Curvature Estimates for Geometric Deformations in Kähler Geometry

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Dedicated to my Mother and the memory of my Father

Statement of Originality

This is to certify that, to the best of my knowledge, the content of this thesis is my own work. This thesis has not been submitted for any degree or other purposes.

I certify that the intellectual content of this thesis is the product of my own work and that all the assistance received in preparing this thesis and sources have been acknowledged.

Hosea Wondo

December 26, 2023

Authorship Attribution Statement

Chapters 4, 5 and 6 of this thesis are based on the following articles:

- Chapter 4 of this thesis appears as a preprint on ArXiv [WZ23] and has been accepted for publication in *Geometry and Topology*. This is a joint work with my advisor Zhou Zhang.
- Chapter 5 of this thesis is published in *Communications in Contemporary Mathematics* Vol. 25 no. 9 (2023) [Won23b] where I am the sole author.
- Chapter 6 of this thesis has been published in *International Journal of Mathematics* Vol. 34, No. 12 (2023) [Won23a] where I am the sole author.

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As the supervisor for the candidature upon which this thesis is based, I can confirm that the authorship attribution statements above are correct.

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Preface

A Kähler manifold is a smooth manifold capable of supporting three structures: a Riemannian metric, a complex structure and a symplectic form. In physics, these manifolds are believed to describe the six extra dimensions in superstring theory. Mathematically, they reside at the intersection of algebraic geometry and differential geometry. Furthermore, Kähler manifolds serve as a launchpad to understand more general Hermitian manifolds, which are currently not well understood.

In the last 40 years, a new tool to study Kähler manifolds has emerged. The celebrated Ricci flow is a geometric evolution equation that deforms a Riemannian metric according to its Ricci curvature and resembles a heat-type equation for curvature. The flow was first introduced by Hamilton in 1982 [Ham82a] and, since then, has found tremendous success in proving results in geometry and topology. The most notable example is Perelman's resolution of the Poincare conjecture [Per02; Per03], thus completing Thurston's geometrisation of 3-manifolds. The flow has also successfully proved the differentiable sphere theorem [Bre10] and reproved the uniformisation theorem [CLT06]. Recently, promising developments have appeared in the study of four-dimensional manifolds [Bam21].

Applying the Ricci flow to a Kähler metric yields a smooth continuum of Kähler metrics. Such a setup was first considered by Cao in 1986 [Cao86], where he reproved the Calabi conjecture, demonstrating that it is indeed a powerful tool for studying Kähler structures. In its normalised variant, the flow is given by

$$\begin{cases} \dfrac{\partial}{\partial t}\omega(t) = -\mathrm{Ric}(\omega(t)) - \omega(t), \\ \omega(0) = \omega_0. \end{cases} \tag{1}$$

We call the above equation the Kähler-Ricci flow. Stationary solutions correspond to canonical Kähler metrics, the Kähler-Einstein metrics, thus opening the possibility of constructing these using the flow. Moreover, the flow takes a central role in the completion of the birational classification program for Kähler manifolds called the *Analytic Minimal Model Program* proposed by Song and Tian [ST06].

Taking a discretisation of the time derivative, one obtains an equation first introduced by Rubinstein in [Rub08]. It takes the form

$$\begin{cases} (1+t)\omega(t) = \omega_0 - t\mathrm{Ric}(\omega(t)), \\ \omega(0) = \omega_0. \end{cases} \tag{2}$$

This equation may also be interpreted as the elliptic version of the Kähler-Ricci flow and, as pointed out by La Nave and Tian in [LT16], provides an alternative to carry out the Analytic Minimal Model program. Similar deformations have been studied in analysing the limit of Ricci flat spaces [Tos10; GTZ13; GTZ20; Li21]. At first glance, the continuity method is a more direct deformation than the flow; the simplicity of equation (2) immediately ensures that solutions admit a lower Ricci curvature bound and thus, geometric comparison theorems can be readily applied. However, the caveat is that the continuity method is not parabolic and therefore, not smoothing.

Due to non-linearities in these equations, the Kähler-Ricci flow and the continuity method will often develop singularities. In the absence of singularities, we expect convergence to stationary solutions corresponding to canonical metrics. On the other hand, singular solutions are expected to reveal the underlying topology of the manifold. Understanding how geometric quantities such as the metric, diameter and curvature behave is essential for this analysis. The overarching aim of this thesis is to understand how curvature behaves at singularities.

This thesis is organised as follows. Chapters 1, 2 and 3 contain background material on Kähler geometry, the Kähler-Ricci flow and the Continuity method. Thematically, these three chapters describe the *behaviour of curvature without singularities*. The techniques presented here are standard and can be skipped by experts. The subsequent chapters, 4, 5 and 6, are based on papers written or co-written by the author. The first shows that the singularity type is independent of the starting metric in the Kähler-Ricci flow. The second describes some transferable curvature estimates from one solution of the continuity method to another. The last classifies curvature blowup rates for an example arising from algebraic geometry. These papers share a common goal of describing the *behaviour of curvature in the presence of singularities*. The appendix contains relevant preliminaries on geometry and partial differential equations.

Contents

Chapter 1

Kähler Geometry

This chapter briefly overviews Kähler manifolds. We begin by giving some initial definitions, examples and basic properties. We then establish a cohomology class on Kähler manifolds, which form a core element of our later analysis. Finally, we discuss a classification paradigm of Kähler manifolds called the "Minimal Model Program", which motivates the study of the Kähler-Ricci flow and continuity method.

1.1 The Kähler Condition

There are several common definitions of Kähler manifolds. The choice comes down to which structures you have available before 'upgrading' to a Kähler structure.

Definition 1.1. A *Kähler Manifold* is given by a triple (X, g, J) comprising of a smooth manifold X, a Riemannian metric g, and complex structure J, such that

$$\omega(u, v) := g(Ju, v) \quad \text{for all } u, v \in TX \tag{1.1}$$

is a closed form[1]. We call ω the Kähler form. We also refer to a Kähler manifold as a smooth manifold that supports a Kähler form.

The Hermitian metric has components given by

$$h(u, v) = g(u, v) + \sqrt{-1}\omega(u, v) \quad \text{for all } u, v \in TX. \tag{1.2}$$

One can check that $\mathrm{Re}(h) = g$ and $\mathrm{Im}(h) = \omega$, that is, the real and imaginary components are Riemannian and symplectic structures respectively.

Remark 1.2. We will follow the convention of calling ω, h and g Kähler metrics.

Despite the seemingly strict conditions[2], we remarkably generate many examples of Kähler manifolds.

Example 1.3. The following classes of manifolds are Kähler.

[1] That is, ω is symplectic.

[2] The set of Kähler manifolds is the intersection of Riemannian complex and symplectic manifolds.

1. Two dimensional smooth manifolds.

2. The complex projective space equipped with the Fubini-Study metric, $(\mathbb{CP}^N, \omega_{FS})$.

3. The complex Euclidean space equipped with the usual hermitian metric $(\mathbb{C}^N, \delta_{i\bar{j}})$.

Proposition 1.4. *Let (X, ω) be a Kähler manifold. Then if $Y \subset X$ is a submanifold of X, then $(Y, \omega|_Y)$ is Kähler manifold.*

We can combine this Proposition with our previous examples to observe the following.

Example 1.5. The following classes of manifolds are Kähler.

1. Smooth algebraic varieties as a submanifold of $\mathbb{CP}^N$.

2. Stein manifolds as submanifolds of $\mathbb{C}^N$.

Several important consequences arise out of (1.1). At the geometric level, we have the following symmetry for the Hermitian metric.

Proposition 1.6. *Let (X, ω) be a Kähler manifold. Then in local coordinates $(z_i)_{i=1}^n$, the metric $g = \sum_{i,j=1}^n g_{i\bar{j}} dz_i \otimes d\bar{z}_j$ satisfies*

$$\frac{\partial g_{i\bar{j}}}{\partial z_k} = \frac{\partial g_{k\bar{j}}}{\partial z_i}, \quad \frac{\partial g_{i\bar{j}}}{\partial \bar{z}_k} = \frac{\partial g_{i\bar{k}}}{\partial \bar{z}_j}. \tag{1.3}$$

As a result of this symmetry, the Chern connection and the Levi-Civita connection on Kähler manifold coincide.

Corollary 1.7. *The Chern connection for the Kähler metric has vanishing torsion. That is, the Chern connection coincides with the Levi-Civita connection.*

The Kähler condition also imposes some restrictions at the topological level.

Proposition 1.8. *If X admits a Kähler manifold, then the odd Betti numbers must be even.*

Example 1.9. The Hopf surface (diffeomorphic to $S^3 \times S^1$) does not support a Kähler metric since $b_1(X) = 1$.

1.2 Cohomology

From now on, we only consider compact Kähler manifold. The set of Kähler metrics live in a cohomology class defined as follows. Recall that the complex structure J splits the tangent bundle into two eigenspaces; the *holomorphic tangent space*, $T_p'X$ consisting of derivations vanishing on antiholomorphic functions and the *antiholomorpic tangent space* $T_p''X$ consisting of derivations that vanish on holomorphic functions.

$$T_{\mathbb{C},p}X = T_p'X \bigoplus T_p''X. \tag{1.4}$$

This induces a spitting of forms

$$\Omega_{\mathbb{C}}X = \Omega^{1,0}X \oplus \Omega^{0,1}X, \tag{1.5}$$

and

$$\Omega^r_{\mathbb{C}}X = \oplus_{p+q=r}\Omega^{(p,q)}X. \tag{1.6}$$

Projecting the exterior derivative onto each component yields the Dolbeault operator, which is a complex variant of the exterior derivative. These operators are given by

$$\partial\alpha := \pi^{p+1,q}(d\alpha), \quad \bar{\partial}\alpha := \pi^{p,q+1}(d\alpha),$$

where $\pi^{p+1,q} : \Omega^{k+1}_{\mathbb{C}}(X) \to \Omega^{p+1,q}(X)$ and $\pi^{p,q+1} : \Omega^{k+1}_{\mathbb{C}}(X) \to \Omega^{p,q+1}(X)$ denote the projection maps. Let $C^\infty(X,\mathbb{R})$ be the ring of real smooth functions on X and define the quotient

$$H^{1,1}(X,\mathbb{R}) := \frac{\{\text{closed real } (1,1) \text{ forms on } X\}}{\sqrt{-1}\partial\bar{\partial}C^\infty(X,\mathbb{R})}. \tag{1.7}$$

The space $H^{1,1}(X,\mathbb{R})$ forms a finite-dimensional vector space. We define a subset of this space containing classes with a Kähler representative.

Definition 1.10. The *Kähler cone* $\mathcal{K}_X \subset H^{1,1}(X,\mathbb{R})$ is

$$\mathcal{K}_X = \{[\alpha] \in H^{1,1}(X,\mathbb{R}) \mid [\alpha] \text{ has a representative Kähler form}\}.$$

This set is open, convex and a cone in the sense that if $[\alpha] \in \mathcal{K}_X$ then for any $\lambda > 0$ we have $\lambda[\alpha] \in \mathcal{K}_X$.

Using the complex structure, we define the Ricci form from the Ricci curvature tensor as follows.

Definition 1.11. The *Ricci form*, ρ, is given by

$$\rho(u,v) := \mathrm{Ric}(Ju,v), \quad \forall u,v \in TX. \tag{1.8}$$

Given holomorphic coordinates, $\{z_i\}_{i=1}^n$, the Ricci form has a local expression given by

$$\rho = -\sqrt{-1}\partial_i\bar{\partial}_{\bar{j}} \log\det(g). \tag{1.9}$$

The class of the Ricci form takes a special role in the $H^{(1,1)}(X,\mathbb{R})$ cohomology. First, we show that the class is independent of the metric, thus a topological invariant.

Proposition 1.12. *Let ω and ω' be Kähler metrics on a compact Kähler manifold X. Then*

$$[\mathrm{Ric}(\omega)] = [\mathrm{Ric}(\omega')]. \tag{1.10}$$

Proof. Suppose ω and $\widetilde{\omega}$ are Kähler forms. Then using the local representation of the Ricci form, we have

$$\mathrm{Ric}(\omega) - \mathrm{Ric}(\widetilde{\omega}) = \sqrt{-1}\partial\bar{\partial}\log\frac{\det\widetilde{g}}{\det g} \tag{1.11}$$

where $\frac{\det\widetilde{g}}{\det g}$ is a globally defined smooth function. □

Miraculously, the class of the Ricci form corresponds to a topological class of the canonical line bundle of the manifold. Recall that $K_X = \Omega^{(n,0)}(X)$ defines the canonical line bundle of X. The first Chern class of the canonical line bundle, $c_1(K_X)$, coincides with the class of the Ricci form. We then defined the first Chern class of a manifold to be

$$c_1(X) := -c_1(K_X) = -[\mathrm{Ric}(\omega)]. \tag{1.12}$$

Furthermore, the position of the first Chern class determines if our Kähler manifold is algebraic.

Definition 1.13. A class $[\alpha]$ is *numerically effective* (nef) if it lies in the closure of the Kähler cone in the $H^{(1,1)}(X, \mathbb{R})$ cohomology, $[\alpha] \in \overline{\mathcal{K}_X}$.

The connection to algebraic varieties is made with the following definition.

Definition 1.14. A line bundle over a compact Kähler manifold (X, ω) is *semiample* if there exists $k > 0$ such that for each point $p \in X$, there exists a global section of $L^{\otimes k}$ that does not vanish at p.

This is to say that global sections of tensor powers of the line bundle over X generate a holomorphic map onto a subset of complex projective space,

$$f : X \to Y \subset \mathbb{CP}^N. \tag{1.13}$$

This map may however contain singular sets, denoted as S. Moreover, the dimension of Y corresponds to an algebraic quantity called the Kodaira dimension and may not necessarily equal the dimension of X.

Definition 1.15. The *Kodaira dimension* of X is given by

$$\kappa(X) = \limsup_{\ell\to\infty} \frac{\log \dim H^0\left(X, K_X^{\otimes \ell}\right)}{\log \ell}. \tag{1.14}$$

Therefore, the map induces a fibration structure for X from which we can distinguish three types of fibrations based on the value of κ.

1. *The General Case.* The volume of the manifold remains uniformly bounded away from 0, which is equivalent to $\kappa = n$. The singularity is contained within $f^{-1}(S)$.

2. *Calabi-Yau Case.* When $\kappa = 0$, the space Y is a point. This is equivalent to X being a Calabi-Yau manifold.

3. *Volume Collapsed Case.* The last reaming case is the intermediate range $1 \leqslant \kappa \leqslant n-1$. Away from the singular set, the map f is a holomorphic submersion with Calabi-Yau fibres. The structure of the singular fibre can become quite complicated (see [Tos10; GTZ13; GTZ20]).

Semiampleness always implies numerical effectiveness, however the converse is only conjecturally true. This is the Kähler version of the famous abundance conjecture.

Conjecture 1.16 (Abundance Conjecture - Kähler extension)**.** *Let X be a Kähler manifold. Then K_X is semiample if and only if it is numerically effective.*

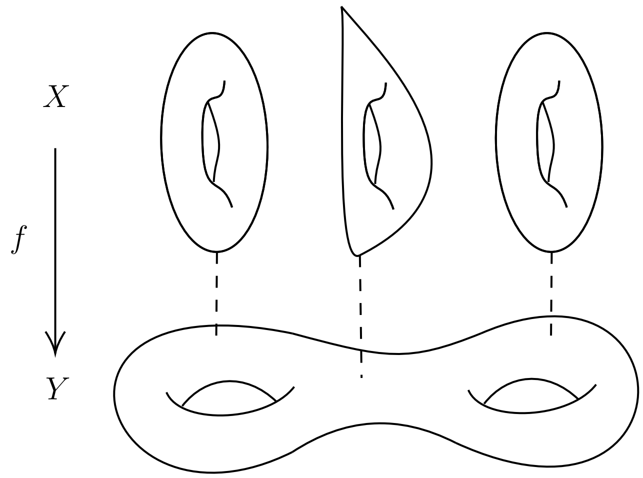

Figure 1.1: Semiample fibration over base space Y with $1 \leqslant \kappa \leqslant n-1$.

1.3 Classification

Since all two-manifolds are Kähler, the classification of Kähler manifolds of complex dimension one is simply given by the uniformisation theorem (see Appendix A.5). Manifolds are classified by Genus, and their relation to curvature is established through the Gauss-Bonnet formula. Unfortunately, this fails to generalise to higher dimensions.

We can aim for a topological classification of compact Calabi-Yau manifolds, which is particularly relevant to physics. The existence of Ricci flat metrics is provided by Yau's solution to the Calabi conjecture. Then grouping by dimension, we have the following scheme. In complex dimensions one, the only examples are elliptic curves due to the uniformisation theorem. In complex dimension two, one has the moduli space of K3 surfaces. However, for complex dimension three, which is the relevant dimension of ten-dimensional superstring theory, classification is an open problem and notoriously difficult.

Returning to the classification of general Kähler manifolds , a natural scheme to pursue is to group by biholomorphisms classes. This also turns out to be an extremely difficult problem, as highlighted by the following example.

Example 1.17. Any elliptic curve is diffeomorphic to $\mathbb{R}^2/\mathbb{Z}$ and hence each other. However, there are infinitely many non-biholomorphic elliptic curves.

Instead, we can relax our classification scheme to look for the best representatives in birational classes. This leads directly to Mori's program, a birational classification program for algebraic varieties. An issue that arises from considering birational classes is the existence of blow-ups. Taking a blow-up of a smooth representative in a birational class produces a different smooth representative. Thus, we must make some canonical choices for a representative. To this end, we give the following definition due to Mori.

Definition 1.18. A compact manifold (X, h) is *minimal* if its canonical line bundle K_X is numerically effective.

Therefore, we can state the minimal model program as follows.

Minimal Model Program: For each bimeromorphic class, find the unique minimal manifold.

Chapter 2

The Kähler-Ricci Flow

This chapter summarises some background results and technical tools for the Kähler-Ricci flow. We first look at the evolution of the metric in the $H^{(1,1)}(X, \mathbb{R})$ cohomology. This provides crucial information about the flow and allows us to reduce the flow to a complex Monge-Ampère equation from which the nonlinear partial differential equations theory can be applied. We then study the evolution of geometric quantities under the Kähler-Ricci flow. By utilising the maximum principle, one can obtain control of geometric quantities of the metric under the flow, as pioneered by Hamilton for the Ricci flow [Ham82b].

2.1 Evolution of the Cohomology Class

Recall that the class of the Ricci form is the Chern class of the canonical line bundle. Taking the class of the flow equation yields

$$\begin{cases} \dfrac{d}{dt}[\omega(t)] = -c_1(X) - [\omega(t)], \\ \quad [\omega(0)] = [\omega_0] \end{cases} \tag{2.1}$$

with solutions given by

$$[\omega(t)] = e^{-t}[\omega_0] - (1 - e^{-t})c_1(X). \tag{2.2}$$

The flow evolves the class of the initial metric ω_0 towards $c_1(K_X)$. However, this class may not be within the interior of the Kähler cone, and thus, the flow will reach a class without a Kähler metric. One may conclude here that the flow must terminate due to a singularity at this point. Indeed, this characterises the maximal time of existence.

Theorem 2.1 (Tian-Zhang [TZ06]). *A solution to the Kähler-Ricci flow can always be extended if $[\omega(t)]$ lies in the interior of the Kähler cone. In other words, the maximal time T is given by*

$$T := \sup\{t > 0 \mid [\omega(t)] > 0\}. \tag{2.3}$$

Remark 2.2. This result distinguishes the Kähler-Ricci flow from the usual Ricci flow in which a precise characterisation does not exist. Instead, the characterisation is provided by the blowup of curvatures [Ham82b; Šeš05; Wan08; Cao10].

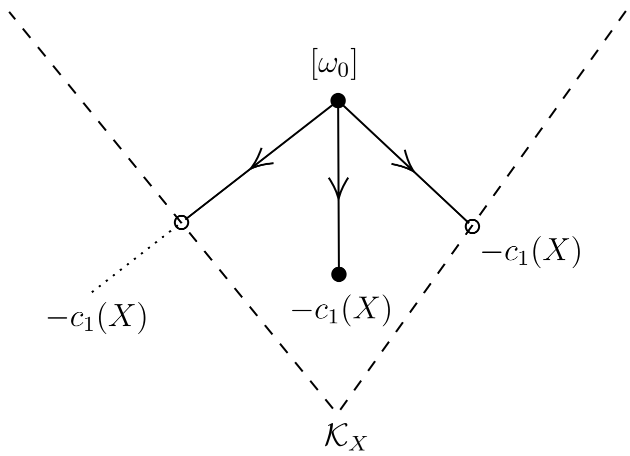

Figure 2.1: Path of the metric's class under the Kähler-Ricci Flow

The position of $-c_1(X)$ in $H^{1,1}(X, \mathbb{R})$ determines whether $T = \infty$ or $T < \infty$. Inspecting (2.2), we observe three cases

1. If $-c_1(X) \in \mathcal{K}_X$, then $T = \infty$ and no singularity develops. The convergence is to the unique Kähler Einstein metric in $-c_1(X)$.
2. If $-c_1(X) \in \partial\mathcal{K}_X$, then $T = \infty$ and an infinite time singularity forms.
3. If $-c_1(X) \in \overline{\mathcal{K}_X}^c$, then $T < \infty$, we call this the finite time singularity case. Similar consideration for the limiting Cohomology class predicts that
$$[\omega_T] = e^{-T}[\omega_0] - (1 - e^{-T})c_1(X)$$
fails to be Kähler.

We can interpret the normalised Kähler-Ricci flow as an equation that seeks out Kähler-Einstein metrics in the manifold's first Chern class.

If the Chern class vanishes, then we are in the setting of the Calabi conjecture ($c_1(K_X) = 0$). In this case, it is convenient to consider the unnormalised Kähler-Ricci flow since the class is static;
$$[\omega(t)] = [\omega_0] - tc_1(X) = [\omega_0]. \tag{2.4}$$
Consequently, Theorem 2.1 implies $T = \infty$, and we expect the solution to converge to the unique Ricci flat metric in $[\omega_0]$. Thus we may interpret the unnormalised Kähler-Ricci flow as a means to construct Calabi-Yau metrics.

Remark 2.3. It is sometimes useful to work between the unnormalised flow and normalised flow. This is done using the following transformation:
$$\omega(s) = e^t\widetilde{\omega}_t, \quad s = e^t - 1. \tag{2.5}$$
For $T < \infty$, solutions to the normalised, $\omega(t)$, and unnormalised flow, $\widetilde{\omega}(t)$, are equivalent. That is, there exists a $C > 0$ independent of t, such that
$$C^{-1}\widetilde{\omega}(t) \leqslant \omega(t) \leqslant C\widetilde{\omega}(t) \tag{2.6}$$
for all $t \in [0, T)$.

2.2 Parabolic Complex Monge-Ampère Equation

Let $\omega(t)$ be a solution to the normalised Kähler-Ricci flow. If we choose $\chi \in -c_1(X)$ then

$$\omega_t = e^{-t}\omega_0 + (1 - e^{-t})\chi \tag{2.7}$$

is an explicit representative of the class $[\omega(t)]$. This implies that for each t, there exists $\varphi(t) : X \to \mathbb{R}$ such that

$$\omega(t) = \omega_t + \sqrt{-1}\partial\overline{\partial}\varphi. \tag{2.8}$$

Taking a time derivative of the above and using (1) gives

$$\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\varphi}{\partial t}\right) = -\mathrm{Ric}(\omega) - \chi - \sqrt{-1}\partial\overline{\partial}\varphi. \tag{2.9}$$

Let Ω be a volume form on X such that $\mathrm{Ric}(\Omega) = -\chi$. By the local expression for Ricci curvature on Kähler manifolds, the above expression becomes

$$\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial\varphi}{\partial t} + \varphi - \log\left(\frac{\omega^n}{\Omega}\right)\right) = 0. \tag{2.10}$$

Harmonic forms on a compact manifold must be constant. Thus

$$\frac{\partial\varphi}{\partial t} + \varphi - \log\left(\frac{\omega^n}{\Omega}\right) = c_t. \tag{2.11}$$

Relabelling $\varphi(t) \to \varphi(t) + \int_0^t c(s)ds + \varphi(0)$ yields

$$\begin{cases} \dfrac{\partial\varphi}{\partial t} = \log\dfrac{\left(\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi\right)^n}{\Omega} - \varphi \\ \varphi(0) = 0. \end{cases} \tag{2.12}$$

Remark 2.4. For the unnormalised flow, one can derive a similar Monge-Ampère equation, which takes the form of (2.12) but without the $-\varphi$ term. However, extra assumptions on the limiting metric ω_∞ must be made regarding the choice of Ω.

Remark 2.5. The parabolicity of the flow can be obtained easily using the Monge-Ampère equation. Indeed, the linearisation of the right-hand side (2.12) is the Laplacian.

Proposition 2.6. *A smooth family of Kähler metrics $\omega(t)$ solves the Kähler-Ricci flow if and only if $\varphi(t)$ defined by $\omega(t) = \omega_t + \sqrt{-1}\partial\overline{\partial}\varphi(t)$ solves* (2.12).

Proof. Suppose φ solves (2.12). Taking a time derivative of $\omega(t) = \omega_t + \sqrt{-1}\partial\overline{\partial}\varphi(t)$ yields

$$\frac{\partial}{\partial t}\omega(t) = e^{-t}(\chi - \omega_0) + \sqrt{-1}\partial\overline{\partial}\log\omega^n - \sqrt{-1}\partial\overline{\partial}\log\Omega - \sqrt{-1}\partial\overline{\partial}\varphi = -\mathrm{Ric}(\omega) - \omega$$

and since $\omega(0) = \omega_0$, we have that $\omega(t)$ solves the Kähler-Ricci flow. For the other direction, we assume $\omega(t)$ solves (1). Define

$$\varphi(t) = e^{-t}\int_0^t e^s \log\left(\frac{\omega(s)^n}{\Omega}\right) ds$$

so that

$$\frac{\partial\varphi}{\partial t} = \log\left(\frac{\omega(t)^n}{\Omega}\right) - \varphi(t), \quad \varphi(0) = 0. \tag{2.13}$$

Thus

$$\frac{\partial}{\partial t}\left(\omega(t) - \omega_t - \sqrt{-1}\partial\bar{\partial}\varphi(t)\right) = -\mathrm{Ric}(\omega(t)) - \chi + \mathrm{Ric}(\omega(t)) - \mathrm{Ric}(\Omega) = 0$$

and

$$\left(\omega(t) - \omega_t - \sqrt{-1}\partial\bar{\partial}\varphi(t)\right)\Big|_{t=0} = 0,$$

so we must have $\omega(t) - \omega_t - \sqrt{-1}\partial\bar{\partial}\varphi(t) \equiv 0$ on $X \times [0, T')$. □

Remark 2.7. This reduction works for more general Hermitian flows, such as the Chern-Ricci flow.

Regularity on the metric can be obtained by showing regularity on the potential function φ due to (2.8). In particular, bounds on the potential lead to bounds on the metric trace.

Proposition 2.8. *Let φ be a solution to* (2.12)*. If there exist $A > 0$ such that*

$$|\varphi| + \left|\frac{\partial\varphi}{\partial t}\right| \leqslant A \tag{2.14}$$

for all $t \in [0, T)$ and $-c_1(X) > 0$. Then there exists $C > 0$ such that

$$C^{-1}\omega_0 \leqslant \omega \leqslant C\omega_0 \tag{2.15}$$

for all $t \in [0, T)$.

Proof. From Proposition 2.18, we have

$$\left(\frac{\partial}{\partial t} - \Delta_\omega\right)\log \mathrm{tr}_\omega\,\omega_0 \leqslant C + C_0\,\mathrm{tr}_\omega\,\omega_0 \tag{2.16}$$

where C_0 depends on the bisectional curvature of ω_0. Since $-c_1(X)$ is a Kähler class, we can choose a representative χ such that $\chi \geqslant C^{-1}\omega_0$ for some $C > 0$. Define

$$Q := \log \mathrm{tr}_\omega\,\omega_0 - A\varphi$$

for $A > 0$ to be chosen later. The evolution of Q is given by

$$\left(\frac{\partial}{\partial t} - \Delta_\omega\right) Q \leqslant C + C_0\,\mathrm{tr}_\omega\,\omega_0 - A\,\mathrm{tr}_\omega\,\omega_0 \leqslant C - \mathrm{tr}_\omega\,\omega_0$$

for A chosen sufficiently large enough. Then the maximum principle yields

$$\mathrm{tr}_\omega\,\omega_0 \leqslant C \tag{2.17}$$

for all $t \in [0, T)$. From the Monge-Ampère equation (2.12), we have

$$C_2^{-1}\omega_0^n \leqslant C_1^{-1}\Omega \leqslant \omega^n \leqslant C_1\Omega \leqslant C_2\omega_0^n, \tag{2.18}$$

where $C_1, C_2 > 0$ uniform of time. Combining this with the eigenvalue estimate

$$\mathrm{tr}_{\omega_0}\,\omega \leqslant n(\mathrm{tr}_\omega\,\omega_0)^{n-1}\frac{\omega^n}{\omega_0^n} \leqslant C$$

yields the conclusion

$$C^{-1}\omega_0 \leqslant \omega \leqslant C\omega_0$$

for all $t \in [0, T)$. □

2.3 Parabolic Evolution of Tensors

As the underlying metric evolves according to the Kähler-Ricci flow, the geometric quantities depending on the metric evolve with it. The resulting evolution equations are nonlinear parabolic equations, thus giving rise to some immediate estimates through the maximum principle.

2.3.1 The Kähler-Ricci Flow as a Parabolic Evolution Equation

Due to the invariance of the Ricci flow under diffeomorphisms, the Ricci flow on a Riemannian manifold is a weakly parabolic equation. Thus, the theory of parabolic PDEs cannot be directly applied to obtain short-time existence and uniqueness. In the Kähler case, the preserved invariant transformations are biholomorphisms, a much smaller gauge group than diffeomorphisms. As a result, the Kähler-Ricci flow is strictly parabolic.

Proposition 2.9. *The Kähler-Ricci flow is a system of strictly parabolic equations. Thus, the flow always exists for short times.*

Proof. Consider the linearisation of the Ricci tensor associated with the Ricci form. Taking a variation of the hermitian metric $g_{i\bar{j}} + \varepsilon h_{i\bar{j}}$, we have

$$\begin{aligned}(D\mathrm{Ric})(h_{i\bar{j}}) &= \frac{\partial}{\partial \varepsilon}\left(\mathrm{Ric}(g_{i\bar{j}} + \varepsilon h_{i\bar{j}})\right)|_{\varepsilon=0} \\ &= -\frac{\partial}{\partial \varepsilon}\left(\frac{\partial \Gamma^k_{ki}}{\partial \bar{z}^j}\right)|_{\varepsilon=0} \\ &= -g^{\bar{q}p}\frac{\partial}{\partial z^i}\frac{\partial}{\partial \bar{z}^j} h_{p\bar{q}} \\ &= \Delta_\omega h_{i\bar{j}}\end{aligned}$$

where the Kähler identities were used to swap the derivative and metric indices in the last line. This results in the Laplace-Beltrami operator for $g_{i\bar{j}}$ which is an elliptic operator. Thus, the Kähler-Ricci flow is strictly parabolic and short time existence is guaranteed (see Theorem B.24). □

Remark 2.10. The linearisation of the Ricci tensor in the Riemannian case involves more terms and results in a more complicated symbol from which a suitable choice of variations h_{ij} results in a degenerate symbol.

Remark 2.11. One can also show strict parabolicity from the Monge-Ampère equation corresponding to the Kähler-Ricci flow.

2.3.2 Scalar Curvature and Volume

Proposition 2.12. *Let $\omega(t)$ be a solution to the Kähler-Ricci flow. The scalar curvature of $\omega(t)$ evolves as*

$$\frac{\partial}{\partial t}R = \Delta_\omega R + \|\mathrm{Ric}(\omega)\|^2_\omega + R, \tag{2.19}$$

where $\|\mathrm{Ric}(\omega)\|_\omega^2 = g^{\bar{\ell}i} g^{\bar{j}k} R_{i\bar{j}} R_{k\bar{\ell}}$.[1]

Remark 2.13. Equation (2.19) is a reaction-diffusion equation for scalar curvature. The diffusion term improves regularity, while the forcing reaction terms drive singularities to form.

Proof. The scalar curvature takes a simple form in local normal coordinates

$$R = -g^{\bar{j}i} \partial_i \partial_{\bar{j}} \log \det g. \tag{2.20}$$

Taking a time derivative yields

$$\begin{aligned} \frac{\partial R}{\partial t} &= -g^{\bar{j}i} \partial_i \partial_{\bar{j}} \left(g^{\bar{\ell}k} \frac{\partial}{\partial t} g_{k\bar{\ell}} \right) - \left(\frac{\partial}{\partial t} g^{\bar{j}i} \right) \partial_i \partial_{\bar{j}} \log \det g \\ &= \Delta_\omega R + g^{\bar{\ell}i} g^{\bar{j}k} R_{k\bar{\ell}} R_{i\bar{j}} + R, \end{aligned}$$

which is just equation (2.19). □

A direct maximum principle leads to a general lower bound on scalar curvature.

Corollary 2.14. *For* $C_0 := -\inf_X R(0) + n$*, we have*

$$R(t) \geqslant -n + C_0 e^{-t}. \tag{2.21}$$

Remark 2.15. Consequently, any blow-up of the scalar curvature must be to $+\infty$.

Proof. The Ricci curvature tensor dominates the scalar curvature since $n\|\mathrm{Ric}(\omega)\|_\omega^2 \geqslant R^2$. Then (2.19) becomes

$$\left(\frac{\partial}{\partial t} - \Delta_\omega \right) R \geqslant \frac{1}{n} R(R+n) = \frac{1}{n}(R+n)^2 = (R+n). \tag{2.22}$$

Consider the quantity $Q := e^t(R+n)$. From (2.22), its evolution is

$$\left(\frac{\partial}{\partial t} - \Delta_\omega \right) Q \geqslant 0. \tag{2.23}$$

This immediately produces a contradiction if the minimum of Q occurs after time $t > 0$ (since X is compact, the parabolic interior is simply the time slice $t = 0$). Thus, $Q(t) \geqslant Q(0)$ and the desired inequality follows. □

It turns out that the scalar curvature plays a fundamental role in the formation of finite time singularities. One can see this in the evolution of volume;

Proposition 2.16. *The volume form evolves as*

$$\omega(t)^n = e^{-\int_0^t R(s)ds - nt} \omega_0^n. \tag{2.24}$$

[1] The norm with respect to the form $\omega(t)$ of a tensor is the usual norm of a tensor contracted with hermitian metric $g_{i\bar{j}}$ associated to $\omega(t)$. We will shorten this notation to $|T|_g$ for ease of notation in proofs.

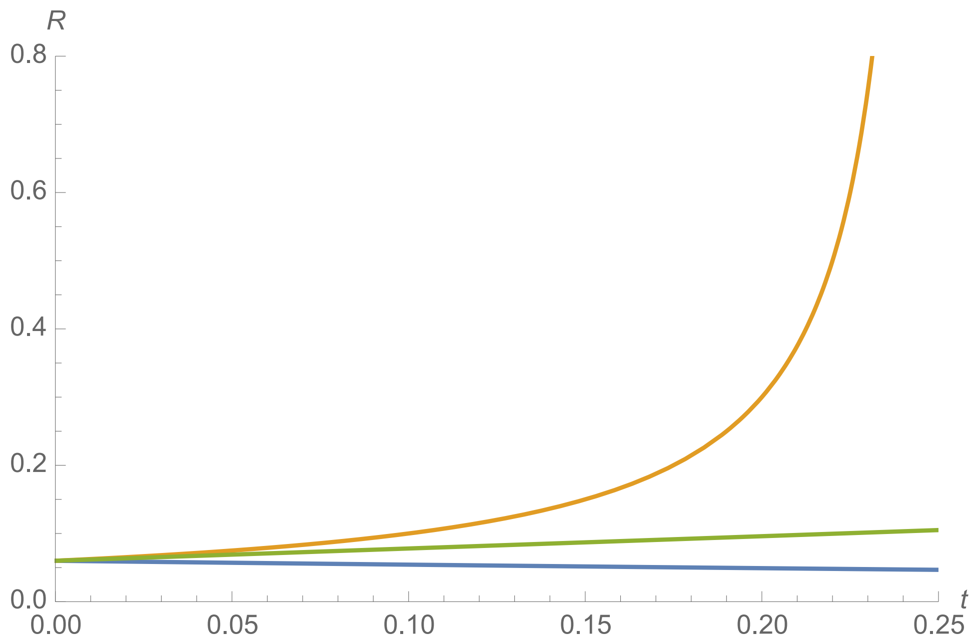

Figure 2.2: Scalar Curvature of Sphere (orange), Ricci flat space (green) and the scalar curvature lower bound (2.21) (blue) for the unnormalised Kähler-Ricci flow.

Proof. We calculate

$$\frac{\partial}{\partial t}\ln(\omega(t)^n) = \mathrm{tr}_\omega\left(\frac{\partial}{\partial t}\omega\right) = -R - n,$$

where the trace identity $\mathrm{tr}_\omega \alpha = n\frac{\alpha\wedge\omega^{n-1}}{\omega^n}$ was used in the first equality. Integrating and solving for $\omega(t)$ yields (2.24) $\square$

In the unnormalised flow (only scalar curvature appears on the right-hand side of the above equation), the scalar curvature's sign determines the manifold's expansion or contraction. If the scalar curvature blows up at time $T < \infty$, we must have $R(t) \to \infty$ as $t \to T$, and thus (2.24) implies that $\omega(t) \to 0$ as $t \to T$. In other words, the geometry degenerates if the scalar curvature blows up.

2.3.3 The Metric Trace

Next, we calculate the evolution of the metric trace. Considering the trace between metrics allows us to apply scalar maximum principles to conclude bounds on the metric tensor. These will involve comparing the solutions metric, $\omega(t)$, to a fixed metric[2], $\hat{\omega}$. We therefore introduce the following notation.

Notation: We carry any symbolic modifier of the metric ω^* to local components $g^*_{i\bar{j}}$, as well as the geometric quantities associated with them, $\nabla^*, \Gamma^*, \mathrm{Rm}(\omega^*), R^*$, etc.

[2]In our later analysis, we will use the refernce metric ω_t or another solution to the flow, $\widetilde{\omega}(t)$

Proposition 2.17. *Let $\widehat{\omega}$ be a fixed metric on X and $\omega(t)$ be a solution to the Kähler-Ricci flow,* (1). *Then*

$$\left(\frac{\partial}{\partial t} - \Delta_\omega\right) \operatorname{tr}_{\widehat{\omega}} \omega = -\operatorname{tr}_{\widehat{\omega}} \omega - g^{\bar{\ell}k} \widehat{R}_{k\bar{\ell}}^{\bar{j}i} g_{i\bar{j}} - \widehat{g}^{\bar{j}i} g^{\bar{q}p} g^{\bar{\ell}k} \widehat{\nabla}_i g_{p\bar{\ell}} \widehat{\nabla}_{\bar{j}} g_{k\bar{q}} \tag{2.25}$$

where the hat denotes operators and quantities associated with the metric $\widehat{\omega}$.

