

The Groups of Order p^6 ($p \neq 3$)

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by

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SUMMARY

Possibly the first thing that anyone asks for in any abstract algebraic system is a list of examples. In the case of groups, a complete list of all those groups G with a fixed number N of elements is still unknown for general values of N although the result has been known for many years in the case where N has only a few prime factors. Also a complete list of all finite abelian groups is well known.

In the case where all the prime factors of N are the same, $N = p^n$ (p prime) and we call G a p - group. A lot more is known about the structure of p - groups than most other groups and so it is hoped that a complete list of these may be determined. In fact the groups of order p^n ($n \leq 5$, p prime > 3) have been known since Schreier's listing of them in 1926. Also, in 1964, M. Hall and J. Senior published a list of the groups of order 2^n ($n \leq 6$).

In this thesis, the number of non-abelian groups of order p^6 (p prime > 3) is found to be $\frac{1}{4}\{13p^2 + 133p + 1346 + 80(p-1,3) + 45(p-1,4) + 8(p-1,5) + 8(p-1,6)\}$ (where $(p-1,n)$ is the greatest common divisor of $p-1$ and n) and the number of non-abelian groups of order 3^6 is found to be 481. The results of O. Schreier have also been checked and extended to the case $p = 3$.

This is done by considering in turn, families of groups which have the same commutator relations and isomorphic central factor groups (called "isoclinism families"). The remaining relations are determined for these groups checking the possibility that two sets of relations define isomorphic groups and exhibiting a canonical set of generators and relations for each isomorphism class.

In chapter 2, the isoclinism families are determined and in chapter 3, the isomorphism classes for each family are determined in terms of corresponding equivalence classes of matrices over $GF(p)$.

The results of this work are contained in a list of the groups of order p^6 ($p \geq 3$) at the end of the thesis (except for one family which is still not determined).

ACKNOWLEDGEMENTS

The results regarding isoclinism families used in this thesis are due to P. Hall, M. Hall Jnr. and J. Senior and lemmas L1, L3 and L5 are due to N. Blackburn. The defining relations for the individual families are due to T. Easterfield and the list of groups of order p^n ($n \leq 5$, $p > 3$) are due to O. Schreier, although in both cases the results have been re-calculated independently by the author.

All the calculations for the individual families (except the metabelian groups of class 5) are original although the methods are adapted from those of N. Blackburn.

Finally, I would like to thank J. Mack for his proof of lemma L7; Professor G. Wall and Dr. D. Barnes for helpful discussions on some of the calculations, and Dr. T. Easterfield for sending me a copy of his Ph.D. dissertation for my perusal. Also I would like to thank Mr. J. Cannon for all those calculations used which involved the use of a computer.

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I INTRODUCTION

In 1940, P. Hall (5) presented a method for classifying p -groups. Soon after T. Easterfield (2) used this classification to obtain some of the properties of the groups of order p^6 but did not produce a complete list of those groups. However, in 1964, M. Hall Jnr. and J. Senior (3) succeeded in producing a complete list of the groups of order 2^6 . Using the methods of the above papers, I have produced a list of the groups of order p^6 ($p \geq 3$).

We say that two groups G and G' (with centres Z, Z' and derived groups H_2 and H_2') are isoclinic if there exist isomorphisms $\theta: G/Z \rightarrow G'/Z'$, and $\varphi: H_2 \rightarrow H_2'$ such that the following diagram commutes (in the category sense):

$$\begin{array}{ccc}
 G/Z \otimes G/Z & \xrightarrow{\theta \otimes \theta} & G'/Z' \otimes G'/Z' \\
 \eta \downarrow & & \downarrow \eta' \\
 H_2 & \xrightarrow{\varphi} & H_2'
 \end{array}$$

where $\theta \otimes \theta(\alpha Z, \beta Z) = (\theta(\alpha Z), \theta(\beta Z))$ and η, η' are the canonical maps taking $(\alpha Z, \beta Z)$ and $(\alpha' Z', \beta' Z')$ to $[\alpha, \beta]$ and $[\alpha', \beta']$ respectively. The pair of isomorphisms (θ, φ) above is called an isoclinism. In particular, if $G = G'$, we have the automorphism $\theta: G/Z \rightarrow G/Z$. In this case, the corresponding automorphism of H_2 will be

denoted by $\bar{\theta}$ and the isoclinism $(\theta, \bar{\theta})$ is called an autologism of G/Z . Also, P. Hall's notation $G \sim G'$ will be used to denote the fact that G is isoclinic to G' .

It follows immediately from the definition that isoclinism is an equivalence relation and the equivalence class Φ^1 will be called an (isoclinism) family.

The following results of P. Hall will be used:

- (1) If A and B are any two abelian groups, $A \sim B$
- (2) If A is any abelian group $G \sim G \times A$ for all groups G .
- (3) For every family, there exist groups of minimal order with respect to belonging to the family,

Such groups are called stem groups and if we write $|G_0| = p^r$ for stem groups, r is called the rank of the family. Also if G is any group of the family Φ and $|G| = p^{r+k}$ ($k > 0$) we say that G belongs to the k -th branch of Φ .

- (4) G is a stem group if and only if $Z(G) \leq H_2(G)$.
- (5) A quantity depending on a variable group which is the same for all groups of a given family is called a family invariant. If we do not distinguish between isomorphic groups, all the members of the lower central series, the central factor group and the (nilpotency) class are family invariants.

¹ We will use Γ when $p = 2$ and Δ when $p = 3$ to distinguish these from the families for general p .

1. Notation

The following notation shall be adopted throughout this thesis unless otherwise stated:

Let A, B be two sets.

$$A \cup B = \{\alpha \mid \alpha \in A \text{ or } \alpha \in B\}$$

$$A \cap B = \{\alpha \mid \alpha \in A \text{ and } \alpha \in B\}$$

$$B \subseteq A \quad B \text{ is a subset of } A$$

$$B \subset A \quad B \text{ is a subset of } A, \quad B \neq A$$

$$\text{If } B \subseteq A, \quad A - B = \{\alpha \mid \alpha \in A, \alpha \notin B\}$$

Upper case latin letters shall denote finite groups with G being used for a finite group in general.

$$|G| \quad \text{the order of } G$$

$$\exp G \quad \text{the exponent of } G$$

$$Z(G) \quad \text{the centre of } G$$

$$H_2(G) \quad \text{the derived group of } G$$

$$1 \quad \text{the unit of a group}$$

$$E \quad \text{the subgroup } \{1\} \text{ of any group}$$

Lower case greek letters shall denote elements of a group. Let $\xi, \eta, \dots$ denote elements of G and $H, K, \dots$ subgroups of G .

$$\alpha^{\tilde{x}} = \alpha_1^{x_1} \alpha_2^{x_2} \dots \alpha_n^{x_n}$$

$$|G:H| \quad \text{the index of } H \text{ in } G$$

$$\langle \xi, \eta, \dots, H, K, \dots \rangle \quad \text{the least subgroup of } G \text{ containing all the elements } \xi, \eta, \dots \text{ of } G \text{ and}$$

all the elements of the subgroups
 $H, K, \dots$ of G .

$$HK = \langle hk \mid h \in H, k \in K \rangle$$

In general, the group G may be written as

$G = \langle \alpha, \beta, \dots \mid w_i(\alpha, \beta, \dots) = 1 \rangle$ where $\alpha, \beta, \dots$ are the generators of G and $w_i(\alpha, \beta, \dots) = 1$ are the defining relations for G . ($i = 1, 2, \dots$)

$G \times H$ the direct product of G and H , the set of all formal products gh ($g \in G, h \in H$) with the defining relations of G and H together with: $gh = hg$ (all $g \in G, h \in H$)

$G \otimes H$ the cartesian product of G and H , regarded as sets.

$G \times_z H$ the central product of G and H , the amalgamated product GH with amalgamated centre (only possible if $Z(G) \cong Z(H)$)

$H \leq K$ H is a subgroup of K

$H < K$ H is a subgroup of K , $H \neq K$

$H \triangleleft K$ H is a normal subgroup of K

$C_H(K)$ the centralizer of K in H

$C(K)$ the centralizer of K in G

$$H^p = \langle \alpha \mid \alpha \in H \rangle$$

$[\alpha, \beta]$ the commutator of α and β , $= \alpha^{-1}\beta^{-1}\alpha\beta$.

$$[\alpha_1, \alpha_2, \dots, \alpha_{n+1}] = [[\alpha_1, \alpha_2, \dots, \alpha_n], \alpha_{n+1}]$$

$[H, K]$ the subgroup of G generated by all commutators $[\alpha, \beta]$, $\alpha \in H$ and $\beta \in K$.

$H_i(G)$ the i -th term in the lower central series of G , defined inductively by $H_{i+1}(G) = [H_i(G), G]$

$Z_i(G)$ the i -th term of the upper central series of G , defined inductively by $Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G))$ and $Z_1(G) = Z(G)$.

$c(G)$ the (nilpotency) class of G .

Lower case latin letters shall denote integers (p being reserved for a prime number) or elements of $GF(p)$ according to the context.

v a (given) quadratic non-residue (mod p).

g a primitive root (mod p).

If G is a regular p -group, we may define the type invariants of G (see (4)). In this case

$m_1, m_2, \dots$ the type invariants of G .

In fact, if there is only one group in the isoclinism family Φ_n with the type invariants $m_1, m_2, \dots, m_r$, we may denote it by $\Phi_n(m_1, \dots, m_r)$.

If there are more than one such group, we define the genus of a group inductively: two abelian groups have the same genus if they are isomorphic and two non-abelian groups G and H have the same genus if there is a 1-1 correspondence between maximal subgroups of G and the maximal subgroups of H such that corresponding subgroups have the same genus. If there is only one group in the isoclinism family Φ_n with the type invariants $m_1, m_2, \dots, m_r$ and of given genus x , we will denote it by $\Phi_n(m_1, m_2, \dots, m_r)x$ where the genus is ordered lexicographically by:

- (i) the family containing the maximal subgroups (in descending order);
- (ii) the type invariants of the maximal subgroups;
- (iii) the genus of the maximal subgroups.

The group $\Phi_n(m_1, m_2, \dots, m_r)x$ will be placed before the group $\Phi_{n'}(m'_1, m'_2, \dots, m'_r)x'$ in our list of groups of order p^l if (a) $n < n'$

(b) $n = n'$, $m_1 = m'_1, \dots, m_i = m'_i, m_{i+1} > m'_{i+1}$ or

(c) $n = n'$, $m_i = m'_i (i = 1, 2, \dots, r)$, x is before x' .

The groups of a family with the same type invariants and the same genus are ordered in an obvious way by the context (or by the type invariants and genus of the maximal factor groups if no other ordering is suggested). For example, the groups of order p^4 and class 3 are denoted by

$\Phi_3(211)a, \Phi_3(211)b_1, \Phi_3(211)b_2$, and $\Phi_3(1^4)$ (see table).

Whenever a non-regular group is obtained in the process of obtaining a regular group, it will be given the same notation as the corresponding regular group. (Thus the groups of order 81 and class 3 are

$\Delta_3(211)a, \Delta_3(211)b_1, \Delta_3(211)b_2, \Delta_3(1^4)$).

For convenience, the following notation will sometimes be used for automorphisms:

As the automorphism θ is completely determined by the images of a minimal set of generators we will write it as

$\{\alpha_1 \rightarrow \theta(\alpha_1), \dots, \alpha_r \rightarrow \theta(\alpha_r)\}$ where $\alpha_1, \dots, \alpha_r$ are the generators (in the given presentation of G) which are not fixed by θ .

Similarly the autologism $(\theta, \bar{\theta})$ will be written

$$\begin{cases} \theta : (\bar{\alpha}_1, \dots, \bar{\alpha}_r) \rightarrow (\theta(\bar{\alpha}_1), \dots, \theta(\bar{\alpha}_r)) \\ \bar{\theta} : (\beta_1, \dots, \beta_s) \rightarrow (\bar{\theta}(\beta_1), \dots, \bar{\theta}(\beta_s)) \end{cases}$$

where $\bar{\alpha}_1, \dots, \bar{\alpha}_r$ are the given generators of G/Z which are not fixed by θ and $\beta_1, \dots, \beta_s$ are the corresponding generators of H_2 not fixed by $\bar{\theta}$.

2. Lemmas

The following results will be used throughout this thesis:

Lemma L1 Let ξ, η, ζ , be elements of the group G . Then

$$(a) [\xi\eta, \zeta] = [\xi, \zeta]^\eta [\eta, \zeta]$$

$$(b) [\xi, \eta\zeta] = [\xi, \zeta][\xi, \eta]^\zeta$$

$$\begin{aligned} \text{Proof - (a) } [\xi, \zeta]^\eta [\eta, \zeta] &= \eta^{-1} [\xi, \zeta] \eta [\eta, \zeta] \\ &= \eta^{-1} \xi^{-1} \zeta^{-1} \xi \zeta \zeta^{-1} \eta \zeta \\ &= (\zeta \eta)^{-1} \zeta^{-1} (\xi \eta) \zeta \\ &= [\xi \eta, \zeta] \end{aligned}$$

(b) is similar.

Lemma L2 Let $H = H_0 \geq H_1 \geq H_2 \geq \dots$ be a series of normal subgroups of G . If the subgroup L is such that

$$[L, H_{i-1}] \leq H_i \quad (i = 1, 2, 3, \dots) \quad \text{then}$$

$$[H_j(L), H_i] \leq H_{i+j} \quad (i = 0, 1, 2, \dots, j = 2, 3, \dots)$$

where $H_j(L)$ is the j 'th member of the lower central series of the group L .

Proof - see Blackburn (1), theorem 1.2.

Corollary 1 $[H_i(G), H_j(G)] \leq H_{i+j}(G) \quad (i, j = 2, 3, \dots)$

Corollary 2 $[H_i(G), Z_j(G)] \leq Z_{j-i}(G) \quad (2 \leq i \leq j)$

Lemma L3 Let G be a metabelian p -group and $\xi, \eta \in G$. Writing

$$\sigma_1 = \eta, \sigma_{i+1} = [\sigma_i, \xi] \quad K = \langle \xi, \eta \rangle,$$

$$(a) (\xi\eta)^2 = \xi^2 \eta^2 \sigma_2 [\sigma_2, \eta]$$

$$(b) (\xi\eta)^3 = \xi^3 \eta^3 \sigma_2^3 \sigma_3 [\sigma_2, \eta]^5 [\sigma_3, \eta]^2 [\sigma_2, \eta, \eta]^2 \pmod{H_5(K)}$$

$$(c) (\xi\eta)^p = \xi^p \eta^p \sigma_2^{(p)} \dots \sigma_p \pmod{[\eta, H_2(K)]^p [\eta, H_{p-1}(K)]} \quad (p > 3)$$

$$\begin{aligned} \text{Proof - (a)} \quad (\xi\eta)^2 &= \xi\eta\xi\eta \\ &= \xi\eta[\eta, \xi]\eta \\ &= \xi^2\eta^2\sigma_2[\sigma_2, \eta] \end{aligned}$$

(b) We may suppose $H_5(K) = E$ and so all commutators of weight 5 are 1.

$$\begin{aligned} (\xi\eta)^3 &= \xi\eta(\xi\eta)^2 \\ &= \xi(\eta\xi^2)\eta^2\sigma_2[\sigma_2, \eta] \\ &= \xi^3\eta[\eta, \xi^2]\eta^2\sigma_2[\sigma_2, \eta] \\ &= \xi^3\eta\sigma_2^2(\sigma_3\eta^2)\sigma_2[\sigma_2, \eta] \quad \text{using L1} \\ &= \xi^3\eta(\sigma_2^2\eta^2)\sigma_3[\sigma_3, \eta]^2\sigma_2[\sigma_2, \eta] \quad \text{since } H_5(K) = 1 \\ &= \xi^3\eta^3\sigma_2^2[\sigma_2, \eta]^4[\sigma_2, \eta, \eta]^2\sigma_3[\sigma_3, \eta]^2\sigma_2[\sigma_2, \eta] \\ &= \xi^3\eta^3\sigma_2^3\sigma_3[\sigma_2, \eta]^5[\sigma_3, \eta]^2[\sigma_2, \eta, \eta]^2 \end{aligned}$$

since G is metabelian.

(c) We use the result of Theorem 1.6 of (1)

Since G is metabelian, we have

(where $Y = \langle \eta, H_2(K) \rangle$)

$$[H_1, H_{p-1}] = 1 \quad (i = 2, 3, \dots)$$

$$[Y, Y] = [Y, H_2]$$

$$[Y, H_1] = [\eta, H_1(K)]$$

and result follows.

Corollary If $\eta \in \mathcal{C}(H_2(K))$ then

$$(\xi\eta)^p = \xi^p \eta^p \sigma_2^{(p)} \dots \sigma_p \quad \text{for all } p.$$

Lemma L4 Let G be a metabelian p -group and $\xi, \eta \in G$. Then

$$[\eta, \xi^n] = \sigma_2^n \sigma_3^{\binom{n}{2}} \dots \sigma_{n+1}$$

where $\sigma_1 = \eta$, $\sigma_{i+1} = [\sigma_i, \xi]$.

Proof - We proceed by induction. The result is obvious for $n = 1$.

$$\text{Suppose } [\eta, \xi^n] = \sigma_2^n \sigma_3^{\binom{n}{2}} \dots \sigma_{n+1}$$

$$\begin{aligned} \text{Then } [\eta, \xi^{n+1}] &= [\eta, \xi] [\eta, \xi^n]^{\xi} \text{ by L1} \\ &= \sigma_2 \sigma_2^n \sigma_3^{\binom{n}{2}} \dots \sigma_{n+1} [\sigma_2^n \sigma_3^{\binom{n}{2}} \dots \sigma_{n+1}, \xi] \\ &= \sigma_2^{n+1} \sigma_3^{\binom{n}{2}} \dots \sigma_{n+1} \sigma_2^n \sigma_3^{\binom{n}{2}} \dots \sigma_{n+2} \quad \text{since} \\ &\quad G \text{ is metabelian.} \\ &= \sigma_2^{n+1} \sigma_3^{\binom{n}{2} + \binom{n}{3}} \dots \sigma_{n+2} \quad \text{since } G \text{ is} \\ &\quad \text{metabelian.} \\ &= \sigma_2^{n+1} \sigma_3^{\binom{n+1}{2}} \dots \sigma_{n+2} \end{aligned}$$

and result follows.

$$\text{Corollary 1 } [\xi^n, \eta] = (\sigma_2^n \sigma_3^{\binom{n}{2}} \dots \sigma_{n+1})^{-1}$$

Corollary 2

$$\begin{aligned} [\xi^m, \eta^n] &= [\xi, \eta]^{mn} [\xi, \eta, \eta]^{\binom{n}{2}} [\xi, \eta, \xi]^{n \binom{m}{2}} [\xi, \eta, \eta, \eta]^{m \binom{3}{3}} \times \\ &\quad \times [\xi, \eta, \eta, \xi]^{\binom{m}{2} \binom{n}{2}} [\xi, \eta, \xi, \xi]^{n \binom{m}{2}} \pmod{H_5(K)} \end{aligned}$$

Lemma L5 Let G be a p -group of class c with

$$|H_i(G) : H_{i+1}(G)| = p \quad (i=2, 3, \dots, c)$$

and let H be a maximal subgroup of G with $[H, H] \leq H_c(G)$.

Then for all $\sigma \in G-H$, $\sigma H_2(G)$ is the conjugacy class of σ .

Proof - See Blackburn (1) theorem 2.15.

Corollary If $(H_2(G))^p \leq Z(G)$ then $(\sigma \xi)^p = \sigma^p$ for all $\xi \in H_2(G)$.

Lemma L6 (a) If H is a characteristic subgroup of G then so is $\ell(H)$

(b) Let $M_1, M_2, \dots, M_r$ be the maximal subgroups of G .

Then $\cap \{\ell(M_i)\}$ and $\langle \ell(M_1), \dots, \ell(M_r) \rangle$ are characteristic subgroups.

Proof (a) Let θ be an automorphism of G and $\alpha \in \ell(H)$.

$$\begin{aligned} \text{Then } 1 &= \theta(1) \\ &= \theta([\alpha, \xi]) \text{ for all } \xi \in H \\ &= [\theta(\alpha), \theta(\xi)] \text{ since } \theta \text{ is an automorphism} \\ &= [\theta(\alpha), \eta], \text{ some } \eta \in H \end{aligned}$$

i.e. $\theta(\alpha) \in \ell(H)$ and so $\ell(H)$ is a characteristic subgroup.

(b) Let θ be an automorphism and $\alpha \in \ell(M_i)$

$$\begin{aligned} \text{Then } E = \theta(E) &= \theta[\alpha, M_i] \\ &= [\theta(\alpha), \theta(M_i)] \end{aligned}$$

Since θ is an automorphism it must permute the maximal subgroups and so $\theta(M_i) = M_j$ and

$$\{\theta(\alpha) \mid \alpha \in \ell(M_i)\} = \ell(M_j) \text{ for some } j \text{ and result follows.}$$

Lemma L7 Let $\left(\frac{a}{p}\right)$ be the Legendre symbol and p be an odd prime.

$$\text{Then, } \sum_{x=0}^{p-1} \left(\frac{x^2 - d}{p} \right) = 1 \text{ all } d \neq 0 \pmod{p}$$

$$\text{Proof - For convenience we will write } \sum_{x=0}^{p-1} \left(\frac{x^2 - d}{p} \right) = \Sigma(d)$$

$$\begin{aligned}
\text{Then } \Sigma(r d) &= \sum_{x=0}^{p-1} \left(\frac{x^2 - r^2 d}{p} \right) \\
&= \sum_{y=0}^{p-1} \left(\frac{r^2 y^2 - r^2 d}{p} \right) && \text{where } x = ry \\
&= \left(\frac{r^2}{p} \right) \sum_{y=0}^{p-1} \left(\frac{y^2 - d}{p} \right) \\
&= \Sigma(d)
\end{aligned}$$

Thus $\Sigma(d) = \Sigma(1)$ if d is a quadratic residue and $\Sigma(d) = \Sigma(v)$ if d is a quadratic non-residue.

$$\begin{aligned}
\text{Now } \Sigma(1) &= \sum_{x=0}^{p-1} \left(\frac{x^2 - 1}{p} \right) \\
&= \sum_{y=1}^{p-1} \left(\frac{y(y+2)}{p} \right) && \text{with } y = x-1 \\
&= \left(\frac{\bar{y}}{p} \right)^2 \sum_{y=1}^{p-1} \left(\frac{y(y+2)}{p} \right) && \text{where } y\bar{y} \equiv 1 \pmod{p} \\
&= \sum_{\bar{y}=1}^{p-1} \left(\frac{y\bar{y}(y\bar{y} + 2\bar{y})}{p} \right) \\
&= \sum_{\bar{y}=0}^{p-1} \left(\frac{1 + 2\bar{y}}{p} \right) - \left(\frac{1}{p} \right)
\end{aligned}$$

But, as $x-1$ runs through a set of non-zero residues mod p ,
 y (and so $\bar{y}$) runs through a set of non-zero residues mod p .

$$\begin{aligned}\text{Hence } \Sigma(1) &= 0 - \left(\frac{1}{p}\right) \\ &= -1.\end{aligned}$$

Finally, (since $\Sigma(0) = p-1$),

$$\begin{aligned}p-1 + \frac{1}{2}(p-1)\{\Sigma(1) + \Sigma(v)\} &= \sum_{d=0}^{p-1} \Sigma(d) \\ &= \sum_{d=0}^{p-1} \sum_{x=0}^{p-1} \left(\frac{x^2 - d}{p} \right) \\ &= \sum_{x=0}^{p-1} \sum_{d=0}^{p-1} \left(\frac{x^2 - d}{p} \right) \\ &= 0\end{aligned}$$

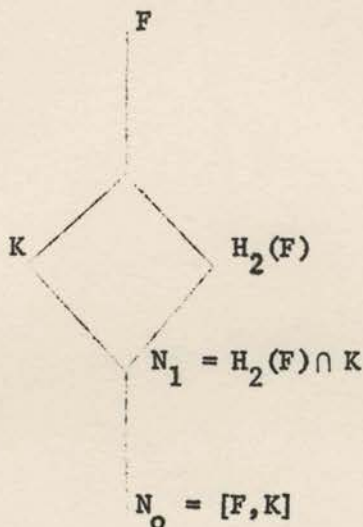
(since, as d runs through a set of residues mod p , so does x^2-d
for each value of x).

Solving this, we get $\Sigma(v) = \Sigma(1) = -1$

i.e. for all $d \neq 0$, $\Sigma(d) = -1$.

II THE FAMILIES OF p-GROUPS (F RANK ≤ 6 ($p > 3$))

1. Let H be an arbitrary group, F a free group and $K \triangleleft F$ such that $F/K \cong H$. We thus have the following diagram:-



We say that H is capable if there exists a group G with centre Z such that $G/Z \cong H$. We may then show:-

²Lemma 1 If H is a capable group and $G/Z \cong H$, then (using the above notation) $G \sim F/N$ if and only if

$$(a) \quad N_0 \leq N \leq N_1$$

$$(b) \quad K/N = Z(F/N)$$

In particular, N_0 may be chosen and is in fact a minimal candidate for such a choice. We call the family containing F/N_0 (which depends only on H) the maximal family for H . Thus

Corollary Every capable group has a maximal family.

Let φ be an automorphism of H . Since $H \cong F/K$, we may consider φ as an automorphism of F/K . With this convention, we may write

²Lemma 2 The outer automorphisms φ of F/K (or H) induce automorphisms $\bar{\varphi}$ of N_1/N_0 in an obvious way

$$(i.e. \bar{\varphi}([\alpha, \beta]N_0) = [\varphi(\alpha K), \varphi(\beta K)])$$

The subgroups N/N_0 of N_1/N_0 (where N satisfies lemma 1) are permuted by the $\bar{\varphi}$'s and $F/N \sim F/N'$ if and only if N and N' are in the same transitive set. The pair $(\varphi, \bar{\varphi})$ is called an autologism of H .

In the case of p -groups, since $|Z| < |G|$, the problem of finding families of rank at most r reduces to finding the maximal families for all the capable groups of order less than p^r , the Schur Multiplier N_1/N_0 of that family and all the subgroups N of N_1 satisfying lemma 1 and then using lemma 2. However, in practice, this method proves to be very unwieldy and so various techniques are used to simplify calculations. The first is to consider which groups can be capable.

²Lemma 3 The group H is incapable if H contains an element $\beta (\neq 1)$ with $\beta \in \langle \alpha_i \rangle$ for each generator α_i of H .

²The proof of Lemmas 1, 2, 3 may be found in (3).

Lemma 4 Let H be a p -group and suppose we can choose generators $\alpha, \alpha_1, \dots, \alpha_r$ such that $|\alpha| > |\alpha_i|$, $\exp H_2(H)$ and $\langle \alpha, \alpha_i \rangle$ is regular for all $i = 1, 2, \dots, r$.

Then H is incapable.

Proof - Let $|\alpha_i| = h_i = p^{m_i}$, $|\alpha| = h = p^m$, $m > m_1 \geq m_2 \dots$

Writing $\alpha'_i = \alpha \alpha_i$ and $h' = \max(h_1, \exp H_2(H))$, we have

$$H = \langle \alpha, \alpha'_1, \dots, \alpha'_r \rangle \quad \text{where}$$

$$\begin{aligned} \alpha_i^{h'} &= (\alpha \alpha_i)^{h'} \\ &= \alpha^{h'} \alpha_i^{h' \gamma_i^{h'}} \end{aligned}$$

with $\gamma_i \in$ derived group of $\langle \alpha, \alpha_i \rangle$.

Since $h_i, \exp H_2(H)$ divide h' , we have

$$\alpha_i^{h'} = \alpha^{h'} = \beta \neq 1 \quad (\text{since } h' < h)$$

Hence, by lemma 3, H is incapable.

Corollary 1 If H is regular with type invariants $m_1 \geq m_2 \geq \dots$, then H is incapable if $m_1 \neq m_2$.

Corollary 2 An abelian p -group is capable if and only if its two largest type invariants are equal.

Proof - Let $H = \langle \alpha_1 \rangle \times \langle \alpha_2 \rangle \times \dots \times \langle \alpha_r \rangle$

where $|\alpha_i| = h_i$, $h_{i+1} | h_i$

If $h_1 \neq h_2$, H is incapable by lemma 4.

If $h_1 = h_2$, define a group G by

$$G = \langle \beta_1, \dots, \beta_r \mid [\beta_i, \beta_j] = 1, \beta_i^{h_j} = \beta_j, [\beta_i, \beta_j] \in Z(G) \ (i > j) \rangle$$

Then $Z(G) = \langle [\beta_i, \beta_j] \mid i, j = 1, 2, \dots, r \rangle$
 and $H \cong G/Z(G)$.

We also make use of a theorem by P. Hall (5) to determine a large class of families:

Lemma 5 (P. Hall) There is a one to one correspondence between partitions $m = \sum m_i$ and families of groups $\mathcal{A}_m$ with maximal abelian subgroups. If G is a stem group of the family $\mathcal{A}_m$ with lower central series $G = H_0 > H_2 > \dots > H_c = E$, then:

$$(i) \quad |H_i : H_{i+1}| = p^{\ell_{i-1}} \quad (i = 2, 3, \dots, c-1)$$

$$(ii) \quad |G| = p^{m+\ell_1+1}, \quad |H_2| = p^m$$

$$(iii) \quad c = m_1 + 1$$

where $(\ell_1, \ell_2, \dots, \ell_{c-2})$ is the conjugate partition of $(m_1, m_2, \dots)$. Thus the families of groups with maximal abelian subgroups and rank ≤ 6 are:

$$\mathcal{A}_{(0)} = \Phi_1, \mathcal{A}_{(1)} = \Phi_2, \mathcal{A}_{(2)} = \Phi_3, \mathcal{A}_{(11)} = \Phi_4, \mathcal{A}_{(3)} = \Phi_9,$$

$$\mathcal{A}_{(21)} = \Phi_{16}, \mathcal{A}_{(4)} = \Phi_{35}.$$

Theorem 1 Let H be the regular group $\mathcal{A}_{(n)} (1^{n+2})$ of maximal class, exponent $p > n+1$ and having a maximal abelian subgroup. Then

(i) If $n = 0$, the maximal family Φ_2 is of rank 3 and so there are no other families.

(ii) If $n = 1$, the maximal family Φ_6 is of rank 5 and there is one family Φ_3 of rank 4.

(iii) If $n > 1$, the maximal family is of rank $2n+2$ with two other families of rank 5 for $n = 2$ and three other families of rank 6 for $n = 3$.

Proof - Let $H = \langle \bar{\alpha}, \bar{\alpha}_1, \dots, \bar{\alpha}_{n+1} \rangle$ where $\bar{\alpha}_{i+1} = [\bar{\alpha}_i, \bar{\alpha}]$,

$$[\bar{\alpha}_i, \bar{\alpha}_j] = 1 \quad (i, j = 1, 2, \dots, n+1), \quad [\bar{\alpha}_{n+1}, \bar{\alpha}] = 1.$$

Then $G = \langle \alpha, \alpha_1, \dots, \alpha_{n+1}, Z(G) \rangle$ corresponds to $\bar{\alpha}_1$.

$$\text{Put } [\alpha_{n+1}, \alpha] = \alpha_{n+2}, \quad [\alpha_1, \alpha_i] = \xi_i \quad (i = 1, 2, \dots, n+1)$$

Clearly $[\alpha_j, \alpha_{n+1}] = 1$ for all $j \geq 2$.

(a) We first show that the Schur Multiplier is generated by

$$\alpha_{n+2} \text{ and } \xi_1, \xi_2, \dots, \xi_{n+1}.$$

$$\text{i.e. } [\alpha_i, \alpha_j] \in \langle \xi_1, \xi_2, \dots, \xi_{n+1}, \alpha_{n+2} \rangle = K \quad \text{all } 1 \leq i, j \leq n+1.$$

Without loss of generality, we may suppose $i < j$.

The proof is by induction on i (increasing) and j (decreasing).

$$\text{Clearly } [\alpha_1, \alpha_j] = \xi_j \in K, \quad [\alpha_i, \alpha_{n+1}] \in \langle \alpha_{n+2} \rangle \leq K$$

We assume $[\alpha_r, \alpha_s], [\alpha_i, \alpha_s], [\alpha_r, \alpha_j] \in K$ for all $r < i$ and $s > j$.

$$\text{Then } [\alpha_i, \alpha_j] = \alpha_i^{-1} \alpha_i^{\alpha_j}$$

$$= \alpha_i^{-1} [\alpha_{i-1}, \alpha]^{\alpha_j}$$

$$= \alpha_i^{-1} [\alpha_{i-1}, \alpha^{\alpha_j}] \text{ since } [\alpha_{i-1}, \alpha_j] \in K \leq Z(G)$$

$$\begin{aligned}
&= \alpha_i^{-1} [\alpha_{i-1}, \alpha_{j+1}^{-1}] \\
&= \alpha_i^{-1} [\alpha_{i-1}, \alpha_{j+1}^{-1}] [\alpha_{i-1}, \alpha] \alpha_{j+1}^{-1} \quad \text{by lemma L1} \\
&= \alpha_i^{-1} [\alpha_{i-1}, \alpha_{j+1}^{-1}] \alpha_{j+1} \alpha_i \alpha_{j+1}^{-1} \\
&= [\alpha_{i-1}, \alpha_{j+1}]^{-1} [\alpha_i, \alpha_{j+1}]^{-1} \text{ since } K \leq Z(G)
\end{aligned}$$

$\in K$ by induction hypothesis.

Thus $[\alpha_i, \alpha_j] \in K$ for all $1 \leq i, j \leq n+1$.

In fact, we note that, when $n \geq 2$,

$$\begin{aligned}
\xi_3 &= [\alpha_1, \alpha_2] \\
&= [\alpha_2, \alpha_2] [\alpha_2, \alpha_3] \quad \text{by the above argument} \\
&= [\alpha_2, \alpha_3]
\end{aligned}$$

$$\begin{aligned}
\text{and } [\alpha_2, \alpha_3] &= [\alpha_1, \alpha_4]^{-1} [\alpha_2, \alpha_4]^{-1} \\
&= \xi_4^{-1} [\alpha_2, \alpha_5] [\alpha_1, \alpha_5]
\end{aligned}$$

$$\in \langle \xi_4, \xi_5, \dots, \xi_{n+1} \rangle$$

Thus $K = \langle \alpha_4, \xi_2 \rangle$ for $n = 2$ and $\langle \alpha_4, \xi_2, \xi_4, \dots \rangle$ for $n > 2$.

(b) We now show that the Schur Multiplier is elementary abelian and has order p (for $n = 0$), p^2 (for $n = 1$) and p^n (for $n > 1$).

By lemma L4 (since H has exponent p),

$$\begin{aligned}\alpha_{n+2}^p &= [\alpha_{n+1}, \alpha]^p = [\alpha_{n+1}^p \alpha] = 1 \\ \xi_i^p &= [\alpha_1, \alpha_i]^p = [\alpha_1, \alpha_i^p] = 1\end{aligned}$$

K is clearly abelian and so elementary abelian.

If $n = 0$, $K = \langle \alpha_2 \rangle$ and so $|K| = p$

If $n = 1$, $K = \langle \alpha_3, \xi_2 \rangle$ and so $|K| = p^2$

If $n = 2$, $K = \langle \alpha_4, \xi_2 \rangle$ and, since $\langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \xi_2 \rangle$

is the split extension of $\langle \alpha, \alpha_2, \alpha_3, \alpha_4 \rangle \times \langle \xi_2 \rangle = \mathcal{A}_{(2)}(1^5)$

by an element α_1 of order p which induces the automorphism

$$\{\alpha \rightarrow \alpha \alpha_2^{-1}, \alpha_2 \rightarrow \alpha_2 \xi_2^{-1}\}, \text{ then}$$

$\langle \alpha, \alpha_1, \alpha_2, \alpha_3, K \rangle$ has order p^6 and so $|K| = p^2$

Similarly, (by induction on $n > 2$) it may be seen that

$\langle \alpha, \alpha_1, \alpha_2, \dots, \alpha_{n+1}, K \rangle$ is the split extension of

$\langle \alpha, \alpha_2, \dots, \alpha_{n+2}, \xi_4, \dots, \xi_{n+1} \rangle \times \langle \xi_2 \rangle$ by an element α_1

of order p which induces the automorphism

$$\{\alpha \rightarrow \alpha \alpha_2^{-1}, \alpha_2 \rightarrow \alpha_2 \xi_2^{-1}, \alpha_i \rightarrow \alpha_i \xi_i^{-1}\} \quad (3 < i < n+2)$$

where we may assume (by induction) that

$\langle \alpha, \alpha_2, \dots, \alpha_{n+2}, \xi_4, \dots, \xi_{n+1} \rangle$ is a group of order p^{2n} .

Thus, for $n > 2$, $|K| = p^n$

(c) For $n = 1$, the Schur Multiplier K is $\langle \alpha_3, \xi_2 \rangle$ and the maximal family is Φ_6 .

Also, since the autologisms

$$\left\{ \begin{array}{l} \bar{\alpha} \rightarrow \bar{\alpha} \bar{\alpha}_1^r \\ \alpha_3 \rightarrow \alpha_3 \xi_2^{-r} \end{array} \right. \quad \text{and} \quad \begin{array}{l} \bar{\alpha} \leftrightarrow \bar{\alpha}_1 \\ \alpha_3 \leftrightarrow \xi_2 \end{array}$$

permute the proper subgroups of K transitively, there is just one family of rank 4.

(d) For $n = 2$, the Schur Multiplier is $\langle \alpha_4, \xi_2 \rangle$ and the maximal family is Φ_{23} .

For subgroups N satisfying lemma 1, we must have $\alpha_4 \notin N$ (otherwise α_3 would be in the centre of G). Thus the only possibilities are $\langle \alpha_4^r \xi_2 \rangle$ ($r = 0, 1, \dots, p-1$) and the autologisms are generated by

$$\begin{cases} \bar{\alpha} \rightarrow \bar{\alpha} \bar{\alpha}_1 \\ \text{identity} \end{cases} \quad \begin{cases} \bar{\alpha} \rightarrow \bar{\alpha}^2 \\ \alpha_4 \rightarrow \alpha_4^8 \end{cases} \quad \text{and} \quad \begin{cases} \bar{\alpha}_1 \rightarrow \bar{\alpha}_1^2 \\ \xi_2 \rightarrow \xi_2^4 \end{cases}$$

Thus we have two families of rank 5: Φ_9 when $r = 0$ and Φ_{10} when $r \neq 0$.

(e) For $n = 3$, the Schur Multiplier N_1 is $\langle \alpha_5, \xi_2, \xi_4 \rangle$. As in (d), we must have α_5 or $\xi_4 \notin N$ for subgroups N satisfying lemma 1. Thus the possible subgroups of index p are

$\langle \xi_2, \xi_4 \rangle$, $\langle \xi_2, \alpha_5, \xi_4^r \rangle$ ($r = 1, 2, \dots, p-1$) and $\langle \alpha_5 \xi_4^s, \xi_2 \xi_4^t \rangle$ ($s, t = 0, 1, \dots, p-1$). The autologisms of H are generated by

$$\begin{cases} \bar{\alpha} \rightarrow \bar{\alpha} \bar{\alpha}_1 \\ \alpha_5 \rightarrow \alpha_5 \xi_4^{-1} \end{cases}, \quad \begin{cases} \bar{\alpha}_i \rightarrow \bar{\alpha}_i \bar{\alpha}_{i+1} \\ \xi_2 \rightarrow \xi_2 \xi_4^{-1} \end{cases}, \quad \begin{cases} \bar{\alpha} \rightarrow \bar{\alpha}_2 \\ (\xi_2, \xi_4, \alpha_5) \rightarrow (\xi_2^2, \xi_4^8, \alpha_5^{16}) \end{cases},$$

$$\begin{cases} \bar{\alpha}_i \rightarrow \bar{\alpha}_i^2 \\ (\xi_2, \xi_4, \alpha_4) \rightarrow (\xi_2^4, \xi_4^4, \alpha_5^2) \end{cases} \quad (i = 1, 2, 3, 4) \quad \text{and} \quad \text{autologisms}$$

which reduce to the identity on N_1 . Thus we have three families of rank 6 : Φ_{35} , Φ_{36} and Φ_{37} .

By lemma 4, we can see that the above theorem gives all the families of rank < 5 and all the families of rank 5 and maximal class.

Theorem 2 Let H be the group $\mathcal{A}_{(n)}(1^{n+3})$, the direct product of the group in theorem 1 with a group of order p .

Then (i) If $n = 0$, the maximal family Φ_{11} is of rank 6 and there is one family Φ_4 of rank 5.

(ii) If $n = 1$, the maximal family is of rank 8 and there are six families (Φ_{16} to Φ_{21}) of rank 6 and one family Φ_7 of rank 5.

(iii) If $n > 1$, the maximal family is of rank $2n+5$, there being two families of rank 6 when $n = 2$.

Proof - Let $H = \langle \bar{\alpha}, \bar{\alpha}_1, \dots, \bar{\alpha}_{n+1} \rangle \times \langle \bar{\beta} \rangle$ where $\bar{\alpha}, \bar{\alpha}_i$

are as in theorem 1 and $\bar{\beta}^p = 1$.

Write $G = \langle \alpha, \alpha_1, \dots, \alpha_{n+1}, \beta, Z(G) \rangle$ where α, α_i, β correspond to $\bar{\alpha}, \bar{\alpha}_i, \bar{\beta}$ respectively ($i = 1, 2, \dots, n+1$).

Put $\alpha_{n+2} = [\alpha_{n+1}, \alpha]$, $[\alpha_1, \alpha_i] = \xi_i$,

$[\beta, \alpha_i] = \eta_i$ ($i = 1, 2, \dots, n+1$), $[\beta, \alpha] = \eta$.

Then, as in theorem 1, we may show that the Schur Multiplier is $\langle \alpha_{n+2}, \eta, \xi_2, \xi_4, \dots, \xi_{n+1}, \eta_1, \eta_2, \dots, \eta_{n+1} \rangle$.

However in this case we have (for $i > 1$)

$$\begin{aligned} [\beta, \alpha_i] &= \alpha_i^{-\beta} \alpha_i \\ &= [\alpha, \alpha_{i-1}]^{\beta} \alpha_i \\ &= [\alpha, \alpha_{i-1}] \alpha_i \text{ since } [\alpha, \beta], [\alpha_{i-1}, \beta] \in Z \\ &= 1 \end{aligned}$$

So Schur Multiplier is $\langle \alpha_{n+2}, \eta, \eta_1, \xi_2, \xi_4, \dots, \xi_{n+1} \rangle$.

As in theorem 1, we may show that this is elementary Abelian and the above forms a minimal basis. Thus, for $n > 1$, the maximal family is of rank $2n+5$.

(i) For $n = 0$, the Schur Multiplier N_1 is $\langle \alpha_2, \eta, \eta_1 \rangle$ of order p^3 . Since $N_1 \cong H$, the automorphisms of H are faithfully represented as automorphisms of N_1 . Thus all subgroups of N_1 with a given order and satisfying lemma 1 give rise to the same family.

Also, we must have at least two of α_2, η, η_1 not in such a subgroup in order to satisfy lemma 1, and so the only families are the maximal family $\Phi_{1,1}$ and one other family Φ_4 of rank 4.

(ii) For $n = 1$, the Schur Multiplier N_1 is $\langle \alpha_3, \xi_2, \eta, \eta_1 \rangle$ and so the maximal family is of rank 8.

For subgroups N satisfying lemma 1 we must have either $\alpha_3 \notin N$ or $\xi_2 \notin N$ (otherwise α_2 would be in the centre of G) and similarly η or $\eta_1 \notin N$. Also, by interchanging the roles of $\bar{\alpha}$ and $\bar{\alpha}_1$ (if necessary), we may suppose that $\alpha_3 \notin N$. Thus we may suppose that all the required subgroups of index p are of the form

$$N = \langle \alpha_3^r \xi_2, \alpha_3^s \eta, \alpha_3^t \eta_1 \rangle \quad (r, s, t, = 0, 1, \dots, p-1, s \text{ or } t \neq 0).$$

The autologisms which permute the subgroups of this form include

$$\left\{ \begin{array}{l} \bar{\alpha}_1 \rightarrow \bar{\alpha}\bar{\alpha}_1 \\ (\xi_2, \eta_1) \rightarrow (\alpha_3^{-1}\xi_2, \eta\eta_1) \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} \bar{\beta} \rightarrow \bar{\beta}\bar{\alpha}_2 \\ (\eta, \eta_1) \rightarrow (\alpha_3^{-1}\eta, \xi_2\eta_1) \end{array} \right.$$

Thus there is only one family of rank 5 and for this family we may suppose $[\alpha_2, \alpha] = [\alpha_1, \beta] = \alpha_3$, $[\alpha_1, \alpha_2] = [\alpha, \beta] = 1$.

For families of rank 6 the above argument becomes quite cumbersome and we use another method.

Let G be a stem group of one of these families. Since G is a stem group and of class 3,

$$H_3(G) \leq Z_1(G) < H_2(G) < Z_2(G) < G$$

$$\begin{array}{c} G \\ | \\ p^2 \\ \times \\ Z_2 \\ | \\ p \\ \times \\ H_2 \\ | \\ p \\ \times \\ Z_1 \\ | \\ p^2 \end{array}$$

Also, since $H \cong G/Z(G)$, we have

$$\begin{aligned} Z(H) &= Z(G/Z(G)) \\ &= Z_2(G)/Z(G) \end{aligned}$$

$$\begin{aligned} \text{Thus } |G : Z_2(G)| &= |G : Z(G)| \div |Z_2(G) : Z(G)| \\ &= |H| \div |Z(H)| \\ &= |H : Z(H)| \\ &= p^2 \end{aligned}$$

Similarly

$$|Z_2(G) : Z(G)| = p^2$$

Also, since $Z_2 > H_2 > Z_1$, we must have

$$|Z_2(G) : H_2(G)| = |H_2(G) : Z_1(G)| = p.$$

Finally, $H_3(G) \leq Z_1(G)$ and so $|H_3(G)| = p$ or p^2 .

(a) If $|H_3(G)| = p$, we may suppose (as in the above) that

$$H_3(G) = \langle \alpha_3 \rangle \text{ and so } \xi_2 \in \langle \alpha_3 \rangle.$$

$$\text{i.e. } \xi_2 = \alpha_3^a$$

By the autologism $\begin{cases} \bar{\alpha}_1 \rightarrow \bar{\alpha}^a \bar{\alpha}_1 \\ \xi_2 \rightarrow \alpha_3^{-a} \xi_2 \end{cases}$ we may suppose

$$\text{that } [\alpha_1, \alpha_2] = 1.$$

If $[\alpha_1, \beta] = 1$, then $\langle \alpha_1, \beta, \alpha_2, Z(G) \rangle$ is a maximal Abelian subgroup of G and so $G \in \Phi_{16}$.

If $[\alpha_1, \beta] \neq 1$, $\in H_3(G)$, then $[\alpha_1, \beta] = \alpha_3^b$ ($b \neq 0$) and so (since $Z(G) > H_3(G)$) $[\alpha, \beta] = \eta \notin H_3(G)$. Also, by the

autologism $\begin{cases} \bar{\beta} \rightarrow \bar{\beta}^b \\ \eta \rightarrow \eta^b \end{cases}$ (where $b\bar{b} \equiv 1 \pmod{p}$), we

may suppose $b = 1$ giving the family Φ_{18} .

If $[\alpha_1, \beta] \notin H_3(G)$, we have $\eta_1 \notin H_3(G)$ and so

$$Z_1(G) = H_3(G) \times \langle \eta_1 \rangle = \langle \alpha_3, \eta_1 \rangle$$

Thus $\eta = \alpha_3^c \eta_1^d$ and, by the autologism

$\begin{cases} \bar{\beta} \rightarrow \bar{\alpha}_2^{-d} \bar{\beta}, \quad \bar{\alpha} \rightarrow \bar{\alpha} \bar{\alpha}_1^{-c} \\ \eta \rightarrow \eta \alpha_3^{-c} \eta_1^{-d} \end{cases}$ we may suppose

$$[\alpha, \beta] = 1, \text{ giving } \Phi_{17}.$$

(b) Suppose $|H_3(G)| = p^2$ and so $H_3(G) = Z(G) = \langle \alpha_3, \xi_2 \rangle$.

Since $\eta, \eta_1 \in Z(G)$, $\eta = \alpha_3^d \xi_2^a$, $\eta_1 = \alpha_3^b \xi_2^c$

where $\eta \neq 1$ or $\eta_1 \neq 1$.

By $\begin{cases} \bar{\beta} \rightarrow \bar{\beta} \bar{\alpha}_2^c \\ (\eta, \eta_1) \rightarrow (\eta \alpha_3^c, \eta_1 \xi_2^{-c}) \end{cases}$ we may suppose $c = 0$, and,

by replacing $\bar{\beta}$ by $\bar{\alpha}_2^{-\frac{1}{2}d}$, $\bar{\alpha}$ by $\bar{\alpha}^x \bar{\alpha}_1^y$ and $\bar{\alpha}_1$ by $\bar{\alpha}^z \bar{\alpha}_1^w$ where $xza + ywb + \frac{1}{2}(xw+yz) d = 0$, $xw \neq yz$, we may suppose $d = 0$.

If $a = 0$, then $b \neq 0$, and by

$$\begin{cases} (\bar{\alpha}, \bar{\beta}) \rightarrow (\bar{\alpha}^b, \bar{\beta}^b) \\ (\alpha_3, \eta_1) \rightarrow (\alpha_3^{b^2}, \eta_1^b) \end{cases} \quad \text{we may suppose } b = 1 \text{ giving } \Phi_{20}.$$

If $b = 0$, then $a \neq 0$, and by interchanging $\bar{\alpha}$ and $\bar{\alpha}_1$ we obtain the above result.

If $a, b \neq 0$ and $a = -r^2 b$ (some $r \neq 0$), then by

$$\begin{cases} (\bar{\alpha}, \bar{\alpha}_1, \bar{\beta}) \rightarrow (\bar{\alpha}_1^{-r\bar{\alpha}}, \bar{\alpha}_1^{-rb\bar{\alpha}-b}, \bar{\beta} \bar{\alpha}_2^{rb}) \\ (\xi_2, \eta, \eta_1) \rightarrow (\xi_2^{-2ab} \alpha_3^{-2rb^2}, 1, \xi_2^{-2ab} \alpha_3^{-2rb^2}) \end{cases}$$

we may suppose $\eta=1$, $\eta_1=\xi_2$ giving Φ_{19} .

If $a, b \neq 0$ and $a = -vr^2 b$ (some $r \neq 0$), then by

$$\begin{cases} (\bar{\alpha}, \bar{\alpha}_1, \bar{\beta}) \rightarrow (\bar{\alpha}^b, \bar{\alpha}_1^{rb}, \bar{\beta}^b) \\ (\alpha_3, \xi_2, \eta, \eta_1) \rightarrow (\alpha_3^{rb^3}, \xi_2^{r^2 b^3}, \eta^{b^2}, \eta_1^{rb^2}) \end{cases}$$

we may suppose $\eta = \xi_2^{-v}$, $\eta_1 = \alpha_3$ giving Φ_{21} .

Finally, the most general automorphism of H is

$$\bar{\alpha}^* = \bar{\alpha}^x \bar{\alpha}_1^{x_1} \bar{\alpha}_2^{x_2} \bar{\beta}^{x_3}$$

$$\bar{\alpha}_1^* = \bar{\alpha}^y \bar{\alpha}_1^{y_1} \bar{\alpha}_2^{y_2} \bar{\beta}^{y_3}$$

$$\bar{\beta}^* = \bar{\beta}^z \bar{\alpha}_2^{z'} \quad z \neq 0$$

$$\bar{\alpha}_2^* = \alpha_2^\Delta \quad \Delta = xy_1 - x_1 y \neq 0$$

$$\begin{aligned}
\text{Inducing in } N, \alpha_3^* &= \alpha_3^{x\Delta} \xi_2^{-\Delta x_1} \\
\xi_2^* &= \alpha_3^{-y\Delta} \xi_2^{y_1\Delta} \\
\eta^* &= \alpha_3^{xz'} \xi_2^{-x_1 z'} \eta^{xz} \eta_1^{x_1 z} \\
\eta_1^* &= \alpha_3^{yz'} \xi_2^{-y_1 z'} \eta^{yz} \eta_1^{y_1 z}
\end{aligned}$$

By substituting each of the above values for η, η_1 , we may show that it is impossible to obtain the other values, showing that the above result yields three distinct families.

(iii) For $n = 2$, the Schur Multiplier is

$$N_1 = \langle \alpha_4, \eta, \eta_1, \xi_2 \rangle$$

and we seek the subgroups N of index p and satisfying lemma 1.

As in theorem 1, we must take $\alpha_4 \notin N$ and so

$$N = \langle \alpha_4^r \eta, \alpha_4^s \eta_1, \alpha_4^t \xi_2 \rangle \quad (r, t = 0, 1, \dots, p-1; s = 1, 2, \dots, p-1)$$

By replacing $\bar{\beta}$ by $\bar{\alpha}^{-r} \bar{\beta}$ we may suppose $r = 0$ and so

$N = \langle \eta, \alpha_4^s \eta_1, \alpha_4^t \xi_2 \rangle$. The other automorphisms of H are generated by $\{\bar{\alpha} \rightarrow \bar{\alpha} \bar{\alpha}_i\}$, $\{\bar{\alpha}_1 \rightarrow \bar{\alpha}_1 \bar{\alpha}_j\}$, $\{\bar{\beta} \rightarrow \bar{\beta} \bar{\alpha}_j\}$

($i = 1, 2, 3$; $j = 2, 3$ - all inducing the identity isomorphism on N) and the autologisms

$$\begin{aligned}
&\left\{ \begin{array}{l} \bar{\alpha} \rightarrow \bar{\alpha}^2 \\ (\alpha_4, \xi_2, \eta) \rightarrow (\alpha_4^8, \xi_2^2, \eta^2) \end{array} \right\} \left\{ \begin{array}{l} \bar{\alpha}_1 \rightarrow \bar{\alpha}_1^2 \\ (\alpha_4, \xi_2, \eta_1) \rightarrow (\alpha_4^2, \xi_2^4, \eta_1^2) \end{array} \right\}, \\
&\left\{ \begin{array}{l} \bar{\beta} \rightarrow \bar{\beta}^2 \\ (\eta, \eta_1) \rightarrow (\eta^2, \eta_1^2) \end{array} \right\}.
\end{aligned}$$

Thus there are two families, corresponding to $t = 0$ (Φ_{24})

and $t \neq 0$ (Φ_{27}).

In Φ_{24} we may suppose $[\alpha_1, \beta] = \alpha_4$, and in Φ_{27} we may suppose $[\alpha_1, \beta] = [\alpha_1, \alpha_2] = \alpha_4$. Q.E.D.

By considering the groups of order p^4 in the light of lemma 4, we see that the only other capable groups of that order are (1^4) , (22) and possibly $\Phi_2(22)$. The first shall be left until later, but the other two may be included in the following:

Theorem 3 Let $H = \langle \bar{\alpha}_1, \bar{\alpha} \mid \bar{\alpha}_1^{p^n} = \bar{\alpha}^{p^n} = 1, [\bar{\alpha}_1, \bar{\alpha}] = \bar{\alpha}_1^{p^r} \rangle$

where $n > 0$, $r \mid n$, $p > n/r$. Then the Schur Multiplier is (r) and the only family is the maximal one of rank $2n+r$.

Proof - Suppose $G = \langle \alpha_1, \alpha, Z(G) \rangle$ where $G/Z(G) \cong H$.

Writing $[\alpha_i, \alpha] = \alpha_{i+1}$ ($i=1, 2, \dots$) we have

$$\alpha_3 = [\alpha_1, \alpha, \alpha] = [\alpha_1^{p^r}, \alpha] = \alpha_2^{p^r} \text{ by lemma L4}$$

$$\text{Similarly } \alpha_{i+2} = [\alpha_1, \alpha]^{p^{ri}} = \alpha_2^{p^{ri}} \quad (i = 1, 2, \dots)$$

(Note that $[\alpha_1, H_2(G)] = 1$)

Also $\alpha_i \in Z(G)$ if and only if $[\alpha_i, \alpha] = \alpha_{i+1} = 1$, and so the Schur Multiplier is $\langle \alpha_{t+1} \rangle$ where $t = n/r$.

$$\text{Now } \alpha_{t+1}^{p^s} = \alpha_2^{p^{rt-r+s}} = \alpha_2^{p^{n-r+s}} \text{ and so } |\alpha_{t+1}| = p^r.$$

Thus the Schur Multiplier is a cyclic group of order p^r and its proper subgroups do not satisfy lemma 1. Finally, G is the extension of the group $\langle \alpha, \alpha_2 \rangle$ by the automorphism $\{\alpha \rightarrow \alpha \alpha_2^{-1}\}$ of order p^n , and so (by induction) is of rank $2n+r$. Q.E.D.

Theorem 3 yields $\Phi_2(n=r=1)$, $\Phi_3(n=2, r=1)$ and $\Phi_{14}(n=r=2)$.

Theorem 4 Let G be a group of class 2 with $|Z(G)| = p$ and $|G:Z(G)| > p^2$. Then G is the central product of several copies of $\Phi_2(111)$ together with at most one copy of $\Phi_2(21)$.

Proof - Since G is a stem group of class 2, $Z(G) = H_2(G)$. Let ξ be an element of $G - Z(G)$ of least order, $X = \langle \xi \rangle, \eta$ be an element of $G - X$ of least order and $Y = \langle \eta \rangle$. Since $[\xi, \eta] \neq 1$, we may write $\langle \gamma \rangle = \langle [\xi, \eta] \rangle = H_2(G) = Z(G)$. If $\zeta \notin X$, then $[\xi, \zeta] = \gamma^a$ for some $a \neq 0$ and so

$$[\xi, \eta^{-a}\zeta] = [\xi, \eta]^{-a} [\xi, \zeta] = 1.$$

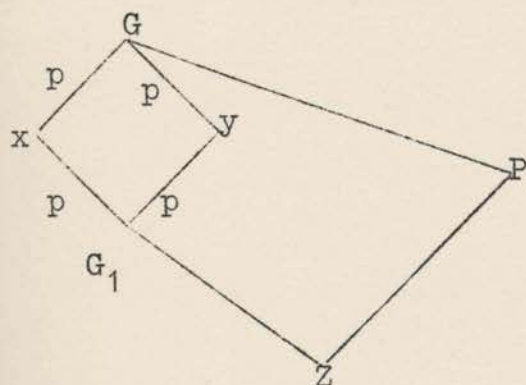
$$\text{i.e. } \zeta \in \eta^a X.$$

Also $[\xi, \eta^p] = [\xi, \eta]^p = 1$ and so $\eta^p \in X$.

Thus $G = X + \eta X + \dots + \eta^{p-1}X$, and so $|G:X| = p$.

Similarly, $|G:Y| = p$.

Since X, Y are maximal subgroups, $XY = G$



and $XY = G_1$ is of index p^2

Writing $P = \langle \xi, \eta \rangle$ we have

$\zeta \in P \cap G_1$ if and only if

$\zeta \equiv \xi^x \eta^y \gamma^z$ and $[\xi, \zeta] = [\eta, \zeta] = 1$

i.e. $[\xi, \eta^y] = [\eta, \xi^x] = 1$

and so $p|y, p|x$

Thus $|P:P \cap G_1| = p^2 = |G:G_1|$

and so $G = G_1 P$ (since $G_1, P \triangleleft G$).

In fact, if $\zeta \in P \cap G_1$ then $\zeta = \xi^{px} \eta^{py} \gamma^z$ and so, for all $\omega \in G$,

$$[\zeta, \omega] = [\xi, \omega]^{px} [\eta, \omega]^{py} = 1$$

i.e. $\zeta \in Z(G)$.

Conversely $Z(G)$ is clearly a subgroup of $P \cap G_1$ and so

$$P \cap G_1 = Z(G) = Z.$$

Thus, since $\mathcal{C}(G_1) = P$,

$$Z(G_1) = P \cap G_1 = Z = Z(P)$$

i.e. $G = P \times_Z G_1$

Since P is $\Phi_2(21)$ or $\Phi_2(111)$ and G_1 is a group satisfying the conditions of the theorem and has order less than G , we may prove by induction on the order of G that

$$G \cong A^r \times_Z B^s$$

where A^r means the central product of r copies of $\Phi_2(21)$ and B^s means the central product of s copies of $\Phi_2(111)$.

It remains to show that r may be taken as nought or one.

Now $A \times_Z A \cong \langle \alpha, \beta', \gamma', \delta \mid \alpha^p = \beta'^p = \delta^p = 1, \gamma'^p = \alpha^p = [\alpha, \beta'] = [\gamma', \delta] \rangle$

By replacing β' by $\beta = \beta' \delta^{-1}$, γ' by $\gamma = \alpha^{-1} \gamma'$

we have $A \times_Z A \cong \langle \alpha, \beta, \gamma, \delta \mid \alpha^p = \beta^p = \gamma^p = \delta^p = 1, [\alpha, \beta] = [\gamma, \delta] = \alpha^p \rangle$

$\cong A \times_Z B$. Thus we may suppose $r = 0$ or 1 .

