

THE UNIVERSITY OF SYDNEY

DOCTORAL THESIS

Transient Chaos in Dissipative Chaotic Scattering

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Applied Mathematics
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Statement of Originality

This is to certify that to the best of my knowledge, the content of this thesis is my own work. This thesis has not been submitted for any degree or other purposes.

I certify that the intellectual content of this thesis is the product of my own work and that all the assistance received in preparing this thesis and sources have been acknowledged.

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Date: January 8, 2024

Declaration of Authorship

This thesis reflects work I, Lachlan BURTON, undertook whilst studying for a PhD degree at the University of Sydney between 2019 and 2023. I declare that this thesis titled, “Transient Chaos in Dissipative Chaotic Scattering” and the work presented in it are my own. I confirm that:

- This work was done wholly while in candidature for a Doctor of Philosophy (PhD) degree at the University of Sydney.
- Where I have consulted the published work of others, or made use of the results therein, this is always clearly attributed.
- I have acknowledged all main sources of help and collaboration.
- Parts of this thesis, appearing mostly in Chapters 5 and 6, have been submitted for publication as “*Probabilistic description of dissipative chaotic scattering*”. A pre-print is available on ArXiv and is cited as [9]. A repository of the Julia code used in the paper is available on GitHub [10]. I contributed to formulating the research project, was the main author on the manuscript, performed all of the numerical experiments and created all of the figures.

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As supervisor for the candidature upon which this thesis is based, I can confirm that the authorship attribution statements above are correct.

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Eduardo G. ALTMANN

Date: January 8, 2024

"Man errs, till he has ceased to strive."

- The Lord; *Faust* - Part One

Abstract

Transient Chaos in Dissipative Chaotic Scattering

by Lachlan BURTON

For many physical systems, it is infeasible to study the underlying physical processes experimentally. This can often be attributed to the experimental inaccessibility of the relevant time and/or length scales. As an example, one of the foundational problems in classical physics concerns the long-term stability of the solar system, whose evolution takes place over billions of years. These issues are exacerbated when the motion is chaotic. In such cases, chaotic transients are the only observations which are available. Dynamical systems theory can offer many insights to such problems through both analytical and computational methods, although much of the existing work concerns idealized scenarios which ignore the kind of perturbations one may observe in reality, such as noise or dissipative effects. Such effects can alter the qualitative behaviour of these chaotic systems in non-trivial and unexpected ways.

In this thesis we consider the influence of dissipative effects on dynamical systems whose unperturbed dynamics are already transiently chaotic. The class of systems we study have been used to model a wide range of physical processes, from celestial-mechanical motions to hydrodynamic flows, which are susceptible to such perturbations. We use existing theory on transient chaos and powerful numerical methods to gain a comprehensive understanding of the dynamics of these scattering systems in both conservative and dissipative settings, in terms of macroscopic quantities like the survival rate of trajectories in particular phase space regions, or the fractal scaling of basin boundaries between final states. We then propose a general framework, inspired by earlier work on pullback attractors, for understanding the qualitative differences and general phenomenology visible in the numerical results for the dissipative system. In particular, we find that features related to conditionally-invariant measures of the unperturbed system can be used to explain the observations of the dissipative system, even when the dynamics of each case are substantially different.

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For my parents.

Chapter 1

Introduction

1.1 Chaos

Deterministic chaos is by now a very well-studied phenomenon. The first evidence of what we now understand as chaotic motion appeared during Poincaré’s famous studies [57] of the three body problem in the late 19th century. The increase in capabilities of scientific computing in the second half of the 20th century revealed that chaotic systems are ubiquitous across many scientific disciplines, and investigations into this phenomenon have only grown in scope and depth since then. It is difficult to point to a universally accepted definition of chaos, but it is usually associated [53] with positive Lyapunov exponents, which describe the expansion (positive exponents) and contraction (negative exponents) characteristics of chaotic sets, and also with positive topological entropy, which is often taken as a quantitative measure of dynamical complexity. Dynamical systems with these properties tend to exhibit exponential separation of nearby trajectories, which is a hallmark of chaotic dynamics. In this thesis when we refer to a dynamical system as *chaotic* we are generally referring to systems with at least one positive Lyapunov exponent. Related to chaotic dynamics is the concept of *hyperbolicity*, which implies a global splitting of the dynamics in phase space into stable and unstable directions, which is described by the notion of stable and unstable manifolds associated to each point in a *hyperbolic set*. Many computations and theoretical formulations rely on the assumption of hyperbolic dynamics, including the *shadowing* [31, 62] property, which justifies considering numerically computed trajectories as representative of the ‘true’ dynamics in spite of the accumulating error. However, many chaotic systems (e.g. generic Hamiltonian systems [53, 71]) are not hyperbolic. In Chapter 3 we discuss the relevance of both settings (hyperbolic and non-hyperbolic) for our purposes.

1.2 Transient chaos

In recent years, a particular type of chaotic behavior known as *transient chaos* has received attention in the literature as it demonstrates remarkable properties not typically seen in other chaotic systems, and finds unique applications in many scientific settings [70, 4]. Transient chaos refers to chaotic dynamics that manifests for finite times only; it is a non-equilibrium state that cannot be understood from the asymptotic behaviour. Trajectories may move around in a complicated way for some transient time before eventually settling to a steady state which may be simple or periodic, or even to a chaotic attractor for which ergodic averages on the transient¹ and the attractor need not agree [71]. In open Hamiltonian systems, such as those we study in this thesis, chaos can only appear in the form of transients as typical orbits will leave the system asymptotically. Transient chaos gives a rich connection between the dynamics of the system and the geometry of certain non-attracting invariant sets in phase space. These *chaotic saddles* are fractal objects of measure zero made up of unstable orbits which can temporarily influence nearby trajectories and are thus responsible for the transient behaviour. Statistical features of the transient behaviour, such as the escape rate of trajectories from any neighbourhood of the saddle, are properties of the saddle itself.

More recently [48, 14, 40], attention to undriven dissipative systems has led to the discovery of a transiently chaotic behavior – termed *doubly-transient chaos* – that is significantly distinct from the usual transient chaos observed in non-dissipative or Hamiltonian systems. In these systems the monotonic decay of energy implies that *all* motion, including the orbits on the chaotic saddles, must eventually cease. It is proposed instead to view the chaotic invariant sets responsible for transient chaos as transients themselves. These systems display peculiar properties such as time-dependent settling rates to attractors as well as asymptotically non-positive Lyapunov exponents [48] and topological entropy [40]. A natural question is to what extent such systems can even be considered ‘chaotic’. The recent literature has presented some exciting results on the topic of doubly-transient chaos, but there are still unanswered questions. For instance, the foundational work [48] on doubly-transient chaos concerned a mechanical system which is permanently chaotic when dissipation is turned off. Open systems with extant transient chaos properties in the absence of dissipation are not obviously comparable to this scenario, and there are many such physical systems, e.g the chaotic motion of astronomical bodies like the

¹These would necessarily be finite-time averages, but the lengths of chaotic transients can be arbitrarily long [71].

three body problem, where dissipative effects are believed to be relevant. One of the motivations of the work in this thesis is to investigate transient chaos in this class of system, for which we use the term *dissipative chaotic scattering*, and to apply or extend the theory of doubly-transient chaos to its description.

1.3 Chaotic scattering

Chaotic scattering is a prevalent phenomenon in open dynamical systems which describes a wide range of physical phenomena, from the aforementioned complex astrophysical motions to hydrodynamical flows and the interactions of charged particles [2, 50, 71, 63]. Such problems are often concerned with the motion of a point particle inside a potential V which is either zero or dynamically negligible outside of some bounded region of interest, the *scattering region*. The motion of the particle outside the region is then trivial (e.g a straight line). The particle, usually incident to the scattering region from outside, interacts with the potential for some time before eventually escaping again and moving off to infinity [54, 21, 35]. This process can also be observed in billiard systems, which involve free particles interacting with obstacles through elastic collisions. In this thesis we use the term ‘chaotic scattering’ to refer to the dynamics of an open chaotic system in a phase space region from which typical trajectories will escape (and not return) after a finite time.

This definition is very general and encompasses both scattering in the usual sense [54, 63] of a potential $V(r)$ that goes to zero with $r \rightarrow \infty$ and also systems where the scattering process is artificially defined, e.g. in terms of a leak [4] or escape of the particle from a specific subset in a naturally open system [65, 64]. The two systems we study in Chapter 5, the Hénon-Heiles [33] system and the Copenhagen problem [49] have been previously studied in this context [50, 78, 65, 64, 77]. Chaotic scattering is well-understood as a phenomenon of transient chaos [71] which explains how hyperbolic dynamics leads to exponential decay of trajectories from the vicinity of a chaotic set inside the scattering region, and non-hyperbolic scattering leads to algebraic decay due to the stickiness around KAM islands.

A prototypical system that also fits this description is the *Hénon map* [32], a map in two variables (x, y) with two parameters a and b . We will use this in Chapter 3 to demonstrate some of the relevant phenomenology. The map is defined by

$$\begin{aligned} x_{n+1} &= 1 - ax_n^2 + y_n \\ y_{n+1} &= bx_n. \end{aligned} \tag{1.1}$$

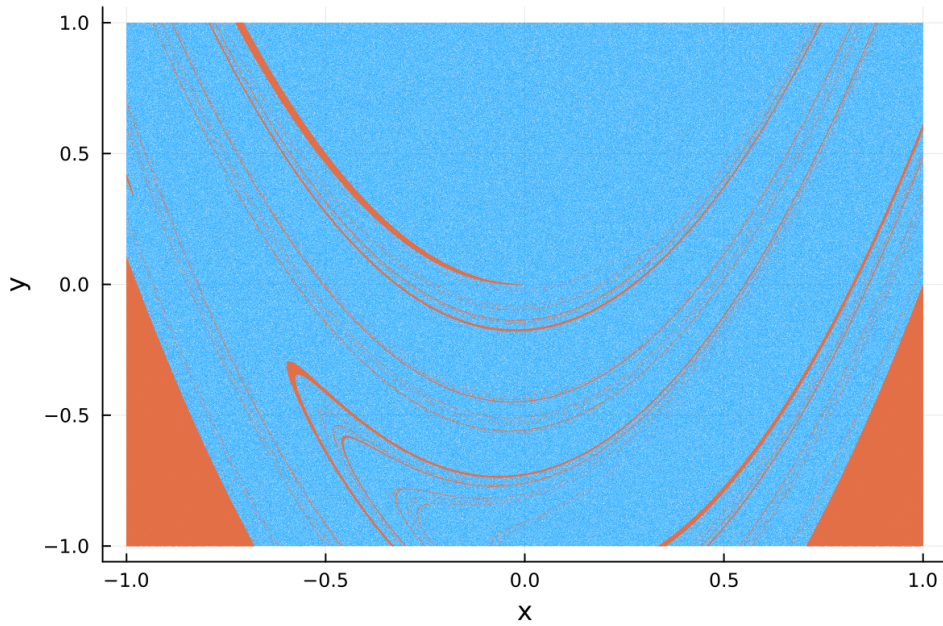


FIGURE 1.1: Coloration of initial conditions in the Hénon map with $(a, b) = (2.0, 0.3)$ according to which direction, left (orange) or right (blue) the trajectory first exited the region $[-1, 1] \times [-1, 1]$.

A generic problem in scattering systems is to relate some quantity associated with a specific initial condition to another quantity associated to its final state. Such an example of a *scattering function* that we will compute several times in this thesis relates initial conditions of the system to their escape times from a bounded region. Another common choice of "output" variable is the angle θ at which the trajectory leaves the scattering region [54], or through which exit in the case of multiple exits, which can be used to construct "exit basins", analogous to basins of attraction, for different trajectory outcomes. These exit basins can display similar fractal properties to basins of attraction [50, 77, 63]. An example for the Hénon map is given as Fig. 1.1. The scattering function here is mapping initial conditions to the sign of the x coordinate when the trajectory leaves the initial square². The interface between initial conditions whose trajectories exit to the left (the orange points) and the right (blue points) suggest a fractal structure, and this is indeed related to the set of initial conditions that remain trapped in the scattering region. We construct these basins for the Hénon-Heiles system and Copenhagen problem as part of our numerical explorations in Chapter 5.

²For the chosen parameter values, there is no attractor in the square and typical points will be mapped out of the square after a short time.

1.4 Dissipative chaotic scattering

Recent studies have focused on the effect of perturbations on the general chaotic scattering picture, with the goal of expanding its applicability to other physical situations [63]. For instance, it has been shown that the effect of white noise leads to enhanced trapping and intermediate-time regimes in the survival probability that depend on the noise strength [63, 11]. Another important type of perturbation, on which we focus in this thesis, is the effect of dissipative forces [65, 63] which appear not only in mechanical systems but also in astrophysical problems (due e.g to frictional passage of the particle through proto-planetary nebulae, or to radiation pressure-like effects, such as the Poynting-Roberston drag [12]) and in advection of particles in fluids (inertial effects [47]).

Dissipation has a strong effect on chaotic scattering as it can lead to new asymptotic dynamics (e.g., attractors) that are not possible in the Hamiltonian setting and it necessarily alters the escape rate (or distribution of trajectory lifetimes) as monotonic energy decay due to dissipation implies that the average speed of any particle is decaying. It also non-trivially affects non-hyperbolic dynamics: dissipation added to an area-preserving map has the effect of stabilising periodic orbits inside the KAM islands or destroying them entirely [47]; dissipation added to a mechanical system in which the energy decayed to a unique minimum [46] led to piecewise-adiabatic invariance of the action integrals, which manifested as complicated tube-like structures in the extended phase space. In any case, the destruction of the KAM islands, the phase space objects responsible for stickiness [53], causes the decay law of the survival probability to lose its algebraic tail. The fractal properties of scattering systems is also strongly affected by dissipation, with the appearance of different scaling regimes at different strengths of the dissipation [47, 65, 63].

1.5 Main contributions

For weak dissipations, it may be natural to expect that the dynamics of the perturbed and unperturbed systems are still “close”, in some sense. We seek to answer the question of whether the transient dynamics in the dissipative system can be understood based on the properties of the unperturbed system, and if so, how and to which extent? This is the main problem addressed in the thesis. The problem is addressed by first performing comprehensive numerical studies of two systems, both of which are motivated by celestial mechanics. We consider each system with

and without dissipation, and confirm even small dissipation causes non-trivial effects on dynamical quantities such as survival probability/escape rate, and fractal dimensions of basin boundaries. We then argue that these differences can be largely understood as the dissipative system evolving along snapshots of the conservative system, and that at any instant the distribution of trajectories in the dissipative system can be quantified by the associated conditionally-invariant measures of the conservative system. Ultimately, we show that from this probabilistic description, our numerical observations of the survival probability/instantaneous escape rate can be recovered. As such, it provides an explanation for the observed effects of dissipation in chaotic scattering systems.

1.6 About this thesis

The thesis is divided as follows:

- In Chapter 2 we revise the essential background material on chaotic Hamiltonian systems.
- In Chapter 3 we discuss the transient chaos description of open chaotic systems, which gives context to our problem of chaotic scattering. We also review existing works on transient chaos in dissipative systems, including both in dissipative chaotic scattering and also in mechanical systems.
- In Chapter 4 we discuss our choices of numerical integration scheme for our experiments in both the conservative and dissipative systems.
- In Chapter 5 the Hénon-Heiles and Copenhagen problems are introduced and framed as scattering systems, and we present some numerical experiments to understand the dynamics and transient chaos phenomenology with and without dissipation. This includes computations of the survival probability, fractal dimension of exit basin boundaries, and approximations of the chaotic sets.
- In Chapter 6 we propose a novel probabilistic description of dissipative chaotic scattering systems to explain these observations, based on the evolving spatio-temporal distribution of trajectories and expressed in terms of the conditionally-invariant measures of the non-dissipative system. The predictions of this description are verified through additional experiments.
- In Chapter 7 we summarise the main results and provide an outlook for future work.

Chapter 2

Hamiltonian Systems

The subject of this thesis concerns open dissipative chaotic dynamical systems that are Hamiltonian in the dissipation-free limit. In the wider context of dynamical systems, Hamiltonian systems form a somewhat special subset. A major appeal of the Hamiltonian formalism is that Hamiltonian mechanics naturally describes many properties of physical systems, such as symmetries and conservation laws.

We will eventually introduce dissipation as small additive terms to the equations of motion of the Hamiltonian systems, and so we begin with a summary of the important details of Hamiltonian mechanics that can be found in most texts on classical mechanics or dynamical systems such as [53, 51, 66]. These concepts will eventually be explored in the Hénon–Heiles system [33] and the planar circular-restricted three body problem, to which we will return several times in this thesis.

2.1 Basic Definitions & Properties

Definition 2.1.1 (Hamiltonian system). A *Hamiltonian dynamical system* is characterized by a smooth scalar function H , called the *Hamiltonian*, which is obtained from another scalar function, a *Lagrangian* L , by a Legendre transform:

$$H(\mathbf{p}, \mathbf{q}, t) = \mathbf{p} \cdot \dot{\mathbf{q}} - L(\mathbf{q}, \dot{\mathbf{q}}, t). \quad (2.1)$$

The Lagrangian is given by the difference between the kinetic (T) and potential (V) energies of the system¹, as expressed in terms of generalized position and velocity coordinates $\mathbf{q}$ and $\dot{\mathbf{q}}$ and time t ². The Legendre transform maps the variables $(\mathbf{q}, \dot{\mathbf{q}})$ to new conjugate variables $(\mathbf{q}, \mathbf{p})$. The generalized momentum variables $\mathbf{p}$ are related to the velocity variables by

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}}. \quad (2.2)$$

¹The definition of the Lagrangian is more general, but this is suitable for our purposes.

²The bold letters indicate $\mathbf{q}$, and $\mathbf{p}$ are N -dimensional vectors.

The dynamics of the system is given by *Hamilton's Equations*:

$$\begin{aligned}\dot{\mathbf{q}} &= \frac{\partial H}{\partial \mathbf{p}} \\ \dot{\mathbf{p}} &= -\frac{\partial H}{\partial \mathbf{q}}.\end{aligned}\tag{2.3}$$

Equation (2.3) is a set of $2N$ first-order differential equations, which are the equations of motion of the system. When the Hamiltonian has no explicit time dependence, the system is said to be *autonomous*. N is the number of *degrees of freedom*. The equations of motion describe a flow in pairs of conjugate variables (q_i, p_i) . The solutions to Equation (2.3) are trajectories which live in a $2N$ -dimensional space Ω called the *phase space*.

Hamilton's equations can be stated more succinctly as

$$\dot{\mathbf{z}} = J \nabla H(\mathbf{z}, t), \quad J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \tag{2.4}$$

where $\mathbf{z} = [\mathbf{q}, \mathbf{p}]^T \in \Omega$, I_n is an $n \times n$ identity matrix and J is called the symplectic matrix. Hamiltonian systems possess a certain kind of symmetric structure; they are known as *symplectic*, which implies³ the following (somewhat informally-stated) result known as Liouville's Theorem:

Theorem 1 (Liouville's Theorem). *Let $A \subseteq \Omega$ be a measurable set in phase space, and let ϕ^t denote the flow generated by Hamilton's equations. The volume $V(A)$ is conserved by the flow, i.e.:*

$$V(A) = V(\phi^t(A)) \text{ for all } t \geq 0. \tag{2.5}$$

The volume measure V on phase space is called the *Liouville measure*, and is defined with respect to a symplectic manifold $\mathcal{M}_{2n}$ with associated symplectic form $d\omega$. For more details see [6].

By evaluating the flow map of a Hamiltonian system at discrete time points, one is led to a discretized analogue of the continuous-time dynamics: a *symplectic map*, a map whose Jacobian is a symplectic matrix.

Definition 2.1.2. A real $2n \times 2n$ matrix B is *symplectic* if

$$B^T J B = J. \tag{2.6}$$

³Symplectic is actually a stronger condition than volume-preserving, but this is one of the most important consequences.

Symplectic matrices have determinant 1, and are invertible with inverse $B^{-1} = J^{-1}B^T J$. Symplectic maps are a useful tool for studying Hamiltonian dynamics, both as simplified settings for more general phenomena (e.g twist maps) and as the basis of numerical schemes to model continuous-time systems (symplectic integrators.) The symplectic character of Hamiltonian systems implies some other peculiar properties, such as the non-existence of attractors, and that non-zero Lyapunov exponents occur in pairs of opposite sign.

Invariance under Hamilton's equations (i.e, along a trajectory in phase space) can also apply to functions of the phase space variables: if we have a function $G(\mathbf{p}, \mathbf{q}, t)$ then its rate of change along a trajectory is, by the chain rule:

$$\begin{aligned} \frac{dG}{dt} &= \frac{\partial G}{\partial t} + \frac{\partial G}{\partial \mathbf{q}} \cdot \frac{\partial \mathbf{q}}{\partial t} + \frac{\partial G}{\partial \mathbf{p}} \cdot \frac{\partial \mathbf{p}}{\partial t} \\ &= \frac{\partial G}{\partial t} + \frac{\partial G}{\partial \mathbf{q}} \cdot \frac{\partial H}{\partial \mathbf{p}} - \frac{\partial G}{\partial \mathbf{p}} \cdot \frac{\partial H}{\partial \mathbf{q}} \\ &= \frac{\partial G}{\partial t} + \{G, H\}. \end{aligned} \tag{2.7}$$

The term $\{G, H\}$ is called the *Poisson Bracket*.

Definition 2.1.3 (Poisson Bracket). For two functions $F(\mathbf{p}, \mathbf{q}, t)$ and $G(\mathbf{p}, \mathbf{q}, t)$, the Poisson bracket is

$$\{F, G\} = \frac{\partial F}{\partial \mathbf{q}} \cdot \frac{\partial G}{\partial \mathbf{p}} - \frac{\partial G}{\partial \mathbf{p}} \cdot \frac{\partial F}{\partial \mathbf{q}}. \tag{2.8}$$

Definition 2.1.4 (First Integral). When there is no explicit time dependence ($\frac{\partial G}{\partial t} = 0$), then G is a *first integral* (or *invariant* or *constant of the motion*) if and only if $\{G, H\} = 0$.

The existence of first integrals imposes constraints on how trajectories move through phase space. If there are M first integrals, then the solutions to Hamilton's equations are constrained to a $2N - M$ -dimensional manifold. If the system is autonomous (or can be made autonomous through an appropriate coordinate change)⁴, then the Hamiltonian itself is a first integral. As the Hamiltonian represents the total energy of the system, this is expressing conservation of energy.

A naturally related concept to constants of motion is that of *invariant measures*, which describe distributions of points that are stationary under the dynamics:

Definition 2.1.5 (Invariant Measure). A measure $\nu : X \rightarrow \mathbb{R}$ is said to be *invariant* under the flow ϕ^t if $\nu(\phi^t(A)) = \nu(A)$ for all measurable sets $A \subseteq X$ and $t \in \mathbb{R}$. For maps $\mathcal{M}$, the invariance condition is $\nu(A) = \nu(\mathcal{M}^{-1}(A))$.

⁴This involves introducing a new degree of freedom (a fictitious time variable) against which the old time becomes a coordinate with its own conjugate momentum. The resulting system will be autonomous with respect to the new time, but the "energy" will not be conserved.

Definition 2.1.6 (Ergodic Measure). An invariant measure ν is *ergodic* if any invariant set A has full or zero measure, i.e $\nu(A) = 0$ or $\nu(A^c) = 0$.

An equivalent statement is that ν is ergodic if the state space cannot be decomposed into $k > 1$ invariant components.

2.2 Integrable Systems and Canonical Transformations

Given a Hamiltonian $H(\mathbf{q}, \mathbf{p})$, it can often be advantageous to study the dynamics in a different coordinate system. The coordinate transform has to be chosen in a particular way in order to preserve the symplectic structure. The mapping from $H(\mathbf{q}, \mathbf{p})$ to a new Hamiltonian $K(\mathbf{Q}, \mathbf{P})$, through the transformations $Q = Q(\mathbf{q}, \mathbf{p}, t)$ and $P = P(\mathbf{q}, \mathbf{p}, t)$ is called a *canonical transformation*. The canonical transformation is defined by a generating function $G : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}$, which takes as inputs combinations of the old and new generalized coordinates and possibly time. There are four kinds of generating functions. We summarize them, and the canonical transformations they induce, in Table 2.1.

Generating Function	Canonical Transformation
$G_1(\mathbf{q}, \mathbf{Q}, t)$	$\mathbf{p} = \frac{\partial G_1}{\partial \mathbf{q}}$ and $\mathbf{P} = -\frac{\partial G_1}{\partial \mathbf{Q}}$
$G_2(\mathbf{q}, \mathbf{P}, t)$	$\mathbf{p} = \frac{\partial G_2}{\partial \mathbf{q}}$ and $\mathbf{Q} = -\frac{\partial G_2}{\partial \mathbf{P}}$
$G_3(\mathbf{p}, \mathbf{Q}, t)$	$\mathbf{q} = \frac{\partial G_3}{\partial \mathbf{p}}$ and $\mathbf{P} = -\frac{\partial G_3}{\partial \mathbf{Q}}$
$G_4(\mathbf{p}, \mathbf{P}, t)$	$\mathbf{q} = \frac{\partial G_4}{\partial \mathbf{p}}$ and $\mathbf{Q} = -\frac{\partial G_4}{\partial \mathbf{P}}$

TABLE 2.1: The four kinds of generating function G_i and their associated canonical transformation.

The relationships between the new and old coordinates are obtained from the canonical transformations. The new and old Hamiltonians are related by

$$K(\mathbf{Q}, \mathbf{P}, t) = H(\mathbf{q}, \mathbf{p}) + \frac{\partial G}{\partial t}. \quad (2.9)$$

Definition 2.2.1 (Integrable System). A Hamiltonian system with N degrees of freedom is said to be *completely integrable* if:

1. the system has N independent first integrals $[G_1, G_2, \dots, G_N]$, one of which can be taken to be the Hamiltonian itself;
2. the Poisson brackets $\{G_i, G_j\}$ vanish for all i, j .

For such systems it is possible (in principle) to find a canonical transformation $(\mathbf{p}, \mathbf{q}) \mapsto (\mathbf{I}, \boldsymbol{\theta})$ for which trajectories trace out orbits on the surface of invariant tori

of dimension N . These tori are spanned by N periodic angle variables. Explicitly, Hamilton's equations are transformed to

$$\begin{aligned}\dot{\mathbf{I}} &= -\frac{\partial H}{\partial \boldsymbol{\theta}} = 0 \\ \dot{\boldsymbol{\theta}} &= \frac{\partial H}{\partial \mathbf{I}} = \boldsymbol{\omega}(\mathbf{I}),\end{aligned}\tag{2.10}$$

where $\boldsymbol{\omega} = [\omega_1, \omega_2, \dots, \omega_N]$ are the frequencies of the rotations. The variables $(\mathbf{I}, \boldsymbol{\theta})$ are called *action-angle* variables. Note that Equation (2.10) implies that the Hamiltonian is a function of the actions only. The result guaranteeing the existence of this canonical transformation, is known as the *Liouville-Arnold Theorem*. For integrable Hamiltonians in action angle coordinates the equations of motion can be trivially solved:

$$\begin{aligned}\mathbf{I} &= \text{constant.} \\ \boldsymbol{\theta} &= \boldsymbol{\omega}t + \boldsymbol{\theta}_0.\end{aligned}\tag{2.11}$$

Completely integrable systems display simple regular motion: trajectories are either periodic or quasiperiodic (depending upon the rational/irrational relations of the frequencies, the latter scenario is generic.) Examples include rigid-body rotations, two-body planetary systems and the simple harmonic oscillator.

2.3 Poincaré Maps

The dimension of phase space in Hamiltonian systems is equal to twice the number of degrees of freedom. Consequently, it very quickly becomes difficult to analyze the dynamics on the full phase space for systems with more than a few degrees of freedom. A convenient tool to reduce the dimension of the studied problem is the *Poincaré map* (or *first return map*), which gives the intersection of an orbit in the full phase space with a lower-dimensional hypersurface (called a *Poincaré surface of section*).

Definition 2.3.1 (Poincaré map). Let Φ^t be the flow of an n -dimensional dynamical system on a phase space Ω , and let $\mathcal{S} \subset \Omega$ be an $(n - 1)$ -dimensional surface. A diffeomorphism $\mathcal{P} : \mathcal{S} \rightarrow \mathcal{S}$ is a Poincaré map if for every $\vec{x} \in \mathcal{S}$, $\exists t' > 0$ such that $\Phi^{t'}(\vec{x}) \in \mathcal{S}$, $\Phi^t(\vec{x}) \notin \mathcal{S} \forall 0 \leq t < t'$ and

$$\mathcal{P}(\vec{x}) = \Phi^{t'}(\vec{x}).\tag{2.12}$$

The continuous dynamics are thus reduced to a discrete map from intersection to intersection. It was proven by Poincaré that the return map preserves the properties of the continuous dynamics, and so studying the return map can yield insights on the original system if the section is chosen well. As an example, periodic orbits can be difficult to identify in the continuous dynamics, but will intersect the section $\mathcal{S}$ at only finitely-many points. Quasi-periodic orbits in two degrees of freedom will densely fill out closed curves in the section, and chaotic orbits will present as a diffuse cloud. In autonomous systems, the Hamiltonian is fixed, and hence $\mathcal{S}$ is of dimension $N - 2$ in this case. The section condition $\vec{x} \in \mathcal{S}$ is usually defined by a function $f(\vec{x})$ that passes through zero at the point of crossing. As this can happen two ways ($(+ \rightarrow 0 \rightarrow -)$ and $(- \rightarrow 0 \rightarrow +)$), it is common to seek only intersections of the same orientation.

In our experiments in Chapter 6 we utilise a modified version of a Poincaré map, called a *true time map*, that also tracks the time between crossings of the Poincaré section in the continuous system. Many of our computations in Chapters 5 and 6 also involve choosing initial conditions from a particular Poincaré section.

2.4 Near-integrable Systems and KAM theory

Chaos emerges in Hamiltonian systems only when the dynamics are not completely integrable. The natural starting point is to consider systems that are “close” to integrable, in that they have been perturbed slightly in a way that introduces irregular motion that scales with perturbation size. These perturbations can violate integrability by destroying first integrals. We will briefly discuss the case for systems with two degrees of freedom.

Consider a perturbation⁵ of an integrable Hamiltonian H_0 by another term H_1 , where the perturbation strength is controlled by a small parameter ε :

$$H(\mathbf{p}, \mathbf{q}) = H_0(\mathbf{p}, \mathbf{q}) + \varepsilon H_1(\mathbf{p}, \mathbf{q}). \quad (2.13)$$

The question first studied by Poincaré [57] and finally answered by Kolmogorov, Arnold and Moser (KAM) [53], is when it should be expected that this perturbed system also admits invariant tori. In essence, this *KAM theory* shows that “most” of the invariant tori in H_0 survive the perturbation, albeit deformed, and the motion is still confined to a two-dimensional subset of phase space. Which tori survive the

⁵The perturbation here is different from the perturbations (dissipation) we consider later - these perturbed systems here are still Hamiltonian.

perturbation is determined by a surprising number-theoretic criterion. The stability of an invariant torus against the perturbation is decided by the irrationality of the ratio of the frequencies ω_1, ω_2 of the motion on the torus. Explicitly, the tori whose frequencies satisfy the Diophantine condition

$$\left| \frac{\omega_1}{\omega_2} - \frac{r}{s} \right| > \frac{K(\varepsilon)}{s^{\frac{5}{2}}} \quad (2.14)$$

for arbitrary integers r, s with $\gcd(r, s) = 1$ still exist in the perturbed system. The term $K(\varepsilon)$ is a constant depending only on ε and goes to zero with $\varepsilon \rightarrow 0$. The set of frequency ratios satisfying Eq. (2.14) for increasing ε are those which are "badly approximable" by rational numbers, a statement well-understood in terms of continued fractions. For more details, see e.g [22, 53], but the gist is that the more closely the frequency ratio of a torus can be approximated by a rational number, the smaller the perturbation size ε required to destroy that torus. Notably excluded by the stability criterion (2.14) are the tori with rational frequency ratios, corresponding to periodic orbits. These are the so-called *resonant* tori, which are destroyed by perturbations of any size. Additionally, any ratio in the interval

$$\left[\frac{r}{s} - \frac{K(\varepsilon)}{s^{\frac{5}{2}}}, \frac{r}{s} + \frac{K(\varepsilon)}{s^{\frac{5}{2}}} \right] \quad (2.15)$$

cannot not satisfy (2.14). The density of the rationals in $\mathbb{R}$ may suggest that (2.14) is only satisfied by null-sets. However, the set of tori that survive has positive measure. To see this, consider the set of frequency ratios lying in $[0, 1]$. Around each rational number r/s , delete an interval of width $K(\varepsilon)/s^{\frac{5}{2}}$. For $s = 1, 2, \dots$, there are s such intervals. The total length deleted is then

$$\begin{aligned} 2 \sum_{s=1}^{\infty} \frac{K(\varepsilon)}{s^{\frac{5}{2}}} s &= 2K(\varepsilon) \sum_{s=1}^{\infty} \frac{1}{s^{\frac{3}{2}}} \\ &= 2K(\varepsilon) \zeta(3/2), \end{aligned} \quad (2.16)$$

where $\zeta(s)$ is the *Riemann zeta function* and $\zeta(3/2) \approx 2.6124$. The sum thus goes to zero with $\varepsilon \rightarrow 0$ and so the phase space volume not foliated by invariant tori can be made arbitrarily small⁶.

⁶Curiously, the original statement of the KAM theorem required the perturbation size ε be extremely small, although this has since been strengthened, and there is plenty of numerical evidence that the result applies for much larger ε than originally considered.

2.5 Chaos in Hamiltonian Systems

KAM theory does not say anything about the manner in which the resonant and nearly-rational tori disintegrate, only that they do. Consider a Poincaré surface of section which transversely cuts an invariant torus with rational frequency ratio r/s . A single orbit on this section would appear as s fixed points along the smooth curve. The *Poincaré-Birkhoff Theorem* [53] states that, under a small perturbation, an even number of fixed points on the torus survive. Moreover, half of them are stable ('elliptic') and the other half are unstable ('hyperbolic'). In the Poincaré section, the points are organised into *island chains*, with the elliptic points appearing at the center of families of invariant tori, with the resulting *KAM islands* inter-spaced with the hyperbolic points. The rational tori in these families are also destroyed in the same way, generating *ad-infinitum* a self-similar structure of island chains at smaller and smaller scales [53].

The dynamics around each hyperbolic point is very complicated, and is related to so-called homoclinic and heteroclinic tangles formed by the infinite intersections of the stable and unstable manifolds of the hyperbolic points. As written by Poincaré [57]:

"The intersections form a kind of lattice, web or network with infinitely tight loops; neither of the two curves must ever intersect itself but it must bend in such a complex fashion that it intersects all the loops of the network infinitely many times. One is struck by the complexity of this figure which I am not even attempting to draw. Nothing can give us a better idea of the complexity... of all the problems in dynamics..."

The first chaotic orbits appear in the neighbourhoods of these objects. As they are not constrained to lie on tori, these orbits tend to fill the available space in the section between surviving tori in a *chaotic sea*. As ε is increased, more and more of the tori will disintegrate and the surviving *noble tori*, associated with the 'most irrational' frequency ratios will form isolated islands of stability in the chaotic sea. For large enough ε , there are no surviving tori, and the chaotic sea forms a single ergodic component.

2.6 The Hénon-Heiles Hamiltonian

To demonstrate the relevant phenomenology of chaotic Hamiltonian dynamics, we introduce the Hénon-Heiles system [33], a Hamiltonian system serving as a toy model of stellar motion in a 2D galactic potential that shows many of the generic

properties discussed in this chapter. The original intent in [33] was to study the existence of additional integrals of motion in models of galactic dynamics. It was found that regular motion occurs only for a subset of initial conditions (now understood to correspond to KAM islands) whereas the rest of phase space is filled by chaotic orbits. The Hamiltonian for a unit mass is given by

$$H = \frac{1}{2}(p_x^2 + p_y^2) + V(x, y) = E \quad (2.17)$$

where the potential function $V(x, y)$ with parameters ω and ε is

$$V(x, y) = \frac{\omega^2}{2}(x^2 + y^2) + \varepsilon \left(x^2 y - \frac{1}{3} y^3 \right), \quad (2.18)$$

which combines a harmonic potential with cubic terms that create non-integrable dynamics. Through Hamilton's equations, we are led to a four dimensional ODE:

$$\dot{x} = \frac{\partial H}{\partial p_x} = p_x, \quad (2.19a)$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = p_y, \quad (2.19b)$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = -\omega^2 x - 2\varepsilon xy, \quad (2.19c)$$

$$\dot{p}_y = -\frac{\partial H}{\partial y} = -\omega^2 y - \varepsilon(x^2 - y^2). \quad (2.19d)$$

The system is often studied in a perturbative setting (i.e, for $\varepsilon \ll 1$) to demonstrate how chaos emerges as the harmonic terms in the potential become less dominant. In this sense it is a near-integrable system of the type in Eq. (2.13). An alternative which we follow and that produces the same observations is to fix $\varepsilon = 1$ and instead increase the value of the energy E from 0. For very small E the harmonic terms dominate and the motion is regular. As E increases, the trajectories begin to explore the regions of the potential dominated by the cubic terms. We set $\omega = 1$. A contour plot of the potential for $\varepsilon = 1$ is given in Fig. 2.1. The contours contain closed curves for $E \leq \frac{1}{6}$, which restricts the motion to a bounded region around the origin.

