,1})$, can be identified with $\mathbb{H}^2$, with isometry group $\mathrm{PSL}_2(\mathbb{R})$ as discussed in Section 1.3.1.

For the remainder of the chapter we use α and β the meridian and longitude, respectively, as the default framing for $S_{1,1}$, $\pi_1(S_{1,1}) = \langle \alpha, \beta \rangle$. We often refer to an equivalence class of simple closed curves $[\gamma]$ by a representative curve γ.

2.1.1 Dehn twists and earthquakes

A common way to study character varieties of manifolds is via the action of the mapping class group.

Definition 2.3. *The* ***mapping class group*** *of a manifold M is the set of all isotopy classes of orientation-preserving homeomorphisms on M to itself,*

$$\mathrm{MCG}^+(M) = \mathrm{Homeo}^+(M)/\sim.$$

Elements of the mapping class group are called ***mapping classes****.*

The group including orientation preserving and reversing homeomorphisms is called the extended mapping class group, $\mathrm{MCG}^{\pm}(M)$, and contains $\mathrm{MCG}^+(M)$ as an index 2 subgroup. For $M = S_g$ a closed surface of genus g, the Dehn-Nielsen-Baer theorem says the extended mapping class group is isomorphic to the group of outer automorphisms on the fundamental group. It is known that the mapping class group of $S_{1,1}$ is $\mathrm{SL}_2(\mathbb{Z})$ (for example, see [FM12]).

The mapping class group acts naturally on the character variety, and can be studied to understand the relationship between different geometric structures. This study is called the dynamics of character varieties and begin with Goldman in [Gol97].

Dehn twists are a special type of mapping class on a surface. Let γ be a simple closed curve on S. A Dehn twist about γ, labelled T_γ twists the surface near γ and keeps the rest of the surface fixed. This can be formalised.

Definition 2.4. *Find an annulus $A \subset S$ such that $\gamma \in \pi_1(S)$ is the core of A. Identify A with the annulus $\hat{A} = \{(t,\theta) \mid t \in [1,2], \theta \in [0,2\pi)\} \subset \mathbb{R}^2$ by an orientation-preserving homeomorphism $\phi\colon A \to \hat{A}$ such that $\phi(\gamma(s)) = \left(\frac{3}{2}, 2\pi s\right)$.*

*The **left-handed Dehn twist** of A is constructed by a map $\hat{T}_A\colon \hat{A} \to \hat{A}$ defined by $(t,\theta) \mapsto (t, \theta + 2\pi(t-1))$. Then $T_A\colon S \to S$ is defined by*

$$T_A(x) = \begin{cases} x & x \in S \setminus A, \\ \phi^{-1}\hat{T}_A\phi & x \in A. \end{cases}$$

This is continuous since $\hat{T}_A$ and ϕ are continuous and $\hat{T}_A \mid_{\partial\hat{A}} = \mathrm{Id}$.

This is a homeomorphism on S and the inverse ${T_A}^{-1}$ (the right-handed Dehn twist) is given by a similar construction where in the interim step we instead take $(t,\theta) \mapsto (t, \theta - 2\pi(t-1))$.

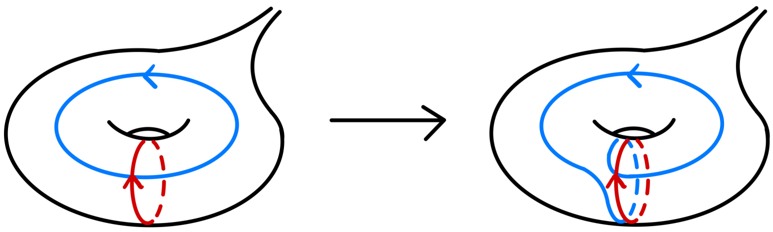

FIGURE 2.1: Example Dehn twist about α the meridian on $S_{1,1}$.

The Dehn twists are well-defined up to different choice of the annulus A and different homotopic curves. We write $T_\gamma = T_{[\gamma]} = T_A$.

Dehn twists are extremely important to the mapping class group. For a closed orientable surface S_g, a finite number of Dehn twists about non-separating simple closed curves generates $\mathrm{MCG}^+(S)$. A proof can be found in [FM12]. The result is due to the cumulative work of Dehn [Deh87], Lickorish [Lic64], and Humphries [Hum79], who include specific sets of generating curves to consider. There is also a similar result for non-closed surfaces (for example, see [FM12]).

For $S_{1,1}$, the simple closed curves whose Dehn twists generate the mapping class group are precisely α and β. Consider the Dehn twists for α, β, $\alpha\beta$,

$$T_\alpha = \begin{cases} \alpha \mapsto \alpha \\ \beta \mapsto \beta\alpha^{-1}, \end{cases} \qquad {T_\alpha}^{-1} = \begin{cases} \alpha \mapsto \alpha \\ \beta \mapsto \beta\alpha, \end{cases} \tag{2.1.1}$$

$$T_\beta = \begin{cases} \alpha \mapsto \alpha\beta \\ \beta \mapsto \beta, \end{cases} \qquad {T_\beta}^{-1} = \begin{cases} \alpha \mapsto \alpha\beta^{-1} \\ \beta \mapsto \beta, \end{cases} \tag{2.1.2}$$

$$T_{\alpha\beta} = \begin{cases} \alpha \mapsto \alpha\beta\alpha \\ \beta \mapsto \alpha^{-1}, \end{cases} \qquad {T_{\alpha\beta}}^{-1} = \begin{cases} \alpha \mapsto \beta^{-1} \\ \beta \mapsto \beta\alpha\beta. \end{cases} \tag{2.1.3}$$

These have have respective representations $A, B, C \in \mathrm{SL}_2(\mathbb{Z})$,

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \qquad A^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{2.1.4}$$

$$B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad B^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \tag{2.1.5}$$

$$C = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}, \qquad C^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}. \tag{2.1.6}$$

The matrices A and B generate $\mathrm{SL}_2(\mathbb{Z})$.

A comparable tool for studying hyperbolic structures and Teichmüller space are earthquakes. Intuitively, an earthquake deformation about a simple closed curve γ twists the surface near γ by some distance s and keeps the rest of the surface fixed. In this way, earthquake deformations can be seen as an extension of Dehn twists, where we shear along the curve. These arose in work of Thurston [Thu80] and can be defined formally like Dehn twists in Definition 2.4.

Definition 2.5 (Earthquakes about simple closed curves)**.** *Let S be a hyperbolic surface. The **left earthquake deformation** about $\gamma \in \pi_1(S)$ by some $r \in \mathbb{R}$ is a variation of the Dehn twist about γ where the map $\widetilde{T_A} : \tilde{A} \to \tilde{A}$ is now defined by*

$$(t, \theta) \mapsto (t, r(\theta + 2\pi(t-1))).$$

Similarly a right earthquake deformation can be obtained using a right twist.

The parameter r represents the fraction of twist completed. A change of parameters $r \mapsto s = r\ell_x(\gamma)$ can be applied to write the shearing in terms of the distance s by using the length of γ under the starting hyperbolic structure, $\ell_x(\gamma)$. In contrast with Dehn twists, earthquakes are not always continuous.

We can define earthquake deformations over a broader class of objects.

Definition 2.6. *Let S be a hyperbolic surface. A **geodesic lamination** λ on S is a foliation of a closed subset of S by complete, simple geodesics. A **measured geodesic lamination** (λ, μ) on S is a geodesic lamination λ equipped with a transverse measure μ.*

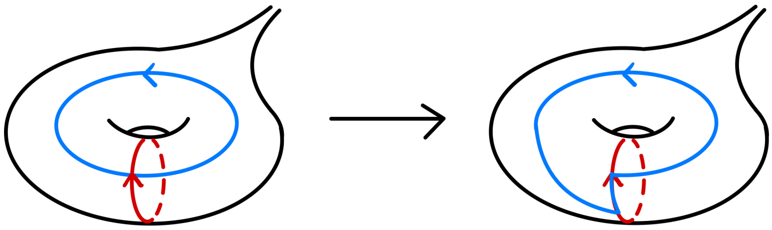

FIGURE 2.2: Example earthquake about α the meridian on $S_{1,1}$ by some $r < 1$.

A *transverse measure* on a geodesic lamination is a measure on transverse arcs to the lamination. The most basic example is simple closed curves, which come with a natural transverse measure in the counting measure. In essence, the transverse measure is an extension of the intersection number with the lamination. More elaborate examples of measured geodesic laminations can be constructed using something called train tracks, which are smoothly embedded trivalent graphs on S with weights for each edge.

An earthquake deformation about a measured geodesic lamination (λ, μ) is defined by taking λ and $r\mu$ in place of γ and r in Definition 2.5. The measure μ determines the "speed" of the twisting. For a simple closed curve, if we ignore the measure, this is precisely the Fenchel-Nielsen twist deformation [FN]. For rest of this paper we use the term earthquake to refer to a left earthquake deformation. Where necessary we will sometimes distinguish between left or right earthquakes by forwards or backwards, respectively.

Let $\mathscr{ML}$ denote the set of measured geodesic laminations on S and $\mathscr{S}$ the set of isotopy classes of simple closed curves on S. Embed $\mathscr{S} \times \mathbb{R}_+$ in $\mathscr{ML}$ by sending (γ, r) to the geodesic isotopic to γ with r times the counting measure.

We can place a topology on $\mathscr{ML}$ by quantifying a measured lamination (λ, μ) in terms of its intersection number and angle with a transverse arc A. This is expanded on in [Ker83] where it is used to construct coordinates for and compare laminations. Using this, we can see a result from Thurston [Thu80] that, for a closed surface S_g, the set $\mathscr{ML}$ is homeomorphic to $\mathbb{R}^{6g-6}$. This also enables a proof of the following result.

Theorem 2.7 (Thurston, [Thu80])**.** *$\mathscr{S} \times \mathbb{R}_+$ is dense in $\mathscr{ML}$.*

We can understand the density result more explicitly on $S_{1,1}$. Let $\gamma, \delta \in \pi_1(S_{1,1})$ be simple closed curves with $i(\gamma, \delta)$ the algebraic intersection number between them and choose framing $\pi_1(S_{1,1}) = \langle \alpha, \beta \rangle$. We say curve γ has **slope**

$$\mathrm{sl}(\gamma) = \frac{i(\gamma, \alpha)}{i(\gamma, \beta)}. \tag{2.1.7}$$

Of course, this depends on the framing α, β. Simple closed curves on $S_{1,1}$ have rational slope. The definition of slope can be extended to general measured geodesic laminations using the density result Theorem 2.7. Measured geodesic laminations on $S_{1,1}$ that are not simple closed curves have irrational slope.

We use Theorem 2.7 to extend the earthquake definition from simple closed curves to measured geodesic laminations. This approach is well-defined as per Kerckhoff [Ker83]. That is, for all sequences of simple closed curves γ_i with respective measures μ_i, if (γ_i, μ_i) converges to (γ, μ) then the sequence of earthquakes about (γ_i, μ_i) converges to the earthquake about (γ, μ).

Earthquakes adjust the hyperbolic structure on the surface by changing the marking. We prefer to think of earthquakes relative to the surface S themselves, where we can view them as: deformations of S; maps from Teichmüller space to itself; and as paths in Teichmüller space as t varies. We could also more generally think of earthquakes on the universal cover of the surface, $\mathbb{H}^2$, like Thurston's approach in [Thu86].

2.1.2 Coordinate systems

We describe two coordinate systems that we use to study earthquakes on the once-punctured torus $S_{1,1}$. The first coordinates were previously presented in Example 1.28.

Definition 2.8. *The* ***trace coordinates*** *for the once-punctured torus are the traces of a representation applied to generators α, β and their product $\alpha\beta$,*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \mathrm{tr}(\rho(\alpha)) \\ \mathrm{tr}(\rho(\beta)) \\ \mathrm{tr}(\rho(\alpha\beta)) \end{pmatrix}. \tag{2.1.8}$$

We saw the polynomial for the trace of the commutator $K = [\alpha, \beta]$ is connected to the geometry around the puncture on $S_{1,1}$. Label the polynomial

$$\kappa(x,y,z) = \mathrm{tr}(\rho(K)) = x^2 + y^2 + z^2 - xyz - 2. \tag{2.1.9}$$

There are some obvious symmetries of $\kappa(x,y,z)$.

Remark 2.9. *The polynomial $\kappa(x,y,z)$ is invariant under the maps*

$$I_{xy}, I_{xz}, I_{yz} \colon \mathbb{C}^3 \to \mathbb{C}^3$$

$$I_{xy} = \begin{cases} x \mapsto -x \\ y \mapsto -y \\ z \mapsto z, \end{cases} \qquad I_{xz} = \begin{cases} x \mapsto -x \\ y \mapsto y \\ z \mapsto -z, \end{cases} \qquad I_{yz} = \begin{cases} x \mapsto x \\ y \mapsto -y \\ z \mapsto -z. \end{cases} \tag{2.1.10}$$

Note the relationship with the Klein four-group

$$K_4 = \langle I_{xy}, I_{xz}, I_{yz} \rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

where the group operation is composition of maps.

Denote level sets of the polynomial $\kappa(x,y,z) = t$ or $\kappa^{-1}(t)$, with $t \in \mathbb{R}$. Consider the specific level set $\kappa^{-1}(-2)$ identified with $\mathscr{T}(S_{1,1})$. This has four connected components as pictured in Figure 1.3. The action of K_4 on these components corresponds to the action of $\mathrm{Hom}(\pi_1(S_{1,1}))$ on $\chi_{\mathrm{SL}_2(\mathbb{C})}(\pi_1(S_{1,1}))$ and hence we may identify the component in the positive octant with the Hitchin component in the $\mathrm{PSL}_2(\mathbb{R})$-character variety of $S_{1,1}$. Label this component $\mathscr{T}_{\mathrm{tr}} \cong \mathscr{T}(S_{1,1})$.

Recall $\mathscr{T}(S_{1,1})$ is itself identified with $\mathbb{H}^2$. We use the Poincaré half-plane model of $\mathbb{H}^2$,

$$\mathbb{H}^2 = \left\{(x,y) \in \mathbb{R}^2 \mid y > 0\right\} \tag{2.1.11}$$

with metric $ds = \frac{1}{y}\sqrt{dx^2 + dy^2}$. Without loss of generality, assume that γ is the unique geodesic associated with the equivalence class $[\gamma]$. We can derive the equation for the length of γ, $\ell(\gamma)$, in terms of the trace,

$$\ell(\gamma) = 2\cosh^{-1}\left(\frac{\mathrm{tr}(\rho(\gamma))}{2}\right). \tag{2.1.12}$$

This enables new coordinates that provide a geometric way to parametrise Teichmüller space.

Definition 2.10. *The* ***triangle lengths*** *[CP24] for the once-punctured torus are the half-lengths of the geodesics associated with generators α, β and their product $\alpha\beta$.*

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} := \begin{pmatrix} \frac{\ell(\alpha)}{2} \\ \frac{\ell(\beta)}{2} \\ \frac{\ell(\alpha\beta)}{2} \end{pmatrix} = \begin{pmatrix} \cosh^{-1}\left(\frac{x}{2}\right) \\ \cosh^{-1}\left(\frac{y}{2}\right) \\ \cosh^{-1}\left(\frac{z}{2}\right) \end{pmatrix} \tag{2.1.13}$$

where (x,y,z) are the trace coordinates.

The triangle lengths (a,b,c) independently satisfy the collar equation (see [CP24]),

$$\cosh^2 a + \cosh^2 b + \cosh^2 c = 2\cosh a \cosh b \cosh c. \tag{2.1.14}$$

This is the equivalent to the polynomial $\kappa(x,y,z) = -2$ in triangle length coordinates. Label the surface corresponding to the collar equation in $\mathbb{R}^3_+$ as $\mathscr{T}_\ell \cong \mathscr{T}(S_{1,1})$.

We study earthquakes by starting first with Dehn twists in these two coordinates.

2.2 Algebraic perspective

We explore the action of Dehn twists on the character variety first. The expression κ from Equation (2.1.9) is invariant under the action of the mapping class group of the once-punctured torus. In fact, the action of $\mathrm{MCG}(S_{1,1})$ on $\chi_{\mathrm{SL}_2(\mathbb{C})}(\pi_1(S_{1,1}))$ is equivalent to the action of the group Γ on $\mathbb{C}^3$ [Hor75], where

$$\Gamma = \left\{\text{polynomial automorphisms } p \text{ on } \mathbb{C}^3 \;\middle|\; \kappa(p(\mathbf{v})) = \kappa(\mathbf{v}), \mathbf{v} \in \mathbb{C}^3\right\}.$$

Denote the polynomial automorphism associated with mapping class f as $\tilde{f}$. Using trace identities, we can write the expressions for T_α, T_β, $T_{\alpha\beta}$ and their inverses in terms of the trace coordinates.

$$\widetilde{T}_\alpha(x,y,z) = (x, xy-z, y), \quad \widetilde{T}_\alpha^{-1}(x,y,z) = (x,z,xz-y), \tag{2.2.1}$$

$$\widetilde{T}_\beta(x,y,z) = (z,y,yz-x), \quad \widetilde{T}_\beta^{-1}(x,y,z) = (xy-z,y,x), \tag{2.2.2}$$

$$\widetilde{T}_{\alpha\beta}(x,y,z) = (xz-y,x,z), \quad \widetilde{T}_{\alpha\beta}^{-1}(x,y,z) = (y,yz-x,z). \tag{2.2.3}$$

Remark 2.11. *The polynomial automorphisms are related by conjugation with elements from the symmetric group on three elements, Σ_3. Specifically, consider rotation and reflections given by*

$$\begin{aligned} \Sigma_{Rot}(x,y,z) &= (z,x,y), \\ \Sigma_{Ref}^1(x,y,z) &= (x,z,y), \\ \Sigma_{Ref}^2(x,y,z) &= (z,y,x), \\ \Sigma_{Ref}^3(x,y,z) &= (y,x,z), \end{aligned}$$

where Σ_{Ref}^i is the reflection that fixes the ith coordinate and switches the other two coordinates. Note that these all preserve the chosen component $\mathscr{T}_{\mathrm{tr}}$.

The polynomial automorphisms $\widetilde{T}_\alpha$, $\widetilde{T}_\beta$, $\widetilde{T}_{\alpha\beta}$ are related by rotations,

$$\widetilde{T}_\alpha(x,y,z) = \Sigma_{Rot}^{-1}\widetilde{T}_\beta\Sigma_{Rot}(x,y,z) = \Sigma_{Rot}\widetilde{T}_{\alpha\beta}\Sigma_{Rot}^{-1}(x,y,z),$$

and the polynomial automorphisms and their inverses are related by reflections,

$$\begin{aligned} \widetilde{T}_\alpha^{-1}(x,y,z) &= \Sigma_{Ref}^1\widetilde{T}_\alpha\Sigma_{Ref}^1(x,y,z), \\ \widetilde{T}_\beta^{-1}(x,y,z) &= \Sigma_{Ref}^2\widetilde{T}_\beta\Sigma_{Ref}^2(x,y,z), \\ \widetilde{T}_{\alpha\beta}^{-1}(x,y,z) &= \Sigma_{Ref}^3\widetilde{T}_{\alpha\beta}\Sigma_{Ref}^3(x,y,z). \end{aligned}$$

As a consequence, any formula for the earthquake deformation or a flow associated with one polynomial automorphism for α, β, or $\alpha\beta$ can be used to derive formulas for all three polynomial automorphisms using the action of Σ_3.

Without loss of generality, consider the polynomial automorphism $\widetilde{T}_\alpha$ associated with the Dehn twist about α. For a point $\mathbf{v} \in \kappa^{-1}(t) \cap \mathbb{R}^3$ we get an orbit of points, given by $\widetilde{T}_\alpha^n(\mathbf{v})$ with $n \in \mathbb{Z}$. Investigating the action of $\widetilde{T}_\alpha^n$ and using results from linear recurrence relations, we get the following theorem.

Theorem 2.12. *For a fixed starting point $\mathbf{v} = (x,y,z) \in \kappa^{-1}(t) \cap \mathbb{R}_+^3$ with $t \in \mathbb{R}$, the Dehn twist about α can be extended to a continuous flow in trace coordinates given by the map $F_\alpha^{\mathrm{tr}}(\mathbf{v})\colon \mathbb{R} \to \mathbb{R}^3$,*

$$\begin{aligned} F_\alpha^{\mathrm{tr}}(\mathbf{v})\colon\ &\mathbb{R} \to \mathbb{R}^3 \\ &r \mapsto \begin{pmatrix} x \\ y_\alpha(r) \\ y_\alpha(r-1) \end{pmatrix} \end{aligned}$$

where

$$\begin{aligned} y_\alpha \colon \mathbb{R} &\to \mathbb{R} \\ r &\mapsto \begin{cases} C^\alpha_+(\mathbf{v}) S^\alpha_+(\mathbf{v})^r + C^\alpha_-(\mathbf{v}) S^\alpha_-(\mathbf{v})^r & \text{if } x \neq 2 \\ y + (y-z)\, r & \text{if } x = 2 \end{cases} \end{aligned}$$

with

$$\begin{aligned} C^\alpha_\pm(\mathbf{v}) &= \frac{y(x^2-4) \pm (xy-2z)\sqrt{x^2-4}}{2(x^2-4)}, \\ S^\alpha_\pm(\mathbf{v}) &= \frac{x \pm \sqrt{x^2-4}}{2}. \end{aligned}$$

Moreover, $F^{\mathrm{tr}}_\alpha(\mathbf{v})(n) = \widetilde{T}^n_\alpha(\mathbf{v})$*, for all* $n \in \mathbb{Z}$*.*

Proof. Take $n \in \mathbb{Z}$. We introduce notation for the three components of the map $\widetilde{T}^n_\alpha$ acting on points $\mathbf{v} = (x,y,z) \in \kappa^{-1}(t) \cap \mathbb{R}^3$,

$$\widetilde{T}^n_\alpha(\mathbf{v}) = \begin{pmatrix} x_\alpha(n) \\ y_\alpha(n) \\ z_\alpha(n) \end{pmatrix}.$$

Note x_α, y_α, z_α are also dependant on $\mathbf{v}$, although this is omitted in the notation for brevity. The polynomial automorphism for $\widetilde{T}_\alpha$ (Equation (2.2.1)) gives a system of recurrence relations,

$$\begin{pmatrix} x_\alpha(n) \\ y_\alpha(n) \\ z_\alpha(n) \end{pmatrix} = \begin{pmatrix} x \\ x y_\alpha(n-1) - y_\alpha(n-2) \\ y_\alpha(n-1) \end{pmatrix}.$$

This gives a linear recurrence relation for y_α,

$$y_\alpha(n) = x y_\alpha(n-1) - y_\alpha(n-2),$$

with initial conditions $y_\alpha(0) = y$ and $y_\alpha(1) = xy - z$.

The recurrence relation can be solved using the characteristic polynomial method for linear recurrence relations. There will be two forms of solution depending on if there are two distinct roots (which occurs if $x \neq 2$) or one repeated root (which occurs if $x = 2$) to the characteristic polynomial. Extending the solutions from $n \in \mathbb{Z}$ to real number $r \in \mathbb{R}$ and considering $\mathbf{v} = (x,y,z) \in \kappa^{-1}(t) \cap \mathbb{R}^3_+$ with $t \in \mathbb{R}$ gives the expression for F^{tr}_α. □

We are particularly interested in the component $\mathscr{T}_{\mathrm{tr}} = \kappa^{-1}(-2) \cap \mathbb{R}^3_+$ mentioned in Section 2.1.2. Note the special case $x = 2$ does not exist on this component.

A similar expression can be derived to extend the Dehn twists about β and $\alpha\beta$ to flows F^{tr}_β and $F^{\mathrm{tr}}_{\alpha\beta}$ respectively. This is done either by following the same derivation or by applying the symmetry relationships from Remark 2.11 to F^{tr}_α.

There are a few remarks we can make about Theorem 2.12.

Remark 2.13. *The results in Theorem 2.12 are given for the positive octant* $\mathbf{v} = (x,y,z) \in \kappa^{-1}(t) \cap \mathbb{R}^3_+$*,* $t \in \mathbb{R}$*. The results could be extended for all starting points* $\mathbf{v} = (x,y,z) \in \kappa^{-1}(t) \cap \mathbb{R}^3$ *using the symmetries given in Remark 2.9.*

Remark 2.14. *The right-handed Dehn twist about* α*,* T_α^{-1}*, can similarly be extended to a continuous flow. This is given by the formula in Theorem 2.12 under the transformation* $r \mapsto -r$*, with*

$$F_\alpha^{\mathrm{tr}}\left(F_\alpha^{\mathrm{tr}}(\mathbf{v})(r)\right)(-r) = \mathbf{v} = F_\alpha^{\mathrm{tr}}\left(F_\alpha^{\mathrm{tr}}(\mathbf{v})(-r)\right)(r),$$

as expected.

Remark 2.15. *Note where imaginary numbers could appear in* y_α*,* $C_\pm^\alpha$*,* $S_\pm^\alpha$*.*

(i) If $r \in \mathbb{Z}$ *then* $y_\alpha(r) \in \mathbb{R}$*.*

Assume $r \in \mathbb{R}$*,* $\mathbf{v} = (x,y,z) \in \kappa^{-1}(t) \cap \mathbb{R}^3_+$ *with* $t \in \mathbb{R}$*,*

(ii) If $x \geq 2$*, then* $C_\pm^\alpha(\mathbf{v}), S_\pm^\alpha(\mathbf{v}) \in \mathbb{R}$ *for all* $y, z \in \mathbb{R}_+$ *and* $y_\alpha \in \mathbb{R}$*, thus* $F_\alpha^{\mathrm{tr}}(\mathbf{v})(r) \in \mathbb{R}^3$*;*

(iii) If $0 \leq x < 2$*, then* $C_\pm^\alpha(\mathbf{v}), S_\pm^\alpha(\mathbf{v}) \notin \mathbb{R}$ *for all* $y, z \in \mathbb{R}_+$*, but* $y_\alpha \in \mathbb{R}$*, and thus* $F_\alpha^{\mathrm{tr}}(\mathbf{v})(r) \in \mathbb{R}^3$*;*

(iv) If $\mathbf{v} = (x,y,z) \in \mathscr{T}_{\mathrm{tr}}$*, then* $F_\alpha^{\mathrm{tr}}(\mathbf{v})(r) \in \mathscr{T}_{\mathrm{tr}}$*.*

Remark 2.16. *Consider the limit of the flow as* $r \to \pm\infty$*,*

$$(x_{\pm\infty}, y_{\pm\infty}, z_{\pm\infty}) := \lim_{r \to \pm\infty} F_\alpha^{\mathrm{tr}}(x,y,z)(r) = (x, \infty, \infty),$$

and define

$$N_\pm(u,v,w) = v \pm \frac{2w - uv}{\sqrt{u^2 - 4}}.$$

Then projectively,

$$[x_{\pm\infty} : y_{\pm\infty} : z_{\pm\infty}] = \left[0 : \frac{N_\mp(x,y,z)}{\sqrt{N_\mp(x,y,z)^2 + N_\pm(x,z,y)^2}} : \frac{N_\pm(x,z,y)}{\sqrt{N_\mp(x,y,z)^2 + N_\pm(x,z,y)^2}}\right].$$

The projective classes provide an idea of the rate of blow up at the limit. In particular, the projective classes as $r \to \infty$ *and* $r \to -\infty$ *are distinct,*

$$[x_\infty : y_\infty : z_\infty] \neq [x_{-\infty} : y_{-\infty} : z_{-\infty}] \text{ for all } (x,y,z) \in \mathbb{R}^3_+.$$

The approach used in the proof of Theorem 2.12 could potentially be applied to other simple closed curves as long as the Dehn twist is known. Whether the recurrence relation can be solved for all simple closed curves is unclear.

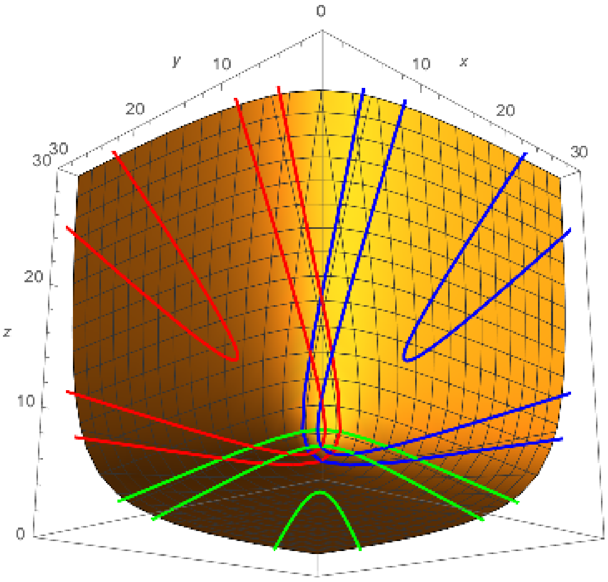

FIGURE 2.3: Example forward and backward earthquake deformations about α (red), β (blue), and $\alpha\beta$ (green) on $\mathscr{T}_{\text{tr}}$. Starting points are given by the respective sets $\mathscr{S}_\alpha$, $\mathscr{S}_\beta$, and $\mathscr{S}_{\alpha\beta}$ from Equation (2.2.4).

See Figure 2.3 for some example flows with starting points,

$$\begin{aligned} \mathscr{S}_\alpha =& \left\{(3,3,3), \left(2\sqrt{2}, 2\sqrt{2}, 4\right), \left(10, 10, -10\left(-5+\sqrt{23}\right)\right)\right\}, \\ \mathscr{S}_\beta =& \Sigma_{\text{Rot}}^{-1}\mathscr{S}_\alpha, \\ \mathscr{S}_{\alpha\beta} =& \Sigma_{\text{Rot}}\mathscr{S}_\alpha. \end{aligned} \tag{2.2.4}$$

The results presented in this section have extended the orbits of the Dehn twists about α, β, and $\alpha\beta$ to a continuous flow in trace coordinates.

2.3 Geometric perspective

We now explore the action of Dehn twists on Teichmüller space using triangle lengths. Unless otherwise stated, use α, β, and $\alpha\beta$ to denote the unique geodesic associated with each equivalence class and let $\overline{T}_\alpha$, $\overline{T}_\beta$, $\overline{T}_{\alpha\beta}$ be the respective Dehn twists in triangle lengths. To analyse the Dehn twists, we turn to maximal collar neighbourhoods as the neighbourhood to twist.

A collar neighbourhood of the geodesic γ is an ε-neighbourhood of γ that is homeomorphic to an open annulus. A **maximal collar neighbourhood** of γ is a collar neighbourhood of γ that is maximal with respect to inclusion of sets. We can generate maximal collar neighbourhoods around α, β, and $\alpha\beta$ on $S_{1,1}$.

For example, the maximal collar neighbourhood for α is shown in Figure 2.4. Here, β and $\alpha\beta$ meet at a point on the boundary δ to ensure they satisfy the collar equation from Equation (2.1.14).

Theorem 2.17 (Buser, [Bus78])**.** *For a maximal collar neighbourhood of a geodesic γ on the once-punctured torus with boundary δ, the length of the boundary curve $d = \ell(\delta)$ and the length of half of* *the neighbourhood ε are given by*

$$
\begin{aligned}
d(\gamma) &= 2\coth\frac{\ell(\gamma)}{2}, \\
\varepsilon(\gamma) &= \sinh^{-1}\left(\frac{1}{\sinh\frac{\ell(\gamma)}{2}}\right).
\end{aligned}
$$

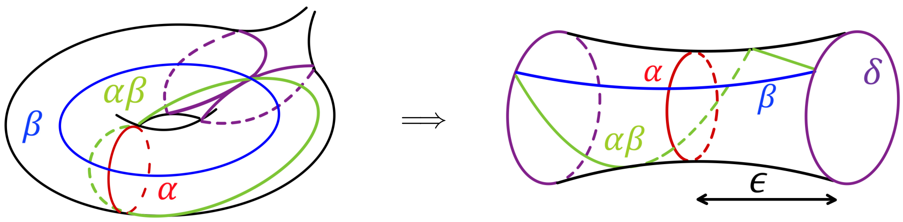

FIGURE 2.4: Maximal collar neighbourhood for α on $S_{1,1}$ (left) and unfolded into a tube (right). The geodesics α, β, and $\alpha\beta$ are given in red, blue, and green respectively while the boundary of the maximal collar neighbourhood δ is given in purple.

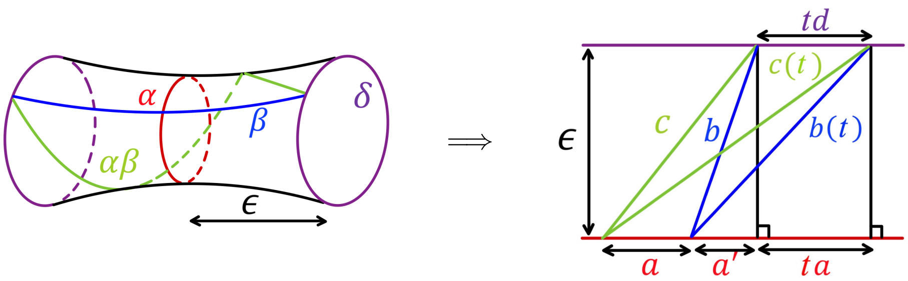

FIGURE 2.5: The maximal collar neighbourhood for α on $S_{1,1}$ unfolded into a tube (left) and half of the neighbourhood unfolded in the universal cover $\mathbb{H}^2$ (right). The geodesics α, β, and $\alpha\beta$ and associated quantities are given in red, blue, and green respectively while the boundary of the maximal collar neighbourhood δ is given in purple.

Consider the image of half of the maximal collar neighbourhood in the universal cover $\mathbb{H}^2$. A representation of this is shown in Figure 2.5. We explain the geometry briefly. When taking a fractional Dehn twist about α of some magnitude $t \in \mathbb{R}$, α will stay the same and the intersection points between α and β and between α and $\alpha\beta$ will stay the same, but the geodesics β and $\alpha\beta$ will change. In $\mathbb{H}^2$, we represent the changes to the associated coordinates b, c as $b(t), c(t)$. Then under the fractional Dehn twist, the triangle formed by a, b, c is transformed to the triangle formed by $a, b(t), c(t)$, with the corner where $b(t)$ and $c(t)$ meet based on δ distance td from the corner where b and c meet. This implies the corresponding distance on α must be ta.

Theorem 2.18. *For a fixed starting point $\mathbf{w} = (a,b,c) \in \mathscr{T}_\ell \subseteq \mathbb{R}^3_+$, the earthquake deformation about α in triangle lengths is given by the map $E^\ell_\alpha(\mathbf{w})\colon \mathbb{R} \to \mathscr{T}_\ell$,*

$$\begin{aligned} E^\ell_\alpha(\mathbf{w})\colon\ \mathbb{R} &\to \mathscr{T}_\ell \\ r &\mapsto \begin{pmatrix} a \\ b_\alpha(r) \\ b_\alpha(r-1)) \end{pmatrix} \end{aligned}$$

where

$$\begin{aligned} b_\alpha\colon\ \mathbb{R} &\to \mathbb{R}_+ \\ r &\mapsto \cosh^{-1}\left(\cosh(b)\cosh(ra) - (\cosh(c) - \cosh(a)\cosh(b))\frac{\sinh(ra)}{\sinh(a)}\right). \end{aligned}$$

Moreover, $E^\ell_\alpha(\mathbf{v})(n) = \overline{T}^n_\alpha(\mathbf{v})$, for all $n \in \mathbb{Z}$.

Proof. The arrangement in Figure 2.5 provides a series of four hyperbolic right-angled triangles. Two triangles represent the initial relative formation of the curves α, β, and $\alpha\beta$ with hypotenuse lengths b, c. Two triangles represent the formation of the curves α, β, and $\alpha\beta$ after taking a fractional Dehn twist of value t, with hypotenuse lengths $b(t)$, $c(t)$. Both triangles have base lengths a. The distance between the corners at b and c and $b(t)$ and $c(t)$ is td, which indicates along the curve α the distance is ta.

Given a hyperbolic right-angled triangle with side lengths p, q and hypotenuse length h, the lengths are related by

$$\cosh(h) = \cosh(p)\cosh(q).$$

Using this relation and those provided in Theorem 2.17, the series of hyperbolic triangles can be solved to first find the variable a' and then find $b(t)$ and $c(t)$ as functions in t and the starting lengths a, b, c. Applying the collar equation from Equation (2.1.14) gives the formula for the earthquake deformation about α. $\square$

A similar expression can be derived for the earthquake deformations about β and $\alpha\beta$, E^ℓ_β and $E^\ell_{\alpha\beta}$ respectively, in triangle lengths. This is done either by following the same derivation process or by applying the symmetry relationships from Remark 2.11 to E^ℓ_α, adjusting for the change of coordinates. See Figure 2.6 for some examples.

Compare the formula for E^ℓ_α as given in Theorem 2.18 to F^{tr}_α from Theorem 2.12. There is a natural homeomorphism between the two coordinate systems,

$$\begin{aligned} \nu\colon\ \mathscr{T}_{\mathrm{tr}} &\to \mathscr{T}_\ell \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} &\mapsto \begin{pmatrix} \cosh^{-1}\left(\frac{x}{2}\right) \\ \cosh^{-1}\left(\frac{y}{2}\right) \\ \cosh^{-1}\left(\frac{z}{2}\right) \end{pmatrix} \end{aligned} \tag{2.3.1}$$

using Equation (2.1.12). We can use this to relate E^ℓ_α and F^{tr}_α.

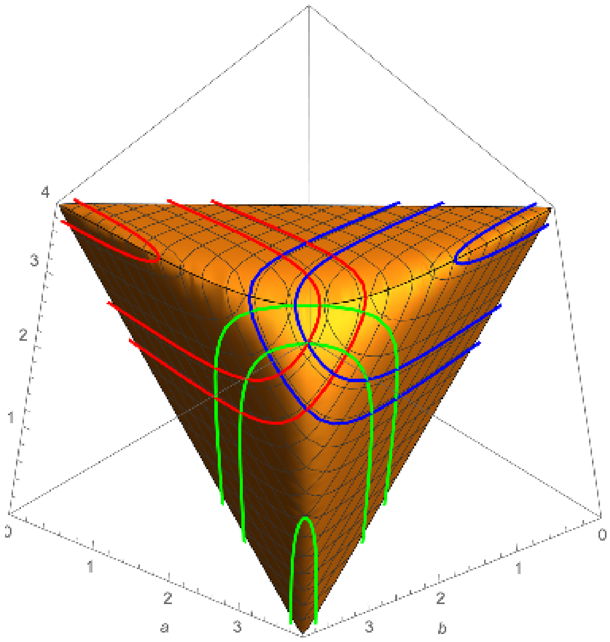

FIGURE 2.6: Example forward and backward earthquake deformations about α (red), β (blue), and $\alpha\beta$ (green) on the image of $\mathscr{T}_{\text{tr}}$ in $\mathbb{R}^3_+$ using triangle lengths. Starting points are given by applying ν from Equation (2.3.1) to the respective sets $\mathscr{S}_\alpha$, $\mathscr{S}_\beta$, and $\mathscr{S}_{\alpha\beta}$ from Equation (2.2.4).

Theorem 2.19. *Take starting point* $\mathbf{w} = (a,b,c) \in \mathscr{T}_\ell \subseteq \mathbb{R}^3_+$, *then*

$$E^\ell_\alpha(\mathbf{w}) = \nu\left(F^{\text{tr}}_\alpha\left(\nu^{-1}(\mathbf{w})\right)\right).$$

Proof. The proof uses the relationship between the trace coordinates and triangle lengths given in Equation (2.1.13), the fact that we are operating on the level set $\kappa(x,y,z) = -2$ (corresponding to component $\mathscr{T}_{\text{tr}}$), and that

$$(\cosh(\theta) \pm \sinh(\theta))^n = \cosh(n\theta) \pm \sinh(n\theta).$$

Recall the special case $x = 2$ in Theorem 2.12 does not lie on Teichmüller space. This corresponds to $a = 0$. □

Corollary 2.20. *The earthquake deformation about* α *in trace coordinates is precisely the continuous flow extension of the Dehn twist about* α *on the component* $\mathscr{T}_{\text{tr}}$,

$$E^{\text{tr}}_\alpha(\mathbf{v}) = F^{\text{tr}}_\alpha(\mathbf{v}) \textit{ for all } \mathbf{v} \in \mathscr{T}_{\text{tr}}.$$

The continuous flow extension of the Dehn twist given in Theorem 2.12 applies beyond just the component $\mathscr{T}_{\text{tr}}$ associated with Teichmüller space.

Question 2.21. *Is there a geometric interpretation of the flow given in Theorem 2.12 beyond* $\mathscr{T}_{\text{tr}}$*?*

We can reparametrise the earthquakes based on what we want to emphasize.

Remark 2.22. *The parameter* $r \in \mathbb{R}$ *represents the fraction or relative length twisted along the curve* α, *with* $r = 1$ *indicating a twist of magnitude the full length of* α. *It can be useful to express the formula* *for earthquakes in terms of the objective length of the curve. This is often used as the more common convention when studying earthquakes.*

In particular, when using the counting measure, the length is precisely the measure of the curve. For the earthquake about α, the coordinate change is $s = r\ell(\alpha) = 2ra$.

After applying $s = 2ra$ the formula from Theorem 2.18 becomes,

$$\begin{aligned} E^{\ell}_{\alpha}(\mathbf{w})\colon\ \mathbb{R} &\to \mathscr{T}_{\ell} \\ s &\mapsto \begin{pmatrix} a \\ b_{\alpha}(s) \\ b_{\alpha}(s-2a) \end{pmatrix} \end{aligned}$$

where

$$\begin{aligned} b_{\alpha}\colon\ \mathbb{R} &\to \mathbb{R}_{+} \\ s &\mapsto \cosh^{-1}\left(\cosh(b)\cosh(s/2) - (\cosh(c) - \cosh(a)\cosh(b))\frac{\sinh(s/2)}{\sinh(a)}\right). \end{aligned}$$

A similar reparametrisation can be done for the result in Theorem 2.12.

The results presented in this section have extended the orbits of the Dehn twists about α, β, and $\alpha\beta$ to get an explicit formula for the earthquake in triangle length coordinates. We have proven this is equivalent to the flow from Section 2.2, which extended the orbits of the Dehn twists about α, β, and $\alpha\beta$ to get a continuous flow in trace coordinates. Thus, this continuous flow is the earthquake in trace coordinates. The formula in trace coordinates gives an algebraic interpretation of earthquakes; the formula in triangle lengths gives a geometric interpretation of earthquakes.

2.4 Change of coordinates algorithm

We present a change of coordinates algorithm that outlines how to find similar formula for earthquakes for all simple closed curves. Given the framing α and β, Sections 2.2 and 2.3 provide a description the earthquakes about α, β, $\alpha\beta$ in trace coordinates and triangle lengths, respectively. The algorithm uses these results to construct the earthquake for any simple closed geodesic γ in the chosen framing. For simplicity we consider only trace coordinates in the description of the algorithm. The definition of triangle lengths in Equation (2.1.13) can be used to convert the expressions.

The algorithm is broken into three steps. Given an input of a simple closed geodesic γ as a word in α and β, $\gamma = w_{\gamma}(\alpha, \beta)$,

1. Compute a simple closed curve δ such that $i(\gamma, \delta) = 1$;
2. Calculate the earthquake for γ in the framing γ and δ using Theorem 2.12;
3. Transform the expression for the earthquake expression to the original framing α and β.

We elaborate on each of these steps in order.

Step (1)

First compute a simple closed curve δ such that $i(\gamma,\delta)=1$.

Such combinations of curves γ, δ can be found via a mapping class applied to α, β. All mapping classes on $S_{1,1}$ are represented by an element

$$M=\begin{pmatrix} m_1 & m_2 \\ n_1 & n_2 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Consider γ an (m_1,n_1)–type curve for some $m_1,n_1\in\mathbb{Z}$ with $(m_1,n_1)\neq(0,0)$. Then to prove existence of δ we need to find an (m_2,n_2)–type curve with $m_2,n_2\in\mathbb{Z}$ that satisfies $m_1n_2-m_2n_1=1$. As (m_1,n_1) is non-trivial, such a vector will always exist. Then there is a mapping class M that takes α to γ and β to δ with $i(\gamma,\delta)=1$. We can write γ and δ as words in α and β,

$$\gamma=w_\gamma(\alpha,\beta),\delta=w_\delta(\alpha,\beta).$$

Step (2)

Second, calculate the earthquake for γ in the framing given by γ and δ.

Define new *local* trace coordinates

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \mathrm{tr}(\rho(\gamma)) \\ \mathrm{tr}(\rho(\delta)) \\ \mathrm{tr}(\rho(\gamma\delta)) \end{pmatrix}. \tag{2.4.1}$$

Substituting (x',y',z') for (x,y,z) in Theorem 2.12 and using Corollary 2.20 gives results for the earthquake about γ in local trace coordinates starting at some point $\mathbf{v}'=(x',y',z')\in\mathscr{T}_{\mathrm{tr}}$,

$$\begin{aligned} E'^{\mathrm{tr}}_{\gamma}(\mathbf{v}') = E^{\mathrm{tr}}_{\alpha}(\mathbf{v}')\colon\ \mathbb{R} &\to \mathscr{T}_{\mathrm{tr}} \\ r &\mapsto \begin{pmatrix} x' \\ y_\gamma(r) \\ y_\gamma(r-1) \end{pmatrix} \end{aligned} \tag{2.4.2}$$

where

$$\begin{aligned} y_\gamma\colon\ \mathbb{R} &\to \mathbb{R} \\ r &\mapsto \begin{cases} C^\gamma_+(\mathbf{v}')S^\gamma_+(\mathbf{v}')^r + C^\gamma_-(\mathbf{v}')S^\gamma_-(\mathbf{v}')^r & \text{if } x'\neq 2 \\ y'+(y'-z')r & \text{if } x'=2 \end{cases} \end{aligned}$$

with

$$\begin{aligned} C^\gamma_\pm(\mathbf{v}') &= \frac{y'(x'^2-4)\pm(x'y'-2z')\sqrt{x'^2-4}}{2(x'^2-4)}, \\ S^\gamma_\pm(\mathbf{v}') &= \frac{x'\pm\sqrt{x'^2-4}}{2}. \end{aligned}$$

Step (3)

Third, transform the expression for the earthquake about γ in the framing γ and δ back to the original framing given by α and β. We do this by relating the local trace coordinates to the original trace coordinates.

We can determine a change of coordinates map ϕ from original to local trace coordinates by expanding γ and δ as words in α and β,

$$\begin{aligned}\phi\colon\ &\mathscr{T}_{\mathrm{tr}} \to \mathscr{T}_{\mathrm{tr}}\\ &\begin{pmatrix}x\\y\\z\end{pmatrix} \mapsto \begin{pmatrix}x'\\y'\\z'\end{pmatrix} = \begin{pmatrix}\mathrm{tr}(\rho(w_\gamma(\alpha,\beta)))\\ \mathrm{tr}(\rho(w_\delta(\alpha,\beta)))\\ \mathrm{tr}(\rho(w_\gamma(\alpha,\beta)w_\delta(\alpha,\beta)))\end{pmatrix}.\end{aligned}$$

The map ϕ is the polynomial automorphism equivalent to the mapping class M from Step (1). Use trace identities to express the right-hand side in terms of $x = \mathrm{tr}(\rho(\alpha))$, $y = \mathrm{tr}(\rho(\beta))$, and $z = \mathrm{tr}(\rho(\alpha\beta))$.

Similarly, as γ and δ provide a framing of $S_{1,1}$, we can write α and β as words in γ, δ,

$$\alpha = w_\alpha(\gamma,\delta), \quad \beta = w_\beta(\gamma,\delta).$$

This can be constructed from the inverse of the mapping class M.

Then we can determine an inverse change of coordinates map ψ from local to original trace coordinates by expanding α and β as words in γ and δ,

$$\begin{aligned}\psi\colon\ &\mathscr{T}_{\mathrm{tr}} \to \mathscr{T}_{\mathrm{tr}}\\ &\begin{pmatrix}x'\\y'\\z'\end{pmatrix} \mapsto \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}\mathrm{tr}(\rho(w_\alpha(\gamma,\delta)))\\ \mathrm{tr}(\rho(w_\beta(\gamma,\delta)))\\ \mathrm{tr}(\rho(w_\alpha(\gamma,\delta)w_\beta(\gamma,\delta)))\end{pmatrix}.\end{aligned}$$

The map ψ is the polynomial automorphism equivalent to the inverse of the mapping class M from Step (1). Use trace identities to expand the right-hand side in terms of $x' = \mathrm{tr}(\rho(\gamma))$, $y' = \mathrm{tr}(\rho(\delta))$, $z' = \mathrm{tr}(\rho(\gamma\delta))$. The maps ϕ and ψ are inverses.

Then use ϕ, ψ, and E^{tr}_α to get the earthquake deformation about γ in original trace coordinates starting at $\mathbf{v} = (x,y,z) \in \mathscr{T}_{\mathrm{tr}}$,

$$E^{\mathrm{tr}}_\gamma(\mathbf{v}) = \psi\left(E^{\mathrm{tr}}_\alpha(\phi(\mathbf{v}))\right).$$

The solution for examples of γ is given in Section 2.6.

2.5 Studying deformations using Fenchel-Nielsen length-twist coordinates

It is useful to study the earthquakes in other coordinate systems. Of particular interest is Fenchel-Nielsen length-twist coordinates.

Fenchel-Nielsen coordinates are based on a pants decomposition of the surface as described in Section 1.3.1.1. Consider the pants decomposition given by α on $S_{1,1}$. The Fenchel-Nielsen coordinates using α, $(\ell_\alpha, \tau_\alpha)$, are related to trace coordinates and triangle lengths by

$$\begin{pmatrix} \ell_\alpha \\ \tau_\alpha \end{pmatrix} = \begin{pmatrix} 2\cosh^{-1}\left(\frac{x}{2}\right) \\ 2\cosh^{-1}\left(\frac{y}{2\coth\left(\cosh^{-1}\left(\frac{x}{2}\right)\right)}\right) \end{pmatrix} = \begin{pmatrix} 2a \\ 2\cosh^{-1}\left(\frac{\cosh(b)}{\coth(a)}\right) \end{pmatrix}. \tag{2.5.1}$$

We could write other coordinate changes for Fenchel-Nielsen coordinates using another simple closed curve γ, $(\ell_\gamma, \tau_\gamma)$. Label the space $\mathscr{T}_{\mathrm{FN}} = \mathbb{R}_+ \times \mathbb{R}$. Each choice of pants decomposition defines an approach to identify $\mathscr{T}_{\mathrm{FN}} \cong \mathscr{T}(S_{1,1})$.

Label the coordinate change from trace coordinates

$$\begin{aligned} \zeta\colon\ \mathscr{T}_{\mathrm{tr}} &\to \mathscr{T}_{\mathrm{FN}} \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} &\mapsto \begin{pmatrix} 2\cosh^{-1}\left(\frac{x}{2}\right) \\ 2\cosh^{-1}\left(\frac{y}{2\coth\left(\cosh^{-1}\left(\frac{x}{2}\right)\right)}\right) \end{pmatrix}. \end{aligned} \tag{2.5.2}$$

These coordinates are no longer symmetric like the trace coordinates or triangle lengths, but they are projected to the plane in a meaningful way. For a fixed starting point $\mathbf{u} = (\ell, \tau) \in \mathscr{T}_{\mathrm{FN}}$, the earthquake about α in Fenchel-Nielsen coordinates for α is given by

$$\begin{aligned} E_\alpha^{\mathrm{FN}}(\mathbf{u})\colon\ \mathbb{R} &\to \mathbb{R}_+ \times \mathbb{R} \\ s &\mapsto \begin{pmatrix} \ell \\ \tau - s \end{pmatrix}. \end{aligned} \tag{2.5.3}$$

Note this earthquake is parametrised by hyperbolic length, as per Remark 2.22.

The expressions for earthquake deformations about β and $\alpha\beta$, $E_\beta^{\mathrm{FN}}(\mathbf{u})$ and $E_{\alpha\beta}^{\mathrm{FN}}(\mathbf{u})$ respectively, are increasingly complex. See Figure 2.7 for examples.

The wider set of measured geodesic laminations, $\mathscr{ML}$, of $S_{1,1}$ is naturally identified with $H_1(S_{1,1};\mathbb{R}) \cong \mathbb{R}^2$. A line through the origin represents a lamination and a vector along this line represents an associated measured lamination. Taking the antipodal map represents the projective laminations. Label the set of earthquakes in Fenchel-Nielsen coordinates for Teichmüller space as $\mathscr{E} \subseteq \mathscr{T}_{\mathrm{FN}}$.

For a given point $\mathbf{u} \in \mathscr{T}_{\mathrm{FN}}$ we can study the map that takes each lamination to their associated earthquake in Fenchel-Nielsen coordinates for α starting at $\mathbf{u}$,

$$\begin{aligned} E_{\mathbf{u}}\colon\ \mathscr{ML} \subseteq H_1(S_{1,1};\mathbb{R}) \cong \mathbb{R}^2 &\to \mathscr{E} \subseteq \mathscr{T}_{\mathrm{FN}} \\ \mathscr{L} &\mapsto E_{\mathscr{L}}^{\mathrm{FN}}(\mathbf{u}), \end{aligned} \tag{2.5.4}$$

with

$$\begin{aligned} E_{\mathscr{L}}^{\mathrm{FN}}(\mathbf{u}, \cdot)\colon\ \mathbb{R} &\mapsto \mathscr{T}_{\mathrm{FN}} \\ s &\mapsto \begin{pmatrix} \ell_\alpha^{\mathscr{L}}(\mathbf{u}, s) \\ \tau_\alpha^{\mathscr{L}}(\mathbf{u}, s) \end{pmatrix}. \end{aligned}$$

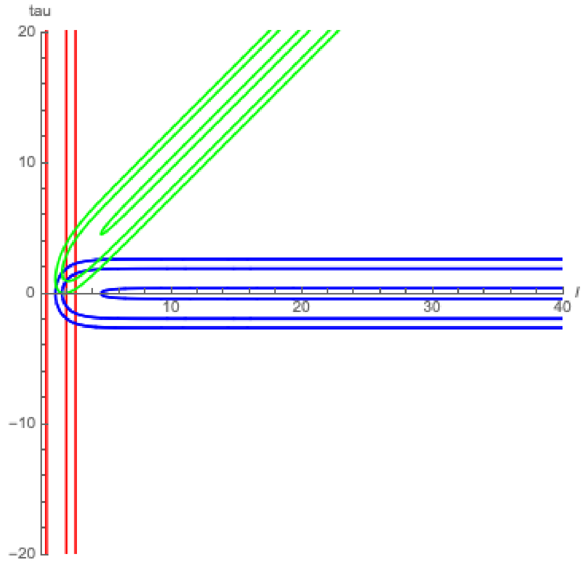

FIGURE 2.7: Example forward and backward earthquake deformations about α (red), β (blue), and $\alpha\beta$ (green) using Fenchel-Nielsen length-twist coordinates. Starting points are given by applying ζ from Equation (2.5.2) to the respective sets $\mathscr{S}_\alpha$, $\mathscr{S}_\beta$, and $\mathscr{S}_{\alpha\beta}$ from Equation (2.2.4).

For a simple closed curve δ, the functions $\ell_\delta^{\mathscr{L}}(\mathbf{u},s)$ and $\tau_\delta^{\mathscr{L}}(\mathbf{u},s)$ represent the length and twist parameter of δ at the point on the earthquake $E_{\mathscr{L}}^{\mathrm{FN}}(\mathbf{u},s)$. Notation for the measure of the lamination is omitted for brevity.

Remark 2.23. *We make some statements about the map $E_{\boldsymbol{u}}$.*

1. *The map is continuous and bijective for each choice of $\boldsymbol{u}$. Continuity is given through work of Kerckhoff (see [Ker83, Lemma 1.2]) and bijectivity is given by this and Thurston's earthquake theorem Theorem 2.1.*
2. *We know $\mathscr{ML}$ provides a foliation of $H_1(S_{1,1};\mathbb{R})$. In particular, any neighbourhood U of a point $\boldsymbol{x} \in \mathbb{R}^2$ is foliated by a subset of $\mathscr{ML}$ and is mapped to a neighbourhood V of a point $\mathbf{y} \in \mathscr{T}_{FN}$ that is foliated by a subset of $\mathscr{E}$.*

Recall the slope of a simple closed curve given in Equation (2.1.7).

3. *Consider a sequence of distinct measured geodesic laminations ordered counter-clockwise by their slope,*
$$\mathscr{L}_1, \mathscr{L}_2, \ldots, \mathscr{L}_n, \qquad sl(\mathscr{L}_1) < sl(\mathscr{L}_2) < \ldots < sl(\mathscr{L}_n).$$
The sequence of associated earthquakes will follow the same order clockwise.

The map is depicted in Figure 2.8.

We can make further observations about the relationship between the slope of laminations and the slope of the associated earthquakes.

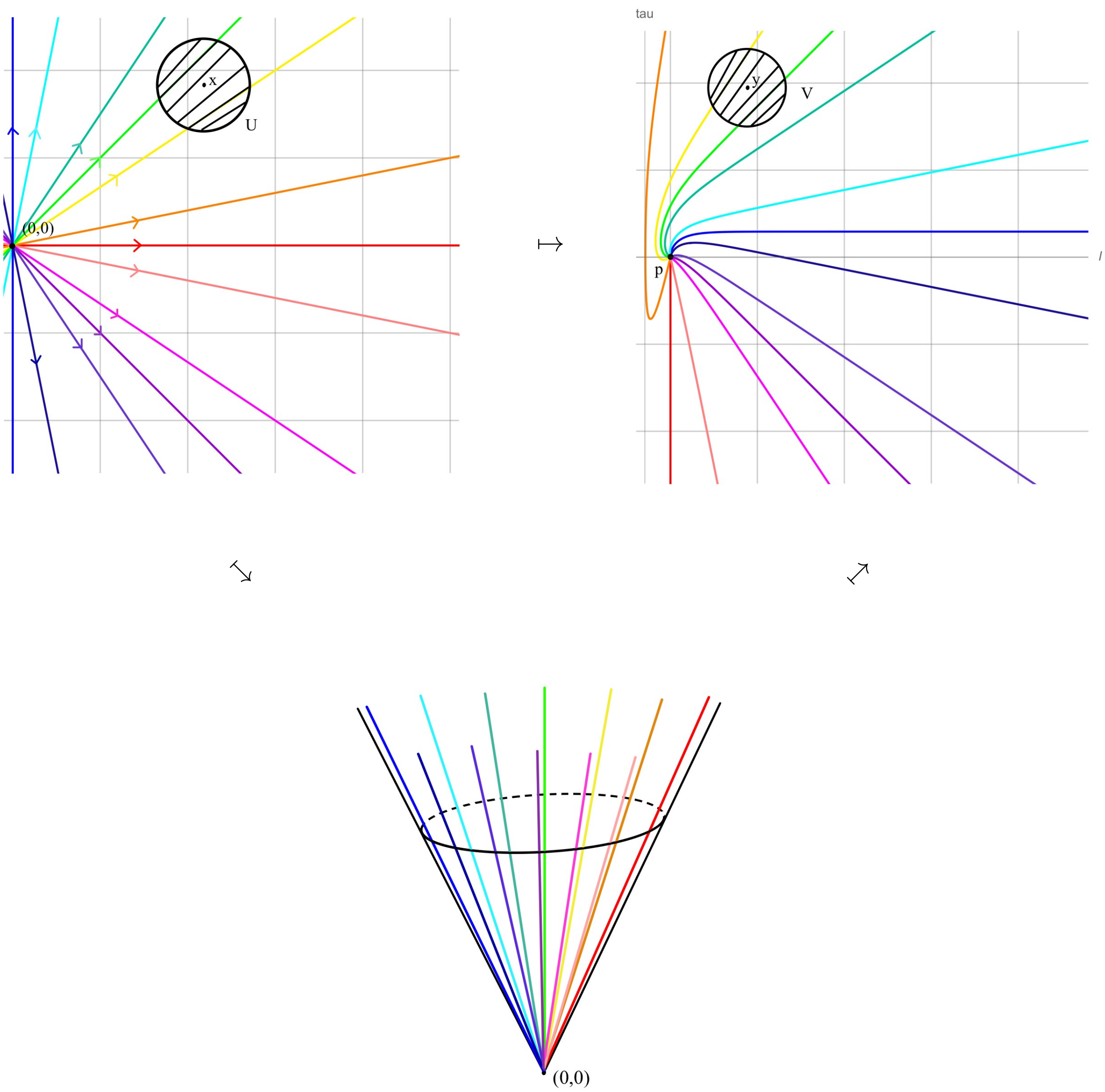

FIGURE 2.8: Illustration of map $E_{\mathbf{p}}$ from Equation (2.5.4) taking laminations with basepoint $(0,0)$ in $\mathbb{R}^2$ (top left) to forward earthquakes with starting point $\mathbf{p}$ in $\mathscr{T}_{\mathrm{FN}}$ (top right) going via the space of projective laminations (bottom). Clockwise, example laminations are given by simple closed curves β (blue), $\alpha\beta^5$, $\alpha^2\beta^3$, $\alpha\beta$, $\alpha^3\beta^2$, $\alpha^5\beta$, α (red), $\alpha^5\beta^{-1}$, $\alpha^3\beta^{-2}$, $\alpha\beta^{-1}$, $\alpha^2\beta^{-3}$, $\alpha\beta^{-5}$. This illustrates the behaviour from Remark 2.23.

Theorem 2.24. *Take simple closed curve γ. The slope of the associated earthquake $E^{FN}_{\gamma}(\mathbf{u},s)$ converges to the inverse of the slope of γ for all starting points $\mathbf{u} \in \mathscr{T}_{FN}$. We have,*

$$\begin{aligned} \frac{\tau^{\alpha}_{\alpha}(\mathbf{u},s)}{\ell^{\alpha}_{\alpha}(\mathbf{u},s)} &\to -\frac{1}{sl(\alpha)} = -\infty \qquad \text{as } s \to \infty \text{ for all } \mathbf{u} \in \mathscr{T}_{FN} \text{ and} \\ \frac{\tau^{\gamma}_{\alpha}(\mathbf{u},s)}{\ell^{\gamma}_{\alpha}(\mathbf{u},s)} &\to \frac{1}{sl(\gamma)} \quad = \frac{i(\gamma,\beta)}{i(\gamma,\alpha)} \ \text{ as } s \to \infty \text{ for all } \mathbf{u} \in \mathscr{T}_{FN}. \end{aligned}$$

The following corollary is apparent after applying the density result in Theorem 2.7.