Clearly $A \times_Z B \not\cong B \times_Z B$ (2nd has no elements of order p^2)

Thus $A \times_Z B^n \not\cong B^{n+1}$ and so there are two possibilities:

$G \cong A \times_Z B^n$ and $G \cong B^{n+1}$ Q.E.D.

Corollary There is just one family $\mathcal{G}_n$ corresponding to each n in the above proof, and its rank is $2n+1$.

(This follows since $A \times_Z B \sim B \times_Z B$).

The families of rank ≤ 6 is defined in this way are $\mathcal{G}_1 = \Phi_2$ and $\mathcal{G}_2 = \Phi_5$.

For families of rank 6 and class 2, those with $|G:Z(G)| \leq p^3$ have been determined and those with $|G:Z(G)| = p^5$ have stem groups with $|Z(G_0)| = p$ and so, by the above theorem, do not exist. The case in which $G/Z(G)$ is of type (22) is given in theorem 3, and so the only remaining families are those with central factor groups of type (1^4) .

Theorem 5 There are three families Φ_{12} , Φ_{13} and Φ_{15} of rank 6 with $G/Z(G)$ of type (1^4) .

Proof - Let G be a stem group of such a family and write

$$H_2 = \langle \beta_1, \beta_2 \rangle, G = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, H_2(G) \rangle$$

Since G is of class 2 and is a stem group, $Z(G) = H_2(G)$.

Thus $G/\langle \beta \rangle$ is a group of order p^5 , class 2 and having central factor group of type (1^4) for all $\beta \in H_2(G)$.

i.e. $G/\langle \beta \rangle \in \Phi_2$ or Φ_5 .

Also, by choice of α 's and β 's, we may suppose $[\alpha_1, \alpha_2] \equiv \beta_1$,

$$[\alpha_r, \alpha_s] = \beta_2 \text{ for some } (r,s) \neq (1,2)$$

(a) Suppose $G/\langle \beta_1 \rangle, G/\langle \beta_2 \rangle \in \Phi_2$

Since $G/\langle \beta_2 \rangle \in \Phi_2$, we have $[\alpha_1, \alpha_2] \equiv \beta_1, [\alpha_i, \alpha_j] \in \langle \beta_2 \rangle (i,j) \neq (1,2)$

Since $G/\langle \beta_1 \rangle \in \Phi_2$, we have $[\alpha_r, \alpha_s] \equiv \beta_2, [\alpha_i, \alpha_j] \in \langle \beta_1 \rangle (i,j) \neq (r,s)$

Thus $[\alpha_1, \alpha_2] \equiv \beta_1, [\alpha_r, \alpha_s] \equiv \beta_2, [\alpha_i, \alpha_j] = 1$ otherwise.

If r or $s = 1$ (say $r = 1$) then $\alpha_j \in Z(G)$ where $j \neq 1, 2$, which is impossible and so $r, s \neq 1$.

Similarly, $r, s \neq 2$, and so without loss of generality, we may suppose $[\alpha_1, \alpha_2] = \beta_1, [\alpha_3, \alpha_4] = \beta_2, [\alpha_i, \alpha_j] = 1$ otherwise (Φ_{12}).

(b) Suppose $G/\langle \beta_1 \beta_2^t \rangle \in \Phi_5, G/\langle \beta_2 \rangle \in \Phi_2$ ($t = 0, 1, \dots, p-1$).

As before, $[\alpha_1, \alpha_2] = \beta_1, [\alpha_i, \alpha_j] \in \langle \beta_2 \rangle$ otherwise.

Since $G/\langle \beta_1 \rangle \in \Phi_5, [\alpha_r, \alpha_s] = \beta_2, [\alpha_m, \alpha_n] = \beta_2^b$ (some $(m, n) \neq (r, s)$) and $[\alpha_i, \alpha_j] = 1$ otherwise.

If $m = r$, by replacing $\bar{\alpha}_n$ by $\bar{\alpha}_s^{-b} \bar{\alpha}_n$ we may suppose

$[\alpha_m, \alpha_n] = 1$ and so $G/\langle \beta_1 \rangle \in \Phi_2$. (Contradiction)

Thus $m \neq r$ and similarly $n \neq s$.

Since $[\alpha_1, \alpha_2] \in \langle \beta_1 \rangle$ (m, n), $(r, s) \neq (1, 2)$,

The only other possibilities are $(m, n) = (1, 3), (r, s) = (2, 4)$ and
 $(m, n) = (1, 4), (r, s) = (2, 3)$.

And so, by rearranging the α 's we may suppose $[\alpha_1, \alpha_2] = \beta_1, [\alpha_1, \alpha_3] =$

$= \beta_2, [\alpha_2, \alpha_4] = \beta_2^b$ ($b \neq 0$).

By replacing $\bar{\alpha}_4^b$ by $\bar{\alpha}_4$, we may suppose

$[\alpha_1, \alpha_2] = \beta_1, [\alpha_1, \alpha_3] = [\alpha_2, \alpha_4] = \beta_2, \text{rest} = 1$ (Φ_{13})

(c) Suppose $G/\langle \beta \rangle \in \Phi_5$, all $\beta \in Z(G)$.

As before we may suppose that

$[\alpha_1, \alpha_2] = [\alpha_3, \alpha_4] = \beta_1, [\alpha_1, \alpha_3] = \beta_2, [\alpha_2, \alpha_4] = \beta_2^b$ ($b \neq 0$)

In replacing $\bar{\alpha}_4$ by $\bar{\alpha}_4^x$ we must replace $\bar{\alpha}_2$ by $\bar{\alpha}_2^x$ and so

$[\alpha_2, \alpha_4]$ will be replaced by $\beta_2^{bx^2}$

Thus we may assume $[\alpha_2, \alpha_4] = \beta_2$ or β_2^v .

If $[\alpha_2, \alpha_4] = \beta_2$, by replacing $\bar{\alpha}_1$ by $\bar{\alpha}_1 \bar{\alpha}_4$, $\bar{\alpha}_4$ by $\bar{\alpha}_4 \bar{\alpha}_1^{-1}$, $\bar{\alpha}_3$ by $\bar{\alpha}_3 \bar{\alpha}_2$ and $\bar{\alpha}_2$ by $\bar{\alpha}_2 \bar{\alpha}_3^{-1}$, we may suppose

$$[\alpha_1, \alpha_2] = \beta_1^2, [\alpha_3, \alpha_4] = \beta_1^2 \beta_2^2, \text{rest} = 1.$$

$$\text{i.e. } G/\langle \beta_1 \rangle \in \Phi_2, \quad G/\langle \beta_2 \rangle \in \Phi_5$$

$$\text{i.e. } G \in \Phi_{13}$$

Thus there is only one new family Φ_{15} with

$$[\alpha_1, \alpha_2] = [\alpha_3, \alpha_4] = \beta_1, [\alpha_1, \alpha_3] = \beta_2, [\alpha_2, \alpha_4] = \beta_2^v \quad \text{Q.E.D.}$$

In order to consider the families of rank 6 and class 3, we consider the possibilities for $G/Z(G)$ in chapter 4 which are in $p^5 \Phi_4$.

Lemma 6 Let H be in $p^5 \Phi_4$ and of the form

$$H = \langle \bar{\alpha}, \bar{\alpha}_1, \bar{\alpha}_2 | \bar{\alpha}_i^p, [\bar{\alpha}_i, \bar{\alpha}] \neq 1 \ (i=1,2), [\bar{\alpha}_1, \bar{\alpha}_2] = \bar{\alpha}^{p=1} \rangle$$

Then there are no groups G with $|G| = p^6$ and $G/Z(G) \cong H$.

Proof - Let $G = \langle \alpha, \alpha_1, \alpha_2, Z(G) \rangle$ where $\alpha Z(G), \alpha_1 Z(G), \alpha_2 Z(G)$

correspond respectively to $\bar{\alpha}, \bar{\alpha}_1, \bar{\alpha}_2$ above.

Then $[\alpha_1, \alpha_2] = \xi \in Z(G), [\alpha_i, \alpha] = \beta_i \eta_i$ where $\beta_i \in G^p, \eta_i \in Z(G)$

Since $\alpha_2^{p^2} \in Z(G), 1 = [\alpha_1, \alpha_2^{p^2}]$

$$= [\alpha_1, \alpha_2^p]^p$$

$$\text{Similarly } [\alpha_1^p, \alpha_2]^p = [\alpha_1^p, \alpha]^p = [\alpha_2^p, \alpha]^p = 1 \dots (1)$$

Thus, for $i = 1$ or $2, [\beta_i, \alpha] = [\alpha_i^p, \alpha]$

$$= [\alpha_i, \alpha]^p [\alpha, \alpha_i, \alpha_i]^{-\binom{p}{2}}$$

$$= [\alpha_i, \alpha]^p \quad (p > 2) \dots (2)$$

(since $[\alpha_i, \alpha] = \beta_i \in Z(G)$ and so $[\beta_i, \alpha_i]^p = 1$)

$$\begin{aligned}
\text{Also, since } \alpha^p \in Z(G), \quad 1 &= [\alpha_1, \alpha^p] \\
&= [\alpha_1, \alpha]^p [\alpha_1, \alpha, \alpha]^{\binom{p}{2}} \\
&= [\alpha_1, \alpha]^p [\beta_1, \alpha]^{\binom{p}{2}} \\
&= [\alpha_1, \alpha]^p \quad \text{by (1)} \\
&= [\alpha_1^p, \alpha] \quad \text{by (2)}
\end{aligned}$$

But $\alpha_1^p \notin Z(G)$ and so we must have $[\alpha_1^p, \alpha_2] \neq 1$

i.e. $\xi^p \neq 1$.

Since $\xi \in Z(G)$, $|Z(G)| > p$ and so $|G| > p^6$.

Corollary None of the groups of $p^5\Phi_4$ (with the possible exception of $\Phi_4(1^5)$, $\Phi_4(221)a$ and $\Phi_4(221)b$) is the central factor group of a group of order p^6 .

Lemma 7 If G is metabelian and $\alpha, \beta, \gamma \in G$ with $[\alpha, \gamma] \in Z(G)$, then

$$[\alpha, \beta, \gamma] = [\gamma, \beta, \alpha]$$

Proof - Writing $\delta = [\gamma, \beta]$, we have

$$\begin{aligned}
[\gamma, \beta, \alpha] &= [\gamma^{-1}\gamma^\beta, \alpha] \\
&= [\gamma^{-1}, \alpha]^{\gamma^\delta} [\gamma^\beta, \alpha] \\
&= [\gamma, \alpha]^{-1} [\beta^{-1}\gamma\beta, \alpha] \quad \text{since } [\gamma, \alpha] \in Z(G) \\
&= [\alpha, \gamma] \left\{ [\beta^{-1}, \alpha]^\gamma [\gamma, \alpha] \right\}^\beta [\beta, \alpha] \\
&= [\beta, \alpha] [\beta^{-1}, \alpha]^{\gamma^\beta} \quad \text{since } G \text{ is metabelian} \\
&= [\alpha, \beta]^{-1} [\alpha, \beta]^{\gamma^\delta} \\
&= [\alpha, \beta, \gamma^\delta] \\
&= [\alpha, \beta, \gamma]^\delta [\alpha, \beta, \delta] \\
&= [\alpha, \beta, \gamma] \quad \text{since } G \text{ is metabelian and } \delta \in H_2(G).
\end{aligned}$$

Corollary The group $\Phi_4(221)a$ is incapable.

Proof - Suppose $G = \langle \alpha, \beta, \gamma, Z(G) \rangle$ where

$$\alpha^{p^2}, \beta^{p^2}, \gamma^p, [\alpha, \gamma], [\alpha, \beta]\alpha^{-p}, [\beta, \gamma]\beta^{-p} \in Z(G)$$

$$\text{Then } [\alpha, \beta, \gamma] = [\alpha^p, \gamma] = [\alpha, \gamma]^p = [\alpha, \gamma^p] = 1$$

and $[\gamma, \beta, \alpha] = [\beta^{-p}, \alpha] = 1$ by above lemma.

Also $[\beta^p, \gamma] = [\beta, \gamma]^p = [\beta, \gamma^p] = 1$ and so $\beta^p \in Z(G)$, contradicting the definition of $G/Z(G)$.

We are now in a position to obtain the families of rank 6 and class 3, which are given by the following two theorems:

Theorem 6 Let G be a group of order p^6 and class 3 with $|Z(G)| = p$ and $Z_2(G)$ abelian. Then G is a stem group of one of the families $\Phi_{31}, \Phi_{32}, \Phi_{33}, \Phi_{34}$.

Proof - G is clearly a stem group since $|G|$ is the least possible for any given $|G/Z(G)|$.

Writing $W = \mathcal{L}(Z_2)$, and noting that G is of class 3, we have

$$G > W \geq Z_2(G) \geq H_2(G) > Z(G) = H_3(G)$$

$$\text{Putting } |G:W| = p^s, |W:Z_2| = p^t, |H_2:Z| = p^r, \text{ we have } p^6 = |G| = \\ = p^{2s+t+1} \quad s \geq r > 0, t \geq 0$$

$$\text{i.e. } 2s+t = 5.$$

[Lemma If H is a normal subgroup of G , $H \leq Z_2(G)$, then

$$|G:\mathcal{L}(H)| = |H:Z(G)|$$

Proof - Let $Z(G) = \langle \gamma | \gamma^p = 1 \rangle$ and suppose $\alpha \in G - \mathcal{L}(H), \beta \in H - Z(G)$

Since $H \leq Z_2(G)$, $[\alpha, \beta] \in Z(G)$ and since $\alpha \notin \mathcal{L}(H), [\alpha, \beta] \neq 1$.

The relation $[\alpha, \beta] = \gamma$ thus defines a one to one correspondence between $G/\mathcal{L}(H)$ and $H/Z(G)$.]

If $r = 1$, take $U = \mathcal{L}(H_2)$ and so $|G:U| = |H_2:Z| = p$

i.e. U is maximal.

Also (as $U \leq \mathcal{L}(U')$) $U' \leq Z$ and so G/Z has a maximal abelian subgroup.

Thus, since $H_2(G/Z)$ is of type (1), $G/Z \in \Phi_2$ and

$$|G:Z| = |G:Z_2| / |Z_2:Z| = p^2$$

i.e. $s + t = 2$ and so $|G| \leq p^5$.

Thus we must take $r = 2$, and so $s = 2$

Hence $H_2 = Z_2$ and H_2/Z is of type (11)

Now $H_2(G/Z)$ is (11) and $|G/Z| \leq p^5$ therefore $G/Z \in \Phi_4$ and so G/Z_2 is of type (111).

i.e. G/Z has a maximal abelian subgroup and so G has a maximal subgroup A with $H_2(A) \leq Z$.

If A is abelian, then $H_2(G) \cong A/Z(G)$ and so $|H_2(G)| = |A:Z(G)| = p^4$

But $|H_2| = |Z_2| = p^{s+1} = p^3$ and so A is non-abelian.

Thus A is a group of order p^5 and class 2, $H_2(A) = Z$

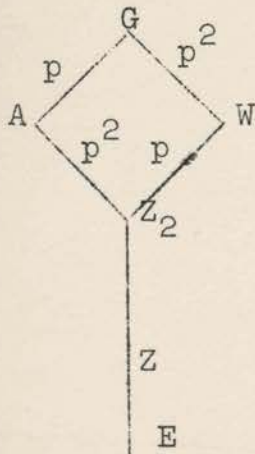
i.e. $A \in \Phi_2$ or Φ_5 .

Case 1

A is a stem group of Φ_5

If $G > A > W$, then (since $|W:Z_2| = p$) A contains a maximal abelian subgroup W and so $A \in \Phi_2$.

Thus $W \cap A < W$ and since $A > Z_2$, $W \cap A = Z_2$



Also, since A is maximal, $W A = G$.

Since $A \in \Phi_5$, $Z(A) = H_2(A) = Z$ and $A/Z = A/Z(A)$ is of type (1^4) . Thus G/Z has a subgroup of type (1^4) . By considering the list on page 191, we see that G/Z is $\Phi_4(1^5)$. Choose $\gamma \in Z$, $\beta_1, \beta_2 \in Z_2 - Z$, $\alpha \in W - Z_2$ and $\alpha_1, \alpha_2 \in A - Z_2$.

Then $\alpha_1^p, \alpha_2^p, \beta_1^p, \beta_2^p \in Z(G)$

By considering $\Phi_4(1^5)$ we may suppose $\beta_i = [\alpha_i, \alpha]$ and, since W is abelian, $[\beta_1, \beta_2] = [\alpha, \beta_i] = 1$ ($i=1,2$)

By the structure of A, $[\alpha_1, \alpha_2], [\alpha_i, \beta_j] \in Z(G)$ ($i, j=1,2$) and, as in the calculations for Φ_5 , we may suppose

$$[\alpha_1, \alpha_2] = [\alpha_1, \beta_2] = [\alpha_2, \beta_1] = 1, [\alpha_1, \beta_1] = \gamma, [\alpha_2, \beta_2] = \gamma^a \quad (a \neq 0)$$

The autologisms are of the form

$$\begin{aligned} \bar{\alpha}^* &= \bar{\alpha}^x \bar{\alpha}_1^{x_1} \bar{\alpha}_2^{x_2} \bar{\beta}_1^{x_3} \bar{\beta}_2^{x_4} \\ \bar{\alpha}_1^* &= \bar{\alpha}_1^{y_1} \bar{\alpha}_2^{y_2} \bar{\beta}_1^{y_3} \bar{\beta}_2^{y_4} \\ \bar{\alpha}_2^* &= \bar{\alpha}_1^{z_1} \bar{\alpha}_2^{z_2} \bar{\beta}_1^{z_3} \bar{\beta}_2^{z_4} \\ \bar{\beta}_1^* &= \bar{\beta}_1^{xy_1} \bar{\beta}_2^{xy_2} \\ \bar{\beta}_2^* &= \bar{\beta}_1^{xz_1} \bar{\beta}_2^{xz_2} \\ \gamma^* &= \gamma^{x(y_1^2 + ay_2^2)} \end{aligned} \quad x \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} \neq 0$$

$$\text{Since } [\alpha_1^*, \beta_2^*] = 1, \quad y_1 z_1 + ay_2 z_2 = 0 \quad (y_1^2 + ay_2^2 \neq 0)$$

From $[\alpha_2^*, \beta_2^*] = \gamma^{*a}$, we get $z_1^2 + az_2^2 = (y_1^2 + ay_2^2)a^*$
 and so $\begin{cases} az_2^2 = y_1^2 a^* \\ z_1^2 = y_2^2 aa^* \end{cases}$

Both of these give us the standard forms $a = 1$ or v^{-1} .

Finally, G is the extension of the group A by the automorphism $\{\alpha_i \rightarrow \alpha_i \beta_i\}$ ($i=1,2$) of order p and so is of order p^6 , and the above two possibilities for $[\alpha_i, \beta_j]$ yield the two families Φ_{31} and Φ_{32} of rank 6.

Case 2

$A \in \Phi_2$ and so $A > W$ (For if $A \not> W$, $A \cap W = Z_2$ and since $W = \mathcal{L}(Z_2)$ all subgroups of A which contain Z_2 would not commute with Z_2 and so $A \in \Phi_5$).

Since Z_2/Z is (11), we may choose $\beta_1, \beta_2 \in Z_2 - Z$ with $[\beta_1, \beta_2] = 1$, $\beta_1^p, \beta_2^p \in Z$.

Take $\alpha_2 \in W - Z_2$ and so $[\alpha_2, \beta_i] = 1$ ($i=1,2$).

Also take $\alpha_1 \in A - W$, $\alpha \in G - A$.

Without loss of generality we may suppose

$$\beta_1 = [\alpha_1, \alpha] \quad (\text{since } \alpha, \alpha_1, \alpha_2 \notin Z_2).$$

Since $W = \mathcal{L}_G(Z_2)[\alpha, \beta_1]$ or $[\alpha, \beta_2] \neq 1$ and $[\alpha_1, \beta_1]$ or $[\alpha_1, \beta_2] \neq 1$, and by structure of A , $[\alpha_1, \beta_1]$ or $[\alpha_1, \beta_2] = 1$.

Suppose $[\alpha_1, \beta_i] = \gamma$ ($\neq 1$). Then $[\alpha_1, \beta_j] = 1$ ($j \neq i$) and so (since $\beta_j \notin Z$) $[\alpha, \beta_j] = \gamma^s$ ($s \neq 0$) and by choice of $\bar{\alpha}_j$ we may suppose $s = 1$.

¹. Easterfield uses the standard forms $a = -1$ and $-v$

Also, suppose $[\alpha, \beta_i] = \gamma^r$

By the autologism $\{\bar{\alpha}_i \rightarrow \bar{\alpha}_i \bar{\alpha}_j^{-r}, \bar{\beta}_i \rightarrow \bar{\beta}_i \bar{\beta}_j^{-r}\}$

we may suppose $[\alpha, \beta_i] = 1$.

Finally, by the autologism $\{\bar{\alpha}_1 \leftrightarrow \bar{\alpha}_2, \bar{\beta}_1 \leftrightarrow \bar{\beta}_2\}$ (if necessary)

we may suppose $[\alpha_1, \beta_1] = [\alpha, \beta_2] = \gamma$, $[\alpha_1, \beta_2] = [\alpha, \beta_1] = 1$.

Taking G/Z as $\Phi_4(1^5)$ we see that G is the extension of the group

$A = \langle \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma \rangle$ by the automorphism $\beta_2 \rightarrow \beta_2 \gamma$ of order p and

so is a group of order p^6 ; and taking G/Z as $\Phi_4(221)b$, we see

that G is the extension of the group

$\langle \alpha, \alpha_1 | \alpha^p = \alpha_1^p = 1, [\alpha_1, \alpha] = \alpha^p \rangle$ by an automorphism $\{\alpha \rightarrow \alpha \alpha_1^p\}$

of order p and so is a group of order p^6 .

Thus case 2 yields two families Φ_{33} and Φ_{34} Q.E.D.

Theorem 7 Let G be a p -group of class 3 with $|Z(G)| = p$. Then G is the central product of copies (possibly none) of groups L of class 2 with $|Z(L)| = p$ together with a group defined in the last theorem.

Proof - As before, take $W = \mathcal{C}_G(Z_2)$, $Z_1(G) = \langle \gamma \rangle$.

Then the relationship $[\xi, \eta] = \gamma$ sets up a one-one correspondence

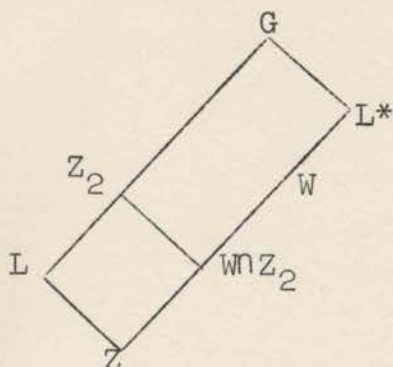
between the subgroups L/Z_1 of Z_2/Z_1 and the factor groups of G/W

with a subgroup L corresponding to its centralizer L^* such that

$L/Z \cong G/L^*$ (see theorem 6).

Choose L so that L/Z is the complement of WZ_2/Z in Z_2/Z

(i.e. $L(WZ_2) = Z_2$, $L \cap W = Z$) and



take $L^* = \ell(L)$

Then $G/L^* \cong Z_2/WN Z_2 = Z_2/L^* \cap Z_2$ and so

$$G = L^* Z_2.$$

$$\begin{aligned} \text{Also } Z(L) &= L \cap L^* = WN L \\ &= Z. \end{aligned}$$

and similarly $Z(L^*) = Z.$

Thus $G = LL^*$, $L \cap L^* = Z = Z(L) = Z(L^*)$ i.e. $G = L \times_Z L^*$.

Also, L is clearly of class 2 or $L = Z(L) = Z$ and in the latter case $L < L^*$ and so $G = L^*$.

Finally, L^* is of class 3, for (since $G = LL^*[L, L^*] = E$)

$$Z_2 = LZ_2(L^*) \text{ and } L \cap Z_2(L^*) = L \cap L^* = Z$$

$$\text{i.e. } Z_2(L^*) = WN Z_2 \neq L^*$$

Thus $3 \leq \text{class } L^* \leq \text{class } G = 3.$ Q.E.D.

For groups of order p^6 , the case where $L = Z$ is done in theorem 6, while for $L \neq Z$, we must have $|L^*| \geq p^4$, $|L| \geq p^3$ and $|L| |L^*| = |G| |Z_1| = p^7.$

Thus, $|L| = p^3$, $|L^*| = p^4$ and so $L \in \Phi_2$, $L^* \in \Phi_3$ and, since all such groups are isoclinic, we have just one family $\Phi_{22}.$

3. Families of rank 6 and class 4 or 5

The families of such groups with $|G:Z(G)| \leq p^4$ have been done. To obtain those with $|G:Z(G)| = p^5$, we consider the suitable groups on page 191.

Theorem 8 Let H be the group defined by ($r \neq 0$)

$$\langle \bar{\alpha}, \bar{\beta}, \bar{\gamma} | \bar{\alpha}^{p^2} \equiv \bar{\beta}^{p^2} \equiv \bar{\gamma}^p \equiv 1, [\bar{\alpha}, \bar{\beta}] \equiv \bar{\gamma}, [\bar{\alpha}, \bar{\gamma}] \equiv \bar{\alpha}^{rp}, [\bar{\beta}, \bar{\gamma}] \equiv \bar{\beta}^{sp} \bar{\alpha}^{tp} \rangle$$

Then H is capable if and only if $r + s \equiv 0 \pmod{p}$ and in this case there is just one family Φ_{42} with $r = 1$, $t = 0$ and this family has only one stem group.

Proof - Suppose H is capable and let

$G = \langle \alpha, \beta, \gamma, Z(G) | G/Z(G) \cong H \rangle$ be a stem group where

$\alpha Z(G), \beta Z(G), \gamma Z(G)$ correspond to $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ above.

Write $\alpha^{p^2} \equiv \xi$, $\beta^{p^2} \equiv \eta$, $\gamma^p \equiv \zeta$, $[\alpha, \beta] \equiv \gamma$, $[\alpha, \gamma] \equiv \alpha^{rp}$ and $[\beta, \gamma] \equiv \beta^{sp} \alpha^{tp}$.

Then, since $\alpha^{p^2}, \beta^{p^2}, \gamma^p \in Z(G)$ and $[\alpha^{rp}, \gamma] \equiv [\alpha, \gamma]^{rp}$,

$$1 = [\alpha^{p^2}, \gamma] \equiv [\alpha, \gamma]^{p^2}$$

$$1 = [\alpha, \gamma^p] \equiv [\alpha, \gamma]^p [\alpha^{rp}, \gamma]^{\binom{p}{2}} \equiv [\alpha, \gamma]^{p(1+r\binom{p}{2})}$$

Hence $[\alpha, \gamma]^p = 1$

Similarly $[\beta, \gamma]^p = 1$.

$$\begin{aligned} \text{Also } 1 &= [\alpha^{p^2}, \beta] \equiv [\alpha, \beta]^{p^2} [\gamma, \alpha]^{\binom{p^2}{2}} \\ &= [\alpha, \beta]^{p^2} \equiv \zeta^p \end{aligned}$$

$$\begin{aligned} \text{Now } [\alpha, \beta^p] &\equiv [\alpha, \beta]^p [\gamma, \beta]^{\binom{p}{2}} \\ &= [\alpha, \beta]^p \end{aligned}$$

and so (since $\beta^p \notin Z(G)$) $\zeta \neq 1$

$$\begin{aligned}
\text{But } [\alpha^{rp}, \beta] &= \alpha^{-rp} (\alpha^{rp})^\beta \\
&= \alpha^{-rp} \tau^{-1} [\alpha, \gamma]^\beta \\
&= \alpha^{-rp} \tau^{-1} [\alpha^\beta, \gamma^\beta] \\
&= \alpha^{-rp} \tau^{-1} [\alpha\gamma, \gamma\beta^{-sp}] \text{ since } \alpha^{tp} \in \langle \alpha, \beta^p, \gamma \rangle \\
&= \alpha^{-rp} \tau^{-1} [\alpha, \beta^{-sp}] [\alpha, \gamma] \\
&= [\alpha, \beta^{-sp}] \\
&= [\alpha^{-sp}, \beta]
\end{aligned}$$

therefore $[\alpha, \beta]^{(r+s)p} = 1$ and so $r + s \equiv 0 \pmod{p}$

Since G is a stem group $Z(G) \leq H_2(G)$

and so $Z(G) \leq \langle [\alpha^p, \beta], [\alpha, \beta^p], [\alpha^p, \gamma], [\beta^p, \gamma] \rangle$

$$= \langle \zeta \rangle \quad \text{where} \quad \zeta^p = 1$$

$$\begin{aligned}
\text{Hence } |G| &= |G:Z(G)| \times |Z(G)| \\
&= p^6
\end{aligned}$$

i.e. the maximal family is of rank 6.

Also since $\tau, \omega \in Z(G)$, put $\tau = \zeta^{tr}$ $\omega = \zeta^{wr}$

By $\{\bar{\alpha} \rightarrow \bar{\alpha} \bar{\gamma}^t, \bar{\beta} \rightarrow \bar{\beta} \bar{\gamma}^w\}$ we may suppose

$$\tau \equiv \omega \equiv 1.$$

Also, by considering the canonical forms given on page¹⁹¹, we may suppose $r = 1, t = 0$.

Thus the only stem group of this family (Φ_{42}) is

$$G = \langle \alpha, \beta, \gamma | \alpha^{p^2} = \beta^{p^2} = \gamma^{p^2} = 1, [\alpha, \beta] = \gamma, [\alpha, \gamma] = \alpha^p, [\beta, \gamma] = \beta^{-p} \rangle$$

Conversely, G is the extension of the group

$$\Phi_4(221)c \cong \langle \alpha, \beta^p, \gamma | [\alpha, \beta^p] = \gamma^p, [\alpha, \gamma] = \alpha^p, [\beta^p, \gamma] = 1 \rangle \text{ by the}$$

automorphism $\theta : (\alpha, \gamma) \rightarrow (\alpha\gamma, \gamma\beta^{-p})$ and, since

$\theta^p(\alpha) = \alpha\gamma^p = \beta^{-p}\alpha\beta^p$, $\theta^p(\gamma) = \gamma\beta^{-p} = \gamma\beta^p$, then G is a group of order p^6 .

Thus for $r + s \equiv 0 \pmod{p}$, H is capable.

Corollary The groups $\Phi_3(221)a$, $\Phi_6(221)a$, $\Phi_6(221)b_k$ ($k \neq -1$)

and $\Phi_6(221)c_i$ are incapable.

Theorem 9 Let H_r be the group of Φ_3 defined by

$$H_r = \langle \bar{\alpha}, \bar{\beta}, \bar{\gamma} | \bar{\alpha}^{p^2} = \bar{\beta}^{p^2} = \bar{\gamma}^p = 1, [\bar{\alpha}, \bar{\beta}] = \bar{\gamma}, [\bar{\alpha}, \bar{\gamma}] = \bar{\beta}^{rp}, [\bar{\beta}, \bar{\gamma}] = 1 \rangle$$

Then H_r is capable for all values of r and if $r \neq 0$ the maximal family is of rank 7 and there are two families of rank 6.

Proof - Let $G = \langle \alpha, \beta, \gamma, Z(G) \rangle$ be a stem group with $G/Z(G) \cong H$ and let $\alpha Z(G)$, $\beta Z(G)$, $\gamma Z(G)$ correspond to $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ above.

Put $[\alpha, \beta] = \gamma$, $[\alpha, \gamma] = \beta^{rp}$, $[\beta, \gamma] = \xi$, $\beta^{p^2} = \eta$, $\gamma^p = \zeta$.

Then since $\alpha^{p^2}, \beta^{p^2}, \gamma^p \in Z(G)$, we have

$$1 = [\beta, \gamma^p] = [\beta, \gamma]^p = [\beta^p, \gamma] = \xi^p.$$

$$1 = [\alpha, \gamma^p] = [\alpha, \gamma]^p [\beta^{rp}, \gamma]^{\binom{p}{2}} = [\alpha, \gamma]^p = [\alpha^p, \gamma]$$

$$[\alpha, \beta^p] = [\alpha, \beta]^p [\gamma, \beta]^{\binom{p}{2}} = \gamma^p = \zeta \quad \text{and so} \quad \zeta^p = 1$$

Similarly, $[\alpha^p, \beta] = \zeta$.

Since G is a stem group, $Z(G) \leq H_4(G)$ and so

$$\begin{aligned} Z(G) &\leq \langle [\alpha^p, \beta], [\alpha, \beta^p], [\alpha^p, \gamma], [\beta, \gamma] \rangle \\ &= \langle \xi, \zeta \rangle \end{aligned}$$

Thus $Z(G)$ is a group of order $\leq p^2$ and exponent p , with

$$\omega, \eta \in \langle \xi, \zeta \rangle$$

But $1 = [\alpha, \gamma]^p = \eta^r \omega^p = \eta^r$ (since $\omega \in Z(G)$) and so $\eta = 1$.

Now the Schur Multiplier has order p^2 for the group

$G = \langle \alpha, \beta, \gamma, \xi, \zeta \rangle$ is the extension of the group

$$\langle \beta, \gamma, \xi, \zeta | \beta^{p^2} = \gamma^{p^2} = \xi^p = 1, [\beta, \gamma] = \xi, \zeta = \gamma^p, [\beta, \xi] = [\gamma, \xi] = 1 \rangle$$

by the automorphism $\{\beta \rightarrow \beta\gamma^{-1}, \gamma \rightarrow \gamma\beta^{-rp}\omega^{-1}\}$ of order p^2 and so G is of order p^7 .

For families of rank 6, we require subgroups N of $\langle \xi, \zeta \rangle$ satisfying lemma 1 with $|N| = p$.
Thus $N = \langle \xi \zeta^s \rangle$ since $\zeta \notin N$.

The autologisms of these families are generated by:

$$\begin{cases} \bar{\alpha} \rightarrow \bar{\alpha} \bar{\beta} \\ \omega \rightarrow \xi \omega \end{cases}, \quad \begin{cases} (\bar{\beta}, \bar{\gamma}) \rightarrow (\bar{\beta} \bar{\gamma}, \bar{\gamma} \bar{\beta}^{rp}) \\ \omega \rightarrow \zeta \omega \end{cases}$$

and autologisms raising elements to some power. Thus the cases $s = 0$ and $s \neq 0$ give rise to two different families with $\xi = 1$ and $\xi = \zeta$ respectively. In both cases we may suppose $\omega = 1$. Q.E.D.

In the case where $\xi = 1$, we have the families Φ_{25} and Φ_{26} defined by:

$$G = \langle \alpha, \alpha_1, Z \mid [\alpha_1, \alpha] = \alpha_{1+1}, [\alpha_1, \alpha_j] = 1 \ (i, j=1, 2, 3), \alpha_3 = \alpha_1^{rp}, \alpha_3^p = \alpha_4^p = 1, \\ \alpha^{p^2}, \alpha_2^p \in Z \rangle \quad (r=1, v)$$

In the case where $\xi = \zeta$, we again have two families (Φ_{28} and Φ_{29}) with the same defining relations except that $[\alpha_1, \alpha_2] = \alpha_4$ instead of 1. The values of 1 and v for r are obtained from the list $p^5 \Phi_3$ on page A4.

As $\Phi_3(1^5)$ was considered in theorem 1, we consider now Φ_6 .

Theorem 10. Let H be the group $\Phi_6(221)d_0$ defined by

$$H = \langle \bar{\alpha}_1, \bar{\alpha}_2 \mid [\bar{\alpha}_1, \bar{\alpha}_2] = \bar{\beta}, \bar{\alpha}_1^p = [\bar{\beta}, \bar{\alpha}_2], \bar{\alpha}_2^p = [\bar{\alpha}_1, \bar{\beta}]^v, \bar{\alpha}_1^{p^2} = \bar{\alpha}_2^{p^2} = \bar{\beta}^p = 1 \rangle$$

Then H is capable and the maximal family Φ_{43} is of rank 6 and has just one stem group (for $p > 3$).

Proof - Let $G = \langle \alpha_1, \alpha_2, Z(G) \rangle$ be a stem group with $G/Z(G) \cong H$ and let α_1, α_2 in G correspond to $\bar{\alpha}_1, \bar{\alpha}_2$ in H .

Put $[\alpha_1, \alpha_2] = \beta, \alpha_i^{p^2} = \eta_i (i=1,2), \beta^p = \zeta, [\beta, \alpha_2] = \alpha_1^p \omega_1$ and

$$[\alpha_1, \beta]^v = \alpha_2^p \omega_2.$$

Then, since $\alpha_i^{p^2}, \beta^p \in Z(G) (i=1,2)$, we have $1 = [\alpha_1^{p^2}, \alpha_2] = [\alpha_1^p, \alpha_2]^p$

$$\begin{aligned} \text{and so } [\alpha_1, \alpha_2^p] &= [\alpha_1, \alpha_2]^p [\beta, \alpha_2]^{\binom{p}{2}} [\alpha_1^p, \alpha_2]^{\binom{p}{3}} \\ &= \zeta (\alpha_1^p \omega_1)^{\frac{1}{2}(p-1)p} \quad \text{for } p > 3. \end{aligned}$$

$$\text{Also } [\beta, \alpha_2^p] = [\beta, \alpha_2]^p [\alpha_1^p, \alpha_2]^{\frac{1}{2}p(p-1)}$$

$$= (\alpha_1^p \omega_1)^p$$

$$\text{Thus } 1 = [\beta, \alpha_2^{p^2}] = [\beta, \alpha_2^p]^p = (\alpha_1^p \omega_1)^{p^2}$$

$$\begin{aligned} \text{and } 1 &= [\alpha_1, \beta^p]^v = [\alpha_1, \beta]^{vp} [\alpha_2^p, \beta]^{\frac{1}{2}vp(p-1)} \\ &= (\alpha_2^p \omega_2)^p \end{aligned}$$

Similarly, by considering $[\alpha_2, \beta^p]$, we get $(\alpha_i^p \omega_i)^p = 1 (i=1,2)$ and

so $[\alpha_1, \alpha_2^p] = \zeta$ and

$$1 = [\alpha_1, \alpha_2^{p^2}] = [\alpha_1, \alpha_2^p]^p = \zeta^p$$

Since G is a stem group, $Z(G) \leq H_4(G)$

$$\text{i.e. } Z(G) \leq \langle [\alpha_1^p, \alpha_2], [\alpha_1, \alpha_2^p] \rangle$$

$$= \langle \zeta \rangle$$

Hence $|G| = p^6$ and so the maximal family is of rank 6.

Also $\omega_i \in Z(G)$ and so $\omega_i = \zeta^{t_i} (i=1,2)$.

Replacing α_i by $\alpha_i \beta^{t_i}$ we may suppose $\omega_i = 1 (i=1,2)$

$$\text{So } G = \langle \alpha_1, \alpha_2 \mid [\alpha_1, \alpha_2] = \beta, \alpha_1^p = [\beta, \alpha_2], \alpha_2^p = [\alpha_1, \beta]^v, \alpha_1^{p^2} = \alpha_2^p = \beta^p = 1 \rangle$$

Finally, G is the extension of the group

$$K = \langle \alpha_1, \alpha_2^p, \beta \mid [\alpha_1, \beta]^p = \alpha_2^p, [\alpha_2^p, \beta] = 1, [\alpha_1, \alpha_2^p] = \beta^p \rangle$$

by the automorphism $\theta: (\alpha_1, \alpha_2^p, \beta) \rightarrow (\alpha_1 \beta, \alpha_2^p, \beta \alpha_1^p)$.

Since $\theta^p(\xi) = \alpha_2^{-p} \xi \alpha_2^p$ for all $\xi \in K$, G must be a group of order p^6 .

Theorem 11. Let H_r be the group defined by:

$$\langle \bar{\alpha}, \bar{\beta}, \bar{\gamma} \mid \bar{\alpha}^{p^2} = \bar{\beta}^{p^2} = \bar{\gamma}^p = 1, [\bar{\alpha}, \bar{\beta}] = \bar{\gamma}, [\bar{\alpha}, \bar{\gamma}] = \bar{\beta}^p, [\bar{\beta}, \bar{\gamma}] = \bar{\alpha}^{rp} \bar{\beta}^p \rangle$$

where $r \neq 0$. Then H_r is incapable for all r .

Proof - Suppose H_r is capable and let

$$G_r = \langle \alpha, \beta, \gamma, Z(G_r) \rangle \text{ be a stem group where } G_r / Z(G_r) \cong H_r$$

and $\alpha Z, \beta Z, \gamma Z$ correspond to $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$

$$\text{Put } \alpha^{p^2} = \xi, \beta^{p^2} = \eta, \gamma^p = \zeta, [\alpha, \beta] = \gamma, [\alpha, \gamma] = \beta^p \omega$$

$$\text{and } [\beta, \gamma] = \alpha^{rp} \beta^p \tau.$$

$$\text{Then } [\beta^p, \gamma] = [\beta, \gamma]^p = [\beta, \gamma^p] = 1.$$

$$\text{Also } 1 = [\alpha, \gamma, \beta]$$

$$= [\alpha, \gamma]^{-1} [\alpha, \gamma]^\beta$$

$$= [\gamma, \alpha] [\alpha \gamma, \gamma \alpha^{-rp} \beta^{-p}]$$

$$= [\gamma, \alpha] [\alpha, \gamma \alpha^{-rp} \beta^{-p}] \text{ since } \gamma \in \mathcal{L}(\gamma, \alpha^p, \beta^p)$$

$$= [\gamma, \alpha] [\alpha, \gamma \beta^{-p}]$$

$$= [\gamma, \alpha] [\alpha, \gamma] [\alpha, \beta^{-p}]$$

$$= [\alpha, \beta^{-p}]$$

Thus $[\alpha, \beta^p] = 1$ and so $\beta^p \in Z(G)$, which contradicts the definition of G_r .

Thus H_r is incapable for all r .

Theorem 12. The group $\Phi_6(1^5)$ is capable. Its maximal family is of rank 8 and there are two families (Φ_{40} and Φ_{41}) of rank 6.

Proof - Let $G = \langle \alpha, \beta, \gamma, \delta, \varepsilon \mid \alpha^p, \beta^p, \gamma^p, \delta^p, \varepsilon^p \in Z(G), [\alpha, \beta] = \gamma, [\alpha, \gamma] = \delta, [\beta, \gamma] = \varepsilon, \text{ rest } \in Z(G) \rangle$ be a stem group.

Then $G/Z(G) \cong \Phi_6(1^5)$ and

$$Z_2(G) = \langle \delta, \varepsilon, Z(G) \rangle, \quad H_2(G) = \langle \gamma, \delta, \varepsilon, Z(G) \rangle$$

Therefore $[\gamma, \delta] = [\gamma, \varepsilon] = [\delta, \varepsilon] = 1$

Also $[\beta, \delta] = \delta^{-\beta} \delta$

$$\begin{aligned} &= [\gamma, \alpha]^\beta \delta \\ &= [\gamma \varepsilon^{-1}, \alpha \gamma] \delta \\ &= [\varepsilon^{-1}, \gamma] [\gamma \varepsilon^{-1}, \alpha] \gamma \delta \\ &= [\varepsilon^{-1}, \gamma] [\gamma, \alpha] [\varepsilon^{-1}, \alpha] \delta \\ &= \delta^{-1} [\alpha, \varepsilon] \delta \\ &= [\alpha, \varepsilon] \end{aligned}$$

Put $\xi = [\alpha, \delta], \eta = [\beta, \delta] = [\beta, \delta] = [\alpha, \varepsilon], \quad \zeta = [\beta, \varepsilon].$

Then, since $\delta^p \in Z(G), \quad 1 = [\alpha, \delta^p]$
 $= [\alpha, \delta]^p$
 $= \xi^p$

and similarly, $\eta^p = \zeta^p = 1.$

Also, putting $K = \langle \beta, \gamma, \delta, \varepsilon, \xi, \eta, \zeta \rangle$ we see that K is the group $\Phi_{16}(1^7) \cong \Phi_{16}(1^6) \times \langle \xi \rangle$ and G is the extension of K by the automorphism $(\beta, \gamma, \delta, \varepsilon) \rightarrow (\beta \gamma^{-1}, \gamma \delta^{-1}, \delta \xi^{-1}, \varepsilon \eta^{-1})$ of order p .

Thus the Schur Multiplier N_1 is of type (111) and so the maximal family is of rank 8.

For families of rank 6, we want $N < N_1, \quad |N| = p^2,$
and $Z(G/N) = Z(G)/N.$

Thus, if $\eta \in N$, ξ and $\zeta \notin N$.

i.e. $N = \langle \xi \eta^r, \zeta \eta^s \rangle$ or $\langle \eta, \xi, \zeta^t \rangle$ ($t \neq 0$).

The autologisms of H are generated by

$$\left\{ \begin{array}{l} (\bar{\alpha}, \bar{\delta}) \leftrightarrow (\bar{\beta}, \bar{\varepsilon}^{-1}) \\ \omega \rightarrow \omega^{-1} \text{ all } \omega \in N_1, \end{array} \right. \quad \left\{ \begin{array}{l} (\bar{\alpha}, \bar{\delta}) \rightarrow (\bar{\alpha}\bar{\beta}, \bar{\delta}\bar{\varepsilon}) \\ (\xi, \eta) \rightarrow (\xi\eta^2\zeta, \eta\zeta) \end{array} \right.,$$

automorphisms inducing the identity map on N_1 and autologisms raising elements to some power. Writing $(\theta, \bar{\theta})$ and $(\varphi, \bar{\varphi})$ for the above autologisms and taking x, y so that

$$2(1-s)x \equiv s \pmod{p}$$

$$(2s-1)y \equiv rs \pmod{p},$$

$$\begin{array}{lcl} \text{we have } \langle \xi, \zeta \rangle & \xrightarrow{\bar{\varphi}^x} & \langle \xi\eta^{2x}, \zeta \rangle \\ & \xrightarrow{\bar{\theta}} & \langle \xi, \zeta\eta^{2x} \rangle \\ & \xrightarrow{\bar{\varphi}^y} & \langle \xi\eta^r, \zeta\eta^s \rangle \end{array}$$

and so the first possibility for N gives one family (Φ_{40}).

In the second case, if $-t = x^2$, by replacing $\bar{\alpha}$ by $\bar{\alpha}^x$ we replace $\xi\zeta^t$ by 1 showing that $|N| = p$, which contradicts the definition of N . If $-t$ is a quadratic non-residue the same autologism gives the standard form $[\alpha, \delta] = \eta$, $[\beta, \varepsilon] = \eta^{-v}$ defining Φ_{41} .

Theorem 13. The group $\Phi_7(1^5)$ is capable. The maximal family is of rank 9 and there is one family Φ_{30} of rank 6.

Proof - Let $G = \langle \alpha, \beta, \gamma, \delta, \varepsilon, Z(G) \rangle$ be a stem group where $[\alpha, \beta] = \gamma$, $[\alpha, \gamma] \equiv [\beta, \delta] \equiv \varepsilon \pmod{Z}$, $\alpha^p \equiv \beta^p \equiv \dots \equiv \varepsilon^p \equiv 1 \pmod{Z}$ and all other commutators are in $Z(G)$.

Then $Z_3(G) = \langle \gamma, \delta, \varepsilon, Z(G) \rangle$, $H_3(G) = \langle \varepsilon, H_4(G) \rangle$ and so

(since $[Z_1(G), H_1(G)] = E$) $[\gamma, \varepsilon] = [\delta, \varepsilon] = 1$.

Also, by lemma 7, $[\gamma, \delta] = [\alpha, \beta, \delta]$

$$= [\delta, \beta, \alpha] = [\varepsilon^{-1}, \alpha].$$

and $[\varepsilon^{-1}, \beta] = [\gamma, \alpha, \beta]$

$$= [\beta, \alpha, \gamma] = 1.$$

Put $[\alpha, \gamma] = \varepsilon$, $[\beta, \delta] = \varepsilon, \omega$, $[\alpha, \delta] = \xi$, $[\alpha, \varepsilon] = [\gamma, \delta] = \eta$

and $\zeta = [\beta, \gamma]$.

Taking $G = \langle \alpha, \beta, \gamma, \delta, \varepsilon, \xi, \eta, \zeta, \omega \rangle$ and

$K = \langle \alpha, \beta, \gamma, \varepsilon, \xi, \eta, \zeta, \omega \rangle$, we see that K is the group

$\Phi_{10}(1^3) = \Phi_{10}(1^6) \times \langle \xi, \omega \rangle$ of order p^3 and G is the extension

of K by the automorphism $(\alpha, \beta, \gamma) \rightarrow (\alpha\xi, \beta\varepsilon\omega, \gamma\eta)$ of order p .

As before $\xi^p = \eta^p = \zeta^p = \omega^p = 1$ and so the Schur Multiplier N_1 is (1^4) .

Hence the maximal family is of rank 9.

For families of rank 6, we want $N < N_1$, $|N| = p^3$ and

$Z(G/N) = Z(G)/N$ and so $\eta \notin N$.

Thus $N = \langle \xi\eta^r, \zeta\eta^s, \omega\eta^t \rangle$ and by repeated applications of the autologisms:

$$\left\{ \begin{array}{l} \theta: \bar{\delta} \rightarrow \bar{\delta} \bar{\varepsilon}, \\ \bar{\theta}: \bar{\xi} \rightarrow \bar{\xi} \eta, \end{array} \right\} \quad \left\{ \begin{array}{l} \varphi: \bar{\beta} \rightarrow \bar{\beta} \bar{\delta}, \\ \bar{\varphi}: \bar{\zeta} \rightarrow \bar{\zeta} \eta^{-1}, \end{array} \right\} \quad \left\{ \begin{array}{l} \psi: \bar{\beta} \rightarrow \bar{\beta} \bar{\gamma} \\ \bar{\psi}: \omega \rightarrow \omega \eta \end{array} \right\}$$

we may see that there is just one family Φ_{30} of rank 6 with

$[\beta, \delta] = \varepsilon$, $[\alpha, \delta] = [\beta, \gamma] = 1$.

Q.E.D.

For groups of class 5, we must have $G/Z \in \Phi_9$ or Φ_{10} and the capable group of Φ_9 was considered in theorem 1.

Theorem 14. The group $\Phi_{10}(1^5)$ is capable. The maximal family is of rank 8 and there are two families Φ_{38} , Φ_{39} of rank 6.

Proof - Let $G = \langle \alpha, \alpha_1 \rangle$ be a stem group, where $[\alpha_i, \alpha] = \alpha_{i+1}$ ($i=1, \dots, 4$),
 $[\alpha_1, \alpha_2] = \alpha_4 \pmod{Z}$ and all p 'th powers and other
commutators are central.

Putting $[\alpha_1, \alpha_2] = \eta$, $[\alpha_1, \alpha_4] = \xi$ we see, as in theorem 1, that the Schur
Multiplier N_1 is $\langle \alpha_5, \xi, \eta \rangle$ of type (111) and so the maximal family
is of rank 8.

Also, since $[H_i(G), H_j(G)] \leq H_{i+j}(G)$ we have $[\alpha_2, \alpha_4] = [\alpha_3, \alpha_4] = 1$,
 $[\alpha_2, \alpha_3] = [\alpha_1, \alpha, \alpha_3] = [\alpha_3, \alpha, \alpha_1] = \xi^{-1}$ (by lemma 7) and

$$\begin{aligned} [\alpha_1, \alpha_3] &= \alpha_3^{-\alpha_1} \alpha_3 \\ &= [\alpha, \alpha_2]^{\alpha_1} \alpha_3 \\ &= [\alpha \alpha_2^{-1}, \alpha_2 \alpha_4^{-1}] \alpha_3 \\ &= [\alpha \alpha_2^{-1}, \alpha_4^{-1}] [\alpha \alpha_2^{-1}, \alpha_2] \alpha_3 \\ &= [\alpha_4, \alpha] [\alpha, \alpha_2] \alpha_3 \\ &= \alpha_5. \end{aligned}$$

For families of rank 6, we want $N < N_1$, $|N| = p^2$ and $Z(G/N) = Z(G)/N$
and so $\alpha_5 \notin N$ or $\xi \notin N$.

i.e. $N = \langle \alpha_5 \xi^r, \eta \xi^s \rangle$ or $\langle \xi, \eta \alpha_5^t \rangle$. By repeated applications of the
autologisms

$$\begin{cases} \bar{\alpha} \rightarrow \bar{\alpha} \bar{\alpha}_1 \\ \alpha_5 \rightarrow \alpha_5 \xi^{-1} \end{cases}, \begin{cases} (\bar{\alpha}_1, \bar{\alpha}_2) \rightarrow (\bar{\alpha}_1 \bar{\alpha}_3, \bar{\alpha}_2 \bar{\alpha}_4) \\ \eta \rightarrow \eta \xi^2 \end{cases}, \begin{cases} \bar{\alpha}_i \rightarrow \bar{\alpha}_i \bar{\alpha}_{i+1} \quad (i=1, 2, 3) \\ \eta \rightarrow \eta \alpha_5 \xi^{-1} \end{cases}$$

we see that each of the above two cases yields one family and these
families are distinct as the first is not metabelian whilst the
second is. Thus there are two families of rank 6 with central factor
groups isomorphic to $\Phi_{10}(1^5)$ and we may take
 $[\alpha_1, \alpha_2] = \alpha_4$, $[\alpha_4, \alpha] = \alpha_5$, $[\alpha_1, \alpha_4] = [\alpha_2, \alpha_3] = 1$ in Φ_{38} and
 $[\alpha_1, \alpha_2] = \alpha_4$, $[\alpha_1, \alpha_4] = \xi$, $[\alpha_4, \alpha] = [\alpha_1, \alpha_3] = 1$ in Φ_{39} .

4. Special Results for $p = 3$

Many of the results for $p > 3$ carry over to the case when $p = 3$. For example, all the calculations for groups of class 1 or 2 are exactly the same. Also the calculations for families of class 3 will be the same as they only depend on the capable groups of class 2.

As lemmas 1 to 7 of chapter 1 do not depend on the value of p , we may use them again here.

Corresponding to Theorem 1, we have

Theorem 1' Let H be the group $\mathcal{A}_{(n)}(1^{n+2})$ defined by
 $H = \langle \bar{\alpha}, \bar{\alpha}_1 \mid [\bar{\alpha}_1, \bar{\alpha}] = \bar{\alpha}_{i+1}, \bar{\alpha}_j^3 \bar{\alpha}_{j+1}^3 \bar{\alpha}_{j+2} = [\bar{\alpha}_1, \bar{\alpha}_j] = 1 \ (i=1,2,\dots,n; j=1,2,\dots,n+1) \rangle$

Then H is capable and the maximal family is of rank $n+4$ with two families of rank $n+3$

Proof Let $G = \langle \alpha, \alpha_1, Z \mid [\alpha_1, \alpha] = \alpha_{i+1}, \alpha_1^3 \alpha_{i+1}^3 \alpha_{i+2}, [\alpha_1, \alpha_j] \in Z \rangle$

(a) As in the case for regular groups, we can show that the Schur Multiplier is

$\langle \alpha_{n+2}, \xi_2, \xi_3, \dots, \xi_n \rangle$ where $\xi_i = [\alpha_1, \alpha_i]$ $\alpha_{n+2} = [\alpha_{n+1}, \alpha]$ and $[\alpha_2, \alpha_3] = \xi_3$.

However, when $p = 3$, we can extend this.

(b) The Schur Multiplier is $\langle \alpha_{n+2} \rangle$ for $n=0$, and $\langle \alpha_{n+2}, \xi_2 \rangle$ (elementary abelian) for $n \geq 1$.

This is proved by induction:

$1 = [\alpha_1, \alpha_1^3 \alpha_2^3 \alpha_3] \equiv \xi_2^3 \xi_3$ and so $\xi_3 \in \langle \xi_2^3 \rangle$

Suppose $\xi_3, \dots, \xi_r \in \langle \xi_2^3 \rangle$

Then $1 = [\alpha_1, \alpha_{r-1}^3 \alpha_r^3 \alpha_{r+1}] = \xi_{r-1}^3 \xi_r^3 \xi_{r+1}$

Hence $\xi_{r+1} \in \langle \xi_{r-1}^3, \xi_r^3 \rangle \leq \langle \xi_2^3 \rangle$ and so

$\xi_r \in \langle \xi_2^3 \rangle$ all $2 < r \leq n+1$

But $1 = [\alpha_1^3 \alpha_2^3 \alpha_3, \alpha_2] = \xi_2^3 \xi_3^{-1}$

and so $\xi_3 = \xi_2^3 = 1$.

Hence $\xi_1 = 1$ ($i > 2$) and $\xi_2^3 = 1$.

Also G is the extension of $\langle \alpha, \alpha_2, \dots, \alpha_{n+2} \rangle \rtimes \langle \xi_2 \rangle$

by an element of order 3 which induces the automorphism

$\{\alpha \rightarrow \alpha \alpha_2^{-1}, \alpha_2 \rightarrow \alpha_2 \xi_2\}$ where we may assume (by induction) that

$\langle \alpha, \alpha_2, \dots, \alpha_{n+2} \rangle$ is a group of order 3^{n+2} .

Thus the Schur Multiplier is of order 9.

For $n = 1$, the maximal family is Δ_6 , and for $n = 2$, the maximal family is Δ_{23} .

(c) For other families, we want subgroups N of the Schur Multiplier satisfying lemma 1 with $|N| = 3$.

Since $[\alpha_1, \alpha_{n+1}] = 1$, $[\alpha_{n+1}, \alpha] = \alpha_{n+2}$, $\alpha_{n+2} \in N$.

Thus $N = \langle \xi_2 \rangle$, $\langle \xi_2 \alpha_{n+2} \rangle$ or $\langle \xi_2^2 \alpha_{n+2} \rangle$.

The automorphisms of H induce the identity in the Schur Multiplier or consist in raising elements to some power.

Thus there are two families with $\xi_2 = 1$ or α_{n+2} .

For $n = 1$, there is just one family Δ_3 ; for $n = 2$, the two families are Δ_9 and Δ_{10} ; and for $n = 3$ they are Δ_{35} and Δ_{36} .

In theorem 2, cases (i) and (ii) still apply for $p = 3$.

Corresponding to (iii) we have

Theorem 2' The group $\Delta_3(1^5)$ given by

$$H = \langle \bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3 \mid [\bar{\alpha}_1, \bar{\alpha}] = \bar{\alpha}_{i+1}, \bar{\alpha}^3 = \bar{\alpha}_1^3 \bar{\alpha}_3 = \bar{\alpha}_2^3 = 1 \rangle \times \langle \beta \rangle$$

is capable. Its maximal family is of rank 9 and it has two families of rank 6.

Proof The calculations are the same as in the case $p \geq 5$.

The families of rank 6 are Δ_{24} and Δ_{27} .

Theorems 3 to 7 remain true for $p = 3$, provided we only consider groups of order 3^n ($n \leq 6$). For convenience we will include the special result for theorem 8 with that for theorem 10. Using the list of groups on page 191, we have (corresponding to theorems 9-14)

Theorem 9' Let H_r be the group of Δ_3 defined by

$$H_r = \langle \bar{\alpha}, \bar{\alpha}_1 \mid [\bar{\alpha}_1, \bar{\alpha}] = \bar{\alpha}_2, [\bar{\alpha}_2, \bar{\alpha}] = \bar{\alpha}_3, \bar{\alpha}_1^3 \bar{\alpha}_3 = \bar{\alpha}_3^r, \bar{\alpha}^9 = \bar{\alpha}_2^3 = \bar{\alpha}_3^3 = 1 \rangle$$

Then H_r is capable for $r \neq 0$, the maximal family is of rank 7 and there are two families of rank 6.

Proof Let $G = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, Z \mid [\alpha_1, \alpha] = \alpha_{i+1}, \alpha_1^3 \alpha_3^{1-r}, \alpha_3^3, \alpha_2^3, \alpha_3^3 \in Z(G) \rangle$

$$\text{and } \alpha_4 = [\alpha_3, \alpha], \quad \xi = [\alpha_1, \alpha_2], \quad \alpha_1^3 \alpha_3 = \alpha_3^r \omega.$$

As in the case for $p > 3$, we may show that

$$\xi^3 = \alpha_4^3 = \alpha_3^3 = 1, \alpha_2^3 \alpha_4 = \alpha_4^r \text{ (replacing } \alpha_1^3 \text{ by } \alpha_1^3 \alpha_3 \text{ in that theorem)}$$

$$\text{and } [\alpha_1, \alpha_3] = [\alpha_2, \alpha_3] = 1.$$

Also as in the case for $p > 3$, Schur Multiplier = $\langle \alpha_4, \xi \rangle$.

For families of rank 6, we want subgroups of order 3 satisfying lemma 1, viz. $\langle \xi \alpha_4^2 \rangle$, $\langle \xi \alpha_4 \rangle$ and $\langle \xi \rangle$.

The autologisms are generated by

$$\left\{ \begin{array}{l} \bar{\alpha} \rightarrow \bar{\alpha} \bar{\alpha}_1 \\ \omega \rightarrow \xi^2 \omega \end{array} \right\}, \left\{ \begin{array}{l} \bar{\alpha}_1 \rightarrow \bar{\alpha}_1 \bar{\alpha}_2 \\ \omega \rightarrow \xi^2 \omega \end{array} \right\}, \left\{ \begin{array}{l} \bar{\alpha} \rightarrow \bar{\alpha}^2 \\ (\xi, \alpha_4) \rightarrow (\xi^2, \alpha_4^2) \end{array} \right\}, \left\{ \begin{array}{l} \bar{\alpha}_1 \rightarrow \bar{\alpha}_1^2 \\ \alpha_4 \rightarrow \alpha_4^2 \end{array} \right\}$$

So we have two families for each r , with $[\alpha_1, \alpha_2] = 1$ or α_4 and we may suppose $\omega = 1$.