The phase space is four-dimensional, but the energy conservation restricts the motion to three-dimensional energy surfaces. To reduce the dimensionality further, we will study two different Poincaré sections of the Hénon-Heiles system. The first of them, denoted $\mathcal{S}_1$, is given simply by the condition $x = 0$. The energy integral is then reduced to

$$E = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}y^2 - \frac{1}{3}y^3, \quad (2.20)$$

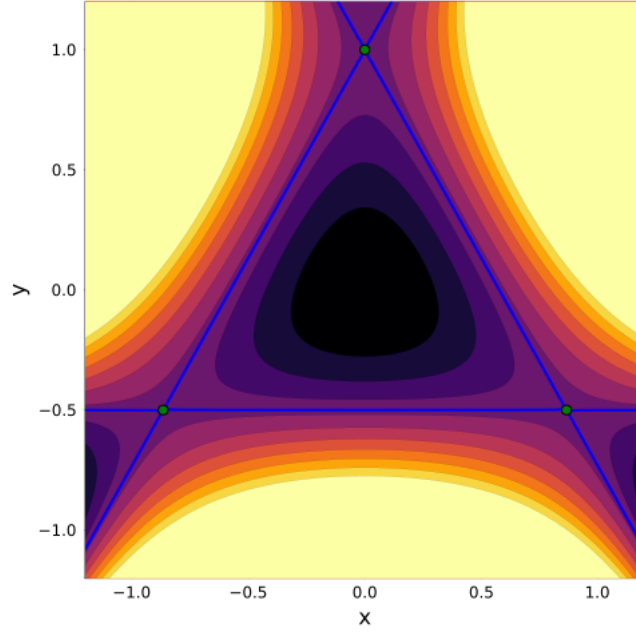


FIGURE 2.1: Contours in the Hénon–Heiles potential (2.18) for 20 values of the energy $E \in (0.0, 0.5)$. The contour corresponding to $E = \frac{1}{6}$ is shown in blue. The saddle points of the potential, are indicated by green dots. Darker colours correspond to smaller values of E .

which we use to eliminate the p_x variable. We consider crossing of the section with $p_x > 0$. The dynamics is thus reduced from a continuous system in a four-dimensional space spanned by (x, y, p_x, p_y) to a discrete map in two dimensions (y, p_y) and parameter E that nonetheless preserves the features of the continuous dynamics. The second choice of Poincaré section $\mathcal{S}_2$ is given in radial coordinates:

$$\mathcal{S}_2 = \{(x, y) \in \mathbb{R}^2 \mid \dot{r} = 0, \dot{\phi} > 0\}, \quad (2.21)$$

where $r = \sqrt{x^2 + y^2}$, $x = r \cos \phi$ and $y = r \sin \phi$. The section condition is $\dot{r} = 0$. It has two disjoint components corresponding to $\dot{\phi} < 0$ and $\dot{\phi} > 0$, from which we choose the latter. The purpose of this choice of section is that it allows one to specify the initial configuration (position) variables (x_0, y_0) freely, an advantage in the chaotic scattering problems we study in a later chapter. The initial velocities $\dot{x}_0$ and $\dot{y}_0$ are chosen to match the value of the Hamiltonian. In Cartesian coordinates they are determined by

$$\begin{aligned} \dot{x}_0 &= -\frac{y_0}{\sqrt{x_0^2 + y_0^2}} \sqrt{2(E - V(x_0, y_0))} \\ \dot{y}_0 &= \frac{x_0}{\sqrt{x_0^2 + y_0^2}} \sqrt{2(E - V(x_0, y_0))}. \end{aligned} \quad (2.22)$$

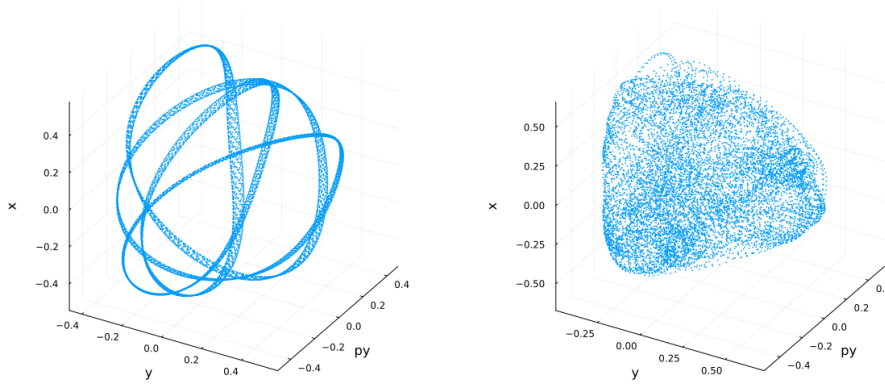


FIGURE 2.2: Quasiperiodic (left) and chaotic (right) orbits in the Hénon-Heiles system (2.19) for $E = 0.14$ in (x, y, p_y) space. The trajectories are integrated until $t = 1000$ and are plotted at increments of $\Delta t = 0.1$.

In Fig. 2.2, two example orbits – one quasiperiodic and one chaotic – are visualized in (x, y, p_y) space. In Fig. 2.3 we show Poincaré maps in the section S_1 for increasing values of E . The destruction of most of the invariant tori and breakdown of integrability is readily apparent. Also visible (up to finite spatial scale) for the higher energy is the qualitative picture described by the Poincaré-Birkhoff theorem: the central elliptic fixed points of the largest KAM islands are surrounded by an alternating set of elliptic and hyperbolic fixed points, with successively smaller island chains forming around the elliptic points.

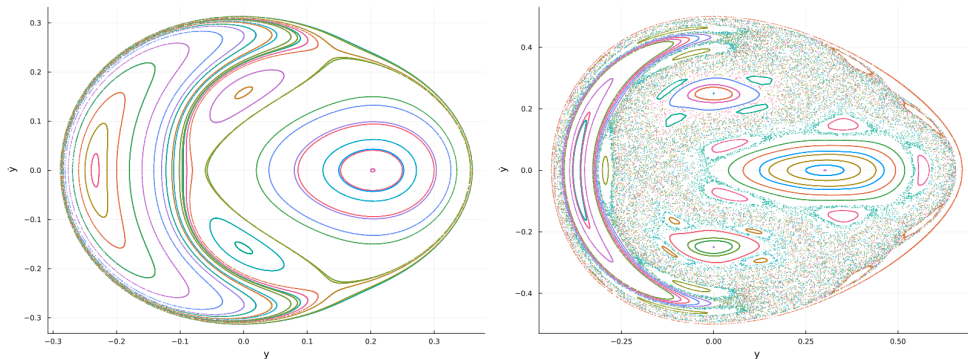


FIGURE 2.3: Regular and chaotic orbits in the Hénon-Heiles system – (2.18) and (2.19) – in the Poincaré section S_1 . **(Left):** $E = 0.05$. The plotted orbits are all quasi-periodic. **(Right):** $E = 0.125$, for which most tori have been destroyed. Island chains of quasiperiodic motion are visibly embedded in a chaotic sea.

Chapter 3

Transient Chaos

3.1 Transient Chaos

There are many examples of dynamical systems, particularly those motivated by the study of physical, chemical or biological processes, which demonstrate qualitatively different behaviors across different timescales. Singularly-perturbed (“*slow-fast*”) systems [74] feature scale dependencies in their equations of motion. Electrical power systems disturbed beyond their normal operating tolerances can fluctuate dramatically before suffering voltage collapse [71, 15] or multistable synchronization behavior [30]. The gravitational interactions of $n > 2$ celestial bodies can exhibit rich scattering dynamics on short timescales alongside stable quasi-periodic motion [50] that persists asymptotically. The main takeaways from these examples is that the dynamics on the finite and asymptotic timescales need not agree, and that the transient behaviour can be just as interesting and important as the asymptotic. The latter two examples are both instances of *transient chaos*: a fascinating phenomenon in which systems display a temporarily chaotic evolution before abruptly changing their behaviour, to either a stable or periodic asymptotic state or even to a different chaotic dynamics. There is a rich body of theory of transient chaos describing the dynamics of open chaotic systems. Comprehensive and modern reviews of the subject can be found in [71, 4]. We begin this chapter by reviewing some of the fundamental results of transient chaos and how it pertains to such systems, along the way giving illustrations in the examples of the Hénon map and Hénon-Heiles system.

3.1.1 Basic concept

In dynamical systems contexts, *transient* refers to the temporal evolution of a trajectory that precedes its asymptotic state. As noted in [71], transients are often more

relevant than asymptotic dynamics for the purposes of observation, modelling, prediction and control of systems. Such purposes are critical in applications of dynamical systems arising in real-world contexts from the sciences, social sciences and economics. A robust understanding of transient behaviour is therefore paramount.

Chaos is also often conceptualized as an asymptotic phenomenon that only manifests after long times. In numerical experiments, it is common to discard a long initial component of a trajectory in order to focus on the asymptotic dynamics. Additionally, indicators of chaos such as Lyapunov exponents and fractal dimensions are usually defined in the $t \rightarrow \infty$ limit. Such definitions are inadequate to quantitatively assess complex transient dynamics. The finite lifetime of transient behaviour thus requires an inherent notion of timescale separation. If t_0 denotes the “characteristic time” of the system (e.g the average return time to a Poincare section, the discrete timestep, etc) and T_0 the maximum time a system is observable (e.g maximum integration time in computer simulations, length of experimental runtime etc) then the notion of a *lifetime* τ for which observations of the system ‘appear’ random or chaotic before their behaviour qualitatively changes requires $t_0 \ll \tau < T_0$. Under such a characterization, permanent (asymptotic) chaos is recovered by taking $\tau, T_0 \rightarrow \infty$.

For demonstration, we consider again the Hénon map (1.1). We will use the Hénon map several times to illustrate most of the phenomena of transient chaos. Depending on the values of the parameters a and b , the map can display sustained (permanent) or transient chaos. To illustrate this, first consider the bifurcation diagram obtained by fixing $b = 0.3$ and varying the parameter a , shown in Fig. 3.1. There is a notable window between $a = 1.2$ and $a = 1.3$ where the map supports a limit cycle of period 7, and chaos at other intermediate values. We select a candidate value from each regime ($a = 1.25, 1.4$ respectively) and plot the x_n component of a trajectory for each.

The trajectory for $a = 1.25$ clearly converges to a periodic attractor, but its transient behaviour is just as, if not *more* irregular than that of the chaotic trajectory. The length of this transient is also sensitive to the choice of initial condition: an arbitrarily small perturbation may lead to dramatic changes in the transient lifetime. This naturally leads one to consider a *distribution* of trajectories lifetimes, which ultimately describes a probability $P(t)$ of finding a trajectory with a lifetime longer than t , called the *survival probability*. The irregularity of the transient in the limit cycle orbit suggests that the mechanism responsible for the transient motion is altogether different from a strange attractor. This is a key observation: the transient behaviour

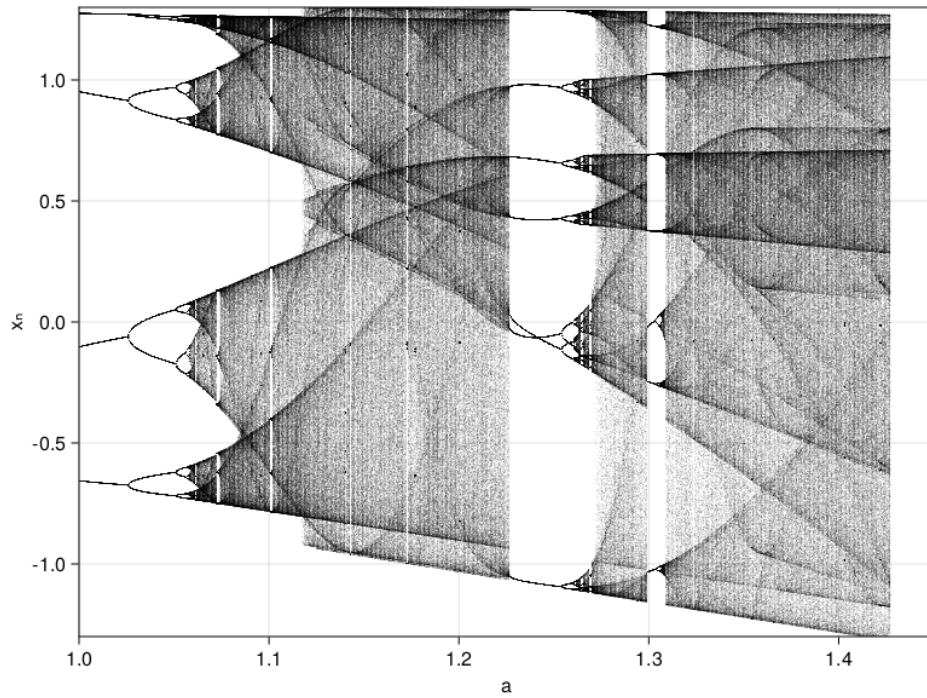


FIGURE 3.1: Bifurcation diagram in a for the Hénon map with $b = 0.3$

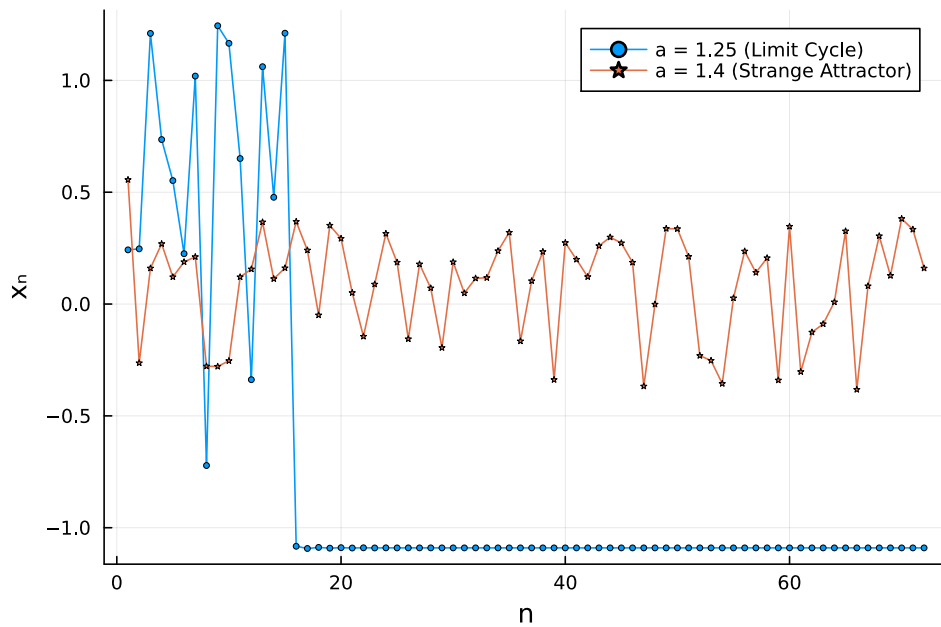


FIGURE 3.2: Chaotic and limit cycle orbits of length 500 in the Hénon map (see legend) for the x_n coordinate. To show the periodicity, only every 7th iterate is shown (the value of n shown on the x -axis is 1/7th of the actual timestep.) $b = 0.3$ in both cases.

can be chaotic even if the asymptotic dynamics are not, and the asymptotic dynamics can also be chaotic but demonstrably different than the transient dynamics.

The principle idea is that there exists a *non-attracting* fractal set in the phase space which is responsible for the behaviour of chaotic transients. Just as strange attractors are the driver of chaotic motion for asymptotic times, these so called *chaotic saddles* (for invertible dynamics) or *repellers* (resp. non-invertible) are the phase space objects underlying transient chaos. A key consequence is that properties of the transient behaviour are universal across most choices of initial conditions. The following working definition of transient chaos is thus offered in [71]: *transient chaos is the form of chaos due to non-attracting chaotic sets in phase space.*

These sets have Lebesgue measure zero and are comprised of unstable orbits with *infinite* transient lifetime. The transient is a result of their (temporary) influence on nearby trajectories: in the case of a chaotic saddle, an orbit may approach the saddle along the saddle's stable manifold and eventually be repelled along the unstable manifold. This is a surprising fact - the existence of a non-attracting set of measure zero is observable in a finite-time observation.

The structure of this object as described gives clues as to its potential visualisation: if a large number of initial conditions are seeded in the vicinity of the saddle, then after sufficient time¹ τ_* , the points which remain in the vicinity will be distributed along the unstable manifold. The initial conditions of these orbits must have been distributed close to the stable manifold. At approximately $\tau_*/2$ is when the trajectories should have been in the vicinity of the saddle itself. This procedure is called the *sprinkler method* [71], and the results for the Hénon map are shown in Fig. 3.3. For the parameter values chosen, (Lebesgue) almost every trajectory diverges (escapes any bounded region in finite time.) The asymptotic dynamics can then be regarded as having an attractor at infinity. There are many existing methods to numerically approximate non-attracting chaotic sets, but the sprinkler method generally suffices for our purposes throughout.

When the dynamics is not invertible, the chaotic set is repulsive in every direction and thus possesses only unstable manifolds. An example of such a system is an iterated quadratic map $z_n \mapsto z_n^2 + c$, in which the repeller object forms a Julia set. In this thesis we are generally considering systems described by differential equations, for which existence and uniqueness results guarantee invertibility of the dynamics. Thus, henceforth when we refer to nonattracting chaotic sets, we are talking about a chaotic saddle.

¹Choosing τ_* to be a small multiple of the average lifetime τ usually suffices.

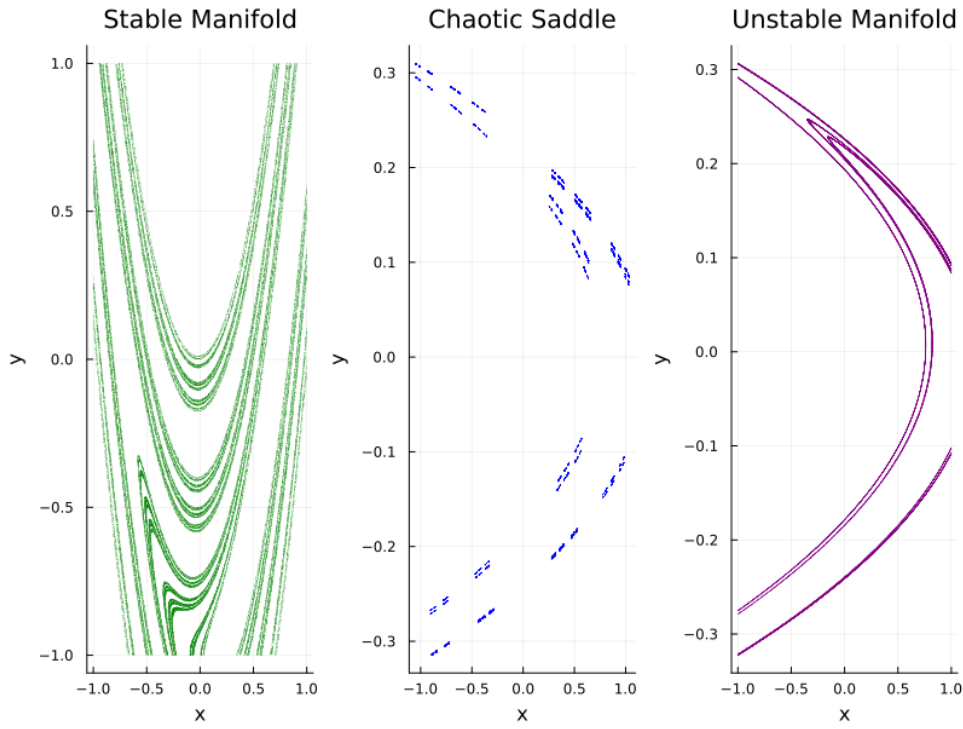


FIGURE 3.3: Stable manifold, chaotic saddle and unstable manifold in the Hénon map for $(a, b) = (2.0, 0.3)$. All three objects have Lebesgue measure zero, but the stable and unstable manifolds have a recognizable filamentary structure: they are expressible as the direct product of a Cantor set with a smooth curve. The saddle itself, as the intersection of the two manifolds, is structured as a Cantor dust. These figures were produced via the sprinkler method [71] by seeding initial conditions into the region $\Gamma = [-1, 1] \times [-1, 1]$ and iterating each trajectory until $\tau_* = 16$.

In the following section we build up the relevant definitions to characterise transient chaos. We generally assume hyperbolicity in the dynamics for now, but discuss how these concepts pertain to non-hyperbolic dynamics in Sec. 3.4.

3.2 Transient chaos in open systems

The preceding discussion was somewhat agnostic about exactly what dynamical behaviour could be described as ‘transient’. Here we give some clear context. Assume that a hole or *leak* $\mathcal{I}$ of positive Lebesgue measure is opened in the phase space Ω of a closed, chaotic dynamical system through which trajectories are allowed to “leave”. Under ergodicity assumptions, almost every trajectory from the chaotic component of Ω will intersect the leak in finite time. The dynamics of these trajectories before they encounter the leak can then be thought of as transient chaos. The introduction of a hole or leak to an otherwise closed system can have an intrinsic physical meaning, or even be a tool to study the chaotic properties of the closed system.

If the dynamics of a trajectory $\vec{x}_0, \vec{x}_1, \dots$ in the closed system is described by a map $\mathcal{M} : \Omega \rightarrow \Omega$, then for the leaking system we have

$$\vec{x}_{n+1} = \begin{cases} \mathcal{M}(\vec{x}_n) & \vec{x}_n \notin \mathcal{I}, \\ \vec{x}_n & \vec{x}_n \in \mathcal{I}, \end{cases} \quad (3.1)$$

so that the leak can be thought of as an invariant component of phase space. It is then possible to define the *escape time* t_{esc} of a trajectory with an initial point $\vec{x}$:

$$t_{esc}(\vec{x}) := \begin{cases} \min\{t > 0 | M^t(\vec{x}) \in \mathcal{I}\}, & \vec{x} \notin \mathcal{I}, \\ 0, & \vec{x} \in \mathcal{I}, \end{cases} \quad (3.2)$$

a definition which extends naturally to flows.

It is important to distinguish leakage, that is, removal or termination of individual trajectories upon meeting some dynamical condition, from dissipation, which implies phase space contraction. One can often exercise control over some properties of the leak, such as its size or position, without fundamentally altering the dynamics from those of the closed system. However, this is not always the case: in some open systems, such as Hénon-Heiles, the position and sizes of the leaks is a property of the potential, and adjusting the controlling parameter (the energy) has dynamical consequences. More generally, many systems are naturally open in that the dynamics are not bounded to any particular region and artificially opening a leak

is unnecessary. We have already seen an example of this in Fig. 3.3: when trajectories exit the set Γ they quickly go to infinity. In this case the "leak" $\mathcal{I}$ as we consider it is the complement of Γ in $\mathbb{R}^2$. The important point is that the 'interesting' dynamics of a system may take place in a bounded region of the phase space, so that trajectories cease to be relevant once they have left this region. This is different from the more intuitive notion of a closed system being opened with a small hole, but most of the general phenomenology of transient chaos still applies. In this thesis we describe this type of behaviour as *chaotic scattering*².

3.2.1 Escape Rate and Survival Probability

It is natural to think geometrically about the collections of points which remain inside Ω for long times. To wit, we define a families of sets $\{\mathcal{S}_t\}_{t \in \mathbb{R}}$ with

$$\mathcal{S}_t = \{\vec{x} \in \Omega | t_{esc}(\vec{x}) \geq t\}. \quad (3.3)$$

In the limit as $t \rightarrow \infty$, we obtain the *forward-invariant* set $\mathcal{S}_\infty$ of points which never leave Ω under the dynamics. We can analogously obtain the *backward-invariant* set $\mathcal{S}_{-\infty}$ by applying the time-reversed dynamics to $\vec{x}$. Points $\vec{x}$ belonging to both sets ($\vec{x} \in \mathcal{S}_{-\infty} \cap \mathcal{S}_\infty$) comprise a subset of Ω invariant in both forward and reverse time. It is this set which forms the chaotic saddle, with the forward-invariant and backward-invariant sets $\mathcal{S}_\infty$ and $\mathcal{S}_{-\infty}$ forming the stable and unstable manifolds respectively.

Since the chaotic saddle has measure zero, a typical (randomly seeded) trajectory will eventually move away from a given neighbourhood of the saddle. As discussed, it is difficult to predict how quickly this occurs for a single trajectory, but one can describe the process statistically with an *escape rate* κ :

Definition 3.2.1 (Escape Rate). Suppose $N_0 \gg 1$ initial conditions are distributed according to a density ρ_0 on a phase-space region R . Let $\Gamma \subseteq R$ ³ be a bounded, compact region which supports the chaotic saddle (i.e., $\mathcal{S}_\infty \subset \Omega$). Let $N(t)$ denote the number of trajectories still inside Γ after time t . The escape rate κ is defined as

$$\kappa := \lim_{t \rightarrow \infty} \frac{1}{t} \log \left(\frac{N_0}{N(t)} \right) \quad (3.4)$$

²Historically, scattering has specifically referred to a process by which trajectories enter Ω from *outside* and then leave again after some time. Here we are talking about starting the particles inside. The difference is relevant mostly in non-hyperbolic systems for reasons we discuss in a later section.

³In scattering systems, R often won't contain the saddle itself, but will at least intersect its stable manifold (the forward-invariant set).

In spite of what the definition suggests, κ is *independent* of the choices of ρ_0 , R , and Γ . It is purely a property of the chaotic set. We remark that it is somewhat unsatisfying to present *another* asymptotic definition in light of the preceding discussion, but the convergence is usually rapid and visible after a short time t_0 so that

$$N(t) \approx N(t_0)e^{-\kappa(t-t_0)} \text{ for } t > t_0. \quad (3.5)$$

The time t_0 and pre-factor $N(t_0)$ ⁴ however, are dependent on ρ_0 , R and Γ .

Definition 3.2.2 (Survival Probability). The survival probability is equal to the probability that a randomly chosen point $\vec{x} \in \Omega$ belongs to $\mathcal{S}_t$. When the space can be normalized under Lebesgue measure μ_L so that $\mu_L(R) = 1$, this is equal to the Lebesgue measure of $\mathcal{S}_t$:

$$P(t) = \mu_L(\mathcal{S}_t). \quad (3.6)$$

If the dynamics is ergodic with respect to μ_L , then $P(t) \rightarrow 0$ almost surely as $t \rightarrow \infty$. The volume of $\mathcal{S}_t$ is well-approximated by the ratio $N(t)/N_0$, and so in light of Eqs. (3.5) and (3.6) it is reasonable to write

$$P(t) \approx P(t_0)e^{-\kappa t}, \text{ for } t > t_0. \quad (3.7)$$

The Hénon map displays this exponential decay, as shown in Fig. 3.4.

The mean lifetime τ is defined from the survival probability as:

$$\tau := \langle t \rangle = \int_0^\infty tP(t)dt. \quad (3.8)$$

For maps, in the case of exponential decay one has the following estimate:

$$\tau \approx \frac{1}{\kappa}. \quad (3.9)$$

3.2.2 Sensitive dependence of transient lifetimes

The escape rate quantifies how quickly trajectories move from the chaotic set. It does not, however, describe the chaoticity of the motion while still inside Γ . Indeed, the transient need not be chaotic at all, as is the case for trajectories which asymptotically approach an attractor (regular or strange) but do not come close to any other chaotic set on the way. The question arises of how to classify a transient as chaotic other than by inspection as in Fig. 3.2. One option to consider Lyapunov exponents computed

⁴For most choices of ρ_0 , the decay will not be exponential immediately from $t = 0$, so $N(t_0)$ differs from N_0 .

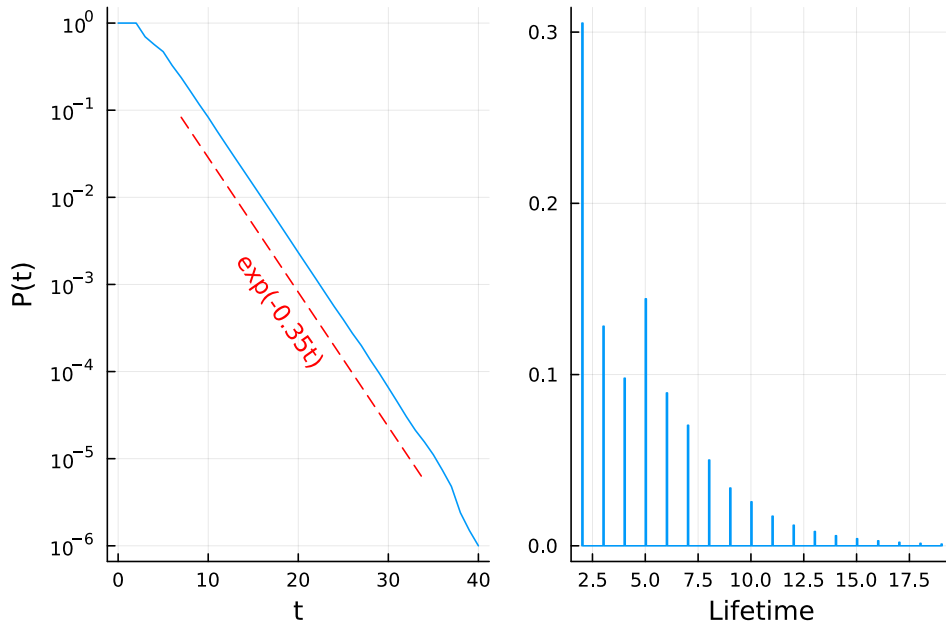


FIGURE 3.4: Survival Probability in the Hénon map for $(a, b) = (2.0, 0.3)$. **(Left):** Survival probability (blue curve) which displays an exponential decay after $t_0 \approx 8$. An escape rate of $\kappa \approx 0.35$ was determined, and the orange dashed line shows the fit to this decay law. **(Right):** Histogram of trajectory lifetimes. $N_0 = 10^7$ initial conditions were used, chosen uniformly in the same region Γ as in Fig. 3.3.

on finite time intervals [67], for which the positivity of at least one finite-time Lyapunov exponent is necessary. An alternative is to examine the sensitive dependence of transient lifetimes on initial conditions. As an illustration we again consider the Hénon map. In the top panel of Fig. 3.5 we sample trajectories initialized on the line $y_0 = -0.5$ and compute the number of iterates n required for the trajectory to leave the region Γ . In some contexts this is called a *scattering function* [41].

It can be seen that the distribution is comprised of continuous segments of constant n and highly irregular regions. The irregular regions correspond to intersections of the line with the chaotic saddle or its stable manifold.⁵ Viewed from bottom to top (increasing n), the top panel of Fig. 3.5 can be considered as showing the construction⁶ of a particular Cantor set. The limiting object is a measure-zero fractal with non-integer dimension. Such an object clearly does not describe the structure of a regular (non-chaotic) non-attracting set. In two dimensions (the bottom panel) we can see that the longest-living trajectories approximate the stable manifold as in the sprinkler method. Between the Cantor filaments are contiguous regions of one

⁵Strictly speaking, what we observe are the escape time of points *close to* the saddle or stable manifold - these sets have measure zero. The lifetimes of the points inside these sets would be infinite. The observed value of n is due, of course, to finite numerical precision.

⁶By removal of the plotted pieces at each n from the interval $[-1, 1]$.

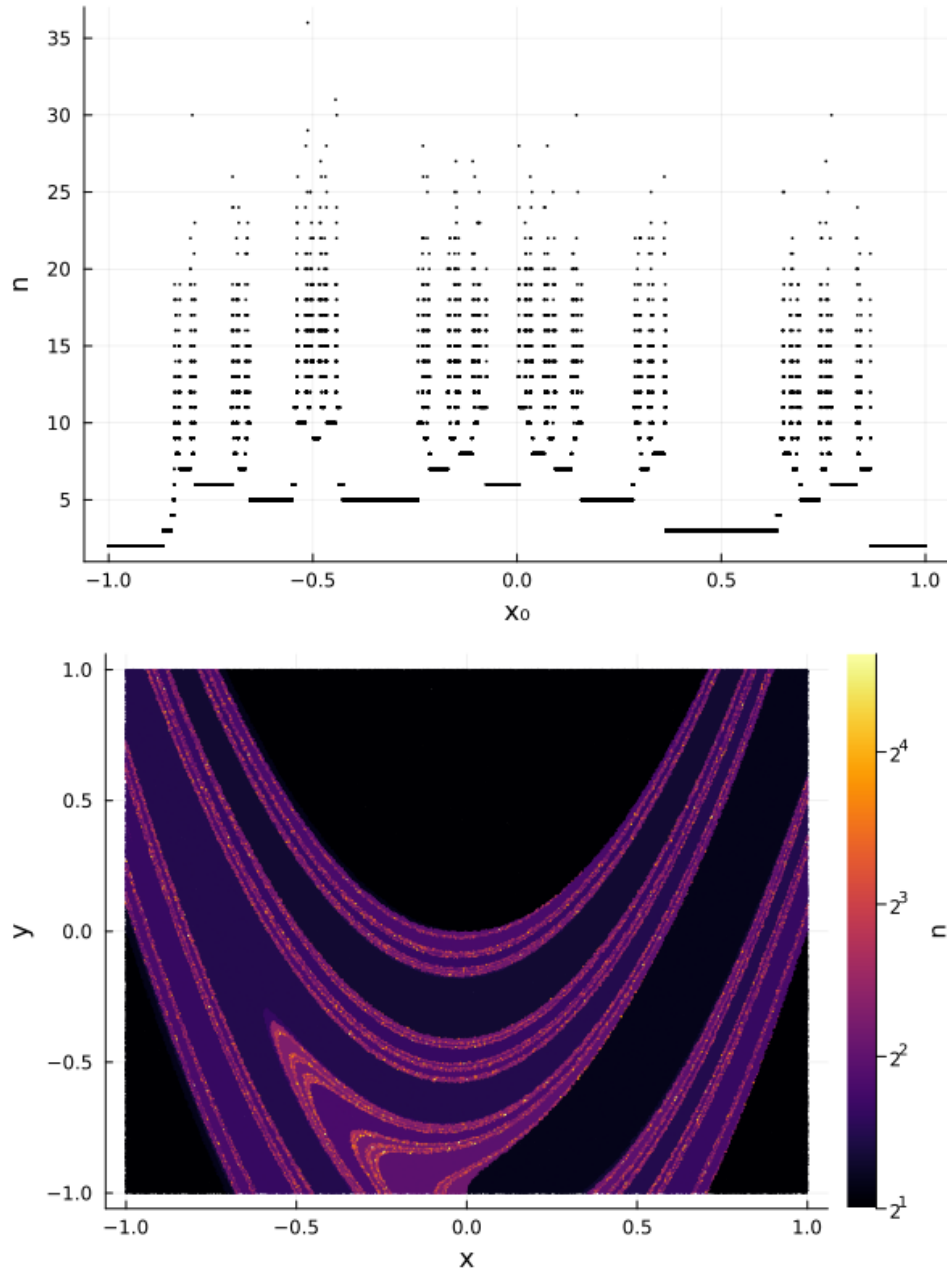


FIGURE 3.5: Transient lifetimes in the Hénon map for $(a, b) = (2.0, 0.3)$. **(Top):** Lifetimes for initial conditions on the line segment $y_0 = -0.5, x_0 \in [-1, 1]$. Each point corresponds to the time taken for the trajectory seeded at that point to leave $\Gamma = [-1, 1] \times [-1, 1]$. **Bottom:** the same experiment for 10^6 initial conditions started uniformly over Γ , where the points are colored according to their escape time.

color, analogous to the intervals of constant n in the top panel. Thus, sensitive dependence on initial conditions in the transient lifetime can be taken as an additional indicator of transient chaos.

3.3 Fractal dimensions in transient chaos

Chaotic dynamics are generally associated with the existence of fractal structures in phase space. We have already discussed some of these, like the chaotic saddles and their stable/unstable manifolds. The *fractal dimension* of these sets provides intriguing connections between the geometric objects and the dynamics of the underlying system, through relations such as the Kaplan-Yorke dimension [23] for strange attractors, or the Kantz-Grassberger relation [37] for chaotic saddles. There are many choices of how to define a fractal dimension, but they generally refer to the scaling behaviour of some quantity as some characteristic size or scale is varied.