Proof. In normal coordinates for $\widehat{g}$ we calculate

$$\begin{aligned}
\Delta_\omega \operatorname{tr}_{\widehat{\omega}} \omega &= g^{\bar{\ell}k} \partial_k \partial_{\bar{\ell}} \left(\widehat{g}^{\bar{j}i} g_{i\bar{j}}\right) \\
&= g^{\bar{\ell}k} \left(\partial_k \partial_{\bar{\ell}} \widehat{g}^{\bar{j}i}\right) g_{i\bar{j}} + g^{\bar{\ell}k} \widehat{g}^{\bar{j}i} \partial_k \partial_{\bar{\ell}} g_{i\bar{j}} \\
&= g^{\bar{\ell}k} \widehat{R}_{k\bar{\ell}}^{\bar{j}i} g_{i\bar{j}} - \widehat{g}^{\bar{j}i} R_{i\bar{j}} + \widehat{g}^{\bar{j}i} g^{\bar{q}p} g^{\bar{\ell}k} \partial_i g_{p\bar{\ell}} \partial_{\bar{j}} g_{k\bar{q}},
\end{aligned} \tag{2.26}$$

and the time derivative of the trace, which is given by

$$\frac{\partial}{\partial t} \operatorname{tr}_{\widehat{\omega}} \omega = -\widehat{g}^{\bar{j}i} R_{i\bar{j}} - \operatorname{tr}_{\widehat{\omega}} \omega. \tag{2.27}$$

Combining these calculations yields equation (2.25). $\square$

Proposition 2.18. *Let $\widehat{\omega}$ be a fixed Kähler metric on M and $\omega(t)$ be a solution to* (1). *Then there exists a constant $\widehat{C}$ depending only on the lower bound of the bisectional curvature for $\widehat{\omega}$ such that*

$$\left(\frac{\partial}{\partial t} - \Delta_\omega\right) \log \operatorname{tr}_{\widehat{\omega}} \omega \leqslant \widehat{C} \operatorname{tr}_\omega \widehat{\omega}. \tag{2.28}$$

Proof. For any positive function $f : X \to \mathbb{R}$,

$$\Delta_\omega \log f = \frac{\Delta_\omega f}{f} - \frac{|\partial f|_g^2}{f^2}.$$

Then, by a Schwartz lemma argument

$$\left(\frac{\partial}{\partial t} - \Delta_\omega\right) \log \operatorname{tr}_{\widehat{\omega}} \omega = \frac{1}{\operatorname{tr}_{\widehat{\omega}} \omega} \left(-\operatorname{tr}_{\widehat{\omega}} \omega - g^{\bar{\ell}k} \widehat{R}_{k\bar{\ell}}^{\bar{j}i} g_{i\bar{j}} + \frac{\left|\partial \operatorname{tr}_{\widehat{\omega}} \omega\right|_g^2}{\operatorname{tr}_{\widehat{\omega}} \omega} - \widehat{g}^{\bar{j}i} g^{\bar{q}p} g^{\bar{\ell}k} \widehat{\nabla}_i g_{p\bar{\ell}} \widehat{\nabla}_{\bar{j}} g_{k\bar{q}} \right). \tag{2.29}$$

By the Cauchy-Schwarz inequality, we can show in normal coordinates that

$$\frac{\left|\partial \operatorname{tr}_{\widehat{\omega}} \omega\right|_g^2}{\operatorname{tr}_{\widehat{\omega}} \omega} - \widehat{g}^{\bar{j}i} g^{\bar{q}p} g^{\bar{\ell}k} \widehat{\nabla}_i g_{p\bar{\ell}} \widehat{\nabla}_{\bar{j}} g_{k\bar{q}} \leqslant 0. \tag{2.30}$$

In these same coordinates and using the lower bound on sectional curvature, we have

$$g^{\bar{\ell}k} \widehat{R}_{k\bar{\ell}}^{\bar{j}i} g_{i\bar{j}} = \sum_{k,i} g^{\bar{k}k} \widehat{R}_{k\bar{k}i\bar{i}} g_{i\bar{i}} \geqslant -\widehat{C} \sum_k g^{\bar{k}k} \sum_i g_{i\bar{i}} = -\widehat{C} \left(\operatorname{tr}_{\widehat{\omega}} \omega\right) \left(\operatorname{tr}_\omega \widehat{\omega}\right). \tag{2.31}$$

Combining these two inequalities with (2.29) yields the desired inequality. $\square$

The inequality is reminiscent of Yau's Schwartz inequality. It turns out that we can derive a parabolic analogue in the form of an evolution inequality.

Proposition 2.19. *Let $f : M \to N$ be a holomorphic map between compact complex manifolds M and N of complex dimension n and κ respectively. Let ω_0 and ω_N be Kähler metrics on M and N respectively and let $\omega = \omega(t)$ be a solution to the Kähler-Ricci flow on $M \times [0, T)$. Then for all points of $M \times [0, T)$ with $\operatorname{tr}_\omega (f^*\omega_N)$ positive we have*

$$\left(\frac{\partial}{\partial t} - \Delta_\omega\right) \log \operatorname{tr}_\omega (f^*\omega_N) \leqslant C_N \operatorname{tr}_\omega (f^*\omega_N)$$

where C_N is an upper bound for the bisectional curvature of ω_N.

Proof. The proof follows a similar calculation done in Proposition 2.18. □

2.3.4 Third Order Estimates

We define

$$\Psi_{ij}^k := \Gamma_{ij}^k - \widehat{\Gamma}_{ij}^k = g^{\bar{\ell}k}\widehat{\nabla}_i g_{j\bar{\ell}} \tag{2.32}$$

and

$$S = g^{\bar{j}i} g^{\bar{l}k} g^{\bar{q}p} \widehat{\nabla}_i g_{k\bar{q}} \overline{\widehat{\nabla}_j g_{l\bar{p}}}. \tag{2.33}$$

Although the Christoffel symbols are not tensor coefficients, the difference in Christoffel symbols are tensorial and thus are coordinate-independent. The tensor S measures the difference between the connections for ∇ and $\widehat{\nabla}$.

Proposition 2.20. *The tensor S evolves along the flow by*

$$\left(\frac{\partial}{\partial t} - \Delta_\omega\right) S = -|\overline{\nabla}\Psi|_g^2 - |\nabla\Psi|_g^2 + |\Psi|_g^2 - 2\operatorname{Re}\left(g^{\bar{j}i} g^{\bar{q}p} g_{k\bar{\ell}} \nabla^{\bar{b}} \widehat{R}_{i\bar{b}p}^k \overline{\Psi_{jq}^\ell}\right). \tag{2.34}$$

Proof. From a direct computation, we have

$$\Delta_\omega S = g^{\bar{j}i} g^{\bar{q}p} g_{k\bar{\ell}} \left(\left(\Delta_\omega \Psi_{ip}^k\right) \overline{\Psi_{jq}^\ell} + \Psi_{ip}^k \left(\overline{\bar{\Delta}_\omega \Psi_{jq}^\ell}\right)\right) + |\overline{\nabla}\Psi|_g^2 + |\nabla\Psi|_g^2$$

Using the commutation formula

$$\bar{\Delta}_\omega \Psi_{jq}^\ell = \Delta_\omega \Psi_{jq}^\ell + R_j^b \Psi_{bq}^\ell + R_q^b \Psi_{jb}^\ell - R_b^\ell \Psi_{jq}^b, \tag{2.35}$$

we rewrite the Laplacian as

$$\begin{aligned}\Delta_\omega S = {} & 2\operatorname{Re}\left(g^{\bar{j}i} g^{\bar{q}p} g_{k\bar{\ell}} \left(\Delta_\omega \Psi_{ip}^k\right) \overline{\Psi_{jq}^\ell}\right) + |\overline{\nabla}\Psi|_g^2 + |\nabla\Psi|_g^2 \\ & + R^{\bar{j}\bar{i}} g^{\bar{q}p} g_{k\bar{\ell}} \Psi_{ip}^k \overline{\Psi_{jq}^\ell} + g^{\bar{j}i} R^{\bar{q}p} g_{k\bar{\ell}} \Psi_{ip}^k \overline{\Psi_{jq}^\ell} - g^{\bar{j}i} g^{\bar{q}p} R_{k\bar{\ell}} \Psi_{ip}^k \overline{\Psi_{jq}^\ell}\end{aligned} \tag{2.36}$$

Taking a time derivative of Ψ yields

$$\frac{\partial}{\partial t}\Psi_{ip}^k = \frac{\partial}{\partial t}\Gamma_{ip}^k = -\nabla_i R_p^k. \tag{2.37}$$

The derivative of the curvature tensor can be rewritten in terms of Ψ. To achieve this, we take a derivative of (2.32) to obtain

$$\nabla_{\bar{b}}\Psi^k_{ip} = \partial_{\bar{b}}\left(\Gamma^k_{ip} - \widehat{\Gamma}^k_{ip}\right) = \widehat{R}_{i\bar{b}p}{}^k - R_{i\bar{b}p}{}^k, \tag{2.38}$$

and then a second derivative, followed by an application of the second Bianchi identity to obtain

$$\Delta_\omega\Psi^k_{ip} = g^{\bar{b}a}\nabla_a\nabla_{\bar{b}}\Psi^k_{ip} = \nabla^{\bar{b}}\widehat{R}_{i\bar{b}p}{}^k - \nabla_i R_p{}^k. \tag{2.39}$$

Combining equation (2.37) and (2.39) yields

$$\frac{\partial}{\partial t}\Psi^k_{ip} = \Delta\Psi^k_{ip} - \nabla^{\bar{b}}\widehat{R}_{i\bar{b}p}{}^k. \tag{2.40}$$

The above allows us to compute

$$\begin{aligned}\frac{\partial}{\partial t}S = {} & 2\,\mathrm{Re}\left(g^{\bar{j}i}g^{\bar{q}p}g_{k\bar{\ell}}\left(\Delta_\omega\Psi^k_{ip} - \nabla^{\bar{b}}\widehat{R}^k_{i\bar{b}p}\right)\overline{\Psi^\ell_{jq}}\right) + R^{\bar{j}i}g^{\bar{q}p}g_{k\bar{\ell}}\Psi^k_{ip}\overline{\Psi^\ell_{jq}} \\ & + g^{\bar{j}i}R^{\bar{q}p}g_{k\bar{\ell}}\Psi^k_{ip}\overline{\Psi^\ell_{jq}} - g^{\bar{j}i}g^{\bar{q}p}R_{k\bar{\ell}}\Psi^k_{ip}\overline{\Psi^\ell_{jq}} + |\Psi|^2_g.\end{aligned} \tag{2.41}$$

To complete the calculation, we combine (2.36) and (2.41). □

It turns out that an equivalence between a solution to the Kähler-Ricci flow, $\omega(t)$, and some fixed metric $\widehat{\omega}$ is sufficient to control S.

Theorem 2.21 (Calabi's Estimate)**.** *Let $\omega(t)$ be a solution to the Kähler-Ricci flow, for $t \in [0, T)$ where $T \in (0, \infty]$. If there exists $C_0 > 0$ such that*

$$\frac{1}{C_0}\widehat{\omega} \leqslant \omega \leqslant C_0\widehat{\omega} \tag{2.42}$$

for all $t \in [0, T)$, then for some $C = C(C_0, \widehat{\omega}) > 0$ uniform of time, we have

$$S := \|\nabla_{\widehat{g}}g\|^2_\omega \leqslant C \tag{2.43}$$

for all $t \in [0, T)$.

Proof. The idea is to use the trace evolution and the evolution of S itself to force a favourable sign in a maximum principle argument. We first modify the last term in (2.34) by rewriting ∇ in terms of $\widehat{\nabla}$ and Ψ as follows

$$\nabla^{\bar{b}}\widehat{R}_{i\bar{b}p}{}^k = g^{\bar{b}r}\widehat{\nabla}_r\widehat{R}_{i\bar{b}p}{}^k - g^{\bar{b}r}\Psi^a_{ir}\widehat{R}_{a\bar{b}p}{}^k - g^{\bar{b}r}\Psi^a_{pr}\widehat{R}_{i\bar{b}a}{}^k + g^{\bar{b}r}\Psi^k_{ar}\widehat{R}_{i\bar{b}p}{}^a. \tag{2.44}$$

By our assumption on the metric (2.42) and the Cauchy-Schwartz inequality, we have

$$\left|2\,\mathrm{Re}\left(g^{\bar{j}i}g^{\bar{q}p}g_{k\bar{\ell}}\nabla^{\bar{b}}\widehat{R}^k_{i\bar{b}p}\overline{\Psi^\ell_{jq}}\right)\right| \leqslant C_1(S + \sqrt{S}) \leqslant 2C_1(S+1). \tag{2.45}$$

Then (2.34) implies

$$\left(\frac{\partial}{\partial t} - \Delta_\omega\right)S \leqslant -|\overline{\nabla}\Psi|^2_g - |\nabla\Psi|^2_g + C_2 S + C_2. \tag{2.46}$$

On the other hand, the last term in the trace evolution (2.25) can be rewritten using

$$\widehat{g}^{\bar{j}i}g^{\bar{q}p}g^{\bar{\ell}k}\widehat{\nabla}_i g_{p\bar{\ell}}\widehat{\nabla}_{\bar{j}}g_{k\bar{q}} \geqslant C^{-1}S,$$

which is an application of the Cauchy-Schwartz inequality in local normal coordinates. This gives us

$$\left(\frac{\partial}{\partial t} - \Delta_\omega\right) \operatorname{tr}_{\widehat{\omega}} \omega \leqslant C - C^{-1}S. \tag{2.47}$$

Now define $Q = S + A \operatorname{tr}_{\widehat{\omega}} \omega$ for some $A > 0$ to be determined later. Combining (2.46) and (2.47), we obtain

$$\left(\frac{\partial}{\partial t} - \Delta_\omega\right) Q \leqslant -S + C_4, \tag{2.48}$$

for $A > 0$ chosen large enough. It follows that S is bounded from above at a point where Q achieves its maximum. Since $\operatorname{tr}_{\widehat{\omega}} \omega$ is uniformly bounded, Q_{max} is also uniformly bounded. Therefore,

$$S = Q - A \operatorname{tr}_{\widehat{\omega}} \omega \leqslant Q_{max} \leqslant C,$$

and the conclusion follows. □

The following Corollary will be useful in deriving curvature estimates.

Corollary 2.22. *There exists $C' = C'(C_0, \widehat{\omega})$ such that*

$$\left(\frac{\partial}{\partial t} - \Delta_\omega\right) S \leqslant -\frac{1}{2}\|\mathrm{Rm}\|_\omega^2 + C'. \tag{2.49}$$

Proof. Observe that

$$|\overline{\nabla}\Psi|_g^2 = \left|\widehat{R}_{i\bar{b}p}{}^k - R_{i\bar{b}p}{}^k\right|_g^2 \geqslant \frac{1}{2}|\mathrm{Rm}|_g^2 - C_5 \tag{2.50}$$

Since S is bounded, we can obtain the desired estimate from (2.46). □

2.3.5 Curvature Estimates

Using the Bianchi identities, one can derive the evolution of the Riemannian curvature tensor under the flow. This expression is quite complicated and is given by

$$\frac{\partial}{\partial t}\mathrm{Rm} = \frac{1}{2}\Delta_{\mathbb{R}}\mathrm{Rm} + \mathrm{Rm} + \mathrm{Rm} * \mathrm{Rm} + \mathrm{Rc} * \mathrm{Rm}. \tag{2.51}$$

Here, $A * B$ represents a linear combination of the tensors A and B contracted by the metric g associated with ω. From the equation above, one can obtain a more manageable inequality

$$\left(\frac{\partial}{\partial t} - \Delta_{\mathbb{R}}\right) \|\mathrm{Rm}\|_\omega^2 \leqslant -\|\nabla\mathrm{Rm}\|_\omega^2 - \left\|\overline{\nabla}\mathrm{Rm}\right\|_\omega^2 + C\|\mathrm{Rm}\|_\omega^3 + \|\mathrm{Rm}\|_\omega^2, \tag{2.52}$$

where $C > 0$. Moreover, for all points of $X \times [0, T)$ where $\|\mathrm{Rm}\|_\omega$ is not zero,

$$\left(\frac{\partial}{\partial t} - \Delta_{\mathbb{R}}\right) \|\mathrm{Rm}\|_\omega \leqslant \frac{C}{2}\|\mathrm{Rm}\|_\omega^2 - \frac{1}{2}\|\mathrm{Rm}\|_\omega. \tag{2.53}$$

The above equation and Calabi estimates permit us to obtain the following Theorem.

Theorem 2.23. *Let $\omega(t)$ be a solution to* (1) *and $\widehat{\omega}$ be a fixed Kähler metric. Assume there exists $C_0 > 0$ such that*

$$\frac{1}{C_0}\widehat{\omega} \leqslant \omega \leqslant C_0\widehat{\omega}. \tag{2.54}$$

Then there exist $C = C(C_0, \widehat{\omega})$ such that

$$\|\mathrm{Rm}(\omega)\|_\omega^2 \leqslant C. \tag{2.55}$$

Proof. Using (2.49) and (2.53), we derive the evolution of the quantity $Q = |\mathrm{Rm}|_g + AS$ for a large enough $A > 0$. This is given by

$$\left(\frac{\partial}{\partial t} - \Delta\right) Q \leqslant -|\mathrm{Rm}|_g^2 + C', \tag{2.56}$$

applying the maximum principle gives us a uniform bound on $|\mathrm{Rm}|$. □

Once we obtain curvature bounds, a result from Hamilton gives us bounds for all derivatives of curvature. Denoting $\nabla_{\mathbb{R}}$ to be the covariant derivative of the Riemannian metric, we have

Theorem 2.24. *If* (2.55) *holds, then there exist uniform constants C_m for $m = 1, 2, \ldots$ such that*

$$|\nabla_{\mathbb{R}}^m \mathrm{Rm}|^2 \leqslant C_m.$$

Proof. The crux of the proof is the following evolution equation

$$\left(\frac{\partial}{\partial t} - \frac{1}{2}\Delta_{\mathbb{R}}\right) |\nabla_{\mathbb{R}}^m \mathrm{Rm}|^2 = -\left|\nabla_{\mathbb{R}}^{m+1}\mathrm{Rm}\right|^2 + \sum_{p+q=m} \nabla_{\mathbb{R}}^p \mathrm{Rm} * \nabla_{\mathbb{R}}^q \mathrm{Rm} * \nabla_{\mathbb{R}}^m \mathrm{Rm} \tag{2.57}$$

which enables an inductive argument via the maximum principle for which the highest-order derivative of curvature has a favourable sign. □

2.3.6 Consequences

The collection of results in the previous sections provides a blueprint for proving curvature bounds given control on the metric. In combination, we see that the results imply that if a solution to the flow remains finite, then higher-order regularity of the metric can be. obtained.

Theorem 2.25. *Let $\omega(t)$ be a solution to the Kähler-Ricci flow for $0 < t < T \in (0, \infty]$. Suppose there exists $\widehat{\omega}$ a Kähler form such that*

$$C^{-1}\widehat{\omega} \leqslant \omega \leqslant C\widehat{\omega}, \tag{2.58}$$

for some $C > 0$. Then for every $m \in \mathbb{N}$, there exists a constant $C_m > 0$ such that

$$\|\omega(t)\|_{C^m(X,\widehat{g})} \leqslant C_m. \tag{2.59}$$

Proof. Theorem 2.23 and Theorem 2.24 imply control for all derivatives of the curvature tensor. Fixing a time $t \in (0, T]$ in local holomorphic coordinates consider the Poisson equation given by

$$\Delta_{\mathrm{E}} g_{i\bar{j}} = -\sum_{k} R_{k\bar{k}i\bar{j}} + \sum_{k.p.q} g^{q\bar{p}} \partial_k g_{i\bar{q}} \partial_{\bar{k}} g_{p\bar{j}} =: Q_{i\bar{j}}. \tag{2.60}$$

The higher order estimates imply $\left\|Q_{i\bar{j}}\right\|_{L^p(B)}$ is uniformly bounded for all $p > 2n$. By standard elliptic estimates we have a bound on $\left\|g_{i\bar{j}}\right\|_{L^p(X,g_{\mathbb{E}})}$. Morrey's embedding theorem gives that $\left\|g_{i\bar{j}}\right\|_{C^{1+\beta}}$ is uniformly bounded for some $0 < \beta < 1$. Since derivatives of $Q_{i\bar{j}}$ are tensor contractions of covariant derivatives of Rm or $m+1$ derivatives of $g_{i\bar{j}}$ we have that Q is uniformly bounded in $C^{m+\beta}$ if g is uniformly bounded in $C^{m+1+\beta}$. For $m = 0$ we have $\|Q_{i\bar{i}}\|_{C^\beta}$ is uniformly bounded from which Shauder estimates imply $\left\|g_{i\bar{j}}\right\|_{C^{2+\beta}}$ is uniformly bounded. Applying a bootstrapping argument completes the proof. □

Furthermore, these estimates can be localised.

Theorem 2.26. *Let $\omega(t)$ be a solution to the Kähler-Ricci flow on $U \times [0, T)$ with $0 \leqslant T \leqslant \infty$, where U is an open subset of X. Assume that there exists a constant C for such that*

$$\frac{1}{C}\widetilde{\omega} \leqslant \omega \leqslant C\widetilde{\omega}.$$

Then for any compact subset $K \subset\subset U$ and for $m = 1, 2, \ldots,$ there exist constants C'_m depending only on $\widetilde{\omega}, K$ and U such that

$$\|\omega(t)\|_{C^m(K,\widetilde{g})} \leqslant C'_m.$$

Using arguments of a similar form, we can generate various well-behaved flows on a Euclidean ball.

Theorem 2.27. *Let $B_1(0)$ be the standard Euclidean unit ball and $\omega(t)$ is a solution to*

$$\frac{\partial}{\partial t}\omega = -\mathrm{Ric}(\omega) - k\omega \tag{2.61}$$

on $B_1(0) \times [0, T]$ so that

$$A^{-1} g_{\mathbb{C}^n} \leqslant g(t) \leqslant A g_{\mathbb{C}^n}$$

for some $A > 1$ and $|k| \leqslant k_0$. Then for all $m \in \mathbb{N}$, there exist $C(n, m, T, A, k_0)$'s such that for all $t \in \left[\frac{1}{2}T, T\right]$,

$$\sup_{B_{1/2}(0)} |\nabla^{m,g_{\mathbb{C}^n}} g(t)| \leqslant C(n, m, T, A, k_0).$$

We can produce a similar result with a perturbation in the complex structure J.

Theorem 2.28. *For all $n, k, l_0 \in \mathbb{N}, \alpha \in (0, 1)$ and $A > 1$, there are $\kappa(n, \alpha)$ and $C(k, n, \alpha, A, l_0) > 0$ such that the following holds. Let $B_1(0)$ be the unit ball in $\mathbb{C}^n$ with the standard Euclidean metric $\omega_{\mathbb{C}^n}$ and J be a complex structure on $B_1(0)$ such that*

$$\|J - J_{\mathbb{C}^n}\|_{C^{1,\alpha}(B_1(0))} < \kappa, \quad \|J - J_{\mathbb{C}^n}\|_{C^{k,\alpha}(B_1(0))} \leqslant A.$$

If $\omega(t)$ is a family of J-Kähler metrics satisfying

$$\frac{\partial}{\partial t}\omega(t) = -\operatorname{Ric}(\omega(t)) - l\omega(t) \tag{2.62}$$

on $Q_1(0) = B_1(0) \times [-1, 0]$ for some $|l| \leqslant l_0$ and

$$A^{-1}\omega_{\mathbb{C}^n} \leqslant \omega(t) \leqslant A\omega_{\mathbb{C}^n} \text{ on } Q_1(0).$$

Then we have

$$\|\omega(t)\|_{C^{k,\alpha}\left(Q_{1/2}(0)\right)} \leqslant C\left(k, n, \alpha, A, l_0\right).$$

In general, solutions to the Kähler-Ricci flow encounter singularities, and therefore, one does not expect to obtain estimates such as (2.58) globally. Given suitable cutoff functions, one can expect to obtain (2.58) away from singular sets. Furthermore, the underlying topology and complex structure are expected to determine the singular set.

Chapter 3

The Continuity Method

The continuity method can be interpreted as the elliptic version of the Kähler-Ricci flow. The equation was introduced by Rubinstein in [Rub08] and its application to the Analytic Minimal model program was first suggested and studied by La Nave and Tian in [LT16]. Interestingly, the continuity method shares many characteristics with the Kähler-Ricci flow, due to their similarity at the level of cohomology. However, unlike the flow, the continuity method is not parabolic. Fortunately, this is compensated with a uniform lower Ricci bound along the deformation, allowing geometric comparison techniques to be readily applied.

3.1 Complex Monge–Ampère Equation

The reduction to a Monge–Ampère equation follows the same procedure as the flow. In fact, we reach the evolution of the cohomology class more immediately. Using the same $H^{(1,1)}(X, \mathbb{R})$ cohomology defined in (1.7), the class of the continuity method is

$$(1+t)[\omega(t)] = [\omega_0] - tc_1(X). \tag{3.1}$$

We choose a representative $\chi \in -c_1(X)$ and define a volume form Ω on X such that $\chi = \sqrt{-1}\partial\overline{\partial}\log\Omega$. Then a representative of $[\omega(t)]$ is

$$\omega_t = \frac{1}{1+t}\omega_0 + \frac{t}{1+t}\chi, \tag{3.2}$$

and using $[\omega(t)] = [\omega_t]$, we have

$$\omega(t) = \omega_t + \sqrt{-1}\partial\overline{\partial}\varphi. \tag{3.3}$$

The continuity method is equivalent to the following Elliptic Monge–Ampère equation.

$$\begin{cases} (1+t)\varphi(t) = t\log\left(\dfrac{\omega^n}{\Omega}\right) \\ \varphi(0) = 0. \end{cases} \tag{3.4}$$

Proposition 3.1. *A smooth family $\omega(t)$ solves the continuity method for $t \in [0, T)$ if and only there exists $\varphi(t)$ a smooth family of functions such that $\omega(t) = \omega_t + \sqrt{-1}\partial\overline{\partial}\,\varphi$ satisfying* (3.4) *for $t \in [0, T)$.*

Proof. Given a solution of (3.4), we define $\omega(t) = \omega_t + \sqrt{-1}\partial\overline{\partial}\varphi$ and find that it satisfies

$$(1+t)\omega(t) = \omega_0 + t\chi - t\mathrm{Ric}(\omega) + t\mathrm{Ric}(\Omega) = \omega_0 - t\mathrm{Ric}(\omega).$$

On the other hand, given $\omega(t)$ a solution to (2), we can define $\varphi(t)$ directly according to (3.4). Then for all t

$$\begin{aligned}\omega(t) - \omega_t - \sqrt{-1}\partial\overline{\partial}\varphi(t) &= -\frac{t}{1+t}\mathrm{Ric}(\omega) - \frac{t}{1+t}\chi - \sqrt{-1}\partial\overline{\partial}\varphi(t) \\ &= -\frac{t}{1+t}\chi + \frac{t}{1+t}\mathrm{Ric}(\Omega) \\ &= 0\end{aligned}$$

for all $t \in [0, T)$. □

Regularity obtained on φ translates to regularity on the metric. As with the Kähler-Ricci flow, uniform bounds on the potential and a limiting Kähler class are sufficient to show full curvature bounds on the curvature tensor and its higher order derivatives.

Proposition 3.2. *Let φ be a solution to* (2.12)*. If there exist $A > 0$ such that*

$$|\varphi| \leqslant A, \tag{3.5}$$

for all $t \in [0, T)$ and $-c_1(X) > 0$, that is, there exists a Kähler representative χ, then there exists $C > 0$ such that

$$C^{-1}\omega_0 \leqslant \omega \leqslant C\omega_0 \tag{3.6}$$

for all $t \in [0, T)$.

Proof. The proof is similar to that in the flow. We have

$$\Delta_\omega \log \mathrm{tr}_\omega\, \omega_0 \geqslant -C - C_0\, \mathrm{tr}_\omega\, \omega_0 \tag{3.7}$$

where C_0 depends on the bisectional curvature of ω_0 and $C > 0$. Since $-c_1(X)$ is a Kähler class, we can choose a representative χ such that $\chi \geqslant C^{-1}\omega_0$ for some $C > 0$. Then define

$$Q := \log \mathrm{tr}_\omega\, \omega_0 + A\varphi$$

for $A > 0$ to be chosen later. Then

$$\Delta_\omega Q \geqslant -C - C_0\, \mathrm{tr}_\omega\, \omega_0 + A\, \mathrm{tr}_\omega\, \omega_0 \geqslant -C + \mathrm{tr}_\omega\, \omega_0$$

for A chosen sufficiently large enough. Then the maximum principle yields

$$\mathrm{tr}_\omega\, \omega_0 \leqslant C. \tag{3.8}$$

From the Monge–Ampère equation (3.4), we have

$$C_2^{-1}\omega_0^n \leqslant C_1^{-1}\Omega \leqslant \omega^n \leqslant C_1\Omega \leqslant C_2\omega_0^n, \tag{3.9}$$

where $C_1, C_2 > 0$ uniform of time. Combining this with the eigenvalue estimate

$$\mathrm{tr}_{\omega_0}\, \omega \leqslant n(\mathrm{tr}_\omega\, \omega_0)^{n-1}\frac{\omega^n}{\omega_0^n} \leqslant C$$

yields the conclusion

$$C^{-1}\omega_0 \leqslant \omega \leqslant C\omega_0$$

for all $t \in [0, T)$. □

3.2 Evolution of Tensors

Rather than having quantities evolve according to a parabolic equation with some time derivative, the evolution is determined by the equation of the continuity method directly. Furthermore, the appropriate operator for the continuity method is the Laplacian; thus, many calculations carry over from the flow.

3.2.1 Scalar Curvature

Proposition 3.3. *Let $\omega(t)$ be a solution to the continuity method. The scalar curvature of $\omega(t)$ evolves as*

$$R(t) = \frac{1}{t}\operatorname{tr}_{\omega(t)}\omega_0 - \frac{n(1+t)}{t}. \tag{3.10}$$

Proof. The result follows from taking the trace of the continuity method. □

Corollary 3.4. *The scalar curvature is bounded from below.*

Remark 3.5. Therefore any blow-up of the scalar curvature must be to $+\infty$.

3.2.2 The Metric Trace

As with the Kähler-Ricci flow, metric trace bounds allow us to conclude metric bounds using a scalar maximum principle. The trace is derived using a Schwartz lemma calculation and replacing the Ricci curvature with the metric according to the continuity method (2).

Proposition 3.6. *Let $\widehat{\omega}$ be a fixed metric on X and ω be the evolving metric satisfying* (2). *Then*

$$\Delta_\omega \operatorname{tr}_{\widehat{\omega}}\omega = g^{\bar{\ell}k}\widehat{R}_{k\bar{\ell}}^{\bar{j}i}g_{i\bar{j}} - \frac{1}{t}\operatorname{tr}_{\widehat{\omega}}\omega_0 + \left(\frac{1+t}{t}\right)\operatorname{tr}_{\widehat{\omega}}\omega + \widehat{g}^{\bar{j}i}g^{\bar{q}p}g^{\bar{\ell}k}\partial_i g_{p\bar{\ell}}\partial_{\bar{j}}g_{k\bar{q}} \tag{3.11}$$

where the hat denotes operators and quantities associated with the metric $\widehat{\omega}$.

Proof. In normal coordinates for $\widehat{g}$ we calculate

$$\begin{aligned}\Delta_\omega \operatorname{tr}_{\widehat{\omega}}\omega &= g^{\bar{\ell}k}\partial_k\partial_{\bar{\ell}}\left(\widehat{g}^{\bar{j}i}g_{i\bar{j}}\right)\\ &= g^{\bar{\ell}k}\left(\partial_k\partial_{\bar{\ell}}\widehat{g}^{\bar{j}i}\right)g_{i\bar{j}} + g^{\bar{\ell}k}\widehat{g}^{\bar{j}i}\partial_k\partial_{\bar{\ell}}g_{i\bar{j}}\\ &= g^{\bar{\ell}k}\widehat{R}_{k\bar{\ell}}^{\bar{j}i}g_{i\bar{j}} - \widehat{g}^{\bar{j}i}R_{i\bar{j}} + \widehat{g}^{\bar{j}i}g^{\bar{q}p}g^{\bar{\ell}k}\partial_i g_{p\bar{\ell}}\partial_{\bar{j}}g_{k\bar{q}},\end{aligned} \tag{3.12}$$

and by replacing $R_{i\bar{j}}$ with the components of the continuity method

$$R_{i\bar{j}} = \frac{1}{t}g^0_{i\bar{j}} - \left(\frac{1+t}{t}\right)g_{i\bar{j}}, \tag{3.13}$$

equation (3.11) follows. □

3.2.3 Third Order Estimates

With the same notation as above, we define

$$\Psi_{ij}^k := \Gamma_{ij}^k - (\Gamma^0)_{ij}^k = g^{\bar{\ell}k}\nabla_i^0 g_{j\bar{\ell}} \tag{3.14}$$

and

$$S = g^{\bar{j}i}g^{\bar{l}k}g^{\bar{q}p}\nabla_i^0 g_{k\bar{q}}\overline{\nabla_j^0 g_{l\bar{p}}}. \tag{3.15}$$

The quantity S measures the difference between the connections for ∇ and ∇^0.

Proposition 3.7. *The tensor S evolves along the flow by*

$$\begin{aligned}\Delta_\omega S = &\frac{1}{t}\left(g_0^{\bar{j}i}g^{\bar{q}p}g_{k\bar{l}} + g^{\bar{j}i}g_0^{\bar{q}p}g_{k\bar{l}} - g^{\bar{j}i}g^{\bar{q}p}g_{k\bar{l}}^0\right)\Psi_{ip}^k\overline{\Psi_{jq}^l} - \frac{1+t}{t}S\\ &+ 2\,\mathrm{Re}\left(g^{\bar{j}i}g^{\bar{q}p}g_{k\bar{l}}\left(\nabla^{\bar{b}}(R_0)_{i\bar{b}p}^k\right)\overline{\Psi_{jq}^l}\right) + \frac{2}{t}\,\mathrm{Re}\left(g^{\bar{j}i}g^{\bar{q}p}g_{n\bar{l}}^0\Psi_{ip}^n\overline{\Psi_{jq}^l}\right)\\ &+ |\overline{\nabla}\Psi|_g^2 + |\nabla\Psi|_g^2\end{aligned} \tag{3.16}$$

Proof. Following the same calculation carried out in the Ricci flow, the laplacian is given by

$$\begin{aligned}\Delta_\omega S = &2\,\mathrm{Re}\left(g^{\bar{j}i}g^{\bar{q}p}g_{k\bar{\ell}}\left(\Delta_\omega\Psi_{ip}^k\right)\overline{\Psi_{jq}^\ell}\right) + |\overline{\nabla}\Psi|_g^2 + |\nabla\Psi|_g^2\\ &+ R^{\bar{j}i}g^{\bar{q}p}g_{k\bar{\ell}}\Psi_{ip}^k\overline{\Psi_{jq}^\ell} + g^{\bar{j}i}R^{\bar{q}p}g_{k\bar{\ell}}\Psi_{ip}^k\overline{\Psi_{jq}^\ell} - g^{\bar{j}i}g^{\bar{q}p}R_{k\bar{\ell}}\Psi_{ip}^k\overline{\Psi_{jq}^\ell}\end{aligned} \tag{3.17}$$

Each Ricci term can be rewritten using the continuity method. The second line in (3.17) which we denote as I is given by

$$I = \frac{1}{t}\left(g_0^{\bar{j}i}g^{\bar{q}p}g_{k\bar{l}} + g^{\bar{j}i}g_0^{\bar{q}p}g_{k\bar{l}} - g^{\bar{j}i}g^{\bar{q}p}g_{k\bar{l}}^0\right)\Psi_{ip}^k\overline{\Psi_{jq}^l} - \frac{1+t}{t}S. \tag{3.18}$$

Using

$$\Delta_\omega\Psi_{ip}^k = \nabla^{\bar{b}}(R_0)_{i\bar{b}p}^k - \nabla_i R_p^k, \tag{3.19}$$

the first term in (3.17) involving the Laplacian is

$$II := 2\,\mathrm{Re}\left(g^{\bar{j}i}g^{\bar{q}p}g_{k\bar{l}}\left(\nabla^{\bar{b}}(R_0)_{i\bar{b}p}^k\right)\overline{\Psi_{jq}^l}\right) - 2\,\mathrm{Re}\left(g^{\bar{j}i}g^{\bar{q}p}g_{k\bar{l}}\left(\nabla_i R_p^k\right)\overline{\Psi_{jq}^l}\right). \tag{3.20}$$

To treat the second term in the above equation, we once again use the continuity equation (2) to obtain

$$R_p^k = \frac{1}{t}g^{\bar{r}k}g_{p\bar{r}}^0 - \frac{1+t}{t}g^{\bar{r}k}g_{p\bar{r}}.$$

Here, g_0 is the metric associated with the form ω_0. Taking the covariant derivative and using metric compatibility yields

$$\nabla_i R_p^k = \frac{1}{t}g^{\bar{r}k}\nabla_i g_{p\bar{r}}^0.$$

Hence, we can rewrite said second term of II as

$$\begin{aligned}-2\,\mathrm{Re}\left(g^{\bar{j}i}g^{\bar{q}p}g_{k\bar{l}}\left(\nabla_i R_p^k\right)\overline{\Psi_{jq}^l}\right) &= -\frac{2}{t}\,\mathrm{Re}\left(g^{\bar{j}i}g^{\bar{q}p}\left(\nabla_i g_{p\bar{l}}^0\right)\overline{\Psi_{jq}^l}\right)\\ &= -\frac{2}{t}\,\mathrm{Re}\left(g^{\bar{j}i}g^{\bar{q}p}\left((\nabla_i - \nabla_i^0)g_{p\bar{l}}^0\right)\overline{\Psi_{jq}^l}\right)\\ &= \frac{2}{t}\,\mathrm{Re}\left(g^{\bar{j}i}g^{\bar{q}p}g_{n\bar{l}}^0\Psi_{ip}^n\overline{\Psi_{jq}^l}\right).\end{aligned} \tag{3.21}$$

Therefore, we have

$$II = 2\,\mathrm{Re}\left(g^{\bar{j}i}g^{\bar{q}p}g_{k\bar{l}}\left(\nabla^{\bar{b}}(R_0)^{k}_{i\bar{b}p}\right)\overline{\Psi^{l}_{jq}}\right)+\frac{2}{t}\,\mathrm{Re}\left(g^{\bar{j}i}g^{\bar{q}p}g^0_{n\bar{l}}\mathring{\Psi}^{n}_{ip}\overline{\Psi^{l}_{jq}}\right). \tag{3.22}$$

This leads to equation (3.16). □

The expression for the laplacian of S is quite complicated partly due to the initial metric appearing in the calculation due to (2). We therefore can significantly simplify our terms if the solution remains controlled by a fixed metric.