In each case $\alpha_1^3 \alpha_3 = \alpha_3^r$ and $\alpha^9 \in \langle \alpha_4 \rangle$.

The families are Δ_{25} , Δ_{26} , Δ_{28} and Δ_{29} .

Theorem 10' Let H_r be the group defined by

$$H_r = \langle \bar{\alpha}_1, \bar{\alpha}_2 \mid [\bar{\alpha}_1, \bar{\alpha}_2] = \bar{\beta}, [\bar{\beta}, \bar{\alpha}_1] = \bar{\beta}_1 (1=1,2), \bar{\alpha}_1^3 = \bar{\beta}_2, \bar{\alpha}_2^3 \bar{\beta}_1^{-1} = \bar{\beta}_1^r \rangle$$

Then H_r is capable for $r \neq 0$ and the maximal families Δ_{42} , Δ_{43} are of rank 6 with just one stem group.

Proof Let $G = \langle \alpha_1, \alpha_2, Z \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, \beta_2 \alpha_1^3, \alpha_2^3 \beta_1^{-1-r}, \beta^3, \beta_1^3 \in Z \rangle$

and $[\alpha_i, \beta_j] = \xi_{ij}$. Then $Z(G) = \langle \{\xi_{ij}\} \mid i, j = 1, 2 \rangle$.

Now $\beta_2 \alpha_1^{-3} \in Z(G)$ and so $[\alpha_1, \beta_2] = 1$.

By the same calculations as in Theorem 12,

$$[\alpha_2, \beta_1] = [\alpha_1, \beta_2] = 1.$$

Also, putting $\alpha_1^3 = \beta_2 \eta$, $\alpha_2^3 \beta_1^{-1} = \beta_1^r \zeta$, we have

$$\begin{aligned} \xi_{22} &= [\alpha_2, \beta_2 \eta] = [\alpha_2, \alpha_1^3] \\ &= [\alpha_2, \alpha_1]^3 [\beta_1^{-1}, \alpha_1]^3 [\beta_1^{-1}, \alpha_1] \\ &= \beta^{-3} \xi_{11} \quad \text{since } \beta_1^3 = 1 \end{aligned}$$

$$\begin{aligned}
\text{and } \xi_{11}^r &= [\alpha_1, \alpha_2^3 \beta_1^{-1}] = [\alpha_1, \alpha_2^3][\alpha_1, \beta_1^{-1}] \\
&= \beta^3 \xi_{22} \xi_{11}^{-1} \quad \text{since } \beta_2^3 = 1. \\
&= 1 \quad \text{using the above result.}
\end{aligned}$$

Thus $\xi_{11} = 1$ and $\xi_{22} = \beta^{-3} = \xi$

As in the case for $p > 3$, the group $\langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2, \xi \rangle$ has order p^6 in each case. Thus the only stem groups are

$$\langle \alpha_1, \alpha_2, \beta \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \alpha_2^{3(1+r)}, [\beta, \alpha_2] = \alpha_1^3, \alpha_1^9 = \alpha_2^9 = \beta^9 = 1 \rangle$$

As in theorem 9', we may suppose $\eta = \xi = 1$.

Theorem 12 Let H be the group defined by

$$H = \langle \bar{\alpha}_1, \bar{\alpha}_2, \bar{\beta}, \bar{\beta}_1, \bar{\beta}_2 \mid [\bar{\alpha}_1, \bar{\alpha}_2] = \bar{\beta}, [\bar{\beta}, \bar{\alpha}_1] = \bar{\beta}_1, \bar{\alpha}_1^3 = \bar{\beta}^3 = \bar{\beta}_1^3 = 1 \rangle$$

Then H has a maximal family of rank 7 and two families Δ_{40} and Δ_{41} of rank 6.

Proof Let $G = \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2, Z(G) \rangle$ where α etc. corresponds to $\bar{\alpha}$ etc. in H.

Since $H_2(G)$ is abelian, $[\beta, \beta_1] = [\beta_1, \beta_2] = 1$ ($i=1,2$).

Also as in theorem 12, $\eta = [\alpha_1, \beta_2] = [\alpha_2, \beta_1], \beta_1^3 = 1$.

$$\text{Now } 1 = [\alpha_1, \alpha_2^3] = [\alpha_1, \alpha_2]^3 [\beta, \alpha_2]^3 [\beta_2, \alpha_2]$$

$$= \beta^3 [\beta_2, \alpha_2]$$

$$\text{and } 1 = [\alpha_2, \alpha_1^3] = [\alpha_2, \alpha_1]^3 [\beta^{-1}, \alpha_1]^3 [\beta_1^{-1}, \alpha_1]$$

$$= \beta^{-3} [\alpha_1, \beta_1]$$

Thus $[\alpha_1, \beta_1] = \beta^3 = [\alpha_2, \beta_2] = \xi$

As before $\xi^3 = \eta^3 = 1$.

Also $K = \langle \alpha_1, \beta, \beta_1, \beta_2, \xi, \eta \rangle$ is $\Delta_{16}(1^6)$ and so of order 3^6 ; and

G is the extension of K by the automorphism $\{(\alpha_1, \beta, \beta_1, \beta_2) \rightarrow (\alpha_1, \beta, \beta\beta_2, \beta_1\xi^{-1}, \beta_2\eta^{-1})\}$ of order 3 and so has order 3^7 .

Thus the Schur Multiplier $N_1 = \langle \xi, \eta \rangle$ has order 9 and the maximal family is of rank 7.

For families of rank 6, we want $N < N_1, |N| = 3$.

If $N = \langle \xi\eta \rangle, [\alpha_i, \beta_1\beta_2] = [\alpha_i, \beta_1][\alpha_i, \beta_2] = \xi\eta \in N (i=1,2)$ and so N does not satisfy lemma 1.

Similarly $N \neq \langle \xi\eta^2 \rangle$. However the subgroups $\langle \xi \rangle$ and $\langle \eta \rangle$ do satisfy lemma 1 and give non-isoclinic families.

These families are given by $[\alpha_1, \beta_2] = [\alpha_2, \beta_1] = 1$ or

$$[\alpha_1, \beta_1] = [\alpha_2, \beta_2] = 1.$$

Theorem 13' The group $\Delta_7(1^5)$ is capable. The maximal family is of rank 9 and there is just one family Δ_{30} of rank 6.

Proof The proof is the same as that for $p > 3$, replacing $\alpha_1^3\alpha_3$ for α_1^p and $\alpha_2^3\alpha_4$ for α_2^p .

Theorem 14' The group $\mathcal{B}_n(1^{n+2})$ ($p = 3$) given by:

$H = \langle \bar{\alpha}, \bar{\alpha}_1, \dots, \bar{\alpha}_{n+1} \mid [\bar{\alpha}_i, \bar{\alpha}] = \bar{\alpha}_{i+1}, [\bar{\alpha}_1, \bar{\alpha}_2] = \bar{\alpha}_{n+1}, \bar{\alpha}_i^3 \bar{\alpha}_{i+1}^3 \bar{\alpha}_{i+2} = 1 \rangle$ is incapable for all $n \geq 3$.

Proof Let G be a group with $G/Z(G) \cong H$ and let

$G = \langle \alpha, \alpha_1, \dots, \alpha_{n+1}, Z(G) \rangle$ where α_i corresponds to $\bar{\alpha}_i$. As in

theorem 1, the Schur Multiplier of H is $\langle \alpha_{n+2} \rangle$ (since $[\alpha_1, \alpha_2] \notin Z(G)$).

But, by theorem 14, $\alpha_{n+2} = [\alpha_1, \alpha_3] = 1$ and so the Schur Multiplier collapses to E .

Corollary There are only two families of 3-groups whose stem groups have maximal class.

This result agrees with Blackburn's result (see (1)).

5. SUMMARY OF FAMILIES OF RANK ≤ 6

Family	Rank	Class	G/Z	H_2	H_3	H_4	$H_2 \cap Z$	d_o
Φ_2	3	2	(11)	(1)			(1)	2
Φ_3	4	3	$\Phi_2(1^3)$	(11)	(1)		(1)	2
Φ_4	5	2	(111)	(11)			(11)	3
Φ_5	5	2	(1^4)	(1)			(1)	4
Φ_6	5	3	$\Phi_2(1^3)$	(111)	(11)		(11)	2
Φ_7	5	3	$\Phi_2(1^4)$	(11)	(1)		(1)	3
Φ_8	5	3	$\Phi_2(22)$	(2)	(1)		(1)	2
Φ_9	5	4	$\Phi_3(1^4)$	(111)	(11)	(1)	(1)	2
Φ_{10}	5	4	$\Phi_3(1^4)$	(111)	(11)	(1)	(1)	2
Φ_{11}	6	2	(111)	(111)			(111)	3
Φ_{12}	6	2	(1^4)	(11)			(11)	4
Φ_{13}	6	2	(1^4)	(11)			(11)	4
Φ_{14}	6	2	(22)	(2)			(2)	2
Φ_{15}	6	2	(1^4)	(11)			(11)	4
Φ_{16}	6	3	$\Phi_2(1^4)$	(111)	(1)		(11)	3
Φ_{17}	6	3	$\Phi_2(1^4)$	(111)	(1)		(11)	3
Φ_{18}	6	3	$\Phi_2(1^4)$	(111)	(1)		(11)	3
Φ_{19}	6	3	$\Phi_2(1^4)$	(111)	(11)		(11)	3
Φ_{20}	6	3	$\Phi_2(1^4)$	(111)	(11)		(11)	3
Φ_{21}	6	3	$\Phi_2(1^4)$	(111)	(11)		(11)	3
Φ_{22}	6	3	$\Phi_2(1^5)$	(11)	(1)		(1)	4

Family	Rank	$H_2/H_2 \cap Z$	Class	$G/Z(G)$	H_2	H_3	H_4	H_5	d_o
Φ_{23}	6	(11)	4	$\Phi_3(1^4)$	(1^4)	(111)	(1)		2
Φ_{24}	6	(11)	4	$\Phi_3(1^5)$	(111)	(11)	(1)		3
Φ_{25}	6		4	$\Phi_3(221)b_1$	(21)	(11)	(1)		2
Φ_{26}	6		4	$\Phi_3(221)b_v$	(21)	(11)	(1)		2
Φ_{27}	6	(11)	4	$\Phi_3(1^5)$	(111)	(11)	(1)		3
Φ_{28}	6		4	$\Phi_3(221)b_1$	(21)	(11)	(1)		2
Φ_{29}	6		4	$\Phi_3(221)b_v$	(21)	(11)	(1)		2
Φ_{30}	6	(11)	4	$\Phi_7(1^5)$	(111)	(11)	(1)		3
Φ_{31}	6	(11)	3	$\Phi_4(1^5)$	(111)	(1)			3
Φ_{32}	6	(11)	3	$\Phi_4(1^5)$	(111)	(1)			3
Φ_{33}	6	(11)	3	$\Phi_4(1^5)$	(111)	(1)			3
Φ_{34}	6		3	$\Phi_4(221)b$	(21)	(1)			3
Φ_{35}	6	(111)	5	$\Phi_9(1^5)$	(1^4)	(111)	(11)	(1)	2
Φ_{36}	6		5	$\Phi_9(1^5)$	(1^4)	(111)	(11)	(1)	2
Φ_{37}	6		5	$\Phi_9(1^5)$	$\Phi_2(1^4)$	(111)	(11)	(1)	2 *
Φ_{38}	6		5	$\Phi_{10}(1^5)$	(1^4)	(111)	(11)	(1)	2 *
Φ_{39}	6		5	$\Phi_{10}(1^5)$	$\Phi_2(1^4)$	(111)	(11)	(1)	2 *
Φ_{40}	6		4	$\Phi_6(1^5)$	(1^4)	(111)	(1)		2
Φ_{41}	6		4	$\Phi_6(1^5)$	(1^4)	(111)	(1)		2
Φ_{42}	6		4	$\Phi_6(221)b_1$	(211)	(111)	(1)		2
Φ_{43}	6		4	$\Phi_6(221)c_o$	(211)	(111)	(1)		2

d_o = number of generators of a stem group.

* The family only exists for $p > 3$.

III Calculations for Individual Families

For a given isoclinism family, the method of finding the isomorphism classes is essentially the same as that used by Blackburn (1) in determining p-groups of maximal class. Let G, G' (with centres Z, Z' respectively) be two isoclinic p-groups. By definition of isoclinism we may suppose without loss of generality that G and G' have the same relations modulo their centres and the same commutator relations. Thus, writing $w(\xi)$ for a word in $\xi_1, \xi_2, \dots, \xi_m$, we may write

$$G = \langle \alpha_1, \alpha_2, \dots, \alpha_m, Z \rangle \text{ for } \alpha_1, \alpha_2, \dots, \alpha_m \in G \text{ and}$$

$$G' = \langle \alpha'_1, \alpha'_2, \dots, \alpha'_m, Z' \rangle \text{ for } \alpha'_1, \alpha'_2, \dots, \alpha'_m \in G'$$

where

$w(\alpha) = 1 \pmod{Z}$ if and only if $w(\alpha') \equiv 1 \pmod{Z'}$ and $w'(\alpha) = 1$ if and only if $w'(\alpha') = 1$ for all commutator words w' .

If $G \cong G'$ and θ is an isomorphism mapping G' onto G , then $Z \cong Z'$ and we may write

$$Z = \langle \gamma_1, \gamma_2, \dots, \gamma_n \rangle \text{ for } \gamma_1, \gamma_2, \dots, \gamma_n \in Z$$

$$Z' = \langle \gamma'_1, \gamma'_2, \dots, \gamma'_n \rangle \text{ for } \gamma'_1, \gamma'_2, \dots, \gamma'_n \in Z'$$

where the order of γ_j equals the order of γ'_j and each γ_j commutes with every element of G and each γ'_j with every element of G' .

Also, since $G \cong G'$, $\gamma_j^{p^{s_j}} = w_0(\alpha)$ if and only if

$\gamma_j^{p^{s_j}} = w_0(\alpha')$ where w_0 is a commutator word.

The only relations for G which have not been determined are those of the form

$$w(\alpha) = \gamma_{\sim}^a.$$

Since G/H_2 is abelian, this reduces to equations of the form

$$\alpha_i^{p^{r_i}} = w_1(\alpha) \gamma_{\sim}^{a_i} \quad (1)$$

where $w_1(\alpha) \in H_2$, $\gamma_{\sim}^{a_i} \in Z$ and $i = 1, 2 \dots m$.

For convenience, we may suppose that no term of $w_1(\alpha)$ is an element of $H_2 \cap Z$. Since $G/Z \cong G'/Z'$, we have (where $a_1^*, a_2^*, \dots, a_m^*$ are the parameters of G')

$$\alpha_i'^{p^{r_i}} = w_1(\alpha') \gamma_{\sim}'^{a_i^*}. \quad (1)'$$

Since $\theta: G' \rightarrow G$ is an isomorphism, we may write

$$G = \langle \alpha_1^*, \alpha_2^*, \dots, \alpha_m^*, Z \rangle$$

$$Z = \langle \gamma_1^*, \gamma_2^*, \dots, \gamma_n^* \rangle$$

where

$$\alpha_i^* = \theta(\alpha_i') \equiv \gamma_{\sim}^{x_i} \pmod{H_2 Z} \quad (2)$$

(where $i = 1, 2 \dots m$),

$$\gamma_j^* = \theta(\gamma_j') = \gamma_{\sim}^{y_j} \quad (j = 1, 2, \dots, n) \quad (3),$$

γ_j^* commutes with every element of G and

$$\gamma_j^{p^{s_j}} = w_0(\alpha), \quad \gamma_j^{*p^{s_j}} = w_0(\alpha^*). \quad (4)$$

Equations (1) and (4), together with the relations mod Z , the commutator relations and the structure of the abelian group Z are the defining relations for the group G .

Using (2) and (3), equation (4) yields

$$(\gamma_j^{y_j})^{p^{s_j}} = w_0(\alpha_{\sim 1}^{x_1}, \alpha_{\sim 2}^{x_2} \dots \alpha_{\sim m}^{x_m}) \quad (5)$$

giving a restriction on $x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n$.

Also under the isomorphism θ , (1)' becomes

$$\alpha_1^{*p^{r_1}} = w_1(\alpha^*) \gamma_{\sim 1}^{*a_1^*}. \quad (1)^*$$

From (2) we get

$$\begin{aligned} \alpha_1^{*p^{r_1}} &\equiv \alpha_{\sim 1}^{p^{r_1} x_{\sim 1}} \pmod{H_2} \\ &\equiv \alpha_1^{p^{r_1} x_{11}} \dots \alpha_1^{p^{r_1} x_{11}} \dots \alpha_n^{p^{r_1} x_{1n}} \pmod{H_2} \end{aligned}$$

(where x_{1j} is the j 'th component of the vector $x_{\sim 1}$)

$$\equiv \alpha_1^{p^{r_1} x_{11}} \dots \gamma_{\sim 1}^{x_{11} a_1} \dots \alpha_n^{p^{r_1} x_{1n}} \pmod{H_2}$$

$\equiv \gamma_{\sim 1}^{f(a_1)}$
for some function $f(a_1)$, since $\gamma_{\sim 1}^{a_1}$ does not occur as any $\alpha_j^{p^{r_1} x_{1j}}$ ($j \neq 1$) and the term involving H_2 in (2) is independent of $\gamma_{\sim 1}^{a_1}$.

Similarly, using (3) we obtain

$$\underline{y}^* \underline{a}_1 = \underline{y} g(\underline{a}_1^*, \underline{a}_2^*, \dots, \underline{a}_n^*)$$

for some function g . Thus, by comparing indices of each y_j on the left and right hand sides of (1)* we obtain a set of equations expressing the a^* 's in terms of the a 's with restrictions on the variables $\underline{x}_1, \underline{x}_2, \dots, \underline{x}_m, \underline{y}_1, \underline{y}_2, \dots, \underline{y}_n$ given by (5). It may be verified that this system of equations actually defines an equivalence relation on the parameters $\underline{a}_1$ and so on the set of matrices

$$A = (\underline{a}_1, \dots, \underline{a}_m).$$

Thus the isomorphism classes of an isoclinism family may be considered to correspond to equivalence classes of such matrices. For p -groups, the calculation of the isomorphism classes is simplified as the matrices are usually over $GF(p)$. However the case of non-regular p -groups must be treated separately from the case of regular p -groups because of complications in calculating the $(\underline{a}_1^*)^p$.

The remainder of this chapter consists in finding the equivalence classes of the above matrices for each family and choosing a suitable canonical form for each equivalence class. (This has been done for all families except Φ_{21} which proved to be too difficult.)

$$\underline{\phi_1 \ (p \geq 2)}$$

In the case of ϕ_1 , the basis theorem for abelian groups tells us that there is a one-to-one correspondence between the partitions of n and the groups of $p^n \phi_1$. In accordance with custom, we shall omit the symbol ϕ_1 for the groups of that family, simply referring to them by their type invariants.

$$\underline{\phi_2 \ (p > 2)}$$

The groups of ϕ_2 are characterized by: $G/Z(G)$ is (11), $H_2(G)$ is (1).

Let $\alpha_1, \alpha_2 \in G - Z(G)$ and $\beta = [\alpha_1, \alpha_2]$.

Then $\alpha_1^p, \alpha_2^p \in Z(G)$ and $H_2(G) = \langle \beta | \beta^p = 1 \rangle$.

Case I If $Z(G)$ is cyclic, $Z(G) = \langle \gamma | \gamma^{p^r} = \beta \rangle$ ($r \geq 0$). For

$i = 1$ or 2 , $\alpha_i^p \in Z(G)$ and so $\alpha_i^p = \gamma^{a_i}$ where we may suppose $a_i \neq 0$ if either $\alpha_i \neq 1$. In this case, by replacing α_2 by $\alpha_1^{-a_2} \alpha_2^{a_1}$ and γ by γ^{a_1} , we may suppose $\alpha_1^p = \gamma$, $\alpha_2^p = 1$.

Thus we have two groups with $\alpha_2^p = 1$ and $\alpha_1^p = \gamma$ or 1 .

Case II If $Z(G)$ is of type (r1) then

$$Z(G) = \langle \gamma_1, \gamma_2 | \gamma_1^{p^r} = \gamma_2^p = 1 \rangle \text{ where } \beta \in Z(G).$$

(a) If $\beta \notin Z^p(G)$, we may suppose $\beta = \gamma_2$.

(i) If both $\alpha_1^p \in H_2(G)$, then G is the direct product of a stem group with a cyclic group of order p^r .

(ii) If α_1^p or $\alpha_2^p \notin H_2(G)$, we may suppose that

$\alpha_1^p \equiv \gamma_1^{a_1} \pmod{H_2}$ where $a_1 \neq 0$. By replacing α_1 by α_1^x ,

α_2 by $\alpha_1^{-a_2} \alpha_2^{a_1}$ (where $xa_1 \equiv 1 \pmod{p^r}$), we may suppose

$$\alpha_1^p = \gamma_1 \gamma_2^{c_1}, \quad \alpha_2^p = \gamma_2^{c_2}.$$

Then, by replacing γ_1 by $\gamma_1 \gamma_2^{c_1}$ and α_1 by $\alpha_1^{c_2}$ (if $c_2 \neq 0$), we may suppose $\alpha_1^p = \gamma_1$ and $\alpha_2^p = \beta$ or 1.

(b) If $\beta \in Z^p(G)$, since $\beta^p = 1$, we may suppose $\beta = \gamma_1^{p^{r-1}}$.

(These groups only exist for $r > 1$.)

(i) If both $\alpha_1^p \in \langle \gamma_1 \rangle$, then G is the direct product of a group in case I with a cyclic group of order p .

(ii) If either $\alpha_1^p \notin \langle \gamma_1 \rangle$, the same reasoning as above shows that we may suppose $\alpha_1^p = \gamma_1$ or 1 and $\alpha_2^p = \gamma_2$.

Case III If $Z(G)$ is elementary abelian, then

$$Z(G) = \langle \gamma_1, \gamma_2, \dots, \gamma_s \rangle \text{ where } \beta = \gamma_s.$$

(a) If both $\alpha_1^p \in H_2(G)$, then G is the direct product of a stem group with an elementary abelian group.

(b) If one of $\alpha_1^p, \alpha_2^p \notin H_2(G)$ we may suppose $\alpha_1^p = \beta^a$, $\alpha_2^p = \gamma_2$. If $a \neq 0$, by replacing α_2 by α_2^a , γ_2 by γ_2^a we may suppose $a = 1$. Thus there are two groups with $\alpha_2^{p^2} = 1$, $\alpha_1^p = [\alpha_1, \alpha_2]$ or 1.

(c) If neither α_1^p nor $\alpha_2^p \in H_2(G)$, we may suppose that $\alpha_2^p \notin \langle \alpha_1^p \rangle$ and so we have just one group (when $s > 2$) with $\alpha_1^{p^2} = \alpha_2^{p^2} = 1$.

Note that for $s = 2$, (a) and (b) reduce to case II with $r = 1$. We now consider the remaining cases of order p^6 .

Case IV $Z(G)$ is of type (22).

i.e. $Z(G) = \langle \gamma_1, \gamma_2 \mid \gamma_1^{p^2} = \gamma_2^{p^2} = 1, \gamma_1^p = \beta \rangle$ with $\alpha_1^p = \gamma_1^{a_1}$.

If $\alpha_1^p = \gamma_1^{a_1 + pb}$ where $\gamma \in Z(G)$, by replacing α_1 by $\alpha_1 \gamma^{-b}$ we

may suppose $\alpha_1^p = \gamma_1^{a_1}$ ($0 \leq a_1 < p$). Similarly we may suppose

$\alpha_2^p = \gamma_1^{a_2}$ ($0 \leq a_2 < p$).

(a) If both $\alpha_1^p, \alpha_2^p \in \langle \gamma_1 \rangle$, then G is the direct product of a group whose centre is of type (2) and a cyclic group of order p^2 .

(b) If some $\alpha_1^p \notin \langle \gamma_1 \rangle$ then by similar arguments we may

suppose $\alpha_1^p = \gamma_2, \alpha_2^p = \gamma_1^{a_{21}} \gamma_2^{a_{22}}$. Replacing α_2 by $\alpha_1^{-a_{12}} \alpha_2$

we may suppose $\alpha_2^p \in \langle \gamma_1 \rangle$ and (if $a_{21} \neq 0$) by replacing α_1, γ_1

by $\alpha_1^{a_{21}}, \gamma_1^{a_{21}}$ we may suppose $a_{21} = 1$ or 0 , giving two groups.

Case V $Z(G)$ is of type (211).

i.e. $Z(G) = \langle \gamma_1, \gamma_2, \gamma_3 \mid \gamma_1^{p^2} = \gamma_2^p = \gamma_3^p = [\gamma_1, \gamma_j] = 1$

$(1, j = 1, 2, 3) \rangle$.

(a) If $\beta \notin Z^p$, we may suppose $\beta = \gamma_2$.

(1) If $\alpha_1^p, \alpha_2^p \in \langle \gamma_2, \gamma_3 \rangle$, then G is the direct product of a group whose centre is of type (11) and a cyclic group of order p^2 .

(ii) If $\alpha_1^p, \alpha_2^p \in \langle \gamma_1, \gamma_2 \rangle$, then G is the direct product of a group whose centre is of type (21) and a cyclic group of order p .

(iii) For any other group in case V (a), one of α_1^p , $\alpha_2^p \notin \langle \gamma_2, \gamma_3 \rangle$ and so (as in case IV) we may suppose $\alpha_1^p = \gamma_1$. Also $\alpha_2^p \notin \langle \gamma_1, \gamma_2 \rangle$ and so we may suppose $\alpha_2^p = \gamma_3$. Thus there is just one group.

(b) If $\beta \in Z^p$, we may suppose $\beta = \gamma_1^p$.

If $\alpha_1^p, \alpha_2^p \in \langle \gamma_1, \gamma \rangle$ for some $\gamma \in Z(G)$, then G is the direct product of a group whose centre is of type (21) and a cyclic group of order p .

If not, $\alpha_1^p \notin \langle \gamma_1 \rangle$ and so we may suppose $\alpha_1^p = \gamma_2$ and $\alpha_2^p \notin \langle \gamma_1, \gamma_2 \rangle$ and so we may suppose $\alpha_2^p = \gamma_3$ giving just one group.

The 2 groups of order p^3 are obtained by taking $r = 1$ in case I.

The 6 groups of order p^4 and class 2 are obtained by taking $r = 2$ in case I and $r = 1$ in case II (a).

The 15 groups of $p^5\phi_2$ are obtained by taking $r = 3$ in case I, $r = 2$ in case II and $s = 3$ in case IV.

The 31 groups of $p^6\phi_2$ are obtained by taking $r = 4$ in case I, $r = 3$ in case II, $s = 4$ in case III and also the groups of cases IV and V.

$$\underline{\alpha_{(n)} \quad (p \geq 3) \quad (\phi_3, \phi_9 \text{ and } \phi_{35})}$$

These families are characterized by: $G/Z(G)$ is the group $\mathcal{A}_{(n-1)}(1^{n+1})$ of theorem 1, $|H_1(G):H_{1+1}(G)| = p$ ($2 \leq 1 \leq n+1$), $H_{n+2}(G)$ is E and G has a maximal abelian subgroup H_1 .

Let $\alpha \in G - H_1$, $\alpha_1 \in H_1 - H_2$, $\alpha_{i+1} = [\alpha_i, \alpha]$ ($i = 1, \dots, n$). Then $H_1(G) = \langle \alpha_1, \dots, \alpha_n \rangle$ and $[\alpha_{n+1}, \alpha] = [\alpha_1, \alpha_j] = 1$ ($i, j = 1, \dots, n$).

Since $G/Z(G)$ is $\mathcal{A}_{(n-1)}(1^{n+1})$, we have

$$\alpha^p, \alpha_1^p \alpha_2^{\binom{p}{2}} \dots \alpha_p \in Z(G) \quad \text{and so}$$

$$\alpha_1^p \alpha_{i+1}^{\binom{p}{2}} \dots \alpha_{i+p-1} = 1 \quad (2 \leq i \leq n+1).$$

Also, by lemma L3, $(\alpha \alpha_1^z)^p = \alpha^p \alpha_1^{zp} \alpha_2^{z \binom{p}{2}} \dots \alpha_p^z$.

Putting $z = x^{-1} x_1$ ($x \neq 0$) and using lemma L5 and the fact that

$(\alpha \alpha_1^z)^x = \alpha^x \alpha_1^{xz} \zeta$, some $\zeta \in H_2(G)$, we have

$$\begin{aligned} (\alpha^x \alpha_1^{x_1})^p &= (\alpha \alpha_1^z)^{px} \\ &= \alpha^{px} (\alpha_1^p \alpha_2^{\binom{p}{2}} \dots \alpha_p)^{x_1}. \end{aligned}$$

Case I If $Z(G)$ is cyclic, $Z(G) = \langle \gamma | \gamma^{p^r} = \alpha_{n+1} \rangle$.

Then $\alpha^p = \gamma^a$, $\alpha_1^p \alpha_2^{\binom{p}{2}} \dots \alpha_p = \gamma^b$, $\alpha^{*p} = \gamma^{a*}$, $\alpha_1^{*p} \dots \alpha_p^* = \gamma^{b*}$

where

$$\alpha^* = \alpha \alpha_1^x \alpha_2^{x_1} \dots \gamma^{x'}$$

$$\alpha^*_1 = \alpha_1^{y_1} \alpha_2^{y_2} \dots \gamma^{y'} \quad xy_1 \neq 0$$

$$\alpha^*_2 = \alpha_2^{xy_1} \dots$$

$$\alpha^*_{n+1} = \alpha_{n+1}^{x^n y_1}$$

and so $\gamma^* = \gamma^{x^n y_1 + pz}$.

By lemma L5 $\alpha^{*p} = (\alpha \alpha_1^x \alpha_1^{x_1})^p \gamma^{px'}$
 $= \gamma^{xa + x_1 b + px'}$ by above

so $(x^n y_1 + pz)a^* = xa + x_1 b + px'$. (1)'

To get a second equation, we note that

$$(\alpha \alpha_1)^p = \alpha^p (\alpha_1^p \alpha_2^{\binom{p}{2}} \dots \alpha_p)$$

$$= \gamma^{a+b}$$

and similarly, $(\alpha^* \alpha^*_1)^p = \gamma^{a^* + b^*}$

where $(\alpha^* \alpha^*_1)^p = (\alpha \alpha_1^x \alpha_1^{x_1} \dots \alpha_1^{y_1} \alpha_2^{y_2} \dots)^p$
 $= (\alpha \alpha_1^{x + x_1 + y_1})^p \gamma^{px' + py'}$ by lemma L5
 $= \alpha^{px} (\alpha_1^p \dots \alpha_p)^{x_1 + y_1} \gamma^{p(x' + y')}$
 $= \gamma^{xa + (x_1 + y_1)b + p(x' + y')}$

So $x^n y(a^* + b^*) = xa + (x_1 + y_1)b + p(x' + y')$.

Using (i)', we have $x^n b^* = b + y_1^{-1} p y' \dots$ (ii)'

By choice of x', y' , we may suppose

$0 \leq a, b \leq p-1$, and equations become

$$x^n y_1 a^* \equiv xa + x_1 b \pmod{p} \quad (i)$$

$$x^n b^* \equiv b \pmod{p}. \quad (ii)$$

If $b \neq 0$, by taking $x_1 = -x^a/b$, we may suppose $a = 0$. Also as x takes all possible values mod p , x^n takes $\frac{p-1}{(p-1, n)}$ values and so we may suppose

$$b = 1, g, \dots, g^s \quad (s+1 = (p-1, n))$$

where g is a primitive root mod p .

If $b = 0$, we see that a may take the values 0 or 1.

Thus we have $2 + (n, p-1)$ groups whose centres are cyclic.

Taking $r = 0$, we obtain the stem groups of ϕ_3, ϕ_9 and ϕ_{35} (for $n = 2, 3, 4$).

Taking $r = 1$, we obtain the groups of ϕ_3 and ϕ_9 with cyclic centres of order p^2 .

Taking $r = 2$, we obtain the groups of $p^6 \phi_3$ with centres of type (3).

In particular, we see that there are $2 + (p-1, 4)$ groups of $p^6 \phi_{35}$ and 4 groups of $3^6 \Delta_{35}$.

Case II If $Z(G)$ is of type (r1), then

$$Z(G) = \langle \gamma_1, \gamma_2 | \gamma_1^{p^r} = \gamma_2^p = 1 \rangle \text{ where } \alpha_{n+1} \in Z(G).$$

(a) If $\alpha_{n+1} \notin Z^p(G)$, we may suppose $\alpha_{n+1} = \gamma_2$.

(i) If $\alpha^p, \alpha_1^p \alpha_2^{\binom{p}{2}} \dots \alpha_p \in H_{n+1}(G)$, then G is the direct product of a stem group with a cyclic group of order p^r .

(ii) If $\alpha_1^p \alpha_2^{\binom{p}{2}} \dots \alpha_p \notin H_{n+1}(G)$, put $\alpha^p = \gamma_1^a \gamma_2^c$,
 $\alpha_1^p \alpha_2^{\binom{p}{2}} \dots \alpha_p = \gamma_1^{a_1} \gamma_2^{c_1}$, where $a_1 \neq 0$. By replacing α
 by $\alpha_1^{a_1 - a}$ and γ_1 by $\alpha_1^{a_1} \gamma_2^{c_1}$, we may suppose $\alpha^p = \gamma_2^c$,
 $\alpha_1^p \alpha_2^{\binom{p}{2}} \dots \alpha_p = \gamma_1$. Then by replacing α_1 by α_1^c (if $c \neq 0$)
 we may suppose $\alpha^p = \alpha_{n+1}$ or 1.

(iii) If $\alpha_1^p \alpha_2^{\binom{p}{2}} \dots \alpha_p \in H_{n+1}(G)$ and $\alpha^p \notin H_{n+1}(G)$ we
 may suppose (as before) $\alpha^p = \gamma_1$, $\alpha_1^p \alpha_2^{\binom{p}{2}} \dots \alpha_p = \alpha_{n+1}^c$.

Taking $\alpha^* = \alpha^x$, $\alpha_1^* = \alpha_1^{x^{1-1}y}$... and $\alpha_1^{*p} \dots \alpha_p^* = \gamma_2^{*c^*}$
 we have $x^n c^* = c$.

As in I, we may suppose $c = 1$, $g \dots g^s$, $0(s+1 = (p-1, n))$.

(b) If $\alpha_{n+1} \in Z^p(G)$ we may suppose $\alpha_{n+1} = \gamma_1^{p^{r-1}}$ (here $r > 1$
 or (b) is impossible).

(i) If $\alpha^p, \alpha_1^p \alpha_2^{\binom{p}{2}} \dots \alpha_p \in \langle \gamma_1 \rangle$, then G is the direct
 product of a group of case I together with a cyclic group
 of order p .

(ii) If $\alpha_1^p \alpha_2^{(p)} \dots \alpha_p \notin \langle \gamma_1 \rangle$, we may write

$$\alpha_1^p \alpha_2^{(p)} \dots \alpha_p = \gamma_2, \quad \alpha^p = \gamma_1^{a_1} \gamma_2^{a_2}.$$

As above, we may suppose $a_2 = 0$, $a_1 = 1$ or 0 .

(iii) If $\alpha_1^p \alpha_2^{(p)} \dots \alpha_p \in \langle \gamma_1 \rangle$ and $\alpha^p \notin \langle \gamma_1 \rangle$ we may write

$$\alpha^p = \gamma_2, \quad \alpha_1^p \alpha_2^{(p)} \dots \alpha_p = \gamma_1^b.$$

As above, we may suppose $b = 0, 1, g \dots g^s$.

Taking $r = 1$ (in (a) only) we obtain the $5 + 2(2, p-1)$ groups of ϕ_3 and the $5 + 2(3, p-1)$ groups of ϕ_9 which have centres of type (11).

Taking $r = 2$, we obtain the $10 + 4(2, p-1)$ groups of ϕ_3 of order p^6 with centres of type (21).

Case III $Z(G)$ is elementary abelian.

If $Z(G)$ is (1) or (11), we may use cases I and II above. So we may suppose

$$Z(G) = \langle \alpha_{n+1}, \gamma_2, \dots, \gamma_s \rangle \quad (s \geq 3).$$

If α^p or $\alpha_1^p \alpha_p \in \langle \alpha_{n+1}, \gamma \rangle$, some $\gamma \in Z - H_{n+1}$, we may suppose

(by choice of γ_2) that $\alpha^p, \alpha_1^p \alpha_2^{(p)} \dots \alpha_p \in \langle \alpha_{n+1}, \gamma_2 \rangle$ and so

G is the direct product of an elementary abelian group together with a group whose centre is of type (11).

Otherwise we may suppose $\gamma_2 = \alpha^p$, $\gamma_3 = \alpha_1^p \alpha_2^{\binom{p}{2}} \dots \alpha_p$ giving only one group which is the direct product of an

elementary abelian group and the group $\langle \alpha, \alpha_1, \dots, \alpha_{n+1} \mid \alpha^{p^2} = \alpha_1^{p^2} \alpha_2^{p^{\binom{p}{2}}} \dots \alpha_p^p = \alpha_1^p \dots \alpha_{p+1-p}^p = 1 \ (i = 2, \dots, n+1),$

$[\alpha_j, \alpha] = \alpha_{j+1} \ (j = 1, \dots, n) \rangle$.

The $2 + (p-1, 2)$ groups of $p^4 \Phi_3$ are obtained by taking $r = 0$ in case I (with $n = 2$).

The $7 + 3(p-1, 2)$ groups of $p^5 \Phi_3$ are obtained by taking $r = 1$ in cases I and II(a) (with $n = 2$).

The $2 + (p-1, 3)$ groups of $p^5 \Phi_9$ are obtained by taking $r = 0$ in case I (with $n = 3$).

The $18 + 7(p-1, 2)$ groups of $p^6 \Phi_3$ are obtained by taking $r = 2$ in cases I and II and $s = 3$ in case III.

The $7 + 3(p-1, 3)$ groups of $p^6 \Phi_9$ are obtained by taking $r = 1$ in cases I and II(a) (with $n = 3$).

The $2 + (p-1, 4)$ groups of $p^6 \Phi_{35}$ are given by case I ($r = 0$, $n = 4$).

$$\underline{\phi_4 \quad (p \geq 3)}$$

The groups of ϕ_4 are characterized by: $G/Z(G)$ is (111), $H_2(G)$ is (11). By theorem 2 of chapter II we may suppose

$$G = \langle \alpha, \alpha_1, \beta, Z(G) \rangle$$

where $[\alpha_1, \alpha] = \alpha_2$, $[\alpha, \beta] = \eta$ and $[\alpha_1, \beta] \in Z(G)$, and we may take $[\alpha_1, \beta] = 1$.

For convenience of notation, we shall write

$$G = \langle \alpha_1, \alpha_2, \alpha, Z(G) \mid [\alpha_1, \alpha] = \beta_1 \in Z(G) \ (i=1,2), [\alpha_1, \alpha_2] = 1 \rangle.$$

I - Stem Groups i.e. $Z(G) = H_2(G) = \langle \beta_1, \beta_2 \rangle$.

Since $G/Z(G)$ has exponent p , $\alpha^p = \beta_1^a$, $\alpha_1^p = \beta_2^a$ ($i = 1, 2$).

$$\text{Let } \alpha^* = \alpha^x \alpha_1^{x_1} \alpha_2^{x_2} \dots$$

$$\alpha_1^* = \alpha_1^{y_1} \alpha_2^{y_2} \dots \quad x \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} = x |y, z| \neq 0$$

$$\alpha_2^* = \alpha_1^{z_1} \alpha_2^{z_2} \dots$$

$$\text{Then } \beta_1^* = \beta_1^{xy}, \quad \beta_2^* = \beta_2^{xz}.$$

By evaluating $\alpha^{*p} = \beta_1^{*a^*}$ etc we get

$$x\alpha_1 + x_1\alpha_{11} + x_2\alpha_{21} = x(y_1\alpha_1^* + z_1\alpha_2^*)$$

$$x\alpha_2 + x_1\alpha_{12} + x_2\alpha_{22} = x(y_2\alpha_1^* + z_2\alpha_2^*)$$

$$y_1\alpha_{11} + y_2\alpha_{21} = x(y_1\alpha_{11}^* + z_1\alpha_{12}^*)$$

$$y_1\alpha_{12} + y_2\alpha_{22} = x(y_2\alpha_{11}^* + z_2\alpha_{12}^*)$$

$$z_1\alpha_{11} + z_2\alpha_{21} = x(y_1\alpha_{21}^* + z_1\alpha_{22}^*)$$

$$z_1\alpha_{12} + z_2\alpha_{22} = x(y_2\alpha_{21}^* + z_2\alpha_{22}^*)$$

$$\text{where } \underline{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \quad \underline{a}_1 = \begin{bmatrix} a_{11} \\ a_{12} \end{bmatrix}.$$

Putting $A^* = (\underline{a}_1^*, \underline{a}_2^*)$, $A = (\underline{a}_1, \underline{a}_2)$, $X = (\underline{y}, \underline{z})$

these equations become

$$\underline{x}\underline{a} + A\underline{x} = \underline{x}X\underline{a}^* \quad \dots (1)$$

$$X^{-1}AX = \underline{x}A^* \quad \dots (2)$$

and so $|A| = x^2 |A^*|$, $\text{tr } A = x \text{tr } A^*$.

(a) If $|A| \neq 0$, take $\underline{x} = -xA^{-1}\underline{a}$ and so we may suppose $\underline{a} = \underline{0}$.

To find canonical forms for A , we consider its characteristic polynomial $c_A(u)$.

(i) If $c_A(u) = (u-a)^2$ ($a \neq 0$) by taking $x = a$ we may suppose $a = 1$.

Since matrices A with $c_A(u) = (u-1)^2$ are similar to one of the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

we may suppose A is one of these, giving two groups.

(ii) If $c_A(u) = (u-a)(u-b)$ ($a \neq b$, $ab \neq 0$) then $X^{-1}AX$ can be taken as $\text{diag}(a, b)$ where

$$a/b = g^k, \quad 1 \leq k < \frac{1}{2}(p-1).$$

So, taking $x = b$, we may suppose

$$A = \text{diag}(a, 1) \quad a = g^k \quad (k = 1, 2, \dots, \frac{1}{2}(p-1))$$

giving $\frac{1}{2}(p-1)$ groups.

(iii) If $c_A(u) = u^2 - g_1u + g_2$ does not have linear factors over $\text{GF}(p)$ and $g_1 = \text{tr } A \neq 0$, we may suppose (by taking $x = g_1$) that $g_1 = 1$ and so we may suppose

$$A^* = \begin{pmatrix} 0 & -g_2 \\ 1 & 1 \end{pmatrix}$$

where $1 - 4g_2$ is a quadratic non-residue

$$\text{i.e. } A^* = \begin{pmatrix} 0 & \frac{1}{2}(g^{1+2k}-1) \\ 1 & 1 \end{pmatrix} \quad k = 0, 1, \dots, \frac{1}{2}(p-3)$$

since any quadratic non-residue may be written as g^{1+2k} where g is a primitive root mod p .

(iv) If $c_A(u) = u^2 + g_2$ does not have linear factors over $GF(p)$, then $-g_2$ is a quadratic non-residue and so, by taking $x^2 = -g_2/v$, we may suppose $g_2 = -v$ and so we may suppose

$$A^* = \begin{pmatrix} 0 & v \\ 1 & 0 \end{pmatrix} \quad (v \text{ a fixed quadratic non-residue}).$$

(b) If $|A| = 0$, then $c_A(u) = u(u - a)$ for some $a \in GF(p)$.

(i) If $a \neq 0$, then A is similar to the matrix $\text{diag}(a, 0)$ and so, by taking $x = a$, we may suppose

$$A^* = \text{diag}(1, 0).$$

Substituting this for A in (2), we see that, under these conditions, $X = \text{diag}(y_1, z_2)$, $x = 1$. Equation (1) becomes

$$\begin{aligned} a_1 + x_1 &= y_1 a_{1.1}^* \\ a_2 &= z_2 a_{2.2}^*. \end{aligned}$$

So we may suppose $a_1 = 0$, $a_2 = 0$ or 1 , giving two groups.

(ii) If $a = 0$, we may suppose $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. If $A = \underline{0}$, equation (1) becomes $\underline{a} = X \underline{a}^*$ and so the standard forms for $\underline{a}$ are $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

If $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, we take $X = \begin{pmatrix} y_1 & z_1 \\ 0 & xy_1 \end{pmatrix}$ and so equation (1) becomes

$$xa_1 + x_2 = xy_1 a_1^* + xz_1 a_2^*$$

$$a_2 = xy_1 a_2^*$$

and so we may suppose $a_1 = 0$, $a_2 = 0$ or 1 .

II - First Branch i.e. $Z(G)$ is (21) or (111) .

(a) If $Z(G)$ is (21) , put $Z(G) = \langle \gamma_1, \gamma_2 \mid \gamma_1^{p^2} = \gamma_2^p = 1 \rangle$.

Since $\beta_i^p = 1$ and $\beta_i \in Z(G)$ ($i = 1, 2$), we may suppose (by suitable choice of $\alpha_1, \gamma_1, \gamma_2$) that

$$\beta_1 = \gamma_1^p, \quad \beta_2 = \gamma_2, \quad \alpha^p = \gamma^a, \quad \alpha_1^p = \gamma^{a_1} \quad (i = 1, 2).$$

Let $\alpha^* = \alpha^x \alpha_1^{x_1} \alpha_2^{x_2} \gamma_1^{x'} \dots$

$$\alpha_1^* = \alpha_1^{y_1} \alpha_2^{y_2} \gamma_1^{y'} \dots \quad x \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} \neq 0$$

$$\alpha_2^* = \alpha_1^{z_1} \alpha_2^{z_2} \gamma_1^{z'} \dots$$

Then $\beta_1^* = \beta^{xy} = \gamma_1^{pxy_1} \gamma_2^{xy_2}$.

Since $\beta_1^* \in Z^p(G)$, we must take $y_2 = 0$ and so

$$\gamma_1^* = \gamma_1^{xy_1+pt} \gamma_2^w, \quad \gamma_2^* = \gamma_1^{pxz_1} \gamma_2^{xz_2}.$$

Also (by choice of x', y', z', t) we may suppose $a_{11}, a_1 < p$.

From $\alpha^*P = y^*a^*$, we get

$$xa_1 + x_1a_{11} + x_2a_{21} = xy_1a_1^*$$

$$xa_2 + x_1a_{12} + x_2a_{22} = wa_1^* + xz_2a_2^*$$

$$a_{11} = xa_{11}^*$$

$$y_1a_{12} = wa_{11}^* + xz_2a_{12}^*$$

$$z_1a_{11} + z_2a_{21} = xy_1a_{21}^*$$

$$z_1a_{12} + z_2a_{22} = wa_{21}^* + xz_2a_{22}^*.$$

(i) If $a_{11} \neq 0$, by taking $x = a_{11}$, we may suppose $a_{11} = 1$.

Also, by taking $z_1 = -z_2a_{21}$, $w = y_1a_{12}$, $x_1 = -(xa_1 + x_2a_{21})$,

we may suppose $a_{21} = a_{12} = a_1 = 0$.

The other equations become $z_2a_{22} = z_2a_{22}^*$ and

$$a_2 = z_2a_2^* - x_2a_{22}$$

with standard forms $a_{22} = \text{any } 0$

$$a_2 = 0 \quad 1.$$

(ii) If $a_{11} = 0$, $a_{21} \neq 0$, by taking $xy_1 = z_2a_{21}$, we may suppose $a_{21} = 1$.

Also, by taking $x_2 = -xa_1$, $w = z_1a_{12} + z_2a_{22}$, we may suppose

$$a_1 = a_{22} = 0.$$

Other equations become $a_{12} = x^2a_{12}^*$

$$xa_2 + x_1a_{12} = x^2y_1a_2^*$$

with standard forms $a_{12} = 1 \quad v \quad 0 \quad 0$

$$a_2 = 0 \quad 0 \quad 1 \quad 0.$$

(iii) If $a_{11} = 0$, $a_{12} \neq 0$, by taking $xz_2 = y_1 a_{12}$, we may suppose $a_{12} = 1$.

Also, by taking $z_1 = -z_2 a_{22}$, $x_1 = -x a_2 - x_2 a_{22} + w a_{11}^*$ we may suppose $a_{22} = a_2 = 0$.

From the other equation, we may suppose $a_1 = 0$ or 1.

(iv) If $a_{11} = a_{12} = 0$, the equations become

$$a_{22} = x a_{22}^*$$

$$a_1 = y_1 a_{11}^*$$

$$x a_2 + x_2 a_{22} = w a_{11}^* + x z_2 a_{12}^*$$

$$\text{with standard forms } a_{22} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$a_1 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$a_2 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

(b) If $Z(G)$ is (111), put $Z(G) = \langle \gamma, \beta_1, \beta_2 \rangle$.

(i) If $\alpha^p, \alpha_1^p, \alpha_2^p \in H_2(G)$, then G is the direct product of a stem group together with a group of type (1).

(ii) If either α_1^p or $\alpha_2^p \notin H_2(G)$, we may suppose (by interchanging α_1 and α_2 if necessary) that $\alpha_1^p \notin H_2(G)$.

$$\text{Take } \gamma = \alpha_1^p, \alpha_2^p = \beta_2^{a_2} \gamma^{b_2}, \alpha^p = \beta_2^a \gamma^b.$$

By $\{\alpha \rightarrow \alpha \alpha_1^{-b}, \alpha_2 \rightarrow \alpha_2 \alpha_1^{-b_2}\}$ we may take $b = b_2 = 0$.

Let $\alpha_2^{*p} = \beta_2^{*a_2^*}$, $\alpha^{*p} = \beta^{*a^*}$ where

$$\begin{aligned}\alpha^* &= \alpha^x \alpha_2^{x_2} \\ \alpha_1^* &= \alpha_1^{y_1} \alpha_2^{y_2} \quad xy_1z_2 \neq 0. \\ \alpha_2^* &= \alpha_2^{z_2}.\end{aligned}$$

Then $\beta_1^* = \beta^{xy}$, $\beta_2^* = \beta^{xz_2}$ and

$$\begin{aligned}xa_1 + x_2a_{21} &= xy_1a_1^* \\ xa_2 + x_2a_{22} &= x(y_2a_1^* + z_2a_2^*) \\ z_2a_{21} &= xy_1a_{21}^* \\ z_2a_{22} &= x(y_2a_{21}^* + z_2a_{22}^*)\end{aligned}$$

whose standard forms are easily seen to be:

$$\begin{aligned}a_{21} &= 1 & 1 & 0 & 0 & 0 & 0 \\ a_{22} &= 0 & 0 & 1 & 1 & 0 & 0 \\ a_1 &= 0 & 0 & 1 & 0 & 1 & 0 \\ a_2 &= 1 & 0 & 0 & 0 & 0 & 1\end{aligned}$$

(iii) If both $\alpha_1^p \in H_2(G)$ and $\alpha^p \notin H_2(G)$, take $\alpha^p = \gamma$.

By an argument similar to that for the stem groups (with $\alpha^p = 1$ there) we see that the standard forms for $(\underline{a}, \underline{a}_1, \underline{a}_2)$, where

$\alpha^p = \gamma\beta^{\underline{a}}$, $\alpha_1^p = \beta^{\underline{a}_1}$, are $(\underline{0}, \underline{0}, \underline{0})$, $(\underline{0}, \underline{e}, \underline{0})$, $(\underline{0}, \underline{0}, \underline{e})$ and $(\underline{0}, \underline{A})$ where $\underline{A}$ is the matrix used in the stem groups with $|\underline{A}| \neq 0$, and $\underline{e} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

The above theory shows that there are $p + 8$ groups of $p^5\Phi_4$ and $3p + 32$ groups of $p^6\Phi_4$.

$\phi_5 \ (p \geq 3)$

The groups of ϕ_5 are characterized by: $G/Z(G)$ is (1^4) , $H_2(G)$ is (1) . By the previous chapter (Theorem 4, Cor.), we may choose elements α_1, β of G such that $[\alpha_1, \alpha_2] = \beta$, $[\alpha_3, \alpha_4] = \beta^b$, other commutators = 1. In fact, by $\{\alpha_3^b \rightarrow \alpha_3\}$ we may suppose $[\alpha_3, \alpha_4] = \beta$. Also, by the same theorem, the only stem groups are $\phi_2(111) \times_Z \phi_2(111)$ and $\phi_2(21) \times_Z \phi_2(111)$.

First Branch i.e. $Z(G)$ is (2) or (11) .

(a) If $Z(G)$ is (2) , put $Z(G) = \langle \gamma | \gamma^p = \beta \rangle$. Then $\alpha_1^p = \gamma^{a_1}$.

If any $a_1 \neq 0$, by permuting the α_1 , we may suppose $a_1 \neq 0$.

By $\{\alpha_2 \rightarrow \alpha_2^{a_1}, \alpha_3 \rightarrow \alpha_3^{a_1}\}$ we may suppose $a_1 = 1$.

By $\{\alpha_2 \rightarrow \alpha_1^{-a_2} \alpha_2\}$ we may suppose $a_2 = 0$.

Similarly we may suppose $a_4 = 0$.

By $\{\alpha_2 \rightarrow \alpha_2 \alpha_4^{a_3}, \alpha_3 \rightarrow \alpha_1^{-a_3} \alpha_3\}$ we may suppose $a_3 = 0$. Thus

there are only two groups of ϕ_5 with $Z(G)$ of type (2) .

(b) If $Z(G)$ is (11) , put $Z(G) = \langle \gamma, \beta \rangle$.

(i) If all $\alpha_1^p \in H_2(G)$, then G is the direct product of a stem group and a group of type (1) .

(ii) If some $\alpha_1^p \notin H_2(G)$, we may suppose $\alpha_1^p \notin H_2(G)$.

Write $\alpha_1^p = \gamma$, $\alpha_1^p = \beta^b \gamma^c$.

By $\{\alpha_2 \rightarrow \alpha_2 \alpha_1^{-c_2}\}$, we may suppose $c_2 = 0$.

Similarly, we may suppose $\alpha_4^p \in H_2(G)$ and so, by

$\{\alpha_2 \rightarrow \alpha_2 \alpha_4^{c_3}, \alpha_3 \rightarrow \alpha_1^{-c_3} \alpha_3\}$, we may suppose $\alpha_3^p \in H_2(G)$.

Also, we may write $\alpha_1^{*p} = \gamma^*$, $\alpha_i^{*p} = \beta^{*b_i^*}$ ($i = 2, 3, 4$) where

$$\alpha_1^* = \alpha_1^x \dots$$

$$\alpha_2^* = \alpha_2^{y_2} \alpha_3^{y_3} \alpha_4^{y_4} \dots$$

$$\alpha_3^* = \alpha_2^{z_2} \alpha_3^{z_3} \alpha_4^{z_4} \dots$$

$$\alpha_4^* = \alpha_2^{w_2} \alpha_3^{w_3} \alpha_4^{w_4} \dots$$

and so
$$\beta^* = \beta^{x_1 y_2 + x_3 y_4 - x_4 y_3}.$$

If either $\alpha_3^p, \alpha_4^p \neq 1$, we may suppose $\alpha_3^p \neq 1$ and so, by suitable choice of α_4^* , we may suppose $\alpha_4^p = 1$.

Thus in the above, we shall take $w_3 = 0$.

From $[\alpha_2^*, \alpha_4^*] = 1$, we get $y_3 w_4 = 0$.

From $[\alpha_2^*, \alpha_3^*] = 1$, we get $y_3 z_4 = y_4 z_3$.

Hence (since the above determinant is non-zero) $y_3 = y_4 = 0$.

From $[\alpha_3^*, \alpha_4^*] = \beta^*$, we have $x_1 y_2 = z_3 w_4$ and so from $\alpha_1^{*p} = \beta^{*b_1^*}$, we have

$$\begin{aligned} y_2 b_2 &= x_1 y_2 b_2^* \\ z_2 b_2 + z_3 b_3 &= x_1 y_2 b_3^* = z_3 w_4 b_3^* \end{aligned}$$

with standard forms $b_2 = 1 \ 0 \ 0$

$$b_3 = 0 \ 1 \ 0.$$

Thus there are seven groups in $p^6\phi_5$.

Φ_6 ($p > 3$)

The groups of Φ_6 are characterized by: $G/Z(G)$ is Φ_2 (111), $H_2(G)$ is (111), $H_3(G)$ is (11), $H_4(G) = E$.

Let $\alpha_1, \alpha_2 \in G - Z_2(G)$, $\beta = [\alpha_1, \alpha_2]$, $\beta_1 = [\beta, \alpha_1]$. Then by theorem 1 of the previous chapter,

$$H_2(G) = \langle \beta, \beta_1, \beta_2 \rangle \quad H_3(G) = \langle \beta_1, \beta_2 \rangle.$$

I Stem Groups i.e. $Z(G) = H_3(G) = \langle \beta_1, \beta_2 \rangle$.

Suppose $\alpha_1^p = \beta_{\sim 1}^{a_1}$, $\alpha_1^{*p} = \beta_{\sim 1}^{*a_1}$ where

$$\alpha_1^* = \alpha_1^{x_1} \alpha_2^{x_2} \beta^x \dots$$

$$\alpha_2^* = \alpha_1^{y_1} \alpha_2^{y_2} \beta^y \dots \quad \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \neq 0$$

and so $\beta^* = \beta^{x_1 y_2 - x_2 y_1} = \beta^{|X|} \dots$

$$\beta_1^* = \beta^{|X| x_{\sim}} \quad \beta_2^* = \beta^{|X| y_{\sim}}.$$

Then, as in the work on Φ_4 , with $A = (a_1, a_2)$,

$A^* = (a_1^*, a_2^*)$ and $X = (x_{\sim}, y_{\sim})$ we have

$$X^{-1}AX = |X|A^*.$$

Note that $|X^{-1}AX| = |A|$ and so $|A| = |X|^2 |A^*|$

i.e. we may reduce $|A^*|$ to 0, 1 or v .

As in Φ_4 , we consider the characteristic polynomial $c_A(u)$.

(i) If $c_A(u) = u^2$, we may suppose

$$X^{-1}AX = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}.$$

Taking $X = \begin{pmatrix} x_1 & y_1 \\ 0 & y_2 \end{pmatrix}$ and $A = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$, we see that we may suppose

$a = 0, 1$ or v .

(ii) If $c_A(u) = (u - a)^2$ ($a \neq 0$) we may take $X^{-1}AX = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$

and so suppose $A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$. Taking $X = \begin{pmatrix} x_1 & y_1 \\ 0 & a/x_1 \end{pmatrix}$, we may

suppose $a = 1$ and, by suitable choice of x_1 , we may suppose $b = 0, 1$ or v .

(iii) If $c_A(u) = (u - a)(u - b)$ ($a \neq b$), we may suppose (as above) that $A = \text{diag}(a, b)$.

If $b = 0$, by taking $X = \text{diag}(a, 1)$, we may suppose $a = 1, b = 0$.

If $b \neq 0$ (since we may take a, b in either order) we may suppose $a/b = g^k$ ($k = 1, 2, \dots, \frac{1}{2}(p-1)$) and, by taking $X = \text{diag}(b, 1)$, we may suppose $b = 1, a = g^k$ ($k = 1, 2, \dots, \frac{1}{2}(p-1)$).

(iv) If $c_A(u) = u^2 - g_1u + g_2$ is irreducible over $\text{GF}(p)$, then, by arguments similar to those in Φ_4 , we may suppose

$$A = \begin{pmatrix} 0 & v \\ 1 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & \frac{1}{4}(g^{2k+1}-1) \\ 1 & 1 \end{pmatrix} \quad k = 0, 1, \dots, \frac{1}{2}(p-3).$$

Thus, in all, there are $p + 7$ groups in $p^5\Phi_6$.

II - First Branch i.e. $Z(G)$ is (21) or (111) .

(a) If $Z(G)$ is (21) , we may suppose (as in ϕ_4)

$$Z(G) = \langle \gamma, \beta_2 \mid \gamma^p = \beta_1, \beta_1^p = 1 \rangle.$$

Take $\alpha_1^p = \gamma^{a_{11}} \beta_2^{a_{12}}$, $\alpha_1^{*p} = \gamma^{a_{11}^*} \beta_2^{a_{12}^*}$ where

$$\alpha_1^* = \alpha_1^{x_1} \beta^{x'} \gamma^x$$

$$\alpha_2^* = \alpha_1^{y_1} \alpha_2^{y_2} \beta^{y'} \gamma^y$$

$$\beta^* = \beta^{x_1 y_2} \dots, \gamma^* = \gamma^{x_1^2 y_2^{+pt}} \beta_2^w.$$

As in ϕ_4 , we may suppose $a_{11} < p$ and we get the analogous equations:

$$a_{11} = x_1 y_2 a_{11}^*$$

$$x_1 a_{12} = w a_{11}^* + x_1 y_2^2 a_{12}^*$$

$$y_1 a_{11} + y_2 a_{21} = x_1^2 y_2 a_{21}^*$$

$$y_1 a_{12} + y_2 a_{22} = w a_{21}^* + x_1 y_2^2 a_{22}^*.$$

If $a_{11} \neq 0$, we may suppose $a_{11} = 1$.

