

A doubly nonlinear fractional diffusive equation

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Statement of originality

I, *Timothy Allen Collier*, declare that this thesis , submitted in fulfilment of the requirements for the conferral of the degree *Doctor of Philosophy (PhD)* from the University of Sydney, is wholly my own work unless otherwise referenced or acknowledged. This document has not been submitted for qualifications at any other academic institution.

I certify that the intellectual content of this thesis is the product of my own work and that all the assistance received in preparing this thesis and sources have been acknowledged.

Statement of authorship attribution

This thesis is based on two papers in preparation written jointly with my supervisor Daniel Hauer.

Timothy Allen Collier

As supervisor for the candidature upon which this thesis is based, I can confirm that the authorship attribution statements above are correct.

Dr. Daniel Hauer

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Abstract

This thesis focuses on the ‘doubly nonlinear fractional diffusive equation’, a doubly nonlinear nonlocal parabolic initial boundary value problem driven by the fractional p -Laplacian equipped with homogeneous Dirichlet boundary conditions on a domain Ω in $\mathbb{R}^d$ and composed with a power-like function. The model problem of this thesis is the equation

$$\begin{cases} \frac{\partial u}{\partial t} + (-\Delta_p)^s u^m + f(x, u) = g(x, t) & \text{in } \Omega \times (0, T), \\ u = 0 & \text{in } \mathbb{R}^d \setminus \Omega \times (0, T), \\ u(x, 0) = u_0 & \text{on } \Omega. \end{cases}$$

We first generalize the nonlinear term u^m , replacing $\cdot^m$ by a continuous, strictly increasing function ϕ . Here we establish well-posedness in L^1 in the sense of mild solutions and a comparison principle. For domains with finite measure and with restricted initial data we obtain that mild solutions of the inhomogeneous evolution problem are strong and distributional.

We then consider the power-like case $\phi(r) = r^m$ where we obtain further regularity properties. In particular, we have an $L^\ell - L^\infty$ regularizing effect for mild solutions (and therefore also for strong solutions), also known as ultracontractivity. We further obtain derivative and energy estimates for this problem. Using these, we extend the previous strong regularity result to obtain strong distributional solutions on general open domains with initial data in L^1 . Moreover, we prove local and global Hölder continuity results in restricted cases as well as a comparison principle that yields extinction in finite time of mild solutions to the homogeneous evolution equation.

We finally restrict to the doubly nonlinear fractional diffusive equation on $\mathbb{R}^d$ without forcing terms, where we investigate self-similarity properties and, in particular, the asymptotic behaviour of solutions for large times. The main result in this case is the existence of Barenblatt solutions, however in finding these we also prove an Aleksandrov symmetry principle for solutions and estimate solutions by global bounding functions which are integrable in space.

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Chapter 1

Introduction

1.1 Motivation

Nonlocal differential equations have gained increasing interest for mathematical modelling in a variety of contexts, particularly for investigating singular, multiscale and long-range interactions of systems. These are applied in a wide range of fields including, for example, physics [42, 58, 59, 88, 97], material sciences [25, 93, 113], statistical mechanics [77, 95, 128], population dynamics [45, 107, 109], finance and stochastic processes [5, 24, 49], neural networks [70, 101] and image processing [35, 68, 130], where we include only a sample of reference material. The analytical complexity and potential intractability of nonlocal models has historically limited their adoption, however computational power and development of the general theory have made such problems more approachable. We refer to [36, 38, 55–57, 84, 96] and references therein for an overview of this extensive topic.

The fractional Laplacian in particular occurs as a nonlocal diffusive operator and as a fractional pseudo-differential operator. As such, it is used to model nonlocal particle interactions such as crystal structures in material science and anomalous diffusion in physical processes, including surface diffusion, such as in the case of the quasi-geostrophic equation [25, 36, 38, 42]. Moreover, the fractional Laplacian models Lévy processes with jumps in stochastic and kinematic models [5, 24, 49, 76, 77, 88, 95, 97, 128] and has been applied in image processing [68]. As a natural nonlocal analogue to the Laplacian, the fractional Laplacian also provides a useful model for developing tools to analyse nonlocal and anomalous diffusion more generally. The fractional Laplacian and related problems have been well-studied by many authors, we refer to [37, 39, 43, 105, 108] for an overview of this operator and some key related problems. Nonlinear variations such as the fractional p -Laplacian have also been considered, motivated by diffusion models and drift-based tug-of-war games [26], while more general integro-differential and variational settings have been applied in [49, 57, 58, 93, 113], for example.

A standard equation involving the fractional p -Laplacian is the asso-

ciated evolution equation,

$$\frac{\partial u}{\partial t}(x, t) + (-\Delta_p)^s u(x, t) = 0, \quad (1.1.1)$$

for x in an appropriate spatial domain, usually a subset of $\mathbb{R}^d$, and t in an interval of time. The mathematical analysis of (1.1.1) has gained significant interest in recent years, for example in [67, 94, 106, 116, 121, 123, 125, 126]. In particular, Mazón et al. [94] (see also, [125]) obtained existence and uniqueness of strong distributional solutions to (1.1.1) equipped with either homogeneous Dirichlet or Neumann boundary conditions. Giacomoni and Tiwari [67] obtained well-posedness in L^∞ of a certain form of strong distributional solutions to equation (1.1.1) equipped with homogeneous Dirichlet boundary data and forcing terms depending on x and the solution u .