Theorem 3.8 (Calabi's Estimate)**.** *Let $\omega(t)$ be a solution to the continuity method, (2), and suppose there exists $C_0 > 0$ such that*

$$\frac{1}{C_0}\omega_0 \leqslant \omega \leqslant C_0\omega_0. \tag{3.23}$$

Then there exists a constant $C = C(C_0, \widehat{\omega})$ such that

$$S := |\nabla_{g_0} g|^2_g \leqslant C. \tag{3.24}$$

Proof. The idea here is to use the trace evolution and the evolution of S itself to force a favourable sign in a maximum principle argument. We first oberve that

$$\begin{aligned}\nabla^{\bar{b}}(R_0)^{k}_{i\bar{b}p} &= g^{\bar{b}a}\nabla_a(R_0)^{k}_{i\bar{b}p}\\ &= g^{\bar{b}a}\left(\nabla_a-\nabla^0_a\right)(R_0)^{k}_{i\bar{b}p}+g^{\bar{b}a}\nabla^0_a(R_0)^{k}_{i\bar{b}p}\\ &= \Psi * \mathrm{Rm}(\omega_0)+g^{\bar{b}a}\nabla^0_a(R_0)^{k}_{i\bar{b}p}.\end{aligned} \tag{3.25}$$

Then using the above, (3.23) and the Cauchy-Schwartz inequality, we have

$$\left|2\,\mathrm{Re}\left(g^{\bar{j}i}g^{\bar{q}p}g_{k\bar{\ell}}\nabla^{\bar{b}}(R_0)^{k}_{i\bar{b}p}\overline{\Psi^{\ell}_{jq}}\right)\right| \leqslant C_1(S+\sqrt{S}) \leqslant 2C_1(S+1). \tag{3.26}$$

The remaining terms in (3.16) are bounded from below by

$$\Delta_\omega S \geqslant -\frac{C}{t}S - S - 2C_1(S+1) - \frac{2}{t}\sqrt{S}\sqrt{S_0} + |\overline{\nabla}\Psi|^2_g + |\nabla\Psi|^2_g. \tag{3.27}$$

On the other hand, the last term in the trace evolution (3.11), where $\hat{g} = g^0$, is bounded by

$$g_0^{\bar{j}i}g^{\bar{q}p}g^{\bar{\ell}k}\nabla^0_i g_{p\bar{\ell}}\nabla^0_{\bar{j}}g_{k\bar{q}} \geqslant C^{-1}S$$

which follows from the Cauchy-Schwartz inequality in local normal coordinates. For the curvature term in (3.11) we have

$$g^{\bar{\ell}k}(R_0)^{\bar{j}i}_{k\bar{\ell}}g_{i\bar{j}} = \sum_{k,i} g^{\bar{k}k}(R_0)_{k\bar{k}i\bar{i}}g_{i\bar{i}} \geqslant -C_0\sum_k g^{\bar{k}k}\sum_i g_{i\bar{i}} = -C_0\left(\mathrm{tr}_{\omega_0}\omega\right)\left(\mathrm{tr}_\omega\omega_0\right). \tag{3.28}$$

The laplacian of the trace is therefore

$$\Delta_\omega \mathrm{tr}_{\omega_0}\omega \geqslant -C_0\left(\mathrm{tr}_{\omega_0}\omega\right)\left(\mathrm{tr}_\omega\omega_0\right) - \frac{1}{t}\mathrm{tr}_\omega\omega_0 + \left(\frac{1+t}{t}\right)\mathrm{tr}_{\omega_0}\omega + {C_1}^{-1}S \geqslant -C_2 + C_1^{-1}S. \tag{3.29}$$

Define $Q = S + A\operatorname{tr}_{\omega_0}\omega$ for some $A > 0$ to be determined later. Combining (3.27) and (3.29), we obtain

$$\Delta_\omega Q \geqslant S - C, \tag{3.30}$$

for $A > 0$ chosen large enough. It follows that S is bounded from above at a point where Q achieves its maximum and since $\operatorname{tr}_{\omega_0}\omega$ is uniformly bounded, Q_{max} is also uniformly bounded. Therefore,

$$S = Q - A\operatorname{tr}_{\omega_0}\omega \leqslant Q_{max} \leqslant C,$$

and the conclusion follows. □

The following will be useful in a later calculation.

Corollary 3.9. *In addition, we have that for some $C' = C'(C_0, \widehat{\omega})$ such that*

$$\Delta_\omega S \geqslant \frac{1}{2}\|\mathrm{Rm}\|^2_\omega + C'. \tag{3.31}$$

Proof. By expanding out the term and applying Hölder's inequality, we obtain

$$|\overline{\nabla}\Psi|^2_g = \left|(R_0)_{i\bar{b}p}{}^k - R_{i\bar{b}p}{}^k\right|^2_g \geqslant \frac{1}{2}|\mathrm{Rm}|^2_g - C_5 \tag{3.32}$$

Since S is bounded, (3.31) follows from (3.27). □

3.2.4 Curvature Estimates

An analogous result to Theorem 2.23 can be obtained for the continuity method.

Theorem 3.10. *Let $\omega(t)$ be a solution to the continuity method,* (2)*, and assume there exists $C_0 > 0$ such that*

$$\frac{1}{C_0}\omega_0 \leqslant \omega \leqslant C_0\omega_0. \tag{3.33}$$

Then there exist $C = C(C_0, \widehat{\omega})$ such that

$$\|\mathrm{Rm}(\omega)\|^2_\omega \leqslant C \tag{3.34}$$

Proof. From Lemma 3.2.10 in [SW13b], we have

$$\Delta_\omega \mathrm{Rm} = \nabla_{\bar{l}}\nabla_k R_{i\bar{j}} + \mathrm{Rm} * \mathrm{Rm} + \mathrm{Rc} * \mathrm{Rm}. \tag{3.35}$$

Once again, we use (2) to rewrite the Ricci term:

$$\nabla_{\bar{l}}\nabla_k R_{i\bar{j}} = \frac{1}{t}\nabla_{\bar{l}}\nabla_k g^0_{i\bar{j}} - \frac{1+t}{t}\nabla_{\bar{l}}\nabla_k g_{i\bar{j}}. \tag{3.36}$$

The second term vanishes due to metric compatibility. The first term, in normal coordinates with respect to g, can be expressed as

$$\begin{aligned}
\frac{1}{t}\nabla_{\bar{l}}\nabla_k g^0_{i\bar{j}} &= \frac{1}{t}\partial_{\bar{l}}\partial_k g^0_{i\bar{j}} - \frac{1}{t}R^{\overline{m}}_{\bar{j}i\bar{l}}g^0_{k\bar{m}}, \\
&= -\frac{1}{t}R^0_{k\bar{l}i\bar{j}} + \frac{1}{t}g_0^{\bar{q}p}(\partial_k g^0_{i\bar{q}})(\partial_{\bar{l}}g^0_{p\bar{j}}) - \frac{1}{t}R^{\overline{m}}_{\bar{j}i\bar{l}}g^0_{k\bar{m}}, \\
&= -\frac{1}{t}R^0_{k\bar{l}i\bar{j}} + \frac{1}{t}g_0^{\bar{q}p}(\nabla_k g^0_{i\bar{q}})(\nabla_{\bar{l}}g^0_{p\bar{j}}) - \frac{1}{t}R^{\overline{m}}_{\bar{j}i\bar{l}}g^0_{k\bar{m}}, \\
&= -\frac{1}{t}R^0_{k\bar{l}i\bar{j}} + \frac{1}{t}g^0_{n\bar{m}}\Psi^n_{ki}\overline{\Psi^m_{lj}} - \frac{1}{t}R^{\overline{m}}_{\bar{j}i\bar{l}}g^0_{k\bar{m}}.
\end{aligned}$$

Thus, for any $t \geqslant 1$, we have in normal coordinates

$$\Delta_\omega R_{i\bar{j}k\bar{l}} = -\frac{1}{t} Rm(\omega_0) + \frac{1}{t} g^0_{n\bar{m}} \Psi^n_{ki} \Psi^m_{lj} - \frac{1}{t} R^{\overline{m}}_{\bar{j}i\bar{l}} g^0_{k\bar{m}} + \mathrm{Rm} * \mathrm{Rm} + \mathrm{Rm} * \mathrm{Ric}. \tag{3.37}$$

Then using (3.37), the metric trace bound, (3.33), and the bound for S, (3.24), we have for some uniform $C > 0$,

$$\begin{aligned} \Delta_\omega \left(|\mathrm{Rm}(\omega)|^2_g \right) \geqslant & - C_1 |\mathrm{Rm}(\omega)|_g - C_2 |\mathrm{Rm}(\omega)|^2_\omega - C_3 |\mathrm{Rm}(\omega)|^3_g \\ & + |\nabla \mathrm{Rm}(\omega)|^2_g + |\overline{\nabla} \mathrm{Rm}(\omega)|^2_g. \end{aligned} \tag{3.38}$$

Set $Q := |\mathrm{Rm}(\omega)|^2_\omega + AS$, for some $A > 0$ to be deduced later. Equation (3.31) and (3.38) imply

$$\Delta_\omega Q \geqslant -C_1 - C_2 |\mathrm{Rm}(\omega)|_g + \left(\frac{A}{2} - C_3 \right) |\mathrm{Rm}(\omega)|^2_g, \tag{3.39}$$

for some positive constants C_1, C_2 and C_3. Choosing $A > 0$ large enough, we obtain some $C > 0$ such that

$$\Delta_\omega Q \geqslant -C + |\mathrm{Rm}(\omega)|^2_g. \tag{3.40}$$

Then by a maximum principle argument, we deduce that at a maximum point $|\mathrm{Rm}(\omega)|_g(p_0) \leqslant C$. Since S is uniformly bounded, we can conclude that

$$\sup_X |\mathrm{Rm}(\omega)|_g \leqslant C, \tag{3.41}$$

for some $C > 0$ independent of t. □

Once we obtain curvature bounds, an adapted calculation by Hamilton [Ham82a] gives us bounds for all derivatives of curvature. Denoting $\nabla_{\mathbb{R}}$ to be the covariant derivative of the Riemannian metric, we have

Theorem 3.11. *If* (3.34) *holds, then there exist uniform constants C_m for $m = 1, 2, \ldots$ such that*

$$|\nabla^m_{\mathbb{R}} \mathrm{Rm}|^2 \leqslant C_m.$$

Proof. Using the identity

$$\frac{1}{2} \Delta_\omega |\nabla^m \mathrm{Rm}|^2_g = |\nabla^{m+1} \mathrm{Rm}|^2_g - \sum_{p+q=m} \nabla^p \mathrm{Rm} * \nabla^q \mathrm{Rm} * \nabla^m \mathrm{Rm}, \tag{3.42}$$

we can apply an induction argument to obtain bounds on the higher-order curvature terms. Defining the quantity

$$Q_m := |\nabla^{m+1} \mathrm{Rm}|^2_g + A |\nabla^m \mathrm{Rm}|^2, \tag{3.43}$$

for some $A > 0$ chosen large enough, we can use (3.42) to setup a maximum principle argument. This implies that the metric and its higher derivatives are bounded □

Chapter 4

Independence of Singularity Type for Long Time Kähler-Ricci Flows

Abstract: In this paper, we show that the singularity type of solutions to the Käher-Ricci flow on a numerically effective manifold (nef) does not depend on the initial metric. More precisely if there exists a type III solution to the Kähler-Ricci flow, then any other solution starting from a different initial metric will also be Type III. This generalises a previous result by Y. Zhang for the semi-ample case.

4.1 Introduction

The Kähler-Ricci flow is a geometric evolution equation used to study Kähler geometry. It has proven itself to be a powerful tool in constructing canonical metrics such as Kähler-Einstein metrics [Cao86; PSS06; MS09; PS06; Tia12].

A primary research direction is the study of the analytic minimal model program in birational geometry [Mat02], where the flow takes a central role in producing minimal models. Initiated by Song and Tian in [ST06], the program proposes to emulate the procedure carried out by Hamilton and Perelmann in resolving the Poincare conjecture. The idea is to run the flow on algebraic manifolds and perform 'algebraic surgeries' such as flips and contractions each time the flow encounters a singularity [SW13b; SW14; SY12]. After finite iterations of this procedure, the resulting manifold would be numerically effective (nef) and hence a minimal model. At this stage, the normalised version of the flow will evolve the metric to a Kähler-Einstein metric or a generalised Kähler-Einstein metric. Significant progress has been made in this direction, see [ST12; ST16; ST17; Tia19; GSW16; FZ15; Zha19; TWY18; TZ16].

Let X be a compact Kähler manifold with numerically effective line bundle which corresponds to the final step of the analytic minimal model program. A family on Kähler metrics $\omega(t)$ is

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said to evolve by the normalised Kähler-Ricci flow if

$$\begin{cases} \dfrac{\partial}{\partial t}\omega(t) = -\mathrm{Ric}(\omega(t)) - \omega(t), \\ \quad \omega(0) = \omega_0. \end{cases} \tag{4.1}$$

Since the flow is weakly parabolic, we have access to interior estimates which smooths the metric for short times. In the long run, non-linearities make the metric non-positive, forming a singularity. The precise time of singularity formation is determined by the class of the evolving metric in the cohomology given by

$$H^{1,1}_{\bar{\partial}}(M, \mathbb{R}) = \frac{\{\bar{\partial}\text{-closed real } (1,1)\text{-forms }\}}{\{\bar{\partial}\text{-exact real } (1,1)\text{-forms }\}}. \tag{4.2}$$

The singular time, T, is characterised by Tian and the second author in [TZ06]:

$$T := \sup \{t \mid [\omega(t)] \text{ is Kähler}\}. \tag{4.3}$$

By formally taking the cohomology class of (4.1) and solving the resulting ODE, we obtain solution

$$[\omega(t)] = e^{-t}[\omega_0] - (1 - e^{-t})c_1(X). \tag{4.4}$$

Since X is numerically effective, $-c_1(X)$ lies in the closure of the Kähler cone, the set of classes with a Kähler representative, and thus the solution exists all time. Furthermore, for any initial metric, the cohomology class converges to the first Chern class, $-c_1(X)$ in the limit as $t \to \infty$. This is independent of any starting metric ω_0.

The singularity formation reveals the underlying topology of the manifold and therefore the curvature blow up should be independent of the starting metric. To describe the curvature blow up, we utilise the following terminology traditionally used in parabolic geometric flows such as the Ricci flow and Mean Curvature flow.

Definition 4.1. We say that a long time solution to the Kähler-Ricci flow develops a type III singularity if

$$\sup_{X \times [0,\infty)} \|\mathrm{Rm}(\omega(t))\|_{\omega(t)} < +\infty, \tag{4.5}$$

and a type IIb singularity if

$$\sup_{X \times [0,\infty)} \|\mathrm{Rm}(\omega(t))\|_{\omega(t)} = +\infty. \tag{4.6}$$

The precise conjecture for metric independence is stated by Tossati in [Tos15].

Conjecture 4.2 (Tosatti [Tos15])**.** *Let X be a compact Kähler manifold with K_X nef, so every solution of the Kähler-Ricci flow exists for all positive time. Then the singularity type at infinity does not depend on the choice of the initial metric ω_0.*

In the semi-ample case, Tossati-YG. Zhang in [TWY18] gives an almost complete classification of singularity type based on the topology of X. More precisely, they show the following.

Theorem 4.3 (Tosatti-YG. Zhang [TZ15])**.** *Let X be a compact Kähler manifold with semi-ample K_X and consider a solution of the Kähler-Ricci flow.*

1. *Suppose* $\mathrm{kod}(X) = 0$.
   - *If X is a finite quotient of a torus, then the solution is of type III.*
   - *If X is not a finite quotient of a torus, then the solution is of type IIb.*
2. *Suppose* $\mathrm{kod}(X) = n$.
   - *If K_X is ample, then the solution is of type III.*
   - *If K_X is not ample, then the solution is of type IIb.*
3. *Suppose* $0 < \mathrm{kod}(X) < n$.
   - *If X_y is not a finite quotient of a torus, then the solution is of type IIb.*
   - *If X_y is a finite quotient of a torus and $V = \emptyset$, then the solution is of type III.*
4. *Suppose $n = 2$ and $\mathrm{kod}(X) = 1$, then the solution is of type III if and only if the only singular fibres on f are of type $mI_0, m > 1$.*

An alternative proof to items (1), (2) and a special case of (3) was proven by Y. Zhang in [Zha20], where he deduces these results by setting up a comparison of some well-behaved metric and an arbitrary metric. Moreover, he showed that for semiample K_X, the singularity type does not depend on the initial metric, thereby partially confirming Conjecture 4.2. Furthermore, Fong and Y. Zhang in [FZ20] showed that the set of singular fibres of the semi-ample fibration on which the Riemann curvature blows up at time-infinity is independent of the choice of the initial Kähler metric. For more results pertaining to the curvature behaviour of semiample Kähler-Ricci flows, see [Zha09; ST16; Jia20].

In this paper, we prove Conjecture 4.2 without assuming K_X is semiample. More precisely, we show that for numerically effective manifolds, if a solution develops a type III singularity, then every other solution develops a type III singularity. Consequently, if a solution develops a type IIb singularity, then any other solution must be type IIb.

Theorem 4.4. *Let $\widetilde{\omega}(t)$ be a solution to the normalised Kähler-Ricci flow* (4.1) *on a manifold X with numerically effective K_X.*

$$\|\mathrm{Rm}(\widetilde{\omega}(t))\|^2_{\widetilde{\omega}(t)} \leqslant C_0. \tag{4.7}$$

Then for any other solution to the Kähler-Ricci flow, $\omega(t)$, there exists a constant $C > 0$ such that

$$C^{-1}\widetilde{\omega}(t) \leqslant \omega(t) \leqslant C\widetilde{\omega}(t), \tag{4.8}$$

for all $t \geqslant 0$.

$$\|\mathrm{Rm}(\omega(t))\|^2_{\omega(t)} \leqslant C. \tag{4.9}$$

Remark 4.5. Any solution to the Kähler-Ricci flow on a numerically effective manifold exists for all time. Indeed, the estimate (4.9) is impossible for finite time singularity due to Z. Zhang's result [Zha10].

The proof builds on the results in [Zha20] but the major difference is that no assumptions on the semi-ampleness of X need to be made.

We end the introduction by outlining the structure of this paper. In Section 2, we show a special case of Theorem 4.4 when the two initial metrics ω_0 and $\widetilde{\omega}_0$ are related by a simple scaling factor $\lambda_0 > 0$. We first derive a Monge–Ampère equation where the potential function relates the two evolving metrics under rescaling of space and time. By deriving bounds on the potential function and because of the curvature assumption on $\widetilde{\omega}$, we derive metric equivalence between the known Type III metric $\widetilde{\omega}$ and the arbitrary metric ω. Curvature estimates are then obtained as done in [Zha20]. In Section 3, we utilise Lemma 4.6 to prove Theorem 4.4. Lemma 4.6 plays a vital role in constructing "comparison" solutions to the arbitrary solution $\omega(t)$ which will be needed to obtain a favourable sign when carrying out maximum principle arguments. As a result, a potential function bound is derived which and used to obtain metric equivalence. With a similar argument as carried out in Section 2, curvature bounds are obtained for the general theorem. Finally, in Section 4, we utilise Theorem 4.4 for examples from [Zha20] in the numerically effective case.

4.2 Preliminary Results

In the first part of this section, we restrict ourselves to the special case of Theorems 4.4 which will be utilised in the proof of the general theorem. More precisely, we show that if there exists a solution to the Kähler-Ricci Flow on a numerically effective manifold X with uniformly bounded curvature, then any scaling of the initial metric produces a solution that also evolves with uniformly bounded curvature.

In the second part of this section, we recall the proof in [Zha19] which allows us to derive a bound on the curvature of a flow $\omega(t)$, given that it is equivalent to a Type III flow $\widetilde{\omega}(t)$.

4.2.1 Flow from a Scaled Initial Metric

We show the following.

Lemma 4.6. *Let X be a numerically effective Kähler manifolds such that there exists $\widetilde{\omega}(t)$ a solution to the Kähler-Ricci Flow with bounded curvature tensor;*

$$\|\mathrm{Rm}(\widetilde{\omega}(t))\|^2_{\widetilde{\omega}(t)} \leqslant C_0. \tag{4.10}$$

Then for any $\lambda_0 > 0$, there exists $C > 0$ such that the solution $\omega(t)$ starting from $\omega(0) = \lambda_0\widetilde{\omega}(0)$ satisfies

$$C^{-1}\widetilde{\omega}(t) \leqslant \omega(t) \leqslant C\widetilde{\omega}(t), \tag{4.11}$$

and

$$\|\mathrm{Rm}(\omega(t))\|^2_{\omega(t)} \leqslant C, \tag{4.12}$$

for all $t \geqslant 0$.

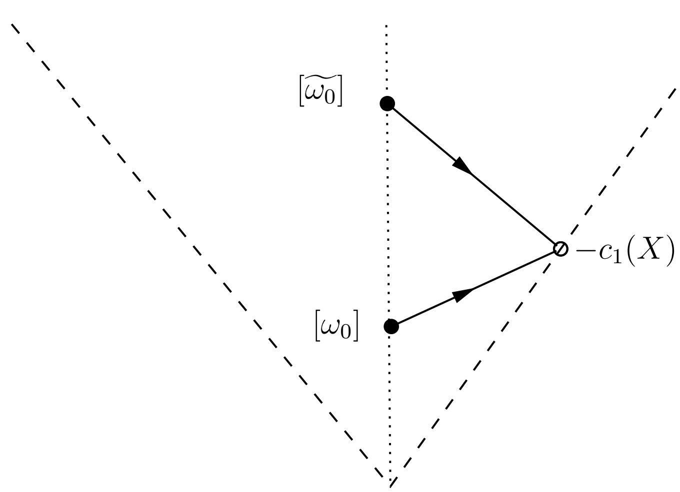

Figure 4.1: Scaling of Initial Metric

We first reduce the problem to a Monge–Ampère equation. The reduction will be slightly different to what is usually done in the literature. Since the initial metrics are equal up to some scaling factor, we scale the solution in space and time by $\lambda(t)$ and $\tau(t)$ respectively so that

$$[\omega(t)] = \lambda(t)[\widetilde{\omega}(\tau(t))], \tag{4.13}$$

for all $t \geqslant 0$ and $\omega_0 = \lambda_0\widetilde{\omega}_0$ where $\lambda_0 > 0$. For any solution to the Kähler-Ricci Flow, the Cohomology class evolves as

$$[\omega(t)] = e^{-t}[\omega_0] - (1 - e^{-t})c_1(X). \tag{4.14}$$

Then from (4.13), we have

$$\lambda_0 e^{-t}\widetilde{\omega}_0 - (1 - e^{-t})c_1(X) = \lambda(t)e^{-\tau(t)}\widetilde{\omega}_0 - (1 - e^{-\tau(t)})\lambda(t)c_1(X),$$

from which we deduce that

$$\begin{cases} \lambda(t)e^{-\tau(t)} = \lambda_0 e^{-t}, \\ \lambda(t) - e^{-\tau(t)}\lambda(t) = 1 - e^{-t}. \end{cases}$$

Solving this system yields

$$\lambda(t) = e^{-t}(\lambda_0 - 1) + 1, \tag{4.15}$$

and

$$\tau(t) = t + \ln\left(\frac{\lambda(t)}{\lambda_0}\right). \tag{4.16}$$

Furthermore, we check that at $t = 0$, the initial class satisfies $[\omega_0] = \lambda_0[\widetilde{\omega}_0]$.

We reduce the flow equation to a Monge–Ampère equation where the rescaled metric takes the role of the reference metric. Using (4.13), there exists $u : X \to \mathbb{R}$ such that

$$\omega(t) = \lambda(t)\widetilde{\omega}(\tau(t)) + \sqrt{-1}\partial\overline{\partial}u(t). \tag{4.17}$$

For ease of notation, we drop the t dependence for τ and λ. Taking a t-time derivative,

$$\frac{\partial \omega}{\partial t} = \lambda'(t)\widetilde{\omega}(\tau) + \lambda \frac{\partial \tau}{\partial t}\frac{\partial \widetilde{\omega}}{\partial \tau} + \sqrt{-1}\partial\overline{\partial}\left(\frac{\partial u}{\partial t}\right).$$

Using $\lambda' + \lambda = 1$ and $\tau' = \lambda^{-1}$ and (4.1) in τ, the above expression becomes

$$\frac{\partial \omega}{\partial t} = \lambda'(t)\widetilde{\omega}(\tau) + \sqrt{-1}\partial\overline{\partial}\log\widetilde{\omega}(\tau)^n - \widetilde{\omega}(\tau) + \sqrt{-1}\partial\overline{\partial}\left(\frac{\partial u}{\partial t}\right).$$

On the other hand, (4.1) in t is

$$\frac{\partial \omega}{\partial t} = \mathrm{Ric}(\omega(t)) - \omega(t) = \sqrt{-1}\partial\overline{\partial}\log\omega(t)^n - \lambda\widetilde{\omega}(\tau) - \sqrt{-1}\partial\overline{\partial}u(t).$$

Combining the previous two equations and using $\lambda' + \lambda = 1$ leads to

$$\sqrt{-1}\partial\overline{\partial}\left(\frac{\partial u}{\partial t}\right) = \sqrt{-1}\partial\overline{\partial}\log\omega(t)^n - \sqrt{-1}\partial\overline{\partial}\log\widetilde{\omega}(\tau)^n = \sqrt{-1}\partial\overline{\partial}u(t).$$

Integrating over the compact manifold X yields the complex Monge–Ampère equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \log\left(\dfrac{\omega(t)^n}{\widetilde{\omega}(\tau)^n}\right) - u(t), \\ u(0) = 0. \end{cases} \tag{4.18}$$

At $t = 0$, we have

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = n\log(\lambda_0), \tag{4.19}$$

which is independent of $x \in X$. Since $u(0) \equiv 0$, then the function $u(t)$ is constant over X, thus (4.18) is an ordinary differential equation and (4.17) is

$$\omega(t) = \lambda(t)\widetilde{\omega}(\tau). \tag{4.20}$$

Proof of Lemma 4.6. Equation (4.20) implies that there exist $C > 0$ such that

$$C^{-1}\widetilde{\omega}(\tau) \leqslant \omega(t) \leqslant C\widetilde{\omega}(\tau). \tag{4.21}$$

Thus to show (4.11), we need to show that for some $C > 0$

$$C^{-1}\widetilde{\omega}(t) \leqslant \widetilde{\omega}(\tau) \leqslant C\widetilde{\omega}(t) \tag{4.22}$$

for all $t \geqslant 0$. To show the above, we use (4.10) to find a $C_0 > 0$ such that

$$-C_0\widetilde{\omega} \leqslant \mathrm{Ric}(\widetilde{\omega}) \leqslant C_0\widetilde{\omega}. \tag{4.23}$$

This implies that (4.1) can be estimated by

$$-C_0\widetilde{\omega} \leqslant \frac{\partial \widetilde{\omega}}{\partial t} + \widetilde{\omega} \leqslant C_0\widetilde{\omega}. \tag{4.24}$$

The lower bound in (4.24) implies

$$0 \leqslant \frac{\partial}{\partial t}\left(e^{(C_0+1)t}\widetilde{\omega}\right)$$

and the upper bound in (4.24) implies

$$\frac{\partial}{\partial t}\left(e^{(1-C_0)t}\widetilde{\omega}\right) \leqslant 0$$

for all $t \geqslant 0$.

We now take cases. Suppose that $0 < \lambda_0 < 1$ and assume that $t \geqslant T$ for T large enough such that $\tau = t + \ln(\lambda(t)/\lambda_0) > t$. Using that $e^{(1-C_0)t}\widetilde{\omega}(t)$ is decreasing in t, we have

$$\widetilde{\omega}(\tau) \leqslant e^{(1-C_0)(t-\tau)}\widetilde{\omega}(t) = e^{C_0-1}\frac{\lambda(t)}{\lambda_0}\widetilde{\omega}(t) \leqslant C\widetilde{\omega}(t),$$

where $C > 0$. On the other hand, $e^{(C_0+1)t}\widetilde{\omega}(t)$ is increasing in t which implies

$$\widetilde{\omega}(\tau) \geqslant e^{(C_0+1)(t-\tau)}\widetilde{\omega}(t) = e^{-(C_0+1)}\frac{\lambda(t)}{\lambda_0}\widetilde{\omega}(t) \geqslant C^{-1}\widetilde{\omega}(t)$$

for some $C > 0$. Combining these gives us (4.22). If $\lambda_0 > 1$ we can derive the estimate using the same method with $\tau < t$.

Combining (4.22) with (4.21) yields (4.11) for all $t \geqslant T$, for some large $T > 0$. Interior estimates imply gives (4.11) for all $t \geqslant 0$.

The curvature bounds, (4.12), follow immediately from (4.20) and the scaling properties of the curvature tensor. □

4.2.2 General Result for Higher Order Estimates

Using the metric equivalence, we derive higher-order estimates for $\omega(t)$ given that the curvature tensor is bounded for $\widetilde{\omega}$. The method is identical to that by Y. Zhang in [Zha20], for convenience we sketch the argument here. When the evolving metric, $\omega(t)$, is equivalent to a fixed metric, ω_0, see Song and Weinkove's notes [SW13a].

Since we have shown the inequality (4.11), we consider the two metrics with the same time scale t and drop the dependence of the metrics on t for notational convenience.

We first show the following Lemma.

Lemma 4.7. *Suppose that ω and $\widetilde{\omega}$ are solutions to* (4.1) *and $|\mathrm{Rm}(\widetilde{\omega})|_{\widetilde{\omega}} \leqslant C$. The evolution of the trace satisfies*

$$\left(\frac{\partial}{\partial t} - \Delta_\omega\right)\operatorname{tr}_\omega \widetilde{\omega} \leqslant C\operatorname{tr}_\omega \widetilde{\omega} + C\left(\operatorname{tr}_\omega \widetilde{\omega}\right)^2 - g^{\bar{j}i}g^{\bar{q}p}\widetilde{g}^{\bar{b}a}\nabla_i\widetilde{g}_{p\bar{b}}\nabla_{\bar{j}}\widetilde{g}_{a\bar{q}}. \tag{4.25}$$

Proof. The time derivative is given by

$$\begin{aligned}\frac{\partial}{\partial t}\left(\operatorname{tr}_\omega \widetilde{\omega}\right) &= \frac{\partial g^{\bar{j}i}}{\partial t}\widetilde{g}_{i\bar{j}} + g^{\bar{j}i}\frac{\partial \widetilde{g}_{i\bar{j}}}{\partial t} \\ &= R^{\bar{j}i}\widetilde{g}_{i\bar{j}} - \operatorname{tr}_\omega \mathrm{Ric}(\widetilde{\omega})\end{aligned} \tag{4.26}$$

The laplacian of the trace is given by the standard calculation carried out in [Yau78a; Lu67].

$$\Delta_\omega \operatorname{tr}_\omega \widetilde{\omega} = R^{\bar{j}i}\widetilde{g}_{i\bar{j}} - g^{\bar{j}i}g^{\bar{q}p}\widetilde{R}_{i\bar{j}p\bar{q}} + g^{\bar{j}i}g^{\bar{q}p}\widetilde{g}^{\bar{b}a}\nabla_i\widetilde{g}_{p\bar{b}}\nabla_{\bar{j}}\widetilde{g}_{a\bar{q}} \tag{4.27}$$

Combining the previous two equations yields

$$\left(\frac{\partial}{\partial t} - \Delta_{\omega(t)}\right)\operatorname{tr}_\omega \widetilde{\omega} = -\operatorname{tr}_\omega \operatorname{Ric}(\widetilde{\omega}) + g^{\bar{j}i}g^{\bar{q}p}\widetilde{R}_{i\bar{j}p\bar{q}} - g^{\bar{j}i}g^{\bar{q}p}\widetilde{g}^{\bar{b}a}\nabla_i\widetilde{g}_{p\bar{b}}\nabla_{\bar{j}}\widetilde{g}_{a\bar{q}}. \tag{4.28}$$

The desired inequality follows from assumption (4.10). □

Proposition 4.8 (Ys. Zhang [Zha20]). *Let $\widetilde{\omega}(\tau)$ be a type III solution to the normalised Kähler-Ricci flow and $\omega(t)$ be a arbitrary solution such that for some $C_0 > 0$,*

$$C_0^{-1}\widetilde{\omega}(t) \leqslant \omega(t) \leqslant C_0\widetilde{\omega}(t), \tag{4.29}$$

for all $t \geqslant 0$. Then there exists $C > 0$ such that

$$\|Rm(\omega(t))\|_{\omega(t)} \leqslant C, \tag{4.30}$$

for all $t \geqslant 0$.

Proof. Let $\Psi = \left(\Psi_{ij}^k\right)$ where $\Psi_{ij}^k := \Gamma_{ij}^k - \widetilde{\Gamma}_{ij}^k$ and $S = |\Psi|_g^2$. In local coordinates, we have

$$S = g^{\bar{j}i}g^{\bar{l}k}g^{\bar{q}p}\widetilde{\nabla}_i g_{k\bar{q}}\overline{\widetilde{\nabla}_j g_{l\bar{p}}}. \tag{4.31}$$

For the last term in (4.25), we have

$$\begin{aligned} g^{\bar{j}i}g^{\bar{q}p}\widetilde{g}^{\bar{b}a}\nabla_i\widetilde{g}_{p\bar{b}}\nabla_{\bar{j}}\widetilde{g}_{a\bar{q}} &= g^{\bar{j}i}g^{\bar{q}p}\widetilde{g}^{\bar{b}a}\left(\nabla_i - \widetilde{\nabla}_i\right)\widetilde{g}_{p\bar{b}}\left(\nabla_{\bar{j}} - \widetilde{\nabla}_{\bar{j}}\right)\widetilde{g}_{a\bar{q}} \\ &= g^{\bar{j}i}g^{\bar{q}p}\widetilde{g}^{\bar{b}a}\left(-\Psi_{ip}^d\right)\widetilde{g}_{d\bar{b}}\left(-\overline{\Psi_{jq}^e}\right)\widetilde{g}_{a\bar{e}} \\ &\geqslant C^{-1}S. \end{aligned} \tag{4.32}$$

Hence, we have the estimate

$$\left(\frac{\partial}{\partial t} - \Delta_\omega\right) tr_\omega\widetilde{\omega} \leqslant C - C^{-1}S. \tag{4.33}$$

The evolution of the tensor S is given by

$$\left(\partial_t - \Delta_\omega\right)S = S - |\nabla\Psi|_g^2 - |\overline{\nabla}\Psi|_g^2 + \langle\widetilde{\nabla}_i\widetilde{R}_p^k - \Psi * Rm(\widetilde{\omega}) - g^{\bar{b}a}\widetilde{\nabla}_a\widetilde{R}_{i\bar{b}p}^k, \Psi\rangle_g, \tag{4.34}$$

where $\langle\cdot,\cdot\rangle_\omega$ is the tensor inner product induced by the metric $g(t)$ associated with $\omega(t)$. Following Hamilton's argument in [Ham82b] (or see [SW13a]), the assumption (4.10) on the curvature tensor of $\widetilde{\omega}$ implies

$$\left\|\nabla_{\widetilde{\omega}(t)}Rm(\widetilde{\omega}(t))\right\|_{\widetilde{\omega}(t)} \leqslant C,$$

for some $C > 0$. In light of this, we estimate (4.34) by

$$\left(\frac{\partial}{\partial t} - \Delta_\omega\right) S \leqslant CS + C - |\nabla\Psi|_g^2 - |\overline{\nabla}\Psi|_g^2. \tag{4.35}$$

We now have all the pieces to set up a maximum principle argument. Let $Q := S + A\,\mathrm{tr}_\omega\,\widetilde{\omega}$ for some sufficiently large $A > 0$. Using (4.33) and (4.35), there exists $C > 0$ such that

$$\left(\frac{\partial}{\partial t} - \Delta_\omega\right) Q \leqslant -S + C. \tag{4.36}$$

By a standard maximum principle argument and noting that $\mathrm{tr}_{\omega(t)}\,\widetilde{\omega}(t) \leqslant C$, we find a constant $C \geqslant 1$ such that $S \leqslant C$.

The last term in (4.35) can be estimated by

$$|\overline{\nabla}\Psi|_g^2 = \left|\widetilde{R}_{i\bar{b}p}^k - R_{i\bar{b}p}^k\right|_g^2 \geqslant \frac{1}{2}|Rm(\omega)|_g^2 - C, \tag{4.37}$$

thus

$$\left(\frac{\partial}{\partial t} - \Delta_\omega\right) S \leqslant C - \frac{1}{2}|Rm(\omega)|_g^2. \tag{4.38}$$

The evolution of curvature can be estimated by

$$\left(\frac{\partial}{\partial t} - \Delta_\omega\right) |Rm(\omega)|_g \leqslant C|Rm(\omega)|_g^2 - \frac{1}{2}|Rm(\omega)|_g. \tag{4.39}$$

Let $Q := |\mathrm{Rm}(\omega)|_g + AS$ for some large constant $A > 0$, we combine the previous two estimates to obtain

$$\left(\frac{\partial}{\partial t} - \Delta_\omega\right) (|Rm(\omega)|_g + AS) \leqslant -\frac{1}{2}|Rm(\omega)|_g + C. \tag{4.40}$$

Then by the maximum principle, there exists $C > 0$ such that

$$\sup_{X\times[0,\infty)} |Rm(\omega)|_g \leqslant C. \tag{4.41}$$

□

4.3 Proof of Theorem 4.4

Using Lemma 4.6, we scale a Type III solution $\widetilde{\omega}(t)$ twice to $\omega^+(t)$ and $\omega^-(t)$, such that $\omega_0^- := \lambda_0^-\widetilde{\omega_0} \leqslant \widetilde{\omega}_0 \leqslant \lambda_0^+\widetilde{\omega}_0 := \omega_0^+$. We then choose λ_0^- small enough and λ_0^+ large enough, such that $\omega_0^- \leqslant \omega_0 \leqslant \omega_0^+$. Furthermore, these solutions starting from these scaled metrics are type III. We treat these new solutions act as reference metrics, where either $\omega^+(t)$ or $\omega^-(t)$ is chosen to obtain a favourable sign when carrying out maximum principle arguments.

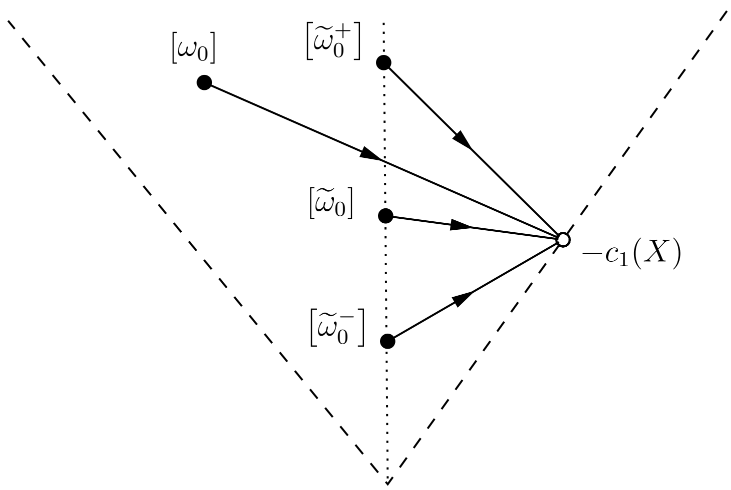

Figure 4.2: Scaling of Type III solutions

4.3.1 Reduction to a Monge–Ampère Equation

Let $\chi \in [\omega_\infty]$ be a representative of the limiting class under the Kähler-Ricci flow. Using the usual reference metrics, we have

$$\begin{aligned} \omega(t) &= e^{-t}\omega_0 + (1-e^{-t})\chi + \sqrt{-1}\partial\overline{\partial}\varphi, \\ \omega^+(t) &= e^{-t}\omega_0^+ + (1-e^{-t})\chi + \sqrt{-1}\partial\overline{\partial}\varphi^+, \qquad \text{and} \\ \omega^-(t) &= e^{-t}\omega_0^- + (1-e^{-t})\chi + \sqrt{-1}\partial\overline{\partial}\varphi^-. \end{aligned} \tag{4.42}$$

In our proof of Theorem 4.4, we require three Monge-Ampère equations relating pairs of $\omega(t)$, $\omega^+(t)$ and $\omega^-(t)$. Taking a time derivative of $\omega(t)$ and $\omega^-(t)$ in (4.42), and using the flow equation, we arrive at a Monge-Ampère equation for $u := \varphi - \varphi^-$ given by

$$\frac{\partial u}{\partial t} = \log\left(\frac{\omega^n}{(\omega^-)^n}\right) - u \tag{4.43}$$

where

$$\omega(t) = \omega^-(t) + e^{-t}(\omega_0 - \omega_0^-) + \sqrt{-1}\partial\overline{\partial}u. \tag{4.44}$$

Equation (4.43) will be the primary Monge-Ampère utilised in the proof of the main Theorem. Note that the construction of ω_0^- ensures $\omega_* := \omega_0 - \omega_0^- > 0$. Similarly for $\psi := \varphi^+ - \varphi^-$ we have

$$\frac{\partial \psi}{\partial t} = \log\left(\frac{(\omega^+)^n}{(\omega^-)^n}\right) - \psi. \tag{4.45}$$

The function ψ relates the two evolving metrics by

$$\omega^+(t) = \omega^-(t) + e^{-t}(\omega_0^+ - \omega_0^-) + \sqrt{-1}\partial\overline{\partial}\psi. \tag{4.46}$$

Finally, for $v := \varphi - \varphi^+$, we have the following Monge-Ampere equation

$$\frac{\partial v}{\partial t} = \log\left(\frac{\omega^n}{(\omega^+)^n}\right) - v \tag{4.47}$$

where

$$\omega(t) = \omega^+(t) + e^{-t}(\omega_0 - \omega_0^+) + \sqrt{-1}\partial\overline{\partial}v. \tag{4.48}$$

4.3.2 Potential Estimates

Our aim is to derive potential estimates for u and its time derivative. We begin by calculating the evolution of u and $\frac{\partial u}{\partial t}$. As before, we drop the t dependence on the metrics and potential functions for ease of notation. To denote geometric quantities associated with $\omega^+(t)$ and $\omega^-(t)$ with their respective symbol in superscripts.