One intuitive and easy-to-compute choice is the box-counting dimension [53], denoted D_0 . The box-counting dimension of a set is defined as the exponent of the minimal number $n(\varepsilon)$ of disjoint boxes of width ε needed to cover the set in the $\varepsilon \rightarrow 0$ limit. Explicitly,

$$D_0 = \lim_{\varepsilon \rightarrow 0} \frac{\ln n(\varepsilon)}{\ln 1/\varepsilon}. \quad (3.10)$$

It is defined similarly⁷ to another common choice of fractal dimension, the *Hausdorff* dimension, the scaling exponent of the number of balls of radius $r > 0$ needed to cover the set [53]. The subscript zero in the box-counting dimension is chosen because the box counting dimension is one of a spectrum of dimensions called the Rényi spectrum [59], defined on probability measures μ :

$$D_q = \frac{1}{1-q} \lim_{n \rightarrow \infty} \frac{\ln \sum_{i=1}^{n(\varepsilon)} \mu_i^q}{\ln 1/\varepsilon}, \quad (3.11)$$

for all $q \in \mathbb{R}$. Here μ_i refers to the measure of the box i . These other dimensions are useful because the box counting dimension fails to take into account how densely each of the boxes is filled by the fractal set. Generally one has

$$N \geq D_q \geq D_{q'} \geq 0 \quad (3.12)$$

where $q' > q$ and N is the dimension of the space in which the fractal object is embedded. The case $q = 1$ ⁸ is called the *information dimension* D_1 , and is related to the Shannon entropy H_ε of the probability distribution associated with the measure

⁷the box counting dimension is an upper bound to the Hausdorff dimension, and asymptotically they are usually equal

⁸Undefined in Eq. (3.11), but the $q \rightarrow 1$ limit is easily shown to exist.

μ :

$$H_\varepsilon := - \sum_{i=1}^{n(\varepsilon)} \mu_i \ln \mu_i. \quad (3.13)$$

The relation between the two quantities is

$$D_1 = \lim_{\varepsilon \rightarrow 0} \frac{H_\varepsilon}{\ln 1/\varepsilon}. \quad (3.14)$$

The information dimension thus describes how the information content grows with the number of boxes. The Kaplan-Yorke formula mentioned earlier relates the information dimension D_1 of a strange attractor to the Lyapunov exponents λ_i of the dynamics:

$$D_1 = k + \frac{\sum_{i=1}^k \lambda_i}{|\lambda_{k+1}|}, \quad (3.15)$$

where the integer k is defined as

$$k := \max \left\{ j \in \mathbb{N} : \sum_{i=1}^j \lambda_i > 0 \right\}. \quad (3.16)$$

For a 2D map to feature a chaotic attractor (e.g the Hénon map) there must be one Lyapunov exponent of each sign, so Eq. (3.15) reduces to

$$D_1 = 1 + \frac{\lambda_1}{|\lambda_2|} \quad (3.17)$$

The spectrum of Rényi dimensions for $q \in [0, 5]$ for the Hénon map is presented in Fig. 3.6.

The Kantz-Grassberger relation for open systems relates the information dimension of the natural measure of the chaotic saddle to the Lyapunov exponents and also to the escape rate κ . For 2D maps it reads [71]:

$$D_1 = (\lambda_1 - \kappa) \left(\frac{1}{\lambda_1} + \frac{1}{|\lambda_2|} \right). \quad (3.18)$$

Eq. (3.18) also generalizes to higher dimensional systems in terms of the information dimensions along the unstable directions of the saddle. In particular, we also have the following relationship between the information and box-counting dimensions of the stable/unstable manifolds $D_{i,s}, D_{i,u}$ ($i = 0, 1$)⁹ and the dimension of the saddle D_i [71]:

$$D_i = D_{i,s} + D_{i,u} - N. \quad (3.19)$$

⁹In Hamiltonian systems, time-reversal symmetry implies $D_{i,s} = D_{i,u}$.

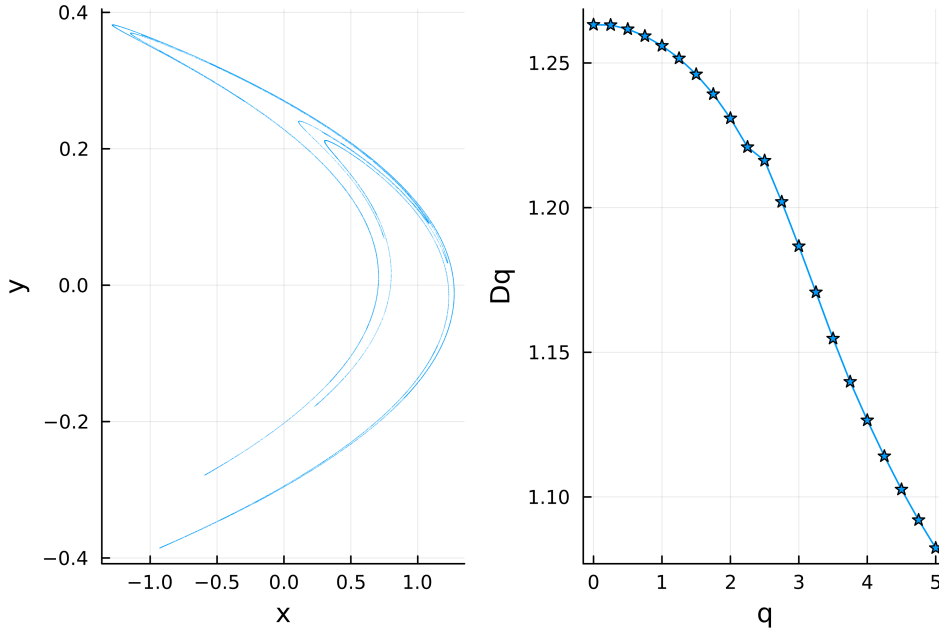


FIGURE 3.6: **(Left):** Chaotic attractor in the Hénon map (1.1) for $(a, b) = (1.4, 0.3)$ traced out by a single trajectory of length 10^6 , discarding a transient of length 10^4 . **(Right):** Approximate Rényi dimensions D_q for this attractor. In particular, we compute Lyapunov exponents of $(\lambda_1, \lambda_2) \approx (0.42, -1.62)$, leading to an information dimension of $D_1 \approx 1.26$ by the Kaplan-Yorke formula. This agrees well with the value for $q = 1$ shown in the figure.

For the Hénon map with $(a, b) = (2.0, 0.3)$ (see Fig. 3.3) we compute $D_{1,s} \approx 1.62$, $D_{1,u} \approx 1.29$ and $N = 2$, so that $D_1 \approx 0.91$. This is close to the information dimension of 0.95 we compute for the set obtained by the sprinkler method in the middle panel of Fig. 3.3.

One more fractal dimension that frequently appears [71, 77, 73, 48, 14] in studies of transient chaos due to its computational convenience over other choices is the *uncertainty dimension* [29]. In systems with more than one final state (e.g. multiple attractors, different holes through which to escape the system, etc), the boundary separating the disjoint basins of initial conditions leading to these different states are often fractal. The reason behind this can be explained as transient chaos related to chaotic sets supported on the boundary. Points belonging to the boundary appear as singularities of any function assigning an initial condition x to one of the final states, an example being the scattering function shown in Fig. 3.5 in which the escape time function t_{esc} is singular on the forward-trapped set S_∞ . A consequence of the fractality is that determining the fate of initial conditions close to the boundary requires exponentially more precision in the specification of that initial condition. The uncertainty dimension quantifies the scaling of this precision requirement in terms of how frequently perturbations of an initial condition by an amount ε lead to

a different final state.

In more detail, a point x is called *uncertain* at the resolution ε if a new point $x' = x + \varepsilon$ ¹⁰ belongs to a different basin than x . The fraction $f(\varepsilon)$ of uncertain points scales with a power law:

$$f(\varepsilon) \propto \varepsilon^\alpha, \quad (3.20)$$

where the extreme case $\alpha = 1$ corresponds to a smooth boundary (no fractality) and the other extreme ($\alpha \rightarrow 0$) to total fractalisation of the space (riddled basins). The *uncertainty exponent* α is related to the dimension D_α of the boundary by

$$D_\alpha = N - \alpha. \quad (3.21)$$

where $N = \dim \Omega$. It was conjectured [29] that the uncertainty dimension is equal to the box-counting dimension in typical dynamical systems, and this has been proven in certain cases [52]. It is the uncertainty dimension we will use in Chapter 5 to quantify the scaling of exit basin boundaries.

3.3.1 Invariant and Conditionally-Invariant Measures

An important dynamical invariant is the *natural measure* μ_N . Several dynamical characteristics such as Lyapunov exponents and fractal dimensions (e.g. box-counting, information dimension) can be defined with respect to this measure. For Hamiltonian systems, the natural measure is the Liouville measure - the volume of phase space.

Definition 3.3.1 (Natural Measure). Let A be a measurable set in state space Ω , and let $\vec{x}_0, \vec{x}_1, \dots$ be an orbit of a dynamical system in a state space Ω . The natural measure μ_N of A is defined by tracking the frequency of visits to A :

$$\mu_N(A) := \int_A d\mu_N = \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{i=0}^{t-1} \mathbb{1}_A(\vec{x}_i), \quad (3.22)$$

where $\mathbb{1}_A$ is an indicator function on A .

In closed systems, ergodicity of the dynamics implies that the density of μ_N will be generated by generic orbits. However, for open chaotic systems, the limiting behaviour of Eq. (3.22) can only be guaranteed on the set $\mathcal{S}_\infty$, i.e., on the chaotic saddle. The support has measure zero, so in practice one can only approximate the measure by either sprinkler method as previous described, or by a coarse-graining

¹⁰This can be a perturbation in a random direction of size $|\varepsilon|$, but a common choice to simplify the computation is to choose points and their perturbations on a line intersecting the boundary.

procedure such as Ulam's method [24], which would involve covering the set with boxes of width $\varepsilon \ll 1$ and track the relative proportion of visits to the boxes by $n \gg 1$ trajectories. The probability measure of box i is then approximated by

$$P_i(\varepsilon) \approx \frac{n_i}{n} \quad (3.23)$$

where n_i is the number of visits to box i . A related concept that is perhaps more useful is the *conditionally-invariant* measure, introduced by Pianigiani and Yorke [55]:

Definition 3.3.2 (Conditionally-invariant measure). A measure μ_c is said to be conditionally-invariant with respect to a map $\mathcal{M}$ if, for any measurable $A \subset \Omega$, we have

$$\mu_c(A) = \frac{\mu_c(\mathcal{M}^{-1}A)}{\mu_c(\mathcal{M}^{-1}\Omega)}. \quad (3.24)$$

The conditionally-invariant measure ('c-measure' for short) is obtained as the limiting distribution of a smooth ensemble of initial conditions in the neighbourhood of $\mathcal{S}_\infty$. For long times ($t \gg 1/\kappa$), the probability of a surviving trajectory being in any region $A \in \Omega$ approaches $\mu_c(A)$ [4, 20, 71]. Thus, the c-measure acts similarly to an attractor for generic ensembles of initial conditions which intersect the stable manifold of the saddle, analogous to Sinai-Ruelle-Bowen (SRB) measures for attractors. This distribution is invariant provided it is re-normalized at each time-step of the dynamics to account for the points which leave during this step (achieved by the denominator in Eq. (3.24)). This amounts to supplying new points to A at the same rate at which they are leaving asymptotically - that is, the number of points is multiplied by e^κ at each step where κ is the escape rate. It is thus possible to uniquely express the escape rate in terms of the c-measure:

$$\kappa = -\log(1 - \mu_c(\mathcal{E})), \quad (3.25)$$

where $\mu_c(\mathcal{E}) = 1 - \mu_c(\mathcal{M}^{-1}(\Omega))$ is the c-measure of the set $\mathcal{E}$ of trajectories that exit Ω in one iteration of the map $\mathcal{M}$. It is worth noting that the same argument implies that the stable manifold supports the c-measure of the time reversed dynamics. The restriction of either c-measure to the support of the saddle yields a natural measure of the saddle (in forward or reverse time), and these will agree if the system exhibits time-reversal symmetry (e.g in Hamiltonian systems.)

3.4 Transient Chaos and Non-Hyperbolicity

Until now we have assumed that the dynamics of the system in question is hyperbolic. This is a very strong statement: hyperbolicity implies exponential separation of nearby trajectories, contributing to efficient mixing and ergodic exploration of the phase space by generic orbits. However, it is generally accepted that typical Hamiltonian systems do *not* have strong hyperbolic properties [56, 71, 4]. They often have a mixed phase space, with both regular (integrable) and chaotic components as we have seen in Chapter 2. KAM tori form transport barriers which divide these components from each other, meaning that a chaotic orbit cannot penetrate into the regular region from outside, and regular orbits will never leave [53]. Remarkably, there is still a possible interaction between the regions: when a chaotic trajectory approaches the boundary between the chaotic and regular components, it can spend a long time “stuck” close to the boundary.

This phenomenon is thus called *stickiness*, and occurs because the local Lyapunov exponents go to zero (the same as on the KAM tori themselves) due to the smoothness of the dynamics. Consequently, during their visit to the *sticky region* the orbit ceases to be chaotic and instead the orbit behaves quasi-periodically. Eventually the orbit will leave the sticky region and rejoin the chaotic sea, potentially to later return to the same or another sticky region after an indeterminate time. This leads one to speak of *intermittent* [45] phases of chaotic and regular behaviour in the evolution of a trajectory.

The length of visits to these sticky regions obeys a power law (heavy-tailed) distribution, which has deep implications for any calculations and descriptions relying on strong chaotic properties like uniform hyperbolicity. A direct consequence for transient chaos is that the expected exponential decay in the survival probability will not hold asymptotically: the trajectories with the longest (finite) lifetimes will be those trapped close to regions of regular motion. Empirically, one instead observes a crossover between exponential and algebraic decay in $P(t)$ after some finite time t_* [71, 4]:

$$P(t) \sim \begin{cases} e^{-\kappa t}, & t_0 < t < t_* \\ t^{-z}, & t \geq t_*. \end{cases} \quad (3.26)$$

In such cases, the chaotic saddle can be regarded as having both hyperbolic (dominant on short times) and non-hyperbolic (resp. long times) components [4]. We demonstrate this phenomenon numerically in two open systems in Chapter 5.

3.5 Dissipation

We consider the evolution of Hamiltonian systems perturbed with a dissipative effect. One way that friction-like dissipation can be added to Hamiltonian systems is through a Rayleigh dissipation function $\mathcal{F}$ [28]. For a single-particle system with two degrees of freedom, it is written as

$$\mathcal{F}(t) = \frac{1}{2}(\gamma_1 \dot{q}_1^2 + \gamma_2 \dot{q}_2^2), \quad (3.27)$$

where γ_1, γ_2 are parameters controlling the strength of the dissipation and $\dot{q}_1, \dot{q}_2$ are the generalized velocities¹¹. The derivatives $-\frac{\partial \mathcal{F}}{\partial \dot{q}_i}$ and $-\frac{\partial \mathcal{F}}{\partial \dot{q}_j}$ represents the frictional forces in the equations of motion. The relationship between $\mathcal{F}$ and energy E is given by

$$\frac{dE}{dt} = -2\mathcal{F}(t). \quad (3.28)$$

When we add dissipation to the Hénon-Heiles system, it will be in the form of Eq. (3.27). The form of dissipation we consider for the Copenhagen problem is more complicated and will be discussed in a later section.

3.6 Doubly-transient chaos

The descriptions of transient chaos we have presented rely on the existence of a chaotic invariant set of orbits. However, there is a broad class of systems which cannot satisfy this requirement. When dissipation is added to an undriven system, the energy must decrease so that all motion eventually ceases. Consequently, there cannot exist any non-trivial invariant sets. A novel theory to describe the motion of trajectories in non-linear systems with dissipation is called *doubly-transient chaos* [48, 71, 73]. As suggested by its name, the idea is to view the chaotic invariant sets underlying transient chaos to be themselves transients due to the loss of energy. This theory has been shown to describe the systems in which the dissipation acts as a non-autonomous element in the dynamics, with mechanical systems like the magnetic pendulum receiving particular attention [48]. The transient chaos properties of such systems are found to be surprisingly distinct from those in the conservative case. Key features include time-dependent settling rates to attractors, and scale-dependent fractal dimensions of attractor basin boundaries. Ref. [40] explored further properties of doubly-transient chaos by analysing the time evolution under

¹¹This term can be added to the Lagrangian, from which the dissipative Hamilton's equations are obtained. Alternatively, eq. (3.27) can be expressed in terms of momenta instead of generalized velocities and then added to Hamilton's equations.

dissipation of phase space structures of the dissipation-free system, proposing new tools to analyse how the complexity of the motion decays in time.

Most existing works on doubly-transient chaos have focused on closed mechanical systems [48, 40, 14], wherein every trajectory of an otherwise permanently chaotic system eventually settles to a fixed point when dissipation is added. It is not clear from these studies what to expect when dissipation is added to open systems whose generic behaviour is already transient. Scattering systems are a natural example, as the only orbits with infinite lifetimes can often form measure-zero fractal sets. An interesting exception is [73], in which the theory of doubly-transient chaos is applied to a Hamiltonian system subject to parameter drift whose unperturbed dynamics is transiently chaotic (a decaying open flow), describing the escape process using time-dependent (effective) escape rates. It was suggested that the changing escape rate could be understood as the decaying system evolving along snapshots of the system without parameter drift. An important simplifying property of the system studied in that work is that all trajectories experience the same dynamics at time t . In more typical dissipation scenarios which we consider in this thesis, the dissipation depends on the trajectory (e.g., on the velocity) and thus different trajectories experience a different unperturbed dynamics (have a different energy) at the same time t .

The larger aim of this thesis is to study open *dissipative* chaotic scattering systems, and to propose new methods to understand and explain the temporal evolution of these systems and their statistical properties. We explore these phenomena numerically for the Hénon-Heiles and Copenhagen systems in the next chapter, and propose a theoretical understanding in Chapter 6.

Chapter 4

Numerical Methods

This thesis relies on computer experiments. The equations of motion for chaotic dynamical systems cannot be solved explicitly, and so solutions need to be approximated numerically using differential equation solvers. The experiments in this thesis were performed using the `DifferentialEquations.jl` [58] and `DynamicalSystems.jl` [19] software libraries for the *Julia* programming language. Additionally, all visualisations were made with the `Plots.jl` [17] package. *Julia* is an exciting new language, particularly for scientific computing applications, owing to its balancing of high-performance (comparable to C) and high-level syntax (comparable to Python). In this chapter we discuss the choices of different solver algorithms and how they are applied to our problem.

4.1 Runge-Kutta Algorithms

The simplest method for solving an initial value problem $\dot{x} = f(x, t)$, $x(0) = x_0$ is the *Euler method* [61]. The solution is computed by a forward finite-difference method with small step size Δt :

$$\begin{aligned} x_{n+1} &= x_n + \Delta t f(x_n, t_n), \\ t_{n+1} &= t_n + \Delta t. \end{aligned} \tag{4.1}$$

There can be several sources of error in the computed solution. Two that are ever-present are *truncation error*, which is the error introduced in a single step of the solver and is controlled by the step size Δt , and *round-off error*, which is the limitation imposed by the finite precision of floating-point representation. In some cases the differential equations are *stiff*, generally referring to instabilities caused by the derivative terms growing extremely large very quickly (e.g near singularities). Circumventing this requires extremely small step sizes, which introduces a performance trade-off.

Euler methods are generally not suitable beyond very short times for the above reasons. *Runge-Kutta* methods generalise the Euler method to better account for how the derivative changes between steps by computing estimates for the slope of $f(x, t)$ at n distinct points in the interval $[t_n, t_n + \Delta t]$ and using a weighted sum of these to estimate x_{n+1} [61]. As an example, a fourth-order ($n = 4$) Runge-Kutta method, abbreviated *RK4*, works as follows:

$$\begin{aligned} x_{n+1} &= x_n + \frac{\Delta t}{6}(k_1 + 2k_2 + 2k_3 + k_4), \\ t_{n+1} &= t_n + \Delta t, \end{aligned} \tag{4.2}$$

where the terms k_1, k_2, k_3, k_4 are given by

$$\begin{aligned} k_1 &= f(x_n, t_n), \\ k_2 &= f\left(x_n + \Delta t \frac{k_1}{2}, t_n + \frac{\Delta t}{2}\right), \\ k_3 &= f\left(x_n + \Delta t \frac{k_2}{2}, t_n + \frac{\Delta t}{2}\right), \\ k_4 &= f(x_n + \Delta t k_3, t_n + \Delta t). \end{aligned} \tag{4.3}$$

More weight is given to estimates of the slope of the derivative from the mid-point of the interval. Runge-Kutta methods do not overcome the above error contributions, but are much more appropriate than simple first-order approximations for most problems. Most computer implementations of Runge-Kutta methods allow for the step size Δt to be adjusted dynamically during integration, so that larger step sizes can be used when the derivatives are small, and vice-versa, for reduced computational cost. The range of allowable step sizes is controlled by user-specified error tolerances in the solution. In this thesis, when we integrate trajectories of dissipative systems, we use a 9th-order Runge-Kutta method [72] with error tolerances of less than 10^{-14} . This is effectively the accuracy limit of double (64-bit) precision. Whenever higher accuracy is desired, for instance when studying fractal dimensions, double-double (128-bit) precision is used and the error tolerances adjusted accordingly. 128-bit precision allows for keeping tracking of up to around 30 decimal places.

4.2 Symplectic Integrators

A key feature of Hamiltonian systems is that the solutions to Hamilton's equations reside on a symplectic manifold in phase space, with a natural 2-form defined by the

position and momentum components of the solution. There exist classes of numerical integration schemes, known as *geometric integrators*, which preserve geometric properties of the solution. A certain subclass of geometric integrators, called *symplectic integrators*, preserve this 2-form.

Many symplectic integrators rely on a technique called *splitting*. The main idea is to decompose ("split") the Hamiltonian into several terms (for instance, into kinetic energy T and potential energy V , or into separate Hamiltonians $H_1, H_2, \dots$) which can be either be solved or approximated independently. As an example, one of the most popular splitting methods is the *leapfrog* or *Verlet* algorithm, a second-order method which alternates between independently updating the position and momentum variables. The idea is to make the flow approximation

$$\phi^t \sim \phi_T^{t/2} \circ \phi_V^t \circ \phi_T^{t/2}, \quad (4.4)$$

where ϕ_T, ϕ_V are the flows associated with the kinetic and potential terms. The resulting relationship between $(\mathbf{q}_n, \mathbf{p}_n)$ and $(\mathbf{q}_{n+1}, \mathbf{p}_{n+1})$ is a symplectic map. Higher-order methods can also be developed; for an in-depth discussion, see e.g [76, 34].

The solutions evolved by symplectic integrators reside on a symplectic manifold, albeit slightly perturbed from the 'true' manifold to the Hamiltonian system. This perturbation has several components. First is the drift that arises from round-off error, which tends to grow linearly in time. Secondly, the truncation error induces a perturbation from the true trajectory of size $\mathcal{O}(\Delta t^n)$, where n is the order of the specific method. The energy is thus not exactly conserved, although the numerical manifold is close to the true manifold, so observed energy differences are small and tend to look periodic. This is demonstrated in Figure 4.1, which displays the energy drift in a 4th-order symplectic method[60] for a trajectory in the Hénon-Heiles system.

Non-symplectic methods generally have quadratic (or worse!) numerical drift, and tend not to display this periodic behavior. The global properties of the phase space path are also not preserved. Observe in Figure 4.2 the difference between the symplectic method from Figure 4.1 and a 5th order Runge-Kutta method due to Tsitouras [69]. Catastrophic energy drift can result over long integration times when error tolerances are high. While this effect can be mitigated by using stricter error tolerances or selecting integrators more optimized for a given problem type, this is often limited by performance concerns.

For integrating trajectories of Hamiltonian systems, we use an explicit 8th-order

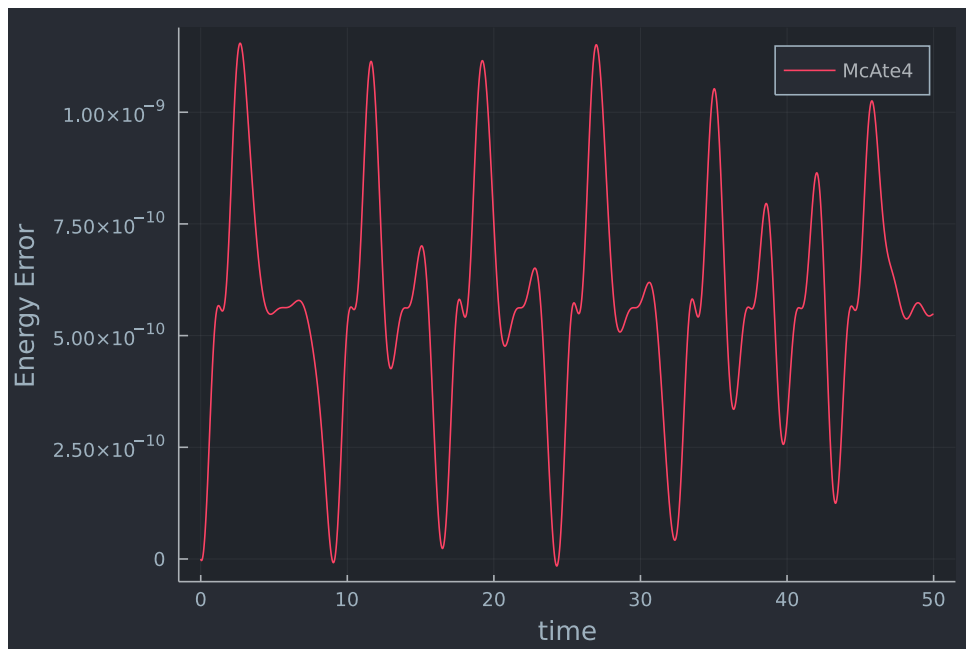


FIGURE 4.1: Relative energy error for a trajectory in the Hénon-Heiles system using a 4th-order symplectic method.

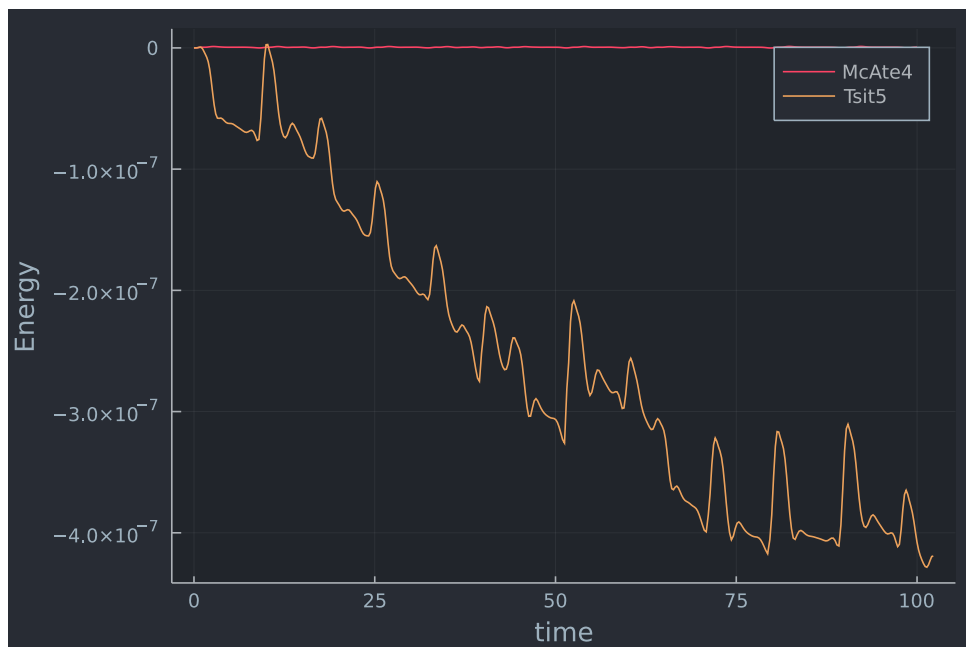


FIGURE 4.2: Relative energy error for trajectories of the same initial condition, using a 4th-order symplectic method and a 5th-order Runge-Kutta method.

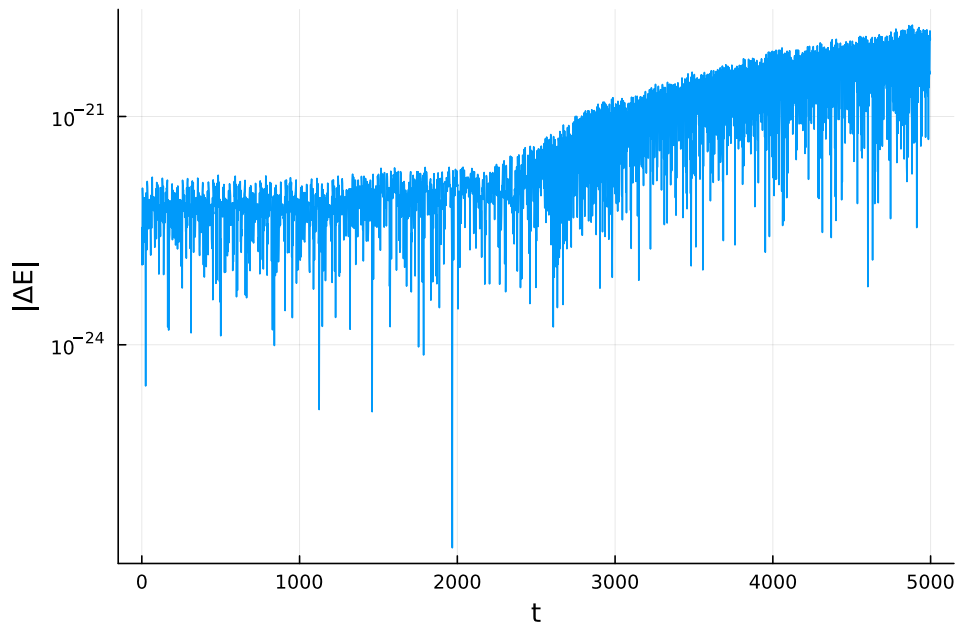


FIGURE 4.3: Absolute value of the energy error over a trajectory of length 5000 in the Hénon-Heiles system, computed using an 8th-order symplectic method [36] with a fixed time step of $\Delta t = 0.01$ and quadruple-precision arithmetic.

symplectic method [36] with a fixed time-step of $\Delta t = 0.01$. This has excellent energy conservation properties, which we demonstrate in Fig. 4.3 for a trajectory in the Hénon-Heiles system. This level of precision is more than sufficient for our purposes.

Chapter 5

Numerical Experiments

In this chapter we present the results of some numerical experiments in two open chaotic systems: the Hénon-Heiles system and the Copenhagen problem. We consider each system with and without dissipation, study their transient chaos properties, and identify the qualitative differences that dissipation causes.

5.1 Transient chaos in the Hénon-Heiles system

We previously introduced the Hénon-Heiles system in Chapter 2 in the context of showing the breakdown of integrability with perturbations. For increasing values of the energy parameter E , we observed a disintegration of the invariant tori inside a bounded potential. At $E_c = 1/6$, which we will call the ‘critical energy’, three openings are created and particles may escape through any of the *exit channels*. We emphasize that it is only for energies above E_c that escape is possible, and thus transient chaos is observable. At $E = E_c$, the potential contours form an equilateral triangle with vertices at saddle points, see Fig. 2.1.

The escape dynamics in the Hamiltonian problem have been extensively studied [2, 77]. A typical orbit moves about the scattering region until it passes through one of the three exit channels and escapes to infinity. The existence of special unstable periodic orbits, called *Lyapunov orbits*, allows for the definition of a precise criterion for escape from the region around the origin. These orbits straddle the exit channel openings, and persist for all energies $E > E_c$. Trajectories which cross these orbits with outward velocity vectors can never return to the central region [18]. For our problem, we thus define the scattering region Ω to be the region of configuration space bounded by the potential contours $V = E$ and the Lyapunov orbits at energy E connecting neighbouring contours.

5.1.1 Exit basins

Here we determine the exit basins in the open Hénon-Heiles system, a concept we described in Chapter 3. We seed initial conditions uniformly inside Ω , with initial velocities chosen according to the section S_2 . In the Hamiltonian setting ($\gamma = 0$) we use a numerical criterion to determine escape. If the orbit's (x, y) position exceeds a fixed radius¹ $r_{esc} > 1$ then the orbit is deemed to have escaped at this time. The distance from the origin to any of the three saddle points is 1, so choosing an r_{esc} slightly larger than 1 ensures that the orbit has actually crossed the threshold determined by the Lyapunov orbit and thus left Ω . Various specific values of r_{esc} were considered, all leading to the same observations. We distinguish between initial conditions that escaped through different channels by plotting them in different colors.

¹With the understanding that all of Ω is contained inside this disk.

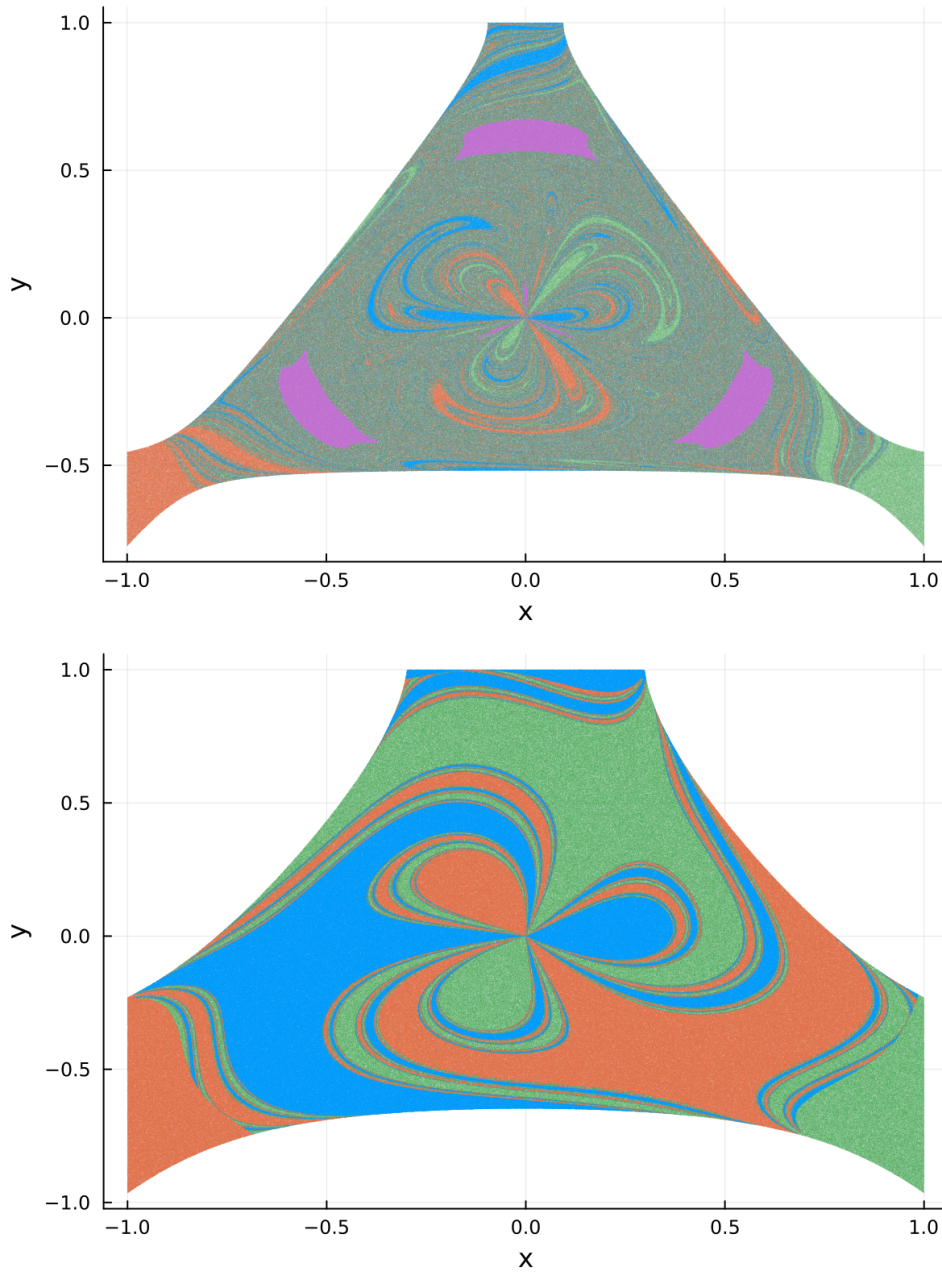


FIGURE 5.1: Basins of escape (green, orange, blue) through different exit channels in the conservative Hénon-Heiles system. **(Top):** Basins for $E = 0.18$. The pink points correspond to KAM islands, whose orbits never escape. **(Bottom):** Basins for $E = 0.3$. $N_0 = 5 \cdot 10^6$ points are used to generate each plot.

