Mean Sojourn Time Problems and Eigenvalue Asymptotics of the Laplacian

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A thesis submitted in fulfilment of
the requirements for the degree of
Doctor of Philosophy

School of Mathematics and Statistics
Faculty of Science
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October, 2023

Statement of originality

This is to certify that to the best of my knowledge, the content of this thesis is my own work. This thesis has not been submitted for any degree or other purposes.

I certify that the intellectual content of this thesis is the product of my own work and that all the assistance received in preparing this thesis and sources have been acknowledged.

William N.K. Trad

Authorship attribution statement

Chapters (2), (3) and (4) are slightly repurposed (albeit essentially identical) versions of the following articles:

1. *Medet Nursultanov. William Trad. Justin Tzou. Leo Tzou. "The narrow capture problem on general Riemannian surfaces." Differential Integral Equations 36 (11/12) 877 - 906, November/December 2023.* ***https://doi.org/10.57262/die036-1112-877***

2. *Nursultanov, M, Trad, W, Tzou, L. Narrow escape problem in the presence of the force field. Math Meth Appl Sci. 2022; 45(16): 10027- 10051. doi:10.1002/mma.8354*

3. *M. Nursultanov, W. Trad, J. Tzou, L. Tzou. Eigenvalue Variations of the Neumann Laplace Operator Due to Perturbed Boundary Conditions.* ***https://arxiv.org/pdf/2306.00491.pdf***

In all three articles, I had major roles in deriving the results as well as preparing the articles for submission. In cases where I am not the corresponding author of a published item, permission to include the published material has been granted by the corresponding author.

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As the supervisor for the candidature upon which this thesis is based, I can confirm that the authorship attribution statements above are correct.

Leo Tzou

Acknowledgements

Firstly, I'd like to extend thanks to my informal supervisor Prof. Leo Tzou, for his role in my development as a mathematician since my undergraduate years and his invaluable feedback regarding my presentation and communication skills. I also would like to thank Leo for his continued support despite the less-than-ideal situations that my candidature ran into. I would also like to thank my direct collaborators Dr. Medet Nursultanov and Dr. Justin Tzou for their guidance in the preparation of the results of my thesis. I would also like to thank my formal supervisor Prof. Ben Goldys for giving me invaluable career advice and taking over the administrative side of my studies since Leo moved to Amsterdam. Additional professional thanks should also be given to Dr. Daniel Hauer and Dr. Haotian Wu. I'd like to thank Daniel for always finding the time to converse with me about essentially anything. I'd also like to thank him for his career advice and his recommendations on problems that I should think about, such as one involving the Dirichlet to Neumann map. I'd similarly like to thank Haotian for being receptive to some of my questions regarding an extension to my thesis that I'd been thinking about for a long time and for providing me with the necessary resources to formulate the extension. I haven't been able to extend the theorem yet, however, I am nonetheless grateful to Haotian for highlighting some resources regarding finite volume hyperbolic cusps. Last, but not least, I would like to thank those from outside my academic life. First, my enduring parents for supporting my education and for their unending love. Second, my dearest and precious friends, Anuj, David, Edgar, Hosea, Lucia, Paul, Rumi, and Tom, all of whom I met throughout my studies, for their sustained support and wonderful memories.

Abstract

Within this thesis, we describe three closely related problems, two of which are inspired by questions that appear ubiquitously throughout biophysics and cellular biology and initially sit squarely within computational and applied mathematics. The third of these problems is of a more theoretical nature and is reminiscent of historical and contemporary questions arising in the study of spectral asymptotics of the Laplacian in connection to ambient geometry. These problems, however, exhibit a degree of similarity via the role of Greens functions, objects which are central in this thesis. It is due to these Greens functions, that the second and third problems can be viewed as mirror images of one another.

The first section of this thesis concerns the narrow capture problem. Here, the average amount of time for a diffusing, surface-bound ion to strike a target (mean sojourn time) is derived. In this section, it is found that the mean sojourn time on the Greens function crucially depends on the rudimentary geometric properties of the underlying geometry of the problem. The second section will address the narrow escape problem. This problem asks a similar question to the first, however, it involves a change in the geometric landscape (closed to compact with boundary) and thus involves a significantly more sophisticated construction of a boundary restriction for the Greens function. There is much work done on the narrow escape problem in two and three dimensions when there is no advection and there are high degrees of symmetry. Thus, it is within this thesis that the question of the influence of drift on the mean sojourn time is addressed in more general geometric situations. In addition, following the works of [Amm+12a] and [NTT21], we also provide a rigorous derivation of the $O(1)$ term of the mean sojourn time in the presence of

advection. Furthermore, the notion of geodesic ellipses, introduced by [SSH06a] in earlier studies of the problem and made rigorous in the setting of compact Riemannian manifolds by [NTT21] are used to describe the influence of the *shape* of the boundary-bound target on the mean sojourn time. As a spoiler, inhomogeneities in the shape will be felt via the difference in the principal curvatures at the centre of the trap. In the final section, we will compare the Neumann eigenvalues of the Laplace-Beltrami operator on a compact Riemannian 3-manifold with the corresponding eigenvalues of a perturbed eigenvalue problem which involves the partitioning of the boundary into a Neumann and Dirichlet "part". Precisely speaking, we examine the behaviour of the perturbed eigenvalues as the Dirichlet part shrinks to a point. It is intuitive that such eigenvalues will approach the Neumann eigenvalues, however, interestingly is a lack of any substantial dependence on the ambient geometry of the Riemannian manifold beyond the local geometry near the Dirichlet part. In addition, strikingly, when comparing the asymptotics of the narrow escape problem and the eigenvalue variation problem, the local geometry of the target or Dirichlet part appears in the same areas of the asymptotic expansions.

Contents

Chapter 1

Introduction

1.1 Immune responses

The narrow capture problem involves the derivation of mean sojourn time asymptotic expansions for a diffusing surface-bound ion, viewed as a Brownian motion. An expanded description of the problem involving the proof of our main result will be contained in Chapter 2. There are many examples in mathematical biology which invoke the narrow capture framework. Within this introductory body, we will describe two extremely similar physiological mechanisms that involve some sort of signalling "cascade" to relay vital information for the facilitation of critical immune responses. The first such process involves the activation of mastocytes and their release of histamines in the presence of allergens [CSW09]. Along the mastocyte membrane are embedded FcεR1 receptors. Such receptors are of key importance in signalling. When the FcεR1 receptors are bound to a common antibody chain (IgE), and there is an antigen (e.g. pollen) present, a "clustering" phenomena occurs along the membrane in which these FcεR1-IgE antibody-linked receptors exhibit "cross-linking". That is, these antibody-linked receptors are pulled together into close proximity by the present multi-valent antigen to form many, separated FcεR1-IgE clusters. This clustering opens up a signalling pathway to the nucleus via the γ chain of the FcεR1 receptors and the Lyn and Syk signalling proteins. This scenario is shown in Figure 1.1(a).

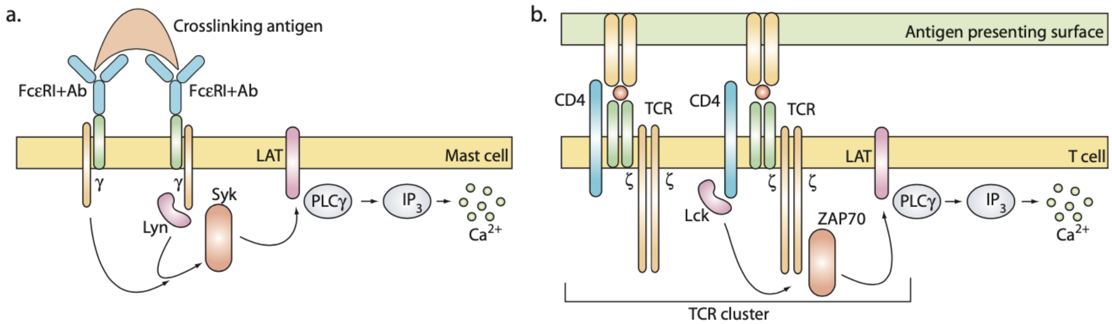

Figure 1.1: The above images were taken from [CSW09]. Copyright ©2009 Society for Industrial and Applied Mathematics. Reprinted with permission. All rights reserved.

Figure 1.1(b) depicts an experiment involving a synthetic material with embedded antigens along the surface. The material is then brought near a T cell. Within this experiment, the T cell receptors (TCR) bind to the antigens present on the surface of the material. A cross-linking mechanism (similar to that of the FcεR1-IgE links) amongst the TCR-antigen links pulls these links into close proximity on the surface of the T cell. Slightly differently, it is no longer the γ chains which facilitate the signal cascade, it is instead the ζ chains of the TCRs. In this situation, the pathway is then built off of the Lck and ZAP70 signalling proteins. For either pathway to be of any use, however, the second aspect of the mechanism needs to be in the correct configuration. That is, the transmembrane LAT proteins need to be at the cluster sites in order to trigger these cascades, which ultimately lead to degranulation. The LAT proteins are surface-bound and exhibit a high degree of freedom in their motion, up to the motion of other small vibrating molecules. Thus, it is reasonable to treat the trajectory of a LAT protein as a sample path of a Brownian motion. If the overarching question is to understand the aforementioned immune response mechanisms, then we require some understanding of the timescales with which important components of these cascades operate. In particular, within Chapter 2 of this thesis, we seek to understand how long it takes *on average* for *one* LAT protein (modelled as a single Brownian particle) to strike *one* FcεR1-IgE cluster. To emphasise, this signalling LAT protein is modelled as a common stochastic process traversing a two-dimensional surface. Mathematically, we will consider a Brownian motion $(X, \mathbb{P}_x)$ taking values in (M, g) which denotes a closed Riemannian surface. Here, $\mathbb{P}_x$ denotes the law of $X = (X_t)_{t\geq 0}$ starting at $x \in M$. Roughly speaking, $\mathbb{P}_x$ is the probability distribution of all possible sample paths starting at x. In the case of Brownian motions, this law coincides with the classical Wiener measure. We will use $\Gamma_\varepsilon := B_\varepsilon(x^*)$ (representative of our FcεR1-IgE cluster) to denote a geodesic ball with respect to g. We are therefore interested in deriving an asymptotic expansion for the average amount of time for a sample path of $(X, \mathbb{P}_x)$ to strike Γ_ε. This quantity is denoted by $u_\varepsilon(x)$. Note, that this quantity is only a function of the starting position of the Brownian motion. In addition, we will use $M_\varepsilon := M \setminus \Gamma_\varepsilon$.

There is an intricate connection between diffusion problems and Brownian motions. In particular, it has been long understood that the associated probability density functions solve the heat equation with various initial and boundary conditions. If $p_\varepsilon(t, x)$ denotes the probability density function for $(X, \mathbb{P}_x)$, then we prescribe vanishing Dirichlet conditions at ∂M_ε so that the Brownian particle is never found at the boundary. Similarly, we introduce an initial condition, $\rho_0(x)$ to denote a non-uniform distribution supported away from ∂M_ε (so as to nullify any potential losses in regularity in $p_\varepsilon(t, x)$ as x approaches ∂M_ε). Our problem is thus

$$\frac{\partial p_\varepsilon}{\partial t} = \Delta_g p_\varepsilon, \quad p_\varepsilon(0, x) = \rho_0(x), \quad p_\varepsilon(t, x)|_{x\in\partial M_\varepsilon} = 0.$$

Through a process of "averaging", one can recover $u_\varepsilon(x)$, heuristically, as

$$u_\varepsilon(x) = \int_{M_\varepsilon}\int_0^\infty p_\varepsilon(t, y \mid x) dt d\mu_g(y).$$

where $p_\varepsilon(t, y \mid x)$ denotes the *transition* density function. Such a function represents the likelihood that a sample path starting at $x \in M$ will be at $y \in M$ at time $t \in [0, \infty)$. Since we are working in the setting of non-Euclidean geometry, a little bit of care needs to be taken here. Namely, y and x need to be sufficiently close to one another (within the diameter of a coordinate neighborhood). Additionally, without being too technical and dispensing with rigor, $d\mu_g$ denotes the natural Riemannian volume form chosen with respect to g and coincides with integration in local coordinates against the Lebesgue measure. For greater details on $d\mu_g$, see [Lee18]. In any case, it follows that u_ε solves the following elliptic Dirichlet boundary value problem for $x \in M_\varepsilon$

$$\Delta_g u_\varepsilon = -1, \quad u_\varepsilon|_{x\in\partial M_\varepsilon} = 0.$$

The implication that u_ε solves the above problem follows from the classical Andronov-Vitt-Pontryagin theorem [Sch80]. This sort of "averaging" is also shown within the appendix of [NTT21] where functional analytic methods are employed as opposed to the probabilistic methods that are usually used. It is then through an observation that the Greens function of Δ_g on a closed manifold can be directly connected to u_ε. This connection results in the conclusion that the leading order singularities of u_ε that manifest as $\varepsilon \to 0$ are given by the leading order singularities in the Greens function that appear near the diagonal of $M \times M$. Furthermore, it will also be seen that the "smaller", asymptotically negligible terms of u_ε originate from the smoother, latter terms of the Greens function expansion. In addition, we also see some elementary geometric information contained in u_ε, such as the local geodesic distance function and the volume of the manifold. This is encapsulated within the following theorem.

Theorem 1.1.1. *Let (M, g) be a closed orientable Riemannian surface. Fix $x^* \in M$ and let $\Gamma_\varepsilon := B_\varepsilon(x^*)$ be a geodesic ball centred at x^* of geodesic radius $\varepsilon > 0$. Furthermore, let $E(x, y) \in \mathcal{D}'(M \times M)$ denote the Greens function of $-\Delta_g$ on (M, g) given by*

$$E(x, y) = -\frac{1}{2\pi} \log d_g(x, y) + P(x, y),$$

with $P \in C^1(M \times M)$. Then, we have that
i) For each $x \notin \Gamma_\epsilon$, the mean sojourn time has the following asymptotic expansion, as $\varepsilon \to 0$,

$$u_\varepsilon(x) = -\frac{|M|}{2\pi} \log \varepsilon + |M|P(x^*, x^*) - |M|E(x, x^*) + r_\varepsilon(x) + C_\varepsilon.$$

where $r_\varepsilon(x) = O_{L^\infty(K)}(\varepsilon)$ for any compact set $K \subset M$ for which $K \cap \Gamma_\varepsilon = \varnothing$ and C_ε is a constant of order $\varepsilon \log \varepsilon$.
ii) Let $M_\varepsilon = M \setminus \Gamma_\varepsilon$, then the spatial average of the mean sojourn time satisfies the asymptotic formula, as $\varepsilon \to 0$,

$$\frac{1}{|M_\epsilon|} \int_{M_\varepsilon} u_\varepsilon(x) d\mu_g(x) = -\frac{|M|}{2\pi} \log \varepsilon + |M|P(x^*, x^*) + C_\varepsilon.$$

where C_ε is a constant of order $\varepsilon \log \varepsilon$.

It should be noted, that the Greens function expansion used is note a particularly refined one. It is *possible* that more refined expansions of u_ε and the associated spatial average can be derived if one uses other kinds of parametrix constructions. One example could be the Hadamard parametrix. This is worth considering since one can derive terms that contain scalar curvature. This would lead to more geometric information appearing in the mean sojourn time. A possible avenue of future research is to study the behaviour of Brownian motion sample paths near regions of high negative curvature. If Γ_ε is in one of these regions, then one would expect for any given initial condition, there would be a "large" spectrum of sample paths that deviate away from the trap. If this is in fact true, then this might be due to the fact that these sample paths move along a collection of infinitesimally small geodesics that may be influenced by the ergodicity of the geodesic flow (for negative curvature). Lastly, one could investigate the relationship between u_ε and the regularity of ∂M_ε. There is a lot of harmonic analysis done in $\mathbb{R}^n$ to study the relationship between harmonic measures, see [Cap05], and the geometric and boundary regularity properties of the Dirichlet problem. Within the language of probability theory, such harmonic measures represent the exit distributions along the boundary of trapped Brownian motions. Naturally, one would expect an intimate connection (albeit potentially trivial) to exist between u_ε and such classes of measures.

1.2 Neurotransmission amongst the synapses

The narrow escape problem (NEP) is a similar problem to the previously mentioned narrow capture problem. Historically, this problem was posed in two dimensions to try to understand dendritic structures and their role in facilitating neurotransmission. Thus, the initial distinguishing feature from the narrow capture problem was the inclusion of a non-empty boundary and the placement of the trap/absorbing window on the boundary. Despite this shift in topological structure, these problems are similar from an algorithmic perspective. Within this thesis, we seek to follow the works of [SSH06b],[CWS10],[NTT21] and [Amm+12a] to solve the narrow escape problem in three dimensions in the presence of advection. According to [SSH08] and [Amm+12a], the increase to three dimensions makes the application of the established algorithms to the narrow escape problem significantly more challenging. Specifically speaking, a crucial element of the algorithm is a relatively explicit expression for the boundary restriction of some Greens function and it is well-known that even in simple geometries, such an expression is quite difficult to get a hold of. Similar to the narrow capture problem, there are many examples in mathematical biology that invoke the narrow escape framework. Here though, we will focus purely on one of the key physiological mechanisms underpinning neurotransmission.

The brain consists of a vast and complex network of neurons. Neurons are cells that consist of three main components, the cell body, the axon, and the dendrites. The most important aspect of neurotransmission involves direct connections between the axon terminal of one neuron and the branch-like dendritic structures of another. These connections are known as biological junctions and there are two types which appear in neurotransmission. The first type is known as an electrical synapse and the second type is known as a chemical synapse. Both synapses have their own respective roles within neurotransmission, however, we will simply focus our discussion on the second kind (see Figure 1.2). The role of dendrites within neurons is to receive signals from the axon terminals of other neurons. The received signal can then initiate a nerve impulse or an electrical signal to be sent along the axon of the neuron. Upon reaching the axon terminal, the nerve impulse triggers the release of a neurotransmitter, a Ca^{2+} ion, at the chemical synapse. The Ca^{2+} ion then strikes one of the dendrites of the target neuron, thus propagating the signal. Our goal is a modest one and it is to simply understand a small component of this complex physiological mechanism. In particular, we are interested in how long it takes for a Ca^{2+} ion to be released *on average* from the axon terminal. There are studies by [GT10] and [GCB02] which suggest that changes in synaptic stability, as well as plasticity, are *correlated* to changes in the rate of Ca^2+ ion diffusion. This is in turn linked to attempted improvements on existing diagnosis timelines for certain degenerative neurological conditions.

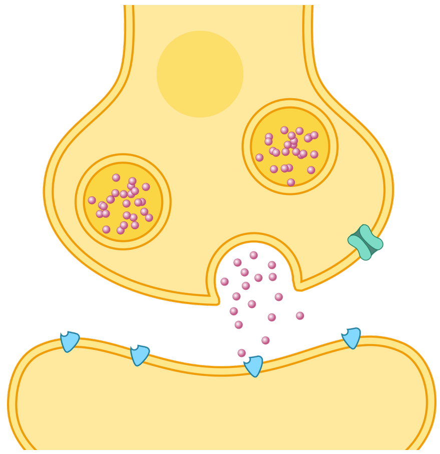

Figure 1.2: The diffusion of neurotransmitters (Ca^{2+} ions) across the synaptic cleft from the axon terminal of one neuron to one of the dendritic spine of the next neuron. This image was contributed by the user carolina-hresjsa on Pixabay.

Similar to the LAT protein of the narrow capture problem, we can treat the trajectory of the Ca^{2+} ion as a sample path of some Brownian motion $(X, \mathbb{P}_x)$ traversing the interior of $(M, g, \partial M)$ which denotes a compact, simply connected, Riemannian manifold of dimension 3 with a non-empty boundary. Much like the previous section, $\mathbb{P}_x$ denotes the law of the Brownian motion $X = (X_t)_{t\geq 0}$. Despite the notation, the Brownian motion considered here is not the same as that considered in the narrow capture problem. As mentioned earlier, this problem will involve the consideration of the effect of advection (X is influenced by a drift term). We will use $F := \nabla_g \phi$ to denote such a force. In addition, within this problem, we are interested in how the *shape* of the ion channel/absorbing window affects the mean sojourn time. As a result, we will, much like [NTT21] and [SSH06a] consider *geodesic ellipses* on the boundary. We will denote such ellipses by $\Gamma_{\varepsilon,a} \subset \partial M$. These will be defined more rigorously in the appendix as well as Chapter 3. All that we need for now is that the parameter a, determines the *eccentricty* of the ellipse. To be precise, the geodesic ellipse becomes a geodesic ball (with respect to the boundary metric) on ∂M when $a = 1$. The influence of the shape of the window can be felt by the mean sojourn time via the inclusion of the *principal curvatures* and the alteration of the magnitudes of various coefficients in the asymptotic expansion. Similar to the narrow capture problem, we can consider a probability density function $p_{\varepsilon,a}(t, x)$ for the

Brownian motion *under the influence of drift*. The advection-diffusion problem is given by

$$\frac{\partial p_{\varepsilon,a}}{\partial t} = \Delta_g p_{\varepsilon,a} - \mathrm{div}_g(F p_{\varepsilon,a}), \quad p_{\varepsilon,a}(0,x) = \rho_0(x),$$

with Dirichlet and Neumann boundary conditions

$$p_{\varepsilon,a}(t,x)|_{x\in\Gamma_{\varepsilon,a}} = 0, \quad \partial_\nu p_{\varepsilon,a}(t,x)|_{x\in\Gamma^c_{\varepsilon,a}} = 0.$$

Similar to the narrow capture problem, we can, heuristically write the mean sojourn time $u_{\varepsilon,a}$ as (note the dependence on the parameter a)

$$u_{\varepsilon,a}(x) = \int_M \int_0^\infty p_{\varepsilon,a}(t, y \mid x) dt d\mu_g(y).$$

Once again, we use $p_{\varepsilon,a}(t, y \mid x)$ to denote the transition density function. Additionally, $d\mu_g$ denotes the Riemannian volume form of M. Note, when integrating along the boundary, we replace $d\mu_g$ with the pull-back of $d\mu_g$ under the inclusion map $\iota : \partial M \hookrightarrow M$. We can employ classical theorems such as the Andronov-Vitt-Pontryagin theorem [Sch80] or employ arguments with a more functional analytic flavour, found in [NTT21] to derive the following problem for $u_{\varepsilon,a}$

$$\Delta_g u_{\varepsilon,a} + g(F, \nabla_g u_{\varepsilon,a}) = -1, \quad u_{\varepsilon,a}|_{x\in\Gamma_{\varepsilon,a}} = 0, \quad \partial_\nu u_{\varepsilon,a}|_{x\in\Gamma^c_{\varepsilon,a}} = 0.$$

Following this setup, it is within Chapter 3 that we will derive the leading order singular terms in the asymptotic expansion for $u_{\varepsilon,a}$. In particular, we will prove the following theorem

Theorem 1.2.1. *Let $(M, g, \partial M)$ be a smooth Riemannian manifold of dimension three with boundary. Fix $x^* \in \partial M$ and let $\Gamma_{\varepsilon,a}$ be a boundary geodesic ellipse centred at x^* given by (3.2) with $\varepsilon > 0$.*
i) For each $x \in M \backslash \Gamma_{\varepsilon,a}$, we have that

$$u_{\varepsilon,a}(x) = C_{\varepsilon,a} + \mathcal{G}(x) - \Phi(x^*) G(x^*, x) + r_\varepsilon(x),$$

with $r_\varepsilon = O_{C^k(K)}(\varepsilon)$ for any integer k and compact set $K \subset \overline{M}$ which does not contain x^*. The function $\mathcal{G}$ is the solution of some Neumann boundary value problem *(to be defined in Chapter 3). The constant $C_{\varepsilon,a}$ is given by*

$$\begin{aligned}
C_{\varepsilon,a} = {} & \frac{K_a\Phi(x^*)}{4\pi^2 a\varepsilon} - \frac{(H(x^*)+g_{x^*}(F,\nu))\Phi(x^*)}{4\pi}\log\varepsilon + R(x^*,x^*)\Phi(x^*) - \mathcal{G}(x^*) \\
& - \frac{(H(x^*)+g_{x^*}(F,\nu))\Phi(x^*)}{16\pi^3}\int_{\mathbb{D}}\int_{\mathbb{D}} \frac{\log\left((t_1-s_1)^2+a^2(t_2-s_2)^2\right)^{1/2}}{\sqrt{(1-|s'|^2)(1-|t'|^2)}}dt'ds' \\
& + \frac{(\kappa_1(x^*)-\kappa_2(x^*))\Phi(x^*)}{64\pi^3}\int_{\mathbb{D}}\int_{\mathbb{D}} \frac{(t_1-s_1)^2-a^2(t_2-s_2)^2}{(t_1-s_1)^2+a^2(t_2-s_2)^2}\frac{1}{\sqrt{(1-|s'|^2)(1-|t'|^2)}}dt'ds' \\
& + O(\varepsilon\log\varepsilon),
\end{aligned}$$

where $R(x,y)$ is the regular part of the ∂M-restricted Neumann Greens function (See Chapter 3), $\mathbb{D}\subset\mathbb{R}^2$ is the unit disk centred at the origin, and

$$\Phi(x) := \int_M e^{\phi(z)-\phi(x)}d\mu_g(z), \qquad K_a := \frac{\pi}{2}\int_0^{2\pi}\left(\cos^2\theta + \frac{\sin^2\theta}{a^2}\right)^{-1/2}d\theta.$$

In addition, $H(x^*)$ denotes the mean curvature at x^* and $\kappa_1(x^*),\kappa_2(x^*)$ denote the principal curvatures at x^*.
ii) The spatial average of the mean sojourn time satisfies the following asymptotic formula, as $\varepsilon\to 0$,

$$\int_M u_{\varepsilon,a}(x)d\mu_g(x) = C_{\varepsilon,a}|M| + \int_M \mathcal{G}(x)d\mu_g(x) - \Phi(x^*)\int_M G(x,x^*)d\mu_g(x).$$

An important feature of the proof of the above theorem is the observation that the restriction of the Neumann Greens function, which solves the adjoint of the problem with which $u_{\varepsilon,a}$ solves, to the boundary exists and is a left pseudo-differential parametrix of the elliptic Dirichlet-to-Neumann operator. We use the derivation of the symbol expansion of the Dirichlet-to-Neumann operator in [LU89a] (adapted to our operator) to then employ a standard iterative procedure to derive the leading order singularities of the boundary restriction of the Neumann Greens function. In turn, we can use Greens identity to connect this construction to $u_{\varepsilon,a}$, thus yielding the given expansions for $u_{\varepsilon,a}$. It is through this connection to the Greens function that the geometric features of (M,g), near the window, are felt in $u_{\varepsilon,a}$. In particular, we see quantities such as the volume of the manifold, the mean curvature at the window and the principal curvatures at the window. It should be noted though, that the principal curvatures are an artefact of the inclusion of a non-trivial eccentricity. If the window is "disk" shaped, then these principal curvature terms would disappear. This can be seen if one sets $a = 1$ and calculates the integral in the sixth term of $C_{\varepsilon,a}$ in the above theorem.

1.3 Perturbation of Neumann eigenvalues

Within this segment of the introduction, we will describe an eigenvalue perturbation problem that was solved using much of the same methodology employed in the derivation of the main theorem of the narrow escape problem with advection. First consider a compact, simply connected, orientable Riemannian 3-manifold with a nonempty boundary which we denote by $(M, g, \partial M)$. We will re-use the notion of a geodesic ellipse $\Gamma_{\varepsilon,a} \subset \partial M$ and maintain much of the same notation. The Neumann eigenvalues that we are interested in are those of the following system

$$\Delta_g u_j + g(F, \nabla_g u_j) = -\lambda_j u_j, \quad \partial_\nu u_j|_{\partial M} = 0.$$

Here, u_j is a sequence a $L^2(M, e^\phi d\mu_g)$-orthonomalized eigenfunctions and $F = \nabla_g \phi$. The overarching problem is to understand what occurs to the eigenvalues λ_j when a small hole is drilled into the boundary of the manifold. If $u_{j,\varepsilon}$ denotes a sequence of $L^2(M, e^\phi d\mu_g)$-orthonomalized eigenfunctions of

$$\Delta_g u_{j,\varepsilon} + g(F, \nabla_g u_{j,\varepsilon}) = -\lambda_{j,\varepsilon} u_{j,\varepsilon}, \quad \partial_\nu u_{j,\varepsilon}|_{\Gamma^c_{\varepsilon,a}} = 0, \quad u_{j,\varepsilon}|_{\Gamma_{\varepsilon,a}} = 0.$$

with $\lambda_{j,\varepsilon}$, the corresponding sequence of eigenvalues, we seek to understand the behaviour of

$$\lambda_{j,\varepsilon} - \lambda_j \text{ as } \varepsilon \to 0.$$

A component of the analysis within chapter (4) also shows us how the inhomogeneity in the shape of the hole in the boundary manifests in the asymptotic expansion of $\lambda_{j,\varepsilon} - \lambda_j$. Using a slew of pseudo-differential and asymptotic methods (to be elaborated on within chapter (4)) we are able to derive the following theorem.

Theorem 1.3.1. *Fix $x^* \in \partial M$, the centre of $\Gamma_{\varepsilon,a}$. If λ_j is a simple eigenvalue, then*

$$\lambda_{j,\varepsilon} - \lambda_j = A\varepsilon + B\varepsilon^2 \log \varepsilon + (C_1 + C_2 + C_3)\varepsilon^2 + O(\varepsilon^3 \log^2 \varepsilon).$$

Here, the constants are defined as follows

$$\begin{aligned} K_a &= \frac{\pi}{2}\int_0^{2\pi} \frac{a}{(a^2\cos^2\theta + \sin^2\theta)^{1/2}} d\theta \\ A &= \frac{4\pi^2 a}{K_a} |u_j(x^*)|^2 e^{\phi(x^*)}, \\ B &= \frac{4\pi^3 a^2 |u_j(x^*)|^2 e^{\phi(x^*)}}{K_a^2}(H(x^*) - g_{x^*}(F,\nu)) \end{aligned}$$

and

$$\begin{aligned} C_1 = \frac{a^2\pi(H(x^*) - g_{x^*}(F,\nu))|u_j(x^*)|^2 e^{\phi(x^*)}}{K_a^2} \times \\ \times \int_{\mathbb{D}} \frac{1}{(1-|s'|^2)^{1/2}} \int_{\mathbb{D}} \frac{\log\left((t_1 - s_1)^2 + a^2(t_2 - s_2)^2\right)^{1/2}}{(1-|t'|^2)^{1/2}} dt' ds', \end{aligned}$$

$$\begin{aligned} C_2 = \frac{a^2\pi\left(\kappa_1(x^*) - \kappa_2(x^*)\right)|u_j(x^*)|^2 e^{\phi(x^*)}}{4K_a^2} \times \\ \times \int_{\mathbb{D}} \frac{1}{(1-|s'|^2)^{1/2}} \int_{\mathbb{D}} \frac{(t_1 - s_1)^2 - a^2(t_2 - s_2)^2}{(t_1 - s_1)^2 + a^2(t_2 - s_2)^2} \frac{1}{(1-|t'|^2)^{1/2}} dt' ds' \end{aligned}$$

$$C_3 = \frac{16\pi^4 a^2 R_{\partial M}^{\lambda_j}(x^*, x^*)|u_j(x^*)|^2 e^{\phi(x^*)}}{K_a^2},$$

where $R_{\partial M}^{\lambda_j}(x^*, x^*)$ is the evaluation at $(x, y) = (x^*, x^*)$ of the kernel $R_{\partial M}^{\lambda_j}(x, y)$ in Proposition 4.1.1. Here $H(x^*)$ denotes the mean curvature at x^* and $\kappa_1(x^*), \kappa_2(x^*)$ denotes the principal curvatures at x^*.

Similar to the asymptotics of the mean sojourn time, which were derived in solving the narrow escape problem, we see that $\lambda_{j,\varepsilon} - \lambda_j$ consists of the same kinds of geometric quantities such as the mean and principal curvatures of the manifold near the ellipse. It is worth noting that the "asymptotic priority" of these quantities is also roughly the same as that of the mean sojourn time. That is, the mean curvature takes precedence over the principal curvature in the asymptotic expansion for $\lambda_{j,\varepsilon} - \lambda_j$. This follows from the unavoidable connection between $\lambda_{j,\varepsilon} - \lambda_j$ and the boundary restriction of the Neumann Greens function, whose second-order singularity carries the mean curvature compared to the third-order term which carries the scalar second fundamental form (these yield the principal curvatures at the centre of the ellipse in an appropriate choice of orthonormal frame). This leads to an interesting question regarding the geometric information contained in the eigenvalue variation if the Dirichlet and Neumann boundary conditions are swapped. Lastly, of a slightly different flavor, it is also interesting to study what sort of topological features (such as Euler characteristic or genus) can be weaved into such an eigenvalue asymptotic expansion. This sort of problem may however require other elements of global analysis beyond the construction of Greens functions.

Chapter 2

The Narrow Capture Problem

2.1 Introduction

We consider a Brownian particle bound to a surface that contains a small trap denoted Γ_ε. The narrow capture problem deals with the time required for such a particle to first encounter the trap. This time is called the sojourn time and is denoted $\tau_{\Gamma_\varepsilon}$. Starting from an initial location x on the surface, the *expected time* that a particle will wander before being captured by the trap is called the mean sojourn time and is denoted $u_\varepsilon(x)$.

In [Pil+10], a matched asymptotic method was employed to determine the $O(1)$ term in the expansion in terms of a certain Greens function that encoded information on the geometry of Ω, the locations of the traps, and the initial position x. This method, developed in [WK93b], effectively summed all logarithmic correction terms in the expansion of u_ε, with the resulting error term being transcendentally small in ε. In [BL15], a more detailed model was considered in which the windows were allowed to open and close stochastically, more closely mimicking the behaviour of cell ion channels. Other quantities of interest aside from the mean sojourn time include the variance of the sojourn time [Kur+15] as well as the so-called extreme sojourn time, the minimum search time achieved by a large group of searchers [ML20].

The narrow capture problem also has wide applications in cellular biology [CW11;

CSW09]. For example, a diffusing molecule must arrive at a localized signaling region within a cell or on its surface before a signaling cascade can be initiated. In another example, a T cell may diffuse in search of an antigen-presenting cell to trigger an immune response. In this latter example, determining the duration of this search is relevant to understanding immune response time [CSW09; DWC15].

The matched asymptotic methods of [WK93b] that were used for the aforementioned narrow escape problems have also been successfully applied to narrow capture problems in Euclidean metrics. In earlier works, the closely related problem of computing the fundamental Neumann eigenvalue λ_0 for the Laplacian in Euclidean two- and three-dimensional domains with small traps was considered in [CW11; KTW05b]. The spatial average of the mean sojourn time was shown to be proportional to λ_0^{-1} (see [KTW05b; WK93b]), and in [RC19], a numerical algorithm was employed to optimize this quantity with respect to configurations of traps located in the domain.

Extensions of these works include computing the full probability distribution (i.e., all moments as opposed to just the first) of the sojourn time [LST16; Bre21a; Bre21b; BS22], stochastic resetting [Bre21a; Bre21b; BS22], moving traps [TK15; LTK17; Iya+21a], partially absorbing traps [LBW17; BS22], traps grouped in clusters [Kur+15; IW21], and the effect of advection [Kur+15; NTT22].

In a non-Euclidean geometry, [CSW09] considered the mean sojourn time of a Brownian particle on a sphere containing small absorbing traps. Explicit results were obtained through the employment of the aforementioned matched asymptotic method along with the known analytic formula for the Neumann Greens function for the Laplacian on the sphere. The spherical geometry considered was meant to approximate the geometry of a cell with receptor clusters on its surface awaiting the arrival of surface-bound signaling molecules. A more detailed model of a cell, however, would be non-spherical. In fact, a cell's geometry can be crucial to the manner in which it serves its function [Fer+19]. It is with this motivation that we develop here a rigorous mathematical framework for narrow capture problems posed on non-Euclidean and non-spherical geometries.

We now mathematically formulate the narrow capture problem. Let (M, g) be a compact, connected, orientable, Riemannian surface with smooth boundary, ∂M.