Lemma 4.9. *For $\omega_* := \omega_0 - \omega_0^- > 0$, any solution u to (4.43) satisfies*

$$\left(\frac{\partial}{\partial t} - \Delta_{\omega(t)}\right) u = \frac{\partial u}{\partial t} - n + \operatorname{tr}_\omega \omega^- + e^{-t} \operatorname{tr}_\omega \omega_*, \tag{4.49}$$

and

$$\left(\frac{\partial}{\partial t} - \Delta_{\omega(t)}\right) \frac{\partial u}{\partial t} = R^- - \operatorname{tr}_\omega \operatorname{Ric}(\omega^-) - \frac{\partial u}{\partial t} - \operatorname{tr}_\omega \omega^- - e^{-t} \operatorname{tr}_\omega \omega_* + n. \tag{4.50}$$

Proof. The first equation, (4.49) follows from (4.42);

$$\left(\frac{\partial}{\partial t} - \Delta_{\omega(t)}\right) u = \frac{\partial u}{\partial t} - \operatorname{tr}_\omega(\omega - \omega^- - e^{-t}\omega_*) = \frac{\partial u}{\partial t} - n + \operatorname{tr}_\omega \omega^- + e^{-t} \operatorname{tr}_\omega \omega_*.$$

Equation (4.50) is derived using (4.43);

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta_{\omega(t)}\right) \frac{\partial u}{\partial t} &= \left(\frac{\partial}{\partial t} - \Delta_{\omega(t)}\right) \log\left(\frac{\omega^n}{\omega^{-n}}\right) - \left(\frac{\partial}{\partial t} - \Delta_{\omega(t)}\right) u \\ &= R^- - \operatorname{tr}_\omega \operatorname{Ric}(\omega^-) - \frac{\partial u}{\partial t} + n - \operatorname{tr}_\omega \omega^- - e^{-t} \operatorname{tr}_\omega \omega_*.\end{aligned}$$

□

Lemma 4.10. *Let u be a solution to (4.43). Then for $0 < \eta < 1/2$, there exists $C > 0$ such that*

$$-Ce^{-t} \leqslant u(t) \leqslant Ce^{-\eta t} \tag{4.51}$$

for all $t \geqslant 0$.

Proof. The lower bound follows from the maximum principle. Indeed, at a minimum point of u,

$$\frac{\partial}{\partial t}\left(e^t u\right) = e^t \log\left(\frac{(\omega^-(t) + e^{-t}(\omega_0 - \omega_0^-) + \sqrt{-1}\partial\overline{\partial} u)^n}{(\omega^-)^n}\right) \geqslant e^t \log(1) = 0.$$

Integrating yields

$$u(t) \geqslant -Ce^{-t}.$$

To derive an upper bound for u, we first observe that at a maximum for v in (4.47), we have

$$\frac{\partial v_{\max}}{\partial t} \leqslant \log\left(\frac{(\omega_+ + e^{-t}(\omega_0 - \omega_0^+))^n}{(\omega^+)^n}\right) - v_{\max} \leqslant -v_{\max}.$$

Then by the maximum principle, we deduce that $v \leqslant 0$. On the other hand, we have $C > 0$ such that

$$|\psi| \leqslant C \tag{4.52}$$

for all $t > 0$. Indeed, the initial metrics ω_0^+, ω_0^- and $\widetilde{\omega}_0$ are scalar multiples of each other, Lemma 4.6 and (4.45) imply

$$\frac{\partial}{\partial t}\left(e^t\psi\right) = e^t \log\left(\frac{(\omega^+)^n}{(\omega^-)^n}\right) \leqslant Ce^t$$

for some $C > 0$. Integrating the above gives us (4.52). It now follows that

$$u(t) = \varphi(t) - \varphi^-(t) = v(t) + \psi(t) \leqslant C$$

where $C > 0$ is independent of time. □

Lemma 4.11. *Let u be a solution to* (4.43)*. Then there exists $C > 0$ such that*

$$\left|\frac{\partial u}{\partial t}\right| \leqslant C, \tag{4.53}$$

for all $t \geqslant 0$.

Proof. We first note that Lemma 4.6 implies that for some $C > 0$, we have

$$C^{-1}\widetilde{\omega}(t) \leqslant \omega^-(t) \leqslant C\widetilde{\omega}(t), \tag{4.54}$$

and

$$|\mathrm{Rm}(\omega^-(t))|^2_{\omega^-(t)} \leqslant C, \tag{4.55}$$

for all $t \geqslant 0$. Let $Q := \frac{\partial u}{\partial u} - Au$, where $A > 0$ a constant that will be set later. By combining (4.49) and (4.50), and using (4.55), we obtain

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta_\omega\right) Q &= R^- - \operatorname{tr}_\omega \mathrm{Ric}(\omega^-) - (A+1)\operatorname{tr}_\omega \omega^- - (A+1)e^{-t}\operatorname{tr}_\omega \omega_* - (A+1)\frac{\partial u}{\partial t} + (A+1)n \\ &\leqslant -(A+1-C_0)\operatorname{tr}_\omega \omega^- - (A+1)e^{-t}\operatorname{tr}_\omega \omega_* - (A+1)\frac{\partial u}{\partial t} + (A+1+C_0)n.\end{aligned}$$

We choose $A = 2 + C_0$ to obtain the estimate

$$\left(\frac{\partial}{\partial t} - \Delta_\omega\right) Q \leqslant -C\frac{\partial u}{\partial t} + C.$$

Applying the parabolic maximum principle yields the upper bound

$$\frac{\partial u}{\partial t} \leqslant C,$$

for some $C > 0$.

For the lower bound, we instead consider the quantity

$$Q := \frac{\partial u}{\partial t} + Au,$$

for some $A > 0$ to be determined later. Once again, we use (4.49), (4.50) and (4.55) to set up a maximum principle argument:

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta_\omega\right) Q &= R^-(t) - \operatorname{tr}_\omega \operatorname{Ric}(\omega^-) + (A-1)\operatorname{tr}_\omega \widetilde{\omega} + (A-1)e^{-t}\operatorname{tr}_\omega \omega_* + (A-1)\frac{\partial u}{\partial t} + n\,(1-A) \\ &\geqslant (C_0 + 1 - A)n + (A-1)\frac{\partial u}{\partial t} + (A - C_0 - 1)\operatorname{tr}_\omega \omega^- + (A-1)e^{-t}\operatorname{tr}_\omega \omega_*.\end{aligned}$$

Applying a similar argument to before, we choose $A = 2 + C_0$ to obtain an estimate

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta_\omega\right) Q &\geqslant -C + C\frac{\partial u}{\partial t} + \operatorname{tr}_\omega \omega^- \\ &\geqslant -C + C\frac{\partial u}{\partial t} + n\left(\frac{(\omega^-)^n}{\omega^n}\right)^{\frac{1}{n}} \\ &= -C + C\frac{\partial u}{\partial t} + ne^{-\frac{1}{n}\left(\frac{\partial u}{\partial t} + u\right)}\end{aligned}$$

Then applying the maximum principle, using that u is uniformly bounded, shows that there exists a $C > 0$ independent of time such that

$$\frac{\partial u}{\partial t} \geqslant -C. \tag{4.56}$$

□

4.3.3 Proof of Theorem 4.4

We now derive a metric equivalence between the Type III solution $\widetilde{\omega}$ and the arbitrary solution ω.

Proof. From a standard calculation, using (4.55), we have for some $C > 0$ and $C_0 > 0$

$$\left(\frac{\partial}{\partial t} - \Delta_\omega\right) \log \operatorname{tr}_\omega \omega^- \leqslant C_0 \operatorname{tr}_\omega \omega^- + C, \tag{4.57}$$

for all times $t \geqslant 0$. The uniform bound on $\frac{\partial u}{\partial t}$ from Lemma 4.11 implies that

$$\left(\frac{\partial}{\partial t} - \Delta_\omega\right) u = \frac{\partial u}{\partial t} - n + \operatorname{tr}_\omega \omega^- + e^{-t}\operatorname{tr}_\omega\left(\omega_0 - \omega_0^-\right) \geqslant -C + \operatorname{tr}_\omega \omega^-. \tag{4.58}$$

Following a similar argument as before, we define $Q := \log \operatorname{tr}_\omega \omega^- - Au$, then using the two inequalities, we choose A large enough such that

$$\left(\frac{\partial}{\partial t} - \Delta_\omega\right) Q \leqslant C - \operatorname{tr}_\omega \omega^-. \tag{4.59}$$

Applying the maximum principle, and again using Lemmas 4.10 and 4.11, we obtain some $C > 0$ such that

$$\operatorname{tr}_{\omega} \omega^{-} \leqslant C, \tag{4.60}$$

for all $t \geqslant 0$. Combining the potential estimates from Lemma 4.10 and 4.11, we derive from the Monge–Ampère equation (4.43) the volume bounds

$$C^{-1} \leqslant \frac{\omega^n}{(\omega^-)^n} \leqslant C, \tag{4.61}$$

for some $C > 0$ for all $t \geqslant 0$. Similar to before, we employ standard eigenvalue estimates to obtain

$$\operatorname{tr}_{\omega^-} \omega \leqslant n \left(\frac{\omega^n}{(\omega^-)^n} \right) \operatorname{tr}_{\omega} \omega^- \leqslant C.$$

Then the following metric equivalence follows from the two trace bounds;

$$C^{-1}\omega^-(t) \leqslant \omega(t) \leqslant C\omega^-(t). \tag{4.62}$$

Then the above inequality and (4.54) yields (4.8). To obtain bounds on the curvature tensor, we apply Proposition 4.8 with $\omega^-(t)$ in place of $\widetilde{\omega}$. This completes the proof of Theorem 4.4. □

4.4 Applications

Immediately, Theorem 4.4 allows us to answer Conjecture 4.2.

Theorem 4.12. *Let X be a Kähler manifold with numerically effective canonical line bundle. Then the singularity type of solutions to the Kähler-Ricci flow is independent of the initial metric.*

Proof. Suppose a solution $\widetilde{\omega}$ is Type III. Then by Theorem 4.4, any other solution starting from a different initial metric must also be Type III. On the other hand, if a type IIb solution exists, then any other solution must be Type IIb as otherwise, Theorem 4.4 implies that all solutions must be Type III, a contradiction. □

We also include some examples where an explicit solution to the normalised Kähler-Ricci Flow is known. This allows us to deduce the singularity type for an arbitrary initial metric for these manifolds. These examples are obtained from [Zha19].

4.4.1 Calabi-Yau Metrics

Suppose that X admits a Calabi-Yau metric. Then under the Kähler-Ricci flow, we check that $\omega(t) = e^{-t}\omega_{CY}$ is a solution with $\omega(0) = \omega_{CY}$. For such solutions, the curvature evolves as

$$\|\mathrm{Rm}(\omega(t))\|_{\omega(t)} = e^t \|\mathrm{Rm}\,(\omega_{CY})\|_{\omega_{CY}}, \tag{4.63}$$

and therefore develops a type III singularity if and only if ω_{CY} is a flat metric, which is only possible if and only if X is a finite quotient of a torus. Therefore, we conclude from Theorem

4.4 that the Kähler-Ricci Flow develops a type III solution on a numerically effective manifold X if and only if it is a finite quotient of a torus. Furthermore, this demonstrates Item (1) of Theorem 4.3 without assuming K_X is semiample.

Remark 4.13. Item (2) in Theorem 4.3 assumes K_X is big. Since nef and big imply semiample, the generalisation to nef is immediate.

4.4.2 Product Manifolds

Consider the product manifold $X := B \times Y$ where Y is a Calabi-Yau manifold. This is a special case of item (3). Suppose that ω is a Kähler-Einstein metric on B, then we check that

$$\widetilde{\omega}(t) = e^{-t}\omega_{CY} + (1 - e^{-t})\omega_B \tag{4.64}$$

is a solution to the Kähler-Ricci flow. The curvature is given by

$$\|\mathrm{Rm}(\omega(t)\|^2_{\omega(t)} = e^{2t}\|\mathrm{Rm}(\omega_{CY})\|^2_{\omega_{CY}} + \frac{1}{(1-e^{-t})^2}\|\mathrm{Rm}(\omega_B)\|^2_{\omega_B}. \tag{4.65}$$

Thus, we have a type III singularity if and only if ω_{CY} is flat which occurs if Y is a finite quotient of a torus. Thus, a type III singularity develops if and only if Y is a finite quotient of a torus. Thus, using Theorem 4.4, any initial metric ω_0, there exists $C > 0$ such that

$$C^{-1}\widetilde{\omega}(t) \leqslant \omega(t) \leqslant C\widetilde{\omega}(t) \tag{4.66}$$

and

$$\|\mathrm{Rm}(\omega(t))\|_{\omega(t)} \leqslant C, \tag{4.67}$$

for all $t \geqslant 0$.

Remark 4.14. Finally, we note that Theorem 4.4 can be interpreted as evidence for the Kähler extension of the abundance conjecture. Indeed, if a counter example to Theorem 4.4 could be constructed, then Y. Zhang's result [Zha20] implies that the manifold cannot be semiample.

Chapter 5

Curvature Estimates for the Continuity Method

Abstract: We obtain curvature estimates for long-time solutions of the continuity method on compact Kähler manifolds with semi-ample canonical line bundles. In this setting, initiated in [LT16] and [Rub08], we adapt arguments from [FZ20] for the Kähler-Ricci flow to this setup. As an application, we derive curvature bounds for general metrics on product manifolds.

5.1 Introduction

Let $\omega(t)$ be a smooth family of Kähler metrics on a Kähler Manifold (X, ω_0), parameterized by $t \geqslant 0$, such that

$$\begin{cases} (1+t)\omega(t) = \omega_0 - t\mathrm{Ric}(\omega), \\ \omega(0) = \omega_0. \end{cases} \tag{5.1}$$

This equation, known as the continuity method, first appeared in [Rub08] by Rubinstein, and was later reintroduced by La Nave and Tian in [LT16] as an alternative to the Kähler-Ricci flow for carrying out the analytic minimal model program [ST12; ST17; ST06; Tia19]. The main advantage the continuity method has over the flow is the immediate lower bound for the Ricci curvature along the deformation. This allows the application of geometric comparison techniques. For the Kähler-Ricci flow, a Ricci bound is conjectured in general, but this problem is still open (for some recent progress, see [FZ20]).

As with the Kähler-Ricci flow, the continuity method should deform any metric on a numerically effective Kähler manifold to its canonical metric, identified as one of Kähler-Einstien type. At the level of Cohomology, (5.1) deforms the initial class $[\omega_0]$ to the manifold's first Chern class $-c_1(X)$. Motivated by the abundance conjecture, we assume that the canonical line bundle, K_X, is semi-ample. Under this assumption, K_X generates a fibre space map

$$f : X \to X_{can} \subset \mathbb{CP}^N. \tag{5.2}$$

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This map may contain singular fibres, thus we denote

$$S := \{s \in X_{can} \mid X_s := f^{-1}(s) \text{ is a singular fibre}\}$$

to be the singular set resulting from f. The convergence of the family of metrics satisfying the continuity method depends on the Kodaria dimension of X, $\mathrm{kod}(X)$. Geometrically, this is the dimension of the possibly singular manifold S away from the singular set S. The Kodaria dimension cannot exceed the complex dimension of X, n. From this, we distinguish the following three cases depending on $\mathrm{kod}(X)$:

1. We say that X is of *General Type* if $\mathrm{kod}(X) = n$. It was shown in [LT16] (after rescaling appropriately) that the metric weakly converges to a Kähler-Einstien metric as $t \to \infty$.

2. If $\mathrm{kod}(X) = 0$, then X is a *Calabi Yau manifold*. For the Kähler-Ricci flow, classical result by Cao in [Cao86] can be carried over to the continuity method to show smooth deformation of $(1+t)\omega(t)$ to the unique Ricci flat metric in the class $[\omega_0]$ as $t \to \infty$.

3. If $0 < \mathrm{kod}(X) < n$, then we are in the *Volume Collapsed Case*. For the Kähler-Ricci flow, the volume collapsed case has been extensively studied [STZ19; TZ21; TZ15; Zha20; CL21; GTZ20; FZ20; GTZ13; JS21; Zha19; TWY18]. When X admits a Fano fibration, Gromov-Hausdorff convergence for (5.1) is obtained in [ZZ20a]. When X is a minimal elliptic surface, similar convergence was obtained in [ZZ19]. In the more general setting where X is a Hermitian manifold admitting an elliptic bundle over a Riemann surface, Gromov-Hausdorff convergence and curvature estimates are established in [SW20].

It was shown in [Zha20] that the singularity type of the Kähler-Ricci flow does not depend on the choice of the initial metric, from which Y. Zhang gives an elegant proof of some classification results by Tossati-YG. Zhang [TZ15]. The authors Fong and Y. Zhang in [FZ20] then obtained an improved result which localizes the estimate and allows for degenerate metric and curvature bounds. The results of this paper are inspired by results from these two papers.

Theorem 5.1. *Suppose that we are in the setup given by* (5.2)*. Assume there exists* $\tau : [0,\infty) \to [1,\infty)$ *and an initial metric* $\widetilde{\omega}_0$ *whose solution* $\widetilde{\omega}(t)$ *to* (5.1) *satisfies*

$$\sup_X |Rm(\widetilde{\omega}(t))|_{\widetilde{\omega}(t)} \leqslant \tau(t). \tag{5.3}$$

Then the solution $\omega(t)$ *to* (5.1) *starting from any other initial metric* $\omega(0) = \omega_0$ *satisfies*

$$C^{-1}e^{-C\tau(t)}\widetilde{\omega}(t) \leqslant \omega(t) \leqslant Ce^{C\tau(t)}\widetilde{\omega}(t), \tag{5.4}$$

for all $t \geqslant 0$ *and for some* $C > 0$ *uniform of* t.

From the metric equivalence (5.4), we can also obtain Ricci curvature bounds for (5.1).

Corollary 5.2. *In the same setting as in Theorem 5.1, there exists* $C > 0$ *uniform of* t *such that*

$$-C\omega(t) \leqslant \mathrm{Ric}(\omega(t)) \leqslant Ce^{C\tau(t)}\omega(t). \tag{5.5}$$

Therefore, if one can find a solution with uniformly bounded curvature, then any other solution starting from a different initial metric will be equivalent and will have bounded Ricci curvature. For the curvature tensor, we have the following theorem.

Theorem 5.3. *Suppose that we are in the setup given by* (5.2). *Assume there exists a family of metrics $\widetilde{\omega}$ solving* (5.1) *and functions $\tau, \sigma, \rho : [0, \infty) \to [1, \infty)$ satisfying the following metric bounds*

$$\sigma(t)^{-1}\widetilde{\omega}(t) \leqslant \omega(t) \leqslant \sigma(t)\widetilde{\omega}(t), \tag{5.6}$$

curvature bounds

$$\sup_X |Rm(\widetilde{\omega}(t))|_{\widetilde{\omega}(t)} \leqslant \tau(t), \tag{5.7}$$

and curvature derivative bounds

$$\sup_X |\widetilde{\nabla} Rm(\widetilde{\omega}(t))|_{\widetilde{\omega}(t)} \leqslant \rho(t). \tag{5.8}$$

Furthermore, assume that there exists a constant $B > 0$, independent of time, such that for all $t \geqslant 1$

$$\frac{1}{t} \operatorname{tr}_{\widetilde{\omega}(t)} \widetilde{\omega}_0 \geqslant B^{-1} > 0. \tag{5.9}$$

Then for any other initial metric ω_0 with solution $\omega(t)$ to (5.1), *there exists $C > 0$ uniform of t such that*

$$\sup_X |Rm(\omega(t))|_{\omega(t)} \leqslant Ct^2\sigma^4\tau^3\rho^2 \tag{5.10}$$

for all $t \geqslant 1$.

Remark 5.4. When (5.9) is a holomorphic submersion, the assumption (5.9) is automatically satisfied. In a note by W. Jian and Y. Zhang, the authors proved that a "twisted" scalar curvature converges to $-kod(X)$ away from singular fibres. As a consequence, one can show (5.9) on compact sets away from singular fibres, thus (5.9) holds globally if (5.2) is a holomorphic submersion.

Remark 5.5. The assumption (5.8) is serves as a substitute for Shi's estimate in the Kähler-Ricci Flow. To the author's knowledge, there is no such analogous result for the continuity method.

The above theorems are particularly useful when considering product manifolds. Product manifolds serve as a prototype in understanding how geometry collapses under evolution equations such as the Kähler-Ricci flow and (5.1). In the volume collapsed case, the manifold X is split into a collapsing component, Y, and base component, B, so that $X = Y \times B$. In such cases, a product metric solution to (5.1) and the Kähler-Ricci flow can be easily constructed. On B, one endow a Kähler-Einstein metric, on Y, a Calabi-Yau metric. The behaviour regarding how curvature blows up for this product metric can be directly computed. However, the geometric behaviour is not well understood for non-product metrics. Theorem 5.1 and Theorem 5.3 allow us to obtain curvature estimates for general solutions to (5.1) on product manifolds.

We end the introduction with a brief description of the organization of the paper. In Section 5.2, we recall various results from the literature on the continuity equation (5.1). In

Section 5.3, we prove the metric equivalence result stated in Theorem 5.1. We begin Section 5.4 by obtaining C^3 estimates for the solution metric $\omega(t)$ in terms of $\sigma(t), \tau(t), \rho(t)$ - the bounds for $\widetilde{\omega}(t)$. Using this, we can complete the proof of Theorem 5.3. The proofs are all maximum principle-type arguments. Finally, we apply our theorems in product manifolds in Section 5.5.

5.2 Background on the Continuity Method

To simplify our notation, we omit the t dependence of the metrics; $\omega, \widetilde{\omega}$, and functions; σ, τ and ρ. We first recall some properties of the continuity equation. Let

$$f : X \to X_{\text{can}} \subset \mathbb{CP}^N \tag{5.11}$$

be the semi-ample fibration induced by the pluricanonical system. The target manifold X_{can} is a $kod(X)$-dimensional projective variety and is called the canonical model of X. In this setting, one obtains χ, a multiple of the Fubini-Study metric on $\mathbb{CP}^N$, such that its pullback $f^*\chi$ is a smooth semi-positive representation of $-2\pi c_1(X)$. From a geometric perspective, $2\pi c_1(X)$ is the class that contains the Ricci form of X. Furthermore, we set Ω to be a smooth volume form on X with $\sqrt{-1}\partial\bar{\partial}\log\Omega = f^*\chi$. The general strategy for studying (5.1) is to use cohomology to reduce the continuity equation to a Monge-Ampère equation. More precisely, for $t \in [0, \infty)$ we define a family of reference metrics given by

$$\omega_t := \frac{1}{1+t}\omega_0 + \frac{t}{1+t} f^*\chi. \tag{5.12}$$

This allows us to define $\varphi : X \times [0, \infty) \to \mathbb{R}$ through

$$\sqrt{-1}\partial\overline{\partial}\varphi = \omega - \omega_t. \tag{5.13}$$

By substituting this expression into equation (5.1), we obtain the following elliptic Monge-Ampère equation for $\varphi(t)$;

$$\omega^n = (1+t)^{-r} e^{\left(\frac{1+t}{t}\right)\varphi}\Omega. \tag{5.14}$$

Here $n = \dim(X)$ and $r = n - kod(X)$, the dimension of a generic collapsing fibre. From the analysis carried out in [ZZ19], we have the following

Lemma 5.6 (Lemma 2.1 in [ZZ19])**.** *Let $\varphi(t)$ be the Kähler potential of $\omega(t)$. Then there exists $C > 0$ such that*

$$|\varphi(t)| \leqslant C, \quad \textit{for all } t \in [0, \infty). \tag{5.15}$$

Although the result in [ZZ19] are for elliptic surfaces, $n = 2$ and $kod(X) = 1$, one can check that the arguments carry through for higher dimensions. As mentioned in [ZZ19], the proof follows from the pluripotential theory of degenerate complex Monge-Ampère equation ([DP10; EGZ08]).

Using Lemma 5.6, we can show the following volume equivalence, which will be useful in the next section.

Lemma 5.7. *Let $\omega(t)$ and $\widetilde{\omega}(t)$ be solutions to the continuity method starting from different initial metrics. Then there exists a uniform $C > 0$ such that*

$$C^{-1}\widetilde{\omega}^n \leqslant \omega^n \leqslant C\widetilde{\omega}^n \tag{5.16}$$

for all $t \geqslant 1$.

Proof. From the Monge-Ampère equation (5.14) we have

$$\frac{1+t}{t}\varphi = \log\left(\frac{(1+t)^r\omega^n}{\Omega}\right) \quad \text{and} \quad \frac{1+t}{t}\widetilde{\varphi} = \log\left(\frac{(1+t)^r\widetilde{\omega}^n}{\Omega}\right).$$

Combining the two equations above, we have

$$\frac{1+t}{t}(\varphi - \widetilde{\varphi}) = \log\left(\frac{\omega^n}{\widetilde{\omega}^n}\right).$$

Then, using Lemma 5.6 in the above, we obtain (5.16). □

Currently, is not known whether the scalar curvature is uniformly bounded along the continuity method. On compact sets away from singular fibres, a twisted scalar curvature converges to $-kod(X)$ as a result due to Jian and Y. Zhang. This is substantially different from the flow, where it was shown in [Jia20] that the scalar curvature converges to $-kod(X)$ on compact sets away from singular fibres. Furthermore, global scalar curvature bounds for the Kähler-Ricci flow are known, see [ST17].

Assuming that our initial metric has negative curvature, one can obtain a bound on scalar curvature. By direct calculation, one can show

$$\Delta_\omega \operatorname{tr}_\omega \omega_0 \geqslant -g^{\bar{j}i}g^{\bar{l}k}R^0_{i\bar{j}k\bar{l}} + \frac{1}{t}|\omega_0|^2_g - \frac{1+t}{t}\operatorname{tr}_\omega \omega_0 + \frac{|\partial \operatorname{tr}_\omega \omega_0|^2_g}{\operatorname{tr}_\omega \omega_0}. \tag{5.17}$$

Suppose that the curvature of the initial metric satisfies

$$R_{i\bar{i}j\bar{j}} \leqslant -A < 0,$$

for some $C > 0$. Then from (5.17), we have

$$\Delta_\omega \operatorname{tr}_\omega \omega_0 \geqslant A(\operatorname{tr}_\omega \omega_0)^2 - \frac{1+t}{t}\operatorname{tr}_\omega \omega_0.$$

Then, applying a maximum principle argument, one obtains

$$\operatorname{tr}_\omega \omega_0 \leqslant C, \tag{5.18}$$

for t away from 0. For times close to 0, we can simply use that ω is close to ω_0 and thus the above trace quantity is bounded. In fact, this shows that the solution is volume non-collapsed. This reflects the coarse correspondence between Kodaria dimension and curvature; a negative Kodaria dimension corresponds to positive curvature, a zero Kodaria dimension corresponds to flatness, and a maximum Kodaria dimension (general type) corresponds to negative curvature.

5.3 Metric Equivalence

In this section, we give a proof for Theorem 5.1 and derive Ricci curvature estimates as stated in Corollary 5.2.

(Proof of Theorem 5.1). For convenience of notation, we drop the time dependence for the metrics and denote $C > 0$ to be some constant which may change in each line but can be fixed at the end of each proof. We also denote the Hermitian metric associated to ω by g and apply the same convention for $\tilde{\omega}$ with $\tilde{g}$ and ω_0 with g_0. Recall the standard identity

$$\Delta_\omega \operatorname{tr}_\omega \widetilde{\omega} = g^{\bar{b}p} g^{\bar{q}a} \widetilde{g}_{p\bar{q}} R_{a\bar{b}} - g^{\bar{j}i} g^{\bar{q}p} \widetilde{R}_{i\bar{j}p\bar{q}} + g^{\bar{j}i} g^{\bar{q}p} \widetilde{g}^{\bar{b}a} \nabla_i \widetilde{g}_{p\bar{b}} \nabla_{\bar{j}} \widetilde{g}_{a\bar{q}}.$$

The first term can be rewritten in terms of metrics using (5.1) and the second term contains a controlled curvature term due to (5.3). Then using the Cauchy-Schwarz inequality, we have

$$\Delta_\omega \operatorname{tr}_\omega \widetilde{\omega} \geqslant -\frac{C}{t} \operatorname{tr}_\omega \omega_0 \operatorname{tr}_\omega \widetilde{\omega} - \frac{1+t}{t} \operatorname{tr}_\omega \widetilde{\omega} - C\tau (\operatorname{tr}_\omega \widetilde{\omega})^2 + g^{\bar{j}i} g^{\bar{q}p} \widetilde{g}^{\bar{b}a} \nabla_i \widetilde{g}_{p\bar{b}} \nabla_{\bar{j}} \widetilde{g}_{a\bar{q}}, \tag{5.19}$$

for all $t \geqslant 1$. Since (5.3) yields a Ricci curvature bound for $\widetilde{\omega}$ which satisfies (5.1), we have

$$\omega_0 \leqslant C\widetilde{\omega}_0 \leqslant Ct\tau\widetilde{\omega},$$

for some $C > 0$ and for all $t \geqslant 1$. Substituting the above inequality into (5.19) yields

$$\Delta_\omega \operatorname{tr}_\omega \widetilde{\omega} \geqslant -C\tau (\operatorname{tr}_\omega \widetilde{\omega})^2 - C \operatorname{tr}_\omega \widetilde{\omega} + g^{\bar{j}i} g^{\bar{q}p} \widetilde{g}^{\bar{b}a} \nabla_i \widetilde{g}_{p\bar{b}} \nabla_{\bar{j}} \widetilde{g}_{a\bar{q}}. \tag{5.20}$$

Then by a standard trick using the Cauchy-Schwarz inequality, we can rewrite (5.20) as

$$\Delta_\omega \log \operatorname{tr}_\omega \widetilde{\omega} \geqslant -C - C\tau \operatorname{tr}_\omega \widetilde{\omega}. \tag{5.21}$$

We divide through by τ, noting that $\tau \geqslant 1$, to obtain

$$\Delta_\omega (\tau^{-1} \log \operatorname{tr}_\omega \widetilde{\omega}) \geqslant -C \operatorname{tr}_\omega \widetilde{\omega} - C. \tag{5.22}$$

On the other hand, taking the trace of (5.13) for both $\widetilde{\omega}$ and ω respectively yields

$$\Delta_\omega \widetilde{\varphi} = \operatorname{tr}_\omega \widetilde{\omega} - \operatorname{tr}_\omega \widetilde{\omega}_t, \tag{5.23}$$

and

$$\Delta_\omega \varphi = n - \operatorname{tr}_\omega \omega_t, \tag{5.24}$$

respectively. Here we define $\widetilde{\omega}_t$ to be the reference metric similarly defined as in (5.12) but with $\widetilde{\omega}_0$ in place of ω_0. Let $A \geqslant 1$ be large enough so that $A\omega_0 > \widetilde{\omega}_0$. From (5.23) and (5.24), we have

$$\Delta_\omega (\widetilde{\varphi} - A\varphi) = \operatorname{tr}_\omega \widetilde{\omega} - An + \frac{1}{1+t} (A \operatorname{tr}_\omega \omega_0 - \operatorname{tr}_\omega \widetilde{\omega}_0) + (A-1) \frac{t}{1+t} \operatorname{tr}_\omega f^* \chi.$$

This gives us the following inequality

$$\Delta_\omega (\widetilde{\varphi} - A\varphi) \geqslant \operatorname{tr}_\omega \widetilde{\omega} - C. \tag{5.25}$$

Now, for a large enough $B > 0$, using (5.22) and (5.25), we derive the estimate

$$\Delta_\omega \left(\tau^{-1} \log \operatorname{tr}_\omega \widetilde{\omega} + B(\widetilde{\varphi} - A\varphi)\right) \geqslant C \operatorname{tr}_\omega \widetilde{\omega} - C.$$

Then applying a standard maximum principle argument with the help of the uniform bounds in Lemma 5.6, we obtain a uniform $C > 0$ such that

$$\operatorname{tr}_\omega \widetilde{\omega} \leqslant C e^{C\tau} \tag{5.26}$$

holds on X for all $t \geqslant 1$. This yields the lower bound in (5.4). To deduce the upper estimate, we use

$$\operatorname{tr}_{\widetilde{\omega}} \omega \leqslant n(\operatorname{tr}_\omega \widetilde{\omega})^{n-1} \frac{\omega^n}{\widetilde{\omega}^n}$$

and Lemma 5.16 to show

$$\operatorname{tr}_{\widetilde{\omega}} \omega \leqslant C e^{C\tau}, \tag{5.27}$$

for all $t \geqslant 1$. The upper estimate immediately follows from the above inequality. □

From the metric equivalence in 5.1, one can immediately obtain Ricci curvature estimates along (5.1).

(Proof of Corollary 5.2). A lower bound for the Ricci curvature always holds for the continuity equation. Indeed,

$$\operatorname{Ric}(\omega(t)) = \frac{1}{t}\omega_0 - \frac{1+t}{t}\omega(t) \geqslant -2\omega(t) \tag{5.28}$$

for all $t \geqslant 1$. To obtain the upper bound, we choose $A > 0$ large enough such that $\omega_0 \leqslant A\widetilde{\omega}_0$ and rewrite

$$\begin{aligned} \operatorname{Ric}(\omega(t)) &= \frac{1}{t}\omega_0 - \frac{1+t}{t}\omega(t) \\ &\leqslant \frac{A}{t}\widetilde{\omega}_0 \\ &\leqslant C\tau\widetilde{\omega} \\ &\leqslant C\tau e^{C\tau}\omega. \end{aligned} \tag{5.29}$$

In the second last line, we used (5.3) to deduce that $\widetilde{\omega}_0 \leqslant Ct\tau\widetilde{\omega}$ from the scalar curvature bound. The last line follows from (5.26). From the Ricci bound, the scalar curvature bounds immediately follow. □

To conclude this section, we make note of an estimate which will be useful throughout the next section. Suppose we are in the setup of Theorem 5.3. Taking the trace of (5.1), the scalar curvature $R(t)$ is given by

$$R(t) = \frac{1}{t} \operatorname{tr}_\omega \omega_0 - n\frac{1+t}{t}. \tag{5.30}$$

An upper bound for scalar curvature implies that there exists $C > 0$ such that

$$\operatorname{tr}_\omega \omega_0 \leqslant Ct\sigma\tau, \tag{5.31}$$

for all $t \geqslant 1$.

5.4 Curvature Estimates

We first obtain Calabi C^3-bounds for the metric ω in terms of the bounds for $\widetilde{\omega}$. This will be crucial when applying the maximum principle to obtain the curvature bound in Theorem 5.3. Define the tensor $\Psi^k_{i\bar{j}} := \Gamma^k_{i\bar{j}} - \widetilde{\Gamma}^k_{i\bar{j}}$ and its norm $S = |\Psi|^2_{\omega(t)}$. In local coordinates,

$$S = g^{\bar{j}i} g^{\bar{l}k} g^{\bar{q}p} \widetilde{\nabla}_i g_{k\bar{q}} \widetilde{\nabla}_{\bar{j}} g_{p\bar{l}}. \tag{5.32}$$

If S and T are tensors, we write $S * T$ for any linear combination of the products of tensors S and T formed by contractions using the metric g.

Lemma 5.8. *Let S be defined as above, the norm difference between Christoffel symbols for the evolving metric ω and a given metric $\tilde{\omega}$. We denote their corresponding geometric quantities similarly; $\tilde{\omega}$, $\tilde{g}$, $\widetilde{\mathrm{Ric}}$, $\widetilde{\mathrm{Rm}}$ and so on. Then*

$$\begin{aligned} \Delta_\omega S \geqslant & - C\sigma\tau S - C|\Psi * Rm(\widetilde{\omega})|_\omega \sqrt{S} - C \left| g^{\bar{b}a} \widetilde{\nabla}_a \widetilde{R}_{i\bar{b}p}{}^k \right|_\omega \sqrt{S} \\ & - 2\sigma\tau\sqrt{S_0}\sqrt{S} + |\nabla\Psi|^2_\omega + |\bar{\nabla}\Psi|^2_\omega. \end{aligned} \tag{5.33}$$

Proof. The laplacian of S is given by

$$\begin{aligned} \Delta_\omega S =& R^{\bar{j}i} g^{\bar{q}p} g_{k\bar{l}} \Psi^k_{ip} \overline{\Psi^l_{jq}} + g^{\bar{j}i} R^{\bar{q}p} g_{k\bar{l}} \Psi^k_{ip} \overline{\Psi^l_{jq}} - g^{\bar{j}i} g^{\bar{q}p} R_{k\bar{l}} \Psi^k_{ip} \overline{\Psi^l_{jq}} \\ & + 2\,\mathrm{Re}\left(g^{\bar{j}i} g^{\bar{q}p} g_{k\bar{l}} \left(\Delta_\omega \Psi^k_{ip}\right) \overline{\Psi^l_{jq}} \right) \\ & + |\nabla\Psi|^2_\omega + |\bar{\nabla}\Psi|^2_\omega \\ =&: I + II + |\nabla\Psi|^2_\omega + |\bar{\nabla}\Psi|^2_\omega. \end{aligned} \tag{5.34}$$

Here, we denote the terms in the first line of (5.34) by I and second line of (5.34) by II. In the Kähler-Ricci flow, the Ricci terms in I cancel with the time evolution. For the continuity method, we instead use (5.1) to rewrite the Ricci terms in terms of ω and ω_0 for $t \geqslant 1$;

$$I = \frac{1}{t}\left(g_0^{\bar{j}i} g^{\bar{q}p} g_{k\bar{l}} + g^{\bar{j}i} g_0^{\bar{q}p} g_{k\bar{l}} - g^{\bar{j}i} g^{\bar{q}p} g^0_{k\bar{l}} \right) \Psi^k_{ip} \overline{\Psi^l_{jq}} - \frac{1+t}{t} S. \tag{5.35}$$

We apply local normal coordinates which diagonalizes ω_0 with respect to ω. Let λ^0_i be the principle eigenvalues of g_0. Using the upper trace bound (5.31) in local coordinates, $\lambda^0_i \leqslant Ct\sigma\tau$, we obtain the estimate

$$\begin{aligned} I &= \frac{2}{t\lambda^0_i} \Psi^k_{ip} \overline{\Psi^k_{ip}} - \frac{\lambda^0_i}{t} \Psi^k_{ip} \overline{\Psi^k_{ip}} - \frac{1+t}{t} S \\ &\geqslant \frac{2}{C\sigma\tau t^2} S - C\sigma\tau S - \frac{1+t}{t} S \\ &\geqslant -C\sigma\tau S \end{aligned} \tag{5.36}$$

for all $t \geqslant 1$. Moving onto the terms in II, we can use

$$\Delta_\omega \Psi^k_{ip} = \nabla^{\bar{b}} \widetilde{R}^k_{i\bar{b}p} - \nabla_i R^k_p, \tag{5.37}$$

to rewrite II as

$$II = 2\,\mathrm{Re}\left(g^{\bar{j}i}g^{\bar{q}p}g_{k\bar{l}}\left(\nabla^{\bar{b}}\widetilde{R}^{k}_{i\bar{b}p}\right)\overline{\Psi^{l}_{jq}}\right) - 2\,\mathrm{Re}\left(g^{\bar{j}i}g^{\bar{q}p}g_{k\bar{l}}\left(\nabla_i R^k_p\right)\overline{\Psi^{l}_{jq}}\right). \tag{5.38}$$

To treat the first term in (5.38), we can use

$$\begin{aligned}\nabla^{\bar{b}}\widetilde{R}^{k}_{i\bar{b}p} &= g^{\bar{b}a}\nabla_a\widetilde{R}^{k}_{i\bar{b}p}\\ &= g^{\bar{b}a}\left(\nabla_a - \widetilde{\nabla}_a\right)\widetilde{R}^{k}_{i\bar{b}p} + g^{\bar{b}a}\widetilde{\nabla}_a\widetilde{R}^{k}_{i\bar{b}p}\\ &= \Psi * Rm(\widetilde{\omega}) + g^{\bar{b}a}\widetilde{\nabla}_a\widetilde{R}^{k}_{i\bar{b}p}.\end{aligned} \tag{5.39}$$

For the second term in (5.38), we once again use the continuity equation (5.1) to obtain

$$R^k_p = \frac{1}{t}g^{\bar{r}k}g^0_{p\bar{r}} - \frac{1+t}{t}g^{\bar{r}k}g_{p\bar{r}}.$$

Here, g_0 is the metric associated with the form ω_0. Taking the covariant derivative and using metric compatibility yields

$$\nabla_i R^k_p = \frac{1}{t}g^{\bar{r}k}\nabla_i g^0_{p\bar{r}}.$$

Hence, we can rewrite said second term of II as

$$\begin{aligned}-2\,\mathrm{Re}\left(g^{\bar{j}i}g^{\bar{q}p}g_{k\bar{l}}\left(\nabla_i R^k_p\right)\overline{\Psi^{l}_{jq}}\right) &= -\frac{2}{t}\,\mathrm{Re}\left(g^{\bar{j}i}g^{\bar{q}p}\left(\nabla_i g^0_{p\bar{l}}\right)\overline{\Psi^{l}_{jq}}\right)\\ &= -\frac{2}{t}\,\mathrm{Re}\left(g^{\bar{j}i}g^{\bar{q}p}\left((\nabla_i - \nabla^0_i)g^0_{p\bar{l}}\right)\overline{\Psi^{l}_{jq}}\right)\\ &= \frac{2}{t}\,\mathrm{Re}\left(g^{\bar{j}i}g^{\bar{q}p}g^0_{n\bar{l}}\mathring{\Psi}^n_{ip}\overline{\Psi^{l}_{jq}}\right)\\ &\geqslant -2\sigma\tau\sqrt{S_0}\sqrt{S}\end{aligned} \tag{5.40}$$

where Ψ^0 and S_0 are defined by

$$(\Psi^0)^k_{i\bar{j}} := \Gamma^k_{i\bar{j}} - (\Gamma_0)^k_{i\bar{j}} \text{ and } S_0 = |\mathring{\Psi}|^2_\omega.$$

Using Cauchy-Schwarz inequality and the above, we can estimate II from below by

$$II \geqslant -C|\Psi * Rm(\widetilde{\omega})|_\omega\sqrt{S} - C\left|g^{\bar{b}a}\widetilde{\nabla}_a\widetilde{R}_{i\bar{b}p}{}^{k}\right|_\omega\sqrt{S} - 2\sigma\tau\sqrt{S_0}\sqrt{S}. \tag{5.41}$$

Thus, combining (5.34), (5.36) and (5.41), we have (5.33). □

Using Lemma 5.8, we obtain a bound for S.