Global $L^\ell - L^\infty$ regularity estimates, $1 \leq \ell < \infty$, for solutions to the parabolic equation (1.1.1) with homogeneous Dirichlet boundary conditions and a forcing term which is Lipschitz continuous with respect to u have been obtained by Coulhon and Hauer in the monograph [50]. Vázquez [121, 126] has applied symmetry properties of the fractional p -Laplacian in order to obtain global bounds for solutions and the existence of Barenblatt solutions for $p > \frac{2d}{d+s}$, and investigated the associated asymptotic behaviour. For the elliptic problem, Iannizzotto, Mosconi and Squassina [73] have obtained global Hölder regularity for bounded domains up to the boundary while Brasco, Lindgren and Schikorra [30] considered local Hölder regularity for $p \geq 2$.

As in the local case, a natural extension is to consider a porous medium type nonlinearity within the diffusive term, that is, equations of the form

$$\frac{\partial u}{\partial t} + (-\Delta)u^m = 0 \quad (1.1.2)$$

where the Laplacian may be replaced by a general diffusive operator such as the fractional Laplacian or, in the case of this thesis, the fractional p -Laplacian. Here we emphasize our notation for powers,

$$r^q := \begin{cases} |r|^{q-1}r & \text{if } r \neq 0, \\ 0 & \text{if } r = 0, \end{cases}$$

for all $r \in \mathbb{R}$ with $q \geq 0$. In particular, when $q \geq 1$, we can write $r^q = |r|^{q-1}r$. In this way we interpret r^q as a signed function which is crucial both for the fractional p -Laplacian and so that the nonlinearity r^m is an increasing function on $\mathbb{R}$. In the case of the Laplacian, (1.1.2) is the *porous medium equation* (equivalently, porous media equation) motivated by the study of fluid flow through a porous medium. We refer to [7, 72, 124, 127] for an overview of the mathematical analysis of this problem. When the Laplacian in (1.1.2) is replaced by the p -Laplacian, we refer to the resulting equation as the *doubly nonlinear diffusive equation* (also

known as the doubly nonlinear porous medium equation). This has been studied in [3, 8–10, 110, 115, 119], with Saá as well as Stan and Vázquez investigating the long-time behaviour and finding convergence to a profile function in the degenerate case. Porzio [104] considered decay estimates of a class of such evolution problems, finding $L^\ell - L^\infty$ estimates.

Similarly, we can consider the fractional Laplacian case which we call the *fractional porous medium equation*. This and related problems have also been studied by various authors, including [27–29, 62, 99, 100, 132, 133]. In particular, the authors in [100] develop existence and uniqueness results for $m > \frac{(d-2s)^+}{d}$ based on the L^1 -contraction semigroup setting. Moreover, Hölder regularity and finite time of extinction is obtained by Kim and Lee in [83] when $\frac{d-2s}{d+2s} < m < 1$.

There are also a number of interesting generalizations to such problems involving the fractional Laplacian or fractional p -Laplacian which we do not consider in this thesis, but for which the methods presented here may be useful. For example, the kernel $|x - y|^{d+sp}$ in the fractional p -Laplacian (see Section 2.3) may be replaced by a more general $K(x, y)$ which satisfies certain two-sided bounds (for example, [81]) so that similar estimates (in particular the key Sobolev embeddings of this setting) may still be obtained. Another important consideration is the case of the fractional p -Laplacian with $p = 1$, the so-called fractional 1-Laplacian. This has been considered for the fractional p -Laplacian evolution equation in [94], using accretivity properties to prove the existence and uniqueness of strong solutions. In this case the term $(u(x) - u(y))^{p-1}$ is replaced by an antisymmetric, bounded function $\eta(x, y)$ which approximates $\text{sign}(u(x) - u(y))$, in particular, satisfying $\|\eta\|_{L^\infty(\Omega) \times L^\infty(\Omega)} \leq 1$, $\eta(x, y) \in \text{sign}(u(x) - u(y))$ and $\eta(x, y) = -\eta(y, x)$ for a.e. x, y in the domain. Due to these additional complications, in this text we restrict to $1 < p < \infty$.

In this thesis we consider as our model problem a natural extension of these nonlocal and nonlinear diffusion equations, the porous medium type equation with diffusion governed by the fractional p -Laplacian. That is,

$$\frac{\partial u}{\partial t} + (-\Delta_p)^s u^m = 0 \quad (1.1.3)$$

for $m > 0$, which we call the *doubly nonlinear fractional diffusive equation* in light of comparable problems. This has also been referred to in the literature more generally as a doubly nonlinear nonlocal evolution equation or a fractional p -Laplacian parabolic problem. Such problems have gained interest much more recently, being considered in [65, 66] in the form

$$\frac{\partial}{\partial t} u^q + (-\Delta_p)^s u = 0. \quad (1.1.4)$$

In particular, Giacomoni, Gouasmia and Mokrane [66] consider (1.1.4) on bounded domains in $\mathbb{R}^d$ with forcing terms of the form $f(x, u)$ and $h(x, t)u^{q-1}$ and find well-posedness of positive solutions in a certain strong

distributional sense. They also prove regularity estimates and asymptotic behaviour with convergence to a stationary solution. Gosh et. al. [65] extend (1.1.4) based on the double phase equation and prove existence of variational solutions.

The setting of (1.1.4) is motivated by the accretivity properties of the operator $(-\Delta_p)^s$ while (1.1.3) is based on properties of the composed operator $(-\Delta_p)^s \varphi$ and the standard setting for evolution equations. These equations are of course equivalent, however each results in a different focus when obtaining regularity results for solutions u .