The basins for the conservative system for two values of the energy, $E = 0.18$ and $E = 0.3$ are shown in Fig. 5.1. Common to both panels is the $2\pi/3$ symmetry in the basins, which is expected from the potential. At the lower energy, the scattering region is heavily fractalized, in which points of one color are located arbitrarily close to points of other colors. There are some larger hook-like structures located near the origin comprised of points that escape very quickly through the same channel, as

well as some thinner structures close to the walls of the potential which also leave quickly. Generally speaking, contiguous regions of one color in these diagrams indicate regions with short escape times, as these points start far from chaotic saddles and perform similar movements in Ω . This is verified in Fig. 5.2, which colors each point in the basins by their time to escape. Apparent from these figures is the reduction in typical escape time (equivalently, the increase in the escape rate) as the energy is increased. The top panel plots every point whose orbit escaped before $t = 500$, and the bottom panel the points whose orbit escaped before $t = 50$. In fact, the escape basins in the Hénon-Heiles system are known to exhibit a stronger condition known as the *Wada property* [2], under which all three of the exit basins share the same boundary. This has even been claimed to persist under weak dissipation [65], although recent work [14] on the geometry of basin boundaries in undriven dissipative systems suggest that this may only hold to finite spatial scales.

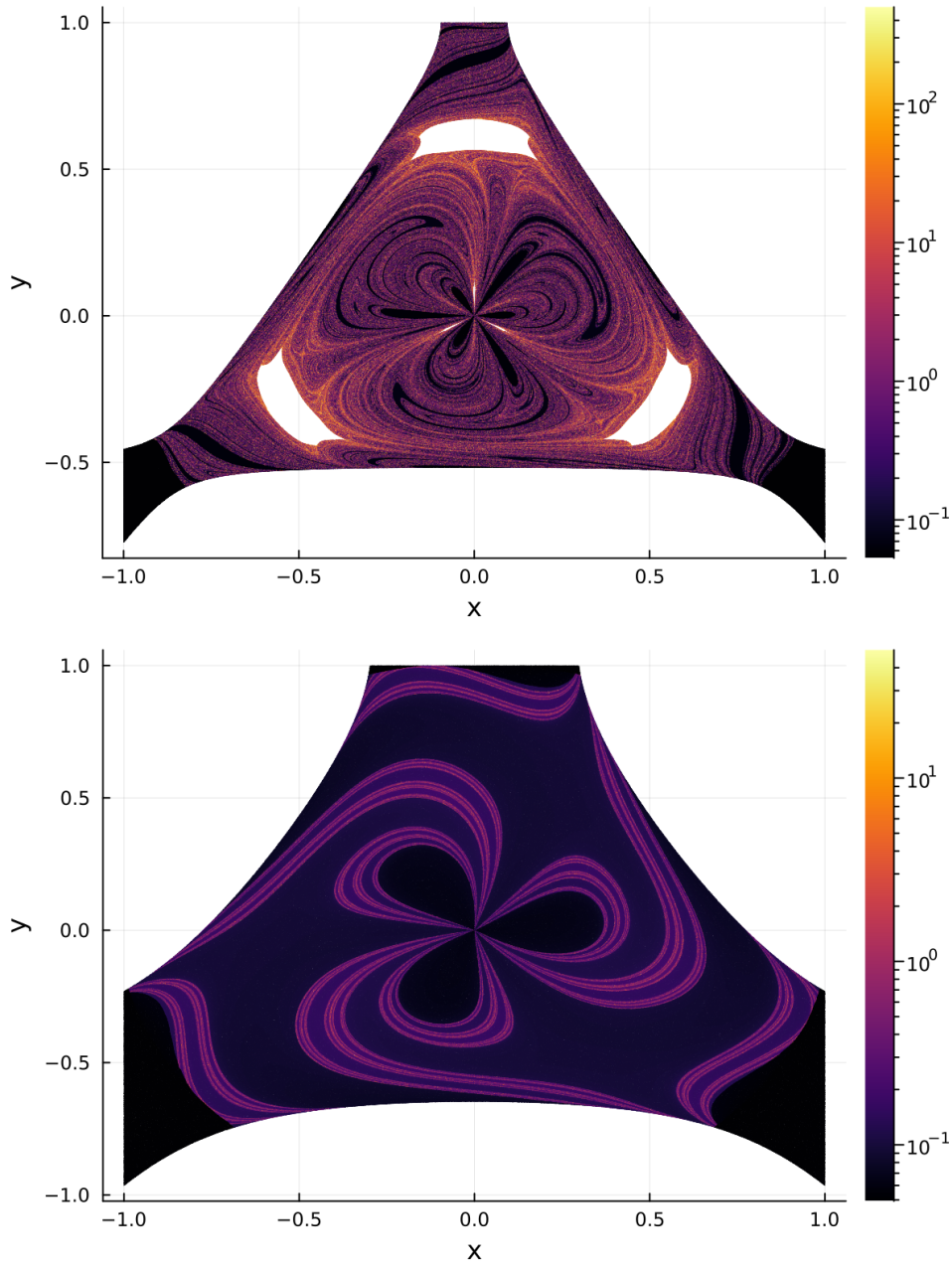


FIGURE 5.2: Coloration of the configuration space coordinates of initial conditions from the Poincaré section S_3 according to their escape time in the conservative Hénon-Heiles system. The top and bottom panels correspond to the same values of the energy E as in Fig. 5.1. The points belonging to KAM islands (top panel) are left uncolored, as their escape time is infinite. $N_0 = 5 \cdot 10^6$ points are used to generate each plot.

5.1.2 Survival Probability

Numerical computations of the survival probability in the Hénon-Heiles system (2.18) are summarized in Fig. 5.3. The top panel confirms the existence of all the generic regimes described in Chapter 3, depending on the initial energy E . Importantly,

even if for small energies the existence of KAM islands hinders the decay to be non-exponential, there is an effective escape rate κ at intermediate times t [47, 4] that governs the escape of the majority of the initial conditions. It is thus possible to investigate the dependency of the (effective) escape rate κ on the energy E for all values of $E > E_c$. The numerical results in Fig. 5.3(b) indicate a continuous dependence of κ on E_0 , with a roughly linear dependency for E in the hyperbolic regime².

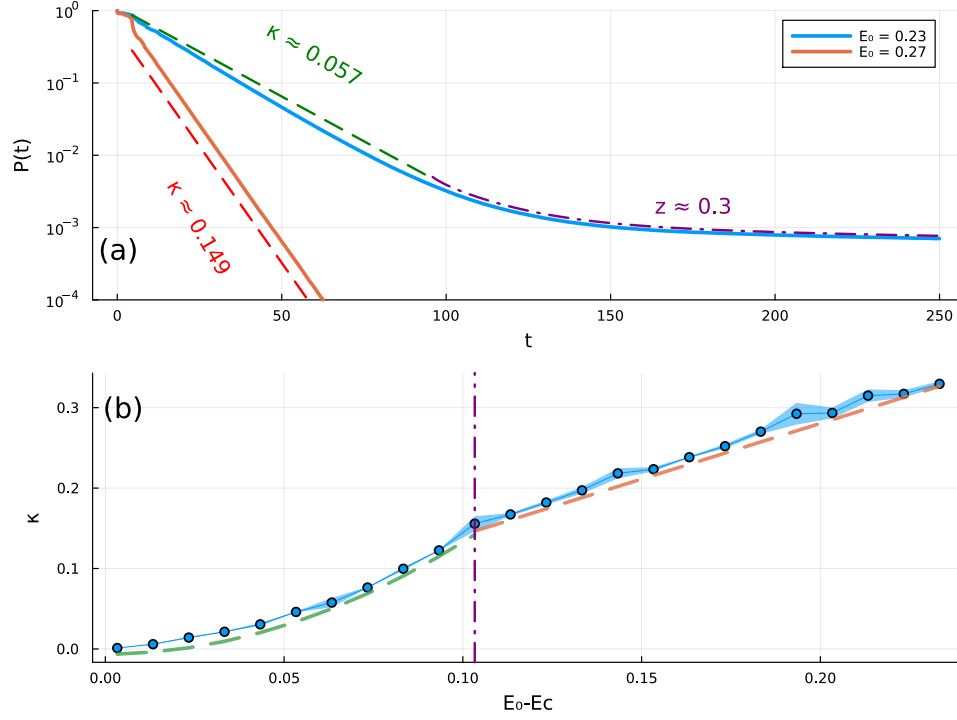


FIGURE 5.3: Survival probability $P(t)$ and escape rate $\kappa = \kappa_0$ in the Hamiltonian system. **(a)**: $P(t)$ for 2 values of the energy $E_0 > E_c = 1/6$ (see legend). For the smaller energy ($E_0 = 0.23$), a crossover occurs between an exponential and a power law decay at $t = \tau^* \approx 100$, as described in Chapter 3 for non-hyperbolic systems. For the larger energy ($E_0 = 0.27$), there is no crossover and the decay is exponential, as observed in uniformly hyperbolic dynamics. The dashed lines have been added as a visual aid, with the approximate decay exponents for the exponential and power law regimes annotated. $N_0 = 10^7$ initial conditions were used for each curve. **(b)**: Dependence of the escape rate κ on the energy E_0 in the Hamiltonian system. Note that the x-axis is translated by E_c . Each point is the mean of 4 separate estimates of κ made by least-square fits of the logarithm of $P(t)$ across different time intervals. There is an observed non-linear scaling which ends at around $E_0 = 0.27$ (marked with the purple dash-dotted line) after which the dependence becomes linear. Parametric fits of the non-linear (green dashes, $\kappa = 13.63(E - E_c)^2 + 0.03(E - E_c)$) and linear (orange dashes, $\kappa = 1.41(E - E_c)$) regimes are overlaid.

²We have avoided speculating on the reason for the change that occurs in κ at around $E = 0.27$, although it is perhaps worth mentioning that this coincides with the point at which the last regions of bounded regular motion bifurcate away [7].

5.1.3 Chaotic sets

The chaotic invariant sets for the conservative system for $E = 0.3$ are visualised in Fig. 5.4. The saddle displays the same azimuthal symmetry as the potential, and the stable and unstable manifolds also show the expected symmetry arising from their equivalence under time-reversal.

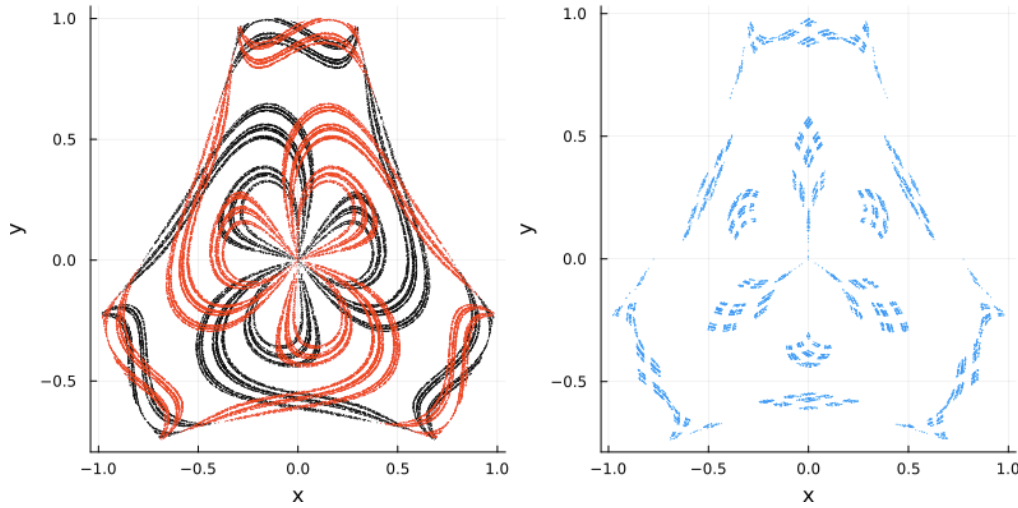


FIGURE 5.4: The Hénon-Heiles chaotic saddle (right) and its stable (left, black) and unstable (left, red) manifolds in the Poincaré section $\mathcal{S}_2$ for energy $E = 0.3$. All three objects were computed using the sprinkler method.

5.2 Hénon-Heiles with dissipation

We study the effects of weak dissipation on the escape dynamics of the open system by introducing an additional term to eqs. (2.19) modelling a drag linearly dependent on the velocity v , which can be represented as a Rayleigh dissipation function like in eq. (3.27). The ODEs of the resulting dissipative system are

$$\dot{x} = p_x, \quad (5.1a)$$

$$\dot{y} = p_y, \quad (5.1b)$$

$$\dot{p}_x = -x - 2xy - \gamma p_x, \quad (5.1c)$$

$$\dot{p}_y = -y - (x^2 - y^2) - \gamma p_y, \quad (5.1d)$$

where $\gamma \geq 0$ controls the strength of the dissipation. The system is no longer Hamiltonian and the energy of a trajectory decays³ as

$$\begin{aligned}\frac{dE}{dt} &= -\gamma(p_x^2 + p_y^2) = -\gamma v^2, \\ &= -2\gamma(E - V(x, y)).\end{aligned}\tag{5.2}$$

The decay rate of the energy will thus differ for each initial condition depending on the potential $V(x, y)$ they experience. For large times t long-living trajectories perform a similar movement in Ω . The variations across such trajectories will be small and the potential can be approximated (on average across t) as a fraction of the total energy $V(x, y) = (1 - \frac{\mu}{2})E$, where $\mu \approx 1$ is a proportionality factor. Using this approximation in Eq. (5.2), we obtain

$$E(t) = E_0 e^{-\mu\gamma t}.\tag{5.3}$$

For damped oscillations around a stable fixed-point, the linearised system has eigenvalues with real part $-\gamma/2$, and thus E should decay at a rate γ and thus $\mu = 1$.

We numerically test the validity of Eq. (5.3) for our system in Fig. 5.5. The decay is well-described by an exponent of $\mu = 0.91$ for the time interval $[0, 200]$, after which the only remaining orbits will be slowly collapsing towards the origin. The energy of such long-lived orbits decays faster than the overall fit (the red-curve slightly overestimates the numerical results for large t). μ approaches 1 for large t because the dynamics becomes increasingly linear.

We make the observation that orbits that are seeded inside KAM islands tend to remain inside the decaying islands under weak dissipation. Evidence for this is given in Figs. 5.1, 5.6, and 5.7, in which the points whose initial conditions are inside the islands in the conservative dynamics do not escape the system under the dissipative dynamics – they remain in the system until escape is impossible and eventually converge to the attractor. We observe the same qualitative behaviour as reported in [46]: the islands followed a complicated structure of bifurcations as the energy decreases, as opposed to the islands being converted to attracting funnels as suggested in [47]. Similar computations were performed in [40] for other systems at the level of individual tori, which generically were shown to persist in a deformed state for a few intersections with the Poincaré surface before ultimately disintegrating.

³Henceforth, $E_0 = E(t = 0)$ will refer to the (constant) initial energy of the ensemble, or to the energy of the conservative system.

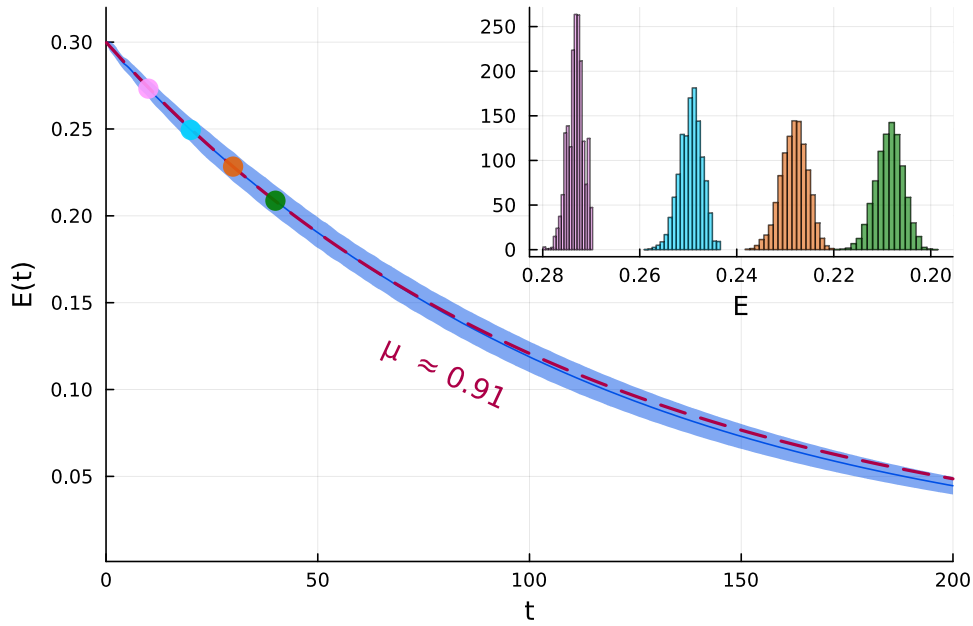


FIGURE 5.5: Energy dependence on time in the dissipative system. The dark blue line shows the mean energy $\langle E \rangle$ computed over an ensemble of trajectories inside Ω at time t . The blue ribbon shows 3 standard deviations. The dashed red line is a fit to Eq. (5.3) obtained via least-squares of the log-transformed data. **(Inset):** Histograms showing the distribution of energies at times corresponding to the coloured dots in the main plot. The parameters are $(E_0, \gamma) = (0.3, 0.01)$.

5.2.1 Exit basins

We again compute the basins of escape, this time for the dissipative system. There are no longer KAM islands in the dissipative system ($\gamma > 0$), but there is now a considerable set of initial conditions that do not escape Ω and eventually converge to the attractor at the origin. In the dissipative setting, the Lyapunov orbit criterion for determining escape cannot be relied upon, so we choose an expedient method to determine the escape condition: we integrate an initial condition until the energy of the orbit drops below the critical energy E_c . An orbit with energy $E < E_c$ cannot leave or re-enter Ω , and so its position at this time will determine its eventual fate. An orbit on the exterior of Ω will be considered to have escaped, and an orbit on the interior will be considered trapped. Integrating further is unnecessary due to the monotonicity of the energy decay. We call the time t taken for the energy of a trapped orbit to reach $E(t) < E_c$ the *settling time*. For an escaping orbit, we consider the actual escape time to be the time when the trajectory left Ω ⁴, with the apriori knowledge that it will not return. Trajectories that leave Ω quickly achieve high velocity, and so rapidly fall below E_c on the outside. The basins are presented in

⁴In the simulation, actually the time the trajectory leaves the disk containing Ω .

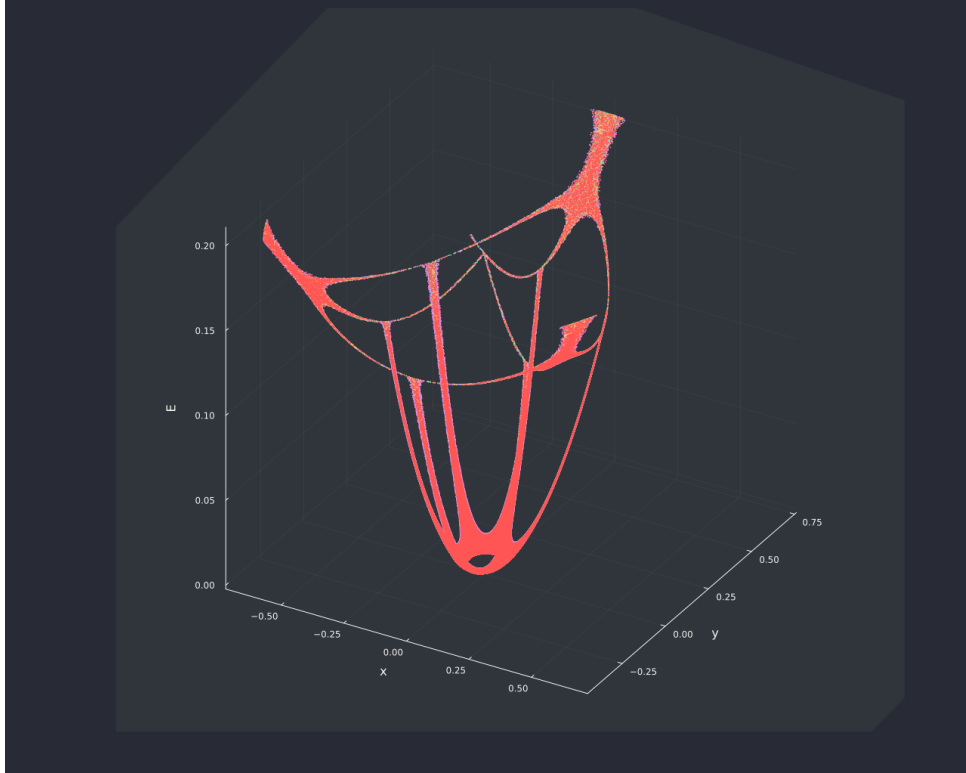


FIGURE 5.6: Bifurcation structure of the KAM islands in the Hénon-Heiles system. The intersections with the Poincaré section are plotted for 7 initial conditions seeded inside KAM islands at $E_0 = 0.21$ with $\gamma = 2 \cdot 10^{-5}$. A maximum integration time of 200,000 was used.

Fig. 5.7.

It can be seen in these figures, and by comparison to the corresponding figures 5.1 and 5.2 for the conservative system that the orbits remaining trapped in Ω under the dissipative dynamics are those starting close to the boundary between basins. The settling times of these points are also typically much longer than the escape times of orbits that manage to leave Ω . We identify these points as starting close to the stable manifold of the chaotic saddle of the conservative system, suggesting a transient chaos description is applicable even though the invariant chaotic sets don't exist in the dissipative system. This is one of the main statements of the theory of doubly-transient chaos [48] that was mentioned in chapter 3. In chapter 6 we propose an extension of this theory.

5.2.2 Survival Probability

We now focus on the survival probability $P(t)$ of trajectories under dissipation. Dissipation creates an attractor at the origin corresponding to a point of zero energy. The basin of attraction of this fixed-point attractor is comprised by initial conditions in or around KAM islands [47, 46] and orbits initialised near the stable manifold

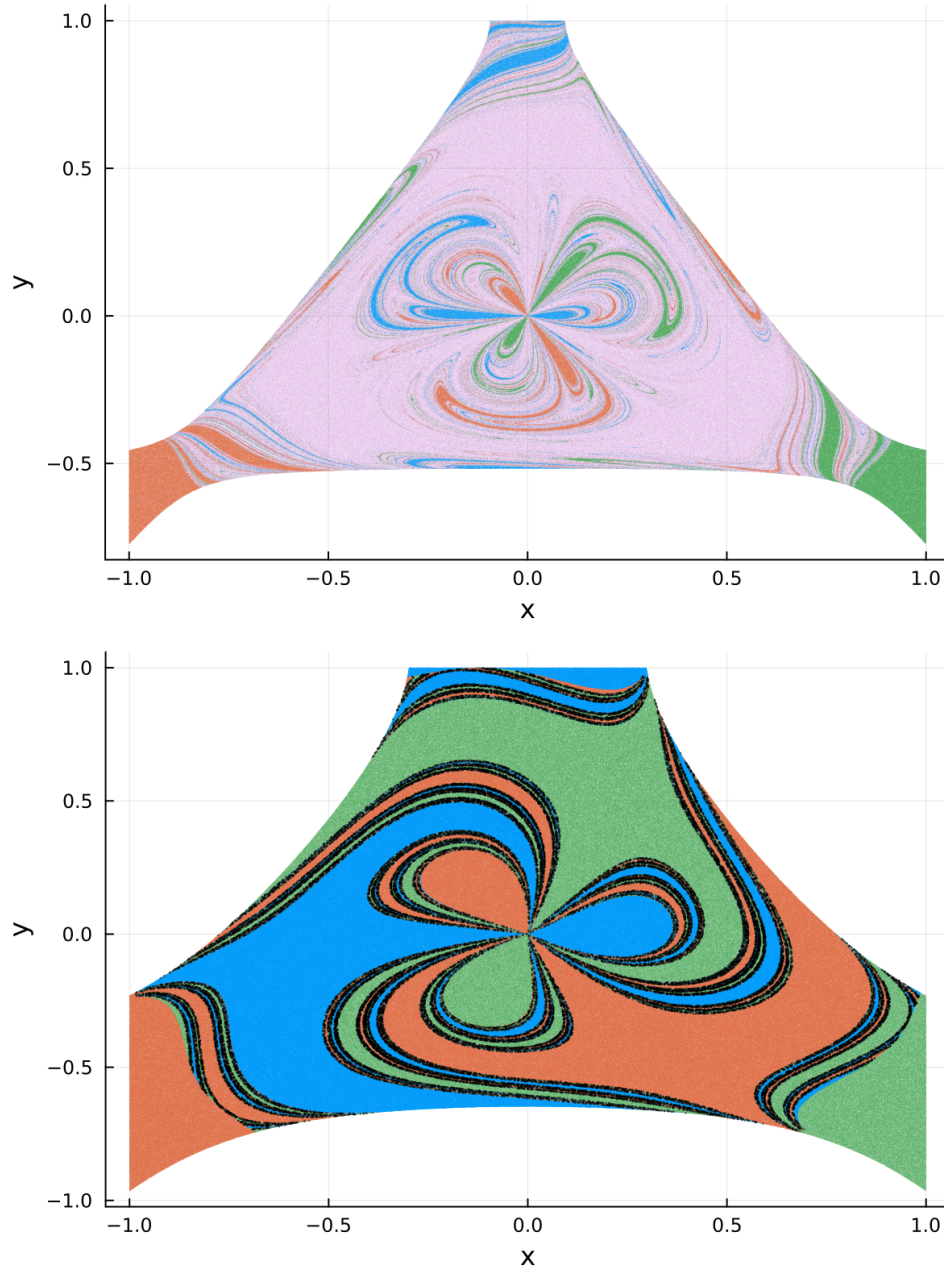


FIGURE 5.7: Basins of escape through different exit channels in the dissipative Hénon-Heiles system. The color scheme for escaping points is the same as for the conservative system. **(Top):** Basins for $E_0 = 0.18$ and $\gamma = 0.001$. The pink points correspond to orbits which remain in the system long enough for the energy to fall below E_c after which escape is not possible. **(Bottom):** Basins for $E_0 = 0.3$ and $\gamma = 0.01$. The trapped points here are colored black, and in a larger marker size to aid visualisation. $N_0 = 5 \cdot 10^6$ points are used to generate each figure.

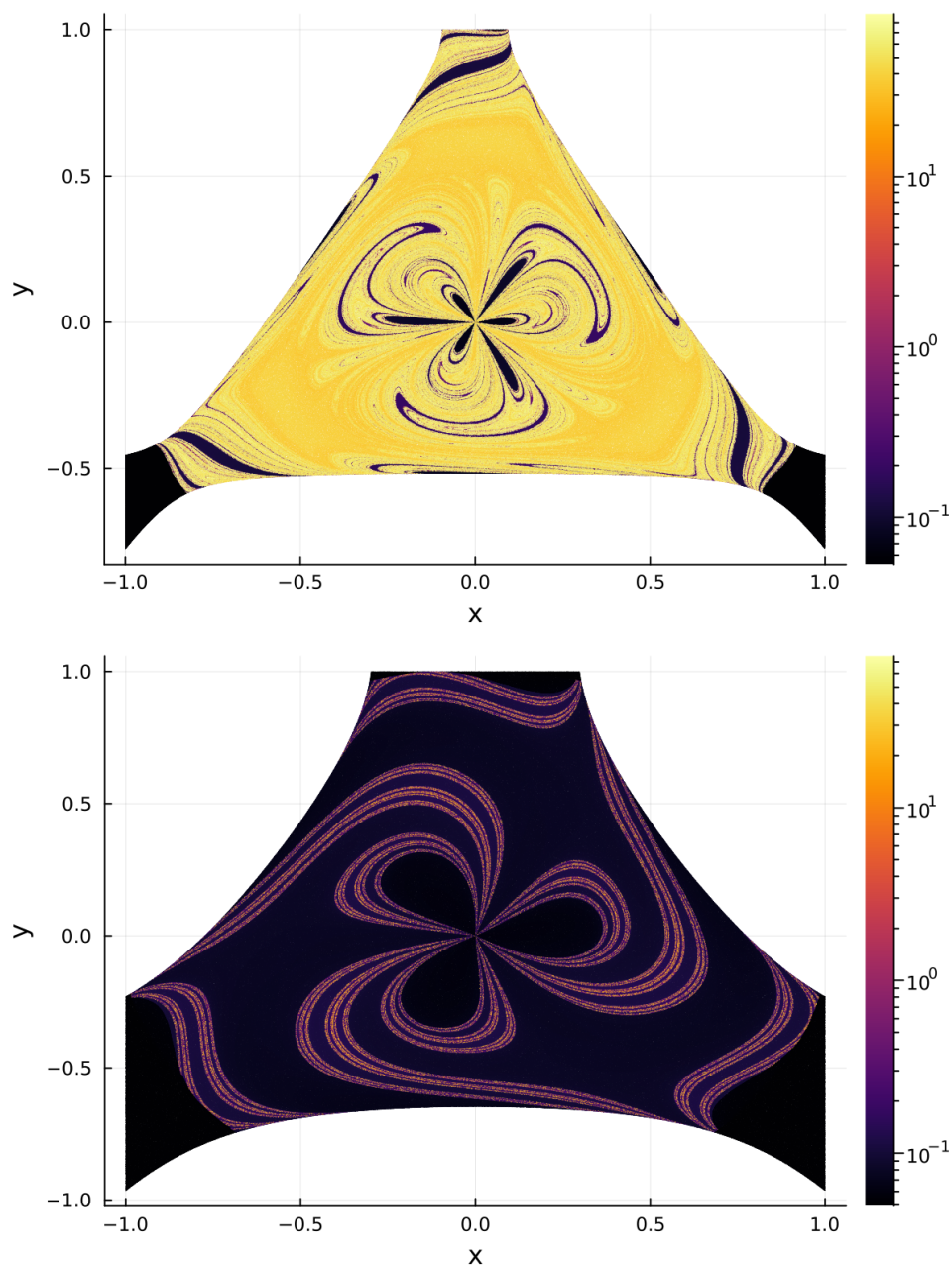


FIGURE 5.8: Coloration of the configuration space coordinates of initial conditions in the Poincaré section S_2 according to their escape/settling time in the dissipative Hénon-Heiles system. The top and bottom panels correspond to the same values of the energy E as in Fig. 5.7. $N_0 = 5 \cdot 10^6$ points are used to generate each plot.

of the chaotic saddle of the conservative system. These are long-living trajectories for which dissipation acts for sufficiently long time to bring their energy to E_c . In comparison to the conservative case, the attractor provides an additional way for an orbit to “leave” the system.

The numerical results for $P(t)$ for various dissipation strengths γ are shown in Fig. 5.9. Different decay regimes and the effects of increasing γ are readily visible. After an initial period, the $P(t)$ curve deviates from the simple exponential of the Hamiltonian system (blue line), showing a slower decay (increased trapping).

In order to characterize the observed survival probability $P(t)$, we introduce three characteristic times τ_0 , τ_1 , and τ_2 . The interval $[0, \tau_0]$ is the period during which the decay is highly-dependent on the choice of initial ensemble. Consequently, τ_0 is effectively independent of γ . The time τ_1 that marks the start of the enhanced trapping regime can be thus defined as

$$\tau_1 := \min\{t \geq \tau_0 | P_{\gamma>0}(t) = \alpha P_{\gamma=0}(t)\}, \quad (5.4)$$

where $\alpha > 1$ is an arbitrary constant that demarcates when significant deviations from the non-dissipative system are observable (we use $\alpha = 2$ for our experiments). When $P(t)$ is almost constant (no escape), there is a sudden and dramatic increase in the decay that continues until there are no points left. We denote by τ_2 the time that marks the end of the enhanced trapping regime, when $P(t)$ decays quickly (exponentially) towards zero. The numerical results show that both τ_1 and τ_2 increase with decreasing γ , consistent with the expectation that $P(t)$ varies continuously with γ . In the next chapter we show how these numerical observations can be theoretically explained.

5.2.3 Uncertainty Dimension

We analyse the fractality of the exit basin boundaries using the uncertainty dimension outlined in Chapter 3. As a reminder, the fraction $f(\varepsilon)$ of uncertain points at a given resolution ε scales as $f(\varepsilon) \sim \varepsilon^\alpha$, where $\alpha = N - D$. This leads to the estimate

$$D \approx N - \left. \frac{d \ln f(\varepsilon')}{d \ln \varepsilon'} \right|_{\varepsilon'=\varepsilon}. \quad (5.5)$$

In general there may be a non-trivial scale dependence with ε in the dimension, with constant α manifesting under hyperbolic conditions. This motivates one to speak of an *effective* dimension at a given ε . It has been established that in non-hyperbolic chaotic scattering, the asymptotic ($\varepsilon \rightarrow 0$) uncertainty dimension of the

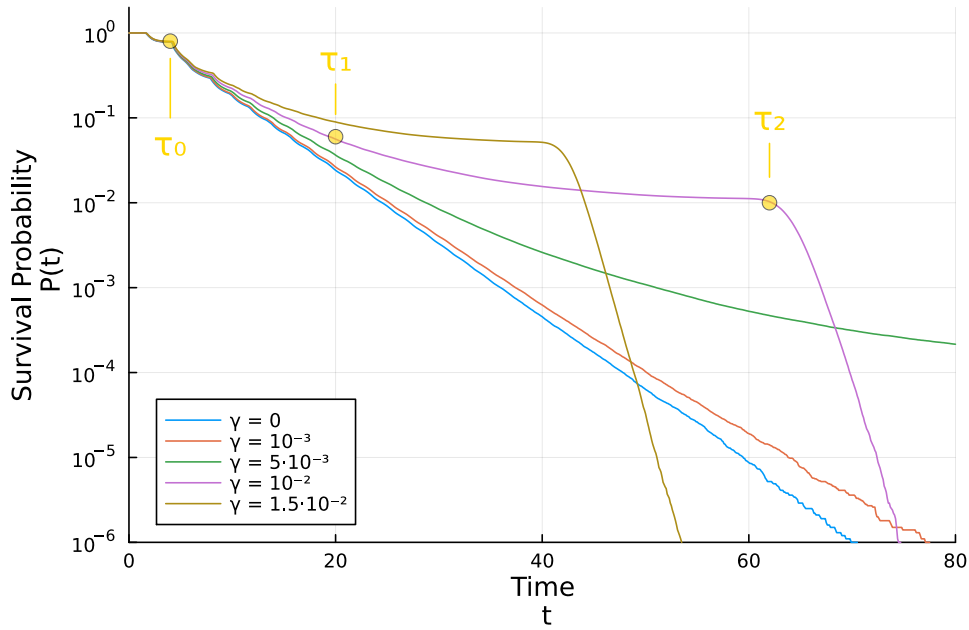


FIGURE 5.9: Distribution of trajectory lifetimes in the dissipative Hénon–Heiles system for various small dissipations γ . The survival probability for the Hamiltonian setting ($\gamma = 0$, blue curve) is added for comparison. For the purple curve ($\gamma = 10^{-2}$), the approximate positions of the timepoints τ_0 , τ_1 and τ_2 are marked.

basin boundaries approaches the dimension of the phase space [41]. This corresponds to $\alpha \rightarrow 0$, or an increasing effective dimension. An effect of dissipation that has been observed in scattering settings is an immediate and considerable *decrease* in the dimension [64, 47], with the boundaries potentially becoming non-fractal below certain length scales [14], implying $\alpha \rightarrow N - 1$. All three of these scaling behaviours can be understood with Cantor set arguments which are outlined in the aforementioned references. A visualisation suggestive of this construction is given in Fig. 3.5. The construction for the dissipative system [48] implies that the convergence of the dimension is logarithmically slow with respect to the length scale. We present some results for the scaling in the Hénon–Heiles system in Fig. 5.10.