Proposition 5.9. *Suppose we are in the setting of Theorem 5.3 and let S be defined as in (5.32). There exists $C > 0$ such that*

$$S \leqslant Ct^2\sigma^4\tau^3\rho^2, \tag{5.42}$$

for all $t \geqslant 1$.

Proof. We begin by observing that the last term in (5.20) can be estimated by

$$\begin{aligned} g^{\bar{j}i}g^{\bar{q}p}\widetilde{g}^{\bar{b}a}\nabla_i\widetilde{g}_{p\bar{b}}\nabla_{\bar{j}}\widetilde{g}_{a\bar{q}} &= g^{\bar{j}i}g^{\bar{q}p}\widetilde{g}^{\bar{b}a}\left(\nabla_i-\widetilde{\nabla}_i\right)\widetilde{g}_{p\bar{b}}\left(\nabla_{\bar{j}}-\widetilde{\nabla}_{\bar{j}}\right)\widetilde{g}_{a\bar{q}} \\ &= g^{\bar{j}i}g^{\bar{q}p}\widetilde{g}^{\bar{b}a}\left(-\Psi^d_{ip}\right)\widetilde{g}_{d\bar{b}}\left(-\overline{\Psi^e_{jq}}\right)\widetilde{g}_{a\bar{e}} \\ &\geqslant \sigma^{-1}S. \end{aligned} \tag{5.43}$$

By assumption (5.6), the remaining terms in (5.20) are controlled. Therefore,

$$\Delta_\omega(\sigma^{-1}\operatorname{tr}_\omega\widetilde{\omega}) \geqslant -C - C\tau\sigma + \sigma^{-2}S. \tag{5.44}$$

On the other hand, (5.7), (5.8) and (5.9), gives us

$$\begin{aligned} &|\Psi * Rm(\widetilde{\omega})|_\omega \leqslant |\Psi|_\omega |Rm(\widetilde{\omega})|_\omega \leqslant C\sigma^2\tau|\Psi|_\omega \quad \text{and} \\ &\left|g^{\bar{b}a}\widetilde{\nabla}_a\widetilde{R}_{i\bar{b}p}{}^k\right|_\omega \leqslant \sigma\left|\widetilde{\nabla}^{\bar{b}}\widetilde{R}_{i\bar{b}p}{}^k\right|_\omega \leqslant \sigma^{\frac{5}{2}}\left|\widetilde{\nabla}^{\bar{b}}\widetilde{R}_{i\bar{b}p}{}^k\right|_{\widetilde{\omega}} \leqslant C\sigma^{\frac{5}{2}}\rho. \end{aligned} \tag{5.45}$$

Thus, we are left to bound the S_0 term in (5.33). To this end, we make the following claim.

Claim: There exists a uniform $C > 0$ such that

$$S_0 \leqslant Ct^2\sigma^4\tau^3 \tag{5.46}$$

for all $t \geqslant 1$.

We assume this for now and proceed with the proof of Proposition 5.9. Combining (5.45) and (5.46) with (5.33), we obtain

$$\Delta_\omega S \geqslant -C\sigma\tau S - C\sigma^2\tau S - C\sigma^{\frac{5}{2}}\rho\sqrt{S} - 2Ct\sigma^3\tau^{\frac{5}{2}}\sqrt{S} + |\nabla\Psi|^2_\omega + |\bar{\nabla}\Psi|^2_\omega$$

for all $t \geqslant 1$. After collecting terms by their power of S, using that $\sigma, \tau, \rho, t \geqslant 1$, and discarding the norms of $\nabla\Psi$, the above inequality simplifies to

$$\Delta_\omega S \geqslant -C\sigma^2\tau S - Ct\sigma^3\tau^{\frac{5}{2}}\rho\sqrt{S}.$$

In order to apply the maximum principle, we rewrite this as

$$\Delta_\omega(\sigma^{-4}\tau^{-1}S) \geqslant -C\sigma^{-2}S - Ct\sigma^{-1}\tau^{\frac{3}{2}}\rho\sqrt{S}. \tag{5.47}$$

Let $Q := \sigma^{-4}\tau^{-1}S + A\sigma^{-1}\operatorname{tr}_\omega\widetilde{\omega}$ for some arbitrary $A > 0$. From (5.44) and (5.47), we have

$$\Delta_\omega Q \geqslant (A - C)\sigma^{-2}S - Ct\sigma^{-1}\tau^{\frac{3}{2}}\rho\sqrt{S} - AC - A\tau\sigma, \tag{5.48}$$

for some constant $C > 0$ and all $t \geqslant 1$. We can always choose $A \geqslant 2C$ large enough to force a sign on the coefficients of S. Moreover, we can assume that at the maximum of point of Q,

$$t\sigma^{-1}\tau^{\frac{3}{2}}\rho\sqrt{S} \leqslant \frac{1}{2}\sigma^{-2}S.$$

Otherwise, we would have $S \leqslant Ct^2\sigma^2\tau^3\rho^2$ at this point, which would lead to (5.42). Putting all this together, we obtain a lower estimate at the maximum of Q:

$$\Delta_\omega Q \geqslant \frac{C}{2}\sigma^{-2}S - A\tau\sigma - AC. \tag{5.49}$$

Then, by the maximum principle, we obtain the estimate

$$S \leqslant C\sigma^4\tau \leqslant Ct^2\sigma^4\tau^3\rho^2 \tag{5.50}$$

on all of X and for all $t \geqslant 1$.

We now prove the claim. From a similar calculation leading to (5.20), we can find some $C > 0$ such that

$$\Delta_\omega\left(\frac{1}{t}\operatorname{tr}_\omega\omega_0\right) \geqslant -\frac{C}{t} - C\sigma\tau - Ct\sigma^2\tau^2 + \frac{1}{t}g^{\bar{j}i}g^{\bar{q}p}g^0_{a\bar{b}}\mathring{\Psi}^d_{ip}\overline{\mathring{\Psi}^e_{jq}}, \tag{5.51}$$

for all $t \geqslant 1$. To estimate the last term, we use (5.6) and (5.9) to obtain

$$t^{-1}\operatorname{tr}_\omega\omega_0 \geqslant \sigma^{-1}t^{-1}\operatorname{tr}_{\widetilde{\omega}}\widetilde{\omega}_0 > C^{-1}\sigma^{-1}.$$

Applying normal coordinates for which ω_0 is diagonalised with respect to ω, we obtain

$$\frac{1}{t}g^{\bar{j}i}g^{\bar{q}p}g^0_{a\bar{b}}\mathring{\Psi}^d_{ip}\overline{\mathring{\Psi}^e_{jq}} = \frac{1}{t}\lambda^0_a S_0 \geqslant C^{-1}\sigma^{-1}S_0. \tag{5.52}$$

As a consequence, we have for some $C > 0$,

$$\Delta_\omega\left(t^{-1}\operatorname{tr}_\omega\omega_0\right) \geqslant -Ct^{-1} - C\sigma\tau - Ct\sigma^2\tau^2 + C^{-1}\sigma^{-1}S_0, \tag{5.53}$$

for all $t \geqslant 1$. We can also compute a lower bound for $\Delta_\omega S_0$ using Lemma 5.8. Here we have $\mathring{\Psi}$ instead of Ψ and Rm^0 in place of $\widetilde{Rm}$. Moreover, by (5.31) we have

$$\begin{aligned}
&|\mathring{\Psi} * Rm(\omega_0)|_\omega \leqslant |\mathring{\Psi}|_\omega |Rm(\omega_0)|_\omega \leqslant Ct^2\sigma^2\tau^2|\mathring{\Psi}|_\omega \quad \text{and} \\
&\left|g^{\bar{b}a}\mathring{\nabla}_a\mathring{R}_{i\bar{b}p}{}^k\right|_\omega \leqslant t\sigma\tau\left|\mathring{\nabla}^{\bar{b}}\mathring{R}_{i\bar{b}p}{}^k\right|_\omega \leqslant (t\sigma\tau)^2\left|\mathring{\nabla}^{\bar{b}}\mathring{R}_{i\bar{b}p}{}^k\right|_{\omega_0} \leqslant Ct^2\sigma^2\tau^2.
\end{aligned}$$

Therefore, for some $C > 0$, we have

$$\Delta_\omega S_0 \geqslant -Ct^2\sigma^2\tau^2 S_0 - Ct^2\sigma^2\tau^2\sqrt{S_0} + |\nabla\mathring{\Psi}|^2_\omega + |\overline{\nabla}\mathring{\Psi}|^2_\omega, \tag{5.54}$$

for all $t \geqslant 1$. For some $A > 0$, we set

$$Q := t^{-2}\sigma^{-4}\tau^{-3}S_0 + At^{-1}\sigma^{-1}\tau^{-1}\operatorname{tr}_\omega\omega_0.$$

We can then combine (5.53) and (5.54) to estimate the Laplacian of Q from below. Then, choosing A large enough, we can force a sign on the coefficient of S_0 to obtain

$$\Delta_\omega Q \geqslant C\sigma^{-2}\tau^{-1}S_0 - C\sigma^{-2}\tau^{-1}\sqrt{S_0} - Ct\sigma\tau, \tag{5.55}$$

for some $C > 0$ and for all $t \geqslant 1$. We assume that $S_0 \geqslant 1$ and apply the maximum principle to show that at a maximum of Q,

$$S_0 \leqslant Ct\sigma^3\tau^2.$$

This bound and the uniform bound of $t^{-1}\sigma^{-1}\tau^{-1}\operatorname{tr}_\omega\omega_0$ establishes (5.46). $\square$

With the proof of Proposition 5.9 completed, we can proceed to prove Theorem 5.3.

Proof of Theorem 5.3. From Lemma 3.2.10 in [SW13b], we have

$$\Delta_\omega \mathrm{Rm} = \nabla_{\bar{l}} \nabla_k R_{i\bar{j}} + \mathrm{Rm} * \mathrm{Rm} + \mathrm{Rc} * \mathrm{Rm}. \tag{5.56}$$

Once again, we use (5.1) to rewrite the Ricci term:

$$\nabla_{\bar{l}} \nabla_k R_{i\bar{j}} = \frac{1}{t} \nabla_{\bar{l}} \nabla_k g^0_{i\bar{j}} - \frac{1+t}{t} \nabla_{\bar{l}} \nabla_k g_{i\bar{j}}. \tag{5.57}$$

The second term is zero by metric compatibility. The first term, expressed in normal coordinates with respect to g, can be expressed as

$$\begin{aligned}
\frac{1}{t} \nabla_{\bar{l}} \nabla_k g^0_{i\bar{j}} &= \frac{1}{t} \partial_{\bar{l}} \partial_k g^0_{i\bar{j}} - \frac{1}{t} R_{\bar{j} i \bar{l}}^{\overline{m}} g^0_{k\bar{m}}, \\
&= -\frac{1}{t} R^0_{k\bar{l} i\bar{j}} + \frac{1}{t} g_0^{\bar{q}p} (\partial_k g^0_{i\bar{q}})(\partial_{\bar{l}} g^0_{p\bar{j}}) - \frac{1}{t} R_{\bar{j} i \bar{l}}^{\overline{m}} g^0_{k\bar{m}}, \\
&= -\frac{1}{t} R^0_{k\bar{l} i\bar{j}} + \frac{1}{t} g_0^{\bar{q}p} (\nabla_k g^0_{i\bar{q}})(\nabla_{\bar{l}} g^0_{p\bar{j}}) - \frac{1}{t} R_{\bar{j} i \bar{l}}^{\overline{m}} g^0_{k\bar{m}}, \\
&= -\frac{1}{t} R^0_{k\bar{l} i\bar{j}} + \frac{1}{t} g^0_{n\bar{m}} \mathring{\Psi}^n_{ki} \overline{\mathring{\Psi}^m_{lj}} - \frac{1}{t} R_{\bar{j} i \bar{l}}^{\overline{m}} g^0_{k\bar{m}}.
\end{aligned}$$

Thus, for any $t \geqslant 1$, we have in normal coordinates

$$\Delta_\omega R_{i\bar{j}k\bar{l}} = -\frac{1}{t} Rm(\omega_0) + \frac{1}{t} g^0_{n\bar{m}} \mathring{\Psi}^n_{ki} \overline{\mathring{\Psi}^m_{lj}} - \frac{1}{t} R_{\bar{j} i \bar{l}}^{\overline{m}} g^0_{k\bar{m}} + \mathrm{Rm} * \mathrm{Rm} + \mathrm{Rm} * \mathrm{Ric}. \tag{5.58}$$

Denote $\Phi_{kilj} := t^{-1} g^0_{n\bar{m}} \mathring{\Psi}^n_{ki} \overline{\mathring{\Psi}^m_{lj}}$. Then, using the above and the metric trace bound (5.31), we have

$$\begin{aligned}
\Delta_\omega \left(|\mathrm{Rm}(\omega)|^2_\omega \right) \geqslant & -\frac{1}{t} |\mathrm{Rm}(\omega_0)|_\omega |\mathrm{Rm}(\omega)|_\omega - |\Phi|_\omega |\mathrm{Rm}(\omega)|_\omega - C\sigma\tau |\mathrm{Rm}(\omega)|^2_\omega \\
& - C|\mathrm{Rm}(\omega)|^3_\omega + |\nabla \mathrm{Rm}(\omega)|^2_\omega + |\overline{\nabla} \mathrm{Rm}(\omega)|^2_\omega.
\end{aligned}$$

In normal coordinates with respect to ω, we can deduce an upper bound for Φ:

$$|\Phi|_\omega = t^{-1} |g^0_{n\bar{m}} \mathring{\Psi}^n_{ki} \overline{\mathring{\Psi}^m_{lj}}|_\omega \leqslant C\sigma\tau S_0 \leqslant C t^2 \sigma^5 \tau^4.$$

Using (5.31), we also have

$$|\mathrm{Rm}(\omega_0)|_\omega \leqslant C t^2 \sigma^2 \tau^2 |\mathrm{Rm}(\omega_0)|_{\omega_0} \leqslant C t^2 \sigma^2 \tau^2.$$

Thus, taking $t \geqslant 1$, using (5.46) and the previous two estimates, we deduce that for some uniform $C > 0$,

$$\begin{aligned}
\Delta_\omega \left(|\mathrm{Rm}(\omega)|^2_\omega \right) \geqslant & - C t \sigma^2 \tau^2 |\mathrm{Rm}(\omega)|_\omega - C t^2 \sigma^5 \tau^4 |\mathrm{Rm}(\omega)|_\omega - C\sigma\tau |\mathrm{Rm}(\omega)|^2_\omega \\
& - C|\mathrm{Rm}(\omega)|^3_\omega + |\nabla \mathrm{Rm}(\omega)|^2_\omega + |\overline{\nabla} \mathrm{Rm}(\omega)|^2_\omega \\
\geqslant & - C t^2 \sigma^5 \tau^4 |\mathrm{Rm}(\omega)|_\omega - C\sigma\tau |\mathrm{Rm}(\omega)|^2_\omega \\
& - C|\mathrm{Rm}(\omega)|^3_\omega + |\nabla \mathrm{Rm}(\omega)|^2_\omega + |\overline{\nabla} \mathrm{Rm}(\omega)|^2_\omega
\end{aligned} \tag{5.59}$$

In the last inequality, we collected terms by their power of curvature using that $\sigma, \tau, \rho, t \geqslant 1$. To simplify our notation, we set

$$\widetilde{S} = t^{-2}\sigma^{-4}\tau^{-3}\rho^{-2}S,$$

which is uniformly bounded for $t \geqslant 1$ due to (5.42). From the proof in Proposition (5.9), there exists $C > 0$ such that

$$\Delta_\omega \widetilde{S} \geqslant -C\sigma^2\tau + t^{-2}\sigma^{-4}\tau^{-3}\rho^{-2}\left(|\nabla\Psi|^2_\omega + |\overline{\nabla}\Psi|^2_\omega\right), \tag{5.60}$$

for all $t \geqslant 1$.

Next, for $A > 0$ large enough such that $A - \tilde{S} > 1$, we define the quantity

$$Q := \frac{|\mathrm{Rm}(\omega)|^2_\omega}{A - \widetilde{S}}. \tag{5.61}$$

Since $\widetilde{S}$ is uniformly bounded, we can take A large enough so that the $A - \widetilde{S}$ is positive. We will now set up a maximum principle argument to complete the proof for our result. Taking the Laplacian and applying Cauchy-Schwarz, we can find some $C > 0$ such that

$$\begin{aligned}\Delta_\omega Q \geqslant & -Ct^2\sigma^5\tau^4\frac{|\mathrm{Rm}(\omega)|_\omega}{A-\widetilde{S}} - C\sigma\tau\frac{|\mathrm{Rm}(\omega)|^2_\omega}{A-\widetilde{S}} - C\frac{|\mathrm{Rm}(\omega)|^3_\omega}{A-\widetilde{S}} + \frac{|\nabla\mathrm{Rm}(\omega)|^2_\omega}{A-\widetilde{S}} + \frac{|\overline{\nabla}\mathrm{Rm}(\omega)|^2_\omega}{A-\widetilde{S}} \\ & - C\sigma^2\tau\frac{|\mathrm{Rm}(\omega)|^2_\omega}{(A-\widetilde{S})^2} + t^{-2}\sigma^{-4}\tau^{-3}\rho^{-2}(|\nabla\Psi|^2_\omega + |\overline{\nabla}\Psi|^2_\omega)\frac{|\mathrm{Rm}(\omega)|^2_\omega}{(A-\widetilde{S})^2} \\ & - 2\frac{|\partial|\mathrm{Rm}(\omega)|^2_\omega|_\omega|\partial\widetilde{S}|_\omega}{(A-\widetilde{S})^2} + 2\frac{|\mathrm{Rm}(\omega)|^2_\omega|\partial\widetilde{S}|_\omega}{(A-\widetilde{S})^3},\end{aligned} \tag{5.62}$$

for all $t \geqslant 1$. The goal here is to produce a positive fourth power curvature term to dominate the negative curvature terms. This will come from the third-last term. To deal with the second-last term, we use

$$|\partial|\mathrm{Rm}(\omega)|^2_\omega|^2_\omega \leqslant 2|\mathrm{Rm}(\omega)|^2_\omega(|\nabla\mathrm{Rm}(\omega)|^2_\omega + |\overline{\nabla}\mathrm{Rm}(\omega)|^2_\omega)$$

in combination with Young's inequality to obtain

$$2\frac{|\partial|\mathrm{Rm}(\omega)|^2_\omega|_\omega|\partial\widetilde{S}|_\omega}{(A-\widetilde{S})^2} \leqslant 4\frac{|\mathrm{Rm}(\omega)|^2_\omega}{(A-\widetilde{S})^3}|\partial\widetilde{S}|^2_\omega + \frac{|\nabla\mathrm{Rm}(\omega)|^2_\omega + |\overline{\nabla}\mathrm{Rm}(\omega)|^2_\omega}{A-\widetilde{S}}.$$

Furthermore, by (5.42) we have

$$\begin{aligned}|\partial\widetilde{S}|^2_\omega &= t^{-4}\sigma^{-8}\tau^{-6}\rho^{-4}|\partial S|^2_\omega \\ &\leqslant 4t^{-4}\sigma^{-8}\tau^{-6}\rho^{-4}S\left(|\nabla\Psi|^2_\omega + |\overline{\nabla}\Psi|^2_\omega\right) \\ &\leqslant 4Ct^{-2}\sigma^{-4}\tau^{-3}\rho^{-2}\left(|\nabla\Psi|^2_\omega + |\overline{\nabla}\Psi|^2_\omega\right).\end{aligned}$$

Putting this all together, we derive

$$\begin{aligned}\Delta_\omega Q \geqslant & -Ct^2\sigma^5\tau^4\frac{|\mathrm{Rm}(\omega)|_\omega}{A-\widetilde{S}} - C\sigma^2\tau\frac{|\mathrm{Rm}(\omega)|^2_\omega}{A-\widetilde{S}} - C\frac{|\mathrm{Rm}(\omega)|^3_\omega}{A-\widetilde{S}} \\ & + \frac{1}{2}t^{-2}\sigma^{-4}\tau^{-3}\rho^{-2}(|\nabla\Psi|^2_\omega + |\overline{\nabla}\Psi|^2_\omega)\frac{|\mathrm{Rm}(\omega)|^2_\omega}{(A-\widetilde{S})^2} \\ & + \frac{1}{2}t^{-2}\sigma^{-4}\tau^{-3}\rho^{-2}(|\nabla\Psi|^2_\omega + |\overline{\nabla}\Psi|^2_\omega)\frac{|\mathrm{Rm}(\omega)|^2_\omega}{(A-\widetilde{S})^2}\left(1 - \frac{16}{A-\widetilde{S}}\right).\end{aligned} \tag{5.63}$$

We can choose A large enough to force a sign on the last term. Then converting the second-last term into quadratic curvature terms via

$$|\overline{\nabla}\Psi|^2_\omega \geqslant \frac{1}{2}|\mathrm{Rm}(\omega)|^2_\omega - C\sigma^4\tau^2, \tag{5.64}$$

we reach

$$\begin{aligned}\Delta_\omega Q \geqslant & -Ct^2\sigma^5\tau^4\frac{|\mathrm{Rm}(\omega)|_\omega}{A-\widetilde{S}} \\ & - C(\sigma^2\tau + t^{-2}\tau^{-1}\rho^{-2})\frac{|\mathrm{Rm}(\omega)|^2_\omega}{A-\widetilde{S}} - C\frac{|\mathrm{Rm}(\omega)|^3_\omega}{A-\widetilde{S}} \\ & + \frac{1}{4}t^{-2}\sigma^{-4}\tau^{-3}\rho^{-2}\frac{|\mathrm{Rm}(\omega)|^4_\omega}{(A-\widetilde{S})^2},\end{aligned} \tag{5.65}$$

for some uniform $C > 0$ and for all $t \geqslant 1$. Observe that at the maximum point of Q, we have

$$C(\sigma^2\tau + t^{-2}\tau^{-1}\rho^{-2})\frac{|\mathrm{Rm}(\omega)|^2_\omega}{(A-\widetilde{S})^2} \leqslant \frac{1}{12}t^{-2}\sigma^{-4}\tau^{-3}\rho^{-2}\frac{|\mathrm{Rm}(\omega)|^4_\omega}{(A-\widetilde{S})^2}$$

since otherwise, there would exists $C > 0$ such that $|\mathrm{Rm}(\omega)|_\omega \leqslant Ct\sigma^3\tau^2\rho$ for all $t \geqslant 1$, leading to a better bound than (5.10). Similarly, we can assume that at the maximum point of Q,

$$Ct^2\sigma^5\tau^4\frac{|\mathrm{Rm}(\omega)|_\omega}{A-\widetilde{S}} \leqslant \frac{1}{12}t^{-2}\sigma^{-4}\tau^{-3}\rho^{-2}\frac{|\mathrm{Rm}(\omega)|^4_\omega}{(A-\widetilde{S})^2}.$$

Otherwise we would obtain $|\mathrm{Rm}(\omega)|_\omega \leqslant Ct^{\frac{4}{3}}\sigma^3\tau^{\frac{7}{3}}\rho^{\frac{2}{3}}$ for all $t \geqslant 1$. Therefore,

$$\Delta_\omega Q \geqslant -C\frac{|\mathrm{Rm}(\omega)|^3_\omega}{A-\widetilde{S}} + \frac{1}{12}t^{-2}\sigma^{-4}\tau^{-3}\rho^{-2}\frac{|\mathrm{Rm}(\omega)|^4_\omega}{(A-\widetilde{S})^2}. \tag{5.66}$$

At the maximum of Q, we have

$$|\mathrm{Rm}(\omega)|_\omega \leqslant Ct^2\sigma^4\tau^3\rho^2.$$

Using the uniform bound for $A - \widetilde{S}$, we obtain (5.10). $\square$

5.5 Applications and Further Remarks

The usefulness of Theorem 5.3 depends on our ability to construct well-behaved solutions to the continuity equation. Unfortunately, it is difficult to construct solutions to the continuity method. In certain cases, however, such as when X is Calabi-Yau or a product manifold, such constructions can be made and thus we can use Theorem 5.3 to understand the continuity method starting from a general initial metric. This was previously done in [Zha20] for the Kähler-Ricci flow where several results from [TZ15] were reproved. In this section, we study the same setups from these papers but in the context of the continuity method.

We first recall the following definition.

Definition 5.10. A long time solution $\omega(t)$ of the continuity method (5.1) on X, is of singularity type III if

$$\limsup_{t\to\infty}\left(\sup_X |Rm(\omega(t))|_{\omega(t)}\right) < \infty.$$

Otherwise, we say the solution is of type IIb.

Example 5.11. Suppose that X is a Calabi-Yau Manifold. Thanks to Yau's solution to the Calabi conjecture, we can guarantee the existence of ω_{CY} on X. One can easily check that the solution to the continuity equation starting from $\widetilde{\omega}(0) = \omega_{CY}$ is

$$\widetilde{\omega}(t) = \frac{\omega_{CY}}{1+t}. \tag{5.67}$$

Moreover

$$\frac{1}{t}\operatorname{tr}_{\widetilde{\omega}}\widetilde{\omega}_0 = \frac{1+t}{t}\operatorname{tr}_{\omega_{CY}}\omega_{CY} = \frac{1+t}{t}n \not\to 0 \text{ as } t\to\infty. \tag{5.68}$$

Since the curvature re-scales with time as

$$|\mathrm{Rm}(\widetilde{\omega}(t))|^2_{\widetilde{\omega}(t)} = (1+t)^2|\mathrm{Rm}(\omega_{CY})|^2_{\omega_{CY}}, \tag{5.69}$$

the solution will be Type III if and only if ω_{CY} is flat. Furthermore,

$$|\nabla_{\widetilde{\omega}(t)}\mathrm{Rm}(\widetilde{\omega}(t))|^2_{\widetilde{\omega}(t)} = (1+t)^2|\nabla_{CY}\mathrm{Rm}(\omega_{CY})|^2_{\omega_{CY}} = 0 \leqslant 1. \tag{5.70}$$

Therefore, for any solution to (5.1) starting from a general metric, we can apply Theorem 5.1 and Theorem 5.3 by $\tau \equiv 1$ and $\rho \equiv 1$. Thus, there exists, $C > 0$ such that

$$|\mathrm{Rm}(\omega(t))|_{\omega(t)} \leqslant Ct^2, \tag{5.71}$$

for $t \geqslant 1$.

Example 5.12. Let X be a product manifold given by

$$X = Y \times Z$$

where Y is a Calabi-Yau manifold and Z is a compact manifold with ample K_Y. Define the product metric

$$\omega_X = \omega_Y + \omega_Z, \tag{5.72}$$

where ω_Y is Calabi-Yau and ω_Z is a Kähler-Einstein metric. We denote the corresponding levi-civita connections by ∇_{ω_Y} and ∇_{ω_Z} respectively. Starting from ω_X, the family of metrics

$$\widetilde{\omega}(t) := \frac{1}{1+t}\omega_Y + \omega_Z \tag{5.73}$$

satisfies the continuity equation. Furthermore, the product structure allows us to compute the Riemannian Curvature tensor using

$$|\mathrm{Rm}(\widetilde{\omega}(t))|^2_{\widetilde{\omega}(t)} = (1+t)^2|\mathrm{Rm}(\omega_Y)|^2_{\omega_Y} + |\mathrm{Rm}(\omega_Z)|^2_{\omega_Z}. \tag{5.74}$$

Therefore, the metric $\omega(t)$ is Type III if and only if ω_Y is flat. Moreover,

$$|\nabla_{\widetilde{\omega}}\mathrm{Rm}(\widetilde{\omega}(t))|^2_{\widetilde{\omega}(t)} = |\nabla_{\omega_Z}\mathrm{Rm}(\omega_Z)|^2_{\omega_Z} \leqslant C. \tag{5.75}$$

Checking our last assumption, we have

$$\frac{1}{t}\operatorname{tr}_{\widetilde{\omega}}\omega_0 = \frac{1+t}{t}\dim(Y) + o(1), \tag{5.76}$$

thus $t^{-1}\operatorname{tr}_{\widetilde{\omega}}\omega_0 \not\to 0$ as $t \to \infty$. By Theorem 5.3, we conclude that any solution of the continuity method on X with Y being a flat tori, there exists $C > 0$ such that

$$|\mathrm{Rm}(\omega(t))|_{\omega(t)} \leqslant Ct^2 \tag{5.77}$$

for all $t \geqslant 1$.

Chapter 6

Calabi Symmetry and the Continuity Method

Abstract: We study the convergence and curvature blow-up of the continuity method on a generalised Hirzebruch surface. We show that the Gromov-Hausdorff convergence is similar to that of the Kähler-Ricci flow and obtain curvature estimates. We also show that a general solution to the continuity method either exists for all times or the scalar curvature blows up. This behaviour is known to be exhibited by the Kähler-Ricci flow.

6.1 Introduction

Let (X, ω_0) be a compact n-dimensional Kähler manifold and $\omega(t)$ be a deformation of ω_0 given by the continuity method

$$\begin{cases} \omega(t) = \omega_0 - t\mathrm{Ric}(\omega(t)), \\ \omega(0) = \omega_0. \end{cases} \tag{6.1}$$

Metric deformation equations have become a central tool in geometric analysis in which a family of metrics deforming to a possibly degenerate target metric is studied. This idea appears in Yau's solution to the Calabi conjecture [Yau78b] in which a continuity method was employed to demonstrate that one could deform an arbitrary metric to one that is Ricci flat. Another direction utilising this idea is in the study degenerating Ricci flat metric (see [Tos10; GTZ13; GTZ20; Li21]) and in the study of non-positive holomorphic sectional curvatures (see [Zha21; ZZ20b]). The study of non-positive sectional curvature has been widely studied due to the profound relationship between holomorphic sectional curvature and the canonical line bundle of the manifold (see [WY16; TY17; DT19]).

We can also consider deformations according to evolution with time derivatives, a famous example being the Kähler-Ricci flow. The success of the Kähler-Ricci flow in solving geometric problems has generated much interest in the Kähler-Ricci flow. As a mathematical tool the Kähler-Ricci flow is hoped to give a geometric classification of algebraic varieties. This classification program is known as Mori's minimal model program and aims to find a suitable

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simplest representative metric within every birational class. The analytic minimal model program initiated in [ST12] proposes to achieve this by evolving a metric to this simplest metric through the Kähler-Ricci flow.

The continuity method, given by (6.1), was introduced in [Rub08] and is proposed in [LT16] as an alternative to the Kähler-Ricci flow in carrying out the minimal model program. This equation can be obtained by formally discretising the Kähler Ricci flow. The main advantage of the continuity method is that is maintains a lower Ricci bound along the deformation and therefore comparison geometry techniques, such as Cheeger–Colding–Tian's compactness theory, can be readily applied. For recent developments pertaining to the continuity method see [LTZ17; ZZ19; Zha19; ZZ20a; FGS20; Won23a]. In the same way as one can generalise the Kähler Ricci Flow to the Chern-Ricci Flow for non-Kähler manifolds (see [Gil10; TWY15; LT20; TW22]), the continuity method can analogously be generalised to the Chern continuity method (see [SW20; LZ21]).

To understand geometric deformations such as the flow and continuity method, it is essential to consider several prototype examples. To this end, we consider the following n-dimensional Kähler manifold which generalises the Hirzebruch surface,

$$X = X_{n,k} = \mathbb{P}(\mathcal{O}_{\mathbb{CP}^{n-1}} \oplus \mathcal{O}_{\mathbb{CP}^{n-1}}(-k)). \tag{6.2}$$

Here we have $\mathcal{O}_{\mathbb{CP}^{n-1}}$ to denote the trivial bundle over $\mathbb{CP}^{n-1}$ and $\mathcal{O}_{\mathbb{CP}^{n-1}}(-k)$ to be k times tensored hyperplane bundle of $\mathbb{CP}^{n-1}$. The manifold $X_{n,k}$ is a $\mathbb{CP}^1$ hyperplane bundle over $\mathbb{CP}^{n-1}$ and forms an interesting family of manifolds. For instance, the space of cohomology classes for these manifolds, $H^{1,1}(X, \mathbb{R})$, are spanned by just two classes of exceptional divisors, $[D_0]$ and $[D_\infty]$, and therefore is a two-dimensional space [SW11]. As a result, the Kähler cone can be written explicitly as

$$\mathcal{K} = \left\{ -\frac{a}{k}\,[D_0] + \frac{b}{k}\,[D_\infty] \mid 0 < a < b \right\}. \tag{6.3}$$

It is therefore possible to obtain a concise picture of the evolution of cohomology classes for the continuity method and for other geometric deformations such as the Kähler Ricci Flow.

In [LT16], it was shown that the solution to the continuity method exists until the Kähler class leaves the Kähler cone, that is up to time

$$T =: \sup \left\{ t \mid [\omega_0] - tc_1(X) > 0 \right\}. \tag{6.4}$$

If the initial metric satisfies the Calabi symmetry, then the solution $\omega(t)$ also satisfies the Calabi symmetry. We aim to study the convergence of such solutions on X and the curvature blow-up rate as the solution approaches this singular time. To describe this, we borrow some terminology from the study of finite time Kähler-Ricci flows.

Definition 6.1. Let $\omega(t)$ be a solution to the continuity method for $t \in [0, T)$. We say that the curvature tensor norm develops a *Type I singularity* if

$$\sup_X |\mathrm{Rm}(\omega(t))|_{\omega(t)} \leqslant \frac{C}{T-t}. \tag{6.5}$$

Otherwise, we say that the curvature tensor norm blows up at a *Type IIa rate*. If at a point $p \in X$, there exists a sequence $t_i \to T$ as $i \to \infty$, such that

$$|Rm(\omega(t_i))|_{\omega(t_i)}(p) \geqslant \frac{C}{T-t_i}, \tag{6.6}$$

then we say that the curvature tensor norm at p blows up at an *essential Type I rate*.

We can also apply the above definition to the norm of the Ricci curvature and scalar curvature.

There have been several studies of the Kähler Ricci flow on $X_{n,k}$ and similar manifolds with Calabi Symmetry. The study of Gromov Hausdorff convergence of the Kähler Ricci flow on $X_{n,k}$ was done in [SW11]. Singularity and curvature analysis for manifolds exhibiting Calabi symmetry was first derived in [Fon14], providing the first non-trivial non-Fano Type I solution to the Kähler-Ricci Flow. A blow-up analysis for the manifold studied in our paper was conducted in [Son15] where it was shown that only Type I singularities can develop. For a related manifold $\mathbb{P}\left(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(m+1)}\right)$, Gromov-Hausdorff convergence was studied in [SY12]. Many arguments and calculations in this manuscript are inspired by these papers.

It will also be convenient to consider the normalised equation for (6.1)

$$\begin{cases} \widetilde{\omega}(t) = \dfrac{1}{T-t}\widetilde{\omega}_0 - \dfrac{t}{T-t}\mathrm{Ric}(\widetilde{\omega}(t)) \\ \omega(0) = \omega_0. \end{cases} \tag{6.7}$$

We now state the first main result of the theorem.

Theorem 6.2. *On the manifold $X_{n,k}$ defined by* (6.2)*, a solution to the continuity method* (6.1) *with an initial metric ω_0 satisfying the Calabi symmetry evolves as follows:*

1. *If $k \geqslant n$ or $1 \leqslant k \leqslant n-1$ with $a_0(n+k) > b_0(n-k)$, then the solution exists until $T = (b_0 - a_0)/2k$ and $(X_{n,k}, g(t))$ converges to $(\mathbb{CP}^{n-1}, a_T g_{\mathrm{FS}})$ as $t \to T$ in the Gromov-Hausdorff sense. The norm of the curvature tensor develops a Type I singularity.*

2. *If $1 \leqslant k \leqslant n-1$ with $a_0(n+k) = b_0(n-k)$, then the re-scaled equation* (6.7) *converges in the Gromov-Hausdorff sense to the unique Kähler Einstein metric ω_{KE} in the class $C_1(X)$, given such a metric exists. The norm of the curvature tensor blows up at a Type I rate.*

3. *If $1 \leqslant k \leqslant n-1$ with $a_0(n+k) < b_0(n-k)$, then the solution converges in the Gromov-Hausdorff sense to a unique weakly Kähler metric ω_T which is smooth outside of a subvariety. Furthermore, the curvature blows up at the rate*

$$|\mathrm{Rm}(\omega)|_\omega \leqslant \frac{C}{(T-t)^2}. \tag{6.8}$$

Before proceeding, we would like to make some remarks regarding the Theorem above.

Remark 6.3. The Gromov-Hausdorff result in Item 1 is a strengthened version of a general result on Fano bundles where the solution to the continuity method converges in the Gromov-Hausdorff sense to a unique Kähler metric on the base space.

Remark 6.4. With regards to Item 2, the question of the existence of Kähler-Einstein metric on Fano manifolds has been extensively studied (for a survey see [Tia12]).

Remark 6.5. Item 3 falls under the finite time volume non-collapsed case in which the Gromov-Hausdorff convergence was obtained in [LTZ17].