We generalize (1.1.3) by replacing the nonlinearity u^m by a function $\varphi(u)$ and including forcing terms f and g . In particular, in this thesis we investigate equations of the form

$$\frac{\partial u}{\partial t}(x, t) + (-\Delta_p)^s \varphi(u(x, t)) + f(x, u(x, t)) = g(x, t) \quad (1.1.5)$$

for $x \in \Omega$, an open subset of $\mathbb{R}^d$, and $t \in (0, T)$ for $T > 0$. In our case, φ is continuous, strictly increasing and satisfies $\varphi(0) = 0$ and f is Lipschitz continuous with respect to u . Moreover, we consider the Dirichlet problem on an open domain in $\mathbb{R}^d$ with further restrictions detailed in each result. Here the operator $(-\Delta_p)^s \varphi$ models a (singular or degenerated) nonlocal diffusion. Due to the structure of the fractional p -Laplacian, the doubly nonlinear fractional diffusive equation fits into the wider context of nonlocal and degenerate parabolic problems as well as the theory of accretive operators which we apply.

In this thesis, we extend results relating to the fractional p -Laplacian evolution equation, doubly nonlinear diffusive equation and the fractional porous medium equation to equations (1.1.3) and (1.1.5), in particular applying the setting of accretive operators to obtain well-posedness and regularity results for (1.1.5) and further using the self-similar scaling properties of (1.1.3) to investigate asymptotic behaviour.

1.2 Thesis outline

This thesis is comprised of two parts forming Chapters 2 and 3, based on two papers submitted for publication. We first focus on obtaining, under appropriate restrictions, the well-posedness of a Dirichlet problem associated with (1.1.5) on an open domain $\Omega \subseteq \mathbb{R}^d$ with a general nonlinearity φ which is continuous, strictly increasing and satisfies $\varphi(0) = 0$. Moreover, we include forcing terms f and g , where f is Lipschitz in u and g is integrable in space and time. In particular, we consider

$$\begin{cases} \frac{\partial u}{\partial t} + (-\Delta_p)^s \varphi(u) + f(x, u) = g(x, t) & \text{in } \Omega \times (0, T), \\ u = 0 & \text{in } \mathbb{R}^d \setminus \Omega \times (0, T), \\ u(x, 0) = u_0(x) & \text{on } \Omega, \end{cases} \quad (1.2.1)$$

with initial data $u_0 \in L^1(\Omega)$. We refer to Chapter 2 for details on the restrictions of φ , f and g . We consider solutions in $L^1(\Omega)$ in the sense of *mild* solutions and *strong distributional* solutions which are strong (in time) and distributional (in space) (see Definition 2.2.4 and Definition 2.2.7). Moreover, we obtain regularity properties of solutions to this equation. Restricting to the power-like case $\varphi(r) = r^m$, $r \in \mathbb{R}$, with $m > 0$ we find further regularity properties and use these to improve the well-posedness results in this setting. The main results of Chapter 2 are presented explicitly in Section 2.1.

Well-posedness is a standard aim for such problems as the existence and uniqueness of solutions may be used to justify the equation from a modelling perspective, and also provides insight into the appropriate analytical setting. Mild solutions, essentially limits of solutions to the time-discretized problem, provide information about convergence of the discretized problem but also provide prospective solutions which may be regularized to obtain stronger solutions, as is the case in this thesis. Importantly, solutions which are strong in time (see Definition 2.2.6) are necessarily mild solutions (see [12]), so that our regularity and comparison results for mild solutions also apply to strong solutions. However, in general this is not the case for solutions which are distributional in space which are not also strong in time.

The existence of mild solutions is based on the general theory of m -accretive operators which we introduce in Section 2.2 (see also [12] and [50]). In the case of the fractional p -Laplacian evolution equation, since $(-\Delta_p)^s$ is in fact m -completely accretive in $L^2(\Omega)$, Mazón, Rossi and Toledo [94] were able to prove the existence of mild solutions in $L^2(\Omega)$ and moreover, extend these to strong distributional solutions. In Section 2.3, we introduce the setting for the fractional p -Laplacian as a subdifferential in $L^2(\Omega)$ and the doubly nonlinear operator $(-\Delta_p)^s \varphi$ as an operator on $L^1(\Omega)$. We then apply the accretivity results of [50] which provide m - T -accretivity of the composed operator $(-\Delta_p)^s \varphi$ in $L^1(\Omega)$. When φ is sufficiently regular we obtain a density result that provides the existence of mild solutions for all initial data in $L^1(\Omega)$. We prove this in the more general context of the subdifferential of a proper, lower semicontinuous, convex functional composed with a continuous, strictly increasing φ . We state the existence result for mild solutions to the fractional p -Laplacian in Theorem 2.1.1.

We then apply the strong chain-rule regularity effect and existence method of [22] which is proved for such degenerate parabolic equations coming from the subdifferential of an energy functional. From this we obtain the existence of strong distributional solutions when Ω has finite Lebesgue measure and the initial data u_0 is restricted to $\hat{D}((-\Delta_p)^s|_{L^1 \cap \infty} \varphi)$, essentially requiring that $u_0 \in L^\infty(\Omega)$ and $\varphi(u_0)$ is in the closure of $(-\Delta_p)^s$ in $L^1(\Omega) \times L^1(\Omega)$ (see Definition 2.0.1 and (2.1.4)). Strong solutions are of natural interest in such a theory and so have been investigated for many related problems. For the fractional p -Laplacian

evolution equation, these have been found in [94] and for the fractional porous medium equation on $\mathbb{R}^d$, in [100]. We will later improve on this result when restricting the nonlinearity φ .