The approximate dimension under the dissipative dynamics is seen to be slowly increasing as ε decreases, in accordance with e.g. [64]. The current numerical resolution is insufficient to observe the finite-scale fractality transition predicted by doubly-transient chaos, but this is not surprising. A practical consequence of the logarithmically slow convergence of D is that accurate estimates require going to extremely small length scales, often beyond reasonable computational limits. This is also reflected in the scaling for the conservative system, which is approximately constant over the specified range in ε . The chosen value of E lies at the upper end of the non-hyperbolic regime, and so we expect the dimension to eventually increase

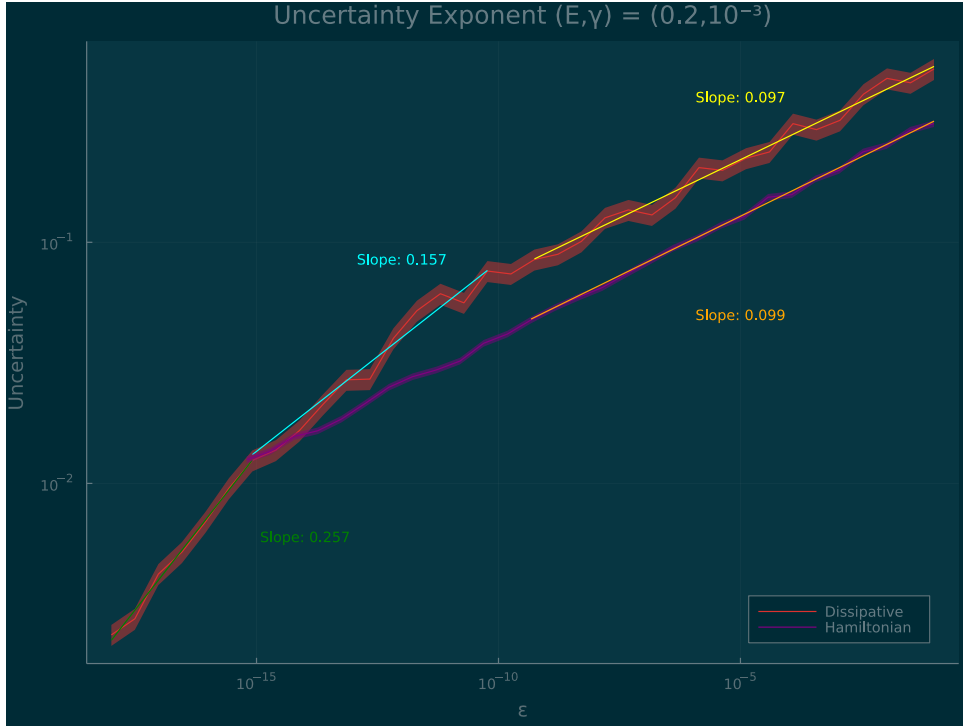


FIGURE 5.10: Scaling of the fraction of uncertain points with ε in the dissipative Hénon-Heiles system. The parameters are $(E, \gamma) = (0.2, 0.001)$. The results for the conservative system at the same energy are added for comparison (see legend). Initial conditions are chosen from a line segment in the configuration space (x, y) intersecting all boundaries except for the KAM islands. The uncertainty algorithm is run until a target number of uncertain points are found, here 100. The number of initial conditions required for this is used in the error ribbons. The approximate slope of $\log f(\varepsilon)$ against $\log \varepsilon$ is overlaid for various intervals in ε .

towards 1. This is more easily observed at larger ε by choosing points in a different coordinate plane/Poincaré section, See e.g. [77].

5.3 The Copenhagen problem

The three-body problem holds a significant place in the annals of classical mechanics. Its origin is rooted in the work of Newton, who famously proposed that the gravitational force F between two point masses is described by an inverse-square law:

$$F = \mathcal{G} \frac{m_1 m_2}{r^2}, \quad (5.6)$$

for masses m_1, m_2 separated by a distance r and $\mathcal{G}$ denoting the universal gravitational constant. Motivated by questions regarding the long-term stability of the solar system, the three-body problem is concerned with determining the motion of three celestial bodies, whose pairwise gravitational interactions are given by eq. (5.6). The simplicity of the problem statement belies the considerable challenge that it offers, which has made it an attractive problem for centuries of mathematicians. A possible formulation of the full three-body problem in Hamiltonian mechanics is specified by a system of 18 differential equations for the positions and momenta of each body, with three position ($\mathbf{r}_i = [x_i, y_i, z_i]$) and momentum ($\mathbf{p}_i = [p_{x_i}, p_{y_i}, p_{z_i}]$) coordinates per body):

$$\begin{aligned} \frac{d\mathbf{r}_i}{dt} &= \frac{\partial H}{\partial \mathbf{p}_i} \\ \frac{d\mathbf{p}_i}{dt} &= -\frac{\partial H}{\partial \mathbf{r}_i}. \end{aligned} \quad (5.7)$$

The Hamiltonian is

$$H(\mathbf{r}, \mathbf{p}) = \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} + \frac{\mathbf{p}_3^2}{2m_3} - \left(\mathcal{G} \frac{m_1 m_2}{|\mathbf{r}_1 - \mathbf{r}_2|} + \mathcal{G} \frac{m_2 m_3}{|\mathbf{r}_3 - \mathbf{r}_2|} + \mathcal{G} \frac{m_3 m_1}{|\mathbf{r}_3 - \mathbf{r}_1|} \right). \quad (5.8)$$

Henri Poincaré famously proved that the system is non-integrable, and these efforts led to the first descriptions of deterministic chaos. The high number of degrees of freedom and low number of conserved quantities motivated many attempts to reduce the dimensionality of the problem. One such proposal, first formulated by Euler, is to consider a simplified version of the system in which one of the three bodies has negligible mass. The corresponding force terms in the equations of motion

are ignored, and so the two massive bodies (the *primaries*) move in *Keplerian ellipses*⁵. This approximation is appropriate to describe many configurations in the solar system, e.g the Sun, Jupiter and comets [49]. The motion of the third body (the *particle*) is also assumed to be co-planar with the orbit of the primaries. Together, these assumptions (negligible mass m_3 , circular motion of the primaries and co-planar motion of the particle) represent a substantial simplification of the original three-body problem, which is often called the *planar circular restricted three-body problem* or PCR3BP for short. We will focus on a rather exotic scenario in which the two primaries are of equal mass, which is called the *Copenhagen problem*.

There are several long-term outcomes in both the PCR3BP and the full three-body problem. One possibility, the most likely, is that one of the three bodies (the particle in the former case) is ejected from the configuration after some time and flies off to infinity. A second possibility is that some or all⁶ of the bodies may collide. A third outcome is that the configuration of the three bodies persists for all time, showing neither escape nor collision. The uncertainty in associating an initial condition with one of these outcomes, as well as the complicated dynamics that unfold before the final outcome, make the three body problem an interesting setting to study transient chaos. In this section we investigate the dynamics of the Copenhagen problem in the context of transient chaos, and of chaotic scattering. In the interest of brevity, we eschew some details which may be important in more focused studies in celestial mechanics, such as discussion of orbital resonances, stability of equilibria (Lagrange points), detailed classification of orbits, and physical characteristics of the bodies. For more details in these directions, see e.g. [68, 49, 50].

5.3.1 Equations of motion

We first consider an inertial, or *sidereal*⁷ reference frame $(\mathcal{O}, \xi, \eta)$ in the plane of rotation of the primaries, with origin $\mathcal{O}$ in the barycentre of the primaries. Let the two primaries be denoted P_1 and P_2 , with masses m_1 and m_2 respectively. The coordinates of the primaries are denoted (ξ_1, η_1) and (ξ_2, η_2) . We define a new mass parameter $\mu = \frac{m_2}{m_1 + m_2}$ and choose units of mass so that $\mathcal{G}(m_1 + m_2) = 1$. The two masses in this unit system are $\mu_1 = \mathcal{G}m_1 = 1 - \mu$ and $\mu_2 = \mathcal{G}m_2 = \mu$. Additionally, we choose units of length so that the distance between P_1 and P_2 is one.

⁵The dynamics of two gravitationally-attracted bodies – the *two-body problem* – is solvable, and the relative motion of one body to the other traces out conic sections. We hereby assume the primaries move in circular orbits around the centre of mass.

⁶Only binary (particle and one primary) collisions are possible in the PCR3BP

⁷A term in astronomy referring to measurements made relative to fixed stars.

The equations of motion for the particle are [49]

$$\begin{aligned}\ddot{\xi} &= \mu \frac{\xi_1 - \xi}{r_1^3} + (1 - \mu) \frac{\xi_2 - \xi}{r_2^3}, \\ \ddot{\eta} &= \mu \frac{\eta_1 - \eta}{r_1^3} + (1 - \mu) \frac{\eta_2 - \eta}{r_2^3},\end{aligned}\tag{5.9}$$

where $r_1 = \sqrt{(\xi_1 - \xi)^2 + (\eta_1 - \eta)^2}$ and $r_2 = \sqrt{(\xi_2 - \xi)^2 + (\eta_2 - \eta)^2}$ are the distances from the particle to each primary. This comes from a Lagrangian L_{inertial}

$$L_{\text{inertial}}(\xi, \eta, \dot{\xi}, \dot{\eta}, t) = \frac{1}{2}(\dot{\xi}^2 + \dot{\eta}^2) + \frac{1 - \mu}{r_1} + \frac{\mu}{r_2}.\tag{5.10}$$

In the sidereal frame frame, the positions of each primary as a function of time are described by a rotation with angular velocity n :

$$\begin{aligned}(\xi_1, \eta_1) &= (-\mu \cos nt, -\mu \sin nt) \\ (\xi_2, \eta_2) &= ((1 - \mu) \cos nt, (1 - \mu) \sin nt),\end{aligned}\tag{5.11}$$

which means that L_{inertial} is time-dependent. The Hamiltonian of the particle in the sidereal frame is

$$H_{\text{sidereal}}(\mathbf{p}, \mathbf{q}, t) = \frac{1}{2}(p_\xi^2 + p_\eta^2) - \frac{1 - \mu}{r_1} - \frac{\mu}{r_2}.\tag{5.12}$$

This Hamiltonian is non-autonomous because of the explicit time dependence in the positions of the primaries. We are only concerned with the relative motion of the particle, so a natural next step is to transform to a rotating, or *synodic*⁸ reference frame. The origin in the synodic frame is the same as in the inertial frame. If the frame rotates at a uniform rate n then the locations of the primaries are now fixed. We choose coordinate axes (x, y) in this reference frame so that the primaries lie on the x -axis at coordinates $(x_1, y_1) = (-\mu, 0)$ and $(x_2, y_2) = (1 - \mu, 0)$. The relation between the coordinates (ξ, η) and (x, y) space is given by

$$\begin{aligned}\xi &= x \cos nt - y \sin nt \\ \eta &= x \sin nt + y \cos nt,\end{aligned}\tag{5.13}$$

which can be inverted to give

$$\begin{aligned}x &= \xi \cos nt + \eta \sin nt \\ y &= -\xi \sin nt + \eta \cos nt.\end{aligned}\tag{5.14}$$

⁸Another astronomical term referring to rotations of a body with respect to its parent star.

This mapping is a canonical transformation from $(\mathbf{q}, \mathbf{p})$ to $(\mathbf{Q}, \mathbf{P})$ where $\mathbf{q} = (\xi, \eta)$, $\mathbf{p} = (p_\xi, p_\eta)$, $\mathbf{Q} = (x, y)$ and $\mathbf{P} = (p_x, p_y)$, and is obtained by the type 2 (see Table 2.1) generating function $G = G(\mathbf{q}, \mathbf{P}, t)$:

$$G(\mathbf{q}, \mathbf{P}, t) = (\xi \cos nt + \eta \sin nt)p_x + (-\xi \sin nt + \eta \cos nt)p_y. \quad (5.15)$$

Taking derivatives of G with respect to P yields eq. (5.14), and taking derivatives with respect to the q gives the relations between the momenta:

$$\begin{aligned} p_\xi &= \frac{\partial G}{\partial \xi} = p_x \cos nt - p_y \sin nt \\ p_\eta &= \frac{\partial G}{\partial \eta} = p_x \sin nt + p_y \cos nt. \end{aligned} \quad (5.16)$$

Observe that $p_\xi^2 + p_\eta^2 = p_x^2 + p_y^2$. Accordingly, the Hamiltonian in the synodic coordinates is given by

$$H_{\text{synodic}} = H_{\text{sidereal}} + \frac{\partial G}{\partial t}, \quad (5.17)$$

where from eqs. (5.14) and (5.15) we have that

$$\frac{\partial G}{\partial t} = n(y p_x - x p_y), \quad (5.18)$$

a term corresponding to the planar angular momentum. The new Hamiltonian is now autonomous, as the positions of the primaries are now fixed and

$$\begin{aligned} r_1 &= \sqrt{(x + \mu)^2 + y^2} \\ r_2 &= \sqrt{(x - 1 + \mu)^2 + y^2}. \end{aligned} \quad (5.19)$$

For convenience of notation, we let

$$\Phi(r_1, r_2) = -\frac{1 - \mu}{r_1} - \frac{\mu}{r_2}, \quad (5.20)$$

so that the Hamiltonian H_{synodic} can be written as

$$H_{\text{synodic}} = \frac{1}{2}(p_x^2 + p_y^2) + \Phi(r_1, r_2) + n(y p_x - x p_y). \quad (5.21)$$

Hamilton's equations then read

$$\begin{aligned}
 \dot{x} &= \frac{\partial H_{\text{synodic}}}{\partial p_x} = p_x + ny, \\
 \dot{y} &= \frac{\partial H_{\text{synodic}}}{\partial p_y} = p_y - nx, \\
 \dot{p}_x &= -\frac{\partial H_{\text{synodic}}}{\partial x} = np_y - \frac{\partial \Phi}{\partial x}, \\
 \dot{p}_y &= -\frac{\partial H_{\text{synodic}}}{\partial y} = -np_x - \frac{\partial \Phi}{\partial y}.
 \end{aligned} \tag{5.22}$$

In the Copenhagen problem we take $\mu = 1/2$, and for simplicity choose units of time so that $n = 1$.

5.3.2 Jacobi integral and Hill regions

It was noted above that the Hamiltonian in the sidereal reference frame was non-autonomous, which implies that energy is not conserved in the PCR3BP. This is a consequence of ignoring the influence of the particle on the primaries, essentially allowing the particle to extract energy from the system. It is also easily shown that the angular momentum is not conserved. Nonetheless, we will refer to H_{synodic} as the “energy”. There is only one first integral in the PCR3BP, called the *Jacobi integral*, which is proportional to H_{synodic} . By rearranging the first two equations in Eq. (5.22), we see that $p_x = \dot{x} - ny$ and $p_y = \dot{y} + nx$. Taking time derivatives and combining with the last two equations in (5.22), we have

$$\begin{aligned}
 \ddot{x} - 2n\dot{y} &= \frac{\partial U}{\partial x} \\
 \ddot{y} + 2n\dot{x} &= \frac{\partial U}{\partial y},
 \end{aligned} \tag{5.23}$$

where the function $U = \frac{n^2}{2}(x^2 + y^2) - \Phi$ can be thought of as a ‘pseudo-potential’⁹ acting as the source of the centrifugal and gravitational forces on the particle in the synodic reference frame, but not of the Coriolis terms. In this way, H_{synodic} (5.21) can be written¹⁰ as

$$H_{\text{synodic}} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - U. \tag{5.24}$$

The Jacobi integral C_J is taken to be $-2H_{\text{synodic}}$:

$$C_J = 2U - (\dot{x}^2 + \dot{y}^2). \tag{5.25}$$

⁹Contrary to some other physical conventions, U is a positive function.

¹⁰Strictly speaking, this isn't appropriate as the Hamiltonian should depend on the momenta, not the velocities.

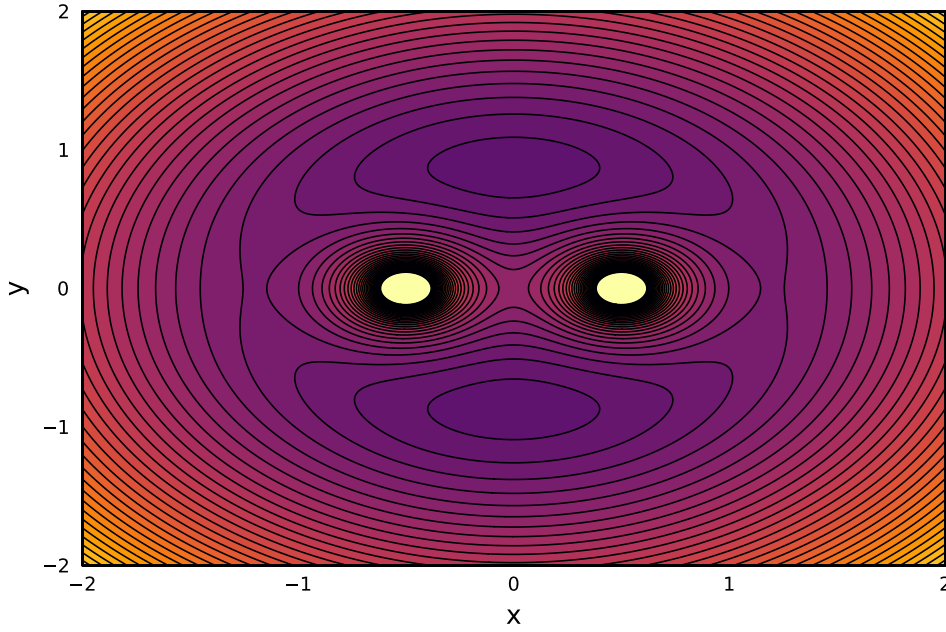


FIGURE 5.11: Contour plot of the pseudo-potential U in the Copenhagen problem. Darker colours correspond to larger values of U , which demonstrate the forbidden regions – particles with an initial energy corresponding to a lighter colour are dynamically forbidden from entering the darker areas. U goes to ∞ very quickly near the primaries, located at $(\pm\frac{1}{2}, 0)$, and more slowly as $|x|, |y| \rightarrow \infty$.

The value of the Jacobi integral constrains the particle to certain regions of the phase space, called *Hill regions*, which we denote R_{Hill} :

$$R_{\text{Hill}} = \{(x, y) \in \mathbb{R}^2 \mid U(x, y) \geq \frac{1}{2}C_J\}. \quad (5.26)$$

$C_J = 2.75$ corresponds to the maxima of U , known as the Lagrange points L_4 and L_5 , forming equilateral triangles with the primaries. For $C_J \leq 2.75$, Hill's region is the entire configuration space $\mathbb{R}^2$. For larger values of C , the motion of the particle is increasingly restricted. The complement of R_{Hill} is called the *forbidden region*. A view of the Hill regions for a range of values of C is given in Fig. 5.11. The physical interpretation of the Hill regions is that, for low enough values of the energy, the particle starting close to a primary will remain orbiting that primary ('*Hill stability*'). As the energy is increased, passage between the gravity wells of the primaries becomes possible. For even larger energies ($C_J \leq 2.75$) the particle can move freely around the phase space, commonly escaping to infinity.

5.3.3 Transient chaos in the Copenhagen problem

There are physically meaningful ways for the particle to "leave" bounded neighbourhoods of the primaries; either by escape to infinity – the gravitational influence of the primaries goes to zero with $1/r$ – or collision with a primary. We may thus study the Copenhagen problem in the context of transient chaos. Generally, we consider motions for which the full configuration space is initially accessible, and so take $C = 2.75$.

We first introduce some criteria for particle escape and collision. We say that the particle has escaped if its distance from the origin in the synodic frame exceeds a fixed radius R_{sys} . In the simulations we choose $R_{sys} = 100$, although much smaller values (e.g, 10) have been considered in the literature [50]. We choose a larger escape radius because an effect of dissipation at the chosen strength is that orbits can at first appear to escape to intermediate distances ($10 < R < R_{sys}$) before being sufficiently slowed by it that they are eventually recaptured by the primaries, a phenomenon that is missed if R_{sys} is chosen too small. The particle will be considered to have collided with a primary if its distance to that primary becomes less than a fixed radius R_i . Along with equal masses, we assume that both primaries are the same size, with $R_1 = R_2 = 10^{-4}$. This condition amounts to opening two small holes in the configuration space through which the particle can leave [4].

A note about numerical stability and performance

Opening holes at the locations of the primaries has the added benefit of helping to avoid potential stability issues in the numerical integrator caused by the divergence of the $1/r_i$ terms. Formally, these issues can be resolved by regularisation of the equations of motion to remove the singularities. Details for this procedure in the PCR3BP can be found in e.g [12]. For our purposes, the singularities do not seem to cause severe stability issues provided the collision radii R_1, R_2 aren't too small. Nonetheless, we do observe somewhat more accurate integrations, as measured by the absolute energy error over the trajectory, in the (non-regularized) PCR3BP using a high-order Runge-Kutta method than when using symplectic methods. This is perhaps related to the requirement of a fixed time-step size with symplectic methods, leading to poorer conservation properties when the problem is stiff. The variable step size of Runge-Kutta methods better avoids the issue in these circumstances. However, these integrations generally took much longer to run because the step size was small for very long times in the case of sticky orbits. In any case, we found the accuracy of both methods adequate for these experiments in the conservative

system, and the choice did not dramatically affect the observations. Even on the longest ($T_{max} = 50,000$) integrations using the Runge-Kutta method, energy errors smaller than 10^{-11} were typical.

Initial conditions

We choose initial conditions according to the Poincaré section we denote by $\mathcal{S}_3$:

$$\mathcal{S}_3 = \{(x, y) \in \mathbb{R}^2 | \dot{r} = 0, \dot{\phi} < 0\}, \quad (5.27)$$

which is similar to the section $\mathcal{S}_2$ we used to study the Hénon-Heiles system, except we take $\dot{\phi}$ (and subsequently, the initial velocities) with opposite sign. The initial velocities are chosen in the same way as in eq. (2.22), but with $E = -1/2C_J$ and $V = -U$.

$$\begin{aligned} \dot{x}_0 &= \frac{y_0}{\sqrt{x_0^2 + y_0^2}} \sqrt{2U(x_0, y_0) - C_J} \\ \dot{y}_0 &= -\frac{x_0}{\sqrt{x_0^2 + y_0^2}} \sqrt{2U(x_0, y_0) - C_J}. \end{aligned} \quad (5.28)$$

Exit basins

We may similarly determine the basins of initial conditions corresponding to each dynamical outcome (escape, collision, bounded motion), and we present the visualisation of this in Fig. 5.12. The choice between $\mathcal{S}_2$ and $\mathcal{S}_3$ corresponds to choosing between pro-grade ($\mathcal{S}_2$) or retro-grade¹¹ ($\mathcal{S}_3$) initial velocities for the particle, which have surprising effects on the resulting dynamics. In fact, it appears in our experiments in $\mathcal{S}_3$ that the chaotic sets associated with transient chaos do not have finite spatial extent, and are numerically detectable out to any chosen R_{sys} . This is perhaps related to the previously studied [43, 44, 75] ‘chaos at infinity’ in the restricted three body problem, and could potentially have consequences for our results. However, we do not observe this for initial conditions chosen in $\mathcal{S}_2$. For most of our numerical results for the Copenhagen problem presented in the figures in this chapter, there is a corresponding figure in Appendix B using initial conditions from $\mathcal{S}_2$.

¹¹These are astronomical terms referring to orbital motion of a body in the same (pro-grade) or opposite (retro-grade) direction as the primary they orbit. In this system we have set the primaries to orbit counter-clockwise, as viewed from above the orbital plane.

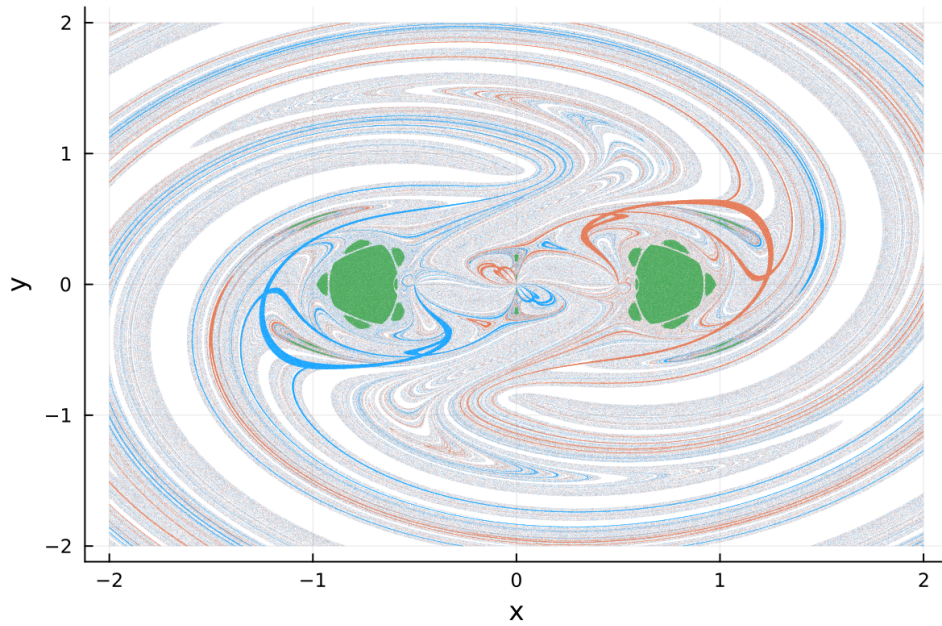


FIGURE 5.12: Basin topology in the section S_3 , visualized in configuration space (x, y) , for the Copenhagen problem. The colours of the pixels correspond to: escape (white), collision with a primary (blue and orange), and KAM islands/regular motion (green). $N_0 = 2 \cdot 10^7$ initial conditions were chosen from S_3 with $C_I = 2.75$. A maximum integration time of 20000 was used.

The basin landscape in Fig. 5.12 demonstrates considerable complexity that is compatible with the generic picture of transient chaos in non-hyperbolic systems. Particle escape is seen to be the most likely outcome for the chosen parameters, with large contiguous spiral arms of initial conditions all leading to escape taking up a large portion of the area. The smooth sets of initial conditions leading to collision with a primary also take intricate shapes, concentrated mostly around the barycenter and close to the KAM islands, but with filamentary tails also following the spiral structure. An orbit classification scheme was developed in [50] to identify the generic behaviour of these orbits. It was found that the points belong to the two main/large islands tend to orbit around their closest primary, and the points belonging to the smaller regular structures (the thin bands above and below the larger islands, and the two components lying on the y-axis close to the origin) execute quasiperiodic motions *between* the primaries. The most notable feature of this figure, however, is the significant fractalization of the section. Away from the contiguous regions of all one color, points of all colors can be found arbitrarily close together. The volume of initial conditions belonging to each basin depends sensitively on the size of the holes R_1, R_2 , which has been studied in e.g [78].

The coloration of these initial conditions by their escape/collision time is given

in Fig. 5.13. As expected, the open sets belonging to each basin all exit the system after similar short times, and the longest-lived orbits start in the fractal region, which supports the stable manifold of the chaotic saddle.

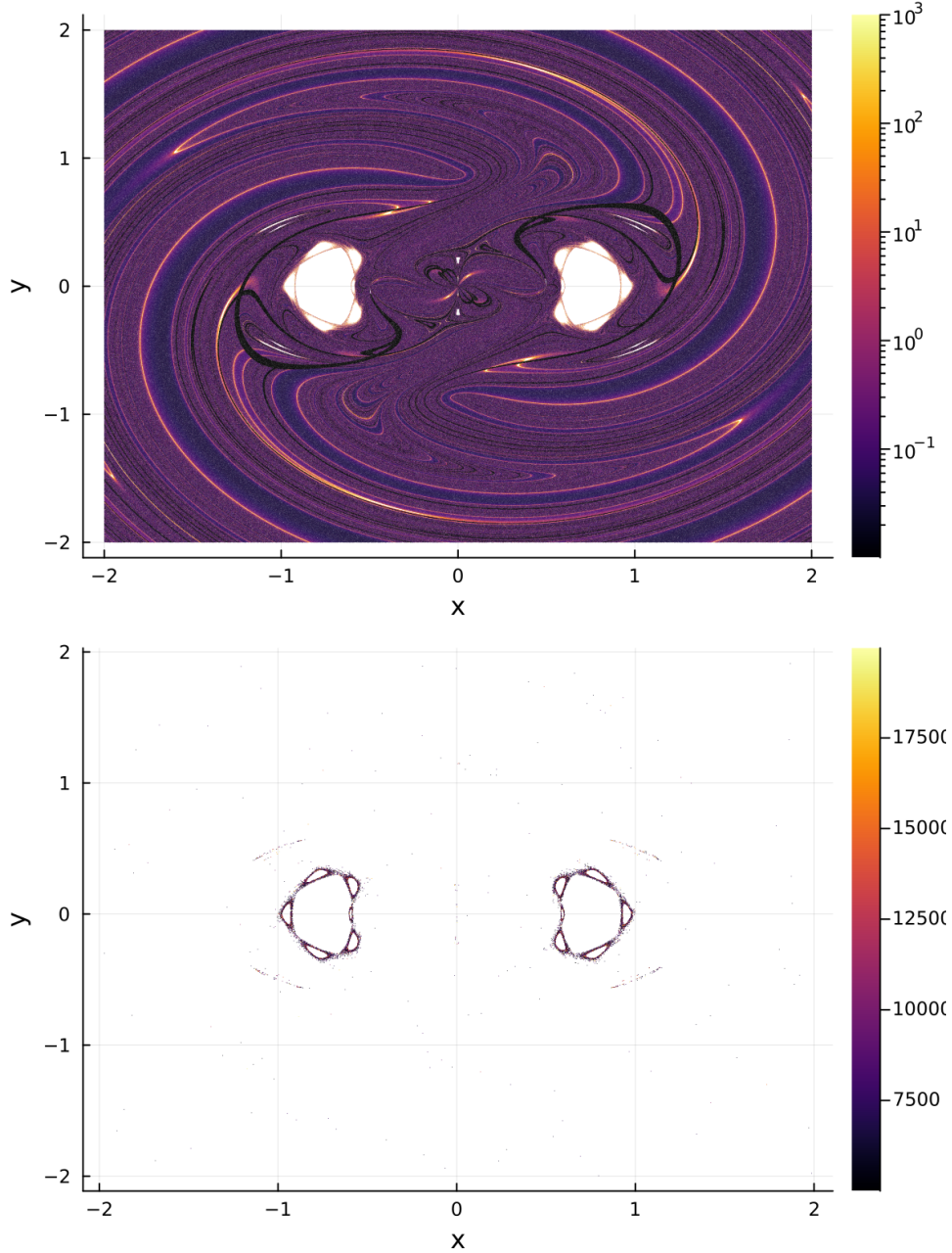


FIGURE 5.13: Spatial distribution of escape/collision times of initial conditions in $\mathcal{S}_3$ in the Copenhagen problem, visualised in configuration coordinates (x, y) . The points belonging to KAM islands have not been plotted, as their escape time is infinite. The value of the Jacobi constant was $C_J = 2.75$. The domain of configuration space is the same as in Fig. 5.12. **(Top):** Initial conditions with escape/collision time less than 1000. **(Bottom):** Initial conditions with escape/collision time greater than 5000.

Survival probability

The presence of KAM islands implies that the saddle possesses non-hyperbolic components, which is supported by evidence of a power-law tail in the survival probability $P(t)$, which we also compute separately¹² for escaping and colliding initial conditions in Fig. 5.14 for two choices of R_{sys} .

A consequence of the choice of large R_{sys} is that typical escapes and collision can occur on significantly different timescales. The minimum escape time is found to be more than 100, while there is a significant set of initial conditions that collide with a primary before $t = 1$. For large times t (approximately, $2500 < t < 15000$ in the figure), the two sets of orbits decay at a similar algebraic rate. The sharp decay to zero after this period is a numerical artefact appearing because of the finite size of the ensemble. The algebraic decay is expected to continue indefinitely with increasing ensemble size or an ensemble initially concentrated around the sticky region. The green curve obtained as the distribution of lifetimes of all orbits, including the KAM islands, is decaying to a constant approximating the volume of the KAM islands. For the smaller R_{sys} , we observe a more obvious exponential decay for shorter times ($t < t_* \approx 200$) and a slower algebraic decay for large times. In general, transient chaos theory predicts that the escape/collision rate scales with the size of the leaks, and that the exponent of the algebraic decay will be the same for escape and collision. This latter point is verified by our survival probability observations.

Chaotic sets

We may again compute the chaotic saddle (assuming its existence) and its invariant manifolds using the sprinkler method, but here we demonstrate an expedient alternative. First, we approximate the stable manifold by filtering for the initial conditions of orbits that remain in the system for at least a long time t . Then we multiply each of the differential equations by -1 and approximate the stable manifold of the time-reversed dynamics. By the time-reversal symmetry of autonomous Hamiltonian systems, this corresponds to the unstable manifold of the original dynamics. The support of the saddle is approximated by where these two sets overlap¹³. The results of this process are presented in Fig. 5.15.

¹²This is because there can potentially be large timescale separation between typical escapes and collisions. This can happen if, e.g, R_{sys} is large (leading to inflated escape times) or R_1, R_2 are too large (meaning more collisions happen and on shorter times).

¹³The two computed sets won't overlap exactly due to numerical error, but this is valid approximation up to some resolution.

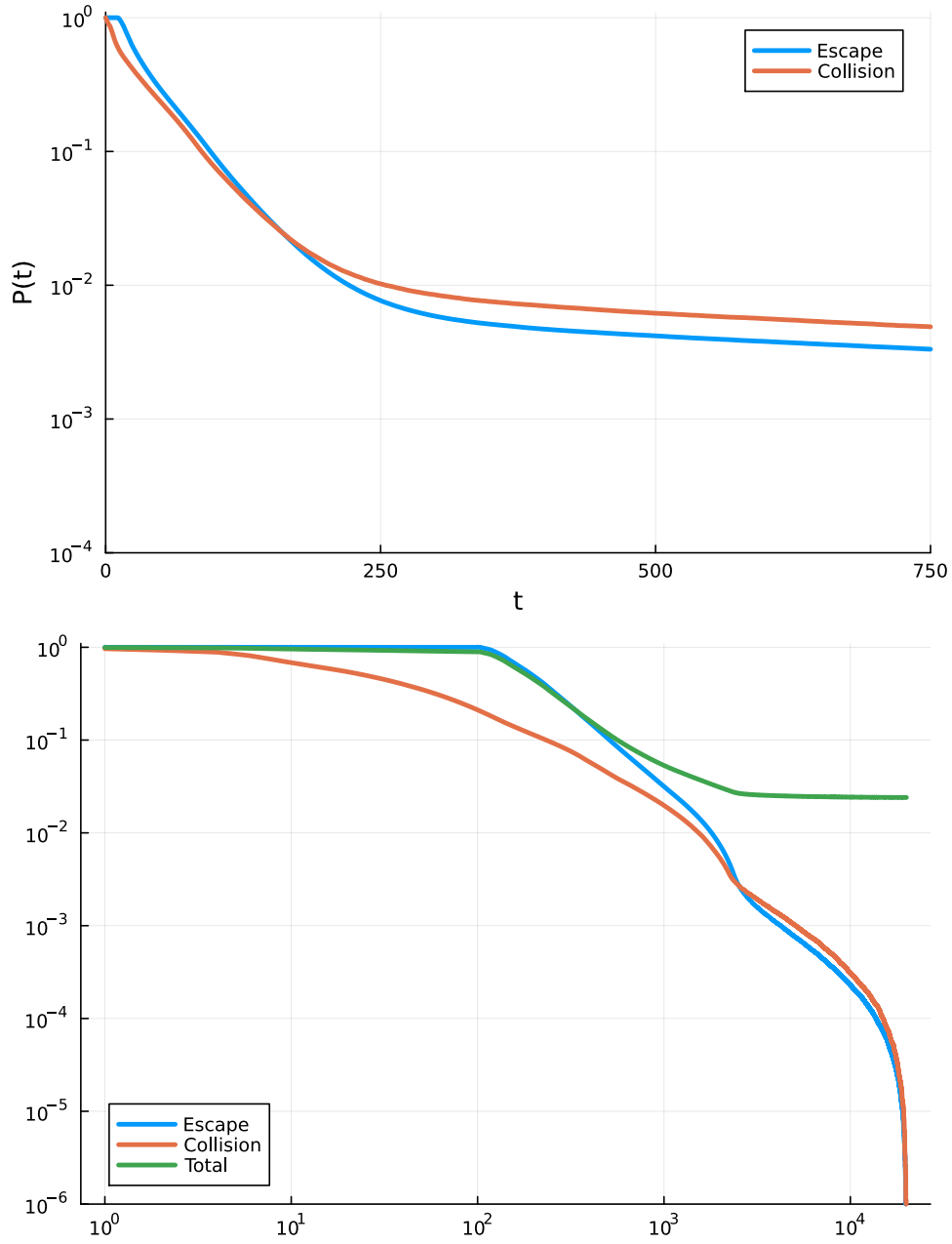


FIGURE 5.14: Survival probability in the Copenhagen problem. The decay in the cumulative distribution of lifetimes for both escaping (blue) and colliding (orange) orbits are shown for escape/collision times up to a maximum integration time of $t = 2 \cdot 10^4$. **(Top):** Survival probability for $R_{sys} = 10$. **(Bottom):** Survival probability for $R_{sys} = 100$ (note the different scale). Also plotted in green in the bottom panel is the "total" survival probability, i.e, the distribution of escape, collision and KAM orbits.