First, we assume the boundary to be empty and calculate the associated mean sojourn time as well as its spatial average. When the manifold is of non-empty boundary, we assume, without loss of generality that M is a connected open subset of a compact orientable, Riemannian manifold $(\widetilde{M}, g)$ without boundary. Let $(X, \mathbb{P}_x)$ be the Brownian motion on M generated by the (negative) Laplace-Beltrami operator Δ_g. Here $\mathbb{P}_x$ denotes the classical Wiener measure or law of the Brownian motion. We use $\Gamma_\varepsilon \subset M$ to denote a geodesic ball centred at $x^* \in M$ with radius $\varepsilon > 0$ and we denote by $\tau_{\Gamma_\varepsilon}$ the first time the Brownian motion X_t hits Γ_ε, that is

$$\tau_{\Gamma_\varepsilon} := \inf\{t \geq 0 : X_t \in \Gamma_\varepsilon\}.$$

Within the narrow capture problem, we wish to derive an asymptotic as $\varepsilon \to 0$ for the mean sojourn time which is defined as the following expected value

$$u_\varepsilon(x) = \mathbb{E}\left(\tau_{\Gamma_\varepsilon} \mid X_0 = x\right).$$

An associated quantity of interest is the spatial average of the mean sojourn time

$$|M_\varepsilon|^{-1} \int_{M_\varepsilon} \mathbb{E}\left(\tau_{\Gamma_\varepsilon} \mid X_0 = x\right) d\mu_g(x),$$

where $M_\varepsilon := M \setminus \Gamma_\varepsilon$ and $|M_\varepsilon|$ is the Riemannian volume of M_ε with respect to the metric g.

Many works have been devoted to this topic, especially in applied mathematics. In [Lin86], the mean sojourn time for diffusing particles on a surface of the sphere with one absorbing trap was considered. They obtained the asymptotic, up to the bounded term, for the mean sojourn time as well as its spatial average. These results were generalized, in [CSW09], for the case of several traps. In [CW11; LBW17], the three-dimensional version of this problem was studied. For domains in $\mathbb{R}^3$, they obtain the asymptotic formulas in terms of capacitance, by using the method of matched asymptotic expansions. We also mention works [BEW08; RC19; TW00; Iya+21b; Bre21a; Bre21b; SWF07], where the authors investigate related problems.

Despite the large number of works on this topic, there are still many questions regarding more general geometries. In this direction, the goal of this chapter is to investigate the narrow capture problem for the Riemannian surface. Similar to

[Amm+12b], we use a layer potential method, however by adjoining this method with techniques originating from geometric microlocal analysis, we can extend the results, as well as the method to more general geometries, similar to the extension to broader classes of geometries for the narrow escape problem in [NTT21; NTT22]. As mentioned previously, we will consider empty and non-empty boundary cases. Within this work, a crucial aspect of the derivation for the mean sojourn time is the Greens function. For the sake of conciseness, we will present the results and required Greens function for the $\partial M = \varnothing$ case here. For the associated *Neumann* Greens function and results for the $\partial M \neq \varnothing$, see Section 5. First, consider the following problem for which the Greens function E of Δ_g on a closed Riemannian surface solves

$$\Delta_g E(x,y) = -\delta_y(x) + \frac{1}{|M|}, \quad E(x,y) = E(y,x), \quad \int_M E(x,y)d\mu_g(y) = 0. \tag{2.1}$$

where $|M|$ denotes the Riemannian volume of M with respect to the metric g. It is commonly known that near the diagonal the Greens function has the following representation (see [Tay11])

$$E(x,y) = -\frac{1}{2\pi}\log d_g(x,y) + P_{-4}(x,y), \tag{2.2}$$

for some $P_{-4}(x,y) \in C^1(M \times M)$ which is infinitely smooth away from the set $\{(x,y) \in M \times M \mid x = y\}$. (In fact, in the language of pseudodifferential operators, $P_{-4}(x,y)$ is the Schwartz kernel of a pseudodifferential operator of degree -4.)

This expansion allows us to obtain the following asymptotic for the narrow capture of Brownian particles in a small trap:

Theorem 2.1.1. *Let (M,g) be a closed orientable Riemannian surface. Fix $x^* \in M$ and let $\Gamma_\varepsilon := B_\varepsilon(x^*)$ be a geodesic ball centred at x^* of geodesic radius $\varepsilon > 0$.*

i) For each $x \notin B_\varepsilon(x^*)$, the mean sojourn time satisfies the following asymptotic formula, as $\varepsilon \to 0$,

$$u_\varepsilon(x) = -\frac{|M|}{2\pi}\log\varepsilon + |M|P_{-4}(x^*,x^*) - |M|E(x,x^*) + r_\varepsilon(x) + O(\varepsilon\log\varepsilon).$$

where $r_\varepsilon(x) = O_{L^\infty(K)}(\varepsilon)$ for any compact set $K \subset M$ for which $K \cap \Gamma_\varepsilon = \varnothing$ and C_ε is a constant of order $\varepsilon\log\varepsilon$.

ii) Let $M_\varepsilon = M \setminus \Gamma_\varepsilon$, then the spatial average of the mean sojourn time satisfies the asymptotic formula, as $\varepsilon \to 0$,

$$\frac{1}{|M_\varepsilon|}\int_{M_\varepsilon} u_\varepsilon(x)d\mu_g(x) = -\frac{|M|}{2\pi}\log\varepsilon + |M|P_{-4}(x^*, x^*) + C_\varepsilon.$$

where C_ε is a constant of order $\varepsilon\log\varepsilon$.

In Section 5 Theorem 2.5.3 we will prove a similar result for $\partial M \neq \varnothing$ with reflection boundary conditions for the Brownian motion. In this setting, we will use instead the Neumann Greens function $E(x, y)$. See Section 2.5 for details.

This chapter is structured in the following manner. In Section 2.2, we introduce some notation and the geometric framework with which we will be operating. Section 2.3 deals with investigating the singular structure of the Greens function on a Riemannian surface without boundary. In Section 2.4, We make use of the derived Greens function to prove Theorem 2.1.1. In Section 2.5, we consider the analogous problem in the setting of a manifold with boundary and impose a reflecting boundary condition for our Brownian motion. The result will be stated in Theorem 2.5.3.

2.2 Preliminaries

Throughout this chapter, (M, g) is a compact connected orientable Riemannian surface with a smooth boundary, ∂M which could be empty. The corresponding volume form and geodesic distance are denoted by $d\mu_g$ and $d_g(\cdot,\cdot)$, respectively. By $|M|$ we denote the volume of M.

For fixed $x \in M$, we will denote by $B_\rho(x)$ the geodesic ball of radius $\rho > 0$ centred at x. In what follows ρ will always be smaller than the injectivity radius of (M, g) and the distance from x to ∂M. We let $\mathbb{D}_\rho$ be the Euclidean ball in $\mathbb{R}^2$ of radius ρ centred at the origin.

In this work, we will often use the geodesic coordinates constructed as follows. For fixed $x^* \in M$ and orthonormal tangent vectors $E_1, E_2 \in T_{x^*}M$, write $t = (t_1, t_2) \in$

$\mathbb{D}_\rho$ and define

$$x(t;x^*) := \exp_{x^*}(t_1E_1 + t_2E_2), \tag{2.3}$$

where $\exp_{x^*}(V)$ denotes the time 1 map of g-geodesics with initial point x^* and initial velocity $V \in T_{x^*}M$. The coordinate $t \in \mathbb{D}_\rho \mapsto x(t;x^*)$ is then an g-geodesic coordinate system for a neighbourhood of x^* on M.

We will also use the re-scaled version of this coordinate system. For $\varepsilon > 0$ sufficiently small we define the (re-scaled) g-geodesic coordinate by the following map

$$x^\varepsilon(\cdot;x^*) : t = (t_1,t_2) \in \mathbb{D} \mapsto x(\varepsilon t;x^*) \in B_\varepsilon(x^*), \tag{2.4}$$

where $\mathbb{D}$ is the unit disk in $\mathbb{R}^2$.

In the subsequent sections, we denote the centre of the "trap" by $x^* \in M$, which will be considered as fixed. We will use the following notations. We set $\Gamma_\varepsilon := B_\varepsilon(x^*)$

$$M_\varepsilon = M_\varepsilon(x^*) := M \setminus \Gamma_\varepsilon,$$

and denote by $h = h(\varepsilon, x^*)$ the metric on ∂M_ε, induced by the trivial embedding of ∂M_ε into M_ε. The corresponding volume form is denoted by $d\mu_h$. Further, we set $z \in \partial M_\varepsilon \mapsto \nu_z$ to be an outward pointing normal for M_ε. Finally, we let $|M_\varepsilon|$, $|\partial\Gamma_\varepsilon|$ be the volumes of M_ε and $\partial\Gamma_\varepsilon$ with respect to g and h.

2.3 Initial Estimates on the Greens function

Within this section, we assume that ∂M is empty and we consider the Greens function on M, which is the fundamental solution to the Laplace equation:

$$\Delta_g E(x,y) = -\delta_y(x) + \frac{1}{|M|}, \quad E(x,y) = E(y,x), \quad \int_M E(x,y)d\mu_g(y) = 0. \tag{2.5}$$

For a fixed $x^* \in M$ and set $\Gamma_\varepsilon = B_\varepsilon(x^*)$ we consider the following function

$$I_\varepsilon(x^*,x) := \int_{\Gamma_\varepsilon} E(x,y)d\mu_g(y).$$

for $x \in M_\epsilon$. We will need to know about the singular behaviour of $\partial_{\nu_x}E(\cdot,\cdot)$, $I_\varepsilon(x^*,\cdot)$, and $\partial_{\nu_x}I_\varepsilon(x^*,\cdot)$ on $\partial\Gamma_\varepsilon$ as we approach neighbourhoods of the diagonal. To investigate these, we recall the singularity structure of $E(\cdot,\cdot)$:

Proposition 2.3.1. *The Greens function $E(x,y)$ and has the following singularity structure near the diagonal*

$$E(x,y) = -\frac{1}{2\pi}\log d_g(x,y) + P_{-4}(x,y),$$

where $P_{-4}(x,y) \in C^1(M\times M)$ is infinitely differentiable off the diagonal $\{x=y\}$.

As the distance function plays a crucial role in the Greens function $E(x,y)$, it is useful to derive asymptotics for them in the appropriate coordinate systems:

Lemma 2.3.2. *Let $d_g^*(s,t) := d_g(x(s,x^*), x(t,x^*))$, where $t=(t_1,t_2)\in\mathbb{D}_\rho$, resp. $s=(s_1,s_2)\in\mathbb{D}_\rho$ map to $x(t,x^*)$ resp. $x(s,x^*)$ are the coordinate systems defined in* (2.3)*. Then we have that*

$$d_g^*(s,t) = |s-t| + |s-t|F\left(t, \frac{s-t}{|s-t|}, |s-t|\right),$$

for some smooth function $F(t,\omega,r)\in C^\infty(\mathbb{D}_\rho\times S^1\times[0,2\rho])$ which is $O(t)+O(r)$.

Proof. By Lemma 4.8 of [Lef], if $t\mapsto x(t,x^*)$ is any coordinate system, there exists a matrix $H_{j,k}(s,t)$ smooth in (s,t) such that

$$d_g^*(s,t)^2 = \sum_{j,k=1}^{2} H_{j,k}(s,t)(s_j-t_j)(s_k-t_k), \tag{2.6}$$

where $H_{j,k}(t,t) = g_{j,k}(t)$ is the coordinate expression for the metric tensor g. Since the coordinate system (2.3) is the geodesic coordinate system, we have that $g_{j,k}(t) = \delta_{j,k}+O(|t|^2)$. So we get $H_{j,k}(t,t) = \delta_{j,k}+O(|t|^2)$. Taylor expanding $H_{j,k}(s,t)$ around $s=t$ and inserting the resulting expression into (2.6) we obtain

$$d_g^*(s,t) = |s-t| + |s-t|F\left(t, \frac{s-t}{|s-t|}, |s-t|\right),$$

for some smooth function $F\in C^\infty(\mathbb{D}_\rho\times S^1\times[0,r_0])$ which is $O(t)+O(r)$. □

The following distance expression in the rescaled normal coordinates given by (2.4) was stated in Corollary 2.6 of [NTT21]:

Lemma 2.3.3. *For the coordinates given by* (2.4),

$$d_g^{-1}(x^\varepsilon(s,x^*),x^\varepsilon(t,x^*)) = \varepsilon^{-1}|t-s|^{-1} + \varepsilon|t-s|^{-1}A(\varepsilon,s,r,\omega),$$

for some smooth function A in the variables $(\varepsilon,s,r,\omega) \in [0,\varepsilon_0]\times\mathbb{D}\times\mathbb{R}\times S^1$, where $r=|t-s|$ and $\omega = \dfrac{t-s}{|t-s|}$.

The next two lemmas describe quantitative bounds for $I_\varepsilon(x^*,x)$ and its normal derivative for $x \in \partial\Gamma_\varepsilon$.

Lemma 2.3.4. *The following estimate holds*

$$I_\varepsilon(x^*,x) = O_{L^\infty(\partial\Gamma_\varepsilon)}(\varepsilon^2\log\varepsilon), \qquad \text{as } \varepsilon\to 0. \tag{2.7}$$

Proof. Due to Proposition 2.3.1 it is sufficient to prove that

$$\int_{\Gamma_\varepsilon} \log d_g(x,y)d\mu_g(y) = O_{L^\infty(\partial\Gamma_\varepsilon)}(\varepsilon^2\log\varepsilon).$$

We consider $\varepsilon>0$ sufficiently small, so that $\log(10\varepsilon)<0$. Then, for $x\in\partial\Gamma_\varepsilon$,

$$\begin{aligned}\left|\int_{\Gamma_\varepsilon}\log d_g(x,y)d\mu_g(y)\right| &= \left|\int_{B_\varepsilon(x^*)}\log d_g(x,y)d\mu_g(y)\right|,\\ &\le \left|\int_{B_{2\varepsilon}(x)}\log d_g(x,y)d\mu_g(y)\right|.\end{aligned}$$

For $\varepsilon>0$ sufficiently small we can find $\rho>3\varepsilon$ which is smaller than the injectivity radius. We will use the coordinate system given by

$$\mathbb{D}_\rho \ni (s_1,s_2)\mapsto x(s_1,s_2;x^*),$$

defined in Section 2.2. We recall that $s=(s_1,s_2)$ and $t=(t_1,t_2)$ and let

$$d_g^*(s,t) := d_g(x(s,x^*),x(t,x^*)).$$

Lemma 2.3.2 tells us that

$$d_g^*(s,t) = |s-t| + |s-t|F\left(t,\frac{s-t}{|s-t|},|s-t|\right),$$

for some smooth function F which is $O(|t|) + O(|s-t|)$. Therefore, for sufficiently small $\varepsilon > 0$, we can choose $\rho > 0$ small enough so that for all $s, t \in \mathbb{D}_\rho$,

$$\frac{1}{2}|s-t| \leq d_g^*(t,s) \leq 2|s-t|.$$

Furthermore, we choose $C > 0$ such that $\sqrt{\det(g_{j,k}(s))} \leq C$ for $s \in \mathbb{D}_\rho$. Therefore, for $x = x(t, x^*) \in \partial\Gamma_\varepsilon$, we estimate

$$\begin{aligned}\left|\int_{B_{2\varepsilon}(x)} \log d_g(x,y) d\mu_g(y)\right| &\leq \left|C \int_{d_g^*(t,s)\leq 2\varepsilon} \log\left(2|s-t|\right) ds\right|, \\ &\leq \left|C \int_{|s-t|\leq 4\varepsilon} \log\left(2|s-t|\right) ds\right|.\end{aligned}$$

Thus, we have that $I_\varepsilon(x^*, x) = O_{L^\infty(\partial\Gamma_\varepsilon)}(\varepsilon^2 \log \varepsilon)$. □

Lemma 2.3.5. *The following estimate holds*

$$\partial_{\nu_x} I_\varepsilon(x^*, x) = O_{L^\infty(\partial\Gamma_\varepsilon)}(\varepsilon), \qquad \text{as } \varepsilon \to 0. \tag{2.8}$$

Proof. Let us use the coordinate system $x(t, x^*)$. Note that in these coordinates the volume form for M is given by

$$d\mu_g(y) = (1 + \varepsilon V_\varepsilon(s)) ds_1 ds_2 \tag{2.9}$$

for some smooth function $V_\varepsilon(s)$ whose derivatives of all orders are bounded uniformly in ε. We also note that in these coordinates, we have

$$d_g(x(t;x^*), x(s;x^*))^2 = \sum_{\alpha,\beta=1}^{2} G_{\alpha,\beta}(s,t)(s_\alpha - t_\alpha)(s_\beta - t_\beta) \tag{2.10}$$

where $t = (t_1, t_2)$, $s = (s_1, s_2)$, and $G_{\alpha,\beta}(s,t)$ is a smooth function on $\mathbb{D} \times \mathbb{D}$ such that $G_{\alpha,\beta}(s,s) = \delta_{\alpha,\beta} + O(|s|^2)$ for s near 0. Then, by Proposition 2.8 in [Tay11], we know that

$$\begin{aligned}E(x(t;x^*), x(s;x^*)) = -\frac{1}{4\pi} \log\left(\sum_{\alpha,\beta=1}^{2} G_{\alpha,\beta}(s,t)(s_\alpha - t_\alpha)(s_\beta - t_\beta)\right) \\ + q_2(s, s-t) + p_2(s, s-t) \log|s-t| + R(s,t).\end{aligned}$$

Here $p_2(x,z)$ is a polynomial homogeneous of degree 2 in z, with the coefficients that are bounded, together with their x-derivatives. A function $q_2(x,z)$ is smooth on $\mathbb{R}^2 \setminus \{0\}$ and homogeneous of degree 2 in z. Finally, $R \in C^2(\mathbb{R}^2_s \times \mathbb{R}^2_t)$.

Let us use the polar coordinates

$$t = (r\cos\theta, r\sin\theta) \qquad s = (r'\cos\theta', r'\sin\theta').$$

We note that $\partial\Gamma_\varepsilon$ is the image under the x-chart of $\{|t| = \varepsilon\}$ and $r \mapsto (r\cos\theta, r\sin\theta)$ is the parametrization of unit speed geodesic issued from the origin. Therefore, since ∂_{ν_x} is the inward normal of $\partial\Gamma_\varepsilon$, it follows from Gauss Lemma that $\Phi_*\partial_{\nu_x} = -\partial_r \in T_{(r\cos\theta, r\sin\theta)}\mathbb{R}^2$. Therefore

$$\begin{aligned}\partial_{\nu_x} E(x,y) = \partial_r \Bigg[-\frac{1}{4\pi} \log\left(\sum_{\alpha,\beta=1}^{2} G_{\alpha,\beta}(s,t)(s_\alpha - t_\alpha)(s_\beta - t_\beta) \right) \\ + q_2(s, s-t) + p_2(s, s-t)\log|s-t| + R(s,t) \Bigg].\end{aligned}$$

Using the definition for $I(x^*, x)$ we have obtain the following expression

$$\begin{aligned}\partial_{\nu_x} I(x^*, x) = \int_{\mathbb{D}_\varepsilon} \partial_r \Bigg[-\frac{1}{4\pi} \log\left(\sum_{\alpha,\beta=1}^{2} G_{\alpha,\beta}(s,t)(s_\alpha - t_\alpha)(s_\beta - t_\beta) \right) \\ + q_2(s, s-t) + p_2(s, s-t)\log|s-t| + R(s,t) \Bigg] (1 + \varepsilon V_\varepsilon(s)) ds.\end{aligned}$$

From the properties of functions q_2, p_2, and R, mentioned above, it follows that

$$\int_{\mathbb{D}_\varepsilon} \partial_r \left(q_2(s, s-t) + p_2(s, s-t)\log|s-t| + R(s,t) \right) (1 + \varepsilon V_\varepsilon(s)) ds = O(\varepsilon^2)$$

uniformly on t. Hence, we have that

$$\begin{aligned}\partial_{\nu_x} I(x^*, x) = -\frac{1}{4\pi} \int_{\mathbb{D}_\varepsilon} \partial_r \log\left(\sum_{\alpha,\beta=1}^{2} G_{\alpha,\beta}(s,t)(s_\alpha - t_\alpha)(s_\beta - t_\beta) \right) ds \\ - \frac{\varepsilon}{4\pi} \int_{\mathbb{D}_\varepsilon} \partial_r \log\left(\sum_{\alpha,\beta=1}^{2} G_{\alpha,\beta}(s,t)(s_\alpha - t_\alpha)(s_\beta - t_\beta) \right) V_\varepsilon(s) ds \\ + O(\varepsilon^2). \qquad (2.11)\end{aligned}$$

The first integral of the right-hand side is equal to

$$\int_0^{2\pi}\int_0^{\varepsilon} \frac{2\begin{bmatrix}\cos\theta & \sin\theta\end{bmatrix} G \begin{bmatrix} r\cos\theta - r'\cos\theta' \\ r\sin\theta - r'\sin\theta'\end{bmatrix}}{\begin{bmatrix} r\cos\theta - r'\cos\theta' & r\sin\theta - r'\sin\theta'\end{bmatrix} G \begin{bmatrix} r\cos\theta - r'\cos\theta' \\ r\sin\theta - r'\sin\theta'\end{bmatrix}} r'dr'd\theta' +$$
$$\int_0^{2\pi}\int_0^{\varepsilon} \frac{\begin{bmatrix} r\cos\theta - r'\cos\theta' & r\sin\theta - r'\sin\theta'\end{bmatrix} \partial_r G \begin{bmatrix} r\cos\theta - r'\cos\theta' \\ r\sin\theta - r'\sin\theta'\end{bmatrix}}{\begin{bmatrix} r\cos\theta - r'\cos\theta' & r\sin\theta - r'\sin\theta'\end{bmatrix} G \begin{bmatrix} r\cos\theta - r'\cos\theta' \\ r\sin\theta - r'\sin\theta'\end{bmatrix}} r'dr'd\theta' \tag{2.12}$$

where $G = G(r,\theta,r',\theta')$ is the two by two matrix with entries $\{G_{\alpha,\beta}(s,t)\}$ with $t = t(r,\theta)$ and $s = s(r',\theta')$. Since $x \in \partial B_\varepsilon(x^*)$, we take $r = \varepsilon$. Then, if we re-scale $r' \mapsto \varepsilon r'$, the last expression becomes

$$\varepsilon\int_0^{2\pi}\int_0^{1} \frac{2\begin{bmatrix}\cos\theta & \sin\theta\end{bmatrix} G \begin{bmatrix} \cos\theta - r'\cos\theta' \\ \sin\theta - r'\sin\theta'\end{bmatrix}}{\begin{bmatrix} \cos\theta - r'\cos\theta' & \sin\theta - r'\sin\theta'\end{bmatrix} G \begin{bmatrix} \cos\theta - r'\cos\theta' \\ \sin\theta - r'\sin\theta'\end{bmatrix}} r'dr'd\theta'$$
$$+\varepsilon^2\int_0^{2\pi}\int_0^{1} \frac{\begin{bmatrix} \cos\theta - r'\cos\theta' & \sin\theta - r'\sin\theta'\end{bmatrix} \partial_r G \begin{bmatrix} \cos\theta - r'\cos\theta' \\ \sin\theta - r'\sin\theta'\end{bmatrix}}{\begin{bmatrix} \cos\theta - r'\cos\theta' & \sin\theta - r'\sin\theta'\end{bmatrix} G \begin{bmatrix} \cos\theta - r'\cos\theta' \\ \sin\theta - r'\sin\theta'\end{bmatrix}} r'dr'd\theta'$$

Lemma 2.3.2 in combination with equation 2.10 suggests that we have an integrable singularity at the point $(r',\theta') = (1,\theta)$ and are uniformly bounded in θ. This is because we can bound the integrals by integrals of the form

$$\int_{\mathbb{D}} \frac{1}{|t-s|} ds$$

where $|t| = 1$. Therefore, the integrals given above are $O_{L^\infty(\partial\Gamma_\varepsilon)}(\varepsilon)$ as $\varepsilon \to 0$ uniformly in θ. Since $V_\varepsilon(s)$ is bounded uniformly in ε, the second term of (2.11) is $O_{L^\infty(\partial\Gamma_\varepsilon)}(\varepsilon^3)$. $\square$

Next, we obtain the singularity structure of $\partial_{\nu_x} E(\cdot,\cdot)$ in a neighbourhood of x^*:

Lemma 2.3.6. *For $x, y \in \partial M_\varepsilon$, we have that*

$$\partial_{\nu_x} E(x,y) = \frac{1}{4\pi\varepsilon} + Q_\varepsilon(x,y),$$

for some function Q_ε such that

$$\int_{\partial\Gamma_\varepsilon} Q_\varepsilon(x,y)\,d\mu_h(y) = O_{L^\infty(\partial\Gamma_\varepsilon)}(\varepsilon).$$

Proof. We begin as in the proof of Lemma 2.3.5. We can repeat all the steps until we derive the following equation

$$\partial_{\nu_x} E(x,y) = \partial_r\Bigg[-\frac{1}{4\pi}\log\left(\sum_{\alpha,\beta=1}^{2} G_{\alpha,\beta}(s,t)(s_\alpha - t_\alpha)(s_\beta - t_\beta)\right) \\ + q_2(s,s-t) + p_2(s,s-t)\log|s-t| + R(s,t)\Bigg]. \quad (2.13)$$

We recall that $p_2(x,z)$ is a polynomial homogeneous of degree 2 in z, with the coefficients that are bounded, together with their $x-$derivatives. A function $q_2(x,z)$ is smooth on $\mathbb{R}^2 \setminus \{0\}$ and homogeneous of degree 2 in z. Finally, $R \in C^2(\mathbb{R}^2_s \times \mathbb{R}^2_t)$. These conditions imply that

$$\int_0^{2\pi} \partial_r\left[q_2(s,s-t) + p_2(s,s-t)\log|s-t| + R(s,t)\right]\Big|_{r=r'=\varepsilon} d\theta' = O(1)$$

as $\varepsilon \to 0$ uniformly on θ.

Next, we investigate the first term of the right-hand side of (2.13), which can be written as follows

$$\frac{1}{4\pi}\frac{2\begin{bmatrix}\cos\theta & \sin\theta\end{bmatrix} G \begin{bmatrix} r\cos\theta - r'\cos\theta' \\ r\sin\theta - r'\sin\theta'\end{bmatrix}}{\begin{bmatrix} r\cos\theta - r'\cos\theta' & r\sin\theta - r'\sin\theta'\end{bmatrix} G \begin{bmatrix} r\cos\theta - r'\cos\theta' \\ r\sin\theta - r'\sin\theta'\end{bmatrix}} \\ + \frac{1}{4\pi}\frac{\begin{bmatrix} r\cos\theta - r'\cos\theta' & r\sin\theta - r'\sin\theta'\end{bmatrix} \partial_r G \begin{bmatrix} r\cos\theta - r'\cos\theta' \\ r\sin\theta - r'\sin\theta'\end{bmatrix}}{\begin{bmatrix} r\cos\theta - r'\cos\theta' & r\sin\theta - r'\sin\theta'\end{bmatrix} G \begin{bmatrix} r\cos\theta - r'\cos\theta' \\ r\sin\theta - r'\sin\theta'\end{bmatrix}}.$$

Since x, $y \in \partial\Gamma_\varepsilon$, we let $r = r' = \varepsilon$, so that the above expression is given by

$$\frac{1}{2\pi\varepsilon}\frac{\begin{bmatrix}\cos\theta & \sin\theta\end{bmatrix} G \begin{bmatrix}\cos\theta-\cos\theta' \\ \sin\theta-\sin\theta'\end{bmatrix}}{\begin{bmatrix}\cos\theta-\cos\theta' & \sin\theta-\sin\theta'\end{bmatrix} G \begin{bmatrix}\cos\theta-\cos\theta' \\ \sin\theta-\sin\theta'\end{bmatrix}}$$
$$+\frac{1}{4\pi}\frac{\begin{bmatrix}\cos\theta-\cos\theta' & \sin\theta-\sin\theta'\end{bmatrix} \partial_r G \begin{bmatrix}\cos\theta-\cos\theta' \\ \sin\theta-\sin\theta'\end{bmatrix}}{\begin{bmatrix}\cos\theta-\cos\theta' & \sin\theta-\sin\theta'\end{bmatrix} G \begin{bmatrix}\cos\theta-\cos\theta' \\ \sin\theta-\sin\theta'\end{bmatrix}}.$$

Note that the last term belongs to $L^\infty(S^1_\theta \times S^1_{\theta'})$ uniformly in ε. Therefore, it remains to show that

$$\frac{1}{2\pi\varepsilon}\frac{\begin{bmatrix}\cos\theta & \sin\theta\end{bmatrix} G \begin{bmatrix}\cos\theta-\cos\theta' \\ \sin\theta-\sin\theta'\end{bmatrix}}{\begin{bmatrix}\cos\theta-\cos\theta' & \sin\theta-\sin\theta'\end{bmatrix} G \begin{bmatrix}\cos\theta-\cos\theta' \\ \sin\theta-\sin\theta'\end{bmatrix}} = \frac{1}{4\pi\varepsilon} + L_\varepsilon(\theta,\theta'), \tag{2.14}$$

for some function L_ε such that

$$\int_0^{2\pi} L_\varepsilon(\theta,\theta')d\theta' = O(1), \qquad \text{as } \varepsilon \to 0,$$

uniformly on θ. Let $J(\theta,\theta')$ be the left-hand side of (2.14). We denote $J_1 := J\chi_{|\theta-\theta'|<\varepsilon}$ and $J_2 := J\chi_{|\theta-\theta'|>\varepsilon}$, where χ is an indicator function of the corresponding set. To investigate J_1, we Taylor expand the numerator and denominator at $\theta' = \theta$. We recall that $G_{j,k}(x,x) = g_{j,k}(x)$, for $r = r' = \varepsilon$, we get

$$G = g + \varepsilon R_\varepsilon(\theta,\theta')(\theta-\theta'), \tag{2.15}$$

where R_ε is a two by two matrix with $C^\infty(S^1_\theta \times S^1_{\theta'})$ entries and $g = \{g_{j,k}(\varepsilon\cos\theta, \varepsilon\sin\theta)\}^2_{k,j}$. Furthermore, we express g as

$$g = I + \Gamma(\varepsilon,\theta), \tag{2.16}$$

where I is two by two identity matrix and $\Gamma(\varepsilon,\theta)$ is a two by two matrix with interiors $O(\varepsilon^2)$. Therefore, by Taylor expanding at $\theta = \theta'$, we obtain

$$\begin{aligned}\begin{bmatrix}\cos\theta & \sin\theta\end{bmatrix} G \begin{bmatrix}\cos\theta - \cos\theta' \\ \sin\theta - \sin\theta'\end{bmatrix} &= \begin{bmatrix}\cos\theta & \sin\theta\end{bmatrix} g \begin{bmatrix}-\sin\theta \\ \cos\theta\end{bmatrix} \\ &+ \begin{bmatrix}\cos\theta & \sin\theta\end{bmatrix} g \begin{bmatrix}K_1(\theta,\theta') \\ K_2(\theta,\theta')\end{bmatrix} (\theta-\theta')^2 + O(\varepsilon)O(|\theta-\theta'|^2),\end{aligned}$$

for some $K = (K_1, K_2) \in L^\infty\left(S^1 \times S^1\right)^2$. Note that the normal vector to $\{|t| = \varepsilon\}$ is given by $\cos\theta\partial_{t_1} + \sin\theta\partial_{t_2}$ at the point $(\varepsilon\cos\theta, \varepsilon\sin\theta)$, while the tangent is given by $-\sin\theta\partial_{t_1} + \cos\theta\partial_{t_2}$. Therefore the first term of the right-hand side of the last equation is zero, so that

$$\begin{aligned}\begin{bmatrix}\cos\theta & \sin\theta\end{bmatrix} G \begin{bmatrix}\cos\theta - \cos\theta' \\ \sin\theta - \sin\theta'\end{bmatrix} &= \begin{bmatrix}\cos\theta & \sin\theta\end{bmatrix} g \begin{bmatrix}K_1 \\ K_2\end{bmatrix} (\theta-\theta')^2 \\ &+ O(\varepsilon)O(|\theta-\theta'|^2).\end{aligned}$$

Similarly, by using (2.15) and (2.16), we have that

$$\begin{aligned}\begin{bmatrix}\cos\theta - \cos\theta' & \sin\theta - \sin\theta'\end{bmatrix} G \begin{bmatrix}\cos\theta - \cos\theta' \\ \sin\theta - \sin\theta'\end{bmatrix} &= 2(\theta-\theta')^2 \\ &+ O(\varepsilon)O(|\theta-\theta'|^2).\end{aligned}$$

The previous two estimates imply that

$$\int_0^{2\pi} J_1 d\theta' = \frac{1}{2\pi\varepsilon}\int_0^{2\pi} \frac{\begin{bmatrix}\cos\theta & \sin\theta\end{bmatrix} G \begin{bmatrix}K_1 \\ K_2\end{bmatrix} + O(\varepsilon)}{2 + O(\varepsilon)} \chi_{|\theta-\theta'|<\varepsilon}(\theta') d\theta' = O(1)$$

as $\varepsilon \to 0$ uniformly in θ. From (2.15) and (2.16), it follows that

$$\begin{aligned}\begin{bmatrix}\cos\theta & \sin\theta\end{bmatrix} G \begin{bmatrix}\cos\theta - \cos\theta' \\ \sin\theta - \sin\theta'\end{bmatrix} &= 1 - \cos(\theta-\theta') + O(\varepsilon^2)O(|\theta-\theta'|) \\ &+ O(\varepsilon)O(|\theta-\theta'|^2).\end{aligned}$$

In the region $\{|\theta-\theta'| > \varepsilon\}$, we can rewrite this

$$\begin{bmatrix}\cos\theta & \sin\theta\end{bmatrix} G \begin{bmatrix}\cos\theta - \cos\theta' \\ \sin\theta - \sin\theta'\end{bmatrix} = 1 - \cos(\theta-\theta') + O(\varepsilon)O(|\theta-\theta'|^2).$$

Similarly, we have that

$$\begin{bmatrix}\cos\theta - \cos\theta' & \sin\theta - \sin\theta'\end{bmatrix} G \begin{bmatrix}\cos\theta - \cos\theta' \\ \sin\theta - \sin\theta'\end{bmatrix} = 2 - 2\cos(\theta - \theta') \\ + O(\varepsilon)O(|\theta - \theta'|^3).$$

As a result, we have that

$$\begin{aligned} J_2 &= \frac{1}{2\pi\varepsilon}\left(\frac{1}{2} + \frac{O(\varepsilon)O(|\theta - \theta'|^2)}{2 - 2\cos(\theta - \theta') + O(\varepsilon)O(|\theta - \theta'|^3)}\right) \\ &= \frac{1}{4\pi\varepsilon} + \frac{O(1)}{\frac{2 - 2\cos(\theta - \theta')}{(\theta - \theta')^2} + O(\varepsilon)O(|\theta - \theta'|)}. \end{aligned}$$

Since $(2 - 2\cos(\theta - \theta'))(\theta - \theta')^{-2}$ is a positive and continuous function of $\theta' \in [0, 2\pi]$, we conclude that

$$\int_0^{2\pi} \left(J_2 - \frac{1}{4\pi\varepsilon}\right) d\theta' = O(1) \qquad \text{as } \varepsilon \to 0,$$

uniformly in θ, thus the lemma is proven. □

2.4 Narrow capture problem on a closed surface

In this section, we prove Theorem 2.1.1. We start by recalling the formulation of the problem. Let $(X, \mathbb{P}_x)$ denote a Brownian motion on a closed Riemannian manifold M starting at x, generated by Δ_g. For $x^* \in M$ and $\varepsilon > 0$, let $\Gamma_\varepsilon = B_\varepsilon(x^*)$ be a small geodesic ball centred at fixed point $x^* \in M$ and $u_\varepsilon(x)$ denote the mean sojourn time of a Brownian particle starting at $x \in M \setminus \Gamma_\varepsilon$. It is known that $u_\varepsilon(x)$ satisfies the following boundary value problem, see for instance Appendix A in [NTT21],

$$\begin{cases} \Delta_g u_\varepsilon = -1 & \text{on } M_\varepsilon; \\ u_\varepsilon = 0 & \text{on } \partial M_\varepsilon = \partial\Gamma_\varepsilon, \end{cases} \tag{2.17}$$

Furthermore, it follows that we have the following compatibility condition

$$\int_{\partial\Gamma_\varepsilon} \partial_\nu u_\varepsilon(y) d\mu_h(y) = -|M_\varepsilon|. \tag{2.18}$$

To prove Theorem 2.1.1, we will need the following auxiliary result.