For the remainder of this chapter we restrict to the doubly nonlinear fractional diffusive equation, taking $\varphi(r) = r^m$, $r \in \mathbb{R}$ for $m > 0$. That is,

$$\begin{cases} \frac{\partial u}{\partial t} + (-\Delta_p)^s u^m + f(x, u) = g(x, t) & \text{in } \Omega \times (0, T), \\ u = 0 & \text{in } \mathbb{R}^d \setminus \Omega \times (0, T), \\ u(x, 0) = u_0(x) & \text{on } \Omega. \end{cases} \quad (1.2.2)$$

This power-like φ provides significant regularity properties compared to (1.2.1). These properties are due, in large part, to the homogeneity of $(-\Delta_p)^{s \cdot m}$ which provides self-similar scaling-type properties for (1.2.2) and the power-like structure of $r \mapsto r^m$, allowing for conversion between estimates of u^m and u in the L^q space setting.

First restricting to the case $m \geq 1$ and requiring either that Ω is bounded or that $sp < d$, we obtain $L^\ell - L^\infty$ regularizing decay estimates. That is, for mild solutions $u(t)$ to (1.2.1) with $u_0 \in L^\ell(\Omega)$, we have the immediate regularizing effect that $u(t) \in L^\infty(\Omega)$ for all $t \in (0, T)$. In the case

$$m(p-1) > 1 - \frac{sp}{d}, \quad (1.2.3)$$

we have $L^1 - L^\infty$ estimates. This type of estimate has been well-explored in the local setting, known as decay estimates or ultracontractive estimates. We refer to [52, 104] and the references therein. In the nonlocal case, this has been considered particularly for problems associated with the fractional Laplacian such as [40]. Also in [48] where Dirichlet problems with operators arising from bilinear, symmetric forms such as the fractional Laplacian are considered. For these estimates we are able to retain both the Lipschitz perturbation f and the forcing term g , where the integrability of g in space and time must be sufficiently regular.

Noting that the doubly nonlinear operator $(-\Delta_p)^{s \cdot m}$ is homogeneous of order $m(p-1)$, we apply the regularizing effect of homogeneous operators [20] to obtain Lipschitz continuity, derivative estimates and energy estimates for (1.2.2). We also apply the previous $L^\ell - L^\infty$ regularity effect to extend these results.

We then apply these *a priori* estimates to the previous strong solutions of equation (1.2.2) on bounded domains to approximate domains with possibly infinite measure, including the full space $\mathbb{R}^d$. Hence we obtain existence and uniqueness of strong distributional solutions for initial data in L^∞ and, when applying the $L^1 - L^\infty$ estimates, for all initial data in L^1 . This result is stated in Theorem 2.1.8.

Finally, we apply the elliptic results of [30] and [73] to obtain local Hölder regularity when $m > s$ and $p \geq 2$ and global Hölder regularity when restricting to the case $m = 1$.

In Chapter 3, we continue to focus on the doubly nonlinear fractional diffusive equation, but now only on the full space $\mathbb{R}^d$ and without forcing or perturbation terms, that is $f \equiv 0$ and $g \equiv 0$. In particular, we consider

$$\begin{cases} \frac{\partial u}{\partial t} + (-\Delta_p)^s u^m = 0 & \text{in } \mathbb{R}^d \times (0, T), \\ u(x, 0) = u_0(x) & \text{on } \mathbb{R}^d. \end{cases} \quad (1.2.4)$$

With no forcing terms and the full-space setting, (1.2.4) satisfies key self-similarity properties which allow us to examine the asymptotic behaviour of solutions. In particular, the aim of this chapter is to prove the existence and uniqueness of Barenblatt solutions to (1.2.4). The methods here are based on two papers by Vázquez in the case of the fractional p -Laplacian evolution equation, [123] and [126] corresponding to the two decay regimes of solutions to (1.2.4) as $|x| \rightarrow \infty$. Here we see the importance of the homogeneity condition (1.2.3) more clearly as it plays a key role in the self-similar scaling transformations of (1.2.4). The importance of Barenblatt solutions is clear in the case of linear parabolic equations due to their role as fundamental solutions, however in nonlinear setting they provide important information about the asymptotic behaviour of solutions. We refer to [15, 47, 54, 120] and further discussion in Chapter 3.

To prove the existence of Barenblatt solutions we first obtain two key results. First an Aleksandrov symmetry principle [2, 46], which states that if the initial data of (1.2.4) can be compared around a reflection mapping $\Pi : \mathbb{R}^d \rightarrow \mathbb{R}^d$, i.e. $u_0(x) \leq u_0(\Pi(x))$ for all x on one half of the hyperplane being reflected around, then the mild solution u satisfies the same comparison for all $0 < t < T$. This symmetry principle has found great use in both the linear and nonlinear setting. For example in the porous medium equation [127] and the fractional p -Laplacian evolution equation [126].

Second, we obtain global barriers for solutions to (1.2.4) with bounded initial data having compact support, again applying methods from the case $m = 1$ given in [123, 126]. That is, we find functions $H(x, t)$ which are bounded and integrable in space for $t > 0$ such that for u a mild solution to (1.2.4) with bounded initial data having compact support, $|u(x, t)| \leq H(x, t)$ almost everywhere in $\mathbb{R}^d \times [0, T]$. We note that H depends on the constants of (1.2.4), the size of $\|u_0\|_\infty$ and the support of u_0 . As in [123, 126], we find different decay regimes as $|x| \rightarrow \infty$ in H depending on the homogeneity $m(p - 1)$ with the critical case when

$$m(p - 1) = \frac{d}{d + sp}.$$

The self-similar scaling properties of (1.2.4) are crucial here and we again see the importance of the homogeneity restriction (1.2.3).