Theorem 6.6. *Solutions to the continuity method on $X_{n,k}$ satisfying the Calabi Symmetry has scalar curvature bound*

$$R(t) \leqslant \frac{C}{T-t}. \tag{6.9}$$

Moreover, the scalar curvature blow-up in Case I and II is essential in the sense that there exists (p_i, t_i) such that as $i \to \infty$, $t_i \to T$ and

$$R(p_i, t_i) \geqslant \frac{C}{T-t_i}. \tag{6.10}$$

The behaviour of scalar curvature for the continuity method is not well understood, even for infinite time solutions to (6.1). On the other hand, we have several results regarding scalar curvature under the Kähler-Ricci flow;

(1) The Kähler-Ricci flow either the flow exists for all time, or the scalar curvature blows up [Zha10].

(2) Solutions to the normalised Kähler-Ricci flow with semi-ample canonical line bundle has bounded scalar curvature [Zha09; ST16].

(3) In the same setup as the previous point, the scalar curvature away from singular divisors converges to $-kod(X)$ [Jia20].

Theorem 6.6 alludes that scalar curvature must blow up for finite time singularities, resembling the result from finite time Kähler-Ricci flows in [Zha10]. We show that solutions to the continuity method (6.1) also satisfy point (1). Point (2) is currently not known globally, however a bound can be obtained on compact sets away from singular fibres [ZZ19]. For point (3), Y. Zhang and Jian proved that the twisted scalar curvature converges to $-kod(X)$.

Theorem 6.7. *Let $\omega(t)$ be a solution to the continuity method,* (6.1). *Then either the solution exists for all time, or the scalar curvature blows up;*

$$\sup_{X\times[0,T)} R\left(\omega(t)\right) = +\infty. \tag{6.11}$$

Remark 6.8. The scalar curvature blowup must be from above since the Ricci curvature of solutions for (6.1) is always bounded from below.

We end the introduction with an outline of the paper. In Section 6.2, we describe the Calabi Symmetry condition and the geometry of $X_{n,k}$ in more detail. This will allow us to introduce the three cases for types of behaviour for solutions of (6.1) at the end of this section. The next section contains three subsections where we analyse each case separately. They will follow the same format; first convergence as the solutions approach singular time established, then curvature estimates are obtained. In Section 6.4, we prove Theorem 6.7.

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6.2 Preliminaries

In this section, we recall the Calabi Symmetry and briefly describe the geometry of $X_{n,k}$. For a detailed construction, see [Cal82] or preliminary/background sections in [SW11; SY12; Son15]. As stated in the introduction, the manifold

$$X = \mathbb{P}\left(\mathcal{O}_{\mathbb{CP}^{n-1}} \oplus \mathcal{O}_{\mathbb{CP}^{n-1}}(-k)\right), \tag{6.12}$$

is a $\mathbb{CP}^1$ bundle over $\mathbb{CP}^{n-1}$ obtained by blowing up $\mathbb{CP}^{n-1}$. Let D_0 be the exceptional divisor of X defined by the image of the section $(1,0)$ of $\mathcal{O}_{\mathbb{CP}^{n-1}} \oplus \mathcal{O}_{\mathbb{CP}^{n-1}}(-k)$ and D_∞ be the divisor of X defined by the image of the section $(0,1)$ of $\mathcal{O}_{\mathbb{CP}^{n-1}} \oplus \mathcal{O}_{\mathbb{CP}^{n-1}}(-k)$. Both the 0-section D_0 and the ∞-section are complex hypersurfaces in X isomorphic to $\mathbb{CP}^{n-1}$. The classes $[D_\infty]$ and $[D_0]$ span $H^{1,1}(X,\mathbb{R})$, and the Kähler cone is given by

$$\mathcal{K} = \left\{ -\frac{a}{k}\left[D_0\right] + \frac{b}{k}\left[D_\infty\right] \mid 0 < a < b \right\}. \tag{6.13}$$

Using the bundle map $\pi : X_{n,k} \to \mathbb{CP}^{n-1}$, we can pull back the Fubini-study metric on $\mathbb{CP}^{n-1}$. Taking its cohomology class gives the the class $[D_H]$, which is related to $[D_0]$ and $[D_\infty]$ by

$$k\left[D_H\right] = \left[D_\infty\right] - \left[D_0\right]. \tag{6.14}$$

We can then rewrite the Kähler cone in a more instructive basis

$$\mathcal{K} = \left\{ \frac{b-a}{k}\left[D_\infty\right] + a\left[D_H\right] \mid 0 < a < b \right\}. \tag{6.15}$$

To introduce the Calabi Symmetry into our analysis, we consider the standard holomorphic coordinates $(x_1, ..., x_n)$ on $\mathbb{C}^n \backslash \{0\}$, the manifold $\mathbb{CP}^{n-1} := (\mathbb{C}^n \backslash \{0\})/\mathbb{C}^*$ is charted by n coordinate charts $U_1, ..., U_n$ where U_i is characterised by $x_i \neq 0$. For fixed i, the holomorphic coordinates are given by $z^j_{(i)} = x_j/x_i$. In this coordinate system, the fibre coordinate $y_{(i)}$ on $\pi^{-1}(U_i)$ transforms by

$$y_{(\ell)} = \left(\frac{x_\ell}{x_i}\right)^k y_{(i)}, \quad \text{ on } \pi^{-1}\left(U_i \cap U_\ell\right), \tag{6.16}$$

where $\pi : X_{n,k} \to \mathbb{CP}^{n-1}$ is the bundle map. Consider Kähler metrics such that $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} u(\rho)$ for a potential $u = u(\rho)$ where

$$\rho = \log \left(\sum_{i=1}^{n} |x_i|^2 \right). \tag{6.17}$$

The metrics defined in this way are invariant under the action of

$$G_k \cong U(n)/\mathbb{Z}_k.$$

The following Lemma gives the condition for which g_{ij} is Kähler.

Lemma 6.9 (Calabi [Cal82])**.** *A smooth convex function $u = u(\rho)$ defined on $(-\infty, \infty)$ generates a Kähler metric $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} u(\rho)$ if*

1. $u'(\rho) > 0$
2. $u''(\rho) > 0$
3. *There exist smooth functions $u_{[D_0]}, u_{[D_\infty]} : [0, \infty) \to \mathbb{R}$ with $u'_{[D_0]}(0) > 0$, $u'_{[D_\infty]}(0) > 0$ such that*
$$u_{[D_0]}\left(e^{k\rho}\right) = u(\rho) - a\rho, \tag{6.18}$$
and
$$u_{[D_\infty]}\left(e^{-k\rho}\right) = u(\rho) - b\rho, \tag{6.19}$$
for all $\rho \in \mathbb{R}$.

The divisor D_0 corresponds to $\rho = -\infty$ where as the D_∞ corresponds to $\rho = \infty$. Furthermore, the coefficients a and b are the values of $u'(\rho)$ at the singular divisors. Indeed, taking the ρ derivative of (6.18) and (6.19) gives

$$a = \lim_{\rho \to -\infty} u'(\rho) < u'(\rho) < \lim_{\rho \to \infty} u'(\rho) = b. \tag{6.20}$$

In the standard holomorhpic coordinates $(x_i)_{i=1}^n$ on $\mathbb{C}^n \backslash \{0\}$, metrics under the Calabi condition can be written locally as

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} u = e^{-\rho} u'(\rho) \delta_{ij} + e^{-2\rho} \bar{x}_i x_j \left(u''(\rho) - u'(\rho)\right). \tag{6.21}$$

The determinant of the matrix $g_{i\bar{j}}$ is given by

$$\det g = e^{-n\rho} \left(u'(\rho)\right)^{n-1} u''(\rho). \tag{6.22}$$

Thus, the Ricci curvature $R_{i\bar{j}} = -\partial_i \overline{\partial}_j \log \det g$ is locally given by

$$R_{i\bar{j}} = e^{-\rho} v'(\rho) \delta_{ij} + e^{-2\rho} \bar{x}_i x_j \left(v''(\rho) - v'(\rho)\right) \tag{6.23}$$

where

$$v = -\log \det g = n\rho - (n-1) \log u'(\rho) - \log u''(\rho). \tag{6.24}$$

Using the expressions above, we rewrite the continuity method (6.1) in terms of u. Denoting the t variable as a subscript, $u_t(\rho)$, we obtain

$$u_t(\rho) = u_0(\rho) + t(n-1)\log(u_t'(\rho)) + t\log(u_t''(\rho)) - tn\rho + c_t, \tag{6.25}$$

where c_t is chosen to be

$$c_t = -t(n-1)\log u_t'(0) - t\log u_t''(0)$$

to ensure $u_t(0) = 0$ for all $t \in [0,T)$. Taking derivatives with respect to ρ yields

$$u_t'(\rho) = {u_0}'(\rho) + t(n-1)\frac{u_t''(\rho)}{u_t'(\rho)} + t\frac{u_t'''(\rho)}{u_t''(\rho)} - tn, \tag{6.26}$$

and

$$u_t''(\rho) = u_0''(\rho) + t(n-1)\frac{u_t'''(\rho)}{u_t'(\rho)} - t(n-1)\frac{u_t''(\rho)^2}{u_t'(\rho)^2} - t\frac{u_t'''(\rho)^2}{u_t''(\rho)^2} + t\frac{u_t^{(4)}(\rho)}{u_t''(\rho)}. \tag{6.27}$$

For clarity of notation, we may omit the dependence of $u_t(\rho)$ on ρ and write $u_t = u_t(\rho)$. We also denote the derivatives in terms of ρ with u_t'.

The rotational invariance simplifies many calculations since we can apply a transformation to send any point on the manifold to coordinates $x = (x_1, 0..., 0)$. Under this coordinate, the evolving metric, initial metric and Ricci curvatures take the form

$$\left\{g_{i\bar{j}}\right\} = e^{-\rho}\,\mathrm{diag}\left\{u_t'', u_t', \ldots, u_t'\right\}, \tag{6.28}$$

$$\left\{g^0_{i\bar{j}}\right\} = e^{-\rho}\,\mathrm{diag}\left\{u_0'', u_0', \ldots, u_0'\right\}, \tag{6.29}$$

and

$$R_{i\bar{j}} = e^{-\rho}\,\mathrm{diag}\left\{v_t'', v_t', \ldots, v_t'\right\}. \tag{6.30}$$

respectively. The anticanonical line bundle is given by

$$K_M^{-1} = 2\left[D_\infty\right] - (k-n)\left[D_H\right] = \frac{(k+n)}{k}\left[D_\infty\right] + \frac{(k-n)}{k}\left[D_0\right]. \tag{6.31}$$

The singular time can be deduced from the cohomology as seen by (6.4). The maximum time of existence is therefore given by

$$T = \sup\left\{t \geqslant 0 \mid \alpha_0 + t\left[K_X\right] > 0\right\}. \tag{6.32}$$

For our setup, the metric's class deforms according to

$$\alpha_t = \frac{b_t}{k}\left[D_\infty\right] - \frac{a_t}{k}\left[D_0\right] = \frac{b_t - a_t}{k}\left[D_\infty\right] + a_t\left[D_H\right], \tag{6.33}$$

where

$$b_t = b_0 - (k+n)t \quad \text{and} \quad a_t = a_0 + (k-n)t. \tag{6.34}$$

Finally, we pick a reference for the Kähler class α_t. Let

$$\hat{u}_t(\rho) = a_t\rho + \frac{(b_t - a_t)}{k}\log\left(e^{k\rho} + 1\right). \tag{6.35}$$

Since $\partial\overline{\partial}\rho = \pi^*\omega_{FS} := \chi \in [D_H]$ and $2\partial\overline{\partial}\log\left(e^{k\rho}+1\right) \in 2[D_\infty]$, we see that the associated form $\hat{\omega}$ lies in the desired class. The evolution of the cohomology class exhibits three distinct behaviours.

1. *Case I*: The coefficient of $[D_\infty]$ degenerates whilst $[D_H]$ remains positive. This occurs when $k \geqslant n$ or when $1 \leqslant k \leqslant n-1$ and $a_0(n+k) > b_0(n-k)$. Indeed, if the class α_t fails to be Kähler due to the coefficient of $[D_\infty]$ vanishing, that is
$$\frac{b_t - a_t}{2k} = \frac{b_0 - a_0}{2k} - t = 0,$$
then
$$T = \frac{b_0 - a_0}{2k}. \tag{6.36}$$
If $k \geqslant n$, then $a_t > a_0 > 0$, but if $1 \leqslant k \leqslant n-1$, then this uniform bound away from 0 only holds when
$$a_T = \frac{a_0(n+k) - b_0(n-k)}{2k} > 0, \tag{6.37}$$
that is when with the additional condition $a_0(n+k) > b_0(n-k)$ holds. The Cohmology class therefore predicts that the continuity method contracts fibres and the resulting limiting space should be $\mathbb{CP}^{n-1}$.

2. *Case II*: Both coefficients in $[D_\infty]$ and $[D_H]$ approach 0 as the same singular time T which occurs when $1 \leqslant k \leqslant n-1$ with $a_0(n+k) = b_0(n-k)$. As a consequence, (6.37) yields $a_T = 0$, and the Kähler class is proportional to the first Chern class.
$$\alpha_t = \left(\frac{a_0}{n-k} - t\right) c_1(M). \tag{6.38}$$
The singular time would then be
$$T = \frac{a_0}{n-k}$$
where we expect a volume collapsed singularity to form.

3. *Case III*: The coefficient of $[D_\infty]$ remains uniformly bounded from 0 and instead the coefficient of $[D_H]$ degenerates. This occurs when $1 \leqslant k \leqslant n-1$ with $a_0(n+k) < b_0(n-k)$. The coefficient, a_t, of $[D_H]$ is zero when
$$T = \frac{a_0}{n-k}. \tag{6.39}$$
The condition for k and n in this case is obtained from imposing $(b_t - a_t)/2k > 0$.

The three cases discussed above can be visualised as in Figure 6.2.

6.3 Continuity Method on $X_{n,k}$

6.3.1 Analysis for Case I

Let $\chi = \pi^*\omega_{FS}$ as previously defined and θ be a semi-positive representative with Calabi symmetry of the class $[D_\infty]$. We can rewrite the reference metric as
$$\hat{\omega}_t = a_t\chi + \frac{(b_t - a_t)}{2k}\theta = a_t\chi + (T-t)\theta \in \alpha_t. \tag{6.40}$$

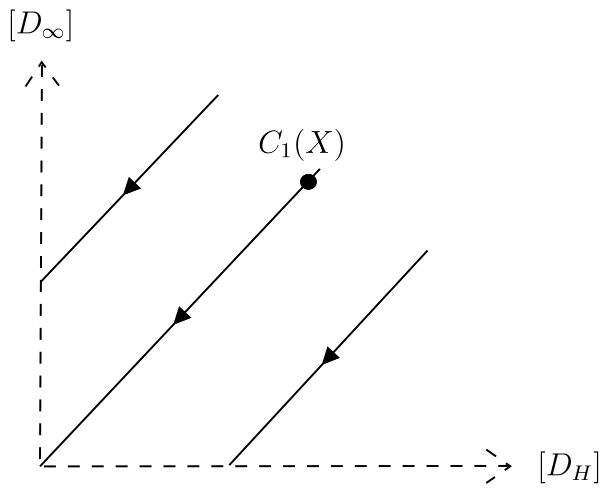

Figure 6.1: Three distinct evolution of the cohomology class of $\omega(t)$ under (6.1).

Moreover, we observe that $\theta - (k-n)\chi \in c_1(X_{n,k})$, thus we choose a form Ω such that

$$\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\Omega = -\theta + (k-n)\chi. \tag{6.41}$$

Reducing to a Monge-Ampere equation in the usual way, we obtain

$$\begin{cases} \varphi(t) = t\log\left(\dfrac{\omega^n}{(T-t)\Omega}\right), \\ \varphi(0) = 0. \end{cases} \tag{6.42}$$

In Case I, the parameter a_t remains uniformly bounded away from zero. Using this property, we obtain the following lemmas.

Lemma 6.10. *Let $\varphi(t)$ be a solution to (6.42). Then there exists $C > 0$ such that for all $t \in [0, T)$,*

$$|\varphi(t)|_{L^\infty(X)} \leqslant C, \tag{6.43}$$

and

$$C^{-1}(T-t)\Omega \leqslant \omega^n \leqslant C(T-t)\Omega. \tag{6.44}$$

Proof. The form χ is a pullback of a $(n-1, n-1)$ form, hence $\chi^n = 0$. Expanding out the volume form using (6.40) gives

$$\hat{\omega}_t^n = \sum_{k=0}^{n-1} a_t^k (T-t)^{n-k}\chi^k \wedge \theta^{n-k}.$$

Since a_t is uniformly bounded away from 0, then for some $C > 0$ independent of t such that

$$C^{-1}(T-t)\Omega \leqslant \hat{\omega}_t^n \leqslant C(T-t)\Omega. \tag{6.45}$$

Applying the maximum principle to (6.42) yields a uniform upper bound for $\varphi(t)$. The lower bound follows from the pluripotential theory of degenerate complex Monge-Ampère equation ([DP10; EGZ08]). It can be observed from (6.42) that (6.43) and (6.44) are equivalent. □

In addition, the well known Schwartz lemma holds.

Lemma 6.11. *Let $\omega(t)$ be a solution to* (6.1)*. There exists $C > 0$ such that*

$$\operatorname{tr}_\omega \chi \leqslant C, \tag{6.46}$$

for all $t \in [0, T)$.

Proof. These estimates are common and utilise a maximum principle argument. By a standard identity and rewriting any Ricci terms for $\omega = \omega(t)$ using (6.1), we obtain

$$\Delta_\omega \log \operatorname{tr}_\omega \chi \geqslant -C_\chi \operatorname{tr}_\omega \chi + \frac{1}{t}\frac{\operatorname{tr}_{\omega_0} \chi}{\operatorname{tr}_\omega \chi} - \frac{1}{t}. \tag{6.47}$$

Furthermore

$$\Delta_\omega \varphi = n - a_t \operatorname{tr}_\omega \chi - (T-t) \operatorname{tr}_\omega \theta. \tag{6.48}$$

Combining the two lines above and using that θ is semi-positive, we can choose $A > 0$ large enough so that there exists a $C = C(A)$ such that

$$\Delta_\omega \left(\log \operatorname{tr}_\omega \chi - A\varphi\right) \geqslant \operatorname{tr}_\omega \chi - C. \tag{6.49}$$

Then we apply the maximum principle and since φ is uniformly bounded by Lemma 6.10, we conclude (6.46). ☐

Lemma 6.12. *The function $u_t = u_t(\rho)$ satisfying* (6.25) *under the range of n, k in Case I satisfies:*

1. *The derivative $u_t'(\rho)$ remains uniformly bounded away form 0 and is controlled from above. More precisely*
$$a_0 < u_t'(\rho) < a_t + 2k(T-t) \tag{6.50}$$
for all $t \in [0, T)$.

2. *Limiting behaviour of $u_t(\rho)$ approaching the singular time,*
$$\lim_{t \to T} \left(u_t(\rho) - a_T \rho\right) = 0. \tag{6.51}$$

3. *There exist $C = C(\omega_0)$ such that*
$$C^{-1}\frac{e^{k\rho}}{(1+e^{k\rho})^2}(T-t) < u_t''(\rho) \leqslant C\frac{e^{k\rho}}{(1+e^{k\rho})^2}(T-t) \tag{6.52}$$
for all $t \in [0, T)$.

4. *There exist $C = C(\omega_0)$ such that*
$$\frac{|u_t'''(\rho)|}{u_t''(\rho)} \leqslant C \tag{6.53}$$
for all $t \in [0, T)$.

Proof. The proofs for (1), (2) and (3) can be readily adapted from [SW11], we include them here for completeness. From (6.20) and the convexity of u_t', we have

$$a_t \leqslant u_t' \leqslant b_t.$$

Since $k \geqslant n$, $a_t > a_0 > 0$, that is u_t' remains uniformly bounded away from 0. Using (6.36), we rewrite b_t in terms of a_t;

$$b_t = a_t + 2k(T-t),$$

to obtain the estimate in (1). To obtain (2), we recall that $u_t(0) = 0$ for all $t \in [0, T)$, and so

$$u_t(\rho) - a_t\rho = \int_0^\rho (u_t'(s) - a_t)\, ds.$$

We then apply (i) which yields

$$|u_t(\rho) - a_t\rho| \leqslant 2k(T-t)|\rho| \longrightarrow 0$$

as $t \to T$.

For (3), we note that Lemma 6.10 implies

$$\det g(t) \leqslant C(T-t) \det \hat{g}_0, \tag{6.54}$$

where $\hat{g}_0$ is the reference metric with associated potential $\hat{u}_0$. Thus, we estimate

$$\begin{aligned}\det \hat{g}_0 &= e^{-n\rho} \left(\hat{u}_0'\right)^{n-1} \hat{u}_0'' \\ &= k\,(b_0 - a_0)\, e^{-n\rho} \left(a_0 + (b_0 - a_0)\, \frac{e^{k\rho}}{1 + e^{k\rho}}\right)^{n-1} \frac{e^{k\rho}}{(1 + e^{k\rho})^2} \\ &\leqslant Ce^{-n\rho},\end{aligned}$$

where the second equality is obtained from (6.35). Moreover, we have

$$\det g = e^{-n\rho} \left(u_t'\right)^{n-1} u_t''.$$

Combining the above with the previous two inequalities yields

$$\left(u_t'\right)^{n-1} u_t'' \leqslant C \frac{e^{k\rho}}{(1 + e^{k\rho})^2} (T-t). \tag{6.55}$$

Since $u_t' \geqslant a_t$ which is bounded uniformly away from 0, we obtain (6.52). Similarly, using the lower bound in (6.52),

$$\det(g) \geqslant C^{-1}(T-t) \det \hat{g}_0 \geqslant C^{-1} \frac{e^{k\rho}}{(1 + e^{k\rho})^2} (T-t).$$

To obtain the two estimates in (4), we first rewrite (6.26) as

$$\frac{u_t'''}{u_t''} = \frac{u_t' - u_0'}{t} - (n-1)\frac{u_t''}{u_t'} + n.$$

Since $u_t'' > 0$, it follows from (6.50) and (6.52) that there exists $C > 0$ uniform of t such that

$$\frac{|u_t'''|}{u_t''} \leqslant C. \tag{6.56}$$

□

As a consequence, we can recover the same results in [SW11] for the continuity method.

Theorem 6.13. *We have*

1. $\omega(t) \geqslant a_t\chi$.
2. $\sup_M \operatorname{tr}_{\hat{g}_0} g \leqslant C$.
3. *For any compact set* $K \subset M\backslash(D_\infty \cup D_0)$,

$$\sup_K |\nabla_{\hat{g}_0} g|_{\hat{g}_0} \leqslant C_K.$$

Proof. In local coordinates

$$g_{i\bar{j}}(t) = e^{-\rho}u'\delta_{ij} + e^{-2\rho}\bar{x}_i x_j (u'' - u'), \quad \text{and} \quad \chi_{i\bar{j}} = e^{-\rho}\delta_{ij} - e^{-2\rho}\bar{x}_i x_j. \tag{6.57}$$

Then it follows easily that

$$g_{i\bar{j}}(t) \geqslant u_t' e^{-\rho}\left(\delta_{ij} - \frac{\bar{x}_i x_j}{\sum_k |x_k|^2}\right) \geqslant a_t e^{-\rho}\left(\delta_{ij} - \frac{\bar{x}_i x_j}{\sum_k |x_k|^2}\right) = a_t\chi_{i\bar{j}},$$

In local coordinates, we use (6.28) and (6.29) to compute

$$\operatorname{tr}_{\hat{g}_0} g = \frac{u_t''}{\hat{u}_0''} + (n-1)\frac{u_t'}{\hat{u}_0'}. \tag{6.58}$$

From Lemma 6.12, we have

$$\frac{u_t'}{\hat{u}_0'} \leqslant \frac{b_0}{a_0} \quad \text{and} \quad \frac{u_t''}{\hat{u}_0''} = \frac{\left(1 + e^{k\rho}\right)^2}{k\left(b_0 - a_0\right)e^{k\rho}}u_t'' \leqslant C,$$

which implies (2). Furthermore,

$$\frac{\partial}{\partial x_k} g_{i\bar{j}} = e^{-2\rho}\left(u_t'' - u_t'\right)\left(\bar{x}_k\delta_{ij} + \bar{x}_i\delta_{jk}\right) + e^{-3\rho}\bar{x}_i x_j \bar{x}_k\left(u_t''' - 3u_t'' + 2u_t'\right). \tag{6.59}$$

Away from the divisors $[D_\infty] \cup [D_0]$, ρ and x_i are bounded which implies (3). □

Following from Theorem 6.13, we can replicate the same argument as in [SW11] verbatim to show Gromov Hausdorff convergence.

Theorem 6.14. $(M, g(t))$ *converges to* $(\mathbb{CP}^{n-1}, a_T g_{FS})$ *in the Gromov-Hausdorff sense as* $t \to T$.

We now show that the norm of the curvature tensor blows up at a Type I rate. To do this, we first prove the following proposition on the scalar curvature.

Proposition 6.15. *For solution to* (6.1) *in Case I, there exists* $C > 0$ *such that*

$$R(t) \leqslant \frac{C}{T-t}, \tag{6.60}$$

for all $t \in [0, T)$. *Furthermore, the blow-up is essential everywhere on* X.

Proof. We first take the trace of (6.1) which yields

$$R(t) = \frac{1}{t}\left(\operatorname{tr}_\omega \omega_0 - n\right).$$

In the coordinates given by (6.28), the trace above becomes

$$R(t) = \frac{u_0''}{tu''} + (n-1)\frac{u_0'}{tu_t'} - \frac{n}{t}.$$

Since a_t is uniformly bounded from below,

$$\frac{{u_0}'}{u_t'} \leqslant C, \tag{6.61}$$

and using (6.50), we have for compact sets $K \subset\subset X \setminus ([D_0] \cup [D_\infty])$ we have

$$\frac{u_0''}{u_t''} \leqslant \frac{C}{(T-t)} \frac{(1+e^{k\rho})^2 u_0''}{e^{k\rho}} \leqslant \frac{C}{T-t}. \tag{6.62}$$

To obtain bounds near the singular divisors, we take derivatives of both (6.18) and (6.19) to obtain

$$u_0'' = k^2 e^{k\rho} u_{[D_0]}'\left(e^{k\rho}, 0\right) + k^2 e^{2k\rho} u_{[D_0]}''\left(e^{k\rho}, 0\right), \tag{6.63}$$

and

$$u_0'' = k^2 e^{-k\rho} u_{[D_\infty]}'\left(e^{-k\rho}, 0\right) + k^2 e^{-2k\rho} u_{[D_\infty]}''\left(e^{-k\rho}, 0\right). \tag{6.64}$$

Thus, we have

$$\lim_{\rho\to-\infty} \frac{(1+e^{k\rho})^2 u_0''}{e^{k\rho}} = k^2 \quad \text{and} \quad \lim_{\rho\to\infty} \frac{(1+e^{k\rho})^2 u_0''}{e^{k\rho}} = k^2, \tag{6.65}$$

Therefore, (6.62) holds on the entire manifold X and Proposition 6.15 is proven.

Finally, we use the upper bound of u_t'' in Lemma 6.12 to obtain

$$R(t) \geqslant \frac{u_0''}{tu_t''} - \frac{n}{t} > \frac{C}{(T-t)} - C, \tag{6.66}$$

for any points away from singular fibres $[D_0] \cup [D_\infty]$. To extend the lower bound across the whole manifold, we compute the limits using (6.63) and (6.64) to obtain (6.66) on the whole of X. Thus, the Type I blow up rate is essential on every point in X. □

From scalar curvature bounds, we obtain bounds on the Riemannian curvature tensor.

Proposition 6.16. *Under the continuity method, we have*

$$|\mathrm{Rm}(\omega(t))|_{\omega(t)} \leqslant \frac{C}{T-t}. \tag{6.67}$$

Remark 6.17. Since we know that the scalar curvature blows up at an essential Type I rate everywhere on X, the curvature norm must also do the same since $R(t) \leqslant C(n)|\mathrm{Rm}(\omega(t))|_{\omega(t)}$ for some C depending on the dimension of X.

Proof. The holomorphic bisectional curvature for metrics with Calabi symmetry condition is calculated in [Cao96] (also see [Son15]),

$$\begin{aligned} R_{i\bar{j}k\bar{l}} =& e^{-2\rho}\left(u_t' - u_t''\right)\left(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj}\right) \\ &+ e^{-2\rho}\left(3u_t'' - 2u_t' - u_t'''\right)\left(\delta_{ij}\delta_{kl1} + \delta_{il}\delta_{kj1} + \delta_{kl}\delta_{ij1} + \delta_{kj}\delta_{il1}\right) \\ &+ e^{-2\rho}\left(6u_t''' - 11u_t'' - u_t^{(4)} + 6u_t' + \frac{\left(u_t'' - u_t'''\right)^2}{u_t''}\right)\delta_{ijkl1} \\ &+ e^{-2\rho}\frac{\left(u_t' - u_t''\right)^2}{u_t'}\left(\delta_{ij\hat{1}}\delta_{kl1} + \delta_{il1}\delta_{kj1} + \delta_{kl1}\delta_{ij1} + \delta_{kj\hat{1}}\delta_{il1}\right) \end{aligned} \tag{6.68}$$

By applying a unitary transformation, we can send any point $p \in \mathbb{C}^n\backslash\{0\}$ to $(x_1, 0, ..., 0)$. Then all the non-vanishing terms of the holomorphic bisectional curvature are given by

$$\begin{aligned} R_{1\bar{1}1\bar{1}} &= e^{-2\rho}\left(-u_t^{(4)} + \frac{\left(u_t'''\right)^2}{u_t''}\right) \\ R_{k\bar{k}k\bar{k}} &= 2e^{-2\rho}\left(u_t' - u_t''\right), k > 1 \\ R_{1\bar{1}k\bar{k}} &= e^{-2\rho}\left(-u_t''' + \frac{\left(u_t''\right)^2}{u_t'}\right), k > 1 \\ R_{k\bar{k}l\bar{l}} &= e^{-2\rho}\left(u_t' - u_t''\right), k > 1, l > 1, k \neq l. \end{aligned} \tag{6.69}$$

In these coordinates, by (6.28), the norm of the curvature tensor is given by

$$\begin{aligned} |\mathrm{Rm}(\omega)|^2_\omega =& \sum_{k,l=1}^n g^{k\bar{k}}g^{k\bar{k}}g^{l\bar{l}}g^{l\bar{l}}R_{k\bar{k}l\bar{l}}\overline{R_{k\bar{k}l\bar{l}}} \\ =& \left(-\frac{u_t^{(4)}}{u_t''^2} + \frac{\left(u_t'''\right)^2}{u_t''^3}\right)^2 + (n-1)(n+3)\left(\frac{1}{u_t'} - \frac{u_t''}{u_t'^2}\right)^2 + (n-1)\left(-\frac{u_t'''}{u_t'u_t''} + \frac{u_t''}{u_t'^2}\right)^2. \end{aligned} \tag{6.70}$$

On the other hand, taking the trace of the Ricci curvature using (6.28) and (6.30) yields

$$R(t) = -\frac{2(n-1)}{u'}\frac{u_t'''}{u_t''} + \frac{n(n-1)}{u_t'} - \frac{(n-1)(n-2)u_t''}{u_t'^2} + \frac{u_t'''^2}{u_t''^3} - \frac{u_t^{(4)}}{u_t''^2} \tag{6.71}$$

Using Lemma 6.15 and the estimates in Lemma 6.12, we obtain

$$-C_1 \leqslant \frac{u_t'''^2}{u_t''^3} - \frac{u_t^{(4)}}{u_t''^2} \leqslant \frac{C_1}{T-t}. \tag{6.72}$$

Once again, we use the estimates in Lemma 6.12, there exists constants $C_i > 0$, $i = 2, 3$ such that

$$-C_2 \leqslant \frac{1}{u_t'} - \frac{u_t''}{u_t'^2} \leqslant C_2,$$

and

$$-C_3 \leqslant -\frac{u_t'''}{u_t'u_t''} + \frac{u_t''}{u_t'^2} \leqslant C_3.$$

Combining the three inequalities with (6.70) give us (6.67). $\square$

6.3.2 Analysis for Case II

Since the continuity method evolves cohomology as

$$\alpha_t = (T-t)\, c_1(X), \tag{6.73}$$

where

$$T = \frac{a_0}{n-k}.$$

We apply a rescaling to obtain a normalised equation that keeps the class of the metric constant. From (6.73) we see that the appropriate rescaling is

$$\widetilde{\omega}(t) = \frac{\omega(t)}{T-t}, \tag{6.74}$$

from which (6.1) becomes

$$\widetilde{\omega}(t) = \frac{\widetilde{\omega}_0}{T-t} - \frac{t}{T-t}\mathrm{Ric}(\widetilde{\omega}(t)). \tag{6.75}$$

This corresponds to a reparameterisation of the classical continuity method proposed by Aubin (see [Aub76; Aub84; ZZ20a]). If the manifold admits a unique Kähler-Einstein metric ω_{KE}, then $\omega(t)$ converges to this Kähler-Einstein metric in the $C^\infty(X, \omega_0)$ topology as $t \to T$.

Returning to (6.1), we take the reference metric based on cohomology to be

$$\hat{\omega}_t = (T-t)((n-k)\chi + \theta). \tag{6.76}$$

Thus, we expect a fully collapsed deformation. Like in Case I, we reduce the continuity method to a Monge-Ampere equation

$$\varphi(t) = t \log\left(\frac{\omega^n}{(T-t)^n \Omega}\right) \quad \text{with } \varphi(0) = 0. \tag{6.77}$$

Lemma 6.18. *The solution, $\varphi(t)$, the* (6.77) *satisfies*

$$|\varphi(t)|_{L^\infty(X)} \leqslant C \tag{6.78}$$

for some $C > 0$. Furthermore

$$C^{-1}(T-t)^n \leqslant \omega^n \leqslant C(T-t)^n \Omega. \tag{6.79}$$

Proof. The proof is similar to that in Case I. Using that (6.76) implies

$$\hat{\omega}_t^n \sim C(T-t)^n \Omega.$$

Then applying a maximum principle argument yields an upper bound. From the results in [DP10; EGZ08], we obtain a uniform lower bound. □

We aim to replicate Lemma 6.12 for Case II. However, we need to make some modifications.

Lemma 6.19. *The estimates for $u = u_t(\rho)$ in Lemma 6.12 hold with the following modifications to (1) and (2);*

1. *Equivalence of $u_t'(\rho)$ with a_t, that is*
$$(n-k)(T-t) < u_t'(\rho) < (n+k)(T-t), \tag{6.80}$$
for all $t \in [0, T)$.

2. *Limiting behaviour of $u_t(\rho)$ approaching the singular time,*
$$\lim_{t \to T} u_t(\rho) = 0.$$

3. *As with Case I, there exists $C > 0$ such that*
$$C^{-1} \frac{e^{k\rho}}{(1+e^{k\rho})^2}(T-t) < u_t''(\rho) \leqslant C \frac{e^{k\rho}}{(1+e^{k\rho})^2}(T-t), \tag{6.81}$$
and
$$\frac{|u_t'''(\rho)|}{u_t''(\rho)} \leqslant C. \tag{6.82}$$

Remark 6.20. A key observation here is that u_t is no longer uniformly bounded from below away from 0.

Proof. The proof only requires slight modification from the proof of Lemma 6.12. Part (1) simplify follows from $a_t < u_t' < b_t$,since $a_t = (n-k)(T-t)$ and similarly $b_t = (n+k)(T-t)$. Then for any $\rho \in \mathbb{R}$
$$|u_t(\rho)| = \left| \int_0^\rho u_t'(s) ds \right| \leqslant (T-t)|\rho| \to 0,$$
as $t \to T$. To obtain (6.81), we again note that (6.52) implies
$$C^{-1}(T-t)^n \det(\hat{g}_0) \leqslant \det(g) \leqslant C(T-t)^n \det(\hat{g}_0).$$
In terms of the potential, u_t, the above inequality becomes
$$C^{-1}(T-t)^n \frac{e^{k\rho}}{(1+e^{k\rho})^2} \leqslant (u_t')^{n-1} u_t'' \leqslant C \frac{e^{k\rho}}{(1+e^{k\rho})^2}(T-t)^n.$$
Using that $u_t \geqslant (n-k)(T-t)$ from (6.80), our desired bound for u_t'' easily follows. The third order estimates only rely on an upper bound for u_t'' and thus can be derived in the same way as before in Lemma 6.12. ☐

With a lower bound for u_t'' identical to Case I, we obtain the same Type I blow up for scalar curvature and consequently

Proposition 6.21. *In Case II, the Riemannain curvature tensor blows up at a Type I rate.*

Proof. Like in Case I, we first bound scalar curvature. This can be done in the same way replacing (6.67) with $u_0/u_t' \leqslant C(T-t)^{-1}$. The bound (6.62) remains unchanged along with the limit calculations for $\rho \to \pm\infty$. Thus the scalar curvature blows up at a Type I rate and (6.71) holds. Furthermore, u_t' and $u_t'' < C(T-t)$ away from singular sets. Thus, the Type I blow-up for scalar curvature is essential everywhere. Returning to the estimate for the curvature tensor, we use Lemma 6.19 to bound terms in (6.70) as follows. There exists $C_i > 0$, $i = 1, 2, 3$ uniformly of t such that

$$-\frac{C_1}{T-t} \leqslant -\frac{u_t^{(4)}}{u_t''^2} + \frac{(u_t''')^2}{u_t''^3} \leqslant \frac{C_1}{T-t},$$

$$-\frac{C_2}{T-t} \leqslant \frac{1}{u_t'} - \frac{u_t''}{u_t'^2} \leqslant \frac{C_2}{T-t},$$

and

$$-\frac{C_3}{T-t} \leqslant -\frac{u_t'''}{u_t' u_t''} + \frac{u_t''}{u_t'^2} \leqslant \frac{C_3}{T-t}.$$

Substituting the above estimate into (6.70) reveals a type I rate.

□

6.3.3 Analysis for Case III

In Case III, the singular time is

$$T = \frac{a_0}{n-k}. \tag{6.83}$$

Here $a_t = (n-k)(T-t)$ and $b_t > 0$. The reference metric is given by

$$\hat{\omega}_t = \omega_0 + t((k-n)\chi - \theta) := \omega_0 + t\eta. \tag{6.84}$$

Since $-\eta \in C_1(X)$, there exists a volume form Ω such that $\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\Omega = \eta$. One then reduces the continuity to the following Monge-ampere equations

$$\varphi(t) = t\log\left(\frac{\omega^n}{\Omega}\right). \tag{6.85}$$

For this equation, we also have

Lemma 6.22. *The solution, $\varphi(t)$ to* (6.85) *satisfies*

$$\|\varphi(t)\|_{L^\infty(X)} \leqslant C \tag{6.86}$$

for some $C > 0$, and hence,

$$C^{-1}\Omega \leqslant \omega^n \leqslant C\Omega. \tag{6.87}$$

The proof is identical to Lemma 6.10. Furthermore, we can see that we are in the finite time volume non-collapsed case. The Gromov-Hausdorff convergence was obtained in [LTZ17] in which the limit space was identified to be a weakly Kähler metric ω_T which is smooth outside a subvariety S_X and satisfies

$$\omega_T = \omega_0 - T\,\mathrm{Ric}\,(\omega_T)\,, \qquad \text{on } M\backslash\mathcal{S}_X. \tag{6.88}$$

Thus, we need to obtain the curvature bound. For the estimates on the potential $u_t(\rho)$, we prove the following Lemma.

Lemma 6.23. *The function $u = u_t(\rho)$ satisfies, for all $\rho \in \mathbb{R}$,*

1. *Bound on $u_t'(\rho)$,*
$$(n-k)(T-t) < u_t'(\rho) < (n+k)(T-t) + b_0, \tag{6.89}$$
for all $t \in [0,T)$.