Corollary 2.25. *Take a measured geodesic lamination $\mathscr{L}$ that is not α. The slope of the associated earthquake $E^{FN}_{\mathscr{L}}(\mathbf{u})$ converges to the inverse of the slope of $\mathscr{L}$ for all starting points $\mathbf{u} \in \mathscr{T}_{FN}$. That is,*

$$\frac{\tau^{\mathscr{L}}_{\alpha}(\mathbf{u},s)}{\ell^{\mathscr{L}}_{\alpha}(\mathbf{u},s)} \to \frac{1}{sl(\mathscr{L})} \text{ as } s \to \infty \text{ for all } \mathbf{u} \in \mathscr{T}_{FN}.$$

Proof of Theorem 2.24. The theorem is proved in three main steps.

(i) First prove the result directly for α and β using explicit formula for the earthquakes.

Consider now a simple closed curve γ that intersects both α and β at least once.

(ii) Second prove that

$$\frac{\ell^{\gamma}_{\beta}(\mathbf{u},s)}{\ell^{\gamma}_{\alpha}(\mathbf{u},s)} \to \left|\frac{i(\gamma,\beta)}{i(\gamma,\alpha)}\right| \text{ as } s \to \infty \text{ for all } \mathbf{u} \in \mathscr{T}_{\mathrm{FN}}.$$

(iii) Third prove that

$$\ell^{\gamma}_{\beta}(\mathbf{u},s) \sim \begin{cases} \tau^{\gamma}_{\alpha}(\mathbf{u},s) & \text{if } \mathrm{sl}(\gamma) > 0 \\ -\tau^{\gamma}_{\alpha}(\mathbf{u},s) & \text{if } \mathrm{sl}(\gamma) < 0 \end{cases} \text{ as } s \to \infty \text{ for all } \mathbf{u} \in \mathscr{T}_{\mathrm{FN}}.$$

Bringing this all together proves the theorem.

Item (i). The result for α is clear from the formula from Equation (2.5.3). Fix $\mathbf{u} \in \mathscr{T}_{\mathrm{FN}}$. We have

$$\frac{\tau^{\gamma}_{\alpha}(\mathbf{u},s)}{\ell^{\gamma}_{\alpha}(\mathbf{u},s)} \to -\infty = -\frac{1}{\mathrm{sl}(\alpha)} \text{ as } s \to \infty.$$

Using the change of coordinates formulas we find an expression for the earthquake about β.

$$\begin{aligned} E^{\mathrm{FN}}_{\beta}(\mathbf{u},s) &= \zeta(\Sigma_{\mathrm{Rot}}(E^{\mathrm{tr}}_{\alpha}(\Sigma^{-1}_{\mathrm{Rot}}(\zeta^{-1}(\mathbf{u})),s)) \\ &= \begin{pmatrix} \ell^{\beta}_{\alpha}(\mathbf{u},s) \\ \tau^{\beta}_{\alpha}(\mathbf{u},s) \end{pmatrix} \end{aligned}$$

with

$$\begin{aligned} F(\ell,\tau) &= \cosh^{-1}\left(\cosh\left(\frac{\tau}{2}\right)\coth\left(\frac{\ell}{2}\right)\right), \\ \cosh\left(\frac{\ell^{\beta}_{\alpha}(\mathbf{u},s)}{2}\right) &= \cosh\left(\frac{\ell}{2}\right)\left(\cosh\left(\frac{s}{2}\right) + \frac{\sinh\left(\frac{s}{2}\right)\sinh\left(\frac{\tau}{2}\right)}{\sinh(F(\ell,\tau))}\right), \\ \cosh\left(\frac{\tau^{\beta}_{\alpha}(\mathbf{u},s)}{2}\right) &= \cosh\left(\frac{\tau}{2}\right)\cosh\left(\frac{\ell}{2}\right)\tanh\left(\frac{\ell_{\beta}(\mathbf{u},s)}{2}\right). \end{aligned}$$

Then the slope is

$$\frac{\tau^{\beta}_{\alpha}(\mathbf{u},s)}{\ell^{\beta}_{\alpha}(\mathbf{u},s)} = \frac{2\cosh^{-1}\left(\cosh\left(\frac{\tau}{2}\right)\cosh\left(\frac{\ell}{2}\right)\tanh\left(\frac{\ell_{\beta}(\mathbf{u},s)}{2}\right)\right)}{\ell_{\beta}(\mathbf{u},s)}.$$

We find
$$\ell_\alpha^\beta(\mathbf{u},s) \to \infty \text{ and } \tanh\left(\ell_\beta^\alpha(\mathbf{u},s)\right) \to 1 \text{ as } s \to \infty$$
and
$$\frac{\tau_\alpha^\beta(\mathbf{u},s)}{\ell_\alpha^\beta(\mathbf{u},s)} \to \frac{2\cosh^{-1}\left(\cosh\left(\frac{\tau}{2}\right)\cosh\left(\frac{\ell}{2}\right)\right)}{\infty} = 0 = \frac{1}{\mathrm{sl}(\beta)} \text{ as } s \to \infty.$$

Item (ii). We exploit the relationship between earthquakes and Dehn twists. Let T_γ be the Dehn twist about γ and δ another simple closed curve. Consider the length of $\ell_{T_\gamma^n(\delta)}(\mathbf{u})$. We claim
$$\ell_{T_\gamma^n(\delta)}(\mathbf{u}) \sim n|i(\gamma,\delta)|\ell_\gamma(\mathbf{u}) \text{ as } n \to \infty. \tag{2.5.5}$$

If $|i(\gamma,\delta)| = k$, we can split δ into k segments, ε_i, $1 \le i \le k$ such that
$$\begin{aligned} \delta &= \varepsilon_1\varepsilon_2\ldots\varepsilon_k, \\ |i(\gamma,\varepsilon_i)| &= 1 \text{ for all } 1 \le i \le k, \text{ and} \\ T_\gamma^n(\delta) &= \gamma^n\varepsilon_1\gamma^n\varepsilon_2\ldots\gamma^n\varepsilon_k. \end{aligned}$$

Note that the second condition implies that the pair γ,ε_i generate $\pi_1(S_{1,1})$ for all $1 \le i \le k$. In particular, that $\mathrm{tr}(\rho(\gamma\varepsilon_i\gamma^{-1}\varepsilon_i^{-1}) = -2$ for all $1 \le i \le k$ and for all representations ρ that represent a hyperbolic structure in $\mathscr{T}(S_{1,1})$.

Take a such a representation ρ such that,
$$\rho(\gamma) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \text{ with } \lambda > 0, \qquad \rho(\varepsilon_i) = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}.$$

Note, if $a_i = 0$, $\rho(\varepsilon_i)$ sends the attracting fixed point of $\rho(\gamma)$ to the repelling fixed point and their invariant axes are distinct. This contradicts work by Goldman [Gol03, Lemma 3.4.5]. Thus, $a_i \neq 0$ for all $1 \le i \le k$.

Then
$$\begin{aligned} \ell_{T_\gamma^n(\delta)}(\mathbf{u}) &= 2\cosh^{-1}\left(\frac{\mathrm{tr}\left(\rho\left(\gamma^n\varepsilon_1\gamma^n\varepsilon_2\ldots\gamma^n\varepsilon_k\right)\right)}{2}\right) \\ &= 2\cosh^{-1}\left(\frac{\lambda^{kn}a_1a_2\ldots a_k + \ldots + \lambda^{-kn}d_1d_2\ldots d_k}{2}\right) \\ &\sim 2\cosh^{-1}\left(\frac{\lambda^{kn}a_1a_2\ldots a_k}{2}\right) \text{ as } n \to \infty \\ &= 2\log\left(\frac{\lambda^{kn}a_1a_2\ldots a_k}{2} + \sqrt{\left(\frac{\lambda^{kn}a_1a_2\ldots a_k}{2}\right)^2 - 1}\right) \\ &\sim 2\log\left(\lambda^{kn}a_1a_2\ldots a_k\right) \text{ as } n \to \infty \\ &= nk2\log(\lambda) + \log(a_1a_2\ldots a_k) \\ &= n|i(\gamma,\delta)|\ell_\gamma(\mathbf{u}) + \log(a_1a_2\ldots a_k) \\ &\sim n|i(\gamma,\delta)|\ell_\gamma(\mathbf{u}) \text{ as } n \to \infty. \end{aligned}$$

The same asymptotic expression can be deduced taking the inverse Dehn twist,

$$\ell_{T_\gamma^{-n}(\delta)}(\mathbf{u}) \sim n|i(\gamma,\delta)|\ell_\gamma(\mathbf{u}) \text{ as } n \to \infty.$$

We get the following consequences to Equation (2.5.5),

$$\ell_\delta\left(T_\gamma^n(\mathbf{u})\right) = \ell_{T_\gamma^{-n}(\delta)}(\mathbf{u}) \sim n|i(\gamma,\delta)|\ell_\gamma(\mathbf{u}) \text{ as } n \to \infty,$$

Then,

$$\ell_\delta\left(E_\gamma^{\mathrm{FN}}(\mathbf{u},2\pi n)\right) = \ell_\delta\left(T_\gamma^n(\mathbf{u})\right) \sim n|i(\gamma,\delta)|\ell_\gamma(\mathbf{u}) \text{ as } n \to \infty.$$

The length function is convex on Teichmüller space (see [Ker85]) and we can deduce

$$\ell_\delta^{\mathscr{L}}(\mathbf{u},s) = \ell_\delta\left(E_\gamma^{\mathrm{FN}}(\mathbf{u},s)\right) \sim \frac{s}{2\pi}|i(\gamma,\delta)|\ell_\gamma(\mathbf{u}) \text{ as } s \to \infty.$$

Then compare the ratio

$$\begin{aligned}\frac{\ell_\beta^{\mathscr{L}}(\mathbf{u},s)}{\ell_\alpha^{\mathscr{L}}(\mathbf{u},s)} &\sim \frac{\frac{s}{2\pi}|i(\gamma,\beta)|\ell_\gamma(\mathbf{u})}{\frac{s}{2\pi}|i(\gamma,\alpha)|\ell_\gamma(\mathbf{u})} \text{ as } s \to \infty \\ &\to \left|\frac{i(\gamma,\beta)}{i(\gamma,\alpha)}\right| \text{ as } s \to \infty.\end{aligned}$$

This concludes the proof of Item (ii).

Item (iii). Consider the following change of pants decomposition formula from [ALPS12, Oka93] in the case of a cusp,

$$\cosh\left(\frac{\ell_\beta^\gamma(\mathbf{u},s)}{2}\right) = \sqrt{\frac{\cosh(\ell_\alpha^\gamma(\mathbf{u},s))+1}{2\sinh\left(\frac{\ell_\alpha^\gamma(\mathbf{u},s)}{2}\right)^2}}\cosh\left(\frac{\tau_\alpha^\gamma(\mathbf{u},s)}{2}\right).$$

We have $\ell_\alpha^\gamma(\mathbf{u},s) \to \infty$ as $s \to \infty$. Then,

$$\begin{aligned}\frac{\cosh(\ell_\alpha^\gamma(\mathbf{u},s))+1}{2\sinh\left(\frac{\ell_\alpha^\gamma(\mathbf{u},s)}{2}\right)^2} &= \frac{\cosh\left(\frac{\ell_\alpha^\gamma(\mathbf{u},s)}{2}\right)^2+\sinh\left(\frac{\ell_\alpha^\gamma(\mathbf{u},s)}{2}\right)^2+1}{2\sinh\left(\frac{\ell_\alpha^\gamma(\mathbf{u},s)}{2}\right)^2} \\ &= \frac{1}{2}\left(1+\coth\left(\frac{\ell_\alpha^\gamma(\mathbf{u},s)}{2}\right)^2+\operatorname{csch}\left(\frac{\ell_\alpha^\gamma(\mathbf{u},s)}{2}\right)^2\right) \\ &\to 1 \text{ as } s \to \infty.\end{aligned}$$

Using this gives

$$\begin{gathered}\cosh\left(\frac{\ell_\beta^\gamma(\mathbf{u},s)}{2}\right) \sim \cosh\left(\frac{\tau_\alpha^\gamma(\mathbf{u},s)}{2}\right) \text{ as } s \to \infty \text{ and} \\ \ell_\beta^\gamma(\mathbf{u},s) \sim \pm\tau_\alpha^\gamma(\mathbf{u},s) \text{ as } s \to \infty.\end{gathered}$$

Where the sign is determined by if γ intersects α positively or negatively and β positively or negatively. This is equivalent to,

$$\ell_\beta^\gamma(\mathbf{u},s) \sim \begin{cases} \tau_\alpha^\gamma(\mathbf{u},s) & \text{if } \mathrm{sl}(\gamma) > 0, \\ -\tau_\alpha^\gamma(\mathbf{u},s) & \text{if } \mathrm{sl}(\gamma) < 0. \end{cases}$$

This finishes the proof. □

We may consider the change in this result if we instead considered the right earthquake.

Remark 2.26. *The result about slope does not rely on the orientation of the earthquake. That is, for each lamination $\mathscr{L}$, forwards and backwards earthquakes will tend to the same slope,*

$$\frac{\tau_\alpha^{\mathscr{L}}(\mathbf{u},-s)}{\ell_\alpha^{\mathscr{L}}(\mathbf{u},-s)} \to \frac{1}{sl(\mathscr{L})} \text{ as } s \to \infty \text{ for all } \mathbf{u} \in \mathscr{T}_{FN}.$$

This is clear from the examples in Figure 2.7 and is proven using the same approach as Theorem 2.24. The only difference arises for $\mathscr{L} = \alpha$, where

$$\frac{\tau_\alpha^{\alpha}(\mathbf{u},-s)}{\ell_\alpha^{\alpha}(\mathbf{u},-s)} \to \frac{1}{sl(\alpha)} = \infty \qquad \text{as } s \to \infty \text{ for all } \mathbf{u} \in \mathscr{T}_{FN}.$$

We make a final remark regarding the relationship between forwards and backwards earthquakes.

Remark 2.27. *It appears that, for a fixed starting point $\mathbf{u} \in \mathscr{T}_{FN}$, any backwards earthquake $E_{\mathscr{L}}^{FN}(\mathbf{u},-s)$ about a lamination $\mathscr{L}$ will intersect all other forwards earthquakes $E_{\mathscr{K}}^{FN}(\mathbf{u},s)$ for all laminations $\mathscr{K}$ in exactly one point (excluding $\mathbf{u}$).*

That is, for each $\mathscr{L}$ and each $\mathscr{K}$, there exists $s_{\mathscr{L},\mathscr{K}} \neq 0$ such that

$$E_{\mathscr{L}}^{FN}(\mathbf{u},-s_{\mathscr{L},\mathscr{K}}) = E_{\mathscr{K}}^{FN}(\mathbf{u},s_{\mathscr{L},\mathscr{K}}).$$

The results from this section help to study general earthquake behaviour, as we will see in examples in the next section.

2.6 Examples of families of earthquakes

We provide examples of the change of coordinates algorithm for two families of curves

1. $\gamma_n = \alpha\beta\alpha^{n-1}$; and
2. $\gamma_n = T^n(\alpha)$ for $T = T_\beta T_\alpha^{-1}$.

The first family of curves is an example of a sequence of simple closed curves that converges to a simple closed curve as $n \to \infty$. The second family of curves is an example of a sequence of simple closed curves that converges to a measured geodesic lamination that is not a simple closed curve as $n \to \infty$.

For both examples we first describe the simple closed curve δ and the maps ϕ and ψ required for the algorithm in both trace and triangle length coordinates. To this end, define new local triangle length coordinates using the local trace coordinates from Equation (2.4.1),

$$\begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} = \begin{pmatrix} \cosh^{-1}\left(\frac{x'}{2}\right) \\ \cosh^{-1}\left(\frac{y'}{2}\right) \\ \cosh^{-1}\left(\frac{z'}{2}\right) \end{pmatrix}.$$

We then examine behaviour at the limit as $n \to \infty$ and investigate other ways to study the earthquakes.

Example 2.28 ($\gamma_n = \alpha\beta\alpha^{n-1}$)**.** *First, compute a sequence of simple closed curves δ_n such that $i(\gamma_n, \delta_n) = 1$ for each $n \in \mathbb{N}$. One such example is $\delta_n = \alpha^{-1}$. The mapping class that takes α, β to γ_n, δ_n is $T_\alpha^{-n+1} T_\beta T_\alpha$. Given δ_n has no reliance on n for this family of curves we exclude the subscript. Then*

$$\begin{aligned} \gamma_n &= \alpha\beta\alpha^{n-1}, \quad \delta = \alpha^{-1}, \\ \alpha &= \delta^{-1}, \qquad\quad \beta = \delta\gamma_n\delta^{n-1}. \end{aligned}$$

Second, calculate the earthquake about γ_n in the framing γ_n and δ as given in Equation (2.4.2).

Third, construct the series of maps ϕ_n and ψ_n to go between the original framing α, β and the new framing γ_n, δ for each n.

Going to local trace coordinates,

$$\begin{aligned} \phi_n \colon \mathscr{T}_{\mathrm{tr}} &\to \mathscr{T}_{\mathrm{tr}} \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} &\mapsto \begin{pmatrix} x'(n) \\ y'(n) \\ z'(n) \end{pmatrix} = \begin{pmatrix} \mathrm{tr}(\rho(\alpha\beta\alpha^{n-1})) \\ \mathrm{tr}(\rho(\alpha^{-1})) \\ \mathrm{tr}(\rho(\alpha\beta\alpha^{n-2})) \end{pmatrix} = \begin{pmatrix} x'(n) \\ x \\ x'(n-1) \end{pmatrix}, \end{aligned}$$

where $x'(n) = xx'(n-1) - x'(n-2)$, with $x'(0) = y$, $x'(1) = z$. The characteristic equation method for solving linear recurrence relations can be used to solve for $x'(n)$.

Going to original trace coordinates,

$$\begin{aligned} \psi_n \colon \mathscr{T}_{\mathrm{tr}} &\to \mathscr{T}_{\mathrm{tr}} \\ \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} &\mapsto \begin{pmatrix} x(n) \\ y(n) \\ z(n) \end{pmatrix} = \begin{pmatrix} \mathrm{tr}(\rho(\delta^{-1})) \\ \mathrm{tr}(\rho(\delta\gamma_n\delta^{n-1})) \\ \mathrm{tr}(\rho(\gamma_n\delta^{n-1})) \end{pmatrix} = \begin{pmatrix} y' \\ y(n) \\ y(n-1) \end{pmatrix}, \end{aligned}$$

where $y(n) = y'y(n-1) - y(n-2)$, with $y(0) = x'$, $y(1) = z'$. Again, the characteristic equation method for solving linear recurrence relations can be used to solve for $y(n)$.

Then given some starting point $\mathbf{v} = (x, y, z) \in \mathscr{T}_{\mathrm{tr}}$, the earthquake about the family γ_n is given by the map $E_n^{\mathrm{tr}}(\mathbf{v}) \colon \mathbb{R} \to \mathscr{T}_{\mathrm{tr}}$,

$$E_n^{\mathrm{tr}}(\mathbf{v}) = \psi_n\left(E_\alpha^{\mathrm{tr}}(\phi_n(\mathbf{v}))\right).$$

In triangle length coordinates the family of maps become

$$\begin{aligned}\overline{\phi}_n\colon\ \mathscr{T}_\ell &\to \mathscr{T}_\ell\\ \begin{pmatrix}a\\ b\\ c\end{pmatrix} &\mapsto \begin{pmatrix}a'(n)\\ b'(n)\\ c'(n)\end{pmatrix} = \begin{pmatrix}\cosh^{-1}(A(n))\\ a\\ \cosh^{-1}(A(n-1))\end{pmatrix}\end{aligned} \tag{2.6.1}$$

with

$$A(n) = \cosh b \cosh(na) + (\cosh c - \cosh a \cosh b)\,\frac{\sinh(na)}{\sinh a}$$

and

$$\begin{aligned}\overline{\Psi}_n\colon\ \mathscr{T}_\ell &\to \mathscr{T}_\ell\\ \begin{pmatrix}a'\\ b'\\ c'\end{pmatrix} &\mapsto \begin{pmatrix}a(n)\\ b(n)\\ c(n)\end{pmatrix} = \begin{pmatrix}b'\\ \cosh^{-1}(B(n))\\ \cosh^{-1}(B(n-1))\end{pmatrix}\end{aligned} \tag{2.6.2}$$

where

$$B(n) = \cosh a' \cosh\left(nb'\right) + \left(\cosh c' - \cosh a' \cosh b'\right)\frac{\sinh(nb')}{\sinh b'}.$$

Then given some starting point $\mathbf{w} = (a,b,c) \in \mathscr{T}_\ell$ *the earthquake about the family* γ_n *is given by the map* $E_n^\ell(\mathbf{w})\colon\ \mathbb{R} \to \mathscr{T}_\ell$,

$$E_n^\ell(\mathbf{w}) = \overline{\Psi}_n\left(E_\alpha^\ell(\overline{\phi}_n(\mathbf{w}))\right).$$

We can calculate the limit as $n \to \infty$ *by expanding the right-hand side expression. The sequence* $\gamma_n = \alpha\beta\alpha^{n-1}$ *asymptotically approaches* α *and thus we expect the earthquake about* γ_n *to converge to the earthquake about* α. *We use hyperbolic trigonometry identities in addition to the collar equation to simplify expressions.*

Rescale using $r \mapsto r/n^2$ *to account for the growing length of the curve and the appropriate sequence of transverse measures in* n. *Using Taylor series expansions to evaluate certain terms in the limit we find,*

$$\lim_{n\to\infty} E_n^\ell(\mathbf{w})\left(\frac{r}{n^2}\right) = E_\alpha^\ell(\mathbf{w})(r).$$

We give details for this calculation in Appendix A.

If we used the hyperbolic length parametrisation as in Remark 2.22 the appropriate rescaling is $s \mapsto s/n$,

$$\lim_{n\to\infty} E_n^\ell(\mathbf{w})\left(\frac{s}{n}\right) = E_\alpha^\ell(\mathbf{w})(s). \tag{2.6.3}$$

Note the rescaling is important, otherwise the limit would tend to ∞ *in multiple coordinates. This result is consistent with Kerckhoff's result [Ker83] that defining earthquakes for general measured geodesic laminations through the limits of sequences is well-defined.*

See Figure 2.9 for examples of the orbits of γ_n *for* $n = 1,2,3,4$.

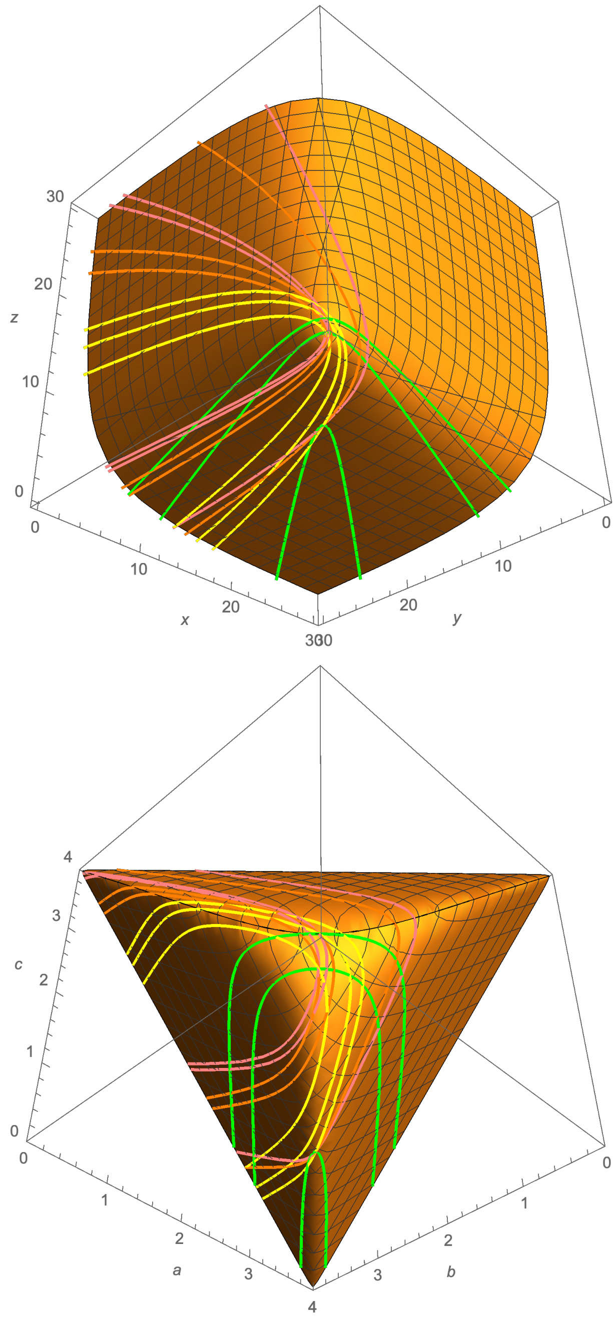

FIGURE 2.9: Some example earthquakes about $\gamma_n = \alpha\beta\alpha^{n-1}$ in trace coordinates (top) and triangle length coordinates (bottom). Examples shown are for $n = 1,2,3,4$ in green, yellow, orange, pink, respectively. Starting points are given by $\mathscr{S}_\alpha$ (top) and $\nu\mathscr{S}_\alpha$ (bottom) from Equations (2.2.4) and (2.3.1).

Example 2.29 ($\gamma_n = T^n(\alpha)$)**.** *Take the following mapping class,*

$$T = T_\beta T_\alpha^{-1} = \begin{cases} \alpha \mapsto \alpha\beta \\ \beta \mapsto \beta\alpha\beta, \end{cases}$$

with matrix representation

$$N = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

This is a pseudo-Anosov mapping class with irrational eigenvalues $\lambda_{\pm} = \left(3 \pm \sqrt{5}\right)/2$*. Repeated applications of* T *to* α *will converge to a lamination* ℓ *with irrational slope* $(1+\sqrt{5})/2$*. That is, a measured geodesic lamination that is not a simple closed curve.*

First, compute a sequence of simple closed curves δ_n *such that* $i(\gamma_n, \delta_n) = 1$ *for each* $n \in \mathbb{N}$*. One such example is* $\delta_n = T^n(\beta)$*. Then*

$$\begin{aligned} \gamma_n &= T^n(\alpha), & \delta_n &= T^n(\beta), \\ \alpha &= T^{-n}(\gamma_n), & \beta &= T^{-n}(\delta_n). \end{aligned}$$

Examples for $n = 1, 2$ *are shown below,*

$$\begin{aligned} \gamma_1 &= \alpha\beta, & \delta_1 &= \beta\alpha\beta, \\ \alpha &= \gamma_1^2\delta_1^{-1}, & \beta &= \delta_1\gamma_1^{-1}, \end{aligned}$$

$$\begin{aligned} \gamma_2 &= \alpha\beta^2\alpha\beta, & \delta_2 &= \beta\alpha\beta\alpha\beta^2\alpha\beta, \\ \alpha &= \gamma_2^2\delta_2^{-1}\gamma_2^2\delta_2^{-1}\gamma_2\delta_2^{-1}, & \beta &= \delta_2\gamma_2^{-1}\delta_2\gamma_2^{-2}. \end{aligned}$$

Note that as $n \to \infty$*,* γ_n *and* δ_n *will converge to the same measured geodesic lamination with the same irrational slope, which we can use to our advantage.*

Second, calculate the earthquake deformation for γ_n *and* δ_n *in the framing* γ_n *and* δ_n *as given in Equation (2.4.2).*

Third, construct the series of maps $\phi_n = \widetilde{T}^n$ *and* $\psi_n = \widetilde{T}^{-n}$ *to go between the original framing* α*,* β *and the new framing* γ_n*,* δ_n *for each* n*. Both* $\widetilde{T}^n$ *and* $\widetilde{T}^{-n}$ *involve a system of recurrence relations that are not linear and we have found no typical method for solving them. We can still understand the underlying maps* $\widetilde{T}$ *and* $\widetilde{T}^{-1}$*,*

$$\begin{aligned} &\widetilde{T}\colon\ \mathscr{T}_{\mathrm{tr}} \to \mathscr{T}_{\mathrm{tr}} \\ &\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \mathrm{tr}(\rho(\alpha\beta)) \\ \mathrm{tr}(\rho(\beta\alpha\beta)) \\ \mathrm{tr}(\rho(\alpha\beta^2\alpha\beta)) \end{pmatrix} = \begin{pmatrix} z \\ yz - x \\ z(yz-x) - y \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} &\widetilde{T}^{-1}\colon\ \mathscr{T}_{\mathrm{tr}} \to \mathscr{T}_{\mathrm{tr}} \\ &\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \mathrm{tr}(\rho(\gamma_n^2\delta_n^{-1})) \\ \mathrm{tr}(\rho(\delta_n\gamma_n^{-1})) \\ \mathrm{tr}(\rho(\gamma_n)) \end{pmatrix} = \begin{pmatrix} x'(x'y'-z') - y' \\ x'y' - z' \\ x' \end{pmatrix}. \end{aligned}$$

Given some starting point $\mathbf{v} = (x,y,z) \in \mathscr{T}_{\mathrm{tr}}$ *we can define the earthquake about the family* γ_n *recursively by* $E_n^{tr}(\mathbf{v})\colon\ \mathbb{R} \to \mathscr{T}_{\mathrm{tr}}$,

$$\begin{aligned} E_n^{tr}(\mathbf{v}) &= \widetilde{T}^{-n}\left(E_\alpha^{\mathrm{tr}}(\widetilde{T}^n(\mathbf{v}))\right), \\ &= \widetilde{T}^{-1}\left(E_{n-1}^{tr}(\widetilde{T}(\mathbf{v}))\right). \end{aligned}$$

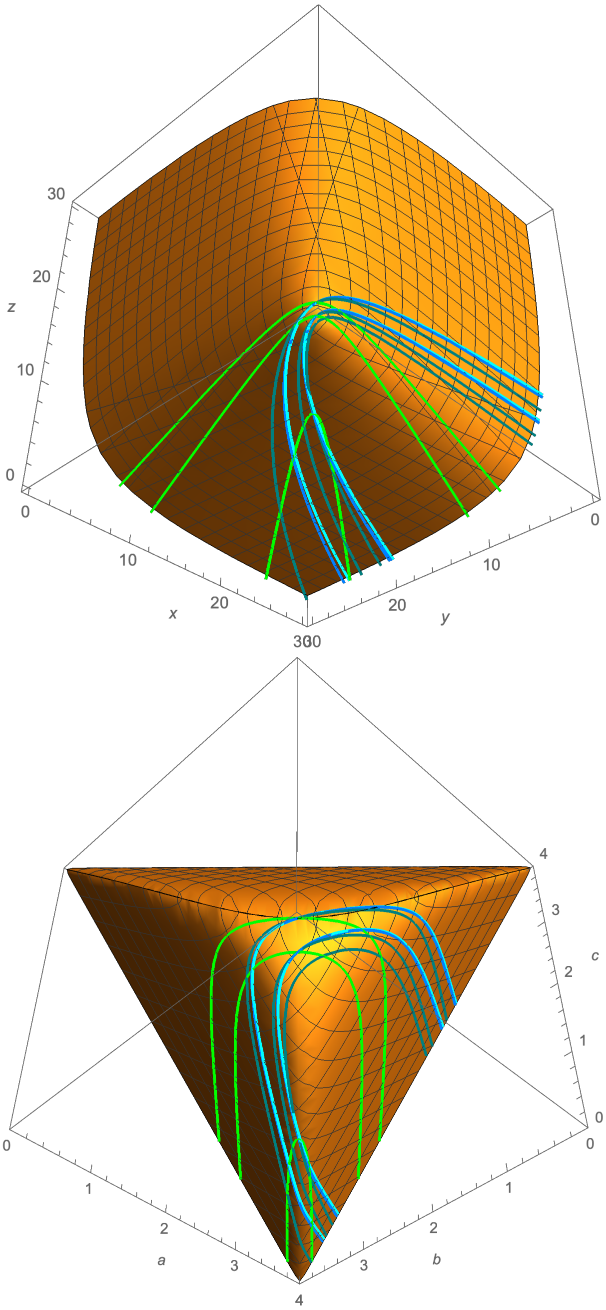

FIGURE 2.10: Some example earthquakes about $\gamma_n = T^n(\alpha)$ in trace coordinates (top) and triangle length coordinates (bottom). Examples shown are for $n = 1,2,3,4$ in green, turquoise, light blue, and cyan, respectively. Starting points are given by $\mathscr{S}_\alpha$ (top) and $\nu\mathscr{S}_\alpha$ (bottom) from Equations (2.2.4) and (2.3.1).

In triangle length coordinates the maps $\widetilde{T}$ *and* $\widetilde{T}^{-1}$ *become,*

$$\begin{aligned}
&\overline{T}\colon \mathscr{T}_\ell \to \mathscr{T}_\ell \\
&\begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} = \begin{pmatrix} c \\ \cosh^{-1}\left(2\cosh b \cosh c - \cosh a\right) \\ \cosh^{-1}\left(2\cosh c\left(2\cosh b \cosh c - \cosh a\right) - \cosh b\right) \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
&\overline{T}^{-1}\colon \mathscr{T}_\ell \to \mathscr{T}_\ell \\
&\begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} \mapsto \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \cosh^{-1}\left(2\cosh a'\left(2\cosh a' \cosh b' - \cosh c'\right) - \cosh b'\right) \\ \cosh^{-1}\left(2\cosh a' \cosh b' - \cosh c'\right) \\ a' \end{pmatrix}
\end{aligned}$$

Given some starting point $\mathbf{w} = (a,b,c) \in \mathscr{T}_\ell$ *we can define the earthquake about the family* γ_n *recursively by* $E_n^\ell(\mathbf{w})\colon \mathbb{R} \to \mathscr{T}_\ell$,

$$\begin{aligned}
E_n^\ell(\mathbf{w}) &= \overline{T}^{-n}\left(E_\alpha^\ell(\overline{T}^n(\mathbf{w}))\right), \\
&= \overline{T}^{-1}\left(E_{n-1}^\ell(\overline{T}(\mathbf{w}))\right).
\end{aligned}$$

We can similarly define the earthquake about the family δ_n *in trace coordinates and triangle lengths.*

Again, to assess the limit as $n \to \infty$ *a rescaling factor is required, otherwise the length and the transverse measure would become unbounded. Recall* λ_+ *is the largest eigenvalue of the matrix representation* N, $\lambda_+ = \left(3+\sqrt{5}\right)/2$. *The appropriate rescaling to take for our original parametrisation is* $r \mapsto r/\lambda_+^{2n}$ *and for our hyperbolic length reparametrisation is* $s \mapsto r/\lambda_+^n$.

See Figure 2.10 for examples of the orbits of γ_n *for* $n = 1,2,3,4$. *Note the variation in the earthquake from* $n = 3$ *to* $n = 4$ *is already getting small. The recursive formula makes it difficult to evaluate terms in the limit directly.*

Compare the difference between γ_n *and* δ_n *as* n *increases. Given Thurston's earthquake theorem Theorem 2.1, we know this narrows the region of* $\mathscr{T}(S_{1,1})$ *where the limiting earthquake can appear. Given the relationship between slope in Theorem 2.24, we know the slope of the limiting earthquake must be* $2/(1+\sqrt{5})$. *We can estimate the limiting earthquake from these two pieces of information. See Figure 2.11 for examples of the forwards and backwards orbits of* γ_n *and* δ_n *with an estimation of the limiting earthquake about* ℓ.

2.7 Pictures of earthquakes

We use the results to generate pictures of earthquakes in multiple different coordinate systems. We focus on earthquakes for the framing α, β, as well as those for the families of simple closed curves in the two

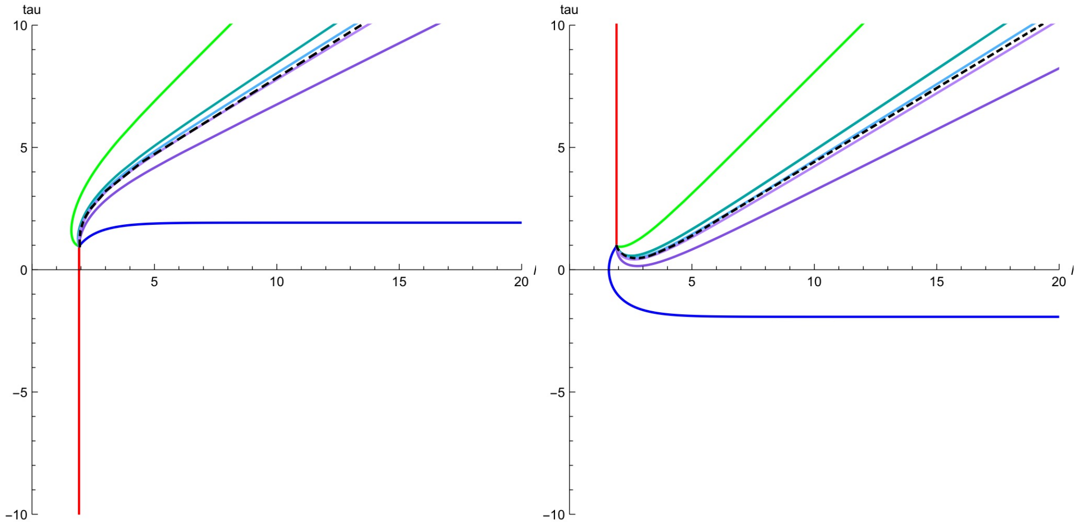

FIGURE 2.11: Example forwards (left) and backwards (right) earthquakes about $\gamma_n = T^n(\alpha)$ for $n = 0,1,2,3$ and $\delta_n = T^n(\beta)$ for $n = 0,1,2$ with starting point $\left(2\cosh^{-1}\left(\frac{3}{2}\right), 2\cosh^{-1}\left(\frac{\sqrt{5}}{2}\right)\right)$. The dashed black line denotes an estimation of the limiting earthquake between γ_3 and δ_2.

examples. Denote the set of all of these curves to be $\mathscr{C}$,

$$\mathscr{C} = \{\alpha, \beta, \alpha\beta, \alpha\beta\alpha, \alpha\beta\alpha^2, \alpha\beta\alpha^3, T^2(\alpha), T^3(\alpha), T^4(\alpha)\}.$$

We look at starting points $(3,3,3)$, $\left(2\sqrt{2}, 2\sqrt{2}, 4\right)$, and $\left(10, 10, -10(-5+\sqrt{23})\right)$, given in trace coordinates. Note $(3,3,3)$ represents the hexagonal torus and $(2\sqrt{2}, 2\sqrt{2}, 4)$ represents the square torus.

We consider the earthquakes in six coordinate systems: trace coordinates on the component $\mathscr{T}_{\mathrm{tr}}$, two modified trace coordinates referred to as *spherical trace coordinates* and *inverted trace coordinates*, triangle length coordinates, Fenchel-Nielsen length-twist coordinates, and the *simplex*. It would be useful to study the earthquakes directly on $\mathscr{T}(S_{1,1}) \cong \mathbb{H}^2$, but it is difficult to calculate change of coordinates between $\mathbb{H}^2$ and the other representations of $\mathscr{T}(S_{1,1})$, see Abe [Abe00]. We discuss this more in Section 2.8.

Recall trace coordinates (x,y,z) on the component $\mathscr{T}_{\mathrm{tr}}$ with $x^2+y^2+z^2-xyz=0$. Pictures of earthquakes about the curves in $\mathscr{C}$ in trace coordinates are given in Figure 2.12.

Spherical trace coordinates are given by applying a spherical coordinate change to (x,y,z),

$$\begin{pmatrix} \theta \\ \phi \\ r \end{pmatrix} = \begin{pmatrix} \tan^{-1}\left(\frac{y}{x}\right) \\ \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) \\ \sqrt{x^2+y^2} \end{pmatrix}$$

with $1 - r\cos(\theta)\sin(\theta)\cos(\phi)\sin(\phi)^2 = 0$. Pictures of earthquakes about the curves in $\mathscr{C}$ in spherical trace coordinates are given in Figure 2.13.

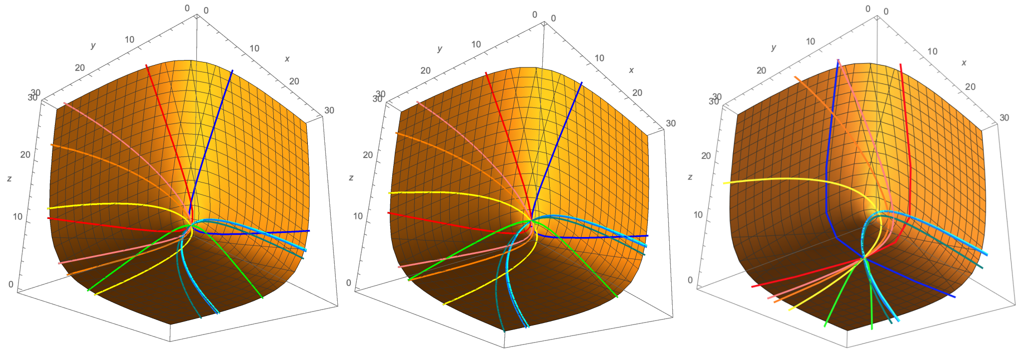

FIGURE 2.12: Forward and backward earthquakes about curves in $\mathscr{C}$ on $\mathscr{T}_{\mathrm{tr}}$ in trace coordinates given starting points $(3,3,3)$ (left), $\left(2\sqrt{2},2\sqrt{2},4\right)$ (centre), and $\left(10,10,-10(-5+\sqrt{23})\right)$ (right).

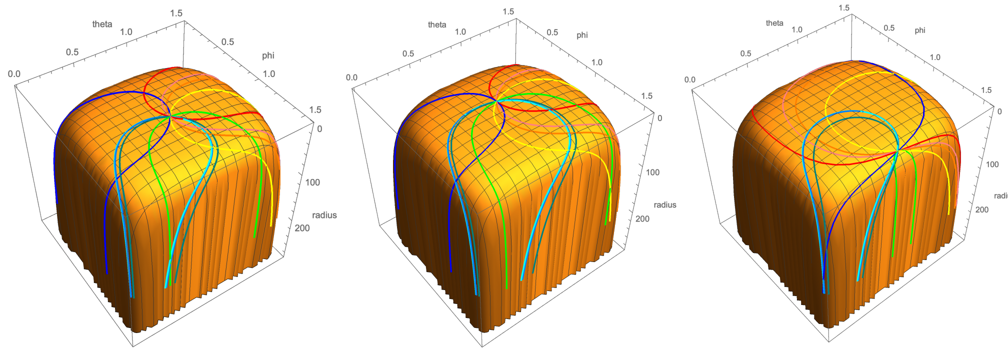

FIGURE 2.13: Forward and backward earthquakes about curves in $\mathscr{C}$ on $\mathscr{T}_{\mathrm{tr}}$ in spherical trace coordinates given starting points corresponding to trace coordinates $(3,3,3)$ (left), $\left(2\sqrt{2},2\sqrt{2},4\right)$ (centre), and $\left(10,10,-10(-5+\sqrt{23})\right)$ (right).

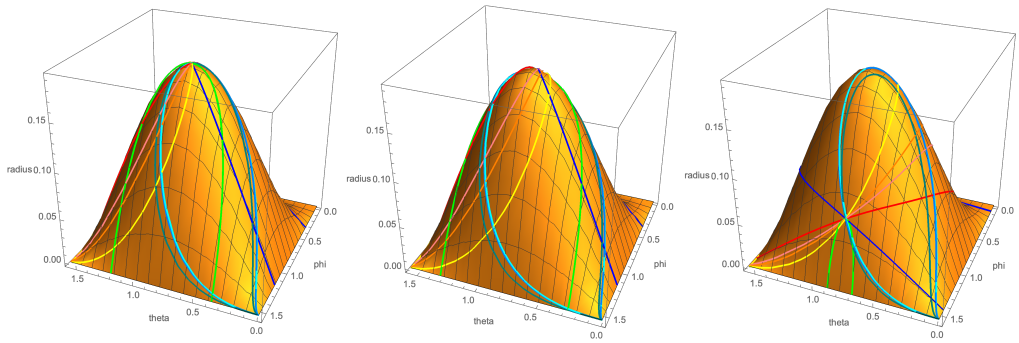

FIGURE 2.14: Forward and backward earthquakes about curves in $\mathscr{C}$ on $\mathscr{T}_{\mathrm{tr}}$ in inverted trace coordinates given starting points corresponding to trace coordinates $(3,3,3)$ (left), $\left(2\sqrt{2},2\sqrt{2},4\right)$ (centre), and $\left(10,10,-10(-5+\sqrt{23})\right)$ (right).

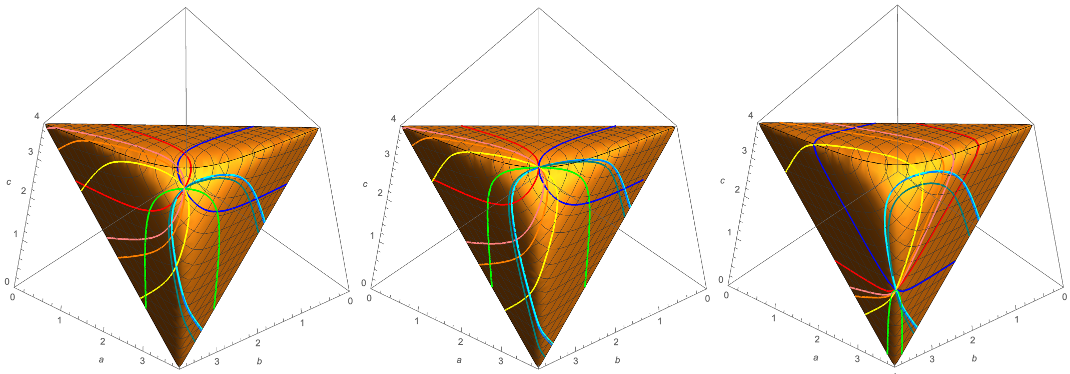

FIGURE 2.15: Forward and backward earthquakes about curves in $\mathscr{C}$ on $\mathscr{T}_\ell$ in triangle length coordinates given starting points corresponding to trace coordinates $(3,3,3)$ (left), $\left(2\sqrt{2},2\sqrt{2},4\right)$ (centre), and $\left(10,10,-10(-5+\sqrt{23})\right)$ (right).

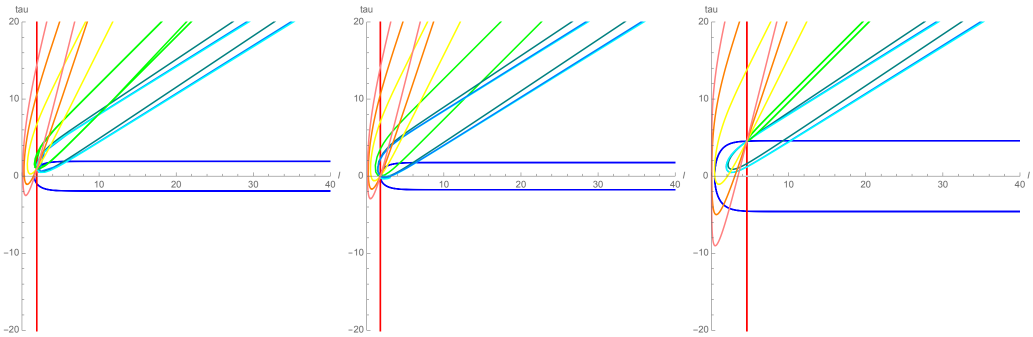

FIGURE 2.16: Forward and backward earthquakes about curves in $\mathscr{C}$ in Fenchel-Nielsen coordinates given starting points corresponding to trace coordinates $(3,3,3)$ (left), $\left(2\sqrt{2},2\sqrt{2},4\right)$ (centre), and $\left(10,10,-10(-5+\sqrt{23})\right)$ (right).

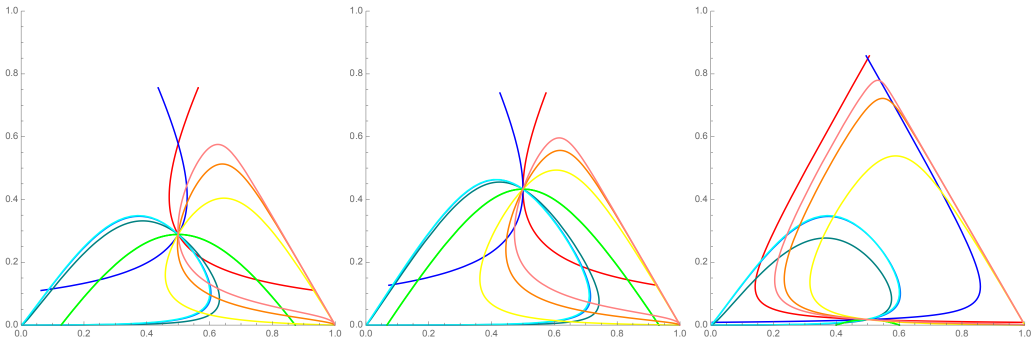

FIGURE 2.17: Forward and backward earthquakes about curves in $\mathscr{C}$ on the simplex given starting points corresponding to trace coordinates $(3,3,3)$ (left), $\left(2\sqrt{2},2\sqrt{2},4\right)$ (centre), and $\left(10,10,-10(-5+\sqrt{23})\right)$ (right).

Inverted trace coordinates are given by inverting the radius r in the spherical trace coordinates,

$$\begin{pmatrix} \theta \\ \phi \\ r \end{pmatrix} = \begin{pmatrix} \tan^{-1}\left(\frac{y}{x}\right) \\ \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) \\ \frac{1}{\sqrt{x^2+y^2}} \end{pmatrix}$$

with $1-\frac{1}{r}\cos(\theta)\sin(\theta)\cos(\phi)\sin(\phi)^2=0$. Pictures of earthquakes about the curves in $\mathscr{C}$ in inverted trace coordinates are given in Figure 2.14.

Recall triangle lengths as defined in Equation (2.1.13). Pictures of earthquakes about the curves in $\mathscr{C}$ in triangle length coordinates are given in Figure 2.15.

Also recall Fenchel-Nielsen coordinates using α are related to trace coordinates and triangle lengths by Equation (2.5.1). Pictures of earthquakes about the curves in $\mathscr{C}$ in Fenchel-Nielsen coordinates are given in Figure 2.16.

Coordinates for the **simplex** can be defined by the transformation,

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} \frac{x}{yz} \\ \frac{y}{xz} \\ \frac{z}{xy} \end{pmatrix}$$

with $p+q+r=1$, $p,q,r>0$. The simplex coordinates can be transformed to the plane (p',q'). Pictures of earthquakes about the curves in $\mathscr{C}$ on the simplex are given in Figure 2.17.

We can compare these pictures to previous work on earthquakes. In particular, Waterman and Wolpert [WW83] used Fenchel-Nielsen coordinates to plot forward earthquakes about simple closed geodesics starting at points $\left(\cosh^{-1}(3),0\right)$, $\left(\cosh^{-1}(3),1\right)$. Plots using our formulas converted to Fenchel-Nielsen coordinates are shown in Figure 2.18 and are consistent with Waterman and Wolpert's results.

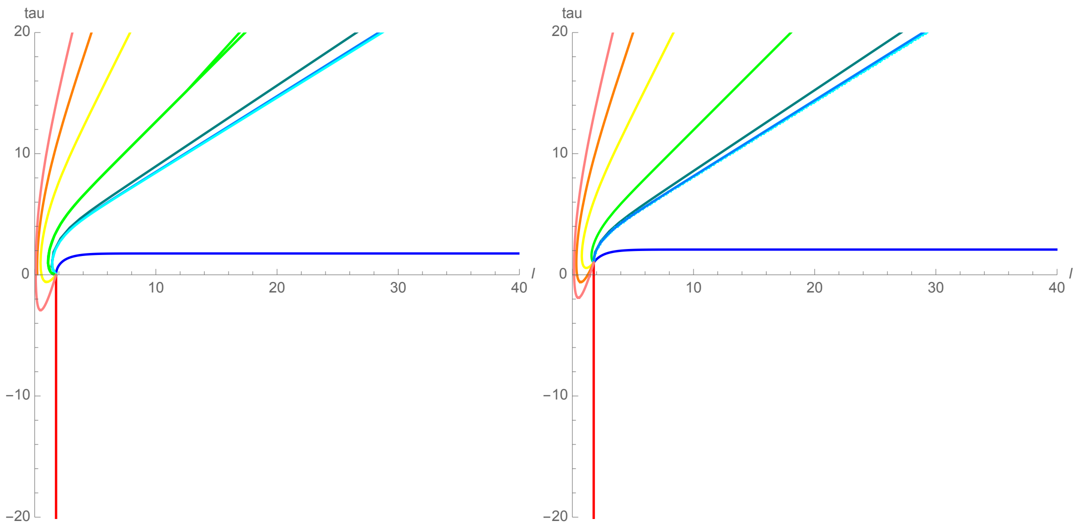

FIGURE 2.18: Forward earthquakes about curves in $\mathscr{C}$ in Fenchel-Nielsen coordinates given starting points $\left(\cosh^{-1}(3),0\right)$ (left), $\left(\cosh^{-1}(3),1\right)$ (right).

2.8 Mapping to the hyperbolic plane

We have visualised earthquakes using many different coordinate systems and perspectives on Teichmüller space. We know the Teichmüller space of the once-punctured torus is the hyperbolic plane, $\mathscr{T}(S_{1,1}) \cong \mathbb{H}^2$ (see the discussion in Section 1.3.1). In [Abe00], Abe constructs a holomorphic map between fundamental domains of $\mathscr{T}_{\mathrm{tr}}$ and $\mathbb{H}^2$ using Weierstrass functions. Previous research studying earthquakes directly on the hyperbolic plane has been shown to be useful [Pfe17, Thu86]. It would be good to better understand the connection between the coordinate systems. Translating the results from Theorems 2.12 and 2.18 to the hyperbolic plane would be the natural first step. We present some initial calculations in this direction.

We again start with the action of the Dehn twists. Each earthquake corresponds to the action of the 1-parameter groups of parabolic elements on Teichmüller space associated with the Dehn twists in $\mathrm{SL}_2(\mathbb{Z})$. The twist about α, T_α, is associated with the action of the isometry $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. This is a parabolic isometry with fixed point ∞ and will translate points to the left (this can be seen as translating points clockwise along circles tangent to ∞). The twist about β, T_β, is associated with the action of the isometry $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. This is a parabolic isometry with fixed point 0 and will translate points clockwise along circles that are tangent to 0. They are visualised in Figure 2.19. (Note the isometries are elements in $\mathrm{PSL}_2(\mathbb{R})$, we have omitted the notation for equivalence classes and just written representatives in $\mathrm{SL}_2(\mathbb{R})$.)

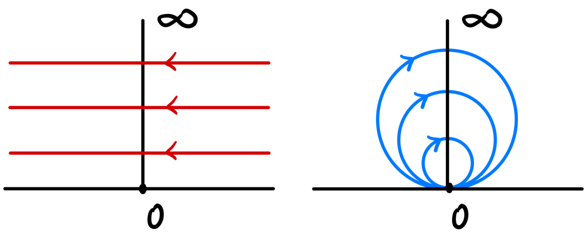

FIGURE 2.19: Isometries on $\mathbb{H}^2$ associated with Dehn twists about α, β.

The action of the Dehn twists T_α and T_β naturally extends to the flows defined by the isometry associated with matrices A_s and B_t.

$$A_s = \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}, \quad B_t = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix},$$

where A_s translates points to the left by $s \in \mathbb{R}$ and B_t translates points clockwise along horocycles at 0 by $t \in \mathbb{R}$.

Dehn twists about a simple closed curve will always be associated with parabolic isometries on $\mathbb{H}^2$, which will always be translations along circles tangent to a point on the boundary. Given two points in the hyperbolic plane $u = a + ib$, $v = c + id$ we can connect them by a circle of this form. If $b = d$ then the circle is a horizontal line that is tangent to ∞ as in the Dehn twist about α. Otherwise, there will be precisely two horocycles $\mathscr{C}_\pm$ connecting that u and v. Depending on the configuration of u and v, $\mathscr{C}_+$ will connect u to v clockwise (or anticlockwise) and $\mathscr{C}_-$ will connect v to u clockwise (or anticlockwise).

The formulae for the horocycles are given by

$$\begin{aligned}
\mathscr{C}_\pm &: (x-g_\pm)^2 + (y-r_\pm)^2 = r_\pm^2,\\
g_\pm &= \frac{bc-ad \pm \sqrt{bd(a^2+b^2+c^2+d^2-2(ac-bd))}}{b-f},\\
r_\pm &= \frac{a^2+b^2-2ag_\pm+g_\pm^2}{2b} = \frac{c^2+d^2-2cg_\pm+g_\pm^2}{2d},
\end{aligned}$$

with radius $r_\pm$ and centre $(g_\pm, r_\pm)$. Note the horocycle $\mathscr{C}_\pm$ is tangent to the boundary at $g_\pm \in \mathbb{R}$. An example is given in Figure 2.20.

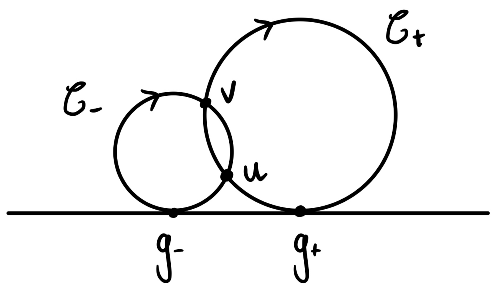

FIGURE 2.20: Example horocycles $\mathscr{C}_\pm$ connecting u and v on $\mathbb{H}^2$.

It is unclear what the correct way is to associate a curve (or lamination) with the either of the horocycles $\mathscr{C}_\pm$. In fact, there are multiple ways this could be done. We can examine the action of α and β explicitly. For example, we can connect any two points in the hyperbolic plane by a combination of the action of T_α and T_β and their inverses. To go from $u = a+ib$ to $v = c+id$ use the following procedure:

1. If $b = d$ then apply A_s for some $s \in \mathbb{R}$ taking u to v.
2. Else construct the circle C_u and C_v that tangent to 0 and going through u and v respectively.
   (a) If $C_u = C_v$ then apply B_t for some $t \in \mathbb{R}$;
   (b) Else, if u is inside C_v find an intersection point between C_v and the line $y = b$, w_1. Then apply A_s for some $s \in \mathbb{R}$ to take u to the intersection point w_1 and then apply B_t for some $t \in \mathbb{R}$ to get to v;
   (c) Else, u will be outside C_v. Find an intersection point between C_u and the line $y = d$, w_2. Then apply B_t for some $t \in \mathbb{R}$ to take u the intersection point w_2 and then apply A_s for some $s \in \mathbb{R}$.

Each of these steps can be formalised with basic calculations; examples are given in Figure 2.21. This process is not unique, but can be made unique by requiring $s, t > 0$ and requiring a maximum of one instance each of A_s and B_t in cases Items 2b and 2c.

Question 2.30. *Given two points $u, v \in \mathbb{H}^2$ can we use the action of A_s and B_t and the horocycles $\mathscr{C}_\pm$ to determine the unique earthquake that takes u to v?*

We initially conjectured the action associated with Dehn twists in $\mathrm{SL}_2(\mathbb{Z})$ would extend to an action of $\mathrm{SL}_2(\mathbb{R})$, but this does not appear to be the case. See also the work of [AHW23]. It appears likely that we need new tools to solve this question.

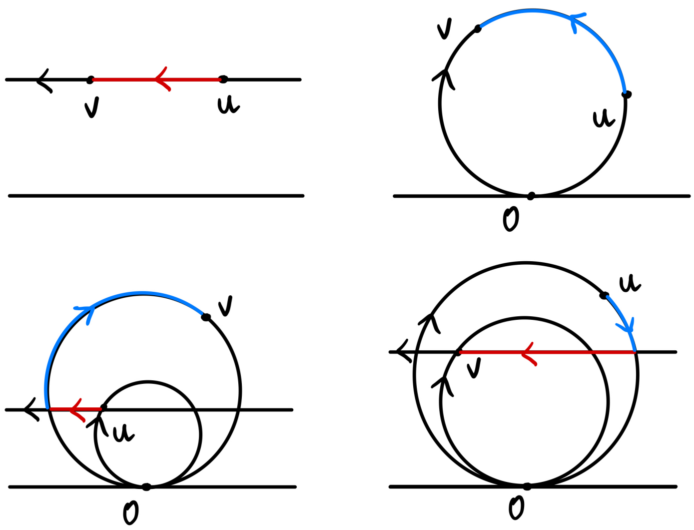

FIGURE 2.21: Example isometries between u to v on $\mathbb{H}^2$ given by the procedure outlined in Item 1 to Item 2c.

Chapter 3

Essential surfaces and varieties of characters

In this chapter, we lay the foundation for finding essential surfaces in a 3–manifold by studying their $\mathrm{SL}_2(\mathbb{F})$–variety of characters for $\mathbb{F}$ an algebraically closed field of arbitrary characteristic. We apply the extension to the *A*–polynomial, detection of the prime decomposition, and the JSJ decomposition. This is research that is in preparation [GT] in collaboration Stephan Tillmann.

Essential surfaces and character varieties were previously connected via the work of Culler and Shalen [CS83] through the geometry of character varieties, group actions on trees and ideal points, and the theory of incompressible surfaces in 3–manifolds. Here, they were really studying the $\mathrm{SL}_2(\mathbb{C})$–variety of characters through trace functions.

The theory has been incredibly influential in the study of 3–manifolds and knot theory and led to the definitions of tools like the *A*–polynomial [CCG$^+$94]. The *A*–polynomial contains information on the boundary slopes of essential surfaces in a manifold M that are detected using the theory of Culler and Shalen.

The techniques developed were used in the proofs of numerous well-known theorems, including the weak Neuwirth conjecture [CS84], the Smith conjecture [Sha99], the cyclic surgery theorem [CGLS87], and the finite surgery theorem [BZ01]. For example, the weak Neuwirth conjecture states:

Theorem 3.1 (Weak Neuwirth)**.** *Let M be a compact, orientable, irreducible* 3*–manifold whose boundary is a single torus. Then either,*

- *M is a solid torus, or*
- *M contains an essential separating annulus, or*
- *M contains an essential non–separating torus, or*
- *M has at least two boundary slopes.*

The last point means that M contains at least two essential surfaces that intersect the boundary non-trivially with different slopes. Together the result means that every non-trivial knot complement in $\mathbb{S}^3$ has a separating essential surface with non-empty boundary.

Not all essential surfaces can be detected by the $\mathrm{SL}_2(\mathbb{C})$–character variety. Much work has been done to further characterise which surfaces are detected by which ideal points (for example, see [Che13, DG12, Seg11, Til04, Til12, Yos91]). The theory has been extended to $\mathrm{SL}_\mathrm{n}(\mathbb{C})$–character varieties [HK21], and it was shown that for every essential surface in a 3–manifold there is some n with the property that the surface is detected by the $\mathrm{SL}_\mathrm{n}(\mathbb{C})$–character variety [FKN18].