Chapter 2

Well-posedness in L^1

Throughout this chapter, we let Ω be an open set in $\mathbb{R}^d$, $d \geq 1$, $0 < T < \infty$, $1 < p < \infty$ and $0 < s < 1$. We focus on the following initial boundary value problem

$$\begin{cases} \frac{\partial u}{\partial t} + (-\Delta_p)^s \varphi(u) + f(x, u) = g(x, t) & \text{in } \Omega \times (0, T), \\ u = 0 & \text{in } \mathbb{R}^d \setminus \Omega \times (0, T), \\ u(x, 0) = u_0(x) & \text{on } \Omega, \end{cases} \quad (1.2.1)$$

for given $u_0 \in L^1$ and $g \in L^1(0, T; L^1)$, where we abbreviate the Lebesgue space $L^q(\Omega)$ by L^q for $1 \leq q \leq \infty$, and impose the following conditions on φ and f :

$$\varphi \in C(\mathbb{R}) \text{ is a strictly increasing function satisfying } \varphi(0) = 0, \quad (2.0.1)$$

and $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ admits the following properties:

$$f \text{ is a } \textit{Lipschitz-continuous Carathéodory} \text{ function; that is,} \quad (2.0.3a)$$

$$\begin{cases} \text{for every } u \in \mathbb{R}, x \mapsto f(x, u) \text{ is measurable on } \Omega, \text{ and there is} \\ \text{an } \omega > 0 \text{ such that} \\ |f(x, u_1) - f(x, u_2)| \leq \omega |u_1 - u_2| & \text{for all } u_1, u_2 \in \mathbb{R}, \\ \text{uniformly for a.e. } x \in \Omega, \end{cases}$$

$$\text{and } f(x, 0) = 0 \text{ for a.e. } x \in \Omega. \quad (2.0.3b)$$

The conditions on f here are natural in the accretive setting as we will consider the Nemytskii operator associated to f , which we denote by $F : L^1 \rightarrow L^1$. Then F will also be Lipschitz continuous with respect to u and the relevant accretivity properties of $(-\Delta_p)^s \varphi$ transfer to quasi-accretive properties for the operator $(-\Delta_p)^s \varphi + F$ (see Section 2.2 and also [12]).

We introduce the function space and operator setting for (1.2.1) in detail in Section 2.2 and Section 2.3. Here we briefly describe the key notions which inform the main results of this chapter, presented in Section 2.1.

The doubly nonlinear nonlocal operator $(-\Delta_p)^s \varphi$ in the evolution problem (1.2.1) is the composition of the (variational) Dirichlet fractional p -Laplacian (see Section 2.3.4) given by

$$\langle (-\Delta_p)^s u, v \rangle := \int_{\mathbb{R}^{2d}} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y)) (v(x) - v(y))}{|x - y|^{d+ps}} dy dx \quad (2.0.3)$$

for every $u, v \in W_0^{s,(2,p)}$ where we write $W_0^{s,(2,p)}$ for the mixed fractional Sobolev space $W_0^{s,(2,p)}(\Omega)$ (see Section 2.3.1 for the details) and a function φ satisfying (2.0.1). In this setting, the fractional p -Laplacian may be viewed as the subdifferential in L^2 of the energy functional,

$$\mathcal{E}(u) = \begin{cases} \frac{1}{2p} [u]_{s,p}^p & \text{if } u \in W_0^{s,(2,p)}, \\ \infty & \text{if } u \in L^2 \setminus W_0^{s,(2,p)}, \end{cases} \quad (2.0.4)$$

where $[\cdot]_{s,p}$ is the Gagliardo seminorm on $\mathbb{R}^d$ given by

$$[u]_{s,p} = \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^p}{|x - y|^{d+sp}} dy dx \right)^{1/p} \quad (2.0.5)$$

(see Proposition 2.3.1). The choice of the mixed Sobolev space $W_0^{s,(2,p)}$, rather than the more common space $W_0^{s,p}$, is important for ensuring lower semicontinuity of (2.0.4) in L^2 and hence the density and accretivity properties of the composition of the subdifferential with φ .

In order to consider equation (1.2.1) in L^1 , we realize the doubly nonlinear operator $(-\Delta_p)^s \varphi$ as an operator in L^1 . For this we restrict $(-\Delta_p)^s$ as a graph to $L^{1 \cap \infty} \times L^{1 \cap \infty}$ where we denote the intersection space $L^{1 \cap \infty} := L^1 \cap L^\infty$. Then we identify $(-\Delta_p)^s \varphi$ with the closure of this composition in $L^1 \times L^1$ with the following definition as a graph. We refer to Section 2.3.4 for more details.

Definition 2.0.1. Let $(-\Delta_p)^s$ be the fractional p -Laplacian in L^2 and $(-\Delta_p)^s|_{L^{1 \cap \infty}}$ be the part of $(-\Delta_p)^s$ in $L^{1 \cap \infty} \times L^{1 \cap \infty}$ given by

$$(-\Delta_p)^s|_{L^{1 \cap \infty}} := \left\{ (u, v) \in L^{1 \cap \infty} \times L^{1 \cap \infty} \mid v = (-\Delta_p)^s u \right\}.$$

Suppose $\varphi : \mathbb{R} \rightarrow \mathbb{R}$. Then, we call the operator $\overline{(-\Delta_p)^s|_{L^{1 \cap \infty}} \varphi}^{L^1}$ given by

$$\left\{ (u, v) \in L^1 \times L^1 \mid \begin{array}{l} \text{there exists } ((u_k, v_k))_{k \geq 1} \text{ such that} \\ v_k = (-\Delta_p)^s|_{L^{1 \cap \infty}} \varphi(u_k) \text{ for all } k \geq 1, \\ \lim_{k \rightarrow \infty} u_k = u \text{ in } L^1 \text{ and } \lim_{k \rightarrow \infty} v_k = v \text{ in } L^1 \end{array} \right\}$$

the nonlocal doubly nonlinear operator in L^1 , since it provides a realization of the operator $(-\Delta_p)^s \varphi$ in L^1 .