Taking $w = x_1 a_{12}$, $y_1 = -y_2 a_{21}$, we may suppose $a_{12} = a_{21} = 0$ and the last equation becomes $a_{22} = a_{22}^*$ showing that a_{22} may have any value from 0 to $p-1$.

If $a_{11} = 0$, the equations become

$$a_{12} = y_2^2 a_{12}^*$$

$$a_{21} = x_1^2 a_{21}^*$$

$$\text{and } y_1 a_{12} + y_2 a_{22} = w a_{21}^* + x_1 y_2^2 a_{22}^*$$

with standard forms

a_{12}	=	1	1	1	v	v	v	0	0	0	0
a_{21}	=	1	v	0	1	v	0	1	v	0	0
a_{22}	=	0	0	0	0	0	0	0	0	1	0.

(b) If $Z(G)$ is (111), put $Z(G) = \langle \gamma, \beta_1, \beta_2 \rangle$.

(i) If $\alpha_1^p, \alpha_2^p \in H_3(G)$, then G is the direct product of a stem group together with a group of type (1).

(ii) If either α_1^p or $\alpha_2^p \notin H_3(G)$, we may suppose (as in Φ_4) that $\alpha_1^p = \gamma$, $\alpha_2^p = \beta^a$.

Taking $\alpha_2^{*p} = \beta^{a*}$ where

$$\alpha_1^* = \alpha_1^{x_1} \alpha_2^{x_2} \beta^x \dots$$

$$\alpha_2^* = \alpha_2^{y_2} \beta^y \dots \quad x_1 y_2 \neq 0$$

$$\beta^* = \beta^{x_1 y_2}, \quad \gamma^* = \gamma^{x_1} \beta^{x_2 a}$$

$$\beta_1^* = \beta^{x_1 y_2 x} \text{ and } \beta_2^* = \beta_2^{x_1 y_2^2}$$

we get

$$a_1 = x_1^2 a_1^*$$

$$a_2 = x_1 x_2 a_1^* + x_1 y_2 a_2^*$$

with standard forms $a_1 = 1 \quad v \quad 0 \quad 0$

$$a_2 = 0 \quad 0 \quad 1 \quad 0.$$

Thus we can see that there are $p + 7$ groups of $p^5 \Phi_6$ and $2p + 21$ groups of $p^6 \Phi_6$ ($p > 3$).

Δ_6

Since the groups of Δ_6 are of class 3, they have the same commutator relations and relations modulo the centre as Φ_6 ($p > 3$). Thus, for $G \in \Delta_6$,

$$G = \langle \alpha_1, \alpha_2, Z(G) \mid [\alpha_1, \alpha_2] = [\beta, \beta, \alpha_1] = \beta_1, \alpha_1^3 \in Z(G) \rangle.$$

Using lemma L5, we have $(\alpha_1^{z_1} \alpha_2^{z_2})^3 = \alpha_1^{3z_1} \alpha_2^{3z_2} \beta_1^{-z_1^2 z_2} \beta_2^{z_1 z_2^2}$ and we may write $-z = 2z$, $1/z = z$ ($z \neq 0$) since all integers may be regarded as elements of $GF(3)$.

I Stem Groups i.e. $Z(G) = H_3(G) = \langle \beta_1, \beta_2 \rangle$.

As in Φ_6 , put $\alpha_1^3 = \beta_1^{a_1}$, $\alpha_1^{*3} = \beta_1^{a_1^*}$ where

$$\begin{aligned} \alpha_1^* &= \alpha_1^{x_1} \alpha_2^{x_2} \beta^x \dots \\ \alpha_2^* &= \alpha_1^{y_1} \alpha_2^{y_2} \beta^y \dots \end{aligned} \quad |X| = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \neq 0$$

and so $\beta^* = \beta^{|X|}$

$$\beta_1^* = \beta^{|X|x}, \quad \beta_2^* = \beta^{|X|y}.$$

Using the above, $\alpha_1^{*3} = (\alpha_1^{x_1} \alpha_2^{x_2})^3 \beta^{3x}$ since $(H_3(G))^3 = E$
 $= (\alpha_1^{x_1} \alpha_2^{x_2})^3$ since $(H_2(G))^3 = E$
 $= \alpha_1^{3x_1} \alpha_2^{3x_2} \beta_1^{-x_1^2 x_2} \beta_2^{x_1 x_2^2}.$

So, from $\alpha_1^{*3} = \beta_1^{*3}$, we get

$$\begin{aligned} |X|(x_1 a_{11}^* + y_1 a_{12}^*) &= x_1 a_{11} + x_2 a_{21} - x_1^2 x_2 \\ |X|(x_2 a_{11}^* + y_2 a_{12}^*) &= x_1 a_{12} + x_2 a_{22} + x_1 x_2^2. \end{aligned}$$

Similarly (by finding α_2^{*3}) we have

$$\begin{aligned} |X|(x_1 a_{21}^* + y_1 a_{22}^*) &= y_1 a_{11} + y_2 a_{21} - y_1^2 y_2 \\ |X|(x_2 a_{21}^* + y_2 a_{22}^*) &= y_1 a_{12} + y_2 a_{22} + y_1 y_2^2. \end{aligned}$$

Writing $A = (\underline{a}_1, \underline{a}_2)$, $A^* = (\underline{a}_1^*, \underline{a}_2^*)$, these equations become

$$|X|XA^* = AX + XX^*$$

$$\text{where } X^* = |X| \begin{pmatrix} -x_1 x_2 (x_1 y_2 + x_2 y_1) & y_1^2 y_2^2 \\ x_1^2 x_2^2 & y_1 y_2 (x_1 y_2 + x_2 y_1) \end{pmatrix}$$

$$\text{i.e. } A^* = |X|(X^{-1}AX + X^*).$$

By evaluating all the matrices A^* corresponding to each A (see (6)), we see that there are seven groups in $3^5\Delta_6$, the defining relations being determined by the equivalence classes on page q2.

II First Branch i.e. $Z(G)$ is (21) or (111).

(a) If $Z(G)$ is (21), write $Z(G) = \langle \gamma_1, \gamma_2 | \gamma_1^9 = \gamma_2^3 = 1 \rangle$.

By $\{\alpha_1 \leftrightarrow \alpha_2\}$ (if necessary) we may suppose $\beta_1 \in Z^3$ and by $\{\alpha_2 \rightarrow \alpha_2^2\}$ (if necessary) we may suppose $\beta_1 = \gamma_1^3 = \gamma^3$.

Then $\beta_2 = \gamma_1^{b_1} \gamma_2^{b_2}$ and so $\beta_2^3 = \beta_1^{3b_1}$.

By $\{\alpha_2 \rightarrow \alpha_1^{-b_1} \alpha_2\}$ we may suppose $\gamma_2 = \beta_2$.

Thus $\alpha_1^3 = \gamma^{a_{11}} \beta_2^{a_{12}}$ $\alpha_1^{*3} = \gamma^{*a_{11}} \beta_2^{*a_{12}}$ where

$$\bar{\alpha}_1^* = \bar{\alpha}_1^{x_1} \quad \bar{\beta}^{x'} \gamma^x$$

$$\bar{\alpha}_2^* = \bar{\alpha}_1^{y_1} \bar{\alpha}_2^{y_2} \bar{\beta}^{y'} \gamma^y$$

$$\frac{\bar{\alpha}^*}{\bar{\beta}} = \frac{x_1 y_2}{\beta} \quad x_1 y_2 \neq 0$$

$$\beta_1^* = \beta_1^{y_2} \text{ and so } \gamma^* = \gamma^{y_2+3w_1} \beta_2^{w_2}$$

$$\beta_2^* = \beta_1^{x_1 y_1 y_2} \beta_2^{x_1}.$$

As in Φ_6 , $\alpha_2^{*3} = \alpha_1^{3y_1} \alpha_2^{3y_2} \beta_1^{-y_1^2 y_2} \beta_2^{y_1} \gamma^{3y}$

$$= \gamma^{y_1 a_{11} + y_2 a_{21} + 3(y - y_1^2 y_2)} \beta_2^{y_1(a_{12}+1) + y_2 a_{22}}.$$

$$\therefore y_1 a_{11} + y_2 a_{21} + 3(y - y_1^2 y_2) = (y_2 + 3w_1) a_{21}^* + 3x_1 y_1 y_2 a_{22}^*$$

$$y_1(a_{12}+1) + y_2 a_{22} = w_2 a_{21}^* + x_1 a_{22}^*$$

Similarly $x_1 a_{11} + 3x = (y_2 + 3w_1) a_{11}^* + 3x_1 y_1 y_2 a_{12}^*$

$$x_1 a_{12} = w_2 a_{11}^* + x_1 a_{12}^*.$$

By choice of x, y we may suppose $0 \leq a_{11} < 3$ and so we have

(with $a_{12}' = a_{12} + 1$)

$$a_{11} = x_1 y_2 a_{11}^*$$

$$x_1 a_{12}' = w_2 a_{11}^* + x_1 a_{12}^{*'}.$$

$$y_1 a_{11} + y_2 a_{21} = y_2 a_{21}^*$$

$$y_1 a_{12}' + y_2 a_{22} = w_2 a_{21}^* + x_1 a_{22}^*.$$

As these are the same equations as were obtained for $p > 3$, the standard forms will be the same

$$\begin{aligned} \text{viz. } a_{11} &= 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{12} &= -1 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ a_{21} &= 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 \\ a_{22} &= \text{any} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{aligned}$$

(b) If $Z(G)$ is (111), we have $Z(G) = \langle \beta_1, \beta_2, \gamma \mid \beta_1^3 = \gamma_3 = 1 \rangle$.

If $\alpha_1^3 H_3(G) G$ is the direct product of a stem group and a cyclic group of order 3.

If not, we may suppose $\alpha_1^3 = \gamma$, $\alpha_2^3 = \beta_1^{a_1} \beta_2^{a_2}$.

In this case, we are only interested in the automorphisms which leave fixed $\langle \alpha_2, H_2(G) \rangle$ and, as this subgroup is regular, we will get the same results as for $p > 3$.

Thus, there are 7 groups of $3^5 \Delta_6$ and 24 groups of $3^6 \Delta_6$.

Equivalence classes for the 2×2 matrices under the relation

$$A^* = |X|(X^{-1}AX + X^*)$$

$$1) \quad 0, \begin{pmatrix} 0 & 1 \\ 0 & r \end{pmatrix}, \begin{pmatrix} r & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$$

$$2) \quad \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}, r \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} u & 2 \\ 0 & u \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & r \end{pmatrix}, \begin{pmatrix} u & 0 \\ 1 & u \end{pmatrix}$$

$$3) \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & r \end{pmatrix}, \begin{pmatrix} r & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} u & 2 \\ u & u+1 \end{pmatrix}, \begin{pmatrix} 2u & 2 \\ u & 2u-1 \end{pmatrix}$$

$$4) \quad \begin{pmatrix} 0 & 2 \\ r & s \end{pmatrix}, \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} u & 0 \\ 1 & s \end{pmatrix}, \begin{pmatrix} s & 2 \\ 0 & u \end{pmatrix}, \begin{pmatrix} r & s \\ s & 3-r \end{pmatrix}, \begin{pmatrix} r & s-1 \\ s-2 & r \end{pmatrix}$$

$$5) \quad \begin{pmatrix} r & s-1 \\ 2s+1 & r \end{pmatrix}, \begin{pmatrix} s & 1 \\ 0 & u \end{pmatrix}, \begin{pmatrix} u & 0 \\ 2 & s \end{pmatrix}$$

$$6) \quad \begin{pmatrix} s & 1 \\ 2 & u \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & r \end{pmatrix}, \begin{pmatrix} u & s \\ s & u \end{pmatrix}, r \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

where $r, s = 1, 2$; $u = 0, 1, 2$ and $u \neq s$ throughout.

$$\phi_7 \ (p \geq 3)$$

The groups of ϕ_7 are characterized by: $G/Z(G)$ is $\phi_2(1^4)$, $H_2(G)$ is (11), $H_3(G)$ is (1).

Take $\alpha \in G - \mathcal{L}(H_2(G))$, $\alpha_1 \in \mathcal{L}(H_2(G)) - Z_2(G)$, $\beta \in Z_2(G) - Z(G)$
 $\alpha_{i+1} = [\alpha_i, \alpha] \ (i = 1, 2)$.

By theorem 2, we may suppose $[\alpha_1, \beta] = \alpha_3 \in Z(G)$, $[\alpha_1, \alpha_2] =$
 $[\alpha, \beta] = [\alpha_2, \beta] = 1$ and $H_3(G) = \langle \alpha_3 \rangle$, $H_2(G) = \langle \alpha_2, \alpha_3 \rangle$,
 $Z_2(G) = \langle \beta, Z \rangle$, $\mathcal{L}(H_2) = \langle \alpha_1, Z_2 \rangle$ are characteristic subgroups.

As in $\mathcal{A}_{(n)}$, we may apply lemmas L3 and L5 and so

$$\alpha_1^p \alpha_3^{\binom{p}{3}} \in Z(G), \quad \alpha_2^p = \alpha_3^p = 1, \quad (\sigma\xi)^p = \sigma^p \text{ all } \xi \in H_2$$

$$\text{and } (\alpha^z \alpha_1^{z_1})^p = \alpha^{pz} (\alpha_1^p \alpha_3^{\binom{p}{3}})^{z_1}.$$

I $Z(G)$ is cyclic i.e. $Z(G) = \langle \gamma | \gamma^{p^r} = \alpha_3 \rangle$.

Put $\alpha^p = \gamma^a$, $\alpha_1^p \alpha_3^{\binom{p}{3}} = \gamma^{a_1}$, $\beta^p = \gamma^b$, $\alpha^{*p} = \gamma^{a^*}$ etc

where (because of the above characteristic groups)

$$\alpha^* = \alpha^x \alpha_1^{x_1} \beta^{x'} \alpha_2^{x_2} \gamma^{x''}$$

$$\alpha_1^* = \alpha_1^{y_1} \beta^{y'} \alpha_2^{y_2} \gamma^{y''}$$

$$\beta^* = \beta^{z'} \alpha_2^{z_2} \gamma^{z''}$$

$$\text{and so } \bar{\alpha}_2^* = \bar{\alpha}_2^{xy_1}$$

$$\gamma^* = \gamma^{x^2 y_1 + pw}$$

Since $[a_1^*, \beta^*] = a_3^*$, we have $y_1 z' = x^2 y_1$
and so $z' = x^2$.

Also, by choice of x'' , y'' , z'' , w , we may suppose $0 \leq a, a_1, b < p$.

By remarks above,

$$\begin{aligned} \alpha^{*p} &= (\alpha^x_{\alpha_1} x_1^x \beta^{x'})^p \\ &= (\alpha^x_{\alpha_1} x_1^x)^p \beta^{px'} \quad \text{since } (H_2(G))^p = E \\ &= \alpha^{px} (\alpha_1^p \alpha_3^{\binom{p}{3}})^{x_1} \beta^{px'}. \end{aligned}$$

So, from $\alpha^{*p} = \gamma^{*a^*}$, we get

$$x^2 y_1 a^* = xa + x_1 a_1 + x'b \quad \dots \quad (1)$$

To get a second equation, we use the same method as in $\mathcal{O}_{(n)}$, getting

$$\begin{aligned} \gamma^{*(a^*+a_1^*)} &= (\alpha^* \alpha_1^*)^p \\ &= (\alpha^x_{\alpha_1} x_1^{x+y_1} \beta^{x'+y'})^p \end{aligned}$$

and, by evaluating both sides, we get

$$x^2 y_1 (a^*+a_1^*) = xa + (x_1+y_1)a_1 + (x'+y')b.$$

Subtracting (1), we have

$$x^2 y_1 a_1^* = y_1 a_1 + y'b \quad \dots \quad (11)$$

Finally, from $\beta^{*p} = \gamma^{*b^*}$, we get

$$x^2 y_1 b^* = x^2 b \quad \dots \quad (111)$$

$$\text{and so } xa + x'b = x^2 y_1 a^*$$

$$b = y_1 b^* \quad \text{with standard forms}$$

$$b = 1 \quad 0 \quad 0$$

$$a = 0 \quad 1 \quad 0.$$

(iv) If $\alpha_1^p \alpha_3^{(p)} \binom{p}{3}$, $\beta^p \in H_3(G)$, $\alpha^p \notin H_3(G)$, we may suppose

$$\alpha^p = \gamma, \alpha_1^p \alpha_3^{(p)} \binom{p}{3} = \alpha_3^{a_1}, \beta^p = \alpha_3^b, \alpha^{*p} = \gamma^* \text{ etc where}$$

$$\alpha^* = \alpha^x \alpha_1^{x_1} \beta^{x'}$$

$$\alpha_1^* = \alpha_1^{y_1} \beta^{y'}$$

$$\beta^* = \beta^{x^2}$$

$$\alpha_2^* = \alpha_2^{xy_1}, \alpha_3^* = \alpha_3^{x^2 y_1}, \gamma^* = \gamma^x \alpha_3^{x_1 a_1 + x' b}$$

$$\text{and so } y_1 a_1 + y' b = x^2 y_1 a_1^*$$

$$b = y_1 b^* \quad \text{with standard forms}$$

$$b = 1 \quad 0 \quad 0 \quad 0$$

$$a_1 = 0 \quad 1 \quad v \quad 0.$$

Thus, in all, there are 21 groups of $p^6 \Phi_7$.

Φ_8 ($p \geq 3$)

The groups of Φ_8 are characterized by: $G/Z(G)$ is $\Phi_2(22)$, $H_2(G)$ is (2), $H_3(G)$ is (1).

By theorem 3, we may suppose

$G = \langle \alpha_1, \alpha_2, Z(G) \mid [\alpha_1, \alpha_2] = \beta, \beta^{p^2} = 1, [\beta, \alpha_1], \alpha_1^{p^2}, \alpha_1^p \beta^{-1} \in Z(G) \text{ (} i = 1, 2 \text{)} \rangle$ and $H_3(G) = \langle \beta^p \rangle$, $H_2(G) = \langle \beta \rangle$ and $\mathcal{C}(H_2) = \langle \alpha_1, \alpha_2^p, \beta, Z \rangle$ are characteristic subgroups.

I Stem Groups i.e. $Z(G) = H_3(G) = \langle \beta^p \rangle$.

Since $\alpha_1^p \equiv \beta \pmod{Z(G)}$, we have $\alpha_1^{p^2} = \beta^p$ and so we may write

$$\alpha_1^p = \beta^{1+pa}, \quad \alpha_2^{p^2} = \beta^{bp}.$$

By replacing α_1 by $\alpha_1 \beta^{-a}$ and α_2 by $\alpha_1^{-b} \alpha_2$, we see that there is just one group:

$$\Phi_8(32) = \langle \alpha, \beta \mid \alpha^{p^3} = \beta^{p^2} = 1, [\alpha, \beta] = \alpha^p \rangle.$$

II First Branch i.e. $Z(G)$ is (2) or (11).

(a) If $Z(G)$ is (2), write $Z(G) = \langle \gamma \mid \gamma^{p^2} = 1 \rangle$.

Since β^p is an element of $Z(G)$ of order p , we may suppose, without loss of generality, that $\gamma^p = \beta^p$.

Since $\alpha_2^{p^2}, \alpha_1^p \beta^{-1} \in Z(G)$, we write $\alpha_1^p = \beta \gamma^{a-1}$, $\alpha_2^{p^2} = \gamma^b$,

$$\alpha_1^{*p} = \beta^* \gamma^{*a-1}, \quad \alpha_2^{*p^2} = \gamma^{*b} \quad \text{where}$$

$$\alpha_1^* \equiv \alpha_1^{x_1} \alpha_2^{px_2} \gamma^x \pmod{H_3(G)}$$

$$\alpha_2^* \equiv \alpha_1^{y_1} \alpha_2^{y_2} \gamma^y \pmod{H_3(G)}$$

$$\text{and so } \beta^* \equiv \beta^{x_1 y_2} \pmod{H_3(G)}$$

From $\alpha_1^{*p} \equiv \beta^* \pmod{Z(G)}$, we have $y_2 \equiv 1 \pmod{p}$.

Also, since $\alpha_1^{*p} = (\alpha_1^{x_1} \alpha_2^{px_2})^p \gamma^{xp}$, we may suppose (by choice of x) that $0 \leq a < p$.

Writing $b = b_1 + pb_2$, $x_1 = x_{11} + px_{12}$, we have

$$\begin{aligned} \alpha_1^{*p} &= \alpha_1^{px_{11} + p^2 x_{12}} \alpha_2^{p^2 x_2} \gamma^{px} \\ &= \beta^{x_{11}} \gamma^{x_{11}(a-1) + x_2 b_1} \text{ by choice of } x. \end{aligned}$$

From $\gamma^{*p} = \beta^{*p}$, we get $\gamma^* = \gamma^{x_{11} + pz}$ and from $\alpha_1^{*p} = \beta^* \gamma^{*a-1}$

$$\text{we get } x_{11} a^* = x_{11} a + x_2 b_1 \quad \dots \quad (1)$$

Similarly, writing $y_1 = y_{11} + py_{12}$ and $y_2 = 1 + py_{22}$,

from $\alpha_2^{*p^2} = \gamma^{*b^*}$ we get

$$x_{11} b_1^* = b_1 \quad \dots \quad (2)$$

$$z b_1^* + x_{11} b_2^* = b_2 + y_{11} a + y_{22} b_1 \quad \dots \quad (3)$$

From (1), (2), (3) we may suppose

$$b_1 = 1 \quad 0 \quad 0$$

$$a = 0 \quad \text{any} \quad 0$$

$$b_2 = 0 \quad 0 \quad 1$$

and so there are $p + 2$ such groups.

(b) If $Z(G)$ is (11), put $Z(G) = \langle \gamma, \beta^p \rangle$.

(i) If $\alpha_2^{p^2}, \alpha_1^p \beta^{-1} \in H_3(G)$, G is the direct product:

$$\langle \alpha, \beta | \alpha^{p^3} = \beta^{p^2} = 1, [\alpha, \beta] = \alpha^p \rangle \times \langle \gamma | \gamma^p = 1 \rangle.$$

(ii) If $\alpha_2^{p^2} \notin H_3(G)$, put $\alpha_2^{p^2} = \gamma$, $\alpha_1^p \beta^{-1} = \beta^{ap} \gamma^b$.

By $\{\alpha_1 \rightarrow \alpha_1 \alpha_2^{-pb}\}$, we may suppose $b = 0$ and by $\{\alpha_2 \rightarrow \alpha_2^{1+pa}\}$ we may suppose $a = 0$.

(iii) If $\alpha_2^{p^2} \in H_3(G)$, $\alpha_1^p \beta^{-1} \notin H_3(G)$, put $\alpha_1^p = \beta \gamma$,

$\alpha_2^{p^2} = \beta^{cp}$. Then $\alpha_1^{p^2} = \beta^p$ and by $\{\alpha_2 \rightarrow \alpha_1^{-c} \alpha_2\}$, we may suppose $c = 0$.

Thus, in all, we have $p + 5$ groups of $p^6 \phi_8$.

For $p = 3$, $\alpha_1^{*p^2}, \alpha_2^{*p^2}, \alpha_1^{*p} \beta^{*-1}$ will have the same values as in the above cases and so the same results hold for $p = 3$. In fact, since $(\alpha_1^{x_1} \alpha_2^{x_2})^3 = \alpha_1^{3x_1} \alpha_2^{3x_2} \beta^{3x_1 x_2} \gamma$

(where $\gamma \in H_3(G)$) $= \alpha_1^{3x_1} \alpha_2^{3x_2} \beta^{3x}$, some x ,

the groups of ϕ_8 ($p = 3$) are regular.



$$\mathcal{B}_n (\phi_{10} \text{ and } \phi_{36}) \quad p \geq 3$$

These families are characterized by: $G/Z(G)$ is $\mathcal{A}_{(n-1)}(1^{n+1})$, $H_2(G)$ is (1^n) , ... $H_{n+1}(G)$ is (1) (in the case of regular groups) and G has a maximal subgroup $H_1(G)$ with $H_1'(G) = H_{n+1}(G)$.

Let $\alpha \in G - H_1(G)$, $\alpha_1 \in H_1(G) - \langle H_2(G), Z(G) \rangle$, $\alpha_{i+1} = [\alpha_i, \alpha]$. Then $[\alpha_1, \alpha_2] = \alpha_{n+1}$, $[\alpha_i, \alpha_j] = 1$ otherwise and $\alpha_{n+2} = 1$.

By lemma L3 we have (for $p > 3$)

$$(\alpha \alpha_1^z)^p = \alpha^p (\alpha_1^p \alpha_2^{\binom{p}{2}} \dots \alpha_p)^z.$$

In the case $p = 3$ we have

$$\begin{aligned} (\alpha \alpha_1^z)^3 &= \alpha^3 (\alpha_1^z)^{\alpha^2} (\alpha_1^z)^{\alpha} \alpha_1^z \\ &= \alpha^3 (\alpha_1^{\alpha^2})^z (\alpha_1^{\alpha})^z \alpha_1^z \\ &= \alpha^3 (\alpha_1 \alpha_2^2 \alpha_3)^z (\alpha_1 \alpha_2)^z \alpha_1^z \\ &= \alpha^3 \alpha_1^z \alpha_2^{2z} \alpha_{n+1}^{-2\binom{z}{2}} \alpha_3^z \alpha_1^z \alpha_2^z \alpha_{n+1}^{-\binom{z}{2}} \alpha_1^z \\ &= \alpha^3 \alpha_1^z (\alpha_2^{2z} \alpha_1^z) (\alpha_2^z \alpha_1^z) \alpha_3^z \quad \text{since } \alpha_{n+1}^3 = 1 \\ &= \alpha^3 \alpha_1^{2z} \alpha_2^{2z} \alpha_{n+1}^{-2z^2} \alpha_1^z \alpha_2^z \alpha_{n+1}^{-z^2} \alpha_3^z \\ &= \alpha^3 \alpha_1^{2z} (\alpha_2^{2z} \alpha_1^z) \alpha_2^z \alpha_3^z \\ &= \alpha^3 \alpha_1^{3z} \alpha_2^{2z} \alpha_{n+1}^{-2z^2} \alpha_2^z \alpha_3^z \\ &= \alpha^3 \alpha_1^{3z} \alpha_2^{3z} \alpha_3^z \alpha_{n+1}^{z^2}. \end{aligned}$$

So, for all $p \geq 3$,

$$(\alpha \alpha_1^z)^p = \alpha^p (\alpha_1^p \alpha_2^{\binom{p}{2}} \dots \alpha_p)^z \alpha_{n+1}^{2 \binom{p}{3}}$$

and so $(\alpha^x \alpha_1^{x_1})^p = \alpha^{px} (\alpha_1^p \alpha_2^{\binom{p}{2}} \dots \alpha_p)^{x_1} \alpha_{n+1}^{xx_1 2 \binom{p}{3}}$ for $x \neq 0$.

As in $\mathcal{O}_p(n)$, we have $\alpha^p, (\alpha \alpha_1)^p \in Z(G)$ and so

$$\alpha_1^p \alpha_2^{\binom{p}{2}} \dots \alpha_p \in Z(G)$$

$$\alpha_1^p \alpha_{i+1}^{\binom{p}{2}} \dots \alpha_{p+i-1} = 1 \quad i \geq 2.$$

Case I If $Z(G)$ is cyclic, put $Z(G) = \langle \gamma | \gamma^{p^r} = \alpha_{n+1} \rangle$.

Then $\alpha^p = \gamma^a$, $\alpha_1^p \alpha_2^{\binom{p}{2}} \dots \alpha_p = \gamma^b$, $\alpha^{*p} = \gamma^{*a}$, $\alpha_1^{*p} \dots \alpha_p^{*p} = \gamma^{*b}$

$$\text{where } \alpha^* = \alpha^x \alpha_1^{x_1} \alpha_2^{x_2} \dots \gamma^{x'}$$

$$\alpha_1^* = \alpha_1^{y_1} \alpha_2^{y_2} \dots \gamma^{y'}$$

$$\alpha_2^* = \alpha_2^{xy_1} \dots$$

$$\alpha_{n+1}^* = \alpha_{n+1}^{x^n y_1}$$

$$\gamma^* = \gamma^{x^n y_1 + pz}$$

From $[\alpha_1^*, \alpha_2^*] = \alpha_{n+1}^*$, we get $y_1 = x^{n-1}$.

From $\alpha^{*p} = \gamma^{*a}$, we get

$$(x^{2n-1} + pz) \alpha^* = x a + x_1 b + p x' + x x_1^{2 \binom{p}{3}} p^r \dots \quad (1)'$$

From $(\alpha\alpha_1)^p = \gamma^{a+b+p} r(\frac{p}{3})$ and $(\alpha^*\alpha_1^*)^p = \gamma^{a^*+b^*+p} r(\frac{p}{3})$

we get $\gamma^{a^*+b^*+p} r(\frac{p}{3}) = (\alpha^{\alpha_1} x_1^{y_1} \gamma^{x'+y'})^p$

and so $(x^{2n-1} + pz)(a^*+b^*+p r(\frac{p}{3})) = xa + (x_1+y_1)b + p r(\frac{p}{3})x(x_1+y_1)^2 + p(x'+y')$.

Using (i)' we have

$$(x^n y_1 + pz)(b^* + p r(\frac{p}{3})) = y_1 b + py' + xy_1(2x_1+y_1)p r(\frac{p}{3}) \dots (ii)'$$

By choice of x' , y' , we may suppose $0 \leq a, b < p$.

For $p = 3$, $r = 0$, (i)' and (ii)' reduce to

$$x^{2n-1} a^* \equiv xa + x_1 b + xx_1^2 \pmod{3}$$

$$x^n(b^* + 1) \equiv b + x(2x_1 + x^{n-1})$$

$$\text{i.e. } x^n b^* \equiv b + 2xx_1 \pmod{3}.$$

Taking $x_1 = bx$, we may suppose $b = 0$ and so

$$a \equiv x^{2(n-1)} a^* \equiv a^* \pmod{3} \text{ since } x^2 \equiv 1 \pmod{3}.$$

Thus, for $p = 3$, there are 3 stem groups.

For $p \neq 3$ or $r \neq 0$, (i)' and (ii)' reduce to

$$x^{2n-1} a^* \equiv xa + x_1 b \dots (i)$$

$$x^n b^* \equiv b \dots (ii)$$

with standard forms $b = 1 \quad g^k \quad 0 \quad 0 \quad 0$

$$a = 0 \quad 0 \quad 1 \quad g^l \quad 0$$

where $k = 1, 2, \dots (p-1, n)-1$ and $l = 1, 2, \dots (p-1, 2n-2)-1$.

Taking $r = 0$, we obtain the $1 + (p-1, n) + (p-1, 2n-2)$ stem groups for $p > 3$.

II If $Z(G)$ is (11), put $Z(G) = \langle \gamma \rangle \times \langle \alpha_{n+1} \rangle$.

(a) If $\alpha^p, \alpha_1^p \in H_{n+1}(G)$, then G is the direct product of a stem group together with a group of type (1).

(b) If $\alpha_1^p \dots \alpha_p \notin H_{n+1}(G)$, put $\alpha_1^p \dots \alpha_p = \gamma$, $\alpha^p = \alpha_{n+1}^a \gamma^b$.
By $\{\alpha \rightarrow \alpha \alpha_1^{-b}\}$ we may suppose $b = 0$.

The other automorphisms are the same as above with $x_1 = 0$ and so we may suppose $a = 0, 1, g^\ell$ where $\ell = 1, 2, \dots (p-1, 2n-2)-1$.

(c) If $\alpha_1^p \alpha_2^{\binom{p}{2}} \dots \alpha_p \in H_{n+1}(G)$ and $\alpha^p \notin H_{n+1}(G)$, put $\alpha^p = \gamma$,
 $\alpha_1^p \alpha_2^{\binom{p}{2}} \dots \alpha_p = \alpha_{n+1}^b$.

Using the same automorphism as in I, with $x' = y' = 0$, $\gamma^* = \alpha^{*p}$, we obtain the same relation for b and so we may suppose $b = 0, 1, g^k$ where $k = 1, 2, \dots (p-1, n)-1$ for $p > 3$ and $b = 0$ for $p = 3$.

The $1 + (p-1, 3) + (p-1, 4)$ groups of $p^5 \Phi_{10}$ ($p > 3$) and the three groups of $3^5 \Delta_{10}$ are obtained by putting $r = 0$, $n = 3$ in I, and the $4 + 3(p-1, 3) + 3(p-1, 4)$ groups of $p^6 \Phi_{10}$ ($p > 3$) and the 11 groups of $3^6 \Delta_{10}$ are obtained by putting $r=1$, $n = 3$ in I and $n = 3$ in II.

The $1 + (p-1, 4) + (p-1, 6)$ groups of $p^6 \Phi_{36}$ ($p > 3$) and the three groups of $3^6 \Delta_{36}$ are obtained by putting $r = 0$, $n = 4$ in I.

$$\underline{\phi_{11} \ (p \geq 3)}$$

The groups of ϕ_{11} are characterized by: $G/Z(G)$ is (111), $H_2(G)$ is (111).

Take $\alpha_1 \in G - Z(G)$, $\alpha_2 \in G - \langle \alpha_1, Z(G) \rangle$, $\alpha_3 \in G - \langle \alpha_1, \alpha_2, Z(G) \rangle$ and $\beta_3 = [\alpha_1, \alpha_2]$, $\beta_1 = [\alpha_2, \alpha_3]$, $\beta_2 = [\alpha_3, \alpha_1]$.

For stem groups, we have $Z(G) = \langle \beta_1, \beta_2, \beta_3 \rangle$ and so $\alpha_1^p = \tilde{\alpha}_1^{a_1}$ and $\alpha_1^{*p} = \tilde{\beta}_1^{*a_1}$ where

$$\begin{aligned} \alpha_1^* &= \alpha_1^{x_1} \alpha_2^{x_2} \alpha_3^{x_3} \\ \alpha_2^* &= \tilde{\alpha}_2^y & |X| \neq 0 \\ \alpha_3^* &= \tilde{\alpha}_3^z \end{aligned}$$

and $\beta_1^* = \tilde{\beta}_1^{y \times z}$, $\beta_2^* = \tilde{\beta}_2^{z \times x}$, $\beta_3^* = \tilde{\beta}_3^{x \times y}$.

Putting $A = (\alpha_1, \alpha_2, \alpha_3)$, $A^* = (\alpha_1^*, \alpha_2^*, \alpha_3^*)$ and $X = (x, y, z)$, we note that

$$\text{adj } X^T = (y \times z, z \times x, x \times y)$$

and so we obtain the transformation

$$\begin{aligned} AX &= \text{adj } X^T A^* \\ X^T AX &= X^T (\text{adj } X^T) A^* = |X| A^* \quad \dots \end{aligned} \tag{1}$$

To find the canonical forms we write $A = S + K$ where

$$S^T = S, \quad K^T = -K.$$

(a) If A is skew-symmetric, we know that A must be of rank 0 or 2 and so the standard forms are $\underline{0}$ and

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = K_0.$$

(b) If A is symmetric, we may suppose

$$A = \text{diag} (s_1, s_2, s_3).$$

If $s_1 \neq 0$, some i , we may suppose $s_1 \neq 0$.

Taking $X = \text{diag} (s_1^{-1}, 1, 1)$, we may suppose

$$s_1 = 1.$$

If $s_1, s_2 \neq 0$, take $X = \text{diag} (xy, x, y)$ and so we have $(xy)^2 s_2^* = x^2 s_2$ with standard forms $s_2 = 1, v$ or 0 .

Similarly we may suppose $s_3 = 1, v$ or 0 .

Since v is any quadratic non-residue (mod p) we may choose it such that $v - 1$ is a quadratic residue (e.g. $v = 2$ if 2 is a quadratic non-residue) and we may suppose

$A = \underline{0}, I, \text{diag} (1, 0, 0), \text{diag} (1, 1, 0), \text{diag} (1, v, 0),$
 $\text{diag} (1, 1, v), \text{or} \text{diag} (1, v, v).$

Taking $X = \begin{pmatrix} -v & 0 & 0 \\ 0 & \sqrt{v-1} & 1 \\ 0 & 1 & -\sqrt{v-1} \end{pmatrix}$ we see that

$$X^T \text{diag} (1, v, v) X = |X| I$$

and taking $X = \begin{pmatrix} \sqrt{v-1} & 1 & 0 \\ 1 & -\sqrt{v-1} & 0 \\ 0 & 0 & -1 \end{pmatrix}$ we see that

$$X^T \text{diag} (1, 1, v) X = |X| I.$$

Thus A may be taken as 0, I, diag (1,0,0), diag (1,1,0) or diag (1,v,0).

Since rank is invariant under the above transformation, the first four of these are canonical forms.

Suppose we can find an X such that

$$X^T \text{diag} (1,v,0) X = |X| \text{diag} (1,1,0) \dots (2)$$

Writing $X = \begin{pmatrix} X_1 & \tilde{a} \\ b^T & c \end{pmatrix}$ and diag (1,v,0) as $A_1 \oplus (0)$, (2) becomes

$$X_1^T A_1 X_1 = |X| I$$

$$X_1^T A_1 \tilde{a} = \tilde{0}$$

$$\tilde{a}^T A_1 \tilde{a} = 0.$$

Taking determinants of the first of these

$$v |X_1|^2 = |X|^2$$

which is impossible, since $|X| \neq 0$.

Thus we have five canonical forms above.

(c) If A is neither symmetric nor skew-symmetric, by (a) we may suppose $A = S + K_0$, where $S^T = S$ and K_0 is as in (a)

(1) becomes $X^T S X + X^T K_0 X = |X| (S^* + K_0)$

or $X^T S X = |X| S^*$

$$X^T K_0 X = |X| K_0.$$

Putting $X = (\underline{x}, \underline{y}, \underline{z})$ the second of these becomes

$$x_1 z_2 = x_2 z_1, \quad y_1 z_2 = y_2 z_1$$

$$\text{and } x_1 y_2 - x_2 y_1 = |X|.$$

If $z_1 \neq 0$, then $z_1 |X| = x_1 y_2 z_1 - x_2 z_1 y_1 = x_1 y_1 z_2 - x_1 z_2 y_1 = 0$
and so $z_1 = 0$.

Similarly $z_2 = 0, z_3 = 1$.

Writing $X_1 = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$, $S = \begin{pmatrix} S_1 & s_3 \\ s_3^T & s_{33} \end{pmatrix}$ where $S_1 = (\underline{s}_1, \underline{s}_2)$ and

$\underline{s}_i = \begin{pmatrix} s_{i1} \\ s_{i2} \end{pmatrix}$ ($i = 1, 2, 3$), the first equation yields

$$|X_1| s_{33}^* = s_{33}$$

$$|X_1| s_{23}^* = y_1 s_{13} + y_2 s_{23} + y_3 s_{33}$$

$$|X_1| s_{13}^* = x_1 s_{13} + x_2 s_{23} + x_3 s_{33}$$

and so the isomorphism classes may be split up into three mutually exclusive collections viz.

$$(i) \quad s_{33} \neq 0, (ii) \quad s_{33} = 0 \text{ and } \underline{s}_3 \neq \underline{0}, (iii) \quad \underline{s}_3 = \underline{0}, s_{33} = 0$$

$$(i) \quad \text{If } s_{33} \neq 0, \text{ by choice of } |X_1|, x_3, y_3 \text{ we may suppose } s_{33} = 1, \underline{s}_3 = \underline{0}.$$

The first equation then reduces to

$$|X_1| = |X| = 1$$

$$X_1^T S X_1 = S_1^* \quad \text{with standard forms}$$

$$S_1 = \underline{0}, \text{diag}(1,0), \text{diag}(\nu,0) \text{ or } \text{diag}(a,1) \quad (a \neq 0).$$

(11) If $s_{33} = 0$, we have $X_1^T s_3 = |X_1| s_3^*$.

Taking $y_1 = -s_{23}$, $y_2 = s_{13}$ (if $s_3 \neq 0$), we may suppose $s_{23} = 0$ and $s_{13} = 1$.

The first equation then reduces to

$$y_1 = 0$$

$$x_1 = |X_1| = x_1 y_2$$

$$s_{22} = x_1 s_{22}^*$$

$$x_1 s_{12} + x_1 y_3 + x_2 s_{22} = x_1 s_{12}^*$$

$$x_1^2 s_{11} + 2x_1 x_2 s_{12} + 2x_1 x_3 + x_2^2 s_{22} = x_1 s_{11}^*$$

and so we may suppose $s_{11} = 0$, $s_{22} = 0$ or 1.

(111) If $s_{13} = s_{23} = s_{33} = 0$, the first equation becomes

$$X_1^T S_1 X_1 = |X_1| S_1^*$$

with standard forms $S_1 = \underline{0}$, $\text{diag}(1,0)$ and $\text{diag}(a,1)$ ($a \neq 0$).

Thus, in all, there are $2p + 10$ groups of $p^6 \phi_{11}$.

$$\underline{\phi_{12}}$$

The groups of ϕ_{12} are characterized by: $G/Z(G)$ is (1^4) , $H_2(G)$ is (11) and we may find two elements β_1, β_2 such that $H_2(G) = \langle \beta_1, \beta_2 \rangle$ and $G/\langle \beta_i \rangle \in \phi_2$ ($i = 1, 2$). Let $\alpha_1 \in G - Z(G)$, $\alpha_{j+1} \in G - \langle \alpha_1, \dots, \alpha_j, Z(G) \rangle$ ($j = 1, 2, 3$). Then by theorem 5 we may suppose

$$[\alpha_1, \alpha_2] = \beta_1, [\alpha_3, \alpha_4] = \beta_2, [\alpha_i, \alpha_j] = 1 \text{ otherwise.}$$

For stem groups, we have $Z(G) = H_2(G) = \langle \beta_1, \beta_2 \rangle$ and

$\alpha_i^p = \alpha_i^{a_i}$ ($i = 1, 2, 3, 4$). Put $A = (a_1, a_2, a_3, a_4)$. If a_{11} or $a_{21} \neq 0$, by $\{\alpha_1 \leftrightarrow \alpha_2\}$ (if necessary) we may suppose

$a_{11} \neq 0$ and by $\{\alpha_2 \rightarrow \alpha_1^{-a_{21}} \alpha_2^{a_{11}}\}$ we may suppose $a_{11} = 1, a_{21} = 0$.

Similarly we may suppose $a_{42} = 0, a_{32} = 0$ or 1 and so

$$A = \begin{pmatrix} a_{11} & 0 & a_3 & a_4 \\ a_1 & a_2 & a_{32} & 0 \end{pmatrix} \text{ where } a_{11}, a_{32} = 0 \text{ or } 1.$$

(a) Suppose $a_{11} = a_{32} = 0$. If some $a_i \neq 0$, we may suppose (by permuting the α_i) that $a_1 \neq 0$. By $\{\alpha_3 \rightarrow \alpha_3^{a_1}\}$ we may suppose

$a_1 = 1$. By $\{\alpha_2 \rightarrow \alpha_1^{-a_2} \alpha_2\}$ we may suppose $a_2 = 0$ and similarly we may suppose $a_3 = 1$ or 0, $a_4 = 0$. As the corresponding groups have different type invariants, they are not isomorphic and so the canonical forms are

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

(b) If one only of a_{11} , a_{32} is non-zero, we may suppose (as before) $a_{11} = 1$, $a_{32} = 0$.

Also, as in (a), we may suppose $a_4 = 0$, $a_3 = 1$ or 0 .

If $a_2 \neq 0$, by $\{\alpha_4 \rightarrow \alpha_4^{a_2}, \alpha_1 \rightarrow \alpha_1 \alpha_2^{-a_1}\}$ we may suppose $a_2 = 1$, $a_1 = 0$.

If $a_2 = 0$, $a_1 \neq 0$, by $\{\alpha_4 \rightarrow \alpha_4^{a_1}\}$ we may suppose $a_1 = 1$. Thus we have reduced A to one of the following:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(c) $a_{11} = a_{32} = 1$.

If $a_2 \neq 0$, by $\{\alpha_3 \rightarrow \alpha_3^{a_2}, \alpha_1 \rightarrow \alpha_1 \alpha_2^{-a_1}\}$ we may suppose $a_2 = 1$, $a_1 = 0$.

Similarly, if $a_4 \neq 0$, we may suppose $a_3 = 0$, $a_4 = 1$. Also, as before, we may suppose that $a_2 = 1$ whenever $a_4 = 1$. We thus have 3 cases for A :

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & a_3 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & a_3 & 0 \\ a_1 & 0 & 1 & 0 \end{pmatrix}.$$

In the second case, if $a_3 \neq 0$, by $\{\alpha_1 \rightarrow \alpha_1^{a_3}\}$ we may suppose $a_3 = 1$.

Similarly, in the third case, we may suppose $a_1 = 0$ or 1 , $a_3 = 0$ or 1 and also (as above) $a_1 = 1$ or $a_1 = a_3 = 0$.

This gives us the following 6 matrices:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

To check that no two of the above 15 matrices give isomorphic groups, we write

$$\alpha_i^p = \beta_{\sim}^{a_i}, \quad \alpha_i^{*p} = \beta_{\sim}^{*a_i} \quad \text{where}$$

$$\begin{aligned} \alpha_1^* &= \alpha_{\sim}^x \\ \alpha_2^* &= \alpha_{\sim}^y \\ \alpha_3^* &= \alpha_{\sim}^z \\ \alpha_4^* &= \alpha_{\sim}^w. \end{aligned}$$

From $[\alpha_1^*, \alpha_3^*] = 1$ etc., we get

$$x_1 z_2 = x_2 z_1 \quad y_1 z_2 = y_2 z_1 \quad x_3 z_4 = x_4 z_3 \quad y_3 z_4 = y_4 z_3$$

$$x_1 w_2 = x_2 w_1 \quad y_1 w_2 = y_2 w_1 \quad x_3 w_4 = x_4 w_3 \quad y_3 w_4 = y_4 w_3$$

$$\text{and so } x_3 = x_4 = y_3 = y_4 = z_1 = z_2 = w_1 = w_2 = 0$$

$$\text{or } x_1 = x_2 = y_1 = y_2 = z_3 = z_4 = w_3 = w_4 = 0$$

i.e. putting $X = (\underline{x}, \underline{y}, \underline{z}, \underline{w})$, we may suppose

$$X = \begin{pmatrix} xX_1 & yX_1 \\ yX_2 & xX_2 \end{pmatrix}$$

where $X_1 = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$, $X_2 = \begin{pmatrix} z_3 & w_3 \\ z_4 & w_4 \end{pmatrix}$ and $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$$\text{Thus } \beta_1^* = \beta_1 \frac{x|X_1|}{\beta_2} \frac{y|X_1|}{\beta_2}$$

$$\beta_2^* = \beta_1 \frac{y|X_1|}{\beta_2} \frac{x|X_2|}{\beta_2}.$$

As for earlier families, the relations $\alpha_1^{*p} = \beta_1^{*a_1^*}$ give

$$AX = YA^* \quad \dots \quad (1)$$

where $A = (a_1, a_2, a_3, a_4)$, $A^* = (a_1^*, a_2^*, a_3^*, a_4^*)$,

$$Y = \begin{pmatrix} x|X_1| & y|X_2| \\ y|X_1| & x|X_2| \end{pmatrix} \text{ and so } Y^{-1} = \frac{1}{|X|} \begin{pmatrix} x|X_2| & -y|X_2| \\ -y|X_1| & x|X_1| \end{pmatrix}.$$

Putting $a_{21} = a_{42} = 0$ (as we have done above), (1) becomes

$$|X|A^* = \begin{pmatrix} x_1(xa_{11}-ya_{32})|X_2| & y_1(xa_{11}-ya_{32})|X_2| & \dots & \dots \\ \dots & \dots & z_3(xa_{32}-ya_{11})|X_1| & w_3(xa_{32}-ya_{11})|X_1| \\ \dots & \dots & \dots & \dots \end{pmatrix} \quad (2)$$

$$a_{21}^* = a_{42}^* = 0 \text{ gives}$$

$$y_1(xa_{11} - ya_{32}) = w_3(xa_{32} - ya_{11}) = 0 \quad \dots \quad (3)$$

If $a_{11} = a_{32} = 0$, (2) gives $a_{11}^* = a_{32}^* = 0$.

If $a_{11} = a_{32} = 1$, then $xa_{11} - ya_{32} = \pm 1$ and so (from (3)),

$$y_1 = w_3 = 0.$$

Thus $x_1y_2z_3w_4 \neq 0$ and (2) gives $a_{11}^*, a_{32}^* \neq 0$.

Thus the three cases (a), (b), (c) above are mutually exclusive.

The three groups of case (a) have been shown to be not isomorphic. From now on, we shall take

$$a_{11} = 1, a_{11}^* \neq 0.$$

Thus, from (2), we have $x_1(xa_{11} - ya_{32}) \neq 0$ and so $y_1 = 0$, $x_1, y_2 \neq 0$.

In case (b), two of the groups are of type (21^4) and so not isomorphic to any of the other groups (nor to each other since one is the direct product of two groups of order p^3). For the other groups of case (b), we take $a_{32} = a_{32}^* = 0$ and so $y = 0$, and we have

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$X A^* = \begin{pmatrix} x_1|X_2| & 0 & 0 & 0 \\ . & . & . & . \end{pmatrix}, \begin{pmatrix} x_1|X_2| & 0 & z_3|X_2| & w_3|X_2| \\ x_2|X_1| & y_2|X_1| & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} x_1|X_2| & 0 & z_3|X_2| & w_3|X_2| \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus no two of these matrices give isomorphic groups.

For case (c), the first matrix gives a group of type (21^4) , the second gives the direct product of two groups of order p^3 and for the others $a_{32}^* \neq 0$ and so (from (2) and (3)) $w_3 = 0$, $x_1y_2z_3w_4 \neq 0$.

Thus:

A	A*	Notes
$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} x_1(x-y) X_2 & 0 & -z_3y X_2 & 0 \\ xx_1 X_1 & 0 & (x-y)z_3 X_1 & 0 \end{pmatrix}$	
$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} x_1(x-y) X_2 & 0 & -yz_4 X_2 , -yw_4 X_2 \\ xx_2 X_1 , xy_2 X_1 , (x-y)z_3 X_1 , 0 \end{pmatrix}$	If $a_{22}^* \neq 0$, then $a_{31}^* =$ $a_{41}^* = 0$
$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} x_1 X_2 & 0 & z_4 X_2 & w_4 X_2 \\ x_2 X_1 & y_2 X_1 & z_3 X_1 & 0 \end{pmatrix}$	

Thus no two of these matrices give isomorphic groups.

Thus there are just 15 groups in $p^6\phi_{12}$.

Φ_{13} ($p \geq 3$)

The groups of Φ_{13} are characterized by: $G/Z(G)$ is (1^4) , $H_2(G)$ is (11), and there exist $\beta_1, \beta_2 \in H_2(G)$ with $G/\langle \beta_2 \rangle \in \Phi_2$, $G/\langle \beta_1 \rangle \in \Phi_5$.

By theorem 5 above we may suppose

$$G = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, Z(G) \mid [\alpha_1, \alpha_2] = \beta_1, [\alpha_1, \alpha_3] = [\alpha_2, \alpha_4] = \beta_2, \text{rest} = 1, \alpha_1^p \in Z(G) \rangle.$$

Also by lemma L6, we see that $\langle \alpha_3, \alpha_4, Z(G) \rangle$ is a characteristic subgroup of G (For $\langle \alpha_4, Z \rangle = \mathcal{C}(\langle \alpha_1, \alpha_3, \alpha_4, Z \rangle)$, $\langle \alpha_3, Z \rangle = \mathcal{C}(\langle \alpha_2, \alpha_3, \alpha_4, Z \rangle)$, Z = centralizer of any other maximal subgroup).

For stem groups $Z(G) = H_2(G) = \langle \beta_1, \beta_2 \rangle$ and so

$$\alpha_1^p = \tilde{\beta}^{a_1} 1, \quad \alpha_1^{*p} = \tilde{\beta}^{*a_1} 1^* \quad \text{where}$$

$$\left. \begin{aligned} \alpha_1^* &\equiv \alpha_1^{x_1} \alpha_2^{x_2} \alpha_3^{x_3} \alpha_4^{x_4} \\ \alpha_2^* &\equiv \alpha_1^{y_1} \alpha_2^{y_2} \alpha_3^{y_3} \alpha_4^{y_4} \\ \alpha_3^* &\equiv \alpha_3^{z_3} \alpha_4^{z_4} \\ \alpha_4^* &\equiv \alpha_3^{w_3} \alpha_4^{w_4} \end{aligned} \right\} \quad (\text{mod } Z(G))$$

$$\beta_1^* = \beta_1^{x_1 y_2 - x_2 y_1} \beta_2^{x_1 y_3 - x_3 y_1 + x_2 y_4 - x_4 y_2} = \beta_1^{x_1 y_2 - x_2 y_1} \beta_2^t$$

$$\beta_2^* = \beta_2^{y_1 w_3 + y_2 w_4}.$$

By considering $[\alpha_i^*, \alpha_j^*]$ we must have

$$x_1 z_3 + x_2 z_4 = y_1 w_3 + y_2 w_4 \neq 0 \quad \dots \quad (1)$$

$$x_1 w_3 + x_2 w_4 = 0 \quad \dots \quad (2)$$

$$y_1 z_3 + y_2 z_4 = 0 \quad \dots \quad (3)$$

Case 1 α_4^p, α_3^p not both in $\langle \beta_2 \rangle$.

By interchanging α_3, α_4 (if necessary) we may suppose $\alpha_3^p \notin \langle \beta_2 \rangle$.

Thus $a_{31} \not\equiv 0 \pmod{p}$ and so we take $a_{31} \bar{a}_{31} \equiv 1 \pmod{p}$.

By $\{\alpha_4 \rightarrow \alpha_3^{-a_{41} \bar{a}_{31}} \alpha_4\}$ we may suppose $\alpha_4^p \in \langle \beta_2 \rangle$.

By $\{\alpha_3 \rightarrow \alpha_3^{\bar{a}_{31}}, \alpha_4 \rightarrow \alpha_4^{\bar{a}_{31}}\}$ we may suppose $\alpha_3^p \equiv \beta_1 \pmod{\langle \beta_2 \rangle}$.

From $\alpha_4^{*p} \in \langle \beta_2^* \rangle$ we have $w_3 = 0$.

In (2), $x_2 w_4 = 0$ and so $x_2 = 0$ (since $w_1 \neq 0$, some 1).

From $\alpha_3^{*p} \equiv \beta_1^* \pmod{\langle \beta_2^* \rangle}$ we have $z_3 = x_1 y_2 - x_2 y_1 = x_1 y_2$.

In (3) $x_1 y_1 y_2 + y_2 z_4 = 0$ and so $z_4 = -x_1 y_1$ (since x_2 or $y_2 \neq 0$).

In (1) $x_1^2 y_2 = y_2 w_4$ and so $w_4 = x_1^2$ (as above).

The remaining equations $\alpha_1^{*p} = \beta_1^{* \bar{a}_{11}}$ become

$$x_1 a_{11} + x_3 = x_1 y_2 a_{11}^*$$

$$x_1 a_{12} + x_3 a_{32} + x_4 a_{42} = t a_{11}^* + x_1^2 y_2 a_{12}^*$$

$$y_1 a_{11} + y_2 a_{21} + y_3 = x_1 y_2 a_{21}^*$$

$$y_1 a_{12} + y_2 a_{22} + y_3 a_{32} + y_4 a_{42} = t a_{21}^* + x_1^2 y_2 a_{22}^*$$

$$x_1 y_2 a_{32} - x_1 y_1 a_{42} = t + x_1^2 y_2 a_{32}^*$$

$$x_1^2 a_{42} = x_1^2 y_2 a_{42}^*.$$

By choice of x_3, y_3, t we may suppose $a_{11} = a_{21} = a_{32} = 0$ and the above reduce to $x_3 = y_3 = 0, t = -x_1 y_1 a_{42}$ and

$$x_1 a_{12} + x_4 a_{42} = x_1^2 y_2 a_{12}^*$$

$$y_1 a_{12} + y_2 a_{22} + y_4 a_{42} = x_1^2 y_2 a_{22}^*$$

$$a_{42} = y_2 a_{42}^*$$

with standard forms: $a_{42} = 1 \ 0 \ 0 \ 0 \ 0$

$$a_{12} = 0 \ 1 \ 0 \ 0 \ 0$$

$$a_{22} = 0 \ 0 \ 1 \ v \ 0$$

and so there are five groups in case 1.

Case 2 $\alpha_3^p, \alpha_4^p \in \langle \beta_2 \rangle$, one of $\alpha_1^p, \alpha_2^p \notin \langle \beta_2 \rangle$.

As in case 1, we may suppose $\alpha_1^p \equiv \beta_1 \pmod{\langle \beta_2 \rangle}, \alpha_2^p \in \langle \beta_2 \rangle$.

From $\alpha_2^p \in \langle \beta_2 \rangle$, we get $y_1 = 0$.

In (3), $y_2 z_4 = 0$ and so $z_4 = 0$.

From $\alpha_1^p \equiv \beta_1$, we get $x_1 = x_1 y_2 - x_2 y_1 = x_1 y_2$ and so $y_2 = 1$.

In (1), $w_4 = x_1 z_3 + x_2 z_4 = x_1 z_3 \neq 0$.

The remaining equations become

$$x_1 a_{12} + x_2 a_{22} + x_3 a_{32} + x_4 a_{42} = x_1 z_3 a_{12}^* + t$$

$$a_{22} + y_3 a_{32} + y_4 a_{42} = x_1 z_3 a_{22}^*$$

$$z_3 a_{32} = x_1 z_3 a_{32}^*$$

$$w_3 a_{32} + x_1 z_3 a_{42} = x_1 z_3 a_{42}^*.$$

By choice of t , we may suppose $a_{12} = 0$ and the standard forms

$$\text{for the rest are } a_{32} = 1 \ 0 \ 0$$

$$a_{22} = 0 \ 1 \ 0$$

$$a_{42} = 0 \ 0 \ \text{any}$$

and so there are $p + 2$ groups in case 2.

Case 3 $\alpha_1^p \in \langle \beta_2 \rangle$. As before we have $\alpha_3^p \neq 1$, $\alpha_1^p = \alpha_2^p = \alpha_4^p = 1$; $\alpha_1^p \neq 1$, $\alpha_2^p = \alpha_3^p = \alpha_4^p = 1$; or $\alpha_1^p = 1$ (all 1).

In the first two cases, by replacing α_3 or α_1 by a suitable power we may suppose $\alpha_3^p = \beta_2$ or $\alpha_1^p = \beta_2$.

Thus case 3 gives rise to 3 groups.

In all, there are $p + 10$ groups in $p^6\phi_{13}$.

ϕ_{14} ($p \geq 3$)

The groups of ϕ_{14} are characterized by: $G/Z(G)$ is (22), $H_2(G)$ is (2). Let $\alpha_1 \in G - \langle Z(G), G^p \rangle$ and $\alpha_2 \in G - \langle \alpha_1, G^p \rangle$ and $[\alpha_1, \alpha_2] = \beta$.

Then $\alpha_1^{p^2} \in Z(G)$, $\beta^{p^2} = 1$.

For stem groups, $Z(G) = H_2(G) = \langle \beta \rangle$ and so $\alpha_1^{p^2} = \beta^{a_1}$.

If a_1 or $a_2 \not\equiv 0 \pmod{p}$ we may suppose, without loss of generality, that $a_1 \not\equiv 0 \pmod{p}$.

By $\{\alpha_2 \rightarrow \alpha_1^{-a_2} \alpha_2^{a_1}\}$ we may suppose $a_1 = 1$, $a_2 = 0$.

If $a_1 \equiv a_2 \equiv 0 \pmod{p}$, put $a_1 = pa'_1$, $a_2 = pa'_2$.

If a'_1 or $a'_2 \not\equiv 0 \pmod{p}$, we may again suppose $a'_1 \not\equiv 0 \pmod{p}$

and by $\{\alpha_2 \rightarrow \alpha_1^{-a'_2} \alpha_2^{a'_1}\}$ we may suppose $a_1 = p$, $a_2 = 0$.

Thus there are just three groups in $p^6\phi_{14}$:

$$\langle \alpha, \beta | \alpha^{p^4} = \beta^{p^2} = 1, [\alpha, \beta] = \alpha^{p^2} \rangle$$

$$\langle \alpha, \beta, \gamma | \alpha^{p^3} = \beta^{p^2} = 1, \gamma^p = \alpha^{p^2}, [\alpha, \beta] = \gamma \rangle$$

$$\langle \alpha, \beta, \gamma | \alpha^{p^2} = \beta^{p^2} = \gamma^{p^2} = 1, [\alpha, \beta] = \gamma \rangle.$$

Note: The second group may also be written as

$$\langle \alpha, \beta, \gamma | \alpha^{p^3} = \beta^{p^2} = \gamma^p = 1, [\alpha, \beta] = \alpha^p \gamma \rangle.$$

ϕ_{15} ($p \geq 3$)

The groups of ϕ_{15} are characterized by: $G/Z(G)$ is (1^4) , $H_2(G)$ is (11) and $G/\langle \beta \rangle \in \phi_5$ for all $\beta \in Z(G)$.