2. *Estimates on $u_t''(\rho)$. For all $\rho \in \mathbb{R}$, there exists $C > 0$ such that*
$$C^{-1}\frac{e^{k\rho}}{(1+e^{k\rho})^2} \leqslant u_t''(\rho) \leqslant \frac{Cu_t'(\rho)}{T-t}, \tag{6.90}$$

3. *There exist $C = C(\omega_0)$ such that*
$$\frac{|u_t'''(\rho)|}{u_t''(\rho)} \leqslant \frac{C}{T-t}. \tag{6.91}$$

Proof. The estimate in (6.89) follows from the definition of a_t and b_t. The lower bound of (6.90) can be derived from the volume estimate (6.87). Similar to the previous cases,

$$C^{-1}\frac{e^{k\rho}}{(1+e^{k\rho})^2} \leqslant (u_t')^{n-1}u_t''$$

Since in Case III, $u_t' \leqslant C$, the lower bound for (6.90) follows. The same method however does not seem to produce effective bounds as in Case II since the volume is non-collapsed. Hence the lower bound for u_t' produces an upper bound for u_t'' of $(T-t)^{-(n-1)}$. To show the upper bound to part (6.90), consider $H_t(\rho) := u_t''(\rho)/u_t'(\rho)$. Using (6.18) and (6.19), we note that

$$\lim_{\rho\to-\infty} H = \lim_{\rho\to-\infty} \frac{k^2e^{k\rho}u_{[D_0]}'(e^{k\rho}) + k^2e^{2k\rho}u_{[D_0]}''(e^{k\rho})}{k^2e^{k\rho}u_{[D_0]}'(e^{k\rho}) + a_t} = 0, \tag{6.92}$$

$$\lim_{\rho\to\infty} H = \lim_{\rho\to\infty} \frac{k^2e^{-k\rho}u_{[D_0]}'(e^{-k\rho}) + k^2e^{-2k\rho}u_{[D_0]}''(e^{-k\rho})}{k^2e^{-k\rho}u_{[D_0]}'(e^{-k\rho}) + b_t} = 0. \tag{6.93}$$

Thus, a maximum exists away from singular divisors. The derivatives of H are given by

$$H' = \frac{u_t'''}{u_t'} - \frac{u_t''^2}{u_t'^2}$$

and

$$H'' = \frac{u_t^{(4)}}{u_t'} - \frac{u_t''u_t'''}{u_t'^2} + 2\frac{u_t''^3}{u_t'^3}.$$

Using (6.27) and the expression for H', we rewrite the previous equality as

$$H'' = \frac{u_t'}{t}H^2 - \frac{u_0''}{t}H + \frac{(H'+H^2)^2}{H} - (n+2)H(H'+H^2) + (n+1)H^3. \tag{6.94}$$

At maximum points p_0, the above equation yields

$$0 \geqslant \frac{u_t'}{t} H(p_0) - \frac{u_0''}{t}.$$

Using (6.89), we obtain

$$H(p_0) \leqslant \frac{u_0''}{u_t'} \leqslant \frac{C}{T-t} \tag{6.95}$$

for some $C > 0$ independent of $t \in [0, T)$. Since u_t' is uniformly bounded, we obtain the upper bound in (6.90). □

Proposition 6.24. *The curvature tensor in Case III satisfies*

$$|\mathrm{Rm}(\omega)|_\omega \leqslant \frac{C}{(T-t)^2}. \tag{6.96}$$

Proof. The scalar curvature is bounded from above by

$$R(t) = \frac{u_0''}{t u_t''} + (n-1)\frac{u_0'}{t u_t'} - \frac{n}{t} \leqslant \frac{C}{T-t},$$

and we always have a lower scalar curvature bound. From (6.71), and using Lemma 6.23, we obtain

$$-\frac{C_1}{(T-t)^2} \leqslant \frac{u_t'''^2}{u_t''^3} - \frac{u_t^{(4)}}{u_t''^2} \leqslant \frac{C_1}{(T-t)^2}. \tag{6.97}$$

The u_t estimates also give us

$$-\frac{C_2}{T-t} \leqslant \frac{1}{u_t'} - \frac{u_t''}{u_t'^2} \leqslant \frac{C_2}{T-t},$$

and

$$-\frac{C_3}{(T-t)^2} \leqslant -\frac{u_t'''}{u_t' u_t''} + \frac{u_t''}{u_t'^2} \leqslant \frac{C_3}{(T-t)^2}.$$

Thus, the worst blow-up term in the curvature tensor is of order $(T-t)^{-2}$. □

6.4 Finite Time Singularity

In this section, we prove Theorem 6.7; if the continuity method encounters a finite time singularity, the scalar curvature must necessarily blow up. We observe this behaviour in the analysis of Cases I and II in the previous section. Furthermore, this result holds for the Kähler Ricci Flow (see [Zha10]).

We need some setup before proving Theorem 6.7. For convenience, we rescale (6.1) by

$$(1+s)\omega(s) = \omega(t) \quad \text{with} \quad s = t. \tag{6.98}$$

The continuity method becomes

$$\begin{cases} (1+t)\omega(t) = \omega_0 - t\mathrm{Ric}(\omega(t)), \\ \omega(0) = \omega_0. \end{cases} \tag{6.99}$$

Choose a form ω_∞ in the canonical class K_X. By Yau's theorem, there exists a Ω such that

$$\omega_\infty = -\operatorname{Ric}(\Omega). \tag{6.100}$$

This allows us to define the reference metric,

$$\omega_t := \frac{1}{1+t}\omega_0 + \frac{t}{1+t}\omega_\infty, \tag{6.101}$$

which is obtained by formally solving the cohomology evolution of (6.99). Then (6.1) reduces to the scalar equation

$$\begin{cases} \dfrac{1+t}{t}\varphi(t) = \log\left(\dfrac{\omega^n}{\Omega}\right) \\ \varphi(0) = 0. \end{cases} \tag{6.102}$$

The characterisation (6.4) of the maximum time of existence still applies here and a solution to the Monge-Ampere equation above generates a solution to the continuity method.

Lemma 6.25. *There exists a constant $C > 0$ such that for any time $t \geqslant 1$*

$$\varphi(t) \leqslant C.$$

Proof. Expanding out ω_t^n, we have

$$\omega_t^n = \sum_{k=r}^{n} \binom{n}{k} \frac{t^{n-k}}{(1+t)^n}\omega_0^k \wedge \omega_\infty^{n-k} \leqslant C. \tag{6.103}$$

Here we have used that $t < T < \infty$. Applying the maximum principle and substituting the above into (6.102), we obtain some $C > 0$ such that

$$\varphi(t) \leqslant C,$$

for all $t \in [0, T)$. □

From (6.102), we immediately have

Corollary 6.26. *There exists $C > 0$ such that for all $t \in [0, T)$,*

$$\omega(t)^n \leqslant C\Omega. \tag{6.104}$$

Proof of Theorem 6.7. Suppose that $T < \infty$ is the maximal time given by (6.4) and assume by contradiction that $R(t) < C$ for some $C > 0$ independent of t. From taking the trace of (6.1), we have

$$R = \frac{1}{t}\operatorname{tr}_\omega \omega_0 - n\left(\frac{1+t}{t}\right),$$

which implies

$$\operatorname{tr}_\omega \omega_0 \leqslant C. \tag{6.105}$$

To obtain a lower estimate for $\omega(t)$, we use the well known eigenvalue estimate and (6.104) to obtain a $C > 0$ such that

$$\operatorname{tr}_{\omega_0} \omega \leqslant (n-1)!(\operatorname{tr}_\omega \omega_0)^{n-1} \frac{\omega^n}{\omega_0^n} \leqslant C, \tag{6.106}$$

for all $t \in [0, T)$. The last inequality holds from our assumption $T < \infty$. Combining (6.105) and (6.105), we conclude that there exists $C > 0$ such that

$$C^{-1}\omega_0 \leqslant \omega(t) \leqslant C\omega_0. \tag{6.107}$$

To obtain the contradiction, we recall Demailly and Paun's characterisation of Kähler metrics:

Theorem 6.27 (Theorem 4.2 in [DP04]). *Let (X, ω_0) be a compact Kähler manifold and let $[\alpha]$ be a $(1,1)$ cohomology class in $H^{1,1}(X, \mathbb{R})$. Then $[\alpha]$ is Kähler if and only if for every irreducible analytic set $Y \subset X, \dim Y = n$, and every $t \geqslant 0$,*

$$\int_Y (\alpha + t\omega_0)^n > 0. \tag{6.108}$$

However, the lower bound in (6.107) reveals that (6.108) holds for the classes $[\omega(t)]$ for all $t \in [0, T)$. From taking the limit, we see that the class $[\omega_T]$ also satisfies (6.108) and thus, the limiting class is Kähler which contradicts the maximal time of existence. □

Alternatively, one can derive higher order estimates for $\omega(t)$ using only maximum principle arguments. These uniform higher order estimates up to the metric at the singular time T provide a contradiction with the maximal time of existence T.

Alternate Proof of Theorem 6.7. Let $g, \nabla, R_{i\bar{j}k\bar{l}}$ be the metric, connection and curvature tensor associated with the form $\omega(t)$. We use $g^0, \nabla^0, (R^0)_{i\bar{j}k\bar{l}}$ to represent the same geometric objects associated with ω_0. Define the tensor $\Psi^k_{i\bar{j}} := \Gamma^k_{i\bar{j}} - (\Gamma^0)^k_{i\bar{j}}$ and its norm $S = |\Psi|^2_{\omega(t)}$. In local coordinates,

$$S = g^{\bar{j}i} g^{\bar{l}k} g^{\bar{q}p} \nabla^0_i g_{k\bar{q}} \nabla^0_{\bar{j}} g_{p\bar{l}}. \tag{6.109}$$

On the other hand, by a standard calculation, we have

$$\Delta_\omega \operatorname{tr}_\omega \omega_0 \geqslant -C(\operatorname{tr}_\omega \omega_0)^2 - C \operatorname{tr}_\omega \omega_0 + g^{\bar{j}i} g^{\bar{q}p} g_0^{\bar{b}a} \nabla_i g^0_{p\bar{b}} \nabla_{\bar{j}} g^0_{a\bar{q}}. \tag{6.110}$$

Using (6.107), the last term in (6.110) can be estimated by

$$\begin{aligned} g^{\bar{j}i} g^{\bar{q}p} g_0^{\bar{b}a} \nabla_i g^0_{p\bar{b}} \nabla_{\bar{j}} g^0_{a\bar{q}} &= g^{\bar{j}i} g^{\bar{q}p} g_0^{\bar{b}a} \left(\nabla_i - \nabla^0{}_i\right) g^0_{p\bar{b}} \left(\nabla_{\bar{j}} - \nabla^0_{\bar{j}}\right) g^0_{a\bar{q}} \\ &= g^{\bar{j}i} g^{\bar{q}p} g_0^{\bar{b}a} \left(-\Psi^d_{ip}\right) g^0_{d\bar{b}} \left(-\overline{\Psi^e_{jq}}\right) g^0{}_{a\bar{e}} \\ &\geqslant C^{-1} S. \end{aligned} \tag{6.111}$$

Using the above and (6.107) in (6.110), we can derive the estimate

$$\Delta_\omega \operatorname{tr}_\omega \omega_0 \geqslant -C + C^{-1} S. \tag{6.112}$$

We now derive a lower bound for the Laplacian of the tensor S using a similar calculation by Phong-Šešum-Sturm for the Kähler-Ricci flow. The Laplacian of S yields Ricci terms.

Using the continuity method (6.1), we can rewrite any Ricci terms as an expression involving the initial and evolving metrics. From this, we compute (for more details see [Won23b])

$$\Delta_\omega S = I + II + |\nabla\Psi|^2_\omega + |\bar{\nabla}\Psi|^2_\omega, \tag{6.113}$$

where

$$I = \frac{1}{t}\left(g_0^{\bar{j}i} g^{\bar{q}p} g_{k\bar{l}} + g^{\bar{j}i} g_0^{\bar{q}p} g_{k\bar{l}} - g^{\bar{j}i} g^{\bar{q}p} g^0_{k\bar{l}}\right) \Psi^k_{ip} \overline{\Psi^l_{jq}} - \frac{1+t}{t} S, \tag{6.114}$$

and

$$II = 2\,\mathrm{Re}\left(g^{\bar{j}i} g^{\bar{q}p} g_{k\bar{l}} \left(\nabla^{\bar{b}} (R^0)^k_{i\bar{b}p}\right) \overline{\Psi^l_{jq}}\right) - 2\,\mathrm{Re}\left(g^{\bar{j}i} g^{\bar{q}p} g_{k\bar{l}} \left(\nabla_i R^k_p\right) \overline{\Psi^l_{jq}}\right). \tag{6.115}$$

Thanks to the metric equivalence (6.107) and using that $t \leqslant T < \infty$, we can easily estimate term I by

$$I \geqslant -CS, \tag{6.116}$$

for some $C > 0$. For the first term in II, we can compute

$$\begin{aligned} \nabla^{\bar{b}} (R^0)^k_{i\bar{b}p} &= g^{\bar{b}a} \nabla_a (R^0)^k_{i\bar{b}p} \\ &= g^{\bar{b}a} \left(\nabla_a - \nabla^0_a\right) (R^0)^k_{i\bar{b}p} + g^{\bar{b}a} \nabla^0_a (R^0)^k_{i\bar{b}p} \\ &= \Psi * Rm(\omega_0) + g^{\bar{b}a} \nabla^0_a (R^0)^k_{i\bar{b}p}. \end{aligned} \tag{6.117}$$

We can rewrite the first term using the Cauchy-Schwartz inequality as

$$\begin{aligned} 2\,\mathrm{Re}\left(g^{\bar{j}i} g^{\bar{q}p} g_{k\bar{l}} \left(\nabla^{\bar{b}} (R^0)^k_{i\bar{b}p}\right) \overline{\Psi^l_{jq}}\right) &= -C|\Psi * Rm(\omega_0)|_\omega \sqrt{S} - C \left| g^{\bar{b}a} \nabla^0_a (R^0)_{i\bar{b}p}{}^k \right|_\omega \sqrt{S} \\ &\geqslant -CS - C\sqrt{S}. \end{aligned} \tag{6.118}$$

Throughout these estimates, we have used (6.107) to estimate $g_{i\bar{j}}$ in terms of $g^0_{i\bar{j}}$ and vice versa. For the second term in (6.115), we use the continuity equation (6.1) to obtain

$$R^k_p = \frac{1}{t} g^{\bar{r}k} g^0_{p\bar{r}} - \frac{1+t}{t} g^{\bar{r}k} g_{p\bar{r}}.$$

Here, g_0 is the metric associated with the form ω_0. Taking the covariant derivative and using metric compatibility yields

$$\nabla_i R^k_p = \frac{1}{t} g^{\bar{r}k} \nabla_i g^0_{p\bar{r}}.$$

Hence, we can rewrite the second term of II as

$$\begin{aligned} -2\,\mathrm{Re}\left(g^{\bar{j}i} g^{\bar{q}p} g_{k\bar{l}} \left(\nabla_i R^k_p\right) \overline{\Psi^l_{jq}}\right) &= -\frac{2}{t}\,\mathrm{Re}\left(g^{\bar{j}i} g^{\bar{q}p} \left(\nabla_i g^0_{p\bar{l}}\right) \overline{\Psi^l_{jq}}\right) \\ &= -\frac{2}{t}\,\mathrm{Re}\left(g^{\bar{j}i} g^{\bar{q}p} \left((\nabla_i - \nabla^0_i) g^0_{p\bar{l}}\right) \overline{\Psi^l_{jq}}\right) \\ &= -\frac{2}{t}\,\mathrm{Re}\left(g^{\bar{j}i} g^{\bar{q}p} g^0_{n\bar{l}} \Psi^n_{ip} \overline{\Psi^l_{jq}}\right) \\ &\geqslant -CS, \end{aligned} \tag{6.119}$$

for some $C > 0$. Thus, we have

$$\Delta_\omega S \geqslant -CS + |\nabla\Psi|^2_\omega + |\overline{\nabla}\Psi|^2_\omega. \tag{6.120}$$

Using (6.120) and (6.112), we can obtain the estimate for the Laplacian of $Q := S + A\,\mathrm{tr}_\omega\,\omega_0$ to obtain, for some $C > 0$, the inequality

$$\Delta_\omega Q \geqslant -C + (A - C)S. \tag{6.121}$$

Choosing $A > 0$ large enough, we can apply a maximum principle argument to deduce that at a maximum point of Q,

$$S \leqslant C, \tag{6.122}$$

for some $C > 0$. Since $\mathrm{tr}_\omega\,\omega_0$ is uniformly bounded, we deduce that $S \leqslant C$ where $C > 0$ is uniform of time. Utilising this bound for S, we can return to (6.120) and use the estimate

$$|\overline{\nabla}\Psi|^2_\omega \geqslant \frac{1}{2}|\mathrm{Rm}(\omega)|^2_\omega - C, \tag{6.123}$$

to obtain

$$\Delta_\omega S \geqslant -C + \frac{1}{2}|\mathrm{Rm}(\omega)|^2_\omega, \tag{6.124}$$

for some $C > 0$. The Laplacian of the curvature tensor is given by

$$\begin{aligned}\Delta_\omega\left(|\mathrm{Rm}(\omega)|^2_\omega\right) \geqslant &- C_1|\mathrm{Rm}(\omega)|_\omega - C_2|\mathrm{Rm}(\omega)|^2_\omega - C_3|\mathrm{Rm}(\omega)|^3_\omega \\ &+ |\nabla\mathrm{Rm}(\omega)|^2_\omega + |\overline{\nabla}\mathrm{Rm}(\omega)|^2_\omega.\end{aligned} \tag{6.125}$$

Using this, we can deduce that

$$\begin{aligned}\Delta_\omega|\mathrm{Rm}(\omega)|_\omega &= \frac{\Delta_\omega|\mathrm{Rm}(\omega)|^2_\omega}{2|\mathrm{Rm}(\omega)|_\omega} - \frac{|\nabla|\mathrm{Rm}(\omega)|^2_\omega|^2_\omega}{4|\mathrm{Rm}(\omega)|^3_\omega} \\ &\geqslant -C_1 - C_2|\mathrm{Rm}(\omega)|_\omega - C_3|\mathrm{Rm}(\omega)|^2_\omega \\ &+ \frac{|\nabla\mathrm{Rm}(\omega)|^2_\omega}{|\mathrm{Rm}(\omega)|_\omega} + \frac{|\overline{\nabla}\mathrm{Rm}(\omega)|^2_\omega}{|\mathrm{Rm}(\omega)|_\omega} - \frac{|\nabla|\mathrm{Rm}(\omega)|^2_\omega|^2_\omega}{4|\mathrm{Rm}(\omega)|^3_\omega} \\ &\geqslant -C_1 - C_2|\mathrm{Rm}(\omega)|_\omega - C_3|\mathrm{Rm}(\omega)|^2_\omega,\end{aligned} \tag{6.126}$$

where the Cauchy-Schwartz inequality is used in the last line;

$$g^{\bar{j}i}\nabla_i|\mathrm{Rm}|^2\nabla_{\bar{j}}|\mathrm{Rm}|^2 \leqslant 2|\mathrm{Rm}|^2\left(|\nabla\mathrm{Rm}|^2 + |\overline{\nabla}\mathrm{Rm}|^2\right). \tag{6.127}$$

Setting $Q := |\mathrm{Rm}(\omega)|^2_\omega + AS$, for some $A > 0$ to be deduced later. We can use the previous estimates to obtain

$$\Delta_\omega Q \geqslant -C_1 - C_2|\mathrm{Rm}(\omega)|_\omega + \left(\frac{A}{2} - C_3\right)|\mathrm{Rm}(\omega)|^2_\omega, \tag{6.128}$$

for some positive constants C_1, C_2 and C_3. Choosing $A > 0$ large enough, we obtain

$$\Delta_\omega Q \geqslant -C + |\mathrm{Rm}(\omega)|^2_\omega. \tag{6.129}$$

Then by a maximum principle argument, we deduce that at a maximum point $|\mathrm{Rm}(\omega)|_\omega(p_0) \leqslant C$. Since S is uniformly bounded, we can conclude that

$$\sup_X |\mathrm{Rm}(\omega)|_\omega \leqslant C, \tag{6.130}$$

for some $C > 0$ independent of t. Using the identity

$$\frac{1}{2}\Delta_\omega|\nabla^m \mathrm{Rm}|^2 = |\nabla^{m+1}\mathrm{Rm}|^2 - \sum_{p+q=m} \nabla^p \mathrm{Rm} * \nabla^q \mathrm{Rm} * \nabla^m \mathrm{Rm}, \tag{6.131}$$

we can apply an induction argument to obtain bounds on the higher order curvature terms. Defining the quantity

$$Q_m := |\nabla^{m+1}\mathrm{Rm}|^2 + A|\nabla^m \mathrm{Rm}|^2, \tag{6.132}$$

for some $A > 0$ chosen large enough, we can use (6.131) to setup a maximum principle argument. This implies that the metric and its higher derivatives are bounded at the singular time yielding a contradiction. $\square$

Appendix A

Geometry

Geometry is one of mathematics's oldest fields, dating back to the ancient Greeks. The famous treatise, 'Euclid's Elements', written in 300 BC, establishes Euclidean geometry. This framework dominated geometry until the 19th century when Nikolai Lobachevsky modified Euclid's fifth parallel postulate to obtain hyperbolic geometry, a geometry outside Euclid's paradigm. This, along with the intrinsic Euclidean geometry developed by Gauss, established the foundations of modern geometry. Today, the modern formulation of geometry, which captures non-Euclidean spaces, is found in many fields outside of mathematics, the predominant example being space-time in Einstein's theory of general relativity.

A.1 Differentiable Manifolds

A manifold is a topological space that resembles open sets of Euclidean space on a local scale but differs significantly globally.

Definition A.1. A *topological manifold*, M, is a topological space that is Hausdorff, second countable and locally Euclidean. We say that a topological space is locally Euclidean if every point has a neighbourhood that is homeomorphic to an open ball in $\mathbb{R}^n$.

These properties guarantee the existence of an atlas.

Definition A.2. An *atlas* is a collection of pairs $(U_\alpha, \varphi_\alpha)_{\alpha \in \mathcal{A}}$ called charts, where $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ is an open cover of M and $\phi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^n$ are homeomorphic maps. Between two charts (U_β, ϕ_β) and (U_γ, ϕ_γ), with $U_\beta \cap U_\gamma \neq \emptyset$, we define the transition map to be the composition

$$\phi_\beta \circ \phi_\gamma^{-1} : V_\beta \to V_\gamma.$$

Charts of an atlas allow us to identify points on the manifold with points in Euclidean space in a globally consistent manner. Furthermore, we can define the ring of continuous functions by composing a function $f : M \to \mathbb{R}$ with a chart $f \circ \phi_\alpha^{-1} : V_\alpha \to \mathbb{R}$. To define higher-order regularity of functions, we must impose regularity on the transition maps of the atlas.

Definition A.3. A *smooth manifold* is a topological manifold that admits an atlas, $(U_\alpha, \phi_\alpha)_{\alpha \in \mathcal{A}}$ with smooth transition maps[1].

[1] We call such an atlas a smooth atlas.

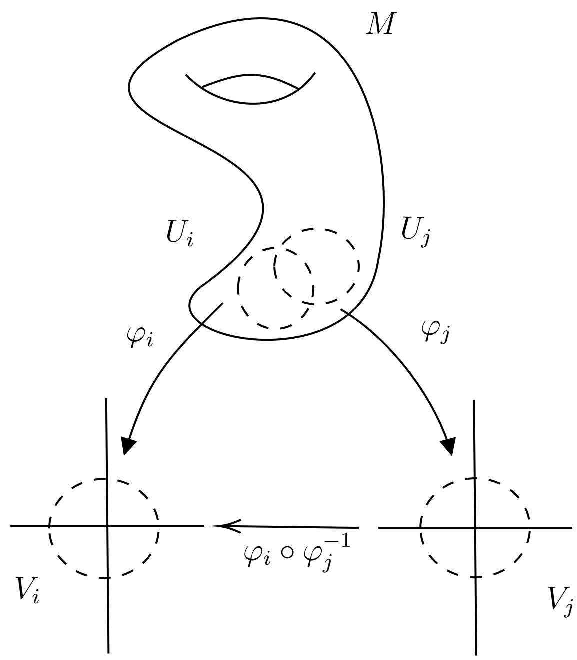

Figure A.1: Chart and transition maps of a manifold.

Remark A.4. A topological manifold may not admit a smooth atlas. For example, E8 is a topological manifold that cannot be charted by a smooth atlas.

Remark A.5. To remove the dependence on the choice of atlas, we define an equivalence class of smooth atlases: Two atlases are equivalent to their union is a smooth atlas. This equivalence class of smooth atlases may not be unique; for example, there are seven smooth atlases for the exotic 7-sphere.

On smooth manifolds, the smooth transition maps ensure that the regularity of a given function does not depend on a chosen chart. Thus, we have a well-defined ring of smooth functions $C^\infty(M)$, where we say that $\varphi \in C^\infty(M)$ if for each $p \in M$, there exists (U_α, ϕ_α) such that $f \circ \phi_\alpha^{-1}$ is smooth.

If we replace $\mathbb{R}^n$ with $\mathbb{C}^n$ in previous definitions, we can define manifolds that support an algebra of holomorphic functions.[2] We call these manifolds complex manifolds.

Definition A.6. A *complex manifold* is a topological manifold that admits an atlas with holomorphic transition maps.

Remark A.7. Once again, we remove the dependence on the choice of atlas by defining an equivalence class of holomorphic atlases: Two atlases are equivalent if their union is a holomorphic atlas.

[2]If M is compact, any holomorphic function is constant by the maximum modulus principle. Therefore, we must use sheaf theory to properly make sense of holomorphic functions on complex manifolds.

Topological Manifolds	Smooth Manifolds	Complex Manifolds
n-simplex E8 manifold	n-sphere Exotic 7-sphere	Riemann sphere

Table A.1: Examples of manifolds

We can construct new manifolds from old manifolds via a Cartesian product or by gluing points through a quotient space.

Proposition A.8. *Let M_1 and M_2 be a smooth/complex manifold. Then $M_1 \times M_2$ is a smooth/complex manifold.*

Proposition A.9. *Let M be a manifold and Γ be a discrete group acting on it. If Γ acts smoothly, freely and discontinuously, then the quotient M/Γ equipped with the quotient topology is a topological manifold.*

So far, we have defined functions on our manifolds. We next describe how to define vector fields. This is relatively straightforward: if our manifold can be embedded into Euclidean space, we simply define fields on the manifold using the vector space structure of the ambient Euclidean space. To do this intrinsically, without an embedding, we introduce vector bundles, which take the role of ambient Euclidean space.

Definition A.10. A *real (or complex) vector bundle* over a manifold M is a triple (E, π, M) comprising of:

1. a collection of real (or complex) vector spaces $\{E_x\}_{x\in M}$ parameterised by $x \in M$ such that the union $E = \cup_{p\in M} E_p$ is a topological manifold,
2. a continuous projection map $\pi : E \to M$ and
3. for each open set U in a covering of M, there exists a map $\varphi_U : \pi^{-1}(U) \to U \times V$ that is a homeomorphism and point-wise an isomorphism. We call φ a local trivialisation.

A vector bundle exhibits a local resemblance to a product vector space, yet its global appearance is notably distinct. This echoes a recurring theme of geometry: locally Euclidean spaces that differ globally.

Definition A.11. A *section* of a vector bundle (E, π, M) is a continuous map $\sigma : M \to E$ such that $\pi(\sigma(p)) = p$ for all $p \in B$.

Sections assign an element of the vector space E_p at each point $p \in M$ and generalise vector fields to manifold. The regularity of a vector field is determined through local trivialisation.

Definition A.12. A *smooth vector bundle* is a real vector bundle (E, p, M) such that E and M are smooth manifolds, $p : E \to M$ is a smooth map, and the local trivialisations are diffeomorphisms.

Definition A.13. A *holomorphic vector bundle* is a complex vector bundle, (E, p, M), such that E and M are complex manifolds, $p : E \to M$ is a holomorphic map, and the local trivialisations are biholomorphisms.

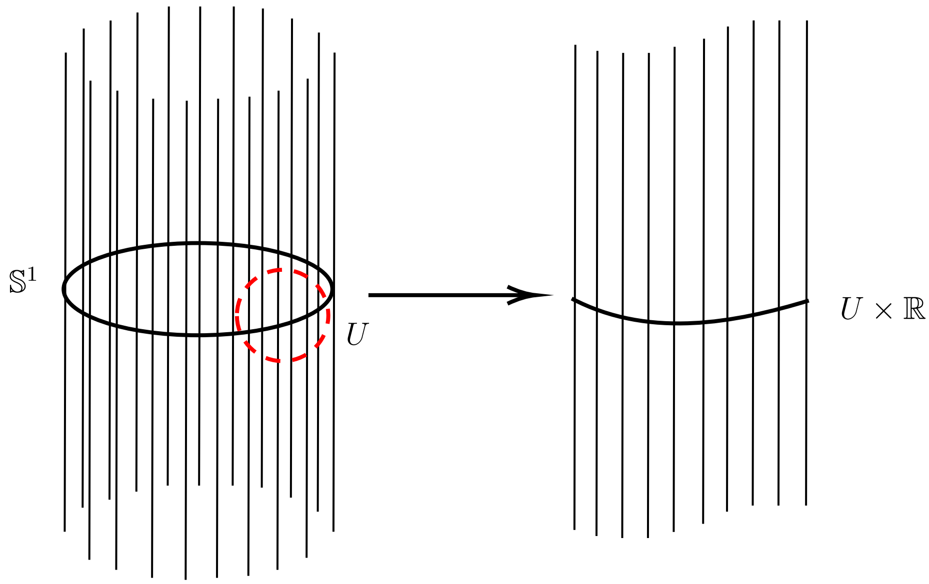

Figure A.2: Local trivialisation of a vector bundles over $\mathbb{S}^1$

There are two important examples of vector bundles: a smooth manifold's tangent bundle and a complex manifold's holomorphic tangent bundle. These are defined through derivations on the ring of functions supported by the manifold, and the coordinate charts guarantee their existence.

Example A.14. Let M be a smooth manifold. A *derivation* at $p \in M$ is a $\mathbb{R}$-linear map $D : C^\infty(M) \to \mathbb{R}$ that satisfies the Leibniz identity:

$$\forall f, g \in C^\infty(M) : \quad D(fg) = D(f) \cdot g(x) + f(x) \cdot D(g). \tag{A.1}$$

The *tangent space* at p, denoted as T_pM, is the set of all derivations at a point $p \in M$. The *tangent bundle* is the disjoint union of tangent spaces

$$TM = \bigsqcup_{p \in M} T_pM. \tag{A.2}$$

Example A.15. Let M be a complex manifold. The *holomorphic tangent space*, T'_pM, is the set of derivations that vanish on antiholomorphic functions. Similarly, define the *antiholomorpic tangent space* T''_pM as the set of all derivations that vanish on holomorphic functions. Then

$$T_{\mathbb{C},p}M = T'_pM \bigoplus T''_pM. \tag{A.3}$$

The complex tangent bundle is the disjoint union of complex tangent spaces

$$T_{\mathbb{C}}M = \bigsqcup_{p \in M} T_{\mathbb{C},p}M. \tag{A.4}$$

Hidden in the definition of a complex tangent space is an endomorphism of the underlying real tangent space that takes the role of multiplication by $\sqrt{-1}$. This endomorphism is important when we begin with an even-dimensional smooth manifold and try to complexify its tangent space to produce a complex manifold.

Definition A.16. Let M be an even-dimensional manifold. An *almost complex structure* is an endomorphism $J : T_{\mathbb{R}}M \to T_{\mathbb{R}}M$ such that $J^2 = -I$. The eigenvectors of J determine the holomorphic and antiholomorphic sub-bundle of the complexified tangent space.

Remark A.17. On an even-dimensional manifold, J_p exists pointwise. However, a globally defined endomorphism is equivalent to the reduction of the structure group of the tangent bundle from $\mathrm{GL}(2n, \mathbb{R})$ to $\mathrm{GL}(n, \mathbb{C})$.

However, an almost complex structure is insufficient to guarantee a complex structure, i.e. a holomorphic atlas. The condition that dictates whether an almost complex structure admits a complex structure is given by the Newlander–Nirenberg theorem.

Theorem A.18 (Newlander–Nirenberg)**.** *Let J be an almost complex structure on M and N_J be the Nijenhuis tensor defined by the equation*

$$N_J(X, Y) := [X, Y] + J([JX, Y] + [X, JY]) - [JX, JY]. \tag{A.5}$$

Then (M, J) admits a complex structure if the Nijenhuis tensor vanishes.

Furthermore, in our two examples of bundles, the tangent bundle for smooth manifolds and the holomorphic tangent bundle for complex manifolds, we define a type of differentiation for each that allows us to make sense of Stoke's theorem. These lead to the De Rham and Dolbeault cohomologies.

We use the following notation for forms: $\Omega^r(M) := \wedge_r(T^*M)$ and $\Omega^r_{\mathbb{C}}(M) := \wedge_r(T^*_{\mathbb{C}}M)$.

Definition A.19. The exterior derivative is the unique $\mathbb{R}$-linear mapping that maps k forms to $k+1$ forms satisfying the following properties.

1. df is the differential of f for a 0 -form f.

2. $d(df) = 0$ for a 0 -form f.

3. $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p(\alpha \wedge d\beta)$ where α is a r-form.

Alternatively, one can define the exterior derivative in local coordinates. Suppose we have a local coordinate system $(x^1, \ldots, x^n)$. The coordinate differentials $dx^1, \ldots, dx^n$ form a basis of the space of one forms. For a multi-index $I = (i_1, \ldots, i_k)$ with $1 \le i_r \le n$ for $1 \le r \le k$ (and denoting $dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ with dx^I), the exterior derivative of a k-form

$$\varphi = g dx^I = g dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}$$

over $\mathbb{R}^n$ is defined as

$$d\varphi = \frac{\partial g}{\partial x^i} dx^i \wedge dx^I$$

The property $d^2 = 0$ leads to the De Rham cohomology, where the exterior derivative is a co-boundary operator.

Definition A.20. The De Rham complex is the cochain complex of differential forms on some smooth manifold M, with the exterior derivative as the differential:

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \Omega^3(M) \to \cdots,$$

where $\Omega^0(M)$ is the space of smooth functions on $M, \Omega^r(M)$ is the space of r-forms. The De Rham cohomology is defined by

$$H^{p,q}_{\bar\partial}(M) := \frac{\{\alpha \in \Omega^r(M) : d\alpha = 0\}}{\{d\beta : \beta \in \Omega^{r-1}(M)\}}. \tag{A.6}$$

In the complex case, the almost complex structure induces a spitting of forms.

$$\Omega_{\mathbb{C}} M = \Omega^{1,0} M \oplus \Omega^{0,1} M, \tag{A.7}$$

and

$$\Omega^r_{\mathbb{C}} M = \oplus_{p+q=r} \Omega^{(p,q)} M. \tag{A.8}$$

Projecting the exterior derivative onto each component defines the Dolbeault operator.

Definition A.21. Let M be a complex manifold with exterior derivative d acting on complex forms. We define $\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M)$ and $\bar\partial : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$ by

$$\partial\alpha := \pi^{p+1,q}(d\alpha), \quad \bar\partial\alpha := \pi^{p,q+1}(d\alpha),$$

where $\pi^{p+1,q} : \Omega^{k+1}_{\mathbb{C}}(M) \to \Omega^{p+1,q}(M)$ and $\pi^{p,q+1} : \Omega^{k+1}_{\mathbb{C}}(M) \to \Omega^{p,q+1}(M)$ denote the projection maps.

We can now describe vector fields on manifolds through sections of vector bundles. However, there is a slight problem in that each point possesses its own vector space and therefore, there is no canonical way to compare sections at different points on the manifold. Such a comparison is vital in calculating rates of change in the definition of the derivative. It turns out that an artificial choice must be made by defining a connection.

Definition A.22. A *connection* on a smooth real vector bundle E over a smooth manifold M is an $\mathbb{R}$-linear map

$$\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E),$$

satisfying the following properties.

1. $\nabla_{fX} Y = f\nabla_X Y$, that is, ∇ is $C^\infty(M)$-linear in the first variable.
2. $\nabla_X(fY) = df(X)Y + f\nabla_X Y$, that is, ∇ satisfies Leibniz rule in the second variable.

In the complex case, an analogous definition can stated by replacing $\mathbb{R}$ with $\mathbb{C}$ and 'smooth' with 'holomorphic'.

A connection on a vector bundle induces a connection on the dual space, tensor powers and direct sums of E.

Proposition A.23. *Let E be a vector bundle and ∇ be a connection. Then the dual connection ∇^* on E^* defined implicitly by*

$$d(\langle \xi, s\rangle)(X) = \langle \nabla^*_X \xi, s\rangle + \langle \xi, \nabla_X s\rangle$$

is a connection on E^*. Here $X \in \Gamma(TM)$ is a smooth vector field, $s \in \Gamma(E)$ is a section of E, $\xi \in \Gamma(E^*)$ a section of the dual bundle. The natural pairing between a vector space and its dual is denoted as $\langle \cdot, \cdot \rangle$ and exterior differentiation is d.

Proposition A.24. *Let ∇^E and ∇^F be connections on vector bundles E and F over M. The product connection is given by*

$$\left(\nabla^E \otimes \nabla^F\right)_X (s \otimes t) = \nabla^E_X(s) \otimes t + s \otimes \nabla^F_X(t)$$

is a connection on $E \otimes F$.

Proposition A.25. *The direct sum connection defined by the equation*

$$\left(\nabla^E \oplus \nabla^F\right)_X (s \oplus t) = \nabla^E_X(s) \oplus \nabla^F_X(t),$$

is a connection on $E \oplus F$.

The connection allows us to take a directional derivative. Any vector $X \in TM$ defines a map $\nabla \to \nabla_X$ where ∇_X measures the change of a section of E in the direction of X. Unlike the canonical directional derivative in $\mathbb{R}^n$, the following quantities may be non-zero.

Definition A.26. Let ∇ be a connection on the tangent bundle TM. The *torsion* is defined to be

$$T^\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \tag{A.9}$$

where $X, Y \in \Gamma(TM)$.

Definition A.27. Let ∇ be a connection on the vector bundle E. The *curvature* is given by the formula

$$R^\nabla_{X,Y} Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{A.10}$$

where $Z \in \Gamma(E)$ and $X, Y \in \Gamma(TM)$.

These two quantities take a central role in geometry. A significant goal of the field is to understand the interaction between torsion and curvature with the underlying topology and additional geometric structures on the manifold.

A.2 Riemannian Geometry

The study of Riemannian geometry concerns the study of smooth manifolds equipped with an additional structure defined on the tangent bundle called the Riemannian metric. The Riemannian metric is responsible for endowing the manifold with angles and distances.

Definition A.28. Let M be a smooth real manifold. A *Riemannian metric* is a smooth section of $T^*M \otimes T^*M$ which is symmetric and positive definite. In local coordinates, the Riemannian metric can be written as

$$g = \sum_{i,j=1}^{n} g_{ij} dx^i \otimes dx^j, \tag{A.11}$$

where $g_{ij} := g\left(\partial_{x_i}, \partial_{x_j}\right)$. A *Riemannian manifold* is a smooth manifold equipped with a Riemannian metric (M, g).

Example A.29.

1. Euclidean space equipped with the usual inner product $(\mathbb{R}^n, \delta_{ij})$.
2. Embedded hypersurfaces in $\mathbb{R}^n$ equipped with the metric obtained by restricting the usual inner product on the tangent space of the hypersurface.
3. The space of normal distributions equipped with the Fisher-Rao information metric.
4. The Gibbs distribution for the grand canonical ensemble in thermodynamics equipped with the Fisher-Rao information metric.
5. Projective space equipped with the Fubini-Study metric.
6. Grassmanians are submanifolds of projective space, thus inheriting a Riemannian metric.
7. A Lie group with any left invariant inner product on the Lie algebra.
8. Space-time from Einstein's theory of general relativity is a pseudo-Riemannian manifold.