The chapter proceeds as follows. We start by establishing key results for the theory of varieties of characters over arbitrary characteristic in Section 3.1 in line with previous results surveyed in Section 1.2. In particular, we prove the trace ring is generated by the trace functions of single and ordered double and triple products of generators unless the field is of characteristic 2. We show that these trace functions nevertheless give a natural coordinate system for the variety of characters in all characteristics. We determine a birational inverse from this parameterisation of the variety of characters to the trace ring in characteristic 2 using new trace identities. In studying the variety of characters, we also present a new characterisation of equations for reducible characters that holds for any characteristic.

We then show the theory of Culler and Shalen follows in Section 3.2 and Section 3.3 by finding ideal points of curves, defining valuations, constructing the Bass-Serre tree with an action of the fundamental group, and using the Stalling-Esptein-Waldhausen construction to associate a surface to the action. The ideal points of curves are defined for all algebraically closed fields and the rest follows as expected. Following Culler and Shalen, we also prove a bound on the dimension of the varieties originally due to [Thu80], with some adjustments to assumptions for characteristic 2.

We present a variety of applications to the extended theory. In Section 3.4, we use the theory to define the A–polynomial over an arbitrary algebraically closed field, for both the single and multiple boundary tori cases. We prove new versions of theorems for the standard A–polynomial in positive characteristic, including the boundary slopes theorem. We then apply the theory to the detection of the prime decomposition in Section 3.5 and JSJ decomposition in Section 3.6. The former looks at detecting splittings of free products of groups, the latter looks at detecting splittings of amalgamated products of groups. An example for prime decomposition proves the existence of essential surfaces detected in a positive characteristic that are not detected in characteristic 0.

Again, we let $\mathbb{F}$ be an algebraically closed field of arbitrary characteristic. We write $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ with the convention $\mathbb{Z}_0 = \mathbb{Z}$ and sometimes use $\mathbb{F}_p$ for an arbitrary algebraically closed field of characteristic p. In general, we may denote terms calculated in a specific characteristic with a subscript p. For brevity, we use the notation $\mathrm{R}(\Gamma, \mathbb{F}) = \mathrm{R}(\Gamma, \mathrm{SL}_2(\mathbb{F}))$ and $\mathrm{X}(\Gamma, \mathbb{F}) = \mathrm{X}(\Gamma, \mathrm{SL}_2(\mathbb{F}))$.

3.1 Character varieties in arbitrary characteristic

We start by expanding on the discussion of the $\mathrm{SL}_2(\mathbb{F})$–representation variety and variety of characters for $\mathbb{F}$ an algebraically closed field that was started in Section 1.2. We recall their definitions, provide some results for traces, and use this to describe coordinates for the variety of characters. This includes both classic results for traces and new results for traces in characteristic 2. We end with a useful result on normal forms and conjugacy classes. Recall Γ is a finitely generated group and E denotes the identity matrix in $\mathrm{SL}_2(\mathbb{F})$.

3.1.1 The variety of characters is defined by trace functions

We establish results on the $\mathrm{SL}_2(\mathbb{F})$–variety of characters that allow allows us to identify the variety of characters with the trace ring. Then the trace coordinates for $\mathrm{T}(\Gamma,\mathbb{F})$ given in Section 1.2.2 are also natural coordinates on the variety of characters $\mathrm{X}(\Gamma,\mathbb{F})$.

Identify $\mathrm{R}(\Gamma,\mathbb{F})$ with its image in $\mathrm{SL}_2(\mathbb{F})^n$. Since there is a natural epimorphism from the free group F_n in n letters to Γ and $\mathrm{R}(\Gamma,\mathbb{F}) \subset \mathrm{R}(F_n,\mathbb{F}) = \mathrm{SL}_2(\mathbb{F})^n$ is an algebraic set consisting of representations of the free group in n letters, it suffices to analyse representation and character varieties of the free group.

We know the representation variety of Γ forms an algebraic set. For the free group F_n, view $\mathrm{SL}_2(\mathbb{F})^n \subset \mathbb{F}^{4n}$ and use the Zariski topology on $\mathbb{F}^{4n}$. Embed $\mathrm{SL}_2(\mathbb{F})^n \subset \mathbb{F}^{4n}$ by setting

$$A_k = \begin{pmatrix} w_k & x_k \\ y_k & z_k \end{pmatrix}.$$

Then using coordinates $(w_1,x_1,y_1,z_1,\ldots,w_n,x_n,y_n,z_n)$ for $\mathbb{F}^{4n}$, $\mathrm{SL}_2(\mathbb{F})^n$ is a Zariski closed subset of $\mathbb{F}^{4n}$ with coordinates defined by the equations $w_k z_k - x_k y_k = 1$ for each $1 \le k \le n$.

Let $(A_1,\ldots,A_n) \in \mathrm{SL}_2(\mathbb{F})^n$ and take the $d = n + \binom{n}{2} + \binom{n}{3}$ traces

$$t_{j_1 j_2 \ldots j_m} = \mathrm{tr}(A_{j_1} \ldots A_{j_m}), \quad 1 \le j_1 < j_2 < \ldots < j_m \le n, \quad m \le 3 \tag{3.1.1}$$

from Corollary 1.12. Define

$$\mathrm{tr_n} \colon \; \mathrm{SL}_2(\mathbb{F})^n \to \mathbb{F}^d \tag{3.1.2}$$

to be the map

$$\mathrm{SL}_2(\mathbb{F})^n \ni (A_1,\ldots,A_n) \mapsto (t_{j_1 j_2 \ldots j_m})_{1 \le j_1 < j_2 < \ldots < j_m \le n,\, m \le 3} \in \mathbb{F}^d.$$

as in Section 1.2.2 for $\Gamma = F_n$. The image of the map is identified with the trace ring $\mathrm{T}(F_n,\mathbb{F})$. The indices are ordered first by length and then lexicographically. We identify the coordinate ring of $\mathbb{F}^d$ with

$$\mathbb{F}[t_1,t_2,\ldots,t_d]$$

and note that this gives natural identifications $t_{n+1} = t_{12}, \ldots, t_d = t_{n-2\,n-1\,n}$.

The group $\mathrm{SL}_2(\mathbb{F})$ acts by conjugation on n–tuples. We write $A = (A_1,\ldots,A_n) \in \mathrm{SL}_2(\mathbb{F})^n$ and the action of $Q \in \mathrm{SL}_2(\mathbb{F})$ is

$$Q \cdot A = QAQ^{-1} = (QA_1Q^{-1},\ldots,QA_nQ^{-1}).$$

Note that $\mathrm{tr_n}(Q \cdot A) = \mathrm{tr_n}(A)$ since traces are invariant under conjugation.

We saw by Lemma 1.17 and Remark 1.18 that the space of reducible representations on Γ is Zariski closed for all $\mathbb{F}$ algebraically closed using the commutator subgroup. We can prove this another way on F_n with a more explicit subset of commutators.

Proposition 3.2 (Reducible is Zariski closed)**.** $(A_1,\ldots,A_n) \in \mathrm{SL}_2(\mathbb{F})^n$ *is reducible if and only if*

1. $\mathrm{tr}[\,A_i,\, A_j\,] = 2$ *for all* $1 \le i < j \le n$*, and*
2. $\mathrm{tr}[\,A_i,\, A_jA_k\,] = 2$ *for all* $1 \le i < j < k \le n$.

In particular, the set R_n^{red} *of all reducible* $(A_1,\dots,A_n) \in \mathrm{SL}_2(\mathbb{F})^n \subset \mathbb{F}^{4n}$ *is a Zariski closed set.*

Proof. If $(A_1,\dots,A_n) \in \mathrm{SL}_2(\mathbb{F})^n$ is reducible, then the equations are clearly satisfied by the previous results. Hence suppose that $(A_1,\dots,A_n) \in \mathrm{SL}_2(\mathbb{F})^n$ is irreducible.

If there is a pair (A_i,A_j) that is irreducible, then $\mathrm{tr}[\,A_i,\,A_j\,] \neq 2$. Hence assume that each pair (A_i,A_j) is reducible but $(A_1,\dots,A_n)$ is irreducible. Then each A_i must have at least two invariant 1–dimensional subspaces.

Note that if $(A_1,\dots,A_n) \in \mathrm{SL}_2(\mathbb{F})^n$ is irreducible and $A_1 = \pm E$, then $(A_2,\dots,A_n) \in \mathrm{SL}_2(\mathbb{F})^{n-1}$ is irreducible. Hence without loss of generality, we may assume that $A_1 \neq \pm E$.

Suppose the 1–dimensional invariant subspaces of A_1 are spanned by e_1 and e_2, and, after relabelling, suppose $A_2,\dots,A_{i_0-1}$ all have invariant subspaces spanned by e_1 and e_2. Since $(A_1,\dots,A_n) \in \mathrm{SL}_2(\mathbb{F})^n$ is irreducible, we have $i_0 - 1 < n$. If all other matrices have an invariant subspace spanned by e_2, then the representation is reducible. Hence there is a matrix A_{i_0} with no invariant subspace spanned by e_2. Since A_1 and A_{i_0} have a common invariant subspace, this is spanned by e_1.

Let $A_{i_0},\dots,A_{j_0-1}$ be the matrices with invariant subspaces spanned by e_1 but not e_2. Again, $j_0 - 1 < n$, since otherwise $(A_1,\dots,A_n) \in \mathrm{SL}_2(\mathbb{F})^n$ is reducible. Hence let $A_{j_0},\dots,A_n$ be those with invariant subspaces spanned by e_2 but not e_1. In particular, $A_k \neq \pm E$ for all $i_0 \leq k \leq n$.

With respect to the basis (e_1,e_2), we have

$$A_1 = \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix} \quad \text{and} \quad A_{i_0} = \begin{pmatrix} s & 1 \\ 0 & s^{-1} \end{pmatrix} \quad \text{and} \quad A_{j_0} = \begin{pmatrix} t & 0 \\ u & t^{-1} \end{pmatrix} \tag{3.1.3}$$

where $u \neq 0$, $v^2 \neq 1$, $s^2 \neq 1$, $t^2 \neq 1$ and $1 < i_0 < j_0$. The condition on A_{i_0} and A_{j_0} having a common invariant 1–dimensional subspace implies

$$u = (s^{-1} - s)(t - t^{-1}).$$

We compute $\mathrm{tr}[\,A_1,\,A_{i_0}A_{j_0}\,] - 2 = (v - v^{-1})^2(1 - s^{-2})(1 - t^{-2}) \neq 0$. This completes the proof. $\square$

We can also better understand irreducible representations and their the image under $\mathrm{tr_n}$.

Proposition 3.3 (Irreducible uniquely determined by traces of at most three)**.** *Suppose* $A = (A_1,\dots,A_n) \in \mathrm{SL}_2(\mathbb{F})^n$ *is irreducible. Then* A *is determined uniquely up to conjugation by* $\mathrm{tr_n}(A)$*. Hence the orbit of* A *under the conjugation action is the Zariski closed set*

$$\mathrm{SL}_2(\mathbb{F}) \cdot A = \{B \in \mathrm{SL}_2(\mathbb{F})^n \mid \mathrm{tr_n}(B) = \mathrm{tr_n}(A)\} \subset \mathbb{F}^{4n}.$$

Proof. The hypothesis implies $n \geq 2$. First suppose that there are two matrices which have no common invariant 1–dimensional subspace. We can identify such a pair A, B amongst the matrices $A_1,\dots,A_n$ using the given traces since they satisfy

$$2 \neq \mathrm{tr}[\,A,\,B\,] = (\mathrm{tr}(A))^2 + (\mathrm{tr}(B))^2 + (\mathrm{tr}(AB))^2 - \mathrm{tr}(A)\,\mathrm{tr}(B)\,\mathrm{tr}(AB) - 2$$

according to Lemma 1.15, Equation (1.2.7), and noting $\mathrm{tr}\,AB = \mathrm{tr}\,BA$.

First assume that $A = A_1$ and $B = A_2$. Up to conjugation, they are of the form

$$A_1 = \begin{pmatrix} s_1 & 1 \\ 0 & s_1^{-1} \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} s_2 & 0 \\ u_2 & s_2^{-1} \end{pmatrix}$$

with $u_2 \neq 0$. The equations $t_1 = \operatorname{tr}(A_1) = s + s^{-1}$ and $t_2 = \operatorname{tr}(A_2) = t + t^{-1}$ have up to four distinct pairs of solutions (s,t), (s,t^{-1}), (s^{-1},t), (s^{-1},t^{-1}). The pair (s_1,s_2) equals one of these four pairs, and for each such choice, u_2 is uniquely determined by $t_{12} = \operatorname{tr}(A_1A_2) = st + u_2 + s^{-1}t^{-1}$. The triple (t_1,t_2,t_{12}) therefore determines up to four triples (s_1,s_2,u_2):

$$\begin{aligned}(s_1,s_2,u_2) \in \{\, &(s,t,t_{12} - st - s^{-1}t^{-1}),\ (s,t^{-1},t_{12} - st^{-1} - s^{-1}t), \\ &(s^{-1},t,t_{12} - st^{-1} - s^{-1}t),\ (s^{-1},t^{-1},t_{12} - st - s^{-1}t^{-1}) \,\}.\end{aligned}$$

Any two of these four triples give conjugate pairs of matrices. This proves the lemma for the case $n = 2$.

Suppose $n \geq 3$, and that we have chosen $(s_1,s_2,u_2) = (s,t,t_{12} - st - s^{-1}t^{-1})$ in the above. This fixes the pair (A_1,A_2) in its conjugacy class. We now need to show that the entries of all other matrices $A_3,\ldots,A_n$ are uniquely determined by the set of traces in Equation (3.1.1).

Let $3 \leq k \leq n$. The four entries of $A_k = \begin{pmatrix} w_k & x_k \\ y_k & z_k \end{pmatrix}$ satisfy a quadratic equation from $\det A_k = 1$, and four linear equations from $\operatorname{tr}(A_k) = t_k$, $\operatorname{tr}(A_1A_k) = t_{1k}$, $\operatorname{tr}(A_2A_k) = t_{2k}$, and $\operatorname{tr}(A_1A_2A_k) = t_{12k}$. The linear equations give:

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ s & 0 & 1 & s^{-1} \\ t & u & 0 & t^{-1} \\ st+u & s^{-1}u & t^{-1} & s^{-1}t^{-1} \end{pmatrix} \begin{pmatrix} w_k \\ x_k \\ y_k \\ z_k \end{pmatrix} = \begin{pmatrix} t_k \\ t_{1k} \\ t_{2k} \\ t_{12k} \end{pmatrix}.$$

This system of linear equations has a unique solution if and only if

$$u(u - (s^{-1} - s)(t - t^{-1})) \neq 0.$$

This is equivalent to $\operatorname{tr}(A_1A_2A_1^{-1}A_2^{-1}) \neq 2$.

Hence if A_1 and A_2 have no common invariant 1–dimensional subspace and $n \geq 3$, then $A_1,\ldots,A_n \in \mathrm{SL}_2(\mathbb{F})$ are uniquely determined up to conjugation by the $n + (n-1) + (n-2) + (n-3) = 4n-6$ values

$$\operatorname{tr}(A_1),\ \operatorname{tr}(A_2),\ \operatorname{tr}(A_k),\ \operatorname{tr}(A_1A_2),\ \operatorname{tr}(A_1A_k),\ \operatorname{tr}(A_2A_k),\ \operatorname{tr}(A_1A_2A_k) \quad 3 \leq k < n.$$

This is clearly a subset of the traces given in Equation (3.1.1).

Now given the pair of matrices A,B with no common invariant 1–dimensional subspace, if $A = A_i$ and $B = A_j$, we may assume $i < j$ and determine the entries as above from the values of

$$\operatorname{tr}(A_k),\ \operatorname{tr}(A_iA_k),\ \operatorname{tr}(A_jA_k),\ \operatorname{tr}(A_iA_jA_k) \quad 1 \leq k < n,\ k \notin \{i,j\}.$$

It follows from Equations (1.2.4) and (1.2.6) that all of these traces are polynomials in integral coefficients in the traces given in Equation (3.1.1). This completes the proof of the lemma if there are at least two matrices without a common invariant 1–dimensional subspace.

Suppose any two matrices have a common invariant 1–dimensional subspace, but there is no such subspace invariant under all matrices. This is exactly the set-up in the second half of the proof of Proposition 3.2. Now Lemma 1.16 and Proposition 3.2 imply that amongst $A_1,\dots,A_n$ there are three matrices A, B and C with the property that $\operatorname{tr}[\,A,\,BC\,] \neq 2$. Again suppose first that $A = A_1$, $B = A_2$ and $C = A_3$. We make the Ansatz

$$A_1 = \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} s & 1 \\ 0 & s^{-1} \end{pmatrix} \quad \text{and} \quad A_3 = \begin{pmatrix} t & 0 \\ u & t^{-1} \end{pmatrix}$$

with $t_1 = \operatorname{tr}(A_1)$, $t_2 = \operatorname{tr}(A_2)$, $t_3 = \operatorname{tr}(A_3)$, $t_{12} = \operatorname{tr}(A_1A_2)$, $t_{13} = \operatorname{tr}(A_1A_3)$, $t_{23} = \operatorname{tr}(A_2A_3)$, $t_{123} = \operatorname{tr}(A_1A_2A_3)$. Here, the left hand sides are the given values for the traces, and the right hand sides are given in terms of the indeterminants v,s,t,u. We need to show that we can express them uniquely in terms of the values of the traces.

As in the proof of Proposition 3.2, $\operatorname{tr}[\,A_2,\,A_3\,] = 2$ implies $u = (s^{-1}-s)(t-t^{-1})$, and then

$$0 \neq \operatorname{tr}[\,A_1,\,A_2A_3\,] - 2 = (v-v^{-1})^2(1-s^{-2})(1-t^{-2}).$$

This implies $v^2 \neq 1$, $s^2 \neq 1$ and $t^2 \neq 1$.

Combining $t_{12} = \operatorname{tr}A_1A_2$ and $t_2 = \operatorname{tr}A_2$ gives

$$s = \frac{v-v^{-1}}{t_2v - t_{12}}.$$

Combining $t_{13} = \operatorname{tr}A_1A_3$ and $t_3 = \operatorname{tr}A_3$ gives

$$t = \frac{v-v^{-1}}{t_3v - t_{13}}.$$

In particular, all entries of A_1 and A_2 are uniquely determined by the traces $t_2, t_3, t_{12}, t_{13}, t_{23}$ and choosing as upper left entry for A_1 a zero v of $t_1 = x + x^{-1}$. We need to identify the correct zero. Note that repeated application of $v^2 = t_1v - 1$ allows us to express all higher powers of v as linear polynomials with coefficients in $\mathbb{Z}[t_1]$. Hence we obtain $0 = t_{123} - \operatorname{tr}(A_1A_2A_3)$ if and only if $0 = f_1 - f_2v$, where $f_1, f_2 \in \mathbb{Z}[t_1,t_2,t_3,t_{12},t_{13},t_{23},t_{123}]$. This uniquely determines v unless $f_1 = 0 = f_2$. In terms of our indeterminants, we have

$$f_1 = v^{-5}(s^2-1)(t^2-1)(v^2-1)^4 \neq 0.$$

This proves the claim for A_1,A_2,A_3.

For every A_k where $k \geq 4$, we again let $A_k = \begin{pmatrix} w_k & x_k \\ y_k & z_k \end{pmatrix}$ and consider the four equations arising from $\operatorname{tr}A_k = t_k$, $\operatorname{tr}A_1A_k = t_{1k}$, $\operatorname{tr}A_2A_k = t_{2k}$, $\operatorname{tr}A_3A_k = t_{3k}$. As above this system of linear equations in the matrix entries,

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ v & 0 & 0 & v^{-1} \\ s & 0 & 1 & s^{-1} \\ t & u & 0 & t^{-1} \end{pmatrix} \begin{pmatrix} w_k \\ x_k \\ y_k \\ z_k \end{pmatrix} = \begin{pmatrix} t_k \\ t_{1k} \\ t_{2k} \\ t_{12k} \end{pmatrix}$$

has a unique solution under our hypotheses on the eigenvalues.

Hence the representative in the conjugacy class is uniquely determined by (a subset of) the given traces. The result in the general case (where we find three matrices amongst the generators satisfying the required inequality on the trace of a commutator) now again follows by application of trace identities.

For the second statement note that Proposition 3.2 implies that $\mathrm{tr_n}(A)$ is not the image of a reducible n–tuple. This proves the result. □

We can elaborate on some of these steps.

Remark 3.4. *It is a pleasant exercise to explicitly determine the n–tuple in the second part of the proof. Up to conjugation, we may assume that all of $A_1,\dots,A_{i_0-1}$ are diagonal (with e_1 representing the first standard basis vector) and such that $A_1^2 \neq E$, all of $A_{i_0},\dots,A_{j_0-1}$ are upper triangular (but not diagonal), and all of $A_{j_0},\dots,A_n$ are lower triangular (but not diagonal). We further fix $(A_1,\dots,A_n)$ in its conjugacy class by stipulating that the off-diagonal entry of A_{i_0} equals one. This completely fixes the matrix entries in $(A_1,\dots,A_n)$ without ambiguity.*

We now show that all matrix entries are uniquely determined by the values of the traces given in Equation (3.1.1). Write

$$A_k = \begin{pmatrix} s_k & 0 \\ 0 & s_k^{-1} \end{pmatrix} \quad \text{and} \quad A_i = \begin{pmatrix} s_i & w_i \\ 0 & s_i^{-1} \end{pmatrix} \quad \text{and} \quad A_j = \begin{pmatrix} s_j & 0 \\ u_j & s_j^{-1} \end{pmatrix}$$

where $1 \le k \le i_0-1$, $i_0 \le i \le j_0-1$, $j_0 \le j \le n$, and $w_i \neq 0 \neq u_j$ and $w_{i_0}=1$, $s_1^2 \neq 1$, $s_i^2 \neq 1$, $s_j^2 \neq 1$. The condition on A_{i_0} and A_j having a common invariant 1–dimensional subspace implies for all $j_0 \le j \le n$,

$$u_j = (s_{i_0}^{-1} - s_{i_0})(s_j - s_j^{-1}).$$

The condition on A_{j_0} and A_i having a common invariant 1–dimensional subspace implies for all $i_0 \le i \le j_0-1$,

$$w_i = \frac{(s_i^{-1}-s_i)(s_{j_0}-s_{j_0}^{-1})}{u_{j_0}} = \frac{(s_i^{-1}-s_i)(s_{j_0}-s_{j_0}^{-1})}{(s_{i_0}^{-1}-s_{i_0})(s_{j_0}-s_{j_0}^{-1})} = \frac{s_i^{-1}-s_i}{s_{i_0}^{-1}-s_{i_0}}.$$

This agrees with $w_{i_0}=1$. In particular, the n–tuple $(A_1,\dots,A_n)$ is completely determined by the values and position of the eigenvalues in the matrices.

The orbit space under the conjugation action $\mathrm{SL}_2(\mathbb{F})^n / \mathrm{SL}_2(\mathbb{F})$ is the character stack of the free group in n letters (see Definition 1.2). We write $[A] = \mathrm{SL}_2(\mathbb{F}) \cdot A \in \mathrm{SL}_2(\mathbb{F})^n / \mathrm{SL}_2(\mathbb{F})$ for each orbit. The map $\mathrm{tr_n}\colon\ \mathrm{SL}_2(\mathbb{F})^n \to \mathbb{F}^d$ is constant on conjugacy classes of n–tuples, and hence factors through the character stack. Let

$$\mathrm{q_n}\colon\ \mathrm{SL}_2(\mathbb{F})^n / \mathrm{SL}_2(\mathbb{F}) \to \mathbb{F}^d$$

denote the intermediate map taking conjugacy classes of n–tuples of matrices to the $d = n + \binom{n}{2} + \binom{n}{3}$ traces in Equation (3.1.1).

It follows from Proposition 3.3, Lemma 1.15 and Equation (1.2.7) that for each $x \in \mathbb{F}^d$, the preimage ${\mathrm{q_n}}^{-1}(x)$ contains either reducible or irreducible conjugacy classes, but not both. Proposition 3.3 implies that $\mathrm{q_n}$ is injective on orbits of irreducible n–tuples. We record this as:

Corollary 3.5 (Irreducible orbit is Zariski closed)**.** *Suppose* $[A],[B] \in \mathrm{SL}_2(\mathbb{F})^n/\mathrm{SL}_2(\mathbb{F})$ *with* $[A]$ *irreducible. If* $\mathrm{q_n}([A]) = \mathrm{q_n}([B])$*, then* $[A] = [B]$*. In particular,* $\mathrm{SL}_2(\mathbb{F})^n \cdot (A_1,\dots,A_n) \subset \mathrm{SL}_2(\mathbb{F})^n$ *is a Zariski closed set, and two such orbits are either disjoint or equal.*

We note that for reducible $[A] \in \mathrm{SL}_2(\mathbb{F})^n/\mathrm{SL}_2(\mathbb{F})$, the preimage of $\mathrm{q_n}([A]) \in \mathbb{F}^d$ may contain infinitely many orbits in $\mathrm{SL}_2(\mathbb{F})^n/\mathrm{SL}_2(\mathbb{F})$.

Let $[A]_c$ be the equivalence class of $A \in \mathrm{SL}_2(\mathbb{F})^n$ in $\mathrm{X}(F_n, \mathrm{SL}_2(\mathbb{F}))$. We also get:

Corollary 3.6 (Irreducible orbit closures)**.** *Let* $A \in \mathrm{SL}_2(\mathbb{F})^n$ *be irreducible. Then* $[A]_c = [A] = \mathrm{tr_n}^{-1}(\mathrm{tr_n}(A))$.

Proof. According to Corollary 3.5, $[A]$ is Zariski closed and if B is irreducible, then either $[A] \cap [B] = \emptyset$ or $[A] = [B]$. Now if B is reducible, then the orbit of B is contained in the Zariski closed set R_n^{red} from Proposition 3.2. Hence the closure of the orbit of B is contained in R_n^{red}. But $[A] \cap R_n^{red} = \emptyset$. Hence $[A]_c$ only contains the orbit of A. □

We can similarly look at reducible orbit closures.

Lemma 3.7 (Reducible orbit closures)**.** *Suppose* $A = (A_1,\dots,A_n) \in \mathrm{SL}_2(\mathbb{F})^n$ *is reducible. Then there are diagonal matrices* $D_1,\dots,D_n \in \mathrm{SL}_2(\mathbb{F})$ *such that* $\mathrm{tr_n}(A) = \mathrm{tr_n}(D)$*, where* $D = (D_1,\dots,D_n)$*. Moreover,* $D \in [A]_c$ *and* $D^{-1} = (D_1^{-1},\dots,D_n^{-1}) \in [A]_c$*. In particular,* $[A]_c = \mathrm{tr_n}^{-1}(\mathrm{tr_n}(A))$.

Proof. If $A = (A_1,\dots,A_n) \in \mathrm{SL}_2(\mathbb{F})^n$ is reducible, then we may conjugate so that each A_k is upper triangular,

$$A_k = \begin{pmatrix} s_k & u_k \\ 0 & s_k^{-1} \end{pmatrix}, \quad s_k \in \mathbb{F}\setminus\{0\},\ u_k \in \mathbb{F}.$$

Define

$$D_k = \begin{pmatrix} s_k & 0 \\ 0 & s_k^{-1} \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Q_u = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}, \quad u \in \mathbb{F}\setminus\{0\}.$$

Let $D = (D_1,\dots,D_n)$. Then $\mathrm{tr_n}(A) = \mathrm{tr_n}(D)$.

We will show $D \in [A]_c$ and $D^{-1} \in [A]_c$. Now $D \in [A]$ if and only if $D^{-1} = QDQ^{-1} \in [A]$, in which case there is nothing to show. Hence suppose $D \notin [A]$.

Now

$$Q_u A_k Q_u^{-1} = A_k = \begin{pmatrix} s_k & u_k u^2 \\ 0 & s_k^{-1} \end{pmatrix} \tag{3.1.4}$$

for every $u \in \mathbb{F}\setminus\{0\}$.

Since $D \notin [A]$, there is an index i_0 with $u_{i_0} \neq 0$. Given the entries s_k, u_k, consider the algebraic subset X of $\mathbb{F}^{4n}$ with variables w_k, x_k, y_k, z_k defined by the equations

$$w_k = s_k,\ u_{i_0} x_k = u_k x_{i_0},\ y_k = 0,\ z_k = s_k^{-1} \quad \text{for all } 1 \le k \le n.$$

These represent entries of some matrix that solves Equation (3.1.4). Then all variables are specified except for x_{i_0} and $X \cong \mathbb{F}$. We also observe that $X = \{Q_u A_k Q_u^{-1} \mid u \in \mathbb{F}\setminus\{0\}\} \cup \{D\}$. This shows that the Zariski closure of $\{Q_u A_k Q_u^{-1} \mid u \in \mathbb{F} \setminus \{0\}\}$ contains $\{D\}$. Since the Zariski closure of a set contains the Zariski closure of all of its subsets, this implies that the Zariski closure of $[A]$ contains D. Hence $D \in [A]_c$. Conjugation by Q shows $D^{-1} \in [A]_c$.

For the last part, we need to show that D and D^{-1} are the only n–tuples of diagonal matrices in $\mathrm{tr_n}^{-1}(\mathrm{tr_n}(A))$. Suppose $\mathrm{tr_n}(A) = (t_1, \ldots, t_{n-2\,n-1\,n})$. We show that there are at most two n–tuples of diagonal matrices with the specified traces.

If $t_k = \varepsilon_k 2$, where $\varepsilon \in \{\pm 1\}$, then let $D_k = \varepsilon_k E$. Note that $D_k = D_k^{-1}$ in this case. If each $t_k = \varepsilon_k 2$, then there is no ambiguity in the choices and there is exactly one n–tuple of diagonal matrices with the specified traces.

Suppose t_i is the first coordinate not equal to ± 2. Then $t_i = x + x^{-1}$ has two distinct roots. Choose a root d_i of $t_i = x + x^{-1}$ to be the upper left entry of D_i. Let t_j be the next coordinate not equal to ± 2. Then $t_j = x + x^{-1}$ has two distinct roots, and we need to choose the upper left entry d_j of D_j amongst them. We also have the requirement on the upper left entry d_j of D_j to satisfy $t_{ij} = d_i d_j + d_i^{-1} d_j^{-1}$. If this is satisfied for both choices of roots, then $d_i d_j + d_i^{-1} d_j^{-1} = d_i d_j^{-1} + d_i^{-1} d_j$, which is equivalent to $(d_i - d_i^{-1})(d_j - d_j^{-1}) = 0$, which contradicts $t_i^2 \neq 4 \neq t_j^2$. Hence the root d_j of $t_j = s_j + s_j^{-1}$ is uniquely determined by $t_{ij} = d_i d_j + d_i^{-1} d_j^{-1}$.

Inductively, each matrix D_k, $i < k \leq n$ is uniquely determined by the first choice of upper left entry in D_i and the values of t_k and t_{ik} or it is determined by $t_k = \pm 2$. It also follows that making a different choice for the upper left entry of D_i results in inverses of all D_k. This completes the proof. □

Recall for each $\rho \in \mathrm{R}(F_n, \mathbb{F})$, we have the trace function $\tau_\rho \colon F_n \to \mathbb{F}$. Corollary 3.6 and Lemma 3.7 imply:

Theorem 3.8 (Traces are isomorphic to characters)**.** *The closure-equivalence classes in* $\mathrm{R}(F_n, \mathbb{F}) = \mathrm{SL}_2(\mathbb{F})^n$ *are precisely the fibres of* $\mathrm{tr_n} \colon \mathrm{SL}_2(\mathbb{F})^n \to \mathbb{F}^d$. *Hence the image of* $\mathrm{tr_n}$ *is naturally isomorphic with* $\mathrm{X}(F_n, \mathbb{F})$ *and*

$$\mathrm{X}(F_n, \mathbb{F}) \cong \mathrm{T}(F_n, \mathbb{F}).$$

We get the following corollary.

Corollary 3.9. *Suppose* $\rho, \sigma \in \mathrm{R}(F_n, \mathbb{F})$. *Then the following four statements are equivalent:*

1. $\tau_\rho = \tau_\sigma$;
2. τ_ρ *and* τ_σ *agree on all ordered products of distinct generators;*
3. τ_ρ *and* τ_σ *agree on all ordered single, double and triple products of distinct generators;*
4. ρ *and* σ *are closure-equivalent.*

Proof. We first note that the equivalence of (1) and (2) is the content of Lemma 1.11. Now (2) implies (3), since we restrict to a smaller set. The equivalence of (3) and (4) is the content of Theorem 3.8. Finally, (4) implies (1) since every trace function is an algebraic function on $\mathrm{R}(F_n, \mathbb{F})$. □

From now on, we will often identify $\mathrm{X}(\Gamma,\mathbb{F})$ with its image in $\mathbb{F}^d$, which in turn is identified with the trace ring $\mathrm{T}(\Gamma,\mathbb{F})$. Then the long trace coordinates and short trace coordinates on the trace ring $\mathrm{T}(\Gamma,\mathbb{F})$ from Section 1.2.2 provide coordinates for the variety of characters $\mathrm{X}(\Gamma,\mathbb{F})$ and we get the equivalent commutative diagram with $\mathrm{tr_n} = r \circ \mathrm{Tr_n}$:

$$\begin{tikzcd}
\mathrm{R}(\Gamma,\mathrm{SL}_2(\mathbb{F})) \arrow[r] \arrow[d, "\varphi"'] & \mathrm{X}(\Gamma,\mathbb{F}) \arrow[d, "\varphi_n"'] \arrow[dr, "\varphi_d"] & \\
\mathrm{SL}_2(\mathbb{F})^n \arrow[r, "\mathrm{Tr_n}"] & \mathbb{F}^{2^n-1} \arrow[r, "r"] & \mathbb{F}^d
\end{tikzcd}$$

Reducible characters form a Zariski closed set since they are the image of a Zariski closed set (see previous results Lemma 1.17 and Remark 1.18 or Proposition 3.2). Specific equations are given in Corollary 3.13, Section 3.1.3. We denote $\mathrm{X}^{\mathrm{red}}(\Gamma,\mathbb{F})$ the union of all reducible characters. The union of irreducible characters may not be Zariski closed. We denote $\mathrm{X}^{\mathrm{irr}}(\Gamma,\mathbb{F})$ the Zariski closure of the union of all irreducible characters.

3.1.2 Traces in characteristic 2

We saw trace identities in Section 1.2.1 that hold over $\mathbb{F}$ an algebraically closed field of characteristic p. Let $\mathbb{F} = \mathbb{F}_2$ an algebraically closed field of characteristic 2. Here we introduce extra trace identities in characteristic 2 and use them to determine the birational inverse for $r\colon \varphi_n(\mathrm{X}(\Gamma,\mathbb{F}_2)) \to \varphi_d(\mathrm{X}(\Gamma,\mathbb{F}_2))$.

Note that in all previously established identities, we have $2 = 0$. In particular, Equation (1.2.10) reads:

$$\begin{aligned}
0 = &\operatorname{tr}(AC)\operatorname{tr}B\operatorname{tr}D + \operatorname{tr}A\operatorname{tr}B\operatorname{tr}(CD) + \operatorname{tr}B\operatorname{tr}C\operatorname{tr}(DA)\\
&+\operatorname{tr}(AC)\operatorname{tr}(BD) + \operatorname{tr}(AB)\operatorname{tr}(CD) + \operatorname{tr}(BC)\operatorname{tr}(DA)\\
&+\operatorname{tr}(ACB)\operatorname{tr}D + \operatorname{tr}A\operatorname{tr}(BCD) + \operatorname{tr}B\operatorname{tr}(CDA) + \operatorname{tr}C\operatorname{tr}(DAB).
\end{aligned} \tag{3.1.5}$$

As before, write $t_{i_0\dots i_m} = \operatorname{tr}(A_{i_0}\cdots A_{i_m})$. We also use the notation $t_A = \operatorname{tr}(A)$ for all $A \in \mathrm{SL}_2(\mathbb{F}_2)$.

Lemma 3.10. *Let $A,B,C,D,E \in \mathrm{SL}_2(\mathbb{F}_2)$. Then*

$$\begin{aligned}
t_At_{BCDE} = &(t_{BD}+t_Bt_D)(t_{ACE}+t_Ct_{AE}) + (t_{CE}+t_Ct_E)(t_{BAD}+t_Bt_{AD})\\
&+t_{BC}t_{ADE} + t_{BE}(t_{ACD}+t_Ct_{AD}+t_Dt_{AC}+t_At_Ct_D)\\
&+t_{CD}(t_{BAE}+t_Bt_{AE}+t_Et_{AB}+t_At_Bt_E) + t_{DE}(t_{BAC}+t_Bt_{AC}+t_Ct_{AB}+t_At_Bt_C).
\end{aligned} \tag{3.1.6}$$

In particular,

$$\begin{aligned}
t_At_{ABCD} = &(t_{AC}+t_At_C)t_{ABD} + t_{AB}t_{ACD} + t_Ct_{AB}t_{AD} + t_{AD}t_{ABC}\\
&+t_Bt_Ct_D + t_Bt_{CD} + t_Ct_{BD} + t_Dt_{BC}.
\end{aligned} \tag{3.1.7}$$

Proof. The first equation can be verified via direct computation in characteristic 2.

The second follows from the first because, in characteristic 2, for all $X,Y \in \mathrm{SL}_2(\mathbb{F}_2)$ we have $t_{XX} = t_X^2$ and $t_{XXY} + t_Xt_{XY} = t_Y$ using Equations (1.2.5) and (1.2.9) respectively. □

It follows from Lemma 3.10 and induction that if $Z \subseteq \mathrm{X}(\Gamma, \mathbb{F}_2)$ is a component with the property that for some generator γ_i of Γ the trace function t_i is not in the ideal defining $\varphi_d(Z)$, then we can define a birational inverse of the projection map when restricted to $\mathbb{F}_2^{2^n-1} \supset \varphi_n(Z) \to \varphi_d(Z) \subset \mathbb{F}_2^d$ of the form

$$f = (f_1, \dots, f_{2^n-1})\colon\ \mathbb{F}_2^d \to \mathbb{F}_2^{2^n-1}$$

with $f_k = t_k$ for all $1 \le k \le d$, and for all $k > d$ we have $f_k \in \mathbb{F}_2[t_1, \dots, t_{i-1}, t_i^{\pm 1}, t_{i+1}, \dots t_d]$. In particular, if Γ is the free group, then we may take $i = 1$ and apply f to all of $\varphi_d(\mathrm{X}(\Gamma, \mathbb{F}_2))$.

Lemma 3.11. *Let $Z \subseteq \mathrm{T}(\Gamma, \mathbb{F}_2)$ be a component with the property that for each generator γ_i, the trace function t_i vanishes on Z. If all trace functions t_{ij} also vanish on Z, then Z only contains the character of the trivial representation. In particular, Z is mapped to the origin in $\mathbb{F}_2^d$.*

Proof. Suppose ρ is a non-trivial representation with character $\chi \in Z$. Then some generator, say γ_i has non-trivial image and since $t_i(\chi) = 0$, it is parabolic and hence has a unique invariant subspace. Now for every $1 \le j \le n$, we have $t_j(\chi) = t_{ij}(\chi) = 0$. By computation modulo 2, it follows that for every $1 \le j, k \le n$,, $t_{ijk} = 0$. Equations (1.2.7) and (1.2.8) and Proposition 3.2 imply that the image under ρ of the subgroup generated by γ_i and γ_j is reducible. Since $\rho(\gamma_j)$ is either parabolic or trivial, this implies that it also leaves the unique 1–dimensional invariant subspace of $\rho(\gamma_i)$ invariant. Hence ρ is reducible and conjugate into the upper triangular group. Since all generators have trace zero, they are either parabolic or the identity. Hence ρ has the same character as the trivial representation. □

We get the following immediate corollary.

Corollary 3.12. *Let $Z \subseteq \mathrm{X}(\Gamma, \mathbb{F}_2)$ be a component that contains at least two (and hence infinitely many) characters. Then at least one of the coordinate functions t_i ($1 \le i \le n$) or t_{ij} ($1 \le i < j \le n$) does not vanish on all of $\varphi_d(Z)$.*

We are now in a position to describe the birational inverse to the projection map in the case where all coordinate functions t_i, $1 \le i \le n$, vanish on an arbitrary component Z. If $\varphi_d(Z) = \{0\} \subset \mathbb{F}_2^d$, then this is mapped to the origin in $\mathbb{F}_2^{2^n-1}$. Otherwise at least one of the functions t_{ij} does not vanish on all of $\varphi_d(Z)$.

Consider $t_{j_1 j_2 \dots j_m}$ where $m \ge 4$. Recall $A_i \in \mathrm{SL}_2(\mathbb{F}_2)$ is the image of the generator γ_i under a representation and $t_i \in \mathbb{F}_2$ is its trace for all $1 \le i \le n$. Suppose for some pair of indices j_a, j_b satisfying $j_1 \le j_a < j_b \le j_m$, the function $t_{j_a j_b}$ does not vanish on all of $\varphi_d(Z)$. If $j_b < j_m$ we apply Equation (3.1.5) with $A = A_{j_1} A_{j_2} \dots A_{j_{m-1}}$, $B = A_{j_m}$, $C = A_{j_a}$ and $D = A_{j_b}$. Since the traces of generators vanish, we obtain the equality on $\varphi_d(Z)$:

$$\operatorname{tr}(CD)\operatorname{tr}(AB) = \operatorname{tr}(AC)\operatorname{tr}(BD) + \operatorname{tr}(AD)\operatorname{tr}(BC) + \operatorname{tr} A \operatorname{tr}(BCD).$$

We note that on $\varphi_d(Z)$, Equation (1.2.6) implies that we may permute images of generators without changing traces, and applying Equation (1.2.9) gives $t_Y = t_{A_k A_k Y} + t_{A_k} t_{A_k Y} = t_{A_k A_k Y}$, for all $1 \le k \le n$ and all $Y \in \mathrm{SL}_2(\mathbb{F}_2)$. This implies that adding or removing even powers of generators does not change the trace. Substituting and simplifying gives:

$$t_{j_a j_b} t_{j_1 j_2 \dots j_m} = t_{j_1 \dots \hat{j}_a \dots j_{m-1}} t_{j_b j_m} + t_{j_1 \dots \hat{j}_b \dots j_{m-1}} t_{j_a j_m} + t_{j_1 j_2 \dots j_{m-1}} t_{j_a j_b j_m} \tag{3.1.8}$$

where the hat means that an index does not appear.

Hence $t_{j_1 j_2 \cdots j_m}$ can be expressed as a rational function in shorter traces and with denominator involving only $t_{j_a j_b}$. If $j_b = j_m$, we use $A = A_{j_m} A_{j_1} A_{j_2} \dots A_{j_{m-2}}$ and $B = A_{j_{m-1}}$ instead, and get a similar expression to Equation (3.1.8).

Suppose for every pair of indices $j_a < j_b$ the function $t_{j_a j_b}$ vanishes on the whole component $\varphi_d(Z)$. Then Lemma 3.11 implies that the character restricted to the subgroup generated by $A_{j_1}, A_{j_2}, \dots, A_{j_m}$ has the same character as the trivial representation and hence all trace functions with indices in this subset vanish identically on $\varphi_d(Z)$. Hence the birational inverse is identically zero on the corresponding coordinates in $\mathbb{F}_2^{2^n-1}$.

Combining the results from this section we can define the birational inverse. If there exists a generator γ_i $1 \le i \le n$ with t_i nonzero on $\varphi_d(Z)$, then we can use Equation (3.1.7) and invert t_i to define the birational inverse for $i = A$. If, for all generators γ_i, $1 \le i \le n$, t_i vanishes on $\varphi_d(Z)$ and there exists $1 \le k < l \le n$ such that t_{kl} is nonzero on $\varphi_d(Z)$, then we can use Equation (3.1.8) and invert t_{kl} to define the birational inverse with $k = j_a, l = j_b$. If, for all generators γ_i, γ_j, $1 \le i \le j \le n$, t_i and t_{ij} vanish on $\varphi_d(Z)$, then the birational inverse is zero on the corresponding coordinates in $\mathbb{F}_2^{2^n-1}$.

3.1.3 Equations for reducible characters

A method to derive equations for the character variety for $\mathbb{F} = \mathbb{C}$ is given in [GM93]. Here, we describe equations for the reducible characters of the free group. The difference with the arguments in the proof of Lemma 3.7 is that there we knew that a point in $\mathbb{F}^d$ is the image of a reducible n–tuple under $\mathrm{tr_n}$. The situation now is that we specify points in $\mathbb{F}^d$ using some equations and then need to show that a suitable n–tuple with the specified image exists. A priori, the set we specify may be too large.

Corollary 3.13. *For $n \ge 3$, the image under $\mathrm{tr_n}\colon\ \mathrm{SL}_2(\mathbb{F})^n \to \mathbb{F}^d$ of the set of all reducible n–tuples is the Zariski closed set*

$$
\begin{aligned}
X_n^{red} = \{(\, t_1, \dots, t_{n-2\,n-1\,n}) \in \mathbb{F}^d \mid \\
&(1) \quad t_i^2 + t_j^2 + t_{ij}^2 - t_i t_j t_{ij} = 4 \ \textit{for all } 1 \le i < j \le n, \\
&(2) \quad t_{ij}^2 + t_{ik}^2 + t_{jk}^2 - 2 t_j t_{jk} t_k + t_j^2 t_k^2 + t_{ij} t_{ik} (t_{jk} - t_j t_k) = 4 \ \textit{for all } 1 \le i < j < k \le n \\
&(3) \quad t_{ij}^2 + t_{ik}^2 + t_{jk}^2 - 2 t_i t_{ik} t_k + t_i^2 t_k^2 + t_{ij} t_{jk} (t_{ik} - t_i t_k) = 4 \ \textit{for all } 1 \le i < j < k \le n \\
&(4) \quad t_{ij}^2 + t_{ik}^2 + t_{jk}^2 - 2 t_i t_{ij} t_j + t_i^2 t_j^2 + t_{ik} t_{jk} (t_{ij} - t_i t_j) = 4 \ \textit{for all } 1 \le i < j < k \le n \\
&(5) \quad t_i^2 + t_{jk}^2 + t_{ijk}^2 - t_i t_{jk} t_{ijk} = 4 \ \textit{for all } 1 \le i < j < k \le n \\
&(6) \quad t_j^2 + t_{ik}^2 + t_{ijk}^2 - t_j t_{ik} t_{ijk} = 4 \ \textit{for all } 1 \le i < j < k \le n \\
&(7) \quad t_k^2 + t_{ij}^2 + t_{ijk}^2 - t_k t_{ij} t_{ijk} = 4 \ \textit{for all } 1 \le i < j < k \le n \\
&(8) \quad t_i^2 + t_{jk}^2 - t_i t_{jk} (t_i t_{jk} + t_j t_{ik} + t_k t_{ij} - t_i t_j t_k - t_{ijk}) \\
&\qquad + (t_i t_{jk} + t_j t_{ik} + t_k t_{ij} - t_i t_j t_k - t_{ijk})^2 = 4 \ \textit{for all } 1 \le i < j < k \le n \\
&(9) \quad t_j^2 + t_{ijk}^2 - t_j t_{ijk} (t_{ik} + t_{ij} t_{jk} - t_i t_k) + (t_{ik} + t_{ij} t_{jk} - t_i t_k)^2 = 4 \ \textit{for all } 1 \le i < j < k \le n \}
\end{aligned}
$$

The sets of equations all arise from traces of commutators. Equation (1) arises from the first criterion of Proposition 3.2; equations (2), (3), (4) come from equations of the form $\mathrm{tr}[\,AB,\,AC\,] = 2$; equations (5), (6), and (7) arise from the second criterion of Proposition 3.2 and cyclic permutation; equation (8) comes the second criterion of Proposition 3.2 and changing the cyclic order; and equation (9) comes from equations of the form $\operatorname{tr}[\,B,\,ABC\,] = 2$. Note that the right-hand side of each equation equates to 0 in characteristic 2. In fact, the last equation (9) is only required in characteristic 2.

If $n = 1$ there is only one generator and all n–tuples are reducible. If $n = 2$ there are only two generators and we only need consider the equation (1). This is covered in Section 1.2.4.

Proof. We start without any assumptions on the characteristic. It follows from the characterisation of reducible n–tuples in Proposition 3.2 and Equations (1.2.7) and (1.2.8) that the image of each reducible n–tuple satisfies all equations and hence lies in X_n^{red}.

Let $\chi = (\,t_1, \dots,\, t_{n-2\,n-1\,n}) \in X_n^{red}$. We need to show there is a reducible n–tuple such that $\operatorname{tr}_n(D_1, \dots, D_n) = \chi$. We construct the n–tuple to be diagonal.

Without loss of generality, we can rearrange the terms so there exists i_0, $1 \leq i_0 \leq n+1$ such that $t_m^2 = 4$ for all $m \in \{1, \dots, i_0 - 1\}$ and $t_m^2 \neq 4$ for all $m \in \{i_0, \dots, n\}$. Then i_0 is the lowest integer such that $t_{i_0}^2 \neq 4$. In particular, if $i_0 = n+1$, then $t_m^2 = 4$ for all m, and if $i_0 = 1$, $t_m^2 \neq 4$ for all m.

We go through the cases for substituting pairs and triples from $1 \leq a \leq b \leq c \leq i_0 \leq i \leq j \leq k \leq n$ into equations (1)–(9).

Consider $1 \leq a \leq i_0 - 1$. There is $\varepsilon_a \in \{\pm 1\}$ such that $t_a = \varepsilon_a 2$. For all $1 \leq a < i_0$, define

$$D_a = \varepsilon_a E. \tag{3.1.9}$$

Then $\operatorname{tr}(D_a) = t_a$.

For each $1 \leq a < b \leq i_0 - 1$, consider t_{ab}. Equation (1) is equivalent to

$$\begin{aligned} (t_{ab} - \varepsilon_a \varepsilon_b 2)^2 &= 0, \\ \implies t_{ab} &= \varepsilon_a \varepsilon_b 2 = \operatorname{tr}(D_a D_b). \end{aligned} \tag{3.1.10}$$

For each $1 \leq a < b < c \leq i_0 - 1$, consider t_{abc}. Equation (5) is equivalent to

$$\begin{aligned} (t_{abc} - \varepsilon_a \varepsilon_b \varepsilon_c 2)^2 &= 0, \\ \implies t_{abc} &= \varepsilon_a \varepsilon_b \varepsilon_c 2 = \operatorname{tr}(D_a D_b D_c). \end{aligned} \tag{3.1.11}$$

Substituting the results into the remaining equations gives $0 = 0$.

Consider i_0. Choose d_{i_0} such that $t_{i_0} = d_{i_0} + d_{i_0}^{-1}$ and define

$$D_{i_0} = \begin{pmatrix} d_{i_0} & 0 \\ 0 & d_{i_0}^{-1} \end{pmatrix}. \tag{3.1.12}$$

Then $\operatorname{tr}(D_{i_0}) = t_{i_0}$.

For $1 \leq i_0 \leq i \leq n$, consider t_i and $t_{i_0 i}$. Let $x \in \mathbb{F}$ be such that $t_i = x + x^{-1}$. Then equation (1) applied to i_0, i gives

$$\left(\left(d_{i_0} x + d_{i_0}^{-1} x^{-1}\right) - t_{i_0 i}\right)\left(\left(d_{i_0} x^{-1} + d_{i_0}^{-1} x\right) - t_{i_0 i}\right) = 0. \tag{3.1.13}$$

Note that $t_{i_0}^2 \neq 4 \neq t_i^2$ implies that only one of these factors is equal to zero. This uniquely determines $d_i \in \{x, x^{-1}\}$ such that

$$\begin{aligned} d_{i_0}d_i + d_{i_0}^{-1}d_i^{-1} - t_{i_0 i} &= 0, \\ \implies t_{i_0 i} &= d_{i_0}d_i + d_{i_0}^{-1}d_i^{-1}. \end{aligned}$$

We let

$$D_i = \begin{pmatrix} d_i & 0 \\ 0 & d_i^{-1} \end{pmatrix}. \tag{3.1.14}$$

Then $\operatorname{tr}(D_i) = t_i$ and $\operatorname{tr}(D_{i_0}D_i) = t_{i_0 i}$. The n–tuple $(D_1, \dots, D_n)$ is now completely defined, and we need to show that the traces of their double and triple products have the correct values.

For each $1 \leq a < i_0 \leq i \leq n$, consider t_{ai}. Equation (1) is equivalent to

$$\begin{aligned} \left((d_i + d_i^{-1}) - \varepsilon_a t_{ai}\right)^2 &= 0, \\ \implies t_{ai} &= \varepsilon_a \left(d_i + d_i^{-1}\right) = \operatorname{tr}(D_a D_i). \end{aligned} \tag{3.1.15}$$

For each $1 \leq a < b < i_0 \leq i \leq n$, consider t_{abi}. Equation (5) is equivalent to

$$\begin{aligned} \left((d_i + d_i^{-1}) - \varepsilon_a \varepsilon_b t_{abi}\right)^2 &= 0, \\ \implies t_{abi} &= \varepsilon_a \varepsilon_b \left(d_i + d_i^{-1}\right) = \operatorname{tr}(D_a D_b D_i). \end{aligned} \tag{3.1.16}$$

Substituting the results into the remaining equations gives $0 = 0$.

For each $1 \leq a < i_0 \leq i < j \leq n$, consider t_{aij}. Equation (5) is equivalent to

$$\begin{aligned} \left(t_{aij} - \varepsilon_a \left(d_i d_j + d_i^{-1} d_j^{-1}\right)\right)^2 &= 0 \\ \implies t_{aij} &= \varepsilon_a \left(d_i d_j + d_i^{-1} d_j^{-1}\right) = \operatorname{tr}(D_a D_i D_j). \end{aligned} \tag{3.1.17}$$

Substituting the results into the remaining equations gives $0 = 0$.

For each $1 \leq i_0 < j < k \leq n$, consider t_{jk}. Equation (1) is equivalent to

$$\left(d_j d_k + d_j^{-1} d_k^{-1} - t_{jk}\right)\left(d_j d_k^{-1} + d_j^{-1} d_k - t_{jk}\right) = 0. \tag{3.1.18}$$

Then (2), (3), and (4) applied to i_0, j, k gives

$$\begin{aligned} \left(d_j d_k + d_j^{-1} d_k^{-1} - t_{jk}\right)\left(\left(1 - d_{i_0}^{-2}\right) d_j^{-1} d_k^{-1} + \left(1 - d_{i_0}^2\right) d_j d_k + d_j^{-1} d_k + d_j d_k^{-1} - t_{jk}\right) &= 0, \\ \left(d_j d_k + d_j^{-1} d_k^{-1} - t_{jk}\right)\left(d_{i_0}^2 d_j d_k^{-1} + d_{i_0}^{-2} d_j^{-1} d_k - t_{jk}\right) &= 0, \\ \left(d_j d_k + d_j^{-1} d_k^{-1} - t_{jk}\right)\left(d_{i_0}^{-2} d_j d_k^{-1} + d_{i_0}^2 d_j^{-1} d_k - t_{jk}\right) &= 0. \end{aligned} \tag{3.1.19}$$

Assume $t_{jk} \neq d_j d_k + d_j^{-1} d_k^{-1}$. Then,

$$\begin{aligned}
t_{jk} &= d_j d_k^{-1} + d_j^{-1} d_k, \\
&= \left(1 - d_i^{-2}\right) d_j^{-1} d_k^{-1} + \left(1 - d_i^2\right) d_j d_k + d_j^{-1} d_k + d_j d_k^{-1}, \\
&= d_i^2 d_j d_k^{-1} + d_i^{-2} d_j^{-1} d_k, \\
&= d_i^{-2} d_j d_k^{-1} + d_i^2 d_j^{-1} d_k.
\end{aligned}$$

Since $t_i^2 \neq 4$, we have $d_i \neq d_i^{-1}$. The equations can be compared to get

$$\begin{aligned}
d_i^2 d_j^2 d_k^2 &= 1, \\
d_i^2 d_j^2 &= d_k^2, \\
d_i^2 d_k^2 &= d_j^2, \\
\left(d_j + d_j^{-1}\right)\left(d_i d_k - d_i^{-1} d_k^{-1}\right) &= 0, \\
\left(d_i + d_i^{-1}\right)\left(d_j d_k^{-1} - d_j^{-1} d_k\right) &= 0.
\end{aligned}$$

The last two equations give multiple possibilities. Together with the first three equations,

$$\begin{aligned}
d_j + d_j^{-1} = 0 \text{ and } d_i + d_i^{-1} = 0 &\implies d_k^2 = 1, \\
d_j + d_j^{-1} = 0 \text{ and } d_j d_k^{-1} - d_j^{-1} d_k = 0 &\implies d_i^2 = 1, \\
d_j d_k^{-1} - d_j^{-1} d_k = 0 \text{ and } d_i + d_i^{-1} = 0 &\implies d_j^2 = 1, \\
d_j d_k^{-1} - d_j^{-1} d_k = 0 \text{ and } d_j d_k^{-1} - d_j^{-1} d_k = 0 &\implies d_i^2 = 1,
\end{aligned} \tag{3.1.20}$$

which all give contradictions to $t_m^2 \neq 4$ for each $m \in \{i, j, k\}$.

Thus to satisfy Equation (3.1.19) we must have $t_{jk} = d_j d_k + d_j^{-1} d_k^{-1} = \operatorname{tr}(D_j D_k)$. This means equations (2), (3), (4) evaluate to 0.

For each $1 \leq i_0 \leq i < j < k \leq n$, now consider t_{ijk}. Equations (5), (6), and (7) are equivalent to

$$\begin{aligned}
\left(d_i d_j^{-1} d_k^{-1} + d_i^{-1} d_j d_k - t_{ijk}\right)\left(d_i^{-1} d_j^{-1} d_k^{-1} + d_i d_j d_k - t_{ijk}\right) &= 0, \\
\left(d_i^{-1} d_j d_k^{-1} + d_i d_j^{-1} d_k - t_{ijk}\right)\left(d_i^{-1} d_j^{-1} d_k^{-1} + d_i d_j d_k - t_{ijk}\right) &= 0, \\
\left(d_i d_j d_k^{-1} + d_i^{-1} d_j^{-1} d_k - t_{ijk}\right)\left(d_i^{-1} d_j^{-1} d_k^{-1} + d_i d_j d_k - t_{ijk}\right) &= 0.
\end{aligned} \tag{3.1.21}$$

Assume $t_{ijk} \neq d_i^{-1} d_j^{-1} d_k^{-1} + d_i d_j d_k$. Then,

$$\begin{aligned}
t_{ijk} &= d_i d_j^{-1} d_k^{-1} + d_i^{-1} d_j d_k, \\
&= d_i^{-1} d_j d_k^{-1} + d_i d_j^{-1} d_k, \\
&= d_i d_j d_k^{-1} + d_i^{-1} d_j^{-1} d_k.
\end{aligned}$$

Since $t_m^2 \neq 4$ for each $m \in \{i, j, k\}$, we have $d_m \neq d_m^{-1}$ for each $m \in \{i, j, k\}$. The equations can be compared to get $d_i^2 = d_j^2 = d_k^2$.

Then equations (8) and (9) applied to i, j, k give

$$\frac{2\left(d_k^2-1\right)^4\left(d_k^2+1\right)^2}{d_k^6}=0, \tag{3.1.22}$$

$$\frac{\left(d_k^2-1\right)^4\left(d_k^2+1\right)^2\left(1+d_k^2+d_k^4\right)}{d_k^8}=0. \tag{3.1.23}$$

Again since $t_k^2 \neq 4$, we have $d_k \neq d_k^{-1}$ and $d_k^2-1 \neq 0$.

In characteristic $p \neq 2$, Equation (3.1.22) implies $d_k^2=-1$. Given $d_i^2=d_j^2=d_k^2$ we also have $d_i^2=-1=d_j^2$. Then $t_{ijk}=0=\operatorname{tr}(D_iD_jD_k)$. Substituting the results into Equation (3.1.23) gives $0=0$.

In characteristic 2, Equation (3.1.22) gives $0=0$. Equation (3.1.23) this implies either $d_k^2=1$, which gives a contradiction, or $1+d_k+d_k^2=0$. Then $d_k^{-1}=d_k+1$ and $t_{ijk}=0=\operatorname{tr}(D_iD_jD_k)$.

Assume instead $t_{ijk}=d_i^{-1}d_j^{-1}d_k^{-1}+d_id_jd_k$ from Equation (3.1.21). Then equations (8) and (9) evaluate to 0 and $t_{ijk}=\operatorname{tr}(D_iD_jD_k)$.

Thus, given $\chi \in X_n^{red}$, for all possible combinations of $1 \leq a \leq b \leq c \leq i_0 \leq i \leq j \leq k \leq n$, we can construct a representation $(D_1,\ldots,D_n)$ such that $\operatorname{tr_n}(D_1,\ldots,D_n)=\chi$. The proves the result. □

3.1.4 Aside: normal forms; conjugacy classes

We record a result on normal forms and conjugacy classes that will be useful later. The following observation is stated in [PP20, Lemma 3.1] in terms of $p \geq 3$ and $\mathrm{PSL}_2(\mathbb{F})$ with reference to the structure of projective special groups over finite fields. We give a different argument for $\mathrm{SL}_2(\mathbb{F})$ using minimal polynomials and Jordan normal form.

Lemma 3.14. *Let $\mathbb{F}$ be an algebraically closed field of characteristic p and $A \in \mathrm{SL}_2(\mathbb{F})$ be an element of finite order. If $p \geq 3$, then either the order of A is coprime with p or it equals p or $2p$. If $p=2$, then the order of A is either odd or equal to two. In addition:*

1. *If the order of A is coprime with p, then A is diagonalisable.*
2. *If the order of A equals p, then A is conjugate to* $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.
3. *If p is odd and the order of A equals $2p$, then A is conjugate to* $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$.

Proof. Suppose $A^n=E$. Then the minimal polynomial of A divides x^n-1 in $\mathbb{F}[x]$. Suppose $n=p^mq$, where q is coprime with p. Applying the Frobenius map gives $x^n-1=x^{p^mq}-1^{p^m}=(x^q-1)^{p^m}$. In particular, the eigenvalues of A are roots of x^q-1. Let λ be an eigenvalue of A.

If $\lambda \neq \lambda^{-1}$, then A is diagonalisable. Since $\lambda^q=1=\lambda^{-q}$, we have $A^q=E$ and hence the order of A divides q, which is co-prime with p. This is the first case of the lemma.