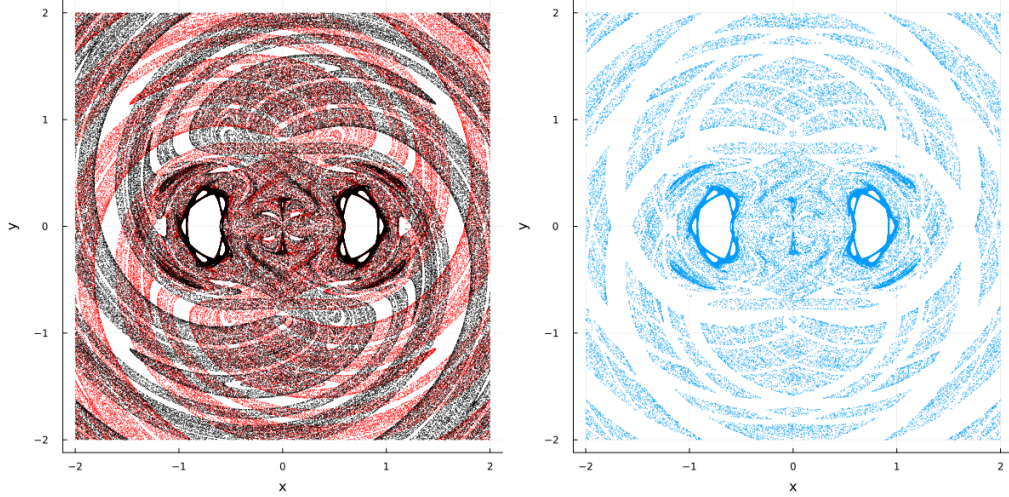


FIGURE 5.15: Approximate chaotic saddle (right) and stable/unstable manifolds (left) in the Copenhagen problem in the Poincaré section S_3 . The color scheme is the same as in Fig. 5.4. The manifolds were approximated by the points that remain in the system until at least $t = 150$. Here $C_f = 2.75$ and $R_{sys} = 100$.

The time chosen to determine "long-living" orbits in this figure is still in the intermediate time regime in which the hyperbolic components dominate and the survival probability decays exponentially (see Fig. 5.14). For longer times, the asymptotic picture is governed by the non-hyperbolic components and the density of surviving orbits will concentrate in the sticky regions at the borders of the KAM islands [4].

5.4 The Copenhagen problem with Poynting-Robertson drag

There are several physically relevant dissipative mechanisms in the dynamics of the solar system. One of the most well-studied effects, the *Poynting-Robertson drag* is related to the incident radiation pressure on small particles [8]. It is caused by anisotropic re-emission of sunlight that the particle absorbs. The main consequence of this is that the particle experiences an effective drag force, the impact of which is to cause the semi-major axis of the orbit to decay over time, leading to the particle slowly spiralling into the sun. Orbital resonances between the particle and other bodies can arrest this decay for a time. The Poynting-Robertston drag has often been considered in studies of the three body problem [12, 16, 42] and more general astronomical contexts [38, 1]. For the PCR3BP in the synodic frame, the drag force has the form [49]

$$F = -\frac{k}{r^2} \left(\dot{x} - y + \frac{x}{r^2}(x\dot{x} + y\dot{y}), \dot{y} + x + \frac{y}{r^2}(x\dot{x} + y\dot{y}) \right). \quad (5.29)$$

Here r refers to the distance from the primary which the source of the radiation, and k is a parameter controlling the strength of the drag, which is itself the product of several physical parameters. In our simulations we choose $k = 10^{-4}$. This force is the result of two contributions, which we label F_1 and F_2 . F_1 is the drag due to the radiation pressure of incident photons, which can be considered a force proportional to the momentum (corresponding to velocity in the sidereal frame) and opposite in sign. The amount of light incident on the particle is inversely proportional to the square of the distance from the star, which introduces a $1/r^2$ dependence:

$$F_1 = -\frac{k}{r^2}(\dot{x} - y, \dot{y} + x). \quad (5.30)$$

The term F_2 is due to the Doppler shift of incident light:

$$F_2 = -\frac{k}{r^4}(x\dot{x} + y\dot{y})(x, y), \quad (5.31)$$

which includes an radial momentum-like term and decreases much faster than F_1 with distance from the star. For this reason we neglect the contribution from the F_2 term. Additionally, we consider both primaries to be sources of the F_1 drag (as in e.g a binary star system), with identical dissipation strengths k . The resulting force from Poynting-Robertson drag on the particle is

$$F = (F_x, F_y) = -k \left(\frac{1}{r_1^2} + \frac{1}{r_2^2} \right) (\dot{x} - y, \dot{y} + x) \quad (5.32)$$

The force is added to eq. (5.23) to give the equations of motion for the dissipative system:

$$\begin{aligned} \ddot{x} - 2n\dot{y} &= \frac{\partial U}{\partial x} + F_x \\ \ddot{y} + 2n\dot{x} &= \frac{\partial U}{\partial y} + F_y. \end{aligned} \quad (5.33)$$

The Jacobi integral is no longer conserved. The ‘energy’ $E = -\frac{1}{2}C_J$ now varies as

$$\frac{dE}{dt} = -k \left(\frac{1}{r_1^2} + \frac{1}{r_2^2} \right) (\dot{x}^2 + x\dot{y} - y\dot{x} + \dot{y}^2), \quad (5.34)$$

which notably is not sign-definite. The sign of Eq. 5.34 depends on the specific trajectory and cannot be determined for arbitrary initial conditions by inspection. This means that the Poynting-Robertson drag is not a ‘true’ dissipation in the synodic frame in the sense that the ‘energy’ is not monotonically decreasing, although this is the case for most trajectories. One could also consider a so-called *nebular* drag

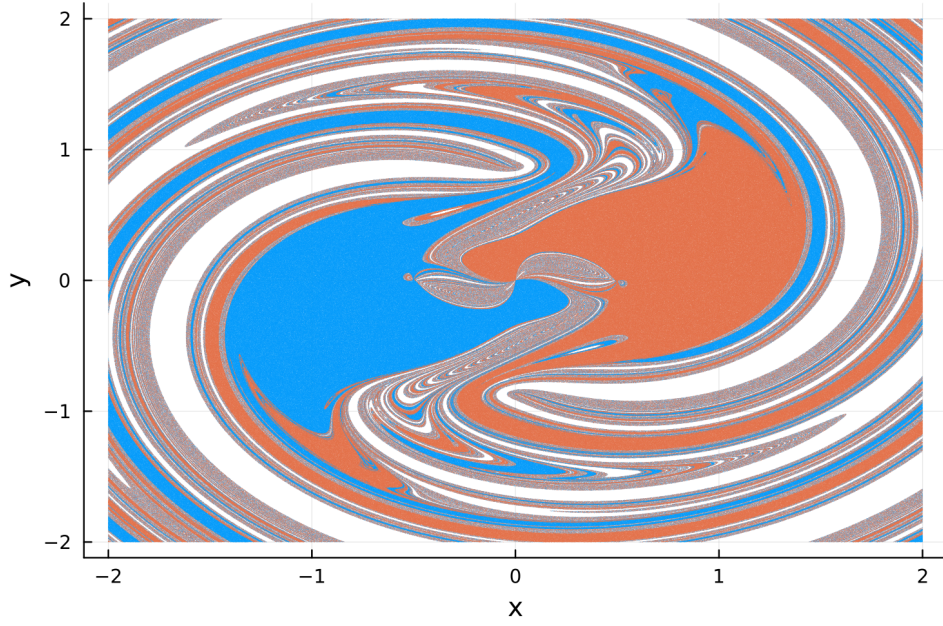


FIGURE 5.16: Basins of escape and collision in the section $\mathcal{S}_3$ for the Copenhagen problem with Poynting-Robertson drag. The color scheme is the same as in Fig. 5.12, with blue and orange points corresponding to collision and white to escape. The initial value of C_I was 2.75 and $k = 10^{-4}$.

force [49], which is proportional to the velocity in the synodic frame so that the time derivative of the energy is non-positive everywhere, but we do not do this here.

5.4.1 Exit basins

Fig. 5.16 shows that Poynting-Robertson dissipation dramatically alters the basin topology in the Copenhagen problem. Collision becomes a much more likely outcome for a random chosen initial condition¹⁴, and similarly to the Hénon-Heiles system, the KAM islands are destroyed entirely, with the main island chains around each primary disappearing into the collision basin associated with that primary. However, despite the significant enlargement of the collision basins there is still a considerable region in which complicated, seemingly fractal basin boundaries appear.

Individually, the trajectories starting inside the islands stay close and slowly tend inwards to the primary, and they tend to be relatively long-lived among collision orbits, as the top panel of Fig. 5.17 shows, suggesting that a similar picture to Fig. 5.6 might be observed. They are not, however, the longest-living collision orbits. The bottom panel of Fig. 5.17 shows that the longest-living points are still found close to

¹⁴collision was the outcome of roughly 60% of trajectories in our simulation when $k = 10^{-4}$, compared to 16% in the conservative system.

the boundary of the different basins. Of the $2 \cdot 10^7$ initial conditions in the simulation, the longest-living orbit escaped at around $t \approx 5800$, and the longest-living collision orbit reaching its fate at around $t \approx 5200$. The collision orbits with large lifetimes often begin to exit the system, and may reach a large radius from the origin before the dissipation acts sufficiently to arrest their escape (bring the radial velocity to zero, and then to negative) after which the particle will slowly spiral back inwards towards the primaries. There are large collision basins near each primary, the orbits starting in which collide extremely quickly ($t < 0.1$). These orbits very quickly fall into the gravity wells of the primaries, where the dissipation strength rapidly diverges.

5.4.2 Survival Probability

The existence of extremely long-living ($t > 5000$) orbits in the dissipative system can be attributed in large part to the non-uniform strength of the dissipation. The dissipation term is highly dependent on distance, with $F_1 \rightarrow 0$ for $r_1, r_2 \rightarrow \infty$ and $F_1 \rightarrow \infty$ as $r_1, r_2 \rightarrow 0$. Another factor is that the KAM islands disappear into the collision basins. The absence of the associated stickiness suggests an exponential rarity in long-living trajectories. The survival probability in the dissipative system is shown in Fig. 5.18. Compared to Fig. 5.14, it can be seen that the addition of Poynting-Robertson drag leads to an increased collision rate, and that the tail of the distribution drops off exponentially for both curves. That the curve for the escaping trajectories in the dissipative system looks almost the same away from the tail as the curve for the conservative system is interesting, but can perhaps be explained by:

- The escape radius R_{sys} is large and the drag strength k is small, so the dissipation term is essentially zero, and can even change sign, when orbits are approaching R_{sys} .
- As shown by the exit basins, the escape orbits which *are* dramatically affected by the dissipation are converted to collision orbits and so are not being counted in this curve.

It is noticed that the escape rate actually *increases* on intermediate times under Poynting-Robertson drag. This surprising observation is because there is now a considerable set of escaping orbits that extract energy from the system, due to the fact that eq. (5.32) can change sign depending on the path of the particle. Under a nebular or other sign-definite drag force we expect that this would not occur and that dissipation would slow the rate at which escaping orbits leave the system at R_{sys} .

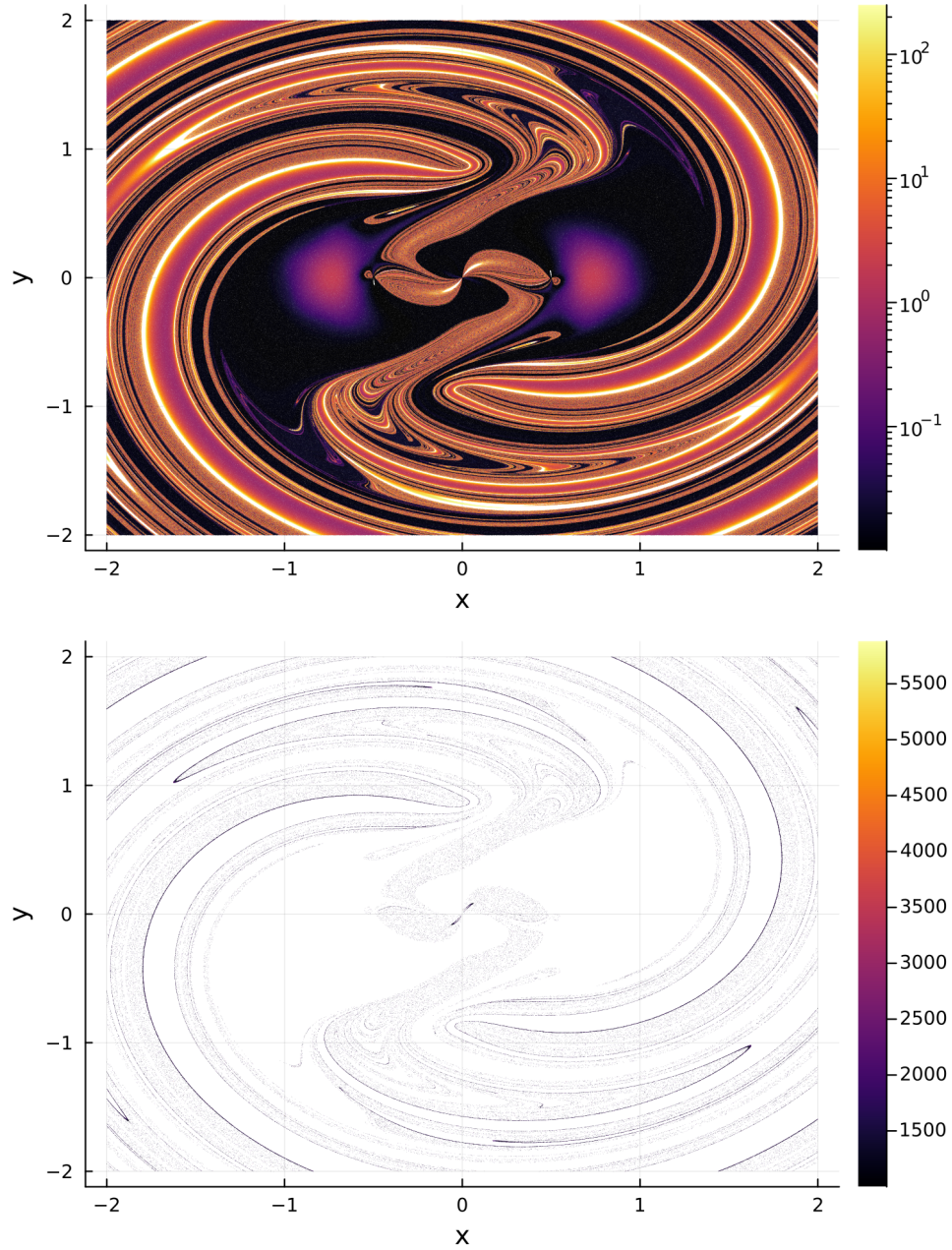


FIGURE 5.17: Spatial distribution of escape/collision times in S_3 for the Copenhagen problem with Poynting-Robertson drag ($k = 10^{-4}$) visualised in configuration coordinates (x, y) . **(Top):** Initial conditions with escape/collision time less than 500. **(Bottom):** Initial conditions with escape/collision time greater than 1000. $N = 2 \cdot 10^7$ initial conditions were used to generate the figure, with a maximum integration time of $t = 20,000$. In this figure, $C_I = 2.75$. In each panel, white points belong to initial conditions with an escape/collision time not in the interval being considered.

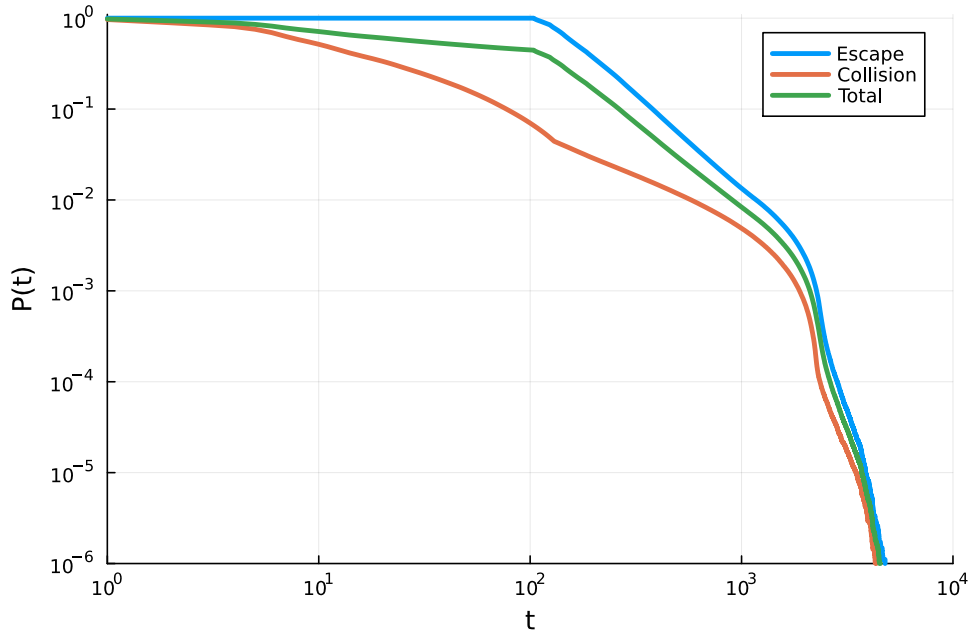


FIGURE 5.18: Survival probability of escape and collision orbits in the Copenhagen problem with Poynting-Robertson drag. The color scheme is the same as in Fig. 5.14. Also plotted is the "total" survival probability, i.e, the distribution of escape *and* collision times (see legend).

5.4.3 Uncertainty Dimension in the Copenhagen problem

We compute the uncertainty dimension along a line of initial conditions intersecting the boundaries of escape and collision. We present the results in Fig. 5.19 in a slightly different way than in Fig. 5.10 for the Hénon-Heiles system. We plot the effective dimension D against ε .

In the top panel, it is seen that the effective dimension is slowly increasing towards one, as expected for non-hyperbolic scattering. In the bottom panel, the effective dimension is seen to decrease away from one. In both cases, as with the Hénon-Heiles system, the convergence is quite slow, emphasizing that finite-scale effective dimensions are perhaps more useful and computationally accessible than the usual asymptotic definition of dimension.

5.4.4 Conclusions

In this chapter we investigated the transient chaos properties of two scattering¹⁵ systems with different dissipation mechanisms. We argue that the general picture of the survival probability in dissipative chaotic scattering is not as simple as exponential or power-law tails, as is the typical case in conservative systems. Our

¹⁵In the sense that we defined scattering in Chapter 1.

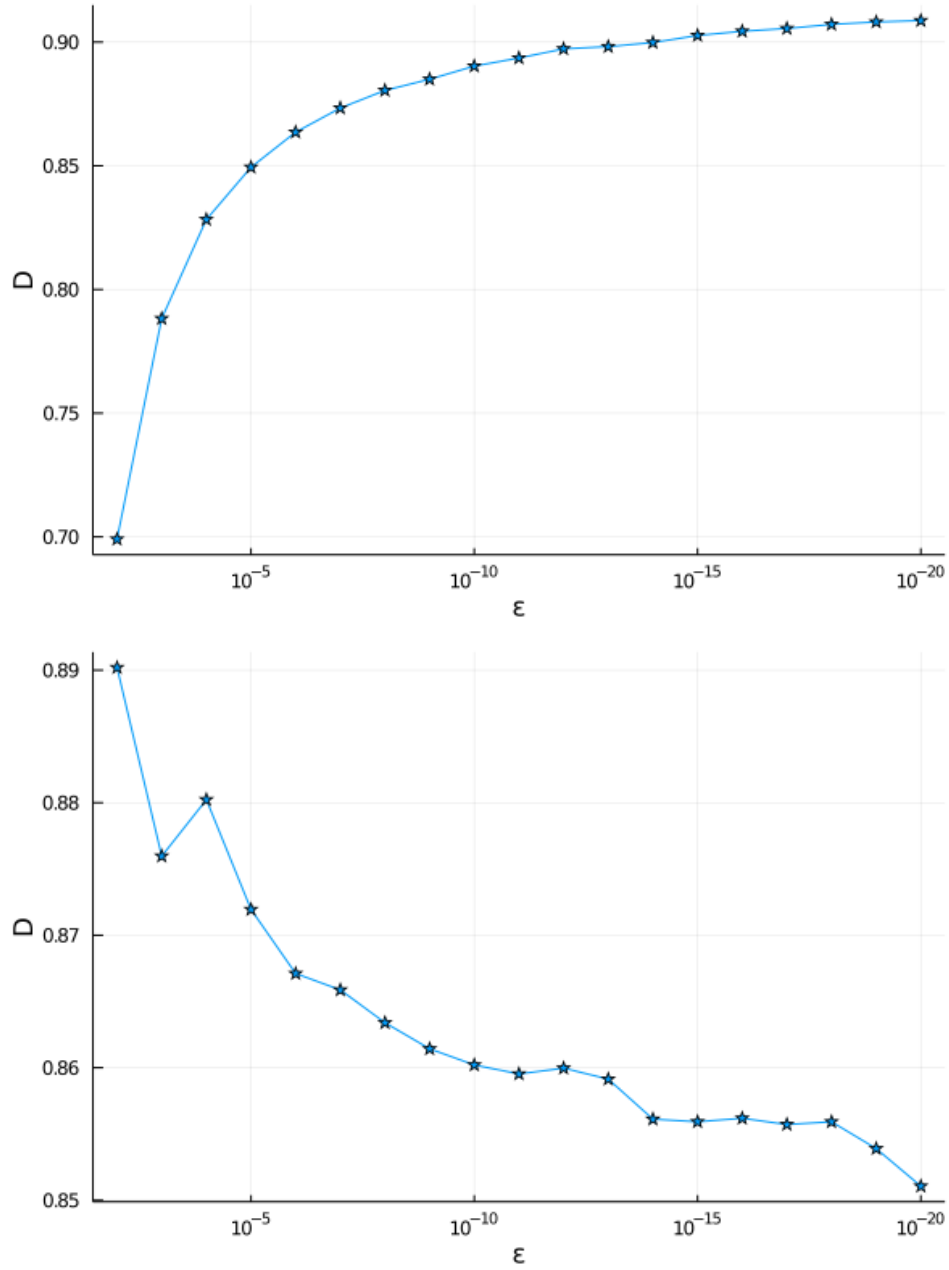


FIGURE 5.19: Scaling of the effective uncertainty dimension in the Copenhagen problem with phase space scale ε . **(Top):** scaling for the conservative system. **(Bottom):** scaling for the dissipative system with $k = 10^{-4}$. Initial conditions were chosen from a line segment in the configuration space in S_3 intersecting all basins except the KAM islands.

numerical results from this chapter confirm the emergence of non-trivial regimes in the escape time distributions of both the Hénon-Heiles and Copenhagen systems. In both systems we interpret the dissipation as leading to enhanced trapping. In the Copenhagen problem with Poynting-Robertson drag, orbits may exceed large radii and still return to the scattering region around the primaries. This leads to large deviations from the exponential decay in $P(t)$ visible on short times in the conservative system and for small escape radii R_{sys} in the dissipative problem.

In the Hénon-Heiles system the effect of dissipation manifested as an exponential decrease in the instantaneous escape rate, which in the framework of doubly-transient chaos [48] is suggestive of a time-dependent chaotic saddle that becomes less repelling/more attracting. However, in the next chapter we argue that such an object is difficult to describe in a quantitative sense and that more precise connections between the conservative and dissipative dynamics can be made by appealing to descriptions focusing on conditionally-invariant measures. In the Copenhagen problem, the effect of Poynting-Robertson drag is more complicated because it is neither spatially uniform nor non-positive. The collision rate is seen to increase under Poynting-Robertson drag. This can be attributed to the fact that most collision orbits remain relatively close to the primaries and so feel a strong dissipation. Collision orbits, which remain relatively close to the primaries and so experience a stronger dissipation, tend to collide faster

Chapter 6

Probabilistic Description of Dissipative Chaotic Scattering

In this chapter we build on the previous works on dissipative scattering and doubly transient chaos to propose a probabilistic description of the effect of dissipation on scattering problems.

6.1 Overview

In light of the material presented in previous chapters, we are ready to restate the main problem addressed in this thesis, which is the subject of this chapter:

How and in which extent can the transient chaos properties of a chaotic scattering system with dissipation be understood from the properties of the dissipation-free system?

It is clear from our numerical results in the previous chapter that the addition of dissipation to chaotic scattering systems causes distinct changes in the macroscopic transient chaos properties of the system. Some of these observed changes (e.g fractal dimension of basin boundaries) are in agreement with previous research into doubly-transient chaos, with the fractality showing an apparent decreasing trend for smaller and smaller length scales¹. In the scattering systems we studied there is a time-dependent escape rate, which can be compared with the time-dependent settling rate to attractors that was observed in [48]. The characterisation of this behaviour given in this and subsequent references on doubly-transient chaos is in terms of a time-dependent chaotic set extant only for a finite time, during which chaotic transients in the dynamics are visible. This object is only loosely defined, as there are no unstable periodic orbits aside from fixed points. As written in [48]:

¹The term *slim fractal* has been coined to describe this behaviour [14].

“...we propose that during the period of rapid separation the trajectories wander erratically in the vicinity of a set that plays the role of a chaotic saddle. This set can be estimated from the positions where the trajectories are while they separate exponentially from each other. The result...is strikingly similar to the usual chaotic saddles governing transient chaos. However, this set consists of only pieces of trajectories in the phase space and as such is not an invariant set of orbits. Moreover, this set manifests itself only during the period of exponential separation, which motivates us to refer to it as a transient chaotic saddle.”

This is certainly an appealing idea, but there is no obvious way to connect this object to the numerical observations, e.g. to the instantaneous settling rate, from which the decreasing fractal dimension is argued to follow as a consequence. We offer a description that can not only explain our results and make further predictions, but is also not incompatible with the transient saddle idea. Instead of focusing on transient chaotic sets or specific dynamics quantities (e.g., escape rates or fractal dimensions of basin boundaries), we focus on the probability of a trajectory being in a phase space region at times t and show how this allow us to derive the observable quantities of interest, such as the survival probability as a function of time $P(t)$ and the phase-space (fractal) distribution of long-living trajectories.

6.2 Probabilistic Description

Here we outline a probabilistic description to understand dissipative scattering systems, which will be compared to the numerical results shown previously. The key inspiration comes from the idea of snapshot saddles and pullback attractors [71], an ensemble-based view of the dynamics allowing for the well-defined computation of statistical quantities at a given instant in time. This is especially appropriate in systems with time-dependent parameters, in which single trajectories are often not representative. In dissipative systems, the energy is one such parameter, with the dissipation strength establishing a timescale on which significant deviations from the conservative system become apparent. These methods have been applied in the study of many physical phenomena from epidemics [39] to climate dynamics [27, 13].

Formally, a *pullback attractor* is defined [39] as a time-dependent set $\mathcal{A}(t)$ with $(-\infty < t < \infty)$ in the phase space. For any instant t , $\mathcal{A}(t)$ is understood to be the limiting ($s \rightarrow t$) object of initial conditions $\vec{x}_0$ starting at times $s \ll t$, which is often expressed in terms of operators $\varphi(s, t)$ acting on initial conditions $\vec{x}_0$. We can then

write

$$\lim_{s \rightarrow -\infty} |\varphi(s, t) \vec{x}_0 - \mathcal{A}(t)| = 0, \quad (6.1)$$

for all t . The set $\mathcal{A}(t)$ at any specific value of t is called a *snapshot attractor* at time t , and is invariant with respect to the dynamics in the sense that $\varphi(s, t)\mathcal{A}(s) = \mathcal{A}(t)$ for all $s \leq t$. By starting initial conditions in the distant ($s \rightarrow -\infty$) past, we are now dealing with an object that exists in ‘actual’ time as opposed to in an asymptotic future sense. Importantly, the instantaneous distribution of trajectories in the set defines a measure analogous to the natural measure, albeit one that varies with time. It would be reasonable to expect that in the dissipation-free limit, this measure increasingly resembles the instantaneous distribution of an ensemble evolving under the conservative dynamics. For the open chaotic systems we have discussed, this distribution corresponds, after sufficient times, to the conditionally-invariant measure. The conditionally-invariant measure of the conservative dynamics can vary with the energy of the system – see Fig. 6.1 – as the value of the energy affects, among other things, properties of the unstable periodic orbits constituting the chaotic sets. We then argue that the instantaneous distribution of orbits in the dissipative system should shadow the c-measure for the time period during which their energies are comparable. We make the following suggestion: *the conditionally-invariant measure of the Hamiltonian system with energy E is a snapshot attractor for initial conditions of the weakly-dissipative system at the time t that their energy is E .* Consequently, we focus on surviving trajectories that at a given large time t have energy $E = E(t)$ and that started at arbitrarily early times with large energies (i.e., in a limiting case, we take $t_0 \rightarrow -\infty$ and $E(t = t_0) \gg E(t = 0)$ ²). In particular, what we propose is that the trajectories at time t and energy $E(t)$ will be distributed according to the conditionally invariant measure of the Hamiltonian system with the same (fixed) energy. In this way we are positing that, through the decay in energy, a sequence of c-measures in the conservative system at lower and lower energies fulfills a similar role to the ‘transient chaotic saddle’ described for doubly-transient chaos.

In this chapter we will assess the validity of this view in describing the dynamics of the dissipative Hénon-Heiles system, showing how it can recover our observations and also discussing possible further implications. Importantly, we specify the ranges of time t , energy E , and dissipation strength γ for which this view can be expected to hold, showing that the description can be made arbitrarily precise even at finite (yet small) dissipation γ .

²There is a temptation to take $E(t = t_0) \rightarrow \infty$, but statistical quantities become poorly defined and model validity is compromised in the high-energy limit. It is therefore more reasonable to say that there is a (possibly large) but *finite* energy horizon.

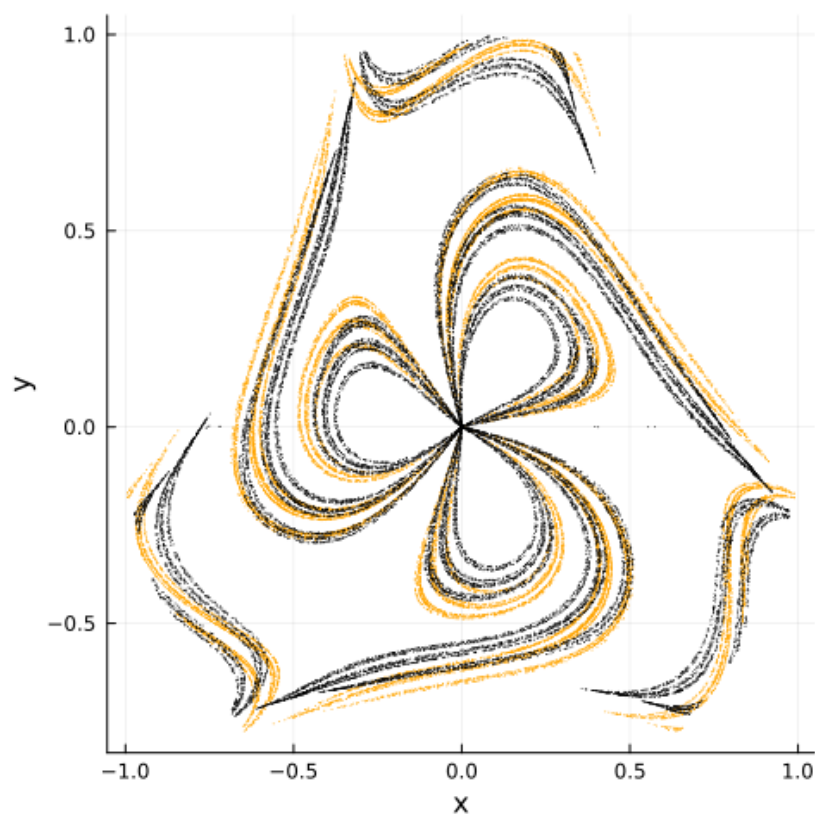


FIGURE 6.1: Support of the conditionally-invariant measure in the Hénon–Heiles system for $E = 0.35$ (orange points) and $E = 0.3$ (black points). These approximations were computed via the sprinkler method [71]: integrating a large uniform ensemble of initial conditions for a long time $t \gg \frac{1}{\kappa}$ and plotting the positions of the points still inside Ω at the next intersection with the Poincaré surface.

This description has important and testable consequences. Namely, dynamical properties of ensembles of trajectories evolving on the dissipative system can be approximated by the properties of the dissipation-free system with the corresponding energy, but differences are expected at small spatial scales. To account for these differences, we describe an entire ensemble at time t as a combination of c-measures over the range of trajectory energies, weighted by the distribution of trajectories at these energies. We also show that the enhanced trapping phenomenon we observed in dissipative scattering systems can be well-approximated from the relevant quantities in the Hamiltonian system. This works particularly well if the underlying conservative dynamics is hyperbolic, for which chaotic invariant sets are the dominant structures in the scattering region, although it will be seen that this description can also hold approximately on intermediate time regimes in non-hyperbolic systems.

To assess this claim we consider the intersection of the flow with a Poincaré surface of section. We choose the Poincaré section $\mathcal{S}_2$ in our experiment. This allows us not only to visualise the trajectories in a 2D space – here the configuration space (x, y) – but it defines a true-time map, i.e., a discrete-time system h mapping consecutive intersections with the section in which the time T between intersections is recorded [3, 4]. As in our previous experiments, we start trajectories in the surface $(x, y) \in \Omega$ at the same energy E . This choice of initial conditions inside the scattering region differs from usual scattering settings, in which particles approach Ω but start far from it. Our choice simplifies numerical computations and is more suitable for settings in which dissipation acts globally – for instance, starting orbits at spatial infinity would be inappropriate as they would never reach Ω . In the conservative setting, this choice affects the stickiness exponent in the intermediate- E (non-hyperbolic) regime [56]. However, it has no significant impact on our simulations (which are typically at larger E) and conclusions.

Dissipation leads to a trajectory-dependent decay of the energy $E(t)$ and the scattering process in the Hénon-Heiles system ends when the critical energy is met $E(t) = E_c$. The expected time at which this happens, which we associate to the transition time τ_2 introduced in Sec. 5.2.2, is obtained by solving eq. (5.3) as

$$\tau_2 = \frac{1}{\mu\gamma} \log \left(\frac{E_0}{E_c} \right). \quad (6.2)$$

The interesting dynamics happens at large times ($t > \tau_1$, when effects of dissipation become relevant) smaller than τ_2 . The key idea is to use the properties of the non-dissipative system to describe the scattering in this regime. We thus assume the unperturbed system to be fully chaotic and to have a well defined (E -dependent)

conditionally-invariant measure μ_c^E in a common scattering region Ω for a wide range of (large) energies E (i.e., $\int_{\Omega} d\mu_c^E = 1$ for any $E > E_c$). The challenge here is that the c-measure is an attractor for surviving trajectories and thus provides a precise description only in the limit $t \mapsto \infty$, which is not applicable in our case since scattering happens only for $t < \tau_2$ (i.e., for $t > \tau_2$ there is no scattering or invariant set).

We posit that, for $\tau_0 \ll t < \tau_2$, the probability of finding such a surviving trajectory in a region of Ω is increasingly well described (for $t \lesssim \tau_2 \rightarrow \infty$ and $\gamma \rightarrow 0$) by the c-measure of the conservative system μ_c^E with $E = E(t)$. From eq. (6.2), τ_2 and thus the time the trajectory has to relax to μ_c can be made arbitrarily large by reducing γ or increasing E_0 . This would suggest that the description can be made arbitrarily precise for any fixed γ or E_0 . However, for γ too large, the dynamics of the dissipative system are no longer comparable to the non-dissipative system, and so we restrict to a small γ regime.

We now consider the consequences of this to an ensemble of trajectories, as investigated in our numerical experiments. After some sufficient time $t > \tau_0$, the energy $E(t)$ of trajectories that have not yet escaped will be distributed within some range $0 \leq E(t)_{\min} \leq E(t) \leq E(t)_{\max} < \infty$ with a probability density $\rho(E = E(t))$ (which depends on the initial ensemble). The probability of finding a surviving trajectory at time t in any region $A \subset \Omega$ is given by the composition of the sub-ensembles of trajectories at a specific energy $E = E(t)$ as

$$\mu_t(A) = \int_{E(t)_{\min}}^{E(t)_{\max}} \mu_c^E(A) \rho(E) dE. \quad (6.3)$$

The convergence of surviving trajectories towards eq. (6.3) depends on the time t available to relax towards μ_c^E , which can be made arbitrarily large varying the initial energy E_0 and escape rate γ as described in eq. (6.2). While this description assumes the existence of μ_c , it is agnostic to the specific dynamics of the system and the form of dissipation considered. For instance, if we consider the case of a discrete Hamiltonian system with parameter drift as in [73], then all trajectories at time t have the same energy³ ($E(t)_{\min} = E(t)_{\max} = E(t)$) and so the density ρ reduces to a Dirac delta function

$$\rho(E) = \delta(E - E(t)), \quad (6.4)$$

³The analogue of the energy in the system in this reference is the parameter η , so these arguments apply to the system in [73] provided E is replaced with η in eq. (6.3).

and eq. (6.3) yields

$$\mu_t(A) = \int_{E(t)_{\min}}^{E(t)_{\max}} \mu_c^E(A) \delta(E - E(t)) dE = \mu_c^{E(t)}(A). \quad (6.5)$$

Given the connection between the c-measure and the escape rate κ in eq. (3.25), we see that our description recovers as a particular case the finding made in [73] about the instantaneous escape rate mirroring that of the Hamiltonian system at the new parameters. We may also account, through the energy density $\rho(E(t))$, for situations where the initial energy is not the same for each orbit, a common scenario in systems where particles are released from rest inside a potential. It should be pointed out that there is nothing about our snapshot c-measure model that restricts its usage to systems with strictly decreasing energy. Even if the energy of trajectories can increase, as was seen to be possible in the Copenhagen problem, the ensemble is still describable by eq. (6.3). It also trivially describes ensembles of different energies in conservative systems. Generally speaking, the more information one has about ρ , the better one can approximate the state of the dissipative system.