By theorem 5, we may suppose all stem groups are of the form

$\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$ where $\alpha_1^p = \tilde{\alpha}_1^{a_1}$ ($i = 1, 2, 3, 4$), $[\alpha_1, \alpha_2] = [\alpha_3, \alpha_4] = \beta_1$, $[\alpha_1, \alpha_3] = \beta_2$, $[\alpha_2, \alpha_4] = \beta_2^v$, rest = 1.

Let

$$\begin{aligned}\alpha_1^* &= \tilde{\alpha}_1^z \\ \alpha_2^* &= \tilde{\alpha}_2^y \\ \alpha_3^* &= \tilde{\alpha}_3^w \\ \alpha_4^* &= \tilde{\alpha}_4^x.\end{aligned}$$

From $[\alpha_1^*, \alpha_4^*] = [\alpha_2^*, \alpha_3^*] = 1$ we get

$$\begin{aligned}z_1x_2 - z_2x_1 + z_3x_4 - z_4x_3 &= 0 \\ z_1x_3 - z_3x_1 + v(z_2x_4 - z_4x_2) &= 0 \\ y_1w_2 - y_2w_1 + y_3w_4 - y_4w_3 &= 0 \\ y_1w_3 - y_3w_1 + v(y_2w_4 - y_4w_2) &= 0\end{aligned}$$

and from $[\alpha_1^*, \alpha_2^*] = [\alpha_3^*, \alpha_4^*]$ and $[\alpha_2^*, \alpha_4^*] = [\alpha_1^*, \alpha_3^*]^v$, we get

$$\begin{aligned}z_1y_2 - z_2y_1 + z_3y_4 - z_4y_3 &= w_1x_2 - w_2x_1 + w_3x_4 - w_4x_3 \\ z_1y_3 - z_3y_1 + v(z_2y_4 - z_4y_2) &= w_1x_3 - w_3x_1 + v(w_2x_4 - w_4x_2) \\ y_1x_2 - y_2x_1 + y_3x_4 - y_4x_3 &= v(z_1w_2 - z_2w_1 + z_3w_4 - z_4w_3) \\ y_1x_3 - y_3x_1 + v(y_2x_4 - y_4x_2) &= v(z_1w_3 - z_3w_1) + v^2(z_2w_4 - z_4w_2).\end{aligned}$$

Solving these, we get (with $\epsilon^2 = 1$)

$$\begin{aligned} z_1 &= \epsilon x_4 & w_1 &= -\epsilon y_4 \\ v z_2 &= -\epsilon x_3 & v w_2 &= \epsilon y_3 \\ z_3 &= -\epsilon x_2 & w_3 &= \epsilon y_2 \\ v z_4 &= \epsilon x_1 & v w_4 &= -\epsilon y_1. \end{aligned}$$

With these values, we have

$$\beta_1^* = \beta_1^{d_{11}} \beta_2^{d_{12}}$$

$$\beta_2^* = \beta_1^{d_{21}} \beta_2^{d_{22}}$$

$$\text{where } d_{21} = v^{-1}(x_4 y_3 - x_3 y_4 + x_2 y_1 - x_1 y_2)$$

$$d_{22} = x_4 y_2 - x_2 y_4 + v^{-1}(x_3 y_1 - x_1 y_3)$$

$$d_{11} = \epsilon d_{22}$$

$$d_{12} = \epsilon v d_{21}$$

From $\alpha_1^{*p} = \beta_1^{*a_1^*}$ we get

$$DA^* = AX \quad |X| \neq 0 \quad \dots \quad (1)$$

where $D = (d_1, d_2)$, $A^* = (a_1^*, a_2^*, a_3^*, a_4^*)$,

$$X = (z, y, w, x), \quad A = (a_1, a_2, a_3, a_4).$$

Case I $\alpha_1^p = 1$ all i .

Thus $A = \underline{0}$ and this is the standard form.

Case II $\alpha_i^p \in \langle \beta_1 \rangle$ ($i = 1, 2, 3, 4$) and some $\alpha_i^p \neq 1$.

By a suitable permutation of the α_i , we may suppose that

$$\alpha_1^p \neq 1 \text{ (i.e. } a_{11} \neq 0).$$

By $\{\alpha_2 \rightarrow \alpha_2^{a_{11}}, \alpha_3 \rightarrow \alpha_3^{a_{11}}\}$ we may suppose $\alpha_1^p = \beta_1$.

By $\{\alpha_3 \rightarrow \alpha_1^{-a_{31}}\alpha_3, \alpha_2 \rightarrow \alpha_2\alpha_4^{a_{31}}\}$ we may assume $\alpha_3^p = 1$.

(a) If $\alpha_4^p = 1$, by $\{\alpha_2 \rightarrow \alpha_1^{-a_{21}}\alpha_2, \alpha_3 \rightarrow \alpha_3\alpha_4^{v^{-1}a_{21}}\}$ we may suppose $\alpha_2^p = 1$.

i.e. standard form is $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

(b) If $\alpha_4^p \neq 1$, by $\{\alpha_1 \rightarrow \alpha_1\alpha_4^{-v^{-1}a_{41}}, \alpha_2 \rightarrow \alpha_1^{-a_{21}}\alpha_2\alpha_3^{-a_{41}}, \alpha_3 \rightarrow \alpha_2^{-v^{-1}a_{41}}\alpha_3\alpha_4^{v^{-1}a_{21}}, \alpha_4 \rightarrow \alpha_1^{-a_{41}}\alpha_4\}$, we may suppose $\alpha_2^p = \alpha_4^p = 1$.

Thus we have (a) again.

Case III $\alpha_1^p \in \langle \beta_2 \rangle$ all i , some $\alpha_i^p \neq 1$.

As in case II, we may suppose $\alpha_1^p = \beta_2, \alpha_2^p = \alpha_3^p = \alpha_4^p = 1$.

By $\{\alpha_2 \rightarrow \alpha_3, \alpha_3 \rightarrow \alpha_2^{v^{-1}}\}$ we may suppose $\alpha_1^p = \beta_1$ and so once again we have case II.

Case IV $\alpha_i^p \notin \langle \beta_1 \rangle, \alpha_j^p \notin \langle \beta_2 \rangle$ some i, j .

By the same reasoning as in case II we may suppose $\alpha_1^p \equiv \beta_1$

(mod $\langle \beta_2 \rangle$) i.e. $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & b & c & d \end{pmatrix}$.

(a) If $b = c = d = 0$, by $\{\alpha_2 \rightarrow \alpha_2 \alpha_3^a, \alpha_3 \rightarrow \alpha_2 v^{-1} \alpha_3\}$ we may assume $\alpha_1^p = \beta_1$ and so we have case II once again.

(b) If $a = b = c = 0, d \neq 0$, (1) becomes

$$y_1 = y_4 = 0$$

and $\epsilon x_4 = d_{11}$

$$\epsilon v^{-1} dx_1 = d_{12}$$

$$x_1 = d_{21} d^*$$

$$dx_4 = d_{22} d^*$$

$$= \epsilon v^{-1} d_{12} d^*$$

$$= \epsilon d_{11} d^*$$

$$= v^{-2} x_1 d d^*$$

$$= x_4 d^*.$$

Thus $d = d^*$ or $dd^* = v^2$.

As we may take v to be any non quadratic residue, we may take $v = g$ (primitive root).

Then $v^2 = g^2 = g^{p+1}$

and so the standard forms are

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & g^k \end{pmatrix} \quad k = 1, 2, \dots, \frac{1}{2}(p+1).$$

These groups are obviously not isomorphic to any earlier ones by considering the type invariants.

(c) If $b = c = 0, a, d \neq 0$, (1) becomes

$$y_1 = y_4 = 0$$

and $\epsilon x_4 = d_{11} + a^* d_{21}$

$$\epsilon(ax_4 - v^{-1} dx_1) = d_{12} + d_{22} a^*$$

$$x_1 = d_{21} d^*$$

$$ax_1 + dx_4 = d_{22} d^*.$$

$$\text{Thus } d_{12}d^* = \epsilon v d_{21}d^* = \epsilon v x_1$$

$$d_{22}d^* = \epsilon d_{11}d^* = x_4d^* - \epsilon x_1a^*$$

$$\text{and so } (v^{-1}dd^* - v + a^{*2})x_1 + d^*(a - \epsilon a^*)x_4 = 0$$

$$(a + \epsilon a^*)x_1 + (d - d^*)x_4 = 0.$$

As $x_1 = x_4 = 0$ would mean $|X| = 0$, these equations must have a non-zero solution.

$$\text{i.e. } d^*(a^2 - a^{*2}) = (d - d^*)(v^{-1}dd^* - v + a^{*2}) \quad \dots \quad (2)$$

Fixing d , we have $a^2 = a^{*2}$ and so we may suppose

$$a = 1, 2, \dots, \frac{p-1}{2}.$$

(Note that $a^* = 0$ only if $a = 0$ and $d^* = 0$ only if $d = 0$.)

$$(2) \text{ becomes } a^2(d - d^*) = v^{-1}(v^2 - dd^*)(d - d^*)$$

$$\text{i.e. } d = d^* \text{ or } dd^* = v(v - a^2).$$

Now, as $v(v - a^2)$ can have $\frac{p-1}{2}$ values as a runs through a complete

set of residues (mod p), lemma L7 tells us that $[\frac{p+1}{4}]$ of these

are non quadratic residues and $[\frac{p-1}{4}]$ are quadratic residues.

Also if $v(v - a^2)$ is a non quadratic residue, the relation

$dd^* = v(v - a^2)$ pairs off all the non-zero residues while if

$v(v - a^2) = m^2$ the relation pairs off all non-zero residues

except m and $-m$ giving $\frac{1}{2}(p - 3) + 2 = \frac{1}{2}(p + 1)$ standard forms.

For $p \equiv 1 \pmod{4}$, $v(v - a^2)$ has $\frac{p-1}{4}$ quadratic values and $\frac{p-1}{4}$

non quadratic values giving $\frac{1}{2}(p + 1)$ and $\frac{1}{2}(p - 1)$ standard forms

respectively.

$$\therefore \text{Number of standard forms} = \frac{1}{4}(p^2 - p).$$

For $p \equiv 3 \pmod{4}$, $v(v - a^2)$ has $\frac{p-3}{4}$ quadratic values giving $\frac{1}{2}(p+1)$ standard forms each and $\frac{p+1}{4}$ non quadratic values giving $\frac{1}{2}(p-1)$ standard forms each.

$$\begin{aligned}\therefore \text{Number of standard forms} &= \frac{1}{2}(p+1)\left(\frac{p-3}{2} + \frac{p-1}{2}\right) \\ &= \frac{1}{2}(p^2 - p - 2).\end{aligned}$$

Thus the standard forms are $\begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 0 & 0 & d \end{pmatrix}$ where $a = 1, 2, \dots, \frac{1}{2}(p-1)$

and d, d^* give isomorphic groups if $d = d^*$ or $dd^* = v(v - a^2)$, there being $\frac{1}{2}(p^2 - p + (p-1, 4) - 4)$ groups altogether.

(d) If b or $c \neq 0$, by interchanging α_1, α_2 with α_4, α_3 respectively (if necessary) we may suppose $b \neq 0$.

By $\{\alpha_1 \rightarrow \alpha_1^b, \alpha_4 \rightarrow \alpha_4^b\}$ we may suppose $b = 1$.

By $\{\alpha_1 \rightarrow \alpha_1 \alpha_3^d, \alpha_4 \rightarrow \alpha_2^{-d} \alpha_4\}$ we may suppose $d = 0$.

By $\{\alpha_2 \rightarrow \alpha_1^{vx} \alpha_2 \alpha_3^a \alpha_4^{-y}, \alpha_3 \rightarrow \alpha_1^a \alpha_2^{v^{-1}a} \alpha_3 \alpha_4^{-x}\}$ we may suppose $a = 0$, where $v(v - a^2)x = a + a^2c$ and $v(v - a^2)y = a^2 + vac$.

By $\{\alpha_1 \rightarrow \alpha_1 \alpha_2^{-u} \alpha_3^{vcu}, \alpha_2 \rightarrow \alpha_1^{-c} \alpha_2^v \alpha_3^u, \alpha_3 \rightarrow \alpha_2^{-cv} \alpha_3^v \alpha_4^{v^{-1}c}, \alpha_4 \rightarrow \alpha_2^{-vcu} \alpha_3^{vu} \alpha_4\}$ we may suppose $c = 0$, where $(1 - vc^2)v = 1$ and $u = vcv$.

Thus the standard form for this case is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Also if $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} X = D \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 0 & 0 & d \end{pmatrix}$ then we must have $\underline{y} = \underline{0}$

contradicting the assumption that $|X| = 0$.

Thus the total number of groups in $p^6\phi_{15}$ is $\frac{1}{4}(p^2 + p + 10 + (p-1,4))$.

Φ_{16} ($p \geq 3$)

The groups of Φ_{16} are characterized by: $G/Z(G)$ is $\Phi_2(1^4)$, $H_2(G)$ is (111), $H_3(G)$ is (1) and G has a maximal abelian subgroup $H_1(G)$.

Let $\alpha \in G - H_1(G)$, $\alpha_1 \in H_1(G) - Z_2(G)$, $\beta \in H_1(G) - \langle \alpha_1, Z(G), H_2(G) \rangle$, $[\alpha_1, \alpha] = \alpha_{1+1}$ ($i = 1, 2$) and $\gamma_2 = [\beta, \alpha]$. By theorem 2, we have $[\alpha_1, \alpha_2] = [\alpha_1, \beta] = 1$ ($i = 1, 2$) and $\alpha_3, \gamma_2 \in Z(G)$.

For stem groups, we have $Z(G) = \langle \alpha_3, \gamma_2 \rangle$, $Z_2(G) = \langle \beta, \alpha_2, \alpha_3, \gamma_2 \rangle$ and $H_j(G)$ ($j = 1, 2, 3$) are characteristic subgroups. Also since $\langle \alpha, \alpha_1 \rangle \in \Phi_3$, we use the same methods as there.

Writing $\gamma_1 = \alpha_3$, we have

$$\alpha^p = \gamma^a, \alpha_1^p \alpha_3^{(p)} = \gamma^{\tilde{a}1}, \beta^p = \gamma^{\tilde{a}2}, \alpha^{*p} = \gamma^{*\tilde{a}*} \text{ etc.}$$

where
$$\alpha^* = \alpha^x \alpha_1^{x_1} \beta^{x_2} \dots$$

$$\alpha_1^* = \alpha_1^{y_1} \beta^{y_2} \dots \quad xy_1 z_2 \neq 0$$

$$\beta^* = \beta^{z_2} \alpha_2^{z'} \dots$$

and so
$$\alpha_2^* = \alpha_2^{xy_1} \dots$$

$$\alpha_3^* = \alpha_3^{x^2 y_1}$$

$$\gamma_2^* = \alpha_3^{xz'} \gamma_2^{xz_2}.$$

Using $(\alpha^x \alpha_1^{x_1})^p = \alpha^{px} (\alpha_1^p \alpha_3^{\binom{p}{3}}) x_1$, we get

$$a_{22} = x a_{22}^*$$

$$z_2 a_{21} = x^2 y_1 a_{21}^* + x z' a_{22}^*$$

$$y_1 a_{12} + y_2 a_{22} = x z_2 a_{12}^*$$

$$y_1 a_{11} + y_2 a_{21} = x^2 y_1 a_{11}^* + x z' a_{12}^*$$

$$x a_2 + x_1 a_{12} + x_2 a_{22} = x z_2 a_2^*$$

$$x a_1 + x_1 a_{11} + x_2 a_{21} = x^2 y_1 a_1^* + x z' a_2^*.$$

(a) If $a_{22} \neq 0$, take $x = a_{22}$ and so we may suppose $a_{22} = 1$.

Also, by taking $x z' = z_2 a_{21}$, $y_2 = -y_1 a_{12}$, $x_2 = -(x a_2 + x_1 a_{12})$,

we may suppose $a_{21} = a_{12} = a_2 = 0$ and so

$$a_{11} = a_{11}^*$$

$$a_1 + x_1 a_{11} = y_1 a_1^*$$

with standard forms $a_{11} = 0$ any

$$a_1 = 1 \quad 0.$$

(b) If $a_{22} = 0$, $a_{21} \neq 0$, take $z_2 a_{21} = x^2 y_1$ and so can assume

$a_{21} = 1$. Also, by taking $y_2 = x z' a_{12}^* - y_1 a_{11}$, $x_2 = x z' a_2^* -$

$x a_1 - x_1 a_{11}$, we may suppose $a_{11} = a_1 = 0$ and so

$$a_{12} = x^3 a_{12}^* \quad (\text{since } z_2 = x^2 y_1)$$

$$x a_2 + x_1 a_{12} = x^3 y_1 a_2^*$$

with standard forms $a_{12} = 1 \quad g^r \quad 0 \quad 0 \quad r = 1, \dots, (p-1, 3) - 1$

$$a_2 = 0 \quad 0 \quad 1 \quad 0.$$

(c) If $\underline{a}_2 = \underline{0}$, $a_{12} \neq 0$, take $y_1 a_{12} = xz_2$ and so can assume $a_{12} = 1$. Taking $xz' = y_1 a_{11}$, $x_1 = -xa_2$, we may suppose $a_{11} = a_2 = 0$ and so $xa_1 = x^3 z_2 a_1^*$

i.e. we may suppose $a_1 = 0$ or 1 .

(d) If $\underline{a}_2 = \underline{0}$, $a_{12} = 0$, we have $a_{11} = x^2 a_{11}^*$

$$xa_2 = xz_2 a_2^*$$

$$xa_1 + x_1 a_{11} = x^2 y_1 a_1^* + xz' a_2^*$$

with standard forms $a_{11} = 1 \vee 1 \vee 0 \ 0 \ 0$

$$a_2 = 1 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0$$

$$a_1 = 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0.$$

Thus there are $p + 12 + (p-1, 3)$ groups of $p^6 \phi_{16}$.

$$\phi_{17} (p \geq 3)$$

The groups of ϕ_{17} are characterized by: $G/Z(G)$ is $\phi_2(1^4)$, $H_2(G)$ is (111), $H_3(G)$ is (1) and $[H_1(G), Z_2(G)] \not\leq H_3(G)$ where $H_1(G) = \mathcal{C}(H_2(G))$.

Let $\alpha \in G - H_1(G)$, $\alpha_1 \in H_1(G) - Z_2(G)$, $\beta \in Z_2(G) - \langle Z(G), H_2(G) \rangle$, $\alpha_{i+1} = [\alpha_i, \alpha]$ ($i = 1, 2$), $\gamma_2 = [\alpha_1, \beta]$. By theorem 2, we may suppose $[\beta, \alpha] = [\beta, \alpha_2] = [\alpha_1, \alpha_2] = 1$ and $\alpha_3, \gamma_2 \in Z(G)$.

For stem groups, we have $Z(G) = \langle \alpha_3, \gamma_2 \rangle$, $Z_2(G) = \langle \beta, \alpha_2, \alpha_3, \gamma_2 \rangle$ and $H_j(G)$ are characteristic subgroups ($j = 1, 2, 3$). Writing $\gamma_1 = \alpha_3$, we have

$$\alpha^p = \gamma^a, \alpha_1^p = \gamma^{a_1}, \beta^p = \gamma^{a_2}, \alpha^{*p} = \gamma^{*a^*} \text{ etc.}$$

$$\begin{aligned} \text{where} \quad \alpha^* &= \alpha^x \alpha_1^{x_1} \beta^{x_2} \dots \\ \alpha_1^* &= \alpha_1^{y_1} \beta^{y_2} \dots \quad xy_1 z_2 \neq 0 \\ \beta^* &= \beta^{z_2} \dots \end{aligned}$$

$$\begin{aligned} \text{and so} \quad \alpha_2^* &= \alpha_2^{xy_1} \dots \\ \alpha_3^* &= \alpha_3^{x^2 y_1} \\ \gamma_2^* &= \gamma_2^{y_1 z_2}. \end{aligned}$$

From $[\alpha^*, \beta^*] = 1$, we get $x_1 z_2 = 0$ and so $x_1 = 0$.

From the above we get $a_{22} = y_1 a_{22}^*$

$$z_2 a_{21} = x^2 y_1 a_{21}^*$$

$$y_1 a_{12} + y_2 a_{22} = y_1 z_2 a_{12}^*$$

$$y_1 a_{11} + y_2 a_{21} = x^2 y_1 a_{11}^*$$

$$x a_2 + x_2 a_{22} = y_1 z_2 a_2^*$$

$$x a_1 + x_2 a_{21} = x^2 y_1 a_1^*.$$

If $a_{22} \neq 0$, take $y_1 = a_{22}$ and so we may suppose $a_{22} = 1$.

Also, taking $y_2 = -y_1 a_{12}$, $x_2 = -x a_2$, we may suppose

$$a_{12} = a_2 = 0 \quad \text{and so}$$

$$z_2 a_{21} = x^2 a_{21}^*$$

$$a_{11} = x^2 a_{11}^*$$

$$a_1 = x a_1^* \quad \text{with standard forms}$$

$$a_1 = 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0$$

$$a_{11} = \text{any} \quad \text{any} \quad 1 \quad 1 \quad v \quad v \quad 0 \quad 0$$

$$a_{21} = 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0.$$

(b) If $a_{22} = 0$, $a_{21} \neq 0$, take $x^2 y_1 = z_2 a_{21}$ and so we may suppose $a_{21} = 1$.

As before, we may suppose $a_{11} = a_1 = 0$

$$\text{and so} \quad a_{12} = x^2 y_1 a_{12}^*$$

$$a_2 = x y_1^2 a_2^* \quad \text{with standard forms}$$

$$a_{12} = 1 \quad 1 \quad 1 \quad 0 \quad 0$$

$$a_2 = 0 \quad 1 \quad g^r \quad 1 \quad 0 \quad r = 1, 2, \dots (p-1, 3) - 1.$$

(c) If $\tilde{a}_2 = 0$, $a_{12} \neq 0$, take $z_2 = a_{12}$ and so we may suppose $a_{12} = 1$.

Thus

$$a_{11} = x^2 a_{11}^*$$

$$a_1 = xy_1 a_1^*$$

$$xa_2 = y_1 a_2^* \quad \text{with standard forms}$$

$$\begin{array}{rcccccccccccc} a_2 & = & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ a_{11} & = & 1 & v & 0 & 0 & 0 & 1 & 1 & v & v & 0 \\ a_1 & = & \text{any} & \text{any} & 1 & v & 0 & 1 & 0 & 1 & 0 & 1 \end{array}$$

(d) If $\tilde{a}_2 = 0$, $a_{12} = 0$, we have

$$a_{11} = x^2 a_{11}^*$$

$$xa_2 = y_1 z_2 a_2^*$$

$$a_1 = xy_1 a_1^*$$

i.e. we may suppose $a_{11} = 0, 1$ or v , $a_2 = 0$ or 1 , $a_1 = 0$ or 1 .

Thus, in all, there are $4p + 30 + (p-1, 3)$ groups of $p^6 \phi_{17}$.

For $p = 3$, the only change comes from considering the products

$\alpha^x \alpha_1^{x_1}$ ($x_1 \neq 0$). So, since $x_1 = 0$, the above results hold for $p = 3$.

Φ_{18} ($p \geq 3$)

The groups of Φ_{18} are characterized by: $G/Z(G)$ is $\Phi_2(1^4)$, $H_2(G)$ is (111), $H_3(G)$ is (1) and $[H_1(G), Z_2(G)] = H_3(G)$ where $H_1(G) = \mathcal{C}(H_2(G))$.

Let $\alpha \in G - H_1(G)$, $\alpha_1 \in H_1(G) - Z_2(G)$, $\beta \in Z_2(G) - \langle Z(G), H_2(G) \rangle$, $\alpha_{i+1} = [\alpha_i, \alpha]$ ($i = 1, 2$), $\gamma_2 = [\beta, \alpha]$ and so (by theorem 2), we have $[\alpha_1, \beta] = \alpha_3$, $[\alpha_2, \alpha_1] = [\alpha_2, \beta] = 1$ and $\alpha_3, \gamma_2 \in Z(G)$.

For stem groups, we have $Z(G) = \langle \alpha_3, \gamma_2 \rangle$, $Z_2(G) = \langle \beta, \alpha_2, \alpha_3, \gamma_2 \rangle$ and $H_j(G)$ are characteristic subgroups ($j = 1, 2, 3$). Writing $\gamma_1 = \alpha_3$, we have

$$\alpha^p = \gamma_1^a, \alpha_1^p \alpha_3^{(p)} = \gamma_1^a, \beta^p = \gamma_2^a, \alpha^{*p} = \gamma_1^{a*} \text{ etc.}$$

where
$$\alpha^* = \alpha^x \alpha_1^{x_1} \beta^{x_2} \dots$$

$$\alpha_1^* = \alpha_1^{y_1} \beta^{y_2} \dots \quad xy_1 z_2 \neq 0$$

$$\beta^* = \beta^z \alpha_2^{z'} \dots$$

and so
$$\alpha_2^* = \alpha_2^{xy_1}$$

$$\alpha_3^* = \alpha_3^{x^2 y_1}$$

$$\gamma_2^* = \alpha_3^{xz' - x_1 z_2} \gamma_2^{xz_2}.$$

From $[\alpha_1^*, \beta^*] = \alpha_3^*$, we get $y_1 z_2 = x^2 y_1$ and so $z_2 = x^2$.

Similarly we get

$$a_{22} = x a_{22}^*$$

$$x a_{21} = x y_1 a_{21}^* + (z' - x x_1) a_{22}^*$$

$$y_1 a_{12} + y_2 a_{22} = x^3 a_{12}^*$$

$$y_1 a_{11} + y_2 a_{21} = x^2 y_1 a_{11}^* + x(z' - x x_1) a_{12}^*$$

$$x a_2 + x_1 a_{12} + x_2 a_{22} = x^3 a_2^*$$

$$x a_1 + x_1 a_{11} + x_2 a_{21} = x^2 y_1 a_1^* + x(z' - x x_1) a_2^*.$$

(a) If $a_{22} \neq 0$, take $x = a_{22}$ and so we may suppose $a_{22} = 1$.

By choice of the other variables we have (as in ϕ_{16})

$$a_{21} = a_{12} = a_2 = 0, \quad a_{11} = 0 \quad \text{any}$$

$$a_1 = 1 \quad 0.$$

(b) If $a_{22} = 0$, $a_{12} \neq 0$, take $y_1 a_{12} = x^3$, and so can assume

$a_{12} = 1$. By choice of the other variables, we may suppose

$$a_{11} = a_2 = 0 \quad \text{and so}$$

$$a_{21} = y_1 a_{21}^* = x^3 a_{21}^*$$

$$x a_1 + x_2 a_{21} = x^5 a_1^* \quad \text{with standard forms}$$

$$a_{21} = 1 \quad g^r \quad 0 \quad 0 \quad 0 \quad r = 1, \dots, (p-1, 3) - 1$$

$$a_1 = 0 \quad 0 \quad 1 \quad g^s \quad 0 \quad s = 1, \dots, (p-1, 4) - 1.$$

(c) If $a_{12} = 0$, $a_{21} \neq 0$, take $y_1 = a_{21}$ and so we may suppose

$$a_{21} = 1.$$

Taking $y_2 = -y_1 a_{11}$, $x_2 = x(z' - x x_1) a_2^* - x a_1 - x_1 a_{11}$, we may

suppose $a_{11} = a_1 = 0$ and so

$$a_2 = x^2 a_2^*$$

i.e. a_2 may be taken to be 0, 1 or v .

(d) If $a_2 = 0$, $a_{12} = 0$, we have $a_{11} = x^2 a_{11}^*$

$$a_2 = x^2 a_2^*$$

$$x a_1 + x_1 a_{11} = x^2 y_1 a_1^* + x(z' - x x_1) a_2^*$$

with standard forms $a_{11} = 1 \quad v \quad 0 \quad 0 \quad 0 \quad 0$

$$a_2 = \text{any any } 1 \quad v \quad 0 \quad 0$$

$$a_1 = 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0.$$

Thus we have $3p + 9 + (p-1,3) + (p-1,4)$ groups of $p^6 \phi_{18}$.

ϕ_{19} ($p \geq 3$)

The groups of ϕ_{19} are characterized by: $G/Z(G)$ is $\phi_2(1^4)$, $H_2(G)$ is (111), $H_3(G)$ is (11) and all the maximal subgroups of G are in ϕ_2 or ϕ_6 .

Let $\alpha_1 \in G - Z_2(G)$, $\alpha_2 \in G - \langle \alpha_1, Z_2(G) \rangle$, $\alpha \in Z_2(G) - \langle Z(G), H_2(G) \rangle$, $\beta = [\alpha_1, \alpha_2]$ and $\beta_1 = [\beta, \alpha_1]$ ($i = 1, 2$). By theorem 2, we have $[\alpha, \alpha_1] = \beta_1$, $[\alpha, \alpha_2] = [\alpha, \beta] = 1$.

For stem groups, $Z(G) = H_3(G) - \langle \beta_1, \beta_2 \rangle$ and so, since $Z_j(G)$, $H_j(G)$ are characteristic ($j = 1, 2, 3$) we have

$\alpha^p = \beta^a$, $\alpha_i^p = \beta_i^a$ ($i = 1, 2$), $\alpha^{*p} = \beta^{*a}$ etc. where

$$\begin{aligned}\alpha_1^* &= \alpha_1^{x_1} \alpha_2^{x_2} \alpha^x \dots \\ \alpha_2^* &= \alpha_1^{y_1} \alpha_2^{y_2} \alpha^y \dots \quad \Delta z \neq 0 \\ \alpha^* &= \alpha^z \beta^{z'} \dots \\ \beta^* &= \beta^\Delta \dots\end{aligned}$$

and so

$$\begin{aligned}\beta_1^* &= \beta^{\Delta x} \\ \beta_2^* &= \beta^{\Delta y}\end{aligned}$$

where $X = (\underline{x}, \underline{y})$, $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $\Delta = |X|$.

From $[\alpha^*, \alpha_2^*] = 1$, we get $(z + z')y_1 = z'y_2 = 0$.

From $[\alpha^*, \alpha_1^*] = \beta_1^*$, we get $(z + z')x_1 = \Delta x_1$, $z'x_2 = \Delta x_2$.

Thus $x_2 = y_1 = z' = 0$, $\Delta = x_1 y_2$; or $x_1 = y_2 = 0$, $z' = \Delta = -x_2 y_1$.

From $\alpha^*p = \beta^*a^*$ etc. we get

$$za_1 = \Delta(x_1a_1^* + y_1a_2^*) \quad \dots \quad (1)$$

$$za_2 = \Delta(x_2a_1^* + y_2a_2^*) \quad \dots \quad (11)$$

$$y_1a_{11} + y_2a_{21} + ya_1 = \Delta(x_1a_{21}^* + y_1a_{22}^*) \quad \dots \quad (111)$$

$$y_1a_{12} + y_2a_{22} + ya_2 = \Delta(x_2a_{21}^* + y_2a_{22}^*) \quad \dots \quad (1v)$$

$$x_1a_{11} + x_2a_{21} + xa_1 = \Delta(x_1a_{11}^* + y_1a_{12}^*) \quad \dots \quad (v)$$

$$x_1a_{12} + x_2a_{22} + xa_2 = \Delta(x_2a_{11}^* + y_2a_{12}^*) \quad \dots \quad (v1)$$

where $x_2 = y_1 = z' = 0$, $z = x_1y_2 = \Delta$

or $x_1 = y_2 = 0$, $z' = -z = -x_2y_1 = \Delta$.

Case 1 $a_1, a_2 \neq 0$.

Taking $x_1 = a_1, y_2 = a_2, x_2 = y_1 = 0$, we may suppose that

$$a_1 = a_2 = 1.$$

Taking $x_2 = 0, x = -x_1a_{12}, y = -y_2a_{21}$, we may suppose that

$$a_{12} = a_{21} = 0.$$

If $x_2 = 0$, (111) becomes $y = 0$

and (1) becomes $z = \Delta x_1$

but $x_2 = 0 \Rightarrow z = \Delta$ and so $x_1 = 1$.

Similarly $x = 0, y_2 = 1$.

Similarly, if $x_1 = 0, x = y = 0$ and $x_2 = y_1 = -1$.

Thus $\text{diag}(a_{11}, a_{22})$ may only be replaced by $\text{diag}(-a_{22}, -a_{11})$,

and so we may suppose that $0 \leq a_{11} < p - a_{22}$.

Thus there are $\frac{1}{2}p(p+1)$ values.

Case 2 Just one of a_1, a_2 is non-zero.

By $\{a_1 \leftrightarrow a_2, \alpha \rightarrow \alpha\beta^{-1}\}$ (if necessary) we may suppose $a_1 \neq 0$, $a_2 = 0$.

From (ii), $x_2 = 0$.

Taking $x_1 = a_1$, we may suppose $a_1 = 1$.

Taking $y = -y_2 a_{21}$, $x = -x_1 a_{11}$, we may suppose $a_{11} = a_{21} = 0$.

The above equations then become

$x_1 = 1, y = x = 0$ and

$$a_{22} = y_2 a_{22}$$

$$a_{12} = y_2^2 a_{12}^*$$

with standard forms $a_{22} = 1 \quad 0 \quad 0 \quad 0$

$$a_{12} = \text{any } 1 \quad v \quad 0$$

giving $p + 3$ groups.

Case 3 $a_1 = a_2 = 0$.

Writing $X = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$, $A = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$, equations (iii) - (vi) become $\Delta A^* = X^{-1}AX$.

$$\text{i.e. } \Delta^2 A^* = \begin{pmatrix} x_1 y_2 a_{11} - x_2 y_1 a_{22} & y_2^2 a_{21} - y_1^2 a_{12} \\ x_1^2 a_{12} - x_2^2 a_{21} & x_1 y_2 a_{22} - x_2 y_1 a_{11} \end{pmatrix}.$$

Taking, in turn, $x_1 = 0$ and $x_2 = 0$ we see that A^* must have the same number of zeroes as A .

(a) If A has 4 zeroes, $A^* = A = \underline{0}$.

(b) Suppose A has 3 zeroes.

$$\text{If } A = \begin{pmatrix} 0 & 0 \\ 0 & a_{22} \end{pmatrix} \text{ then } A^* = \frac{1}{x_1 y_2} \begin{pmatrix} 0 & 0 \\ 0 & a_{22} \end{pmatrix} \text{ or}$$

$$A^* = -\frac{1}{x_2 y_1} \begin{pmatrix} a_{22} & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus we may suppose $a_{11} = 1, a_{12} = a_{21} = a_{22} = 0$.

Similarly, if $a_{11} \neq 0, a_{12} = a_{21} = a_{22} = 0$, we may suppose $a_{11} = 1$.

If $a_{21} \neq 0$, others = 0, then, taking $x_2 = 0$, we have

$$a_{21} = x_1^2 a_{21}^*$$

with standard forms $a_{21} = 1$ or v .

Taking $y_1^2 = -1$ or $-v$ (depending on p), we may suppose

$$a_{12} = 1 \text{ or } v, \quad a_{11} = a_{21} = a_{22} = 0.$$

Similarly, if $a_{12} \neq 0, a_{11} = a_{21} = a_{22} = 0$, we may suppose

$$a_{12} = 1 \text{ or } v.$$

(c) Suppose A has two zeroes.

$$(1) \quad \text{If } A = \begin{pmatrix} 0 & 0 \\ a_{12} & a_{22} \end{pmatrix}, \text{ taking } x_2 = 0, \text{ we have}$$

$$a_{12} = y_2^2 a_{12}^*$$

$$a_{22} = x_1 y_2 a_{22}^*$$

with standard forms $a_{22} = 1, a_{12} = 1$ or v .

As above, taking $x_2 y_1 = -1, x_2^2 = -1$ or $-v$, we may suppose

$$a_{11} = 1, a_{21} = 1 \text{ or } v, a_{12} = a_{22} = 0.$$

Similarly, if $a_{11}, a_{21} \neq 0$, we may suppose $a_{11} = 1, a_{21} = 1$ or v .

(ii) If $A = \begin{pmatrix} 0 & a_{21} \\ 0 & a_{22} \end{pmatrix}$ or $\begin{pmatrix} a_{11} & 0 \\ a_{12} & 0 \end{pmatrix}$, using the same arguments

as in (i), we may suppose

$$a_{11} = 1, a_{12} = 1 \text{ or } v, a_{21} = a_{22} = 0.$$

(iii) If both $a_{11} \neq 0, a_{12} = a_{21} = 0$, then

$$a_{11} = \Delta a_{11}^* \text{ or } \Delta a_{22}^*$$

$$a_{22} = \Delta a_{22}^* \text{ or } \Delta a_{11}^*.$$

Taking $x_1 y_2 = a_{11}$, we may suppose $a_{11} = 1$.

Taking $x_1 = 0$, we have $a_{22} = -x_2 y_1$ and $1 = -x_2 y_1 a_{22}^* = a_{22} a_{22}^*$.

So we may replace a_{22} by a_{22}^{-1}

i.e. we may suppose $a_{22} = 1, g, \dots, g^{\frac{p-1}{2}}$

where g is a primitive root mod p . ($\frac{p+1}{2}$ forms).

(iv) If both $a_{11} = 0, a_{12}, a_{21} \neq 0$, taking $x_1^2 = a_{21}$ or va_{21} and $y_2^2 = a_{12}$ or va_{12} , we may suppose $a_{12} = 1$ or $v, a_{21} = 1$ or v .

Taking $x_1 = 0$, we have $a_{21} = -y_1^2 a_{12}^*$

$$a_{12} = -x_2^2 a_{21}^*.$$

So we may replace $\begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix}$ by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ if and only if $p \equiv 3 \pmod{4}$

and $\begin{pmatrix} 0 & 1 \\ v & 0 \end{pmatrix}$ by $\begin{pmatrix} 0 & v \\ 1 & 0 \end{pmatrix}$ if and only if $p \equiv 1 \pmod{4}$.

Thus there are 3 groups in each case.

(d) Suppose A has just one zero.

If $A = \begin{pmatrix} 0 & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$, taking $x_2 = 0$, we have

$$a_{21} = x_1^2 a_{21}^*$$

$$a_{12} = y_2^2 a_{12}^*$$

$$a_{22} = x_1 y_2 a_{22}^*$$

with standard forms $a_{22} = 1$, $a_{21} = 1$ or v , $a_{12} = \text{any}$.

Taking $y_1 x_2 = -1$, $y_1^2 = -1$ or $-v$, we may suppose $a_{11} = 1$, $a_{12} = 1$ or v , $a_{21} = \text{any}$, $a_{22} = 0$.

Similarly, if $a_{22} = 0$, we get the same standard forms.

If $A = \begin{pmatrix} a_{11} & a_{21} \\ 0 & a_{22} \end{pmatrix}$ taking $x_2 = 0$, we have

$$a_{11} = x_1 y_2 a_{11}^*$$

$$a_{21} = x_1^2 a_{21}^*$$

$$a_{22} = x_1 y_2 a_{22}^*$$

with standard forms $a_{22} = 1$, $a_{21} = 1$ or v , $a_{11} = \text{any}$.

Taking $x_2 y_1 = -1$, $y_1^2 = -1$ or $-v$, we may suppose $a_{11} = 1$, $a_{12} = 1$ or v , $a_{22} = \text{any}$.

Similarly, if $a_{21} = 0$, we get the same standard forms.

Thus there are $4p - 4$ groups for case 3(d).

(e) Suppose A has no zero terms. Taking $x_2 = 0$, $x_1 y_2 = a_{11}$ and $x_1^2 = a_{21}$ or $v a_{21}$ we may suppose $a_{11} = 1$, $a_{21} = 1$ or v .

(i) If $a_{21} = 1$, taking $x_1 = 0$, we have

$$1 = -x_2 y_1 a_{22}^*$$

$$1 = -y_1^2 a_{12}^*$$

$$a_{22} = -x_2 y_1$$

$$a_{12} = -x_2^2 \text{ (and so } -a_{12} \text{ must be a}$$

quadratic residue mod p).

If $-a_{12}$ is a non-quadratic residue, we cannot reduce the possibilities for a_{22} and so we have $\frac{1}{2}(p-1)^2$ standard forms.

If $-a_{12}$ is a quadratic residue, the above equations reduce to $a_{22} a_{22}^* = 1$, $-x_2 y_1 = a_{22}$ and so we may replace a_{22} by $1/a_{22}$.

Thus we may suppose $a_{22} = 1, g, \dots, g^{\frac{p-1}{2}}$ giving $\frac{1}{2}(p-1)(p+1)$ standard forms.

(ii) If $a_{21} = v$, taking $x_1 = 0$, the equations become

$$1 = -x_2 y_1 a_{22}^*$$

$$v = -y_1^2 a_{12}^*$$

$$a_{22} = -x_2 y_1$$

$$a_{12} = -x_2^2 a_{21}^*.$$

If $-a_{12}$ is a quadratic residue, by taking $x_2^2 = -a_{12}$, we may suppose $a_{21} = 1$, giving the above case.

If $-a_{12}$ is a non-quadratic residue, we take $x_2^2 = -va_{12}$ and so a_{21}^* is again v .

Once again we have $a_{22}a_{22}^* = 1$, $-x_2y_1 = a_{22}$ and so we may suppose $a_{22} = 1, g, \dots, g^{(p-1)/2}$, $a_{21} = v$ whenever $-a_{12}$ is a non-quadratic residue.

Thus case 3(e) gives $\frac{1}{2}(p-1)^2 + \frac{1}{2}(p^2-1) = (p-1)p$ groups.

Hence there are $\frac{1}{2}(3p^2 + 10p + 21)$ groups in $p^6\phi_{19}$.

For $p = 3$, the only change comes from considering the products

$\alpha_1^{z_1} \alpha_2^{z_2}$ ($z_1 z_2 \neq 0$) which are excluded.

Thus the same results hold for $p = 3$.

$$\underline{\phi_{20} \ (p > 3)}$$

The groups of ϕ_{20} are characterized by: $G/Z(G)$ is $\phi_2(1^4)$, $H_2(G)$ is (111), $H_3(G)$ is (11) and G has a maximal subgroup M in ϕ_4 .

Let $\alpha_2 \in G - M$, $\alpha_1 \in M - Z_2(G)$, $\alpha \in Z_2(G) - \langle Z(G), H_2(G) \rangle$, $\beta = [\alpha_1, \alpha_2]$, $\beta_1 = [\beta, \alpha_1]$ ($i = 1, 2$). Then, by theorem 2, we have $[\alpha, \alpha_1] = \beta_2$, $[\alpha, \alpha_2] = [\alpha, \beta] = 1$.

For stem groups, $Z(G) = H_3(G) = \langle \beta_1, \beta_2 \rangle$, $H_2(G) = \langle \beta, \beta_1, \beta_2 \rangle$, $Z_2(G) = \langle \alpha, \beta, \beta_1, \beta_2 \rangle$ are characteristic. Also $\mathcal{L}(\langle \alpha_1, \alpha_2, H_2 \rangle) = \mathcal{L}(M) = Z$, $\mathcal{L}(\langle \alpha, \alpha_2, H_2 \rangle) = \langle Z, \alpha \rangle$ and so, by lemma L7, $\langle Z, \alpha \rangle$ and $\langle \alpha, \alpha_2, H_2 \rangle = \mathcal{L}(\langle Z, \alpha \rangle)$ are characteristic.

Thus $\alpha^p = \tilde{\beta}^a$, $\alpha_1^p = \tilde{\beta}^{a_1}$, $\alpha^{*p} = \beta^{*a^*}$ etc. where

$$\alpha_1^* = \alpha_1^{x_1} \alpha_2^{x_2} \alpha^x \dots$$

$$\alpha_2^* = \alpha_2^{y_2} \alpha^y \dots \quad x_1 y_2 z \neq 0$$

$$\overline{\alpha}^* = \overline{\alpha}^z$$

and so

$$\overline{\beta}^* = \overline{\beta}^{x_1 y_2}$$

$$\beta_1^* = \beta_1^{x_1^2 y_2} \beta_2^{x_1 x_2 y_2}$$

$$\beta_2^* = \beta_2^{x_1 y_2^2}.$$

From $[\alpha^*, \alpha_1^*]$, we get $z = y_2^2$.

From $\alpha * p = \beta * a^*$ etc., we get

$$y_2 a_1 = x_1^2 a_1^* \quad \dots \quad (i)$$

$$y_2 a_2 = x_1 (x_2 a_1^* + y_2 a_2^*) \quad \dots \quad (ii)$$

$$y_2 a_{21} + y a_1 = x_1^2 y_2 a_{21}^* \quad \dots \quad (iii)$$

$$y_2 a_{22} + y a_2 = x_1 y_2 (x_2 a_{21}^* + y_2 a_{22}^*) \quad \dots \quad (iv)$$

$$x_1 a_{11} + x_2 a_{21} + x a_1 = x_1^2 y_2 a_{11}^* \quad \dots \quad (v)$$

$$x_1 a_{12} + x_2 a_{22} + x a_2 = x_1 y_2 (x_2 a_{11}^* + y_2 a_{12}^*) \quad \dots \quad (vi)$$

(a) If $a_1 \neq 0$, take $y_2 a_1 = x_1^2$ and so we may suppose $a_1 = 1$.

Taking $y = -y_2 a_{21}$, $x_2 = x_1 a_2$ and $x = -x_1 a_{11} - x_2 a_{21}$, we may suppose $a_{21} = a_2 = a_{11} = 0$.

In (i) - (vi) we have $a_{22} = x_1^3 a_{22}^*$

$$a_{12} = x_1^4 a_{12}^*$$

i.e. $a_{12}/a_{22} = x_1 (a_{12}/a_{22})^*$.

Thus if $a_{22} \neq 0$, taking $x_1 = a_{12}/a_{22}$ we may suppose $a_{12} = a_{22}$ or 0.

So the standard forms are

$$a_{22} = \text{any} \quad g^r \quad 0$$

$$a_{12} = a_{22} \quad 0 \quad g^s$$

where $r + 1 = 1, \dots (p-1, 3)$ and $s + 1 = 1, \dots (p-1, 4)$ and so the number of groups with $a_1 \neq 0$ is $p + (p-1, 3) + (p-1, 4)$.

(b) If $a_1 = 0$, $a_2 \neq 0$, take $x_1 = a_2$ and so we may suppose $a_2 = 1$.

Taking $y = y_2 (x_2 a_{21}^* - a_{22})$ and $x = x_1 x_2 y_2 a_{11} - x_1 a_{12} - x_2 a_{22}$ we may suppose $a_{22} = a_{12} = 0$.

In (i) - (vi) we have $a_{21} = a_{21}^*$

$$a_{11} + x_2 a_{21} = y_2 a_{11}^*$$

with standard forms $a_{21} = \text{any } 0$

$$a_{11} = 0 \quad 1 \quad \text{giving } p + 1 \text{ groups.}$$

(c) If $a_1 = a_2 = 0$, the remaining equations become

$$X^{-1}AX = |X|A^* \text{ where } A = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}, X = \begin{pmatrix} x_1 & 0 \\ x_2 & y_2 \end{pmatrix}$$

$$\text{i.e. } a_{21} = x_1^2 a_{21}^*$$

$$a_{22} = x_1(x_2 a_{21}^* + y_2 a_{22}^*)$$

$$x_1 a_{11} + x_2 a_{21} = x_1^2 y_2 a_{11}^*$$

$$x_1 a_{12} + x_2 a_{22} = x_1 y_2 (x_2 a_{11}^* + y_2 a_{12}^*).$$

If $a_{21} \neq 0$, take $x_1^2 = a_{21}$ or $v a_{21}$ and so we may suppose

$$a_{21} = 1 \text{ or } v.$$

Taking $x_2 a_{21} = -x_1 a_{11}$, we may suppose $a_{11} = 0$ and so $x_2 = 0$,

$$x_1^2 = 1 \text{ and } a_{22} = x_1 y_2 a_{22}^*$$

$$a_{12} = y_2^2 a_{12}^*$$

$$\text{with standard forms } a_{22} = 1 \quad 0 \quad 0 \quad 0$$

$$a_{12} = \text{any } 1 \quad v \quad 0.$$

$$\text{If } a_{21} = 0, \text{ then } a_{22} = x_1 y_2 a_{22}^*$$

$$a_{11} = x_1 y_2 a_{11}^*$$

$$x_1 a_{12} + x_2 a_{22} = x_1 y_2 x_2 a_{11}^* + y_2 a_{12}^*)$$

$$\text{i.e. } x_1 a_{12} + x_2 (a_{22} - a_{11}) = x_1 y_2^2 a_{12}^*$$

with standard forms $a_{11} = \begin{matrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{matrix}$

$a_{22} = \text{any} \begin{matrix} 1 & 1 & 1 & 0 & 0 & 0 \end{matrix}$

$a_{12} = \begin{matrix} 0 & 1 & v & 0 & 1 & v & 0. \end{matrix}$

So the number of groups with $a = 0$ is $3p + 12$.

$\therefore$ Total number of groups in $p^6\phi_{20}$ is $5p + 13 + (p-1,3) + (p-1,4)$.

$$\Delta_{20}$$

By the previous chapter, Δ_{20} has the same defining relations (with $p = 3$) as ϕ_{20} . Thus

$$G = \langle \alpha, \alpha_1, \alpha_2, Z \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1] = \beta_2, \text{rest} = 1 \rangle.$$

As in Δ_6 , we have $(\alpha_1^{z_1} \alpha_2^{z_2})^3 = \alpha_1^{3z_1} \alpha_2^{3z_2} \beta_1^{-z_1^2 z_2} \beta_2^{z_1 z_2^2}$ and we may write $-z = 2z$, $1/z = z$ ($z \neq 0$). Also, since $\alpha \in Z_2(G)$, $(\sigma\alpha)^3 = \sigma^3 \alpha^3$ for all $\sigma \in G$.

As in ϕ_{20} , $H_1(G)$, $Z_1(G)$, $\langle Z, \alpha \rangle$ and $\langle \alpha, \alpha_2, H_2 \rangle$ are characteristic subgroups of a stem group.

$$\text{Thus } \alpha^p = \tilde{\beta}^a, \alpha_1^p = \tilde{\beta}^{a_1}, \alpha^{*p} = \tilde{\beta}^{*a^*}, \alpha_1^{*p} = \tilde{\beta}^{*a_1^*}$$

$$\text{where } \alpha_1^* = \alpha_1^{x_1} \alpha_2^{x_2} \alpha^x \dots$$

$$\alpha_2^* = \alpha_2^{y_2} \alpha^y \dots \quad x_1 y_2 z \neq 0$$

$$\bar{\alpha}^* = \bar{\alpha}^z$$

$$\bar{\beta}^* = \bar{\beta}^{x_1 y_2}$$

$$\beta_1^* = \beta_1^{x_1} \beta_2^{x_1 x_2 y_2}$$

$$\beta_2^* = \beta_2^{x_1}.$$

$$\text{From } [\alpha^*, \alpha_1^*] = \beta_2^*, \text{ we have } z = y_2^2 = 1.$$

From the other relations $y_2 a_1 = a_1^*$

$$a_2 = x_1 x_2 y_2 a_1^* + x_1 a_2^*$$

$$y_2 a_{21} + y a_1 = y_2 a_{21}^*$$

$$y_2 a_{22} + y a_2 = x_1 (x_2 y_2 a_{21}^* + a_{22}^*)$$

$$x_1 a_{11} + x_2 a_{21} + x a_1 - x_2 = y_2 a_{11}^*$$

$$x_1 a_{12} + x_2 a_{22} + x a_2 + x_1 x_2^2 = x_1 (x_2 y_2 a_{11}^* + a_{12}^*)$$

the last two equations following from the fact that

$$\begin{aligned} \alpha_1^*{}^3 &= (\alpha_1^{x_1} \alpha_2^{x_2})^3 \alpha^{3x} \\ &= \alpha_1^{3x_1} \alpha_2^{3x_2} \alpha^{3x} \beta_1^{-x_2} \beta_2^{x_1 x_2^2}. \end{aligned}$$

(a) If $a_1 \neq 0$, take $y_2 = a_1$, and so we may suppose $a_1 = 1$.

Taking $x_2 = x_1 a_2$, $y = -y_2 a_{21}$, $x = x_2 - x_1 a_{11} - x_2 a_{21}$, we may suppose $a_2 = a_{21} = a_{11} = 0$.

Thus

$$a_{22} = x_1 a_{22}^*$$

$$a_{12} = a_{12}^*$$

and so we may suppose $a_{22} = 0$ or 1.

(b) If $a_1 = 0$, $a_2 \neq 0$, take $x_1 = a_2$ and so we may suppose $a_2 = 1$.

Taking $y = x_2 y_2 a_{21}^* - y_2 a_{22}$, $x = x_2 y_2 a_{11}^* - x_1 a_{12} - x_2 a_{22} - x_1 x_2^2$, we may suppose $a_{12} = a_{22} = 0$.

Thus

$$a_{21}' = (a_{21}^*)'$$

$$a_{11} + x_2 a_{21}' = y_2 a_{11}^* \quad \text{where } a_{21}' = a_{21} - 1$$

with standard forms $a_{21}' = 1 \quad 2 \quad 0 \quad 0$

$$a_{11} = 0 \quad 0 \quad 0 \quad 1.$$

(c) If $\tilde{a} = 0$, the equations become

$$a_{21} = a_{21}^*$$

$$y_2 a_{22} = x_1 (x_2 y_2 a_{21}^* + a_{22}^*)$$

$$x_1 a_{11} + x_2 (a_{21} - 1) = y_2 a_{11}^*$$

$$x_1 a_{12} + x_2 a_{22} + x_1 x_2^2 = x_1 (x_2 y_2 a_{11}^* + a_{12}^*)$$

and so $y_2 (a_{11} - a_{22}) - x_1 x_2 y_2 = x_1 (a_{11}^* - a_{22}^*)$.

Taking $x_1 x_2 = a_{11} - a_{22}$, we may suppose $a_{11} = a_{22}$ and so

$$a_{21} = a_{21}^*$$

$$y_2 a_{22} = x_1 a_{22}^*$$

$$a_{12} = a_{12}^*.$$

Standard forms are: $a_{22} = 0$ or 1 , a_{12} , $a_{21} = \text{any}$.

Thus there are 28 groups in $3^6 \Delta_{20}$.

ϕ_{21}

The groups of ϕ_{21} are characterized by: $G/Z(G)$ is $\phi_2(1^4)$, $H_2(G)$ is (111), $H_3(G)$ is (11) and $\ell(Z_2(G)) = Z_2(G)$.

Let $\alpha_1 \in G - Z_2(G)$, $\alpha_2 \in G - \langle \alpha_1, Z_2(G) \rangle$,
 $\alpha \in Z_2(G) - \ell(\langle \alpha_1, \alpha_2 \rangle)$, $\beta = [\alpha_1, \alpha_2]$ and $\beta_1 = [\beta, \alpha_1]$ ($i = 1, 2$).
 By theorem 2, we may suppose $[\alpha, \alpha_1] = \beta_2$, $[\alpha, \alpha_2] = \beta_1^v$ and
 $[\alpha, \beta] = [\alpha, \beta_1] = 1$ ($i = 1, 2$).

For stem groups, $H_2(G) = \langle \beta, \beta_1, \beta_2 \rangle$, $Z(G) = H_3(G) = \langle \beta_1, \beta_2 \rangle$ and $Z_2(G) = \langle \alpha, \beta, \beta_1, \beta_2 \rangle$ are characteristic.

Thus $\alpha^p = \tilde{\beta}^a$, $\alpha_1^p = \tilde{\beta}^{\tilde{a}1}$, $\alpha^{*p} = \tilde{\beta}^{*a*}$, etc. where

$$\alpha_1^* = \alpha_1^{x_1} \alpha_2^{x_2} \alpha^x \dots$$

$$\alpha_2^* = \alpha_1^{y_1} \alpha_2^{y_2} \alpha^y \dots \quad z|X| = z(x_1 y_2 - x_2 y_1) \neq 0$$

$$\alpha^* = \alpha^z \dots$$

and so

$$\beta^* = \beta^{|X|} \dots$$

$$\beta_1^* = \beta_1^{|X|x_1} \beta_2^{|X|x_2}$$

$$\beta_2^* = \beta_1^{|X|y_1} \beta_2^{|X|y_2}.$$

From $[\alpha^*, \alpha_1^*] = \beta_2^*$, we get $zx_1 = |X|y_2$, $vzx_2 = |X|y_1$.

From $[\alpha^*, \alpha_2^*] = \beta_1^{*v}$, we get $zy_1 = v|X|x_2$, $zy_2 = |X|x_1$

and so $y_1 = \epsilon vx_2$, $y_2 = \epsilon x_1$,

$$z = \epsilon|X| = x_1^2 - vx_2^2 \quad \text{where } \epsilon^2 = 1.$$

From $\alpha^{*p} = \beta^{*a^*}$ etc., we get

$$\epsilon a_1 = x_1 a_1^* + y_1 a_2^*$$

$$\epsilon a_2 = x_2 a_1^* + y_2 a_2^*$$

$$y_1 a_{11} + y_2 a_{21} + y a_1 = |X| (x_1 a_{21}^* + y_1 a_{22}^*)$$

$$y_1 a_{12} + y_2 a_{22} + y a_2 = |X| (x_2 a_{21}^* + y_2 a_{22}^*)$$

$$x_1 a_{11} + x_2 a_{21} + x a_1 = |X| (x_1 a_{11}^* + y_1 a_{12}^*)$$

$$x_1 a_{12} + x_2 a_{22} + x a_2 = |X| (x_2 a_{11}^* + y_2 a_{12}^*).$$

Writing $A = (\underline{a}_1, \underline{a}_2)$, $A^* = (\underline{a}_1^*, \underline{a}_2^*)$, $X = (\underline{x}, \underline{y})$ these equations become

$$\underline{a} = \epsilon X \underline{a}^* \quad \dots \quad (1)$$

$$AX + (x \underline{a}, y \underline{a}) = |X| X A^* \quad \dots \quad (2)$$

Case I

If $\underline{a} \neq 0$, by taking $X = \begin{pmatrix} \epsilon a_1 & v a_2 \\ \epsilon a_2 & a_1 \end{pmatrix}$ we may suppose $a_1 = 1, a_2 = 0$

$a_1 = 1, a_2 = 0$ and so $X = \text{diag}(1, \epsilon)$.

Taking $x = -\epsilon a_{11}, y = -\epsilon a_{21}$, we may suppose $a_{11} = a_{21} = 0$ and so

$$a_{22} = \epsilon a_{22}^*$$

$$a_{12} = a_{12}^*.$$

Thus we may suppose $0 \leq a_{22} \leq \frac{1}{2}(p-1)$ giving $\frac{1}{2}p(p+1)$ groups.

Case II If $\underline{a} = 0$, (2) reduces to

$$|X| A^* = X^{-1} A X \quad \dots \quad (3)$$

For convenience, we shall write

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & v \\ 1 & 0 \end{pmatrix}, \Delta = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix}, \lambda_X = x_1^2 - v x_2^2.$$

Then $X = (x_1 E_1 + x_2 E_2) \Delta$, $|X| = \epsilon \lambda_X$ and $\Delta^{-1} = \Delta$, $E_2^{-1} = (1/v) E_2$,
 $\lambda_X X^{-1} = \Delta(x_1 E_1 - x_2 E_2)$.

In order to find the number of groups (i.e. isomorphism classes of groups) we need to find the number of equivalence classes of matrices under the equivalence relation (3). This reduces to finding the number of distinct A^* 's which may be obtained from each A and working systematically through the choices for A until all 2×2 matrices have been found. Also if there are s matrices Y such that $\frac{1}{|Y|} Y^{-1} A Y = \frac{1}{|X|} X^{-1} A X$ for fixed A and X and s is independent of X , then there are $\frac{2(p^2-1)}{s}$ A^* 's obtainable from this A .

$$\text{Suppose } \frac{1}{|X|} X^{-1} A X = \frac{1}{|Y|} Y^{-1} A Y.$$

$$\text{Then } |X|^2 |A| = |Y|^2 |A|$$

$$|X| \operatorname{tr} A = |Y| \operatorname{tr} A$$

$$\lambda_X \operatorname{tr}' A = \lambda_Y \operatorname{tr}' A \quad \text{where } \operatorname{tr}' A = a_{21} + v a_{12}.$$

(a) If $\operatorname{tr} A = \operatorname{tr}' A = |A| = 0$, then $A = \underline{0}$ and so there is only one A^* .

(b) If $\operatorname{tr} A = \operatorname{tr}' A = 0$, $|A| \neq 0$, then

$$A = \begin{pmatrix} a & -vc \\ c & -a \end{pmatrix} \quad a \text{ or } c \neq 0.$$

Consider the matrices $\begin{pmatrix} 0 & -v \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} a & -v \\ 1 & -a \end{pmatrix}$ where a is such that

$v - a^2$ is a quadratic residue.