Proposition 2.4.1. *Let u_ε be the solution of* (2.17), *then for $x \in \partial M_\varepsilon$, we have that*

$$\partial_\nu u_\varepsilon = -\frac{|M_\varepsilon|}{2\pi\varepsilon} + W_\varepsilon.$$

for some $W_\varepsilon \in O_{L^\infty(\partial\Gamma_\varepsilon)}(1)$ as $\varepsilon \to 0$.

Proof. By using Greens identity, we obtain

$$\frac{1}{|M|}\int_{M_\varepsilon} u_\varepsilon(y)d\mu_g(y) - u_\varepsilon(x) + \int_{\partial\Gamma_\varepsilon} E(x,y)\partial_\nu u_\varepsilon(y)d\mu_h(y) = I_\varepsilon(x^*,x). \tag{2.19}$$

We take ∂_{ν_x} and restrict to $\partial\Gamma_\varepsilon$

$$-\partial_{\nu_x} u_\varepsilon(x) + \partial_{\nu_x}\int_{\partial\Gamma_\varepsilon} E(x,y)\partial_\nu u_\varepsilon(y)d\mu_h(y) = \partial_{\nu_x} I_\varepsilon(x^*,x),$$

and hence, by Lemma 2.3.5, we derive

$$-\partial_{\nu_x} u_\varepsilon(x) + \partial_{\nu_x}\int_{\partial\Gamma_\varepsilon} E(x,y)\partial_\nu u_\varepsilon(y)d\mu_h(y) = O_{L^\infty(\partial\Gamma_\varepsilon)}(\varepsilon).$$

By Proposition 11.3 of [Tay11],

$$-\frac{1}{2}\partial_\nu u_\varepsilon(x) + \int_{\partial\Gamma_\varepsilon} \partial_{\nu_x} E(x,y)\partial_\nu u_\varepsilon(y)d\mu_h(y) = O_{L^\infty(\partial\Gamma_\varepsilon)}(\varepsilon).$$

Therefore, from Lemma 2.3.6, it follows

$$\begin{aligned}\frac{1}{2}\partial_\nu u_\varepsilon(x) = \frac{1}{4\pi\varepsilon}\int_{\partial\Gamma_\varepsilon} \partial_\nu u_\varepsilon(y)d\mu_h(y) + \int_{\partial\Gamma_\varepsilon} Q_\varepsilon(x,y)\partial_\nu u_\varepsilon(y)d\mu_h(y) \\ + O_{L^\infty(\partial\Gamma_\varepsilon)}(\varepsilon).\end{aligned}$$

Hence, the compatibility condition (2.18) gives

$$\varepsilon\partial_\nu u_\varepsilon(x) = -\frac{|M_\varepsilon|}{2\pi} + 2\varepsilon\int_{\partial\Gamma_\varepsilon} Q_\varepsilon(x,y)\partial_\nu u_\varepsilon(y)d\mu_h(y) + O_{L^\infty(\partial\Gamma_\varepsilon)}(\varepsilon^2). \tag{2.20}$$

Next, we estimate

$$\begin{aligned}\sup_{x\in\partial\Gamma_\varepsilon}\left|\varepsilon\int_{\partial\Gamma_\varepsilon} Q_\varepsilon(x,y)\partial_\nu u_\varepsilon(y)d\mu_h(y)\right| &\leq \sup_{x\in\partial\Gamma_\varepsilon}|\varepsilon\partial_\nu u_\varepsilon| \sup_{x\in\partial\Gamma_\varepsilon}\int_{\partial\Gamma_\varepsilon}|Q_\varepsilon(x,y)|\,d\mu_h(y) \\ &\leq C\varepsilon^2\sup_{x\in\partial\Gamma_\varepsilon}|\partial_\nu u_\varepsilon|\end{aligned}$$

The last estimate comes from Lemma 2.3.6. Combine this estimate with (2.20) we obtain that

$$\varepsilon\partial_{\nu_x} u_\varepsilon(x) = -\frac{|M_\varepsilon|}{2\pi} + O_{L^\infty(\partial\Gamma_\varepsilon)}(\varepsilon).$$

This completes the proof. □

Proof of Theorem 2.1.1. We first prove ii) then proceed with i). By Proposition 2.4.1, we can express

$$\partial_\nu \, u_\varepsilon|_{\partial\Gamma_\varepsilon} = -\frac{|M_\varepsilon|}{2\pi\varepsilon} + W_\varepsilon$$

$$W_\varepsilon = O_{L^\infty(\partial\Gamma_\epsilon)}(1). \tag{2.21}$$

Then, for $x \in M_\varepsilon \setminus \partial\Gamma_\varepsilon$, (2.19) gives

$$\begin{aligned} I_\varepsilon(x^*, x) = \frac{1}{|M|}\int_{M_\varepsilon} u_\varepsilon(y)d\mu_g(y) - u_\varepsilon(x) - \frac{|M_\varepsilon|}{2\pi\varepsilon}\int_{\partial\Gamma_\varepsilon} E(x,y)d\mu_h(y) \\ + \int_{\partial\Gamma_\varepsilon} E(x,y)W_\varepsilon(y)d\mu_h(y), \end{aligned}$$

or equivalently

$$\begin{aligned} \frac{1}{|M|}\int_{M_\varepsilon} u_\varepsilon(y)d\mu_g(y) = u_\varepsilon(x) &+ \frac{|M_\varepsilon|}{2\pi\varepsilon}\int_{\partial\Gamma_\varepsilon} E(x,x^*)d\mu_h(y) \\ &+ \frac{|M_\varepsilon|}{2\pi\varepsilon}\int_{\partial\Gamma_\varepsilon} (E(x,y) - E(x,x^*))d\mu_h(y) \\ &- \int_{\partial\Gamma_\varepsilon} E(x,y)W_\varepsilon(y)d\mu_h(y) + I_\varepsilon(x,x^*). \end{aligned} \tag{2.22}$$

To compute the left-hand side, we restrict this to $\partial\Gamma_\varepsilon$ where $u_\varepsilon = 0$. We note that Proposition 2.3.1 combined with Lemma 2.6 shows that in the coordinates (2.3) the singularity structure of the Greens function $E(x(s,x^*,),x(t,x^*))$ is of the form

$$E(x(s,x^*),x(t,x^*)) = C\log|t-s| + L^\infty(\mathbb{D}_\rho \times \mathbb{D}_\rho).$$

Combining this with Lemma 2.3.4 and (2.21) gives

$$\begin{aligned} \int_{\partial\Gamma_\varepsilon} E(x,y)W_\varepsilon(y)d\mu_h(y) &= O_{L^\infty(\partial\Gamma_\varepsilon)}(\varepsilon\log\varepsilon), \\ I_\varepsilon(x^*,x) &= O_{L^\infty(\partial\Gamma_\varepsilon)}(\varepsilon^2\log\varepsilon), \end{aligned}$$

as $\varepsilon \to 0$. Therefore, restricting (2.22) to $\partial\Gamma_\varepsilon$ and using Proposition 2.3.1, we obtain

$$\begin{aligned} \frac{1}{|M|}\int_{M_\varepsilon} u_\varepsilon(y)d\mu_g(y) = &-\frac{|M_\varepsilon||\partial\Gamma_\varepsilon|}{4\pi^2\varepsilon}\log\varepsilon + \frac{|M_\varepsilon||\partial\Gamma_\varepsilon|}{2\pi\varepsilon}P_{-4}(x^*,x^*) \\ &+ \frac{|M_\varepsilon|}{2\pi\varepsilon}\int_{\partial\Gamma_\varepsilon}(E(x,y)-E(x,x^*))d\mu_h(y)\Big|_{x\in\partial\Gamma_\varepsilon} \\ &+ O_{L^\infty(\partial\Gamma_\varepsilon)}(\varepsilon\log\varepsilon). \end{aligned} \tag{2.23}$$

Let us examine the third term of the right-hand side

$$\int_{\partial\Gamma_\varepsilon}(E(x,y)-E(x,x^*))d\mu_h(y) = \int_{\partial\Gamma_\varepsilon}(\log d_g(x,y)-\log d_g(x,x^*))d\mu_h(y) + \int_{\partial\Gamma_\varepsilon}(P_{-4}(x,y)-P_{-4}(x,x^*))d\mu_h(y), \tag{2.24}$$

where $x\in\partial\Gamma_\varepsilon$. Joint differentiability of P_{-4} gives

$$\sup_{x\in\partial\Gamma_\varepsilon}\left|\int_{\partial\Gamma_\varepsilon}(P_{-4}(x,y)-P_{-4}(x,x^*))d\mu_h(y)\right|\le C\varepsilon^2. \tag{2.25}$$

To investigate the first term of the right-hand side of (2.24), we use the coordinate system $x^\varepsilon(\cdot,x^*)$ with $x=x^\varepsilon(t,x^*)$ and $y=x^\varepsilon(s,x^*)$. Let $d\sigma(s)$ be the pull-back of the volume form $d\mu_h(y)$ under $s\mapsto x^\varepsilon(s,x^*)$, then

$$d\mu_h(y)=\varepsilon(1+v_\varepsilon(s))d\sigma(s)$$

for some smooth function $\|v_\varepsilon\|_{L^\infty(\partial\Gamma_\varepsilon)}\le C\varepsilon$. By Lemma 2.3.3 we have

$$d_g^{-1}(x^\varepsilon(t,x^*),x^\varepsilon(s,x^*))=\varepsilon^{-1}|t-s|^{-1}+\varepsilon|t-s|^{-1}A(\varepsilon,s,r,\omega)$$

for some smooth function $A(\varepsilon,s,r,\omega)$ in the variables $(\varepsilon,s,r,\omega)\in[0,\varepsilon_0]\times\mathbb{D}\times\mathbb{R}\times S^1$, where $r=|t-s|$ and $\omega=\frac{t-s}{|t-s|}$. Therefore, the first term of (2.24) becomes

$$\varepsilon\int_{\partial\mathbb{D}}\log\left(\frac{\varepsilon|t-s|}{1+\varepsilon^2A}\right)(1+v_\varepsilon(s))d\sigma(s). \tag{2.26}$$

It easily follows that

$$\varepsilon\left|\int_{\partial\mathbb{D}}\log\left(\frac{\varepsilon}{1+\varepsilon^2A(\varepsilon,s,r,\omega)}\right)(1+v_\varepsilon(s))d\sigma(s)\right|\le C\varepsilon\log\varepsilon \tag{2.27}$$

and

$$\left|\varepsilon\int_{\partial\mathbb{D}}\log|t-s|v_\varepsilon(s)d\sigma(s)\right|\le\varepsilon\int_{\partial\mathbb{D}}\left|\log|t-s|v_\varepsilon(s)\right|d\sigma(s)\le C\varepsilon^2 \tag{2.28}$$

for all $t\in\partial\mathbb{D}$. Therefore, inserting the estimates (2.27) and (2.28) into (2.26) gives that for $t\in\partial\mathbb{D}$,

$$\int_{\partial\Gamma_\varepsilon}(\log d_g(x,y)-\log d_g(x,x^*))d\mu_h(y)=O_{L^\infty(\partial\Gamma_\varepsilon)}(\varepsilon\log\varepsilon)$$

This combined with (2.25) and (2.24) gives

$$\left|\int_{\partial\Gamma_\varepsilon}(E(x,y)-E(x,x^*))d\mu_h(y)\right| \leq C\varepsilon\log\varepsilon.$$

Inserting this estimate into (2.23) implies

$$\frac{1}{|M|}\int_{M_\varepsilon} u_\varepsilon(y)d\mu_g(y) = -\frac{|M_\varepsilon||\partial\Gamma_\varepsilon|}{4\pi^2\varepsilon}\log\varepsilon + \frac{|M_\varepsilon||\partial\Gamma_\varepsilon|}{2\pi\varepsilon}P_{-4}(x^*,x^*) + O_{L^\infty(\partial\Gamma_\varepsilon)}(\varepsilon\log\varepsilon).$$

We take the supremum norm over $\partial\Gamma_\varepsilon$ to obtain

$$\frac{1}{|M|}\int_{M_\varepsilon} u_\varepsilon(y)d\mu_g(y) \;=\; -\frac{|M_\varepsilon||\partial\Gamma_\varepsilon|}{4\pi^2\varepsilon}\log\varepsilon \;+\; \frac{|M_\varepsilon||\partial\Gamma_\varepsilon|}{2\pi\varepsilon}P_{-4}(x^*,x^*) \;+\; C_\varepsilon. \tag{2.29}$$

where C_ε is a constant of order $\varepsilon\log\varepsilon$. Next, we note that $|\Gamma_\varepsilon| = |B_\varepsilon(x^*)| = O(\varepsilon^2)$ and

$$|\partial\Gamma_\varepsilon| = |\partial B_\varepsilon(x^*)| = \int_{\partial B_\varepsilon(x^*)} d\mu_h(y) = \int_{\partial\mathbb{D}}\varepsilon(1+v_\varepsilon(s))d\sigma(s) = 2\pi\varepsilon + O(\varepsilon^2).$$

This gives us part ii) of Theorem 2.1.1.

Let us put (2.29) into (2.19) to we obtain

$$u_\varepsilon(x) = -\frac{|M|}{2\pi}\log\varepsilon + |M_\varepsilon|P_{-4}(x^*,x^*) + \int_{\partial M_\varepsilon} E(x,y)\partial_\nu u_\varepsilon(y)d\mu_h(y) - I_\varepsilon(x^*,x) + O(\varepsilon\log\varepsilon).$$

as $\varepsilon \to 0$. It remains to show that for a fixed compact set $K \subset\subset M_\varepsilon$,

$$\int_{\partial\Gamma_\varepsilon} E(x,y)\partial_\nu u_\varepsilon(y)d\mu_h(y) - I_\varepsilon(x^*,x) = -|M|E(x,x^*) + r_\varepsilon(x). \tag{2.30}$$

for some $r_\varepsilon(x)$ whose $L^\infty(K)$ norm is of order ε.

To do this, let us fix any compact set $K \subset M$ which does not contain x^*, so that $K \cap \Gamma_\varepsilon$ is empty for sufficiently small $\varepsilon > 0$. Then $E(\cdot,\cdot)$ is smooth in $K \times \Gamma_\varepsilon$, and hence we estimate

$$\sup_{x\in K} I(x^*,x) = \sup_{x\in K}\int_{\Gamma_\varepsilon} E(x,y)d\mu_g(y) = O(\varepsilon^2) \qquad \text{as } \varepsilon \to 0.$$

Next, we write

$$\int_{\partial\Gamma_\varepsilon} E(x,y)\partial_\nu u_\varepsilon(y)d\mu_h(y) = \int_{\partial\Gamma_\varepsilon} (E(x,y) - E(x,x^*))\partial_\nu u_\varepsilon(y)d\mu_h(y) + E(x,x^*)\int_{\partial\Gamma_\varepsilon} \partial_\nu u_\varepsilon(y)d\mu_h(y).$$

The first integral can be estimated using Proposition 2.4.1 and the smoothness of $E(\cdot,\cdot)$ in $K \times \Gamma_\varepsilon$to give

$$\int_{\partial\Gamma_\varepsilon} E(x,y)\partial_\nu u_\varepsilon(y)d\mu_h(y) = O_{L^\infty(K)}(\varepsilon) + E(x,x^*)\int_{\partial\Gamma_\varepsilon} \partial_\nu u_\varepsilon(y)d\mu_h(y).$$

Now use the compatibility condition (2.18) to get for $x \in K$,

$$\begin{aligned}\int_{\partial\Gamma_\varepsilon} E(x,y)\partial_\nu u_\varepsilon(y)d\mu_h(y) &= -|M_\varepsilon|E(x,x^*) + O_{L^\infty(K)}(\varepsilon)\\ &= -|M|E(x,x^*) + O_{L^\infty(K)}(\varepsilon).\end{aligned}$$

This gives Part i) of Theorem 2.1.1.

□

2.5 Narrow capture problem on a surface with boundary

Here, we consider the same problem for the case when the surface has a smooth boundary , ∂M, which reflects the particle. Without loss of generality, we assume that M is an connected open subset of a compact orientable Riemannian manifold $(\widetilde{M}, g)$ without boundary. Let $\widetilde{E}(x,y)$ be the Greens function on $\widetilde{M}$, given by (2.5). The Neumann Greens function $E(x,y)$ is given by, for $x \in M^o$,

$$\begin{cases} \Delta_{g,y}E(x,y) = -\delta_x(y) + \dfrac{1}{|M|}, & \text{for } y \in M,\\ \partial_{\nu_y}E(x,y) = 0, & \text{for } y \in \partial M,\\ \int_M E(x,y)d\mu_g(y) = 0. & \end{cases} \tag{2.31}$$

We can obtain this function by setting $E = \widetilde{E} - C$, where the correction term $C(x,y)$ is the solution to the boundary value problem, for $x \in M^0$,

$$\begin{cases} \Delta_{g,y} C(x,y) = \dfrac{1}{|\widetilde{M}|} - \dfrac{1}{|M|}, & \text{for } y \in M, \\ \partial_{\nu_y} C(x,y) = \partial_{\nu_y} \widetilde{E}(x,y), & \text{for } y \in \partial M, \\ \int_M C(x,y) d\mu_g(y) = \int_M \widetilde{E}(x,y) d\mu_g(y). \end{cases}$$

Therefore, for $U \subset\subset M$ away from the boundary, that is $\mathrm{dist}_g(U, \partial M) > 0$, it follows

$$C = \widetilde{E} - E \in C^\infty(\bar{U} \times \bar{U}).$$

Hence, we can decompose

$$E(x,y) = -\frac{1}{2\pi} \log d_g(x,y) + P_{-4}(x,y), \tag{2.32}$$

where $P_{-4}(x,y) \in C^1(\bar{U} \times \bar{U})$. Moreover, since we are considering the centre of the trap, x^*, to be fixed in the interior of M with a sufficiently small $\varepsilon > 0$, we have that

$$\begin{aligned} \int_{\Gamma_\varepsilon} C(x,y) d\mu_g(y) &= O_{L^\infty(\partial\Gamma_\varepsilon)}(\varepsilon^2), \\ \partial_{\nu_x} \int_{\Gamma_\varepsilon} C(x,y) d\mu_g(y) &= O_{L^\infty(\partial\Gamma_\varepsilon)}(\varepsilon^2), \\ \partial_{\nu_x} \int_{\partial\Gamma_\varepsilon} C(x,y) d\mu_h(y) &= O_{L^\infty(\partial\Gamma_\varepsilon)}(\varepsilon), \end{aligned}$$

as $\varepsilon \to 0$. Note that the difference between the second and third identities are that we are integrating with respect to $d\mu_h$ over $\partial\Gamma_\varepsilon$ as opposed to $d\mu_g$ over Γ_ε. As a result, we obtain the following analogues of Lemmas 2.3.4-2.3.6 for the function

$$I_\varepsilon(x^*, x) := \int_{\Gamma_\varepsilon} E(x,y) d\mu_g(y),$$

Lemma 2.5.1. *As $\varepsilon \to 0$, we have that*

$$I_\varepsilon(x^*, x) = O_{L^\infty(\partial\Gamma_\varepsilon)}(\varepsilon^2 \log \varepsilon), \qquad \partial_{\nu_x} I(x^*, x) = O_{L^\infty(\partial\Gamma_\varepsilon)}(\varepsilon),$$

$$\partial_{\nu_x} E(x,y) \mid_{x,y \in \partial\Gamma_\varepsilon} = \frac{1}{4\pi\varepsilon} + Q_\varepsilon(x,y),$$

for some function Q_ε such that

$$\int_{\partial\Gamma_\varepsilon} Q_\varepsilon(x,y) d\mu_h(y) = O_{L^\infty(\partial\Gamma_\varepsilon)}(\varepsilon).$$

Note that, in this case, $\partial M \neq \varnothing$, the mean sojourn time $u_\varepsilon(x)$ satisfies the following mixed boundary value problem, see Appendix in [NTT21],

$$\begin{cases} \Delta_g u_\varepsilon = 1 & \text{on } M_\varepsilon, \\ u_\varepsilon = 0, & \text{on } \partial\Gamma_\varepsilon, \\ \partial_\nu u = 0, & \text{on } \partial M. \end{cases} \tag{2.33}$$

which gives the compatibility condition

$$\int_{\partial\Gamma_\varepsilon} \partial_\nu u_\varepsilon(y) d\mu_h(y) = -|M_\varepsilon|. \tag{2.34}$$

As a result, we have the following analogue of Proposition 2.4.1

Proposition 2.5.2. *Let u_ε be the solution of* (2.33), *then*

$$\partial_\nu \, u_\varepsilon|_{\partial\Gamma_\varepsilon} = -\frac{|M_\varepsilon|}{2\pi\varepsilon} + W_\varepsilon.$$

for some $W_\varepsilon \in O_{L^\infty(\partial\Gamma_\varepsilon)}(1)$ as $\varepsilon \to 0$.

Proof. Greens identity yields the following equation,

$$\begin{aligned} &\frac{1}{|M|}\int_{M_\varepsilon} u_\varepsilon(y) d\mu_g(y) - u_\varepsilon(x) - I_\varepsilon(x^*, x) \\ &\qquad = \int_{\partial M_\varepsilon} \partial_{\nu_y} E(x,y) u_\varepsilon(y) d\mu_h(y) - \int_{\partial M_\varepsilon} E(x,y)\partial_\nu u_\varepsilon(y) d\mu_h(y). \end{aligned}$$

Using the boundary conditions for E and u_ε, we obtain

$$\frac{1}{|M|}\int_{M_\varepsilon} u_\varepsilon(y) d\mu_g(y) - u_\varepsilon(x) + \int_{\partial\Gamma_\varepsilon} E(x,y)\partial_\nu u_\varepsilon(y) d\mu_h(y) = I_\varepsilon(x^*, x).$$

Next, as in Proposition 2.4.1, we take ∂_{ν_x}, restrict to ∂M_ε, and use Lemma 2.5.1 to obtain

$$-\partial_\nu u_\varepsilon(x) + \partial_\nu \int_{\partial\Gamma_\varepsilon} E(x,y)\partial_\nu u_\varepsilon(y) d\mu_h(y) = O_{L^\infty(\partial\Gamma_\varepsilon)}(\varepsilon).$$

By Proposition 11.3 of [Tay11], we have that

$$\begin{aligned} -\frac{1}{2}\partial_\nu u_\varepsilon(x) &+ \int_{\partial\Gamma_\varepsilon} \partial_{\nu_x} \widetilde{E}(x,y)\partial_\nu u_\varepsilon(y) d\mu_h(y) \\ &- \int_{\partial\Gamma_\varepsilon} \partial_{\nu_x} C(x,y)\partial_\nu u_\varepsilon(y) d\mu_h(y) = O_{L^\infty(\partial\Gamma_\varepsilon)}(\varepsilon) \end{aligned}$$

which implies that

$$-\frac{1}{2}\partial_\nu u_\varepsilon(x) + \int_{\partial\Gamma_\varepsilon} \partial_{\nu_x} E(x,y)\partial_\nu u_\varepsilon(y) d\mu_h(y) = O_{L^\infty(\partial\Gamma_\varepsilon)}(\varepsilon).$$

We can then repeat the proof of Proposition 2.4.1 by replacing Lemma 2.3.6 with 2.5.1. $\square$

It follows that repeating the proof of Theorem 2.1.1 by replacing Proposition 2.4.1 with 2.5.2 and Lemmas 2.3.4-2.3.6 with 2.5.1 yields the following theorem

Theorem 2.5.3. *Let $(M, g, \partial M)$ be a compact, connected, and orientable Riemannian surface with a smooth boundary. Fix $x^* \in M^o$ and let $\Gamma_\varepsilon := B_\varepsilon(x^*)$ be a geodesic ball centred at x^* of geodesic radius $\varepsilon > 0$ such that $\partial\Gamma_\varepsilon \cap \partial M = \varnothing$.*

i) For each $x \in M_\varepsilon$, the mean sojourn time satisfies the following asymptotic formula, as $\varepsilon \to 0$,

$$u_\varepsilon(x) = -\frac{|M|}{2\pi}\log\varepsilon + |M|P_{-4}(x^*, x^*) - |M|E(x, x^*) + r_\varepsilon(x) + C_\varepsilon.$$

for some function $r_\varepsilon(x) = O_{L^\infty(K)}(\varepsilon)$ for any compact $K \subset M$ for which $K \cap \Gamma_\varepsilon = \varnothing$. The Neumann Greens function $E(x, y)$ is given by (2.31) and $P_{-4}(x^*, x^*)$ is the evaluation at $(x, y) = (x^*, x^*)$ of the kernel $P_{-4}(x, y)$ in (2.32). In addition, C_ε denotes a constant of order $\varepsilon\log\varepsilon$.

ii) Let $M_\varepsilon = M \setminus \Gamma_\varepsilon$, then the spatial average of the mean sojourn time satisfies the asymptotic formula, as $\varepsilon \to 0$,

$$\frac{1}{|M|}\int_{M_\varepsilon} u_\varepsilon(x) d\mu_g(x) = -\frac{|M|}{2\pi}\log\varepsilon + |M|P_{-4}(x^*, x^*) + C_\varepsilon.$$

where C_ε denotes a constant of order $\varepsilon\log\varepsilon$.

Chapter 3

The Narrow Escape Problem with Advection

3.1 Introduction

Let us consider a diffusing ion under the influence of drift confined to a bounded domain by a reflecting boundary, except for a small absorbing part. The narrow escape problem deals with computing the average amount of time it takes for the diffusing ion to strike the absorbing part of the boundary. This quantity is known as the mean sojourn time and will be formally introduced in subsequent paragraphs. The geometric landscape of this article involves a compact, connected, orientable Riemannian 3-manifold with a non-empty smooth boundary, denoted by $(M, g, \partial M)$. We will use $(X_t, \mathbb{P}_x)$ to denote the Brownian motion, representative of a diffusing ion experiencing drift in M generated by

$$\Delta_g \cdot + g(F, \nabla_g \cdot)$$

where $\Delta_g = -d^*d$ is the (negative) Laplace-Beltrami operator, ∇_g is the gradient operator, F is a force field given by a smooth potential ϕ, that is $F = \nabla_g \phi$. Initially, this problem was mentioned in the context of acoustics in [Ray45] (1945). Much later, in 2004, interest in this problem was renewed due to the relation between mean sojourn time problems and molecular biology as well as biophysics; see [HS04]. Many problems in cellular biology may be formulated as mean sojourn time problems; a collection of analytical methods, results, applications, and references may be found in [HS15] and [BN13]. For example, cells have been modeled as simply connected two-dimensional domains with small absorbing windows on the boundary representing ion channels or target binding sites; the quantity sought is then the average time for a diffusing ion or receptor to exit through an ion channel or reach a binding site [SSH07; HS04; Pil+10]. All of this has led to the narrow escape theory in applied mathematics and computational biology; see [SSH06a; Sin+06; SSH06b].

There has been much progress regarding this problem in the setting of planar domains, and we refer the readers to [HS04; Pil+10; Sin+06; Amm+12c] and references therein for a complete bibliography. An important contribution was made in the planar case by [Amm+12c] which involved the introduction of rigor into the computation of [Pil+10]. The use of single and double layer potentials within [Amm+12c] also cast this problem into the mainstream language of elliptic PDE and facilitates a large portion of the methodology employed within this article.

Few results exist for three-dimensional domains in $\mathbb{R}^n$ or Riemannian manifolds; see [CWS10; SSH06a; SSH08; GC15; NTT21] and references therein. The additional difficulties introduced by higher dimensions are highlighted in the introduction of [Amm+12c] and the challenges in geometry are outlined in [SSH08]. In the case when M is a domain in $\mathbb{R}^3$ with Euclidean metric and Γ_ε is a single small disk absorbing window, [SSH06a; SSH08] gave an expansion for the average of the expected sojourn time, averaged over M, up to an unspecified $O(1)$ term.

These results were improved upon in [NTT21], by using geometric microlocal analysis. Namely, the authors derived the bounded term and estimated the remaining term, moreover, they obtained these results for general Riemannian manifolds. The case when Γ_ε is a small elliptic window was also addressed in [SSH06a; SSH08; NTT21]. We also mention [Sch12], where the author gave a short review of related works (up to 2012).

When M is a three-dimensional ball with multiple circular absorbing windows on the boundary, an expansion capturing the explicit form of the $O(1)$ correction in terms of the Neumann Greens function and its *regular part* was done in [CWS10].

The method of matched asymptotic was used and required the explicit computation of the Neumann Greens function, which is only possible in special geometries with high degrees of symmetry/homogeneity. In the aforementioned results, one does not see the full effects of local geometry. This result was also rigorously proved in [CF11] but with a better estimate for the error term.

Much less has been done for the case of non-zero force fields. For instance, all works we mentioned above, except [Amm+12c], deal with diffusion in the absence of advection. We could find two works concerning this case: [SS07] and [Amm+12c]. Both these works consider M being a domain in $\mathbb{R}^2$ or $\mathbb{R}^3$. In [SS07], the authors generalize the method of [SSH06a; Sin+06; SSH06b] to obtain the leading-order term of the average of the expected sojourn time for two and three-dimensional cases. For planar domains, the authors in [Amm+12c], by using layer potential techniques, derive an asymptotic expansion of the mean sojourn time up to $O(\varepsilon)$ term.

3.1.1 Main Results

Within this article, we derive all the main terms of the expected value of the mean sojourn time for Brownian motions under the influence of drift on $(M, g, \partial M)$. The window or target, $\Gamma_{\varepsilon,a}$, is considered to be a small geodesic ellipse of eccentricity $\sqrt{1-a^2}$ and size $\varepsilon \to 0^+$ (to be made precise later). To investigate the mean sojourn time, we will investigate the singularity structure of the Neumann Greens function which solves

$$\begin{cases} \Delta_{g,z} G(x,z) - \operatorname{div}_{g,z}(F(z)G(x,z)) = -\delta_x(z), & \text{for } z \in M \\ \partial_{\nu_z} G(x,z) - F(z)\cdot \nu_z G(x,z) \mid_{z\in\partial M} = -\frac{1}{|\partial M|}, & \text{for } z \in \partial M \\ \int_{\partial M} e^{-\phi(z)} G(x,z) d\mu_h(z) = 0. \end{cases} \tag{3.1}$$

when *restricted* to $\partial M \times \partial M$. Such a restriction will be denoted by $G_{\partial M} \in \mathcal{D}'(\partial M \times \partial M)$ and will be defined rigorously later. For the case $F = 0$, the authors in [SSH08] highlighted the difficulty in obtaining informative asymptotic expansions for $G_{\partial M}$ when M is a bounded domain in $\mathbb{R}^n$, but it turns out that even when M is a general Riemannian manifold this question can be treated via pseudo-differential techniques, which is done in [NTT21]. Here, we generalize the result of [NTT21] for the case with advection. Knowledge of the singularity structure allows us to derive the mean sojourn time of a Brownian particle wandering within the interior of a Riemannian manifold with a single absorbing window which is a small geodesic ellipse. Our method extends to multiple windows but we present the single window case for notational simplicity. To state our results in completion, we begin by formulating the problem. Let $(X, \mathbb{P}_x)$ be the Brownian motion on M starting at x with law $\mathbb{P}_x$, generated by the differential operator

$$\Delta_g \cdot + g(F, \nabla_g \cdot),$$

where F is a force given by a smooth potential ϕ, that is $F = \nabla_g \phi$. For $x^* \in \partial M$ and $\varepsilon > 0$, let $\Gamma_{\varepsilon,a} \subset \partial M$ be a small geodesic ellipse defined as

$$\Gamma_{\varepsilon,a} := \{\exp_{x^*;h}(\varepsilon t_1 E_1(x^*) + \varepsilon t_2 E_2(x^*)) \mid t_1^2 + a^{-2} t_2^2 \leq 1\}. \tag{3.2}$$

Denote by $\tau_{\Gamma_{\varepsilon,a}}$ the first time the Brownian motion X_t hits $\Gamma_{\varepsilon,a}$, that is

$$\tau_{\Gamma_{\varepsilon,a}} := \inf\{t \geq 0 : X_t \in \Gamma_{\varepsilon,a}\}.$$

We aim to investigate the mean sojourn time, that is the expected value

$$u_{\varepsilon,a}(x) := \mathbb{E}\left(\tau_{\Gamma_{\varepsilon,a}} \mid X_0 = x\right),$$

and its spatial average

$$|M|^{-1} \int_M \mathbb{E}\left(\tau_{\Gamma_{\varepsilon,a}} \mid X_0 = x\right).$$

Namely, we want to derive an asymptotic expansion for these quantities as $\varepsilon \to 0$. Within the appendix, it is shown that the mean sojourn time, $u_{\varepsilon,a}$, satisfies the following elliptic mixed boundary value problem

$$\begin{cases} \Delta_g u_{\varepsilon,a} + g(F, \nabla_g u_{\varepsilon,a}) = -1; \\ u_{\varepsilon,a}|_{\Gamma_{\varepsilon,a}} = 0; \\ \partial_\nu u_{\varepsilon,a}|_{\partial M \setminus \Gamma_{\varepsilon,a}} = 0. \end{cases} \tag{3.3}$$

Let $G(\cdot,\cdot) \in \mathcal{D}'(M \times M)$ solve (3.1). If $x \in M^{\mathrm{int}}$, then, we can use Greens identity in conjunction with (3.1) and (3.3) yields the following integral formula for the mean sojourn time $u_{\varepsilon,a}$

$$u_{\varepsilon,a}(x) = \mathcal{G}(x) + \int_{\partial M} G(x,z)\partial_{\nu_z} u_{\varepsilon,a}(z) d\mu_h(z) + C_{\varepsilon,a} \tag{3.4}$$

where

$$C_{\varepsilon,a} := \frac{1}{|\partial M|} \int_{\partial M} u_{\varepsilon,a}(z) d\mu_h(z), \qquad \mathcal{G}(x) := \int_M G(x,z) d\mu_g(z).$$

and $\mathcal{G}$ satisfies the following boundary value problem

$$\begin{cases} \Delta_g \mathcal{G}(x) + g(F(x), \nabla_g \mathcal{G}(x)) = -1; \\ \partial_\nu \mathcal{G}(x) \mid_{x \in \partial M} = -\frac{\Phi(x)}{|\partial M|}; \\ \int_{\partial M} \mathcal{G}(x) d\mu_h(x) = 0. \end{cases} \tag{3.5}$$

Within (3.5), Φ denotes a weighted volume to be defined in Theorem (3.1.2).

In order to derive an asymptotic expansion for $u_{\varepsilon,a}$, we need to derive asymptotics for $C_{\varepsilon,a}$. The first step within said program is to exploit the vanishing Dirichlet boundary condition of $u_{\varepsilon,a}$ on $\Gamma_{\varepsilon,a}$. In doing so, we restrict $x \in M^{\text{int}}$ to $x \in \Gamma_{\varepsilon,a}$ which yields

$$0 = \mathcal{G}(x)|_{\Gamma_{\varepsilon,a}} + \left(\int_{\partial M} G(x,z)\partial_{\nu_z} u_{\varepsilon,a}(z) d\mu_h(z) \right)\Bigg|_{\Gamma_{\varepsilon,a}} + C_{\varepsilon,a}.$$

What follows, is the definition of the restricted Neumann Greens function, defined as the Schwartz kernel of the operator

$$G_{\partial M} : f \mapsto \left(\int_{\partial M} G(x,y) f(y) d\mu_h(y) \right)\Bigg|_{\partial M}.$$

Here $G_{\partial M} : C^\infty(\partial M) \to C^\infty(\partial M)$ can be extended to $H^k(\partial M)$. Using a parametrix construction, we can show that $G_{\partial M}$ attains the following form for $x, y \in \partial M$ near the diagonal.