Our probabilistic description also provides predictions for the fractal properties of scattering systems with dissipation, a problem previously studied through different approaches [48, 40, 14, 64, 73]. These studies employed a variety of different theoretical and numerical approaches, often based on the uncertainty algorithm [29]) to evaluate the fractality of basin geometry as a function of the dissipation. Our proposed description is that the evolution of the dissipative system can be viewed as a time-evolving mixture of snapshots of the Hamiltonian system and that a well-defined fractal distribution exists (for arbitrarily small spatial scales) in these systems in suitable limits of E_0 , γ , and t that allow for a relaxation towards μ_c^E . The nature of the relaxation time and how it depends on the dissipation and energy is an interesting question for future work.

Instead of focusing on the basin boundaries of initial conditions, one could consider the fractal dimension of the instantaneous distribution of surviving trajectories, which evolves in time. For homogeneous energy loss, or for a sub-ensemble of points in a narrow energy range, we expect this observed dimension would essentially agree up to the numerical resolution with computations of the dimension of the conditionally-invariant measure in the conservative system at the same energies. In our theory, the snapshot attractor of the dissipative dynamics approximates the c-measure in the conservative system, whose fractal dimension increases with decreasing E . The set supporting the c-measure achieves full dimension in non-hyperbolic energy regimes [41], and the c-measure only exists at energies where the

scattering process is defined. The fractal properties of this object will then certainly be different to those of the basin boundaries in dissipative systems, which generally decrease with resolution scale ε , possibly even becoming non-fractal [14]. A loss of fractality in the instantaneous distributions we have considered should be expected for $E < E_c$, when the c-measure ceases to be defined and the saddle is comprised of perhaps a single point. In more general settings, a cut-off in the scaling underlying fractal analysis (in time or space) of the instantaneous distribution is expected due to the combination of μ_c^E s, each with a slightly different fractal support.

6.3 Case Study: Hénon-Heiles

6.3.1 Conditionally-invariant measures

Neglecting the variation in the energy of trajectories – see Fig. 5.5 –, our expectation is that the distribution of long-living trajectories at time t will be well-approximated by the c-measure μ_c^E , with $E = E(t)$ given by eq. (5.3). To verify this, we initialise an ensemble of initial conditions in the dissipative system at an energy $E_2 > E_1$ and, at the time t^* that eq. (5.3) predicts $E(t^*) = E_1$, we plot the position (x, y) of the next crossing with the Poincaré surface alongside the support of $\mu_c^{E_1}$ (computed using the conservative dynamics). The results of one such numerical experiment are reported in the top panel of Fig. 6.2, and an alternative figure is given in Appendix A.

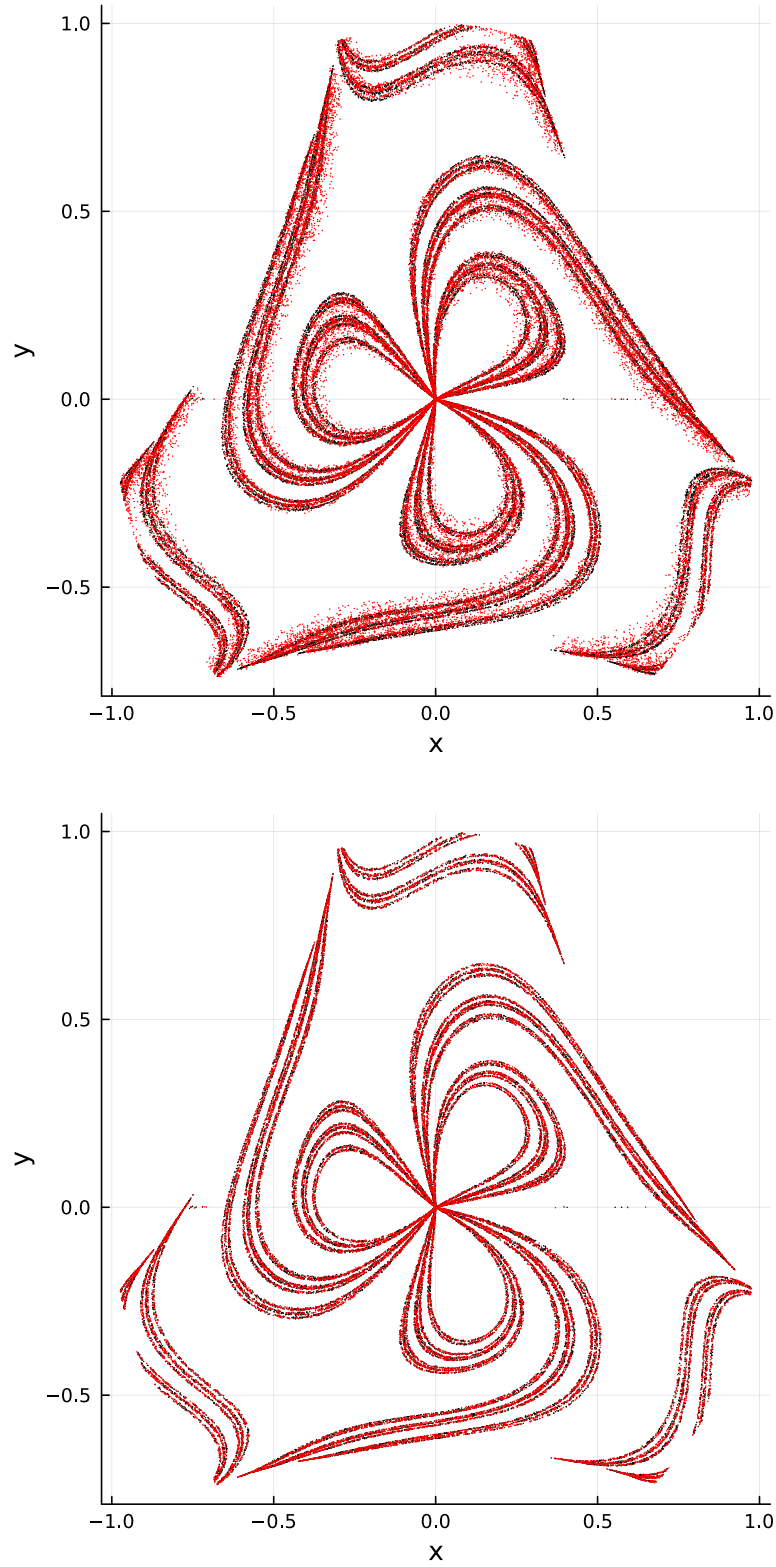


FIGURE 6.2: Comparison of the c-measure of the Hamiltonian system with $E = 0.3$ (black points) with the distribution of trajectories in the dissipative system ($E = 0.35, \gamma = 0.01$) at $t = t^* \approx 17 \gg \tau_0$ (red points.) **(Top)**: all points that survived until t^* (the first intersection for $t \geq t^*$ with the surface of section is shown). **(Bottom)**: only the surviving points whose energy at time of crossing $t \geq t^*$ was in the interval $[0.295, 0.305]$, that is, within 1.67% of E_1 . $N = 10^7$ initial conditions were used and approximately $3 \cdot 10^4$ are plotted in each case.

Our choice of E_1, E_2 , corresponds to the two c-measures shown previously in Fig. 6.1 and γ ensures that the time t^* for the average energy to reach E_1 was $\tau_0 \ll t^* < \tau_2$, allowing enough time for the initially uniform ensemble to relax. We find that the distribution of points in the dissipative system indeed mostly coincides with the c-measure $\mu_c^{E=E_1}$, which provides a better description than the non-dissipative measure $\mu_c^{E=E_2}$ (shown in Fig. 6.1). Still, the points of the dissipative system show a more noisy distribution and often do not sharply overlap with (the support of) $\mu_c^{E=E_1}$. Under eq. (6.3) the explanation for this is that these points experience a different average decay in energy, implying that their energy at $t \approx t^*$ is far enough from E_1 that they are distributed according to different measures (i.e., the disagreement is due to the approximation of neglecting fluctuations in $E(t)$). To test this expectation, in the bottom panel of Fig. 6.2 we further restrict the distribution to points whose energy is $E(t) \approx E_1$, effectively re-weighting the density ρ towards E_1 . The reduction in the dispersion of points is clear: the remaining points almost exactly coincide with the support of the measure $\mu_c^{E=E_1}$. The agreement is expected to hold up to a finite spatial scale, which becomes increasingly small as the time for the distributions to relax increases (e.g., by increasing the initial energy E_2 or reducing the dissipation rate γ). For sufficient relaxation time, we posit that the sub-ensemble with energy E converges *exactly* to the c-measure μ_c^E .

6.3.2 Survival Probability

Interpreting the macroscopic dynamics of the dissipative system in terms of the conservative system also allows us to propose a model for the survival probability for the time window $t \leq \tau_2$. Our c-measure theory predicts that an instantaneous escape rate κ_E applies for the sub-ensemble at each given energy $E(t) = E$ at time t . To compute the leading effect of the varying κ_E in time, we neglect the dispersion in orbit energies in our system and assume that most trajectories experience the same dynamics (with associated escape rate κ_0) at any given time t . The escape rate, as we have shown in Fig. 5.3, depends intimately on the energy, which in turn decays over time. By combining this information, we can approximate the decay in survival probability $P(t)$ for the entire dissipative ensemble.

The first step is to compute the time-dependent escape rate of the dissipative system, $\kappa_\gamma(t)$, by introducing the exponential decay of the energy as a function of time in eq. (5.3) into our parametric model for $\kappa_0(E)$ (see Fig. 5.3), so that

$$\kappa_\gamma(t) = \kappa_0(E(t)). \quad (6.6)$$

The expression for the approximate survival probability $P_\gamma(t)$ of escaping orbits is then given as

$$P_\gamma(t) \sim e^{-\kappa_\gamma t}, \quad (6.7)$$

which is similar in form to the exponential decay of hyperbolic scattering in the Hamiltonian system, although now with a time-dependent escape rate $\kappa_\gamma(t)$ as opposed to a fixed rate $\kappa_0(E)$.⁴ This description applies to escaping trajectories only, and thus to times $t < \tau_2$. The behavior of $P(t)$ for $t > \tau_2$ is exponential, dominated by trajectories settling towards the attractor, and thus our theoretical model for the survival probability $P(t)$ is

$$P_\gamma(t) = \begin{cases} \text{irregular} & t \in [0, \tau_0] \\ f_{ne} + f_e \cdot \exp(-\kappa_\gamma(t) \cdot (t - \tau_0)) & t \in [\tau_0, \tau_2] \\ f_{ne} \cdot \exp(-g(t, \gamma) \cdot (t - \tau_2)) & t > \tau_2. \end{cases} \quad (6.8)$$

The term f_{ne} refers to the fraction of trajectories in the initial ensemble that remain trapped, which at fixed E_0 increases with γ (resp. decreases with increasing E_0 at fixed γ .) Likewise, f_e refers to the fraction of orbits that escape at a time $t > \tau_0$. We have that $f_{ne} + f_e = P(\tau_0)$. The function $g(t, \gamma)$ is introduced because the observed transition in $P(t)$ between the escape-dominated and settling-dominated decay regimes is smooth⁵. We thus require $g(t, \gamma)$ to be such that $g(t) \geq 0$ for $t \geq 0, \gamma > 0$ and to increase monotonically with t towards a constant L that depends on the dissipation strength γ . In practice, we found that a sigmoid function of the following form provides a satisfying fit

$$g(t, \gamma) = \frac{L}{(1 + \exp(-b(t - \tau_2)))^2}. \quad (6.9)$$

Finally, we evaluate eq. (6.8) using the parametric fits of $\kappa_0(E)$ – in Fig. 5.3 – and $E(t)$ – in Fig. 5.5.

Direct comparison of simulation data for select parameter values with the predictions from eq. (6.8) is shown in Fig. 6.3(a). In general, our model shows a satisfying agreement with the data, despite relying on the approximation of homogeneous energy decay. Fig. 6.3(b) shows the comparison between the observed occurrences of the timepoints τ_1 and τ_2 with their predicted occurrences under our model while

⁴This turns out to be different from the usual definition of an instantaneous escape rate – the negative logarithmic derivative of $P(t)$ as used in e.g. [48].

⁵It should be noted that, as presented, $P_\gamma(t)$ is not continuous at $t = \tau_2$. This is because there exist orbits that escape after τ_2 . However the proportion of such orbits is small and decays quickly. Thus, the form presented is a reasonable approximation.

varying γ . There are deviations visible at large γ in the data for τ_2 , but in this regime $\tau_2 \ll 1$ and the dynamics of the system are essentially trivial; there is essentially no escape/scattering. For such large γ the terms involving γ in the equations of motion dominate and so the dynamics cease to be comparable to the conservative system. There are also small deviations in τ_1 visible for smaller γ . The reason for this is the assumption that trajectories are losing energy at the same rate. An improved agreement would be obtained by avoiding the approximation of homogeneous energy decay and better taking into account properties of the energy density $\rho(E(t))$. As an example, this could take the form of a mixture model for $P(t)$ involving multiple sub-ensembles, each with their own energy decay exponent μ . Overall, these results suggest that the time interval $\tau_1 \leq t \leq \tau_2$ of the dissipative scattering process in which our probabilistic description applies becomes arbitrarily large with reducing γ .

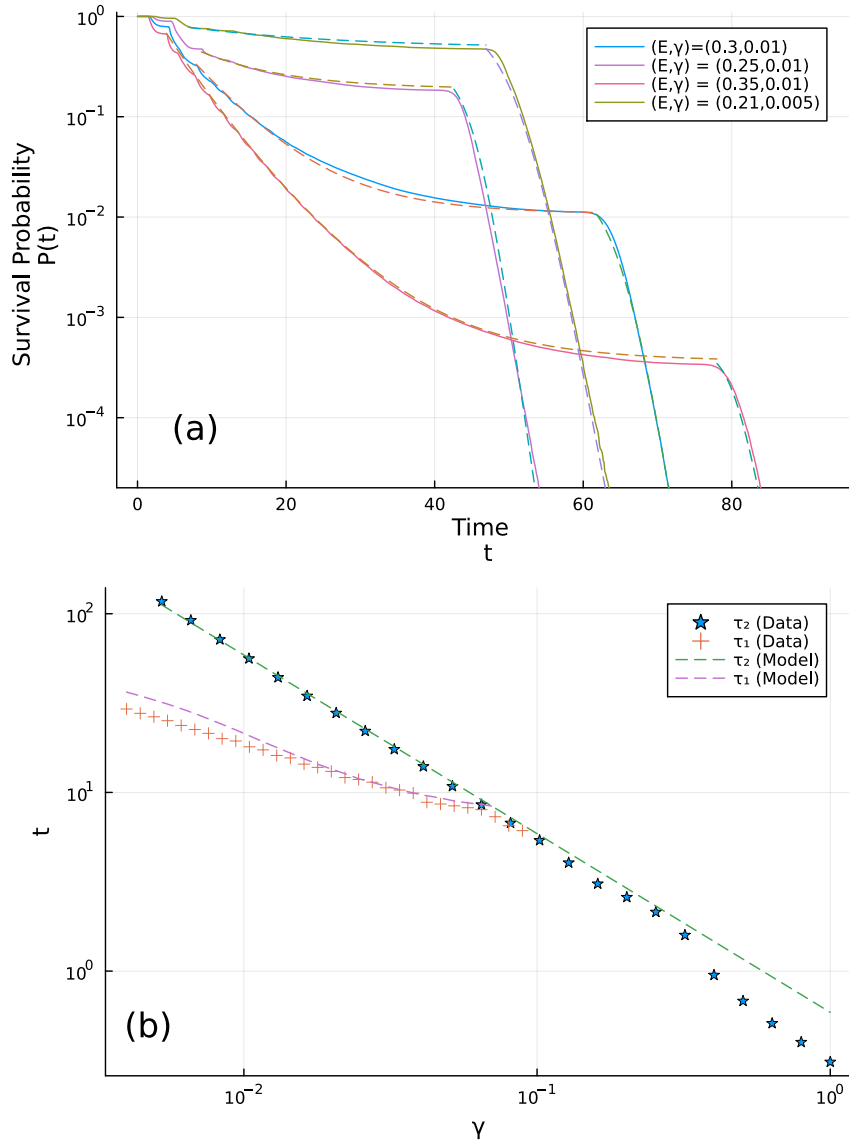


FIGURE 6.3: Comparison between theoretical model and numerical simulations of the survival probability $P(t)$. **(a)**: Comparison of eq. (6.8) (dashed lines) to the simulation data (solid lines) for several choices of (E, γ) for $t > \tau_0$. $N_0 = 10^7$ initial conditions were used for each curve. **(b)**: Comparison of the predicted occurrences (dashed lines) for the transition times τ_1 and τ_2 (see legend) with their observed occurrences (symbols) in the simulations (for $E_0 = 0.3$). For τ_1 , the observed occurrence is the first time eq. (5.4) was satisfied in the simulation, while the predicted occurrence comes from numerically solving eq. (6.8) with the relevant parameters. For τ_2 , the observed occurrence is the first time at which the energy of a non-escaping orbit dropped below E_c , and the predicted occurrence $\tau_2 \propto 1/\gamma$ comes from eq. (6.2). For increasing γ , the volume of escaping orbits goes to 0 and τ_1 cannot be determined.

6.3.3 Fractal dimensions

Our earlier statements about fractal dimensions are supported by simple computations of the box-counting dimension (3.10) of the three sets in Fig. 6.2, as estimated

by the slope of the logarithm of the scaling for ε in the range $[2^{-16}, 2^1]$. For the support of the conditionally-invariant measure $\mu_c^{E=0.3}$, we compute an approximate box-counting dimension of $D_0 \approx 1.56 \dots$, which agrees to two decimal places with the box-counting dimension of the instantaneous distribution of the dissipative system in the bottom panel of Fig. 6.2, providing further quantitative evidence that the two sets have the same distribution. This calculation also agrees with previous estimates of the dimension for the unstable manifold in the conservative system reported in [77] in which the uncertainty algorithm was used⁶. For the “noisy” distribution from the top panel of Fig. 6.2 we compute $D_0 \approx 1.65$. That the dimension of the noisy distribution is closer to two – the dimension of the space in which it is embedded – is not unexpected, for aforementioned reasons.

⁶In this work the dimension of the *full* manifold, which is between 2 and 3, was computed. We are studying an intersection of this object with a 2D plane, hence our computed dimension is one less.

Chapter 7

Conclusion

In this thesis we have considered the problem of chaotic scattering systems subject to dissipation, through exploration of two specific systems, the Copenhagen problem and the Hénon-Heiles system. The transient chaos properties of both systems under the effect of dissipation were investigated. Some of these results (e.g uncertainty dimensions of basin boundaries) are comparable to previous results for undriven dissipative systems [48, 14, 64] while our results for instantaneous escape rate and survival probability show surprising finite-time regimes not reported for the systems in those references. The closest comparison that can be found in the literature to our observations for the Hénon-Heiles system would be [73]. Abstractly, the time-dependent sinks of the system in [73] are comparable to the narrowing escape channels in Hénon-Heiles as both are controlled by the energy, which influences the escape rate at any instant. Moreover, the energy decays exponentially in both systems, which together with the previous point serves to justify the similarity in observations. This leads us to suggest that comparable systems will demonstrate the same finite-time regimes. Ultimately, we have been able to understand and describe [73] as a particular case of our theory presented in Chapter 6, and expect that it should be applicable more generally.

Our results from the Copenhagen problem also illustrate some of the peculiar properties of systems experiencing non-uniform dissipation. Many initial conditions were able to extract energy from the system, something that is not possible in the conservative dynamics. The choice of pro-grade versus retro-grade initial velocities – the Poincaré sections $\mathcal{S}_2$ and $\mathcal{S}_3$, see Appendix B – was also seen to make an impressive difference in the numerical observations. Most notably the chaotic sets appear to extend to spatial infinity for retro-grade velocities, something that does not hold for the pro-grade velocities. More generally, we believe that weak dissipation – uniform for not – generically causes time-dependencies in the escape rate in a system-specific way, that can be understood through our probabilistic description.

It would be interesting to explore this question further.

We have presented a generally-applicable model for the study of dissipative chaotic scattering and illustrated its validity with numerical studies of the Hénon–Heiles system. This model, inspired by the concepts of pullback/snapshot attractors, proposes that long-living trajectories in the dissipative system can be described, for an intermediate but arbitrarily large time interval $t < \tau_2$, by conditionally-invariant measures μ_c of the non-dissipative system. Our description includes the dependence of the measures and time interval on dissipation rate γ , initial energy E_0 , and time t . The temporal evolution of ensembles of trajectories in the dissipative system can then be understood by considering weighted sub-ensembles of the conservative system, with each sub-ensemble at a given energy $E(t)$ described by the corresponding c-measure. This theoretical model has shown to be effective at explaining transient chaos phenomena like the survival probability $P(t)$, which can be described using time-dependent escape rates, and the distributions of trajectories in phase space. This evolving snapshot model expressed in terms of conditionally-invariant measures of the conservative system can perhaps be considered an extension of doubly-transient chaos to systems that already show transiently chaotic behaviour before dissipation is added. A possible direction for future work would be deepening the theoretical connections to pullback attractors, with particular regard to the understanding the relaxation time required for convergence to the c-measure.

An interesting observation made in our experiments involving the survival probability is that the theoretically derived curves apply also for times at which trajectories are in energy ranges $E_c < E \leq E_{KAM}$ in which the conservative system shows the existence of KAM islands. This is a regime in which there is no c-measure μ_c^E or escape rate κ and therefore our description should not strictly apply. The reason for its unexpected success is that the hyperbolic components of the invariant sets generate an effective escape rate that drives the numerical observations [47, 5, 4]. For smaller dissipations γ , trajectories have time to relax towards the non-hyperbolic components of the saddle in these energy ranges and deviations become relevant. In particular, we expect a concentration of the combined c-measure in Eq. (6.3) towards non-hyperbolic regions and that $P(t)$ will converge to a constant earlier in t (i.e., when the energy reaches E_{KAM} instead of E_c). Another direction would be to investigate the fractal properties of these snapshot objects, especially in non-hyperbolic regimes.

A final suggested area of future work is to extend the probabilistic description for

the dynamics of ensembles of initial conditions using an operator-theoretic formalism. Our results showing that the evolution depends mainly on the time-dependent energy density $\rho(E(t))$ suggest that further insights can be obtained through spectral approaches considering Perron-Frobenius (or similar) operators acting on these densities [26, 25, 4].

Appendix A

Alternate presentation of Fig. 6.2

Here we give an alternate presentation of Fig. 6.2 that may more concisely illustrate the observations in the numerical experiments in Ch. 6. The figure is divided into four components, three corresponding to (portions of) the plotted objects in Fig. 6.2 and one a portion of the c-measure for $E = 0.35$ shown in Fig. 6.1. This alternate presentation contains a lot of information revealed through comparing different quadrants:

- Comparing the blue and purple quadrants reveals that the c-measure $\mu_c^{E_1}$ is a good descriptor of the dissipative trajectories with comparable energy.
- Comparing the purple and red quadrants highlights that many trajectories in the ensemble – those with energies ‘far’ from E_1 – are not so well described by $\mu_c^{E_1}$.
- Comparing the green quadrant with any other quadrant (notably, the broken symmetry) shows the evolution of the fractal object with the energy.

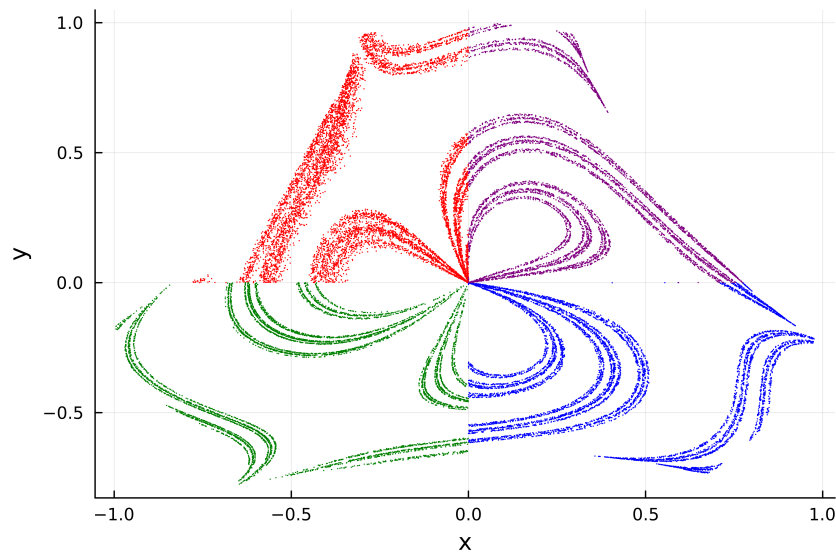


FIGURE A.1: Comparison of the c-measure in the conservative Hénon-Heiles system with an instantaneous distribution of the dissipative system (see Fig. 6.2). From top left, clockwise: (red): instantaneous distribution at $t^* \approx 17$, (purple): the same distribution restricted to points with energy $0.295 < E(t^*) < 0.305$. (blue): c-measure of conservative system, $E = 0.3$. (green): c-measure of conservative system, $E = 0.35$.

Appendix B

Experiments in the Copenhagen problem with pro-grade initial conditions

Here we present numerical results for the Copenhagen problem similar to those presented in Chapter 5. The difference between those figures and the ones shown here is that initial conditions are chosen here in the Poincaré section $\mathcal{S}_2$, while they are chosen from $\mathcal{S}_3$ in Chapter 5. The experiments to produce the following figures all used 10^6 random initial conditions from $\mathcal{S}_2$ with a maximum integration time of 10^4 . The color schemes in the figures are generally the same as their counterparts in Chapter 5. In all figures, $R_{sys} = 100$ and $C_f = 2.75$.

Some general observations about the results:

- In $\mathcal{S}_2$, the fractal region of the escape and collision basins is limited to a small area around the primaries. Collision seems to be a possible outcome only close to the primaries, while at large radii there are only escaping orbits. In $\mathcal{S}_3$, the fractal region extends as far as R_{sys} with points of all basins. This is true with or without dissipation, making it a feature of the section.
- There is a significant set of initial conditions in the dissipative system that escape with a higher energy than they started with – see Fig. B.6. Some orbits started in $\mathcal{S}_3$ can also extract energy from the system, but they were not visualized earlier. This must be related to the direction in which they leave the neighbourhood of the primaries – see eq. (5.32).
- In $\mathcal{S}_2$ the escape and collision rates are changed significantly - as reflected in the survival probability. For long times the set of escape orbits decays faster than the set of collision orbits. The opposite was true for $\mathcal{S}_3$.

- In $\mathcal{S}_2$, collision accounts for a much smaller proportion of trajectory outcomes than for $\mathcal{S}_3$. One of the reasons for this is that the initial dissipation felt by each trajectory is weaker $\mathcal{S}_2$ than in $\mathcal{S}_3$, which can be seen by comparing eq. (5.28) to eq. (5.32).

B.1 Conservative system

B.1.1 Exit basins

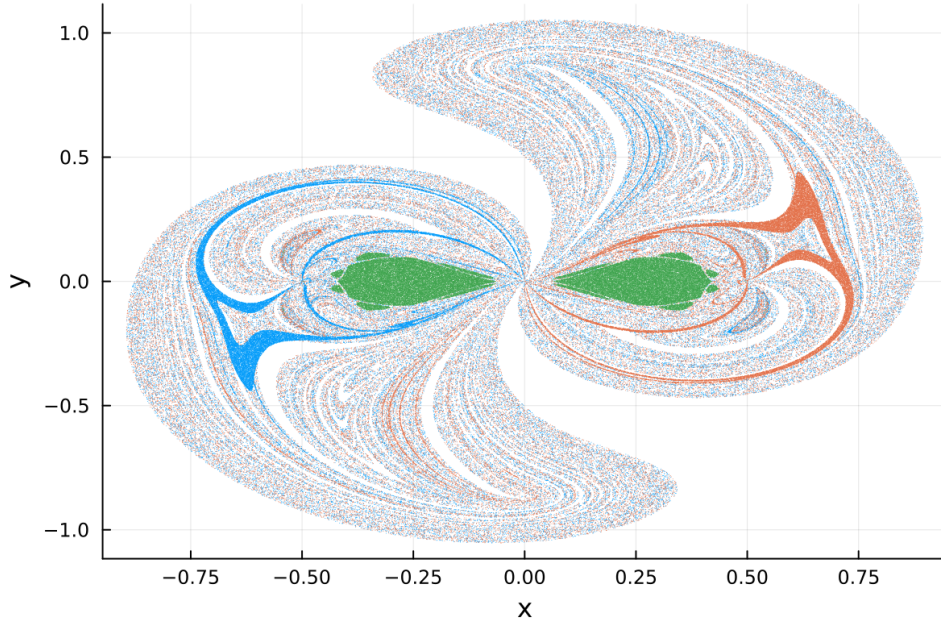


FIGURE B.1: Basin topology in the Copenhagen problem visualised in the configuration space (x, y) . The initial conditions of trajectories ending in collision, and those in KAM islands, are colored. The uncolored points correspond to escape orbits.

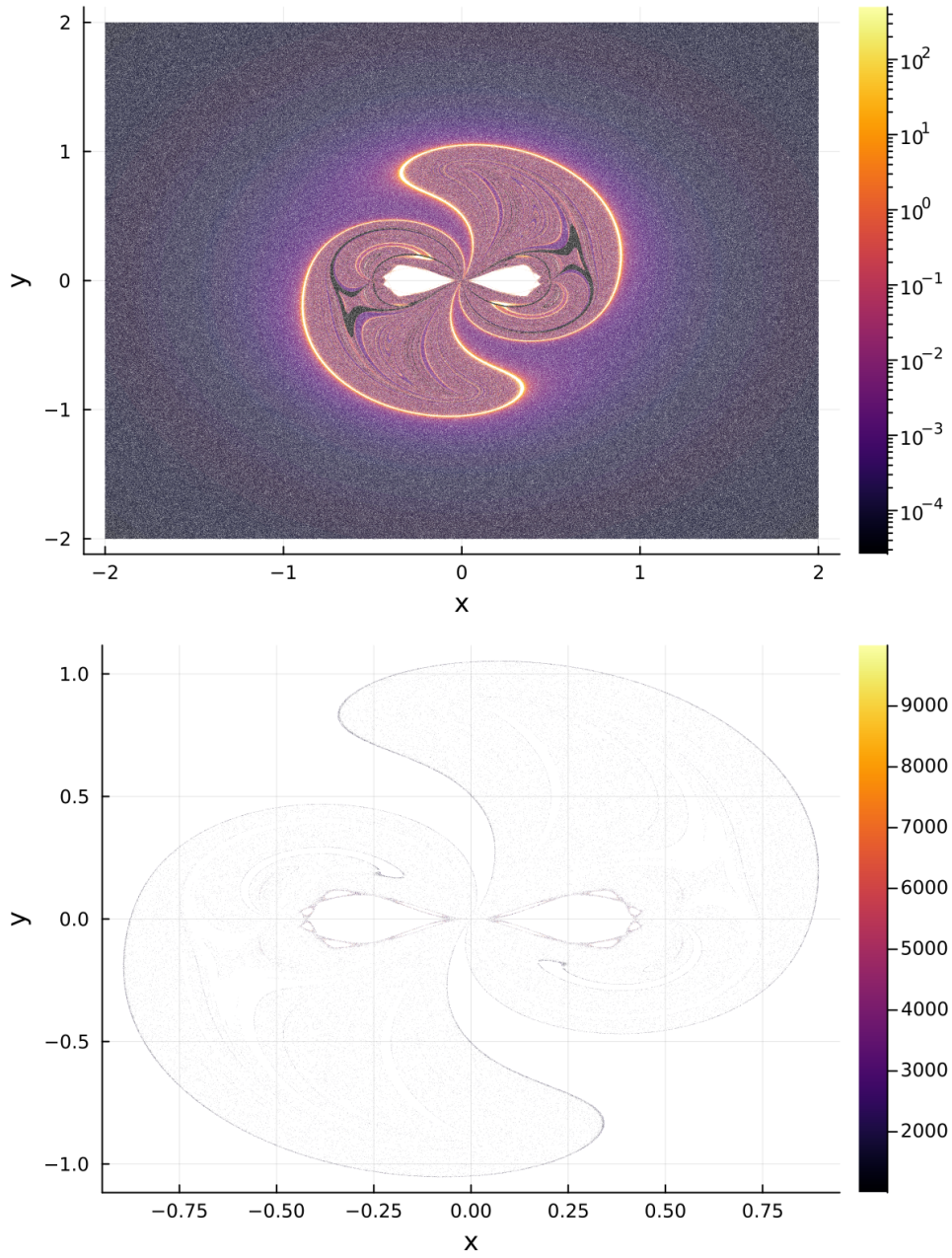


FIGURE B.2: Spatial distribution of escape/collision times in the Copenhagen problem, visualised in configuration coordinates (x, y) . The points belonging to KAM islands have not been plotted, as their escape time is infinite. **(Top):** initial conditions with escape/collision times less than 500. **(Bottom):** initial conditions with escape/collision times greater than 1000.

B.1.2 Survival probability

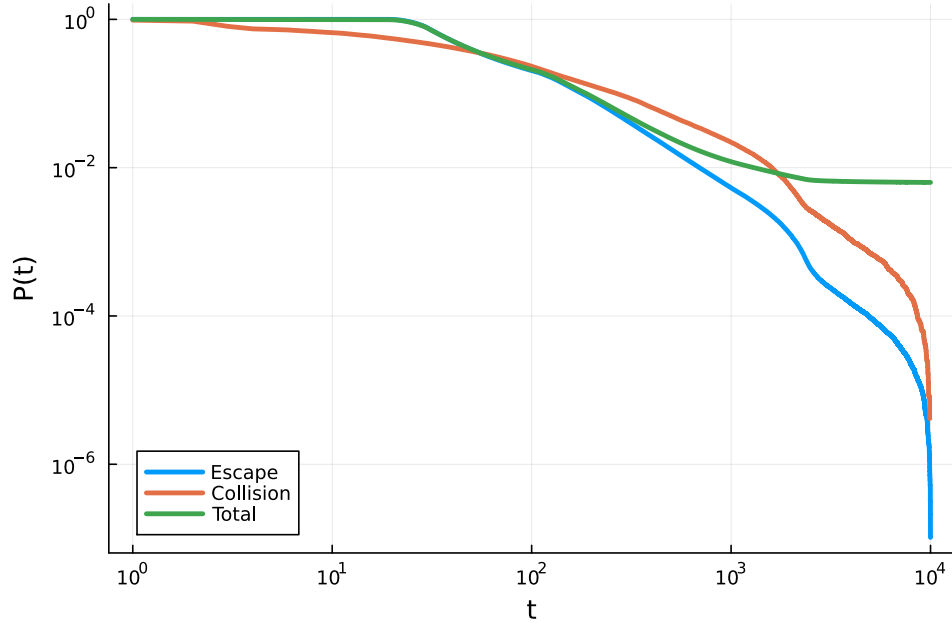


FIGURE B.3: Survival probability in the Copenhagen problem. The decay in the cumulative distribution of lifetimes for both escaping (blue) and colliding (orange) orbits are shown for escape/collision times up to $t = 10^4$. Also plotted in green is the "total" survival probability, i.e, the distribution for escape, collision and KAM orbits.

B.1.3 Chaotic sets

We present approximations of the chaotic sets in $\mathcal{S}_2$ using the same method as for Fig. 5.15. We filter for points that remain in the system for $t > 150$ and $t > 1000$. The longer-lived points in both temporal directions seem to lie near regions of maximum potential and immediately around the primaries, see Fig. 5.11.