A Riemannian metric can be interpreted as an inner product on the tangent space that a smoothly varying as you move around in the base manifold. It allows us to define distance and geodesics.

Definition A.30. Let $\gamma : I \to M$ be a curve on (M, g). The the *length* of γ is

$$\|\gamma\|_g := \int_a^b \sqrt{g(\gamma'(s), \gamma'(s))} ds. \tag{A.12}$$

Definition A.31. Let (M^n, g) be a Riemannian manifold. We say that a curve $\gamma : [t_0, t_1] \to M$ is a geodesic if γ is a (local) minimum of the length functional $\|\cdot\|_g$.

The norm induces a metric space structure on the topological space M.

Definition A.32. The *metric* $d : M \times M \to \mathbb{R}_{\geqslant 0}$ *induced by the Riemannian metric* g si give by the following formula

$$d(p, q) := \inf\{\|\gamma\|_g \mid \gamma : [0, 1] \to M, \text{ piece wise smooth curves with } \gamma(0) = p \text{ and } \gamma(1) = q\}.$$

If a curve attains this infimum, it is called a segment between p and q.

Remark A.33. Defined as above, (M, d) becomes a metric space with an induced topology that is equivalent to the topology of the underlying manifold (M, τ).

Definition A.34. Let (M^n, g) be a Riemannian manifold and $p \in M$ be a point. The *exponential map* $\exp_p : \mathcal{U} \to M$ is defined by sending $v_p \in T_pM$ to the endpoint $\gamma(1) \in M$ of the unique geodesic γ with $\gamma(0) = p$ and $\dot{\gamma}(0) = v_p$.

If the exponential map can be extended to all of TM, then we say that the manifold is geodesically complete. This relates to the completeness of the induced metric space through the Hopf-Rinow theorem.

Theorem A.35 (Hopf-Rinow)**.** *Let (M, g) be a connected and smooth Riemannian manifold. Then the following statements are equivalent:*

1. *The closed and bounded subsets of M are compact;*
2. *M is a complete metric space;*
3. *M is geodesically complete.*

Definition A.36. Let (M, g) be a Riemannian manifold. A *Levi-Civita connection* is an affine map that is:

1. Compatible with $g : X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$.
2. Torsion free: $\nabla_X Y - \nabla_Y X = [X, Y]$.

Theorem A.37 (Fundamental Theorem of Riemannian Geometry)**.** *Every Riemannian manifold admits a unique Levi-Civita connection.*

Let $\theta_X := g(X, \cdot)$. An equivalent way to define the Levi-Civita symbol without invoking the fundamental theorem is the following.

Definition A.38. Let (M, g) be a Riemannian manifold. The Levi-Civita connection can be written as

$$2g(\nabla_Y X, Z) := (L_X g)(Y, Z) + (d\theta_X)(Y, Z). \tag{A.13}$$

The expression for the Levi-Civita connection above tells us that the connection in the X field depends on a Lie derivative of g with respect to X and the divergence of X. Although theoretically nice, this formula is difficult to unpack. The Kozul Formula provides a more convenient expression of the Levi-Civita connection.

Proposition A.39 (The Koszul Formula)**.** *The Levi-Civita connection for (M, g) is given by*

$$\begin{aligned} 2g\left(\nabla_X Y, Z\right) = {} & \partial_X(g(Y, Z)) + \partial_Y(g(X, Z)) - \partial_Z(g(X, Y)) \\ & + g([X, Y], Z) - g([X, Z], Y) - g([Y, Z], X). \end{aligned}$$

Definition A.40. The curvature tensor for the Levi-Civita connection is called the Riemannian curvature tensor. It is sometimes more convenient to identify the curvature tensor with the following tensor;

$$RM(X, Y, Z, W) = g(W, R^{\nabla}_{X,Y} Z) \tag{A.14}$$

Proposition A.41. *The Riemannian curvature tensor satisfies the following properties.*

1. *Symmetry:*
$$RM(X,Y,Z,W) = -RM(Y,X,Z,W) = -RM(X,Y,W,Z).$$
2. *The First Bianchi Identity:*
$$RM(X,Y,Z,T) + RM(Y,Z,X,T) + RM(Z,X,Y,T) = 0$$
3. *The Second Bianchi Identity*
$$(\nabla_X RM)(Y,Z,T,W) + (\nabla_Y RM)(Z,X,T,W) + (\nabla_Z RM)(X,Y,T,W) = 0$$

There are several ways we can reformulate the curvature tensor. Due to the symmetries in Proposition A.41, we can write the curvature tensor as a bilinear map on the alternating space $\bigwedge^2 M$.

Definition A.42. The curvature operator is a symmetric bilinear form on
$$\mathcal{R} : \bigwedge^2 M \times \bigwedge^2 M \to \mathbb{R},$$
and is defined by the expression
$$\mathcal{R}(X \wedge Y, Z \wedge W) = RM(X,Y,Z,W).$$

We can also reformulate the curvature tensor in a way that directly generalises Gaussian curvature.

Definition A.43. The *sectional curvature* is given by the expression
$$\sec(X,Y) = \frac{RM(X,Y,Y,X)}{|X \wedge Y|}. \tag{A.15}$$

Sectional curvature is the Gaussian curvature of two planes passing through a point p. Moreover, the sectional curvature at each point on the manifold gives sufficient information to recover the full Riemannian curvature tensor.

The definition of the curvature tensor is quite abstract. Reading off from the definition, the curvature tensor measures the failure of second-order covariant derivatives to commute. The next theorem gives us a more geometric interpretation of the curvature tensor. We first recall the following definition,

Definition A.44. We call a Riemannian manifold (M,g) locally flat if, for every $p \in M$, there is a neighbourhood $p \in \mathcal{U} \subset M$ and an embedding $\varphi : \mathcal{U} \to \mathbb{R}^n$ such that $\varphi^* g_0 = g$.

Theorem A.45. *A Riemannian manifold (M,g) is locally flat if and only if its Riemann tensor vanishes identically.*

Therefore, the Riemannian curvature tensor measures obstruction for finding coordinates charts that are isometrically Euclidean.

The curvature tensor contains all the information regarding the geometry of our manifold. Working with the full tensor is unwieldy, thus it is useful to define weaker but more tractable notions of curvature.

Definition A.46. The Ricci Curvature is defined by

$$\operatorname{Ric}(X,Y) = \operatorname{tr}\left(Z \mapsto R(Z,X)Y\right). \tag{A.16}$$

Remark A.47. The Ricci curvature measures the difference in the volume of the unit ball when compared to the unit ball in Euclidean space.

Definition A.48. The *scalar curvature* is defined by the equation

$$R = \operatorname{tr}_g \operatorname{Ric}(g). \tag{A.17}$$

One of the central goals of Riemannian geometry is to determine how the curvature of the Levi-Civita connection interacts with the distance it induces and the underlying topology. For example, a positive curvature imposes the following geometric restriction on the manifold.

Theorem A.49 (Hopf-Rinow 1931 and Myers 1932)**.** *Suppose (M,g) is complete and satisfies* $\sec \geqslant k > 0$*. Then M is compact and satisfies* $\operatorname{diam}(M,g) \leqslant \frac{\pi}{\sqrt{k}} = \operatorname{diam}(\mathbb{S}^n_k)$*. In particular, M has finite fundamental group.*

Remark A.50. Completeness is a crucial property for this theorem and others of a similar type. Without completeness, the curvature does not seem to impose any restrictions. For example, Gromov showed that any noncompact Riemannian manifold that is not geodesically complete admits metrics that display any range of sectional curvature (any open set on the real line).

The restriction holds with a positive sign on the Ricci curvature.

Theorem A.51 (Myers 1941)**.** *Suppose (M,g) is a complete Riemannian manifold with* $\operatorname{Ric}(g) \geqslant (n-1)kg > 0$*. Then* $\operatorname{diam}(M,g) \leqslant \pi/\sqrt{k}$ *and (M,g) has finite fundamental group.*

Positive curvature may restrict the underlying topology in more subtle ways.

Theorem A.52 (Synge, 1936)**.** *Let M be a compact manifold with* $\sec > 0$.

1. *If M is even-dimensional and orientable, then M is simply connected.*
2. *If M is odd-dimensional, then M is orientable.*

Theorem A.53 (Berger, 1962 and Grove-Shiohama, 1977)**.** *If (M,g) is a closed Riemannian manifold with* $\sec \geqslant 1$ *and* $\operatorname{diam} > \pi/2$*, then M is homeomorphic to a sphere.*

Theorem A.54. *Suppose (M,g) is simply connected of dimension n with* $1 \leqslant \sec \leqslant 4+\varepsilon$.

1. *(Berger, 1983) If n is even, then there is $\varepsilon(n) > 0$ such that M must be homeomorphic to a sphere or diffeomorphic to one of the spaces $\mathbb{CP}^{n/2}, \mathbb{HP}^{n/4}$ $\mathbb{CP}^2$.*

2. *(Abresch-Meyer, 1994) If n is odd, then there is an $\varepsilon > 0$, which can be chosen independently of n, such that M is homeomorphic to a sphere.*

If one relaxes the assumption of the manifold being simply connected, we get the following classification.

Theorem A.55. *(Grove-Gromoll, 1987 Wilking, 2001) Suppose (M, g) is closed and satisfies* $\sec \geqslant 1$, *diam* $\geqslant \frac{\pi}{2}$ *Then one of the following cases holds:*

1. *M is homeomorphic to a sphere.*

2. *M is isometric to a finite quotient $\mathbb{S}^n(1)\backslash\Gamma$, where the action of Γ is reducible (has an invariant subspace).*

3. *M is isometric to one of $\mathbb{CP}^{n/2}, \mathbb{HP}^{n/4}, \mathbb{CP}^{n/2}/\mathbb{Z}_2$ for $n = 2 \bmod 4$.*

4. *M is isometric to $\mathbb{OP}^2$.*

The sphere theorems give a nice description of positively curved closed manifolds. For manifolds with non-negative sectional curvatures, we have the famous soul theorem.

Theorem A.56. *(Cheeger-Gromoll-Meyer, 1969, 1972) If (M, g) is a complete noncompact Riemannian manifold with* $\sec \geqslant 0$, *then M contains a soul $S \subset M$ which is a closed totally convex submanifold, such that M is diffeomorphic to the normal bundle over S. Moreover, when* $\sec > 0$, *the soul is a point and M is diffeomorphic to $\mathbb{R}^n$.*

A.3 Hermitian Geometry

The study of Hermitian geometry concerns the study of complex manifolds, connections and their interaction with an additional structure defined on the tangent bundle called the Hermitian metric. A Hermitian metric is smoothly varying Hermitian inner product on the complex tangent space, which splits into real and imaginary components.

Definition A.57. Let M be a complex manifold of dimension n. A Hermitian metric on TM is given by a positive definite inner product on the holomorphic tangent space at $z \in M$ depending smoothly on z. In local coordinates on M, the Hermitian metric takes the form

$$h = \sum_{i,j} h_{ij} dz_i \otimes d\bar{z}_j$$

where $h_{i,j}$ is a smooth function is z. A Hermitian manifold is a complex manifold equipped with a Hermitian metric (M, h).

The real and imaginary components of the Hermitian metric induce a Riemannian metric and a quadratic form. Using the $\mathbb{R}$-linear isomorphism

$$T_{z,\mathbb{R}}M \to T'_z M,$$

we have

$$\mathrm{Re}h : T_{\mathbb{R},z}(M) \otimes T_{\mathbb{R},z}(M) \to \mathbb{R}$$

which is a Riemannian metric on M. We call this the induced Riemannian metric. The imaginary component of the Hermitian metric defines an alternating quadratic form

$$\mathrm{Im}h : T_{\mathbb{R},p}(M) \otimes T_{\mathbb{R},p}(M) \to \mathbb{R}.$$

We can see this by writing the Hermitian metric explicitly. If $(\varphi_1, ..., \varphi_n)$ is a co-frame for ds^2, we write

$$\varphi_i = \alpha_i + \sqrt{-1}\beta_i$$

where α_i and φ_j are real differential forms.

$$\begin{aligned} h &= \left(\sum \left(\alpha_1 + \sqrt{-1}\beta_i\right)\right) \otimes \left(\sum \left(\alpha_i - \sqrt{-1}\beta_i\right)\right) \\ &= \sum_i \left(\alpha_i \otimes \alpha_i + \beta_i \otimes \beta_i\right) + \sqrt{-1} \sum_j \left(-\alpha_i \otimes \beta_i + \beta_i \otimes \alpha_i\right). \end{aligned}$$

The induced Riemannian metric is given by the formula

$$\mathrm{Re}h = \sum \left(\alpha_i \otimes \alpha_i + \beta_i \otimes \beta_i\right)$$

and the associated $(1,1)$ form of the alternating quadratic form of the Hermitian metric is given by

$$\omega = -\frac{1}{2}\mathrm{Im}h = \sum \alpha_i \wedge \beta_i = \frac{\sqrt{-1}}{2} \sum \varphi_i \wedge \bar{\varphi}_i.$$

It follows from the last representation that given any $(1,1)$ form ω, we can construct a Hermitian form $H(,)$ on each tangent space $T'_z M$. The Wringer theorem highlights the interplay between a Hermitian metric's real and imaginary components.

Theorem A.58. *For a complex submanifolds S (dimension d) of a manifold M with $(1,1)$ form ω, we have*

$$\mathrm{vol}(S) = \frac{1}{d!} \int_S \omega^d.$$

One can define various types of natural connections on Hermitian vector bundles over complex manifolds called Hermitian connections. Given a Hermitian manifold (M, h, J), the natural connection to study is one that is compatible with the complex structure J and Hermitian metric h.

Definition A.59. Let (E, π, M) be a complex vector bundle over a smooth manifold equipped with a Hermitian metric h and a complex structure J. A Hermitian connection is a connection such that

$$\nabla h = 0. \tag{A.18}$$

Given a Hermitian vector bundle, a connection compatible with both the Hermitian and holomorphic structure is called the Chern connection.

Definition A.60. Let (E, h) be a Hermitian vector bundle over a complex manifold M. The Chern connection is a Hermitian connection such that for any smooth sections s of E, $\pi_{0,1}\nabla s = \bar{\partial}_E s$ where $\pi_{0,1}$ takes the $(0, 1)$-component of an E-valued 1-form.

Remark A.61. There exists a unique Chern connection on every Hermitian holomorphic vector bundle (E, h).

On the other hand, one can ask for a connection that preserves the metric and complex structure. However, in general, one has to forfeit zero torsion.

Definition A.62. Let (M, h) be a complex Hermitian manifold. The Bismut connection is the unique connection that satisfies the following conditions,

1. The connection, metric and complex structure satisfy $\nabla g = \nabla J = 0$.
2. The torsion $T(X, Y)$ contracted with the metric, i.e. $T(X, Y, Z) = g(T(X, Y), Z)$, is totally skew-symmetric.

Gaudachon showed that one can generate a line of Hermitian connections in the following way.

$$\left\{\nabla^s := \left(1 - \frac{s}{2}\right)\nabla^c + \frac{s}{2}\nabla^b : s \in \mathbb{R}\right\} \tag{A.19}$$

where $\nabla^c = \nabla^0$ and $\nabla^b = \nabla^2$ are the Chern and Strominger-Bismut connections.

Hermitian manifolds are far less understood than Riemannian manifolds. However, when the Levi-Civita and Chern connection coincides, we gain access to the theorems developed in Riemannian geometry. Such manifolds are called Kähler manifolds, and despite the strong condition, there are many natural examples of these.

A.4 The Embedding Theorems

In this section, we mention some embedding theorems which provide a solid grounding on the abstract definition of manifold. Each embedding theorem states that each type of manifold can represented as a hypersurface in some canonical space $\mathbb{R}^N, \mathbb{C}^N$ and $\mathbb{CP}^N$. However, the dimension N will not coincide with the manifold's dimension and is often a much larger quantity. We can interpret the intrinsic notion of a manifold as a more condensed representation of a higher dimensional hypersurface.

For smooth structures, we have Whitney's embedding theorem.

Theorem A.63 (Whitney Embedding Theorem)**.** *Let M be a smooth manifold of real dimension $n \in \mathbb{N}$. There exists a smooth embedding*

$$\iota : M \hookrightarrow \mathbb{R}^{2n} \tag{A.20}$$

Remark A.64. The dimension of the target space is sharp.

For Riemannian structures, we have the Nash embedding theorem.

Theorem A.65 (Nash embedding theorem)**.** *Let (M, g) be a compact Riemannian manifold. There exists an isometric embedding*

$$\iota : M \to \mathbb{R}^N \tag{A.21}$$

where $\mathbb{R}^N$ is equipped with the standard Euclidean metric.

For compact complex manifolds, no holomorphic embedding exists into $\mathbb{C}^N$. This is a consequence of the maximum modulus principle for holomorphic functions. If we instead consider embedding into complex projective space, then we have the following theorem.

Theorem A.66 (Kodaria's Embedding Theorem)**.** *Let M be a compact complex manifold, and $L \to M$ be a positive line bundle. There exist integers $N, m > 0$ and a holomorphic embedding*

$$\iota : M \to \mathbb{CP}^N \tag{A.22}$$

with $\varphi^*\mathcal{O}_{\mathbb{CP}}(1) = L^{\otimes m}$. Moreover, the converse holds.

A.5 Classification

Now that we have defined topological manifolds and placed them on solid footing via Whitney's embedding theorem, we can now describe a classification theory of manifolds. We start with defining a useful topological concept, the universal cover.

Definition A.67. A universal cover of a connected topological space M is a simply connected space $\widetilde{M}$ with a map $f : \widetilde{M} \to M$ that is a covering map; an open surjective and locally homeomorphic map.

The following proposition dictates the existence of a universal cover.

Proposition A.68. *Let M be a connected, locally simply connected topological space; then, there exists a universal covering $p : \widetilde{M} \to M$.*

Every manifold admits a universal cover. Indeed, every manifold is locally homeomorphic to an open subset of $\mathbb{R}^n$, which implies that the manifold is locally contractible and thus locally path connected and locally simply connected. Furthermore, we can construct manifolds by taking quotients of simply connected spaces (see Proposition A.9). We may be able to generate all closed manifolds using this procedure. This naturally leads to the idea of a classification of closed surfaces through the universal cover.

Definition A.69. A *model geometry* is a simply connected smooth manifold M with a transitive action of a Lie group G. A *geometric structure* on a manifold M is a diffeomorphism from M to M/Γ for some model geometry M, where Γ is a discrete subgroup of G acting freely on M.

It turns out that there are only three model geometries modelling all closed surfaces and eight for 3-manifolds.

Genus	Surface	Model Geometry
0	$\mathbb{S}^2$	Spherical
1	$\mathbb{T}^2$	Euclidean
2	$\mathbb{T}^2 \# \mathbb{T}^2$	Hyperbolic
3	$\mathbb{T}^2 \# \mathbb{T}^2 \# \mathbb{T}^2$	Hyperbolic
$\vdots$	$\vdots$	$\vdots$

Table A.2: classification of closed surfaces.

Uniformisation Theorem

We first recall the following definitions before stating the classification of closed surfaces.

Definition A.70. A connected sum of two m-dimensional manifolds is a manifold formed by deleting a ball inside each manifold and gluing together the resulting boundary spheres.

Definition A.71. Given a triangulation of a closed surface S with vertices V, faces F and edges E the Euler characteristic is

$$\chi(S) = |V| - |E| + |F|. \tag{A.23}$$

Remark A.72. The Euler characteristic is independent of the triangulation.

The Euler characteristic relates to genus by

$$\chi(S) = 2 - 2g. \tag{A.24}$$

Curvature enters the picture through the Gauss-Bonnet theorem. The formula states that

$$\int_S K dS = 2\pi\chi(S) \tag{A.25}$$

where K is the Gaussian curvature for S. If S is modelled on a smooth manifold M with Riemannian metric g, then K coincides with the curvature of (M, g). There are three common constant curvature geometries. These are the spherical geometry with curvature 1, Euclidean with curvature 0 and hyperbolic geometries with curvature -1. The uniformisati on theorem states that these are the only three model geometries needed to describe any closed surface.

Theorem A.73 (Uniformisation Theorem)**.** *Any closed surface admits a geometric structure modelled by the Euclidean, the hyperbolic, or the spherical space.*

These geometries admit Einstein metrics, which are self-similar solutions to the Ricci flow. Running the flow from these metrics, we see that spherical geometries shrink, Euclidean geometries are stationary and hyperbolic geometries expand. In this light, it should not be surprising that the Ricci flow can provide an alternative proof to the uniformisation theorem (Chen-Lu-Tian [CLT06]).

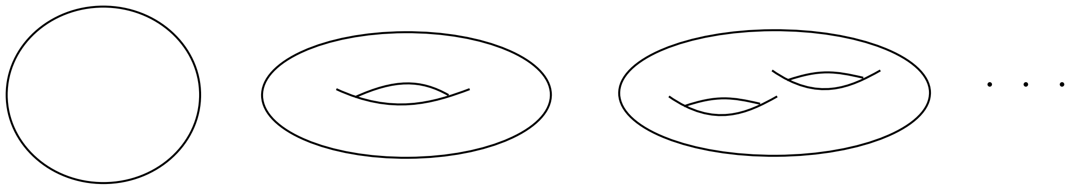

Figure A.3: Surfaces modelled on spherical, flat and hyperbolic spaces.

Geometrisation Theorem

The Geometrization Theorem is the three-dimensional analogue of the uniformisation theorem. A straightforward classification does not exist in two dimensions. However, Thurston proposed that closed 3-manifolds can be canonically decomposed into one of 8 model geometries. There are two stages of this decomposition: prime decomposition and JSJ decomposition.

Definition A.74 (Prime Decomposition)**.** A manifold is prime if it cannot be represented as a connected sum of more than one manifold, none of which is the 3-sphere. A prime decomposition of a manifold is the representation of the manifold as a connected sum of 3-prime manifolds.

Remark A.75*.* Every 3-manifold can be written as a unique connected sum of 3-prime manifolds

Definition A.76 (JSJ Decomposition)**.** Irreducible orientable closed 3-manifolds have a unique (up to isotopy) minimal collection of disjointly embedded incompressible tori such that each component of the 3-manifold obtained by cutting along the tori is either atoroidal or Seifert-fibered.

Applying the prime decomposition followed by a JSJ-decomposition yields complements that admit one of Thurston's eight geometries. These are

$$\mathbb{R}^3, \mathbb{S}^3, \mathbb{H}^3, \mathbb{S}^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R}, \mathrm{Sol}, \mathrm{Nil}, \text{ and } \widetilde{\mathrm{SL}_2(\mathbb{R})}. \tag{A.26}$$

Theorem A.77 (Geometerisation Theorem)**.** *Let M be a compact, orientable, prime 3-manifold. There is a finite collection of disjoint, embedded, incompressible tori in M so that each component of the complement admits a geometric structure modelled on one of the eight Thurston's geometries.*

Three of the Thurston geometries are spaces as the three-dimensional versions of the model geometries in the uniformisation theorem: $\mathbb{R}, \mathbb{S}^3$ and $\mathbb{H}^3$. In addition to these, we have two product model geometries. The remaining three are Lie Groups. Some properties of these spaces are listed in Table A.3. Each space is characterised by the first fundamental group. The groups listed here are a subset of the set of all fundamental groups, and again, we see that the most abundant model space is the hyperbolic space $\mathbb{H}^3$. As in the case of surfaces, the model geometries $\mathbb{R}, \mathbb{S}^3$ and $\mathbb{H}^3$ admit Einstein metrics and therefore are self-similar solutions to the Ricci flow. Under the flow, $\mathbb{R}^3$ is static, $\mathbb{S}^3$ shrinks and $\mathbb{H}^3$ expands. Thus, given an arbitrary initial metric on these spaces, we expected the flow to

Geometry	Metric	Ricci Curvature	$\pi_1(M)$
$\mathbb{R}^3$	$ds^2 = dx^2 + dy^2 + dz^2$	$R_{ij} = 0$	Virtually abelian
$\mathbb{S}^3$	$ds^2 = dx^2 + \sin^2 x dy^2 + (dz + \cos x dy)^2$	$R_{ij} = \frac{1}{2} g_{ij}$	Finite
$\mathbb{H}^3$	$ds^2 = \frac{1}{x^2}\left(dx^2 + dy^2 + dz^2\right)$	$R_{ij} = -2g_{ij}$	Other
$\mathbb{S}^2 \times \mathbb{R}$	$ds^2 = dx^2 + \sin^2 x dy^2 + dz^2$	$R_{11} = 1, R_{22} = \sin^2 x$	Virtually cyclic
$\mathbb{H}^2 \times \mathbb{R}$	$ds^2 = \frac{1}{x^2}\left(dx^2 + dy^2\right) + dz^2$	$R_{11} = R_{22} = -\frac{1}{x^2}$	$N \triangleleft \pi_1(M)$ N cyclic
Sol	$ds^2 = e^{2z} dx^2 + e^{-2z} dy^2 + dz^2$	$R_{33} = -2$	Virtually solvable
Nil	$ds^2 = dx^2 + dy^2 + (dz + x dy)^2$	$R_{11} = -\frac{1}{2}, R_{22} = \frac{x^2-1}{2}$ $R_{33} = \frac{1}{2}, R_{23} = \frac{x}{2}$	Virtually nilpotent
$SL(2, \mathbb{R})$	$ds^2 = \frac{1}{x^2}\left(dx^2 + dy^2\right) + \left(dz + \frac{1}{x} dy\right)^2$	$R_{11} = -\frac{3}{2x^2}, R_{22} = -\frac{1}{x^2}$ $R_{23} = \frac{1}{2x}, R_{33} = \frac{1}{2}$	$N \triangleleft \pi_1(M)$ N cyclic

Table A.3: Thurston's eight model geometries, their Ricci curvature and fundamental groups.

converge to these Einstein metrics. In this way, the role of the Ricci flow in Thurston's geometerisation is clear. Hamilton initiated the use of the Ricci flow to resolve the Poincare conjecture, [Ham82a], which was vital in completing the proof of Theorem A.77. Much progress was made by Hamilton and others. This led to the breakthrough by Perelman [Per02; Per03], who introduced the analysis required to understand the singularity formation of the Ricci flow in three dimensions. Furthermore, he extended the work of Hamilton on the surgery of 3-manifolds, which permitted him to complete the proof of the geometrisation conjecture.

Appendix B

The Maximum Principle

In Riemannian geometry, the underlying manifold is smooth and therefore supports a differentiable structure. In this setting, smooth functions and derivatives are well-defined; therefore, one can write down partial differential equations (PDEs) on the manifold. These equations are important in modelling phenomena on non-Euclidean geometries. Moreover, partial differential equations describe many geometric structures. Many geometric problems can be reduced to finding solutions of partial differential equations.

B.1 Second Order Equations

Definition B.1. A scalar linear partial differential operator of order k is an operator $P : C^\infty(M) \to C^\infty(M)$ of the form

$$Pu = \sum_{|\alpha| \leqslant k} a^\alpha \partial^\alpha u. \tag{B.1}$$

where α is a multi-index and $a^\alpha : M \to \mathbb{R}$.

Definition B.2. The *symbol* $\sigma_P(x, \xi)$ of P at a point $x \in M$ is a polynomial in the cotangent bundle given by

$$\hat{\sigma}_P(x, \xi) = \sum_{|\alpha|=k} a^\alpha \xi_\alpha \tag{B.2}$$

under some local trivialisation.

Second order partial differential equations are classified based on their principle symbol.

Definition B.3. Let (M, g) be a Riemannian manifold and $u : M \to \mathbb{R}$ be a smooth function. An *elliptic equation* is a partial differential equation of the form

$$L[u] = 0 \tag{B.3}$$

where L is a second-order linear differential operator with a positive principle definite principle symbol ($\hat{\sigma}_L(x, \xi) > 0$ for all trivialisations $(x, \xi) \in T^*M$ with $\xi \neq 0$). We call L an *elliptic operator*.

Definition B.4. Let (M, g) be a Riemannian manifold and $u : M \times [0, T] \to \mathbb{R}$ be a smooth function on the manifold over some time interval $[0, T]$. A *parabolic equation* is a partial differential equation of the form

$$\frac{\partial u}{\partial t} = L[u] \tag{B.4}$$

L is a second-order linear differential operator with a positive principle definite principle symbol ($\hat{\sigma}_L(x, \xi) > 0$ for all trivialisations $(x, \xi) \in T^*M$ with $\xi \neq 0$).

Definition B.5. A *weakly parabolic equation* is defined in the same way as parabolic equation, but allow $\hat{\sigma}_L(x, \xi) = 0$ for some $x \in M$ and $\xi \neq 0$.

Example B.6. Let (M, g) be a Riemannian manifold and Δ_g be the laplace-beltrami oprator associated with g. The heat equation

$$\frac{\partial u}{\partial t} = \Delta_g u \tag{B.5}$$

is a parabolic equation on (M, g).

Example B.7. Let (M, g) be a Riemannian manifold with scalar curvature R and Δ_g be the laplace-beltrami oprator associated with g. The backwards heat equation

$$\frac{\partial u}{\partial t} = \Delta_g u + R \tag{B.6}$$

is a parabolic equation on (M, g).

Linear operators exhibit nice behaviours but are rare, especially in geometry. Instead, partial differential equation in geometry tend to be nonlinear.

Definition B.8. A *quasi-linear operator* of order k is an operator $F : C^\infty(M) \to C^\infty(M)$ of the form

$$F\left(x, u, \partial u, \dots \partial^k u\right) = \sum_{|\alpha| = k} a^\alpha \left(x, u, \partial u, \dots, \partial^{k-1} u\right) \partial^\alpha u + f\left(x, u, \partial u, \dots, \partial^{k-1} u\right). \tag{B.7}$$

Remark B.9. Quasilinear operators are therefore linear in its highest derivative of u. If the coefficients also depend on $\partial^k u$, then the operator is called *fully non-linear.*

Example B.10. Let $u : \mathbb{R}^n \to \mathbb{R}$ be a graph over a domain $\Omega \subset \mathbb{R}^n$ and $K(x)$ be some function. Then

$$\det D^2 u - K(\mathbf{x}) \left(1 + |Du|^2\right)^{(n+2)/2} = 0 \tag{B.8}$$

is a second order fully nonlinear equation.

We can still make sense of parabolic and elliptic classifications for nonlinear equations by first linearising them.

Definition B.11. Let F be a nonlinear differential oeprator. Then its linearisation, denoted as L, is defined to be

$$L(v) = (D\tilde{F})(v) = \frac{d}{d\varepsilon} \tilde{F}(x, u + \varepsilon v)\bigg|_{\varepsilon = 0} \tag{B.9}$$

.

Through linearisation, second order nonlinear operators can be categorised.

Definition B.12. Let F be a nonlinear second order operator F and L be its linearisation. The second order equation

$$F(x, u, \partial_x u, \partial_x^2 u) = 0 \tag{B.10}$$

is an elliptic equation if

$$L(u) = 0 \tag{B.11}$$

is elliptic.

Definition B.13. Let F be a nonlinear operator F and L be its linearisation. The second order equation

$$\frac{\partial u}{\partial t} = F(x, u, \partial_x u, \partial_x^2 u) \tag{B.12}$$

is a parabolic equation if

$$\frac{\partial u}{\partial t} = L(u) \tag{B.13}$$

is parabolic.

Often in geometry, we work on vector bundles of the manifold. Indeed, important geometric quantities such as the metric and curvature tensors live in vector bundles. Therefore, we need an analogous notion of parabolic and elliptic equations.

The same definitions can be recycled but rather than coefficients taking values in $C^\infty(M)$, they instead take values in $\mathrm{Hom}(E)$ where $\pi : R \to M$ is a vector bundle over M. A linear operator of order k is therefore an operator $L : \Gamma(E) \to \Gamma(E)$ of the form

$$L(v) = \sum_{|\alpha|<k} L^\alpha \partial^\alpha v \tag{B.14}$$

where $L^\alpha \in \mathrm{Hom}(E)$. Its symbol is given by

$$\sigma_L(x, \xi) = \sum_{|\alpha|\leq k} L^\alpha \xi_\alpha. \tag{B.15}$$

Elliptic and parabolic equations are defined in the same way for scalar functions through the symbol of the operator. A quasi-linear operator takes the form

$$F\left(x, v, \partial v, \ldots \partial^k v\right) = \sum_{|\alpha|=k} A^\alpha \left(x, v, \partial v, \ldots, \partial^{k-1} v\right) \partial^\alpha v + f\left(x, v, \partial v, \ldots, \partial^{k-1} v\right) \tag{B.16}$$

where $A^\alpha : \Gamma(E) \to \Gamma(E)$. Similarly, second-order nonlinear operators can be classified by first linearising the equation and taking the principle symbol.

B.2 Estimates

A wealth of methods and techniques are available to study partial differential equations. Second-order equations contain only first and second derivatives, which lands us immediately in the realm of maximum principle arguments. These arguments are fundamentally based on the observation that the first derivative vanishes at maximum points and the second admits a sign. As a result, maximum principle arguments are particularly effective for second-order equations, since maximum points impose very strong conditions on the equation, and may result in bounds on solutions or the ruling out of non-trivial solutions.

Theorem B.14 (Maximum Principle)**.** *Let (M, g) be a compact Riemannian manifold with boundary and $u \in C^\infty(M)$ such that*

$$L(u) \geqslant 0 \quad in \quad M. \tag{B.17}$$

Then

$$\max_M u = \max_{\partial M} u. \tag{B.18}$$

Theorem B.15 (Weak Maximum Principle)**.** *Suppose that (M, g) is a compact Riemannian manifold with boundary ∂M and $u \in C^\infty(M \times (0, T])$ satisfies*

$$\left(\frac{\partial}{\partial t} - L\right) u \leqslant 0 \qquad in \qquad M \times (0, T] \tag{B.19}$$

where L is an elliptic operator. Then

$$\max_{M \times [0,T]} u = \max_{\Gamma_T} u$$

where $\Gamma_T = (\partial M \times [0, T]) \cup (M \times \{0\})$.

Theorem B.16 (Strong maximum principle)**.** *Suppose that (M, g) is a connected Riemannian manifold with boundary ∂M and $u \in C^\infty(M \times (0, T])$ satisfies*

$$\left(\frac{\partial}{\partial t} - L\right) u \leqslant 0 \qquad in \qquad M \times (0, T] \tag{B.20}$$

where L is an elliptic operator. If u attains a non-negative maximum over $M \times [0, T]$ at a point $(x_0, t_0) \in (M \backslash \partial M) \times (0, T]$, then u is constant on $(M \backslash \partial M) \times (0, t_0]$.

Theorem B.17 (Comparison Maximum Principle)**.** *Suppose that (M, g) is a closed Riemannian manifold and $u \in C^\infty(M \times (0, T])$ is a subsolution to the parabolic equation, that is*

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - L\right) u \leqslant f(u, t) \qquad in \qquad M \times (0, T] \\ \qquad u(\cdot, 0) \leqslant C \in \mathbb{R} \end{cases} \tag{B.21}$$

where L is an elliptic operator and $f : \mathbb{R} \times [0, T] \to \mathbb{R}$ is a smooth function. Suppose further that $\phi : [0, T] \to \mathbb{R}$ solves the following ODE:

$$\begin{cases} \dfrac{d\phi}{dt} = f(\phi(t), t) \\ \phi(0) = C \end{cases}$$

Then, for all $t \in [0,T]$, *we have*

$$u(\cdot,t) \leqslant \phi(t)$$

Swapping the inequalities and looking at minima points in the maximum principle gives us similar results for minima.

Theorem B.18 (Minimum Principle)**.** *The maximum principle also for minimum points if we replace* $\geqslant$ *with* $\leqslant$ *in the previous three theorems.*

For vector bundle values PDEs, we need Hamilton's extension of the maximum principle.

Theorem B.19 (Hamilton's maximum principle)**.** *Let* $t \mapsto (M, g(t))$ *be a smooth flow of compact Riemannian manifolds on a time interval* $[0,T]$*. Let* V *be a vector bundle over* M *with connection* ∇*, and let* $u : [0,T] \mapsto \Gamma(V)$ *be a smoothly varying family of sections that obeys the nonlinear PDE*

$$\frac{d}{dt}u(t,x) = \nabla^\alpha \nabla_\alpha u + F(t,x,u)$$

where for each $(t,x) \in [0,T] \times M$,

$$F(t,x) : V_x \to V_x \tag{B.22}$$

is a locally Lipschitz function (using the metric on V_x *induced by* g*) which is continuous in* t, x *with uniformly bounded Lipschitz constant in the 1-neighbourhood (say) of* K_x*. For each time* $t \in [0,T]$*, let* $K(t) \subset V$ *a closed fibrewise convex parallel set varying continuously in* t*. We assume that* K *is preserved by* F *in the sense that for each* $(t,x) \in [0,T) \times M$ *and each boundary point* $v \in \partial K_x(t) \subset V_x$*, the spacetime vector* $(1, F(t,x,v)) \in \mathbb{R} \times V_x$ *is an inward or tangential vector to the spacetime body*

$$K_x := \{(t',v') : t' \in [0,T], v' \in K_x(t')\}$$

at the boundary point (t, v)*. Suppose also that* $u(0,x) \in K_x(0)$ *for all* $x \in M$*. Then* $u(t,x) \in K_x(t)$ *for all* $(t,x) \in [0,T] \times M$*.*

To end this section, we mention some inequalities in Kähler geometry that are crucial in setting up maximum principle arguments. These are also known as Schwartz lemmas and allow one to calculate the laplacian of the metric tensor.

Proposition B.20 (Aubin-Yau)**.** *Let* ω, ω' *be two Kähler forms on a complex manifold* X*. If the holomorphic bisectional curvature of* ω *is bounded below by a constant* $B \in \mathbb{R}$ *on* X*, then*

$$\Delta_{\omega'} \log \operatorname{tr}_\omega(\omega') \geqslant -\frac{\operatorname{tr}_\omega \operatorname{Ric}(\omega')}{\operatorname{tr}_\omega(\omega')} + B \operatorname{tr}_{\omega'}(\omega).$$

Remark B.21. For this inequality to be valid, we need a lower bound on bisectional curvature for ω and an upper bound on Ricci curvature for ω'.

Proposition B.22 (Chern-Lu)**.** *Let* ω' *and* ω *be Kähler metrics. Then*

$$\Delta_\omega \log \operatorname{tr}_\omega \omega' \geqslant \frac{1}{\operatorname{tr}_\omega \omega'} \left(g^{k\bar{\ell}} g^{p\bar{q}} \operatorname{Ric}(\omega)_{k\bar{q}} (g')_{p\bar{\ell}} - g^{k\bar{\ell}} g^{p\bar{q}} (\operatorname{Rm}(\omega'))_{k\bar{\ell}p\bar{q}} \right) \tag{B.23}$$

B.3 Existence and Regularity for Second Order Linear Equations

Second order elliptic and parabolic equations have nice existence and regularity properties due to the estimates available for these equations. Consider the following equation.

$$\begin{cases} \dfrac{\partial u}{\partial t} = L(u) \quad \text{in} \quad M \times (0, T] \\ u(\cdot, 0) = 0 \end{cases} \tag{B.24}$$

A classical result due to Aubin and Hamilton (for manifolds with boundary) guarantees the existence and uniqueness of smooth solutions to (B.24).

Theorem B.23. *There exists a unique and smooth solution to* (B.24).

For quasilinear parabolic PDEs, we only have short time existence. Suppose we have the following equation.

$$\begin{cases} \dfrac{\partial u}{\partial t} = F\left(x, t, u, \partial_x u, \partial_x^2 u\right) \\ u(\cdot, 0) = u_0(x) \end{cases} \tag{B.25}$$

Theorem B.24 (Short time existence)**.** *Suppose that $u_0 \in C^\infty(M)$, then there exists $\varepsilon > 0$ such that* (B.25) *has a unique smooth solution $u(x,t)$ for $(x,t) \in M \times [0, \varepsilon]$.*

This is quite informative when working with the Ricci flow. The evolution of various geometric quantities are parabolic but non-linear, thus reflecting the fact that the flow exists for short times but in the long run, singularities often develop.

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