Proposition 3.1.1. *There exists an open neighbourhood of the diagonal*

$$\text{Diag} := \{(x,y) \in \partial M \times \partial M \mid x = y\}$$

such that in this neighbourhood, the singularity structure of $G_{\partial M}(x,y)$ is given by:

$$\begin{aligned} G_{\partial M}(x,y) = &\frac{1}{2\pi} d_g(x,y)^{-1} - \frac{H(x)}{4\pi}\log d_h(x,y) + \frac{g_x(F,\nu)}{4\pi}\log d_h(x,y) \\ &+ \frac{1}{16\pi}\left(\mathrm{II}_x\left(\frac{\exp_x^{-1}(y)}{|\exp_x^{-1}(y)|_h}\right) - \mathrm{II}_x\left(\frac{\star\exp_x^{-1}(y)}{|\exp_x^{-1}(y)|_h}\right)\right) &(3.6) \\ &+ \frac{1}{4\pi}h_x\left(F^{\|}(x), \frac{\exp_x^{-1}(x)}{|\exp_x^{-1}(y)|_h}\right) + R_{\partial M}(x,y) &(3.7) \end{aligned}$$

where $F^{\|}$ is the tangential part of F and $R_{\partial M}(\cdot,\cdot) \in C^{0,\mu}(\partial M \times \partial M)$, for all $\mu < 1$, is called the regular part *of the Greens function and $\star$ is the Hodge-star operator (i.e. anti-clockwise rotation by $\pi/2$ on the surface ∂M).*

We will use the formula in Proposition 3.1.1 to derive the mean sojourn time of a Brownian particle on a Riemannian manifold with a single absorbing window which is a small geodesic ellipse. As mentioned earlier, our method extends to multiple windows but we present the single window case to simplify the exposition. We first state the result when the window is a geodesic *ball* of the boundary ∂M around a fixed point to provide a concise glimpse into the full result

Theorem 3.1.2. *Let $(M, g, \partial M)$ be a smooth compact Riemannian manifold of dimension three with boundary. Fix $x^* \in \partial M$ and let Γ_ε be a boundary geodesic ball centred at x^* of geodesic radius $\varepsilon > 0$.*
i) For each $x \notin \Gamma_\varepsilon$,

$$u_\varepsilon(x) = C_\varepsilon + \mathcal{G}(x) - \Phi(x^*)G(x^*, x) + r_\varepsilon(x),$$

with $\|r_\varepsilon\|_{C^k(K)} \le C_{k,K}\varepsilon$ for any integer k and compact set $K \subset \overline{M}$ which does not contain x^*. The function $\mathcal{G}$ is the solution of (3.5). The constant C_ε is given by

$$\begin{aligned} C_\varepsilon = &\frac{\Phi(x^*)}{4\varepsilon} - \frac{(H(x^*) + g_{x^*}(F,\nu))\Phi(x^*)}{4\pi}\log\varepsilon + R(x^*, x^*)\Phi(x^*) - \mathcal{G}(x^*) \\ &- \frac{(H(x^*) + g_{x^*}(F,\nu))\Phi(x^*)}{4\pi}\left(2\log 2 - \frac{3}{2}\right) + O(\varepsilon\log\varepsilon), \end{aligned}$$

where $R(x^*, x^*)$ is the evaluation at $(x,y) = (x^*, x^*)$ of the kernel $R(x,y)$ in Proposition 3.1.1 and

$$\Phi(x) := \int_M e^{\phi(z)-\phi(x)} d\mu_g(z).$$

ii) One has that the integral of u_ε over M satisfies

$$\int_M u_\varepsilon(x)d\mu_g(x) = C_\varepsilon|M| + \int_M \mathcal{G}(x)d\mu_g(x) - \Phi(x^*)\int_M G(x,x^*)d\mu_g(x).$$

Remark 3.1.3. We use the convention $u_\varepsilon(x) := u_{\varepsilon,1}(x)$ within this theorem as well as this chapter.

The geometric insights revealed in the boundary restriction of the Greens function constructed within Proposition 3.1.1 are not realized to their full potential in Theorem 3.1.2 as the choice of trivial eccentricity kills the principal curvatures which appear in 3.1.1 as an artifact of an inhomogeneous local geometry of the local geometry at x^*. If we replace geodesic balls with geodesic *ellipses* as our targets, all the geometric terms in the Greens function will appear in the mean sojourn time. This is seen in the following theorem (when compared to theorem 3.1.2).

Theorem 3.1.4. *Let $(M, g, \partial M)$ be a smooth Riemannian manifold of dimension three with a boundary. Fix $x^* \in \partial M$ and let $\Gamma_{\varepsilon,a}$ be a boundary geodesic ellipse given by (3.2) with $\varepsilon > 0$.*
i) For each $x \in M\backslash\Gamma_{\varepsilon,a}$,

$$u_{\varepsilon,a}(x) = C_{\varepsilon,a} + \mathcal{G}(x) - \Phi(x^*)G(x^*,x) + r_\varepsilon(x),$$

with $\|r_\varepsilon\|_{C^k(K)} \le C_{k,K}\varepsilon$ for any integer k and compact set $K \subset \overline{M}$ which does not contain x^*. The function $\mathcal{G}$ is the solution of (3.5). The constant $C_{\varepsilon,a}$ is given by

$$\begin{aligned} C_{\varepsilon,a} = {} & \frac{K_a\Phi(x^*)}{4\pi^2 a\varepsilon} - \frac{(H(x^*) + g_{x^*}(F,\nu))\Phi(x^*)}{4\pi}\log\varepsilon + R(x^*,x^*)\Phi(x^*) - \mathcal{G}(x^*) \\ & - \frac{(H(x^*) + g_{x^*}(F,\nu))\Phi(x^*)}{16\pi^3}\int_{\mathbb{D}}\int_{\mathbb{D}} \frac{\log\left((t_1-s_1)^2 + a^2(t_2-s_2)^2\right)^{1/2}}{\sqrt{(1-|s'|^2)(1-|t'|^2)}}dt'ds' \\ & + \frac{(\kappa_1(x^*) - \kappa_2(x^*))\Phi(x^*)}{64\pi^3}\int_{\mathbb{D}}\int_{\mathbb{D}} \frac{(t_1-s_1)^2 - a^2(t_2-s_2)^2}{(t_1-s_1)^2 + a^2(t_2-s_2)^2}\frac{1}{\sqrt{(1-|s'|^2)(1-|t'|^2)}}dt'ds' \\ & + O(\varepsilon\log\varepsilon), \end{aligned}$$

where $R(x^*,x^*)$ is the evaluation at $(x,y) = (x^*,x^*)$ of the kernel $R(x,y)$ in Proposition 3.1.1, $\mathbb{D}$ is the two-dimensional unit disk centred at the origin, and

$$\Phi(x) := \int_M e^{\phi(z)-\phi(x)}d\mu_g(z), \qquad K_a := \frac{\pi}{2}\int_0^{2\pi}\left(\cos^2\theta + \frac{\sin^2\theta}{a^2}\right)^{-1/2}d\theta$$

ii) One has that the integral of $u_{\varepsilon,a}(x)$ over M satisfies

$$\int_M u_{\varepsilon,a}(x)d\mu_g(x) = C_\varepsilon|M| + \int_M \mathcal{G}(x)d\mu_g(x) - \Phi(x^*)\int_M G(x,x^*)d\mu_g(x).$$

3.2 The Neumann Greens Function

Here, we investigate the boundary restriction of Neumann Greens function. Namely, we derive a representative singular asymptotic expansion near the diagonal, as a subset of $\partial M \times \partial M$. By the Neumann Greens function, $G(\cdot,\cdot) \in \mathcal{D}'(M \times M)$, we mean the distribution solving the following problem

$$\begin{cases} \Delta_{g,z}G(x,z) - \mathrm{div}_{g,z}(F(z)G(x,z)) = -\delta_x(z), & \text{for } z \in M \\ \partial_{\nu_z}G(x,z) - F(z)\cdot\nu_z G(x,z)\mid_{\partial M} = -\frac{1}{|\partial M|}, & \text{for } z \in \partial M \\ \int_{\partial M} e^{-\phi(z)}G(x,z)d\mu_h(z) = 0. \end{cases} \tag{3.8}$$

We begin our analysis by observing the following identity, which follows from the definition of G

$$e^{-\phi(y)}G(z,y) = -\int_M e^{-\phi(x)}G(z,x)\left(\Delta_g G(y,x) - \mathrm{div}_g(F(x)G(y,x))\right)d\mu_g(x),$$

Applying Greens identity to the right-hand side of the above equation yields the following formula

$$e^{-\phi(y)}G(z,y) - e^{-\phi(z)}G(y,z) = \frac{1}{|\partial M|}\left(\int_{\partial M} e^{-\phi(x)}G(y,x)d\mu_h(x) - \int_{\partial M} e^{-\phi(x)}G(z,x)d\mu_h(x)\right).$$

Thus, it follows, due to the last condition on G in (3.8), that

$$G(z,y) = e^{\phi(y)-\phi(z)}G(y,z). \tag{3.9}$$

Once again, invoking the last condition on G in (3.8), we have that the integral of G in the first argument vanishes along the boundary, that is

$$\int_{\partial M} G(z,y)d\mu_h(z) = \int_{\partial M} e^{\phi(y)-\phi(z)}G(y,z)d\mu_h(z) = 0,$$

In addition, using the Neumann boundary condition on G results in the following expression for the normal derivative of G in the first argument

$$\partial_{\nu_z} G(z,y) = e^{\phi(y)-\phi(z)}\big(\partial_{\nu_z} G(y,z) - F(z)\cdot \nu_z G(y,z)\,|_{\partial M}\big) = -\frac{e^{\phi(y)-\phi(z)}}{|\partial M|}$$

for $z \in \partial M$, and finally, using (3.9), we have that

$$\begin{aligned}\Delta_{g,z} G(z,y) + g(F(z), \nabla_{g,z} G(z,y)) = e^{\phi(y)-\phi(x)}\Big(& g(\nabla_{g,z}\phi(z), \nabla_{g,z}\phi(z))G(y,z) \\ &- \Delta_{g,z}\phi(z)G(y,z) - 2g(\nabla_{g,z}\phi(z), \nabla_{g,z}G(y,z)) + \Delta_{g,z}G(y,z) \\ &- g(F(z), \nabla_{g,z}\phi(z))G(y,z) + g(F(z), \nabla_{g,z}G(y,z))\Big).\end{aligned}$$

Therefore, we conclude that $G(z,y)$ satisfies the differential equation (with respect to the first variable)

$$\begin{cases} \Delta_{g,z} G(z,y) + g(F(z), \nabla_{g,z} G(z,y)) = -\delta_y(z); \\ \partial_{\nu_z} G(z,y)\,|_{z\in\partial M} = -\frac{e^{\phi(y)-\phi(z)}}{|\partial M|}; \\ \int_{\partial M} G(z,x) d\mu_h(z) = 0. \end{cases} \tag{3.10}$$

This indicates that $\mathcal{G}$ solves the problem (3.5). Now, we consider some $f \in C^\infty(\partial M)$. Using this f, we introduce u_f, the solution to the following auxiliary Dirichlet boundary value problem

$$\Delta_g u_f(z) + g_z(F(z), \nabla_g u_f(z)) = 0, \quad u_f|_{z\in\partial M} = f \in C^\infty(\partial M). \tag{3.11}$$

The definition of $G(x,z)$ implies that we have the following expression for u_f

$$u_f(x) = -\int_M u_f(z)\left(\Delta_{g,z} G(x,z) - \mathrm{div}_{g,z}(F(z)G(x,z))\right) d\mu_g(z), \tag{3.12}$$

Greens identity and the divergence theorem imply that

$$\begin{aligned} u_f(x) = &-\int_M \Delta_g u_f(z) G(x,z) d\mu_g(z) - \int_{\partial M} \left(u_f(z)\partial_{\nu_z} G(x,z) - \partial_{\nu_z} u_f(z) G(x,z)\right) d\mu_h(z) \\ &- \int_M G(x,z) g_z(F(z), \nabla_g u_f(z)) d\mu_g(z) + \int_{\partial M} u_f(z) G(x,z) F(z)\cdot \nu_z d\mu_h(z). \end{aligned}$$

Since functions u_f, $G(x,z)$ satisfy (3.11), (3.8) respectively, we conclude that

$$u_f(x) = \int_{\partial M} G(x,z)\partial_{\nu_z} u_f(z) d\mu_h(z) + \frac{1}{|\partial M|}\int_{\partial M} f(z) d\mu_h(z).$$

Restricting x to ∂M and invoking the Dirichlet boundary condition of (3.11), we have that

$$f(x) = \left(\int_{\partial M} G(x,z) \partial_{\nu_z} u_f(z) d\mu_h(z) \right)\Big|_{\partial M} + \frac{1}{|\partial M|} \int_{\partial M} f(z) d\mu_h(z) \tag{3.13}$$

Let $\Lambda_F \in \Psi^1_{cl}(\partial M)$ be the Dirichlet-to-Neumann map associated to the boundary value problem (3.11) and $G_{\partial M}(x,y)$ be the Schwartz kernel of the operator

$$f \to \left(\int_{\partial M} f(y) G(x,y) d\mu_h(y) \right)\Big|_{\partial M}$$

which takes $C^\infty(\partial M) \to C^\infty(\partial M)$. Then we can rewrite (3.13) in the following way

$$f(x) = G_{\partial M} \Lambda_F f + P f$$

where P is a smoothing operator, that is $P \in \Psi^{-\infty}(\partial M)$. In operator form this is

$$I = G_{\partial M} \Lambda_F + P. \tag{3.14}$$

Since Λ_F is a first order, elliptic pseudo-differential operator, we can construct $G_{\partial M}$ via a standard pseudo-differential parametrix construction. The details of the derivation of $G_{\partial M}$ are contained in Appendix D.

3.3 Proof of Theorems 3.1.2 and 3.1.4

In this section we give a proof for Theorems 3.1.2 and 3.1.4. We recall that the mean sojourn time, $u_{\varepsilon,a}$, satisfies the elliptic mixed boundary value problem (3.3), this is proved in the Appendix of [NTT21]. Recall that Greens identity and a restriction to $x \in \Gamma_{\varepsilon,a}$ yields the following equation

$$0 = \mathcal{G}(x) + \int_{\partial M} G_{\partial M}(x,z) \partial_{\nu_z} u_{\varepsilon,a}(z) d\mu_h(z) + C_{\varepsilon,a}.$$

Therefore, Proposition 3.1.1 in conjunction with the above identity gives

$$
\begin{aligned}
-\mathcal{G}(x) - C_{\varepsilon,a} = & \frac{1}{2\pi}\int_{\Gamma_{\varepsilon,a}} d_g(x,y)^{-1}\partial_\nu u_{\varepsilon,a}(y)d\mu_h(y) \\
& - \frac{(H(x)+g_x(F,\nu))}{4\pi}\int_{\Gamma_{\varepsilon,a}} \log d_h(x,y)\partial_\nu u_{\varepsilon,a}(y)d\mu_h(y) \\
& + \frac{1}{16\pi}\int_{\Gamma_{\varepsilon,a}} \left(\mathrm{II}_x\left(\frac{\exp_x^{-1}(y)}{|\exp_x^{-1}(y)|_h}\right) - \mathrm{II}_x\left(\frac{\star\exp_x^{-1}(y)}{|\exp_x^{-1}(y)|_h}\right)\right)\partial_\nu u_{\varepsilon,a}(y)d\mu_h(y) \\
& + \frac{1}{4\pi}\int_{\Gamma_{\varepsilon,a}} h_x\left(F^{\parallel}(x), \frac{\exp_{x;h}(y)}{|\exp_{x;h}(y)|_h}\right)\partial_\nu u_{\varepsilon,a}(y)d\mu_h(y) \\
& + \int_{\Gamma_{\varepsilon,a}} R(x,y)\partial_\nu u_{\varepsilon,a}(y)d\mu_h(y).
\end{aligned}
\tag{3.15}
$$

Since $F = \nabla_g\phi$, the fact that $u_{\varepsilon,a}$ satisfies (3.3) implies

$$\mathrm{div}_g(e^\phi\nabla_g u_{\varepsilon,a}) = e^\phi(\Delta_g u_{\varepsilon,a} + F\cdot\nabla_g u_{\varepsilon,a}) = -e^\phi.$$

The divergence theorem indicates that

$$\int_M \mathrm{div}_g(e^\phi\nabla_g u_{\varepsilon,a})(z)d\mu_g(z) = \int_{\partial M} e^\phi\partial_\nu u_{\varepsilon,a}(z)d\mu_h(z).$$

Therefore, by integrating the above equation, we derive the following compatibility condition

$$\int_{\partial M} e^{\phi(z)}\partial_\nu u_{\varepsilon,a}(z)d\mu_h(z) = -\int_M e^{\phi(z)}d\mu_g(z). \tag{3.16}$$

We will use the coordinate system given by

$$\mathbb{D} \ni (s_1,s_2) \mapsto x^\varepsilon(s_1, as_2; x^*) \in \Gamma_{\varepsilon,a}, \tag{3.17}$$

where $x^\varepsilon(\cdot;x^*)$ is the coordinate defined in Appendix B. To simplify notation we will drop the x^* in the notation and denote $x^\varepsilon(\cdot;x^*)$ by simply $x^\varepsilon(\cdot)$.

Note that in these coordinates the volume form for ∂M is given by

$$d_h = a\varepsilon^2(1+\varepsilon^2 Q_\varepsilon(s'))ds_1ds_2,\ \ s'\in\mathbb{D} \tag{3.18}$$

for some smooth function $Q_\varepsilon(s')$ whose derivatives of all orders are bounded uniformly in ε. We denote

$$\psi_\varepsilon(s') := \partial_\nu u_\varepsilon(x^\varepsilon(s_1,as_2)). \tag{3.19}$$

Then, in this coordinate system, the compatibility condition becomes

$$\int_{\mathbb{D}} e^{\phi(x^{\varepsilon}(s'))}\psi_{\varepsilon}(s')(1+\varepsilon^2 Q_{\varepsilon}(s'))ds' = -\frac{1}{a\varepsilon^2}\int_M e^{\phi(z)}d\mu_g(z). \tag{3.20}$$

We can re-write this as follows

$$e^{\phi(x^*)}\int_{\mathbb{D}}\psi(x)dx+\int_{\mathbb{D}}\left[(e^{\phi(x^{\varepsilon}(s'))}-e^{\phi(x^*)})+\varepsilon^2 e^{\phi(x^*)}Q_{\varepsilon}(s')\right]\psi_{\varepsilon}(s')ds' = -\frac{1}{a\varepsilon^2}\int_M e^{\phi(z)}d\mu_g(z).$$

Since ϕ is smooth, we conclude that

$$\int_{\mathbb{D}}\psi(x)dx = -\frac{\Phi(x^*)}{a\varepsilon^2}+\varepsilon\int_{\mathbb{D}}\tilde{Q}_{\varepsilon}(s')\psi_{\varepsilon}(s')ds', \tag{3.21}$$

where Φ is the function defined in Theorem (3.1.2) and $\tilde{Q}_{\varepsilon}(s')$ is some smooth function whose derivatives of all orders are bounded uniformly in ε.

Next, we re-write (3.15) in this coordinate system given by (3.17). To do this, let us first introduce the following operators. Consider

$$L_a f = a\int_{\mathbb{D}}\frac{f(s')}{((t_1-s_1)^2+a^2(t_2-s_2)^2)^{1/2}}ds' \tag{3.22}$$

acting on functions of the disk $\mathbb{D}$. By [SSH06a] we have that

$$L_a\left({K_a}^{-1}(1-|t'|^2)^{-1/2}\right) = 1, \tag{3.23}$$

on $\mathbb{D}$ where

$$K_a = \frac{\pi}{2}\int_0^{2\pi}\frac{1}{\left(\cos^2\theta+\frac{\sin^2\theta}{a^2}\right)^{1/2}}d\theta.$$

By (4.4) in [NTT21], this is the unique solution in $H^{1/2}(\mathbb{D})^*$ to $L_a u = 1$. Next, we introduce three operators $R_{\log,a}$, $R_{\infty,a}$ and $R_{F,a}$ which are defined as follows

$$R_{\log,a}f(t') := a\int_{\mathbb{D}}\log\left((t_1-s_1)^2+a^2(t_2-s_2)^2\right)^{1/2}f(s')ds',$$

$$R_{\infty,a}f(t') := a\int_{\mathbb{D}}\frac{(t_1-s_1)^2-a^2(t_2-s_2)^2}{(t_1-s_1)^2+a^2(t_2-s_2)^2}f(s')ds',$$

$$R_{F,a}f(t') := a\int_{\mathbb{D}}\frac{F_1(0)(t_1-s_1)+aF_2(0)(t_2-s_2)}{((t_1-s_1)^2+a^2(t_2-s_2)^2)^{1/2}}f(s')ds'.$$

Here, $F_1(0)$ and $F^2(0)$ denote the components of F at the centre of the window.

Remark 3.3.1. In [NTT21], it was shown that the operators $R_{\log,a}$ and $R_{\log,a}$ are bounded maps from $H^{1/2}(\mathbb{D})^*$ to $H^{3/2}(\mathbb{D})$. By repeating the arguments, one can show that this is also true for $R_{F,a}$.

We unwrap the right-hand side of (3.15) term by term in the following five lemmas. Note that the first four of them are proved in [NTT21]. We repeat them here for the convenience of the readers.

Lemma 3.3.2. *We have the following identity*

$$\int_{\Gamma_{\varepsilon,a}} d_g(x,y)^{-1}\partial_\nu u_{\varepsilon,a}(y)d\mu_h(y) = \varepsilon L_a\psi_\varepsilon(t') + \varepsilon^3\mathcal{A}_\varepsilon\psi_\varepsilon(t')$$

with $x = x^\varepsilon(t')$ *for some* $\mathcal{A}_\varepsilon : H^{1/2}(\mathbb{D}; ds')^* \to H^{1/2}(\mathbb{D}; ds')$ *with operator norm bounded uniformly in* ε*.*

Proof. By using Proposition D.0.1 and (3.18), we see that in the coordinate system (3.17) it follows

$$\begin{aligned}\int_{\Gamma_{\varepsilon,a}} d_g(x,y)^{-1}\partial_\nu u_{\varepsilon,a}(y)d\mu_h(y) =& a\varepsilon\int_{\mathbb{D}}\frac{1}{((s_1-t_1)^2+a(s_2-t_2)^2)^{1/2}}\psi_\varepsilon(s')ds' \\ &+ a\varepsilon^3\int_{\mathbb{D}}\frac{A^1(s',t')(1+\varepsilon^2Q_\varepsilon(s'))+Q_\varepsilon(s')}{((s_1-t_1)^2+a(s_2-t_2)^2)^{1/2}}\psi_\varepsilon(s')ds'.\end{aligned}$$

Therefore, by Lemma 2.11 of [NTT21], the second term of the right-hand side can be written as $\varepsilon^3\mathcal{A}_\varepsilon\psi$, for operator $\mathcal{A}_\varepsilon$ which satisfies the requirement of the statement. □

From now on, we will denote by $\mathcal{A}_\varepsilon$ any operator which takes $H^{1/2}(\mathbb{D}; ds')^* \to H^{1/2}(\mathbb{D}; ds')$ whose operator norm is bounded uniformly in ε. For the second term of the right-hand side of (3.15), the following lemma holds.

Lemma 3.3.3. *We have the following identity*

$$\begin{aligned}(H(x)-\partial_\nu\phi(x))\int_{\Gamma_{\varepsilon,a}}\log d_h(x,y)\partial_\nu u_{\varepsilon,a}(y)d\mu_h(y) &= -(H(x^*)-\partial_\nu\phi(x^*))\Phi(x^*)\log\varepsilon \\ &+ \varepsilon^2(H(x^*)-\partial_\nu\phi(x^*))R_{log,a}\psi_\varepsilon(t') + O_{H^{1/2}(\mathbb{D})}(\varepsilon\log\varepsilon) + \varepsilon^3\log\varepsilon\mathcal{A}_\varepsilon\psi_\varepsilon.\end{aligned}$$

where $x = x^\varepsilon(t')$ *and* Φ *is the function defined in Theorem* (3.1.2)*.*

Proof. By using Proposition (D.0.1) and (3.18), we obtain

$$
\begin{aligned}
&(H(x)-\partial_\nu\phi(x))\int_{\Gamma_{\varepsilon,a}}\log d_h(x,y)\partial_\nu u_{\varepsilon,a}(y)d\mu_h(y)\\
&= a\varepsilon^2\log\varepsilon(H(x^\varepsilon(t'))-\partial_\nu\phi(x^\varepsilon(t')))\int_{\mathbb{D}}\psi_\varepsilon(s')ds'\\
&\quad + a\varepsilon^2(H(x^\varepsilon(t'))-\partial_\nu\phi(x^\varepsilon(t')))\int_{\mathbb{D}}\log\left[((t_1-s_1)^2+a(t_2-s_2)^2)^{1/2}\right]\psi_\varepsilon(s')ds'\\
&\quad - a\varepsilon^2(H(x^\varepsilon(t'))-\partial_\nu\phi(x^\varepsilon(t')))\int_{\mathbb{D}}\log\left(1+\varepsilon^2A(s',t')\right)\psi_\varepsilon(s')ds'\\
&\quad - a\varepsilon^4(H(x^\varepsilon(t'))-\partial_\nu\phi(x^\varepsilon(t')))\int_{\mathbb{D}}\log\left[\varepsilon((t_1-s_1)^2+a(t_2-s_2)^2)^{1/2}\right]Q_\varepsilon(s')\psi_\varepsilon(s')ds'\\
&\quad - a\varepsilon^4(H(x^\varepsilon(t'))-\partial_\nu\phi(x^\varepsilon(t')))\int_{\mathbb{D}}\log\left(1+\varepsilon^2A(s',t')\right)Q_\varepsilon(s')\psi_\varepsilon(s')ds'.
\end{aligned}
$$

Apart from the first and second terms, all terms on the right-hand side can be written as $\varepsilon^3\mathcal{A}_\varepsilon\psi_\varepsilon$. The first term, by (3.21), is equal to

$$
\begin{aligned}
-(H(x^*)-\partial_\nu\phi(x^*))\Phi(x^*)\log\varepsilon+\varepsilon^3\log\varepsilon(H(x^\varepsilon(t'))-\partial_\nu\phi(x^\varepsilon(t')))\int_{\mathbb{D}}\tilde{Q}_\varepsilon(s')\psi_\varepsilon(s')ds'\\
+(H(x^*)-\partial_\nu\phi(x^*)-H(x^\varepsilon(t')+\partial_\nu\phi(x^\varepsilon(t'))))\log\varepsilon\Phi(x^*),
\end{aligned}
$$

consequently, by Lemma 2.11 of [NTT21], it is equal to

$$
-(H(x^*)-\partial_\nu\phi(x^*))\Phi(x^*)\log\varepsilon+O_{H^{1/2}(\mathbb{D})}(\varepsilon\log\varepsilon)+\varepsilon^3\log\varepsilon\mathcal{A}_\varepsilon\psi_\varepsilon.
$$

While the second term is

$$
\begin{aligned}
&\varepsilon^2(H(x^*)-\partial_\nu\phi(x^*))R_{log,a}\psi_\varepsilon(t')\\
&\qquad+\varepsilon^2(H(x^\varepsilon(t'))-\partial_\nu\phi(x^\varepsilon(t'))-H(x^*)+\partial_\nu\phi(x^*))R_{log,a}\psi_\varepsilon(t').
\end{aligned}
$$

Since H and ϕ are smooth functions, it follows that the last term of the above expression is $\varepsilon^3\mathcal{A}_\varepsilon\psi_\varepsilon$. □

For the fourth term of (3.15), we have the following lemma.

Lemma 3.3.4. *We have the following identity*

$$
\begin{aligned}
\int_{\Gamma_{\varepsilon,a}}\left(\mathrm{II}_x\left(\frac{\exp_x^{-1}(y)}{|\exp_x^{-1}(y)|_h}\right)-\mathrm{II}_x\left(\frac{\star\exp_x^{-1}(y)}{|\exp_x^{-1}(y)|_h}\right)\right)\partial_\nu u_{\varepsilon,a}(y)d\mu_h(y)\\
=\varepsilon^2(\kappa_1(x^*)-\kappa_2(x^*))R_{\infty,a}\psi_\varepsilon(t')+\varepsilon^3\mathcal{A}_\varepsilon\psi_\varepsilon.
\end{aligned}
$$

Proof. By Proposition D.0.1 and (3.18), we see that

$$\begin{aligned}
&\int_{\Gamma_{\varepsilon,a}} \left(\mathrm{II}_x\left(\frac{\exp_x^{-1}(y)}{|\exp_x^{-1}(y)|_h}\right) - \mathrm{II}_x\left(\frac{\star\exp_x^{-1}(y)}{|\exp_x^{-1}(y)|_h}\right)\right)\partial_\nu u_{\varepsilon,a}(y)d\mu_h(y)\\
&\quad= a(\kappa_1(x^*)-\kappa_2(x^*))\varepsilon^2\int_{\mathbb{D}}\frac{(s_1-t_1)^2-a^2(s_2-t_2)^2}{(s_1-t_1)^2+a^2(s_2-t_2)^2}\psi_\varepsilon(s')ds'\\
&\quad\quad+a\varepsilon^3\int_{\mathbb{D}}\left(R^1(t',s')-R^2(t',s')\right)(1+\varepsilon^2Q(s'))\psi_\varepsilon(s')ds'\\
&\quad\quad+a(\kappa_1(x^*)-\kappa_2(x^*))\varepsilon^4\int_{\mathbb{D}}\frac{(s_1-t_1)^2-a^2(s_2-t_2)^2}{(s_1-t_1)^2+a^2(s_2-t_2)^2}Q(s')\psi_\varepsilon(s')ds'.
\end{aligned}$$

We use Lemma 2.11 of [NTT21] to complete the proof. □

Next, we study the forth term of (3.15).

Lemma 3.3.5. *The following is true*

$$\int_{\Gamma_{\varepsilon,a}} h_x\left(F^{\|}(x), \frac{\exp_{x;h}(y)}{|\exp_{x;h}(y)|_h}\right)\partial_\nu u_{\varepsilon,a}(y)d\mu_h(y) = \varepsilon^2 R_{F,a}\psi_\varepsilon + \varepsilon^3\mathcal{A}_\varepsilon\psi_\varepsilon.$$

Proof. By Proposition D.0.1 and (3.18), we get

$$\begin{aligned}
&\int_{\Gamma_{\varepsilon,a}} h_x\left(F^{\|}(x), \frac{\exp_{x;h}(y)}{|\exp_{x;h}(y)|_h}\right)\partial_\nu u_{\varepsilon,a}(y)d\mu_h(y)\\
&\quad= a\varepsilon^2\int_{\mathbb{D}}\frac{F_1(t')(s_1-t_1)+aF_2(t')(s_2-t_2)}{((s_1-t_1)^2+a^2(s_2-t_2)^2)^{1/2}}\psi_\varepsilon(s')ds'\\
&\quad\quad+a\varepsilon^3\int_{\mathbb{D}}R^3(t',s')(1+\varepsilon^2Q(s'))\psi_\varepsilon(s')ds'\\
&\quad\quad+a\varepsilon^4\int_{\mathbb{D}}\frac{F_1(t')(s_1-t_1)+aF_2(t')(s_2-t_2)}{((s_1-t_1)^2+a^2(s_2-t_2)^2)^{1/2}}Q(s')\psi_\varepsilon(s')ds'
\end{aligned}$$

Finally, Lemma 2.11 of [NTT21] implies that the last two terms are $\varepsilon^3\mathcal{A}_\varepsilon\psi_\varepsilon$. □

Finally, let us look to the last term of (3.15). By Lemma 5.1 in [NTT21], we know that for operator $T_\varepsilon : C_c^\infty(\mathbb{D}) \to \mathcal{D}'(\mathbb{D})$ defined by the integral kernel

$$R(x^\varepsilon(t';x^*), x^\varepsilon(s';x^*)) - R(x^*, x^*),$$

we have $\|T_\varepsilon\|_{H^{1/2}(\mathbb{D})^* \to H^{1/2}(\mathbb{D})} = O(\varepsilon \log \varepsilon)$. Therefore, by using Lemmas 4.3.5-4.3.8, we re-write (3.15) in the following way

$$\begin{aligned}
-\mathcal{G}(x^*) - C_{\varepsilon,a} =& \frac{\varepsilon}{2\pi} L_a \psi_\varepsilon + \frac{1}{4\pi}(H(x^*) - \partial_\nu \phi(x^*))\Phi(x^*) \log \varepsilon \\
& - \frac{\varepsilon^2}{4\pi}\left((H(x^*) - \partial_\nu \phi(x^*)) R_{\log,a} - \frac{\kappa_1(x^*) - \kappa_2(x^*)}{4\pi} R_{\infty,a} - R_{F,a} \right) \psi_\varepsilon \\
& - R(x^*, x^*)\Phi(x^*) + \varepsilon^2 a T_\varepsilon \psi_\varepsilon + O_{H^{1/2}(\mathbb{D})}(\varepsilon \log \varepsilon) + \varepsilon^3 \log \varepsilon \mathcal{A}_\varepsilon \psi_\varepsilon.
\end{aligned}$$

This is equivalent to

$$\begin{aligned}
\frac{2\pi}{\varepsilon}\left(R(x^*, x^*)\Phi(x^*) - \mathcal{G}(x^*) - C_{\varepsilon,a} - \frac{(H(x^*) - \partial_\nu \phi(x^*))\Phi(x^*) \log \varepsilon}{4\pi} \right) &= (L_a - \varepsilon \mathcal{R})\psi_\varepsilon \\
&+ \varepsilon a T_\varepsilon \psi_\varepsilon + O_{H^{1/2}(\mathbb{D})}(\log \varepsilon) + \varepsilon^2 \log \varepsilon \mathcal{A}_\varepsilon \psi_\varepsilon,
\end{aligned}$$

where

$$\mathcal{R} := \frac{H(x^*) - \partial_\nu \phi(x^*)}{2} R_{\log,a} - \frac{\lambda_1 - \lambda_2}{8} R_{\infty,a} - \frac{1}{2} R_{F,a}.$$

Applying L_a^{-1} to both sides, we obtain

$$\begin{aligned}
\frac{2\pi}{\varepsilon}\left(R(x^*, x^*)\Phi(x^*) - \mathcal{G}(x^*) - C_{\varepsilon,a} - \frac{(H(x^*) - \partial_\nu \phi(x^*))\Phi(x^*) \log \varepsilon}{4\pi} \right) L_a^{-1} 1 \\
= (I - \varepsilon L_a^{-1} \mathcal{R} + \varepsilon L_a^{-1} T'_\varepsilon)\psi_\varepsilon + O_{H^{1/2}(\mathbb{D})^*}(\log \varepsilon), \quad (3.24)
\end{aligned}$$

for some $T'_\varepsilon : H^{1/2}(\mathbb{D})^* \to H^{1/2}(\mathbb{D})^*$ with operator norm $O(\varepsilon \log \varepsilon)$. As we mentioned in Remark 3.3.1, $R_{\log,a}$, $R_{\infty,a}$, and $R_{F,a}$ are bounded maps from $H^{1/2}(\mathbb{D})^*$ to $H^{3/2}(\mathbb{D})$. Therefore, the right side can be inverted by Neumann series to deduce

$$\psi_\varepsilon = -\frac{2\pi C_{\varepsilon,a}}{\varepsilon} L_a^{-1} 1 + C_{\varepsilon,a} O_{H^{1/2}(\mathbb{D})^*}(1) + O_{H^{1/2}(\mathbb{D})^*}(\varepsilon^{-1} \log \varepsilon). \quad (3.25)$$