If these play the rôle of the matrix A above, we have

$$A^* = \frac{1}{\lambda^2} \begin{pmatrix} -2v\epsilon x_1 x_2 & -v(x_1^2 + vx_2^2) \\ x_1^2 + x_2^2 & 2v\epsilon x_1 x_2 \end{pmatrix} \text{ or } \frac{1}{\lambda^2} \begin{pmatrix} \epsilon(ax_1^2 - 2vx_1 x_2 + vax_2^2) & -v(x_1^2 + vx_2^2 - 2ax_1 x_2) \\ x_1^2 + vx_2^2 - 2ax_1 x_2 & -\epsilon(ax_1^2 - 2vx_1 x_2 + vax_2^2) \end{pmatrix}$$

and in both cases $\frac{1}{|X|} X^{-1}AX = \frac{1}{|Y|} Y^{-1}AY$ yields

$$|X| = \delta |Y| \quad \text{where } \delta^2 = 1.$$

(i) In the case of the first matrix, if $\delta = 1$, we have

$$\epsilon_X x_1 x_2 = \epsilon_Y y_1 y_2$$

$$x_1^2 + vx_2^2 = y_1^2 + vy_2^2$$

$$\epsilon_X(x_1^2 - vx_2^2) = \epsilon_Y(y_1^2 - vy_2^2)$$

and so $2vx_2^2 = y_1^2(1 - \epsilon_X\epsilon_Y) + vy_2^2(1 + \epsilon_X\epsilon_Y)$.

If $\epsilon_X = -\epsilon_Y$, then $vx_2^2 = y_1^2$ and so $x_2 = y_1 = 0$.

Similarly, $x_1 = y_2 = 0$ contradicting the definition of X . Thus $\epsilon_X = \epsilon_Y$ and so $x_1 = \pm y_1$, $x_2 = \pm y_2$, the same sign being used each time.

Similarly, if we take $\delta = -1$, we get $x_1 = \pm y_1$, $x_2 = \mp y_2$.

Thus there are four matrices Y such that $\frac{1}{|X|} X^{-1}AX = \frac{1}{|Y|} Y^{-1}AY$

for each Γ . Since there are $2(p^2 - 1)$ matrices of the form X , we get $\frac{1}{2}(p^2 - 1)$ distinct values A^* .

(ii) In the case of the second matrix, if $\delta = 1$,

$$\epsilon_X \{a(x_1^2 + vx_2^2) - 2vx_1x_2\} = \epsilon_Y \{a(y_1^2 + vy_2^2) - 2vy_1y_2\}$$

$$x_1^2 + vx_2^2 - 2ax_1x_2 = y_1^2 + vy_2^2 - 2ay_1y_2$$

$$\text{and so } \epsilon_X(x_1^2 + vx_2^2)(a^2 - v) = (y_1^2 + vy_2^2)(a^2\epsilon_Y - v\epsilon_X) - 2vy_1y_2a(\epsilon_Y - \epsilon_X)$$

$$-2\epsilon_X(v - a^2)x_1x_2 = a(y_1^2 + vy_2^2)(\epsilon_Y - \epsilon_X) - 2y_1y_2(v\epsilon_Y - a^2\epsilon_X).$$

$$\text{Also } \epsilon_X(x_1^2 - vx_2^2) = \epsilon_Y(y_1^2 - vy_2^2).$$

If $\epsilon_X = -\epsilon_Y$ we have

$$(x_1^2 + vx_2^2)(a^2 - v) = -(y_1^2 + vy_2^2)(a^2 + v) + 4vy_1y_2a$$

$$x_1^2 - vx_2^2 = -(y_1^2 - vy_2^2)$$

$$\begin{aligned} \text{and so } 2vx_2^2(a^2 - v) &= (y_1^2 - vy_2^2)(a^2 - v) - (y_1^2 + vy_2^2)(a^2 + v) + 4vy_1y_2a \\ &= -2vy_1^2 - 2vy_2^2a^2 + 4vy_1y_2a. \end{aligned}$$

$$\text{Hence } x_2\sqrt{(v - a^2)} = \pm(y_1 - ay_2)$$

$$x_1\sqrt{(v - a^2)} = \mp(ay_1 - vy_2).$$

If $\epsilon_X = \epsilon_Y$, the equations are the same as in (i) and so

$$x_1 = \pm y_1, \quad x_2 = \pm y_2.$$

For $\delta = -1$, the above reasoning yields (with $\epsilon_X = -\epsilon_Y$)

$$2vx_2^2(a^2 - v) = -(y_1^2 - vy_2^2)(a^2 - v) - (y_1^2 + vy_2^2)(a^2 + v) + 4vy_1y_2a$$

$$\text{and so } vx_2^2(v - a^2) = (y_1a - vy_2)^2.$$

Thus $x_2 = 0$, $y_2 = v^{-1}ay_1$ and so

$$x_1^2 = y_1^2(v^{-1}a^2 - 1).$$

Thus $x_1 = 0$ contradicting the definition of X.

Similarly $\delta = -1$, $\epsilon_X = \epsilon_Y$ leads to a contradiction and so, once again, the second matrix yields $\frac{1}{2}(p^2-1)$ distinct A^* 's.

The two sets are disjoint as a matrix in the first set has a quadratic residue as its determinant while a matrix in the second set has a quadratic non-residue as its determinant. As there are $p^2 - 1$ matrices of the required form, the above two matrices will serve as canonical forms.

(c) $\text{tr } A = 0$, $\text{tr}' A \neq 0$.

For convenience of notation, we shall write the matrix A as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(i) Suppose A is of the form $\begin{pmatrix} 0 & vc \\ c & 0 \end{pmatrix}$ ($c \neq 0$).

Then $\lambda A^* = A$.

Taking $\lambda = c$, we may suppose $c = 1$.

(ii) If A is not of this form, we use an argument similar to the above.

If $\frac{1}{|X|} X^{-1}AX = \frac{1}{|Y|} Y^{-1}AY$, then
 $|X| = \pm |Y|$

i.e. $XY^{-1} = Z \in \mathcal{L}(A)^{1)} \text{ or } \mathcal{L}'(A)$, where $\mathcal{L}'(A)$ denotes the set $\{X | AX = -XA\}$.

1) $\mathcal{L}(A)$ is the centralizer in the ring of 2×2 matrices.

If $Z \in \mathcal{L}(A)$, then (using the original equations)

$$z_2 b = \epsilon v z_2 c$$

$$(1-\epsilon)z_1 b = -2v z_2 a$$

$$(1-\epsilon)z_1 c = 2z_2 a$$

and so $(1-\epsilon)z_1 = 0$ (since $\text{tr}'A \neq 0$).

But $z_1 = 0 \Rightarrow z_2 \neq 0$ and so $a = 0$, $b = \epsilon v c$

i.e. $\text{tr}'A = 0$ or A is of the form above (in (1)) contradicting the definition of A .

Thus $\epsilon = 1$, $z_2 = 0$ and so $\mathcal{L}(A) = \{\lambda E_1 \mid \lambda \in \text{GF}(p)\}$

i.e. $\frac{1}{|X|} X^{-1} A X = \frac{1}{|Y|} Y^{-1} A Y$ and $|X| = |Y| \Rightarrow X = \pm Y$.

If $Z \in \mathcal{L}'(A)$ then (using the original equations again)

$$2z_1 a + z_2(b + \epsilon v c) = 0$$

$$-2z_1 a + z_2(\epsilon b + v c) = 0$$

$$(1+\epsilon)z_1 c = (1+\epsilon)z_1 b = 0$$

and so $(1+\epsilon)z_2 \text{tr}'A = (1+\epsilon)z_1 \text{tr}'A = 0$.

Hence $\epsilon = -1$ and so $\mathcal{L}'(A) = \{x \begin{pmatrix} v c - b & -2v a \\ 2a & b - v c \end{pmatrix}\}$.

Thus $\frac{1}{|X|} X^{-1} A X = \frac{1}{|Y|} Y^{-1} A Y$ and $|X| = -|Y| \Rightarrow |XY^{-1}| = 1$ and

$XY^{-1} \in \mathcal{L}'(A)$

i.e. $x^2 \{(b - v c)^2 - 4v a^2\} = 1$.

If $a = 0$, this is always possible. If $a \neq 0$, this is possible if and only if $y^2 - v = \text{quadratic residue}$ where $y = \frac{b - v c}{2a}$.

Since (by lemma L7) y can take $\frac{1}{2}(p-1)$ values, a can take $(p-1)$ values (for $a \neq 0$) and c can take p values we have $\frac{1}{2}p(p-1)^2$ matrices of this form with $a \neq 0$ and so $(\frac{1}{2}p+1)(p-1)^2$ such matrices altogether. Of these, there are $\frac{1}{2}(p-1)^2$ matrices with $\text{tr}'A = 0$, leaving $\frac{1}{2}(p^2-1)(p-1)$ such matrices with $\text{tr} A = 0$, $\text{tr}'A \neq 0$.

For each of these matrices, we obtain $\frac{1}{2}(p^2-1)$ matrices A^* and so there must be $p-1$ equivalence classes.

Since there are $(p+1)(p-1)^2$ matrices with $\text{tr} A = 0$, $\text{tr}'A \neq 0$, a or $b - \sqrt{c} \neq 0$, the number of such matrices with $(b - \sqrt{c})^2 - 4\sqrt{c}a^2 = \text{quadratic non-residue}$ is $\frac{1}{2}(p^2-1)(p-1)$. Each of these matrices yields p^2-1 matrices A^* and so there are $\frac{1}{2}(p-1)$ equivalence classes.

(d) $\text{tr} A \neq 0$.

In this case, if $\frac{1}{|X|} X^{-1}AX = \frac{1}{|Y|} Y^{-1}AY$, then $|X| = |Y|$ and so $XY^{-1} = Z \in \mathcal{L}(A)$.

Since $x_1 E_1 \in \mathcal{L}(A)$ for all A , $E_2 \in \mathcal{L}(A) \Rightarrow x_1 E_1 + x_2 E_2 \in \mathcal{L}(A)$ and $X, Y \in \mathcal{L}(A) \Rightarrow XY, xX \in \mathcal{L}(A)$ we have

$$|\mathcal{L}(A)| = 2p^2, p^2, 2p \text{ or } p.$$

(i) If $|\mathcal{L}(A)| = 2p^2$, then $E_2 A = A E_2$ and $A \Delta = \Delta A$

$$\text{i.e. } \begin{pmatrix} a & -b \\ c & -d \end{pmatrix} = \begin{pmatrix} a & b \\ -c & -d \end{pmatrix} \text{ and } \begin{pmatrix} b & \sqrt{c}a \\ d & \sqrt{c} \end{pmatrix} = \begin{pmatrix} \sqrt{c} & \sqrt{c}d \\ a & b \end{pmatrix}.$$

Hence $A = aE_1$ and so E_1 is the standard form.

(ii) If $|\mathcal{L}(A)| = p^2$, then $\mathcal{L}(A) = \{x_1E_1 + x_2E_2 | x_1 \in GF(p)\}$ and so $AE_2 = E_2A$.

As in (i), we must have $A = \begin{pmatrix} a & vc \\ c & a \end{pmatrix}$ ($a, c \neq 0$).

$$\begin{aligned} \text{In this case } \epsilon \lambda A^* &= \Delta^{-1}(x_1E_1 + x_2E_2)^{-1}A(x_1E_1 + x_2E_2)\Delta \\ &= \Delta^{-1}A\Delta \\ &= \begin{pmatrix} a & \epsilon vc \\ \epsilon c & a \end{pmatrix}. \end{aligned}$$

Since there are 2 choices for ϵ and $p-1$ choices for λ , each matrix A yields $2(p-1)$ matrices A^* , and so (as there are $(p-1)^2$ matrices of this form) there are $\frac{1}{2}(p-1)$ equivalence classes.

(iii) If $|\mathcal{L}(A)| = 2p$, then $E_2 \notin \mathcal{L}(A)$ but there is an $(x_1E_1 + x_2E_2)\Delta$ ($x_1 \neq 0$) such that

$$\begin{aligned} A(x_1E_1 + x_2E_2)\Delta &= (x_1E_1 + x_2E_2)\Delta A \\ \text{i.e. } \begin{pmatrix} ax_1 + bx_2 & -(vax_2 + bx_1) \\ cx_1 + dx_2 & -(vcx_2 + dx_1) \end{pmatrix} &= \begin{pmatrix} ax_1 - vcx_2 & bx_1 - vdx_2 \\ ax_2 - cx_1 & bx_2 - dx_1 \end{pmatrix}. \end{aligned}$$

$$\begin{aligned} \text{Thus } x_2 &= 0 & b &= -vc \\ & & \text{or} & \\ b = c &= 0 & cx_1 + dx_2 &= ax_2 - cx_1. \end{aligned}$$

In either case, $\text{tr}'A = 0$ and $2cx_1 = (a-d)x_2$.

$$\text{Thus } \frac{1}{|X|} X^{-1}AX = \frac{1}{|Y|} Y^{-1}AY \text{ and } X \neq \pm Y \Rightarrow XY^{-1} = x \begin{pmatrix} a-d & -2vc \\ 2c & d-a \end{pmatrix} \text{ and}$$

$$|XY^{-1}| = 1$$

$$\text{i.e. } x^2(4vc^2 - (a-d)^2) = 1$$

where $a \neq -d$, and either $c \neq 0$ or $a \neq d$.

However, if $a = d$, we have $4vc^2x^2 = 1$ which is impossible and so $a \neq d$. Thus this is possible if and only if

$$y^2 - v = \text{quadratic non-residue where } y = \frac{2vc}{a-d}.$$

If $d = 0$ (by lemma L7) there are $p-1$ choices for a and so $p-1$ choices of A for each value of y .

If $d \neq 0$, by lemma L7, there are $p-1$ choices for d and (as $a \neq d$) $p-2$ choices for a . Thus, since y can have $\frac{1}{2}(p+1)$ values, the total number of matrices of the required form is $\frac{1}{2}(p^2-1)(p-1)$.

As before, each A yields $\frac{1}{2}(p^2-1)$ matrices A^* and so there are $p-1$ equivalence classes.

The remaining matrices with $|\mathcal{L}(A)| = 2p$ will be dealt with below.

$$(iv) \quad \frac{1}{|X|} X^{-1}AX = \frac{1}{|Y|} Y^{-1}AY \Rightarrow X = \pm Y.$$

The only remaining matrices are those with $|\mathcal{L}(A)| = 2p$ and $4vc - (a-d)^2 = \text{quadratic non-residue}$; and $|\mathcal{L}(A)| = p$.

In both these cases, each A yields p^2-1 matrices A^* , and since there are p^4 possibilities for A , the number in this last section is

$$\begin{aligned} p^4 - \{1 + (p^2-1) + (p-1) + (p^2-1)(p-1) + (p-1) + (p-1)^2 + \\ \frac{1}{2}(p^2-1)(p-1)\} \\ = \frac{1}{2}(p^2-1)(2p^2-3p+1). \end{aligned}$$

Thus there are $\frac{1}{2}(2p^2-3p+1)$ equivalence classes in this last section.

In all, there are $\frac{1}{2}(3p^2 + 4p + 5)$ groups in $p^6\phi_{21}$ for $p > 3$. Canonical forms have not been found although equivalence classes for A have been found on a computer for $p \leq 23$ the above figure being found to be correct for those values. Due to economy of space the equivalence classes for $p = 3$ only have been given (See Δ_{21} .)

$$\Delta_{21}$$

The groups of Δ_{21} have the same defining relations (with $p = 3$) as Φ_{21} .

Thus $G = \langle \alpha_1, \alpha_2, \alpha, [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1 \in Z(G), [\alpha, \alpha_1] = \beta_2, [\alpha, \alpha_2] = \beta_1^2, [\alpha, \beta] = 1 \rangle$.

By lemma L2, we have

$$\begin{aligned} (\alpha_1^{w_1} \alpha_2^{w_2} \alpha^w)^3 &= (\alpha_1^{w_1} \alpha_2^{w_2})^3 \alpha^{3w} \\ &= \alpha_1^{3w_1} \alpha_2^{3w_2} \alpha^{3w} \beta_1^{-w_1^2 w_2} \beta_2^{w_1 w_2^2}. \end{aligned}$$

For stem groups, we have

$$\alpha^3 = \beta^a, \alpha_1^3 = \beta^a_1, \alpha^{*3} = \beta^{*a}, \alpha_1^{*3} = \beta^{*a}_1 \text{ where}$$

$$\alpha_1^* = \alpha_1^{x_1} \alpha_2^{x_2} \alpha^x \dots$$

$$\alpha_2^* = \alpha_1^{y_1} \alpha_2^{y_2} \alpha^y \dots \quad z|X| = z(x_1 y_2 - x_2 y_1) \neq 0$$

$$\alpha^* = \alpha^z \dots$$

$$\text{and so} \quad \beta^* = \beta^{|X|}$$

$$\beta_1^* = \beta^{|X| x}$$

$$\beta_2^* = \beta^{|X| y}.$$

As in Φ_{21} ($p > 3$) we have

$$y_1 = 2\epsilon x_2, y_2 = \epsilon x_1, z = \epsilon |X| = x_1^2 + x_2^2 \quad (\epsilon \neq 0).$$

From $\alpha^{*3} = \beta^{*a^{*}}$ etc. we get (with $a_{12}' = a_{12}-1$, $a_{21}' = a_{21}+1$)

$$\epsilon a_1 = x_1 a_1^{*} - \epsilon x_2 a_2^{*}$$

$$\epsilon a_2 = x_2 a_1^{*} + \epsilon x_1 a_2^{*}$$

$$x_1 a_{11} + x_2 a_{21}' - x_2(1+x_1^2 - \epsilon|X|) + x a_1 = |X|(x_1 a_{11}^{*} - \epsilon x_2 a_{12}'^{*})$$

$$x_1 a_{12}' + x_2 a_{22} + x_1(1+x_2^2 - \epsilon|X|) + x a_2 = |X|(x_2 a_{11}^{*} + \epsilon x_1 a_{12}'^{*})$$

$$\epsilon\{-x_2 a_{11} + x_1 a_{21}' - x_1(1+x_2^2 - \epsilon|X|)\} + y a_1 = |X|(x_1 a_{21}'^{*} - \epsilon x_2 a_{22}^{*})$$

$$\epsilon\{-x_2 a_{12}' + x_1 a_{22} - x_2(1+x_1^2 - \epsilon|X|)\} + y a_2 = |X|(x_2 a_{21}'^{*} + \epsilon x_1 a_{22}^{*}).$$

$$\text{Now } x_2(1 + x_1^2 - \epsilon|X|) = x_2(1 - x_2^2) = 0$$

$$\text{and similarly } x_1(1 + x_2^2 - \epsilon|X|) = 0.$$

Thus the above equations reduce to the same equations as for $p > 3$ (with a_{12} replaced by a_{12}' and a_{21} replaced by a_{21}') and so the number of groups in Δ_{21} is obtained by substituting $p = 3$ in the formula for ϕ_{21} ($p > 3$).

The isomorphism classes when $a \neq 0$ may be read off from the computer results for A as follows:

1) $\underline{0}$

2) $\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & r \end{pmatrix}, \begin{pmatrix} r & s \\ s & r \end{pmatrix}$

3) $\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$

4) $\begin{pmatrix} r & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & r \end{pmatrix}, \begin{pmatrix} r & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & r \end{pmatrix}$

$$5) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$$

$$6) \begin{pmatrix} r & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 2 & r \end{pmatrix}, \begin{pmatrix} r & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & r \end{pmatrix}$$

$$7) r \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, r \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$8) \begin{pmatrix} r & 2 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & r \end{pmatrix}$$

$$9) r \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$$

$$10) \begin{pmatrix} r & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & r \end{pmatrix}, \begin{pmatrix} r & 0 \\ 2 & r \end{pmatrix}, \begin{pmatrix} r & 1 \\ 0 & r \end{pmatrix}$$

$$11) \begin{pmatrix} r & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & r \end{pmatrix}, \begin{pmatrix} r & 0 \\ 1 & r \end{pmatrix}, \begin{pmatrix} r & 2 \\ 0 & r \end{pmatrix}$$

$$12) \begin{pmatrix} r & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 2 & r \end{pmatrix}$$

$$13) rI$$

$$14) \begin{pmatrix} 2 & r \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & r \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ r & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ r & 1 \end{pmatrix}$$

$$15) \begin{pmatrix} 1 & r \\ r & 2 \end{pmatrix}, \begin{pmatrix} 2 & r \\ r & 1 \end{pmatrix}$$

$$16) \begin{pmatrix} r & 1 \\ 2 & r \end{pmatrix}, \begin{pmatrix} r & 2 \\ 1 & r \end{pmatrix}$$

where $r, s = 1, 2$.

$$\phi_{22}$$

By theorem 7, all stem groups of ϕ_{22} are central products of groups in $p^3\phi_2$ and groups in $p^4\phi_3$. Thus

$$G = \langle \alpha, \alpha_1, \beta_1, \beta_2 \rangle \text{ where } [\alpha_1, \alpha] = \alpha_{1+1} \ (i = 1, 2), [\beta_1, \beta_2] = \alpha_3 \in Z(G)$$

$$\text{and } \alpha^p = 1 \quad \alpha_3 \quad 1 \quad \alpha_3 \quad 1 \quad 1 \quad 1 \quad 1$$

$$\alpha_1^p \alpha_3^{(p)} = 1 \quad 1 \quad 1 \quad 1 \quad \alpha_3 \alpha_3^v \quad \alpha_3 \alpha_3^v$$

$$\beta_1^p = 1 \quad 1 \quad \alpha_3 \quad \alpha_3 \quad 1 \quad 1 \quad \alpha_3 \alpha_3$$

$$\beta_2^p = 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1.$$

By replacing α by $\alpha\beta_1$ and β_2 by $\beta_2\alpha_2$ in the third group, we see that it is isomorphic to the fourth group. Also by considering the maximal subgroups contained in ϕ_3 (see below) we may see that there are seven non-isomorphic groups in $p^6\phi_{22}$.

α^p	$\alpha_1^p \alpha_3^{(p)}$	β_1^p	$\langle \alpha, \alpha_1, \beta_2 \rangle$	$\langle \alpha\beta_1^r, \alpha_1, \beta_2 \rangle$ ($r \neq 0$)	$\langle \alpha\beta_2^r, \alpha_1, \beta_1\beta_2^s \rangle$
1	1	1	$\phi_3(1^5)$	$\phi_3(1^5)$	$\phi_3(1^5)$
α_3	1	1	$\phi_3(2111)a$	$\phi_3(2111)a$	$\phi_3(2111)a$
1	1	α_3	$\phi_3(1^5)$	$\phi_3(2111)a$	$\phi_3(2111)c$
1	α_3	1	$\phi_3(2111)b_1$	$\phi_3(2111)b_1$	$\phi_3(2111)b_1$
1	α_3^v	1	$\phi_3(2111)b_v$	$\phi_3(2111)b_v$	$\phi_3(2111)b_v$
1	α_3	α_3	$\phi_3(2111)b_1$	$\phi_3(2111)c$	$\phi_3(2111)c$
1	α_3^v	α_3	$\phi_3(2111)b_v$	$\phi_3(2111)c$	$\phi_3(2111)c$

$$\underline{\phi_{23} \ (p \geq 3)}$$

The groups of ϕ_{23} are characterized by: $G/Z(G)$ is $\phi_3(1^4)$, $H_2(G)$ is (1^4) , $H_3(G)$ is (111) and $H_4(G)$ is (1) . By theorem 1, we note that $|G:\mathcal{L}(H_3)| = p$.

Let $\alpha \in G - \mathcal{L}(H_3)$, $\alpha_1 \in \mathcal{L}(H_3) - \mathcal{L}(H_2)$, $\alpha_{i+1} = [\alpha_i, \alpha]$ ($i = 1, 2, 3$), $[\alpha_1, \alpha_2] = \gamma_2$ and so $\alpha_4, \gamma_2 \in Z(G)$, $[\alpha_1, \alpha_3] = [\alpha_2, \alpha_3] = 1$.

For stem groups we have $Z(G) = \langle \alpha_4, \gamma_2 \rangle$. Writing $\alpha_4 = \gamma_1$, we have

$$\gamma_2^p = \alpha_4^p = \alpha_3^p = \alpha_2^p \alpha_4^{\binom{p}{3}} = 1, \alpha^p = \gamma^a, \alpha_1^p \alpha_3^{\binom{p}{3}} = \gamma^b, \alpha^{*p} = \gamma^{*a} \text{ etc. where}$$

$$\alpha^* = \alpha^x \alpha_1^{x_1} \alpha_2^{x_2} \dots$$

$$\alpha_1^* = \alpha_1^{y_1} \alpha_2^{y_2} \dots$$

$$\text{and so } \alpha_2^* = \alpha_2^{xy_1} \dots$$

$$\alpha_3^* = \alpha_3^{x^2 y_1}$$

$$\alpha_4^* = \alpha_4^{x^3 y_1}$$

$$\gamma_2^* = \gamma_2^{xy_1^2}$$

As is β_n , we have

$$(\alpha^z \alpha_1^{z_1})^p = \alpha^{pz} (\alpha_1^p \alpha_3^{\binom{p}{3}})^{z_1} \gamma^{zz_1^2 \binom{p}{3}} \quad (z \neq 0).$$

From $\alpha^*{}^p = \gamma^*{}^a$, we get

$$x^3 y_1 a_1^* = x a_1 + x_1 b_1 \quad \dots \quad (i)$$

$$x y_1^2 a_2^* = x a_2 + x_1 b_2 + x x_1^{2(p)} \quad \dots \quad (ii)$$

From $(\alpha^* \alpha_1^*)^p = \gamma^*{}^{a+b} \gamma_2^*{}^{(p)}$, we get

$$x^3 y_1 (a_1^* + b_1^*) = x a_1 + (x_1 + y_1) b_1$$

$$x y_1^2 (a_2^* + b_2^* + \binom{p}{3}) = x a_2 + (x_1 + y_1) b_2 + x (x_1 + y_1)^{2(p)}$$

and so $x^3 b_1^* = b_1 \quad \dots \quad (iii)$

$$x y_1 b_2^* = b_2 + 2 \binom{p}{3} x x_1 \quad \dots \quad (iv)$$

If $p \neq 3$, the standard forms are

$$\begin{array}{llllllllll} b_1 = g^r & 0 & 0 & g^r & g^r & g^r & 0 & 0 & 0 & 0 & 0 \\ b_2 = 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_1 = 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ a_2 = 0 & 0 & 0 & 1 & v & 0 & g^s & 0 & 1 & v & 0 \end{array}$$

where $r + 1 = 1, \dots, (p-1, 3)$ and $s + 1 = 1, \dots, (p-1, 4)$

and so there are $(p-1, 4) + 4(p-1, 3) + 6$ groups.

If $p = 3$, we may suppose $b_2 = 0$ and so we take $x_1 = 0$.

Equations (i) - (iii) become $y_1 a_1^* = a_1$

$$a_2^* = a_2$$

$$x b_1^* = b_1$$

and so we may suppose $a_1 = 0$ or 1 , $b_1 = 0$ or 1 , $a_2 = 0, 1$ or 2 , giving 12 groups.

Thus, for all p , there are $(p-1, 4) + 4(p-1, 3) + 6$ groups in $p^6 \Phi_{23}$.

ϕ_{24} and ϕ_{27} ($p > 3$)

These families are characterized by: $G/Z(G)$ is $\phi_3(1^5)$, $H_2(G)$ is (111), $H_3(G)$ is (11), $H_4(G)$ is (1). By theorem 2, there is a characteristic subgroup $H_1(G)$ of maximal order with $[H_1(G), H_2(G)] \leq H_4(G)$.

Let $\alpha \in G - H_1$, $\alpha_1 \in H_1 - Z_2(G)$, $\beta \in Z_2 - \mathcal{C}(H_1) \cap Z_2$, $\alpha_2 = [\alpha_1, \alpha]$, $\alpha_3 = [\alpha_2, \alpha]$, $\alpha_4 = [\alpha_3, \alpha] \in Z(G)$.

By theorem 2, we may suppose $[\alpha, \beta] = [\alpha_1, \alpha_3] = [\alpha_2, \alpha_3] = 1$, $[\alpha_1, \beta] = \alpha_4$, $[\alpha_1, \alpha_2] = \alpha_4^r$ and $\alpha^p, \alpha_1^p, \beta^p \in Z(G)$ where $r = 0$ for ϕ_{24} and $r = 1$ for ϕ_{27} .

For stem groups, we have $Z(G) = \langle \alpha_4 \rangle$, $Z_2(G) = \langle \beta, H_3 \rangle$, and $H_1(G)$ are characteristic subgroups.

Thus $\alpha_i^p = 1$ ($i \geq 2$), $\alpha^p = \alpha_4^a$, $\alpha_1^p = \alpha_4^{a_1}$, $\beta^p = \alpha_4^b$, $\alpha^{*p} = \alpha_4^{*a^*}$, etc. where

$$\alpha^* = \alpha^x \alpha_1^{x_1} \beta^{x'} \alpha_2^{x_2} \dots$$

$$\alpha_1^* = \alpha_1^{y_1} \beta^{y'} \alpha_2^{y_2} \dots$$

$$\beta^* = \beta^{z'} \dots$$

$$\alpha_2^* = \alpha_2^{xy_1} \dots$$

$$\alpha_3^* = \alpha_3^{x^2 y_1}$$

$$\alpha_4^* = \alpha_4^{x^3 y_1}.$$

From $[\alpha_1^*, \beta^*] = \alpha_4^*$, we get $z' = x^3$.

From $[\alpha_1^*, \alpha_2^*] = \alpha_4^{*r}$, we get $ry_1 = rx^2$.

From $\alpha^{*p} = \alpha_4^{*a^*}$ etc., we get $b = y_1 b^*$

$$y_1 a_1 + y' b = x^3 y_1 a_1^*$$

$$x a + x_1 a_1 + x' b = x^3 y_1 a^*.$$

In the case of ϕ_{24} , $r = 0$ and the standard forms are:

$$b = 1 \quad 0 \quad 0 \quad 0$$

$$a_1 = 0 \quad g^r \quad 0 \quad 0 \quad r + 1 = 1, \dots (p-1, 3)$$

$$a = 0 \quad 0 \quad 1 \quad 0.$$

In the case of ϕ_{27} , $r = 1$, $y = x^2$ and so the equations

become

$$b = x^2 b^*$$

$$x^2 a_1 + y' b = x^5 a_1^*$$

$$x a + x_1 a_1 + x' b = x^5 a^*$$

with standard forms

$$b = 1 \quad v \quad 0 \quad 0 \quad 0$$

$$a_1 = 0 \quad 0 \quad g^r \quad 0 \quad 0 \quad r + 1 = 1, \dots (p-1, 3)$$

$$a = 0 \quad 0 \quad 0 \quad g^s \quad 0 \quad s + 1 = 1, \dots (p-1, 4).$$

Thus there are $3 + (p-1, 3)$ groups in $p^6 \phi_{24}$ ($p > 3$) and $3 + (p-1, 3) + (p-1, 4)$ groups in $p^6 \phi_{27}$ ($p > 3$).

Δ_{24} and Δ_{27}

These families have the same defining relations (with $p = 3$) as ϕ_{24} and ϕ_{27} except that " $\alpha_1^p \in Z(G)$ " is to be replaced by " $\alpha_1^3 \alpha_3 \in Z(G)$ ".

Also $\alpha_2^3 \alpha_4 = [\alpha_1, \alpha]^3 [\alpha_3, \alpha] = [\alpha_1^3 \alpha_3, \alpha] = 1$.

As in ϕ_{24}, ϕ_{27} , we write $\alpha^3 = \alpha_4^a$, $\alpha_1^3 \alpha_3 = \alpha_4^{a_1}$, $\beta^3 = \alpha_4^b$.

Since $\langle \alpha, \alpha_1 \rangle \in \Delta_9$, we have

$$\begin{aligned} (\alpha_1 \alpha_2^{a_1})^3 \alpha_3 &= \alpha_1^3 \alpha_3 \alpha_2^{3a_1} \\ &= \alpha_4^{a_1} \alpha_4^{-a_1} = 1 \end{aligned}$$

and so, by $\{\alpha_1 \rightarrow \alpha_1 \alpha_2^{a_1}\}$ we may suppose $a_1 = 0$.

$$\begin{aligned} \text{Similarly } (\alpha \alpha_1^a)^3 &= \alpha^3 (\alpha_1^3 \alpha_2^3 \alpha_3)^a \\ &= \alpha^3 \alpha_2^{3a} = 1 \end{aligned}$$

and so, by $\{\alpha \rightarrow \alpha \alpha_1^a\}$ we may suppose $a = 0$.

Also, since $\langle \beta, H_3 \rangle$ is a characteristic subgroup, the only automorphisms which will affect the value of b are of the form

$$\begin{aligned} \beta^* &= \beta^x \alpha_3^x \alpha_4^x \\ \alpha^* &= \alpha^x \dots \\ \alpha_1^* &= \alpha_1^y \dots \\ \text{and so } \alpha_4^* &= \alpha_4^{xy}. \end{aligned}$$

Thus, if $\beta^*{}^3 = \alpha_4^*{}^{b^*}$, we have

$$b = yb^*.$$

In the case of Δ_{24} , we may suppose $b = 0$ or 1 .

In the case of Δ_{27} , the relation $[\alpha_1^*, \alpha_2^*] = \alpha_4^*$ means that $y = 1$ and so b can take any value.

Thus there are two groups in $3^6\Delta_{24}$ and three groups in $3^6\Delta_{27}$.

ϕ_{25} and ϕ_{26} ($p \geq 3$)

These families are characterized by: $G/Z(G)$ is $\phi_3(221)b_r$, $H_2(G)$ is (21), $H_3(G)$ is (11), $H_4(G)$ is (1) and $\mathcal{L}(H_2(G)) \neq H_2(G)$ where $r = 1$ for ϕ_{25} and $r = v$ for ϕ_{26} .

Let $\alpha \in G - \mathcal{L}(H_2)$, $\alpha_1 \in \mathcal{L}(H_2) - Z \cap H_2$,

$$\alpha_{i+1} = [\alpha_i, \alpha] \quad (i = 1, 2, 3).$$

Then, by theorem 9, $[\alpha_1, \alpha_2] = 1$, $\alpha_1^p \alpha_3^{\binom{p}{3}} = \alpha_3^r$, $\alpha_2^p \alpha_4^{\binom{p}{3}} = \alpha_4^r$,

$$\alpha_3^p = \alpha_4^p = 1, \alpha^{p^2} \in Z(G) \text{ and } H_2 = \langle \alpha_2, \alpha_3 \rangle = \langle \alpha_2, \alpha_1^p \rangle,$$

$$H_3 = \langle \alpha_1^p, \alpha_2^p \rangle, H_4 = \langle \alpha_2^p \rangle.$$

For stem groups, $Z(G) = H_4$ and so $\alpha^{p^2} = \alpha_4^a$, $\alpha^{*p^2} = \alpha_4^{*a}$

where

$$\alpha^* = \alpha \alpha_1^x \alpha_2^{x_1} \alpha_3^{x_2} \alpha_4^{x_3}$$

$$\alpha_1^* = \alpha_1^y$$

and so

$$\alpha_2^* = \alpha_2^{xy_1}$$

$$\alpha_3^* = \alpha_3^{x^2 y_1}$$

$$\alpha_4^* = \alpha_4^{x^3 y_1}.$$

From $\alpha_2^{*p} \alpha_4^{*\binom{p}{3}} = \alpha_4^{*r}$, we get $x^3 y_1 = xy_1$ and so $x^2 = 1$.

From $\alpha^{*p^2} = \alpha_4^{*a}$, we get

$$xa = xy_1 a^*.$$

Thus we may suppose $\alpha^{p^2} = 1$ or α_2^p giving two stem groups in each family.

ϕ_{28} and ϕ_{29} ($p \geq 3$)

These families have the same properties as ϕ_{25} and ϕ_{26} except that (with the notation of these families) $[\alpha_1, \alpha_2] = \alpha_4$.

Using the same automorphisms as for ϕ_{25} and ϕ_{26} , we get, from $[\alpha_1^*, \alpha_2^*] = \alpha_4^*$, that $y_1 = 1$.

Thus the automorphism does not affect the value of a which may take any value between 0 and p .

Thus there are p stem groups in each of these families.

$$\phi_{30} \ (p > 3)$$

The groups of ϕ_{30} are characterized by: $G/Z(G)$ is $\phi_7(1^5)$, $H_2(G)$ is (111), $H_3(G)$ is (11) and $H_4(G)$ is (1). As is ϕ_{24} , these groups have a maximal subgroup $H_1(G)$ with $[H_1(G), H_2(G)] = 1$ and $H_1(G)$ is characteristic.

Let $\alpha \in G - H_1(G)$, $\alpha_1 \in H_1(G) - Z_2(G)$, $\beta \in Z_2(G) - \langle H_2, Z_1 \rangle$, $\alpha_2 = [\alpha_1, \alpha]$, $\alpha_3 = [\alpha_2, \alpha]$, $\alpha_4 = [\alpha_3, \alpha] \in Z_1(G)$.

By theorem 13, we may suppose $[\alpha, \beta] = [\alpha_1, \alpha_2] = [\alpha_1, \alpha_2] = [\alpha_2, \alpha_3] = 1$, $[\alpha_1, \beta] = \alpha_3$, $[\alpha_2, \beta] = \alpha_4$.

For stem groups we have $Z(G) = \langle \alpha_4 \rangle$, $Z_2(G) = \langle \alpha_3, \alpha_4 \rangle$, $Z_3(G) = \langle \beta, H_2 \rangle$ and $\mathcal{L}(H_2) = \langle \alpha_1, H_2 \rangle$ are characteristic.

Thus $\alpha_4^p = \alpha_2^p = \alpha_3^p = 1$, $\alpha^p = \gamma^a$, $\alpha_1^p = \gamma^{a_1}$, $\beta^p = \gamma^b$, $\alpha^{*p} = \gamma^{*a^*}$, $\alpha_1^{*p} = \gamma^{*a_1^*}$, $\beta^{*p} = \gamma^{*b^*}$ where $\gamma = \alpha_4$ and

$$\alpha^* = \alpha^x \alpha_1^{x_1} \beta^{x'} \alpha_2^{x_2} \dots$$

$$\alpha_1^* = \alpha_1^{y_1} \alpha_2^{y_2} \dots \quad xy_1 z' \neq 0$$

$$\beta^* = \beta^{z'} \alpha_2^{z_2} \dots$$

$$\alpha_2^* = \alpha_2^{xy_1} \dots$$

$$\alpha_3^* = \alpha_3^{x^2 y_1} \dots$$

$$\gamma^* = \gamma^{x^3 y_1}.$$

From $[\alpha_1^*, \beta^*] = \alpha_3^*$, we get $z' = x^2$.

From the rest we get $b = xy_1 b^*$

$$a_1 = x^3 a_1^*$$

$$xa + x_1 a_1 + x' b = x^3 y_1 a^*$$

with standard forms:

$$b = 1 \quad 1 \quad 0 \quad 0 \quad 0$$

$$a_1 = g^r \quad 0 \quad g^r \quad 0 \quad 0 \quad r = 0, \dots, (p-1, 3) - 1$$

$$a = 0 \quad 0 \quad 0 \quad 1 \quad 0.$$

Thus, in all, there are $2(p-1, 3) + 3$ groups of p^6_{30} .

$$\Delta_{30}$$

This family is defined by: $G/Z(G)$ is $\Delta_7(1^5)$, $H_2(G)$ is (111), $H_3(G)$ is (11) and $H_4(G)$ is (1). Using the notation of Φ_{30} , we have for stem groups

$$G = \langle \alpha, \alpha_1, \beta \mid [\alpha_1, \alpha] = \alpha_{1+1}, [\alpha_1, \beta] = \alpha_{1+2} \quad (i = 1, 2, 3) \rangle \text{ where } \alpha_3^3 = \alpha_4^3 = 1, \alpha_3^3, \alpha_1^3 \alpha_3^3, \beta^3 \in Z(G).$$

$$\text{Thus } 1 = [\alpha_1^3 \alpha_3, \alpha] = \alpha_2^3 \alpha_4, \alpha^3 = \alpha_4^a, \alpha_1^3 \alpha_{1+2}^3 = \alpha_4^a \alpha_1^b, \beta^3 = \alpha_4^b.$$

$$\text{Now if we put } \sigma = \alpha \alpha_1^{z_1},$$

$$\begin{aligned} \text{then } [\beta, \sigma] &= [\beta, \alpha \alpha_1^{z_1}] \\ &= [\beta, \alpha_1^{z_1}] \\ &= \alpha_3^{-z_1} \end{aligned}$$

$$\text{and } [\beta, \sigma, \sigma] = \alpha_4^{-z_1}.$$

So $\langle \sigma, \beta \rangle \in \Delta_3$ if $z_1 \neq 0$ and abelian if $z_1 = 0$.

$$\begin{aligned} \text{Thus } (\sigma^x \beta^{x'})^3 &= \sigma^{3x} (\beta^3 [\beta, \sigma, \sigma]^{z_1^2})^{x'} \\ &= \sigma^{3x} \beta^{3x'} \alpha_4^{-z_1 x'}. \end{aligned}$$

Also, as $\langle \alpha, \alpha_1 \rangle \in \Delta_9$,

$$\begin{aligned} \sigma^3 &= (\alpha \alpha_1^{z_1})^3 \\ &= \alpha^3 (\alpha_1^3 \alpha_3)^{z_1}. \end{aligned}$$

Putting $x_1 = xz_1$, we have

$$\begin{aligned} (\alpha^x \alpha_1^{x_1} \beta^{x'})^3 &= \alpha^{3x} (\alpha_1^3 \alpha_3)^{x_1} (\beta^3 \alpha_4^{-xx_1})^{x'} \\ &= \alpha_4^{xa+x_1a_1+x'b-xx_1x'}. \end{aligned}$$

If $a_1 \neq 0$, take $x = a$, $x_1 = -a_1$, $x' = 0$.

If $b \neq 0$, take $x = a$, $x_1 = 0$, $x' = -b$.

If $a_1 = b = 0$, take $x_1 x' = a$.

Then, by replacing α by $\alpha^x \alpha_1^{x_1} \beta^{x'}$, we may suppose $\alpha^3 = 1$.

The other automorphisms are of the form

$$\begin{aligned} \alpha_1^* &= \alpha_1^{y_1} \alpha_2^{y_2} \dots \\ \beta^* &= \beta \alpha_2^{z_2} \dots \\ \alpha_2^* &= \alpha_2^{y_1} \dots \\ \alpha_3^* &= \alpha_3^{y_1} \dots \\ \alpha_4^* &= \alpha_4^{y_1} \end{aligned}$$

and since $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$ and $\langle \beta, H_2(G) \rangle$ are regular, the result follows by considering the case where $\alpha^3 = 1$ for $p > 3$.

Thus we may suppose $a_1, b = 0$ or 1 and so there are 4 groups of $3^6 \Delta_{30}$.

ϕ_{31} and ϕ_{32} ($p \geq 3$)

These families are characterized by: $G/Z(G)$ is $\phi_4(1^5)$, H_2 is (111), H_3 is (1) and G has a maximal subgroup $A \in \phi_5$.

Let $\alpha \in G - A$, $\alpha_1 \in A - Z_2$, $\alpha_2 \in A - \langle \alpha_1, Z_2 \rangle$, $\beta_1 = [\alpha_1, \alpha]$, $[\alpha_1, \beta_1] = \gamma$. By theorem 6, we may suppose $[\alpha, \beta_1] = [\alpha_1, \alpha_2] = [\alpha_1, \beta_2] = [\alpha_2, \beta_1] = 1$, $[\alpha_2, \beta_2] = \gamma^r$ ($i = 1, 2$).¹⁾

For stem groups, $Z_1 = \langle \gamma \rangle$, $Z_2 = \langle \beta_1, \beta_2, \gamma \rangle$, A and $\mathcal{L}(H_2(G)) = \langle \alpha, \beta_1, \beta_2, \gamma \rangle$ are characteristic subgroups.

Thus $\alpha^p = \gamma^a$, $\alpha_i^p = \gamma^{a_i}$, $\alpha^{*p} = \gamma^{*a}$, $\alpha_i^{*p} = \gamma^{*a_i}$ where

$$\alpha^* = \alpha^x \quad \beta_1^{x'} \dots$$

$$\alpha_1^* = \alpha_1^{y_1} \alpha_2^{y_2} \beta_1^{y'} \dots \quad x(y_1 z_2 - y_2 z_1) \neq 0$$

$$\alpha_2^* = \alpha_1^{z_1} \alpha_2^{z_2} \beta_1^{z'} \dots$$

$$\bar{\beta}_1^* = \bar{\beta}_1^{xy_1} \bar{\beta}_2^{xy_2}$$

$$\bar{\beta}_2^* = \bar{\beta}_1^{xz_1} \bar{\beta}_2^{xz_2}$$

$$\gamma^* = \gamma^{x(y_1^2 + ry_2^2)}.$$

Since $[\alpha_1^*, \beta_2^*] = 1$, we have $y_1 z_1 + r y_2 z_2 = 0$.

Since $[\alpha_2^*, \beta_2^*] = \gamma^{*r}$, we have $z_1^2 + r z_2^2 = r(y_1^2 + r y_2^2)$.

Hence $z_2 = \epsilon y_1$, $z_1 = -\epsilon r y_2$ ($\epsilon^2 = 1$).

From $a^{*p} = \gamma^{*a^*}$ etc., we have

$$a = (y_1^2 + ry_2^2)a^*, \quad y_1^2 + ry_2^2 \neq 0$$

$$y_1 a_1 + y_2 a_2 = x(y_1^2 + ry_2^2)a_1^*$$

$$\epsilon(y_1 a_2 - ry_2 a_1) = x(y_1^2 + ry_2^2)a_2^*$$

and so $ra_1^2 + a_2^2 = x^2(y_1^2 + ry_2^2)(ra_1^{*2} + a_2^{*2})$.

If $ra_1^2 + a_2^2 = 0^{(1)}$, by choice of y_1, y_2 we may suppose $a = 0$ or 1 , and by choice of x , we may suppose $a_1 = 0$ or $1^{(1)}$.

If $ra_1^2 + a_2^2 \neq 0$, we may suppose $ra_1^2 + a_2^2 = 1$.

Taking $y_1 a_1 = -y_2 a_2$, we may suppose $a_1 = 0, a_2 = 1$.

Thus $y_2 = 0, xy_1 = \epsilon$ and the first equation becomes

$$a = y_1^2 a^*.$$

Thus we may suppose $a = 0, 1$ or v .

Standard forms are:

$$\begin{array}{lllllll} a_1 = 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ a_2 = 0 & 0 & \sqrt{-r} & \sqrt{-r} & 1 & 1 & 1 \\ a = 0 & 1 & 0 & 1 & 0 & 1 & v \quad 1) \end{array}$$

giving $3 + (p-1, 4)$ groups in $p^6 \phi_{31}$ and $9 - (p-1, 4)$ groups in $p^6 \phi_{32}$.

As all the characteristic subgroups are regular for $p = 3$, this is the same as the case for $p > 3$.

- 1) For ϕ_{31} ($r = 1$), $a_1^2 + a_2^2 = 0 \Rightarrow a_1 = a_2 = 0$ iff $p \equiv 3 \pmod{4}$
 and for ϕ_{32} ($r = v$), $va_1^2 + a_2^2 = 0 \Rightarrow a_1 = a_2 = 0$ iff
 $p \equiv 1 \pmod{4}$.

ϕ_{33} ($p > 3$)

ϕ_{33} is characterized by: $G/Z(G)$ is $\phi_4(1^5)$, H_2 is (111), H_3 is (1) and G has a maximal subgroup $A \in \phi_2$.

Let $\alpha \in G - A$, $\alpha_1 \in A - \mathcal{L}(Z_2)$, $\alpha_2 \in \mathcal{L}(Z_2) - Z_2$, $\beta_1 = [\alpha_1, \alpha]$, ($i = 1, 2$) $[\beta_2, \alpha] = \gamma$. By theorem 6, we may suppose $[\alpha_1, \beta_1] = \gamma$ $[\beta_1, \alpha] = [\alpha_1, \alpha_2] = [\beta_2, \alpha_1] = [\beta_1, \beta_2] = 1$.

For stem groups, $Z(G) = \langle \gamma \rangle$, $H_2(G) = \langle \beta_1, \beta_2, \gamma \rangle$, $\mathcal{L}(H_2) = \langle \alpha_2, \beta_1, \beta_2, \gamma \rangle$ and $A = \langle \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma \rangle$ are characteristic subgroups.

Thus $\alpha^p = \gamma^a$, $\alpha_1^p = \gamma^{a_1}$, $\alpha^{*p} = \gamma^{*a^*}$, $\alpha_1^{*p} = \gamma^{*a_1^*}$ where

$$\alpha^* = \alpha^x \alpha_1^{x_1} \alpha_2^{x_2} \dots$$

$$\alpha_1^* = \alpha_1^{y_1} \alpha_2^{y_2} \dots \quad xy_1z_2 \neq 0$$

$$\alpha_2^* = \alpha_2^{z_2} \dots$$

$$\bar{\beta}_1^* = \bar{\beta}_1^{xy_1} \bar{\beta}_2^{xy_2}$$

$$\bar{\beta}_2^* = \bar{\beta}_2^{xz_2}$$

$$\gamma^* = \gamma^{x^2z_2}.$$

From $[\beta_1^*, \alpha^*] = 1$, we get $x^2y_2 - xx_1y_1 = 0$ and so $xy_2 = x_2y_1$.

From $[\alpha_1^*, \beta_1^*] = \gamma^*$, we get $xy_1^2 = x^2z_2$ and so $xz_2 = y_1^2$.

From $\alpha^{*p} = \gamma^{*a^{*}}$ etc. we get

$$a_2 = x^2 a_2^{*}$$

$$x a_1 + x_1 a_2 = x^2 y_1 a_1^{*}$$

$$x a + x_1 a_1 + x_2 a_2 = x y_1^2 a^{*}.$$

If $a_2 \neq 0$, by taking $x^2 = a_2$ or $v^{-1} a_2$, $x_1 a_2 = -x a_1$ and

$x_2 a_2 = -(x a + x_1 a_1)$, we may suppose $a_2 = 1$ or v , $a = a_1 = 0$.

If $a_2 = 0$, equations become $a_1 = x y_1 a_1^{*}$

$$x a + x_1 a_1 = x y_1^2 a^{*}$$

with standard forms $a_1 = 1 \ 0 \ 0 \ 0$

$$a = 0 \ 1 \ v \ 0.$$

Thus there are 6 groups in $p^6 \phi_{33}$ ($p > 3$).

Δ_{33}

This family has the same defining relations as for ϕ_{33} ($p > 3$). Using the same notation as for $p > 3$, consider $\sigma = \alpha \alpha_1^{z_1}$.

Since $[\alpha_2, \sigma] = \beta_2$, $[\beta_2, \sigma] = \gamma$, we see that $\langle \sigma, \alpha_2 \rangle \in \Delta_3$ and similarly $\langle \alpha_1, \sigma \rangle \in \Delta_3$.

$$\text{Thus } \sigma^3 = \begin{cases} \alpha_1^{3z_1} (\alpha^3 \gamma) & \text{for } z_1 \neq 0 \\ \alpha_3 & \text{for } z_1 = 0 \end{cases}$$

$$\text{i.e. } \sigma^3 = \alpha_1^{3z_1} \alpha^3 \gamma^{z_1^2}.$$

$$\begin{aligned} \text{So } (\sigma^{\alpha_2^{x_2}})^3 &= \sigma^{3x_2} (\alpha_2^3 \gamma)^{x_2} \\ &= \alpha_1^{3z_1 x_2} \alpha^{3x_2} \alpha_2^{3x_2} \gamma^{z_1^2 x_2 + x_2}. \end{aligned}$$

Putting $x_1 = xz_1$ (and so $z_1 = xx_1$) we have

$$(\alpha^{\alpha_1^{x_1}} \alpha_2^{x_2})^3 = \alpha^{3x_1} \alpha_1^{3x_1} \alpha_2^{3x_2} \gamma^{x_1^2 x + x_2}.$$

Thus the equations for $p > 3$ become

$$a_2 = a_2^*$$

$$xa_1 + x_1 a_2 = y_1 a_1^*$$

$$xa + x_1 a_1 + x_2 a_2 + x_1^2 x + x_2 = xa^*.$$

If $a_2 = 1$, we may suppose $a_1 = a = 0$.

If $a_2 = 2$, we may suppose $a_1 = 0$ and so $a = a^*$.

If $a_2 = 0$, we may suppose $a = 0$ and so

$$xa_1 = y_1 a_1^*.$$

So the standard forms are

$$a_2 = \begin{matrix} 1 & 2 & 2 & 2 & 0 & 0 \end{matrix}$$

$$a_1 = \begin{matrix} 0 & 0 & 0 & 0 & 0 & 1 \end{matrix}$$

$$a = \begin{matrix} 0 & 0 & 1 & 2 & 0 & 0 \end{matrix}$$

giving 6 groups in $3^6\Delta_{33}$.

$$\underline{\phi_{34} \ (p \geq 3)}$$

ϕ_{34} is characterized by: $G/Z(G)$ is $\phi_4(221)c$, $H_2(G)$ is (111), $H_3(G)$ is (1).

By theorem 6, ϕ_{34} has the same relations as ϕ_{33} except that (using the same notation) $\beta_1 = \alpha^p$, $\beta_2 = \alpha_1^p$.

Thus

$$\begin{aligned} \gamma &= [\beta_2, \alpha] \\ &= [\alpha_1^p, \alpha] \\ &= [\alpha_1, \alpha]^p [\beta_1, \alpha_1]^{-\binom{p}{2}} \\ &= \beta_1^p \\ &= \alpha^{p^2}. \end{aligned}$$

Also $\alpha_1^{p^2} = \beta_2^p = 1$ and $\alpha_2^p = \gamma^{a_2}$.

Replacing α by $\alpha^* = \alpha^x$ and α_2 by $\alpha_2^{z_2}$ we get $\gamma^* = \gamma^x$ and

$$z_2 a_2 = x a_2^* \quad \text{where } x z_2 = 1.$$

So we may suppose $a_2 = 0, 1$ or v , giving 3 groups in $P^6 \phi_{34}$.

ϕ_{37} to ϕ_{39} ($p \geq 5$)

These families are discussed by Blackburn in (1), whose results I have used below.

The groups of ϕ_{38} are given by:

$G = \langle a, a_1, \dots, a_5 \rangle$ where $[a_1, a] = a_{i+1}$ ($i = 1, \dots, 4$) $[a_5, a] = 1$
 $[a_1, a_2] = a_4 a_5^{-1}$, $[a_2, a_3] = a_5$, $[a_1, a_j] = 1$ otherwise and

$$a^p = a_5^d, a_1^p a_5^{(p)} = a_5^c, a_i^p = 1 \quad (i > 1) \text{ where we may suppose}$$

$$c = \text{any } g^k \quad 0$$

$$d = c \quad 0 \quad g^1$$

$$k + 1 = 1, 2, \dots, (p-1, 4) \quad b + 1 = 1, 2, \dots, (p-1, 5).$$

Thus there are $p + (p-1, 4) + (p-1, 5)$ groups of $p^6 \phi_{38}$.

The groups of ϕ_{37} and ϕ_{39} are given by

$G = \langle a, a_1, \dots, a_5 \rangle$ where $[a_1, a] = a_{i+1}$ ($i = 1, 2, 3$),
 $[a_4, a_1] = [a_2, a_3] = [a_3, a_1] = a_5 \in Z(G)$, $[a_1, a_2] = a_4^a$ and
 $a^p = a_5^d, a_1^p = a_5^c, a_i^p = 1 \quad (i > 2).$

If $a \neq 0$ (ϕ_{39}) we may suppose $a = 1$ and

$$c = \text{any } g^1 \quad 0$$

$$d = c \quad 0 \quad g^m$$

$$l + 1 = 1, 2, \dots, (p-1, 5), \quad m + 1 = 1, 2, \dots, (p-1, 6).$$

If $a = 0$ (ϕ_{37}) we may suppose

$$c = 1 \quad 1 \quad 0 \quad 0 \quad 0$$

$$d = g^k \quad 0 \quad 1 \quad v \quad 0 \quad k + 1 = 1, 2, \dots, (p-1, 6)$$

Thus there are $4 + (p-1, 4)$ groups in $p^6 \phi_{37}$ and $p + (p-1, 5) + (p-1, 6)$ groups in $p^6 \phi_{39}$.

ϕ_{40} ($p \geq 3$)

This family is characterized by: $G/Z(G)$ is $\phi_5(1^5)$,

$H_2(G)$ is (1^4) , $H_3(G)$ is (111) , $H_4(G)$ is (1) .

Let $\alpha_1 \in G - H_2(G)$, $\alpha_2 \in G - \langle \alpha_1, H_2, Z \rangle$, $\beta = [\alpha_1, \alpha_2]$,
 $\beta_i = [\beta, \alpha_i]$ ($i = 1, 2$), $\gamma = [\beta_1, \alpha_2]$. By theorem 12, we may
suppose $[\beta_2, \alpha_1] = \gamma$, $[\alpha_1, \beta_1] = [\beta, \beta_1] = 1$ ($i = 1, 2$).

For stem groups, we have $Z = \langle \gamma \rangle$ and so

$$\beta^p = \beta_1^p = 1, \alpha_1^p = \gamma^{a_1}, \alpha_1^{*p} = \gamma^{*a_1^*} \text{ where}$$

$$\alpha_1^* = \alpha_1^{x_1} \alpha_2^{x_2} \beta^x \dots$$

$$\alpha_2^* = \alpha_1^{y_1} \alpha_2^{y_2} \beta^y \dots \Delta = x_1 y_2 - x_2 y_1 \neq 0$$

$$\beta^* = \beta^\Delta \dots$$

$$\bar{\beta}_1^* = \bar{\beta}_1^{\Delta x_1} \bar{\beta}_2^{\Delta x_2}$$

$$\bar{\beta}_2^* = \bar{\beta}_1^{\Delta y_1} \bar{\beta}_2^{\Delta y_2}$$

$$\gamma^* = \gamma^{\Delta(x_1 y_2 + x_2 y_1)}.$$

From $[\alpha_1^*, \beta_1^*] = 1$, we get $x_1 x_2 = 0$.

From $[\alpha_2^*, \beta_2^*] = 1$, we get $y_1 y_2 = 0$.

From $\alpha_1^{*p} = \gamma^{*a_1^*}$, we get

$$a_1 = x_1 y_2^2 a_1^*$$

$$a_2 = x_1^2 y_2 a_2^*$$

or

$$a_2 = -x_2 y_1^2 a_1^*$$

$$a_1 = -x_2^2 y_1 a_2^*.$$

If $a_1 a_2 = 0$, we may suppose $a_1 = 0$ and so

$$a_2 = x_1^2 y_2 a_2^*.$$

Thus we may suppose $a_2 = 0$ or 1.

If $a_1 a_2 \neq 0$, by taking $x_1 y_2^2 = a_1$, we may suppose

$$a_1 = 1 \text{ and so } x_1 = y_2^{-2} \text{ or } a_2 = -x_2 y_1^2, 1 = -x_2^2 y_1 a_2^*.$$

$$\text{Thus } y_2^3 a_2 = a_2^* \text{ or } a_2 = x_2^3 y_1^3 a_2^*$$

with standard forms $a_2 = 1, g \dots g^r \quad r+1 = (p-1, 3).$

Since the above automorphisms do not affect the calculation when $p = 3$, the results there are the same as for $p > 3$.

Thus there are $2 + (p-1, 3)$ groups in $p^6 \phi_{40}$ ($p \geq 3$).

$$\phi_{41} \ (p > 3)$$

This family has the same defining relations for $G/Z(G)$

and $H_2(G)$ as ϕ_{40} . Using the same notation we have $[\alpha_1, \beta_2] =$

$$[\alpha_2, \beta_1] = 1, [\alpha_1, \beta_1] = \gamma, [\alpha_2, \beta_2] = \gamma^{-v}.$$

With the same automorphisms as ϕ_{40} :

$$\gamma^* = \gamma^{\Delta(x_1^2 - vx_2^2)}.$$

From $[\alpha_1^*, \beta_2^*] = 1$, we have $x_1 y_1 = vx_2 y_2$.

From $[\alpha_2^*, \beta_2^*] = \gamma^{*-v}$, we have $y_1^2 - vy_2^2 = -v(x_1^2 - vx_2^2)$.

Solving these, $y_1 = \epsilon vx_2$

$$y_2 = \epsilon x_1 \quad (\epsilon^2 = 1).$$

From $\alpha_1^{*p} = \gamma^{*a_1^*}$, we get

$$\begin{aligned} x_1 a_1 + x_2 a_2 &= \epsilon(x_1^2 - vx_2^2)^2 a_1^* \\ vx_2 a_1 + x_1 a_2 &= (x_1^2 - vx_2^2)^2 a_2^*. \end{aligned}$$

Taking $x_1 a_2 = -vx_2 a_1$, we may suppose $a_2 = 0$.

Thus $x_2 a_1 = 0$ and so

$$a_1 = x_1^3 \epsilon a_1^* = (\epsilon x_1)^3 a_1^*.$$

So we may suppose $a_1 = 0, 1, g, \dots, g^r$, $r+1 = (p-1, 3)$ giving

$1 + (p-1, 2)$ groups in $p^6 \phi_{41}$ ($p > 3$).

$$\Delta_{41}$$

As in ϕ_{41} ($p > 3$), a stem group of Δ_{41} is of the form $G = \langle \alpha_1, \alpha_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_i] = \beta_i \ (i = 1, 2), [\alpha_1, \beta_1] = [\alpha_2, \beta_2] = \gamma, [\alpha_1, \beta_2] = [\alpha_2, \beta_1] = 1 \rangle$ where $H_2(G)$ is abelian and $\alpha_i^3 \in Z(G)$.