Notation 2.0.2. For convenience, we use the notation $(-\Delta_p)^s \varphi$ for $\overline{(-\Delta_p)^s|_{L^{1 \cap \infty}} \varphi}^{L^1}$.

Importantly, with Definition 2.0.1, solutions to (1.2.1) do not immediately satisfy the regularity property $\varphi(u(t)) \in D((-\Delta_p)^s) \subset W_0^{s,(2,p)}$. In particular, this will be the case for mild solutions. However, in the case of strong distributional solutions, for a.e. $t \in (0, T)$, u is differentiable at t , $\varphi(u(t)) \in W_0^{s,(2,p)}$ and $(-\Delta_p)^s \varphi(u(t)) = g(\cdot, t) - f(\cdot, u(t)) - u_t(t)$ in L^1 by definition. We similarly have this regularity for distributional solutions (see Definition 2.2.7). We now state the main results of this chapter.

2.1 Main results

We begin by stating our well-posedness results in the sense of *mild solutions* in L^1 and a comparison principle. Here we use the notation $[u]^+$ to denote $\max\{u, 0\}$, the positive part of u , and $[u]^1 = u$. We also use the notation of q -brackets defined by Notation 2.2.2.

Theorem 2.1.1 (Well-posedness & comparison principle in L^1). *Let Ω be an open domain in $\mathbb{R}^d$, $d \geq 1$ and $T > 0$. Let $1 < p < \infty$, $0 < s < 1$, and suppose that φ , f satisfy (2.0.1) and (2.0.3a)-(2.0.3b), respectively. Then the following statements hold.*

- (1) *If $\varphi \in W_{\text{loc}}^{1,q}(\mathbb{R})$ for $q \in (\frac{1}{1-s}, \infty]$, then one has that*

$$\overline{D((-\Delta_p)^s_{|L^{1 \cap \infty}} \varphi)}^{L^1} = L^1. \quad (2.1.1)$$

- (2) *For every initial value $u_0 \in \overline{D((-\Delta_p)^s_{|L^{1 \cap \infty}} \varphi)}^{L^1}$ and $g \in L^1(0, T; L^1)$, there is a unique mild solution $u \in C([0, T]; L^1)$ of the initial boundary value problem (1.2.1). Moreover, for all $1 \leq q \leq \infty$, one has that*

$$\|u(t)\|_q \leq e^{\omega t} \|u_0\|_q + \int_0^t e^{\omega(t-r)} \|g(r)\|_q \, dr \quad (2.1.2)$$

for every $u_0 \in \overline{D((-\Delta_p)^s_{|L^{1 \cap \infty}} \varphi)}^{L^1} \cap L^q$ and $g \in L^1(0, T; L^1 \cap L^q)$.

- (3) *For every $y_1, y_2 \in L^1$, $g_1, g_2 \in L^1(0, T; L^1)$, and corresponding mild solutions u_1, u_2 of (1.2.1) with initial data y_1, y_2 respectively, one has*

$$\begin{aligned} \|[u_1(t) - u_2(t)]^\nu\|_1 &\leq e^{\omega t} \|[y_1 - y_2]^\nu\|_1 \\ &\quad + \int_0^t e^{\omega(t-r)} [u_1(r) - u_2(r), g_1(r) - g_2(r)]_\nu \, dr \end{aligned} \quad (2.1.3)$$

for all $0 \leq t \leq T$, and $\nu \in \{+, 1\}$.

The statements of Theorem 2.1.1 follow as an application of the existence theory developed by [22], in the monograph [50] by Coulhon and

Hauer, and by classical nonlinear semigroup theory (cf. [12, 21]). Since [21] may not be readily available, we give corresponding references, particularly to [12], where relevant. We give the details of the proof in Section 2.4. In the case $\varphi(r) = r^m$ for all $r \in \mathbb{R}$, the restriction $\varphi \in W_{\text{loc}}^{1,q}(\mathbb{R})$ for $q \in (\frac{1}{1-s}, \infty]$, which we use for the density result (2.1.1), is satisfied for $m > s$. Importantly, this means that our approximation results and results with initial data in L^1 also rely on this condition.

By restricting Ω and the domain of the initial data u_0 , we improve the regularity of these solutions in the next result. Introducing the restriction on the domain of u_0 , for $1 < p < \infty$ and $0 < s < 1$, we set

$$\tilde{D}((-\Delta_p)^s_{|L^{1 \cap \infty}}) = \left\{ u \in L^{1 \cap \infty} \left| \begin{array}{l} \exists (u_n, h_n)_{n \geq 1} \subseteq (-\Delta_p)^s_{|L^{1 \cap \infty}} \text{ s.t.} \\ u_n \rightarrow u \text{ in } L^1 \text{ and} \\ (u_n, h_n)_{n \geq 1} \text{ is bounded in } L^\infty \times L^1. \end{array} \right. \right\}$$

and for every continuous φ , we let

$$\hat{D}((-\Delta_p)^s_{|L^{1 \cap \infty}} \varphi) = \left\{ u \in L^1 \mid \varphi(u) \in \tilde{D}((-\Delta_p)^s_{|L^{1 \cap \infty}}) \right\}. \quad (2.1.4)$$

Then, by taking advantage of [22, Theorem 4.1], we show that for every $u_0 \in \hat{D}((-\Delta_p)^s_{|L^{1 \cap \infty}} \varphi)$, the corresponding mild solutions u of the initial boundary value problem (1.2.1) is strong and distributional. In this theorem, the term $[\cdot]_{s,p}$ denotes the Gagliardo semi-norm (2.0.5). We prove this result in Section 2.5.