Hence suppose $\lambda = \lambda^{-1}$. Then $\lambda \in \{\pm 1\}$ and A has normal form either $\pm E$ or $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$.

Now if $p = 2$, then $E = -E$ is diagonal of order one which is coprime with 2 and hence is in the first case. If $p \geq 3$, then E has order one and $-E$ has order two; both are diagonal with order coprime with p and are again covered by the first case.

For any prime p, the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has order equal to p, and hence this is the second case of the lemma. If $p = 2$, then $-1 = 1$ and we are done. If $p \geq 3$, then $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ has order $2p$ and we are in the third case.

This completes the proof of the lemma. □

3.2 Splittings of groups detected by ideal points

Following the treatment in [CS83], we define ideal points of curves in the variety of characters for arbitrary algebraically closed fields in such a way that we can apply Bass-Serre theory [Bas93, Ser03]. The up-shot is that an ideal point of a curve in $\mathrm{X}(\Gamma, \mathbb{F})$ gives rise to a **splitting** of Γ, that is, an isomorphism of Γ with the fundamental group of a graph of groups.

3.2.1 Ideal points and valuations

We call a 1–dimensional irreducible subvariety a **curve**. We briefly recall the definitions of ideal points of a curve and their relationship with non-trivial discrete rank 1 valuation on the function field of the curve.

Let $\mathbb{K}$ be a field. A **discrete rank 1 valuation** on $\mathbb{K}$ is a map $v\colon\ \mathbb{K} \to \mathbb{Z} \cup \{\infty\}$ that is the extension of a map $\mathbb{K} \setminus \{0\} \to \mathbb{Z}$ by letting $v(0) = \infty$ satisfying

1. $v(ab) = v(a) + v(b)$ for all $a, b \in \mathbb{K} \setminus \{0\}$,
2. $v(a+b) \geq \min\{v(a), v(b)\}$,

where $z \leq \infty$ and $z + \infty = \infty = \infty + z$ for all $z \in \mathbb{Z}$.

Two varieties V, W are **birationally equivalent** is there are rational, dense maps $\varphi\colon\ V \to W$, $\psi\colon\ W \to V$, such that $\varphi \circ \psi = 1 = \psi \circ \varphi$ where defined. The **projective completion** of a variety V over $\mathbb{F}$ is the closure of V under the map $J : \mathbb{F}^m \to \mathbb{F}P^m$ defined by $J(z_1, \ldots, z_m) = [1, z_1, \ldots, z_m]$. When V is a 1–dimensional irreducible variety there is a unique non-singular projective variety $\widetilde{V}$, which is birationally equivalent to the projective completion $\overline{J(V)}$. This is called the **smooth projective closure** of V, and exists over all algebraically closed fields $\mathbb{F}$ (and more generally, for all perfect fields).

Let Γ be a finitely generated group and consider a curve in the variety of characters, $C \subset \mathrm{X}(\Gamma, \mathbb{F})$. The **ideal points** of C are the points of the smooth projective closure $\widetilde{C}$ that correspond to the set $\overline{J(C)} - J(C)$

under the birational equivalence. Intuitively, these are the points added to C by the process of taking the smooth projective closure. Note $\mathbb{F}(\widetilde{C}) \cong \mathbb{F}(C)$.

Take an ideal point $\xi \in \widetilde{C}$. This defines a non-trivial discrete rank 1 valuation on $\mathbb{F}(C)$ via,

$$\begin{aligned} v_\xi :\ & \mathbb{F}(C) \to \mathbb{Z} \cup \{\infty\} \\ v_\xi(f) = & \begin{cases} q & f \text{ has a zero of order } q \text{ at } \xi, \\ \infty & f = 0, \\ -q & f \text{ has a pole of order } q \text{ at } \xi. \end{cases} \end{aligned} \tag{3.2.1}$$

Take $R_C \subset \mathrm{R}(\Gamma, \mathbb{F})$ an irreducible subvariety with the property that $q(R_C) = C$ under the natural map $q : \mathrm{R}(\Gamma, \mathbb{F}) \to \mathrm{X}(\Gamma, \mathbb{F})$. We can extend the valuation to $\mathbb{F}(R_C)$ using the following result from Alperin and Shalen [AS82].

Lemma 3.15 (Alperin-Shalen)**.** *Let E be a discretely valued field with valuation v. If F is a finitely generated extension field of E then there is an extension of v to a discrete valuation $\tilde{v}$ of F.*

The natural algebraic map $q : \mathrm{R}(\Gamma, \mathbb{F}) \to \mathrm{X}(\Gamma, \mathbb{F})$ induces

$$\mathbb{F}(\mathrm{X}(\Gamma, \mathbb{F})) = \mathbb{F}[t_i]/I \subseteq \mathbb{F}[z_i]/I = \mathbb{F}(\mathrm{R}(\Gamma, \mathbb{F}))$$

for ideal I and z_i coordinates for $\mathrm{R}(\Gamma, \mathbb{F})$ such that $t_i \in \mathrm{span}\{z_j\}$ for all i. We can restrict this to

$$\mathbb{F}(C) = \mathbb{F}[t_i]/I_C \subseteq \mathbb{F}[z_i]/I_C = \mathbb{F}(R_C)$$

for the ideal I_C defining the curve C, with $I \subset I_C$. Then setting $E = \mathbb{F}(C)$, $v = v_\xi$ and $F = \mathbb{F}(R_C)$, we can use Lemma 3.15 to extend the valuation v_ξ to a discrete rank 1 valuation on $\mathbb{F}(R_C)$,

$$\tilde{v}_\xi :\ \mathbb{F}(R_C) \to \mathbb{Z} \cup \{\infty\}. \tag{3.2.2}$$

Let $\mathbb{K} = \mathbb{F}(R_C)$. The **tautological representation** is

$$\begin{aligned} \mathscr{P} :\ & \pi_1(M) \to \mathrm{SL}_2(\mathbb{K}), \\ \gamma \mapsto \mathscr{P}(\gamma) = & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ for } \rho(\gamma) = \begin{pmatrix} a(\rho) & b(\rho) \\ c(\rho) & d(\rho) \end{pmatrix}. \end{aligned} \tag{3.2.3}$$

Here, a, b, c, d are polynomial functions in the ambient coordinates defining $\mathrm{R}(M, \mathbb{F})$ and can hence be viewed as elements in $\mathbb{K} = \mathbb{F}(R_C)$.

3.2.2 Actions of groups on trees

Let $\mathbb{K} = \mathbb{F}(R_C)$, ξ an ideal point of C and $v = v_\xi$ the associated valuation. We recall the construction in [CS83, Section 2] of the **Bass-Serre tree $\mathbf{T}_\xi$** with an action of $\mathrm{SL}_2(\mathbb{K})$. More details on the construction can be found in [Bau93, Ser03, Sha02].

The **valuation ring** is

$$\mathscr{O} = \{a \in \mathbb{K} \mid v(a) \geq 0\}. \tag{3.2.4}$$

The valuation ring is a principal ideal domain and will contain a maximal ideal $\mathscr{M}$ consisting of the non-units of $\mathscr{O}$ with $\mathscr{M} = \langle \pi \rangle$ for $v(\pi) = 1$.

Consider $V = \mathbb{K}^2$ a left $\mathscr{O}$-module. An **$\mathscr{O}$-lattice** L is an $\mathscr{O}$-submodule of the form

$$L = \mathscr{O}x + \mathscr{O}y \tag{3.2.5}$$

for $x, y \in V$ linearly independent. The multiplicative group $\mathbb{K}^\times$ acts on $\mathscr{O}$-lattices by multiplication via

$$aL = \mathscr{O}ax + \mathscr{O}ay \tag{3.2.6}$$

with orbits of this action defining equivalence classes $\Lambda = [L]$.

The key observation is that given $\mathscr{O}$-lattices L_1 and L_2, these is $m \in \mathbb{Z}$ such that $\pi^m L_2 \subset L_1$. Hence for $\pi^m L_2 \subset L_1$, $m \in \mathbb{Z}$, there is a basis $\{x, y\}$ for L_1 such that $\{\pi^f x, \pi^g y\}$ is a basis for $\pi^m L_2$ for some $f, g \in \mathbb{Z}$. One obtains a well-defined distance between the equivalence classes of $\mathscr{O}$-lattices $\Lambda_1 = [L_1]$ and $\Lambda_2 = [L_2]$ by letting

$$d\,(\Lambda_1, \Lambda_2) = |f - g|. \tag{3.2.7}$$

This defines a metric on the set of all equivalence classes of $\mathscr{O}$-lattices.

We construct a graph (which turns out to be a tree) $\mathbf{T}_\xi$ by the following set of vertices $\mathscr{V}$ and edges $\mathscr{E}$,

$$\begin{aligned} \mathscr{V} &= \{\Lambda = [L] \mid L \text{ an } \mathscr{O}\text{-lattice}\}, \\ \mathscr{E} &= \{(\Lambda_1, \Lambda_2) \mid d\,(\Lambda_1, \Lambda_2) = 1\}. \end{aligned} \tag{3.2.8}$$

The group $\mathrm{GL}(V)$ acts on $\mathscr{O}$-lattices via

$$A \circ L = A \circ (\mathscr{O}x + \mathscr{O}y) = \mathscr{O}Ax + \mathscr{O}Ay \tag{3.2.9}$$

for $A \in \mathrm{GL}(V)$. The action is well-defined on equivalence classes, giving an action on the tree $\mathbf{T}_\xi$. In fact, this action is via isometries. That is, $d\,(A \circ \Lambda_1, A \circ \Lambda_2) = d\,(\Lambda_1, \Lambda_2)$, for all $A \in \mathrm{GL}(V)$ and for all Λ_1, Λ_2.

If we restrict the action to $\mathrm{SL}_2(\mathbb{K})$, it can be proven that the action of $\mathrm{SL}_2(\mathbb{K})$ on $\mathbf{T}_\xi$ is non-trivial, simplicial, and without inversions. We use the tautological representation $\mathscr{P}$ to pull back the action of $\mathrm{SL}_2(\mathbb{K})$ on $\mathbf{T}_\xi$ to an action of Γ on $\mathbf{T}_\xi$ by $\gamma \circ \Lambda = \mathscr{P}(\gamma) \circ \Lambda$ for all $\gamma \in \Gamma$. As per the action of $\mathrm{SL}_2(\mathbb{K})$, the action of Γ is simplicial and without inversions.

We use the tautological representation to define a function $I_\gamma = \mathrm{tr}(\mathscr{P}(\gamma)) \in \mathbb{F}(\mathrm{R}(M, \mathbb{F}))$ for each $\gamma \in \pi_1(M)$:

$$\begin{aligned} I_\gamma\colon\ \mathrm{R}(M, \mathbb{F}) &\to \mathbb{F} \\ \rho &\mapsto \mathrm{tr}(\rho(\gamma)). \end{aligned} \tag{3.2.10}$$

We say that I_γ **blows up** (at ξ) if $v_\xi(I_\gamma) < 0$. Otherwise we say that I_γ is **bounded** (at ξ).

The proofs of [Sha02, Property 5.4.2] and [CS83, Theorem 2.2.1] apply verbatim with the current set-up, and hence we have the following result:

Theorem 3.16 (Culler-Shalen)**.** *Let C be a curve in $\mathrm{X}(\Gamma,\mathbb{F})$. To each ideal point ξ of $\widetilde{C}$, one can associate a splitting of Γ with the property that for each element $\gamma \in \Gamma$, the following are equivalent:*

1. $v_\xi(I_\gamma) \geq 0$;
2. *A vertex of the Bass-Serre tree $\mathbf{T}_\xi$ is fixed by γ.*

Thus, in particular, the action of Γ on $\mathbf{T}_\xi$ is non-trivial and the associated splitting of Γ is non-trivial.

3.3 Essential surfaces in 3–manifolds detected by ideal points

Assume M is a compact, orientable 3–manifold that is not necessarily irreducible. We outline the tools and approach used by Culler and Shalen [CS83] now over $\mathbb{F}$ to associate essential surfaces in 3–manifolds to ideal points of curves in the $\mathrm{SL}_2(\mathbb{F})$–variety of characters. We give a brief outline following [Sha02], where further details and proofs can be found. We conclude with remarks on bounds on the dimension of the character variety, which were first provided in [CS83] based on work by Thurston [Thu80]. These show that curves must always exist for hyperbolic 3–manifolds with at least one boundary torus. Recall $\mathrm{R}(M,\mathbb{F}) = \mathrm{R}(\pi_1(M),\mathbb{F})$ and $\mathrm{X}(M,\mathbb{F}) = \mathrm{X}(\pi_1(M),\mathbb{F})$.

3.3.1 Splittings and surfaces

Stallings [Sta65] previously showed that a free product decomposition of a fundamental group of a 3–manifold gives rise to a system of essential 2–spheres in the manifold and Epstein [Eps61] and Waldhausen [Wal67] previously showed that a decomposition of the fundamental group as a free product with amalgamation or HNN extension gives rise to a system of incompressible surfaces. Theorem 3.16 states that an ideal point of a curve in the character variety of $\pi_1(M)$ gives a non-trivial splitting of $\pi_1(M)$.

As a consequence of this, Culler and Shalen [CS83, Proposition 2.3.1] distill the earlier work of Stallings [Sta65], Epstein [Eps61] and Waldhausen [Wal67] into the following result.

Proposition 3.17 (Culler-Shalen)**.** *[CS83, Proposition 2.3.1] Let M be a compact, orientable 3–manifold. For any non-trivial splitting of $\pi_1(M)$ there exists a non-empty system $S = S_1 \cup \ldots \cup S_m$ of orientable incompressible surfaces in N, none of which is boundary parallel, such that $\mathrm{im}(\pi_1(S_i) \to \pi_1(M))$ is contained in an edge group for $i = 1,\ldots,m$, and $\mathrm{im}(\pi_1(M_j) \to \pi_1(M))$ is contained in a vertex group for each connected component M_j of $M \setminus S$.*

Moreover, if $K \subset \partial M$ is a subcomplex such that $\mathrm{im}(\pi_1(K) \to \pi_1(M))$ is contained in a vertex group for each connected component of K, we may take S to be disjoint from K.

3.3.2 Detecting essential surfaces

From now onwards, we assume that M is irreducible. Following Culler and Shalen, we now obtain finer information on surfaces detected by the ideal point ξ of a curve C in the variety of characters. We outline the general approach and refer the reader again to [CS83, Sha02] for a detailed account.

The action by $\pi_1(M)$ on the Bass-Serre tree $\mathbf{T} = \mathbf{T}_\xi$ is used to find an essential surface in M via the following construction from Stallings [Sta65]. Given a triangulation on M, lift the triangulation to the universal cover $\widetilde{M}$ so that $\pi_1(M)$ acts simplicially on $\widetilde{M}$. Construct a simplicial, $\pi_1(M)$–equivariant map $f\colon \widetilde{M} \to \mathbf{T}$. Denote the set of midpoints of edges of $\mathbf{T}$ to be E. Then $f^{-1}(E)$ is a surface $\widetilde{S}$ in $\widetilde{M}$ that descends to a surface S in M.

The surface associated with the ideal point ξ depends on the choice of triangulation of M and the choice of map f. A surface found in this way is called **dual to the action** of $\pi_1(M)$ on $\mathbf{T}$. The surface may contain finitely many parallel copies of some of its components, which we implicitly discard. The definition of a surface being dual to the action combined with Proposition 3.17 has the following consequence.

Corollary 3.18. *If S is a surface dual to the action of $\pi_1(M)$ on $\mathbf{T}$, then,*

(i) for each component S_i of S, the subgroup $im(\pi_1(S_i) \to \pi_1(M))$ of $\pi_1(M)$ is contained in the stabiliser of an edge of $\mathbf{T}$; and

(ii) for each component M_j of $M - S$, the subgroup $im(\pi_1(M_j) \to \pi_1(M))$ of $\pi_1(M)$ is contained in the stabiliser of a vertex of $\mathbf{T}$.

If the surface S dual to the action is not essential we can find a map homotopic to f such that it is (see [CS83, Sha02]). An essential surface that arises in this way is said to be **detected** by an ideal point.

3.3.3 Boundary slopes and essential surfaces

A **boundary slope** is the slope of the boundary curve of an essential surface S in M that has non-empty intersection with ∂M. The relationship between boundary slopes of essential surfaces in M and valuations of trace functions of peripheral elements is given in the following reformulation of [CGLS87, Proposition 1.3.9].

Proposition 3.19. *Let M be a compact, orientable, irreducible 3–manifold with ∂M a torus. Suppose there is an element $\gamma \in \pi_1(M)$ such that $v_\xi(I_\gamma) < 0$. Let $\alpha \in \mathrm{im}(\pi_1(T) \to \pi_1(M))$ such that $v_\xi(I_\alpha) \geq 0$. Then either*

1. *$v_\xi(I_\beta) \geq 0$ for all $\beta \in \mathrm{im}(\pi_1(T) \to \pi_1(M))$ and there is a closed essential surface in M, or*
2. *α determines a boundary slope of M.*

Proof. Since $v_\xi(I_\alpha) \geq 0$, we know that α stabilises a vertex in the associated Bass-Serre tree. Thus, there is an associated essential surface S disjoint from any curve C on $T \subseteq \partial M$ representing α. If $\partial S \neq \emptyset$ then C is a boundary slope. Otherwise $\mathrm{im}(\pi_1(\partial M) \to \pi_1(M))$ stabilises a vertex and the first part is true. □

Corollary 3.20. *Let M be a compact, orientable, irreducible 3–manifold with ∂M a torus. Let C be a curve in the variety of characters with an ideal point ξ. We have the following mutually exclusive cases.*

1. *If there is an element γ in $\operatorname{im}(\pi_1(\partial M) \to \pi_1(M))$ such that $v_\xi(I_\gamma) < 0$, then up to inversion there is a unique primitive element $\alpha \in \operatorname{im}(\pi_1(\partial M) \to \pi_1(M))$ such that $v_\xi(I_\alpha) \geq 0$. Then every essential surface S detected by ξ has non-empty boundary and α is parallel to its boundary components. In this case, α is called a* ***strongly detected boundary slope****.*
2. *If $v_\xi(I_\gamma) \geq 0$ for all $\gamma \in \operatorname{im}(\pi_1(\partial M) \to \pi_1(M))$, then an essential surface S detected by ξ may be chosen that is disjoint from ∂M. Moreover, if S is detected by ξ and has non-empty boundary, then its boundary slope is* ***weakly detected*** *by ξ.*

3.3.4 Bounds on dimension

Based on Thurston [Thu80, Theorem 5.6], Culler and Shalen [CS83, Proposition 3.2.1] provided a bound on the dimension of the $\mathrm{SL}_2(\mathbb{C})$–character variety. We state some preliminary results first, which are used to prove the bound. The proofs apply almost verbatim in our context.

Lemma 3.21. *[CS83, Lemma 1.5.1] Let $\mathbb{F}$ an algebraically closed field of characteristic $p \neq 2$. Suppose $\rho\colon \pi_1(M) \to \mathrm{SL}_2(\mathbb{F})$ is an irreducible representation and $\alpha \in \pi_1(M)$ such that $\rho(\alpha) \neq \pm E$. Then there exists $\gamma \in \pi_1(M)$ such that the restriction of ρ to the subgroup generated by α and γ is irreducible and* $\operatorname{tr}(\rho(\gamma))^2 \neq 4$.

This result over $\mathbb{C}$ was previously used to prove Proposition 1.19.

Proof. Take $\alpha \in \pi_1(M)$ such that $\rho(\alpha) \neq \pm E$. As ρ is irreducible, by definition there exists $\gamma \in \pi_1(M)$ such that ρ restricted to the subgroup generated by α and γ is irreducible. We show γ can be chosen so that $\operatorname{tr}(\rho(\gamma))^2 \neq 4$.

Suppose $\gamma_0 \in \pi_1(M)$ is such that ρ restricted to the subgroup generated by α and γ_0 is irreducible and $\operatorname{tr}(\rho(\gamma_0))^2 = 4$. Then $\rho(\gamma_0) \neq \pm E$ and we may assume $\rho(\gamma_0)$ is parabolic. Up to conjugation we may assume

$$\rho(\gamma_0) = \pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \rho(\alpha) = \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix}, \quad c \neq 0. \tag{3.3.1}$$

Then for all $n \in \mathbb{Z}$ we have $\operatorname{tr}(\rho(\alpha\gamma_0^n)) = (\pm 1)^n \left(a + a^{-1} + cn\right)$.

For characteristic 3 then there are only two options for $n = \pm 1$. We have for $n = 1$,

$$\operatorname{tr}(\rho(\alpha\gamma_0)) = (\pm 1)\left(a + a^{-1} + c\right)$$

and for $n = -1$,

$$\operatorname{tr}(\rho(\alpha\gamma_0^{-1})) = (\pm 1)\left(a + a^{-1} - c\right).$$

If at least one of $\operatorname{tr}(\rho(\alpha\gamma_0^n)) \neq \pm 1$ then take $\gamma = \alpha\gamma_0^n$. Otherwise we have $a + a^{-1} = 0$ and $c^2 = 1$. In this case, consider instead the commutator

$$\gamma = \alpha\gamma_0\alpha^{-1}\gamma_0^{-1} = \begin{pmatrix} 1 - ac & ac \\ -1 & -1 + ac \end{pmatrix}.$$

Then $\mathrm{tr}(\rho(\gamma)) = 0 \neq \pm 1$. It remains to prove that $\rho(\alpha)$ and $\rho(\gamma)$ form an irreducible pair. We calculate $\rho([\alpha,\gamma]) = -E$ and $\mathrm{tr}(\rho([\alpha,\gamma])) = 1 \neq 2 \pmod 3$. By Lemma 1.15, the restriction of ρ to the subgroup generated by α and γ is irreducible and γ satisfies the requirements. Note this only works in characteristic 3 assuming $c^2 = 1$ and $a + a^{-1} = 0$, otherwise the conditions may not be satisfied. □

The issue with the result of Lemma 3.21 in characteristic 2 is that we only have one option for $n \in \mathbb{Z}_p \setminus \{0\}$, $n = 1$, to get combinations $\alpha\gamma_0^n$. In order to refine the result for characteristic 2 we use the infinite dihedral group $\mathrm{D}_\infty \cong \mathbb{Z}_2 * \mathbb{Z}_2 \cong \mathbb{Z} \rtimes \mathbb{Z}_2 = \langle s,t \mid s^2 = 1,\ sts^{-1} = t^{-1}\rangle$. We say a group G is **D_∞–like** if

$$G \cong \mathrm{D}_\infty^{\mathrm{n}} = \langle s, t_1, \dots t_n \mid s^2 = 1,\ st_is^{-1} = t_i^{-1},\ t_it_j = t_jt_i \ \text{ for all } \ i,j = 1,\dots,n\rangle$$

for some $n \in \mathbb{N}$.

Lemma 3.22. *Let $\mathbb{F}$ be an algebraically closed field of characteristic 2. Suppose $\rho\colon\ \pi_1(M) \to \mathrm{SL}_2(\mathbb{F})$ is an irreducible representation and $\alpha \in \pi_1(M)$ such that $\rho(\alpha) \neq E$. Assume* $\mathrm{im}(\rho)$ *is not* D_∞*–like. Then there is a $\gamma \in \pi_1(M)$ such that the restriction of ρ to the subgroup generated by α and γ is irreducible and* $\mathrm{tr}(\rho(\gamma)) \neq 0$.

Proof. We follow the same initial steps as Lemma 3.21.

Take $\alpha \in \pi_1(M)$ such that $\rho(\alpha) \neq E$. As ρ is irreducible, by definition there exists $\gamma_0 \in \pi_1(M)$ such that ρ restricted to the subgroup generated by α and γ_0 is irreducible. Without loss of generality, we can consider γ_0 to be one of the finite generators

$$\pi_1(M) = \langle \gamma_1, \dots \gamma_n \mid r_1, \dots r_m\rangle,$$

say $\gamma_0 = \gamma_1$.

If $\mathrm{tr}(\rho(\gamma_1)) \neq 0$ then we are done. Assume that $\mathrm{tr}(\rho(\gamma_1)) = 0$. Then up to conjugation we may assume

$$\rho(\gamma_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \rho(\alpha) = \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix}, \quad c \neq 0. \tag{3.3.2}$$

Consider $\mathrm{tr}(\rho(\alpha\gamma_1)) = a + a^{-1} + c$ and assume $\mathrm{im}(\rho)$ is not D_∞–like. If $\mathrm{tr}(\rho(\alpha\gamma_1)) \neq 0$ then we can again take $\gamma = \alpha\gamma_1$ and we are done. If $\mathrm{tr}(\rho(\alpha\gamma_1)) = 0$ then $c = a + a^{-1}$ and

$$\mathrm{im}\left(\rho\left(\langle \alpha, \gamma_1\rangle\right)\right) = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a & 0 \\ a + a^{-1} & a^{-1} \end{pmatrix} \right\rangle \cong \mathrm{D}_\infty .$$

The isomorphism is constructed by sending s to the first matrix, $\rho(\gamma_1)$, and t to the second matrix, $\rho(\alpha)$, for s and t the generators of D_∞.

Then $\mathrm{im}\left(\rho\left(\langle \alpha, \gamma_1\rangle\right)\right)$ cannot be the whole image, otherwise this would contradict our assumption that $\mathrm{im}(\rho)$ is not D_∞–like. Therefore there is an element $\delta \in \pi_1(M)$ such that $\rho(\delta) \notin \mathrm{im}\left(\rho\left(\langle \alpha, \gamma_0\rangle\right)\right)$. Say without loss of generality $\delta = \gamma_2$,

$$\rho(\gamma_2) = \begin{pmatrix} e & f \\ g & h \end{pmatrix}, \quad eh - fg = 1.$$

We get $\operatorname{tr}(\rho(\gamma_2\alpha\gamma_2^{-1}\alpha^{-1})) = (a+a^{-1})^2 f(e+f+g+h)$. Assume the restriction of ρ to the subgroup generated by α and γ_2 is reducible. Then $(a+a^{-1})^2 f(e+f+g+h) = 0$. Given $\operatorname{tr}(\rho(\alpha)) \neq 0$, we must have $f = 0$ or $f = e+g+h$. A simple calculation shows the restriction of ρ to the subgroup generated by α and $\gamma_1\gamma_2$ is irreducible with trace $\operatorname{tr}(\rho(\gamma_1\gamma_2)) = e+g+h$. If $\operatorname{tr}(\rho(\gamma_1\gamma_2)) = 0$ then $f = 0$ and $g = e+e^{-1}$, implying

$$\operatorname{im}(\rho(\langle\alpha,\gamma_1,\gamma_2\rangle)) = \left\langle \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \begin{pmatrix}a & 0\\ a+a^{-1} & a^{-1}\end{pmatrix}, \begin{pmatrix}e & 0\\ e+e^{-1} & e^{-1}\end{pmatrix} \right\rangle \cong D_\infty^2.$$

The isomorphism is constructed by sending s to the first matrix, $\rho(\gamma_1)$, t_1 to the second matrix, $\rho(\alpha)$, and t_2 to the third matrix, $\rho(\gamma_2)$, for s, t_1, and t_2 the generators of D_∞^2.

We can repeat this process over each generator γ_i. Given our assumption that $\operatorname{im}(\rho)$ is not D_∞–like, there must be a generator γ_k, $2 \leq k \leq n$ such that $\rho(\gamma_k) \notin \operatorname{im}(\rho(\langle\alpha,\gamma_1\rangle))$ and either the restriction of ρ to the subgroup generated by α and γ_k is reducible and $\operatorname{tr}(\rho(\gamma_1\gamma_k)) \neq 0$ or the restriction of ρ to the subgroup generated by α and γ_k is irreducible. Again let

$$\rho(\gamma_k) = \begin{pmatrix}e & f\\ g & h\end{pmatrix}, \quad eh - fg = 1.$$

In the first case take $\gamma = \gamma_1\gamma_k$. In the second case consider the options $\gamma = \gamma_k$, $\gamma = \gamma_1\gamma_k$ or $\gamma = \alpha\gamma_k$. The restriction of ρ to the subgroup generated by α and γ is irreducible for each of these choices of γ. We have $\operatorname{tr}(\rho(\gamma_k)) = e+h$, $\operatorname{tr}(\rho(\gamma_1\gamma_k)) = e+g+h$ and $\operatorname{tr}(\rho(\alpha\gamma_k)) = ae+(a+a^{-1})f+a^{-1}h$. If $\operatorname{tr}(\rho(\gamma_k)) = \operatorname{tr}(\rho(\gamma_1\gamma_k)) = \operatorname{tr}(\rho(\alpha\gamma_k)) = 0$ then $h = e$, $g = 0$ and $f = e$. Combining with the determinant condition this would give $\rho(\gamma_k) = \rho(\gamma_1)$, which contradicts $\rho(\gamma_k) \notin \operatorname{im}(\rho(\langle\alpha,\gamma_1\rangle))$. Thus one of $\operatorname{tr}(\rho(\gamma_k))$, $\operatorname{tr}(\rho(\gamma_1\gamma_k))$, $\operatorname{tr}(\rho(\alpha\gamma_k))$ is nonzero and we can choose γ accordingly. □

In fact, the converse statement of Lemma 3.22 is true if we also assume $\operatorname{tr}(\rho(\alpha)) \neq 0$.

Lemma 3.23. *Let $\mathbb{F}$ be an algebraically closed field of characteristic 2. Suppose $\rho\colon \pi_1(M) \to \mathrm{SL}_2(\mathbb{F})$ is an irreducible representation and $\alpha \in \pi_1(M)$ such that $\rho(\alpha) \neq E$ and $\operatorname{tr}(\rho(\alpha)) \neq 0$. Assume $\gamma \in \pi_1(M)$ such that the restriction of ρ to the subgroup generated by α and γ is irreducible and $\operatorname{tr}(\rho(\gamma)) \neq 0$. Then $\operatorname{im}(\rho)$ is not D_∞–like.*

Proof. Assume for the purposes of a contradiction that $\operatorname{im}(\rho)$ is D_∞–like and assume first $\operatorname{im}(\rho) \cong \mathrm{D}_\infty = \langle s,t \mid s^2 = 1,\ sts^{-1} = t^{-1}\rangle$.

As ρ is irreducible there exists some $\gamma \in \pi_1(M)$ such that the restriction of ρ to the subgroup generated by α and γ is irreducible. Assign $\mu, \nu \in \pi_1(M)$ to be the elements such that $\rho(\mu)$ and $\rho(\nu)$ are mapped to s,t respectively in the isomorphism. Then the restriction of ρ to the subgroup generated by μ and ν must be irreducible and we can conjugate their image to

$$\rho(\mu) = \begin{pmatrix}x & 1\\ 0 & x^{-1}\end{pmatrix}, \quad \rho(\nu) = \begin{pmatrix}y & 0\\ u & y^{-1}\end{pmatrix}, \quad u \neq 0.$$

Then the isomorphism forces $\rho(\mu)^2 = E$ and $\rho(\mu\nu\mu^{-1}\nu) = E$, which implies $x = 1$ and $u = y+y^{-1}$.

We will see that any element δ in $\pi_1(M)$ then either has trace 0 or the restriction of ρ to the subgroup generated by ν and δ is reducible. Given $\mathrm{tr}(\rho(\alpha)) \neq 0$, the restriction of ρ to the subgroup generated by ν and α must be reducible and the only option for an element γ that is irreducible with α must have $\mathrm{tr}(\rho(\gamma)) = 0$.

To see all elements must be one of these two options, take an arbitrary element $\delta \in \pi_1(M)$. Then $\rho(\delta) = \rho(\nu)^{k_1}\rho(\mu)^{l_1}\dots\rho(\nu)^{k_n}\rho(\mu)^{l_n}$ for some sequences $k_i, l_i \in \mathbb{Z}$, $i = 1,\dots,n$. Without loss of generality, assume $k_i \neq 0$ for all $i \in \{2,\dots,n\}$ and $l_i \neq 0$ for all $i \in \{1,\dots,n-1\}$. The terms $\rho(\mu)^{l_i}$ are defined by the parity of l_i, so we can take $l_i = 1$ for all $i \in \{1,\dots,n-1\}$. Then the terms in $\rho(\delta)$ satisfy

$$\rho(\nu)^{k_i}\rho(\mu) = \begin{pmatrix} y^{k_i} & y^{k_i} \\ y^{k_i}+y^{-k_i} & y^{k_i} \end{pmatrix},$$

$$\rho(\nu)^{k_i}\rho(\mu)\rho(\nu)^{k_j}\rho(\mu) = \begin{pmatrix} y^{k_i-k_j} & 0 \\ y^{k_i-k_j}+y^{-k_i+k_j} & y^{-k_i+k_j} \end{pmatrix}.$$

We can deduce

$$\rho(\delta) = \begin{pmatrix} y^{k_1-k_2+\dots-k_{n-1}+k_n} & 0 \\ y^{k_1-k_2+\dots-k_{n-1}+k_n}+y^{-k_1+k_2-\dots+k_{n-1}-k_n} & y^{-k_1+k_2-\dots+k_{n-1}-k_n} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{l_n} \tag{3.3.3}$$

for n odd or

$$\rho(\delta) = \begin{pmatrix} y^{k_1-k_2+\dots+k_{n-1}-k_n} & y^{k_1-k_2+\dots+k_{n-1}-k_n} \\ y^{k_1-k_2+\dots+k_{n-1}-k_n}+y^{-k_1+k_2-\dots-k_{n-1}+k_n} & y^{k_1-k_2-\dots+k_{n-1}-k_n} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{l_n} \tag{3.3.4}$$

for n even.

There are four cases. If n is odd and $l_n = 1 \pmod 2$ or if n is even and $l_n = 0 \pmod 2$ then $\mathrm{tr}(\rho(\delta)) = 0$. If n is odd and $l_n = 0 \pmod 2$ or if n is even and $l_n = 1 \pmod 2$ then the restriction of ρ to the subgroup generated by ν and δ is reducible.

Assume now $\mathrm{im}(\rho) \cong D_\infty^q = \langle s, t_1, \dots t_q \mid s^2 = 1,\ st_is^{-1} = t_i^{-1},\ t_it_j = t_jt_i \text{ for all } i,j = 1\dots q\rangle$. Assign $\mu, \nu_i \in \pi_1(M)$, $i = 1,\dots,q$ to be the elements such that $\rho(\mu)$ and $\rho(\nu_i)$ are mapped to s, t_i respectively in the isomorphism. Then similar to the case for $q = 1$, we can conjugate their image to,

$$\rho(\mu) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \rho(\nu_i) = \begin{pmatrix} y_i & 0 \\ y_i + y_i^{-1} & y_i^{-1} \end{pmatrix}, \quad \text{for all } i = 1,\dots,q.$$

Again we will get that all elements $\delta \in \pi_1(M)$ must be of one of the forms Equations (3.3.3) and (3.3.4), with the entries now in terms of combinations of powers of $y_1,\dots,y_q$. The same case analysis will show that any element δ in $\pi_1(M)$ then either has trace 0 or the restriction of ρ to the subgroup generated by ν_i and δ is reducible for all $i = 1,\dots,q$. Given $\mathrm{tr}(\rho(\alpha)) \neq 0$, the restriction of ρ to the subgroup generated by ν_i and α must be reducible for all $i = 1,\dots,q$ and the only option for an element γ that is irreducible with α must have $\mathrm{tr}(\rho(\gamma)) = 0$, which gives the contradiction. □

We also use the following result relating the dimension of components of $\mathrm{R}(M,\mathbb{F})$ and $\mathrm{X}(M,\mathbb{F})$.

Lemma 3.24 (Culler-Shalen). *[CS83, Lemma 1.5.3] Let M be a compact orientable 3–manifold. If $R_0 \subseteq \mathrm{R}(M,\mathbb{F})$ is an irreducible component containing an irreducible representation $\rho_0 \in \mathrm{R}_0$, and if*

$X_0 = \mathrm{tr_n}(R_0)$, *then*

$$\dim(X_0) = \dim(R_0) - 3.$$

Proof. We follow the proof of [CS83, Lemma 1.5.3] verbatim. By Corollary 3.13, the set of reducible representations has the form ${\mathrm{tr_n}}^{-1}(X_n^{red})$ for the closed algebraic set $X_n^{red} \subseteq \mathbb{F}^d$.

Set $U = X_0 \cap (\mathbb{F}^d - X_n^{red})$. For each $\chi \in U$, ${\mathrm{tr_n}}^{-1}(\chi)$ consists of irreducible representations. By Corollary 3.6, ${\mathrm{tr_n}}^{-1}(\chi)$ must be a single equivalence class of representations.

Fix $\rho \in {\mathrm{tr_n}}^{-1}(\chi)$. Define $\sigma_\rho\colon \mathrm{SL}_2(\mathbb{F}) \to {\mathrm{tr_n}}^{-1}(\chi)$ by $\sigma_\rho(M) = M\rho M^{-1}$. This is a covering map and for characteristic $p \neq 2$ this will be a two-sheeted covering map. This is because an irreducible representation has centraliser $\{\pm E\}$ in characteristic $p \neq 2$ and $\{E\}$ in characteristic 2.

Then $\dim({\mathrm{tr_n}}^{-1}(\chi)) = \dim(\mathrm{SL}_2(\mathbb{F})) = 3$ for every point χ in a Zariski open set $U \subset X_0$ and thus $\dim({\mathrm{tr_n}}^{-1}(X_0)) = 3$. We also have $\dim({\mathrm{tr_n}}^{-1}(X_0)) = \dim(R_0) - \dim(X_0)$. Together these give the result. □

We combine Lemmas 3.21, 3.22 and 3.24 as in [CS83, Proposition 3.2.1] to get the following bound on dimension initially due to Thurston now over $\mathbb{F}$.

Proposition 3.25 (Thurston)**.** *Let M be a compact orientable 3–manifold and $\mathbb{F}$ an algebraically closed field of characteristic p. Let $\rho_0\colon \pi_1(M) \to \mathrm{SL}_2(\mathbb{F})$ be an irreducible representation such that for each torus component T of ∂M, $\rho_0(\mathrm{im}(\pi_1(T) \to \pi_1(M))) \not\subseteq \{\pm E\}$. In characteristic 2, also assume that $\mathrm{im}(\rho_0)$ is not D_∞–like.*

Let R_0 be the irreducible component of $R(M)$ containing ρ_0. Then $X_0 = \mathrm{tr_n}(R_0)$ has dimension $\geq s - 3\chi(M)$, where s is the number of torus components of ∂M.

Proof. We again almost follow the proof of [CS83, Proposition 3.2.1] verbatim, with the exception of characteristic 2. By Lemma 3.24, $\dim(X_0) \geq s - 3\chi(M) \Leftrightarrow \dim(R_0) \geq s - 3\chi(M) + 3$. Continue by induction on s.

Assume $s = 0$. We assume the existence of an irreducible representation ρ_0. If $\partial M = \emptyset$ then $\chi(M) = 0$ and the result is trivially satisfied. Hence we can assume $\partial M \neq \emptyset$.

Then M will have the same homotopy type as a finite 2–dimensional CW-complex K with one 0–cell, m 1–cells, and n 2–cells. Then $\pi_1(M)$ has a presentation

$$\pi_1(M) = \langle g_1, \ldots, g_m \mid r_1, \ldots r_n \rangle$$

with $\chi(M) = \chi(K) = 1 - m + n$.

Define $f\colon \mathrm{SL}_2(\mathbb{F})^m \to \mathrm{SL}_2(\mathbb{F})^n$ to be the regular map $f(x_1, \ldots, x_m) = (r_1(x_1, \ldots, x_m), \ldots, r_m(x_1, \ldots, x_m))$. Then $\mathrm{R}(M, \mathbb{F}) = f^{-1}(E, \ldots, E)$ and $\mathrm{R}(M, \mathbb{F})$ is the inverse image of a point under a regular map from a $3m$–dimensional affine variety to a $3n$–dimensional affine variety with irreducible components. This gives

$$\dim(R_0) \geq 3m - 3n = -3\chi(M) + 3$$

which proves the result for $s = 0$.

Consider $s > 0$. Let T be a torus component of ∂M. Then $\rho_0(\mathrm{im}(\pi_1(T) \to \pi_1(M))) \not\subseteq \{\pm E\}$ means there exists $\alpha \in \pi_1(M)$ represented by a simple closed curve on T such that $\rho_0(\alpha) \neq \pm E$. In characteristic 2, we also required $\mathrm{im}(\rho_0)$ is not D_∞–like. Then by Lemma 3.21 and Lemma 3.22 there exists $\gamma \in \pi_1(M)$ such that ρ_0 restricted to the subgroup generated by α and γ is irreducible and $\mathrm{tr}(\rho_0(\gamma))^2 \neq 4$.

We can construct a manifold M' with one less torus boundary component than M by removing a neighbourhood of an embedded loop on T that represents $\gamma \in \pi_1(M)$. Then T will become a genus two boundary component T'. Our original manifold M can be obtained from M' by adding a 2–handle to T'.

We may choose a standard basis $\alpha', \beta', \gamma', \delta'$ of $\pi_1(T')$ such that α', γ' are mapped to α, γ under the natural surjection $i_*\colon\ \pi_1(M') \to \pi_1(M)$. The 2–handle is attached to T' along a simple closed curve that represents δ'. We write $\sigma = [\alpha', \beta'] = [\gamma', \delta']$.

Define a representation $\rho_0' = \rho_0 \circ i_* \in \mathrm{R}(M', \mathbb{F})$. This is irreducible, as ρ_0 is irreducible and i_* is surjective. Since the kernel of i_* is the normal closure of δ', we can identify $\mathrm{R}(M, \mathbb{F})$ with $W \subset \mathrm{R}(M', \mathbb{F})$ the closed algebraic set of representations ρ such that $\rho(\delta') = E$. Let R_0' be the irreducible component of $\mathrm{R}(M', \mathbb{F})$ containing ρ_0'. Then R_0 is an irreducible component of $R_0' \cap W$.

By the induction hypothesis, M' has $s-1$ torus components and

$$\dim(R_0') \geq (s-1) - 3\chi(M') + 3 = s - 3\chi(M) + 5.$$

Let $g\colon\ R_0' \to \mathbb{F}^2$ be defined by $g(\rho) = (\mathrm{tr}(\rho(\delta')), \mathrm{tr}(\rho(\sigma)))$. Then $g(R_0) = (2,2)$ and $R_0 \subset g^{-1}(2,2)$.

Let Y be the set of all representations $\rho \in R_0'$ whose restriction to the subgroup in $\pi_1(M')$ generated by α', γ' is reducible. By Corollary 3.13, Y is a closed algebraic set. Let Z be the closed algebraic set consisting of all $\rho \in R_0'$ such that $\mathrm{tr}(\rho(\gamma')) = \pm 2$. Clearly, $\rho_0 \notin Y \cup Z$.

Take $\rho \in g^{-1}(2,2)$. Then $\mathrm{tr}(\rho(\delta')) = \mathrm{tr}(\rho(\sigma)) = 2$. By Lemma 1.15, the restrictions of ρ to the subgroups generated by α', β' and by γ', δ' are reducible and $\rho(\sigma)$ has an eigenvector in common with $\rho(\alpha'), \rho(\gamma')$. If $\rho(\sigma) \neq E$, then it has a unique, 1–dimensional invariant subspace and the pair $\rho(\alpha'), \rho(\gamma')$ is reducible. Then $\rho \in Y$. If $\rho(\sigma) = E$, then $\rho(\gamma')$ and $\rho(\delta')$ commute. Since $\mathrm{tr}(\rho(\delta')) = 2$, we either have $\rho(\delta') = E$, which implies $\rho \in W$, or $\mathrm{tr}(\rho(\gamma')) = \pm 2$, which implies $\rho \in Z$.

Putting this all together, $g^{-1}(2,2) \subset (W \cap R_0') \cup Y \cup Z$. Then

$$\begin{aligned} \dim(R_0) &= \dim(R_0') - \dim(\mathbb{F}^2) \\ &\geq s - 3\chi(M) + 3. \end{aligned}$$

This finishes the proof. $\square$

Culler and Shalen use this result to find examples of curves. For example, a hyperbolic 3–manifold with at least one boundary torus has a curve in the $\mathrm{SL}_2(\mathbb{C})$–character variety and we can apply the theory of Culler and Shalen to detect at least one essential surface.

3.4 Applications to the A–polynomial

The A–polynomial was introduced in [CCG+94] over $\mathrm{SL}_2(\mathbb{C})$, with an alternative, equivalent, definition presented in [CL96] using the eigenvalue map and eigenvalue varieties. We define the A–polynomials and eigenvalue varieties over $\mathbb{F}$ and revise the theory in this setting for both a single boundary component following following [CL96] and multiple boundary components following [Til05]. We compare the A–polynomial in arbitrary characteristic p to the standard A–polynomial over $\mathbb{C}$.

3.4.1 A single boundary component

Consider manifold M with boundary a torus T. We can define the A–polynomial and eigenvalue variety over a general field $\mathbb{F}$. We have fundamental group

$$\pi_1(M) = \langle \gamma_1, \dots \gamma_n \mid r_1, \dots, r_m \rangle.$$

Call generators for $\pi_1(T)$ a meridian $\mathscr{M}$ and longitude $\mathscr{L}$ and identify them with their images in $\pi_1(M)$ under the inclusion homomorphism. Let $\mathrm{R}_U(M,\mathbb{F})$ be the subvariety of $\mathrm{R}(M,\mathbb{F})$ defined so that the lower left entries of $\rho(\mathscr{M})$ and $\rho(\mathscr{L})$ are zero,

$$\mathrm{R}_U(M,\mathbb{F}) = \left\{ \rho\colon\ \pi_1(M) \to \mathrm{SL}_2(\mathbb{F}) \;\middle|\; \rho(\mathscr{M}) = \begin{pmatrix} m & * \\ 0 & m^{-1} \end{pmatrix}, \rho(\mathscr{L}) = \begin{pmatrix} l & * \\ 0 & l^{-1} \end{pmatrix} \right\}.$$

Every representation in $\mathrm{R}(M,\mathbb{F})$ is conjugate to such a representation as the peripheral subgroup is abelian.

Define the **eigenvalue map** to be

$$\begin{aligned} e\colon\ \mathrm{R}_U(M,\mathbb{F}) &\to (\mathbb{F}\setminus\{0\})^2 \\ \rho &\mapsto (m,l) \end{aligned}$$

taking representation ρ to the eigenvalues of $\rho(\mathscr{M})$ and $\rho(\mathscr{L})$ corresponding to the common invariant subspace spanned by the first basis vector.

We may assume that $\gamma_1 = \mathscr{M}$, $\gamma_2 = \mathscr{L}$ in the presentation of the fundamental group. Then

$$\mathrm{R}_U(M,\mathbb{F}) = \left\{ (A_1,\dots,A_n) \in \mathrm{SL}_2(\mathbb{F})^n \;\middle|\; R_1,\dots,R_m \text{ and } A_1 = \begin{pmatrix} m & * \\ 0 & m^{-1} \end{pmatrix}, A_2 = \begin{pmatrix} l & * \\ 0 & l^{-1} \end{pmatrix} \right\} \subseteq \mathbb{F}^{4n} \tag{3.4.1}$$

where $A_i = \rho(\gamma_i) = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$ and $R_j = r_j(A_1,\dots,A_n)$. The entries of R_j are polynomials in $\mathbb{Z}_p[a_1,\dots,d_n]$ where $m = a_1$ and $l = a_2$. Note we have $c_1 = 0 = c_2$ and the determinant conditions $a_id_i - b_ic_i = 1$.

Then the map e is precisely the projection

$$\begin{aligned} e\colon\ \mathrm{R}_U(M,\mathbb{F}) &\to (\mathbb{F}\setminus\{0\})^2 \\ (a_1,\dots,d_n) &\mapsto (a_1,a_2) = (m,l). \end{aligned}$$

The **eigenvalue variety** is the closure of the image of this map, $\overline{e(\mathrm{R}_U(M,\mathbb{F}))}$. By [Til05], the dimension of the eigenvalue variety is at most one. We write

$$\overline{e(\mathrm{R}_U(M,\mathbb{F}))} = V_0 \cup V_1$$

where V_0 is the union of all 0–dimensional components and V_1 is the union of all 1–dimensional components.

If V_1 is non-empty, then it is defined by a principal ideal generated by some polynomial $\widetilde{A}_{\mathbb{F}}$ in variables m and l. A generator for the radical of the ideal is the A**–polynomial**, written $A_{\mathbb{F}}$.

The A–polynomial contains information about the incompressible surfaces in the manifold M. In particular, the well-known boundary slopes theorem from [CCG+94] asserts that the slopes in the Newtown polygon of the A–polynomial of M are precisely the slopes of incompressible surfaces in M associated to an action of $\pi_1(M)$ on a tree τ. Recall the Newtown polygon of a polynomial $f(x,y)$ is the convex hull of integer lattice points (s,t) in the plane that are degrees of monomials $x^s y^t$ that appear in $f(x,y)$.

The A–polynomial can be calculated using elimination theory by way of resultants or **Gröbner bases** (see [CL96, CLO15]), the latter of which give a standard basis for an ideal that behaves well with polynomial division. Once a basis has been chosen for the boundary torus, the generator of the principal ideal $\widetilde{A}_{\mathbb{F}}$ and the A–polynomial are both well-defined up to multiplication by units in $\mathbb{F}(m^{\pm 1}, l^{\pm 1})$.

We make some remarks about Gröbner bases and how they are calculated.

Remark 3.26. *The Gröbner basis of a polynomial ideal $I \subset k[x_1,\dots,x_n]$ is calculated using* Buchberger's algorithm *(see [CLO15, Theorem 2]). Using the ascending chain condition for ideals we know we can do this using finitely many polynomials.*

For our purposes, we use a modified version of Buchberger's algorithm where there is no division by coefficients in k. In particular, if an element has coefficients in a ring $R \subset k$ then they will remain in R under our algorithm. We describe the differences in the versions of the algorithm below.

The main ingredients of Buchberger's algorithm are the division algorithm *(given in[CLO15, Theorem 3]) and the S–polynomial. Given a non-zero polynomial $f \in k[x_1,\dots x_n]$, we can write f as a sum of monomials*

$$f = \sum_\alpha a_\alpha x^\alpha, \quad a_\alpha \in k$$

for $x = (x_1,\dots,x_n)$, $\alpha = (\alpha_1,\dots \alpha_n) \in \mathbb{Z}^n_{\geq 0}$, $x^\alpha = x_1^{\alpha_1}\dots x_n^{\alpha_n}$. For a given monomial ordering $>$, we say the multidegree of f is

$$\operatorname{multideg}(f) = \max\{\alpha \in \mathbb{Z}^n_{\geq 0} \mid a_\alpha \neq 0\}.$$

Then the ***leading monomial***, ***leading coefficient***, *and* ***leading term*** *of f are respectively given by*

$$\mathrm{LM}(f) = x^{\operatorname{multideg}(f)}, \quad \mathrm{LC}(f) = a_{\operatorname{multideg}(f)}, \quad \mathrm{LT}(f) = a_{\operatorname{multideg}(f)} x^{\operatorname{multideg}(f)}.$$

Given a set of polynomials $f_1,\dots,f_m$ and a monomial ordering, the ***division algorithm*** *finds a way to write a polynomial f in terms of $f_1,\dots,f_m$,*

$$f = a_1 f_1 + \ldots + a_m f_m + r, \quad a_i, r \in k[x_1,\dots,x_n]$$

with $r = 0$ or no monomial in r being divisible by $\mathrm{LT}(f_i)$ for all $1 \le i \le m$. The algorithm involves dividing terms by $\mathrm{LT}(f_i)$. For example, in order to find the remainder r in the steps of the algorithm, if $\mathrm{LT}(f_i)$ divides $\mathrm{LT}(f)$ then we take iterations of

$$f \mapsto f - \frac{\mathrm{LT}(f)}{\mathrm{LT}(f_i)} f_i. \tag{3.4.2}$$

*Given a pair of polynomials $f, g \in k[x_1, \dots x_n]$ the **S–polynomial** of f and g cancels the leading terms in both f and g to get a polynomial of lower degree. Let β_i be the powers of x_i in g. Then*

$$S(f,g) = \frac{x^\gamma}{\mathrm{LT}(f)} f - \frac{x^\gamma}{\mathrm{LT}(g)} g \quad \textit{for} \quad \gamma = (\gamma_1, \dots, \gamma_n),\ \gamma_i = \max(\alpha_i, \beta_i). \tag{3.4.3}$$

*Approximately, **Buchberger's algorithm** starts with a generating set for the ideal, takes each pair of polynomials in the set, calculates the S–polynomial for this pair, and then performs the division algorithm of the S–polynomial with the generating set. If the remainder is non-zero this is added to the generating set, until all pairs have an S–polynomial with zero remainder under the division algorithm. The Gröbner basis can then be deduced from this.*

In order to apply Buchberger's algorithm over $\mathbb{Z}$ we replace every instance of division by $\mathrm{LT}(h)$ for a polynomial h to be instead division by $\mathrm{LM}(h)$ and multiplying all other terms by $\mathrm{LC}(h)$. For example Equation (3.4.2) becomes

$$f \mapsto \mathrm{LC}(f_i) f - \frac{\mathrm{LT}(f)}{\mathrm{LM}(f_i)} f_i$$

and Equation (3.4.3) becomes

$$S(f,g) = \frac{\mathrm{LC}(g) x^\gamma}{\mathrm{LM}(f)} f - \frac{\mathrm{LC}(f) x^\gamma}{\mathrm{LM}(g)} g.$$

The result will be the same as the previous algorithm up to multiplication by a constant in k. In particular the degrees of polynomials remain the same but no terms are ever divided by a coefficient. This makes it easier to compare the results of the algorithm across characteristics, which we will see later.

Note when we refer to the Gröbner basis, we are typically thinking of the *reduced* Gröbner basis, although this should not change any mathematical results. Typically, this is a Gröbner basis such that:

1. Every polynomial in the Gröbner basis has leading coefficient 1, and
2. For all elements p of the Gröbner basis G, no monomial of p lies in $\langle \mathrm{LT}(G \setminus \{p\}) \rangle$.

When using the modified Buchberger's algorithm described above we waive the requirement for Item 1. Any nonzero polynomial ideal has a unique reduced Gröbner basis (see [CLO15, Proposition 6]), and under the modified algorithm this will be unique up to a common coefficient.

Lemma 3.27. *Let $\mathbb{F}$ be an algebraically closed field of characteristic p. A generator for the radical of the principal ideal of V_1 can be chosen in $\mathbb{Z}_p[m^{\pm 1}, l^{\pm 1}]$. Moreover, $A_{\mathbb{F}}$ only depends on p, not $\mathbb{F}$.*

Proof. Let $\mathbb{F}$ be an algebraically closed field of characteristic p. The argument is similar to a result in [CCG+94], where it is shown we may multiply $A_{\mathbb{C}}$ by a suitable nonzero constant so that the coefficients are integers. We may similarly multiply $A_{\mathbb{F}}$ by a suitable nonzero constant so that the coefficients are elements in $\mathbb{Z}_p$.

Consider the polynomial ideals associated with $\mathrm{R}_U(M,\mathbb{F})$. Let $I_{\mathbb{F}} = \langle f_1,\dots,f_k\rangle$, $f_i \in \mathbb{Z}_p[a_1,\dots,d_n]$ be the ideal defining the variety $\mathrm{R}_U(M,\mathbb{F})$ such that

$$\mathrm{R}_U(M,\mathbb{F}) = V(I_{\mathbb{F}}) = \left\{(w_1,\dots,w_{4n}) \in \mathbb{F}^{4n} \mid f_i(w_1,\dots,w_{4n}) = 0 \text{ for all } 1 \le i \le k\right\}. \tag{3.4.4}$$

Here, the polynomials f_i are determined by the entries of the relations R_i from Equation (3.4.1) together with the determinant conditions. By definition, the closure of the image under the eigenvalue map is $\overline{e(V(I_{\mathbb{F}}))} = V(\langle \widetilde{A}_{\mathbb{F}}\rangle)$.

Elimination theory shows us that for any field $\mathbb{F}$ the polynomial $\widetilde{A}_{\mathbb{F}}$ will appear as an entry in the Gröbner bases calculated with respect to lexicographic ordering of $I_{\mathbb{F}}$ (see [CLO15]). In fact, the polynomial will be the unique entry with only variables m and l. Following Buchberger's algorithm, we know we can calculate the Gröbner basis with finitely many polynomials.

We follow Buchberger's algorithm so that all operations are done over $\mathbb{Z}$ as described in Remark 3.26. Following this process, $\widetilde{A}_{\mathbb{F}}$ will appear in the Gröbner basis as a the unique entry in $\mathbb{Z}_p[m^{\pm 1}, l^{\pm 1}]$. For all distinct algebraically closed fields $\mathbb{F}$ and $\mathbb{F}'$ both of characteristic p, $\widetilde{A}_{\mathbb{F}}$ and $\widetilde{A}_{\mathbb{F}'}$ are calculated the same way and will be the same. Then the A–polynomials also agree. This proves the result. □

We omit the specific field and denote the different A–polynomials by characteristic, $\widetilde{A}_p = \widetilde{A}_{\mathbb{F}}$ and $A_p = A_{\mathbb{F}}$ for $\mathbb{F}$ a field of characteristic p. If we require the A–polynomial has no common integer factor, then we can consider the polynomial unique up to sign.

We have a version of the boundary slopes theorem for characteristic p, which follows, as in characteristic 0, from Corollary 3.20 and the relationship between valuations and boundary slopes of the Newton polygon:

Theorem 3.28. *Let $\mathbb{F}$ be an algebraically closed field of characteristic p. The boundary slopes of the Newton polygon for A_p are the strongly detected boundary slopes for* $\mathrm{X}(M,\mathbb{F})$.

Another well-known result is the ones in the corners theorem from [CL97]. This asserts that if the A–polynomial is normalised so that the greatest common divisor of the coefficients is one, the corner vertices in the Newtown polygon of the A–polynomial of M have associated coefficient ± 1. In this way we can think of the A–polynomial as being monic, in the sense that the dominant terms have coefficient ± 1.

Considering fields of positive characteristic, a corollary to the ones in the corners theorem immediately follows. Let $\overline{A_0} = A_0 \pmod{p}$ and more generally for f a polynomial with integer coefficients, let $\overline{f} = f \pmod{p}$ for a given prime p be the polynomial with coefficients modulo p. We will use this notation when p is fixed. Since the coefficients in the corners will remain unchanged modulo p, we have:

Corollary 3.29. *The Newton polygon of $\overline{A_0} = A_0 \pmod{p}$ is the same as the Newton polygon of A_0 for all primes p.*

Recall $\widetilde{A}_p$ is the polynomial that generates the ideal that defines the eigenvalue variety in characteristic p (whilst A_p is a generator of the radical of the ideal defined by $\widetilde{A}_p$).

Lemma 3.30. *For all but finitely many primes p,*

$$\overline{\widetilde{A}_0} = \widetilde{A}_p. \tag{3.4.5}$$

Proof. Let $\mathbb{F}$ be an algebraically closed field of characteristic p. By Lemma 3.27 it is sufficient to only consider the characteristic of the field. Consider the polynomial ideals associated with $\mathrm{R}_U(M,\mathbb{F})$ in characteristic 0 and characteristic p.

Let $I_0 = \langle f_1,\dots,f_k\rangle$, $f_i \in \mathbb{Z}[a_1,\dots,d_n]$ be the ideal associated with the variety $\mathrm{R}_U(M,\mathbb{F}_0)$ such that

$$\mathrm{R}_U(M,\mathbb{F}_0) = V(I_0) = \left\{(w_1,\dots,w_{4n}) \in \mathbb{F}_0^{4n} \;\middle|\; f_i(w_1,\dots,w_{4n}) = 0 \text{ for all } 1 \le i \le k\right\}. \tag{3.4.6}$$

Here, the polynomials f_i are determined by the relations in Equation (3.4.1). By definition, the closure of the image under the eigenvalue map is $\overline{e(V(I_0))} = V(\langle \widetilde{A}_0\rangle)$.

Then in characteristic p, write $I_p = \langle \overline{f_1},\dots,\overline{f_k}\rangle$, $\overline{f_i} \in \mathbb{Z}_p[a_1,\dots,d_n]$ for the corresponding ideal I_0 modulo p. We have

$$\mathrm{R}_U(M,\mathbb{F}_p) = V(I_p) = \left\{(w_1,\dots,w_{4n}) \in \mathbb{F}_p^{4n} \;\middle|\; \overline{f_i}(w_1,\dots,w_{4n}) = 0 \text{ for all } 1 \le i \le k\right\}. \tag{3.4.7}$$

Again by definition, the closure of the image under the eigenvalue map is $\overline{e(V(I_p))} = V(\langle \widetilde{A}_p\rangle)$.

Elimination theory shows us that the respective polynomials $\widetilde{A}_0$ and $\widetilde{A}_p$ will appear as an entry in the Gröbner bases calculated with respect to lexicographic ordering of I_0 and I_p (see [CLO15]). In fact, they will be the unique entry with only variables m and l.

Follow the modified Buchberger's algorithm from Remark 3.26 to calculate the Gröbner bases, G_0 and G_p respectively, of I_0 and I_p over $\mathbb{Z}$ and $\mathbb{Z}_p$ with respect to lexicographic ordering. Suppose that the prime p does not divide any of the leading terms of polynomials that appear in the Gröbner basis calculation for G_0. Then the Gröbner basis G_0 modulo p, $\overline{G_0}$, is the same as the Gröbner basis G_p. In this case, we get

$$\overline{\widetilde{A}_0} = \widetilde{A}_p.$$

This proves the result for all primes p that do not divide the leading term of a polynomial in Buchberger's algorithm over $\mathbb{Z}$. Since there are only finitely many leading terms, the proof is complete. □

This leads to the following.

Lemma 3.31. *For all but finitely many primes p, the A–polynomial in characteristic p divides the A–polynomial in characteristic 0 modulo p. Moreover,*

$$\overline{A_0}(m,l) = g(m,l)A_p(m,l), \tag{3.4.8}$$

where $g \in \mathbb{Z}_p[m,l]$ consists only of polynomial factors that appear in A_p.

Proof. This follows directly from Lemma 3.30 and the definition of the A–polynomial. Let us break this down further. As in the proof of Lemma 3.30, consider the finitely many primes p that do not divide any of the leading terms of polynomials that appear in the Gröbner basis calculations over $\mathbb{Z}$.