Let us integrate this over $\mathbb{D}$ and use (3.21), then we derive

$$-\frac{2\pi C_{\varepsilon,a}}{\varepsilon} \int_{\mathbb{D}} L_a^{-1} 1(s') ds' + C_{\varepsilon,a} O(1) + O(\varepsilon^{-1} \log \varepsilon) = -\frac{\Phi(x^*)}{a\varepsilon^2}.$$

Note that

$$\int_{\mathbb{D}} L_a^{-1} 1(s') ds' = \frac{1}{K_a} \int_{\mathbb{D}} \frac{1}{(1 - |s'|^2)^{1/2}} ds' = \frac{2\pi}{K_a}, \quad (3.26)$$

and hence, the previous equation gives

$$C_{\varepsilon,a} = \frac{K_a \Phi(x^*)}{4\pi^2 a\varepsilon} + C'_{\varepsilon,a}, \quad (3.27)$$

with $C'_{\varepsilon,a} = O(\log \varepsilon)$. We put this into (3.25), to obtain

$$\psi_\varepsilon = -\frac{K_a \Phi(x^*)}{2\pi a \varepsilon^2} L_a^{-1} 1 + \psi'_\varepsilon, \tag{3.28}$$

where $\|\psi'_\varepsilon\|_{H^{1/2}(\mathbb{D};ds')^*} = O(\varepsilon^{-1} \log \varepsilon)$. Let us insert this into (3.21), then we obtain

$$\int_{\mathbb{D}} \psi'_\varepsilon(s')ds' = -\frac{K_a \Phi(x^*)}{2\pi a \varepsilon} \int_{\mathbb{D}} \tilde{Q}(s') L_a^{-1} 1(s')ds' + \varepsilon \int_{\mathbb{D}} \tilde{Q}(s') \psi'_\varepsilon(s')ds' \tag{3.29}$$

where

$$\tilde{Q}(s') = \frac{1}{\varepsilon} \left((e^{\phi(x^\varepsilon(s'))} - e^{\phi(x^*)}) + \varepsilon^2 e^{\phi(x^*)} Q_\varepsilon(s') \right)$$

and Q is the function involved into volume form (3.18). Taylor expanding e^ϕ at $x = x^*$ yields the following

$$e^\phi = e^{\phi(x^*)} + \varepsilon g_{x^*}(e^{\phi(x^*)} \left.\nabla_g \phi(x)\right|_{x=x^*}, t_1 E_1 + t_2 E_2) + O(\varepsilon^2).$$

Therefore

$$\begin{aligned}
\int_{\mathbb{D}} e^{\phi(x^\varepsilon(t))}[L_a^{-1}1](t)dt =& e^{\phi(x^*)} \int_{\mathbb{D}} [L_a^{-1}1](t)dt \\
&+ \varepsilon g_{x^*} \left(e^{\phi(x^*)} \left.\nabla_g \phi(x)\right|_{x=x^*}, E_1(x^*) \right) \int_{\mathbb{D}} t_1 [L_a^{-1}1](t)dt \\
&+ \varepsilon g_{x^*} \left(e^{\phi(x^*)} \left.\nabla_g \phi(x)\right|_{x=x^*}, E_2(x^*) \right) \int_{\mathbb{D}} t_2 [L_a^{-1}1](t)dt \\
&+ O(\varepsilon^2).
\end{aligned}$$

Noting that

$$\int_{\mathbb{D}} t_1 [L_a^{-1}](t)dt = \int_{\mathbb{D}} t_2 [L_a^{-1}1](t)dt = 0,$$

we obtain

$$\int_{\mathbb{D}} e^{\phi(x^\varepsilon(t))}[L_a^{-1}1](t)dt = e^{\phi(x^*)} \int_{\mathbb{D}} [L_a^{-1}1](t)dt + O(\varepsilon^2). \tag{3.30}$$

Therefore, from (3.29) it follows that

$$\int_{\mathbb{D}} \psi'_\varepsilon(s')ds' = O(1) + O(\log \varepsilon) = O(\log \varepsilon). \tag{3.31}$$

Next, we put (3.27) and (3.28) into (3.24) to obtain

$$\begin{aligned}
&\frac{2\pi}{\varepsilon} \left(R(x^*, x^*)\Phi(x^*) - \mathcal{G}(x^*) - C'_{\varepsilon,a} - \frac{(H(x^*) - \partial_\nu \phi(x^*))\Phi(x^*) \log \varepsilon}{4\pi} \right) L_a^{-1} 1 \\
&\qquad = \frac{K_a \Phi(x^*)}{2\pi a \varepsilon} L_a^{-1} \mathcal{R} L_a^{-1} 1 - \frac{K_a \Phi(x^*)}{2\pi a \varepsilon} L_a^{-1} T'_\varepsilon L_a^{-1} 1 \\
&\qquad\quad + \psi' - \varepsilon \left(L_a^{-1} \mathcal{R} - L_a^{-1} T'_\varepsilon \right) \psi' + O_{H^{1/2}(\mathbb{D})^*}(\log \varepsilon).
\end{aligned}$$

Therefore, recalling that

$$\|T'_\varepsilon\|_{H^{1/2}(\mathbb{D})^* \to H^{1/2}(\mathbb{D})^*} = O(\varepsilon \log \varepsilon), \quad \|\mathcal{R}\|_{H^{1/2}(\mathbb{D})^* \to H^{3/2}(\mathbb{D})^*} = O(1),$$

$$\|\psi'_\varepsilon\|_{H^{1/2}(\mathbb{D};ds')^*} = O(\varepsilon^{-1}\log\varepsilon),$$

we derive

$$\begin{aligned}\frac{2\pi}{\varepsilon}\left(R(x^*,x^*)\Phi(x^*) - \mathcal{G}(x^*) - C_{\varepsilon,a} - \frac{(H(x^*) - \partial_\nu\phi(x^*))\Phi(x^*)\log\varepsilon}{4\pi}\right) L_a^{-1}1 \\ = \frac{K_a\Phi(x^*)}{2\pi a\varepsilon} L_a^{-1}\mathcal{R}L_a^{-1}1 + \psi'_\varepsilon + O_{H^{1/2}(\mathbb{D})^*}(\log\varepsilon).\end{aligned}$$

Integrating this over $\mathbb{D}$ and using (3.31), (3.26) yields

$$\begin{aligned}C'_{\varepsilon,a} = &- \frac{(H(x^*) - \partial_\nu\phi(x^*))\Phi(x^*)}{4\pi}\log\varepsilon \\ &+ R(x^*,x^*)\Phi(x^*) - \mathcal{G}(x^*) - \frac{K_a^2\Phi(x^*)}{8\pi^3 a}\int_{\mathbb{D}} L_a^{-1}\mathcal{R}L_a^{-1}1(t)dt + O(\varepsilon\log\varepsilon).\end{aligned}$$

Since L_a^{-1} is self-adjoint, we can express the last integral more explicitly:

$$\int_{\mathbb{D}} L_a^{-1}\mathcal{R}L_a^{-1}1(s')ds' = K_a^{-2}\langle(1-|s'|^2)^{-1/2}, \mathcal{R}(1-|s'|^2)^{-1/2}\rangle.$$

Moreover, we have that the following integral is zero

$$\int_{\mathbb{D}} \frac{s_1}{(1-|s'|^2)^{1/2}} \int_{\mathbb{D}} \frac{1}{((s_1-t_1)^2 + a^2(s_2-t_2)^2)^{1/2}} \frac{1}{(1-|t'|^2)^{1/2}} dt'ds'$$

since plugging the following sets of variables into the integral

$$\begin{aligned}(s_1,s_2,t_1,t_2) &= (r\cos\alpha, r\sin\alpha, \rho\cos\beta, \rho\sin\beta), \\ (s_1,s_2,t_1,t_2) &= (-r\cos\alpha, r\sin\alpha, -\rho\cos\beta, \rho\sin\beta).\end{aligned}$$

lead to a difference by a factor of -1, which implies that the integral is 0. Therefore, we know that

$$\int_{\mathbb{D}} L_a^{-1}R_{F,a}L_a^{-1}1(s')ds' = 0.$$

Finally, recalling the definition of $\mathcal{R}$ and the relation between $C'_{\varepsilon,a}$ and $C_{\varepsilon,a}$, we obtain

$$\begin{aligned}C_{\varepsilon,a} =&\frac{K_a\Phi(x^*)}{4\pi^2 a\varepsilon} - \frac{(H(x^*)-\partial_\nu\phi(x^*))\Phi(x^*)}{4\pi}\log\varepsilon \\ &+ R(x^*,x^*)\Phi(x^*) - \mathcal{G}(x^*) \\ &- \frac{(H(x^*)-\partial_\nu\phi(x^*))\Phi(x^*)}{16\pi^3}\int_{\mathbb{D}}\frac{1}{(1-|s'|^2)^{1/2}}\int_{\mathbb{D}}\frac{\log\left((t_1-s_1)^2+a^2(t_2-s_2)^2\right)^{1/2}}{(1-|t'|^2)^{1/2}}dt'ds' \\ &+ \frac{(\kappa_1(x^*)-\kappa_2(x^*))\Phi(x^*)}{64\pi^3}\int_{\mathbb{D}}\frac{1}{(1-|s'|^2)^{1/2}}\int_{\mathbb{D}}\frac{(t_1-s_1)^2-a^2(t_2-s_2)^2}{(t_1-s_1)^2+a^2(t_2-s_2)^2}\frac{1}{(1-|t'|^2)^{1/2}}dt'ds' \\ &+ O(\varepsilon\log\varepsilon).\end{aligned}$$

In case of the disc, that is $a=1$, the last two terms were explicitly calculated in Lemmas 4.5 and 4.6 in [NTT21]. Therefore, we have

$$C_\varepsilon := C_{\varepsilon,1} = \frac{\Phi(x^*)}{4a\varepsilon} - \frac{(H(x^*)-\partial_\nu\phi(x^*))\Phi(x^*)}{4\pi}\log\varepsilon + R(x^*,x^*)\Phi(x^*) - \mathcal{G}(x^*) \tag{3.32}$$

$$-\frac{(H(x^*)-\partial_\nu\phi(x^*))\Phi(x^*)}{4\pi}\left(2\log 2 - \frac{3}{2}\right) + O(\varepsilon\log\varepsilon). \tag{3.33}$$

Next, let us recall that

$$u_{\varepsilon,a}(x) = \mathcal{G}(x) + C_{\varepsilon,a} - \Phi(x^*)G(x^*,x) + r_\varepsilon(x)$$

for each $x\in M\backslash\Gamma_{\epsilon,a}$. Here the remainder r_ε is given by

$$r_\varepsilon(x) = \int_{\partial M}(G(x,y)-G(x,x^*))\partial_\nu u_{\varepsilon,a}(y)d\mu_h(y). \tag{3.34}$$

Let $K\subset\overline{M}$ be a compact subset of $\overline{M}$ which has a positive distance from x^* and consider $x\in K$. Writing out this integral in the $x^\varepsilon(\cdot;x^*)$ coordinate system and using (3.19), (3.18), and the expression of ψ_ε derived in (3.28), we get

$$\begin{aligned} r_\varepsilon(x) &= \varepsilon\int_{\mathbb{D}}\frac{-|M|}{2\pi(1-|s'|^2)^{1/2}}L(x,\varepsilon s')(1+\varepsilon^2Q_\varepsilon(s'))ds' \\ &+ a\varepsilon^3\int_{\mathbb{D}}\psi'_\varepsilon(s')L(x,\varepsilon s')(1+\varepsilon^2Q_\varepsilon(s'))ds'\end{aligned} \tag{3.35}$$

for some function $L(x,s')$ jointly smooth in $(x,s')\in K\times\mathbb{D}$. The second integral formally denotes the duality between $H^{1/2}(\mathbb{D})^*$ and $H^{1/2}(\mathbb{D})$. The estimate for ψ'_ε

derived in (3.28) now gives for any integer k and any compact set K not containing x^*, $\|r_\varepsilon\|_{C^k(K)} \le C_{k,K}\varepsilon$. This gives us the first parts of Theorems 3.1.2 and 3.1.4.

Finally, we compute the average expected value over M. Let us writte

$$v(x) = \int_{\Gamma_{\varepsilon,a}} G(x,y)\partial_\nu u_{\varepsilon,a}(y)d\mu_h(y).$$

Then

$$v(x) = u_{\varepsilon,a}(x) - \mathcal{G}(x) - C_{\varepsilon,a},$$

so that

$$\begin{cases} \Delta_g v(x) + g(F(x), \nabla_g v(x)) = 0; \\ v(x)\mid_{\partial M} = \int_{\Gamma_{\varepsilon,a}} G_{\partial M}(x,y)\partial_\nu u_{\varepsilon,a}(y)d\mu_h(y). \end{cases}$$

Sicne $G_{\partial M} \in \Psi^{-1}_{cl}(\partial M)$ we know that $v(x)\mid_{\partial M} \in H^{1/2}(\partial M)$.

Let $\{f_j\}_{j=1}^\infty$ be a sequence of smooth functions such that $f_j \to \partial_\nu u_{\varepsilon,a}$ in $H^{-1/2}(\partial M)$. Let $\{v_j\}_{j=1}^\infty$ be functions which satisfy

$$\begin{cases} \Delta_g v_j(x) + g(F(x), \nabla_g v_j(x)) = 0; \\ v_j(x)\mid_{\partial M} = \int_{\Gamma_{\epsilon,a}} G_{\partial M}(x,y)f_j(y)d\mu_h(y). \end{cases}$$

Then $v_j \to v$ in $H^1(M)$. Therefore, we compute

$$\begin{aligned} \int_M \int_{\Gamma_{\varepsilon,a}} G(x,y)\partial_\nu u_{\varepsilon,a}(y)d\mu_h(y)d\mu_g(x) &= \lim_{j\to\infty} \int_M \int_{\partial M} G(x,y)f_j(y)d\mu_h(y)d\mu_g(x) \\ &= \lim_{j\to\infty} \int_{\partial M} f_j(y) \int_M G(x,y)d\mu_g(x)d\mu_h(y) \\ &= \int_{\Gamma_{\varepsilon,a}} \partial_\nu u_{\varepsilon,a}(y) \int_M G(x,y)d\mu_g(x)d\mu_h(y). \end{aligned}$$

Recalling (3.9), we derive

$$\begin{aligned} \int_{\Gamma_{\varepsilon,a}} \partial_\nu u_{\varepsilon,a}(y) \int_M G(x,y)d\mu_g(x)d\mu_h(y) &= \int_{\Gamma_{\varepsilon,a}} \partial_\nu u_{\varepsilon,a}(y) \int_M e^{\phi(y)-\phi(x)}G(y,x)d\mu_g(x)d\mu_h(y) \\ &= -\int_M e^{\phi(z)}d\mu_g(z) \int_M e^{-\phi(x)}G(x^*,x)d\mu_g(x) + O(\varepsilon) \\ &= -\Phi(x^*) \int_M e^{\phi(x^*)-\phi(x)}G(x^*,x)d\mu_g(x) + O(\varepsilon) \\ &= -\Phi(x^*) \int_M G(x,x^*)d\mu_g(x) + O(\varepsilon) \end{aligned}$$

Therefore, integrating the expansion for $u_{\varepsilon,a}$ yields

$$\int_M u_{\varepsilon,a}(x)d\mu_g(x) = \int_M \mathcal{G}(x)d\mu_g(x) - \Phi(x^*)\int_M G(x,x^*)d\mu_g(x) + |M|C_{\varepsilon,a}.$$

This gives us the second parts of Theorems 3.1.2 and 3.1.4.

Chapter 4

Eigenvalue Variations

4.1 Introduction

Comprehending the disturbances within physical fields caused by inhomogeneities in a known environment is crucial for various purposes. It helps to understand the robustness of the body's behaviour under small perturbations of its constituent material; see [AK04; BDV22] for more such applications. Mathematically, this task involves performing an asymptotic analysis of the solution of the partial differential equation, when defining domain or properties of the material are slightly perturbed. Many examples of this general question have been studied, including investigations into the conductivity equation [CV03; AS03; CMV98] and other areas, such as linearized elasticity [Amm+02; Ber+12], Maxwell equations [AVV01; Gri11], and cellular biology [HS15; BN13].

Here, we investigate the eigenvalues of the weighted Laplace operator under mixed Dirichlet-Neumann boundary conditions when the Dirichlet region disappears. More precisely, we formulate the problem as follows. Let (M, g) be a compact, connected, orientable Riemannian manifold with smooth non-empty boundary ∂M. Consider the eigenvalue problem

$$-\Delta_g u - g(F, \nabla_g u) = \lambda u, \qquad u|_{\Gamma_{\varepsilon,a}} = 0, \qquad \partial_\nu u|_{\partial M \setminus \Gamma_{\varepsilon,a}} = 0, \tag{4.1}$$

where Δ_g is the negative Laplace-Beltrami operator, ∇_g is the gradient, ν is the outward pointing normal vector field, F is a force field, and $\Gamma_{\varepsilon,a} \subset \partial M$ is the geodesic ellipse considered in the previous chapter and defined in Appendix B. We denote the corresponding operator by $-\Delta^F_{Mix,\varepsilon}$. The objective is to derive an asymptotic of the eigenvalues $\{\lambda_{j,\varepsilon}\}_{j\in\mathbb{N}}$ of $-\Delta^F_{Mix,\varepsilon}$ as ε tends to zero, that is, as $\Gamma_{\varepsilon,a}$ shrinks to a point. We will do this in terms of the spectral parameters of the unperturbed operator, which is the weighted Neumann Laplacian, denoted by $-\Delta^F_N$.

The problem at hand is closely related to the "narrow escape problem," which has gained significant attention in recent years due to its relevance in cellular biology [HS15; BN13]. One has a cavity with a reflecting boundary, except for a small absorbing window. The particles in are modelled as Brownian motions that exit only through this window. The mean sojourn time, which represents the expected duration a particle will wander before escaping, is a crucial metric in this context. The narrow escape problem concerns the asymptotic behaviour of the mean sojourn time as the size of the window tends towards zero.

4.1.1 Previous results

The investigation of the behaviour of eigenvalues of elliptic boundary value problems under the singular perturbation of boundary conditions has a long history, see for instance [AFL20; Amm+12c; Bor03; Bor04; Che05; FNO21; FNO22; Gad99; Gad93; Gad98; GRS16; Nur+22; Pla05; EK22]. Detailed analysis of the two-dimensional planar domain has been performed in [Gad93], where the author provided the full asymptotic expansion of the perturbed eigenvalues. Moreover, a complete pointwise expansion of the perturbed eigenfunctions is provided as well.

In [Che05], the perturbed eigenvalues in a three-dimensional Euclidean domain with a smooth boundary were studied. The author derived the asymptotic behaviour of the perturbed eigenvalues up to an unspecified $o(\varepsilon)$ term:

$$\lambda_{j,\varepsilon} = \lambda_j + 4\pi |u_j(x^*)|^2 c\varepsilon + o(\varepsilon),$$

where $\{\lambda_j\}_{j\in\mathbb{N}}$ and $\{u_j\}_{j\in\mathbb{N}}$ are unperturbed eigenvalues and the corresponding normalized eigenfunctions, c is the constant depending on geometry of Γ_ε.

The problem of perturbed eigenvalues in a domain with a Lipschitz boundary in the Euclidean space was examined by the authors in [FNO21]. They obtained the asymptotic behaviour of the perturbed eigenvalues, expressed in terms of the unperturbed eigenvalues and the relative Sobolev u_j-capacity of Γ_ε:

$$\lambda_{j,\varepsilon} = \lambda_j + \mathrm{Cap}_M(\Gamma_\varepsilon, u_j) + o\left(\mathrm{Cap}_M(\Gamma_\varepsilon, u_j)\right).$$

The case of a two-dimensional planar domain in the presence of a force field was considered in [Amm+12c]. The authors used layer potential techniques to derive the asymptotic expansion

$$\lambda_{j,\varepsilon} = \lambda_j + \pi |u_j(x^*)|^2 e^{\phi(x^*)} \log \varepsilon + O(|\log \varepsilon|^{-2}),$$

where ϕ is a potential, that is $F = \nabla\phi$.

Additionally, we refer to the studies [WK93a; KTW05a; CSW09; CW11; PIW22], which employed the method of matched asymptotic expansions.

4.1.2 Main results

Despite the large number of works on this topic, there are still many questions regarding more general geometries. The present work is devoted to answering this question. In our setting, the Dirichlet region, $\Gamma_{\varepsilon,a}$, is considered to be a small geodesic ellipse centred at $x^* \in \partial M$, with eccentricity $\sqrt{1-a^2}$ and size $\varepsilon \to 0$. Moreover, the force field is given by a smooth up to the boundary potential ϕ, that is $F = \nabla_g \phi$. Our main objective is to investigate how the geometric characteristics of the manifold M at a specific point x^* impact the asymptotic expansion of $\lambda_{j,\varepsilon} - \lambda_j$. We use a combination of the method of layer potentials and the theory of pseudo-differential operators.

In our analysis, it is crucial to understand the singularity structure of the boundary restriction of the Neumann Greens function. The Neumann Greens function in question solves the following problem

$$\begin{cases} \Delta_g G_M^\omega(x,y) - \mathrm{div}_g(F(y) G_M^\omega(x,y)) + \omega^2 G_M^\omega(x,y) = -\delta_x(y), \\ \partial_\nu G_M^\omega(x,y) - g_y(F(y),\nu) G_M^\omega(x,y)\big|_{y\in\partial M} = 0. \end{cases}$$

where ω^2 is an element of the resolvent set of the eigenvalue problem (4.1). Let us consider a formal restriction of G^ω_M to the boundary ∂M. We denote this by the symbol $G^\omega_{\partial M}$. For an exact definition of this restriction, refer to Section 4.3. We obtain the singularity structure of $G^\omega_{\partial M}$ near the diagonal and in a neighbourhood of an eigenvalue of (4.1), when w^2 approaches it. To state it, let us describe the necessary notions. For $x,\ y \in \partial M$, let $H(x)$ denote the mean curvature of the boundary at x, $d_g(x,y)$ the geodesic distance given by metric g, $d_h(x,y)$ the geodesic distance given by induced metric h on the boundary ∂M, and

$$\mathrm{II}_x(V) := \mathrm{II}(V,V), \qquad V \in T_x\partial M,$$

the scalar second fundamental form (see pages 235 and 381 of [Lee18] for definitions).

Proposition 4.1.1. *Let $(M, g, \partial M)$ be a compact connected orientable Riemannian manifold of dimension three with a non-empty smooth boundary. Let λ_j be a simple eigenvalue of $-\Delta^F_N$ and V_j be a neighbourhood of λ_j which does not contain any other eigenvalue of $-\Delta^F_N$. Then there exists a neighbourhood of*

$$\mathrm{Diag} := \{(x,x) \in \partial M \times \partial M\}$$

where the singularity structure of $G^\omega_{\partial M}$ given by:

$$\begin{aligned} G^\omega_{\partial M}(x,y) &= \frac{1}{2\pi} d_g(x,y)^{-1} - \frac{H(x)}{4\pi}\log d_h(x,y) + \frac{g_x(F,\nu)}{4\pi}\log d_h(x,y) \\ &+ \frac{1}{16\pi}\left(\mathrm{II}_x\left(\frac{\exp^{-1}_{x;h}(y)}{|\exp^{-1}_{x;h}(y)|_h}\right) - \mathrm{II}_x\left(\frac{\star\exp^{-1}_{x;h}(y)}{|\exp^{-1}_{x;h}(y)|_h}\right)\right) \\ &+ \frac{1}{4\pi} h_x\left(F^{||}(x), \frac{\exp^{-1}_{x;h}(y)}{|\exp^{-1}_{x;h}(y)|_h}\right) \\ &+ \frac{u_j(x)u_j(y)}{\lambda_j - \omega^2} e^{\phi(y)} + R^\omega_{\partial M}(x,y), \end{aligned} \tag{4.2}$$

where $\omega^2 \in V_j \setminus \{\lambda_j\}$, $R^\omega_{\partial M}(x,y) \in C^{0,\alpha}(\partial M \times \partial M)$ for $\alpha \in (0,1)$, $F^{||}$ denotes the tangential component of F and $\star$ denotes the Hodge star operator.

In deriving the singularity structure, we employed a standard pseudo-differential parametrix construction as described in [NTT21; NTT22] by observing that $G^\omega_{\partial M}$ is an approximate inverse to a Dirichlet-to-Neumann map. To determine the singularity with respect to ω near the eigenvalues of $-\Delta_N^F$, we have used an approach similar to that of [AKL09]. Using the aforementioned proposition, we derive an asymptotic expression for the eigenvalues $\lambda_{j,\varepsilon}$ as $\varepsilon \to 0$. For the sake of clarity, we begin by presenting the main result for $\Gamma_{\varepsilon,a}$ being a geodesic ball, that is $a = 1$:

Theorem 4.1.2. *Let $(M, g, \partial M)$ be a compact connected orientable Riemannian manifold of dimension three with a non-empty smooth boundary. Fix $x^* \in \partial M$ and let Γ_ε be the boundary geodesic ball centred at x^* of geodesic radius $\varepsilon > 0$. Assume that $F = \nabla_g \phi$ for a potential ϕ smooth up to the boundary. Let $\{\lambda_{j,\varepsilon}\}_{j\in\mathbb{N}}$ be the eigenvalues of $-\Delta_{Mix,\varepsilon}^F$. If λ_j is a simple eigenvalue of $-\Delta_N^F$ and u_j is the corresponding eigenfunction normalized in $L^2(M, e^\phi d\mu_g)$ (weighted L^2 space with a weight e^ϕ), then*

$$\lambda_{j,\varepsilon} - \lambda_j = A\varepsilon + B\varepsilon^2 \log \varepsilon + C\varepsilon^2 + O(\varepsilon^3 \log^2 \varepsilon),$$

where

$$A = 4|u_j(x^*)|^2 e^{\phi(x^*)},$$

$$B = 4\pi |u_j(x^*)|^2 e^{\phi(x^*)} (H(x^*) - \partial_\nu \phi(x^*)),$$

$$C = |u_j(x^*)|^2 e^{\phi(x^*)} \left(\frac{8 \log 2 - 6}{\pi} (H(x^*) - \partial_\nu \phi(x^*)) - 16 R_{\partial M}^{\lambda_j}(x^*, x^*) \right).$$

Here, H is the mean curvature of the boundary, $R_{\partial M}^{\lambda_j}(x^*, x^*)$ is the evaluation at $(x, y) = (x^*, x^*)$ of the kernel $R_{\partial M}^{\lambda_j}(x, y)$ in Proposition 4.1.1.

Theorem 4.1.2 does not realize the full power of Proposition 4.1.1 as it does not see the inhomogeneity of the local geometry at x^*, only the mean curvature shows up. This limitation arises from the fact that the shrinking subset of ∂M assigned with the Dirichlet boundary condition is a geodesic ball. However, by considering geodesic ellipses instead of geodesic balls, we observe that the inclusion of the scalar second fundamental form term in Proposition 4.1.1 contributes to the appearance of the difference of principal curvatures in the asymptotic expansion of $\lambda_{j,\varepsilon} - \lambda_j$. The ellipse, we consider, is defined as follows. Let $E_1(x^*)$, $E_2(x^*) \in T_{x^*}\partial M$ be the unit eigenvectors of the shape operator at x^* corresponding respectively to principal curvatures $\kappa_1(x^*)$ and $\kappa_2(x^*)$. For $a \in (0,1]$ fixed, we set

$$\Gamma_{\varepsilon,a} := \{\exp_{x^*;h}(\varepsilon t_1 E_1 + \varepsilon t_2 E_2) \mid t_1^2 + a^{-2} t_2^2 \leq 1\}. \tag{4.3}$$

Now, we are ready to state the main result:

Theorem 4.1.3. *Let $(M, g, \partial M)$ be a compact connected orientable Riemannian manifold of dimension three with a non-empty smooth boundary. Fix $x^* \in \partial M$ and let $\Gamma_{\varepsilon,a}$ be the boundary geodesic ellipse given by (4.3). Assume that $F = \nabla_g \phi$ for a potential ϕ smooth up to the boundary. Let $\{\lambda_{j,\varepsilon}\}_{j\in\mathbb{N}}$ be the eigenvalues of $-\Delta^F_{Mix,\varepsilon}$. If λ_j is a simple eigenvalue of $-\Delta^F_N$ and u_j is the corresponding eigenfunction normalized in $L^2(M, e^\phi d\mu_g)$, then*

$$\lambda_{j,\varepsilon} - \lambda_j = A\varepsilon + B\varepsilon^2 \log \varepsilon + (C_1 + C_2 + C_3)\varepsilon^2 + O(\varepsilon^3 \log^2 \varepsilon).$$

Here, the constants are defined as follows

$$\begin{aligned} K_a &= \frac{\pi}{2} \int_0^{2\pi} \frac{a}{(a^2 \cos^2\theta + \sin^2\theta)^{1/2}} d\theta \\ A &= \frac{4\pi^2 a}{K_a} |u_j(x^*)|^2 e^{\phi(x^*)}, \\ B &= \frac{4\pi^3 a^2 |u_j(x^*)|^2 e^{\phi(x^*)}}{K_a^2} (H(x^*) - \partial_\nu \phi(x^*)) \end{aligned}$$

and

$$\begin{aligned} C_1 = \frac{a^2\pi(H(x^*) - \partial_\nu\phi(x^*))|u_j(x^*)|^2 e^{\phi(x^*)}}{K_a^2} \times \\ \times \int_{\mathbb{D}} \frac{1}{(1-|s'|^2)^{1/2}} \int_{\mathbb{D}} \frac{\log\left((t_1-s_1)^2 + a^2(t_2-s_2)^2\right)^{1/2}}{(1-|t'|^2)^{1/2}} dt' ds', \end{aligned}$$

$$\begin{aligned} C_2 = \frac{a^2\pi\left(\kappa_1(x^*) - \kappa_2(x^*)\right)|u_j(x^*)|^2 e^{\phi(x^*)}}{4K_a^2} \times \\ \times \int_{\mathbb{D}} \frac{1}{(1-|s'|^2)^{1/2}} \int_{\mathbb{D}} \frac{(t_1-s_1)^2 - a^2(t_2-s_2)^2}{(t_1-s_1)^2 + a^2(t_2-s_2)^2} \frac{1}{(1-|t'|^2)^{1/2}} dt' ds' \end{aligned}$$

$$C_3 = \frac{16\pi^4 a^2 R^{\lambda_j}_{\partial M}(x^*, x^*)|u_j(x^*)|^2 e^{\phi(x^*)}}{K_a^2},$$

where $R^{\lambda_j}_{\partial M}(x^*, x^*)$ is the evaluation at $(x, y) = (x^*, x^*)$ of the kernel $R^{\lambda_j}_{\partial M}(x, y)$ in Proposition 4.1.1.

From this result, we can conclude that the shape of the Dirichlet region is important. The eccentricity of the ellipse affects the main term of the asymptotic. Moreover, we observe a difference in the principal curvatures, which is not visible in the $a = 1$ case.

4.2 Formulation of the problem

Now we are ready to state the problem. Let us consider the operator

$$u \to \Delta_g u + g(F, \nabla_g u),$$

where F is a force field, which is given by $F = \nabla_g \phi$ for a smooth up to the boundary potential ϕ. We can re-write this operator in the following way

$$\Delta_g^F \cdot := \Delta_g \cdot + g(F, \nabla_g \cdot) = \frac{1}{e^{\phi}} \mathrm{div}_g(e^{\phi} \nabla_g \cdot).$$

According to [GS02], the operator Δ_g^F is called a weighted Laplacian and the pair $(M, e^{\phi} d\mu_g)$ is called a weighted manifold. Note that e^{ϕ} is bounded and strictly positive on M. Therefore, $L^2(M) = L^2(M, e^{\phi} d\mu_g)$ as sets with equivalent norms. We also note that the operator Δ_g^F with initial domain $C_0^\infty(M)$ is essentially self-adjoint in $L^2(M, e^{\phi} d\mu_g)$ and non-positive definite. We want to study the operator Δ_g^F with Dirichlet and Neumann boundary conditions on $\Gamma_{\varepsilon,a}$ and $\partial M \setminus \Gamma_{\varepsilon,a}$, respectively. This operator can be defined via quadratic form as follows. Consider the quadratic form

$$a_\varepsilon(u, v) := \int_M e^{\phi(z)} g(\nabla_g u(z), \nabla_g u(z)) d\mu_g(z),$$

with the domain

$$\mathrm{D}(a_\varepsilon) := \{u \in H^1(M) : \mathrm{supp}(u|_{\partial M} \subset \partial M \setminus \Gamma_{\varepsilon,a}\}.$$

Since e^{ϕ} is strictly positive and bounded on M, this quadratic form is non-negative, closed, symmetric and densely defined in $L^2(M, e^{\phi} d\mu_g)$, and hence, generates the self-adjoint non-negative operator; see Theorem 2.6 in [Kat95, Ch. 6.2]. We denote this operator by $-\Delta^F_{Mix,\varepsilon}$ and call it a weighted Laplace operator corresponding to the aforementioned mixed boundary conditions.

Since $H^1(M)$ is compactly embedded in $L^2(M, e^{\phi} d\mu_g)$, the spectrum of $-\Delta^F_{Mix,\varepsilon}$ is discrete and consists of the eigenvalues with finite multiplicity accumulating at infinity. We denote them, taking into account their multiplicities, as follows

$$0 \leq \lambda_{1,\varepsilon} \leq \lambda_{2,\varepsilon} \leq \cdots < \infty.$$

The corresponding normalized, in the $L^2(M, e^{\phi} d\mu_g)$ sense, eigenfunctions are denoted by $\{u_{j,\varepsilon}\}_{j\in\mathbb{N}}$.

We also consider the operator $-\Delta^F_g$ with a Neumann boundary condition, which is generated by the quadratic form

$$a_N(u,u) := \int_M e^{\phi(z)} g(\nabla_g u(z), \nabla_g u(z)) d\mu_g(z), \qquad \text{with } \mathrm{D}(a_N) = H^1(M).$$

We denote this operator by $-\Delta^F_N$. By the same arguments above, we conclude that the spectrum $\mathrm{spec}(-\Delta^F_N)$ is discrete and consists of eigenvalues with finite multiplicity accumulating at infinity. We donate them by

$$0 = \lambda_1 \leq \lambda_2 \leq \cdots < \infty.$$

By $\{u_j\}_{j\in\mathbb{N}}$, we denote the corresponding normalized, in the $L^2(M, e^{\phi} d\mu_g)$ sense, eigenfunctions. We aim to derive an asymptotic expansion for $\lambda_{j,\varepsilon}$ as $\varepsilon \to 0$ in terms of λ_j and u_j.

4.3 Neumann Greens function

In this section, we consider the Greens function G^ω_M defined as the solution (in the distributional sense) to the following boundary value problem

$$\begin{cases} \Delta_g G^\omega_M(x,y) - \mathrm{div}_g(F(y) G^\omega_M(x,y)) + \omega^2 G^\omega_M(x,y) = -\delta_x(y) \ \text{ for } \ y \in M, \\ \partial_\nu G^\omega_M(x,y) - g_y(F(y), \nu) G^\omega_M(x,y) = 0 \ \text{ for } \ y \in \partial M. \end{cases}$$

where ω^2 is a parameter belonging to the resolvent set of the operator $-\Delta_N^F$. We seek the singularity structure of ∂M-restriction of G_M^ω near the diagonal. A more precise definition of this restriction will be given later. Our main aim is to obtain Proposition 4.1.1.