Using the same automorphism as in ϕ_{40} , we have

$$\alpha_1^* = \alpha_1^{x_1} \alpha_2^{x_2} \beta^x \dots$$

$$\alpha_2^* = \alpha_1^{y_1} \alpha_2^{y_2} \beta^y \dots$$

Since $G/Z(G) \in \phi_6$, we have

$$\alpha_1^{*3} \equiv \alpha_1^{3x_1} \alpha_2^{3x_2} \beta_1^{-x_1^2 x_2} \beta_2^{x_1 x_2^2} \pmod{Z}.$$

However, $\alpha_1^{*3} \in Z(G)$ and so $x_1 x_2 = 0$.

Similarly $y_1 y_2 = 0$, and equations become

$$\begin{array}{ll} a_1 = y_2 a_1^* & a_2 = -y_1 a_1^* \\ a_2 = x_1 a_2^* & a_1 = -x_2 a_2^*. \end{array} \quad \text{or}$$

If $a_1 a_2 = 0$, we may suppose (as in Δ_{40}) that

$$a_1 = 0, \quad a_2 = 0 \text{ or } 1.$$

If $a_1 a_2 \neq 0$, we may suppose (by taking $y_2 = a_1, x_1 = a_2$) that

$$a_1 = a_2 = 1.$$

Thus there are 3 groups in $3^6 \Delta_{41}$.

Φ_{42} and Φ_{43}

By theorem 8, there is just one group in $p^6\Phi_{42}$:

$$\langle \alpha, \beta, \gamma | \alpha^{p^2} = \beta^{p^2} = \gamma^{p^2} = 1, [\alpha, \beta] = \gamma, [\alpha, \gamma] = \alpha^p, [\beta, \gamma] = \beta^{-p} \rangle.$$

Note: Taking $\alpha_5 = \alpha\beta^{\frac{1}{2}}$, $\alpha_6 = \alpha^{-1}\beta^{\frac{1}{2}}$ we see that this group is isomorphic to that given by Easterfield in (2).

By theorem 10, there is just one group in $p^6\Phi_{43}$:

$$\langle \alpha, \beta, \gamma | \alpha^{p^2} = \beta^{p^2} = \gamma^{p^2} = 1, [\alpha, \beta] = \gamma, [\alpha, \gamma] = \beta^p, [\beta, \gamma] = \alpha^{vp} \rangle.$$

By theorem 10', the groups in Δ_{42} , Δ_{43} are:

$$3^6\Delta_{42} = \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, \alpha_1^3 = \beta_2, \alpha_2^3 = 1, \beta^9 = \beta_1^3 = 1 \rangle.$$

$$3^6\Delta_{43} = \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, \alpha_1^3 = \beta_2, \alpha_2^3 = \beta_1, \beta^9 = \beta_1^3 = 1 \rangle.$$

III Possible Choices for Capable Groups

The following is a list of non-abelian groups of order p^5 ($p \geq 5$) whose two largest type invariants are the same. By lemma 4, all the capable groups of order p^5 must be taken from this list.

$$\phi_2(1^5) = \langle \alpha, \beta, \gamma, \phi, \epsilon \mid |\xi| = p, \text{ for all } \xi \in G (\xi \neq 1), [\alpha, \beta] = \gamma, \text{rest} = 1 \rangle$$

$$\phi_2(221)a = \langle \alpha, \beta, \gamma \mid \alpha^{p^2} = \beta^{p^2} = \gamma^p = 1, [\alpha, \beta] = \alpha^p, \text{rest} = 1 \rangle$$

$$\phi_2(221)b = \langle \alpha, \beta, \gamma \mid \alpha^{p^2} = \beta^{p^2} = \gamma^p = 1, [\alpha, \gamma] = \alpha^p, \text{rest} = 1 \rangle$$

$$\phi_2(221)c = \langle \alpha, \beta, \gamma \mid \alpha^{p^2} = \beta^{p^2} = \gamma^p = 1, [\alpha, \gamma] = \beta^p, \text{rest} = 1 \rangle$$

$$\phi_2(221)d = \langle \alpha, \beta, \gamma \mid \alpha^{p^2} = \beta^{p^2} = \gamma^p = 1, [\alpha, \beta] = \gamma, \gamma \in Z(G) \rangle$$

$$\phi_3(1^5) = \langle \alpha, \beta, \gamma, \delta, \epsilon \mid |\xi| = p, \text{ all } \xi (\neq 1) \in G, [\alpha, \delta] = \beta, [\beta, \delta] = \gamma, \text{rest} = 1 \rangle$$

$$\phi_3(221)a = \langle \alpha, \beta, \gamma \mid \alpha^{p^2} = \beta^{p^2} = \gamma^p = 1, [\alpha, \beta] = \gamma, [\alpha, \gamma] = \alpha^p, \text{rest} = 1 \rangle$$

$$\phi_3(221)b_1 = \langle \alpha, \beta, \gamma \mid \alpha^{p^2} = \beta^{p^2} = \gamma^p = 1, [\alpha, \beta] = \gamma, [\alpha, \gamma] = \beta^p, \text{rest} = 1 \rangle$$

$$\phi_3(221)b_2 = \langle \alpha, \beta, \gamma \mid \alpha^{p^2} = \beta^{p^2} = \gamma^p = 1, [\alpha, \beta] = \gamma, [\alpha, \gamma] = \beta^{vp}, \text{rest} = 1 \rangle$$

$$\phi_4(1^5) = \langle \alpha, \beta, \delta, \epsilon, \mid |\xi| = p, \text{ all } \xi (\neq 1) \in G, [\alpha, \beta] = \delta, [\alpha, \gamma] = \epsilon \rangle$$

$$\phi_4(221)a = \langle \alpha, \beta, \gamma \mid \alpha^{p^2} = \beta^{p^2} = \gamma^p = 1, [\alpha, \beta] = \alpha^p, [\beta, \gamma] = \beta^p, [\alpha, \gamma] = 1 \rangle$$

$$\phi_4(221)b_0 = \langle \alpha, \beta, \gamma | \alpha^{p^2} = \beta^{p^2} = \gamma^p = 1, [\alpha, \gamma] = \alpha^p, [\beta, \gamma] = \beta^p, [\alpha, \beta] = 1 \rangle$$

$$\phi_4(221)b_k = \langle \alpha, \beta, \gamma | \alpha^{p^2} = \beta^{p^2} = \gamma^p = 1, [\alpha, \gamma] = \alpha^{k^p}, [\beta, \gamma] = \beta^p, k=1, 2, \dots, \frac{p-1}{2} \rangle$$

$$\phi_4(221)c = \langle \alpha, \beta, \gamma | \alpha^{p^2} = \beta^{p^2} = \gamma^p = 1, [\alpha, \beta] = \alpha^p, [\alpha, \gamma] = \beta^p, [\beta, \gamma] = 1 \rangle$$

$$\phi_4(221)d = \langle \alpha, \beta, \gamma | \alpha^{p^2} = \beta^{p^2} = \gamma^p = 1, [\alpha, \gamma] = \alpha^p, [\beta, \gamma] = \alpha^{-p}\beta^p, [\alpha, \beta] = 1 \rangle$$

$$\phi_4(221)e_0 = \langle \alpha, \beta, \gamma | \alpha^{p^2} = \beta^{p^2} = \gamma^p = 1, [\alpha, \gamma] = \beta^p, [\beta, \gamma] = \alpha^{vp}, [\alpha, \beta] = 1 \rangle$$

$$\phi_4(221)d_k = \langle \alpha, \beta, \gamma | \alpha^{p^2} = \beta^{p^2} = \gamma^p = 1, [\alpha, \gamma] = \beta^p, [\beta, \gamma] = \alpha^{\frac{1}{4}(4^k v - 1)p} \beta^p \rangle$$

$$k = 1, 2, \dots, \frac{p-1}{2}$$

$$\phi_5(1^5) = \langle \alpha, \beta, \gamma, \delta, \epsilon | |\xi| = p, \text{ all } \xi (\neq 1) \in G, [\alpha, \beta] = [\gamma, \delta] = \epsilon \rangle$$

$$\phi_6(1^5) = \langle \alpha, \beta, \gamma, \delta, \epsilon | |\xi| = p, \text{ all } \xi (\neq 1) \in G, [\alpha, \beta] = \gamma, [\alpha, \gamma] = \delta, [\beta, \gamma] = \epsilon \rangle$$

$$\phi_6(221)a_k = \langle \alpha, \beta, \gamma | \alpha^{p^2} = \beta^{p^2} = \gamma^p = 1, [\alpha, \beta] = \gamma, [\alpha, \gamma] = \alpha^{kp}, [\beta, \gamma] = \beta^p \rangle$$

$$\phi_6(221)b_1 = \langle \alpha, \beta, \gamma | \alpha^{p^2} = \beta^{p^2} = \gamma^p = 1, [\alpha, \beta] = \gamma, [\alpha, \gamma] = \alpha^p, [\beta, \gamma] = \alpha^p \beta^p \rangle$$

$$\phi_6(221)b_v = \langle \alpha, \beta, \gamma | \alpha^{p^2} = \beta^{p^2} = \gamma^p = 1, [\alpha, \beta] = \gamma, [\alpha, \gamma] = \alpha^p, [\beta, \gamma] = \alpha^{vp} \beta^p \rangle$$

$$\phi_6(221)c_0 = \langle \alpha, \beta, \gamma | \alpha^{p^2} = \beta^{p^2} = \gamma^p = 1, [\alpha, \beta] = \gamma, [\alpha, \gamma] = \beta^p, [\beta, \gamma] = \alpha^{vp} \rangle$$

$$\phi_6(221)c_k = \langle \alpha, \beta, \gamma | \alpha^{p^2} = \beta^{p^2} = \gamma^p = 1, [\alpha, \beta] = \gamma, [\alpha, \gamma] = \beta^p, [\beta, \gamma] = \alpha^{\frac{1}{4}(4^k - 1)p} \beta^p \rangle$$

$$\phi_7(1^5) = \langle \alpha, \beta, \gamma, \delta, \epsilon | |\xi| = p \text{ all } \xi (\neq 1) \in G, [\alpha, \beta] = \gamma, [\alpha, \gamma] = [\beta, \delta] = \epsilon \rangle$$

$$\phi_9(1^5) = \langle \alpha, \beta, \gamma, \delta, \epsilon \mid |\xi|=p, \text{ all } \xi (\neq 1) \in G, [\alpha, \beta]=\gamma, [\alpha, \gamma]=\delta, [\alpha, \delta]=\epsilon \rangle$$

$$\phi_{10}(1^5) = \langle \alpha, \beta, \gamma, \delta, \epsilon \mid |\xi|=p, \text{ all } \xi (\neq 1) \in G, [\alpha, \beta]=\gamma, [\alpha, \gamma]=\delta, [\alpha, \delta]=$$

$$[\beta, \gamma]=\epsilon \rangle$$

Notes - (i) Unless otherwise stated, all commutators are 1.

(ii) v is a fixed quadratic residue mod p .

Possible choices for Capable Groups ($p = 3$)

The following is a list of groups of order 3^4 or 3^5 and class 3 or 4 which do not satisfy the conditions of lemma 4 or the

Lemma Let H be a group of class 3 with lower central series

$H > H_2 > H_3 > E$ where $|H_2:H_3| = |H_3| = 3$, $H/Z(H) \in \Delta_2$ and $\beta^3, \alpha^9 \in H_3$ all $\beta \in Z_2(H)$, $\alpha \in H$.

Then H is incapable if there is an $\alpha \notin \mathcal{L}(H_2)$ with $H_3 \leq \langle \alpha \rangle$.

Proof Let $\alpha_1 \in \mathcal{L}(H_2) - Z_2(H)$, $\alpha_2 = [\alpha_1, \alpha]$, $\alpha_3 = [\alpha_2, \alpha]$.

Then $H_2 = \langle \alpha_2, \alpha_3 \rangle$, $H_3 = \langle \alpha_3 \rangle$, $H = \langle \alpha, \alpha_1, Z_2(H) \rangle$ where α is such that $H_3 \leq \langle \alpha \rangle$. Also let β be an arbitrary element of $Z_2(H)$.

(a) Suppose $\alpha_3 = \alpha^{9a}$ ($a \not\equiv 0 \pmod{3}$).

Then $\alpha^9 = \alpha_3^a$, $\alpha_1^9 = \alpha_3^r$.

Also $(\alpha^{a-ra}\alpha_1)^9 = \alpha^{9(a-ra)}\alpha_1^9$
 $= \alpha_3^a$

and $(\alpha\beta)^9 = \alpha^9 = \alpha_3^a$ all $\beta \in Z_2(H)$.

So, by lemma 3, $H = \langle \alpha, \alpha^{a-ra}\alpha_1, \{\alpha\beta\} \rangle$ is incapable.¹⁾

(b) Suppose $\alpha_3 = \alpha^{3a}$ ($a \not\equiv 0 \pmod{3}$).

By similar work to that in Δ_3 , Δ_7 , we may suppose $a = 1$.

Also $\alpha_3 \in \langle \beta \rangle$ or $\beta^3 = 1$ (and so $(\alpha\beta)^3 = \alpha^3$).

(1) If $\alpha_1^3 \in H_3$, $\alpha_3 \in \langle \alpha_1 \rangle$ or $\alpha_1^3 = 1$ (and so $(\alpha^2\alpha_1^2)^3 = \alpha^6\alpha_1^6\alpha_3^2 = \alpha_3$).

Thus H is incapable by lemma 3.

1) $\{\alpha\beta\}$ is the set of elements of the form $\alpha\beta, \beta \in Z_2(H)$.

(11) If $\alpha_1^3 = \gamma$, $\gamma^3 \in H_3$, then $\alpha_3 \in \langle \alpha_1 \rangle$ or $\gamma^3 = 1$.

In the second case

$$(\alpha^2 \alpha_1)^3 = \alpha^6 \alpha_1^3 \alpha_3 = \gamma = \alpha_1^3.$$

Also, if $\beta^3 = \alpha_3^r$, then

$$\begin{aligned} (\alpha^{r^2} \alpha_1 \beta^{2-(r+1)^2})^3 &= \alpha^{3r^2} \alpha_1^3 \alpha_3 \beta^{6-3(r+1)^2} \\ &= \alpha_3^{r^2+1+2r-(r+1)^2} \gamma \\ &= \gamma \quad \text{for all } \beta \in Z_2(G). \end{aligned}$$

So $\gamma \in \langle \alpha_1 \rangle$, $\langle \alpha^2 \alpha_1 \rangle$, $\langle \alpha^{r^2} \alpha_1 \beta^{2-(r+1)^2} \rangle$ and so by lemma 3,

H is incapable.

The groups of class 2 are regular and so are included in the list for $p \geq 5$. For those groups which satisfy the conditions of lemma 3, the suitable generators are shown.

$$\Delta_3(1^4) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha_3, \alpha_1^3 \alpha_3 = \alpha^3 = \alpha_2^3 = \alpha_3^3 = 1 \rangle$$

$$\Delta_3(1^5) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha_3, \alpha_1^3 \alpha_3 = \alpha^3 = \alpha_2^3 = \alpha_3^3 = \beta^3 = 1 \rangle$$

$$\begin{aligned} \Delta_3(221)a &= \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha_3, \alpha^3 = \alpha_1^9 = \alpha_2^3 = \alpha_3^3 = 1 \rangle \\ &= \langle \alpha \alpha_1, \alpha^2 \alpha_1^2 \mid (\alpha \alpha_1)^3 = (\alpha^2 \alpha_1^2)^3 = \alpha_1^3 \alpha_3 \rangle \end{aligned}$$

$$\Delta_3(221)b_1 = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha_3, \alpha^9 = \alpha_1^3 = \alpha_2^3 = \alpha_3^3 = 1 \rangle$$

$$\Delta_3(221)b_2 = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha_3, \alpha_1^3 \alpha_3 = \alpha^2, \alpha^9 = \alpha_2^3 = \alpha_3^3 = 1 \rangle$$

$$\Delta_6(1^5) = \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, \alpha_1^3 = \alpha_2^3 = \beta^3 = \beta_1^3 = 1 \rangle$$

$$\Delta_6(221)b = \langle \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, \alpha_1^3 = \beta_1, \alpha_2^3 = \beta^3 = \beta_1^3 = 1 \rangle$$

$$\Delta_6(221)c = \langle \alpha_1, \alpha_2, \beta \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \alpha_2^6, [\beta, \alpha_2] = \alpha_1^3, \alpha_1^9 = \alpha_2^9 = \beta^3 = 1 \rangle.$$

(The other groups of Δ_6 are incapable by considering possible choices from the computer calculations.)

$$\Delta_7(1^5) = \langle \alpha, \alpha_1, \beta, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = [\alpha_1, \beta] = \alpha_3, \alpha^3 = \alpha_2^3 = \beta^3 = \alpha_3^3 = \alpha_1^3 \alpha_3 = 1 \rangle$$

$$\Delta_9(1^5) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_1, \alpha] = \alpha_{1+1}, \alpha^3 = \alpha_1^3 \alpha_2^3 \alpha_3 = \alpha_2^3 \alpha_4 = \alpha_3^3 = \alpha_4^3 = 1 \rangle$$

$$\Delta_9(2111)a_1 = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_1, \alpha] = \alpha_{1+1}, \alpha_1^3 \alpha_2^3 \alpha_3 = \alpha_4, \alpha^3 = \alpha_2^3 \alpha_4 = \alpha_3^3 = \alpha_4^3 = 1 \rangle$$

$$= \langle \alpha \alpha_1, \alpha^2 \alpha_1 \mid (\alpha \alpha_1)^3 = (\alpha^2 \alpha_1)^3 = \alpha_4 \rangle$$

$$\Delta_9(2111)a_2 = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_1, \alpha] = \alpha_{1+1}, \alpha_1^3 \alpha_2^3 \alpha_3 = \alpha_4^2, \alpha^3 = \alpha_2^3 \alpha_4 = \alpha_3^3 = \alpha_4^3 = 1 \rangle$$

$$= \langle \alpha \alpha_1, \alpha^2 \alpha_1 \mid (\alpha \alpha_1)^3 = (\alpha^2 \alpha_1)^3 = \alpha_4^2 \rangle$$

$$\Delta_9(2111)b = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_1, \alpha] = \alpha_{1+1}, \alpha^3 = \alpha_4, \alpha_1^3 \alpha_2^3 \alpha_3 = \alpha_2^3 \alpha_4 = \alpha_3^3 = \alpha_4^3 = 1 \rangle$$

$$= \langle \alpha, \alpha \alpha_1 \mid \alpha^3 = (\alpha \alpha_1)^3 = \alpha_4 \rangle.$$

Similarly, the only possible choice in Δ_{10} is

$$\Delta_{10}(1^5) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_1, \alpha] = \alpha_{1+1}, [\alpha_1, \alpha_2] = \alpha_4, \alpha^3 = \alpha_1^3 \alpha_2^3 \alpha_3 = \alpha_2^3 \alpha_4 = \alpha_3^3 = \alpha_4^3 = 1 \rangle.$$

APPENDICES

A list of Non-Abelian p-groups.

NOTES:

1. $\langle \xi_1, \xi_2, \dots, \xi_n \mid w_1(\xi) = w_2(\xi) = w_m(\xi) = 1 \rangle$ means the group generated by $\xi_1, \xi_2, \dots, \xi_n$ with the defining relations $w_1(\xi) = 1, \dots, w_m(\xi) = 1$.
2. Unless otherwise stated, all commutators are 1.
3. Throughout, v will be taken to be a (fixed) non-quadratic residue (mod p) and g will be taken to be a (fixed) primitive root (mod p).
4. $\Phi_2(321)a_i = \Phi_2(32)a_i \times (1)$ means that $\Phi_2(321)a_i$ is the direct product of a cyclic group of order p together with the group $\Phi_2(32)a_i$. Similarly for other such expressions.
5. In the maximal subgroups column, $\Phi_2(32)^{p^2-1} (311)ac^{p-1} (221)b$ means that the group has p^2-1 maximal subgroups isomorphic to $\Phi_2(32)$, one maximal subgroup isomorphic to $\Phi_2(311)a$ and $p-1$ maximal subgroups isomorphic to $\Phi_2(311)c$. Similarly for other such expressions.

APPENDIX I

The Non-Abelian Groups of Order p^n ($p > 2$, $n < 6$)

1. $G \in p^3 \Phi_2$

$$\Phi_2(21) = \langle \alpha, \alpha_1 | \alpha^{p^2} = \alpha_1^p = 1, [\alpha_1, \alpha] = \alpha^p \rangle$$

$$\Phi_2(111) = \langle \alpha, \alpha_1 | \alpha^p = \alpha_1^p = \alpha_2^p = 1, [\alpha_1, \alpha] = \alpha_2 \rangle$$

2.(a) $G \in p^4 \Phi_2$

$$\Phi_2(31) = \langle \alpha, \alpha_1 | \alpha^{p^3} = \alpha_1^p = 1, [\alpha_1, \alpha] = \alpha^{p^2} \rangle$$

$$\Phi_2(22) = \langle \alpha, \alpha_1 | \alpha^{p^2} = \alpha_1^{p^2} = 1, [\alpha_1, \alpha] = \alpha^p \rangle$$

$$\Phi_2(211)a = \Phi_2(21) \times (1)$$

$$\Phi_2(211)b = \langle \alpha, \alpha_1, \gamma | \alpha^p = \alpha_1^p = \gamma^{p^2} = 1, [\alpha_1, \alpha] = \gamma^p \rangle$$

$$\Phi_2(211)c = \langle \alpha, \alpha_1 | \alpha^{p^2} = \alpha_1^p = \alpha_2^p = 1, [\alpha_1, \alpha] = \alpha_2 \rangle$$

Note: The last one has no non-abelian maximal subgroups)

$$\Phi_2(1^4) = \Phi_2(111) \times (1)$$

2.(b) $G \in p^4 \Phi_3$

$$\Phi_3(211) = \langle \alpha, \alpha_1 \rangle \text{ where } [\alpha_i, \alpha] = \alpha_{i+1} \ (i = 1, 2), \alpha_2^p = \alpha_3^p = 1$$

Symbol	α^p	$\alpha_1^p \alpha_3^{(p)}$	Maximal Subgroups	
			Φ_2	Φ_1
a	α_3	1	$(21)^p$	(111)
b_1	1	α_3	$(21)^{p-1}(111)$	(21)
b_v	1	α_3^v	$(21)^{p-1}(111)$	(21)

$$\Phi_3(1^4) = \langle \alpha, \alpha_1 | [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_1^p \alpha_3^{(p)} = \alpha_{i+1}^p \ (i = 1, 2) \rangle$$

3(a) $G \in p^5 \Phi_2$

$$\Phi_2(41) = \langle \alpha, \alpha_1 | \alpha^{p^4} = \alpha_1^p = 1, [\alpha_1, \alpha] = \alpha^{p^3} \rangle$$

$$\Phi_2(32) = \langle \alpha, \alpha_1 | \alpha^{p^3} = \alpha_1^{p^2}, [\alpha_1, \alpha] = \alpha_2 \rangle \text{ where } \gamma^p = 1 \text{ and}$$

Symbol	α^{p^2}	α_1^p	Maximal Subgroups	Maximal Quotients in Φ_2
α_1	α_2	γ	$(31)^p (22)$	$(31)^p$
α_2	γ_2	α_2	$(31)^p (22)$	$(31)^{p-1} (22)$

$$\Phi_2(311) = \langle \alpha, \alpha_1, \alpha_2, \gamma_1 \rangle \text{ where } \alpha_1^p = \gamma_1^{p^2} = 1, [\alpha_1, \alpha] = \alpha_2 \text{ and}$$

Symbol	γ_2^p	α^p	α_2	Maximal Subgroups in Φ_2
a	1	γ_1	γ_1^p	$(31)^{p^2}$
b	γ_1	1	γ_1^p	$(31)^{p^2-1} (211)$
c	1	γ_1	γ_2	-

$= \Phi_2(31) \times (1)$

$$\Phi_2(221) = \langle \alpha, \alpha_1, \alpha_2, \gamma_1, \gamma_2 \rangle \text{ where } [\alpha_1, \alpha] = \alpha_2, \alpha_2^p = \gamma_2^p = 1 \text{ and}$$

Symbol	α^p	α_1^p	γ_1^p	Maximal Subgroups in Φ_2
a	γ_2	α_2	1	$(22)^{p^2}$
b	1	α_2	γ_2	$(22)^{p(p-1)} (211) a c^{p-1}$
c	γ_2	1	α_2	$(22)^{p(p-1)} (211) c^p$
d	γ_2	γ_1	1	-

$= \Phi_2(22) \times (1)$

$= \Phi_2(21) \times (2)$

$$\Phi_2 (2111)a = \Phi_2 (211)a \times (1)$$

$$\Phi_2 (2111)b = \Phi_2 (211)b \times (1)$$

$$\Phi_2 (2111)c = \Phi_2 (211)c \times (1)$$

$$\Phi_2 (2111)d = \Phi_2 (111) \times (2)$$

$$\Phi_2 (1^5) = \Phi_2 (1^4) \times (1)$$

$$3(b) \quad \underline{G \in p^5 \Phi_3}$$

$$\Phi_3(311) = \langle \alpha, \alpha_1 \rangle \text{ where } [\alpha_i, \alpha] = \alpha_{i+1}, \quad (i = 1, 2)$$

$$\alpha_2^p = \alpha_3^p = 1 \text{ and the remaining results are the same as for}$$

$$\Phi_3(211) \text{ replacing } \alpha_3 \text{ by } \gamma \text{ where } \gamma^p = \alpha_3.$$

$$\Phi_3(221) = \langle \alpha, \alpha_1 \rangle \text{ where } [\alpha_i, \alpha] = \alpha_{i+1}, \quad (i=1, 2)$$

$$\alpha_2^p = \alpha_3^p = \gamma^p = 1 \text{ and}$$

Symbol	α^p	$\alpha_1^p \alpha_3^{\binom{p}{3}}$	Maximal Subgroups in Φ_2 .
a	α_3	γ	$(211)ac^{p-1}$
b_1	γ	α_3	$(211)c^p$
b_v	γ	α_3^v	$(211)c^p$

$$\Phi_3 (2111)a = \Phi_3 (211)a \times (1)$$

$$\Phi_3 (2111)b_1 = \Phi_3 (211)b_1 \times (1)$$

$$\Phi_3 (2111)b_v = \Phi_3 (211)b_v \times (1). \text{ Others are given by}$$

$$\Phi_3 (2111) = \langle \alpha, \alpha_1, \gamma \rangle \text{ where } [\alpha_i, \alpha] = \alpha_{i+1}, \quad (i = 1, 2), \alpha_2^p = \alpha_3^p = 1.$$

and

Symbol	α^p	$\alpha_1^p \alpha_3^{(p)}$	γ^p	Maximal Subgroups	
				Φ_2	Φ_3
c	1	1	α_3	$(211)b^p$	$(211)b^{p-1}(1^4)$
d	γ	1	1	$(211)c^p$	-
e	1	γ	1	$(211)c^{p-1}(1^4)$	-

$$\Phi_3(1^5) = \Phi_3(1^4) \times (1).$$

$$3(c) \quad \underline{G \in p^5 \Phi_4}$$

The groups of $p^5 \Phi_4$ of exponent p^2 are given by

$$G = \langle \alpha, \alpha_1, \alpha_2 \rangle \text{ where } [\alpha_i, \alpha] = \beta_i \quad (i = 1, 2) \quad \beta_i^p = 1 \quad \text{and}$$

Type Invariants	Symbol	α^p	α_1^p	α_2^p	Maximal Subgroups in Φ_2
(221)	a	β_2	β_1	1	$(22)^{p^2}(211)ac^{p-1}$
(221)	b	β_2	1	β_1	$(22)^{p^2}(211)c^p$
(221)	c	1	β_1	β_2	$(22)^{p^2-1}(211)a^{p+1}$
(221)	d_r	1	β_1^k	β_2	$(22)^{p^2-1}(211)a^2c^{p-1}$
(221)	e	1	β_1	$\beta_1\beta_2$	$(22)^{p^2-1}(211)ac^p$
(221)	f_o	1	β_2	β_1^{-v}	$(22)^{p^2-1}(211)c^{p+1}$
(221)	f_r	1	β_2	$\beta_1^k\beta_2$	$(22)^{p^2-1}(211)c^{p+1}$
(2111)	a	β_1	1	1	$(211)c^{p^2}a^p$
(2111)	b	1	β_1	1	$(211)c^{p^2-1}a^p(1^4)$
(2111)	c	1	1	β_1	$(211)c^{p^2}a^{p-1}(1^4)$

$$\hat{\alpha}_4(1^5) = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \beta_1, \quad \alpha^p = \alpha_1^p = \beta_1^p = 1 \ (i = 1, 2) \rangle$$

$$3(d) \quad \underline{G \in p^5 \Phi_5}$$

$$\Phi_5(2111) = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_1, \alpha_2] = [\alpha_3, \alpha_4] = \beta, \quad \alpha_1^p = \beta, \alpha_2^p = \alpha_3^p = \alpha_4^p = 1 \rangle$$

$$\Phi_5(1^5) = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_1, \alpha_2] = [\alpha_3, \alpha_4] = \beta, \quad \alpha_i^p = \beta^p = 1 \ (i = 1, \dots, 4) \rangle$$

$$3(e) \quad \underline{G \in p^5 \Phi_6} \ (p > 3)$$

The groups of $p^5 \Phi_6$ of exponent p^2 are given by:

$G = \langle \alpha_1, \alpha_2 \rangle$ where $[\alpha_1, \alpha_2] = \beta, [\beta, \alpha_i] = \beta_i, \alpha_i^p = \beta^p = \beta_i^p = 1 (i=1,2)$

and

Type Invariants	Symbol	α_1^p	α_2^p	Maximal Subgroups	
(221)	a	β_1	β_2	$\Phi_2(211)a^{p+1}$	
(221)	b_r	$\beta_1^{g^r}$	β_2	$\Phi_2(211)a^2c^{p-1}$	$r=1,2,\dots,\frac{1}{2}(p-1)$
(221)	c_1	β_1	$\beta_1\beta_2$	$\Phi_2(211)ac^p$	
(221)	c_v	β_1	$\beta_1^v\beta_2$	$\Phi_2(211)ac^p$	
(221)	d_o	β_2	β_1^{-v}	$\Phi_2(211)c^{p+1}$	
(221)	d_r	β_2	$\beta_1^k\beta_2$	$\Phi_2(211)c^{p+1}$	$4k=g^{2r+1}-1;$ $r=1,2,\dots,\frac{1}{2}(p-1)$
(2111)	a	β_1	1	$\Phi_2(211)ac^{p-1}\Phi_2(1^4)$	
(2111)	b_1	1	β_1	$\Phi_2(211)c^p\Phi_2(1^4)$	
(2111)	b_v	1	β_1^v	$\Phi_2(211)c^p\Phi_2(1^4)$	

$\Phi_6(1^5) = \langle \alpha_1, \alpha_2 | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_i] = \beta_i, \alpha_i^p = \beta^p = \beta_i^p = 1 (i=1,2) \rangle.$

$3(f) G \in p^5 \Phi_7$

$\Phi_7(2111) = \langle \alpha, \beta, \alpha_1 \rangle$ where $[\alpha_i, \alpha] = \alpha_{i+1} (i=1,2), [\alpha_1, \beta] = \alpha_3,$

$\alpha_2^p = \alpha_3^p = 1$ and

Symbol	α^p	β^p	$\alpha_1^p \alpha_3^{(p)}$	Maximal Subgroups	
				Φ_2	Φ_3
a	α_3	1	1	$(211)ab^{p-1}(1^4)$	$(211)a^{p^2}$
b_1	1	1	α_3	$(211)ab^{p-1}(1^4)$	$(211)b^{p^2}$
b_v	1	1	α_3^v	$(211)ab^{p-1}(1^4)$	$(211)b^{p^2}$
c	1	α_3	1	$(211)ab^p$	$(211)a^{p-1}b^{p(p-1)}(1^4)$

$$\Phi_7(1^5) = \langle \alpha, \beta, \alpha_1 \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3 \in Z(G), \alpha_1^p = \alpha_{i+1}^p = \beta^p = 1 (i=1, 2) \rangle$$

$$3(g)G \in p^5\Phi_8$$

$$\Phi_8(32) = \langle \alpha_1, \alpha_2 \mid \alpha_1^{p^3} = \alpha_2^{p^2} = 1, [\alpha_1, \alpha_2] = \alpha_1^p \rangle$$

$$3(h)G \in p^5\Phi_9$$

$$\Phi_9(2111)a = \langle \alpha, \alpha_1 \mid [\alpha_i, \alpha] = \alpha_{i+1} (i=1, 2, 3), \alpha^p = \alpha_4, \alpha_1^p \alpha_3^{(p)} = \alpha_2^p \alpha_4^{(p)} = \alpha_3^p = \alpha_4^p = 1 \rangle$$

$$\Phi_9(2111)b_o = \langle \alpha, \alpha_1 \mid [\alpha_i, \alpha] = \alpha_{i+1} (i=1, 2, 3), \alpha_1^p \alpha_3^{(p)} = \alpha_4, \alpha^p = \alpha_2^p \alpha_4^{(p)} = \alpha_3^p = \alpha_4^p = 1 \rangle$$

$$\Phi_9(2111)b_r = \langle \alpha, \alpha_1 \mid [\alpha_i, \alpha] = \alpha_{i+1} (i=1, 2, 3), \alpha_1^p \alpha_3^{(p)} = \alpha_4^k, \alpha^p = \alpha_2^p \alpha_4^{(p)} = \alpha_3^p = \alpha_4^p = 1 \rangle$$

$$k = g^r, \quad r = 1, 2, \dots, (p-1, 3)-1$$

(The first has an elementary abelian maximal subgroup).

$$\Phi_9(1^5) = \langle \alpha, \alpha_1 \mid [\alpha_i, \alpha] = \alpha_{i+1} (i=1, 2, 3), \alpha^p = \alpha_1^p \alpha_3^{(p)} = \alpha_2^p \alpha_4^{(p)} = \alpha_3^p = \alpha_4^p = 1 \rangle$$

$$3(i)G \in p^5\Phi_{10}$$

$$\Phi_{10}(2111)a_r = \langle \alpha, \alpha_1 \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_4, \alpha_1^p \alpha_{i+2}^{(p)} = \alpha_4^p = 1 (i=1, 2, 3), \alpha^p = \alpha_4^k \rangle$$

$$k = g^r, \quad r + 1 = 1, 2, \dots (p-1, 4).$$

$$\begin{aligned} \Phi_{10}(2111) \, b^r &= \langle \alpha, \alpha_1 | [\alpha_i, \alpha] = \alpha_{i+1} \ (i=1, 2, 3), [\alpha_1, \alpha_2] = \alpha_4, \alpha^p = \alpha_4^r, \dots \alpha_1^p \alpha_3^{(p)} = \\ &= \alpha_2^p \alpha_4^{(p)} = \alpha_3^p = \alpha_4^p = 1 \rangle \end{aligned}$$

$$k = g^r, \quad r+1 = 1, 2, \dots (p-1, 3)$$

$$\Phi_{10}(1^5) = \langle \alpha, \alpha_1 | [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_4, \alpha_i^p \alpha_{i+2}^{(p)} = \alpha^p = \alpha_4^p = 1 \ (i=1, 2, 3) \rangle$$

APPENDIX II

The Groups of Order p^6 ($p > 3$)

$G \in \Phi_2$

$$\Phi_2(51) = \langle \alpha, \alpha_1 | \alpha^{p^5} = \alpha_1^{p-1}, [\alpha_1, \alpha] = \alpha^{p^4} \rangle$$

As in the groups of order p^5 , we have:

$$\Phi_2(42)a_1 = \langle \alpha, \alpha_1 | \alpha^{p^4} = \alpha_1^{p-1}, [\alpha_1, \alpha] = \alpha^{p^3} \rangle$$

$$\Phi_2(42)a_2 = \langle \alpha, \alpha_1 | \alpha^{p^4} = \alpha_1^{p-1}, [\alpha_1, \alpha] = \alpha_1^{p^3} \rangle$$

$$\Phi_2(411)a = \Phi_2(41) \times (1)$$

$$\Phi_2(411)b = \langle \alpha, \alpha_1, \gamma | \alpha^p = \alpha_1^p = \gamma^{p^4} = 1, [\alpha_1, \alpha] = \gamma^{p^3} \rangle$$

$$\Phi_2(411)c = \langle \alpha, \alpha_1 | \alpha^{p^4} = \alpha_1^{p-1} \alpha_2^{p-1}, [\alpha_1, \alpha] = \alpha_2 \rangle$$

$$\Phi_2(321)a_1 = \Phi_2(32)a_1 \times (1)$$

$$\Phi_2(321)a_2 = \Phi_2(32)a_2 \times (1)$$

$$\Phi_2(321)b = \Phi_2(31) \times (2) \text{ Others are given by}$$

$$\Phi_2(321) = \langle \alpha, \alpha_1, \gamma_3 \rangle \text{ where } [\alpha_1, \alpha] = \alpha_2, \alpha_2^p = \gamma_1^{p^2} = \gamma_2^p = 1 \text{ and}$$

Symbol	α^p	α_1^p	γ_3^p	α_2	Maximal Subgroups in Φ_2
c	γ_1	1	γ_2	γ_2	$(32)^{p(p-1)}(311)c^p$
d	γ_2	1	γ_1	γ_1^p	$(32)^{p(p-1)}(311)a^{p-1}(221)c$
e	γ_2	1	γ_1	γ_2	$(32)^{p(p-1)}(311)c^{p-1}(221)b = \Phi_2(21) \times (3)$
f	γ_2	γ_1	1	γ_3	—

$$\Phi_2(3111)a = \Phi_2(311)a \times (1) = \Phi_2(31) \times (11)$$

$$\Phi_2(3111)b = \Phi_2(311)b \times (1)$$

$$\Phi_2(3111)c = \Phi_2(311)c \times (1)$$

$$\Phi_2(3111)d = \Phi_2(111) \times (3)$$

$$\Phi_2(222)a = \Phi_2(22) \times (2)$$

$$\Phi_2(222)b = \langle \alpha, \alpha_1, \gamma | \alpha^2 = \alpha_1^2 = \gamma^2 = 1, [\alpha_1, \alpha] = \gamma^2 \rangle$$

$$\Phi_2(2211)a = \Phi_2(221)a \times (1) = \Phi_2(22) \times (11)$$

$$\Phi_2(2211)b = \Phi_2(221)b \times (1) = \Phi_2(211)a \times (2) = \Phi_2(21) \times (21)$$

$$\Phi_2(2211)c = \Phi_2(221)c \times (1)$$

$$\Phi_2(2211)d = \Phi_2(221)d \times (1)$$

$$\Phi_2(2211)e = \Phi_2(211)b \times (2)$$

$$\Phi_2(2211)f = \Phi_2(211)c \times (2)$$

$$\Phi_2(21^4)a = \Phi_2(2111)a \times (1) = \Phi_2(21) \times (111)$$

$$\Phi_2(21^4)b = \Phi_2(2111)b \times (1) = \Phi_2(211)b \times (11)$$

$$\Phi_2(21^4)c = \Phi_2(2111)c \times (1) = \Phi_2(211)c \times (11)$$

$$\Phi_2(21^4)d = \Phi_2(2111)d \times (1) = \Phi_2(111) \times (21)$$

$$\Phi_2(1^6) = \Phi_2(1^5) \times (1) = \Phi_2(111) \times (111)$$

$$\underline{G \in \Phi_3}$$

$$\Phi_3(411)a = \langle \alpha, \alpha_1 \mid [\alpha_i, \alpha] = \alpha_{i+1}, \alpha_{i+1}^p = 1 (i=1,2), \alpha^p = \alpha_3, \alpha_1^p = 1 \rangle$$

$$\Phi_3(411)b_1 = \langle \alpha, \alpha_1 \mid [\alpha_i, \alpha] = \alpha_{i+1}, \alpha_{i+1}^p = 1 (i=1,2), \alpha^p = 1, \alpha_1^p = \alpha_3 \rangle$$

$$\Phi_3(411)b_v = \langle \alpha, \alpha_1 \mid [\alpha_i, \alpha] = \alpha_{i+1}, \alpha_{i+1}^p = 1 (i=1,2), \alpha^p = 1, \alpha_1^p = \alpha_3^v \rangle$$

(Last two have abelian maximal subgroups of exponent p^4)

$$\Phi_3(321) = \langle \alpha, \alpha_1 \rangle \text{ where } [\alpha_i, \alpha] = \alpha_{i+1}, \alpha_{i+1}^p = 1 (i=1,2) \gamma_2^p = 1 \text{ and}$$

Symbol	α^p	α_1^p	γ_1^p	Maximal Subgroups in Φ_2
a	γ_1	γ_2	α_3	$(311)a^p$
b_r	γ_1	α_3^r	γ_2	$(311)c^p$
c_r	γ_2	γ_1^r	α_3	$(311)a^{p-1}(221)c$
d	α_3	γ_1	γ_2	$(311)c^{p-1}(221)b$

$r=1, v$

$r=1, v$

$$\Phi_3(3111)a = \Phi_3(311)a \times (1)$$

$$\Phi_3(3111)b = \Phi_3(311)b \times (1)$$

$$\Phi_3(3111)b_v = \Phi_3(311)b_v \times (1)$$

$$\Phi_3(3111)c = \langle \alpha, \alpha_1, \gamma \mid [\alpha_i, \alpha] = \alpha_{i+1}, \alpha_{i+1}^p = 1 (i=1,2), \alpha^p = \alpha_1^p = 1, \gamma^p = \alpha_3 \rangle$$

$$\Phi_3(3111)d = \langle \alpha, \alpha_1 \mid [\alpha_i, \alpha] = \alpha_{i+1}, \alpha_{i+1}^p = 1 (i=1,2), \alpha^p = \alpha_1^p = 1 \rangle$$

$$\Phi_3(3111)e = \langle \alpha, \alpha_1 \mid [\alpha_i, \alpha] = \alpha_{i+1}, \alpha_{i+1}^p = 1 (i=1,2), \alpha^p = \alpha_1^p = 1 \rangle$$

(The first is the only one with maximal subgroups in Φ_3 and the last is the only one with a maximal subgroup in Φ_2 and exponent p^2).

$$\Phi_3(2211)a = \Phi_3(221)a \times (1)$$

$$\Phi_3(2211)b_1 = \Phi_3(221)b_1 \times (1)$$

$$\Phi_3(2211)b_v = \Phi_3(221)b_v \times (1) \quad \text{Others are given by}$$

$$\Phi_3(2211) = \langle \alpha, \alpha_1, \gamma_1 \rangle \text{ where } [\alpha_i, \alpha] = \alpha_{i+1}, \alpha_{i+1}^p = 1 (i=1,2), \gamma_2^p = 1 \text{ and}$$

Symbol	α^p	α_1^p	γ_1^p	Maximal Subgroups in Φ_3
c	α_3	1	γ_2	$(221)a^{p(p-1)}(2111)a \quad = \Phi_3(211)a \times (2)$
d_r	1	α_3^r	γ_2	$(221)a^{(p-1)^2}b^{p-1}(2111)be^{p-1} \quad = \Phi_3(211)b_r \times (2)$
e	γ_2	1	α_3	$(221)b^{(p-1)^2}a^{p-1}(2111)d^p$
f	1	γ_2	α_3	$(221)b^{(p-1)^2}a^{p-1}(2111)e^p$
g	γ_1	γ_2	1	_____

$$\Phi_3(21^4)a = \Phi_3(2111)a \times (1) = \Phi_3(211)a \times (11)$$

$$\Phi_3(21^4)b_1 = \Phi_3(2111)b_1 \times (1) = \Phi_3(211)b_1 \times (11)$$

$$\Phi_3(21^4)b_v = \Phi_3(2111)b_v \times (1) = \Phi_3(211)b_v \times (11)$$

$$\Phi_3(21^4)c = \Phi_3(2111)c \times (1)$$

$$\Phi_3(21^4)d = \Phi_3(2111)d \times (1)$$

$$\Phi_3(21^4)e = \Phi_3(2111)e \times (1)$$

$$\Phi_3(21^4)f = \Phi_3(1^4) \times (2)$$

$$\Phi_3(1^6) = \Phi_3(1^5) \times (1) = \Phi_3(1^4) \times (11)$$

$$G \in \Phi_4$$

The groups in $p^6\Phi_4$ of exponent p^3 are given by:

$$G = \langle \alpha, \alpha_1, \alpha_2 \rangle \text{ where } [\alpha_i, \alpha] = \beta_i, \beta_i^{p-1} = 1 (i=1, 2), \gamma^p = \beta_1 \text{ and}$$

Type Invariants	Sym- bol	α^p	α_1^p	α_2^p	Maximal Subgroups in Φ_2
(321)	a	γ	1	β_2	$(32)^{p^2} (311) a^p$
(321)	b	γ	β_2	1	$(32)^{p^2} (311) c^p$
(321)	c	β_2	1	γ	$(32)^{p^2} (311) a^{p-1} (221) c$
(321)	d	β_2	γ	1	$(32)^{p^2} (311) c^{p-1} (221) b$
(321)	e_r	1	γ	β_2^r	$(32)^{p^2-1} (311) a c^{p-1} (221) b$
(321)	f_1	1	β_2	γ	$(32)^{p^2-1} (311) c^p (221) c$
(321)	f_v	1	β_2^v	γ	$(32)^{p^2-1} (311) c^p (221) c$
(3111)	a	γ	1	1	$(311) a^p c^{p^2}$
(3111)	b	1	γ	1	$(311) a^p c^{p^2-1} (2111) d$
(3111)	c	1	1	γ	$(311) a^{p-1} c^{p^2} (2111) b$

$r \neq 0$

$$\Phi_4(222) = \langle \alpha, \alpha_1, \alpha_2 \rangle \text{ where } [\alpha_i, \alpha] = \beta_i, \gamma^p = \beta_i^{p-1} = 1 (i=1, 2) \text{ and}$$

Symbol	α^p	α_1^p	α_2^p	Maximal Subgroups in Φ_2
a	γ	β_1	β_2	$(221) a^{p^2+p}$
b_r	γ	$\beta_1^{g^r}$	β_2	$(221) a^{2p} d^{p(p-1)}$
c	β_1	γ	β_2	$(221) a^{p+1} d^{p^2-1}$
d	γ	β_1	$\beta_1 \beta_2$	$(221) a^p d^{p^2}$
e	β_2	γ	β_1	$(221) a d^{p^2+p-1}$
f_o	γ	β_2	β_1^{-v}	$(221) d^{p^2+p}$
f_r	γ	β_2	$\beta_1^k \beta_2$	$(221) d^{p^2+p}$

$r=1, 2, \dots$

$$\begin{cases} 4k = g^{2r+1} - 1 \\ r=1, 2, \dots, \frac{1}{2}(p-1) \end{cases}$$

$$\Phi_4(2211)a = \Phi_4(221)a \times (1)$$

$$\Phi_4(2211)b = \Phi_4(221)b \times (1)$$

$$\Phi_4(2211)c = \Phi_4(221)c \times (1)$$

$$\Phi_4(2211)d_r = \Phi_4(221)d_r \times (1)$$

$$\Phi_4(2211)e = \Phi_4(221)e \times (1)$$

$$\Phi_4(2211)f_r = \Phi_4(221)f_r \times (1)$$

$$\Phi_4(2211)g = \langle \alpha, \alpha_1, \alpha_2, \gamma | [\alpha_1, \alpha] = \beta_1; \beta_1, \gamma \in Z(G), \gamma^p = \beta_2, \alpha^p = \beta_1, \alpha_1^p = \beta_1^{p-1} \rangle$$

$$\Phi_4(2211)h = \langle \alpha, \alpha_1, \alpha_2, \gamma | [\alpha_1, \alpha] = \beta_1; \beta_1, \gamma \in Z(G), \gamma^p = \beta_2, \alpha_1^p = \beta_1, \alpha^p = \alpha_2^p = \beta_1^{p-1} \rangle$$

$$\Phi_4(2211)i = \langle \alpha, \alpha_1, \alpha_2, \gamma | [\alpha_1, \alpha] = \beta_1; \beta_1, \gamma \in Z(G), \gamma^p = \beta_2, \alpha_2^p = \beta_1, \alpha^p = \alpha_1^p = \beta_1^{p-1} \rangle$$

(Note: All of the above have maximal subgroups in Φ_4 , g has $p^2(p-1)$ maximal subgroups $\Phi_4(221)a$, while i has none.)

For the rest, we have

$$\Phi_4(2211) = \langle \alpha, \alpha_1, \alpha_2 \rangle \text{ where } [\alpha_1, \alpha] = \beta_i \in Z(G), \beta_i^p = \gamma^{p-1} \ (i=1,2) \text{ and}$$

Symbol	α^p	α_1^p	α_2^p	Maximal Subgroups in Φ_2	Maximal Quotients in Φ_4
j_1	γ	β_1	1	$(221)a^p d^{p^2-p} (2111)c^p$	$(221)a^{p^2-p} (2111)b^p$
j_2	β_1	γ	1	$(221)a^p d^{p^2-p} (2111)c^p$	$(221)b^{p^2-p} (2111)ab^{p-1}$
k	β_2	γ	1	$(221)d^{p^2} (2111)ac^{p-1}$	
l	γ	1	β_1	$(221)d^{p^2} (2111)c^p$	
m	1	γ	β_2	$(221)a^{p-1} d^{p^2-p} (2111)ac^p$	
n	1	γ	β_1	$(221)a^{p-1} d^{p^2-p} (2111)c^{p+1}$	

$$\Phi_4(21^4)a = \Phi_4(2111)a \times (1)$$

$$\Phi_4(21^4)b = \Phi_4(2111)b \times (1)$$

$$\Phi_4(21^4)c = \Phi_4(2111)c \times (1)$$

$$\Phi_4(21^4)d = \langle \alpha, \alpha_1, \alpha_2, \gamma \mid [\alpha_1, \alpha] = \beta_1, \gamma = \beta_1, \alpha_i^p = \beta_i^p = 1 (i = 1, 2) \rangle$$

(This is the only one, except for the direct products above, with proper subgroups in Φ_4). The others are $\Phi_4(21^4) = \langle \alpha, \alpha_1, \alpha_2 \rangle$

where $[\alpha_1, \alpha] = \beta_1, \gamma^p = \alpha_2^p = \beta_1^p = 1 (i=1, 2)$ and

Symbol	α^p	α_1^p	Maximal Subgroups in Φ_2
e	γ	1	$(2111)c^{p^2+p}$
f	1	γ	$(2111)c^{p^2+p-1}(1^5)$

$$\Phi_4(1^6) = \Phi_4(1^5) \times (1)$$

$$G \in \Phi_5$$

$$\Phi_5(311) = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_1, \alpha_2] = [\alpha_3, \alpha_4] = \beta, \alpha_1^p = \beta, \alpha_2^p = \alpha_3^p = \alpha_4^p = \beta^p = 1 \rangle$$

$$\Phi_5(2211)a = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_1, \alpha_2] = [\alpha_3, \alpha_4] = \beta, \alpha_2^p = \beta, \alpha_1^p = \alpha_3^p = \alpha_4^p = 1 \rangle$$

$$\Phi_5(2211)b = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_1, \alpha_2] = [\alpha_3, \alpha_4] = \beta, \alpha_3^p = \beta, \alpha_1^p = \alpha_2^p = \alpha_4^p = \beta^p = 1 \rangle$$

(The first has a subgroup of the form $\Phi_2(221)a$, but the second does not.)

$$\Phi_5(21^4)a = \Phi_5(2111) \times (1)$$

$$\Phi_5(21^4)b = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \gamma \mid [\alpha_1, \alpha_2] = [\alpha_3, \alpha_4] = \beta, \gamma = \beta, \alpha_i^p = \beta^p = 1 (i=1, 2, 3, 4) \rangle$$

$$\Phi_5(21^4)c = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_1, \alpha_2] = [\alpha_3, \alpha_4] = \beta, \alpha_i^p = \alpha_2^p = \alpha_3^p = \alpha_4^p = \beta^p = 1 \rangle$$

(The second has subgroups in Φ_5 but the third does not.)

$$\Phi_5(1^6) = \Phi_5(1^5) \times (1)$$

$$\underline{G \in \Phi_6}$$

The groups of $p^6\Phi_6$ of exponent p^3 are given by:

$G = \langle \alpha_1, \alpha_2 \rangle$ where $[\alpha_1, \alpha_2] = \beta, [\beta, \alpha_i] = \beta_i, \gamma^p = \beta_1, \beta^p = \beta_i, \beta_i^p = 1$ ($i=1,2$) and

Type Invariants	Symbol	α_1^p	α_2^p	Maximal Subgroups in Φ_2	
(321)	a_r	γ	β_2^r	$(311)ac^{p-1}(221)b$	$r=1,2,\dots,p-1$
(321)	b_{11}	β_2	γ	$(311)c^p(221)c$	
(321)	b_{1v}	β_2	γ^v	$(311)c^p(221)c$	
(321)	b_{v1}	β_2^v	γ	$(311)c^p(221)c$	
(321)	b_{vv}	β_2^v	γ^v	$(311)c^p(221)c$	
(3111)	a	γ	1	$(311)ac^{p-1}(2111)d$	
(3111)	b_1	1	γ	$(311)c^p(2111)b$	
(3111)	b_v	1	γ^v	$(311)c^p(2111)b$	

$$\Phi_6(2211)a = \Phi_6(221)a \times (1)$$

$$\Phi_6(2211)b_r = \Phi_6(221)b_r \times (1)$$

$$\Phi_6(2211)c_r = \Phi_6(221)c_r \times (1)$$

$$\Phi_6(2211)d_r = \Phi_6(221)d_r \times (1)$$

$$\Phi_6(2211) = \langle \alpha_1, \alpha_2, \gamma \rangle \text{ where } [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_i] = \beta_i, \beta^p = \beta_i, \beta_i^p = 1 (i=1,2) \text{ and}$$

$$\Phi_6(21^4)a = \Phi_6(2111)a \times (1)$$

$$\Phi_6(21^4)b_r = \Phi_6(2111)b_r \times (1)$$

$$\Phi_6(21^4)c = \langle \alpha_1, \alpha_2, \gamma \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_i] = \beta_i \in Z(G), \gamma \in Z(G), \gamma^p = \beta_1, \alpha_i^p = \beta_i^{p-1} \rangle$$

Symbol	α_1^p	α_2^p	γ^p	Maximal Subgroups Φ_6	Maximal Subgroups Φ_2
e	β_1	1	β_2	$(2111)a^{p^2}$	$(221)b^p(2111)b$
f_1	1	β_1	β_2	$(2111)b^{p^2}$	$(221)b^{p-1}c(2111)d$
f_v	1	β_1^v	β_2	$(2111)b^{p^2}$	$(221)b^{p-1}c(2111)d$
g	γ	β_2	1	—	$(2111)ac^p$
h_1	γ	β_1	1	—	$(2111)c^{p+1}$
h_v	γ	β_1^v	1	—	$(2111)c^{p+1}$

$\Phi_6(21^4)d = \langle \alpha_1, \alpha_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_i] = \beta_i \in Z(G), \alpha_1^p = \gamma, \alpha_2^p = \beta, \beta_i^p = \beta_i, \gamma^p = 1 \rangle$
(This is the only group of $\Phi_6(21^4)$ with no subgroups in Φ_6).

$$\Phi_6(1^6) = \Phi_6(1^5) \times (1)$$

$G \in \Phi_7$

As in the case of $p^5\Phi_7$, we write:

$$\Phi_7(3111)a = \langle \alpha, \alpha_1, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, \alpha^{p^2} = \alpha_3, \alpha_{i+1}^p = \beta^p, \alpha_1^p = 1 \rangle$$

$$(i=1, 2)$$

$$\Phi_7(3111)b_1 = \langle \alpha, \alpha_1, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, \alpha_1^{p^2} = \alpha_3, \alpha_i^p = \beta^p, \alpha_{i+1}^p = 1 \rangle$$

$$(i=1, 2)$$

$$\Phi_7(3111)b_v = \langle \alpha, \alpha_1, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, \alpha_1^{p^2} = \alpha_3^v, \alpha^p = \beta^p, \alpha_{i+1}^p = 1 \rangle$$

$$(i=1, 2)$$

$$\Phi_7(3111)c = \langle \alpha, \alpha_1, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, \beta^{p^2} = \alpha_3, \alpha^p = \alpha_1^p, \alpha_{i+1}^p = 1 \rangle$$

$$(i=1, 2)$$

Symbol	α_1^p	α_2^p	γ^p	Maximal Subgroups Φ_6	Maximal Subgroups Φ_2
e	β_1	1	β_2	$(2111)a^{p^2}$	$(221)b^p(2111)b$
f_1	1	β_1	β_2	$(2111)b^{p^2}$	$(221)b^{p-1}c(2111)d$
f_v	1	β_1^v	β_2	$(2111)b^{p^2}$	$(221)b^{p-1}c(2111)d$
g	γ	β_2	1	—	$(2111)ac^p$
h_1	γ	β_1	1	—	$(2111)c^{p+1}$
h_v	γ	β_1^v	1	—	$(2111)c^{p+1}$

$$\Phi_6(21^4)a = \Phi_6(2111)a \times (1)$$

$$\Phi_6(21^4)b_r = \Phi_6(2111)b_r \times (1)$$

$$\Phi_6(21^4)c = \langle \alpha_1, \alpha_2, \gamma \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1 \in Z(G), \gamma \in Z(G), \gamma^p = \beta_1, \alpha_1^p = \beta_1^{p-1} \rangle$$

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As in the case of $p^5\Phi_7$, we write:

$$\Phi_7(3111)a = \langle \alpha, \alpha_1, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, \alpha^{p^2} = \alpha_3, \alpha_{i+1}^p = \beta^p \alpha_1^{p-1} = 1 \quad (i=1,2) \rangle$$

$$\Phi_7(3111)b_1 = \langle \alpha, \alpha_1, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, \alpha_i^{p^2} = \alpha_3, \alpha^{p^2} = \beta^p \alpha_{i+1}^p = 1 \quad (i=1,2) \rangle$$

$$\Phi_7(3111)b_v = \langle \alpha, \alpha_1, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, \alpha_1^{p^2} = \alpha_3^v, \alpha^p = \beta^p \alpha_{i+1}^p = 1 \quad (i=1,2) \rangle$$

$$\Phi_7(3111)c = \langle \alpha, \alpha_1, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, \beta^{p^2} = \alpha_3, \alpha^p = \alpha_1^p = \alpha_{i+1}^p = 1 \quad (i=1,2) \rangle$$

$$\Phi_7(2211) = \langle \alpha, \alpha_1, \beta \rangle \text{ where } [\alpha_i, \alpha] = \alpha_{i+1} (i=1,2) [\alpha_1, \beta] = \alpha_3 \alpha_2^p = \alpha_3^p = \gamma^p = 1$$

and

Sym- bol	α^p	α_1^p	β^p	Maximal	Subgroups
				Φ_2	Φ_3
a	α_3	γ	1	$(2111)ac^p$	$(221)a^{p^2}$
b_1	γ	α_3	1	$(2111)ac^p$	$(221)b^{p^2}$
b_v	γ	α_3^v	1	$(2111)ac^p$	$(221)b^{p^2}$
c	α_3	1	γ	$(221)b^p(2111)c$	$(221)a^{p^2-p}(2111)ad^{p-1}$
d_1	1	α_3	γ	$(221)ab^{p-1}(2111)d$	$(221)a^{(p-1)^2}b^{p-1}(2111)be^{p-1}$
d_v	1	α_3^v	γ	$(221)ab^{p-1}(2111)d$	$(221)a^{(p-1)^2}b^{p-1}(2111)be^{p-1}$
e	γ	1	α_3	$(221)c^p(2111)a$	$(221)^{p^2-p}(2111)d^p$
f	1	γ	α_3	$(221)ac^{p-1}(2111)b$	$(221)b^{p^2-p}(2111)e^p$

$$\Phi_7(21^4)a = \Phi_7(2111)a \times (1)$$

$$\Phi_7(21^4)b_r = \Phi_7(2111)b_r \times (1)$$

$$\Phi_7(21^4)c = \Phi_7(2111)c \times (1)$$

$$\Phi_7(21^4)d = \langle \alpha, \alpha_1, \beta, \gamma | [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, \gamma^p = \alpha_3, \alpha^p = \beta^p = \alpha_1^p = \alpha_{i+1}^p = 1 (i=1,2) \rangle$$

$$\Phi_7(21^4)e = \langle \alpha, \alpha_1, \beta | [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, \alpha^p = \alpha_1^p = \beta^p = \alpha_{i+1}^p = 1 (i=1,2) \rangle$$

$$\Phi_7(21^4)f = \langle \alpha, \alpha_1, \beta | [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, \alpha^p = \alpha_1^p = \beta^p = \alpha_{i+1}^p \neq 1 (i=1,2) \rangle$$

$$\Phi_7(21^4)g = \langle \alpha, \alpha_1, \beta | [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, \alpha^p = \alpha_1^p = \beta^p = \alpha_{i+1}^p \neq 1 (i=1,2), \rangle$$