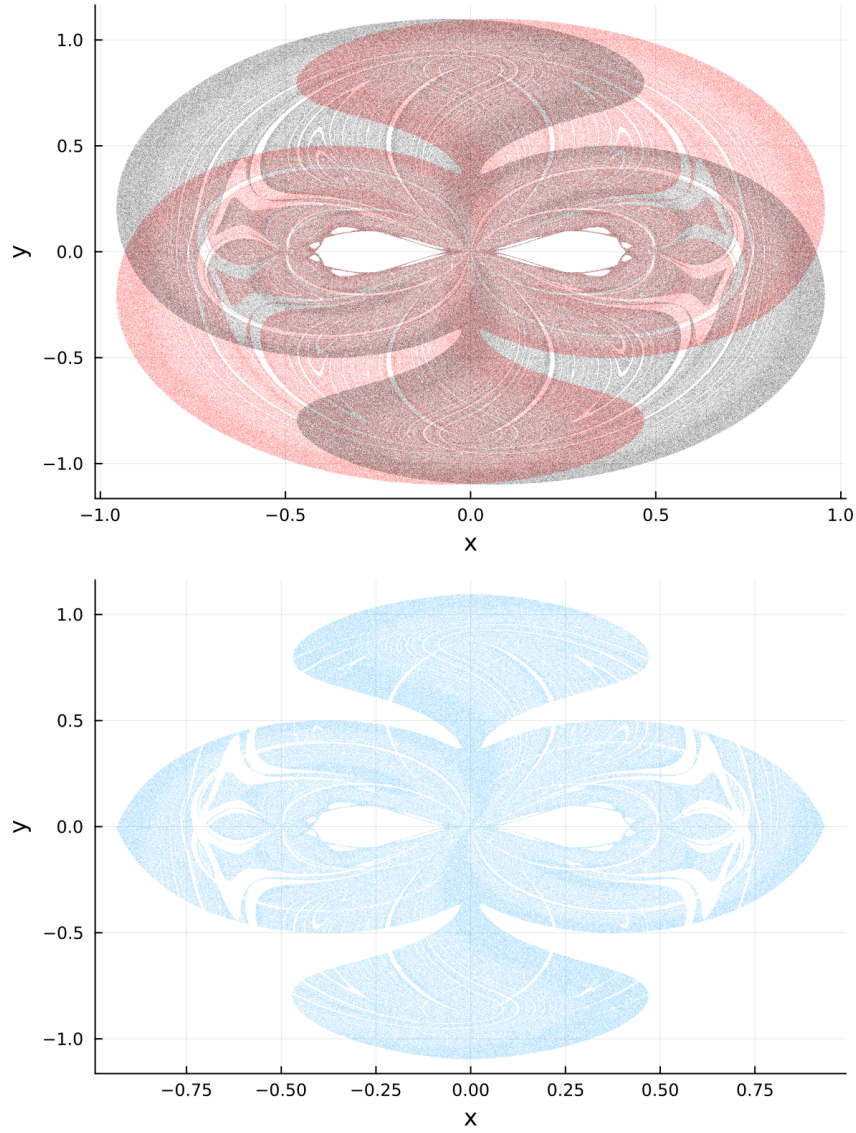


FIGURE B.4: Approximation of the chaotic sets in the Copenhagen problem with pro-grade initial conditions ($\mathcal{S}_2$), visualised in configuration space (x, y) . This figure is formed by orbits that remained in the system until at least $t = 150$. **(Top):** Approximate stable (black) and unstable (red) manifold. **(Bottom):** Approximate chaotic saddle. 10^6 points for each set are plotted in the top panel, and approximately $7.5 \cdot 10^5$ are plotted in the bottom panel.

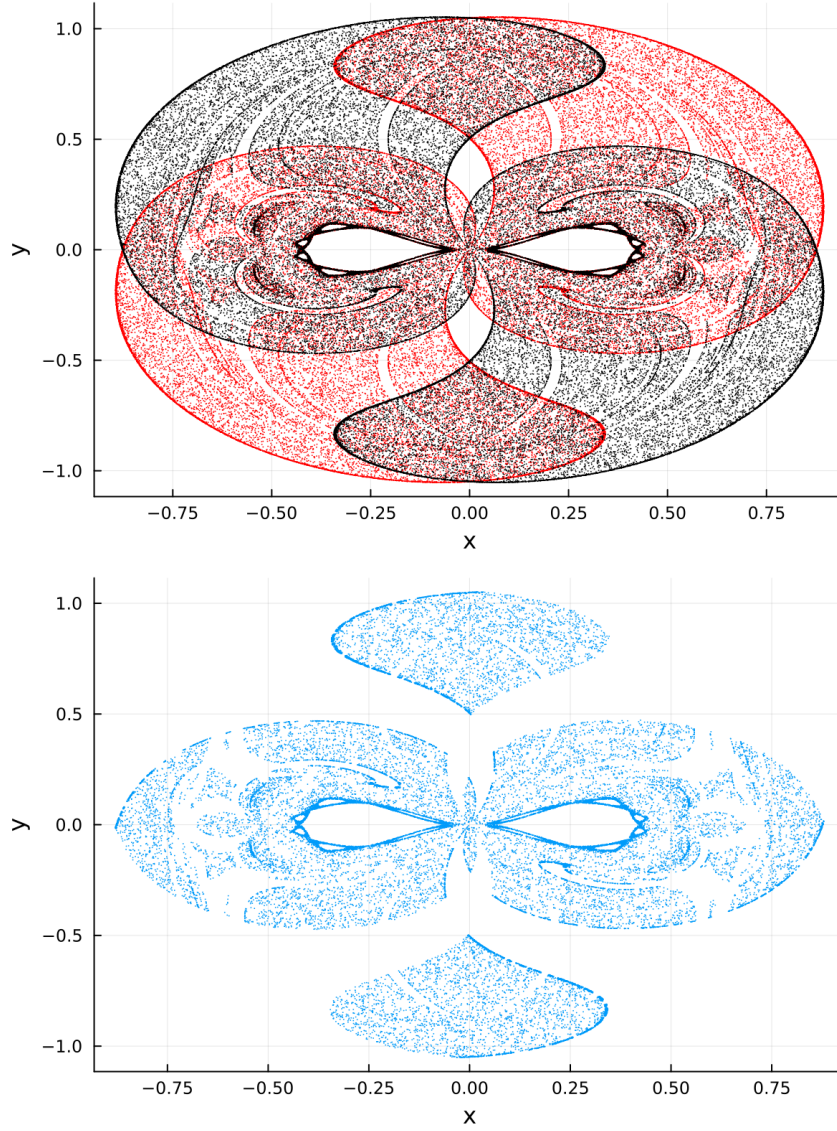


FIGURE B.5: Approximation of the chaotic sets in the Copenhagen problem with pro-grade initial conditions (S_2), visualised in configuration space (x, y) . This figure is formed by orbits that remained in the system until at least $t = 1000$. **(Top):** Approximate stable (black) and unstable (red) manifold. **(Bottom):** Approximate chaotic saddle. Approximately $6 \cdot 10^4$ points for each set are plotted in the top panel, and approximately $3 \cdot 10^4$ are plotted in the bottom panel.

B.2 Dissipative system

B.2.1 Exit basins

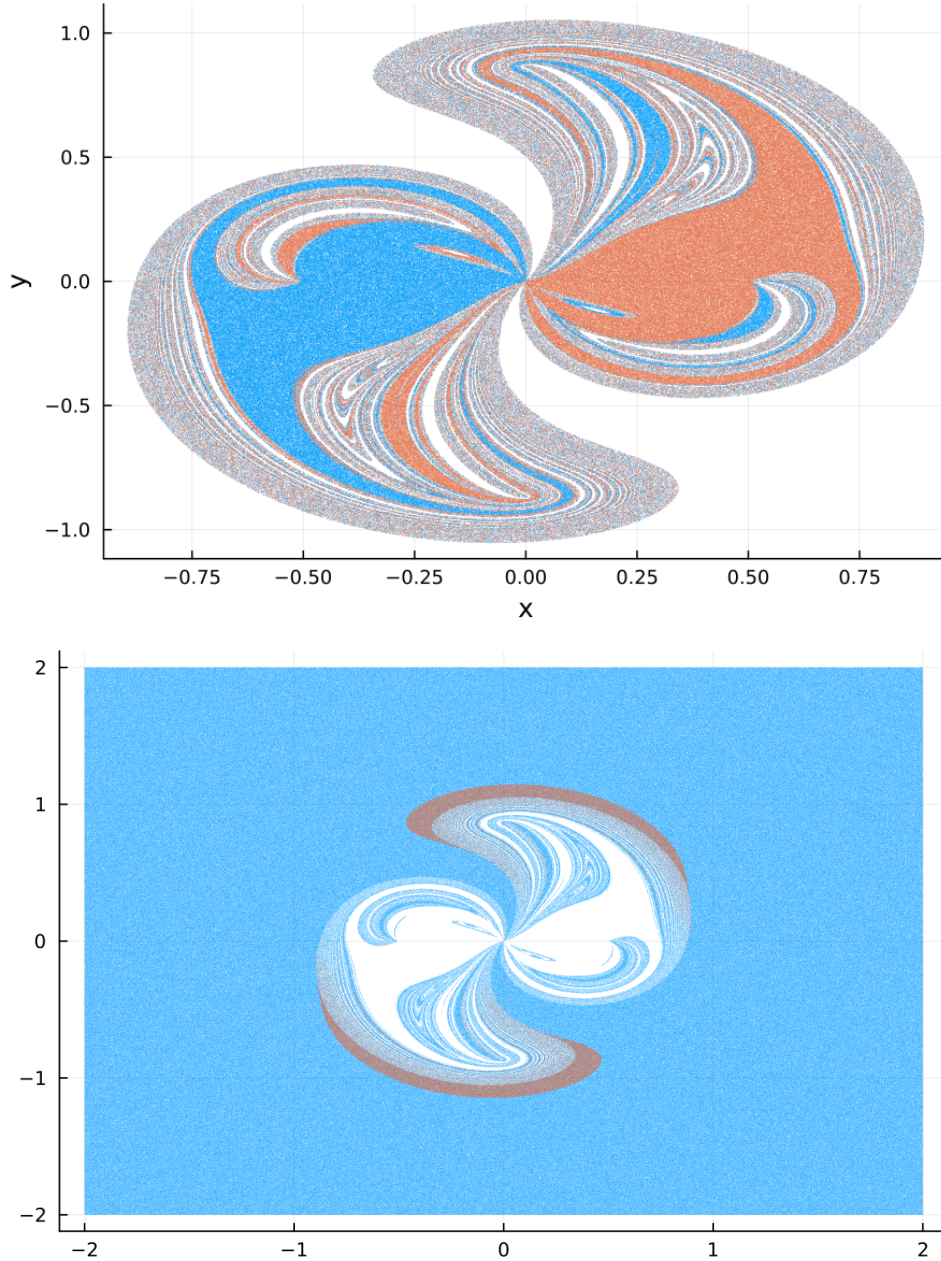


FIGURE B.6: Basin topology in the Copenhagen problem with Poynting-Robertson drag, visualised in the configuration space (x, y) . **(Top):** Initial conditions of trajectories ending in collision (blue and orange). **(Bottom):** Initial conditions of trajectories ending in escape (blue). Overlaid in orange are the initial conditions of escape orbits that managed to extract energy from the system.

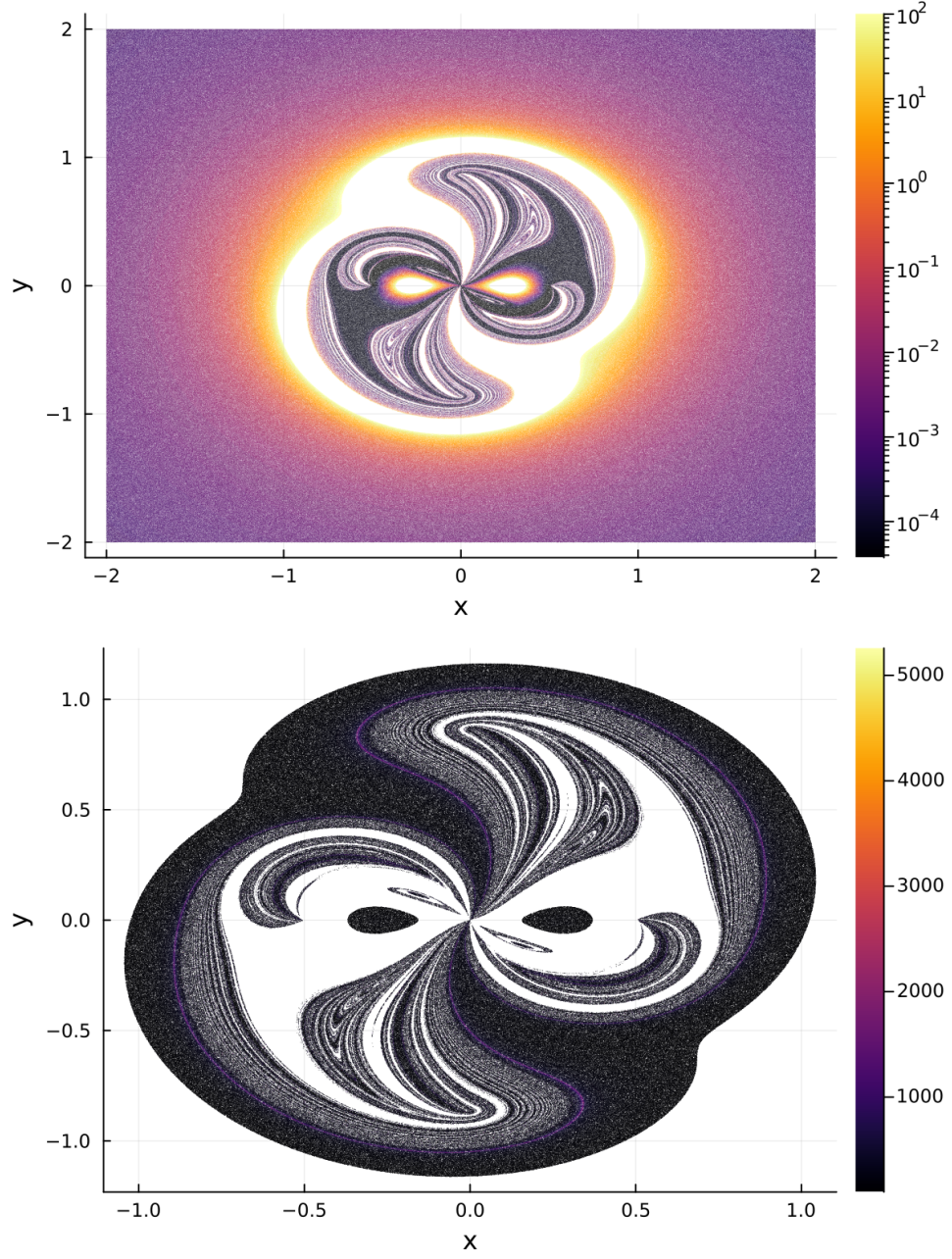


FIGURE B.7: Spatial distribution of escape/collision times in the Copenhagen problem with Poynting-Robertson drag, visualised in configuration coordinates (x, y) . **(Top):** initial conditions with escape/collision times less than 100. **(Bottom):** initial conditions with escape/collision times greater than 100.

B.2.2 Survival probability

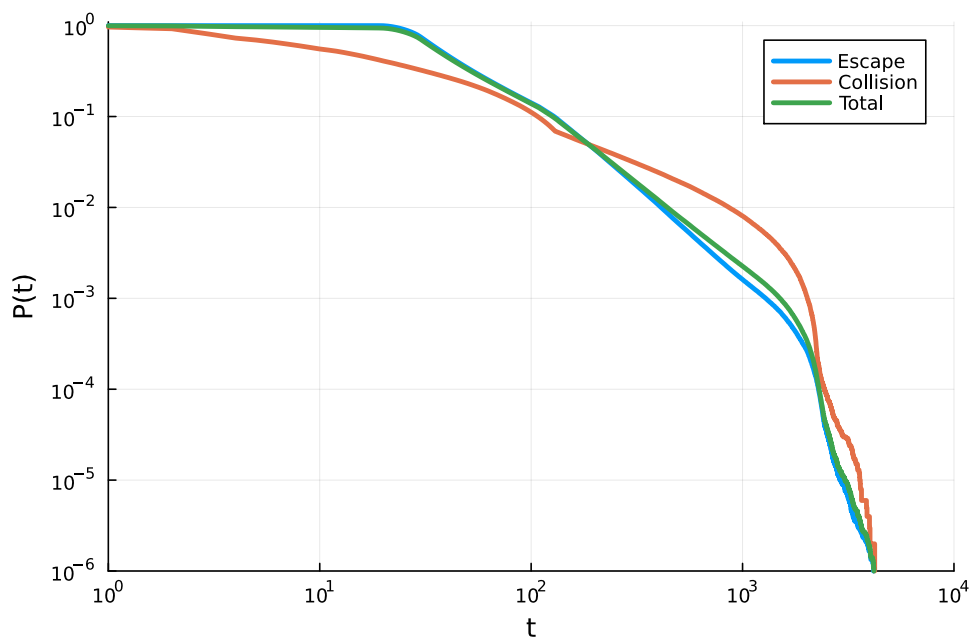


FIGURE B.8: Survival probability in the Copenhagen problem with Poynting-Robertson drag. The decay in the cumulative distribution of lifetimes for both escaping (blue) and colliding (orange) orbits are shown for escape/collision times up to $t = 10^4$. Also plotted in green is the "total" survival probability, i.e., the distribution for escape *and* collision orbits.

Bibliography

- [1] F.A. Abd El-Salam. “The effects of Poynting–Robertson drag on solar sails”. In: *Results Phys* 9 (2018), pp. 897–903. ISSN: 2211-3797.
- [2] Jacobo Aguirre, Juan C. Vallejo, and Miguel A. F. Sanjuán. “Wada basins and chaotic invariant sets in the Hénon-Heiles system”. In: *Phys. Rev. E* 64 (6 2001), p. 066208. DOI: [10.1103/PhysRevE.64.066208](https://doi.org/10.1103/PhysRevE.64.066208).
- [3] Eduardo G. Altmann, Jefferson S. E. Portela, and Tamás Tél. “Chaotic Systems with Absorption”. In: *Phys. Rev. Lett.* 111 (14 2013), p. 144101. DOI: [10.1103/PhysRevLett.111.144101](https://doi.org/10.1103/PhysRevLett.111.144101).
- [4] Eduardo G. Altmann, Jefferson S. E. Portela, and Tamás Tél. “Leaking chaotic systems”. In: *Rev. Mod. Phys.* 85 (2 2013), pp. 869–918. DOI: [10.1103/RevModPhys.85.869](https://doi.org/10.1103/RevModPhys.85.869).
- [5] Eduardo G. Altmann and Tamás Tél. “Poincaré recurrences and transient chaos in systems with leaks”. In: *Phys. Rev. E* 79 (1 2009), p. 016204. DOI: [10.1103/PhysRevE.79.016204](https://doi.org/10.1103/PhysRevE.79.016204).
- [6] V.I Arnold. *Mathematical Methods of Classical Mechanics*. Second Edition. Vol. 60. Graduate Texts in Mathematics. New York, NY: Springer New York, 1989. ISBN: 9780387968902.
- [7] R Barrio, F Blesa, and S Serrano. “Fractal structures in the Hénon-Heiles Hamiltonian”. In: *Europhysics letters* 82.1 (2008), pp. 10003–10003/6. ISSN: 0295-5075.
- [8] Joseph A. Burns, Philippe L. Lamy, and Steven Soter. “Radiation forces on small particles in the solar system”. In: *Icarus (New York, N.Y. 1962)* 40.1 (1979), pp. 1–48. ISSN: 0019-1035.
- [9] Lachlan Burton, Holger Dullin, and Eduardo G. Altmann. *Probabilistic description of dissipative chaotic scattering*. 2023. arXiv: [2308.13555](https://arxiv.org/abs/2308.13555) [[cond-mat.stat-mech](https://arxiv.org/archive/cond)].
- [10] Lachlan G. Burton. *Dissipative Chaotic Scattering Code*. github.com/LnBn/DissipativeScattering. 2023.

- [11] Tamás Bódai, Eduardo G. Altmann, and Antonio Endler. “Stochastic perturbations in open chaotic systems: Random versus noisy maps”. In: *Phys. Rev. E* 87.4 (Apr. 2013). 00003, p. 042902. ISSN: 1539-3755. DOI: [10.1103/PhysRevE.87.042902](https://doi.org/10.1103/PhysRevE.87.042902).
- [12] Alessandra Celletti et al. “Some results on the global dynamics of the regularized restricted three-body problem with dissipation”. In: *Celest Mech Dyn Astr* 109.3 (2011), pp. 265–284. ISSN: 0923-2958. DOI: [10.1007/s10569-010-9326-y](https://doi.org/10.1007/s10569-010-9326-y).
- [13] Mickaël D. Chekroun, Eric Simonnet, and Michael Ghil. “Stochastic climate dynamics: Random attractors and time-dependent invariant measures”. In: *Phys. D: Nonlinear Phenom.* 240.21 (2011), pp. 1685–1700. ISSN: 0167-2789.
- [14] Xiaowen Chen, Takashi Nishikawa, and Adilson E. Motter. “Slim Fractals: The Geometry of Doubly Transient Chaos”. In: *Phys. Rev. X* 7 (2 2017), p. 021040. DOI: [10.1103/PhysRevX.7.021040](https://doi.org/10.1103/PhysRevX.7.021040).
- [15] H.-D. Chiang et al. “On voltage collapse in electric power systems”. In: *IEEE Trans. Power Syst.* 5.2 (1990), pp. 601–611. ISSN: 0885-8950.
- [16] G. F. Choerny. “Quasiintegrals of the photogravitational eccentric restricted three-body problem with Poynting-Robertson drag”. In: *Celestial mechanics and dynamical astronomy* 97.4 (2007), pp. 229–248. ISSN: 0923-2958.
- [17] Simon Christ et al. “Plots.jl – a user extendable plotting API for the Julia programming language”. In: *JORS* 11 (1 2023), p. 5. DOI: [10.5334/jors.431](https://doi.org/10.5334/jors.431).
- [18] G. Contopoulos. “Asymptotic curves and escapes in Hamiltonian systems”. In: *Celest Mech Dyn Astr* 231.1 (May 1990), pp. 41–55.
- [19] George Datseris. “DynamicalSystems.jl: A Julia software library for chaos and nonlinear dynamics”. In: *JORS* 3.23 (2018), p. 598. DOI: [10.21105/joss.00598](https://doi.org/10.21105/joss.00598).
- [20] Mark F Demers and Lai-Sang Young. “Escape rates and conditionally invariant measures”. In: *Nonlinearity* 19.2 (2006), pp. 377–397. ISSN: 0951-7715. DOI: [10.1088/0951-7715/19/2/008](https://doi.org/10.1088/0951-7715/19/2/008).
- [21] Bruno Eckhardt. “Irregular scattering”. In: *Phys. D: Nonlinear Phenom.* 33.1-3 (1988), pp. 89–98.
- [22] Manfred Einsiedler. *Ergodic Theory with a view towards Number Theory*. 1st ed. 2011. Graduate Texts in Mathematics, 259. London: Springer London, 2011. ISBN: 0-85729-021-5.
- [23] Paul Frederickson et al. “The Liapunov dimension of strange attractors”. In: *J. Differ. Equ.* 49.2 (1983), pp. 185–207. ISSN: 0022-0396.

- [24] G. Froyland. "Using Ulam's method to calculate entropy and other dynamical invariants". In: *Nonlinearity* 12.1 (1999), p. 79. DOI: [10.1088/0951-7715/12/1/006](https://doi.org/10.1088/0951-7715/12/1/006).
- [25] Gary Froyland et al. "A transfer operator based numerical investigation of coherent structures in three-dimensional Southern Ocean circulation". In: *IEICE Proc. Ser.* (Jan. 2008). DOI: [10.34385/proc.42.b31-b1](https://doi.org/10.34385/proc.42.b31-b1).
- [26] Gary Froyland et al. "Detection of coherent oceanic structures via transfer operators". In: *Phys. Rev. Lett.* 98.22 (2007), pp. 224503–224503. ISSN: 0031-9007. DOI: [10.1103/PhysRevLett.98.224503](https://doi.org/10.1103/PhysRevLett.98.224503).
- [27] Michael Ghil, Mickaël D. Chekroun, and Eric Simonnet. "Climate dynamics and fluid mechanics: Natural variability and related uncertainties". In: *Phys. D: Nonlinear Phenom.* 237.14 (2008). Euler Equations: 250 Years On, pp. 2111–2126. ISSN: 0167-2789. DOI: <https://doi.org/10.1016/j.physd.2008.03.036>.
- [28] Herbert Goldstein and Vic Twersky. "Classical Mechanics". In: *Phys. Today* 5.9 (1952), pp. 19–20. ISSN: 0031-9228.
- [29] Celso Grebogi et al. "Final state sensitivity: An obstruction to predictability". In: *Phys Lett. A.* 99.9 (1983), pp. 415–418. ISSN: 0375-9601. DOI: [10.1016/0375-9601\(83\)90945-3](https://doi.org/10.1016/0375-9601(83)90945-3).
- [30] Lukas Halekotte, Anna Vanselow, and Ulrike Feudel. "Transient chaos enforces uncertainty in the British power grid". In: *J. Phys: Complex* 2.3 (2021), pp. 35015–. ISSN: 2632-072X.
- [31] Stephan M. Hammel, James A. Yorke, and Celso Grebogi. "Numerical orbits of chaotic processes represent true orbits". In: *Bull. Am. Math. Soc.* 19.2 (1988), pp. 465–469.
- [32] M. Hénon. "A two-dimensional mapping with a strange attractor". In: *Commun. Math. Phys.* 50.1 (1976), pp. 69–77.
- [33] Michel Henon and Carl Heiles. "The applicability of the third integral of motion: Some numerical experiments". In: *Astron. J.* 69 (1964), p. 73. DOI: [10.1086/109234](https://doi.org/10.1086/109234).
- [34] Jialin Hong. *Symplectic integration of stochastic Hamiltonian systems*. 1st ed. 2022. Lecture notes in mathematics ; Volume 2314. Singapore: Springer, 2022. ISBN: 981-19-7670-8.
- [35] C Jung and HJ Scholz. "Cantor set structures in the singularities of classical potential scattering". In: *J. Phys. A: Math* 20.12 (1987), p. 3607.

- [36] William Kahan and Ren-Cang Li. "Composition constants for raising the orders of unconventional schemes for ordinary differential equations". In: *Math. Comput.* 66.219 (1997), 1089–1099. DOI: [10.1090/S0025-5718-97-00873-9](https://doi.org/10.1090/S0025-5718-97-00873-9).
- [37] H. Kantz and P. Grassberger. "Repellers, semi-attractors, and long-lived chaotic transients". In: *Phys. D: Nonlinear Phenom.* 17.1 (1985), pp. 75–86. ISSN: 0167-2789.
- [38] Hiroshi Kobayashi et al. "Dust ring formation due to sublimation of dust grains drifting radially inward by the Poynting–Robertson drag: An analytical model". In: *Icarus (New York, N.Y. 1962)* 201.1 (2009), pp. 395–405. ISSN: 0019-1035.
- [39] T. Kovács. "How can contemporary climate research help understand epidemic dynamics? Ensemble approach and snapshot attractors". In: *J. R. Soc. Interface.* 17.173 (2020), pp. 20200648–. ISSN: 1742-5689.
- [40] György Károlyi and Tamás Tél. "New features of doubly transient chaos: complexity of decay". In: *J. Phys. Complex* 2.3 (2021), p. 035001. DOI: [10.1088/2632-072X/abedc3](https://doi.org/10.1088/2632-072X/abedc3).
- [41] Yun-Tung Lau, John M. Finn, and Edward Ott. "Fractal dimension in nonhyperbolic chaotic scattering". In: *Phys. Rev. Lett.* 66.8 (1991), pp. 978–981. ISSN: 0031-9007.
- [42] C. Lhotka and A. Celletti. "The effect of Poynting–Robertson drag on the triangular Lagrangian points". In: *Icarus (New York, N.Y. 1962)* 250.1 (2015), pp. 249–261. ISSN: 0019-1035.
- [43] Jaume Llibre and Carles Simó. "Oscillatory solutions in the planar restricted three-body problem". In: *Math. Ann.* 248.2 (1980), pp. 153–184. ISSN: 0025-5831.
- [44] Jaume Llibre and Carles Simó. "Some homoclinic phenomena in the three-body problem". In: *J. Differ. Equ.* 37.3 (1980), pp. 444–465. ISSN: 0022-0396.
- [45] James D. Meiss and Edward Ott. "Markov tree model of transport in area-preserving maps". In: *Phys. D: Nonlinear Phenom.* 20.2 (1986), pp. 387–402. ISSN: 0167-2789. DOI: [10.1016/0167-2789\(86\)90041-2](https://doi.org/10.1016/0167-2789(86)90041-2).
- [46] RAÚL J. Mondragón C. and PETER H. Richter. "Weak dissipation in a Hamiltonian map". In: *Int. J. Bifurc. Chaos.* 4.4 (1994), pp. 921–932. ISSN: 0218-1274. DOI: [10.1142/S0218127494000654](https://doi.org/10.1142/S0218127494000654).

- [47] Adilson E. Motter and Ying-Cheng Lai. “Dissipative chaotic scattering”. In: *Phys. Rev. E* 65.1 (2001), 015205(R). ISSN: 1539-3755. DOI: [10.1103/PhysRevE.65.015205](https://doi.org/10.1103/PhysRevE.65.015205).
- [48] Adilson E. Motter et al. “Doubly transient chaos: Generic form of chaos in autonomous dissipative systems”. In: *Phys. Rev. Lett.* 111.19 (2013), pp. 194101–194101. ISSN: 0031-9007. DOI: [10.1103/PhysRevLett.111.194101](https://doi.org/10.1103/PhysRevLett.111.194101).
- [49] Carl D. Murray. *Solar system dynamics*. Cambridge ; Cambridge University Press, 1999. ISBN: 0521572959.
- [50] Jan Nagler. “Crash test for the Copenhagen problem”. In: *Phys. Rev. E* 69 (6 2004), p. 066218. DOI: [10.1103/PhysRevE.69.066218](https://doi.org/10.1103/PhysRevE.69.066218).
- [51] David D Nolte. *Introduction to Modern Dynamics: Chaos, Networks, Space and Time*. Oxford: Oxford University Press, 2015. ISBN: 0199657033.
- [52] H. E Nusse and J. A Yorke. “The equality of fractal dimension and uncertainty dimension for certain dynamical systems”. In: *Commun. Math. Phys.* 150.1 (1992), pp. 1–21. ISSN: 0010-3616.
- [53] Edward Ott. *Chaos in Dynamical Systems*. 2nd ed. Cambridge University Press, 2002. DOI: [10.1017/CB09780511803260](https://doi.org/10.1017/CB09780511803260).
- [54] Edward Ott and Tamás Tél. “Chaotic scattering: An introduction”. In: *Chaos (Woodbury, N.Y.)* 3.4 (1993), pp. 417–426. ISSN: 1054-1500.
- [55] Giulio Pianigiani and James A. Yorke. “Expanding Maps on Sets Which are Almost Invariant: Decay and Chaos”. In: *Trans. Am. Math. Soc.* 252 (1979), pp. 351–366. ISSN: 0002-9947. DOI: [10.2307/1998093](https://doi.org/10.2307/1998093).
- [56] A.S Pikovsky. “Escape exponent for transient chaos and chaotic scattering in nonhyperbolic Hamiltonian systems”. In: *J. Phys. A: Math. Gen.* 25.8 (1992), pp. L477–L481. ISSN: 0305-4470. DOI: [10.1088/0305-4470/25/8/016](https://doi.org/10.1088/0305-4470/25/8/016).
- [57] Henri Poincaré. *Les méthodes nouvelles de la mécanique céleste, Volume 1*. Paris: Gauthier-Villars, 1892.
- [58] Christopher Rackauckas and Qing Nie. “DifferentialEquations.jl—a performant and feature-rich ecosystem for solving differential equations in Julia”. In: *JORS* 5.1 (2017), p. 15. DOI: [10.5334/jors.151](https://doi.org/10.5334/jors.151).
- [59] Alfréd Rényi. “On a new axiomatic theory of probability”. In: *Acta Math. Hungarica*. 6.3-4 (1955), pp. 285–335. ISSN: 0001-5954.
- [60] Jesús Maria Sanz-Serna and Mari-Paz Calvo. “Symplectic numerical methods for Hamiltonian problems”. In: *Int. J. Mod. Phys. C*. 4.02 (1993), pp. 385–392.

- [61] Tim Sauer. *Numerical analysis*. Third edition. Hoboken, New Jersey: Pearson, 2018. ISBN: 9780134696454.
- [62] Timothy D. Sauer. "Shadowing breakdown and large errors in dynamical simulations of physical systems". In: *Phys. Rev. E* 65 (3 2002), p. 036220. DOI: [10.1103/PhysRevE.65.036220](https://doi.org/10.1103/PhysRevE.65.036220).
- [63] Jesús M. Seoane and Miguel A. F. Sanjuán. "New developments in classical chaotic scattering". In: *Rep. Prog. Phys.* 76.1 (Dec. 2012), p. 016001. ISSN: 0034-4885. DOI: [10.1088/0034-4885/76/1/016001](https://doi.org/10.1088/0034-4885/76/1/016001).
- [64] Jesús M. Seoane, Miguel A. F. Sanjuán, and Ying-Cheng Lai. "Fractal dimension in dissipative chaotic scattering". In: *Phys. Rev. E*. 76.1 (2007), pp. 016208–016208. ISSN: 1539-3755.
- [65] Jesús M. Seoane et al. "Basin topology in dissipative chaotic scattering". In: *Chaos* 16.2 (2006), pp. 023–101. ISSN: 1054-1500. DOI: [10.1063/1.2173342](https://doi.org/10.1063/1.2173342).
- [66] Bhimsen K. Shivamoggi. *Nonlinear Dynamics and Chaotic Phenomena: An Introduction*. 2nd ed. 2014. Fluid Mechanics and Its Applications, 103. Dordrecht: Springer Netherlands, 2014. ISBN: 94-007-7094-4.
- [67] K. Stefański, K. Buszko, and K. Piecyk. "Transient chaos measurements using finite-time Lyapunov exponents". In: *Chaos* 20.3 (Sept. 2010), p. 033117. ISSN: 1054-1500. DOI: [10.1063/1.3483877](https://doi.org/10.1063/1.3483877).
- [68] V Szebehely and F. T Geyling. "Theory of Orbits: The Restricted Problem of Three Bodies". In: *J. App. Mech.* 35.3 (1968), pp. 624–624. ISSN: 0021-8936.
- [69] C Tsitouras. "Runge–Kutta pairs of order 5 (4) satisfying only the first column simplifying assumption". In: *Comput. Math. Appl.* 62.2 (2011), 770–775.
- [70] Tamás Tél. "The joy of transient chaos". In: *Chaos* 25.9 (2015). ISSN: 1054-1500.
- [71] Tamás Tél and Ying-Cheng Lai. *Transient Chaos: Complex Dynamics on Finite Time Scales*. 1st ed. Vol. 173. New York, NY: Springer Science, 2011. ISBN: 1441969861. DOI: [10.1007/978-1-4419-6987-3](https://doi.org/10.1007/978-1-4419-6987-3).
- [72] James H Verner. "Numerically optimal Runge–Kutta pairs with interpolants". In: *Numer. Algorithms* 53.2-3 (2010), 383–396. DOI: [10.1007/s11075-009-9290-3](https://doi.org/10.1007/s11075-009-9290-3).
- [73] Rafael D Vilela. "Doubly transient chaos in a decaying open flow". In: *J. Phys. Complex.* 2.3 (2021), p. 035013. DOI: [10.1088/2632-072X/ac0326](https://doi.org/10.1088/2632-072X/ac0326).

- [74] Martin. Wechselberger. *Geometric Singular Perturbation Theory Beyond the Standard Form*. 1st ed. 2020. Frontiers in Applied Dynamical Systems: Reviews and Tutorials, 6. Cham: Springer International Publishing, 2020. ISBN: 3-030-36399-6.
- [75] Zhihong Xia. "Melnikov method and transversal homoclinic points in the restricted three-body problem". In: *J. Differ. Equ.* 96.1 (1992), pp. 170–184. ISSN: 0022-0396.
- [76] Haruo Yoshida. "Construction of higher order symplectic integrators". In: *Phys. Lett. A* 150.5 (1990), pp. 262–268. ISSN: 0375-9601. DOI: [https://doi.org/10.1016/0375-9601\(90\)90092-3](https://doi.org/10.1016/0375-9601(90)90092-3).
- [77] Euaggelos E. Zotos. "An overview of the escape dynamics in the Hénon–Heiles Hamiltonian system". In: *Meccanica* 52.11-12 (2017), pp. 2615–2630. DOI: [10.1007/s11012-017-0647-8](https://doi.org/10.1007/s11012-017-0647-8).
- [78] Euaggelos E. Zotos. "Crash test for the Copenhagen problem with oblateness". In: *Celest. Mech. Dyn. Astron.* 122.1 (2015), pp. 75–99. ISSN: 0923-2958.