4.3.1 Singularity in the spectral parameter

The first step in our analysis is to express G_M^ω as a decomposition of Neumann eigenfunctions. The following result is similar to a result of [AKL09]. We modify it here for our setting

Proposition 4.3.1. *Let $\{\lambda_j\}_{j\in\mathbb{N}}$ be the eigenvalues of $-\Delta_N^F$ and $\{u_j\}_{j\in\mathbb{N}}$ be the corresponding $L^2(M, e^\phi d\mu_g)$-orthonormalized eigenfunctions. Then, for $x \neq y$ and $\omega^2 \in \mathbb{C} \setminus \operatorname{spec}(-\Delta_N^F)$, it follows that*

$$G_M^\omega(x,y) = \sum_{j=1}^{\infty} \frac{u_j(x)u_j(y)}{\lambda_j - \omega^2} e^{\phi(y)}.$$

Proof. Since $(M, g, \partial M)$ is assumed to be compact, it follows that e^ϕ is bounded and strictly positive on M. Therefore, $L^2(M) = L^2(M, e^\phi d\mu_g)$ with equivalent norms. Since $G_M^\omega(x,\cdot) \in L^2(M, e^\phi d\mu_g)$, for any $x \in M$, we can express

$$G_M^\omega(x,y) = \sum_{j=1}^{\infty} v_j(x)u_j(y)e^{\phi(y)}.$$

Since $e^\phi > 0$ is bounded, it follows that $f \in L^2(M)$ if and only if $e^\phi f \in L^2(M)$. Thus, for fixed $x \in M$, the above expression for $G_M^\omega(x,y)$ is unique. Greens identity in conjunction with the divergence theorem as well as the boundary condition on u_j yields the following calculation

$$\begin{aligned}
u_j(x) &= \int_M \left(-\Delta_g G_M^\omega(x,y) + \operatorname{div}_y(F(y)G_M^\omega(x,y)) - \omega^2 G_M^\omega(x,y)\right) u_j(y)d\mu_g(y), \\
&= (\lambda_j - \omega^2)\sum_{k=1}^{\infty} \int_M v_k(x)u_k(y)u_j(y)e^{\phi(y)}d\mu_g(y), \\
&= (\lambda_j - \omega^2)v_j(x)\int_M |u_j(y)|^2 e^{\phi(y)}d\mu_g(y).
\end{aligned}$$

Recall that u_j are $L^2(M, e^{\phi} d\mu_g)$-orthonormalized so that $v_j(x) = u_j(x)/(\lambda_j - \omega^2)$, which implies that

$$G_M^\omega(x,y) = \sum_{j=1}^{\infty} \frac{u_j(x)u_j(y)}{\lambda_j - \omega^2} e^{\phi(y)},$$

as required. $\square$

Let λ_j be a simple eigenvalue of $-\Delta_N^F$ and V_j be an open bounded neighborhood of λ_j in $\mathbb{C}$ such that $V_j \cap \operatorname{spec}(-\Delta_N^F) = \{\lambda_j\}$. Within Section 4.3, we are interested in deriving explicit asymptotics for the *trace* of G_M^ω. To this end, we write G_M^ω as

$$\begin{aligned} G_M^\omega(x,y) &= \sum_{k \neq j} \frac{u_k(x)u_k(y)}{\lambda_k - \omega^2} e^{\phi(y)} + \frac{u_j(x)u_j(y)}{\lambda_j - \omega^2} e^{\phi(y)}, \\ &:= N_M^\omega(x,y) + \frac{u_j(x)u_j(y)}{\lambda_j - \omega^2} e^{\phi(y)}, \end{aligned}$$

for $\omega^2 \in V_j$. From here we can then define $G_{\partial M}^\omega$ and $N_{\partial M}^\omega$ as the boundary restrictions of G_M^ω and N_M^ω as Schwartz kernels of the trace of the integral operators G_M^ω and N_M^ω respectively. That is, for $f \in C^\infty(\partial M)$, we have

$$\begin{aligned} G_{\partial M}^\omega : f &\mapsto \left(\int_{\partial M} G_M^\omega(x,y) f(y) d\mu_h(y) \right)\Big|_{x \in \partial M}, \\ N_{\partial M}^\omega : f &\mapsto \left(\int_{\partial M} N_M^\omega(x,y) f(y) d\mu_h(y) \right)\Big|_{x \in \partial M}. \end{aligned}$$

It can also be seen from the perspective of microlocalization that the above restrictions to $\partial M \times \partial M$ are well-defined since G_M^ω is the Schwartz kernel of a pseudo-differential operator with $\mathrm{WF}(G_M^\omega) \cap N^*(\partial M \times \partial M) = \varnothing$ and the difference between G_M^ω and N_M^ω is a $C^\infty(\partial M \times \partial M)$ term. Thus, we write the trace of G_M^ω as

$$G_{\partial M}^\omega(x,y) = N_{\partial M}^\omega(x,y) + \frac{u_j(x)u_j(y)}{\lambda_j - \omega^2} e^{\phi(y)}. \tag{4.4}$$

Our choice of N_M^ω suggests that any terms which depend on ω^2 in $N_{\partial M}^\omega$ are negligible, since, by definition $\omega^2 \in V_j \setminus \{\lambda_j\}$ and V_j is judiciously chosen such that there are no other Neumann eigenvalues in V_j. This implies that there is only one significant singularity in ω^2 which is given by the second term of (4.4).

4.3.2 Singularities along the diagonal

When considering (4.4), it is apparent that $\frac{u_j(x)u_j(y)}{\lambda_j-\omega^2}e^{\phi(y)}$ is jointly smooth on $\partial M \times \partial M$, thus the only difficulty in deriving asymptotics for $G^\omega_{\partial M}$ for x near y lies in the derivation of $N^\omega_{\partial M}$. We will show that $N^\omega_{\partial M}$ is a left parametrix for a Dirichlet-to-Neumann map, which is associated to the following auxiliary Dirichlet boundary value problem

$$\begin{cases} \Delta_g u_f + g(F, \nabla_g u_f) + \omega^2 u_f = 0, \\ u_f|_{\partial M} = f \in C^\infty(\partial M). \end{cases} \tag{4.5}$$

The Dirichlet-to-Neumann map is given by

$$\Lambda^\omega_{g,F} : H^{1/2}(\partial M) \ni f \mapsto \partial_\nu u_f \in H^{1/2}(\partial M)^*.$$

Using the same arguments as those in Appendix C, it can be easily shown that $\Lambda^\omega_{g,F}$ is an elliptic, first-order, classical pseudo-differential operator. In addition we can construct a symbol for $\Lambda^\omega_{g,F}$, which is essentially identical to that of $\Lambda_{g,F}$, the DtN map considered in Chapter 4.

Lemma 4.3.2. *The Dirichlet-to-Neumann map $\Lambda^\omega_{g,F}$ and $N^\omega_{\partial M}$ satisfy the following operator equation*

$$I = N^\omega_{\partial M}\Lambda^\omega_{g,F} + \Psi^{-\infty}, \tag{4.6}$$

where $\Psi^{-\infty}$ denotes the class of smoothing operators. In particular, $N^\omega_{\partial M} \in \Psi^{-1}_{cl}$ is an elliptic pseudo-differential operator.

Proof. We prove the above lemma by integrating by parts (4.5) against the Neumann Greens function $G^\omega_M(x, y)$

$$\begin{aligned} -u_f(x) = &\int_M G^\omega_M(x,z)\Delta_g u_f d\mu_g(z) + \int_{\partial M} \left(u_f(z)\partial_\nu G^\omega_M(x,z) - G^\omega_M(x,z)\partial_\nu u_f(z)\right) d\mu_h(z) \\ &- \int_M u_f(z)\mathrm{div}_g(F(z)G^\omega_M(x,z))d\mu_g(z) + \omega^2 \int_M u_f(z)G^\omega_M(x,z)d\mu_g(z). \end{aligned}$$

The divergence theorem yields

$$\begin{aligned}-u_f(x) &= \int_M G_M^\omega(x,z)\Delta_g u_f d\mu_g(z) + \int_{\partial M} u_f(z)\partial_\nu G_M^\omega(x,z) - G_M^\omega(x,z)\partial_\nu u_f(z)d\mu_h(z)\\ &+ \int_M g_z(F(z),\nabla_g u_f)G_M^\omega(x,z)d\mu_g(z) - \int_{\partial M} u_f(z)G_M^\omega(x,z)F(z)\cdot\nu d\mu_h(z)\\ &+ \omega^2\int_M u_f(z)G_M^\omega(x,z)d\mu_g(z).\end{aligned}$$

Employing the prescribed Neumann boundary conditions on G_M^ω, we conclude

$$u_f(x) = \int_{\partial M} G_M^\omega(x,z)\partial_\nu u_f(z)d\mu_h(z).$$

Furthermore, restricting $x \in \partial M$ and invoking the prescribed Dirichlet boundary condition from (4.5) using our definition involving the trace (4.4), we have that

$$\begin{aligned}f(x) &= \int_{\partial M} G_{\partial M}^\omega(x,z)\Lambda_{g,F}^\omega f(z)d\mu_h(y),\\ &= \int_{\partial M} N_{\partial M}^\omega(x,z)\Lambda_{g,F}^\omega f(z)d\mu_h(z) + \int_{\partial M} e^{\phi(z)}\frac{u_j(x)u_j(z)}{\lambda_j-\omega^2}\Lambda_{g,F}^\omega f(z)d\mu_h(z).\end{aligned}$$

Since the Neumann eigenfunctions are smooth, the rightmost integral in the above expression consists of a smooth Schwartz kernel and thus gives rise to a smoothing operator. That is, we have

$$I = N_{\partial M}^\omega \Lambda_{g,F}^\omega + \Psi^{-\infty}$$

Finally, since $\Lambda_{g,F}^\omega \in \Psi_{cl}^1$ is an elliptic operator, we conclude that $N_{\partial M}^\omega \in \Psi_{cl}^{-1}$. □

Using the first two symbols, derived from the arguments used in Appendix C, in conjunction with 4.3.2, we can prove the following proposition by iteratively determining the terms in the Borel summation associated with the parametrix $N_{\partial M}^\omega$ modulo smoothing terms.

Proposition 4.3.3. *Let $x, y \in \partial M$ such that $x \neq y$ and $\omega^2 \in \mathbb{C}\setminus \operatorname{spec}(-\Delta_N^F)$. In an open neighbourhood of* $\mathrm{Diag} := \{(x,x) \in \partial M \times \partial M\}$, *we have that*

$$\begin{aligned}N_{\partial M}^\omega(x,y) &= \frac{1}{2\pi}d_g(x,y)^{-1} - \frac{H(x)}{4\pi}\log d_h(x,y) + \frac{g_x(F,\nu)}{4\pi}\log d_h(x,y)\\ &+ \frac{1}{16\pi}\left(\mathrm{II}_x\left(\frac{\exp_x^{-1}(y)}{|\exp_x^{-1}(y)|_h}\right) - \mathrm{II}_x\left(\frac{\star\exp_x^{-1}(y)}{|\exp_x^{-1}(y)|_h}\right)\right)\\ &+ \frac{1}{4\pi}h_x\left(F^{||}(x), \frac{\exp_x^{-1}(y)}{|\exp_x^{-1}(y)|_h}\right) + R_{\partial M}^\omega(x,y),\end{aligned}$$

where $R^{\omega}_{\partial M}(x,y) \in C^{0,\alpha}(\partial M \times \partial M)$ *for* $\alpha \in (0,1)$, *and* $F^{||}$ *denotes the tangential component of* F *and* $\star$ *denotes the Hodge star operator.*

Within the above expression for $N^{\omega}_{\partial M}$, we obtain a clear image as to the structure of the singularity in $x, y \in \partial M$ near the diagonal (where $x = y$) as well as the singularity in $\omega^2 \in \mathbb{C} \setminus \operatorname{spec}(-\Delta^F_N)$ for ω^2 near λ_j. For the sake of clarity, we will include an outline for the proof of Proposition 4.3.3. Since we have already determined the nature of the leading order singularity in ω, all that is left is to reveal the nature of the singularity of the leading order terms for x near y on ∂M.

Proof of Proposition 4.3.3. There are infinitely many additional terms in the asymptotic series of $N^{\omega}_{\partial M}$, these are formed via an iterative argument on the level of symbols. However, for our purposes, we only need the first two elements of the kernel expansion and thus, we only need the first two symbols in the Borel expansion of $\sigma(N^{\omega}_{\partial M})$. Upon deriving $\sigma_{-1}(N^{\omega}_{\partial M})$ and $\sigma_{-2}(N^{\omega}_{\partial M})$ An expression for the asymptotic series of the Schwartz kernel is given by the fourier transform of $\sigma_{-1}(N^{\omega}_{\partial M}) + \sigma_{-2}(N^{\omega}_{\partial M})$. To begin this iterative process, we view the operator equation (4.6) on the level of symbols.

$$1 = \sigma(N^{\omega}_{\partial M}) \# \sigma(\Lambda^{\omega}_{g,F})(x,\xi') + S^{-\infty}, \tag{4.7}$$

where $\#$ denotes the standard composition of symbols in correspondence to the composition of pseudo-differential operators. Furthermore, we write $\sigma(N^{\omega}_{\partial M})(t,\xi')$ in terms of the following asymptotic series, whose existence is guaranteed by Borel's lemma

$$\sigma(N^{\omega}_{\partial M})(x,\xi') \sim \sum_{j\geq 1} \sigma_{-j}(N^{\omega}_{\partial M})(t,\xi'), \quad \sigma_{-j}(N^{\omega}_{\partial M})(t,\xi') \in S^{-j}_{1,0}.$$

Equation (4.7) becomes the following in accordance to the formula for the #-product

$$1 = \sum_{\gamma} \frac{1}{\gamma!} \partial^{\gamma}_{\xi'} \sigma(N^{\omega}_{\partial M}) D^{\gamma}_t \sigma(\Lambda^{\omega}_{g,F}) + S^{-\infty}.$$

where $\gamma \in \mathbb{N}^n$ denotes a multi-index. In the first iteration, we have that for a smooth function χ and $R > 0$ with $\chi(\xi') = 0$ for $|\xi'| \leq R$ and $\chi(\xi') = 1$ for $|\xi'| \geq 2R$

$$1 = \sigma_{-1}(N^{\omega}_{\partial M})(t,\xi')\sigma_1(\Lambda^{\omega}_{g,F})(t,\xi') + S^{-1}_{1,0} \implies \sigma_{-1}(N^{\omega}_{\partial M})(t,\xi') = \frac{\chi(\xi')}{\sigma_1(\Lambda^{\omega}_{g,F})(t,\xi')}.$$

We can further iterate for the second term by forming the following equation

$$\begin{aligned} 1 &= \sigma_{-1}(N^{\omega}_{\partial M})(t,\xi')\sigma_1(\Lambda^{\omega}_{g,F})(t,\xi') + \sigma_{-1}(N^{\omega}_{\partial M})(t,\xi')\sigma_0(\Lambda^{\omega}_{g,F})(t,\xi'), \\ &+ \sigma_{-2}(N^{\omega}_{\partial M})(t,\xi')\sigma_1(\Lambda^{\omega}_{g,F})(t,\xi') + \nabla_{\xi'}\sigma_{-1}(N^{\omega}_{\partial M}) \cdot D_{t'}\sigma_1(\Lambda^{\omega}_{g,F}) + S^{-2}_{1,0}. \end{aligned}$$

We now equate terms of symbol order -1 to obtain the following equation

$$\begin{aligned} 0 &= \sigma_{-1}(N^{\omega}_{\partial M})(t,\xi')\sigma_0(\Lambda^{\omega}_{g,F})(t,\xi') + \sigma_{-2}(N^{\omega}_{\partial M})(t,\xi')\sigma_1(\Lambda^{\omega}_{g,F})(t,\xi') \\ &+ \nabla_{\xi'}\sigma_{-1}(N^{\omega}_{\partial M}) \cdot D_{t'}\sigma_1(\Lambda^{\omega}_{g,F}). \end{aligned}$$

So, we choose $\sigma_{-2}(N^{\omega}_{\partial M})(t,\xi')$ as follows

$$\begin{aligned} \sigma_{-2}(N^{\omega}_{\partial M})(t,\xi') = &- \frac{\chi(\xi')}{\sigma_1(\Lambda^{\omega}_{g,F})(t,\xi')}\sigma_{-1}(N^{\omega}_{\partial M})(t,\xi')\sigma_0(\Lambda^{\omega}_{g,F})(t,\xi') \\ &- \frac{\chi(\xi')}{\sigma_1(\Lambda^{\omega}_{g,F})(t,\xi')}\nabla_{\xi'}\sigma_{-1}(N^{\omega}_{\partial M}) \cdot D_{t'}\sigma_1(\Lambda^{\omega}_{g,F}). \end{aligned}$$

Thus, we have that the Schwartz kernel of the ∂M-restricted Greens function, when evaluated at the center of the h-geodesic disc $\Gamma_{\varepsilon,1}$, up to Ψ^{-3}, when written in local co-ordinates is given by

$$\frac{1}{4\pi^2}\left(\int_{\mathbb{R}^2} e^{-i\xi'\cdot t'}\sigma_{-1}(N^{\omega}_{\partial M})(0,\xi')d\xi' + \int_{\mathbb{R}^2} e^{-i\xi'\cdot t'}\sigma_{-2}(N^{\omega}_{\partial M})(0,\xi')d\xi'\right).$$

This results in the desired singular expansion for the boundary-restricted Greens function fixed at a central point x^*. It can then be extended via a series of estimates derived in [NTT21] to $N^{\omega}_{\partial M}(x,y)$ for $x \neq y$, but suitably close (See Appendix D for full calculation). $\square$

Finally, we note that Proposition 4.1.1 follows as a trivial consequence of Proposition 4.3.3 and relation (4.4)

4.3.3 Schwartz kernel estimates

Within this section, we investigate $G^{\omega}_{\partial M}$ near x^*, in the local coordinates given by (B.1). We introduce several integral operators related to the terms on the right-hand side of (4.2). First, we consider a weighted variant of the normal operator

$$L_a f = a\int_{\mathbb{D}} \frac{f(s')}{((t_1-s_1)^2 + a^2(t_2-s_2)^2)^{1/2}}ds' \tag{4.8}$$

acting on functions on the disk $\mathbb{D}$. It is known that L_a is a self-adjoint operator; see for instance Section 4 in [NTT21]. Moreover, by [SSH06a], it follows that

$$L_a\left({K_a}^{-1}(1-|t'|^2)^{-1/2}\right)=1, \quad K_a=\frac{\pi}{2}\int_0^{2\pi}\left(\cos^2\theta+\frac{\sin^2\theta}{a^2}\right)^{-1/2}d\theta. \tag{4.9}$$

By (4.4) in [NTT21], it was shown that $u(t')=K_a^{-1}(1-|t'|^2)^{-1/2}$ is the unique solution to $L_a u=1$ in $H^{1/2}(\mathbb{D})^*$. Next, we introduce the operators $R_{\log,a}$, $R_{\infty,a}$, $R_{F,a}$ and $R_{I,a}$. These are defined as follows

$$\begin{aligned}
R_{\log,a}f(t') &:= a\int_{\mathbb{D}}\log\left((t_1-s_1)^2+a^2(t_2-s_2)^2\right)^{1/2}f(s')ds',\\
R_{\infty,a}f(t') &:= a\int_{\mathbb{D}}\frac{(t_1-s_1)^2-a^2(t_2-s_2)^2}{(t_1-s_1)^2+a^2(t_2-s_2)^2}f(s')ds',\\
R_{F,a}f(t') &:= a\int_{\mathbb{D}}\frac{F_1(0)(t_1-s_1)+aF_2(0)(t_2-s_2)}{((t_1-s_1)^2+a^2(t_2-s_2)^2)^{1/2}}f(s')ds',\\
R_{I,a}f(t') &:= a\int_{\mathbb{D}}f(s')ds'.
\end{aligned}$$

Remark 4.3.4. In [NTT21], it was shown that the operators $R_{\log,a}$ and $R_{\infty,a}$ are bounded maps from $H^{1/2}(\mathbb{D})^*$ to $H^{3/2}(\mathbb{D})$. Repeating the arguments shows that this is also true for $R_{F,a}$.

Note that these lemmas are proved in [NTT21; NTT22]. We state them here for the convenience of the reader.

Lemma 4.3.5. *We have the following identity*

$$\int_{\Gamma_{\varepsilon,a}} d_g(x,y)^{-1}v(y)d\mu_h(y)=\varepsilon L_a\tilde{v}(t')+\varepsilon^3\mathcal{A}_\varepsilon\tilde{v}(t'),$$

where $x=x^\varepsilon(t')$, $\tilde{v}(t')=v(x^\varepsilon(t'))$, *for some* $\mathcal{A}_\varepsilon : H^{1/2}(\mathbb{D};ds')^*\to H^{1/2}(\mathbb{D};ds')$ *with operator norm bounded uniformly in* ε.

From now on, we will denote by $\mathcal{A}_\varepsilon$ any operator which takes

$$\mathcal{A}_\varepsilon : H^{1/2}(\mathbb{D};ds')^*\to H^{1/2}(\mathbb{D};ds'),$$

whose operator norm is bounded uniformly in ε.

Lemma 4.3.6. *The following identity holds*

$$(H(x) - g_x(F,\nu)) \int_{\Gamma_{\varepsilon,a}} \log d_h(x,y) v(y) d\mu_h(y)$$
$$= \varepsilon^2 \log\varepsilon (H(x^*) - \partial_\nu \phi(x^*)) R_I \tilde{v}(t') + \varepsilon^2 (H(x^*) - \partial_\nu \phi(x^*)) R_{\log,a} \tilde{v}(t') + \varepsilon^3 \log\varepsilon \mathcal{A}_\varepsilon \tilde{v}(t'),$$

where $x = x^\varepsilon(t')$ *and* $\tilde{v}(t') = v(x^\varepsilon(t'))$.

Lemma 4.3.7. *The following identity holds*

$$\int_{\Gamma_{\varepsilon,a}} \left(\mathrm{II}_x \left(\frac{\exp_x^{-1}(y)}{|\exp_x^{-1}(y)|_h} \right) - \mathrm{II}_x \left(\frac{\star \exp_x^{-1}(y)}{|\exp_x^{-1}(y)|_h} \right) \right) v(y) d\mu_h(y)$$
$$= \varepsilon^2 (\kappa_1(x^*) - \kappa_2(x^*)) R_{\infty,a} \tilde{v}(t') + \varepsilon^3 \mathcal{A}_\varepsilon \tilde{v}(t'),$$

where $x = x^\varepsilon(t')$ *and* $\tilde{v}(t') = v(x^\varepsilon(t'))$. *Recall that* $\kappa_1(x^*)$ *and* $\kappa_2(x^*)$ *denote the principal curvatures of the boundary* ∂M *at* x^*.

Lemma 4.3.8. *The following identity holds*

$$\int_{\Gamma_{\varepsilon,a}} h_x \left(F^{\parallel}(x), \frac{\exp_{x;h}(y)}{|\exp_{x;h}(y)|_h} \right) v(y) d\mu_h(y) = \varepsilon^2 R_{F,a} \tilde{v}(t') + \varepsilon^3 \mathcal{A}_\varepsilon \tilde{v}(t'),$$

where $x = x^\varepsilon(t')$ *and* $\tilde{v}(t') = v(x^\varepsilon(t'))$.

Finally, we need to know the behaviour of the last term on the right-hand side of equation (4.2) as the parameter ω converges towards the spectrum of $-\Delta_N^F$.

Proposition 4.3.9. *Let* λ_k *be a simple eigenvalue of* $-\Delta_g^F$ *and let* V_k *be its neighborhood which is open, bounded, and does not contain any other eigenvalue of* $-\Delta_g^F$. *For* $\lambda \in V_k$, *let*

$$R_{\lambda_k,\lambda} : C^\infty(\partial M) \mapsto \mathcal{D}'(\partial M) \tag{4.10}$$

be the operator defined by the integral kernel

$$R^{\lambda_k}(x,y) - R^\lambda(x,y),$$

then

$$\|R_{\lambda_k,\lambda}\|_{H^{1/2}(\partial M)^* \mapsto H^{1/2}(\partial M)} = O(|\lambda_k - \lambda|).$$

Remark 4.3.10. Due to Proposition 4.3.1, for any fixed $x,\ y \in \partial M$, $R_{\partial M}^{\lambda_k}(x,y)$ is well defined.

Proof of Proposition 4.3.9. Throughout the proof, we write $x \lesssim y$ or $y \gtrsim x$ to mean that $x \leq Cy$, where $C > 0$ is some constant. The dependencies of C will be clear from the context. By $x \approx y$ we mean that $x \lesssim y$ and $x \gtrsim y$.

For $\psi \in H^{1/2}(\partial M)^*$, we have the following estimate

$$\left\| \int_{\partial M} (R^{\lambda_k}(x,y) - R^{\lambda}(x,y))\psi(y) d\mu_h(y) \right\|_{H^{1/2}(\partial M)} \leq \|U\|_{H^1(M)},$$

where

$$U(x) = \sum_{j \neq k} \frac{(\lambda_k - \lambda) u_j(x) \langle u_j e^{\phi}, \psi \rangle}{(\lambda_j - \lambda_k)(\lambda_j - \lambda)}$$

and $\langle \cdot, \cdot \rangle$ denotes the pairing between $H^{1/2*}$ and $H^{1/2}$. Let us consider the following Neumann boundary value problem

$$\begin{cases} (-\Delta_g^F + 1)u = 0 & \text{on } M, \\ \partial_\nu u = \psi & \text{on } \partial M. \end{cases} \tag{4.11}$$

The corresponding Neumann-to-Dirichlet map is defined by

$$\begin{aligned} &\mathcal{N} : H^{1/2}(\partial M)^* \mapsto H^{1/2}(\partial M), \\ &\mathcal{N}\psi = u^{\psi}|_{\partial M}, \end{aligned}$$

where u^{ψ} is the solution to (4.11). Using Greens identity, we obtain

$$\begin{aligned} \lambda_j (u_j, u^{\psi})_{L^2(M, e^{\phi} d\mu_g)} &= -\int_M \Delta_g^F u_j(x) u^{\psi}(x) e^{\phi(x)} d\mu_g(x) \\ &= \int_M u_j(x) \Delta_g^F u^{\psi}(x) e^{\phi(x)} d\mu_g(x) + \langle u_j e^{\phi}|_{\partial M}, \psi \rangle. \end{aligned}$$

Therefore,

$$\langle u_j e^{\phi}|_{\partial M}, \psi \rangle = (\lambda_j - 1)(u_j, u^{\psi})_{L^2(M, e^{\phi} d\mu_g)}.$$

Then,

$$U(x) = (\lambda_k - \lambda) \sum_{j \neq k} \frac{\lambda_j - 1}{(\lambda_j - \lambda_k)(\lambda_j - \lambda)} (u_j, u^{\psi})_{L^2(M, e^{\phi} d\mu_g)} u_j(x).$$

Let us set

$$I_1(x) := \sum_{j\neq k} \frac{1}{\lambda_j - \lambda}(u_j, u^\psi)_{L^2(M,e^\phi d\mu_g)} u_j(x)$$

and

$$I_2(x) := \sum_{j\neq k} \frac{\lambda_k - 1}{(\lambda_j - \lambda_k)(\lambda_j - \lambda)}(u_j, u^\psi)_{L^2(M,e^\phi d\mu_g)} u_j(x),$$

so that

$$U(x) = (\lambda_k - \lambda)\left(I_1(x) + I_2(x)\right).$$

By the spectral theorem, we know that

$$(-\Delta_N^F - \lambda)^{-1} u^\psi = \sum_{j=1}^{\infty} \frac{1}{\lambda_j - \lambda}(u_j, u^\psi)_{L^2(M,e^\phi d\mu_g)} u_j(x).$$

Hence,

$$I_1(x) = (Id - P_k)(-\Delta_N^F - \lambda)^{-1} u^\psi,$$

where Id is the identity operator and P_k is the spectral projection to $\{u_k\}$. Therefore,

$$\|I_1\|_{H^1(M)} \lesssim \|\nabla_g u^\psi\|_{L^2(M,e^\phi d\mu_g)} + \|u^\psi\|_{L^2(M,e^\phi d\mu_g)}.$$

Furthermore, using the divergence theorem, we compute

$$\begin{aligned} 0 &= \int_M (-\Delta_g^F u^\psi(x) + u^\psi(x)) u^\psi(x) e^{\phi(x)} d\mu_g(x) \qquad (4.12)\\ &= \|\nabla_g u^\psi\|^2_{L^2(M,e^\phi d\mu_g)} + \|u^\psi\|^2_{L^2(M,e^\phi d\mu_g)} - \langle e^\phi \mathcal{N}\psi, \psi\rangle, \end{aligned}$$

which implies that

$$\|I_1\|_{H^1(M)} \lesssim \sqrt{\langle e^\phi \mathcal{N}\psi, \psi\rangle}.$$

Next, we estimate I_2:

$$\begin{aligned} &\|I_2\|_{H^1(M)} \approx \|\nabla_g I_2\|_{L^2(M,e^\phi d\mu_g)} + \|I_2\|_{L^2(M,e^\phi d\mu_g)}\\ &\lesssim \left(\sum_{j\neq k}\left(\frac{(\lambda_k - 1)}{(\lambda_j - \lambda_k)(\lambda_j - \lambda)}\right)^2 \lambda_j + \left(\frac{(\lambda_k - 1)}{(\lambda_j - \lambda_k)(\lambda_j - \lambda)}\right)^2\right)^{\frac{1}{2}} (u_j, u^\psi)_{L^2(M,e^\phi d\mu_g)}. \end{aligned}$$

Therefore,

$$\|I_2\|_{H^1(M)} \lesssim \left(\sum_{j\neq k} \frac{1}{\lambda_j^3}\right)^{\frac{1}{2}} (u_j, u^\psi)_{L^2(M,e^\phi d\mu_g)} \lesssim \|u^\psi\|_{L^2(M,e^\phi d\mu_g)}$$

Here, as a consequence of Weyl's law [Wey11], we used that $\lambda_j \approx j^{\frac{2}{3}}$. See also [BNR21] for weighted Laplace operators. Due to (4.12), we derive

$$\|I_2\|_{H^1(M)} \lesssim \sqrt{\langle e^{\phi}\mathcal{N}\psi, \psi\rangle},$$

and hence,

$$\|U\|_{H^1(M)} \lesssim |\lambda_k - \lambda|\sqrt{\langle e^{\phi}\mathcal{N}\psi, \psi\rangle}.$$

Since $-1 \notin \operatorname{spec}(-\Delta_N^F)$, it follows that $\mathcal{N}$ is a bounded operator from $H^{1/2}(\partial M)^*$ to $H^{1/2}(\partial M)$, see for instance [BE15]. Therefore, we conclude

$$\|R_{\lambda_k,\lambda}\psi\|_{H^{1/2}(M)} \leq \|U\|_{H^1(M)} \lesssim |\lambda_k - \lambda|\|\psi\|_{H^{1/2}(\partial M)^*}.$$

This completes the proof.

$\square$

4.4 Proof of the main result

In this section, we prove our main result. We first begin with auxiliary lemmas which will be used subsequently. The following lemma is well-known in spectral theory. We state it here for the reader's convenience.

Lemma 4.4.1 (Glazman Lemma)**.** *Let A be a lower-semibounded self-adjoint operator in a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ with corresponding closed sesquilinear form a and form domain $\mathrm{D}(a)$. Then, it holds*

$$N(\lambda, A) = \sup\{\dim L \mid \ L \textit{ subspace of } \mathrm{D}(a) \textit{ s.th. } a(u,u) < \lambda\langle u,u\rangle \textit{ for } u \in L \setminus \{0\}\},$$

where $N(\lambda, A)$ is the spectral distribution function.

The proofs of the following two lemmas are based on proofs of Lemma 3.1 and Theorem 3.2 in [Roh14], respectively.

Lemma 4.4.2. *Let $\lambda \in \mathbb{R}$ and $u \in H^1(M)$ such that $-\Delta_g^F u = \lambda u$. Let $x^* \in \partial M$. If $u\,|_{B_h(x^*,\varepsilon)} = 0$ and $\partial_\nu u\,|_{B_h(x^*,\varepsilon)} = 0$, then $u = 0$ identically on M. (Recall that we consider ε being smaller than injectivity radius.)*

Proof. Let us extend M to a compact connected smooth Riemannian manifold $\widetilde{M}$ such that $\overline{\widetilde{M} \setminus M}$ is compact with non-empty interior and $\overline{\widetilde{M} \setminus M} \cap M = B_h(x^*, \varepsilon)$. Let $\widetilde{u}$ be the extension by zero of u to $\widetilde{M}$. Let $\widetilde{F}$ be any smooth extension, up to the boundary, of F to $\widetilde{M}$, so that we get a new weighted Laplacian $-\widetilde{\Delta}_{\widetilde{g}}^{\widetilde{F}}$. Since $u\,|_{B_h(x^*,\varepsilon)} = 0$ and $\partial_\nu u\,|_{B_h(x^*,\varepsilon)} = 0$, it follows that $\widetilde{u} \in H^1(\widetilde{M})$, moreover $-\widetilde{\Delta}_{\widetilde{g}}^{\widetilde{F}} \widetilde{u} = \lambda \widetilde{u}$. Since $\widetilde{u}\,|_{\widetilde{M} \setminus M} = 0$, the unique continuation arguments imply that $u = 0$ identically on M. □

The previous lemma can be used to derive the following strict monotonicity principle, which will be used later.

Lemma 4.4.3. *Assume that* $0 < \varepsilon_1 < \varepsilon_2$, *then*

$$\lambda_{j,\varepsilon_1} < \lambda_{j,\varepsilon_2}, \qquad j \in \mathbb{N}.$$

Proof. Since $\varepsilon_1 < \varepsilon_2$, it follows that $\mathrm{D}(a_{\varepsilon_2}) \subset \mathrm{D}(a_{\varepsilon_1})$, and hence, by Lemma 4.4.1, we know that $\lambda_{j,\varepsilon_1} \leq \lambda_{j,\varepsilon_2}$. We are now required to show that the previous estimate is strict.

Let $\lambda = \lambda_{j,\varepsilon_2}$ and $\delta > 0$ be sufficiently small such that

$$(\lambda, \lambda + \delta) \cap \operatorname{spec}(-\Delta^F_{Mix,\varepsilon_i}) = \varnothing, \qquad \text{for } i = 1, 2.$$

We denote

$$k = N\left(\lambda + \delta, -\Delta^F_{Mix,\varepsilon_2}\right) \quad \text{and} \quad L := \operatorname{span}\{u_{1,\varepsilon_2}, \cdots, u_{k,\varepsilon_2}\}.$$

Then $k \geq j$ and

$$a_{\varepsilon_1}(u, u) = a_{\varepsilon_2}(u, u) < (\lambda + \delta)(u, u)_{L^2(M, e^\phi d\mu_g)}, \qquad \text{for } u \in L. \tag{4.13}$$

Since $\Gamma_{\varepsilon_2,a} \setminus \Gamma_{\varepsilon_1,a}$ has a non-empty interior in ∂M, by Lemma 4.4.2, it follows that if $v \in \operatorname{Ker}(-\Delta^F_{Mix,\varepsilon_1} - \lambda)$ then $v \notin L$. Therefore, we obtain

$$\dim\left(\operatorname{Ker}(-\Delta^F_{Mix,\varepsilon_1} - \lambda) \oplus L\right) = \dim\left(\operatorname{Ker}\left(-\Delta^F_{Mix,\varepsilon_1} - \lambda\right)\right) + \dim L. \tag{4.14}$$

Furthermore, since $\mathrm{Ker}\left(-\Delta^F_{Mix,\varepsilon_1} - \lambda\right) \subset \mathrm{D}(-\Delta^F_{Mix,\varepsilon_1})$ and $L \subset \mathrm{D}(a_{\varepsilon_2}) \subset \mathrm{D}(a_{\varepsilon_1})$, Theorem 2.1 in [Kat95, Chapter 6] and estimate (4.13) give

$$\begin{aligned} a_{\varepsilon_1}(v+u, v+u) &< (\lambda+\delta)(v,v)_{L^2(M,e^\phi d\mu_g)} + (\lambda+\delta)(u,u)_{L^2(M,e^\phi d\mu_g)} + 2a_{\varepsilon_1}(v,u) \\ &< (\lambda+\delta)\left((v,v)_{L^2(M,e^\phi d\mu_g)} + (u,u)_{L^2(M,e^\phi d\mu_g)}\right) + 2\lambda(v,u)_{L^2(M,e^\phi d\mu_g)} \\ &\leq (\lambda+\delta)(v+u, v+u)_{L^2(M,e^\phi d\mu_g)}, \end{aligned}$$

for $v \in \mathrm{Ker}\left(-\Delta^F_{Mix,\varepsilon_1} - \lambda\right)$ and $u \in L$. Therefore, by Lemma 4.4.1, it follows

$$N\left(\lambda+\delta, -\Delta^F_{Mix,\varepsilon_1}\right) \geq \dim\left(\mathrm{Ker}\left(-\Delta^F_{Mix,\varepsilon_1} - \lambda\right) \oplus L\right),$$

and hence, by (4.14),

$$N\left(\lambda+\delta, -\Delta^F_{Mix,\varepsilon_1}\right) \geq \dim\left(\mathrm{Ker}\left(-\Delta^F_{Mix,\varepsilon_1} - \lambda\right)\right) + \dim L.$$

Then

$$N\left(\lambda, -\Delta^F_{Mix,\varepsilon_1}\right) = N\left(\lambda+\delta, -\Delta^F_{Mix,\varepsilon_1}\right) - \dim\left(\mathrm{Ker}\left(-\Delta^F_{Mix,\varepsilon_1} - \lambda\right)\right) \geq k \geq j,$$

so that $\lambda_{j,\varepsilon_1} < \lambda_{j,\varepsilon_2}$. □

Since $\mathrm{D}(a_\varepsilon) \subset \mathrm{D}(a_N)$, it follows that $\lambda_{j,\varepsilon}$ is bounded from below by λ_j and decreases as $\varepsilon \to 0$, by the last lemma. Therefore, we can define the following limit

$$\lambda_{j,0} := \lim_{\varepsilon\to 0} \lambda_{j,\varepsilon}.$$

Next, we show that $\{\lambda_{j,0}\}_{j\in\mathbb{N}}$ coincides with the sequence of eigenvalues of $-\Delta^F_N$:

Lemma 4.4.4. *For any $j \in \mathbb{N}$, the equality $\lambda_{j,0} = \lambda_j$ holds.*

Proof. For $\varepsilon > 0$, we know that $\mathrm{D}(a^N) \subset \mathrm{D}(a_\varepsilon)$. Therefore, by Lemma 4.4.1, $\lambda_j \leq \lambda_{j,\varepsilon}$. Recalling our definition of $\lambda_{j,0}$, we conclude that $\lambda_j \leq \lambda_{j,0}$, or equivalently

$$N(\lambda, -\Delta^F_N) \geq \#\{\lambda_{j,0} : \lambda_{j,0} < \lambda\}, \qquad \lambda > 0.$$

Therefore, to prove $\lambda_{j,0} = \lambda_j$, it suffices to show that if $\lambda \in \mathrm{spec}(-\Delta^F_N)$ with multiplicity l, then λ appears in $\{\lambda_{j,0}\}_{j\in\mathbb{R}}$ at least l times.