The A–polynomial $A_0(m,l)$ is the radical of $\widetilde{A}_0(m,l)$. Then looking at their irreducible factorisation over $\mathbb{Z}[m,l]$ we find,

$$\begin{aligned} \widetilde{A}_0 &= a_0 g_1^{k_1} \dots g_r^{k_r}, \quad a_0 \in \mathbb{Z} \setminus \{0\},\, k_i \in \mathbb{Z}_{>0} \\ A_0 &= g_1 \dots g_r. \end{aligned}$$

for $g_i \in \mathbb{Z}[m,l]$ irreducible and pairwise distinct ($g_i \neq g_j$ for all $i \neq j$). Note Equation (3.4.5) implies $\overline{a_0} \neq 0 \pmod p$. Consider how this changes modulo p. We could find that there is some i such that $\overline{g_i}$ is no longer irreducible, for example, $\overline{g_i} = h_{i_1} \dots h_{i_s}$ for $h_{i_k} \in \mathbb{Z}[m,l]$ irreducible, or we could find that the collection of $\overline{g_i}$, $i \in \{1,\dots,r\}$ is no longer distinct, for example, $\overline{g_i} = \overline{g_j}$ for $i \neq j$, or both. Then,

$$\begin{aligned} \overline{\widetilde{A}_0} &= \overline{a_0 g_1}^{k_1} \dots \overline{g_r}^{k_r}, \\ &= \overline{a_0 g_{\beta_1}}^{b_1'} \dots \overline{g_{\beta_s}}^{b_s'} h_1^{c_1'} \dots h_t^{c_t'}, \quad b_i', c_i' \in \mathbb{Z}_p \setminus \{0\}, \\ \overline{A_0} &= \overline{g_1} \dots \overline{g_r}, \\ &= \overline{g_{\beta_1}}^{b_1} \dots \overline{g_{\beta_s}}^{b_s} h_1^{c_1} \dots h_t^{c_t}, \quad b_i, c_i \in \mathbb{Z}_p \setminus \{0\}. \end{aligned}$$

Here, $\{\beta_1,\dots,\beta_s\}$ is the subset of $\{1,\dots,r\}$ such that $\overline{g_{\beta_1}},\dots,\overline{g_{\beta_s}}$ are irreducible and distinct, and h_j are the collection of irreducible and distinct factors of the remaining $\overline{g_i}$. We have $1 \leq b_i \leq b_i'$ and $1 \leq c_i \leq c_i'$.

Then using Equation (3.4.5), taking the radical of $\overline{\widetilde{A}_0}$ and removing common integer factors gives,

$$A_p = \overline{g_{\beta_1}} \dots \overline{g_{\beta_s}} h_1 \dots h_t. \tag{3.4.9}$$

Then it is clear that

$$\overline{A_0} = \overline{g_{\beta_1}}^{b_1 - 1} \dots \overline{g_{\beta_s}}^{b_s - 1} h_1^{c_1 - 1} \dots h_t^{c_t - 1} A_p$$

and the polynomial

$$g = \overline{g_{\beta_1}}^{b_1 - 1} \dots \overline{g_{\beta_s}}^{b_s - 1} h_1^{c_1 - 1} \dots h_t^{c_t - 1} \in \mathbb{Z}[m,l] \tag{3.4.10}$$

satisfies the statement.

This proves the result for all primes p that do not divide the leading term of a polynomial in Buchberger's algorithm over $\mathbb{Z}$. □

We can make a few remarks about the result.

Remark 3.32. 1. *The factor $g(m,l)$ is important and the A–polynomial in characteristic p is not necessarily the same as reducing the A–polynomial in characteristic 0 modulo p.*

2. *There may be examples where the prime p does divide the leading term of a polynomial in the Gröbner basis calculations over $\mathbb{Z}$ and the result still holds.*

We will see an example where the A–polynomial changes under different characteristics in Section 4.2.1 (although the detected boundary slopes does not change). Lemma 3.31 is not applicable in the example because the prime p divides leading terms when calculating the Gröbner basis.

Theorem 3.33. *Consider M with a single boundary torus $\partial M = T$. For all but finitely many primes p, the boundary curve of an essential surface in M is strongly detected by A_0 if and only if it is strongly detected by A_p.*

Proof. This follows from Lemmas 3.27, 3.30 and 3.31. The Newton polygon of a product of polynomials $f(x,y) = h(x,y)i(x,y)$ is the sum of the Newton polygons of h and i. In particular, if neither $h(x,y)$ and $i(x,y)$ are monomials, then the slopes of the Newton polygon of f will consist of the slopes of the Newton polygon of h and the slopes of the Newton polygon of i and, if one of $h(x,y)$ and $i(x,y)$ is a monomial, then the slopes of the Newton polygon of f will consist of the slopes of the Newton polygon of the other polynomial. We have Equation (3.4.8) from Lemma 3.31 of this form. Recall by Lemma 3.27, the A–polynomial depends only on the characteristic of $\mathbb{F}$.

Consider a slope on ∂M that is strongly detected by $\mathrm{X}(M,\mathbb{F})$. By the boundary slopes theorem in characteristic p, the slope corresponds to a boundary slope of the Newton polygon of $A_p(m,l)$. By Equation (3.4.8) and above, for all but finitely many primes p, this must also be a boundary slope of the Newton polygon of $\overline{A_0}(m,l)$. By Corollary 3.29, this is the same as a boundary slope of the Newton polygon of $A_0(m,l)$. Then, by Lemma 3.27 and the boundary slopes theorem in characteristic 0, this must be strongly detected by $\mathrm{X}(M,\mathbb{C})$.

Consider a slope on ∂M that is strongly detected by $\mathrm{X}(M,\mathbb{C})$. By the boundary slopes theorem in characteristic 0, the slope corresponds to a boundary slope of the Newton polygon of $A_0(m,l)$. By Corollary 3.29, this is a boundary slope of the Newton polygon of $\overline{A_0}(m,l)$. Then all slopes of the Newton polygon of $\overline{A_0}(m,l)$ will be either a slope of the Newton polygon of $g(m,l)$ or a slope of the Newton polygon of $A_p(m,l)$. But $g(m,l)$ consists only of polynomial factors already present in $A_p(m,l)$. Thus the slope must be a slope of the Newton polygon of $A_p(m,l)$. By the boundary slopes theorem in characteristic p, this must be strongly detected by $\mathrm{X}(M,\mathbb{F})$. This proves the statement. □

3.4.2 The case of multiple boundary components

Consider M with h boundary tori $T_1,\dots,T_h$. We can extend the definitions of the A–polynomial and the eigenvalue variety over a general algebraically closed field $\mathbb{F}$ for a manifold with multiple boundary tori. Assume $\mathbb{F}$ is an algebraically closed field of characteristic p. We follow the approach in [Til05].

Assume we have fundamental group

$$\pi_1(M) = \langle \gamma_1,\dots\gamma_n \mid r_1,\dots,r_m \rangle.$$

Label the meridian and longitude for each boundary torus $\mathscr{M}_i$, $\mathscr{L}_i$. This forms a basis of $\pi_1(T_i)$ for each $i = 1,\dots,h$. Consider the representation variety,

$$\mathrm{R}(M,\mathbb{F}) = \{(A_1,\dots,A_n) \in \mathrm{SL}_2(\mathbb{F})^n \mid R_1,\dots,R_m\} \subseteq \mathbb{F}^{4n} \tag{3.4.11}$$

where $A_j = \rho(\gamma_j) = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}$ and $R_k = r_k(A_1, \dots, A_n)$. The entries of R_k are polynomials in $\mathbb{Z}_p[a_1, \dots, d_n]$ and we also have the determinant conditions $a_j d_j - b_j c_j = 1$.

For each i, we can identify $\mathscr{M}_i$, $\mathscr{L}_i$ with generators of $\operatorname{im}(\pi_1(T_i) \to \pi_1(M))$. Then $\mathscr{M}_i$ and $\mathscr{L}_i$ are words in the generators for $\pi_1(M)$. Label the eigenvalues of $\mathscr{M}_i$ and $\mathscr{L}_i$ as $m_i^{\pm 1}$ and $l_i^{\pm 1}$, respectively, such that m_i and l_i correspond to a common 1–dimensional invariant subspace.

Define the **eigenvalue map** to be

$$\begin{aligned} e\colon\ \mathrm{R}(M,\mathbb{F}) &\to (\mathbb{F} \setminus \{0\})^{2h} \\ \rho &\mapsto (m_1, l_1, \dots, m_h, l_h) \end{aligned}$$

taking representation ρ to eigenvalues of $\rho(\mathscr{M}_i)$ and $\rho(\mathscr{L}_i)$ for $i = 1, \dots, h$.

We can frame this another way. The representation variety is the affine variety of some ideal $\mathrm{R}(M,\mathbb{F}) = V(J_{\mathbb{F}})$, $J_{\mathbb{F}} = \langle f_1, \dots, f_k \rangle$, $f_i \in \mathbb{Z}_p[a_1, \dots, d_n]$. For each $\gamma \in \pi_1(M)$, there is an element $I_\gamma \in \mathbb{Z}_p[a_1, \dots, d_n]$ such that $I_\gamma(\rho) = \operatorname{tr}(\rho(\gamma))$. In particular, this holds for each $\mathscr{M}_i$, $\mathscr{L}_i$, for all $i = 1, \dots, h$. Consider the ideal

$$\begin{aligned} J'_{\mathbb{F}} &= \langle m_i + m_i^{-1} - I_{\mathscr{M}_i},\ l_i + l_i^{-1} - I_{\mathscr{L}_i},\ m_i l_i + m_i^{-1} l_i^{-1} - I_{\mathscr{M}_i \mathscr{L}_i} \text{ for all } i = 1, \dots, h \rangle \\ &\subseteq \mathbb{Z}_p[a_1, \dots, d_n, m_1^{\pm 1}, l_1^{\pm 1}, \dots, m_h^{\pm 1}, l_h^{\pm 1}]. \end{aligned}$$

We combine ideals $J_{\mathbb{F}}$ and $J'_{\mathbb{F}}$ to get the variety

$$\mathrm{R}_E(M,\mathbb{F}) = V(\langle J_{\mathbb{F}}, J'_{\mathbb{F}} \rangle) \subseteq \mathbb{F}^{4n} \times (\mathbb{F} \setminus \{0\})^{2h}. \tag{3.4.12}$$

Then the map e can also be seen as the projection

$$\begin{aligned} e\colon\ \mathrm{R}_E(M,\mathbb{F}) &\to (\mathbb{F} \setminus \{0\})^{2h} \\ (a_1, \dots, d_n, m_1, l_1, \dots, m_h, l_h) &\mapsto (m_1, l_1, \dots, m_h, l_h). \end{aligned}$$

The **eigenvalue variety** is the closure of the image of this map, $\overline{e(\mathrm{R}_E(M,\mathbb{F}))}$. Again, the proofs in [Til05] apply verbatim to show that the dimension of the eigenvalue variety is at most the number of torus boundary components.

The eigenvalue variety is defined by an ideal $K_{\mathbb{F}} \subseteq \mathbb{F}[m_1, l_1, \dots, m_h, l_h]$, $\overline{e(\mathrm{R}_E(M,\mathbb{F}))} = V(K_{\mathbb{F}})$. The ideal $K_{\mathbb{F}}$ can be calculated using elimination theory by way of resultants or Gröbner bases and is the multiple boundary component analogue to the A–polynomial. As described in the previous section, the Gröbner basis can be calculated using Buchberger's algorithm (or a modified version of this described in Remark 3.26) using finitely many polynomials. Once a basis has been chosen for each boundary torus, $K_{\mathbb{F}}$ is well-defined up to multiplication by units in $\mathbb{F}(m_i^{\pm 1}, l_i^{\pm 1})$.

Lemma 3.34. *Let $\mathbb{F}$ be an algebraically closed field of characteristic p. The defining ideal for the eigenvalue variety can be chosen such that the generating set for $K_{\mathbb{F}}$ is inside $\mathbb{Z}_p[m_1^{\pm 1}, l_1^{\pm 1}, \dots, m_h^{\pm 1}, l_h^{\pm 1}]$.*

Proof. Let $\mathbb{F}$ be an algebraically closed field of characteristic p. Similar to the proof in [CCG+94] for a single torus, we may multiply the generators for $K_{\mathbb{C}}$ by a suitable nonzero constant so that the coefficients are integers. We may similarly multiply the generators for $K_{\mathbb{F}}$ by a suitable nonzero constant so that the coefficients are elements in $\mathbb{Z}_p$.

Consider the polynomial ideals associated with $\mathrm{R}_E(M,\mathbb{F})$, $J_{\mathbb{F}}$ and $J'_{\mathbb{F}}$. Here, the f_j in $J_{\mathbb{F}}$ are determined by the entries of the relations R_j in Equation (3.4.1) together with the determinant conditions and the equations in $J'_{\mathbb{F}}$ are trace relations for the meridian and longitude of each boundary torus. By definition, the closure of the image under the eigenvalue map is $\overline{e(V(I_{\mathbb{F}}))} = V(K_{\mathbb{F}})$.

Elimination theory shows us that for any field $\mathbb{F}$ the elements in $K_{\mathbb{F}}$ will appear as entries in the Gröbner basis calculated with respect to lexicographic ordering of $J_{\mathbb{F}}$ and $J'_{\mathbb{F}}$ (see [CLO15]). Following Buchberger's algorithm, we know we can calculate the Gröbner basis with finitely many polynomials.

We follow Buchberger's algorithm so that all operations are done over $\mathbb{Z}$ as described in Remark 3.26. Then $K_{\mathbb{F}}$ is the resulting Gröbner basis. For all distinct algebraically closed fields $\mathbb{F}$ and $\mathbb{F}'$ both of characteristic p, the generating set for $K_{\mathbb{F}}$ and $K_{\mathbb{F}'}$ are calculated the same way and will be the same. This proves the result. □

Note the generating set for both $K_{\mathbb{F}}$ and $K_{\mathbb{F}'}$ can be chosen in $\mathbb{Z}_p[m_1^{\pm 1}, l_1^{\pm 1}, \ldots, m_h^{\pm 1}, l_h^{\pm 1}]$ but we still consider the ideals in separate rings with $K_{\mathbb{F}} \subseteq \mathbb{F}[m_1^{\pm 1}, l_1^{\pm 1}, \ldots, m_h^{\pm 1}, l_h^{\pm 1}]$ and $K_{\mathbb{F}'} \subseteq \mathbb{F}'[m_1^{\pm 1}, l_1^{\pm 1}, \ldots, m_h^{\pm 1}, l_h^{\pm 1}]$.

We recover the information about boundary slopes by taking the logarithmic limit set of the eigenvalue variety. Given an affine variety $V = V(I)$, we list three definitions for the **logarithmic limit set** following [Ber71]. Let $\mathbb{F}[X^{\pm}] = \mathbb{F}[X_1^{\pm 1}, \ldots, X_m^{\pm 1}]$, $(x_1, \ldots, x_m) \in V$.

- An absolute value function on $\mathbb{F}$ is a function $|\cdot|\colon \mathbb{F} \to \mathbb{R}_{\geq 0}$ such that $|0| = 0$, $|1| = 1$, $|xy| = |x||y|$, $|x+y| \leq |x| + |y|$ and there exists $x \in \mathbb{F}$ such that $0 < |x| < 1$. If $\mathbb{F}$ has an absolute value function, define $V_\infty^{(a)}(I)$ as the set of limit points on S^{m-1} of points of the form
$$\frac{(\log|x_1|, \ldots, \log|x_m|)}{\sqrt{1 + \sum (\log|x_i|)^2}}.$$
- Define $V_\infty^{(b)}(I)$ to be the set of m-tuples $(-v(X_1), \ldots, -v(X_m))$ as v ranges over all real-valued valuations on $\mathbb{F}[X^{\pm}]/I$ with $\sum v(X_i)^2 = 1$.
- For $f \in \mathbb{F}[X^{\pm 1}]$, let the support of f be
$$s(f) = \left\{ \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}^m \;\middle|\; x_1^{\alpha_1} \ldots x_m^{\alpha_m} \text{ occurs with nonzero coefficient in } f \right\}.$$
Define $V_\infty^{(c)}(I)$ to be the set of points $\zeta \in S^{m-1}$ such that, for all $0 \neq f \in I$, $\max\limits_{\alpha \in s(f)} (\zeta \cdot \alpha)$ is assumed at least twice.

The last definition gives another interpretation of the logarithmic limit set as the intersection of the spherical duals of the Newtown polytopes of nonzero elements of I. That is, the intersection of all outward pointing unit normal vectors to the support places of the Newton polytopes of nonzero elements of I (see [Til05]). We write,

$$V_\infty^{(c)}(I) = \bigcap_{f \in I} \mathrm{Sph}(\mathrm{Newt}(f))$$

for $\mathrm{Newt}(f)$ the Newton polytope of f and $\mathrm{Sph}(P)$ the spherical dual of a polytope P. Recall, the **Newton polytope** of a polynomial $f(x_1,\dots,x_m)$ is the convex hull of the support of f. This is a generalisation of the Newton polygon.

Bergman [Ber71, Theorem 2] proves that $V_\infty^{(b)}(I) = V_\infty^{(c)}(I)$. We write $V_\infty(I) = V_\infty^{(b)}(I) = V_\infty^{(c)}(I)$. Furthermore, if $\mathbb{F}$ is algebraically closed and equipped with an absolute value function, then $V_\infty^{(a)}(I)$ is a closed subset of $V_\infty(I)$, containing all points therein with rational coordinate ratios. This implies the definition of $V_\infty^{(a)}(I)$ does not depend on the choice of absolute value function and the definition of $V_\infty^{(b)}(I)$ does not depend on the choice of valuation.

A useful tool to understand and compute the logarithmic limit set through the third definition is *tropical bases*. An introduction to these concepts is in [MS15]. The tropical basis of an ideal over $\mathbb{F}[x_1^\pm,\dots,x_n^\pm]$ is the analogue of the *universal* Gröbner basis of an ideal over $\mathbb{F}[x_1,\dots,x_n]$ (which is a Gröbner basis over every monomial ordering). Not every universal Gröbner basis is a tropical basis.

Consider an ideal $I \subseteq \mathbb{F}[x_1^\pm,\dots,x_n^\pm]$. Over $\mathbb{C}$, Bogart et al. [BJS$^+$07] prove that I has a finite tropical basis $\{f_1,\dots,f_k\}$ with the property that

$$\bigcap_{f\in I} \mathrm{Sph}(\mathrm{Newt}(f)) = \bigcap_{1\le i\le k} \mathrm{Sph}(\mathrm{Newt}(f_i)).$$

Maclagan and Sturmfels prove that this result holds over a field $\mathbb{F}$ with non-trivial valuation, see [MS15, Theorem 2.6.5 and Corollary 3.2.3]. In [JS18, Section 2] it is stated that this works over fields of positive characteristic. This can be obtained by using the results of Maclagan and Sturmfels and observing the results hold for a field with trivial valuation. In particular, an algebraically closed field $\mathbb{F}$ always has a trivial valuation and can be embedded into another valued field with a non-trivial valuation so that the new valuation extends the trivial valuation. For example, consider $\mathbb{F}$ embedded in $\mathbb{F}(t)$ with the valuation defined by $v(t^k) = k$. Then the results of [MS15] can be applied using the extended field and will give the desired tropical basis over $\mathbb{F}$.

Recall for a polynomial f with integer coefficients, $\overline{f} = f \pmod p$ is the polynomial with the same coefficients modulo p. Consider an ideal $I = \langle g_1,\dots g_l\rangle \subseteq \mathbb{C}[x_1,\dots,x_n]$ with generators $g_i \in \mathbb{Z}[x_1,\dots,x_n]$. We denote the corresponding ideal over an algebraically closed field $\mathbb{F}$ with characteristic p as $\overline{I}^{\mathbb{F}} = \langle \overline{g_1},\dots \overline{g_l}\rangle \subseteq \mathbb{F}[x_1,\dots,x_n]$. Here, $\overline{g_i} \in \mathbb{Z}_p[x_1,\dots,x_n]$. Then we can prove the following.

Lemma 3.35. *Let I be a finitely generated ideal in $\mathbb{C}[x_1,\dots,x_n]$ with generators over the integers such that the greatest common divisor of all the coefficients is* 1. *Let $\mathbb{F}$ be an algebraically closed field of characteristic p. For all but finitely many p, the logarithmic limit set of I is the same as the logarithmic limit set of $\overline{I}^{\mathbb{F}}$,*

$$V_\infty(I) = V_\infty(\overline{I}^{\mathbb{F}}).$$

Proof. Consider I given by

$$I = \langle g_1,\dots,g_l\rangle \subseteq \mathbb{C}[x_1,\dots,x_n],\ g_i \in \mathbb{Z}[x_1,\dots,x_n]$$

and assume the greatest common divisor of the coefficients of g_i for all i is 1.

By [BJS+07, MS15], I has a finite tropical basis $\{f_1,\dots,f_k\}$. This is done using initial forms. For $f \in \mathbb{Z}[x_1,\dots,x_n]$, $\mathbf{w} \in \mathbb{Z}^n$, the *initial form* $\mathrm{in}_{\mathbf{w}}(f)$ is the terms in f of lowest $\mathbf{w}$-weight (see [BJS+07, MS15] for formal definitions and examples). The construction of the tropical basis takes the original generating set g_i and constructs additional polynomials f such that their initial form $\mathrm{in}_{\mathbf{w}}(f)$ is a unit for specific weight vectors $\mathbf{w} \in \mathbb{Z}^n$. This can be done with operations for coefficients over $\mathbb{Z}$ similar to the modified version of Buchberger's algorithm discussed in Remark 3.26.

Using the third definition for the logarithmic limits and the existence of a tropical basis, $V_\infty(I)$ is precisely the intersection of the spherical duals of the Newton polytopes of elements of the tropical basis for I.

$$V_\infty(I) = \bigcap_{1\le i\le k} \mathrm{Sph}(\mathrm{Newt}(f_i)).$$

The Newton polytopes of nonzero elements of I will only change modulo p if the coefficient of a leading term of at least one of the elements of the tropical basis is divisible by p. Because there are finitely many elements in the basis, there are finitely many primes p that divide coefficients of the leading term.

Excluding these finitely many primes, the Newton polytopes of nonzero elements of $\overline{I}^{\mathbb{F}}$ will remain the same as the Newton polytopes of nonzero elements of I and $V_\infty(I) = V_\infty(\overline{I}^{\mathbb{F}})$. $\square$

Following [Til05], consider the logarithmic limit set of the eigenvalue variety, $V_\infty(K_{\mathbb{F}})$. The eigenvalue variety has symmetries in the choice of eigenvalues $\left(m_i^{\pm1}, l_i^{\pm1}\right)$ for each torus boundary component. This leads to symmetries in the logarithmic limit set. If $(x_1,\dots,x_{2i-1},x_{2i},\dots,x_{2h}) \in V_\infty(K)$ then $(x_1,\dots,-x_{2i-1},-x_{2i},\dots,x_{2h}) \in V_\infty(K)$. A quotient of the logarithmic limit set can be found by factoring by these symmetries inside $\mathbb{RP}^{2h-1}/\mathbb{Z}_2^{h-1} \cong S^{2h-1}$. The quotient map extends to a map $\varphi\colon\ S^{2h-1} \to S^{2h-1}$ of degree 2^h. Let N_h be the $2h \times 2h$ matrix

$$N_h = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}$$

with 2×2 blocks given along the diagonal and 0 otherwise.

The corresponding boundary slopes theorem for multiple boundary tori proceeds loosely as follows. If $\zeta \in V_\infty(K)$ is a point with rational coordinate ratios then there is an essential surface with boundary in M whose projectivised boundary curve coordinate is $\varphi(N_h\zeta)$. Boundary slopes detected this way are again called strongly detected boundary slopes. When M has a single boundary torus, this is precisely the same boundary slopes theorem as stated in Section 3.4.1. We have a version of this theorem over multiple boundary tori for all algebraically closed fields $\mathbb{F}$.

Theorem 3.36. *Let $\mathbb{F}$ be an algebraically closed field. If $\zeta \in V_\infty(K_{\mathbb{F}})$ is a point with rational coordinate ratios then there is an essential surface with boundary in M whose projectivised boundary curve coordinate is $\varphi(N_h\zeta)$.*

The proof follows the same steps as that over $\mathbb{C}$ from [Til05] also using the aforementioned theory for detecting essential surfaces through ideal points.

We now compare the defining ideal $K_{\mathbb{F}}$ across characteristics.

Proposition 3.37. *Let $\mathbb{F}$ be an algebraically closed field of characteristic p. For all but finitely many primes p,*

$$\overline{K_{\mathbb{C}}}^{\mathbb{F}} = K_{\mathbb{F}}. \tag{3.4.13}$$

The proof follows the same steps as Lemma 3.30, applied to the ideals $K_{\mathbb{C}}$ and $K_{\mathbb{F}}$, rather than the polynomials $\widetilde{A}_0$ and $\widetilde{A}_p$. This proves the result for an algebraically closed field $\mathbb{F}$ of characteristic p for all p that do not divide the leading term of a polynomial in Buchberger's algorithm over $\mathbb{Z}$. This leads to the following.

Theorem 3.38. *Let M be an orientable, irreducible, compact 3–manifold with non-empty boundary consisting of a disjoint union of h tori. For all but finitely many primes p, the boundary curve of an essential surface in M is strongly detected by $\mathrm{X}(M,\mathbb{F})$ for $\mathbb{F}$ an algebraically closed field of characteristic p if and only if it is strongly detected by $\mathrm{X}(M,\mathbb{C})$.*

Proof. Let $\mathbb{F}$ be an algebraically closed field of characteristic p.

Consider a slope on ∂M that is strongly detected by $\mathrm{X}(M,\mathbb{F})$. By Theorem 3.36, there is a point $\zeta \in V_\infty(K_{\mathbb{F}})$ with rational coordinate ratios associated with the slope with projectivised boundary curve coordinate is $\varphi(N_h\zeta)$. By Proposition 3.37, for all but finitely many primes p, $K_{\mathbb{F}} = \overline{K_{\mathbb{C}}}^{\mathbb{F}}$ and thus $\zeta \in V_\infty(\overline{K_{\mathbb{C}}}^{\mathbb{F}})$. By Lemma 3.35, for all but finitely many primes p, $V_\infty(K_{\mathbb{C}}) = V_\infty(\overline{K_{\mathbb{C}}}^{\mathbb{F}})$ and thus $\zeta \in V_\infty(K_{\mathbb{C}})$. Then, by Lemma 3.34 and the boundary slopes theorem in characteristic 0, the slope must also be strongly detected by $\mathrm{X}(M,\mathbb{C})$.

Consider a slope on ∂M that is strongly detected by $\mathrm{X}(M,\mathbb{C})$. By Theorem 3.36, there is a point $\zeta \in V_\infty(K_{\mathbb{C}})$ with rational coordinate ratios associated with the slope with projectivised boundary curve coordinate is $\varphi(N_h\zeta)$. By Lemma 3.35, for all but finitely many primes p, $V_\infty(K_{\mathbb{C}}) = V_\infty(\overline{K_{\mathbb{C}}}^{\mathbb{F}})$ and thus $\zeta \in V_\infty(\overline{K_{\mathbb{C}}}^{\mathbb{F}})$. By Proposition 3.37, for all but finitely many primes p, $K_{\mathbb{F}} = \overline{K_{\mathbb{C}}}^{\mathbb{F}}$ and thus $\zeta \in V_\infty(K_{\mathbb{F}})$. Then, by Lemma 3.34 and the boundary slopes theorem in characteristic p, the slope must also be strongly detected by $\mathrm{X}(M,\mathbb{F})$. This proves the statement.

Note the finitely many primes p excluded in Lemma 3.35 are those that divide the leading terms in certain elements of the Gröbner basis and thus a subset of the finitely many primes excluded by Proposition 3.37. $\square$

3.5 Applications to prime decomposition

We can characterise a certain type of group that admits curves in the character variety in positive characteristic: free products of groups that admit non-central representations.

Let $Z(H)$ be the centre of a group H and consider a group G. We say G **admits a non-central representation** into H if there exists a representation $\rho : G \to H$ such that $\mathrm{im}(\rho)$ is not central. That is, there exists $\gamma \in G$ such that $\rho(\gamma) \notin Z(H)$.

Lemma 3.39 (Free product of groups)**.** *Let $\mathbb{F}$ be an algebraically closed field and let G_i, $i = 0, \ldots, k$ be groups that admit non-central representations into* $\mathrm{SL}_2(\mathbb{F})$*. Then the variety of characters* $\mathrm{X}(G_0 * \ldots * G_k, \mathbb{F})$ *contains a curve with the property that each character on the curve restricted to G_i is constant for all i. Moreover, the ideal points of the curve detect the splitting.*

Proof. Consider non-central representations $\rho_i \colon G_i \to \mathrm{SL}_2(\mathbb{F})$ for each $i = 0, \ldots, k$. That is, there exists $\gamma_i \in G_i$ such that $\rho_i(\gamma_i) \notin Z(\mathrm{SL}_2(\mathbb{F})) = \left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \right\}$. Then $\rho_i(\gamma_i)$ has either a unique 1–dimensional eigenspace or two 1–dimensional eigenspaces. Hence we can conjugate the representations so that no pair $\rho_i(\gamma_i)$, $\rho_j(\gamma_j)$, $i \neq j$, has a common eigenspace. Moreover, we may assume that none of the 1–dimensional eigenspaces of each $\rho_i(\gamma_i)$ is spanned by $(1,0)^\intercal$ or $(0,1)^\intercal$. In particular, writing

$$\rho_i(\gamma_i) = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$$

we have $b_i \neq 0 \neq c_i$.

We now construct a parameterised curve $\{\rho_t \mid t \in \mathbb{F} \setminus \{0\}\} \subseteq \mathrm{R}(G_0 * \ldots * G_k, \mathbb{F})$ with a natural valuation defined by $\nu(t^l) = l$ that detects the splitting.

Let $t \in \mathbb{F} \setminus \{0\}$. Construct a representation $\rho_t \colon G_0 * \ldots * G_k \to \mathrm{SL}_2(\mathbb{F})$ by setting

$$\rho_t(\alpha_j) = \begin{pmatrix} t^{-j} & 0 \\ 0 & t^j \end{pmatrix} \rho_j(\alpha_j) \begin{pmatrix} t^j & 0 \\ 0 & t^{-j} \end{pmatrix}$$

for each $\alpha_j \in G_j$ and extending over $G_0 * \ldots * G_k$. Note that since $\rho_j(\gamma_j)$ did not have a 1–dimensional eigenspace spanned by $(1,0)^\intercal$ or $(0,1)^\intercal$, the same is true for $\rho_t(\gamma_j)$.

For every $\alpha_j \in G_j$, we have $\mathrm{tr}(\rho_t(\alpha_j)) = \mathrm{tr}(\rho_j(\alpha_j))$, which is constant in t.

For every pair $\gamma_i \in G_i$, $\gamma_j \in G_j$, $i \neq j$,

$$\begin{aligned} \rho_t(\gamma_i\gamma_j) &= \begin{pmatrix} a_i & t^{-2i}b_i \\ t^{2i}c_i & d_i \end{pmatrix} \begin{pmatrix} a_j & t^{-2j}b_j \\ t^{2j}c_j & d_j \end{pmatrix} \\ &= \begin{pmatrix} a_ia_j + t^{2(j-i)}b_ic_j & \star \\ \star & t^{2(i-j)}c_ib_j + d_id_j \end{pmatrix} \end{aligned}$$

with

$$\mathrm{tr}\left(\rho_t(\gamma_i\gamma_j)\right) = a_ia_j + t^{2(j-i)}b_ic_j + t^{2(i-j)}c_ib_j + d_id_j.$$

Since $b_ic_j \neq 0 \neq c_ib_j$, we have $\nu(\mathrm{tr}\left(\rho_t(\gamma_i\gamma_j)\right)) < 0$ for each such pair $i, j \in \{0, 1, \ldots k\}$ with $i \neq j$. This then gives the desired 1–parameter family of representations. It is indeed a curve, as it can be viewed as the image under an algebraic map of the variety $\{(t,s) \in \mathbb{F}^2 \mid ts = 1\} \cong \mathbb{F} \setminus \{0\}$. $\square$

We can consider the special case of cyclic groups.

Corollary 3.40 (Free product of cyclic groups)**.** *Let $\mathbb{F}$ be an algebraically closed field of characteristic p.*

Let $n_0,\dots,n_k \geq 2$ be integers and C_n the cyclic group order n. Then for $p=2$, $\mathrm{X}(C_{n_0} * \dots * C_{n_k}, \mathbb{F})$ contains a curve that detects the splitting.

If $n_0,\dots,n_k \geq 3$, then for every prime p, $\mathrm{X}(C_{n_0} * \dots * C_{n_k}, \mathbb{F})$ contains a curve that detects the splitting.

Proof. This follows from Lemmas 3.14 and 3.39. For all $n \geq 2$, when $p=2$, C_n admits a non-central representation into $\mathrm{SL}_2(\mathbb{F})$ and the first result for $\mathrm{X}(C_{n_0} * \dots * C_{n_k}, \mathbb{F})$ when $p=2$ follows. For all $n \geq 3$ and primes p, C_n admits a non-central representation into $\mathrm{SL}_2(\mathbb{F})$ and the second result for $\mathrm{X}(C_{n_0} * \dots * C_{n_k}, \mathbb{F})$ follows. □

We can explain further why this difference occurs between characteristic 2 and characteristic $p \geq 3$.

Remark 3.41. *Consider characteristic $p \geq 3$. If there exists i such that $n_i = 2$ the obstacle that arises is that $C_{n_i} = C_2$ admits no non-central representations in $\mathrm{SL}_2(\mathbb{F})$. The only element of order two in $\mathrm{SL}_2(\mathbb{F})$ is the central element $-E$. Hence any homomorphism $C_{n_0} * \dots * C_{n_k} \to \mathrm{SL}_2(\mathbb{F})$ maps the generator of C_{n_i} to $\pm E$, and the generator of the other factors C_{n_j} to an element of order dividing n_j. In this case, there could still be a curve in $\mathrm{X}(C_{n_0} * \dots * C_{n_k}, \mathbb{F})$, but it would not detect the splitting.*

We have the following immediate consequence to Corollary 3.40.

Corollary 3.42 (Sufficient criterion for curves)**.** *Let $\mathbb{F}$ be an algebraically closed field of characteristic p. If the finitely generated group G admits an epimorphism to $G_0 * \dots * G_k$, where G_i admits a non-central representation into $\mathrm{SL}_2(\mathbb{F})$, then $\mathrm{X}(G,\mathbb{F})$ contains a curve that detects the splitting for all primes p.*

The above results combined with Proposition 3.17 allow us to detect the prime decomposition of certain 3–manifolds.

Corollary 3.43. *Let $\mathbb{F}$ be an algebraically closed field of characteristic p.*

Suppose M has prime decomposition $M = M_0 \# \dots \# M_k$ along the union of essential spheres $S = S_1 \cup \dots \cup S_k$. If $\pi_1(M_i)$ admits a non-central representation into $\mathrm{SL}_2(\mathbb{F})$ for all $i = 0,\dots,k$, then S is detected by an ideal point of a curve in $\mathrm{X}(M,\mathbb{F})$.

In particular, if $\pi_1(M_i)$ admits an epimorphism into C_{n_i} the cyclic group of order $n_i \geq 2$ for all $i = 0,\dots,k$, then S is detected by an ideal point of a curve in $\mathrm{X}(M,\mathbb{F})$ if $p=2$. If $n_i \geq 3$ for all $i=0,\dots,k$, then S is detected by an ideal point of a curve in $\mathrm{X}(M,\mathbb{F})$ for all primes p.

For a Lens space $L(q,r)$ we have $\pi_1(L(q,r)) = C_q$ and we can get some interesting examples.

Example 3.44 (Connected sum of lens spaces)**.** *Let $\mathbb{F}$ be an algebraically closed field of characteristic p. Consider $M = L(q,r)\#L(2,1)$ for $q \geq 3$. Then $\pi_1(M) \cong C_q * C_2$.*

By Corollary 3.43, the essential sphere in M is detected by $\mathrm{X}(M,\mathbb{F})$ for $p=2$. In characteristic $p \neq 2$, any homomorphism $C_q * C_2 \to \mathrm{SL}_2(\mathbb{F})$ maps the generator of the first factor to an element of order dividing q and the second factor to $\pm E$. Since there are at most finitely many conjugacy classes of elements of a given finite order, this implies that $\mathrm{X}(M,\mathbb{F})$ has dimension zero. This shows the essential sphere in M is only detected by $\mathrm{X}(M,\mathbb{F})$ when $p=2$ and we cannot detect the prime decomposition in every characteristic p.

A natural question is then the following.

Question 3.45. *Can we always detect the prime decomposition in $\mathbb{F}$ an algebraically closed field of characteristic 2?*

Remark 3.46. *Every prime manifold with non-trivial first homology group will admit a non-central representation into $\mathrm{SL}_2(\mathbb{F})$ for $\mathbb{F}$ a field of characteristic 2. Thus, the prime decomposition is always detected in characteristic 2 if each prime manifold in the decomposition has non-trivial first homology group.*

Then a positive answer to the next question gives a positive answer to Question 3.45.

Question 3.47. *Does every 3–manifold (other than the 3–sphere) with trivial homology admit a non-central representation into $\mathrm{SL}_2(\mathbb{F})$ for $\mathbb{F}$ an algebraically closed field of characteristic 2?*

Example 3.48 (Connected sum of Poincaré homology spheres)**.** *Consider the Poincaré homology sphere, PH. We have*

$$\pi_1(PH) = \langle a, b \mid a^5 = abab = b^3 \rangle .$$

This admits a non-central representation into $\mathrm{SL}_2(\mathbb{F})$ for $\mathbb{F}$ a field of characteristic p conjugate to the form

$$\rho(a) = \begin{pmatrix} x & 1 \\ 0 & x^{-1} \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} y & 0 \\ -y(x-x^{-1}) - x^{-1} & 1-y \end{pmatrix}, \quad x, y \in \mathbb{F},$$

with

$$\begin{aligned} 1 - y + y^2 &= 0, \\ 1 - x + x^2 - x^3 + x^4 &= 0. \end{aligned}$$

Take $M = PH \# \ldots \# PH$ the connected sum of some number of Poincaré homology spheres. By Corollary 3.43 the union of essential spheres in M is detected by an ideal point of a curve in $\mathrm{X}(M, \mathbb{F})$ for all p. Similarly for the connected sum of PH with other manifolds M_i that admit non-central representations, $M = PH \# M_1 \# \ldots \# M_n$.

3.6 Applications to JSJ decomposition

Similarly to free products of groups, we can characterise another type of group that admits curves in the character variety in positive characteristic: amalgamated free products of groups along $\mathbb{Z}^2$ that satisfy certain non-central and irreducibility properties.

Consider a group G with subgroup $\mathbb{Z}^2$. Consider a representation $\rho\colon G \to \mathrm{SL}_2(\mathbb{F})$. The subgroup $\mathbb{Z}^2$ must be mapped to an abelian subgroup of $\mathrm{SL}_2(\mathbb{F})$, $A = \mathrm{im}(\rho|_{\mathbb{Z}^2})$. The abelian subgroup A must be conjugate to either the diagonal matrices or parabolic matrices, or a subset therein,

$$D = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \;\middle|\; t \in \mathbb{F} \setminus \{0\} \right\} \text{ or } P = \left\{ \begin{pmatrix} \pm 1 & t \\ 0 & \pm 1 \end{pmatrix} \;\middle|\; t \in \mathbb{F} \right\}. \tag{3.6.1}$$

We say such a representation ρ of G is $\mathbb{Z}^2$**–trivial** if $A \subseteq \{\pm E\}$. If ρ is not $\mathbb{Z}^2$–trivial, we say ρ is $\mathbb{Z}^2$**–non-central** if A does not centralise $\mathrm{im}(\rho)$. That is, there exists $\gamma \in G$ such that $\rho(\gamma) \notin Z(A)$.

Lemma 3.49. *Let $\mathbb{F}$ be an algebraically closed field and let $G = G_0 *_{\mathbb{Z}^2} G_1$, the amalgamated free product of groups G_i, $i = 0,1$ along $\mathbb{Z}^2$. Assume either*

1. *G admits a representation into $\mathrm{SL}_2(\mathbb{F})$ that is $\mathbb{Z}^2$–trivial and restricts to a non-central representation on G_i for each $i = 0,1$, or*
2. *G admits an irreducible representation into $\mathrm{SL}_2(\mathbb{F})$ that restricts to a $\mathbb{Z}^2$–non-central representation on G_i for each $i = 0,1$.*

Then the variety of characters $\mathrm{X}(G,\mathbb{F})$ contains a curve with the property each character on the curve restricted to G_i is constant for all i. Moreover, the ideal points of the curve detect the splitting.

Proof. Consider Item 1. We have G admits a representation ρ into $\mathrm{SL}_2(\mathbb{F})$ that is $\mathbb{Z}^2$–trivial. Follow the same steps as in the proof of Lemma 3.39. There are no issues conjugating across the splitting along $\mathbb{Z}^2$ because ρ is $\mathbb{Z}^2$–trivial.

Consider Item 2. We have G admits an irreducible representation ρ into $\mathrm{SL}_2(\mathbb{F})$ that is $\mathbb{Z}^2$–non-central on G_i for $i = 0,1$. Then $A = \mathrm{im}(\rho|_{\mathbb{Z}^2})$ is conjugate to diagonal matrices D or parabolic matrices P as in Equation (3.6.1). Consider the centraliser of each,

$$Z(D) = \left\{ \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} \;\middle|\; s \in \mathbb{F} \setminus \{0\} \right\}, \quad Z(P) = \left\{ \begin{pmatrix} \pm 1 & s \\ 0 & \pm 1 \end{pmatrix} \;\middle|\; s \in \mathbb{F} \right\}.$$

An element of $\mathrm{SL}_2(\mathbb{F})$ is centralised by D or P if and only if they have the same eigenspaces.

We show first that there exists $\gamma_0 \in G_0$ and $\gamma_1 \in G_1$ with $\rho(\gamma_0)$ and $\rho(\gamma_1)$ such that neither is not centralised by A and they do not have a common 1–dimensional subspace. There are two cases to consider.

If both $\rho|G_0$ and $\rho|G_1$ are reducible, then ρ irreducible implies they each fix a different 1–dimensional subspace. Both of these 1–dimensional subspaces must be fixed by A. We can choose elements $\gamma_i \in G_i$, $i = 0,1$, such that $\rho(\gamma_i)$ have at most one eigenspace in common with A. Then $\rho(\gamma_i)$ are not centralised by A and they will not have a common 1–dimensional subspace.

If at least one of $\rho|G_0$ and $\rho|G_1$ is irreducible (say $\rho|G_1$) then $\rho|G_1$ does not have a fixed 1–dimensional subspace. Choose $\gamma_0 \in G_0$ such that $\rho(\gamma_0)$ is not centralised by A. As $\rho|G_1$ is irreducible, we can find $\gamma_1 \in G_1$ such that $\rho(\gamma_1)$ does not share an eigenspace with A (and thus is not centralised by A). If $\rho(\gamma_1)$ shares an eigenspace with $\rho(\gamma_0)$ then we can conjugate along A until this does not happen. To see how this works, consider the two cases of A conjugate to the diagonals or to the parabolics in turn.

If A is conjugate to the diagonals D, there exists $a_t \in \mathbb{Z}^2 \subset G$ with $A_t = \rho(a_t)$ and some $U \in \mathrm{SL}_2(\mathbb{F})$ such that

$$U^{-1} A_t U = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \quad U^{-1} \rho(\gamma_1) U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{F}), \quad bc \neq 0, t \neq \pm 1. \tag{3.6.2}$$

Then there exists $k \neq 0$ with $a_t^{-k}\gamma_1 a_t^k \in G_1$ such that $A_t^{-k}\rho(\gamma_1)A_t^k = \rho(a_t^{-k}\gamma_1 a_t^k)$ does not share an eigenspace with $\rho(\gamma_0)$. We relabel γ_1 to be this element. This will still not be centralised by A.

Similarly, if A is conjugate to the parabolics P, there exists $a_t \in \mathbb{Z}^2 \subset G$ with $A_t = \rho(a_t)$ and some $U \in \mathrm{SL}_2(\mathbb{F})$ such that

$$U^{-1}A_tU = \begin{pmatrix} \pm 1 & t \\ 0 & \pm 1 \end{pmatrix}, \quad U^{-1}\rho(\gamma_1)U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{F}), \quad c, t \neq 0. \tag{3.6.3}$$

Then there exists $k \neq 0$ with $a_t^{-k}\gamma_1 a_t^k \in G_1$ such that $A_t^{-k}\rho(\gamma_1)A_t^k = \rho(a_t^{-k}\gamma_1 a_t^k)$ does not share an eigenspace with $\rho(\gamma_0)$. We relabel γ_1 to be this element. This will still not be centralised by A.

Then in all cases we can conjugate the matrices $\rho(\gamma_0)$ and $\rho(\gamma_1)$ to

$$V^{-1}\rho(\gamma_0)V = \begin{pmatrix} a_0 & 1 \\ 0 & a_0^{-1} \end{pmatrix}, \quad V^{-1}\rho(\gamma_1)V = \begin{pmatrix} a_1 & 0 \\ u_1 & a_1^{-1} \end{pmatrix}, \quad a_0, a_1, u_1 \in \mathbb{F} \setminus \{0\}, \tag{3.6.4}$$

for $V \in \mathrm{SL}_2(\mathbb{F})$.

Construct a representation $\rho_t : G_0 *_{\mathbb{Z}^2} G_1 \to \mathrm{SL}_2(\mathbb{F})$ by setting

$$\begin{aligned} \rho_t(\delta_0) &= \rho(\delta_0) \\ \rho_t(\delta_1) &= V^{-1}U^{-1}A_t^{-1}UV\rho(\delta_1)V^{-1}U^{-1}A_tUV \end{aligned}$$

for $\delta_i \in G_i$, $i = 0, 1$, and extend over $G_0 *_{\mathbb{Z}^2} G_1$. The matrices U and V from Equations (3.6.2) to (3.6.4) are required to ensure the gluing relations along $\mathbb{Z}^2$ are preserved while conjugating by A_t.

The proof then follows similarly to that of Lemma 3.39. For every $\delta_i \in G_i$, we have $\mathrm{tr}(\rho_t(\delta_i)) = \mathrm{tr}(\rho(\delta_i))$, $i = 0, 1$, which is constant in t. Let

$$UV = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{F}).$$

For the pair of elements $\gamma_0\gamma_1$,

$$\mathrm{tr}(\rho_t(\gamma_0\gamma_1)) = \begin{cases} a_0a_1 + a_0^{-1}a_1^{-1} + u_1 + \left(t^{-2} - 1\right)E_D + \left(t^2 - 1\right)F_D & \text{if } A \text{ conjugate to } D, \\ a_0a_1 + a_0^{-1}a_1^{-1} + u_1 + tE_P + t^2F_P & \text{if } A \text{ conjugate to } P, \end{cases}$$

with

$$\begin{aligned} E_D &= -bc\left(-c + d\left(a_0 - a_0^{-1}\right)\right)\left(a\left(a_1 - a_1^{-1}\right) + bu_1\right), \\ F_D &= ad\left(a - b\left(a_0 - a_0^{-1}\right)\right)\left(c\left(a_1 - a_1^{-1}\right) + du_1\right), \\ E_P &= c^2\left(a_1 - a_1^{-1}\right) + 2cdu_1 - d^2u_1\left(a_0 - a_0^{-1}\right), \\ F_P &= cd\left(c - d\left(a_0 - a_0^{-1}\right)\right)\left(c\left(a_1 - a_1^{-1}\right) + du_1\right). \end{aligned}$$

It is straightforward to verify that

- $E_D = 0 = F_D$ if and only if $U^{-1}A_tU$ is diagonal as in Equation (3.6.2) and at least one of $\rho_t(\gamma_0)$, $\rho_t(\gamma_1)$ is centralised by A, and
- $E_P = 0 = F_P$ if and only if $U^{-1}A_tU$ is parabolic as in Equation (3.6.3) and at least one of $\rho_t(\gamma_0)$ and $\rho_t(\gamma_1)$ is centralised by A.

Since both $\rho|_{G_0}$ and $\rho|_{G_1}$ are $\mathbb{Z}^2$–non-central, this cannot occur, and $\operatorname{tr}(\rho_t(\gamma_0\gamma_1))$ will blow up in t.

This then gives a curve of representations defined by ρ_t, $t \in \mathbb{F}$, which is constant when restricted to G_i for all $i = 0, 1$, and blows up otherwise. □

We note the following regarding the assumptions.

Remark 3.50. *A representation ρ of G being irreducible implies ρ is $\mathbb{Z}^2$–non-central. Thus, ρ being $\mathbb{Z}^2$–non-central when restricted to G_0 and G_1 is only used when at least one of $\rho|_{G_0}$ and $\rho|_{G_1}$ is reducible.*

We get the following corollary to Lemma 3.49.

Corollary 3.51. *Let $\mathbb{F}$ be an algebraically closed field and let G be the graph of groups with graph a tree such that vertex groups are G_i, $i = 0, \ldots, k$ and all edges groups are $\mathbb{Z}^2$. Consider $i, j \in \{1, \ldots, k\}$, $i \neq j$ with an edge between G_i and G_j in G. Assume either*

1. *$G_i *_{\mathbb{Z}^2} G_j$ admits a representation into $\mathrm{SL}_2(\mathbb{F})$ that is $\mathbb{Z}^2$–trivial and restricts to a non-central representation on G_i and G_j, or*
2. *$G_i *_{\mathbb{Z}^2} G_j$ admits an irreducible representation into $\mathrm{SL}_2(\mathbb{F})$ that restricts to a $\mathbb{Z}^2$–non-central representation on G_i and G_j.*

Then the variety of characters $\mathrm{X}(G, \mathbb{F})$ contains a curve with the property each character on the curve restricted to G_l is constant for all $l \in \{1, \ldots, k\}$ and the ideal points of the curve detect the splitting between G_i and G_j.

In particular, if this holds for all pairs i, j with an edge between them, then the ideal points of the curve detect the whole splitting.

The results along with Proposition 3.17 allows us to detect the JSJ decomposition of certain 3–manifolds.

Suppose M has JSJ decomposition with JSJ pieces $M_0, \ldots, M_k$ along the union of essential tori T. We label the torus between M_i and M_j, T_{ij}, if it exists. Note $T_{ij} = T_{ji}$. Then $T = \bigcup T_{ij}$. We say M has a **tree-like** JSJ decomposition if the dual graph of T in M is a tree. This will occur if and only if the number of essential tori is one less than the number of JSJ pieces. For example, all satellite knots have a tree-like JSJ decomposition (see [Bud06]).

Corollary 3.52. *Let $\mathbb{F}$ be an algebraically closed field. Suppose M has a tree-like JSJ decomposition into JSJ pieces $M_0, \ldots M_k$ along the union of essential tori $T = \bigcup T_{ij}$ where T_{ij} is the torus between M_i and M_j, if it exists. If, for some $i, j \in \{1, \ldots, k\}$ with torus T_{ij}, either*

1. *$\pi_1(M)$ admits a representation into $\mathrm{SL}_2(\mathbb{F})$ that is $\mathbb{Z}^2$–trivial and restricts to a non-central representation on $\pi_1(M_i)$ and $\pi_1(M_j)$, or*

2. *$\pi_1(M)$ admits an irreducible representation into $\mathrm{SL}_2(\mathbb{F})$ that restricts to a $\mathbb{Z}^2$–non-central representation on $\pi_1(M_i)$ and $\pi_1(M_j)$,*

then T_{ij} is detected by an ideal point of a curve in $\mathrm{X}(M,\mathbb{F})$.

In particular, if this holds for all i,j pairs with torus T_{ij}, then the union T is detected by an ideal point of a curve in $\mathrm{X}(M,\mathbb{F})$.

This follows for any M with a tree-like JSJ decomposition by applying Corollary 3.51 using $G=\pi_1(M)$.

Of particular interest are satellite knots. We noted they have tree-like JSJ decomposition and they also admit irreducible representations in characteristic 0 by [KM04] (see also [DG04]). To use the result it remains to show that there is a representation that satisfies the remaining requirements for Corollary 3.52.

Example 3.53 (Connected sum of two trefoils)**.** *Consider two right-handed trefoils, K_1 and K_2 and their connected sum $K_1\#K_2$ (also called the granny knot). Let N be the complement of the connected sum in $\mathbb{S}^3$. We have*

$$\pi_1(N)=\langle x,y,z \mid xyx=yxy,\ xzx=xzx\rangle\,.$$

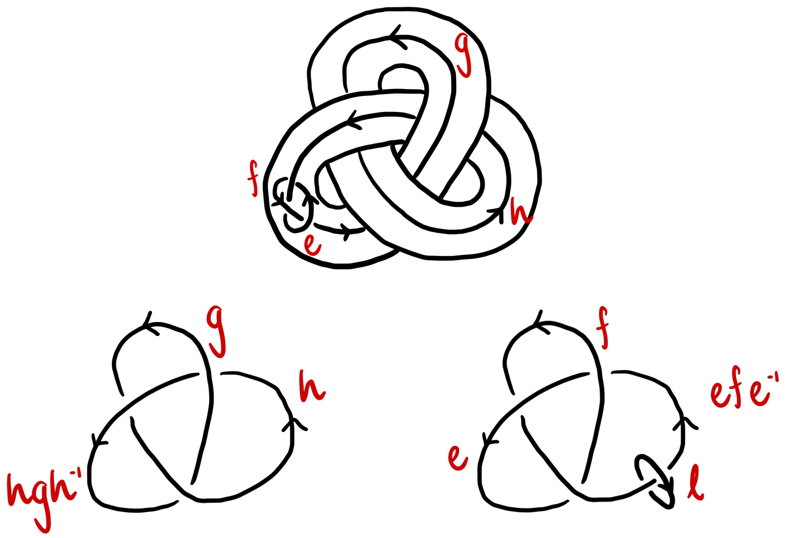

FIGURE 3.1: The connected sum of two trefoils $K_1\#K_2$ as a satellite knot in a knotted torus (top). The knot K_1 associated M_1 (bottom left) and the link $\widetilde{K_1}$ associated with $\widetilde{M_2}$ (bottom right).

We can view the connected sum as a satellite knot with the essential torus occurring along the splitting of the fundamental group as below

$$\begin{aligned}\pi_1(N)&=\pi_1(M_1)*_{\mathbb{Z}^2}\pi_1(\widetilde{M_2}),\\ &=\langle g,h \mid ghg=hgh\rangle *_{efe^{-1}=hgh^{-1},\, l=hg^{-4}hg^2}\left\langle e,f,l \mid efe=fef,\ efe^{-1}l=lefe^{-1}\right\rangle.\end{aligned}$$

The manifolds M_1 and $\widetilde{M_2}$ are the JSJ pieces with M_1 the complement of the trefoil K_1 and $\widetilde{M_2}$ the complement of the trefoil K_2 in the solid torus. The latter can also be seen as the complement of the link $\widetilde{K_2}$ formed by adjoining the trefoil with an extra copy of $\mathbb{S}^1$ around one of the arcs. This is represented in the fundamental group by the extra generator l that commutes with the arc efe^{-1}. The two presentations for $\pi_1(N)$ are related by $x = g$, $y = h$, $z = h^{-1}eh$.

Let $\mathbb{F}$ be an algebraically closed field. Take the irreducible representation $\rho\colon\ \pi_1(N) \to \mathrm{SL}_2(\mathbb{F})$ defined by

$$\rho(x) = \begin{pmatrix} w & 1 \\ 0 & w^{-1} \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} w^{-1} & 0 \\ -1 & w \end{pmatrix}, \quad \rho(z) = \begin{pmatrix} w & 0 \\ -w^2+1-w^{-2} & w^{-1} \end{pmatrix}, \quad w \in \mathbb{F}\setminus\{0\}$$

up to conjugation.

Then adjusting between the presentations we have

$$\begin{aligned} A = \mathrm{im}(\rho|_{\mathbb{Z}^2}) &= \left\langle \rho(hgh^{-1}),\, \rho(hg^{-4}hg^2) \right\rangle, \\ &= \left\langle \rho(yxy^{-1}),\, \rho(yx^{-4}yx^2) \right\rangle, \end{aligned}$$

with

$$\rho(yxy^{-1}) = \begin{pmatrix} w^4+w^2+1+w^{-2}+w^{-4} & w^{-2}\left(w^5+w^3+w+w^{-1}+w^{-3}+w^{-5}\right) \\ -w^2\left(w^5+w^3+w+w^{-1}+w^{-3}+w^{-5}\right) & -\left(w^6+w^4+w^2+1+w^{-2}+w^{-4}+w^{-6}\right) \end{pmatrix},$$
$$\rho(yx^{-4}yx^2) = \begin{pmatrix} w+w^{-1} & w^{-2} \\ -w^2 & 0 \end{pmatrix}, \quad w \in \mathbb{F}\setminus\{0\}.$$

If $w = \pm 1$, we can conjugate ρ so that A is parabolic, $A \subseteq P$, otherwise we can conjugate ρ so that A is diagonal, $A \subseteq D$. In particular, the representation ρ is not $\mathbb{Z}^2$–trivial, and is $\mathbb{Z}^2$–non-central on $\pi_1(M_1)$ and $\pi_1(\widetilde{M_2})$ with $\rho(g) \notin Z(A)$ and $\rho(e) \notin Z(A)$. Therefore, we can use this representation to detect the JSJ torus.

This does not work for every irreducible representation on N. For example, we could have specified $\rho'\colon\ \pi_1(N) \to \mathrm{SL}_2(\mathbb{F})$ defined by

$$\rho'(x) = \begin{pmatrix} w & 1 \\ 0 & w^{-1} \end{pmatrix} = \rho'(z), \quad \rho'(y) = \begin{pmatrix} w^{-1} & 0 \\ -1 & w \end{pmatrix}, \quad w \in \mathbb{F}\setminus\{0\}$$

up to conjugation.

The restriction of ρ' to $\pi_1(M_1)$ is the same as ρ with $\rho'|_{\pi_1(M_1)} = \rho|_{\pi_1(M_1)}$ and $\rho'|_{\mathbb{Z}^2} = \rho|_{\mathbb{Z}^2}$. Then as before, the representation ρ' is not $\mathbb{Z}^2$–trivial and is $\mathbb{Z}^2$–non-central on $\pi_1(M_1)$. However, we get that ρ' is not *$\mathbb{Z}^2$–non-central on $\pi_1(\widetilde{M_2})$. Indeed, we can easily verify that, for generators e, f, l, $\rho'(e), \rho'(f), \rho'(l) \in Z(\mathrm{im}(\rho'|_{\mathbb{Z}^2}))$ and so for all $\gamma \in \pi_1(\widetilde{M_2})$, $\rho'(\gamma) \in Z(\mathrm{im}(\rho|_{\mathbb{Z}^2}))$. Then ρ' cannot be used with Corollary 3.52 to detect the JSJ torus.*

It is unclear if these techniques can be used to detect essential tori in satellite knots generally.

Question 3.54. *Is the JSJ decomposition of a satellite knot (with knot complement M) always detected by an ideal point in the variety of characters $\mathrm{X}(M, \mathbb{F})$ for $\mathbb{F}$ an algebraically closed field?*

Perhaps, if not in generality, this can be answered for certain classes of satellite knots.

Chapter 4

Examples of detected and undetected essential surfaces

In this chapter, we use the theory presented in Chapter 3 to explore examples of detected and undetected surfaces across fields of different characteristics. We continue to assume throughout $\mathbb{F}$ is an algebraically closed field. This is part of research that is in preparation in [GT] in collaboration with Stephan Tillmann and [GMT] in collaboration Ben Martin and Stephan Tillmann.

The motivating question for extending the theory of Culler and Shalen was: given an essential surface S in a compact 3–manifold M, is there a characteristic p such that S is detected by an ideal point of the $\mathrm{SL}_2(\mathbb{F})$–variety of characters of M? We explore this further with examples and show the answer is unfortunately no. In particular, we will see the following:

Theorem 4.1. *Let $\mathbb{F}$ be an algebraically closed field of arbitrary characteristic p.*

1. *There exists a manifold M with an essential surface $S \subset M$ that is not detected by the $\mathrm{SL}_2(\mathbb{C})$–variety of characters but is detected by the $\mathrm{SL}_2(\mathbb{F})$–variety of characters for some p.*
2. *There exists a manifold M with an essential surface $S \subset M$ that is not detected by the $\mathrm{SL}_2(\mathbb{F})$–variety of characters for all p.*

The chapter proceeds as follows. We start in Section 4.1 with an initial analysis of changes in dimension across characteristics for varieties of characters of manifolds from the closed Haken census from [BT18]. In Section 4.2 we focus on examples of three manifolds, `m137` from the cusped census in `SnapPea`, and `m019(3,4)` and `m188(2,3)` from the closed Haken census. We see changes in the A–polynomial and changes in dimension of the varieties. In particular, `m019(3,4)` is an example of a closed, Haken, hyperbolic 3–manifold with no detected essential surface.

Essential surfaces in closed hyperbolic 3–manifolds have genus at least two. We initially speculated that, as essential spheres and tori have abelian fundamental group, they may therefore be easier to detect because representations restricted to the surface are centralised by 1–parameter groups. In Section 4.3, we will construct an infinite family of manifolds consisting of the twisted I–bundle of the Klein bottle glued to the complement of the right-handed trefoil. We comprehensively analyse the essential surfaces in the manifolds and curves in their character varieties. We will see that there are essential tori in infinitely many of these glued manifolds that are not detected by any ideal point of the $\mathrm{SL}_2(\mathbb{F})$–character variety for any algebraically closed field $\mathbb{F}$. Examples given throughout will complete the proof of Theorem 4.1.

4.1 Observations from the closed Haken census

A valuable tool for studying the theory of essential surfaces is the closed Haken census from [BT18]. We analyse the variety of characters of manifolds from the closed Haken census across fields of different characteristics. We have constructed code in `SageMath` to compute changes in dimension for manifolds with two-generator two-relator fundamental groups with a focus on components containing irreducible representations. Most of the manifolds in the census are of this form. The remaining manifolds in the census have three-generator fundamental group and they are not considered here. The main part of the code is as follows.

```
INPUT:  characteristic n;
        two-generator relations for the fundamental group of a manifold M.
OUTPUT: dimension of the character variety of M over an algebraically closed
        field of characteristic n.

if n==0:
        R.<x,y,u,v,w,z> = PolynomialRing(QQ, 6, order='lex')
else:
        R.<x,y,u,v,w,z> = PolynomialRing(GF(n), 6, order='lex')
a=matrix(2, 2, [x,1,0,u])
A=inv(a)
b=matrix([[y,0],[w,v]])
B=inv(b)
Relations=[substitute a,A,b,B into relations from the fundamental group]
rel_1=eval(Relations[0])
rel_2=eval(Relations[1])
P =     [rel_1[0,0].numerator(),
        rel_1[0,1].numerator(),
        (rel_1[1,0]/w).numerator(),
        rel_1[1,1].numerator()]
Q =     [rel_2[0,0].numerator(),
        rel_2[0,1].numerator(),
        (rel_2[1,0]/w).numerator(),
        rel_2[1,1].numerator()]
T=[x*u-1, y*v-1,z*w-1]
I=(P+Q+T)*R
return I.dimension()
```

The ideals P and Q contain the equations from the relations for the fundamental group. The ideal T contains equations $xu-1=0$, $yv-1=0$ to ensure the matrices have determinant 1 and equation $zw-1=0$ to ensure that $w \neq 0$. The last equation means the variety defined by I must contain a Zariski dense set of irreducible representations.