(same as in $p^5\Phi_7$)

$$\Phi_7(1^6) = \Phi_7(1^5) \times (1)$$

$$G \in \Phi_3$$

$$\Phi_3(42) = \langle \alpha_1, \alpha_2 | \alpha_1^{p^3} = 1, \alpha_2^{p^3} = \alpha_1^{p^2}, [\alpha_1, \alpha_2] = \alpha_1^p \rangle$$

$$\Phi_3(33) = \langle \alpha_1, \alpha_2 | \alpha_1^{p^3} = \alpha_2^{p^3} = 1, [\alpha_1, \alpha_2] = \alpha_1^p \rangle$$

$$\Phi_3(321) = \Phi_3(32) \times (1)$$

$$\Phi_3(321)b = \langle \alpha_1, \alpha_2, \gamma | \alpha_1^{p^3} = \alpha_2^{p^2} = 1, \gamma^p = \alpha_1^{p^2}, [\alpha_1, \alpha_2] = \alpha_1^p \rangle$$

The others are given by:

$$\Phi_3(321) = \langle \alpha_1, \alpha_2 \rangle \text{ where } [\alpha_1, \alpha_2] = \beta, \beta^{p^2} = 1 \text{ and}$$

Sym- bol	α_1^p	α_2^p	γ^p	Maximal Subgroups	Maximal Quotients
c_o	$\beta\gamma$	1	1	$\Phi_2(311)a^p(221)a$	$\Phi_6(32)^p\Phi_2(221)d$
c_r	$\beta\gamma^r$	1	β^p	"	$\Phi_2(221)d$ $r=1,2,\dots$ $p-2$
c_{p-1}	$\beta\gamma^{-1}$	β^p	β^p	"	$\Phi_2(221)d$

$$\Phi_3(222) = \langle \alpha_1, \alpha_2 | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_2] = \beta^p, [\beta, \alpha_1] = 1, \alpha_1^{p^2} = \alpha_2^{p^2} = \beta^{p^2} = 1 \rangle$$

$$G \in \Phi_9$$

As in the case of Φ_3 , the groups of Φ_9 are:

$$\Phi_9(3111)a = \langle \alpha, \alpha_1 | [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^{p^2} = \alpha_4, \alpha_1^p = \alpha_{i+1}^p = 1 (i=1,2,3) \rangle$$

$$\Phi_9(3111)b_k = \langle \alpha, \alpha_1 | [\alpha_i, \alpha] = \alpha_{i+1}, \alpha_1^{p^2} = \alpha_4^r, \alpha^p = \alpha_{i+1}^p = 1 (i=1,2,3) \rangle$$

$$\text{where } r = g^k, k+1 = 1, 2, \dots, (p-1, 3)$$

$$\Phi_9(2211)a = \langle \alpha, \alpha_1 | [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_4, \alpha_1^{p^2} = \alpha_{i+1}^p = 1 (i=1,2,3) \rangle$$

$$\Phi_9(2211)b_k = \langle \alpha, \alpha_1 | [\alpha_i, \alpha] = \alpha_{i+1}, \alpha_1^p = \alpha_4^r, \alpha^{p^2} = \alpha_{i+1}^p = 1 (i=1,2,3) \rangle$$

$$\text{where } r = g^k, k+1 = 1, 2, \dots, (p-1, 3)$$

$$\Phi_9(21^4)a = \Phi_9(2111)a \times (1)$$

$$\Phi_9(21^4)b_k = \Phi_9(2111)b_k \times (1)$$

$$\Phi_9(21^4)c = \langle \alpha, \alpha_1, \gamma | [\alpha_1, \alpha] = \alpha_{i+1}, \gamma^p = \alpha_4, \alpha^p = \alpha_1^p = \alpha_{i+1}^p = 1 (i=1, 2, 3) \rangle$$

$$\Phi_9(21^4)d = \langle \alpha, \alpha_1, | [\alpha_1, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_1^p = \alpha_{i+1}^p = 1 (i=1, 2, 3) \rangle$$

$$\Phi_9(21^4)e = \langle \alpha, \alpha_1, | [\alpha_1, \alpha] = \alpha_{i+1}, \alpha_1^{p^2} = \alpha^p = \alpha_{i+1}^p = 1 (i=1, 2, 3) \rangle$$

$$\Phi_9(1^6) = \Phi_9(1^5) \times (1)$$

$$\underline{G \in \Phi_{10}}$$

As in the case of Φ_9 , the groups of Φ_{10} are:

$$\Phi_{10}(3111)a_m = \langle \alpha, \alpha_1 | [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_4, \alpha^{p^2} = \alpha_4^r, \alpha_1^p = \alpha_{i+1}^p = 1 (i=1, 2, 3) \rangle$$

$$\text{where } r = g^m, m+1 = 1, 2, \dots, (p-1, 4)$$

$$\Phi_{10}(3111)b_n = \langle \alpha, \alpha_1 | [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_4, \alpha_1^{p^2} = \alpha_4^s, \alpha^p = \alpha_{i+1}^p = 1 (i=1, 2, 3) \rangle$$

$$\text{where } s = g^n, n+1 = 1, 2, \dots, (p-1, 3)$$

$$\Phi_{10}(2211)a_m = \langle \alpha, \alpha_1 | [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_4, \alpha^p = \alpha_4^r, \alpha_1^{p^2} = \alpha_{i+1}^p = 1 (i=1, 2, 3) \rangle$$

$$\text{where } r = g^m, m+1 = 1, 2, \dots, (p-1, 4)$$

$$\Phi_{10}(2211)b_n = \langle \alpha, \alpha_1 | [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_4, \alpha_1^{p^2} = \alpha_4^s, \alpha^p = \alpha_{i+1}^p = 1 (i=1, 2, 3) \rangle$$

$$\text{where } s = g^n, n+1 = 1, 2, \dots, (p-1, 3)$$

$$\Phi_{10}(21^4)a_m = \Phi_{10}(2111)a_m \times (1)$$

$$\Phi_{10}(21^4)b_n = \Phi_{10}(2111)b_n \times (1)$$

$$\Phi_{10}(21^4)c = \langle \alpha, \alpha_1, \gamma | [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_4, \gamma^p = \alpha_4, \alpha^p = \alpha_1^p = \alpha_{i+1}^p = 1 (i=1, 2) \rangle$$

$$\Phi_{10}(21^4)d = \langle \alpha, \alpha_1 | [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_4, \alpha^{p^2} = \alpha_1^p = \alpha_{i+1}^p = 1 (i=1, 2, 3) \rangle$$

$$\Phi_{10}(21^4)e = \langle \alpha, \alpha_1 | [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_4, \alpha_1^{p^2} = \alpha^p = \alpha_{i+1}^p = 1 (i=1, 2, 3) \rangle$$

$$\Phi_{10}(1^6) = \Phi_{10}(1^5) \times (1)$$

$$\underline{G \in \Phi_{11}}$$

$$\Phi_{11}(222) = \langle \alpha_1, \alpha_2, \alpha_3 \rangle \text{ where } [\alpha_1, \alpha_2] = \beta_3, [\alpha_2, \alpha_3] = \beta_1, [\alpha_3, \alpha_1] = \beta_2, \text{ and}$$

$$\begin{aligned}\alpha_1^p &= \beta_1 & \beta_2^{-1} & \beta_1\beta_2^{-1} & \beta_1^v\beta_2^{-1} & \beta_1^r\beta_2^{-1} & \beta_2^{-1}\beta_3 \\ \alpha_2^p &= \beta_2 & \beta_1 & \beta_1 & \beta_1 & \beta_1\beta_2 & \beta_1\beta_2 \\ \alpha_3^p &= \beta_3 & \beta_3 & \beta_3 & \beta_3 & \beta_3 & \beta_1\end{aligned}$$

where $r = 1, 2, \dots, p-2$

$$\Phi_{11}(2211) = \langle \alpha_1, \alpha_2, \alpha_3 \rangle \text{ where } [\alpha_1, \alpha_2] = \beta_3, [\alpha_2, \alpha_3] = \beta_1, [\alpha_3, \alpha_1] = \beta_2, \alpha_3^p = 1$$

and

$$\begin{aligned}\alpha_1^p &= \beta_1 & \beta_1 & \beta_1 & \beta_2^{-1} & \beta_1\beta_2^{-1} & \beta_2^{-1}\beta_3^2 & \beta_1^r\beta_2^{-1} \\ \alpha_2^p &= \beta_2 & \beta_2^v & \beta_2\beta_3^{-2} & \beta_1 & \beta_1 & \beta_1 & \beta_1\beta_2\end{aligned}$$

where $r = 1, 2, \dots, p-2$

$$\Phi_{11}(21^4) = \langle \alpha_1, \alpha_2, \alpha_3 \rangle \text{ where } [\alpha_1, \alpha_2] = \beta_3, [\alpha_2, \alpha_3] = \beta_1, [\alpha_3, \alpha_1] = \beta_2,$$

$$\alpha_2^p = \alpha_3^p = 1 \text{ and } \alpha_1^p = \beta_1 \text{ or } \beta_2.$$

$$\Phi_{11}(1^6) = \langle \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha_2] = \beta_3, [\alpha_2, \alpha_3] = \beta_1, [\alpha_3, \alpha_1] = \beta_2, \alpha_i^p = \beta_i^p = 1 (i=1, 2, 3) \rangle$$

$$\underline{G \in \Phi_{12}}$$

$$\Phi_{12}(2211) = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle \text{ where } [\alpha_1, \alpha_2] = \beta_1, [\alpha_3, \alpha_4] = \beta_2, \text{ and}$$

$$\alpha_1^p = \beta_1^{*1} \beta_1 \beta_1 \beta_1 \beta_1 \beta_1 \beta_1 \beta_1 \beta_2 \beta_2 \beta_1 \beta_2$$

$$\alpha_2^p = 1 \beta_2 \beta_2 \beta_2 \beta_2 \beta_2 \beta_2 1 1 1$$

$$\alpha_3^p = \beta_2 \beta_2 \beta_2 1 \beta_1 \beta_1 \beta_2 \beta_2 \beta_1 \beta_1$$

$$\alpha_4^p = 1 1 \beta_1 1 1 1 1 1 1 1$$

$$\Phi_{12}(21^4) = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle \text{ where } [\alpha_1, \alpha_2] = \beta_1, [\alpha_3, \alpha_4] = \beta_2, \alpha_2^p = \alpha_4^p = 1 \text{ and}$$

$$\alpha_1^p = \beta_1^{*2} \beta_2 \beta_1 \beta_1 \beta_2 \beta_1 \beta_2$$

$$\alpha_3^p = 1 1 \beta_1 1 \beta_1 \beta_2$$

*1 This is the group $\Phi_2(21) \times \Phi_2(21)$

*2 This is the group $\Phi_2(21) \times \Phi_2(1^3)$

$$\Phi_{12}(1^6) = \Phi_2(1^3) \times \Phi_2(1^3)$$

$$\underline{G \in \Phi_{13}}$$

$$\Phi_{13}(2211) = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle \text{ where } [\alpha_1, \alpha_2] = \beta_1, [\alpha_1, \alpha_3] = [\alpha_2, \alpha_4] = \beta_2, \text{ and}$$

$$\alpha_1^p = \begin{matrix} \beta_1 & \beta_1 & \beta_1 & 1 & 1 & 1 & \beta_2 \end{matrix}$$

$$\alpha_2^p = \begin{matrix} 1 & \beta_2 & 1 & \beta_2 & \beta_2^v & 1 & 1 \end{matrix}$$

$$\alpha_3^p = \begin{matrix} \beta_2 & 1 & 1 & \beta_1 & \beta_1 & \beta_1 & \beta_1 \end{matrix}$$

$$\alpha_4^p = \begin{matrix} 1 & 1 & \beta_2^r & 1 & 1 & \beta_2 & 1 \end{matrix}$$

$$\text{where } r = 1, 2, \dots, p-1$$

$$\Phi_{13}(21^4) = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle \text{ where } [\alpha_1, \alpha_2] = \beta_1, [\alpha_1, \alpha_3] = [\alpha_2, \alpha_4] = \beta_2,$$

$$\alpha_2^p = \alpha_4^p = 1 \text{ and}$$

$$\alpha_1^p = \begin{matrix} \beta_1 & \beta_2 & 1 & 1 \end{matrix}$$

$$\alpha_3^p = \begin{matrix} 1 & 1 & \beta_2 & \beta_1 \end{matrix}$$

$$\Phi_{13}(1^6) = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_1, \alpha_2] = \beta_1, [\alpha_1, \alpha_3] = [\alpha_2, \alpha_4] = \beta_2,$$

$$\alpha_i^p = \beta_j^p = 1 (i=1, \dots, 4; j=1, 2) \rangle$$

$$\underline{G \in \Phi_{14}}$$

$$\Phi_{14}(42) = \langle \alpha, \beta \mid [\alpha, \beta] = \alpha^p{}^2, \alpha^p{}^4 = \beta^p{}^2 = 1 \rangle$$

$$\Phi_{14}(321) = \langle \alpha, \beta, \gamma \mid [\alpha, \beta] = \alpha^p \gamma, \alpha^p{}^3 = \beta^p{}^2 = \gamma^p{}^2 = 1 \rangle$$

$$\Phi_{14}(222) = \langle \alpha, \beta, \gamma \mid [\alpha, \beta] = \gamma, \alpha^p{}^2 = \beta^p{}^2 = \gamma^p{}^2 = 1 \rangle$$

$$\underline{G \in \Phi_{15}}$$

$$\Phi_{15}(2211) = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle \text{ where } [\alpha_1, \alpha_2] = [\alpha_3, \alpha_4] = \beta_1, [\alpha_2, \alpha_4] = [\alpha_1, \alpha_2]^v = \beta_2^v, \alpha_3^p = 1 \text{ and}$$

$$\begin{aligned}\alpha_1^p &= \beta_1 & \beta_1 & \beta_1 \beta_2^a \\ \alpha_2^p &= \beta_2 & 1 & \beta_2^d \quad (d \neq 0) \\ \alpha_4^p &= 1 & \beta_2^r & 1\end{aligned}$$

where $r = g^k$ ($k=1,2,\dots,\frac{1}{2}(p+1)$), $a = 1,2,\dots,\frac{1}{2}(p-1)$ and, for a given value of a , d and d^* determine isomorphic groups whenever $d = d^*$ or $dd^* = v(v-a^2)$.

$$\begin{aligned}\Phi_{15}(21^4) &= \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_1, \alpha_2] = [\alpha_3, \alpha_4] = \beta_1, [\alpha_2, \alpha_4] = [\alpha_1, \alpha_3]^v = \beta_2^v, \alpha_1^p = \\ &= \beta_1, \alpha_1^p = \beta_1^p = \beta_2^p = 1 \quad (i=1,2,3,4)\end{aligned}$$

$$\begin{aligned}\Phi_{15}(1^6) &= \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_1, \alpha_2] = [\alpha_3, \alpha_4] = \beta_1, [\alpha_2, \alpha_4] = [\alpha_1, \alpha_3]^v = \beta_2^v, \\ &\alpha_1^p = \beta_1^p = \beta_2^p = 1 \quad (i=1,2,3,4) \rangle\end{aligned}$$

$$\underline{G \in \Phi_{16}}$$

$$\Phi_{16}(2211) = \langle \alpha, \alpha_1, \beta \rangle \text{ where } [\alpha_i, \alpha] = \alpha_{i+1}, [\beta, \alpha] = \gamma, \alpha_{i+1}^p = \gamma^p = 1 \quad (i=1,2)$$

and,

$$\begin{aligned}\alpha^p &= \alpha_3 & 1 & \gamma & \gamma & \alpha_3 & \gamma & 1 \\ \alpha_1^p &= 1 & \alpha_3^a & \alpha_3^v & \alpha_3^v & \gamma & 1 & \gamma^r \\ \beta^p &= \gamma & \gamma & 1 & 1 & 1 & \alpha_3 & \alpha_3\end{aligned}$$

where $r = g^k$ ($k+1 = 1,2,\dots,(p-1, \dots)$) and $a = 1,2,\dots,p-1$.

$$\Phi_{16}(21^4) = \langle \alpha, \alpha_1, \beta \rangle \text{ where } [\alpha_i, \alpha] = \alpha_{i+1}, [\beta, \alpha] = \gamma, \alpha_{i+1}^p = \gamma^p = 1 \quad (i=1,2) \text{ and}$$

$$\begin{aligned}\alpha^p &= \alpha_3 & \gamma & 1 & 1 & 1 & 1 & 1 \\ \alpha_1^p &= 1 & 1 & \alpha_3 & \alpha_3^v & 1 & \gamma & 1 \\ \beta^p &= 1 & 1 & 1 & 1 & \gamma & 1 & \alpha_3\end{aligned}$$

$$\Phi_{16}(1^6) = \langle \alpha, \alpha_1, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\beta, \alpha] = \gamma, \alpha^p = \alpha_1^p = \beta^p = \gamma^p = \alpha_{i+1}^p = 1 \quad (i=1,2) \rangle$$

$$\underline{G \in \Phi_{17}}$$

$$\Phi_{17}(2211) = \langle \alpha, \alpha_1, \beta \rangle \text{ where } [\alpha_i, \alpha] = \alpha_{i+1}, [\beta, \alpha_1] = \gamma, \alpha_{i+1}^p = \gamma^p = 1 (i=1, 2) \text{ and}$$

$$\alpha^p = \alpha_3 \quad \alpha_3 \quad \alpha_3 \quad 1 \quad 1 \quad \alpha_3 \quad \alpha_3 \quad \alpha_3 \quad \alpha_3 \quad 1 \quad 1$$

$$\alpha_1^p = \alpha_3^a \quad 1 \quad \gamma \quad \alpha_3 \quad \alpha_3^v \quad \alpha_3^a \quad 1 \quad \alpha_3 \gamma \quad \alpha_3^v \gamma \quad \alpha_3 \quad \alpha_3^v$$

$$\beta^p = \gamma \quad \gamma \quad 1 \quad \gamma \quad \gamma \quad \alpha_3 \gamma \quad \alpha_3 \gamma \quad 1 \quad 1 \quad \alpha_3 \gamma \quad \alpha_3 \gamma$$

$$\alpha^p = \alpha_3 \gamma \quad \alpha_3 \gamma \quad \gamma \quad \gamma \quad 1 \quad \gamma^r \quad \gamma \quad \alpha_3 \gamma \quad \alpha_3^v \gamma \quad \alpha_3^{a+1} \gamma \quad \alpha_3^{a+v} \gamma$$

$$\alpha_1^p = \alpha_3 \quad \alpha_3^v \quad \alpha_3 \quad \alpha_3^v \quad \gamma \quad \gamma \quad 1 \quad \gamma \quad \gamma \quad \alpha_3 \gamma \quad \alpha_3^v \gamma$$

$$\beta^p = 1 \quad 1 \quad 1 \quad 1 \quad \alpha_3 \quad \alpha_3 \quad \alpha_3 \quad 1 \quad 1 \quad 1 \quad 1$$

$$\text{where } a = 1, 2, \dots, p-1; r = g^k, k+1 = 1, 2, \dots, (p-1, 3)$$

$$\Phi_{17}(21^4) = \langle \alpha, \alpha_1, \beta \rangle \text{ where } [\alpha_i, \alpha] = \alpha_{i+1}, [\beta, \alpha_1] = \gamma, \alpha_{i+1}^p = \gamma^p = 1 (i=1, 2) \text{ and}$$

$$\alpha^p = \alpha_3 \quad 1 \quad 1 \quad 1 \quad \alpha_3 \quad \alpha_3 \quad 1 \quad \alpha_3 \gamma \quad \gamma \quad \gamma \quad 1 \quad 1 \quad \alpha_3 \gamma \quad \alpha_3^v \gamma$$

$$\alpha_1^p = 1 \quad 1 \quad \alpha_3 \quad \alpha_3^v \quad \alpha_3 \quad \alpha_3^v \quad \gamma \quad 1 \quad 1 \quad \gamma \quad 1 \quad 1 \quad \alpha_3 \gamma \quad \alpha_3^v \gamma$$

$$\beta^p = 1 \quad \gamma \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad \alpha_3 \gamma \quad \alpha_3 \quad 1 \quad 1$$

$$\Phi_{17}(1^6) = \langle \alpha, \alpha_1, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\beta, \alpha_1] = \gamma, \alpha^p = \alpha_1^p = \beta^p = \alpha_{i+1}^p = \gamma^p = 1 (i=1, 2) \rangle$$

$$\underline{G \in \Phi_{18}}$$

$$\Phi_{18}(2211) = \langle \alpha, \alpha_1, \beta \rangle \text{ where } [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, [\alpha, \beta] = \gamma, \alpha_{i+1}^p = \gamma^p = 1$$

and

$$\alpha^p = 1 \quad \alpha_3 \quad \alpha_3^s \quad 1 \quad \gamma \quad \gamma^v \quad \gamma^a \quad \gamma^a$$

$$\alpha_1^p = \alpha_3^a \quad 1 \quad \gamma \quad \gamma \quad 1 \quad 1 \quad \alpha_3 \quad \alpha_3^v$$

$$\beta^p = \gamma \quad \gamma \quad 1 \quad \alpha_3^r \quad \alpha_3 \quad \alpha_3 \quad 1 \quad 1$$

where $a = 1, 2, \dots, p-1$; $r = g^k$, $k+1 = 1, 2, \dots, (p-1, 3)$; $s = g^1$

$l+1 = 1, 2, \dots, (p-1, 4)$

$\Phi_{19}(21^4) = \langle \alpha, \alpha_1, \beta \rangle$ where $[\alpha_1, \alpha] = \alpha_{i+1}$, $[\alpha_1, \beta] = \alpha_3$, $[\alpha, \beta] = \gamma$, $\alpha_{i+1}^p = \gamma^p = 1$

and

$$\begin{aligned} \alpha^p &= \alpha_3 & \gamma & \gamma^v & 1 & 1 & 1 & 1 & 1 \\ \alpha_1^p &= 1 & 1 & 1 & \alpha_3 & \alpha_3^v & \gamma & 1 & 1 \\ \beta^p &= 1 & 1 & 1 & 1 & 1 & 1 & \gamma & \alpha_3 \end{aligned}$$

$\Phi_{19}(1^6) = \langle \alpha, \alpha_1, \beta \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, [\alpha, \beta] = \gamma, \alpha^p = \alpha_1^p = \beta^p = \alpha_{i+1}^p = \gamma^p = 1$
 $(i=1, 2) \rangle$

$G \in \Phi_{19}$

$\Phi_{19}(2211) = \langle \alpha, \alpha_1, \alpha_2 \rangle$ where $[\alpha_1, \alpha_2] = \beta$, $[\beta, \alpha_1] = \beta_1$ ($i=1, 2$), $[\alpha, \alpha_1] = \beta_1$,

$\beta^p = \beta_1^p = 1$ and

$$\begin{aligned} \alpha^p &= 1 & 1 & \beta_1 \beta_2 & \beta_1 & \beta_1 & 1 & 1 & \beta_1 & \beta_1 & 1 & 1 \\ \alpha_1^p &= \beta_1 & \beta_1 & \beta_1^a & \beta_2^a & 1 & \beta_1 \beta_2, \beta_1 \beta_2^v, \beta_2 & \beta_2^v & \beta_2 & \beta_2 & \beta_2 \\ \alpha_2^p &= \beta_2 & \beta_2^{g^k} & \beta_2^b & \beta_2 & \beta_2 & \beta_2^a & \beta & 1 & 1 & \beta_1 & \beta_1^v \end{aligned}$$

$$\begin{aligned} \alpha^p &= 1 & 1 & 1 & 1 & 1 & 1 \\ \alpha_1^p &= \beta_2^v & \beta_1 \beta_2 & \beta_1 \beta_2^v & \beta_1 \beta_2^{-vk^2} & \beta_1 \beta_2^{-k^2} & \beta_1 \beta_2^{-vk^2} \\ \alpha_2^p &= \beta_1^{-v} & \beta_1^a & \beta_1^a & \beta_1 \beta_2^a & \beta_1 \beta_2^{g^1} & \beta_1^v \beta_2^{g^1} \end{aligned}$$

where $b = 1, 2, \dots, p-1$, $a+b = b+1, b+2 \dots p$

$k, l = 1, 2, \dots, \frac{1}{2}(p-1)$ and in the last three cases we have

$a + v k^2 \neq p$, $g^1 + k^2 \neq p$, $g^1 + v^2 k^2 \neq p$ respectively.

$\Phi_{19}(21^4) = \langle \alpha, \alpha_1, \alpha_2 \rangle$ where $[\alpha_1, \alpha_2] = \beta$, $[\beta, \alpha_1] = \beta_1$ ($i=1, 2$) $[\alpha, \alpha_1] =$

$= \beta_1, \beta^p = \beta_1^p = 1$ and

$$\begin{aligned}
\alpha^p &= 1 & 1 & 1 & \beta_1 & \beta_1\beta_2 & 1 & 1 & 1 & 1 & 1 \\
\alpha_1^p &= \beta_1 & \beta_1 & \beta_1 & 1 & 1 & \beta_2 & \beta_2^v & \beta_1\beta_2 & \beta_1\beta_2^v & \beta_1\beta_2^{-vk^2} \\
\alpha_2^p &= 1 & \beta_1 & \beta_1^v & 1 & 1 & 1 & 1 & 1 & 1 & \beta_1\beta_2^{-vk^2}
\end{aligned}$$

$$\begin{aligned}
\alpha^p &= 1 & 1 \\
\alpha_1^p &= \beta_1\beta_2^r & \beta_1\beta_2^{r/v} \\
\alpha_2^p &= \beta_1\beta_2^{g^l} & \beta_1^v\beta_2^{g^l}
\end{aligned}$$

where $r = -k^2$, $k = 1, 2, \dots, \frac{1}{2}(p-1)$, $0 < l < \frac{1}{2}p$ and $g^{l+k^2} = p$.

$$\begin{aligned}
\Phi_{19}(1^6) &= \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1 \in Z(G) \ (i=1,2), [\alpha, \alpha_1] = \beta_1, \\
&\alpha_1^p = \beta^p = \beta_1^p = 1 \rangle
\end{aligned}$$

$$G \in \Phi_{20}$$

$$\Phi_{20}(2211) = \langle \alpha, \alpha_1, \alpha_2 \rangle \text{ where } [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\beta, \alpha_2] = [\alpha, \alpha_1] = \beta_2,$$

$$\beta^p = \beta_1^p = \beta_2^p = 1 \text{ and}$$

$$\begin{aligned}
\alpha^p &= 1 & \beta_1 & \beta_1 & \beta_2 & 1 & 1 & \beta_1 & \beta_2 & 1 \\
\alpha_1^p &= \beta_1 & 1 & \beta_2^a & \beta_1 & \beta_1\beta_2 & \beta_1\beta_2^v & \beta_2^{g^s} & 1 & \beta_2 \\
\alpha_2^p &= \beta_2^a & \beta_2^{g^r} & \beta_2^a & 1 & \beta_2 & \beta_2 & 1 & \beta_1^a & \beta_1
\end{aligned}$$

$$\begin{aligned}
\alpha^p &= 1 & 1 & 1 & 1 & 1 \\
\alpha_1^p &= \beta_2^v & \beta_2 & \beta_2^v & \beta_2^a & \beta_2^a \\
\alpha_2^p &= \beta_1 & \beta_1^v & \beta_1^v & \beta_1\beta_2 & \beta_1^v\beta_2
\end{aligned}$$

where $a = 1, 2, \dots, p-1$, $r = 0, 1, \dots, (p-1, 3)-1$ $s = 0, 1, \dots, (p-1, 4)-1$

$$\Phi_{20}(21^4) = \langle \alpha, \alpha_1, \alpha_2 \rangle \text{ where } [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\beta, \alpha_2] = [\alpha, \alpha_1] = \beta_2, \\ \beta^p = \beta_1^p = \beta_2^p = 1 \text{ and}$$

$$\begin{array}{cccccccccccc} \alpha^p & = & 1 & 1 & \beta_2 & \beta_1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \alpha_1^p & = & \beta_1 & 1 & 1 & 1 & \beta_1 \beta_2 & \beta_1^v \beta_2 & 1 & 1 & \beta_2 & \beta_2^v \\ \alpha_2^p & = & 1 & \beta_2 & 1 & 1 & \beta_2^a & \beta_2^a & \beta_1 & \beta_1^v & 1 & 1 \end{array}$$

where $a = 1, 2, \dots, p-1$.

$$\Phi_{20}(1^6) = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\beta, \alpha_2] = [\alpha, \alpha_1] = \beta_2, \alpha^p = \alpha_1^p = \beta_1^p = \beta_2^p = 1 \rangle$$

$$\underline{G \in \Phi_{21}}$$

The groups of this family have not been determined individually for general p , although the isomorphism classes for small values of p may be determined on a computer (see (6)) and the number of isomorphism classes is known.

$$\underline{G \in \Phi_{22}}$$

$$\Phi_{22}(21^4) = \langle \alpha, \alpha_1, \beta_1, \beta_2 \rangle \text{ where } [\alpha_1, \alpha] = \alpha_{i+1}, [\beta_1, \beta_2] = \alpha_3, \beta_2^p = \alpha_{i+1}^p = 1 \\ (i=1, 2) \text{ and}$$

$$\begin{array}{ccccccccc} \alpha^p & = & \alpha_3 & 1 & \alpha_3 & 1 & 1 & 1 & 1 \\ \alpha_1^p & = & 1 & 1 & 1 & \alpha_3 & \alpha_3^v & \alpha_3 & \alpha_3^v \\ \beta_1^p & = & 1 & \alpha_3 & \alpha_3 & 1 & 1 & \alpha_3 & \alpha_3 \end{array}$$

$$\Phi_{22}(1^6) = \Phi_3(1^4) \times_{\mathbb{Z}} \Phi_2(111)$$

$$G \in \Phi_{23}$$

$$\Phi_{23}(2211) = \langle \alpha, \alpha_1 \rangle \text{ where } [\alpha_i, \alpha] = \alpha_{i+1} (i=1,2,3), [\alpha_1, \alpha_2] = \gamma, \text{ rest } = 1, \\ \alpha_2^p = \alpha_3^p = \alpha_4^p = \gamma^p = 1 \text{ and,}$$

$$\alpha^p = \begin{matrix} & \alpha_4 & \alpha_4 & \gamma & \gamma^v \end{matrix} \\ \alpha_1^p = \begin{matrix} \gamma & \alpha_4^a \gamma & \alpha_4^r & \alpha_4^r \end{matrix}$$

$$\text{where } a = 1, 2, \dots, p-1; r = g^k, k = 0, 1, \dots, (p-1, 3)-1$$

$$\Phi_{23}(21^4) = \langle \alpha, \alpha_1 \rangle \text{ where } [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \gamma, \alpha_{i+1}^p = \gamma^p = 1 (i=1, 2, 3)$$

$$\alpha^p = \begin{matrix} \alpha_4 & \alpha_4 & \alpha_4 \gamma^s & \gamma & \gamma^v & 1 & 1 \end{matrix} \\ \alpha_1^p = \begin{matrix} \alpha_4^r & 1 & 1 & 1 & 1 & \gamma & \alpha_4^r \gamma \end{matrix}$$

$$\text{where } r = g^k, k+1 = 1, 2, \dots, (p-1, 3); s = g^\ell, \ell+1 = 1, 2, \dots, (p-1, 4)$$

$$\Phi_{23}(1^6) = \langle \alpha, \alpha_1 | [\alpha_i, \alpha] = \alpha_{i+1}, (i=1, 2, 3), [\alpha_1, \alpha_2] = \gamma, \alpha_j^p = \alpha_j^p = \gamma^p = 1 \\ (j=1, \dots, 4) \rangle$$

$$G \in \Phi_{24}$$

$$\Phi_{24}(21^4) = \langle \alpha, \alpha_1, \beta \rangle \text{ where } [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_4, \alpha_{i+1}^p = 1 (i=1, 2, 3) \text{ and}$$

$$\alpha^p = \begin{matrix} \alpha_4 & 1 & 1 \end{matrix} \\ \alpha_1^p = \begin{matrix} 1 & \alpha_4^r & 1 \end{matrix} \quad r = g^k, k+1 = 1, 2, \dots, (p-1, 3)$$

$$\beta^p = \begin{matrix} 1 & 1 & \alpha_4 \end{matrix} \\ \Phi_{24}(1^6) = \langle \alpha, \alpha_1, \beta | [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_4, \alpha_1^p = \alpha_1^p = \beta^p = \alpha_{i+1}^p = 1 (i=1, 2, 3) \rangle$$

$$G \in \Phi_{25} \text{ or } \Phi_{26}$$

$$\Phi_{25}(321) = \langle \alpha, \alpha_1 | [\alpha_i, \alpha] = \alpha_{i+1} (i=1, 2, 3), \alpha^p = \alpha_2^p = \alpha_4^p, \alpha_1^p = \alpha_3^x, \alpha_3^p = \alpha_4^{p-1} \rangle$$

$$\Phi_{25}(222) = \langle \alpha, \alpha_1 | [\alpha_i, \alpha] = \alpha_{i+1} (i=1, 2, 3), \alpha_1^p = \alpha_3^x, \alpha_2^p = \alpha_4^p, \alpha^p = \alpha_3^p = \alpha_4^p = 1 \rangle$$

$$\text{where we take } x = 1 \text{ for } \Phi_{25} \text{ and } x = v \text{ for } \Phi_{26}.$$

$$\underline{G \in \Phi_{27}}$$

$$\Phi_{27}(21^4) = \langle \alpha, \alpha_1 \rangle \text{ where } [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = [\alpha_1, \alpha_2] = \alpha_4, \alpha_{i+1}^p = 1, (i=1, 2, 3) \\ \text{and}$$

$$\alpha^p = \alpha_4^s \quad 1 \quad 1 \quad 1 \quad s=g^l, \quad l+1 = 1, 2, \dots, (p-1, 4)$$

$$\alpha_1^p = 1 \quad \alpha_4^r \quad 1 \quad 1 \quad r=g^k, \quad k+1 = 1, 2, \dots, (p-1, 3)$$

$$\beta^p = 1 \quad 1 \quad \alpha_4 \quad \alpha_4^v$$

$$\Phi_{27}(1^6) = \langle \alpha, \alpha_1 \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = [\alpha_1, \alpha_2] = \alpha_4, \alpha^p = \beta^p = \alpha_1^p = \alpha_{i+1}^p = 1 \\ (i=1, 2, 3) \rangle$$

$$\underline{G \in \Phi_{28} \text{ or } \Phi_{29}}$$

$$\Phi_{28}(222)a_r = \langle \alpha, \alpha_1 \mid [\alpha_1, \alpha] = \alpha_{i+1} (i=1, 2, 3), [\alpha_1, \alpha_2] = \alpha_4, \alpha_1^p = \alpha_3^x, \alpha_2^p = \alpha_4^x, \\ \alpha^p = \alpha_4^r, \alpha_3^p = \alpha_4^p = 1 \rangle (r=1, 2, \dots, p)$$

$$\Phi_{28}(2211) = \langle \alpha, \alpha_1 \mid [\alpha_1, \alpha] = \alpha_{i+1}, (i=1, 2, 3), [\alpha_1, \alpha_2] = \alpha_4, \alpha_1^p = \alpha_3^x, \alpha_2^p = \alpha_4^x, \\ \alpha^p = \alpha_3^p = \alpha_4^p = 1 \rangle$$

where we take $x = 1$ for Φ_{28} and $x = v$ for Φ_{29} .

$$\underline{G \in \Phi_{30}}$$

$$\Phi_{30}(21^4) = \langle \alpha, \alpha_1, \beta \rangle \text{ where } [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, [\alpha_2, \beta] = \alpha_4, \alpha_{i+1}^p = 1 \\ (i=1, 2, 3) \text{ and}$$

$$\alpha^p = \alpha_4 \quad 1 \quad 1 \quad 1$$

$$\alpha^p = \alpha_4 \quad 1 \quad 1 \quad 1$$

$$\alpha_1^p = 1 \quad \alpha_4^r \quad \alpha_4^r \quad 1 \quad r=g^k, k+1=1, 2, \dots, (p-1, 3)$$

$$\beta^p = 1 \quad \alpha_4 \quad 1 \quad \alpha_4$$

$$\Phi_{30}(1^6) = \langle \alpha, \alpha_1, \beta \mid [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, [\alpha_2, \beta] = \alpha_4, \alpha^p = \beta^p = \alpha_1^p = \alpha_{i+1}^p = 1 \\ (i=1, 2, 3) \rangle$$

$$\underline{G \in \Phi_{31}}$$

$$\Phi_{31}(21^4) = \langle \alpha, \alpha_1, \alpha_2 \rangle \text{ where } [\alpha_i, \alpha] = \beta_i, [\alpha_i, \beta_i] = \gamma, \beta_i^p = \beta_i^{p-1} \ (i=1,2) \text{ and}$$

$$\alpha^p = \begin{matrix} & \gamma & \gamma & \gamma^v & \gamma & 1 & 1 \end{matrix}$$

$$\alpha_1^p = \begin{matrix} \gamma & \gamma & \gamma & 1 & \gamma & \gamma \end{matrix}$$

$$\alpha_2^p = \begin{matrix} \gamma^i & 1 & 1 & 1 & \gamma^i & 1 \end{matrix}$$

where $i^2 \equiv -1$ (only exists when $p \equiv 1 \pmod{4}$).

$$\Phi_{31}(1^6) = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_i, \alpha] = \beta_i, [\alpha_i, \beta_i] = \gamma, \alpha_i^p = \alpha_i^{p-1} = \beta_i^{p-1} \ (i=1,2) \rangle$$

$$\underline{G \in \Phi_{32}}$$

$$\Phi_{32}(21^4) = \langle \alpha, \alpha_1, \alpha_2 \rangle \text{ where } [\alpha_i, \alpha] = \beta_i, [\alpha_1, \beta_1] = \gamma, [\alpha_2, \beta_2] = \gamma^v, \beta_i^p = \beta_i^{p-1} \ (i=1,2) \text{ and}$$

$$\alpha^p = \begin{matrix} & \gamma & \gamma & \gamma^v & \gamma & 1 & 1 \end{matrix}$$

$$\alpha_1^p = \begin{matrix} \gamma & \gamma & \gamma & 1 & \gamma & \gamma \end{matrix}$$

$$\alpha_2^p = \begin{matrix} \gamma^j & 1 & 1 & 1 & \gamma^j & 1 \end{matrix}$$

where $j^2 \equiv -v$ (only exists when $p \equiv 3 \pmod{4}$).

$$\Phi_{32}(1^6) = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_i, \alpha] = \beta_i, [\alpha_1, \beta_1] = \gamma, [\alpha_2, \beta_2] = \gamma^v, \alpha_i^p = \alpha_i^{p-1} = \beta_i^{p-1} = \gamma^{p-1} \ (i=1,2) \rangle$$

$$\underline{G \in \Phi_{33}}$$

$$\Phi_{33}(21^4) = \langle \alpha, \alpha_1, \alpha_2 \rangle \text{ where } [\alpha_i, \alpha] = \beta_i, [\beta_2, \alpha] = [\alpha_1, \beta_1] = \gamma, \beta_i^p = \gamma^{p-1} \ (i=1,2) \text{ and}$$

$$\alpha^p = \begin{matrix} & \gamma & \gamma^v & 1 & 1 & 1 \end{matrix}$$

$$\alpha_1^p = \begin{matrix} 1 & 1 & \gamma & 1 & 1 \end{matrix}$$

$$\alpha_2^p = \begin{matrix} 1 & 1 & 1 & \gamma & \gamma^v \end{matrix}$$

$$\Phi_{33}(1^6) = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_i, \alpha] = \beta_i, [\beta_2, \alpha] = [\alpha_1, \beta_1] = \gamma, \alpha_i^p = \alpha_i^{p-1} = \beta_i^{p-1} = \gamma^{p-1} \ (i=1,2) \rangle$$

$$G \in \Phi_{34}$$

$$\Phi_{34}(222)a_1 = \langle \alpha, \alpha_1, \alpha_2 | \alpha^p = \alpha_1^p = \alpha_2^p = 1, [\alpha_1, \alpha] = \alpha^p, [\alpha_2, \alpha] = \alpha_1^p, [\alpha_1, \alpha^p] = [\alpha_1^p, \alpha] = \alpha_2^p \rangle$$

$$\Phi_{34}(222)a_v = \langle \alpha, \alpha_1, \alpha_2 | \alpha^p = \alpha_1^p = \alpha_2^p = 1, [\alpha_1, \alpha] = \alpha^p, [\alpha_2, \alpha] = \alpha_1^p, [\alpha_1, \alpha^p] = [\alpha_1^p, \alpha] = \alpha_2^p \rangle$$

$$\Phi_{34}(2211) = \langle \alpha, \alpha_1, \alpha_2 | \alpha^p = \alpha_1^p = \alpha_2^p = \gamma^p = 1, [\alpha_1, \alpha] = \alpha^p, [\alpha_2, \alpha] = \alpha_1^p, [\alpha_1, \alpha^p] = [\alpha_1^p, \alpha] = \gamma \rangle$$

$$G \in \Phi_{35}$$

$$\Phi_{35}(21^4)a = \langle \alpha, \alpha_1 | [\alpha_1, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_5, \alpha_1^p \alpha_5^{(p)} = \alpha_{i+1}^p = 1 (i=1, 2, 3, 4) \rangle$$

$$\Phi_{35}(21^4)b_r = \langle \alpha, \alpha_1 | [\alpha_1, \alpha] = \alpha_{i+1}, \alpha_1^p \alpha_5^{(p)} = \alpha_5^r, \alpha^p = \alpha_{i+1}^p = 1 (i=1, 2, 3, 4) \rangle$$

(r = g^k, k+1 = 1, 2, ..., (p-1, 4))

$$\Phi_{35}(1^6) = \langle \alpha, \alpha_1 | [\alpha_1, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_1^p \alpha_5^{(p)} = \alpha_{i+1}^p = 1 (i=1, 2, 3, 4) \rangle$$

$$G \in \Phi_{36}$$

$$\Phi_{36}(21^4)a_s = \langle \alpha, \alpha_1 | [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_5, \alpha^p = \alpha_5^s, \alpha_1^p \alpha_5^{(p)} = \alpha_{i+1}^p = 1$$

(i=1, 2, 3, 4) \rangle

$$\Phi_{36}(21^4)b_r = \langle \alpha, \alpha_1 | [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_5, \alpha_1^p \alpha_5^{(p)} = \alpha_5^r, \alpha^p = \alpha_{i+1}^p = 1$$

(i=1, 2, 3, 4) \rangle

(r = g^k, k+1 = 1, 2, ..., (p-1, 4); s = g^l, l+1 = 1, 2, ..., (p-1, 3))

$$\Phi_{36}(1^6) = \langle \alpha, \alpha_1 | [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_5, \alpha^p = \alpha_1^p \alpha_5^{(p)} = \alpha_{i+1}^p = 1 (i=1, 2, 3, 4) \rangle$$

$$\underline{G \in \Phi_{37}}$$

$$\Phi_{37}(21^4)a_r = \langle \alpha, \alpha_1 | [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_4, \alpha_1] = [\alpha_2, \alpha_3] = [\alpha_3, \alpha_1] = \alpha_5, \alpha^p = \alpha_5^r,$$

$$\alpha_1^p \alpha_5^{(p)} = \alpha_5, \alpha_{i+1}^p = 1 (i=1, 2, 3, 4) \rangle$$

$$(r = g^k, k+1 = 1, 2, \dots, (p-1, 4))$$

$$\Phi_{37}(21^4)b_1 = \langle \alpha, \alpha_1 | [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_4, \alpha_1] = [\alpha_2, \alpha_3] = [\alpha_3, \alpha_1] = \alpha_5, \alpha^p = \alpha_5, \alpha_1^p \alpha_5^{(p)} =$$

$$= \alpha_5, \alpha_{i+1}^p = 1 (i=1, 2, 3, 4) \rangle$$

$$\Phi_{37}(21^4)b_v = \langle \alpha, \alpha_1 | [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_4, \alpha_1] = [\alpha_2, \alpha_3] = [\alpha_3, \alpha_1] = \alpha_5, \alpha^p = \alpha_5^v,$$

$$\alpha_1^p \alpha_5^{(p)} = \alpha_{i+1}^p = 1 (i=1, 2, 3, 4) \rangle$$

$$\Phi_{37}(21^4)c = \langle \alpha, \alpha_1 | [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_4, \alpha_1] = [\alpha_2, \alpha_3] = [\alpha_3, \alpha_1] = \alpha_5, \alpha_1^p \alpha_5^{(p)} =$$

$$= \alpha_5, \alpha^p = \alpha_{i+1}^p = 1 (i=1, 2, 3, 4) \rangle$$

$$\Phi_{37}(1^6) = \langle \alpha, \alpha_1 | [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_4, \alpha_1] = [\alpha_2, \alpha_3] = [\alpha_3, \alpha_1] = \alpha_5, \alpha^p = \alpha_1^p \alpha_5^{(p)} =$$

$$= \alpha_{i+1}^p = 1 (i=1, 2, 3, 4) \rangle$$

$$\underline{G \in \Phi_{38}}$$

$$\Phi_{38}(21^4)a_m = \langle \alpha, \alpha_1 | [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_4 \alpha_5^{-1}, [\alpha_1, \alpha_3] = \alpha_5, \alpha^p = \alpha_1^p \alpha_5^{(p)} =$$

$$= \alpha_5^m, \alpha_{i+1}^p = 1 (i=1, 2, 3, 4) \rangle$$

$$(m = 1, 2, \dots, p-1)$$

$$\Phi_{38}(21^4)b_s = \langle \alpha, \alpha_1 | [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_4 \alpha_5^{-1}, [\alpha_1, \alpha_3] = \alpha_5, \alpha^p = \alpha_5^s,$$

$$\alpha_1^p \alpha_5^{(p)} = \alpha_{i+1}^p = 1 (i=1, 2, 3, 4) \rangle$$

$$\Phi_{38}(21^4)c_r = \langle \alpha, \alpha_1 | [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_4 \alpha_5^{-1}, [\alpha_1, \alpha_3] = \alpha_5, \alpha_1^p \alpha_5^{(p)} =$$

$$= \alpha_5^r, \alpha^p = \alpha_{i+1}^p = 1 (i=1, 2, 3, 4) \rangle$$

$$(s = p^\ell, \ell+1 = 1, 2, \dots, (p-1, 5); r = g^k, k+1 = 1, 2, \dots, (p-1, 4)).$$

$$\Phi_{39}(1^6) = \langle \alpha, \alpha_1 | [\alpha_1, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_4 \alpha_5^{-1}, [\alpha_1, \alpha_3] = \alpha_5, \alpha^p = \alpha_1^p \alpha_5^{\binom{p}{5}}, \\ = \alpha_{i+1}^p = 1 \quad (i=1, 2, 3, 4) \rangle$$

$$\underline{G \in \Phi_{39}}$$

The groups of Φ_{39} are the same as those for Φ_{37} except that $[\alpha_1, \alpha_2] = \alpha_4$ instead of 1.

$$\underline{G \in \Phi_{40}}$$

$$\Phi_{40}(21^4)_{a_s} = \langle \alpha_1, \alpha_2 | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\beta_1, \alpha_{3-1}] = \gamma, \alpha_1^p = \gamma, \alpha_2^p = \gamma^s, \\ \beta^p = \beta_1^p = \gamma^p = 1 \quad (i=1, 2) \rangle$$

$$(s = g^k, k+1 = 1, 2, \dots, (p-1, 3)).$$

$$\Phi_{40}(21^4)_b = \langle \alpha_1, \alpha_2 | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\beta_1, \alpha_{3-1}] = \gamma, \alpha_1^p = \gamma, \alpha_2^p = \beta^p = \\ = \beta_1^p = \gamma^p = 1 \quad (i=1, 2) \rangle$$

$$\Phi_{40}(1^6) = \langle \alpha_1, \alpha_2 | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\beta_1, \alpha_{3-1}] = \gamma, \alpha_1^p = \beta^p = \beta_1^p = \gamma^p = 1 \\ (i=1, 2) \rangle$$

$$\underline{G \in \Phi_{41}}$$

$$\Phi_{41}(21^4)_{a_s} = \langle \alpha_1, \alpha_2 | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha_1, \beta_1] = \gamma, [\alpha_2, \beta_2] = \gamma^{-v}, \\ \alpha_1^p = \gamma^s, \alpha_2^p = \beta^p = \beta_1^p = \gamma^p = 1 \quad (i=1, 2) \rangle$$

$$(s = g^k, k+1 = 1, 2, \dots, (p-1, 3))$$

$$\Phi_{41}(1^6) = \langle \alpha_1, \alpha_2 | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha_1, \beta_1] = \gamma, [\alpha_2, \beta_2] = \gamma^{-v}, \alpha_1^p = \beta^p = \\ = \beta_1^p = \gamma^p = 1 \quad (i=1, 2) \rangle$$

$$\underline{G \in \Phi_{42}}$$

$$\Phi_{42}(222) = \langle \alpha, \beta | [\alpha, \beta] = \gamma, [\alpha, \gamma] = \alpha^p, [\beta, \gamma] = \beta^{-p}, \alpha^{p^2} = \beta^{p^2} = \gamma^{p^2} = 1 \rangle$$

$$\underline{G \in \Phi_{43}}$$

$$\Phi_{43}(222) = \langle \alpha_1, \alpha_2 | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, \alpha_1^p = \beta_2, \alpha_2^p = \beta_1^{-p} = \beta_1^p = \beta^{p^2} = 1$$

$$(i = 1, 2) \rangle$$

APPENDIX III

The Groups of Order 3^n ($n \leq 6$)

Except for the following families, the groups of order 3^n may be found by putting $p = 3$ in the list given in Appendix I or II.

$$\underline{G \in \Delta_3}$$

The groups of Δ_3 may be found by putting $p = 3$ in the list Φ_3 and replacing α_1^p by $\alpha_1^3 \alpha_3$ there.

$$\underline{G \in 3^5 \Delta_6}$$

$$\Delta_6(221)a = \langle \alpha_1, \alpha_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \alpha_1^3, \alpha_1^9 = \beta^3 = \beta_1^3 = 1 (i=1,2) \rangle$$

$$\Delta_6(221)b = \langle \alpha_1, \alpha_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, \alpha_1^3 = \beta_1^2, \alpha_2^3 = \beta_2, \beta^3 = \beta_1^3 = 1$$

$$(i = 1, 2) \rangle$$

$$\Delta_6(221)c = \langle \alpha_1, \alpha_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, \alpha_1^3 = \beta_1, \alpha_2^3 = \beta_1 \beta_2, \beta^3 = \beta_1^3 = 1 (i=1,2) \rangle$$

$$\Delta_6(221)d_0 = \langle \alpha_1, \alpha_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, \alpha_1^3 = \beta_2, \alpha_2^3 = \beta_1^2, \beta^3 = \beta_1^3 = 1 (i=1,2) \rangle$$

$$\Delta_6(221)d_1 = \langle \alpha_1, \alpha_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, \alpha_1^3 = \beta_2^2, \alpha_2^3 = \beta_1 \beta_2^2, \beta^3 = \beta_1^3 = 1$$

$$(i = 1, 2) \rangle$$

$$\Delta_6(2111) = \langle \alpha_1, \alpha_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, \alpha_2^3 = \beta_1, \alpha_1^3 = \beta^3 = \beta_1^3 = 1 (i=1,2) \rangle$$

$$\Delta_6(1^5) = \langle \alpha_1, \alpha_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, \alpha_1^3 = \beta^3 = \beta_1^3 = 1 (i = 1, 2) \rangle$$

NOTE: The above symbols do not imply any properties of the groups. They are only chosen on analogy with Φ_6 .

$$\underline{G \in 3^6 \Delta_6}$$

The groups of $3^6 \Delta_6$ which are not direct products may be found by putting $p = 3$ in the list $p^6 \Phi_6$.

$$\underline{G \in \Delta_7}$$

The groups of Δ_7 may be found by putting $p = 3$ in the list Φ_7 and replacing α_1^p by $\alpha_1^3 \alpha_3$ there.

$$\underline{G \in \Delta_9 \text{ or } \Delta_{10}}$$

The groups of Δ_9 and Δ_{10} may be found by putting $p = 3$ in the lists Φ_9 and Φ_{10} and replacing α_1^p by $\alpha_1^3 \alpha_3$ and α_2^p by $\alpha_2^p \alpha_4$ there.

$$\underline{G \in \Delta_{16} \text{ or } \Delta_{18}}$$

The groups of Δ_{16} and Δ_{18} may be found by putting $p=3$ in the lists Δ_{16} and Δ_{18} and replacing α_1^p by $\alpha_1^3 \alpha_3$ there.

$$\underline{G \in \Delta_{20}}$$

The groups of Δ_{20} are given by:

$$G = \langle \alpha, \alpha_1, \alpha_2 \rangle \text{ where } [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1] = \beta_2, \beta^3 = \beta_1^3 = 1 (i=1,2)$$

$$\begin{array}{cccccc} \alpha^3 = \beta_1 & \beta_1 & \beta_2 & \beta_2 & 1 & 1 \\ \alpha_1^3 = \beta_2^a & \beta_2^a & \beta_1 & 1 & \beta_1 \beta_2^a & \beta_2^a \\ \alpha_2^3 = \beta_2 & 1 & \beta_1 & \beta_1^a & \beta_1^b \beta_2 & \beta_1^b \end{array}$$

where $a, b = 0, 1$ or 2 .

$$\underline{G \in \Delta_{21}}$$

The groups of Δ_{21} are given by:

$$\Delta_{21}(2211) = \langle \alpha, \alpha_1, \alpha_2 \rangle \text{ where } [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1] = \beta_2, \dots$$

$$[\alpha, \alpha_2] = \beta_1^2, \beta^3 = \beta_1^3 = 1 \quad (i=1,2) \text{ and}$$

$$\begin{aligned} \alpha^3 &= \beta_1 & \beta_1 & & \beta_1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \alpha_1^3 \beta_2^2 &= \beta_2^a & \beta_2^a & & 1 & \beta_2^a & \beta_2^a & \beta_1 & \beta_1^2 \beta_2 & \beta_1 \beta_2 & \beta_1 \beta_2 \\ \alpha_2^3 \beta_1 &= \beta_2 & 1 & & \beta_2 & \beta_1 & \beta_1^b \beta_2 & \beta_2 & \beta_2 & \beta_1 \beta_2^2 & \beta_1^2 \beta_2 \end{aligned}$$

where $a, b = 1 \text{ or } 2$.

$$\Delta_{21}(21^4) = \langle \alpha, \alpha_1, \alpha_2 \rangle \text{ where } [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1] = \beta_2, [\alpha_1, \alpha_2] = \beta_1^2, \beta^3 = \beta_1^3 = 1 \quad (i=1,2) \text{ and}$$

$$\begin{aligned} \alpha^3 &= \beta_1 & 1 & & 1 & & 1 & & 1 & & 1 \\ \alpha_1^3 \beta_2^2 &= 1 & \beta_1 \beta_2 & & \beta_2 & & \beta_1 & & 1 & & 1 \\ \alpha_2^3 \beta_1 &= 1 & 1 & & 1 & & 1 & & \beta_1 \beta_2 & & \beta_1 \end{aligned}$$

$$\Delta_{21}(21^4) = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha, \alpha_1] = \beta, [\alpha, \alpha_2] = \beta_1^2, \alpha_1^3 \beta_2^2 = \alpha_2^3 \beta_1 = \beta^3 = \beta_1^3 = 1 \quad (i=1,2) \rangle$$

$$\underline{G \in \Delta_{22}}$$

The groups of Δ_{22} may be found by putting $p = 3$ in the list Φ_{22} and replacing α_1^p by $\alpha_1^3 \alpha_3$ there.

$$\underline{G \in \Delta_{23}}$$

The groups of Δ_{23} are given by:

$$G = \langle \alpha, \alpha_1 \rangle \text{ where } [\alpha_1, \alpha] = \alpha_{i+1} \quad (i=1,2,3) \quad [\alpha_1, \alpha_2] = \gamma, \gamma^3 = \alpha_4^3 = \alpha_3^3 = \alpha_2^3 \alpha_4 = 1 \text{ and}$$

$$\begin{aligned} \alpha^3 &= \alpha_3 \gamma^a & \gamma^a & & \alpha_3 \gamma^a & & \gamma^a \\ \alpha_1^3 \alpha_3 &= \alpha_3 & \alpha_3 & & 1 & & 1 \end{aligned}$$

where $a = 0, 1, 2$.

$$\underline{G \in \Delta_{24}}$$

$$\begin{aligned} \Delta_{24}(21^4) &= \langle \alpha, \alpha_1, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1} \ (i=1,2,3), [\alpha_1, \beta] = \alpha_4, \beta^3 = \alpha_4, \alpha^3 = \alpha_1^3 \alpha_3^3 = \\ &= \alpha_2^3 \alpha_4^3 = \alpha_3^3 = \alpha_4^3 = 1 \rangle \\ \Delta_{24}(1^6) &= \langle \alpha, \alpha_1, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1} \ (i=1,2,3), [\alpha_1, \beta] = \alpha_4, \alpha^3 = \beta^3 = \alpha_1^3 \alpha_3^3 = \alpha_2^3 \alpha_4^3 = \\ &= \alpha_3^3 = \alpha_4^3 = 1 \rangle \end{aligned}$$

$$\underline{G \in \Delta_{25} \text{ or } \Delta_{26}}$$

The groups of Δ_{25} and Δ_{26} may be found by putting $p = 3$ in the lists Φ_{25} and Φ_{26} and replacing α_1^p by $\alpha_1^3 \alpha_3$ and α_2^p by $\alpha_2^3 \alpha_4$ there.

$$\underline{G \in \Delta_{27}}$$

$$\begin{aligned} \Delta_{27}(21^4)_{a_r} &= \langle \alpha, \alpha_1, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1} \ (i=1,2,3), [\alpha_1, \alpha_2] = [\alpha_1, \beta] = \alpha_4, \beta^3 = \alpha_4^r, \\ &\alpha^3 = \alpha_1^3 \alpha_3^3 = \alpha_2^3 \alpha_4^3 = \alpha_3^3 = \alpha_4^3 = 1 \rangle \ r=1 \text{ or } 2. \end{aligned}$$

$$\begin{aligned} \Delta_{27}(1^6) &= \langle \alpha, \alpha_1, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1} \ (i=1,2,3), [\alpha_1, \alpha_2] = [\alpha_1, \beta] = \alpha_4, \alpha^3 = \beta^3 = \\ &= \alpha_1^3 \alpha_3^3 = \alpha_2^3 \alpha_4^3 = \alpha_3^3 = \alpha_4^3 = 1 \rangle \end{aligned}$$

$$\underline{G \in \Delta_{28} \text{ or } \Delta_{29}}$$

The groups of Δ_{28} and Δ_{29} may be found by putting $p = 3$ in the lists Φ_{28} and Φ_{29} and replacing α_1^p by $\alpha_1^3 \alpha_3$ and α_2^p by $\alpha_2^3 \alpha_4$ there.

$$\underline{G \in \Delta_{30}}$$

The groups of Δ_{30} may be found by putting $p = 3$ in the list Φ_{30} for which $\alpha^p = 1$ and replacing α_1^p by $\alpha_1^3 \alpha_3$ and α_2^p by $\alpha_2^3 \alpha_4$ there.

$$\underline{G \in \Delta_{33}}$$

The groups of Δ_{33} are given by:

$G = \langle \alpha, \alpha_1, \alpha_2 \rangle$ where $[\alpha_1, \alpha] = \beta_1, [\beta_2, \alpha] = [\alpha_1, \beta_1] = \gamma, \gamma^3 = \beta_1^3 = 1$ ($i=1,2$) and

$$\begin{array}{rcccccc} \alpha^3 = & \gamma & \gamma^2 & 1 & 1 & 1 & 1 \\ \alpha_1^3 = & 1 & 1 & \gamma & 1 & 1 & 1 \\ \alpha_2^3 = & \gamma^2 & \gamma^2 & 1 & \gamma & \gamma^2 & 1 \end{array}$$

$$\underline{G \in \Delta_{35} \text{ or } \Delta_{36}}$$

The groups of Δ_{35} and Δ_{36} may be found by putting $p = 3$ in the lists Φ_{35} and Φ_{36} and replacing α_1^p by $\alpha_1^3 \alpha_3$, α_2^p by $\alpha_2^3 \alpha_4$ and α_3^p by $\alpha_3^3 \alpha_5$ there.

Note: The families Δ_{37} , Δ_{38} and Δ_{39} do not exist.

$$\underline{G \in \Delta_{41}}$$

The groups of Δ_{41} are given by:

$G = \langle \alpha_1, \alpha_2 \rangle$ where $[\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, [\alpha_1, \beta_1] = \gamma, \beta_1^3 = \gamma^3 = 1$ ($i=1,2$) and

$$\begin{array}{rcc} \alpha_1^3 = & \gamma & 1 \\ \alpha_2^3 = & \gamma & 1 \end{array}$$

$$\underline{G \in \Delta_{42}}$$

$$\Delta_{42}(222) = \langle \alpha_1, \alpha_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \beta_1, \alpha_1^3 = \beta_1, \alpha_2^3 = \beta^9 = \beta_1^3 = 1 \text{ (} i=1,2 \text{)} \rangle$$

$$\underline{G \in \Delta_{43}}$$

$$\Delta_{43}(222) = \langle \alpha_1, \alpha_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_1] = \alpha_2^6, [\beta, \alpha_2] = \alpha_1^3, \alpha_1^9 = \beta^9 = 1 \text{ (} i=1,2 \text{)} \rangle$$

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