Let $\lambda \in \operatorname{spec}(-\Delta_N^F)$ and l be its multiplicity. Then there exists $k \in \mathbb{N}$ such that $\lambda < \lambda_{k+1}$ and

$$\lambda_{k-l+1} = \cdots = \lambda_k = \lambda.$$

Therefore, there exists $\alpha_0 > 0$ such that $N(\lambda + \alpha, -\Delta_N^F) = k$ for any $\alpha \in (0, \alpha_0)$. For any $\alpha \in (0, \alpha_0)$, we aim to find a small $\varepsilon > 0$ so that $N(\lambda + \alpha, -\Delta_{Mix,\varepsilon}^F) = k$.

Let $\chi_\varepsilon \in C^\infty(M)$ denote a smooth cutoff function such that

$$\chi_\varepsilon(x) = \begin{cases} 1 & \text{for } x \in M \setminus B_g(x^*, 3\varepsilon), \\ 0 & \text{for } x \in B_g(x^*, 2\varepsilon) \end{cases}$$

and

$$\|\nabla_g \chi_\varepsilon\|_{L^2(M, e^\phi d\mu_g)} \to 0, \text{for } \varepsilon \to 0.$$

Consider the set of functions

$$L_\varepsilon := \{u_1\chi_\varepsilon, \cdots, u_k\chi_\varepsilon\}.$$

Since $\{u_j\}_{j=1}^k$ are linearly independent in L^2 and $u_j\chi_\varepsilon \to u_j$ in L^2, it follows that $\{u_j\chi_\varepsilon\}_{j=1}^k$ are also linearly independent in L^2 for sufficiently small $\varepsilon > 0$, so that $\dim(L_\varepsilon) = k$. The definition of χ_ε implies that $L_\varepsilon \subset \mathrm{D}(a_\varepsilon)$ and

$$a(u_j\chi_\varepsilon, u_j\chi_\varepsilon) \to a(u_j, u_j),$$

$$(u_j\chi_\varepsilon, u_j\chi_\varepsilon)_{L^2(M, e^\phi d\mu_g)} \to (u_j, u_j)_{L^2(M, e^\phi d\mu_g)},$$

as $\varepsilon \to 0$. Therefore, for $\alpha \in (0, \alpha_0)$, there exists $\varepsilon > 0$ such that

$$\frac{a(u, u)}{(u, u)_{L^2(M, e^\phi d\mu_g)}} < \lambda + \alpha \qquad \text{for } u \in L_\varepsilon,$$

which implies that $N(\lambda + \alpha, -\Delta_{Mix,\varepsilon}^F) = k$. Therefore, $\operatorname{spec}(-\Delta_{Mix,\varepsilon}^F) \cap (\lambda, \lambda + \alpha)$ is not empty. Moreover, since $\lambda_j < \lambda_{j,\varepsilon}$, we conclude

$$\{\lambda_{k-l+1,\varepsilon}, \cdots, \lambda_{k,\varepsilon}\} \subset \operatorname{spec}(-\Delta_{Mix,\varepsilon}^F) \cap (\lambda, \lambda + \alpha).$$

Since α_0 is an arbitrary sufficiently small number, we conclude that λ appears in $\{\lambda_{j,0}\}_{j \in \mathbb{N}}$ at least l times. This completes the proof. $\square$

Next, we show that the eigenfunctions of $-\Delta^F_{Mix,\varepsilon}$ and $-\Delta^F_N$ are close to each other in the following sense:

Lemma 4.4.5. *Assume that λ_j is a simple eigenvalue of $-\Delta^F_N$. Then there exists $C > 0$ such that*

$$\left|(u_j, u_{j,\varepsilon})_{L^2(M,e^\phi d\mu_g)}\right| > C,$$

for sufficiently small $\varepsilon > 0$.

Proof. Recall that $\{u_k\}_{k\in\mathbb{N}}$ forms an orthonormal basis on $L^2(M, e^\phi d\mu_g)$, so that we can express

$$u_{j,\varepsilon}(x) = \sum_{k\in\mathbb{N}} c^\varepsilon_k u_k.$$

Assume that the lemma is false, which means that there is a sequence of positive numbers $\{\varepsilon_l\}_{l\in\mathbb{N}}$ such that

$$\varepsilon_l \to 0, \qquad c^{\varepsilon_l}_j \to 0, \tag{4.15}$$

as $l \to \infty$. Since λ_j is simple we can choose $\alpha > 0$ such that $\lambda_j + \alpha < \lambda_{j+1}$. Let us define

$$\omega_{j,\varepsilon_l} := \sum_{k\neq j} c^{\varepsilon_l}_k u_k.$$

Then

$$\begin{aligned}(\omega_{j,\varepsilon_l}, \omega_{j,\varepsilon_l})_{L^2(M,e^\phi d\mu_g)} &= (u_{j,\varepsilon_l} - c^{\varepsilon_l}_j u_j, u_{j,\varepsilon_l} - c^{\varepsilon_l}_j u_j)_{L^2(M,e^\phi d\mu_g)} \\ &= (u_{j,\varepsilon_l}, u_{j,\varepsilon_l})_{L^2(M,e^\phi d\mu_g)} - 2(u_{j,\varepsilon_l}, c^{\varepsilon_l}_j u_j)_{L^2(M,e^\phi d\mu_g)} + (c^{\varepsilon_l}_j u_j, c^{\varepsilon_l}_j u_j)_{L^2(M,e^\phi d\mu_g)},\end{aligned}$$

and hence,

$$(\omega_{j,\varepsilon_l}, \omega_{j,\varepsilon_l})_{L^2(M,e^\phi d\mu_g)} = (u_{j,\varepsilon_l}, u_{j,\varepsilon_l})_{L^2(M,e^\phi d\mu_g)} + o(1)$$

as $l \to \infty$. Similarly, we obtain

$$\begin{aligned}a^N(\omega_{j,\varepsilon_l}, \omega_{j,\varepsilon_l}) &= a^N(u_{j,\varepsilon_l}, u_{j,\varepsilon_l}) - 2a^N(u_{j,\varepsilon_l}, c^{\varepsilon_l}_j u_j) + a^N(c^{\varepsilon_l}_j u_j, c^{\varepsilon_l}_j u_j) \\ &= \lambda_{j\varepsilon_l}(u_{j,\varepsilon_l}, u_{j,\varepsilon_l})_{L^2(M,e^\phi d\mu_g)} - \lambda_j(c^{\varepsilon_l}_j)^2 = \lambda_{j\varepsilon_l}(\omega_{j,\varepsilon_l}, \omega_{j,\varepsilon_l})_{L^2(M,e^\phi d\mu_g)} + o(1)\end{aligned} \tag{4.16}$$

as $l \to \infty$. Since $\lambda_{j,\varepsilon_l} \to \lambda_j$ as $l \to \infty$, this implies that

$$a^N(\omega_{j,\varepsilon_l}, \omega_{j,\varepsilon_l}) < (\lambda_j + \alpha)(\omega_{j,\varepsilon_l}, \omega_{j,\varepsilon_l})_{L^2(M,e^\phi d\mu_g)}$$

for sufficiently large $l \in \mathbb{N}$. Let us show that

$$\omega_{j,\varepsilon_l} \notin \operatorname{span}\{u_1, \cdots, u_j\} \tag{4.17}$$

for sufficiently large $l \in \mathbb{N}$. Assume this is not true. Without loss of generality, we assume that (4.17) is false for all $l \in \mathbb{N}$, otherwise consider a subsequence. Due to the definition of $\omega_{\omega,\varepsilon_l}$, this implies that

$$\omega_{j,\varepsilon_l} \notin \operatorname{span}\{u_1, \cdots, u_{j-1}\}.$$

In this case, we would have

$$a^N(\omega_{j,\varepsilon_l}, \omega_{j,\varepsilon_l}) \leq \lambda_{j-1}(\omega_{j,\varepsilon_l}, \omega_{j,\varepsilon_l})_{L^2(M,e^\phi d\mu_g)}.$$

Since λ_j is simple, this contradicts to (4.16). Therefore, (4.17) holds.

We now let $u \in \operatorname{span}\{u_1, \cdots, u_j\}$. Then

$$\begin{aligned} &a^N(\omega_{j,\varepsilon_l} + u, \omega_{j,\varepsilon_l} + u) \\ &\quad \leq (\lambda_j + \alpha)(\omega_{j,\varepsilon_l}, \omega_{j,\varepsilon_l})_{L^2(M,e^\phi d\mu_g)} + \lambda_j(u,u)_{L^2(M,e^\phi d\mu_g)} + 2a^N(\omega_{j,\varepsilon_l}, u), \end{aligned} \tag{4.18}$$

for sufficiently large $l \in \mathbb{N}$. Let us estimate, the last term of the right-hand side. Let χ_{ε_l} be the function described in the proof of Lemma 4.4.4, then

$$\begin{aligned} &a^N(\omega_{j,\varepsilon_l}, u) = a^N(u_{j,\varepsilon_l}, u) - c_j^{\varepsilon_l} a^N(u_j, u) \\ &\quad = a^N(u_{j,\varepsilon_l}, \chi_{\varepsilon_l} u) + a^N(u_{j,\varepsilon_l}, u - \chi_{\varepsilon_l} u) - c_j^{\varepsilon_l} a^N(u_j, u) = a^N(u_{j,\varepsilon_l}, \chi_{\varepsilon_l} u) + o(1) \end{aligned}$$

as $l \to \infty$. Since $u_{j,\varepsilon_l} \in \mathrm{D}(-\Delta^N_{Mix,\varepsilon_l})$ and $u\chi_\varepsilon \in \mathrm{D}(a_{\varepsilon_l})$, it follows that

$$a^N(u_{j,\varepsilon_l}, \chi_{\varepsilon_l} u) = \lambda_{j,\varepsilon_l}(u_{j,\varepsilon_l}, \chi_{\varepsilon_l} u)_{L^2(M,e^\phi d\mu_g)} = \lambda_{j,\varepsilon_l}(\omega_{j,\varepsilon_l}, u)_{L^2(M,e^\phi d\mu_g)} + o(1),$$

as $l \to \infty$. Therefore,

$$a^N(\omega_{j,\varepsilon_l}, u) \leq (\lambda_j + \alpha)(\omega_{j,\varepsilon_l}, u)_{L^2(M,e^\phi d\mu_g)}.$$

Therefore, (4.18) gives

$$a^N(\omega_{j,\varepsilon_l} + u, \omega_{j,\varepsilon_l} + u) \leq (\lambda_j + \alpha)(\omega_{j,\varepsilon_l} + u, \omega_{j,\varepsilon_l} + u)_{L^2(M,e^\phi d\mu_g)},$$

for sufficiently large $l \in \mathbb{N}$. Due to (4.17) and Lemma 4.4.1, we obtain

$$N(\lambda_j + \alpha, \Delta^F_N) \geq j + 1.$$

This contradicts to $\lambda_j + \alpha < \lambda_{j+1}$. □

Now, we are ready to prove the main result.

Proof of Theorem 4.1.3. Let $V_j \subset \mathbb{C}$ be an open neighbourhood of λ_j which does not contain any other eigenvalues of $-\Delta_N^F$. Since λ_j is simple, Theorem 4.4.4 implies that, for sufficiently small $\varepsilon > 0$, $\lambda_{j,\varepsilon}$ is the only eigenvalue of $-\Delta_{Mix,\varepsilon}^F$ in V_j. For $\omega \in V_j$ and $x, y \in \partial M$, Proposition 4.1.1 gives

$$\begin{aligned}
G_{\partial M}^{\omega}(x,y) =& \frac{1}{2\pi} d_g(x,y)^{-1} - \frac{H(x)}{4\pi}\log d_h(x,y) + \frac{g_x(F,\nu)}{4\pi}\log d_h(x,y) \\
&+ \frac{1}{16\pi}\left(\mathrm{II}_x\left(\frac{\exp_x^{-1}(y)}{|\exp_x^{-1}(y)|_h}\right) - \mathrm{II}_x\left(\frac{\star\exp_x^{-1}(y)}{|\exp_x^{-1}(y)|_h}\right)\right) \\
&+ \frac{1}{4\pi} h_x\left(F^{||}(x), \frac{\exp_x^{-1}(y)}{|\exp_x^{-1}(y)|_h}\right) \\
&+ \frac{u_j(x)u_j(y)}{\lambda_j - \omega^2} e^{\phi(y)} + R_{\partial M}^{\lambda_{j,\varepsilon}}(x,y).
\end{aligned}$$

From Greens identity, we know that

$$u_{j,\varepsilon} = (\lambda_{j,\varepsilon} - \omega^2)\int_M G_M^{\omega}(x,y)u_{j,\varepsilon}(y)d\mu_g(y) + \int_{\Gamma_{\varepsilon,a}} G_M^{\omega}(x,y)\partial_\nu u_{j,\varepsilon}(y)d\mu_h(y).$$

We choose $\omega^2 = \lambda_{j,\varepsilon}$ and restrict the last identity to $\Gamma_{\varepsilon,a}$, to obtain

$$\int_{\Gamma_{\varepsilon,a}} G_{\partial M}^{\sqrt{\lambda_{j,\varepsilon}}}(x,y)\partial_\nu u_{j,\varepsilon}(y)d\mu_h(y) = 0. \tag{4.19}$$

Next, we will use the coordinate system given by (B.1). We denote

$$\tilde{u}_j(t') := u_j(x^\varepsilon(t_1, at_2)), \quad \tilde{\phi}(t') := \phi(x^\varepsilon(t_1, at_2))$$

and

$$v_\varepsilon := \partial_\nu u_{j,\varepsilon}, \quad \tilde{v}_\varepsilon(t') := v_\varepsilon(x^\varepsilon(t_1, at_2)).$$

Note that in these coordinates the volume form for ∂M is given by

$$d\mu_h(y) = a\varepsilon^2(1 + \varepsilon^2 Q_\varepsilon(s'))ds_1 ds_2, \tag{4.20}$$

for some smooth function Q_ε whose derivatives of all orders are bounded uniformly in ε. Therefore, if we put the expression for $G_{\partial M}^{\sqrt{\lambda_{j,\varepsilon}}}$ into (4.19) and use Lemmas

4.3.5-4.3.8, we obtain

$$
\begin{aligned}
0 = &\frac{1}{2\pi}\varepsilon L_a \tilde{v}_\varepsilon + a\varepsilon^2 \frac{\tilde{u}_j(t')}{\lambda_j - \lambda_{j,\varepsilon}} \int_{\mathbb{D}} \tilde{u}_j(s')\tilde{v}_\varepsilon(s') e^{\tilde{\phi}(s')} ds' \\
&- \frac{1}{4\pi}\varepsilon^2 (H(x^*) - \partial_\nu \phi(x^*)) R_{\log,a}\tilde{v}_\varepsilon + \frac{1}{16\pi}\varepsilon^2(\kappa_1(x^*) - \kappa_2(x^*)) R_{\infty,a}\tilde{v}_\varepsilon \\
&+ \frac{1}{4\pi}\varepsilon^2 R_{F,a}\tilde{v}_\varepsilon + \varepsilon^2 R_a^{\lambda_{j,\varepsilon}}\tilde{v}_\varepsilon - \frac{1}{4\pi}\varepsilon^2 \log\varepsilon (H(x^*) - \partial_\nu\phi(x^*)) R_{I,a}\tilde{v}_\varepsilon + \varepsilon^3 \log\varepsilon \mathcal{A}_\varepsilon \tilde{v}_\varepsilon,
\end{aligned} \tag{4.21}
$$

where $R_a^\omega : C_c^\infty(\mathbb{D}) \mapsto D'(\mathbb{D})$ is the operator given by the kernel

$$
aR^\omega_{\partial M}(x^\varepsilon(t_1, at_2), x^\varepsilon(s_1, as_2))
$$

for $\omega \in V_j$, and $\mathcal{A}_\varepsilon : H^{1/2}(\mathbb{D}; ds')^* \mapsto H^{1/2}(\mathbb{D}; ds')$ is an operator with the norm bounded uniformly in ε. From now on, we will denote by $\mathcal{A}_\varepsilon$ any operator which takes $H^{1/2}(\mathbb{D}; ds')^* \mapsto H^{1/2}(\mathbb{D}; ds')$ whose operator norm is bounded uniformly in ε.

Let us denote

$$
\mathcal{R}^{\lambda_j,\lambda_{j,\varepsilon}} := \varepsilon^2 R_a^{\lambda_{j,\varepsilon}} - \varepsilon^2 R_a^{\lambda_j}
$$

$$
\begin{aligned}
\mathcal{R}_\varepsilon := &-\frac{1}{4\pi}\varepsilon^2 \log\varepsilon (H(x^*) - \partial_\nu\phi(x^*)) R_{I,a} - \frac{1}{4\pi}\varepsilon^2 (H(x^*) - \partial_\nu\phi(x^*)) R_{\log,a} \\
&+ \frac{1}{16\pi}\varepsilon^2 (\kappa_1(x^*) - \kappa_2(x^*)) R_{\infty,a} + \frac{1}{4\pi}\varepsilon^2 R_{F,a} + \varepsilon^2 R^{\lambda_j}_{\partial M}(x^*, x^*) R_{I,a} + \varepsilon^3 \log\varepsilon \mathcal{A}_\varepsilon.
\end{aligned}
$$

By Lemma 5.1 in [NTT21], we know that

$$
\left\| R_a^{\lambda_j} - R^{\lambda_j}_{\partial M}(x^*, x^*) R_{I,a} \right\|_{H^{1/2}(\mathbb{D}; ds')^* \mapsto H^{1/2}(\mathbb{D}; ds')} = O(\varepsilon \log\varepsilon),
$$

and hence, (4.21) becomes

$$
0 = \frac{1}{2\pi}\varepsilon L_a \tilde{v}_\varepsilon + a\varepsilon^2 \frac{\tilde{u}_j(t')}{\lambda_j - \lambda_{j,\varepsilon}} \int_{\mathbb{D}} \tilde{u}_j(s')\tilde{v}_\varepsilon(s') e^{\tilde{\phi}(s')} ds' + \mathcal{R}_\varepsilon \tilde{v}_\varepsilon + \mathcal{R}^{\lambda_j,\lambda_{j,\varepsilon}} \tilde{v}. \tag{4.22}
$$

Assume that

$$
\int_{\mathbb{D}} \tilde{u}_j(s')\tilde{v}_\varepsilon(s') e^{\tilde{\phi}(s')} ds' = 0. \tag{4.23}
$$

Then, recalling (4.20), we get

$$
\int_{\Gamma_{\varepsilon,a}} u_j(x) \partial_\nu u_{j,\varepsilon}(x) e^{\phi(x)} d\mu_h(x) = O(\varepsilon^4).
$$

Using Greens identity, we derive

$$(\lambda_{j,\varepsilon} - \lambda_j)(u_j, u_{j,\varepsilon})_{L^2(M,e^{\phi}d\mu_g)} = \int_M (\Delta_g^F u_j(x) u_{j,\varepsilon}(x) - u_j(x)\Delta_g^F u_{j,\varepsilon}(x)) e^{\phi(x)} d\mu_g(x)$$
$$= -\int_{\Gamma_{\varepsilon,a}} u_j(x)\partial_\nu u_{j,\varepsilon}(x) e^{\phi(x)} d\mu_g(x) = O(\varepsilon^4).$$

Then, by Lemma 4.4.5, it follows

$$|\lambda_{j,\varepsilon} - \lambda_j| = O(\varepsilon^4).$$

Therefore, by using Proposition 4.3.9 and recalling Remark 3.3.1, we estimate

$$\|\mathcal{R}_\varepsilon + \mathcal{R}^{\lambda_j,\lambda_{j,\varepsilon}}\|_{H^{1/2}(\mathbb{D})^* \mapsto H^{1/2}(\mathbb{D})} = O(\varepsilon^2 \log \varepsilon).$$

Since L_a is invertable as an operator from $H^{1/2}(\mathbb{D})^*$ to $H^{1/2}(\mathbb{D})$, see Section 4 in [NTT21], it follows that

$$L_a + \frac{2\pi}{\varepsilon}\left(\mathcal{R}_\varepsilon + \mathcal{R}^{\lambda_j,\lambda_{j,\varepsilon}}\right) : H^{1/2}(\mathbb{D})^* \mapsto H^{1/2}(\mathbb{D})$$

is an invertable operator. Therefore, (4.22) and (4.23) imply that $\tilde{v}_\varepsilon = 0$ on $\mathbb{D}$, and hence, $\partial_\nu u_{j,\varepsilon} = 0$ on $\Gamma_{\varepsilon,a}$. Then, by Lemma 4.4.2, we would have $u_{j,\varepsilon} =$ on M, and hence $\lambda_{j,0} = 0$. This contradicts to $\lambda_{j,\varepsilon} > \lambda_j \geq 0$. Therefore,

$$\int_{\mathbb{D}} \tilde{u}_j(s')\tilde{v}_\varepsilon(s') e^{\tilde{\phi}(s')} ds' \neq 0.$$

Therefore, we can define

$$\tilde{\psi}_\varepsilon := \frac{\tilde{v}_\varepsilon}{\int_{\mathbb{D}} \tilde{u}_j(s')\tilde{v}_\varepsilon(s') e^{\tilde{\phi}(s')} ds'}.$$

Then, (4.22) becomes

$$0 = \frac{1}{2\pi}\varepsilon L_a \tilde{\psi}_\varepsilon + a\varepsilon^2 \frac{\tilde{u}_j(t')}{\lambda_j - \lambda_{j,\varepsilon}} + \left(\mathcal{R}_\varepsilon + \mathcal{R}^{\lambda_j,\lambda_{j,\varepsilon}}\right)\tilde{\psi}_\varepsilon.$$

Let us hit both sides by $\frac{2\pi}{\varepsilon} L_a^{-1}$, to obtain

$$0 = \tilde{\psi}_\varepsilon + 2\pi a\varepsilon \frac{L_a^{-1}\tilde{u}_j}{\lambda_j - \lambda_{j,\varepsilon}} + \frac{2\pi}{\varepsilon} L_a^{-1}\left(\mathcal{R}_\varepsilon + \mathcal{R}^{\lambda_j,\lambda_{j,\varepsilon}}\right)\tilde{\psi}_\varepsilon. \tag{4.24}$$

Since

$$\left\|L_a^{-1}\left(\mathcal{R}_\varepsilon + \mathcal{R}^{\lambda_j,\lambda_{j,\varepsilon}}\right)\right\|_{H^{1/2}(\mathbb{D})^* \mapsto H^{1/2}(\mathbb{D})} = O(\varepsilon^2 \log \varepsilon),$$

relation (4.24), implies that

$$\|\tilde{\psi}\|_{H^{1/2}(\mathbb{D})^*} = \frac{1}{\lambda_j - \lambda_{j,\varepsilon}} O(\varepsilon). \tag{4.25}$$

This will be used later. Now, we multiply (4.24) by $\tilde{u}_j e^{\tilde{\phi}}$ and integrate over $\mathbb{D}$ to derive

$$0 = 1 + 2\pi a\varepsilon \frac{\langle L_a^{-1}[\tilde{u}_j], \tilde{u}_j e^{\tilde{\phi}}\rangle}{\lambda_j - \lambda_{j,\varepsilon}} + \frac{2\pi}{\varepsilon} \langle L_a^{-1} \left(\mathcal{R}_\varepsilon + \mathcal{R}^{\lambda_j, \lambda_{j,\varepsilon}}\right) \tilde{\psi}_\varepsilon, \tilde{u}_j e^{\tilde{\phi}}\rangle.$$

Equivalently, we write

$$\lambda_{j,\varepsilon} - \lambda_j = 2\pi a\varepsilon \langle L_a^{-1} \tilde{u}_j, \tilde{u}_j e^{\tilde{\phi}}\rangle + \frac{2\pi}{\varepsilon} (\lambda_j - \lambda_{j,\varepsilon}) \langle L_a^{-1} \left(\mathcal{R}_\varepsilon + \mathcal{R}^{\lambda_j, \lambda_{j,\varepsilon}}\right) \tilde{\psi}_\varepsilon, \tilde{u}_j e^{\tilde{\phi}}\rangle.$$

Let us put (4.24) into the equation above to obtain

$$\lambda_{j,\varepsilon} - \lambda_j = 2\pi a\varepsilon \langle L_a^{-1} \tilde{u}_j, \tilde{u}_j e^{\tilde{\phi}}\rangle - 4\pi^2 a \langle L_a^{-1} \left(\mathcal{R}_\varepsilon + \mathcal{R}^{\lambda_j, \lambda_{j,\varepsilon}}\right) L_a^{-1} \tilde{u}_j, \tilde{u}_j e^{\tilde{\phi}}\rangle \tag{4.26}$$
$$- \frac{4\pi^2}{\varepsilon^2} (\lambda_j - \lambda_{j,\varepsilon}) \langle L_a^{-1} \left(\mathcal{R}_\varepsilon + \mathcal{R}^{\lambda_j, \lambda_{j,\varepsilon}}\right) L_a^{-1} \left(\mathcal{R}_\varepsilon + \mathcal{R}^{\lambda_j, \lambda_{j,\varepsilon}}\right) \tilde{\psi}_\varepsilon, \tilde{u}_j e^{\tilde{\phi}}\rangle. \tag{4.27}$$

Taking into account (4.25), the last identity implies that

$$\lambda_{j,\varepsilon} - \lambda_\varepsilon = O(\varepsilon),$$

so that Proposition 4.3.9 gives

$$\left\| \mathcal{R}^{\lambda_j, \lambda_{j,\varepsilon}} \right\|_{H^{1/2}(\mathbb{D})^* \mapsto H^{1/2}(\mathbb{D})} = O(\varepsilon^4).$$

Hence, we can put $\mathcal{R}^{\lambda_j, \lambda_{j,\varepsilon}}$ into $\varepsilon^2 \log \varepsilon \mathcal{A}_\varepsilon$ term in the definition of $\mathcal{R}_\varepsilon$. Further, from (4.25), we obtain

$$\langle L_a^{-1} \mathcal{R}_\varepsilon L_a^{-1} \mathcal{R}_\varepsilon \tilde{\psi}_\varepsilon, \tilde{u}_j e^{\tilde{\phi}}\rangle = \frac{1}{\lambda_j - \lambda_{j,\varepsilon}} O\left(\varepsilon^5 \log^2 \varepsilon\right).$$

Therefore, equation (4.26) gives

$$\lambda_{j,\varepsilon} = \lambda_j + 2\pi \varepsilon a \langle L_a^{-1} \tilde{u}_j, \tilde{u}_j e^{\tilde{\phi}}\rangle - 4\pi^2 a \langle L_a^{-1} \mathcal{R}_\varepsilon L_a^{-1} \tilde{u}_j, \tilde{u}_j e^{\tilde{\phi}}\rangle + O(\varepsilon^3 \log^2 \varepsilon).$$

Recalling the definition of $\mathcal{R}_\varepsilon$ and the boundedness of $\mathcal{A}_\varepsilon : H^{1/2}(\mathbb{D}; ds')^* \to H^{1/2}(\mathbb{D}; ds')$ we obtain

$$\begin{aligned}
\lambda_{j,\varepsilon} - \lambda_j =& 2\pi\varepsilon a \int_{\mathbb{D}} L_a^{-1}\tilde{u}_j(s')\tilde{u}_j(s')e^{\tilde{\phi}(s')}ds' \\
&+ \varepsilon^2 \log\varepsilon a\pi(H(x^*) - \partial_\nu\phi(x^*))\langle L_a^{-1}R_{I,a}L_a^{-1}\tilde{u}_j, \tilde{u}_j e^{\tilde{\phi}}\rangle \\
&+ \varepsilon^2 a\pi(H(x^*) - \partial_\nu\phi(x^*))\langle L_a^{-1}R_{log,a}L_a^{-1}\tilde{u}_j, \tilde{u}_j e^{\tilde{\phi}}\rangle \\
&- \varepsilon^2 a\frac{\pi}{4}\left(\kappa_1(x^*) - \kappa_2(x^*)\right)\langle L_a^{-1}R_{\infty,a}L_a^{-1}\tilde{u}_j, \tilde{u}_j e^{\tilde{\phi}}\rangle \\
&- \varepsilon^2 a\pi\langle L_a^{-1}R_{F,a}L_a^{-1}\tilde{u}_j, \tilde{u}_j e^{\tilde{\phi}}\rangle \\
&- \varepsilon^2 4\pi^2 a R^{\lambda_j}_{\partial M}(x^*, x^*)\langle L_a^{-1}R_{I,a}L_a^{-1}\tilde{u}_j, \tilde{u}_j e^{\tilde{\phi}}\rangle \\
&+ O(\varepsilon^3\log^2\varepsilon).
\end{aligned} \tag{4.28}$$

We recall the definitions of $\tilde{u}_j$, $\tilde{\phi}$ and use Taylor series, to obtain

$$\begin{aligned}
\int_{\mathbb{D}} L_a^{-1}\tilde{u}_j(s')\tilde{u}_j(s')e^{\tilde{\phi}(s')}ds' &= \int_{\mathbb{D}} L_a^{-1}\tilde{u}_j(s')u_j(x(\varepsilon s_1, a\varepsilon s_2))e^{\phi(x(\varepsilon s_1, a\varepsilon s_2))}ds' \\
&= \int_{\mathbb{D}} L_a^{-1}\tilde{u}_j(s')\left(u_j(x^*)e^{\phi(x^*)} + \varepsilon(c_1 s_1 + c_2 s_2)\right)ds' \\
&+ \int_{\mathbb{D}} L_a^{-1}\tilde{u}_j(s')R_2^1(\varepsilon s')ds'
\end{aligned}$$

where R_2^1 is the reminder term of the Taylor series for $\tilde{u}_j e^{\tilde{\phi}}$ near zero and c_1, c_2 are appropriate constants. Using (4.4) in [NTT21], we derive

$$\begin{aligned}
\int_{\mathbb{D}} L_a^{-1}\tilde{u}_j(s')\tilde{u}_j(s')e^{\tilde{\phi}(s')}ds' = \int_{\mathbb{D}} &\left(u_j(x^*) + \varepsilon(b_1 s_1 + b_2 s_2) + R_2^2(\varepsilon s')\right) \times \\
&\times L_a^{-1}\left(u_j(x^*)e^{\phi(x^*)} + \varepsilon(c_1 s_1 + c_2 s_2) + R_2^1(\varepsilon s')\right)ds'
\end{aligned}$$

where R_2^2 is the reminder term of the Taylor series for $\tilde{u}_j$ near zero and b_1, b_2 are appropriate constants. Next, we note that

$$\int_{\mathbb{D}} L_a^{-1}[1](s')s_j ds' = 0, \qquad \|R_2^j(\varepsilon\cdot)\|_{H^1(\mathbb{D})} = O(\varepsilon^2),$$

for $j = 1$, 2. Therefore, the penultimate identity gives

$$\begin{aligned}
\int_{\mathbb{D}} L_a^{-1}\tilde{u}_j(s')\tilde{u}_j(s')e^{\tilde{\phi}(s')}ds' &= |u_j(x^*)|^2 e^{\phi(x^*)}\int_{\mathbb{D}} L_a^{-1}[1](s')ds' + O(\varepsilon^2) \\
&= \frac{2\pi}{K_a}|u_j(x^*)|^2 e^{\phi(x^*)} + O(\varepsilon^2).
\end{aligned}$$

Since u_j is smooth on ∂M, it follows that $\tilde{u}_j(t) - \tilde{u}_j(0) = O_{H^{1/2}}(\varepsilon)$ as $\varepsilon \to 0$. Additionally, we recall that $R_{I,a}$, $R_{log,a}$, and $R_{\infty,a}$ are bounded operators from $H^{1/2}(\mathbb{D})$ to $H^{1/2}(\mathbb{D})^*$. Therefore, using (4.4) in [NTT21], we write

$$\begin{aligned}\langle L_a^{-1} R_{log,a} L_a^{-1} \tilde{u}_j, \tilde{u}_j e^{\tilde{\phi}} \rangle &= \langle R_{log,a} L_a^{-1} \tilde{u}_j, L_a^{-1}[\tilde{u}_j e^{\tilde{\phi}}] \rangle \\ &= |u_j(x^*)|^2 e^{\phi(x^*)} \langle R_{log,a} L_a^{-1}[1], L_a^{-1}[1] \rangle + O(\varepsilon).\end{aligned}$$

Furthermore, using (4.9), we obtain

$$\begin{aligned}&\langle R_{log,a} L_a^{-1} \tilde{u}_j, L_a^{-1}[\tilde{u}_j e^{\tilde{\phi}}] \rangle \\ &= |u_j(x^*)|^2 e^{\phi(x^*)} \frac{a}{K_a^2} \int_{\mathbb{D}} \frac{1}{(1-|s'|^2)^{1/2}} \int_{\mathbb{D}} \frac{\log\left((t_1-s_1)^2 + a^2(t_2-s_2)^2\right)^{1/2}}{(1-|t'|^2)^{1/2}} dt' ds' + O(\varepsilon).\end{aligned}$$

Similarly, we collect expressions for $\langle L_a^{-1} R_{I,a} L_a^{-1} \tilde{u}_j, \tilde{u}_j e^{\tilde{\phi}} \rangle$, $\langle R_{\infty,a} L_a^{-1} \tilde{u}_j, L_a^{-1}[\tilde{u}_j e^{\tilde{\phi}}] \rangle$ and $\langle R_{F,a} L_a^{-1} \tilde{u}_j, L_a^{-1}[\tilde{u}_j e^{\tilde{\phi}}] \rangle$ below

$$\begin{aligned}\langle L_a^{-1} R_{I,a} L_a^{-1} \tilde{u}_j, \tilde{u}_j e^{\tilde{\phi}} \rangle &= a |u_j(x^*)|^2 e^{\phi(x^*)} \int_{\mathbb{D}} L_a^{-1}[1](t') dt' \int_{\mathbb{D}} L_a^{-1}[1](t') dt' + O(\varepsilon) \\ &= \frac{4\pi^2 a}{K_a^2} |u_j(x^*)|^2 e^{\phi(x^*)} + O(\varepsilon),\end{aligned}$$

$$\begin{aligned}\langle R_{\infty,a} L_a^{-1} \tilde{u}_j, L_a^{-1}[\tilde{u}_j e^{\tilde{\phi}}] \rangle &= |u_j(x^*)|^2 e^{\phi(x^*)} \frac{a}{K_a^2} \times \\ &\times \int_{\mathbb{D}} \frac{1}{(1-|s'|^2)^{1/2}} \int_{\mathbb{D}} \frac{(t_1-s_1)^2 - a^2(t_2-s_2)^2}{(t_1-s_1)^2 + a^2(t_2-s_2)^2} \frac{1}{(1-|t'|^2)^{1/2}} dt' ds' + O(\varepsilon),\end{aligned}$$

and finally (see page 10045 in [NTT22]),

$$\langle R_{F,a} L_a^{-1} \tilde{u}_j, L_a^{-1}[\tilde{u}_j e^{\tilde{\phi}}] \rangle = O(\varepsilon).$$

Using the identities above and (4.28), the proof is complete. □

Appendix A

Greens function of Δ_g on closed surface

This section is the appendix of the chapter 2 paper.