Initial results are presented in Table 4.1 for the first 200 manifolds in the closed Haken census with two-generator fundamental groups. Results for manifolds are only included if the dimension is nonzero for at least one example characteristic. Dimension -1 is used to denote the empty set. Across the characteristics $p=0,2,3,5,7$, the only differences we see are in characteristic $p=2$. This only occurs in the examples when the dimension in characteristic 0 is already 0. There are examples where the dimension increases and examples where the dimension decreases. We hope further study of the census will lead to a deeper understanding of when and why changes in dimension occur across characteristic.

It would be useful to examine changes in other features of the variety of characters. Paoluzzi and Porti [PP13, PP18, PP20] have similarly studied the character varieties of knot complements in odd characteristic and found discrepancies across characteristics (for example, in the number of components, dimension, intersection points). Paoluzzi and Porti say $X(\Gamma,\mathbb{C})$ has a **ramification of order-0 in characteristic** p if $X(\Gamma,\mathbb{F})$ either has more components or components of larger dimension than $X(\Gamma,\mathbb{C})$ and $X(\Gamma,\mathbb{C})$ has a **ramification of order-1 in characteristic** p if in $X(\Gamma,\mathbb{F})$ either the type of intersection between two components changes or the singularity types of points on the varieties change.

4.2 Examples from the cusped and closed Haken census

We survey several examples of changes in the variety of characters over fields of arbitrary characteristic. The first example `m137` is from the cusped census in `SnapPea` and shows a change of the A–polynomial, although the boundary slopes that are detected remain the same. The second example `m019(3,4)` shows a decrease of the dimension in characteristic 2. This also exhibits a hyperbolic, Haken, 3–manifold with none of its essential surfaces detected by the variety of characters in any characteristic. The third example `m188(2,3)` shows an increase of the dimension in characteristic 2. This also exhibits changes in the number of connected components of the variety of characters.

In the examples we let $\mathbb{F}=\mathbb{F}_p$ be an algebraically closed field of characteristic p and let $\mathrm{X}_p=\mathrm{X}^{\mathrm{irr}}(M,\mathbb{F})$ denote the components of the variety of characters that contain irreducible representations.

4.2.1 Extended example: `m137`

Take M to be the manifold `m137` in the cusped census of `SnapPea`. In this section we compare the variety of characters and A–polynomial for M in characteristics 0, 2 and 3.

Our manifold M is hyperbolic, one-cusped, and is the complement of a knot in $S^2\times S^1$. It can be obtained by 0 Dehn surgery on either component of the link 7_1^2 in S^3. `SnapPea` computes the following fundamental group and peripheral system,

$$\pi_1(M)=\langle \alpha,\beta \mid \alpha^3\beta^2\alpha^{-1}\beta^{-3}\alpha^{-1}\beta^2\rangle, \quad \{\mathscr{M},\mathscr{L}\}=\{\alpha^{-1}\beta^{-1},\alpha^{-1}\beta^2\alpha^4\beta^2\}.$$

We adjust the presentation to get

$$\pi_1(M)=\langle \mathscr{M},\beta \mid \beta^{-1}\mathscr{M}^{-1}\beta^{-1}\mathscr{M}^{-1}\beta^2\mathscr{M}=\mathscr{M}\beta^{-2}\mathscr{M}^{-1}\beta^2\rangle,$$
$$\{\mathscr{M},\mathscr{L}\}=\{\mathscr{M},\beta^2\mathscr{M}^{-1}\beta^{-3}\mathscr{M}^{-1}\beta^2\}.$$

TABLE 4.1: Dimension of the irreducible components of the variety of characters over different algebraically closed fields $\mathbb{F}$, with -1 indicating the empty set. Manifolds are sampled from the closed Haken census with two-generator fundamental groups. Here, $X_p = X^{\mathrm{irr}}(M,\mathbb{F})$ for $\mathbb{F}$ an algebraically closed field of characteristic $p = 0,2,3,5,7$.

Manifold	Homology	dim(X_0)	dim(X_2)	dim(X_3)	dim(X_5)	dim(X_7)
m019(3,4)	$\mathbb{Z}/50\mathbb{Z}$	0	−1	0	0	0
m188(2,3)	$\mathbb{Z}/2\mathbb{Z}+\mathbb{Z}/26\mathbb{Z}$	0	1	0	0	0
m147(3,2)	$\mathbb{Z}/46\mathbb{Z}$	0	−1	0	0	0
m188(4,1)	$\mathbb{Z}/2\mathbb{Z}+\mathbb{Z}/32\mathbb{Z}$	0	1	0	0	0
m148(6,1)	$\mathbb{Z}/2\mathbb{Z}+\mathbb{Z}/8\mathbb{Z}$	0	1	0	0	0
m149(−2,3)	$\mathbb{Z}/2\mathbb{Z}+\mathbb{Z}/24\mathbb{Z}$	0	1	0	0	0
m303(−3,1)	$\mathbb{Z}/2\mathbb{Z}+\mathbb{Z}/14\mathbb{Z}$	0	1	0	0	0
m294(2,3)	$\mathbb{Z}/2\mathbb{Z}+\mathbb{Z}/6\mathbb{Z}$	0	1	0	0	0
m188(4,3)	$\mathbb{Z}/2\mathbb{Z}+\mathbb{Z}/40\mathbb{Z}$	0	1	0	0	0
m188(−2,3)	$\mathbb{Z}/2\mathbb{Z}+\mathbb{Z}/2\mathbb{Z}$	0	1	0	0	0
m303(−1,3)	$\mathbb{Z}/2\mathbb{Z}+\mathbb{Z}/50\mathbb{Z}$	0	1	0	0	0
s296(5,1)	$\mathbb{Z}/2\mathbb{Z}+\mathbb{Z}/16\mathbb{Z}$	0	1	0	0	0
s297(−1,3)	$\mathbb{Z}/2\mathbb{Z}+\mathbb{Z}/48\mathbb{Z}$	0	1	0	0	0
m304(5,1)	$\mathbb{Z}/2\mathbb{Z}+\mathbb{Z}/10\mathbb{Z}$	0	1	0	0	0
s297(1,3)	$\mathbb{Z}/2\mathbb{Z}+\mathbb{Z}/42\mathbb{Z}$	0	1	0	0	0
m303(1,3)	$\mathbb{Z}/2\mathbb{Z}+\mathbb{Z}/52\mathbb{Z}$	0	1	0	0	0
m290(−1,4)	$\mathbb{Z}/2\mathbb{Z}+\mathbb{Z}/70\mathbb{Z}$	0	1	0	0	0
s254(5,1)	$\mathbb{Z}/4\mathbb{Z}+\mathbb{Z}/12\mathbb{Z}$	1	1	1	1	1
m290(−3,4)	$\mathbb{Z}/2\mathbb{Z}+\mathbb{Z}/74\mathbb{Z}$	0	1	0	0	0
s296(−5,1)	$\mathbb{Z}/2\mathbb{Z}+\mathbb{Z}/14\mathbb{Z}$	0	1	0	0	0
s448(−3,2)	$\mathbb{Z}/2\mathbb{Z}+\mathbb{Z}/16\mathbb{Z}$	0	1	0	0	0
m376(−3,2)	$\mathbb{Z}/2\mathbb{Z}+\mathbb{Z}/2\mathbb{Z}$	0	1	0	0	0
m290(3,2)	$\mathbb{Z}/2\mathbb{Z}+\mathbb{Z}/28\mathbb{Z}$	0	1	0	0	0
m358(3,1)	$\mathbb{Z}/4\mathbb{Z}+\mathbb{Z}/32\mathbb{Z}$	1	1	1	1	1
s297(−5,1)	$\mathbb{Z}/2\mathbb{Z}+\mathbb{Z}/30\mathbb{Z}$	0	1	0	0	0
m303(−5,1)	$\mathbb{Z}/2\mathbb{Z}+\mathbb{Z}/12\mathbb{Z}$	0	1	0	0	0
m303(5,1)	$\mathbb{Z}/2\mathbb{Z}+\mathbb{Z}/22\mathbb{Z}$	0	1	0	0	0
m400(2,3)	$\mathbb{Z}/2\mathbb{Z}+\mathbb{Z}/28\mathbb{Z}$	0	1	0	0	0
s296(−1,3)	$\mathbb{Z}/2\mathbb{Z}+\mathbb{Z}$	0	1	0	0	0
s297(5,1)	$\mathbb{Z}/6\mathbb{Z}+\mathbb{Z}$	0	1	0	0	0
m376(3,2)	$\mathbb{Z}/2\mathbb{Z}+\mathbb{Z}/22\mathbb{Z}$	0	1	0	0	0
m358(1,3)	$\mathbb{Z}/4\mathbb{Z}+\mathbb{Z}/40\mathbb{Z}$	1	1	1	1	1
v2678(2,1)	$\mathbb{Z}/7\mathbb{Z}+\mathbb{Z}/7\mathbb{Z}$	1	1	1	1	1
s500(4,1)	$\mathbb{Z}/4\mathbb{Z}+\mathbb{Z}/20\mathbb{Z}$	1	1	1	1	1
m343(3,2)	$\mathbb{Z}/82\mathbb{Z}$	0	−1	0	0	0
s773(−3,2)	$\mathbb{Z}/2\mathbb{Z}+\mathbb{Z}/32\mathbb{Z}$	0	1	0	0	0
m358(−5,1)	$\mathbb{Z}/4\mathbb{Z}+\mathbb{Z}/24\mathbb{Z}$	1	1	1	1	1
m410(3,2)	$\mathbb{Z}/2\mathbb{Z}+\mathbb{Z}/2\mathbb{Z}$	0	1	0	0	0
m358(−5,3)	$\mathbb{Z}/2\mathbb{Z}+\mathbb{Z}/4\mathbb{Z}$	0	1	0	0	0
s518(−4,1)	$\mathbb{Z}/2\mathbb{Z}+\mathbb{Z}/68\mathbb{Z}$	0	1	0	0	0

Consider the components of the variety of characters containing irreducible representations. Up to conjugation, irreducible representations $\rho : \pi_1(M) \to \mathrm{SL}_2(\mathbb{F})$ have the form

$$\rho(\mathscr{M}) = \begin{pmatrix} m & 1 \\ 0 & m^{-1} \end{pmatrix}, \quad \rho(\beta) = \begin{pmatrix} b & 0 \\ u & b^{-1} \end{pmatrix}$$

for $u \neq 0, -(m-m^{-1})(b-b^{-1})$.

We will see the only non-trivial coefficients that appear in the characteristic 0 case are multiples of 2 and 3 and thus those characteristics are the only situation where the terms vary. Assessing the group relation we find in all characteristics,

$$\begin{aligned} u &= -\frac{\left(1-m^3\right)\left(1-mb^4\right)+b^2\left(1-m+m^2-m^3+m^4\right)}{m\left(1+m+m^2\right)b\left(1+b^2\right)}, \\ 0 = f(m,b) &= \left(1-2m^3+m^6\right)\left(1+b^8\right)+\left(3-m+m^2-6m^3+m^4-m^5+3m^6\right)\left(b^2+b^6\right) \\ &\quad + \left(4-2m+2m^2-9m^3+2m^4-2m^5+4m^6\right)b^4. \end{aligned}$$

As there are two generators, the variety of characters is generated by three trace coordinates $\mathrm{tr}(\rho(\mathscr{M}))$, $\mathrm{tr}(\rho(\beta))$, and $\mathrm{tr}(\rho(\mathscr{M}\beta))$. Writing $s = \mathrm{tr}(\rho(\mathscr{M})) = m+m^{-1}$, $t = \mathrm{tr}(\rho(\beta)) = b+b^{-1}$, and $r = \mathrm{tr}(\rho(\mathscr{M}\beta))$ gives in all characteristics,

$$\begin{aligned} g(s,t) &= \left(4+4s-s^2-s^3\right)t^2-\left(2+3s-s^3\right)t^4-1=0, \\ h(s,t,r) &= r(s+1)t-(s+1)t^2-1=0. \end{aligned}$$

The respective irreducible components of the variety of characters are given by

$$\begin{aligned} X_p &= \left\{(s,t,r) \in \mathbb{F}_p^3 \mid g(s,t)=0, h(s,t,r)=0\right\} \\ &\cong \left\{(s,t) \in \mathbb{F}_p^2 \mid g(s,t)=0\right\}. \end{aligned} \tag{4.2.1}$$

In this example, the equations in characteristic 2 and 3 can be obtained from the equations in characteristic 0 by simply reducing the integer coefficients modulo 2 or modulo 3 respectively. This is because in the calculation of X_p, no terms are divided by 2 or 3.

We now turn to the A–polynomial, where such a reduction is not possible. Assessing the matrix associated with the longitude $\mathscr{L}$ in each characteristic and using Gröbner bases in `Singular` produces

$$\begin{aligned} A_0(m,l) = \widetilde{A}_0(m,l) &= m^4+2m^5+3m^6+m^7-m^8-3m^9-2m^{10}-m^{11} \\ &\quad + l^2\left(-1-3m-2m^2-m^3+2m^4+4m^5+m^6+4m^7\right. \\ &\qquad \left. +m^8+4m^9+2m^{10}-m^{11}-2m^{12}-3m^{13}-m^{14}\right) \\ &\quad + l^4\left(-m^3-2m^4-3m^5-m^6+m^7+3m^8+2m^9+m^{10}\right), \\ A_2(m,l) = \widetilde{A}_2(m,l) &= m^4+m^5+m^7+m^9+m^{10} \\ &\quad + l^2\left(1+m^3+m^4+m^5+m^8+m^9+m^{10}+m^{13}\right)+l^4\left(m^3+m^4+m^6+m^8+m^9\right), \\ A_3(m,l) = \widetilde{A}_3(m,l) &= m^4-m^5+m^7-m^8+m^{10}-m^{11} \\ &\quad + l^2\left(-1+m^2-m^3-m^4+m^5+m^6+m^7+m^8+m^9-m^{10}-m^{11}+m^{12}-m^{14}\right) \\ &\quad + l^4\left(-m^3+m^4-m^6+m^7-m^9+m^{10}\right). \end{aligned}$$

If we reduce A_0 modulo 2, the result gives A_2 times a factor $(m-1)$. There is a chance that in characteristic 2, we have a second component containing irreducible representations where we map:

$$\rho(\mathscr{M}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \rho(\beta) = \begin{pmatrix} b & 0 \\ u & b^{-1} \end{pmatrix}.$$

However, there is no representation conjugate into this form. Hence A_2 is obtained from A_0 by reducing modulo 2 and deleting a factor of the resulting polynomial. In characteristic 3 the result is more straightforward and A_3 is obtained from A_0 by reducing modulo 3.

Although there is a change in the polynomial in characteristic 2, we see there is no change in the strongly detected boundary slopes as the slopes of the Newton polygons remain the same across all characteristics.

4.2.2 Extended example: `m019(3,4)`

We focus on an example of a closed manifold where the dimension of the variety of characters decreases in characteristic 2. Take M to be the manifold `m019(3,4)`. Our manifold M is closed and hyperbolic and appears in the census of [BT18] of closed Haken hyperbolic 3–manifolds.

`Regina` computes the following fundamental group,

$$\pi_1(M) = \left\langle \alpha, \beta \mid \alpha\beta^2\alpha\beta^2\alpha\beta^{-1}\alpha^3\beta^{-1}, \alpha^3\beta^{-3}\alpha^3\beta^{-1}\alpha^4\beta^{-1} \right\rangle.$$

Note that $H_1(M) \cong \mathbb{Z}_{40}$ is finite cyclic, and hence there are only finitely many reducible characters.

We will show that in characteristic $p \neq 2$, the components of the variety of characters that contain irreducible representations consist of points, and in characteristic 2, there are no irreducible representations. That is,

$$\dim(X_p^{\mathrm{irr}}(M)) = 0, \quad \dim(X_2^{\mathrm{irr}}(M)) = -1.$$

In particular, this means that no essential surfaces in M are detected by the variety of characters in any characteristic.

Consider the components of the variety of characters containing irreducible representations. Up to conjugation, irreducible representations $\rho : \pi_1(M) \to \mathrm{SL}_2(\mathbb{F})$ have the form

$$\rho(\alpha) = \begin{pmatrix} a & 1 \\ 0 & a^{-1} \end{pmatrix}, \quad \rho(\beta) = \begin{pmatrix} b & 0 \\ u & b^{-1} \end{pmatrix} \tag{4.2.2}$$

for $u \neq 0, -(a-a^{-1})(b-b^{-1})$.

Substitute the irreducible representation Equation (4.2.2) into the group relations. In characteristic 2, we find no such representations exist as the expressions either force a contradiction, with $a = 0$ or $b = 0$, or force that $u = 0$, and the representation is reducible.

In characteristic $p \neq 2$,

$$\begin{aligned}
u &= \frac{-a^2+a^4+b^2+a^4b^2+b^4-a^2b^4}{a\,(1+a^2)\,b\,(1+b^2)},\\
0 = f(a,b) &= \left(1+a^4\right)\left(1+b^8\right)+2\left(1+a^4\right)\left(b^2+b^6\right)+2\left(2+a^2+2a^4\right)b^4,\\
0 = g(a,b) &= \left(1+a^2+a^4\right)\left(1+b^2\right)-a^2b,\\
0 = h(a,b) &= a^2\left(1+b^6\right)+\left(1-a^2+a^4\right)\left(b+b^5\right)+2a^2\left(b^2-b^3+b^4\right).
\end{aligned}$$

Note that the equations given for u, $f(a,b)$, $g(a,b)$, $h(a,b)$ modulo 2 do not alone satisfy the group relations in characteristic 2.

As there are two generators, the variety of characters is generated by three trace coordinates, $\mathrm{tr}(\rho(\alpha))$, $\mathrm{tr}(\rho(\beta))$, and $\mathrm{tr}(\rho(\alpha\beta))$. Writing $s=\mathrm{tr}(\rho(\alpha))=a+a^{-1}$, $t=\mathrm{tr}(\rho(\beta))=b+b^{-1}$, and $r=\mathrm{tr}(\rho(\alpha\beta))$ gives in characteristic $p\neq 2$,

$$\begin{aligned}
k(s,t,r) &= str-t^2\left(s^2-2\right)-2=0,\\
l(s,t) &= \left(s^2-1\right)t-1=0,\\
m(s,t) &= t^4\left(s^2-2\right)-2t^2\left(s^2-2\right)+2\left(s^2-1\right)=0,\\
n(s,t) &= t^3+t^2\left(s^2-3\right)-t-2\left(s^2-2\right)=0,
\end{aligned}$$

which solves to

$$\begin{aligned}
r &= \frac{s\left(2s^2-3\right)}{s^2-1},\\
t &= \frac{1}{s^2-1},\\
q(s) &= 2s^8-10s^6+18s^4-12s^2+1=0.
\end{aligned}$$

Solving the system of equations we find the the respective irreducible components of the character varieties for $p\neq 2$ are given by

$$\begin{aligned}
X_p &= \left\{(s,t,r)\in\mathbb{F}_p^3 \mid k(s,t,r)=0,\ l(s,t)=0,\ m(s,t)=0,\ n(s,t)=0\right\}\\
&\cong \left\{s\in\mathbb{F}_p \mid q(s)=0\right\}.
\end{aligned}\tag{4.2.3}$$

Each version of $q(s)=0$ modulo $p\neq 2$ has eight distinct solutions over $\mathbb{F}_p$, so each X_p contains eight points.

Then M is an example of a hyperbolic, Haken, 3–manifold with none of its essential surfaces detected by the variety of characters in any characteristic. This gives an example of the second part of Theorem 4.1.

4.2.3 Extended example: `m188(2,3)`

We now focus on an example of a closed manifold where the dimension of the variety of characters increases in characteristic 2. Take M to be the manifold `m188(2,3)`. Our manifold M is closed, Haken, and hyperbolic. As in the previous example, it is known to be Haken because it appears in the census of [BT18]. We will show that in characteristic 0, the components of the variety of characters that contain irreducible representations consist of points, and in characteristic 2, there is a curve. That is,

$$\dim(X_0^{\mathrm{irr}}(M)) = 0, \quad \dim(X_2^{\mathrm{irr}}(M)) = 1.$$

In particular, this gives an independent proof of the Haken property.

`Regina` computes the following fundamental group,

$$\pi_1(M) = \left\langle \alpha^3\beta^{-1}\alpha\beta^2\alpha\beta^2\alpha\beta^{-1},\ \alpha^{-2}\beta^3\alpha^{-2}\beta^3\alpha\beta^{-1}\alpha\beta^3 \right\rangle.$$

Note that $H_1(M) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{26}$ is finite cyclic, and hence there are only finitely many reducible characters.

Consider the components of the variety of characters containing irreducible representations. Up to conjugation, irreducible representations $\rho : \pi_1(M) \to \mathrm{SL}_2(\mathbb{F})$ have the form

$$\rho(\alpha) = \begin{pmatrix} a & 1 \\ 0 & a^{-1} \end{pmatrix}, \quad \rho(\beta) = \begin{pmatrix} b & 0 \\ u & b^{-1} \end{pmatrix} \tag{4.2.4}$$

for $u \neq 0, -(a-a^{-1})(b-b^{-1})$.

Substitute the irreducible representation Equation (4.2.4) into the group relations.

In characteristic 2 either,

$$\begin{aligned} u_2 &= ab(1+b)^8, \\ 0 = f_2(b) &= 1+a^2, \\ 0 = g_2(b) &= 1+b+b^2+b^3+b^4, \end{aligned}$$

or

$$\begin{aligned} u_2 &= \frac{1+a^2}{ab}, \\ 0 = h_2(b) &= 1+b^2. \end{aligned}$$

In characteristic $p \neq 2$,

$$\begin{aligned} u_p &= \frac{-a^2+a^4+b^2+a^4b^2+b^4-a^2b^4}{a(1+a^2)b(1+b^2)}, \\ 0 = f_p(a,b) &= \left(1+a^4\right)\left(1+b^8\right)+2\left(1+a^4\right)\left(b^2+b^6\right)+2\left(2+a^2+2a^4\right)b^4, \\ 0 = g_p(b) &= \left(1+b^4\right)\left(1+3b^2+5b^4+3b^6+b^8\right). \end{aligned}$$

If we took the solution in characteristic $p \neq 2$ and reduced modulo 2, we would get $u_p = 0$, contradicting irreducibility of the representation.

As there are two generators, the variety of characters is generated by three trace coordinates, $\mathrm{tr}(\rho(\alpha))$, $\mathrm{tr}(\rho(\beta))$, and $\mathrm{tr}(\rho(\alpha\beta))$. Write $s = \mathrm{tr}(\rho(\alpha)) = a + a^{-1}$, $t = \mathrm{tr}(\rho(\beta)) = b + b^{-1}$, and $r = \mathrm{tr}(\rho(\alpha\beta))$.

In characteristic 2 either,

$$\begin{aligned} s &= 0, \\ r &= 0, \\ m_2(t) &= t^2 + t + 1 = 0, \end{aligned}$$

or

$$\begin{aligned} t &= 0, \\ r &= 0. \end{aligned}$$

In characteristic $p \neq 2$,

$$\begin{aligned} k_p(s,t,r) &= str - t^2\left(s^2-2\right) - 2 = 0, \\ l_p(s,t) &= -2\left(t^2-1\right)^2 + s^2\left(t^4 - 2t^2 + 2\right) = 0, \\ m_p(t) &= \left(t^2-2\right)\left(t^4 - t^2 + 1\right) = 0. \end{aligned}$$

Then the respective irreducible components of the variety of characters are given by the following.

In characteristic 2,

$$\begin{aligned} X_2 &= \left\{(s,t,r) \in \mathbb{F}_2^3 \mid s=0,\ r=0,\ m_2(t)=0\right\} \cup \left\{(s,t,r) \in \mathbb{F}_2^3 \mid t=0,\ r=0\right\} \\ &\cong \{t \in \mathbb{F}_2 \mid m_2(t)=0\} \cup \{s \in \mathbb{F}_2\}. \end{aligned}$$

The first component has two points, the second forms a line.

In characteristic $p \neq 2$,

$$\begin{aligned} X_p &= \left\{(s,t,r) \in \mathbb{F}_p^3 \mid k_p(s,t,r)=0,\ l_p(s,t)=0,\ m_p(t)=0\right\} \\ &\cong \left\{(s,t) \in \mathbb{F}_p^2 \mid l_p(s,t)=0,\ m_p(t)=0\right\}. \end{aligned}$$

This has eight points in characteristic 3 and twelve points otherwise. The change in number of points in characteristic 3 is a result of the factorisation of $m_3(t)$.

4.3 Essential surfaces not detected in any characteristic

We complete the suite of examples by analysing an infinite family of manifolds N_Φ that consist of a twisted I–bundle over the Klein bottle (Section 4.3.1) glued to the complement of the right-handed trefoil (Section 4.3.2). The family of manifolds is presented in Section 4.3.3. These manifolds are either graph manifolds or Seifert fibered with base orbifolds $S^2(2,2,2,3)$ or $\mathbb{RP}^2(2,3)$. The essential surfaces in N_Φ are classified (up to isotopy) in Section 4.3.4, and we determine which surfaces are detected in which characteristic in Sections 4.3.5 to 4.3.9.

Essential surfaces in Seifert fibered manifolds can be described more explicitly. In a connected, compact, irreducible Seifert fibered manifold M, Hatcher shows [Hat07, Proposition 1.11] that each essential surfaces is isotopic to either a **vertical** surface (a union of regular fibres) or a **horizontal** surface (transverse to the fibration, giving a branched covering of the base orbifold with branch points corresponding to the intersections with singular fibres).

Let the base orbifold of M be Q and let $\hat{Q}$ denote the surface obtained by removing regular neighbourhoods of cone points in Q. By a result of Schultens, [Sch18, Lemma 39], vertical essential surfaces are in bijective correspondence with the isotopy classes of simple closed curves in $\hat{Q}$.

Observations that follow from this analysis are:

Theorem 4.2. *There are infinitely many graph manifolds with the property that they each contain (up to isotopy) a single connected essential surface, the single connected essential surface is a torus, and the torus is not detected by the* $\mathrm{SL}_2(\mathbb{F})$*–variety of characters for any algebraically closed field* $\mathbb{F}$*.*

Theorem 4.3. *There are infinitely many graph manifolds with the property that they each contain (up to isotopy) a single connected essential surface, the single connected essential surface is a torus, and the torus is detected by the* $\mathrm{SL}_2(\mathbb{F})$*–variety of characters if and only if the characteristic of* $\mathbb{F}$ *is 2.*

Theorem 4.4. *There are infinitely many graph manifolds that contain (up to isotopy) two essential surfaces, a torus and a genus two surface, each of which is detected by the* $\mathrm{SL}_2(\mathbb{F})$*–variety of characters if and only if the characteristic of* $\mathbb{F}$ *is 2.*

Theorem 4.5. *There are infinitely many graph manifolds that contain (up to isotopy) two essential surfaces, a torus and a non-separating genus two surface, and with:*

- *the torus not detected by the* $\mathrm{SL}_2(\mathbb{F})$*–character variety for any algebraically closed field* $\mathbb{F}$*, and*
- *the non-separating genus two surface detected by the* $\mathrm{SL}_2(\mathbb{F})$*–variety of characters for every algebraically closed field* $\mathbb{F}$*.*

The second part of the above theorem follows because each non-separating orientable surface is Poincaré dual to an epimorphism from the fundamental group of the ambient 3–manifold to the integers, and hence detected by a curve of reducible representations.

Theorem 4.6. *There are infinitely many Seifert fibered 3–manifolds with base* $\mathbb{RP}^2(2,3)$ *and that contain (up to isotopy) exactly two connected essential surfaces, both vertical tori, with:*

- *one torus is not detected by the* $\mathrm{SL}_2(\mathbb{F})$*–variety of characters for any algebraically closed field* $\mathbb{F}$*, and*
- *the other torus is detected by the* $\mathrm{SL}_2(\mathbb{F})$*–variety of characters if and only if the characteristic of* $\mathbb{F}$ *is 2.*

Theorem 4.7. *There are infinitely many Seifert fibered 3–manifolds with base* $S^2(2,2,2,3)$ *and that contain infinitely many pairwise non-isotopic connected essential tori, all of which are detected by the* $\mathrm{SL}_2(\mathbb{F})$*–variety of characters for every algebraically closed field* $\mathbb{F}$*.*

For Theorem 4.7, we prove that there are infinitely many such essential tori detected by $\mathrm{SL}_2(\mathbb{F})$–variety of characters, and conjecture this is true for all others that appear.

These results prove Theorem 4.1. In particular, Theorems 4.3, 4.4 and 4.6 give examples of the first part for $p = 2$ and Theorems 4.2, 4.5 and 4.6 give examples for the second part.

4.3.1 The Klein bottle group

We consider the Klein bottle group via a well-known presentation,

$$\Gamma_K = \langle a, b \mid aba^{-1}b = 1 \rangle. \tag{4.3.1}$$

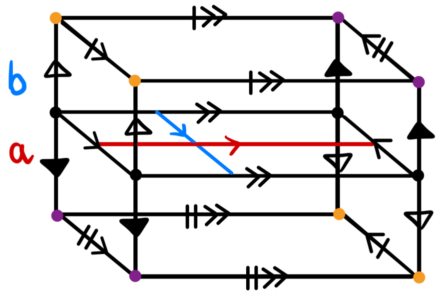

FIGURE 4.1: Generators for the twisted I–bundle over the Klein bottle.

The following result is proven in [FGT18].

Proposition 4.8 ([FGT18]). *Let $\mathbb{K}$ be a field and Γ_K the Klein bottle group as in Equation (4.3.1). Suppose $\sigma\colon \Gamma_K \to \mathrm{SL}_2(\mathbb{K})$ is a representation. If $\sigma(b) \neq \pm E$, then one of the following occurs:*

(i) σ *factors through the abelianization of* Γ_K.

(ii) $\sigma(a^2) = \pm E$.

In particular, Γ_K does not admit faithful representations into $\mathrm{SL}_2(\mathbb{K})$ for any field $\mathbb{K}$.

Note, $\sigma(a^2) = \sigma(a)^2 = \pm E$ implies either $\sigma(a) = \pm E$ or $\mathrm{tr}(\sigma(a)) = 0$.

Consider the character variety $\mathrm{X}(K, \mathbb{F})$. We can easily compute the subvarieties $\mathrm{X}^{\mathrm{irr}}(K, \mathbb{F})$ and $\mathrm{X}^{\mathrm{red}}(K, \mathbb{F})$ respectively. The main observation is that in characteristic 2 there are infinitely many elements in $\mathrm{SL}_2(\mathbb{F})$ with order two and, as a result, we find infinitely many non-conjugate abelian representations for Γ_K.

Each irreducible representation is conjugate to the form

$$\rho(a) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}, \quad y \in \mathbb{F} \setminus \{0\}. \tag{4.3.2}$$

We note that this representation is indeed irreducible if and only if

$$0 \neq \mathrm{tr}[\rho(a), \rho(b)] - 2 = y^2 - 2 + y^{-2}.$$

Hence the representation is irreducible if and only if $y^2 \neq 1$. Letting $t = y + y^{-1}$, the map

$$\rho \mapsto (\mathrm{tr}(\rho(a)), \mathrm{tr}(\rho(b)), \mathrm{tr}(\rho(ab))) \in \mathbb{F}^3$$

gives

$$\mathrm{X}^{\mathrm{irr}}(K,\mathbb{F}) = \{(0,t,0) \mid t \in \mathbb{F}\}.$$

From the trace condition on the commutator, we know $\mathrm{X}^{\mathrm{irr}}(K,\mathbb{F}) \cap \mathrm{X}^{\mathrm{red}}(K,\mathbb{F}) = \{(0,\pm 2,0)\}$. We now determine all reducible representations up to conjugacy. A reducible representation is conjugate to either

$$\rho(a) = \begin{pmatrix} x & 1 \\ 0 & x^{-1} \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}, \quad x \in \mathbb{F}\setminus\{0\}, \text{ or} \tag{4.3.3}$$

$$\rho(a) = \begin{pmatrix} x & u \\ 0 & -x \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix}, \quad u \in \mathbb{F}, x \in \mathbb{F}\setminus\{0\}, x^2 = -1. \tag{4.3.4}$$

Here we have $\rho(b) = \pm E$ for the representations in Equation (4.3.3) and $\mathrm{tr}(\rho(a)) = 0$ for the representations in Equation (4.3.4).

In characteristic $p = 2$,

$$\mathrm{X}^{\mathrm{red}}(K,\mathbb{F}) = \{(s,0,s) \mid s \in \mathbb{F}\}$$

and the matrix pair in Equation (4.3.4) splits into infinitely many conjugacy classes,

$$\rho(a) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad u \in \mathbb{F}. \tag{4.3.5}$$

In characteristic $p \neq 2$,

$$\mathrm{X}^{\mathrm{red}}(K,\mathbb{F}) = \{(s,2,s) \mid s \in \mathbb{F}\} \cup \{(s,-2,-s) \mid s \in \mathbb{F}\}$$

and the matrix pair in Equation (4.3.4) is conjugate to

$$\rho(a) = \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix}, \quad x \in \mathbb{F}\setminus\{0\}, x^2 = -1. \tag{4.3.6}$$

Let $K\tilde{\times}I$ be a twisted I-bundle over the Klein bottle with $\Gamma_{K\tilde{\times}I}$ the associated fundamental group,

$$\Gamma_{K\tilde{\times}I} = \Gamma_K = \langle a,b \mid aba^{-1}b \rangle.$$

Then the character variety of $K\tilde{\times}I$ is the same as that of K,

$$\mathrm{X}^{\mathrm{irr}}(K\tilde{\times}I,\mathbb{F}) = \mathrm{X}^{\mathrm{irr}}(K,\mathbb{F}),$$
$$\mathrm{X}^{\mathrm{red}}(K\tilde{\times}I,\mathbb{F}) = \mathrm{X}^{\mathrm{red}}(K,\mathbb{F}).$$

We note from the above calculations that in any characteristic $p \geq 0$, the variety $\mathrm{X}^{\mathrm{irr}}(K\tilde{\times}I,\mathbb{F})$ is a curve with a single ideal point, and $\mathrm{X}^{\mathrm{red}}(K\tilde{\times}I,\mathbb{F})$ consists of either a single curve with a single ideal point (if $p = 2$) or two disjoint curves, each with a single ideal point (if $p \neq 2$).

The peripheral subgroup for $\Gamma_{K\tilde{\times}I}$ is

$$P_{K\tilde{\times}I} = \langle a^2, b \rangle.$$

The twisted I–bundle over the Klein bottle has two Seifert fibrations. See [AFW15, Section 1.5] for a succinct discussion; we summarise here the points needed for our application and use the above description of the variety of characters to determine which components of $\mathrm{X}(K\tilde{\times}I,\mathbb{F})$ detect which essential surface.

The two Seifert fibrations of $K\tilde{\times}I$ have base orbifolds $D^2(2,2)$ and the Möbius band (with no cone points) respectively. We know there are two types of connected surfaces in $K\tilde{\times}I$: vertical and horizontal. The fibration over the Möbius band shows that there is one connected horizontal surface that arises from an unbranched double cover of the Möbius band (a separating annulus $\mathscr{A}$ with boundary slope a^2) and one connected vertical surface (a non-separating annulus $\mathscr{B}$ with boundary slope b). Note that these surfaces cannot be isotoped to be disjoint. It follows that an essential surface in $K\tilde{\times}I$ consists either of parallel copies of $\mathscr{A}$ or of parallel copies of $\mathscr{B}$. Moreover, an essential surface in $K\tilde{\times}I$ is uniquely determined by its boundary curves.

The slope a^2 is a regular fibre in the Seifert fibration of $K\tilde{\times}I$ with base orbifold $D^2(2,2)$. The annulus $\mathscr{A}$ is foliated by regular fibres and decomposes $K\tilde{\times}I$ into two solid tori with Seifert structures over the base orbifold $D^2(2)$. Hence, $\mathscr{A}$ is an essential annulus in the boundary of the two solid tori. The presentation of the fundamental group corresponding to this is a free product with amalgamation of the form

$$\langle c\rangle *_{c^2=d^2} \langle d\rangle.$$

Here, $c^2=d^2$ corresponds to the regular fibre. An isomorphism to the presentation given in Equation (4.3.1) is:

$$c\mapsto a \qquad \text{and} \qquad d\mapsto b^{-1}a.$$

Note that this gives $c^2=d^2\mapsto a^2$ as claimed.

For any characteristic $p\geq 0$, the restriction of I_a (and hence of I_{a^2}) to the curve

$$\mathrm{X}^{\mathrm{irr}}(K\tilde{\times}I,\mathbb{F})=\{(0,t,0)\mid t\in\mathbb{F}\}$$

is constant equal to zero while the trace function I_b has a pole of order one at the ideal point. It follows from Corollary 3.20 and the above classification of essential surfaces that $\mathscr{A}$ is detected by the curve $\mathrm{X}^{\mathrm{irr}}(K\tilde{\times}I,\mathbb{F})$ and it also follows that $\mathscr{B}$ is not detected by this curve. Also note that the fundamental groups of the complementary solid tori are generated by $\langle a\rangle$ and $\langle b^{-1}a\rangle$ respectively. Now the trace functions I_a and $I_{b^{-1}a}$ are constant equal to zero on $\mathrm{X}^{\mathrm{irr}}(K\tilde{\times}I,\mathbb{F})$, thus verifying the splitting detected at the ideal point.

The slope b is a regular fibre in a Seifert fibration with base orbifold the Möbius band (with no cone points). The non-separating annulus $\mathscr{B}$ is foliated by regular fibres in this fibration, hence $\operatorname{im}(\pi_1(\mathscr{B})\to\pi_1(K\tilde{\times}I))=\langle b\rangle$. Now the trace function I_b is constant on each component of $\mathrm{X}^{\mathrm{red}}(K\tilde{\times}I,\mathbb{F})$ (there is one curve for $p=2$ and two curves for $p\neq 2$) and the trace function I_a has a pole. In particular, Corollary 3.20 implies that $\mathscr{B}$ is detected by each curve in $\mathrm{X}^{\mathrm{red}}(K\tilde{\times}I,\mathbb{F})$ and $\mathscr{A}$ is not detected by the curves in $\mathrm{X}^{\mathrm{red}}(K\tilde{\times}I,\mathbb{F})$. This is related to the fact that $\mathscr{B}$ is dual to the epimorphism $\pi_1(K\tilde{\times}I)\to\mathbb{Z}$ defined by $a\mapsto 1$ and $b\mapsto 0$.

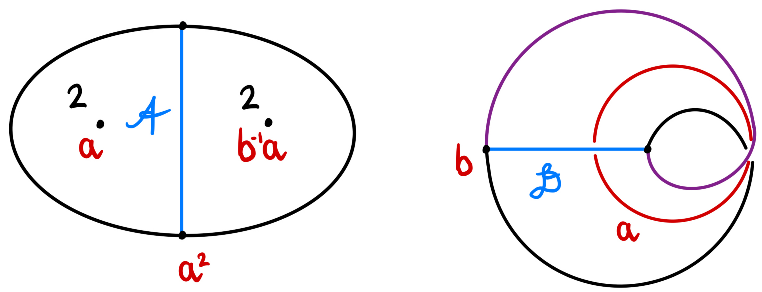

FIGURE 4.2: Essential annuli in the twisted I–bundle over the Klein bottle, with: $\mathscr{A}$ in $D^2(2,2)$ (left), $\mathscr{B}$ in the Möbius band (right).

4.3.2 The right-handed trefoil group

We consider the complement of the right-handed trefoil, M, and the associated fundamental group via a well-known presentation,

$$\Gamma_M = \langle g, h \mid ghg = hgh \rangle .$$

The peripheral subgroup for Γ_M is

$$P_M = \langle g, g^{-4}hg^2h \rangle$$

with g a standard meridian and $g^{-4}hg^2h$ a standard longitude.

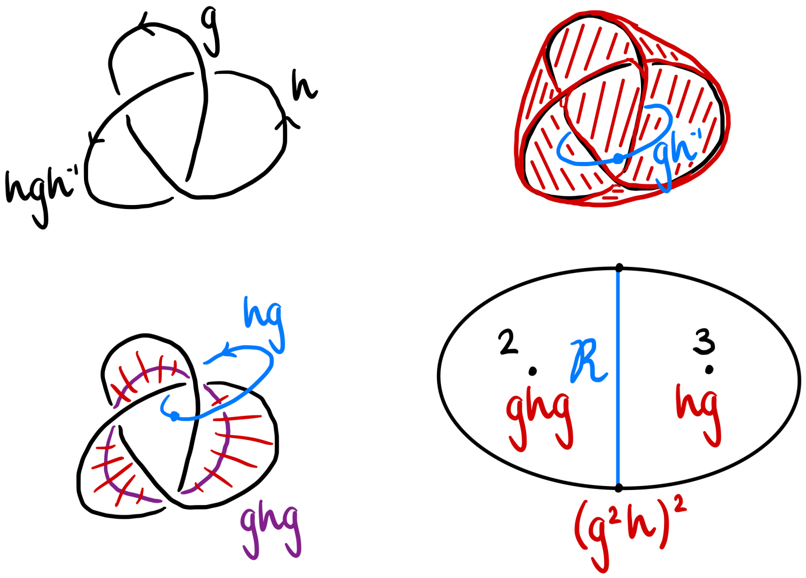

FIGURE 4.3: Generators of the trefoil complement (top left), elements in complementary region of the Seifert surface (top right) and the Möbius band (bottom left), and the Seifert fibration (bottom right).

As for the previous manifold, we apply [Hat07, Proposition 1.11] to show that that there are exactly two connected essential surfaces in M. The right-handed trefoil is Seifert fibered with the curve of slope g^2hg^2h a regular fibre and base orbifold $D^2(2,3)$. In this case, there is a vertical annulus $\mathscr{R}$ with boundary slope g^2hg^2h, separating the trefoil knot complement into two solid tori with fibrations $D^2(2)$ and $D^2(3)$ respectively. This is the only connected vertical surface. The unique connected horizontal surface is a once-punctured torus arising from a branched covering of $D^2(2,3)$. This is a Seifert surface $\mathscr{S}$ for the trefoil knot with boundary slope $g^{-4}hg^2h$. Since the slopes of the two surfaces are not parallel, any essential surface in M either consists of parallel copies of $\mathscr{R}$ or of parallel copies of $\mathscr{S}$. Moreover, the surfaces are again uniquely determined by their boundary curves.

Consider the character variety $\mathrm{X}(M,\mathbb{F})$. We compute the subvarieties $\mathrm{X}^{\mathrm{irr}}(M,\mathbb{F})$ and $\mathrm{X}^{\mathrm{red}}(M,\mathbb{F})$ respectively.

Each irreducible representation is conjugate to

$$\rho(g) = \begin{pmatrix} x & 1 \\ 0 & x^{-1} \end{pmatrix}, \quad \rho(h) = \begin{pmatrix} x^{-1} & 0 \\ -1 & x \end{pmatrix}, \quad x \in \mathbb{F}\setminus\{0\}. \tag{4.3.7}$$

We note that this representation is irreducible if and only if

$$0 \neq \mathrm{tr}[\rho(g),\rho(h)] - 2 = x^2 - 1 + x^{-2}.$$

Letting $s = x + x^{-1}$, we map $\rho \mapsto (\mathrm{tr}(\rho(g)), \mathrm{tr}(\rho(h)), \mathrm{tr}(\rho(gh))) \in \mathbb{F}^3$ giving

$$\mathrm{X}^{\mathrm{irr}}(M,\mathbb{F}) = \{(s,s,1) \mid s \in \mathbb{F}\}.$$

This is a curve with a single ideal point. For each $\rho \in \mathrm{X}^{\mathrm{irr}}(M,\mathbb{F})$, we have $\rho(g^2hg^2h) = -E$. In particular, $I_{g^2hg^2h}$ is constant. Now

$$I_{g^{-4}hg^2h} = -(x^6 + x^{-6}) = -s^6 + 6s^4 - 9s^2 + 2$$

has a pole of order 6 at the ideal point of $\mathrm{X}^{\mathrm{irr}}(M,\mathbb{F})$. Hence Corollary 3.20 implies that $\mathscr{R}$ is detected by this ideal point and $\mathscr{S}$ is not. The order of the pole is related to the fact that the boundary slopes of $\mathscr{R}$ and $\mathscr{S}$ have intersection number 6.

The reducible representations are conjugate to the forms

$$\rho(g) = \begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix} = \rho(h), \tag{4.3.8}$$

$$\rho(g) = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} = \rho(h), \quad x \in \mathbb{F}\setminus\{0\}, \tag{4.3.9}$$

$$\rho(g) = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}, \quad \rho(h) = \begin{pmatrix} x & 1 \\ 0 & x^{-1} \end{pmatrix}, \quad x \in \mathbb{F},\ x^2 + x^{-2} = 1. \tag{4.3.10}$$

The representations in Equations (4.3.8) and (4.3.9) are abelian, and Equation (4.3.10) defines reducible non-abelian representations.

In characteristic $p = 2$,

$$\mathrm{X}^{\mathrm{red}}(M,\mathbb{F}) = \{(s,s,s^2) \mid s \in \mathbb{F}\}.$$

Here, $\mathrm{X}^{\mathrm{irr}}(M,\mathbb{F}) \cap \mathrm{X}^{\mathrm{red}}(M,\mathbb{F}) = \{(1,1,1)\}$. The character in the intersection represents reducible representations conjugate to any of the above forms in Equations (4.3.7), (4.3.9) and (4.3.10) with $x^2 + x^{-2} = 1$.

In characteristic $p \neq 2$,

$$\mathrm{X}^{\mathrm{red}}(M,\mathbb{F}) = \{(s,s,s^2-2) \mid s \in \mathbb{F}\}.$$

Here, $\mathrm{X}^{\mathrm{irr}}(M,\mathbb{F}) \cap \mathrm{X}^{\mathrm{red}}(M,\mathbb{F}) = \{(\pm\sqrt{3},\pm\sqrt{3},1)\}$. The characters in the intersection represents reducible representations conjugate to any of the above forms in Equations (4.3.7), (4.3.9) and (4.3.10) with $x^2 + x^{-2} = 1$.

The Seifert surface $\mathscr{S}$ is Poincaré dual to the epimorphism $\pi_1(M) \to \mathbb{Z}$ and hence detected by the curve $\mathrm{X}^{\mathrm{red}}(M,\mathbb{F})$ for any characteristic $p \geq 0$. This is consistent with the observation that the restriction of $I_{g^{-4}hg^2h}$ to $\mathrm{X}^{\mathrm{red}}(M,\mathbb{F})$ is constant equal to 2 (since all characters on $\mathrm{X}^{\mathrm{red}}(M,\mathbb{F})$ are reducible and the boundary curve of $\mathscr{S}$ is a commutator). Moreover,

$$I_{g^2hg^2h} = x^6 + x^{-6} = s^6 - 6s^4 + 9s^2 - 2$$

and hence $I_{g^2hg^2h}$ has a pole at the ideal point of $\mathrm{X}^{\mathrm{red}}(M,\mathbb{F})$. Thus Corollary 3.20 implies that $\mathscr{S}$ is detected by the ideal point of $\mathrm{X}^{\mathrm{red}}(M,\mathbb{F})$ and $\mathscr{R}$ is not.

4.3.3 Families of graph manifolds and Seifert fibered manifolds

Suppose

$$\Phi = \begin{pmatrix} k & l \\ m & n \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}). \tag{4.3.11}$$

Let N_Φ be the closed orientable 3–manifold obtained by gluing $K\tilde{\times}I$ to M via Φ so that

$$g = \left(a^2\right)^k b^l \quad \text{and} \quad g^{-4}hg^2h = \left(a^2\right)^m b^n. \tag{4.3.12}$$

Then $\Gamma_\Phi = \pi_1(N_\Phi) = \langle\, a,b,g,h \mid ghg = hgh,\ aba^{-1}b = 1,\ g = \left(a^2\right)^k b^l,\ g^{-4}hg^2h = \left(a^2\right)^m b^n \,\rangle$.

Note that the abelianisation of Γ_Φ is

$$\begin{aligned} H_1(N_\Phi) &= \langle a,b,g,h \mid ab = ba, b^2 = 1, g = h, g = a^{2k}b^l, 1 = a^{2m}b^n \rangle, \\ &= \langle a,b \mid ab = ba, b^2 = 1, 1 = a^{2m}b^n \rangle, \\ &= \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_{2m} & \text{if } n = 0 \pmod 2 \\ \mathbb{Z}_{4m} & \text{if } n = 1 \pmod 2 \end{cases} \end{aligned}$$

where $\mathbb{Z}_0 = \mathbb{Z}$. In particular, $H_1(N_\Phi)$ is infinite if and only if $m = 0$ and $k = n = \pm 1$.

4.3.4 Essential surfaces in N_Φ

We determine all the possible essential surfaces in N_Φ.

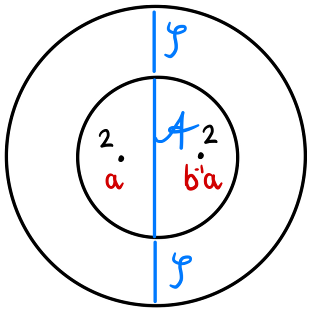

FIGURE 4.4: Separating genus-2 surface arising in $S^2(2,2,2,3)$ from S_2.

(S1) The splitting torus $\mathscr{T}$ between M and $K\tilde{\times}I$. This occurs in N_Φ for all $\Phi \in \mathrm{SL}_2(\mathbb{Z})$ and is an essential separating surface since both M and $K\tilde{\times}I$ have incompressible boundary.

We claim that all other essential surfaces S can be moved via isotopy such that they only have non-trivial intersection with the splitting torus. Consider S with no intersections with the splitting torus. Then it is a closed, orientable, incompressible surface either in M or in $K\tilde{\times}I$. This implies that its components are parallel to $\mathscr{T}$. Thus every other essential surface S must meet the splitting torus in at least one essential curve. Since we can remove annuli parallel to annuli on $\mathscr{T}$ by isotopy, it follows that if S is isotoped to have minimal number of intersection curves with $\mathscr{T}$, then $S \cap (K\tilde{\times}I)$ must be an essential surface in $K\tilde{\times}I$ and $S \cap M$ must be an essential surface in M. The only options for essential surfaces in each manifold $K\tilde{\times}I$ and M are given in Sections 4.3.1 and 4.3.2 respectively.

Using this we can list all connected essential surfaces in N_Φ up to isotopy by considering how the boundary curves match up. Since each of $K\tilde{\times}I$ and M has two boundary slopes of essential surfaces, this gives four possibilities. Matching up one boundary curve from each boundary component of $K\tilde{\times}I$ and M leaves one degree of freedom in the gluing map. We thus have:

(S2) $[\partial\mathscr{S} \leftrightarrow \partial\mathscr{A}]$ Matching up the boundary slope $g^{-4}hg^2h$ with the boundary slope a^2 forces

$$\Phi = \begin{pmatrix} k & \pm 1 \\ \mp 1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

where $k \in \mathbb{Z}$ is arbitrary. All resulting manifolds N_Φ are graph manifolds and not Seifert fibered. Any copy of $\mathscr{A}$ in $K\tilde{\times}I$ matches up with two parallel copies of the Seifert surface in M giving a connected separating genus-2 surface. Any two surfaces obtained this way are isotopic in N_Φ since $M \setminus \mathscr{S} \cong \mathscr{S} \times (0,1)$. We write $S_2 = (2\mathscr{S}) \cup \mathscr{A}$ for the separating genus-2 surface.

(S3) $[\partial\mathscr{S} \leftrightarrow \partial\mathscr{B}]$ Matching up the boundary slope $g^{-4}hg^2h$ with the boundary slope b forces

$$\Phi = \begin{pmatrix} \pm 1 & l \\ 0 & \pm 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

where $l \in \mathbb{Z}$ arbitrary. All resulting manifolds N_Φ are graph manifolds and not Seifert fibered. Any copy of $\mathscr{B}$ in $K\tilde{\times}I$ matches up with two parallel copies of the Seifert surface in M giving a

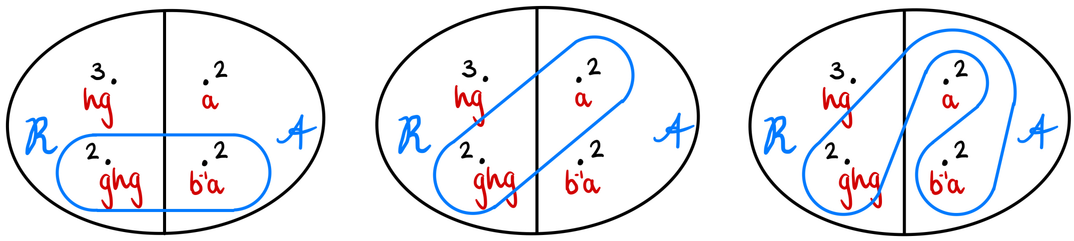

FIGURE 4.5: Example tori arising in $S^2(2,2,2,3)$ from $S_4(q,r)$.

connected surface. Since $\mathscr{B}$ is non-separating in $K\tilde{\times}I$, this is a non-separating genus-2 surface, written $S_3 = (2\mathscr{S})\cup\mathscr{B}$. Any two such surfaces are parallel since $M\setminus\mathscr{S}\cong\mathscr{S}\times(0,1)$.

(S4) $[\partial\mathscr{R}\leftrightarrow\partial\mathscr{A}]$ Recall that the slopes of $\partial\mathscr{R}$ and $\partial\mathscr{A}$ are regular fibres in Seifert fibrations of M and $K\tilde{\times}I$ respectively. The identification forces

$$\Phi = \begin{pmatrix} k & \pm 1 \\ \mp 1 - 6k & \mp 6 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

where $k\in\mathbb{Z}$ is arbitrary. This is a family of Seifert fibered manifolds with base orbifold $S^2(2,2,2,3)$.

Any combination of surfaces $\mathscr{R}$ and $\mathscr{A}$ will be vertical in the Seifert fibration. Schultens [Sch18] says vertical essential surfaces are in bijective correspondence with isotopy classes of simple closed curves on the four-punctured sphere, which are, in turn, in bijection with isotopy classes of simple closed curves on the torus (see [FM12, Section 2.2.5]).

Fix meridian α and longitude β for the torus. For $q,r\in\mathbb{Z}$, a simple closed curve on the torus is a (q,r)-curve if it intersects α q-times and β r-times. We can construct a similar classification of simple closed curves on the four-punctured sphere using the bijection, which proves the simple closed curves are defined by primitive pairs $(q,r)\in\mathbb{Z}^2$. Thus, the vertical essential surfaces are defined by primitive $(q,r)\in\mathbb{Z}^2$ and will correspond to some combination of copies of $\mathscr{R}$ and $\mathscr{A}$.

We assume that the basis is chosen such that $\mathscr{T}$ corresponds to the $(1,0)$-curve under the map $N_\Phi\to S^2(2,2,2,3)$. Our convention for the $(0,1)$-curve is given in Figure 4.5. We write $S_4(q,r) = \mathscr{R}\cup\mathscr{A}(q,r)$ for the essential surface $\mathscr{R}\cup\mathscr{A}$ associated with the (q,r)-curve. This will be a connected, separating torus and for distinct coprime pairs (q_1,r_1) and (q_2,r_2) the surfaces will be distinct and not isotopic. No horizontal essential surfaces arise in $S^2(2,2,2,3)$.

(S5) $[\partial\mathscr{R}\leftrightarrow\partial\mathscr{B}]$ Recall that the slopes of $\partial\mathscr{R}$ and $\partial\mathscr{B}$ are regular fibres in Seifert fibrations of M and $K\tilde{\times}I$ respectively. The identification forces

$$\Phi = \begin{pmatrix} \pm 1 & l \\ \mp 6 & \pm 1 - 6l \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

where $l\in\mathbb{Z}$ is arbitrary. This is a family of Seifert fibered manifolds with base orbifold $\mathbb{RP}^2(2,3)$.

Any combination of surfaces $\mathscr{R}$ and $\mathscr{B}$ will be vertical in the Seifert fibration. Note that $\mathscr{R}\cup\mathscr{B}$ gives an embedded Klein bottle in N_Φ and hence is not an essential surface. Taking two copies

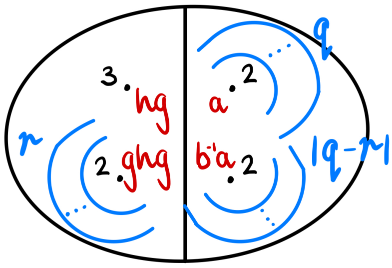

FIGURE 4.6: Example arcs for a general tori arising in $S^2(2,2,2,3)$ from $S_4(q,r)$. There are r copies of $\mathscr{R}$ and r copies of $\mathscr{A}$ twisted so that q arcs go around the cone point labelled a and $|q-r|$ arcs go around the cone point labelled $b^{-1}a$.

of each gives a separating torus, written S_5. Any two such tori are parallel. Now the core Klein bottle in $K\tilde{\times}I$ and the Klein bottle $\mathscr{R}\cup\mathscr{B}$ represent the same element in $H^2(N_\Phi,\mathbb{Z}_2)$. However, the tori $\mathscr{T}$ and $S_5=(2\mathscr{R})\cup(2\mathscr{B})$ are nevertheless not isotopic. This can be seen by considering the geometric intersection number with a curve represented by a. No horizontal essential surfaces arise in $\mathbb{RP}^2(2,3)$.

4.3.5 Curves in the character variety of N_Φ

Let $r_K\colon\ \mathrm{X}(N_\Phi,\mathbb{F})\to\mathrm{X}(K\tilde{\times}I,\mathbb{F})$ and $r_M\colon\ \mathrm{X}(N_\Phi,\mathbb{F})\to\mathrm{X}(M,\mathbb{F})$ be the restriction maps arising from the inclusions $\pi_1(K\tilde{\times}I)\to\pi_1(N_\Phi)$ and $\pi_1(M)\to\pi_1(N_\Phi)$ respectively. Suppose $C\subset\mathrm{X}(N_\Phi,\mathbb{F})$ is a curve. We have the following possibilities for C:

(C1) Both $r_K(C)$ and $r_M(C)$ are curves. It follows from the description of the curves in $\mathrm{X}(K\tilde{\times}I,\mathbb{F})$ and $\mathrm{X}(M,\mathbb{F})$ that for each ideal point ξ of C, there is a unique primitive class $\alpha\in\mathrm{im}(\pi_1(\mathscr{T})\to\pi_1(N_\Phi))$ with $v_\xi(I_\alpha)\geq 0$ and all other primitive classes $\gamma\in\mathrm{im}(\pi_1(\mathscr{T})\to\pi_1(N_\Phi))$ satisfy $v_\xi(I_\gamma)<0$. Hence none of these classes γ are contained in an edge stabiliser of the action on the Bass-Serre tree associated with the ideal point, and hence none of these are homotopic to a curve on the surface detected by ξ. It follows that any essential surface detected by C is as listed in (S2)–(S5). Except for the case (S4), the detected surface is uniquely determined by this information (up to taking parallel copies).

(C2) One of $r_K(C)$ and $r_M(C)$ is a curve and the other is a point. In this case, the curve again forces that for each ideal point ξ of C there is a primitive class $\gamma\in\mathrm{im}(\pi_1(\mathscr{T})\to\pi_1(N_\Phi))$ with $v_\xi(I_\gamma)<0$. However, since the other projection is a point, all representations with character on C have constant trace functions I_β for all $\beta\in\mathrm{im}(\pi_1(\mathscr{T})\to\pi_1(N_\Phi))$. This implies $v_\xi(I_\gamma)\geq 0$, a contradiction.

(C3) Each of $r_K(C)$ and $r_M(C)$ is a point. In this case, all representations with character on C have constant trace functions I_γ for all $\gamma\in\mathrm{im}(\pi_1(\mathscr{T})\to\pi_1(N_\Phi))$, for all $\gamma\in\mathrm{im}(\pi_1(K\tilde{\times}I)\to\pi_1(N_\Phi))$, and for all $\gamma\in\mathrm{im}(\pi_1(M)\to\pi_1(N_\Phi))$. It follows that we may choose the splitting torus $\mathscr{T}$ to be an essential surface detected by ξ. To rule out that any of the essential surfaces S listed in (S2)–(S5) is also detected by C, according to Theorem 3.16 we need to exhibit an element $\gamma\in\mathrm{im}(\pi_1(N'_\Phi)\to\pi_1(N_\Phi))$ with $v_\xi(I_\gamma)<0$ for N'_Φ a component of $N_\Phi\setminus S$.

In the following sections we determine when essential surfaces are detected by curves in the character variety of N_Φ, broken into cases for detecting: the splitting torus as in (S1); additional surfaces in the graph manifold as in (S2) and (S3); additional surfaces in the Seifert fibration with base orbifold $S^2(2,2,2,3)$ as in (S4); and additional surfaces in the Seifert fibration with base orbifold $\mathbb{RP}^2(2,3)$ as in (S5).

The approach for proving each result is similar. For ease of notation write

$$(A,B,G,H) = (\rho(a),\rho(b),\rho(g),\rho(h)) \in \mathrm{SL}_2(\mathbb{F})^4.$$

We analyse the variety of characters of N_Φ for each case and outline when a curve appears, up to conjugacy. We start with the options for representations of $K\tilde{\times}I$ outlined in Section 4.3.1. To extend the representations to N_Φ, the matrices must satisfy the gluing equations

$$G = A^{2k}B^l, \quad G^{-4}HG^2H = A^{2m}B^n. \tag{4.3.13}$$

If $B = \pm E$, as in Equation (4.3.3), the equations simplify to

$$G = A^{2k}(\pm 1)^l, \quad HA^{4k}H = (\pm 1)^n A^{8k+2m}.$$

If $B \neq \pm E$, from Proposition 4.8 either ρ factors through the abelianisation of N_Φ or $A^2 = \pm E$. We have $A^2 = \pm E$ in the case of Equations (4.3.2) and (4.3.4). Here, the equations simplify to

$$G = (\pm 1)^k B^l, \quad HB^{2l}H = (\pm 1)^m B^{4l+n}.$$

The resulting matrices must also satisfy group relation for M,

$$GHG = HGH. \tag{4.3.14}$$

The general procedure is as follows. Matrix G is defined by Equation (4.3.13). Write

$$H = \begin{pmatrix} e & f \\ g & h \end{pmatrix}, \quad eh - fg = 1. \tag{4.3.15}$$

Expanding the powers of A and B, Equation (4.3.13) gives four equations for the entries of H in terms of the other variables in addition to the determinant condition. Outside of the cases where there is a curve, we either find a contradiction in these equations, a contradiction when considering the group relation Equation (4.3.14), a contradiction to the gluing matrix $\Phi \in \mathrm{SL}_2(\mathbb{Z})$, or that all variables are specified and so we have at most some 0–dimensional components.