Theorem 2.1.2 (Mild solutions are strong and distributional). *Let Ω be an open domain in $\mathbb{R}^d$, $d \geq 1$, of finite Lebesgue measure and $T > 0$. Let $1 < p < \infty$, $0 < s < 1$, $\varphi \in C(\mathbb{R})$ be a strictly increasing function such that $\varphi^{-1} \in AC_{\text{loc}}(\mathbb{R})$. Suppose $f(\cdot, u)$ satisfies (2.0.3a)-(2.0.3b) and let F be the Nemystkii operator of f . Further suppose that $g \in BV(0, T; L^1) \cap L^1(0, T; L^\infty)$. Then for every $u_0 \in \hat{D}((-\Delta_p)^s_{|L^{1 \cap \infty}} \varphi)$, the mild solution u of (1.2.1) is a strong distributional solution in L^1 having the regularity*

$$u \in W^{1,\infty}((0, T); L^1) \cap L^\infty([0, T]; L^\infty) \cap C([0, T]; L^q)$$

for every $1 \leq q < \infty$. Moreover, $\varphi(u)$ has a weak derivative given by

$$\frac{d}{dt} \varphi(u(t)) = \varphi'(u(t)) \frac{du}{dt} \quad \text{in } L^2 \text{ for a.e. } t \in (0, T) \quad (2.1.5)$$

and the function $t \mapsto \mathcal{E}(\varphi(u(t))) = \frac{1}{2p} [\varphi(u(t))]_{s,p}^p$ has derivative given by

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(\varphi(u(t))) &= -\left\| \sqrt{\varphi'(u(t))} \frac{du}{dt}(t) \right\|_2^2 \\ &\quad - \left\langle F(u(t)) - g(t), \frac{du}{dt}(t) \varphi'(u(t)) \right\rangle \end{aligned} \quad (2.1.6)$$

for a.e. $t \in (0, T)$.

This well-posedness result complements and generalizes the recent result by Giacomoni et al. [66] by allowing general, strictly increasing functions φ , by considering sign-changing solutions and considering domains with finite measure rather than bounded domains. We note that the restriction on u_0 for [66] is a similar subset of L^∞ . The notion of well-posedness in [66] is a combination of strong and distributional notions which differ to those presented here. In particular, this is due to their setting with the nonlinearity in the time derivative. Comparing Theorem 2.1.2 with [66] in the case $\varphi(r) = r^m$ for all $r \in \mathbb{R}$, we note that a different regime is studied with $\frac{1}{2p-1} \leq m < 1$ for non-negative solutions compared to $m > 0$.

Theorem 2.1.1 also provides well-posedness for general open domains and for all initial data u_0 in L^1 when $m > s$ but in the much weaker sense of mild solutions. However, in the case $\varphi(r) = r^m$ with $m > s$, by approximating solutions given by Theorem 2.1.1 by solutions from Theorem 2.1.2 on bounded domains, we will extend such a strong distributional result to general open domains (see Theorem 2.1.8). For this, we first obtain regularity properties for solutions of equation (1.2.2).

Hence for the remainder of this chapter we consider the case $\varphi(r) = r^m$ for all $r \in \mathbb{R}$ with $m > 0$. That is,

$$\begin{cases} \frac{\partial u}{\partial t} + (-\Delta_p)^s u^m + f(x, u) = g(x, t) & \text{in } \Omega \times (0, T), \\ u = 0 & \text{in } \mathbb{R}^d \setminus \Omega \times (0, T), \\ u(x, 0) = u_0(x) & \text{on } \Omega. \end{cases} \quad (1.2.2)$$

Our next result is concerned with global $L^\ell - L^\infty$ regularity estimates, $1 \leq \ell < \infty$, for mild solutions u of the initial boundary value problem (1.2.2), implying an immediate smoothing effect. For these regularity estimates, the Sobolev embedding (see, for instance, [53, 90])

$$W^{s,p} \hookrightarrow L^{p_s} \quad \text{with} \quad p_s = \begin{cases} \left(\frac{1}{p} - \frac{s}{d}\right)^{-1} & \text{if } p < \frac{d}{s}, \\ \tilde{p} & \text{if } p = \frac{d}{s}, \\ \infty & \text{if } p > \frac{d}{s}, \end{cases} \quad (2.1.7)$$

and $\tilde{p} \in [p, \infty)$, is crucial, where we write $W^{s,p}$ for $W^{s,p}(\Omega)$. Theorem 2.1.3 is a special case of Theorem 2.7.1 in Section 2.7 as illustrated in Section 2.7.1. Theorem 2.7.1 applies to abstract operators A acting on L^q and satisfying an abstract Sobolev inequality (as introduced in [50]). In particular we apply a De Giorgi iteration (cf. [40]) to obtain an $L^{m+1} - L^\infty$ estimate which is then extrapolated to $L^\ell - L^\infty$ for $1 \leq \ell < m+1$. Here we refine the methods in [40] by Caffarelli and Vasseur, [104] by Porzio and [50] by Coulhon and Hauer.