In this section, we offer a derivation of the Greens function for the Laplace-Beltrami operator on a *closed Riemannian surface*. The theorem which is proven is well-known and the contents of this appendix are not novel. We will use a straight-forward pseudo-differential parametrix construction. We will show the following theorem, which is used heavily throughout chapter 2.

Theorem A.0.1. *Let (M, g) be a closed Riemannian surface, then there is a distribution $E \in \mathcal{D}'(M \times M)$ such that*

$$\Delta_g E = -\delta + \frac{1}{|M|}, \quad E(x, y) = E(y, x) \quad \textit{and} \quad \int_M E(x, y) d\mu_g(x) = 0.$$

In particular, we have that in an open neighborhood of $\{x = y\}$

$$E(x, y) = -\frac{1}{2\pi} \log d_g(x, y) + P_{-4}(x, y).$$

where $P_{-4} \in \Psi^{-4}(M)$.

Proof. Since $\Delta_g \in \Psi^2_{cl}(M)$ elliptic, we have that there is a parametrix $P \in \Psi^{-2}_{cl}(M)$

satisfying the following equation

$$\Delta_g P = I + \Psi^{-\infty}(M)$$

Furthermore, as a corollary of Borel's lemma, we can express the Schwartz kernel of P as

$$P(x,y) \sim \sum_{j=0}^{\infty} P_{-2-j}(x,y)$$

where $P_{-2-j} \in \Psi_{cl}^{-2-j}(M)$. A standard first-order parametrix construction indicates that we can choose for x near y

$$P_{-2}(x,y) = -\frac{1}{2\pi} \log d_g(x,y)$$

So, our claim is that $P_{-3} = 0$. This problem reduces to showing that

$$P - P_{-2} \in \Psi_{cl}^{-4}(M) \tag{A.1}$$

Left composition of Δ_g with (A.1) results in the following equivalent formulation

$$\Delta_g P_{-2} - I \in \Psi_{cl}^{-2}(M) \tag{A.2}$$

Self-adjointness of P_{-2} and $\Delta_g$imply that (A.2) is equivalent to

$$P_{-2}\Delta_g - I \in \Psi_{cl}^{-2}(M) \tag{A.3}$$

Should (A.3) be true, then we would infer that the expansion for $P_{-2}\Delta_g$ consists of no -1 order pseudo-differential operator. This is equivalent to requiring the principle symbol, which is homogeneous of degree -1 satisfy the following

$$\sigma_{-1}(P_{-2}\Delta_g - I)(y,\eta) = 0, \text{ for all } (y,\eta) \in T^*M$$

To attain the above requirement, we show that $\sigma_{-1}(P_{-2}\Delta_g - I)$ can be bounded from above in the following manner

$$|\sigma_{-1}(P_{-2}\Delta_g - I)(y_0, \tau\eta_0)| \lesssim \tau^{-2} \tag{A.4}$$

for $\tau \to \infty$ and fixed $(y_0, \eta_0) \in S^*M$. Since the decay is radially symmetric and is independent of the choice of y_0, (A.4) implies that $\sigma_{-1}(P_{-2}\Delta_g - I)$ vanishes on T^*M.

Now, we let $\Phi : V \to U$ be a Riemann normal co-ordinate chart, centred at y_0 for which $\Phi(0) = y_0 \in U \subset M$. Let $A : C_c^\infty(\mathbb{R}^2) \to \mathcal{D}'(\mathbb{R}^2)$ and $B : C_c^\infty(\mathbb{R}^2) \to \mathcal{D}'(\mathbb{R}^2)$ denote the pull-back operators for P_{-2} and Δ_g by Φ respectively. Then, by the invariance of principle symbols under symplectomorphism, we have that

$$\sigma_{-1}(P_{-2}\Delta_g - I)(y_0, \eta_0) = \sigma_{-1}(AB - I)(0, \xi)$$

If $a(t, \xi)$ and $b(t, \xi)$ denote the symbols of A and B, then we have that

$$(a\#b)(t, \xi) = a(t, \xi)b(t, \xi) - i \sum_{|\mu|=1} D_\xi^\mu a(t, \xi) D_t^\mu b(t, \xi) + S_{cl}^{-2}(T^*\mathbb{R}^2)$$

Furthermore, if we restrict $t = 0$, since we are working in Riemannian normal co-ordinates, we have that $D_t^\mu b(t, \xi)|_{t=0} = 0$ and thus

$$(a\#b)(0, \xi) = a(0, \xi)b(0, \xi) + S_{cl}^{-2}(T^*\mathbb{R}^2)$$

Since the symbol $a(0, \xi)$ is given by the Schwartz kernel of A, where

$$a(0, \xi) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-i\xi\cdot t} \log|t| dt = |\xi|^{-2}$$

we have that

$$(a\#b)(0, \xi) = 1 + S_{cl}^{-2}(T^*\mathbb{R}^2)$$

This implies that

$$|(a\#b)(0, \xi) - 1| \lesssim \langle\xi\rangle^{-2} \implies \sigma_{-1}(AB - I)(0, \xi) = 0$$

The last equality thus implies that $\sigma_{-1}(P_{-2}\Delta_g - I)(y, \eta) = 0$ for all $(y, \eta) \in T^*M$. $\square$

Appendix B

Geometric preliminaries

This section consists of a preliminary section contained in the chapter 3 and chapter 4 paper.

In this section, we introduce the basic geometric framework of chapter 3 and chapter 4. We use $(M, g, \partial M)$ to denote a compact connected orientable Riemannian manifold of dimension three with a non-empty smooth boundary. The corresponding volume form and geodesic distance are denoted by $d\mu_g(\cdot)$ and $d_g(\cdot,\cdot)$, respectively. Let $\iota : \partial M \hookrightarrow M$ be the trivial embedding of the boundary ∂M into M. This allows us to define the boundary metric $h := \iota^*_{\partial M} g$ inherited by g. We similarly use $d\mu_h(\cdot)$ and $d_h(\cdot,\cdot)$ to denote respectively the volume form on the boundary and the geodesic distance on the boundary given by metric h. We denote the Laplace-Beltrami operator by $\Delta_g = -d^*d$.

For $x \in \partial M$, let $E_1(x), E_2(x) \in T_x\partial M$ be the unit eigenvectors of the shape operator at $x \in \partial M$ corresponding respectively to the principal curvatures $\kappa_1(x),\ \kappa_2(x)$. We will drop the dependence in x from our notation when there is no ambiguity. We choose E_1 and E_2 such that $E_1^\flat \wedge E_2^\flat \wedge \nu^\flat$ is a positive multiple of the volume form $d\mu_g$ (see p.26 of [Lee18] for the "musical isomorphism" notation of $^\flat$ and $^\sharp$). Here we use ν to denote the outward-pointing normal vector field. By $H(x)$, we denote the mean curvature of ∂M at x. We also set

$$\mathrm{II}_x(V) := \mathrm{II}_x(V, V), \quad V \in T_x\partial M,$$

to be the scalar second fundamental form. Note that, in defining II and the shape operator, we will follow the standard literature in geometry (e.g. [Lee18]) and use the inward-pointing normal so that the sphere embedded in $\mathbb{R}^3$ would have positive mean curvature in our convention.

In this article, we will often use boundary normal coordinates. Therefore, we briefly recall its construction. For a fixed $x^* \in \partial M$, we will denote by $B_h(\rho; x^*) \subset \partial M$ the geodesic disk of radius $\rho > 0$ (with respect to the metric h) centred at x^* and $\mathbb{D}_\rho$ to be the Euclidean disk in $\mathbb{R}^2$ of radius ρ centred at the origin. In what follows ρ will always be smaller than the injectivity radius of $(\partial M, h)$. Letting $t = (t_1, t_2, t_3) \in \mathbb{R}^3$, we will construct a coordinate system $x(t; x^*)$ by the following procedure:

Write $t \in \mathbb{R}^3$ near the origin as $t = (t', t_3)$ for $t' = (t_1, t_2) \in \mathbb{D}_\rho$. Define first

$$x((t', 0); x^*) := \exp_{x^*;h}(t_1 E_1 + t_2 E_2),$$

where $\exp_{x^*;h}(V)$ denotes the time 1 map of h-geodesics with initial point x^* and initial velocity $V \in T_{x^*}\partial M$. The coordinate $t' \in \mathbb{D}_\rho \mapsto x((t', 0); x^*)$ is then an h-geodesic coordinate system for a neighborhood of x^* on the boundary surface ∂M. We can then construct a coordinate system for a neighbourhood of $x^* \in M$ by considering g-geodesic rays $\gamma_{x^*,-\nu} : [0, \rho) \to M$ emanating from points in ∂M orthogonal to ∂M. In particular, we can then smoothly extend t' to U by setting t' to be constant functions along $\gamma_{x^*,-\nu}$. If we then define t_3 to be the unit speed parameter of $\gamma_{x^*,-\nu}$, then (t_1, t_2, t_3) form coordinates for M in some neighborhood of $x^* \in M$. As a consequence, t_3 is a boundary-defining function, that is $t_3 > 0$ away from ∂M and $t_3 = 0$ on ∂M. We will call these local coordinates, *boundary normal coordinates*. For convenience we will write $x(t'; x^*)$ in place of $x((t', 0); x^*)$. Readers wishing to know more about boundary normal coordinates can refer to [Lee18] for a brief recollection of the basic properties we use here for detailed construction. We will also use the rescaled version of this coordinate system. For $\varepsilon > 0$ sufficiently small we define the (rescaled) h-geodesic coordinate by the following map

$$x^\varepsilon(\cdot; x^*) : t' = (t_1, t_2) \in \mathbb{D} \mapsto x(\varepsilon t'; x^*) \in B_h(\varepsilon; x^*), \tag{B.1}$$

where $\mathbb{D}$ is the unit disk in $\mathbb{R}^2$. Given the boundary normal co-ordinate construction, we define the geodesic ellipse $\Gamma_{\varepsilon,a}$ as the following subset of ∂M

$$\Gamma_{\varepsilon,a} := \{\exp_{x^*;h}(\varepsilon t_1 E_1 + \varepsilon t_2 E_2) \mid t_1^2 + a^{-2} t_2^2 \leq 1\}. \tag{B.2}$$

Appendix C

The Dirichlet-to-Neumann map

Within this section, we will introduce the Dirichlet-to-Neumann (DtN) map as well as show that it can be viewed as an elliptic pseudo-differential operator, in addition, the proof offered is constructive in nature and thus will allow us to recover the principle symbol. This is of vital importance in chapter 3 and chapter 4. We will define the DtN map for a common elliptic boundary value problem involving a Laplace-type operator with a drift term. From here, a lot of the results described within this section can be easily generalised to other operators such as Helmholtz operators with drift terms. First, suppose that u_f is a solution to the following *auxiliary* Dirichlet boundary value problem

$$\begin{cases} \Delta_g u_f + g(F, \nabla_g u_f) = 0 \ \text{ on } \ M, \\ u_f = f \in C^\infty(\partial M). \end{cases} \tag{C.1}$$

It should be noted that the assumption that $f \in C^\infty(\partial M)$ is for simplicity and that the DtN map can be defined for more irregular boundary conditions. Given (C.1), we can define the DtN map, denoted by $\Lambda_{g,F}$, as

$$\Lambda_{g,F} : C^\infty(\partial M) \ni f \mapsto \partial_\nu u_f|_{\partial M} \in C^\infty(\partial M).$$

Generally speaking, upon application of the DtN map to a boundary condition, there is a loss of 1 Sobolev order. That is, if $f \in H^{1/2}(\partial M)$, then $\Lambda_{g,F} f \in H^{-1/2}(\partial M)$. The first result that we collect is a classical result which is proven in [LLS16] and

[LU89b]. We offer an adapted sketch of the proof for our special case and in order to keep the exposition as self-contained as possible. The purpose of the following lemma is to show that the DtN map is a nice pseudo-differential operator. Precisely speaking, it's possession of ellipticity and homogeneity in the symbols. Furthermore, the strength of the proof of the following lemma is that it is constructive, this means that it allows us to get our hands on the principle symbol $\sigma_1(\Lambda_{g,F})$.

Lemma C.0.1. *The Dirichlet-to-Neumann map $\Lambda_{g,F}$ is an elliptic pseudo-differential operator of order* 1*. In addition, the first two terms of the symbol, $\sigma(\Lambda_{g,F})(t,\xi')$ are*

$$\begin{aligned}\sigma_1(\Lambda_{g,F}) &= -\sqrt{\widetilde{q}_2},\\ \sigma_0(\Lambda_{g,F}) &= \frac{1}{2\sqrt{\widetilde{q}_2}}(\nabla_{\xi'}\sqrt{\widetilde{q}_2}\cdot D_{t'}\sqrt{\widetilde{q}_2} - \widetilde{q}_1 - \partial_{t_3}\sqrt{\widetilde{q}_2} + \widetilde{E}\sqrt{\widetilde{q}_2}),\end{aligned}$$

where $\widetilde{E}$, $\widetilde{q}_1$ and $\widetilde{q}_2$ are given by (in boundary normal co-ordinates)

$$\begin{aligned}\widetilde{E}(t) &:= -\frac{1}{2}\sum_{\alpha,\beta} h^{\alpha\beta}(t)\partial_{t_3}h_{\alpha\beta}(t) - F_3(t),\\ \widetilde{q}_1(t,\xi') &:= -i\sum_{\alpha,\beta}\left(\frac{1}{2}h^{\alpha\beta}(t)\partial_{t_\alpha}\log\delta(t) + \partial_{t_\alpha}h^{\alpha\beta}(t) - F_\alpha(t)h^\beta_\alpha(t)\right)\xi_\beta,\\ \widetilde{q}_2(t,\xi') &:= \sum_{\alpha,\beta} h^{\alpha\beta}(t)\xi_\alpha\xi_\beta,\end{aligned}$$

Here $\alpha,\beta\in\{1,2\}$, $F = F_1(t)\partial_{t_1} + F^2(t)\partial_{t_2} + F_3(t)\partial_{t_3}$ and $\delta(t) = \det(g_{\alpha\beta})$.

Proof. Within our choice of co-ordinates, we begin with the following decomposition

$$-\Delta_g - g(F,\nabla_g\cdot) = D^2_{t_3} + i\widetilde{E}(t)D_{t_3} + \widetilde{Q}(t,D_{t'}), \tag{C.2}$$

where $\widetilde{Q}(t,D_{t'})$ is

$$\begin{aligned}\widetilde{Q}(t,D_{t'}) :=& \sum_{\alpha,\beta} h^{\alpha\beta}(t)D_{t_\alpha}D_{t_\beta}\\ &- i\sum_{\alpha,\beta}\left(\frac{1}{2}h^{\alpha\beta}(t)\partial_{t_\alpha}\log\delta(t) + \partial_{t_\alpha}h^{\alpha\beta}(t) - F_\alpha(t)h^\beta_\alpha(t)\right)D_{t_\beta}.\end{aligned}$$

It should be noted that the total symbol $\sigma(\widetilde{Q})(t,\xi')$ of $\widetilde{Q}(t,D_{t'})$ is given by

$$\sigma(\widetilde{Q})(t,\xi') = \widetilde{q}_1(t,\xi') + \widetilde{q}_2(t,\xi').$$

It follows that we can construct a first order, classical, pseudo-differential operator $A_F^\omega(t, D_{t'})$ such that

$$-\Delta_g - g(F, \nabla_g \cdot) = (D_{t_3} + i\widetilde{E}(t) - iA_F(t, D_{t'}))(D_{t_3} + iA_F(t, D_{t'})), \tag{C.3}$$

by equating (C.2) and (C.3). This yields the following equation, up to a smoothing operator

$$A_F(t, D_{t'})^2 + i[D_{t_3}, A_F(t, D_{t'})] - \widetilde{Q}(t, D_{t'}) - \widetilde{E}(t)A_F(t, D_{t'}) = 0. \tag{C.4}$$

The standard pseudo-differential calculus allows us to write (C.4) equivalently in terms of symbols associated with the relevant operators (ones which involve some action on functions defined over ∂M) as follows, up to a smoothing symbol

$$\sum_\gamma \frac{1}{\gamma!}\partial_{\xi'}^\gamma \sigma(A_F) D_t^\gamma \sigma(A_F) - \partial_{t_3}\sigma(A_F) - \sigma(\widetilde{Q}) - \widetilde{E}(t)\sigma(A_F) = 0.$$

where $\gamma \in \mathbb{N}^n$ denotes some multi-index. Collecting homogeneous terms of degree 2 and then 1 yields the first two terms of the Borel expansion for the symbol of the pseudo-differential operator $A_F(t, D_{t'})$. These terms are

$$\begin{aligned}
\sigma_1(A_F) &= -\sqrt{\widetilde{q}_2}, \\
\sigma_0(A_F) &= \frac{1}{2\sqrt{\widetilde{q}_2}}(\nabla_{\xi'}\sqrt{\widetilde{q}_2} \cdot D_{t'}\sqrt{\widetilde{q}_2} - \widetilde{q}_1 - \partial_{t_3}\sqrt{\widetilde{q}_2} + \widetilde{E}\sqrt{\widetilde{q}_2}).
\end{aligned}$$

It should be noted that $\sigma_1(A_F) = 0$ *only if* $\xi' = 0$ (which corresponds to the zero section). Thus, it follows that $A_F(t, D_{t'})$ is an elliptic operator. Furthermore, by construction, $\sigma_1(A_F)$ and $\sigma_0(A_F)$ are homogeneous symbols, as are the residual symbols $\sigma_j(A_F)$ for $j \leq -1$. Therefore $A_F(t, D_{t'})$ is a first-order, elliptic, classical pseudo-differential operator. Within the following section of the proof, it is shown that $A_F(t, D_{t'})$ coincides with $\Lambda_{g,F}$ up to a smoothing operator. This is done by first considering the region $\{0 \leq t_3 \leq T\}$. The authors in [LU89b], exploited (C.3) in order to write (4.5) as a system of forward and backward heat equations:

$$\begin{gathered}
(D_{t_3} + iA_F)u_f = v, \quad u_f|_{t_3=0} = f, \\
(D_{t_3} + i\widetilde{E} - iA_F)v = w \in C^\infty([0, T]; \mathcal{D}'(\mathbb{R}^2)),
\end{gathered}$$

where $u_f, v \in C^\infty([0,T]; \mathcal{D}'(\mathbb{R}^2))$. Since $\sigma_1(A_F^\omega) < 0$ for $\xi' \neq 0$, the following heat equation is well-posed

$$\partial_{t_3} v + A_F v - \widetilde{E} v = iw,$$

and thus $v \in C^\infty([0,T] \times \mathbb{R}^2)$. Thus, restricting the forward heat equation to $\{t_3 = 0\}$ yields

$$D_{t_3} u_f|_{t_3=0} + i\, A_F u_f|_{t_3=0} = v|_{t_3=0}\,.$$

Consequently, we have that

$$\sigma(\Lambda_{g,F})(t,\xi') = \sigma(A_F)(t,\xi').$$

$\square$

Appendix D

Boundary restriction of Greens function

This section consists of a calculation contained in the chapter 3 paper.

The ∂M-restricted Greens function, $G_{\partial M}(x,y) \in \mathcal{D}'(\partial M \times \partial M)$, when evaluated at the center of the h-geodesic disc $\Gamma_{\varepsilon,1}$, up to Ψ^{-3}, when written in local co-ordinates is given by

$$\frac{1}{4\pi^2}\left(\int_{\mathbb{R}^2} e^{-i\xi'\cdot t'}\sigma_{-1}(G_{\partial M})(0,\xi')d\xi' + \int_{\mathbb{R}^2} e^{-i\xi'\cdot t'}\sigma_{-2}(G_{\partial M})(0,\xi')d\xi'\right). \tag{D.1}$$

Within this section, we will offer an explicit computation for $G_{\partial M}$ near the diagonal of $\partial M \times \partial M$. Such an asymptotic expansion will appear in Chapters 3 and 4. We begin by evaluating the first term as follows

$$\frac{1}{4\pi^2}\int_{\mathbb{R}^2} e^{-i\xi'\cdot t'}\sigma_{-1}(G_{\partial M})(0,\xi')d\xi' = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} e^{-i\xi'\cdot t'}\frac{\chi(\xi')}{|\xi'|_h}d\xi'.$$

Furthermore, we can split the above integral into a singular and regular part as follows

$$\frac{1}{4\pi^2}\int_{\mathbb{R}^2} e^{-i\xi'\cdot t'}\sigma_{-1}(G_{\partial M})(0,\xi')d\xi' = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} e^{-i\xi'\cdot t'}\frac{1}{|\xi'|_h}d\xi' + I_{\text{reg}}.$$

The regular part is of no interest to us and will be lost in the error as this computation is done in order to isolate the most prevalent singularities occurring in $G_{\partial M}$, henceforth, we will treat our computations as Fourier transforms of homogeneous distributions. This idea can be carried forth in the computation for the second integral. Since we are working in boundary normal co-ordinates at the origin, we have that $h^{\alpha\beta}(0) = \delta^{\alpha\beta}$ and thus $|\xi'|_h = \sqrt{\xi_1^2 + \xi_2^2}$. This implies that

$$\frac{1}{4\pi^2} \int_{\mathbb{R}^2} e^{-i\xi' \cdot t'} \frac{1}{|\xi'|_h} d\xi' = \mathcal{F}(|\xi'|^{-1})(t') = \frac{1}{2\pi} |t'|^{-1}.$$

Next, we calculate the second integral in (D.1) (up to a regular term), the integral is given by

$$-\frac{1}{4\pi^2} \left(\int_{\mathbb{R}^2} \frac{\sigma_{-1}(G_{\partial M})\sigma_0(\Lambda_{g,F}) + \nabla_{\xi'}\sigma_{-1}(G_{\partial M}) \cdot D_{t'}\sigma_1(\Lambda_{g,F})}{\sigma_1(\Lambda_{g,F})} \bigg|_{t=0} e^{-i\xi' \cdot t'} d\xi' \right)$$

Recall that we have that $\sigma_1(\Lambda_{g,F})(t,\xi') = -\sqrt{h^{\alpha\beta}(t)\xi_\alpha\xi_\beta}$. Thus, we have the following identity

$$D_{t'}\sigma_1(\Lambda_{g,F}) = -\frac{D_{t'}h^{\alpha\beta}(t)}{2\sqrt{h^{\alpha\beta}(t)\xi_\alpha\xi_\beta}}.$$

In boundary normal co-ordinates, when $t = 0$, the right hand side is to 0. Thus, it follows that the second integral is vanishing, that is

$$\int_{\mathbb{R}^2} \frac{\nabla_{\xi'}\sigma_{-1}(G_{\partial M}) \cdot D_{t'}\sigma_1(\Lambda_{g,F})}{\sigma_1(\Lambda_{g,F})} \bigg|_{t=0} e^{-i\xi' \cdot t'} d\xi' = 0.$$

We are now left to compute the following integral

$$I(t') := -\frac{1}{4\pi^2} \int_{\mathbb{R}^2} \frac{\sigma_{-1}(G_{\partial M})\sigma_0(\Lambda_{g,F})}{\sigma_1(\Lambda_{g,F})} \bigg|_{t=0} e^{-i\xi' \cdot t'} d\xi'$$

Furthermore, since $\widetilde{q}_1(0,\xi') = iF_\alpha(0)\xi_\alpha$ in boundary normal co-ordinates, we have that I is given by

$$\begin{aligned} I(t') &= \frac{1}{8\pi^2} \int_{\mathbb{R}^2} \frac{1}{|\xi'|^3} \left(\widetilde{E}|\xi'| - \partial_{t^3}\sqrt{h^{\alpha\beta}(t)\xi_\alpha\xi_\beta} - iF_\alpha(0)\xi_\alpha \right) \bigg|_{t=0} e^{-i\xi' \cdot t'} d\xi' \\ &= \frac{\widetilde{E}(0)}{8\pi^2} \int_{\mathbb{R}^2} e^{-i\xi' \cdot t'} \frac{1}{|\xi'|^2} d\xi' - \frac{1}{8\pi^2} \int_{\mathbb{R}^2} \frac{\partial_{t^3}\sqrt{h^{\alpha\beta}(t)\xi_\alpha\xi_\beta}}{|\xi'|^3} \bigg|_{t=0} e^{-i\xi' \cdot t'} d\xi' \\ &- \frac{i}{8\pi^2} F_\alpha(0) \int_{\mathbb{R}^2} e^{-i\xi' \cdot t'} \frac{\xi_\alpha}{|\xi'|^3} d\xi'. \end{aligned}$$

We calculate each term in the integral in order of increasing difficulty. Using proposition 8.17 [Lee18] we have the following

$$\widetilde{E}(0) = -\frac{1}{2}\sum_{\alpha,\beta} h^{\alpha\beta}(0)\partial_{x^3}h_{\alpha\beta}(t)\Big|_{t=0} - F_3(0) = 2H(x^*) - g_{x^*}(F,\nu).$$

Where $H(x^*)$ denotes the mean curvature of $(M, g, \partial M)$ at x^*, the center of the geodesic window $(t = 0)$. Thus, we have that

$$\frac{\widetilde{E}(0)}{8\pi^2}\int_{\mathbb{R}^2} e^{-i\xi'\cdot t'}\frac{1}{|\xi'|^2}d\xi' = \frac{2H(x^*) - g_{x^*}(F,\nu)}{2}\mathcal{F}(|\xi'|^{-2}) = -\frac{2H(x^*) - g_{x^*}(F,\nu)}{4\pi}\log|t'|.$$

Next, we calculate the third term by making the following observation

$$\partial_{\xi_\alpha}|\xi'|^{-1} = -\frac{\xi_\alpha}{|\xi'|^3}.$$

We can re-write the third term as follows

$$\begin{aligned}
-\frac{i}{8\pi^2}\sum_\alpha F_\alpha(0)\int_{\mathbb{R}^2} e^{-i\xi'\cdot t'}\frac{\xi_\alpha}{|\xi'|^3}d\xi' &= \frac{i}{8\pi^2}\sum_\alpha F_\alpha(0)\int_{\mathbb{R}^2} e^{-i\xi'\cdot t'}\partial_{\xi_\alpha}|\xi'|^{-1}d\xi' \\
&= \frac{1}{8\pi^2}\sum_\alpha t_\alpha F_\alpha(0)\int_{\mathbb{R}^2} e^{-i\xi'\cdot t'}|\xi'|^{-1}d\xi' \\
&= \frac{1}{2}\sum_\alpha t_\alpha F_\alpha(0)\mathcal{F}(|\xi'|^{-1})(t') \\
&= \sum_\alpha F_\alpha(0)\frac{t_\alpha}{4\pi|t'|}.
\end{aligned}$$

Lastly, we compute the second term. It should be noted that via rotation with basis $E_1^\flat \wedge E_2^\flat \wedge \nu^\flat$, we find that the scalar second fundamental form is diagonalised in our co-ordinate system. Thus, we have the following

$$\begin{aligned}
&-\frac{1}{8\pi^2}\int_{\mathbb{R}^2} e^{-i\xi'\cdot t'}\frac{\partial_{t^3}|\xi'|_h}{|\xi'|^3}d\xi' \\
&= -\frac{1}{8\pi^2}\int_{\mathbb{R}^2} e^{-i\xi'\cdot t'}\frac{(\partial_{t^3}h^{11} + 2\partial_{t^3}h^{12} + \partial_{t^3}h^{22})|_{t=0}}{|\xi'|^4}d\xi' \\
&= -\frac{1}{8\pi^2}\int_{\mathbb{R}^2} e^{-i\xi'\cdot t'}\frac{\kappa_1(x^*)\xi_1^2 + \kappa_2(x^*)\xi_2^2}{|\xi'|^4}d\xi' \\
&= \frac{1}{4\pi^2}\left(\frac{\kappa_1(x^*) + \kappa_2(x^*)}{2}\pi\log|t'| - \frac{\pi}{4}\left(\frac{\kappa_2(x^*)t_1^2 + \kappa_1(x^*)t_2^2}{|t'|^2} - \frac{\kappa_1(x^*)t_1^2 + \kappa_2(x^*)t_2^2}{|t'|^2}\right)\right)
\end{aligned}$$

Here λ_1, λ_2 denote the associated principal curvatures. Therefore, we have that $I(t')$ is given by

$$\begin{aligned} I(t') = &- \frac{2H(x^*) - g_{x^*}(F,\nu)}{4\pi} \log|t'| \\ &+ \frac{1}{4\pi^2}\left(H(x^*)\pi\log|t'| - \frac{\pi}{4}\left(\frac{\kappa_2(x^*)t_1^2 + \kappa_1(x^*)t_2^2}{|t'|^2} - \frac{\kappa_1(x^*)t_1^2 + \kappa_2(x^*)t_2^2}{|t'|^2}\right)\right) \\ &+ \frac{1}{4\pi}\sum_\alpha \frac{F_\alpha(0)t_\alpha}{|t'|}. \end{aligned}$$

We can re-write this above expression in a co-ordinate free manner using the following proposition

Proposition D.0.1. *Let $x^* \in \partial M$ and $\kappa_1(x^*)$ and $\kappa_2(x^*)$ be the eigenvalues of the shape operator at x^*. Assume that $x = x^\varepsilon(s'; x^*)$, $y = x^\varepsilon(t'; x^*)$, $r = |s' - t'|$ and $t' = s + r\omega$. Then, for $\varepsilon > 0$ sufficiently small, we have that*

$$d_g(x,y)^{-1} = \varepsilon^{-1}r^{-1} + \varepsilon r^{-1}A^1(\varepsilon, s', r, \omega),$$

$$d_h(x,y)^{-1} = \varepsilon^{-1}r^{-1} + \varepsilon r^{-1}A^2(\varepsilon, s', r, \omega),$$

$$\mathrm{II}_x\left(\frac{\exp^{-1}_{x;h}y}{|\exp^{-1}_{x;h}y|_h}, \frac{\exp^{-1}_{x;h}y}{|\exp^{-1}_{x,h}y|_h}\right) = \left(\kappa_1(x^*)\frac{(s_1-t_1)^2}{r^2} + \kappa_2(x^*)\frac{(s_2-t_2)^2}{r^2}\right) + \varepsilon R^1_\varepsilon(t', \omega, r),$$

$$\mathrm{II}_x\left(\star\frac{\exp^{-1}_{x;h}y}{|\exp^{-1}_{x;h}y|_h}, \star\frac{\exp^{-1}_{x;h}y}{|\exp^{-1}_{x,h}y|_h}\right) = \left(\kappa_2(x^*)\frac{(s_1-t_1)^2}{r^2} + \kappa_1(x^*)\frac{(s_2-t_2)^2}{r^2}\right) + \varepsilon R^2_\varepsilon(t', \omega, r),$$

$$h_x\left(F^{\|}(x), \frac{\exp_{x;h}(y)}{|\exp_{x;h}(y)|_h}\right) = \frac{F_1(t')(s_1-t_1) + F_2(t')(s_2-t_2)}{r} + \varepsilon R^3_\varepsilon(t', \omega, r),$$

where A^1, A^2, and A^3 are smooth in $[0,\varepsilon_0] \times \mathbb{D} \times \mathbb{R} \times S^1$ and R^1_ε, R^2_ε, R^3_ε are smooth in $\mathbb{D} \times S^2 \times [0,\rho]$ with derivatives of all orders uniformly bounded in ε.

Proof. The first two statements were proved in Corollaries 2.6 and 2.3 of [NTT21], respectively. The next two results were proved in Corollary 2.9 of [NTT21]. The last one follows from Lemma 2.8. $\square$

Therefore, we have that

$$
\begin{aligned}
I(x) = \Phi^* I(t') = &- \frac{H(x^*)}{4\pi} \log d_h(x^*, x) + \frac{g_{x^*}(F, \nu)}{4\pi} \log d_h(x^*, x) \\
&+ \frac{1}{16\pi} \left(\mathrm{II}_{x^*} \left(\frac{\exp^{-1}_{x^*;h}(x)}{|\exp^{-1}_{x^*;h}(x)|_h} \right) - \mathrm{II}_{x^*} \left(\frac{\star \exp^{-1}_{x^*;h}(x)}{|\exp^{-1}_{x^*;h}(x)|_h} \right) \right) \\
&+ \frac{1}{4\pi} h_{x^*} \left(F^{\|}(x^*), \frac{\exp^{-1}_{x^*;h}(x)}{|\exp^{-1}_{x^*;h}(x)|_h} \right).
\end{aligned}
$$

Corollary 2.5 from [NTT21] additionally yields a further refinement for the asymptotic expansion of $G_{\partial M}(x^*, x)$

$$
\begin{aligned}
G_{\partial M}(x^*, x) = &\frac{1}{2\pi} d_g(x^*, x)^{-1} - \frac{H(x^*)}{4\pi} \log d_h(x^*, x) + \frac{g_{x^*}(F, \nu)}{4\pi} \log d_h(x^*, x) \\
&+ \frac{1}{16\pi} \left(\mathrm{II}_{x^*} \left(\frac{\exp^{-1}_{x^*}(x)}{|\exp^{-1}_{x^*}(x)|_h} \right) - \mathrm{II}_{x^*} \left(\frac{\star \exp^{-1}_{x^*}(x)}{|\exp^{-1}_{x^*}(x)|_h} \right) \right) \\
&+ \frac{1}{4\pi} h_{x^*} \left(F^{\|}(x^*), \frac{\exp^{-1}_{x^*}(x)}{|\exp^{-1}_{x^*}(x)|_h} \right) + \Psi^{-3}(\partial M).
\end{aligned}
$$

The above work yields an expression for $G_{\partial M}$ centred at the origin of the window $x^* \in \Gamma_{\varepsilon,a}$. We however need an expansion for $G_{\partial M}(x, y)$ for x, y close in $\Gamma_{\varepsilon,a}$. In particular, we need an expression given by (B.1). In order to do so, we simply invoke Proposition D.0.1 once more to obtain the following

$$
\begin{aligned}
G_{\partial M}(x, y) = &\frac{1}{2\pi} d_g(x, y)^{-1} - \frac{H(x)}{4\pi} \log d_h(x, y) + \frac{g_x(F, \nu)}{4\pi} \log d_h(x, y) \\
&+ \frac{1}{16\pi} \left(\mathrm{II}_x \left(\frac{\exp^{-1}_{x;h}(y)}{|\exp^{-1}_{x;h}(y)|_h} \right) - \mathrm{II}_x \left(\frac{\star \exp^{-1}_{x;h}(y)}{|\exp^{-1}_{x;h}(y)|_h} \right) \right) \\
&+ \frac{1}{4\pi} h_x \left(F^{\|}(x), \frac{\exp^{-1}_{x;h}(y)}{|\exp^{-1}_{x;h}(y)|_h} \right) + \Psi^{-3}(\partial M).
\end{aligned}
$$

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