Where a curve is possible, we write for ease of notation $t_\gamma = I_\gamma(\rho) = \mathrm{tr}(\rho(\gamma))$ for $\gamma \in \pi_1(N_\Phi)$ and $\rho \in \mathrm{R}(N_\Phi,\mathbb{F})$. We know from Section 3.1 that the variety $\mathrm{X}(N_\Phi,\mathbb{F})$ has coordinates the traces of ordered single, double and triple products of the generators in any characteristic. These coordinates are

$$\left(t_a,t_b,t_g,t_h,t_{ab},t_{ag},t_{ah},t_{bg},t_{bh},t_{gh},t_{abg},t_{abh},t_{agh},t_{bgh}\right) \in \mathbb{F}^{14}.$$

In order to determine what surface is detected by a curve, we appeal to Theorem 3.16 and Corollary 3.20, the discussions of essential surfaces in $K\tilde{\times}I$ and M in Sections 4.3.1 and 4.3.2, the classification of connected essential surfaces in N_Φ given by (S1)–(S5), and the classification of the types of curves that appear in (C1)–(C3). More details for the calculations of curves of representations are given in Appendix B.

4.3.6 Detecting the splitting torus

The manifold N_Φ always contains the splitting torus $\mathscr{T}$ for all $\Phi \in \mathrm{SL}_2(\mathbb{Z})$. We get two results for when the splitting torus is detected over characteristic $p \neq 2$ and over characteristic 2.

Lemma 4.9. *Let $\mathbb{F}$ be an algebraically closed field of characteristic $p \neq 2$. There is a curve in $\mathrm{X}(N_\Phi, \mathbb{F})$ that detects the splitting torus $\mathscr{T}$ if and only if $\Phi = \begin{pmatrix} k & \pm 1 \\ \mp 1 - 6k & \mp 6 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ for $k \in \mathbb{Z}$. Moreover, there are infinitely many such curves whenever Φ is of this form.*

Lemma 4.10. *Let $\mathbb{F}$ be an algebraically closed field of characteristic 2. There is a curve in $\mathrm{X}(N_\Phi, \mathbb{F})$ that detects the splitting torus $\mathscr{T}$ if and only if $\Phi = \begin{pmatrix} k & 1 \\ 1 & 0 \end{pmatrix} \pmod 2 \in \mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$ for $k \in \{0,1\}$.*

Moreover, the curve is unique unless $\Phi = \begin{pmatrix} k & \pm 1 \\ \mp 1 - 6k & \mp 6 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ for $k \in \mathbb{Z}$, in which case there are infinitely many curves.

Theorem 4.2 now follows by taking (for instance) the family $\Phi_k = \begin{pmatrix} 1+6k & k \\ 6 & 1 \end{pmatrix}$ and Theorem 4.3 by taking (for instance) the family $\Phi'_k = \begin{pmatrix} k & 1+6k \\ -1 & -6 \end{pmatrix}$ for $k \in \mathbb{N}$. These families have been chosen such that no other essential surfaces are present. We have Φ_k clearly does not satisfy the requirement for Lemmas 4.9 and 4.10 and Φ'_k only satisfies Lemma 4.10 but not Lemma 4.9.

Proof of Lemma 4.9. Assume $p \neq 2$. We follow the approach described in Section 4.3.5. The splitting torus $\mathscr{T}$ can only be detected by a curve of type (C3) where the restrictions to the character varieties of $K\tilde{\times}I$ and M are points. We will see that there are two ways to get such a curve in characteristic $p \neq 2$, one of which comes from a 2–dimensional component that will give infinitely many curves.

Consider the options for matrix pairs (A,B) from Equations (4.3.2), (4.3.3) and (4.3.6) in turn.

We find one way to extend the representation from Equation (4.3.2) to N_Φ with a curve of type (C3). For gluing matrix $\Phi = \begin{pmatrix} k & \pm 1 \\ \mp 1 - 6k & \mp 6 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ we get a 2–dimensional component of representations

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix},$$

$$G = (-1)^k \begin{pmatrix} y^{\pm 1} & 0 \\ 0 & y^{\mp 1} \end{pmatrix}, \quad H = \begin{pmatrix} \frac{\mp(-1)^k y^{\mp 2}}{y - y^{-1}} & f \\ \frac{-y^2+1-y^{-2}}{f(y-y^{-1})^2} & \frac{\pm(-1)^k y^{\pm 2}}{y-y^{-1}} \end{pmatrix}, \quad f, y \in \mathbb{F} \setminus \{0\}.$$

The corresponding traces give a 2-dimensional component of the variety of characters,

$$C_\Phi = \Bigg\{ \Bigg(0, y+y^{-1}, (-1)^k(y+y^{-1}), (-1)^k(y+y^{-1}), 0, 0, \frac{y^2-1+y^{-2}}{f(y-y^{-1})^2} + f,$$
$$(-1)^k(y^{1\pm1}+y^{-1\mp1}), \pm(-1)^k \frac{y^{-1\pm2}-y^{1\mp2}}{y-y^{-1}}, 1, 0, \frac{y^2-1+y^{-2}}{fy(y-y^{-1})^2} + fy, \tag{4.3.16}$$
$$(-1)^k \left(\frac{y^2-1+y^{-2}}{fy^{\pm1}(y-y^{-1})^2} + fy^{\pm1} \right), \pm \frac{y^{-1\pm1}-y^{1\mp1}}{y-y^{-1}} \Bigg) \;\Bigg|\; f, y \in \mathbb{F}\setminus\{0\} \Bigg\}$$

for $l = \pm1$ and $k \in \mathbb{Z}$ even or odd.

Fix $y \in \mathbb{F}\setminus\{0\}$ to be constant. This defines curves $C_y^{\pm}(k)$ inside C_Φ since I_{ah} is not constant. The restrictions to $K\tilde{\times}I$ and M are points,

$$r_K\left(C_y^{\pm}(k)\right) = (0, y+y^{-1}, 0) \in \mathrm{X}^{\mathrm{irr}}(K\tilde{\times}I, \mathbb{F}_p),$$
$$r_M\left(C_y^{\pm}(k)\right) = \left\{ \left((-1)^k(y+y^{-1}), (-1)^k(y+y^{-1}), 1 \right) \mid k \in \mathbb{Z} \right\} \subset \mathrm{X}^{\mathrm{irr}}(M, \mathbb{F}_p),$$

so it is of is of type (C3). There are infinitely many such curves for $y \in \mathbb{F}\setminus\{0\}$ that are distinct for each $y+y^{-1}$. We deduce the curve detects $\mathscr{T}$ whenever $\Phi = \begin{pmatrix} k & \pm1 \\ \mp1-6k & \mp6 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

We find no ways to extend the representation from Equation (4.3.3) to get a curve of type (C3) when $p \neq 2$.

We find one way to extend the representation from Equation (4.3.6) to N_Φ with a curve of type (C3). For gluing matrix $\Phi = \begin{pmatrix} k & \pm1 \\ \mp1-6k & \mp6 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ we get the curves of representations

$$A = \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}, \quad B = \begin{pmatrix} s & 1 \\ 0 & s \end{pmatrix}, \quad x \in \mathbb{F}\setminus\{0\},\ x^2 = -1$$
$$G = (-1)^k \begin{pmatrix} s & \pm1 \\ 0 & s \end{pmatrix}, \quad H = \begin{pmatrix} e & \mp(-1)^k(-1+2s(-1)^k e - e^2) \\ \mp(-1)^k & 2s(-1)^k - e \end{pmatrix}, \ s \in \{-1,1\}, e \in \mathbb{F}.$$

The condition on Φ arises again from the gluing equation, which requires $n = -6l$ when $p \neq 2$.

The corresponding traces give curves

$$C_{p\neq2} = \Big\{ \Big(0, 2s, 2(-1)^k s, 2(-1)^k s, 0, 0, \mp(-1)^k + 2(e-(-1)^k s)x, 2(-1)^k, (-1)^k(2\mp1), 1, 0,$$
$$2(-(-1)^k + es)x \mp (-1)^k(s+x), -(-1)^k(\pm(-1)^k s + 3(-1)^k x - 2esx), s(1\mp1) \Big) \;\Big|\; e \in \mathbb{F} \Big\} \tag{4.3.17}$$

for $k \in \mathbb{Z}$ even or odd, $l = \pm1$, $s = \pm1$, and $x \in \mathbb{F}\setminus\{0\}$, $x^2 = -1$.

These clearly define curves since I_{ah} is not constant. The restrictions to $K\tilde{\times}I$ and M are points,

$$r_K\left(C_{p\neq2}\right) = \{(0, 2s, 0) \mid s = \pm1\} \subset X^{\mathrm{irr}}(K\tilde{\times}I, \mathbb{F}_p) \cap X^{\mathrm{red}}(K\tilde{\times}I, \mathbb{F}_p),$$
$$r_M\left(C_{p\neq2}\right) = \left\{ (2(-1)^k s, 2(-1)^k s, 1) \mid k \in \mathbb{Z},\ s = \pm1 \right\} \in X^{\mathrm{irr}}(M, \mathbb{F}_p),$$

so it is of type (C3). We deduce the curves $C_{p\neq 2}$ detect $\mathscr{T}$ whenever $\Phi = \begin{pmatrix} k & \pm 1 \\ \mp 1 - 6k & \mp 6 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

No other curves of type (C3) are found for any gluing matrix and other representations when $p \neq 2$. This proves the result. □

The proof of the result over characteristic 2 is similar.

Proof of Lemma 4.10. Assume $p = 2$. We follow the approach described in Section 4.3.5. The splitting torus $\mathscr{T}$ can only be detected by a curve of type (C3) where the restrictions to the character varieties of $K\tilde{\times}I$ and M are points. We will see that there are two ways to get such a curve in characteristic 2, one arises when $\Phi = \begin{pmatrix} k & \pm 1 \\ \mp 1 - 6k & \mp 6 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, which comes from a 2–dimensional component that will give infinitely many curves, and the other arises when $\Phi = \begin{pmatrix} k & 1 \\ 1 & 0 \end{pmatrix} \pmod 2 \in \mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$.

Consider the options for matrix pairs (A, B) from Equations (4.3.2), (4.3.3) and (4.3.5) in turn.

We find one way to extend the representation from Equation (4.3.2) to N_Φ with a curve of type (C3) using the 2–dimensional component C_Φ given in Equation (4.3.16) found in the proof of Lemma 4.9. The same result follows and we get there are infinitely many such curves that detect $\mathscr{T}$ whenever $\Phi = \begin{pmatrix} k & \pm 1 \\ \mp 1 - 6k & \mp 6 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

We find no ways to extend the representation from Equation (4.3.3) to get a curve of type (C3) when $p = 2$.

We find one way to extend the representation from Equation (4.3.5) to N_Φ with a curve of type (C3). For gluing matrix is $\Phi = \begin{pmatrix} k & l \\ m & 2q \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, we get the curve of representations

$$A = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$
$$G = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad u \in \mathbb{F}.$$

Note the determinant condition in $\mathbb{Z}$ forces $l = 1 \pmod 2$ and $m = 1 \pmod 2$. The condition on Φ arises from the gluing equation. We have

$$HB^{2l}H - B^{n+4l} = \begin{pmatrix} e(e+h) & f(e+h)+n \\ g(e+h) & h(e+h) \end{pmatrix}$$

evaluating to the zero matrix, which forces $n = 0 \pmod 2$.

The representation is conjugate to the form,

$$\begin{aligned} A &= \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \\ G &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad H = \begin{pmatrix} e & 1+e^2 \\ 1 & e \end{pmatrix}, \quad u \in \mathbb{F}, \end{aligned} \tag{4.3.18}$$

which is useful when comparing to the $p \neq 2$ case.

The corresponding traces give the curve

$$C_{p=2} = \{(0,0,0,0,0,0,u,0,1,1,0,u+1,u+1,0) \mid u \in \mathbb{F}\}. \tag{4.3.19}$$

This clearly defines a curve since I_{ah} is not constant. The restrictions to $K\tilde{\times}I$ and M are points,

$$\begin{aligned} r_K(C_{p=2}) &= (0,0,0) \in X^{\text{irr}}(K\tilde{\times}I,\mathbb{F}_2) \cap X^{\text{red}}(K\tilde{\times}I,\mathbb{F}_2), \\ r_M(C_{p=2}) &= (0,0,1) \in X^{\text{irr}}(M,\mathbb{F}_2), \end{aligned}$$

so it is of is of type (C3). We deduce the curve detects $\mathscr{T}$ whenever $\Phi = \begin{pmatrix} k & 1 \\ 1 & 0 \end{pmatrix} \pmod 2 \in \mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$ for $k \in \{0,1\}$.

The representations and associated curves correspond to $C_{p\neq 2}$ in Equation (4.3.17) found when $p \neq 2$ in the proof of Lemma 4.9, just for more restrictive gluing matrices. Reducing the above representations modulo 2 is the same as the representation in Equation (4.3.18) with $u = 0$ and reducing the curve $C_{p\neq 2}$ modulo 2 is the same as the curve $C_{p=2}$ with $u = 1$. We find that if $p \neq 2$ the remaining degree of freedom, e, cannot be removed by conjugation and is in fact necessary to find a curve in the variety of characters.

No other curves of type (C3) are found for any gluing matrix and other representations when $p = 2$. This proves the result. □

4.3.7 Curves in the graph manifolds

The manifold N_Φ forms a graph manifold if and only if

$$\Phi = \begin{pmatrix} k & l \\ m & n \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \quad m \neq \pm 6 \neq n.$$

The classification of essential surfaces in N_Φ in Section 4.3.4 shows only the cases (S1), (S2), and (S3) can appear. The case (S1) is already covered in Lemmas 4.9 and 4.10. The other cases (S2) and (S3) and related curves in the character variety of the graph manifold are covered in the next result.

Lemma 4.11. *Let $\mathbb{F}$ be an algebraically closed field of characteristic p. Consider N_Φ a graph manifold with gluing matrix $\Phi = \begin{pmatrix} k & l \\ m & n \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, $m \neq \pm 6 \neq n$. Curves that detect (S2) and (S3) in $\mathrm{X}(N_\Phi,\mathbb{F})$ are characterised by the following.*

- *In case (S2), for* $\Phi = \begin{pmatrix} k & \pm 1 \\ \mp 1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$*, there is a curve that detects* S_2 *if* $p = 2$ *and no curve that detects* S_2 *if* $p \neq 2$*;*
- *In case (S3), for* $\Phi = \begin{pmatrix} \pm 1 & l \\ 0 & \pm 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$*, there is a curve that detects* S_3*.*

In particular, S_2 *is only detected in characteristic* 2 *and* S_3 *is detected whenever it appears.*

Recall from Lemmas 4.9 and 4.10, the splitting torus $\mathscr{T}$ is detected in case (S2) in characteristic $p = 2$ and is never detected otherwise. Thus, Lemmas 4.9 to 4.11 give the statement of Theorem 4.4. Similarly, Lemmas 4.9 to 4.11 give the statement of Theorem 4.5.

Proof. We follow the approach described in Section 4.3.5. The essential surface S_2 can only be detected by either a curve of type (C1) where the restriction to the character variety of $K\tilde{\times}I$ detects $\mathscr{S}$ and the restriction to the variety of characters of M detects $\mathscr{A}$ or a curve of type (C3) that has non-negative valuation on all simple closed curves in components of $N_\Phi \setminus S_2$. Similarly, the essential surface S_3 can only be detected by either a curve of type (C1) where the restriction to the variety of characters of $K\tilde{\times}I$ detects $\mathscr{S}$ and the restriction to the variety of characters of M detects $\mathscr{B}$ or a curve of type (C3) that has non-negative valuation on all simple closed curves in components of $N_\Phi \setminus S_3$.

Consider first the options for matrix pairs (A,B) from Equations (4.3.2), (4.3.3), (4.3.5) and (4.3.6) in turn that could give appropriate curves of type (C1) that detect S_2.

For characteristic 2 we find one way to extend the representation from Equation (4.3.2) to N_Φ with a curve of type (C1). For gluing matrix $\Phi = \begin{pmatrix} k & \pm 1 \\ \mp 1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ we get the curve of representations

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix},$$
$$G = \begin{pmatrix} y^{\pm 1} & 0 \\ 0 & y^{\mp 1} \end{pmatrix}, \quad H = \begin{pmatrix} y^{\pm 1} & 0 \\ 0 & y^{\mp 1} \end{pmatrix}, \quad y \in \mathbb{F} \setminus \{0\}.$$

The corresponding traces give curves

$$C_{S_2} = \big\{ \big(0, y+y^{-1}, y+y^{-1}, y+y^{-1}, 0,0,0, y^{1\pm 1} + y^{-1\mp 1}, \\ y^{1\pm 1} + y^{-1\mp 1}, y^2+y^{-2}, 0,0,0, y^{1\pm 2} + y^{-1\mp 2}\big) \mid y \in \mathbb{F} \setminus \{0\} \big\}$$

for $l = \pm 1$. The restrictions to $K\tilde{\times}I$ and M are curves,

$$r_K(C_{S_2}) = \mathrm{X}^{\mathrm{irr}}(K\tilde{\times}I, \mathbb{F}_2) = \{(0,t,0) \mid t = y+y^{-1},\ y \in \mathbb{F}_2\},$$
$$r_M(C_{S_2}) = \mathrm{X}^{\mathrm{red}}(M, \mathbb{F}_2) = \{(s,s,s^2) \mid s = y+y^{-1},\ y \in \mathbb{F}_2\},$$

so C_{S_2} is of type (C1).

In Section 4.3.1, $r_K(C_{S_2})$ is shown to detect $\mathscr{A}$; in Section 4.3.2, $r_M(C_{S_2})$ is shown to detect $\mathscr{S}$. We deduce both curves detect S_2 whenever $\Phi = \begin{pmatrix} k & \pm 1 \\ \mp 1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

No other curves of type (C1) that could give appropriate curves are found for any gluing matrix and other representations.

Now consider the options for matrix pairs (A,B) from Equations (4.3.2), (4.3.3), (4.3.5) and (4.3.6) that could give appropriate curves of type (C3). We show there are simple closed curves in components of $N_\Phi \setminus S_2$ that give negative valuation for each curve, proving the surface cannot be detected.

We have already classified these options in the proof of Lemmas 4.9 and 4.10 and only $C_{p=2}$ could appear for the case (S2). Consider $[gh^{-1},a] \in \mathrm{im}(\pi_1(N_\Phi \setminus S_2) \to \pi_1(N_\Phi))$.

Using $C_{p=2}$ from Equation (4.3.19) with ξ the ideal point $u \to \infty$ gives

$$I_{[gh^{-1},a]} = u^2 \text{ and } v_\xi(I_{[gh^{-1},a]}) = -2 < 0,$$

which proves S_2 is not detected by $C_{p=2}$. This proves the result for S_2.

Consider now the options for matrix pairs (A,B) from Equations (4.3.2), (4.3.3), (4.3.5) and (4.3.6) in turn that could give appropriate curves of type (C1) that detect S_3.

We find no ways to extend the representation from Equation (4.3.2) to get a curve of type (C1) that detect S_3 in any characteristic.

For general characteristic p we find one way to extend the representation from Equation (4.3.3) to N_Φ with a curve of type (C1). For gluing matrix $\Phi = \begin{pmatrix} \pm 1 & l \\ 0 & \mp 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ we get the curve of representations

$$A = \begin{pmatrix} x & 1 \\ 0 & x^{-1} \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$G = \begin{pmatrix} x^{\pm 2} & \pm(x+x^{-1}) \\ 0 & x^{\mp 2} \end{pmatrix}, \quad H = \begin{pmatrix} x^{\pm 2} & \pm(x+x^{-1}) \\ 0 & x^{\mp 2} \end{pmatrix}, \quad x \in \mathbb{F} \setminus \{0,1,-1\}.$$

The corresponding traces give curves

$$C_{S_3} = \big\{ \big(x+x^{-1}, 2, x^2+x^{-2}, x^2+x^{-2}, x+x^{-1}, x^{1\pm 2}+x^{-1\mp 2}, x^{1\pm 2}+x^{-1\mp 2}, x^2+x^{-2}, x^2+x^{-2},$$
$$x^4+x^{-4}, x^{1\pm 2}+x^{-1\mp 2}, x^{1\pm 2}+x^{-1\mp 2}, x^{1\pm 4}+x^{-1\mp 4}, x^4+x^{-4}\big) \mid x \in \mathbb{F} \setminus \{0,1,-1\} \big\}$$

for $k = \pm 1$. The restrictions to $K\tilde{\times}I$ and M are curves,

$$r_K(C_{S_3}) = \mathrm{X}^{\mathrm{red}}(K\tilde{\times}I, \mathbb{F}_p) = \{(s,2,s) \mid s = x+x^{-1}, x \in \mathbb{F}_p\},$$
$$r_M(C_{S_3}) = \mathrm{X}^{\mathrm{red}}(M, \mathbb{F}_p) = \{\big(s,s,s^2-2\big) \mid s = x^2+x^{-2}, x \in \mathbb{F}_p\},$$

so C_{S_3} is of type (C1).

In Section 4.3.1, $r_K(C_{S_3})$ is shown to detect $\mathscr{B}$; in Section 4.3.2, $r_M(C_{S_3})$ is shown to detect $\mathscr{S}$. We deduce both curves detect S_3 whenever $\Phi = \begin{pmatrix} \pm 1 & l \\ 0 & \pm 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

No other curves of type (C1) that could give appropriate curves are found for any gluing matrix and other representations.

Now consider the options for matrix pairs (A,B) from Equations (4.3.2), (4.3.3), (4.3.5) and (4.3.6) that could give appropriate curves of type (C3). We have already classified these options in the proof of Lemmas 4.9 and 4.10, which do not appear in case (S3). Therefore they could not detect S_3. This proves the result. $\square$

4.3.8 Curves in the Seifert fibered space with base orbifold $S^2(2,2,2,3)$

The manifold N_Φ forms a Seifert fibered space with base orbifold $S^2(2,2,2,3)$ if and only if

$$\Phi = \begin{pmatrix} k & \pm 1 \\ \mp 1 - 6k & \mp 6 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

The classification of essential surfaces in N_Φ in Section 4.3.4 shows only the cases (S1) and (S4) can appear. The case (S1) is already covered in Lemmas 4.9 and 4.10. The remaining case (S4) and related curves in the character variety are covered in the next result.

Lemma 4.12. *Let $\mathbb{F}$ be an algebraically closed field of characteristic p. Consider N_Φ a Seifert fibered space with base orbifold $S^2(2,2,2,3)$ and gluing matrix $\Phi = \begin{pmatrix} k & \pm 1 \\ \mp 1 - 6k & \mp 6 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. There is a 2–dimensional component $C_\Phi \subset \mathrm{X}(N_\Phi, \mathbb{F})$ and for each $q \in \mathbb{Z}$, C_Φ contains a curve that detects $S_4(q,1)$.*

Recall from Lemmas 4.9 and 4.10, the splitting torus $\mathscr{T}$ is always detected in the Seifert fibered space with base orbifold $S^2(2,2,2,3)$. Then Lemmas 4.9, 4.10 and 4.12 prove Theorem 4.7.

Proof. We follow the approach described in Section 4.3.5. We previously saw the existence of the 2–dimensional component C_Φ in Equation (4.3.16) as part of the proof of Lemma 4.9. We had, for gluing matrix $\Phi = \begin{pmatrix} k & \pm 1 \\ \mp 1 - 6k & \mp 6 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, the 2–dimensional components given by

$$\begin{aligned} C_\Phi = \Bigg\{ \Bigg(& 0, y+y^{-1}, (-1)^k(y+y^{-1}), (-1)^k(y+y^{-1}), 0, 0, \frac{y^2-1+y^{-2}}{f(y-y^{-1})^2} + f, \\ & (-1)^k(y^{1\pm 1} + y^{-1\mp 1}), \pm(-1)^k \frac{y^{-1\pm 2} - y^{1\mp 2}}{y-y^{-1}}, 1, 0, \frac{y^2-1+y^{-2}}{fy(y-y^{-1})^2} + fy, \\ & (-1)^k \left(\frac{y^2-1+y^{-2}}{fy^{\pm 1}(y-y^{-1})^2} + fy^{\pm 1} \right), \pm \frac{y^{-1\pm 1} - y^{1\mp 1}}{y-y^{-1}} \Bigg) \;\Bigg|\; f, y \in \mathbb{F} \setminus \{0\} \Bigg\} \end{aligned} \tag{4.3.20}$$

for $k \in \mathbb{Z}$ even or odd, $l = \pm 1$.

We previously considered the case where y is constant and obtained a curve that detects the splitting torus. The restrictions of the component C_Φ to $K\tilde{\times}I$ and M are curves,

$$\begin{aligned} r_K(C_\Phi) &= \mathrm{X}^{\mathrm{irr}}(K\tilde{\times}I,\mathbb{F}_p) = \left\{(0,s,0) \mid s = y+y^{-1},\ y\in\mathbb{F}_p\right\}, \\ r_M(C_\Phi) &= \mathrm{X}^{\mathrm{irr}}(M,\mathbb{F}_p) = \left\{(s,s,1) \mid s = (-1)^k(y+y^{-1}),\ y\in\mathbb{F}_p,\ k\in\mathbb{Z}\right\}. \end{aligned}$$

In Section 4.3.1, $r_K(C_\Phi)$ is shown to detect $\mathscr{A}$; in Section 4.3.2, $r_M(C_\Phi)$ is shown to detect $\mathscr{R}$. We deduce that each curve in C_Φ with the property that I_b is non-constant detects $S_4(q,r)$ for some (q,r).

Now let $f = t^u$, $y = t^v$ for $t\in\mathbb{F}\setminus\{0\}$, $u,v\in\mathbb{Z}$. For each pair (u,v) of co-prime integers, this gives a curve $C(u,v)$ inside C_Φ with different valuations. The curves are given by

$$\begin{aligned} C(u,v) = \Bigg\{ \Bigg(& 0, t^v+t^{-v}, (-1)^k(t^v+t^{-v}), (-1)^k(t^v+t^{-v}), 0, 0, \frac{t^{2v}-1+t^{-2v}}{t^u(t^v-t^{-v})^2} + t^u, \\ & (-1)^k(t^{v(1\pm1)}+t^{-v(1\pm1)}), \pm(-1)^k\frac{t^{v(-1\pm2)}-t^{-v(-1\pm2)}}{t^v-t^{-v}}, 1, 0, \frac{t^{2v}-1+t^{-2v}}{t^{u+v}(t^v-t^{-v})^2} + t^{u+v}, \\ & (-1)^k\left(\frac{t^{2v}-1+t^{-2v}}{t^{u\pm v}(t^v-t^{-v})^2} + t^{u\pm v}\right), \pm\frac{t^{v(-1\pm1)}-t^{-v(-1\pm1)}}{t^v-t^{-v}} \Bigg) \ \Bigg|\ t\in\mathbb{F}\setminus\{0\} \Bigg\}. \end{aligned} \tag{4.3.21}$$

Consider the recurrence relation

$$\alpha_k = \alpha_{k-1}^{-1}\alpha_{k-2}\alpha_{k-1} \text{ with initial values } \alpha_0 = b^{-1}a,\ \alpha_1 = a. \tag{4.3.22}$$

The projection of the surface $S_4(q,1)$ onto the base orbifold $S^2(2,2,2,3)$ is the curve $ghg\alpha_q$ (see Figure 4.5 for examples that give an idea of the pattern). For $S_4(q,1)$ to be detected at the ideal point ξ, $ghg\alpha_q$ must have non-negative valuation at ξ. Consider the curve $C(u,v)$ with ideal point ξ, $t+t^{-1}\to\infty$. We have

$$\begin{aligned} I_{ghg\alpha_q} &= t^u t^{v(-q+1)} + \frac{t^{v(q-1)}}{t^u}\left(\frac{t^{-2v}-1+t^{2v}}{t^{2v}-2+t^{-2v}}\right) \\ \text{and } v_\xi(I_{ghg\alpha_q}) &\geq 0 \text{ if } u = v(q-1). \end{aligned}$$

For example, $v=1$, $u=q-1$ suffices. Therefore the curve $C(q-1,1)$ in C_Φ defined by taking $f = t^{q-1}$, $y=t$, $t\in\mathbb{F}$, detects $S_4(q,1)$. □

We conjecture that we can extend this result to detect all of the surfaces $S_4(q,r)$ using this 2–dimensional component and have made some additional calculations in this direction.

Conjecture 4.13. *Let $\mathbb{F}$ be an algebraically closed field of characteristic p. Consider N_Φ a Seifert fibered space with base orbifold $S^2(2,2,2,3)$ and gluing matrix $\Phi = \begin{pmatrix} k & \pm1 \\ \mp1-6k & \mp6 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. There is a curve in $\mathrm{X}(N_\Phi,\mathbb{F})$ that detects $S_4(q,r)$. Specifically, $S_4(q,r)$ is detected by $C(q-r,r)$ given in Equation (4.3.21).*

Remark 4.14. *We have calculated several (coprime) families (q,r) where we can prove Conjecture 4.13, including: $(2a+1,2)$, $(3a+1,3)$, $(3a+2,3)$. This is done using similar techniques as in the proof of Lemma 4.12.*

4.3.9 Curves in the Seifert fibered space with base orbifold $\mathbb{RP}^2(2,3)$

The manifold N_Φ forms a Seifert fibered space with base orbifold $\mathbb{RP}^2(2,3)$ if and only if

$$\Phi = \begin{pmatrix} \pm 1 & l \\ \mp 6 & \pm 1 - 6l \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

The classification of essential surfaces in N_Φ in Section 4.3.4 shows only the cases (S1) and (S5) can appear. The case (S1) is already covered in Lemmas 4.9 and 4.10. The remaining case (S5) and related curves in the character variety are covered in the next result.

Lemma 4.15. *Let $\mathbb{F}$ be an algebraically closed field of characteristic p. Consider N_Φ a Seifert fibered space with base orbifold $\mathbb{RP}^2(2,3)$ and gluing matrix $\Phi = \begin{pmatrix} \pm 1 & l \\ \mp 6 & \pm 1 - 6l \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. There is a curve in $\mathrm{X}(N_\Phi, \mathbb{F})$ that detects S_5 if $p = 2$ and no curve that detects S_5 if $p \neq 2$.*

Recall from Lemmas 4.9 and 4.10, the splitting torus $\mathscr{T}$ is never detected in the Seifert fibered space with base orbifold $\mathbb{RP}^2(2,3)$. Thus, Lemmas 4.9, 4.10 and 4.15 give the statement of Theorem 4.6.

Proof. We follow the approach described in Section 4.3.5. The essential surface S_5 can only be detected by either a curve of type (C1) where the restriction to the variety of characters of $K \tilde{\times} I$ detects $\mathscr{R}$ and the restriction to the variety of characters of M detects $\mathscr{B}$ or a curve of type (C3) that has non-negative valuation on all simple closed curves in components of $N_\Phi \setminus S_5$.

Consider first the options for matrix pairs (A,B) from Equations (4.3.2), (4.3.3), (4.3.5) and (4.3.6) in turn that could give appropriate curves of type (C1).

We find no ways to extend the representation from Equation (4.3.2) to get a curve of type (C1) that detect S_5 in any characteristic.

For characteristic 2 we find one way to extend the representation from Equation (4.3.3) to N_Φ with a curve of type (C1). For gluing matrix $\Phi = \begin{pmatrix} \pm 1 & l \\ \mp 6 & \pm 1 - 6l \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ we get the curve of representations

$$A = \begin{pmatrix} x & 1 \\ 0 & x^{-1} \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad x \in \mathbb{F} \setminus \{0,1\}$$
$$G = \begin{pmatrix} x^{\pm 2} & x + x^{-1} \\ 0 & x^{\mp 2} \end{pmatrix}, \quad H = \begin{pmatrix} x^{\mp 2} & 0 \\ \frac{1}{x+x^{-1}} & x^{\pm 2} \end{pmatrix}, \quad x \in \mathbb{F} \setminus \{0,1\}.$$

The corresponding traces give curves

$$C_{S_5} = \Bigg\{ \Bigg(x + x^{-1}, 0, x^2 + x^{-2}, x^2 + x^{-2}, x + x^{-1}, x^{1\pm 2} + x^{-1\mp 2}, x^{1\mp 2} + x^{-1\pm 2} + \frac{1}{x+x^{-1}}, x^2 + x^{-2},$$
$$x^2 + x^{-2}, 1, x^{1\pm 2} + x^{-1\mp 2}, x^{1\mp 2} + x^{-1\pm 2} + \frac{1}{x+x^{-1}}, \frac{x^{\mp 2}}{x+x^{-1}} + x^{-1}, 1 \Bigg) \,\Bigg|\, x \in \mathbb{F} \setminus \{0,1\} \Bigg\}$$

for $k = \pm 1$. The restrictions to $K\tilde{\times}I$ and M are curves,

$$r_K(C_{S_5}) = \mathrm{X}^{\mathrm{red}}(K\tilde{\times}I, \mathbb{F}_2) = \{(s, 0, s) \mid s = x + x^{-1}, x \in \mathbb{F}_2\},$$
$$r_M(C_{S_5}) = \mathrm{X}^{\mathrm{irr}}(M, \mathbb{F}_2) = \{(s, s, 1) \mid s = x^2 + x^{-2}, x \in \mathbb{F}_2\},$$

so C_{S_5} is of type (C1).

In Section 4.3.1, $r_K(C_{S_5})$ is shown to detect $\mathscr{B}$; in Section 4.3.2, $r_M(C_{S_5})$ is shown to detect $\mathscr{R}$. We deduce both curves detect S_5 whenever $\Phi = \begin{pmatrix} \pm 1 & l \\ \mp 6 & \pm 1 - 6l \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

No other curves of type (C1) that could give appropriate curves are found for any gluing matrix and other representations.

Now consider the options for matrix pairs (A, B) from Equations (4.3.2), (4.3.3), (4.3.5) and (4.3.6) that could give appropriate curves of type (C3). We have already classified these options in the proof of Lemmas 4.9 and 4.10, which do not appear in case (S5). Therefore they could not detect S_5. This proves the result. $\square$

Appendix A

Family of earthquakes $\gamma_n = \alpha\beta\alpha^{n-1}$

In this appendix we expand on calculations for Example 2.28 from Chapter 2. We calculate the formula for the earthquakes of a family of curves $\gamma_n = \alpha\beta\alpha^{n-1}$, $n > 0$, and evaluate the limit of the earthquakes as $n \to \infty$. All calculations use the triangle length coordinates. We will see that the calculations get quite involved for even a simple example like this, where $\gamma_n \to \alpha$ as $n \to \infty$.

Recall we have the mapping class

$$T_\alpha^{-n+1} T_\beta T_\alpha = \begin{cases} \alpha \mapsto \gamma_n \\ \beta \mapsto \delta \end{cases}$$

with change of framing

$$\begin{aligned} \gamma_n &= \alpha\beta\alpha^{n-1}, \quad & \delta &= \alpha^{-1}, \\ \alpha &= \delta^{-1}, & \beta &= \delta\gamma_n\delta^{n-1}. \end{aligned}$$

We use this mapping class and the coordinates to calculate the family of earthquakes about γ_n. We present useful hyperbolic trigonometry identities and change of coordinate maps in Appendix A.1, give the family of earthquakes in Appendix A.2, and find the limiting earthquake in Appendix A.3.

A.1 Set-up

A few hyperbolic trigonometry identities that are useful for future calculations are

$$\cosh^2\theta - \sinh^2\theta = 1, \tag{A.1.1}$$

$$(\cosh\theta \pm \sinh\theta)^n = \cosh(n\theta) \pm \sinh(n\theta), \tag{A.1.2}$$

$$\sinh(p \pm q) = \sinh p \cosh q \pm \cosh p \sinh q, \tag{A.1.3}$$

$$\cosh(p \pm q) = \cosh p \cosh q \pm \sinh p \sinh q. \tag{A.1.4}$$

Also recall $\cosh 0 = 1$, $\sinh 0 = 0$, $\cosh(-\theta) = \cosh\theta$ and $\sinh(-\theta) = -\sinh\theta$. In addition we have the collar equation Equation (2.1.14), which is satisfied by triangle length coordinates (a, b, c) and local triangle length coordinates (a', b', c'). These will also be useful in evaluating the limit of the earthquake as $n \to \infty$.

Define change of coordinate maps $\overline{\phi}_n$ and $\overline{\psi}_n$ as the maps going between the triangle length coordinates (a, b, c) and the local triangle length coordinates (a', b', c') as in Equations (2.6.1) and (2.6.2). We rewrite them below,

$$\begin{aligned} \overline{\phi}_n &: \mathscr{T}_\ell \to \mathscr{T}_\ell \\ \begin{pmatrix} a \\ b \\ c \end{pmatrix} &\mapsto \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} = \begin{pmatrix} \cosh^{-1}(F(n)) \\ a \\ \cosh^{-1}(F(n-1)) \end{pmatrix} \end{aligned} \tag{A.1.5}$$

where

$$F(n) = \cosh b \cosh(na) + (\cosh c - \cosh a \cosh b) \frac{\sinh(na)}{\sinh a}$$

and

$$\begin{aligned} \overline{\psi}_n &: \mathscr{T}_\ell \to \mathscr{T}_\ell \\ \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} &\mapsto \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b' \\ \cosh^{-1}(G(n)) \\ \cosh^{-1}(G(n-1)) \end{pmatrix} \end{aligned} \tag{A.1.6}$$

where

$$G(n) = \cosh a' \cosh\left(nb'\right) + \left(\cosh c' - \cosh a' \cosh b'\right) \frac{\sinh(nb')}{\sinh b'}.$$

Note the first term of $\overline{\phi}_n$, $\cosh^{-1}(F(n))$, is the half length of γ_n.

A.2 Expressions for family of earthquakes

The change of coordinates algorithm gives an expression for the earthquake about γ_n using $\overline{\phi}_n$, $\overline{\psi}_n$, and the equation for E^ℓ_α from Theorem 2.18. For a fixed starting point $\mathbf{w} = (a, b, c) \in \mathscr{T}_\ell$, the left earthquake deformation about $\gamma_n = \alpha\beta\alpha^{n-1}$ in triangle lengths is given by the map $E_n(\mathbf{w}) : \mathbb{R} \to \mathscr{T}_\ell$ with

$$E_n(\mathbf{w})(r) = \overline{\psi}_n \left(E^\ell_\alpha(\overline{\phi}_n(\mathbf{w}))(r) \right). \tag{A.2.1}$$

We expand the right hand side of the equation, making use of the hyperbolic trigonometry identities given in Equations (A.1.3) and (A.1.4). Let

$$\begin{aligned} C &= \frac{\cosh c - \cosh a \cosh b}{\sinh a}, \\ x(n) &= \cosh b \cosh(na) + C \sinh(na), \\ y(n) &= \cosh b \sinh(na) + C \cosh(na). \end{aligned} \tag{A.2.2}$$

We get

$$\overline{\phi}_n(\mathbf{w}) = \begin{pmatrix} \cosh^{-1}\left(\cosh b \cosh(na) + C\sinh(na)\right) \\ a \\ \cosh^{-1}\left(\cosh b \cosh((n-1)a) + C\sinh((n-1)a)\right) \end{pmatrix},$$

$$E^{\ell}_{\alpha}\left(\overline{\phi}_n(\mathbf{w})\right)(r) = \begin{pmatrix} \cosh^{-1}(x(n)) \\ \cosh^{-1}\Big[\cosh a \cosh\left(r\cosh^{-1}(x(n))\right) \\ \qquad + \frac{y(n)\sinh a}{\sqrt{x(n)^2-1}} \sinh\left(r\cosh^{-1}(x(n))\right)\Big] \\ \cosh^{-1}\Big[\cosh\left(r\cosh^{-1}(x(n))\right)x(n-1) \\ \qquad + \sinh\left(r\cosh^{-1}(x(n))\right)\frac{x(n)y(n)\sinh a - \cosh a\left(x(n)^2-1\right)}{\sqrt{x(n)^2-1}}\Big] \end{pmatrix}.$$

The expression $\sqrt{x(n)^2-1}$ comes from the term $\sinh a$ from E^{ℓ}_{α} from Theorem 2.18 with

$$\sinh\left(\cosh^{-1}(z)\right) = \sqrt{x^2-1}.$$

Then for a fixed starting point $\mathbf{w} = (a,b,c) \in \mathbb{R}^3_+$, the left earthquake deformation about $\gamma_n = \alpha\beta\alpha^{n-1}$ is

$$\begin{aligned} E_n(\mathbf{w})(r) &= \overline{\psi}_n\left(E^{\ell}_{\alpha}(\overline{\phi}_n(\mathbf{w}))(r)\right), \\ &= \begin{pmatrix} \cosh^{-1}(z(n,r)) \\ \cosh^{-1}(H(n,n,r)) \\ \cosh^{-1}(H(n-1,n,r)) \end{pmatrix}, \end{aligned} \tag{A.2.3}$$

where

$$\begin{aligned} z(n,r) &= \cosh a \cosh\left(r\cosh^{-1}(x(n))\right) + \frac{y(n)\sinh a}{\sqrt{x(n)^2-1}} \sinh\left(r\cosh^{-1}(x(n))\right), \\ H(m,n,r) &= x(n)\cosh\left(m\cosh^{-1}(z(n,r))\right) \\ &\quad - \left[y(n)\sinh a \cosh\left(r\cosh^{-1}(x(n))\right)\right. \\ &\quad \left. + \sqrt{x(n)^2-1}\cosh a \sinh\left(r\cosh^{-1}(x(n))\right)\right] \frac{\sinh\left(m\cosh^{-1}(z(n,r))\right)}{\sqrt{z(n,r)^2-1}}. \end{aligned} \tag{A.2.4}$$

Note n and $m = n, n-1$ are parameters for the sequence of curves (n associated with ϕ_n and m associated with ψ_n) and r is the earthquake parameter.

We expect that

- If $n = 0$, $E_0(\mathbf{w})(r)$ is the expression for the earthquake about β;
- If $n = 1$, $E_1(\mathbf{w})(r)$ is the expression for the earthquake about $\alpha\beta$;
- If $r = 0$, $E_n(\mathbf{w})(0) = \mathbf{w}$.

These can be verified with the above expression for $E_n(\mathbf{w})(s)$.

A.3 Limiting earthquake

We know the set $\mathscr{S} \times \mathbb{R}_+$ of measured geodesic laminations foliated by a single simple closed curve is dense in $\mathscr{ML}$, previously recorded as Theorem 2.7, and that earthquakes are well-defined under taking limits by Kerckhoff [Ker83]. Then, given a sequence of curves γ_i converging to γ and a sequence of respective measures μ_i converging to μ then the sequence of earthquakes about (γ_i, μ_i) converge to the earthquake about (γ, μ).

We evaluate the limits of $E_n(\mathbf{w})(r)$ as $n \to \infty$ in order to understand how the sequence of earthquakes converges. We split the details of the limit across two sections, with the results presented using broader strokes first, and details given afterwards for those interested. Note $\gamma_n \to \alpha$ as $n \to \infty$ and so we expect if we make appropriate adjustments to the measure and/or the parametrisation of the earthquake that $E_n(\mathbf{w})(r) \to E^\ell_\alpha(\mathbf{w})(r)$ as $n \to \infty$. In order to evaluate terms in the limit we approximate them around $n \to \infty$ by using Taylor series expansions of terms about $n = \infty$.

A.3.1 Overview of results

First note the following limits, which are proved in Appendix A.3.2.

1. $x(n), y(n) \to \infty$ as $n \to \infty$;
2. $\frac{y(n)}{x(n)} \to 1$ and $\frac{y(n)}{\sqrt{x(n)^2-1}} \to 1$ as $n \to \infty$;
3. $y(n) - x(n) \to 0$ and $y(n) - \sqrt{x(n)^2-1} \to 0$ as $n \to \infty$;
4. $\frac{\cosh^{-1}(x(n))}{n} \to a$ as $n \to \infty$;
5. $\frac{\cosh^{-1}(x(n))}{n^2} \to 0$ as $n \to \infty$.

Reparametrise using $r \mapsto \frac{r}{n^2}$. Then,

$$E_n(\mathbf{w})\left(\frac{r}{n^2}\right) = \begin{pmatrix} \cosh^{-1}\left(z\left(n, \frac{r}{n^2}\right)\right) \\ \cosh^{-1}\left(H\left(n, n, \frac{r}{n^2}\right)\right) \\ \cosh^{-1}\left(H\left(n-1, n, \frac{r}{n^2}\right)\right) \end{pmatrix}. \tag{A.3.1}$$

We have

$$\begin{aligned} z\left(n, \frac{r}{n^2}\right) &= \cosh a \cosh\left(r\frac{\cosh^{-1}(x(n))}{n^2}\right) + \frac{y(n)\sinh a}{\sqrt{x(n)^2-1}} \sinh\left(r\frac{\cosh^{-1}(x(n))}{n^2}\right), \\ \lim_{n\to\infty} z\left(n, \frac{r}{n^2}\right) &= \cosh a. \end{aligned}$$

In order to evaluate the limit for $H\left(n, n, \frac{r}{n^2}\right)$ and $H\left(n-1, n, \frac{r}{n^2}\right)$ we first relabel and rearrange the components for $H(m, n, r)$. The idea is that we can split $H\left(n, n, \frac{r}{n^2}\right)$ and $H\left(n-1, n, \frac{r}{n^2}\right)$ into components that will converge to 0 as $n \to \infty$ plus some component that can be rewritten in a nice fashion that also converges.

Relabelling the expressions in $H(m,n,r)$ we write

$$A(m,n,r) = x(n)\cosh\left(m\cosh^{-1}(z(n,r))\right) - y(n)\sinh\left(m\cosh^{-1}(z(n,r))\right), \tag{A.3.2}$$

$$B_1(m,n,r) = y(n)\sinh\left(m\cosh^{-1}(z(n,r))\right)\left(\frac{\sinh\left(r\cosh^{-1}(x(n))+a\right)}{\sqrt{z(n,r)^2-1}} - 1\right), \tag{A.3.3}$$

$$B_2(m,n,r) = \left(\sqrt{x(n)^2-1} - y(n)\right)\sinh\left(r\cosh^{-1}(x(n))\right)\cosh a\frac{\sinh\left(m\cosh^{-1}(z(n,r))\right)}{\sqrt{z(n,r)^2-1}}, \tag{A.3.4}$$

$$H(m,n,r) = A(m,n,r) - B_1(m,n,r) - B_2(m,n,r). \tag{A.3.5}$$

This can be verified by expanding brackets and using Equation (A.1.3).

Expand $x(n)$ and $y(n)$ in $A(m,n,r)$ using Equation (A.2.2) and rearrange,

$$\begin{aligned}
A(m,n,r) &= x(n)\cosh\left(m\cosh^{-1}(z(n,r))\right) - y(n)\sinh\left(m\cosh^{-1}(z(n,r))\right),\\
&= (\cosh b\cosh(na) + C\sinh(na))\cosh\left(m\cosh^{-1}(z(n,r))\right)\\
&\quad - (\cosh b\sinh(na) + C\cosh(na))\sinh\left(m\cosh^{-1}(z(n,r))\right),\\
&= \cosh b\left(\cosh(na)\cosh\left(m\cosh^{-1}(z(n,r))\right) - \sinh(na)\sinh\left(m\cosh^{-1}(z(n,r))\right)\right)\\
&\quad + C\left(\sinh(na)\cosh\left(m\cosh^{-1}(z(n,r))\right) - \cosh(na)\sinh\left(m\cosh^{-1}(z(n,r))\right)\right),\\
&= \cosh b\cosh\left(na - m\cosh^{-1}(z(n,r))\right) + C\sinh\left(na - m\cosh^{-1}(z(n,r))\right),
\end{aligned}$$

where the last step used the hyperbolic trigonometry identities given in Equations (A.1.3) and (A.1.4).

Then consider the terms of $H\left(n,n,\frac{r}{n^2}\right)$ and $H\left(n-1,n,\frac{r}{n^2}\right)$. We have

$$\begin{aligned}
H\left(n,n,\frac{r}{n^2}\right) &= A\left(n,n,\frac{r}{n^2}\right) - B_1\left(n,n,\frac{r}{n^2}\right) - B_2\left(n,n,\frac{r}{n^2}\right),\\
H\left(n-1,n,\frac{r}{n^2}\right) &= A\left(n-1,n,\frac{r}{n^2}\right) - B_1\left(n-1,n,\frac{r}{n^2}\right) - B_2\left(n-1,n,\frac{r}{n^2}\right),
\end{aligned} \tag{A.3.6}$$

with

$$\begin{aligned}
A\left(n,n,\frac{r}{n^2}\right) &= \cosh b\cosh\left(n\left(a-\cosh^{-1}\left(z\left(n,\frac{r}{n^2}\right)\right)\right)\right) + C\sinh\left(n\left(a-\cosh^{-1}\left(z\left(n,\frac{r}{n^2}\right)\right)\right)\right),\\
A\left(n-1,n,\frac{r}{n^2}\right) &= \cosh b\cosh\left(n\left(a-\cosh^{-1}\left(z\left(n,\frac{r}{n^2}\right)\right)\right) + \cosh^{-1}\left(z\left(n,\frac{r}{n^2}\right)\right)\right)\\
&\quad + C\sinh\left(n\left(a-\cosh^{-1}\left(z\left(n,\frac{r}{n^2}\right)\right)\right) + \cosh^{-1}\left(z\left(n,\frac{r}{n^2}\right)\right)\right),\\
B_1\left(n,n,\frac{r}{n^2}\right) &= y(n)\sinh\left(n\cosh^{-1}\left(z\left(n,\frac{r}{n^2}\right)\right)\right)\left(\frac{\sinh\left(\frac{r}{n^2}\cosh^{-1}(x(n))+a\right)}{\sqrt{z(n,\frac{r}{n^2})^2-1}} - 1\right),\\
B_1\left(n-1,n,\frac{r}{n^2}\right) &= y(n)\sinh\left((n-1)\cosh^{-1}\left(z\left(n,\frac{r}{n^2}\right)\right)\right)\left(\frac{\sinh\left(\frac{r}{n^2}\cosh^{-1}(x(n))+a\right)}{\sqrt{z(n,\frac{r}{n^2})^2-1}} - 1\right),\\
B_2\left(n,n,\frac{r}{n^2}\right) &= \left(\sqrt{x(n)^2-1} - y(n)\right)\sinh\left(\frac{r}{n^2}\cosh^{-1}(x(n))\right)\cosh a\frac{\sinh\left(n\cosh^{-1}\left(z(n,\frac{r}{n^2})\right)\right)}{\sqrt{z(n,\frac{r}{n^2})^2-1}},\\
B_2\left(n-1,n,\frac{r}{n^2}\right) &= \left(\sqrt{x(n)^2-1} - y(n)\right)\sinh\left(\frac{r}{n^2}\cosh^{-1}(x(n))\right)\cosh a\frac{\sinh\left((n-1)\cosh^{-1}\left(z(n,\frac{r}{n^2})\right)\right)}{\sqrt{z(n,\frac{r}{n^2})^2-1}}.
\end{aligned}$$

Reparametrising $n \to \frac{1}{q}$ we can take the limit as $q \to 0$ instead of $n \to \infty$. Each of A, B_1, B_2 have some expressions that go the indeterminate form $\infty \times 0$ as $q \to 0$. Using Taylor series expansions about $q = 0$ for various terms we can evaluate the limits to find

$$\begin{aligned}
\lim_{q\to 0} A\left(\frac{1}{q}, \frac{1}{q}, q^2 r\right) &= \cosh b \cosh(ra) - C\sinh(ra), \\
&= \cosh(b_\alpha(r)), \\
\lim_{q\to 0} A\left(\frac{1}{q} - 1, \frac{1}{q}, q^2 r\right) &= \cosh b \cosh((r-1)a) - C\sinh((r-1)a), \\
&= \cosh(b_\alpha(r-1)),
\end{aligned} \tag{A.3.7}$$

and

$$\begin{aligned}
\lim_{q\to 0} B_1\left(\frac{1}{q}, \frac{1}{q}, q^2 r\right) &= \lim_{n\to\infty} B_1\left(\frac{1}{q} - 1, \frac{1}{q}, q^2 r\right) = 0, \\
\lim_{q\to 0} B_2\left(\frac{1}{q}, \frac{1}{q}, q^2 r\right) &= \lim_{n\to\infty} B_2\left(\frac{1}{q} - 1, \frac{1}{q}, q^2 r\right) = 0.
\end{aligned} \tag{A.3.8}$$

The details of these limit calculations can be found in Appendix A.3.2.

Therefore,

$$\begin{aligned}
\lim_{q\to 0} H\left(\frac{1}{q}, \frac{1}{q}, q^2 r\right) &= \cosh(b_\alpha(r)) - 0 - 0 = \lim_{n\to\infty} H\left(n, n, \frac{r}{n^2}\right), \\
\lim_{q\to 0} H\left(\frac{1}{q} - 1, \frac{1}{q}, q^2 r\right) &= \cosh(b_\alpha(r-1)) - 0 - 0 = \lim_{n\to\infty} H\left(n-1, n, \frac{r}{n^2}\right),
\end{aligned} \tag{A.3.9}$$

and

$$\begin{aligned}
\lim_{n\to\infty} E_n(\mathbf{w})\left(\frac{r}{n^2}\right) &= \begin{pmatrix} \lim_{n\to\infty} \cosh^{-1}\left(z(n, \frac{r}{n^2})\right) \\ \lim_{n\to\infty} \cosh^{-1}\left(H(n, n, \frac{r}{n^2})\right) \\ \lim_{n\to\infty} \cosh^{-1}\left(H(n-1, n, \frac{r}{n^2})\right) \end{pmatrix}, \\
&= \begin{pmatrix} a \\ b_\alpha(r) \\ b_\alpha(r-1) \end{pmatrix} = E^\ell_\alpha(\mathbf{w})(r).
\end{aligned} \tag{A.3.10}$$

A.3.2 Details for results

We provide additional detail for limits used in Appendix A.3.1. Recall $a, b, c \in \mathbb{R}^+$ and $n > 0$. Note a few useful relations and results used in calculations.

Hyperbolic trigonometric functions can be defined using the exponential function and their inverses using the natural logarithm,

$$\begin{aligned}
\cosh x &= \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2}, \\
\sinh x &= \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2}, \\
\cosh^{-1} x &= \ln(x + \sqrt{x^2 - 1}), \\
\sinh^{-1} x &= \ln(x + \sqrt{x^2 + 1}).
\end{aligned} \tag{A.3.11}$$

A few useful Taylor series expansions about $x = 0$ given using their first two terms are:

- $\sqrt{1+x} = 1 + \frac{x}{2} + \mathscr{O}(x^2)$, converges when $|x| < 1$;
- $(1+x)^{-1} = 1 - x + \mathscr{O}(x^2)$, converges when $|x| < 1$;
- $\ln(1+x) = x - \frac{x^2}{2} + \mathscr{O}(x^3)$, converges when $|x| < 1$;
- $\cosh x = 1 + \frac{x^2}{2} + \mathscr{O}(x^4)$;
- $\sinh x = x + \frac{x^3}{6} + \mathscr{O}(x^5)$.

We use these expansions for terms with variable q (or n), where the x-component of the term goes to 0 as q goes to 0 (or n goes to ∞). Excess terms that go to 0 fast are omitted where possible.

Limits comparing $x(n)$ and $y(n)$

Recall

$$C = \frac{\cosh c - \cosh a \cosh b}{\sinh a},$$

and write

$$\begin{aligned}
C_1 &= \cosh b + C, \\
C_2 &= \cosh b - C.
\end{aligned}$$

Using the definitions of $x(n), y(n)$ and the exponential definitions for $\cosh, \sinh$,

$$\begin{aligned}
x(n) &= \cosh b \cosh(na) + C\sinh(na), \\
&= \frac{e^{na}}{2}(\cosh b + C) + \frac{e^{-na}}{2}(\cosh b - C), \\
&= \frac{e^{na}}{2}C_1 + \frac{e^{-na}}{2}C_2, \\
y(n) &= \cosh b \sinh(na) + C\cosh(na), \\
&= \frac{e^{na}}{2}(\cosh b + C) - \frac{e^{-na}}{2}(\cosh b - C), \\
&= \frac{e^{na}}{2}C_1 - \frac{e^{-na}}{2}C_2.
\end{aligned} \tag{A.3.12}$$

Using the exponential forms of $x(n)$ and $y(n)$ we have

$$\begin{aligned}
\frac{y(n)}{x(n)} &= \frac{\frac{e^{na}}{2}C_1 - \frac{e^{-na}}{2}C_2}{\frac{e^{na}}{2}C_1 + \frac{e^{-na}}{2}C_2}, \\
&= \frac{\frac{e^{na}}{2}C_1\left(1 - e^{-2na}\frac{C_2}{C_1}\right)}{\frac{e^{na}}{2}C_1\left(1 + e^{-2na}\frac{C_2}{C_1}\right)}, \\
&= \frac{1 - e^{-2na}\frac{C_2}{C_1}}{1 + e^{-2na}\frac{C_2}{C_1}}.
\end{aligned}$$

Then

$$\lim_{n\to\infty}\frac{y(n)}{x(n)} = 1. \tag{A.3.13}$$

Similarly,

$$\begin{aligned}
\frac{y(n)}{\sqrt{x(n)^2 - 1}} &= \frac{\frac{e^{na}}{2}C_1 - \frac{e^{-na}}{2}C_2}{\sqrt{\left(\frac{e^{na}}{2}C_1 + \frac{e^{-na}}{2}C_2\right)^2 - 1}}, \\
&= \frac{\frac{e^{na}}{2}C_1 - \frac{e^{-na}}{2}C_2}{\sqrt{\frac{e^{2na}C_1^2}{4} + \frac{C_1C_2}{2} + \frac{e^{-2na}C_2^2}{4} - 1}}, \\
&= \frac{\frac{e^{na}}{2}C_1\left(1 - e^{-2na}\frac{C_2}{C_1}\right)}{\frac{e^{na}}{2}C_1\sqrt{1 + 2e^{-2na}\frac{C_1C_2 - 2}{C_1^2} + e^{-4na}\left(\frac{C_2}{C_1}\right)^2}}, \\
&= \frac{\left(1 - e^{-2na}\frac{C_2}{C_1}\right)}{\sqrt{1 + 2e^{-2na}\frac{C_1C_2 - 2}{C_1^2} + e^{-4na}\left(\frac{C_2}{C_1}\right)^2}},
\end{aligned}$$

and

$$\lim_{n\to\infty}\frac{y(n)}{\sqrt{x(n)^2 - 1}} = 1. \tag{A.3.14}$$

Again using the exponential forms of $x(n)$ and $y(n)$ we have

$$\lim_{n\to\infty}(y(n)-x(n)) = -\lim_{n\to\infty} \mathrm{e}^{-na} C_2 = 0$$

and

$$\lim_{n\to\infty}\left(y(n)-\sqrt{x(n)^2-1}\right) = \lim_{n\to\infty}\left[\frac{\mathrm{e}^{na}C_1}{2}\left(1-\sqrt{1+2\,\mathrm{e}^{-2na}\frac{C_1C_2-2}{C_1^2}+\mathrm{e}^{-4na}\left(\frac{C_2}{C_1}\right)^2}\right)-\frac{\mathrm{e}^{-na}C_2}{2}\right].$$

The first term in the limit is indeterminate of the form $\infty\times 0$. Note the terms $2\,\mathrm{e}^{-2na}\frac{C_1C_2-2}{C_1^2}+\mathrm{e}^{-4na}\left(\frac{C_2}{C_1}\right)^2$ go to 0 as $n\to\infty$ so we can use the Taylor series expansion for $\sqrt{1+x}$ about $x=0$. Expanding the square root using the Taylor series gives

$$\lim_{n\to\infty}\left(y(n)-\sqrt{x(n)^2-1}\right) = \lim_{n\to\infty}\left[-\mathrm{e}^{-na}\frac{C_1C_2-2}{2C_1}-\frac{\mathrm{e}^{-3na}C_2^2}{4C_1}-\frac{\mathrm{e}^{-na}C_2}{2}\right]=0. \tag{A.3.15}$$

Note the length of γ_n is given by $2\cosh^{-1}(x(n))$ and the length of α by $2a$. We have

$$\lim_{n\to\infty}\frac{\cosh^{-1}(x(n))}{n} = \lim_{n\to\infty}\frac{\cosh^{-1}(\cosh b\cosh(na)+C\sinh(na))}{n}.$$

This is indeterminate of the form ∞/∞. We find

$$\frac{d}{dn}\cosh^{-1}(x(n))) = \frac{ay(n)}{\sqrt{x(n)^2-1}}.$$

Appling L'Hôpital's rule gives

$$\begin{aligned}\lim_{n\to\infty}\frac{\cosh^{-1}(x(n))}{n} &= \lim_{n\to\infty}\frac{\frac{d}{dn}\cosh^{-1}(x(n)))}{\frac{d}{dn}n},\\ &= \lim_{n\to\infty}\frac{ay(n)}{\sqrt{x(n)^2-1}},\\ &= a.\end{aligned} \tag{A.3.16}$$

Similarly we can also apply L'Hôpital's rule to find

$$\begin{aligned}\lim_{n\to\infty}\frac{\cosh^{-1}(x(n))}{n^2} &= \lim_{n\to\infty}\frac{\frac{d}{dn}\cosh^{-1}(x(n)))}{\frac{d}{dn}n^2},\\ &= \lim_{n\to\infty}\frac{ay(n)}{2n\sqrt{x(n)^2-1}},\\ &= 0.\end{aligned} \tag{A.3.17}$$

Both results use the limit from Equation (A.3.14).

Limits for H

To calculate the limit for H we break H into multiple functions, A, B_1, and B_2, as in Equations (A.3.2) to (A.3.5) and substitute $q = \frac{1}{n}$. Each function has indeterminate limits of type $\infty \times 0$ appear. In order to evaluate the rate at which terms go to ∞ or 0 we approximate terms using the first couple of terms of the Taylor series expansions around $q = 0$ and previous results we have established.

In order to understand how each of A, B_1, and B_2 approaches the limit as $q \to 0$ using Taylor series we need to understand how the terms various terms related to $x\left(\frac{1}{q}\right)$, $y\left(\frac{1}{q}\right)$, and $z\left(\frac{1}{q}, q^2 r\right)$ approach the limit.

We consider, in order:

1. $x\left(\frac{1}{q}\right)$, $\sqrt{x\left(\frac{1}{q}\right)^2 - 1}$;
2. $\cosh^{-1}\left(x\left(\frac{1}{q}\right)\right) = \ln\left(x\left(\frac{1}{q}\right) + \sqrt{x\left(\frac{1}{q}\right)^2 - 1}\right)$, $q^2 r \cosh^{-1}\left(x\left(\frac{1}{q}\right)\right)$;
3. $\cosh\left(q^2 r \cosh^{-1}\left(x\left(\frac{1}{q}\right)\right)\right)$, $\sinh\left(q^2 r \cosh^{-1}\left(x\left(\frac{1}{q}\right)\right)\right)$;
4. $\frac{y\left(\frac{1}{q}\right)}{\sqrt{x\left(\frac{1}{q}\right)^2 - 1}}$;
5. $z\left(\frac{1}{q}, q^2 r\right)$, $\sqrt{z\left(\frac{1}{q}, q^2 r\right)^2 - 1}$;
6. $\cosh^{-1}\left(z\left(\frac{1}{q}, q^2 r\right)\right) = \ln\left(z\left(\frac{1}{q}, q^2 r\right) + \sqrt{z\left(\frac{1}{q}, q^2 r\right)^2 - 1}\right)$.

We will then consider the expressions for A, B_1, B_2 separately.

We have for Item 1

$$x\left(\frac{1}{q}\right) = \mathrm{e}^{\frac{a}{q}} \frac{C_1}{2} + \mathrm{e}^{-\frac{a}{q}} \frac{C_2}{2},$$
$$\sqrt{x\left(\frac{1}{q}\right)^2 - 1} = \mathrm{e}^{\frac{a}{q}} \frac{C_1}{2} \sqrt{1 + 2\mathrm{e}^{-2\frac{a}{q}} \frac{C_1 C_2 - 2}{C_1^2} + \mathrm{e}^{-4\frac{a}{q}} \left(\frac{C_2}{C_1}\right)^2}.$$

Note the terms $2\mathrm{e}^{-2\frac{a}{q}} \frac{C_1 C_2 - 2}{C_1^2} + \mathrm{e}^{-4\frac{a}{q}} \left(\frac{C_2}{C_1}\right)^2$ go to 0 as $q \to 0$. Like in Equation (A.3.15), use the Taylor series expansion of the square root about $q = 0$. Then

$$\begin{aligned} \sqrt{x\left(\frac{1}{q}\right)^2 - 1} &= \mathrm{e}^{\frac{a}{q}} \frac{C_1}{2} \left(1 + \mathrm{e}^{-2\frac{a}{q}} \frac{C_1 C_2 - 2}{C_1^2} + \frac{\mathrm{e}^{-4\frac{a}{q}}}{2} \left(\frac{C_2}{C_1}\right)^2\right), \\ &\approx \mathrm{e}^{\frac{a}{q}} \frac{C_1}{2} \left(1 + \mathrm{e}^{-2\frac{a}{q}} \frac{C_1 C_2 - 2}{C_1^2}\right). \end{aligned} \tag{A.3.18}$$

For Item 2

$$
\begin{aligned}
\cosh^{-1}\left(x\left(\frac{1}{q}\right)\right) &= \ln\left(x\left(\frac{1}{q}\right)+\sqrt{x\left(\frac{1}{q}\right)^2-1}\right),\\
&= \ln\left(\mathrm{e}^{\frac{a}{q}}\frac{C_1}{2}+\mathrm{e}^{-\frac{a}{q}}\frac{C_2}{2}+\mathrm{e}^{\frac{a}{q}}\frac{C_1}{2}\left(1+\mathrm{e}^{-2\frac{a}{q}}\frac{C_1C_2-2}{C_1^2}\right)\right),\\
&= \ln\left(\mathrm{e}^{\frac{a}{q}}C_1\left(1+\mathrm{e}^{-2\frac{a}{q}}\frac{C_1C_2-1}{C_1^2}\right)\right),\\
&= \frac{a}{q}+\ln C_1+\ln\left(1+\mathrm{e}^{-2\frac{a}{q}}\frac{\operatorname{csch}(a)^2}{C_1^2}\right),
\end{aligned}
$$

using $C_1C_2-1=\operatorname{csch}(a)^2$. The term $\mathrm{e}^{-2\frac{a}{q}}\frac{\operatorname{csch}(a)^2}{C_1^2}$ goes to 0 as $q\to 0$. Use the Taylor series expansion of the logarithm about $q=0$. Then

$$
\begin{aligned}
\cosh^{-1}\left(x\left(\frac{1}{q}\right)\right) &= \frac{a}{q}+\ln C_1+\mathrm{e}^{-2\frac{a}{q}}\frac{\operatorname{csch}(a)^2}{C_1^2},\\
q^2r\cosh^{-1}\left(x\left(\frac{1}{q}\right)\right) &= qra+q^2r\left(\ln C_1+\mathrm{e}^{-2\frac{a}{q}}\frac{\operatorname{csch}(a)^2}{C_1^2}\right),\\
&\approx qra.
\end{aligned}
\tag{A.3.19}
$$

Then for Item 3 we have

$$
\begin{aligned}
\cosh\left(q^2r\cosh^{-1}\left(x\left(\frac{1}{q}\right)\right)\right) &\approx \cosh(qra),\\
\sinh\left(q^2r\cosh^{-1}\left(x\left(\frac{1}{q}\right)\right)\right) &\approx \sinh(qra).
\end{aligned}
\tag{A.3.20}
$$

For Item 4, using the calculation from Appendix A.3.2 and substituting the results from Equation (A.3.18) we get

$$
\frac{y\left(\frac{1}{q}\right)}{\sqrt{x\left(\frac{1}{q}\right)^2-1}} = \frac{1-\mathrm{e}^{-2\frac{a}{q}}\left(\frac{C_2}{C_1}\right)}{1+\mathrm{e}^{-2\frac{a}{q}}\frac{C_1C_2-2}{C_1^2}}.
\tag{A.3.21}
$$

The term $\mathrm{e}^{-2\frac{a}{q}}\frac{C_1C_2-2}{C_1^2}$ goes to 0 as $q\to 0$. Use the Taylor series expansion of the multiplicative inverse about $q=0$. Then

$$
\begin{aligned}
\frac{y\left(\frac{1}{q}\right)}{\sqrt{x\left(\frac{1}{q}\right)^2-1}} &= \left(1-\mathrm{e}^{-2\frac{a}{q}}\frac{C_2}{C_1}\right)\left(1-\mathrm{e}^{-2\frac{a}{q}}\frac{C_1C_2-2}{C_1^2}\right),\\
&\approx 1-2\,\mathrm{e}^{-2\frac{a}{q}}\frac{\operatorname{csch}(a)^2}{C_1^2},
\end{aligned}
$$

again using $C_1C_2-1=\operatorname{csch}(a)^2$.

Then for Item 5

$$\begin{aligned}
z\left(\frac{1}{q}, q^2 r\right) &= \cosh a \cosh\left(q^2 r \cosh^{-1}\left(x\left(\frac{1}{q}\right)\right)\right) + \frac{y\left(\frac{1}{q}\right)\sinh a}{\sqrt{x\left(\frac{1}{q}\right)^2 - 1}} \sinh\left(q^2 r \cosh^{-1}\left(x\left(\frac{1}{q}\right)\right)\right), \\
&= \cosh a \cosh(qra) + \left(1 - 2\,\mathrm{e}^{-2\frac{a}{q}} \frac{\operatorname{csch}(a)^2}{C_1^2}\right) \sinh a \sinh(qra), \\
&= \cosh(qra + a) - 2\,\mathrm{e}^{-2\frac{a}{q}} \frac{\sinh(qra)}{C_1^2 \sinh a},
\end{aligned}$$

$$\sqrt{z\left(\frac{1}{q}, q^2 r\right)^2 - 1} = \sinh(qra + a)\left[1 - \frac{2\,\mathrm{e}^{-2\frac{a}{q}}\sinh(qra)}{C_1^2 \sinh a \sinh(qra+a)^2}\left(2\cosh(qra+a) - \frac{2\,\mathrm{e}^{-2\frac{a}{q}}\sinh(qra)}{C_1^2 \sinh a}\right)\right]^{\frac{1}{2}}.$$

The terms $\frac{2\mathrm{e}^{-2\frac{a}{q}}\sinh(qra)}{C_1^2\sinh a\sinh(qra+a)^2}\left(2\cosh(qra+a) - \frac{2\mathrm{e}^{-2\frac{a}{q}}\sinh(qra)}{C_1^2\sinh a}\right)$ go to 0 as $q \to 0$. Use the Taylor series expansion of the square root about $q = 0$. Then

$$\begin{aligned}
\sqrt{z\left(\frac{1}{q}, q^2 r\right)^2 - 1} = \sinh(qra + a)\Bigg[1 - \frac{\mathrm{e}^{-2\frac{a}{q}}\sinh(qra)}{C_1^2 \sinh a \sinh(qra+a)^2} \\
\left(2\cosh(qra+a) - \frac{2\,\mathrm{e}^{-2\frac{a}{q}}\sinh(qra)}{C_1^2 \sinh a}\right)\Bigg].
\end{aligned} \tag{A.3.22}$$

Then for Item 6

$$\begin{aligned}
\cosh^{-1}\left(z\left(\frac{1}{q}, q^2 r\right)\right) &= \ln\left(z\left(\frac{1}{q}, q^2 r\right) + \sqrt{z\left(\frac{1}{q}, q^2 r\right)^2 - 1}\right), \\
&= \ln\Bigg(\cosh(qra + a) - 2\,\mathrm{e}^{-2\frac{a}{q}} \frac{\sinh(qra)}{C_1^2 \sinh a} \\
&\quad + \sinh(qra + a)\Bigg[1 - \frac{\mathrm{e}^{-2\frac{a}{q}}\sinh(qra)}{C_1^2 \sinh a \sinh(qra+a)^2} \\
&\quad \left(2\cosh(qra+a) - \frac{2\,\mathrm{e}^{-2\frac{a}{q}}\sinh(qra)}{C_1^2 \sinh a}\right)\Bigg]\Bigg), \\
&\approx \ln(\cosh(qra + a) + \sinh(qra + a)), \\
&= qra + a.
\end{aligned} \tag{A.3.23}$$

Now combine these results to assess the limits for A, B_1, and B_2.

Limits for A

Consider

$$\begin{aligned} A\left(\frac{1}{q},\frac{1}{q},q^2r\right) &= \cosh b \cosh\left(\frac{1}{q}\left(a-\cosh^{-1}\left(z\left(\frac{1}{q},q^2r\right)\right)\right)\right) \\ &\quad + C\sinh\left(\frac{1}{q}\left(a-\cosh^{-1}\left(z\left(\frac{1}{q},q^2r\right)\right)\right)\right), \\ A\left(\frac{1}{q}-1,\frac{1}{q},q^2r\right) &= \cosh b \cosh\left(\frac{1}{q}\left(a-\cosh^{-1}\left(z\left(\frac{1}{q},q^2r\right)\right)\right)+\cosh^{-1}\left(z\left(\frac{1}{q},q^2r\right)\right)\right) \\ &\quad + C\sinh\left(\frac{1}{q}\left(a-\cosh^{-1}\left(z\left(\frac{1}{q},q^2r\right)\right)\right)+\cosh^{-1}\left(z\left(\frac{1}{q},q^2r\right)\right)\right). \end{aligned}$$

Using Equation A.3.23,

$$\begin{aligned} \frac{1}{q}\left(a-\cosh^{-1}\left(z\left(\frac{1}{q},q^2r\right)\right)\right) &= -ra, \\ \frac{1}{q}\left(a-\cosh^{-1}\left(z\left(\frac{1}{q},q^2r\right)\right)\right)+\cosh^{-1}\left(z\left(\frac{1}{q},q^2r\right)\right) &= qra-a(r-1). \end{aligned}$$

Then

$$\begin{aligned} A\left(\frac{1}{q},\frac{1}{q},q^2r\right) &= \cosh b \cosh(ra) - C\sinh(ra), \\ A\left(\frac{1}{q}-1,\frac{1}{q},q^2r\right) &= \cosh b \cosh(qra-a(r-1)) + C\sinh(qra-a(r-1)) \end{aligned} \tag{A.3.24}$$

and

$$\begin{aligned} \lim_{q\to 0} A\left(\frac{1}{q},\frac{1}{q},q^2r\right) &= \cosh b \cosh(ra) - C\sinh(ra), \\ &= b_\alpha(r), \\ \lim_{q\to 0} A\left(\frac{1}{q}-1,\frac{1}{q},q^2r\right) &= \cosh b \cosh(a(r-1)) - C\sinh(a(r-1)), \\ &= b_\alpha(r-1). \end{aligned} \tag{A.3.25}$$

Limits for B_1

Consider

$$\begin{aligned} B_1\left(\frac{1}{q},\frac{1}{q},q^2r\right) &= y\left(\frac{1}{q}\right)\sinh\left(\frac{1}{q}\cosh^{-1}\left(z\left(\frac{1}{q},q^2r\right)\right)\right) \\ &\qquad \left(\frac{\sinh\left(q^2r\cosh^{-1}\left(x\left(\frac{1}{q}\right)\right)+a\right)}{\sqrt{z\left(\frac{1}{q},q^2r\right)^2-1}}-1\right), \\ B_1\left(\frac{1}{q}-1,\frac{1}{q},q^2r\right) &= y\left(\frac{1}{q}\right)\sinh\left(\left(\frac{1}{q}-1\right)\cosh^{-1}\left(z\left(\frac{1}{q},q^2r\right)\right)\right) \\ &\qquad \left(\frac{\sinh\left(q^2r\cosh^{-1}\left(x\left(\frac{1}{q}\right)\right)+a\right)}{\sqrt{z\left(\frac{1}{q},q^2r\right)^2-1}}-1\right). \end{aligned}$$

Consider the term $\frac{\sinh\left(q^2r\cosh^{-1}\left(x\left(\frac{1}{q}\right)\right)+a\right)}{\sqrt{z\left(\frac{1}{q},q^2r\right)^2-1}}-1$. From Equations A.3.19 and A.3.22 we have

$$\begin{aligned} \frac{\sinh\left(q^2r\cosh^{-1}\left(x\left(\frac{1}{q}\right)\right)+a\right)}{\sqrt{z\left(\frac{1}{q},q^2r\right)^2-1}} &= \frac{\sinh\left(qra+a\right)}{\sinh(qra+a)}\left[1-\frac{e^{-2\frac{a}{q}}\sinh(qra)}{C_1^2\sinh a\sinh(qra+a)^2}\right. \\ &\qquad \left.\left(2\cosh\left(qra+a\right)-\frac{2e^{-2\frac{a}{q}}\sinh(qra)}{C_1^2\sinh a}\right)\right]^{-1}. \end{aligned}$$

The term $\frac{e^{-2\frac{a}{q}}\sinh(qra)}{C_1^2\sinh a\sinh(qra+a)^2}\left(2\cosh\left(qra+a\right)-\frac{2e^{-2\frac{a}{q}}\sinh(qra)}{C_1^2\sinh a}\right)$ goes to 0 as $q\to 0$. We use the Taylor series expansion for the multiplicative inverse about $q=0$. Then

$$\frac{\sinh\left(q^2r\cosh^{-1}\left(x\left(\frac{1}{q}\right)\right)+a\right)}{\sqrt{z\left(\frac{1}{q},q^2r\right)^2-1}} = 1+\frac{e^{-2\frac{a}{q}}\sinh(qra)}{C_1^2\sinh a\sinh(qra+a)^2}\left(2\cosh\left(qra+a\right)-\frac{2e^{-2\frac{a}{q}}\sinh(qra)}{C_1^2\sinh a}\right) \tag{A.3.26}$$

and

$$\frac{\sinh\left(q^2r\cosh^{-1}\left(x\left(\frac{1}{q}\right)\right)+a\right)}{\sqrt{z\left(\frac{1}{q},q^2r\right)^2-1}}-1 \approx \frac{2e^{-2\frac{a}{q}}\sinh(qra)\cosh\left(qra+a\right)}{C_1^2\sinh a\sinh(qra+a)^2}.$$

Consider the terms $y\left(\frac{1}{q}\right)\sinh\left(\frac{1}{q}\cosh^{-1}\left(z\left(\frac{1}{q},q^2r\right)\right)\right)$ and $y\left(\frac{1}{q}\right)\sinh\left(\left(\frac{1}{q}-1\right)\cosh^{-1}\left(z\left(\frac{1}{q},q^2r\right)\right)\right)$. From Equation A.3.23 and using the exponential definition for hyperbolic trigonometry we have

$$\begin{aligned}
\sinh\left(\frac{1}{q}\cosh^{-1}\left(z\left(\frac{1}{q},q^2r\right)\right)\right) &= \sinh\left(ra+\frac{a}{q}\right),\\
&= \frac{\mathrm{e}^{\frac{a}{q}}}{2}(\cosh(ra)+\sinh(ra)) - \frac{\mathrm{e}^{-\frac{a}{q}}}{2}(\cosh(ra)-\sinh(ra)),\\
\sinh\left(\left(\frac{1}{q}-1\right)\cosh^{-1}\left(z\left(\frac{1}{q},q^2r\right)\right)\right) &= \sinh\left((ra+\frac{a}{q})(1-q)\right),\\
&\approx \sinh\left(a(r-1)+\frac{a}{q}\right),\\
&= \frac{\mathrm{e}^{\frac{a}{q}}}{2}(\cosh((r-1)a)+\sinh((r-1)a))\\
&\quad -\frac{\mathrm{e}^{-\frac{a}{q}}}{2}(\cosh((r-1)a)-\sinh((r-1)a)).
\end{aligned}$$

Label

$$\begin{aligned}
C_3(r) &= \cosh(ra)+\sinh(ra),\\
C_4(r) &= \cosh(ra)-\sinh(ra).
\end{aligned} \tag{A.3.27}$$

Then

$$\begin{aligned}
y\left(\frac{1}{q}\right)\sinh\left(\frac{1}{q}\cosh^{-1}\left(z\left(\frac{1}{q},q^2r\right)\right)\right) &= \left(\frac{\mathrm{e}^{\frac{a}{q}}}{2}C_1-\frac{\mathrm{e}^{-\frac{a}{q}}}{2}C_2\right)\left(\frac{\mathrm{e}^{\frac{a}{q}}}{2}C_3(r)-\frac{\mathrm{e}^{-\frac{a}{q}}}{2}C_4(r)\right),\\
&= \frac{\mathrm{e}^{\frac{2a}{q}}C_1C_3(r)-(C_1C_4(r)+C_2C_3(r))+\mathrm{e}^{-\frac{2a}{q}}C_2C_4(r)}{4},\\
&\approx \frac{\mathrm{e}^{\frac{2a}{q}}C_1C_3(r)}{4},\\
y\left(\frac{1}{q}\right)\sinh\left(\left(\frac{1}{q}-1\right)\cosh^{-1}\left(z\left(\frac{1}{q},q^2r\right)\right)\right) &= \left(\frac{\mathrm{e}^{\frac{a}{q}}}{2}C_1-\frac{\mathrm{e}^{-\frac{a}{q}}}{2}C_2\right)\left(\frac{\mathrm{e}^{\frac{a}{q}}}{2}C_3(r-1)-\frac{\mathrm{e}^{-\frac{a}{q}}}{2}C_4(r-1)\right),\\
&= \frac{\mathrm{e}^{\frac{2a}{q}}C_1C_3(r-1)-(C_1C_4(r-1)+C_2C_3(r-1))+\mathrm{e}^{-\frac{2a}{q}}C_2C_4(r-1)}{4},\\
&\approx \frac{\mathrm{e}^{\frac{2a}{q}}C_1C_3(r-1)}{4}.
\end{aligned}$$

Then we get

$$\begin{aligned}
B_1\left(\frac{1}{q},\frac{1}{q},q^2r\right) &= \frac{e^{\frac{2a}{q}}\,C_1C_3(r)}{4}\frac{2\,e^{-2\frac{a}{q}}\sinh(qra)\cosh(qra+a)}{C_1^2\sinh a\sinh(qra+a)^2},\\
&= \frac{C_1C_3(r)\sinh(qra)\cosh(qra+a)}{2C_1^2\sinh a\sinh(qra+a)^2},\\
&\to 0 \text{ as } q\to 0,\\
B_1\left(\frac{1}{q}-1,\frac{1}{q},q^2r\right) &= \frac{e^{\frac{2a}{q}}\,C_1C_3(r-1)}{4}\frac{2\,e^{-2\frac{a}{q}}\sinh(qra)\cosh(qra+a)}{C_1^2\sinh a\sinh(qra+a)^2},\\
&= \frac{C_1C_3(r-1)\sinh(qra)\cosh(qra+a)}{2C_1^2\sinh a\sinh(qra+a)^2},\\
&\to 0 \text{ as } q\to 0,
\end{aligned} \tag{A.3.28}$$

and

$$\begin{aligned}
\lim_{q\to 0} B_1\left(\frac{1}{q},\frac{1}{q},q^2r\right) &= 0,\\
\lim_{q\to 0} B_1\left(\frac{1}{q}-1,\frac{1}{q},q^2r\right) &= 0.
\end{aligned} \tag{A.3.29}$$

Limits for B_2

Consider

$$\begin{aligned}
B_2\left(\frac{1}{q},\frac{1}{q},q^2r\right) &= \left(\sqrt{x\left(\frac{1}{q}\right)^2-1}-y\left(\frac{1}{q}\right)\right)\sinh\left(q^2r\cosh^{-1}\left(x\left(\frac{1}{q}\right)\right)\right)\\
&\quad \cosh a\frac{\sinh\left(\frac{1}{q}\cosh^{-1}\left(z\left(\frac{1}{q},q^2r\right)\right)\right)}{\sqrt{z\left(\frac{1}{q},q^2r\right)^2-1}},\\
B_2\left(\frac{1}{q}-1,\frac{1}{q},q^2r\right) &= \left(\sqrt{x\left(\frac{1}{q}\right)^2-1}-y\left(\frac{1}{q}\right)\right)\sinh\left(q^2r\cosh^{-1}\left(x\left(\frac{1}{q}\right)\right)\right)\\
&\quad \cosh a\frac{\sinh\left((\frac{1}{q}-1)\cosh^{-1}\left(z\left(\frac{1}{q},q^2r\right)\right)\right)}{\sqrt{z\left(\frac{1}{q},q^2r\right)^2-1}}.
\end{aligned}$$

Consider the term $\sqrt{x\left(\frac{1}{q}\right)^2-1}-y\left(\frac{1}{q}\right)\sinh\left(q^2r\cosh^{-1}\left(x\left(\frac{1}{q}\right)\right)\right)$. From Equation A.3.15 we have

$$\begin{aligned}
\sqrt{x\left(\frac{1}{q}\right)^2-1}-y\left(\frac{1}{q}\right) &= e^{-\frac{a}{q}}\frac{\operatorname{csch}(a)^2}{C_1}+\frac{e^{-3\frac{a}{q}}C_2^2}{4C_1},\\
&\approx e^{-\frac{a}{q}}\frac{\operatorname{csch}(a)^2}{C_1},
\end{aligned}$$

using $C_1C_2 - 1 = \operatorname{csch}(a)^2$.

From Equation (A.3.19) we have

$$\sinh\left(q^2 r \cosh^{-1}\left(x\left(\frac{1}{q}\right)\right)\right) = \sinh(qra).$$

Consider the terms $\frac{\sinh\left(\frac{1}{q}\cosh^{-1}\left(z\left(\frac{1}{q},q^2r\right)\right)\right)}{\sqrt{z\left(\frac{1}{q},q^2r\right)^2-1}}$ and $\frac{\sinh\left(\left(\frac{1}{q}-1\right)\cosh^{-1}\left(z\left(\frac{1}{q},q^2r\right)\right)\right)}{\sqrt{z\left(\frac{1}{q},q^2r\right)^2-1}}$. From Equation (A.3.23) and using the exponential definition for hyperbolic trigonometry we have

$$\begin{aligned}
\sinh\left(\frac{1}{q}\cosh^{-1}\left(z\left(\frac{1}{q},q^2r\right)\right)\right) &= \sinh\left(ra+\frac{a}{q}\right),\\
&= \frac{\mathrm{e}^{\frac{a}{q}}}{2}C_3(r) - \frac{\mathrm{e}^{-\frac{a}{q}}}{2}C_4(r),\\
&\approx \frac{\mathrm{e}^{\frac{a}{q}}}{2}C_3(r),\\
\sinh\left(\left(\frac{1}{q}-1\right)\cosh^{-1}\left(z\left(\frac{1}{q},q^2r\right)\right)\right) &= \sinh\left(\left(ra+\frac{a}{q}\right)(1-q)\right),\\
&\approx \sinh\left(a(r-1)+\frac{a}{q}\right),\\
&= \frac{\mathrm{e}^{\frac{a}{q}}}{2}C_3(r-1) - \frac{\mathrm{e}^{-\frac{a}{q}}}{2}C_4(r-1),\\
&\approx \frac{\mathrm{e}^{\frac{a}{q}}}{2}C_3(r-1).
\end{aligned}$$

From calculations used in Equations (A.3.22) and (A.3.26) we have

$$\begin{aligned}
\frac{1}{\sqrt{z\left(\frac{1}{q},q^2r\right)^2-1}} &= \frac{1}{\sinh(qra+a)}\left(1+\frac{\mathrm{e}^{-2\frac{a}{q}}\sinh(qra)}{C_1^2\sinh a\sinh(qra+a)^2}\right.\\
&\qquad \left.\left(2\cosh(qra+a) - \frac{2\,\mathrm{e}^{-2\frac{a}{q}}\sinh(qra)}{C_1^2\sinh a}\right)\right),\\
&\approx \frac{1}{\sinh(qra+a)}\left(1+\frac{2\,\mathrm{e}^{-2\frac{a}{q}}\sinh(qra)\cosh(qra+a)}{C_1^2\sinh a\sinh(qra+a)^2}\right).
\end{aligned}$$

Then

$$
\begin{aligned}
\frac{\sinh\left(\frac{1}{q}\cosh^{-1}\left(z\left(\frac{1}{q},q^2r\right)\right)\right)}{\sqrt{z\left(\frac{1}{q},q^2r\right)^2-1}} &= \frac{C_3(r)\,\mathrm{e}^{\frac{a}{q}}}{2\sinh(qra+a)}\left(1+\frac{2\,\mathrm{e}^{-2\frac{a}{q}}\sinh(qra)\cosh(qra+a)}{C_1^2\sinh a\sinh(qra+a)^2}\right),\\
&\approx \frac{C_3(r)\,\mathrm{e}^{\frac{a}{q}}}{2\sinh(qra+a)},\\
\frac{\sinh\left(\left(\frac{1}{q}-1\right)\cosh^{-1}\left(z\left(\frac{1}{q},q^2r\right)\right)\right)}{\sqrt{z\left(\frac{1}{q},q^2r\right)^2-1}} &= \frac{C_3(r-1)\,\mathrm{e}^{\frac{a}{q}}}{2\sinh(qra+a)}\left(1+\frac{2\,\mathrm{e}^{-2\frac{a}{q}}\sinh(qra)\cosh(qra+a)}{C_1^2\sinh a\sinh(qra+a)^2}\right),\\
&\approx \frac{C_3(r-1)\,\mathrm{e}^{\frac{a}{q}}}{2\sinh(qra+a)}.
\end{aligned}
$$

Then

$$
\begin{aligned}
B_2\left(\frac{1}{q},\frac{1}{q},q^2r\right) &= \mathrm{e}^{-\frac{a}{q}}\,\frac{\operatorname{csch}(a)^2}{C_1}\sinh(qra)\cosh a\frac{\mathrm{e}^{\frac{a}{q}}}{2}C_3(r),\\
&= \frac{\sinh(qra)\cosh aC_3(r)\operatorname{csch}(a)^2}{2C_1},\\
&\to 0 \text{ as } q\to 0,\\
B_2\left(\frac{1}{q}-1,\frac{1}{q},q^2r\right) &= \mathrm{e}^{-\frac{a}{q}}\,\frac{\operatorname{csch}(a)^2}{C_1}\sinh(qra)\cosh a\frac{\mathrm{e}^{\frac{a}{q}}}{2}C_3(r-1).\\
&= \frac{\sinh(qra)\cosh aC_3(r-1)\operatorname{csch}(a)^2}{2C_1},\\
&\to 0 \text{ as } q\to 0,
\end{aligned}
\tag{A.3.30}
$$

and

$$
\begin{aligned}
\lim_{q\to 0} B_2\left(\frac{1}{q},\frac{1}{q},q^2r\right) &= 0,\\
\lim_{q\to 0} B_2\left(\frac{1}{q}-1,\frac{1}{q},q^2r\right) &= 0.
\end{aligned}
\tag{A.3.31}
$$

Appendix B

Curves of representations of the infinite family of manifolds N_Φ

In this appendix we expand on calculations for the character variety of the family of manifolds N_Φ that were used in the proof of Lemmas 4.9 to 4.12 and 4.15 in Chapter 4. Recall the infinite family of manifolds N_Φ from Section 4.3 formed by gluing the twisted I–bundle over the Klein bottle, $K\tilde{\times}I$, to the complement of the right-handed trefoil, M, along

$$\Phi = \begin{pmatrix} k & l \\ m & n \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

We start with the representations on $K\tilde{\times}I$ (as analysed in Section 4.3.1) and methodically look at how they could extend to N_Φ in a way that satisfies the group relations for M (as analysed in Section 4.3.2) and could give a curve, up to conjugacy. The procedure of analysing the representations is the same as described in Section 4.3.5, we restate some details for convenience here.

Let $\mathbb{F}$ be an algebraically closed field of characteristic p. For ease of notation write

$$(A,B,G,H) = (\rho(a),\rho(b),\rho(g),\rho(h)) \in \mathrm{SL}_2(\mathbb{F})^4.$$

The possible types of representations that arise in $K\tilde{\times}I$ are given by the matrix pairs (A,B) from Equations (4.3.2) and (4.3.3) and Equation (4.3.5) (when $p=2$) and Equation (4.3.6) (when $p \neq 2$).

To extend the representations to N_Φ, the matrices must satisfy the gluing equations

$$G = A^{2k}B^{l}, \quad G^{-4}HG^{2}H = A^{2m}B^{n}. \tag{B.0.1}$$

The first gluing equation will define G. Write

$$H = \begin{pmatrix} e & f \\ g & h \end{pmatrix}, \quad eh - fg = 1.$$

The second gluing equation will give four equations for the entries of H in terms of the other variables in addition to the determinant condition.

The resulting matrices must also satisfy group relation for M,

$$GHG = HGH. \tag{B.0.2}$$

We methodically go through the cases, first for $p = 2$ then for $p \neq 2$.

B.1 In characteristic $p = 2$

Assume $\mathbb{F}$ is an algebraically closed field of characteristic $p = 2$. Consider the three options for matrix pairs (A, B) from Equations (4.3.2), (4.3.3) and (4.3.5) in turn to see where curves of representations could arise.

The curves of representations used in Lemma 4.10 are given in Appendices B.1.1 and B.1.3, the curves of representations used in Lemma 4.11 are given in Appendices B.1.1 and B.1.2, the curves of representations used in Lemma 4.12 are given in Appendix B.1.3, and the curves of representations used in Lemma 4.15 are given in Appendix B.1.2.

B.1.1 Extending the representation from Equation (4.3.2)

Consider (A, B) irreducible as in Equation (4.3.2) with

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}.$$

Then $A^2 = E$ and the gluing equations give

$$G = B^l, \quad HB^{2l}H = B^{n+4l}.$$

Computation in characteristic two shows

$$HB^{2l}H - B^{n+4l} = \begin{pmatrix} fgy^{-2l} + e^2y^{2l} + y^{n+4l} & fy^{-2l}(h + ey^{4l}) \\ gy^{-2l}(h + ey^{4l}) & y^{-n-4l} + h^2y^{-2l} + fgy^{2l} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{B.1.1}$$

Solving the equations for the entries plus the determinant for H gives five possible solutions:

1. $e = y^{\frac{n+2l}{2}}$, $f = 0$, $g = 0$, $h = y^{-\frac{n+2l}{2}}$, or
2. $e = y^{-2l}$, $f = 0$, $h = y^{2l}$, $n = -6l$, or
3. $e = 1$, $f = 0$, $h = 1$, $y = 1$, or
4. $f \neq 0$, $g = \frac{1+y^{4l}e^2}{f}$, $h = y^{4l}e$, $n = -6l$, or
5. $f \neq 0$, $g = \frac{1+e^2}{f}$, $h = e$, $y = 1$.

Consider the group relation for M for each option. This will give the following possibilities for each case:

1. (a) $e = y^l$, $f = 0$, $g = 0$, $h = y^{-l}$, $n = 0$, or
   (b) $e = 1$, $f = 0$, $g = 0$, $h = 1$, $y = 1$, which cannot produce a curve as there are no remaining degrees of freedom,
2. $e = 1$, $f = 0$, $g = 0$, $h = 1$, $y = 1$, $n = -6l$, which cannot produce a curve as there are no remaining degrees of freedom,
3. $e = 1$, $f = 0$, $g = 0$, $h = 1$, $y = 1$, which cannot produce a curve as there are no remaining degrees of freedom,
4. $e = \frac{y^{-l}}{1+y^{2l}}$, $f \neq 0$, $g = \frac{1+y^{2l}+y^{4l}}{f(1+y^{2l})^2}$, $h = \frac{y^{3l}}{1+y^{2l}}$, $y \neq 1$, $l \neq 0$, $n = -6l$,
5. $f \neq 0$ and $f = 0$, $g = \frac{1+e^2}{f}$, $h = e$, $y = 1$, which gives a contradiction.

This gives two options for curves of representations stemming from Item 1 (a) and Item 4.

In Item 1, the gluing matrix is $\Phi = \begin{pmatrix} k & l \\ m & 0 \end{pmatrix} = \begin{pmatrix} k & \pm 1 \\ \mp 1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix},$$
$$G = \begin{pmatrix} y^{\pm 1} & 0 \\ 0 & y^{\mp 1} \end{pmatrix}, \quad H = \begin{pmatrix} y^{\pm 1} & 0 \\ 0 & y^{\mp 1} \end{pmatrix}, \quad y \in \mathbb{F} \setminus \{0\}.$$

This is used in Lemma 4.11.

In Item 4, the gluing matrix is $\Phi = \begin{pmatrix} k & l \\ m & -6l \end{pmatrix} = \begin{pmatrix} k & \pm 1 \\ \pm 1 + 6k & -6l \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix},$$
$$G = \begin{pmatrix} y^{\pm 1} & 0 \\ 0 & y^{\mp 1} \end{pmatrix}, \quad H = \begin{pmatrix} \frac{y^{\mp 2}}{y+y^{-1}} & f \\ \frac{y^2+1+y^{-2}}{f(y^2+y^{-2})} & \frac{y^{\pm 2}}{y+y^{-1}} \end{pmatrix}, \quad f, y \in \mathbb{F} \setminus \{0\}.$$

This is used in Lemma 4.10.

B.1.2 Extending the representation from Equation (4.3.3)

Consider (A, B) reducible as in Equation (4.3.3) with

$$A = \begin{pmatrix} x & 1 \\ 0 & x^{-1} \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $B = E$ and the gluing equations give

$$G = A^{2k}, \quad HA^{4k}H = A^{2m+8k}.$$

Two cases arise for if $x = 1$ or $x \neq 1$.

Assume $x = 1$

Computation in characteristic two shows

$$HA^{4k}H - A^{2m+8k} = \begin{pmatrix} e(e+h) & f(e+h) \\ g(e+h) & h(e+h) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The equations together force $h = e$. The determinant condition for H then gives $e^2 + fg = 1$, which implies either:

1. $e = 1, f = 0, h = 1$, or
2. $f \neq 0, g = \frac{1+e^2}{f}, h = e$.

The group relation for M gives:

1. $e = 1,\ f = 0,\ g = 0,\ h = 1$, which cannot specify a curve as there are no remaining degrees of freedom,
2. $f \neq 0$ and $f = 0, g = \frac{1+e^2}{f}, h = e$, which gives a contradiction.

Thus for $x = 1$, a curve cannot appear.

Assume $x \neq 1$

Computation in characteristic two shows

$$\begin{aligned} HA^{4k}H - A^{2m+8k} &= \frac{1}{x^2+1} \begin{pmatrix} \begin{matrix} (x^2+1)(x^{8k+2m}+x^{4k}e^2+x^{-4k}fg) \\ +xeg(x^{4k}+x^{-4k}) \end{matrix} & \begin{matrix} (x^2+1)(x^{4k}ef+x^{-4k}fh) \\ +xeh(x^{4k}+x^{-4k}) \\ +x(x^{8k+2m}+x^{-8k-2m}) \end{matrix} \\ \begin{matrix} g((x^2+1)(x^{4k}e+x^{-4k}h) \\ +xg(x^{4k}+x^{-4k})) \end{matrix} & \begin{matrix} (x^2+1)(x^{-8k-2m}+x^{4k}fg+x^{-4k}h^2) \\ +xgh(x^{4k}+x^{-4k})) \end{matrix} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{B.1.2}$$

Solving the equations for the entries plus the determinant for H gives three possible solutions:

1. $e = x^{2k+m}, f = \frac{x^{1-2k-m}(1+x^{4k+2m})}{1+x^2}, g = 0, h = x^{-2k-m}$, or
2. $e = x^{-4k}, g = 0, h = x^{4k}, m = -6k$, or

3. $f=\frac{xeg(x^{-4k}+x^{4k})+(1+x^2)(x^{-4k}+x^{4k}e^2)}{(1+x^2)x^{-4k}g}$, $g\neq 0$, $h=\frac{x^{4k}(x^{4k}e(1+x^2)+xg(x^{-4k}+x^{4k}))}{1+x^2}$, $m=-6k$.

Consider the group relation for M for each option. This will give the following possibilities for each case:

1. $e=x^{2k}$, $f=\frac{x^{1-2k}(1+x^{4k})}{1+x^2}$, $g=0$, $h=x^{-2k}$, $m=0$,
2. $e=1$, $g=0$, $h=1$, $k=0$, $m=0$, which gives a contradiction to the gluing matrix $\Phi\in\mathrm{SL}_2(\mathbb{Z})$,
3. (a) $e=x^{-2k}$, $f=0$, $g=\frac{x^{-1+2k}(1+x^2)}{1+x^{4k}}$, $h=x^{2k}$, $k\neq 0$, $m=-6k$,
   (b) $e\neq x^{-2k}$, $f=\frac{x(e+x^{2k}+e^2x^{2k}+x^{6k}+e^2x^{6k}+ex^{8k})}{(1+x^2)(1+ex^{2k}+ex^{6k})}$, $g=\frac{x^{-1-2k}(1+x^2)(1+e^2x^{4k}+ex^{6k}+e^2x^{8k})}{(1+ex^{2k})(1+x^{4k})}$,
   $h=\frac{x^{-4k}+ex^{-2k}+ex^{2k}+x^{4k}}{x^{-2k}+x^{2k}}$, $k\neq 0$, $m=-6k$.

Item 3 (a) and (b) are conjugate. This gives two options for curves of representations stemming from Item 1 and Item 3.

In Item 1 the gluing matrix is $\Phi=\begin{pmatrix}k & l\\ 0 & n\end{pmatrix}=\begin{pmatrix}\pm1 & l\\ 0 & \pm1\end{pmatrix}\in\mathrm{SL}_2(\mathbb{Z})$ and

$$A=\begin{pmatrix}x & 1\\ 0 & x^{-1}\end{pmatrix},\quad B=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$
$$G=\begin{pmatrix}x^{\pm2} & \frac{x^2+x^{-2}}{x+x^{-1}}\\ 0 & x^{\mp2}\end{pmatrix},\quad H=\begin{pmatrix}x^{\pm2} & \frac{x^2+x^{-2}}{x+x^{-1}}\\ 0 & x^{\mp2}\end{pmatrix},\quad x\in\mathbb{F}\setminus\{0,1\}.$$

This is used in Lemma 4.11.

In Item 3 the gluing matrix is $\Phi=\begin{pmatrix}k & l\\ -6k & n\end{pmatrix}=\begin{pmatrix}\pm1 & l\\ \mp6 & \pm1-6k\end{pmatrix}\in\mathrm{SL}_2(\mathbb{Z})$ and

$$A=\begin{pmatrix}x & 1\\ 0 & x^{-1}\end{pmatrix},\quad B=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$
$$G=\begin{pmatrix}x^{\pm2} & \frac{x^2+x^{-2}}{x+x^{-1}}\\ 0 & x^{\mp2}\end{pmatrix},\quad H=\begin{pmatrix}x^{\pm2} & 0\\ \frac{x+x^{-1}}{x^2+x^{-2}} & x^{\mp2}\end{pmatrix},\quad x\in\mathbb{F}\setminus\{0,1\}.$$

This is used in Lemma 4.15.

B.1.3 Extending the representation from Equation (4.3.5)

Consider (A,B) reducible as in Equation (4.3.5) with

$$A=\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix},\quad B=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}.$$

Then $A^2=E$ and the gluing equations give

$$G=B^l,\quad HB^{2l}H=B^{n+4l}.$$

Computation in characteristic two shows

$$HB^{2l}H - B^{n+4l} = \begin{pmatrix} e(e+h) & f(e+h)+n \\ g(e+h) & h(e+h) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{B.1.3}$$

Solving the equations for the entries plus the determinant for H gives two possible solutions:

1. $e = 1$, $f = 0$, $h = 1$, $n = 2a$, $a \in \mathbb{Z}$, or
2. $f \neq 0$, $g = \frac{1+e^2}{f}$, $h = e$, $n = 2a$, $a \in \mathbb{Z}$.

Consider the group relation for M for each option. This will give the following possibilities for each case:

1. $e = 1$, $f = 0$, $g = l^{-1}$, $h = 1$, $n = 2a$,
2. (a) $e = 1$, $f = l$, $g = 0$, $h = 1$, $n = 2a$, $a \in \mathbb{Z}$, or
   (b) $f = l(1+e^2)$, $g = l^{-1}$, $h = e$, $n = 2a$, $a \in \mathbb{Z}$.

Item 1 (a) is a special case of Item 2 (a) and Item 2 (a) and (b) are conjugate, which means we get one option. Given n is even l must be odd, which simplifies the case to

2. $e = 1$, $f = 1$, $g = 0$, $h = 1$, $n = 2a$, $a \in \mathbb{Z}$.

This gives one option for a curves of representations, Item 2. The gluing matrix is $\Phi = \begin{pmatrix} k & l \\ m & 2a \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. The determinant condition in $\mathbb{Z}$ forces $l = 1 \pmod 2$ and $m = 1 \pmod 2$ and

$$A = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$
$$G = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad u \in \mathbb{F} \setminus \{0\}.$$

This is used in Lemmas 4.10 and 4.12.

B.2 In characteristic $p \neq 2$

Assume $\mathbb{F}$ is an algebraically closed field of characteristic $p \neq 2$. Consider the three options for matrix pairs (A, B) from Equations (4.3.2), (4.3.3) and (4.3.6) in turn to see where curves of representations could arise.

The curves of representations used in Lemma 4.9 are given in Appendices B.2.1 and B.2.3, the curves of representations used in Lemma 4.11 are given in Appendix B.2.2, and the curves of representations used in Lemma 4.12 are given in Appendix B.2.3.

B.2.1 Extending the representation from Equation (4.3.2)

Consider (A,B) irreducible as in Equation (4.3.2). We have

$$A_p = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B_p = \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}.$$

Then $A^2 = -E$ and the gluing equations give

$$G = (-1)^k B^l, \quad HB^{2l}H = (-1)^m B^{n+4l}.$$

Expand the entries of H as in Appendix B. Computation in arbitrary characteristic gives

$$HB^{2l}H - (-1)^m B^{n+4l} = \begin{pmatrix} \substack{fgy^{-2l}+e^2y^{2l} \\ +(-1)^{1+m}y^{4l+n}} & fy^{-2l}(h+ey^{4l}) \\ gy^{-2l}(h+ey^{4l}) & \substack{h^2y^{-2l}+fgy^{2l} \\ +(-1)^{1+m}y^{-4l-n}} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{B.2.1}$$

Solving the equations for the entries plus the determinant for H gives seven possible solutions:

1. $e = \pm(-1)^{\frac{m}{2}} y^{\frac{n+2l}{2}}$, $f = 0$, $g = 0$, $h = \pm(-1)^{-\frac{m}{2}} y^{-\frac{n+2l}{2}}$, or
2. $e = \pm(-1)^{a+\frac{1}{2}}$, $f = 0$, $h = \pm(-1)^{-a-\frac{1}{2}}$,$y = 1$, $m = 2a+1$, $a \in \mathbb{Z}$, or
3. $e = \pm(-1)^{\frac{2b+1+2l}{2}}$, $f = 0$, $h = \pm(-1)^{-\frac{2b+1+2l}{2}}$, $y = -1$, $m+n$ odd, $n = 2b+1-m$, $b \in \mathbb{Z}$, or
4. $e = \pm(-1)^{a+\frac{1}{2}} y^{-2l}$, $f = 0$, $h = \pm(-1)^{-a-\frac{1}{2}} y^{2l}$, $m = 2a+1$, $a \in \mathbb{Z}$, $n = -6l$, or
5. $f \neq 0$ $g = \frac{-1-e^2}{f}$, $h = -e$, $y = 1$, $m = 2a+1$, $a \in \mathbb{Z}$, or
6. $f \neq 0$, $g = \frac{-1-e^2}{f}$, $h = -e$, $y = -1$, $m+n$ odd, $n = 2b+1-m$, $b \in \mathbb{Z}$, or
7. $f \neq 0$, $g = \frac{-1-y^{4l}e^2}{f}$, $h = -y^{4l}e$, $m = 2a+1$, $a \in \mathbb{Z}$, $n = -6l$.

Consider the group relation for M for each option. This will give the following possibilities for each case:

1. (a) $e = \pm(-1)^{\frac{m}{2}} y^{\frac{n+2l}{2}}$, $f = 0$, $g = 0$, $h = \pm(-1)^{-\frac{m}{2}} y^{-\frac{n+2l}{2}}$, with y defined by $y^{\frac{n}{2}} = \pm(-1)^{-k-\frac{m}{2}}$, which cannot specify a curve as there are no remaining degrees of freedom, or
   (b) $e = \pm(-1)^a y^l$, $f = 0$, $g = 0$, $h = \pm(-1)^{-a} y^{-l}$, $m = 2a$, $a \in \mathbb{Z}$, $n = 0$, which contradicts the gluing matrix $\Phi \in \mathrm{SL}_2(\mathbb{Z})$,
2. $(-1)^{a+k} = \mp(-1)^{\frac{1}{2}}$, which gives a contradiction as no values for $a, k \in \mathbb{Z}$ could give the result,
3. $(-1)^{b+l} = \mp(-1)^{\frac{1}{2}}$, which gives a contradiction as no values for $b, l \in \mathbb{Z}$ could give the result,
4. (a) $e = \pm(-1)^{a+\frac{1}{2}} y^{-2l}$, $f = 0$, $h = \pm(-1)^{-a-\frac{1}{2}} y^{2l}$, $m = 2a+1$, $a \in \mathbb{Z}$, $n = -6l$, with y defined by $y^{-3l} = \mp(-1)^{\frac{1}{2}-a-k}$, which cannot specify a curve as there are no more degrees of freedom, or

   (b) $e = \pm(-1)^{a+\frac{1}{2}} y^{-2l}$, $f = 0$, $h = \pm(-1)^{-a-\frac{1}{2}} y^{2l}$, $l = 0$, $m = 2a+1$, $a \in \mathbb{Z}$, $n = -6l$, and $\mp(-1)^{\frac{1}{2}-a-k} = 1$, which gives a contradiction as no values for $a, k \in \mathbb{Z}$ could give the result,

5. $f \neq 0$ and $f = 0$, $g = \frac{-1-e^2}{f}$, $h = -e$, $y = 1$, $m = 2a+1$, $a \in \mathbb{Z}$, which gives a contradiction,
6. $f \neq 0$ and $f = 0$, $g = \frac{-1-e^2}{f}$, $h = -e$, $y = -1$, $m+n$ odd, $n = 2b+1-m$, $b \in \mathbb{Z}$, which gives a contradiction,
7. $e = \frac{-(-1)^{-k}}{y^l(y^{2l}-1)}$, $f \neq 0$, $g = \frac{-(y^{2l}-1)^2 - y^{2l}}{f(y^{2l}-1)^2}$, $h = \frac{(-1)^{-k} y^{3l}}{y^{2l}-1}$, $m = 2a+1$, $a \in \mathbb{Z}$, $n = -6l$.

This gives one option for a curve of representations stemming from Item 7 . Here the gluing matrix is $\Phi = \begin{pmatrix} k & l \\ 2a+1 & -6l \end{pmatrix} = \begin{pmatrix} k & \mp 1 \\ \pm 1 - 6k & -6l \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix},$$
$$G = (-1)^k \begin{pmatrix} y^{\pm 1} & 0 \\ 0 & y^{\mp 1} \end{pmatrix}, \quad H = \begin{pmatrix} \frac{\mp(-1)^k y^{\mp 2}}{y - y^{-1}} & f \\ \frac{-y^2+1-y^{-2}}{f(y-y^{-1})^2} & \frac{\pm(-1)^k y^{\pm 2}}{y-y^{-1}} \end{pmatrix}, \quad f, y \in \mathbb{F} \setminus \{0\}.$$

This is used in Lemma 4.9.

B.2.2 Extending the representation from Equation (4.3.3)

Consider (A, B) reducible as in Equation (4.3.3). We have

$$A_p = \begin{pmatrix} x & 1 \\ 0 & x^{-1} \end{pmatrix}, \quad B_p = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}, \quad x \in \mathbb{F} \setminus \{0\}.$$

Then $B = \pm E$ and the gluing equations give

$$G = (\pm 1)^l A^{2k}, \quad H A^{4k} H = (\pm 1)^n A^{2m+8k}$$

where the ± 1 factor arises from B. Two cases arise for if $x = \pm 1$ or $x \neq \pm 1$.

Assume $x = \pm 1$

Computation in arbitrary characteristic gives

$$HA^{4k}H - (\pm 1)^n A^{8k+2m} = \begin{pmatrix} \begin{matrix} e(e+h+4gk) \\ -1-(\pm 1)^n \end{matrix} & \begin{matrix} f(e+h) \\ +4ehk-(\pm 1)^n(8k+2m) \end{matrix} \\ g(e+h+4gk) & \begin{matrix} h(e+h+4gk) \\ -1-(\pm 1)^n \end{matrix} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{B.2.2}$$

Solving the equations for the entries plus the determinant for H gives four possible solutions:

1. $B = E$, $e = 1$, $f = 2k+m$, $g = 0$, $h = 1$, which cannot specify a curve as there are no remaining degrees of freedom, or
2. $B = -E$, $e = -1$, $f = -(2k+m)$, $g = 0$, $h = -1$, $n = 2b$, $b \in \mathbb{Z}$, which cannot specify a curve as there are no remaining degrees of freedom, or
3. $B = -E$, $e = -(-1)^{\frac{1}{2}}$, $g = 0$, $h = (-1)^{\frac{1}{2}}$, $m = -6k$, $n = 2b+1$, $b \in \mathbb{Z}$, or
4. $B = -E$, $f = \frac{-e^2-4egk-1}{g}$, $g \neq 0$, $h = -e-4gk$, $m = -6k$, $n = 2b+1$, $b \in \mathbb{Z}$.

Consider the group relation for M for Items 3 and 4. This will give the following possibilities for each case:

3. $-1+(-1)^{\frac{1}{2}} = 0$ and $-1-(-1)^{\frac{1}{2}} = 0$, which gives a contradiction,
4. $1-4gk = 0$ and $-1+e(1-4gk) = 0$, which gives a contradiction.

Thus for $x = \pm 1$ a curve cannot appear.

Assume $x \neq \pm 1$

Computation in arbitrary characteristic shows

$$\begin{aligned} HA^{4k}H - (\pm 1)^n A^{2m+8k} &= \frac{1}{x^2-1} \begin{pmatrix} \begin{matrix} (x^2-1)\left(x^{4k}e^2+x^{-4k}fg-(\pm 1)^n x^{2(4k+m)}\right) \\ -xeg\left(x^{-4k}-x^{4k}\right) \end{matrix} & \begin{matrix} (x^2-1)\left(x^{4k}ef+x^{-4k}fh\right) \\ +(\pm 1)^n x\left(x^{-2(4k+m)}-x^{2(4k+m)}\right)-x\left(x^{-4k}-x^{4k}\right)eh \end{matrix} \\ \begin{matrix} g\left((x^2-1)\left(x^{4k}e+x^{-4k}h\right) \right.\\ \left. -xg\left(x^{-4k}-x^{4k}\right)\right) \end{matrix} & \begin{matrix} (x^2-1)\left(x^{4k}fg+x^{-4k}\left(h^2-(\pm 1)^n x^{-2(2k+m)}\right)\right) \\ +xgh\left(x^{4k}-x^{-4k}\right) \end{matrix} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{B.2.3}$$

Solving the equations for the entries plus the determinant for H gives three possible solutions:

1. $B = E$, $e = x^{2k+m}$, $f = \frac{x^{1-2k-m}(x^{4k+2m}-1)}{x^2-1}$, $g = 0$, $h = x^{-2k-m}$, or
2. $B = -E$, $e = -x^{2k+m}$, $f = -\frac{x^{1-2k-m}(x^{4k+2m}-1)}{x^2-1}$, $g = 0$, $h = -x^{-2k-m}$, $n = 2b$, $b \in \mathbb{Z}$, or
3. $B = -E$, $e = -(-1)^{\frac{1}{2}} x^{2k+m}$, $f = -\frac{(-1)^{\frac{1}{2}} x^{1-2k-m}(x^{4k+2m}+1)}{x^2-1}$, $g = 0$, $h = (-1)^{\frac{1}{2}} x^{-2k-m}$, $n = 2b+1$, $b \in \mathbb{Z}$.

Consider the group relation for M for each option. This will give the following possibilities for each case:

1. $B = E$, $e = x^{2k}$, $f = \frac{x^{1-2k}(x^{4k}-1)}{x^2-1}$, $g = 0$, $h = x^{-2k}$, $m = 0$,

2. $B=-E$, $e=-x^{2k}$, $f=-\frac{x^{1-2k}(x^{4k}-1)}{x^2-1}$, $g=0$, $h=-x^{-2k}$, $m=0$, $n=2b$, $b\in\mathbb{Z}$, which gives a contradiction to the gluing matrix $\Phi\in\mathrm{SL}_2(\mathbb{Z})$,

3. $B=-E, e=-(-1)^{\frac{1}{2}}x^{2k+m}, f=-\frac{(-1)^{\frac{1}{2}}x^{1-2k-m}(x^{4k+2m}+1)}{x^2-1}, g=0, h=(-1)^{\frac{1}{2}}x^{-2k-m}l=2c+1, c\in\mathbb{Z}$, $n=2b+1$, $b\in\mathbb{Z}$, with x defined by $x^m\pm(-1)^{\frac{1}{2}}=0$, which cannot specify a curve as there are no remaining degrees of freedom.

This gives one option for a curve of representations stemming from Item 1. Here the gluing matrix is $\Phi=\begin{pmatrix}k & l\\ 0 & n\end{pmatrix}=\begin{pmatrix}\pm1 & l\\ 0 & \mp1\end{pmatrix}\in\mathrm{SL}_2(\mathbb{Z})$ and

$$A=\begin{pmatrix}x & 1\\ 0 & x^{-1}\end{pmatrix},\quad B=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$
$$G=\begin{pmatrix}x^{\pm2} & \pm(x+x^{-1})\\ 0 & x^{\mp2}\end{pmatrix},\quad H=\begin{pmatrix}x^{\pm2} & \pm(x+x^{-1})\\ 0 & x^{\mp2}\end{pmatrix},\quad x\in\mathbb{F}\setminus\{0,1,-1\}.$$

This is used in Lemma 4.11.

B.2.3 Extending the representation from Equation (4.3.6)

Consider (A,B) reducible as in Equation (4.3.6). We have

$$A_p=\begin{pmatrix}x & 0\\ 0 & -x\end{pmatrix},\quad B_p=\begin{pmatrix}s & 1\\ 0 & s\end{pmatrix},\quad x\in\mathbb{F}\setminus\{0\}, x^2=-1, s\in\{\pm1\}.$$

Then $A^2=-E$ and the gluing equations give

$$G=(-1)^kB^l,\quad HB^{2l}H=(-1)^mB^{n+4l}.$$

Computation in arbitrary characteristic shows

$$\begin{aligned}HB^{2l}H-(-1)^mB^{n+4l}&=\begin{pmatrix}(-1)^{m+1}s^n+e^2+fg+2segl & fh+e(f+2shl)+(-1)^ms^n(4l+n)\\ g(e+h+2sgl) & (-1)^{m+1}s^n+h^2+fg+2sghl\end{pmatrix}\\&=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\end{aligned}\tag{B.2.4}$$

where the instances of ±1 comes from the choice of ±1 in the diagonal of B.

Solving the equations for the entries plus the determinant for H gives ten possible solutions:

1. $s=1$, $e=(-1)^{\frac{2a+1}{2}}$, $g=0$, $h=(-1)^{\frac{-(2a+1)}{2}}$, $m=2a+1$, $a\in\mathbb{Z}$, $n=-6l$, or

2. $s=-1$, $e=(-1)^{\frac{2a+1}{2}}$, $g=0$, $h=(-1)^{\frac{-(2a+1)}{2}}$, $m=2a+1-n$, $a\in\mathbb{Z}$,$n=-6l$, or

3. $s=1$, $e=(-1)^{\frac{2a}{2}}$, $f=\frac{1}{2}(-1)^a(2l+n)$, $g=0$, $h=(-1)^{\frac{-(2a)}{2}}$, $m=2a$, $a\in\mathbb{Z}$, or
4. $s=-1$, $e=(-1)^{\frac{2a}{2}}$, $f=-\frac{1}{2}(-1)^a(2l+n)$, $g=0$, $h=(-1)^{\frac{-(2a)}{2}}$, $m=2a-n$, $a\in\mathbb{Z}$, or
5. $s=1$, $e=-(-1)^{\frac{2a+1}{2}}$, $g=0$, $h=-(-1)^{\frac{-(2a+1)}{2}}$, $m=2a+1$, $a\in\mathbb{Z}$, $n=-6l$, or
6. $s=-1$, $e=-(-1)^{\frac{2a+1}{2}}$, $g=0$, $h=-(-1)^{\frac{-(2a+1)}{2}}$, $m=2a+1-n$, $a\in\mathbb{Z}$, $n=-6l$, or
7. $s=1$, $e=-(-1)^{\frac{2a}{2}}$, $f=-\frac{1}{2}(-1)^a(2l+n)$, $g=0$, $h=-(-1)^{\frac{-(2a)}{2}}$, $m=2a$, $a\in\mathbb{Z}$, or
8. $s=-1$, $e=-(-1)^{\frac{2a}{2}}$, $f=\frac{1}{2}(-1)^a(2l+n)$, $g=0$, $h=-(-1)^{\frac{-(2a)}{2}}$, $m=2a-n$, $a\in\mathbb{Z}$, or
9. $s=1$, $f=\frac{-e^2-1-2egl}{g}$, $g\neq 0$, $h=-e-2gl$, $m=2a+1$, $a\in\mathbb{Z}$, $n=-6l$ or
10. $s=-1$, $f=\frac{-e^2-1+2egl}{g}$, $g\neq 0$, $h=-e+2gl$, $m=2a+1-n$, $a\in\mathbb{Z}$, $n=-6l$.

Consider the group relation for M for each option. This will give the following possibilities for each case:

1. $(-1)^k-(-1)^{\frac{1+2a}{2}}=0$ and $(-1)^k+(-1)^{\frac{1+2a}{2}}=0$, which gives a contradiction,
2. $(-1)^{k+1}-(-1)^{\frac{1+2a}{2}}=0$ and $(-1)^{k+l}+(-1)^{\frac{1+2a}{2}})=0$, which gives a contradiction,
3. k and a must have the same parity and $n=0$, which contradicts the gluing matrix $\Phi\in\mathrm{SL}_2(\mathbb{Z})$,
4. $a+k$ and l must have the same parity and $n=0$, which contradicts the gluing matrix $\Phi\in\mathrm{SL}_2(\mathbb{Z})$,
5. $(-1)^k-(-1)^{\frac{1+2a}{2}}=0$ and $(-1)^k+(-1)^{\frac{1+2a}{2}}=0$, which gives a contradiction,
6. $(-1)^{k+1}-(-1)^{\frac{1+2a}{2}}=0$ and $(-1)^{k+l}+(-1)^{\frac{1+2a}{2}}=0$, which gives a contradiction,
7. $a+k$ odd and $n=0$, which contradicts the gluing matrix $\Phi\in\mathrm{SL}_2(\mathbb{Z})$,
8. $a+k-l$ odd and $n=0$, which contradicts the gluing matrix $\Phi\in\mathrm{SL}_2(\mathbb{Z})$,
9. $s=1$, $f=(-1)^{-1+k}(-1-2(-1)^{1-k}e-e^2)l$, $g=\frac{(-1)^{1-k}}{l}$, $h=-2(-1)^{1-k}-e$, $m=2a+1$, $a\in\mathbb{Z}$, $n=-6l$,
10. $s=-1$, $f=(-1)^{k-l}(-1+2(-1)^{-k+l}e-e^2)l$, $g=\frac{(-1)^{-k+l}}{l}$, $h=2(-1)^{-k+l}-e$, $m=2a+1+6l$, $a\in\mathbb{Z}$, $n=-6l$.

This gives two options stemming from Items 9 and 10. For both options the gluing matrix is $\Phi=\begin{pmatrix} k & l \\ 2a+1 & -6l\end{pmatrix}=\begin{pmatrix} k & \pm1 \\ -6k\mp1 & \mp6\end{pmatrix}\in\mathrm{SL}_2(\mathbb{Z})$ and

$$A=\begin{pmatrix} x & 0 \\ 0 & -x\end{pmatrix},\quad B=\begin{pmatrix} s & 1 \\ 0 & s\end{pmatrix},\quad x\in\mathbb{F}\setminus\{0\}, x^2=-1,$$
$$G=(-1)^k\begin{pmatrix} s & \pm1 \\ 0 & s\end{pmatrix},\quad H=\begin{pmatrix} e & \mp(-1)^k(-1+2s(-1)^ke-e^2) \\ \mp(-1)^k & 2s(-1)^k-e\end{pmatrix}.$$

This is used in Lemmas 4.9 and 4.12.

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