Theorem 2.1.3 (Global $L^\ell - L^\infty$ estimates). *Let Ω be an open domain in $\mathbb{R}^d$, $d \geq 1$ and $T > 0$. Let $p > 1$, $0 < s < 1$, $m \geq 1$ and $1 \leq \ell < m+1$*

such that

$$m(p-1) > 1 - \frac{\ell sp}{d}. \quad (2.1.8)$$

Suppose that $sp < d$ or Ω is bounded. Further, let $\varphi(r) = r^m$ for $r \in \mathbb{R}$, and $f(\cdot, u)$ satisfy (2.0.3a)-(2.0.3b). Suppose $\rho \geq m+1$ and take $q_s = p_s$ if $p \neq \frac{d}{s}$ and $q_s > \max \left\{ p, 1 + \frac{1}{m}, \frac{p\ell}{m(p-1)+\ell-1}, \frac{p\rho}{m(p-1)+\rho-1} \right\}$ if $p = \frac{d}{s}$. For $\psi > 1$ satisfying

$$\begin{cases} \frac{1}{\rho} < \left(1 - \frac{1}{\psi}\right) p \left(\frac{1}{p} - \frac{1}{q_s}\right) & \text{if } m(p-1) \geq 1, \\ \frac{1}{\rho} \leq \left(1 - \frac{1}{\psi}\right) p \left(\frac{m}{m+1} - \frac{1}{q_s}\right) & \text{if } m(p-1) < 1, \end{cases} \quad (2.1.9)$$

let $g \in L^\psi(0, T; L^\rho) \cap L^1(0, T; L^1 \cap L^{1+m+\delta})$ for some $\delta > 0$. Then for every $u_0 \in L^\ell \cap L^1$ and all $\varepsilon \geq 0$, the mild solution u of (1.2.1) in L^1 satisfies

$$\begin{aligned} \|u(t)\|_\infty &\leq C \max \left\{ e^{\omega\beta_1 t} \left(\frac{1}{t} + \omega\right)^\alpha, e^{\omega\beta_2 t} \|g\|_{L^\psi(0,t;L^\rho)}^\eta \right\}^{\frac{1}{\theta}} (1 + N(t)^\gamma) \\ &\quad \times \left(e^{\omega t} \|u_0\|_\ell + \int_0^t e^{\omega(t-\tau)} \|g(\tau)\|_\ell d\tau + \varepsilon \right)^{\frac{\ell\gamma}{(m+1)\theta}} \end{aligned} \quad (2.1.10)$$

for all $t \in (0, T]$, where we set

$$\begin{aligned} N(t) &= \sup_{s \in (0,t]} \frac{M(\frac{s}{2}) \|g\|_{L^1(\frac{s}{2}, s; L^{m+1})} + e^{\frac{\omega\beta_2 s}{2\gamma}} \|g\|_{L^\psi(0, \frac{s}{2}; L^\rho)}^\eta}{M(s)^{\frac{1}{\theta}} (e^{\omega s} \|u_0\|_\ell + \int_0^s e^{\omega(s-\tau)} \|g(\tau)\|_\ell d\tau + \varepsilon)^{\frac{\ell}{(m+1)\theta}}}, \\ M(t) &= \max \left\{ e^{\omega\beta_1 t} \left(\frac{1}{t} + \omega\right)^\alpha, e^{\omega\beta_2 t} \|g\|_{L^\psi(0,t;L^\rho)}^\eta \right\}^{\frac{1}{\gamma}}, \end{aligned} \quad (2.1.11)$$

for the constants given by

$$\begin{aligned} \alpha &= \frac{1}{(m+1)p \left(\frac{m}{m+1} - \frac{1}{q_s}\right)}, \quad \gamma = \frac{\frac{1}{p} - \frac{1}{q_s}}{\frac{m}{m+1} - \frac{1}{q_s}}, \\ \theta &= 1 - \gamma \left(1 - \frac{\ell}{m+1}\right), \\ \eta &= \frac{1}{1 - \frac{m+1}{\rho} + mp \left(1 - \frac{1}{\psi}\right) \left(1 - \frac{m+1}{mq_s}\right)}, \\ \beta_1 &= \begin{cases} \frac{\frac{1}{mp} - \frac{1}{m+1}}{\frac{1}{m+1} - \frac{1}{mq_s}} & \text{if } m(p-1) < 1, \\ 0 & \text{if } m(p-1) \geq 1, \end{cases} \\ \beta_2 &= \begin{cases} \eta(1 - m(p-1)) \left(1 - \frac{1}{\psi}\right) & \text{if } m(p-1) < 1, \\ 0 & \text{if } m(p-1) \geq 1, \end{cases} \end{aligned} \quad (2.1.12)$$

and where $C > 0$ depends on $m, p, s, d, q_s, \ell, \rho$ and ψ (and $|\Omega|$ when $sp \geq d$).

Note that the right-hand side of (2.1.10) is possibly infinite when $\varepsilon = 0$ and $\|u_0\|_\ell = 0$ due to $N(t)$.

Remark 2.1.4. *We note three key restrictions for this result.*

- (1) *The condition (2.1.8) on p , m , ℓ , s , and d is needed in order to get the De Giorgi iteration started in the proof of Theorem 2.1.3. In the case $\ell = 1$, this corresponds to the restriction for Barenblatt solutions.*
- (2) *Our proof of Theorem 2.1.3 given in Lemma 2.7.7 (see Section 2.7.1) requires the condition $m \geq 1$. We note that the local case avoids this restriction via the available chain rule (see [104]). Thus, $L^\ell - L^\infty$ regularity estimates 2.1.10 satisfied by mild solutions u of (1.2.1) for the case $0 < m < 1$ remains at the moment an open problem.*
- (3) *The restriction $sp < d$ in the unbounded case comes from the embedding $\|u\|_{p_s} \leq C[u]_{s,p}$, whereas in other cases we rely on a Poincaré inequality together with the embedding (2.1.7).*

In the case of no forcing term, $g \equiv 0$, this simplifies to the following $L^\ell - L^\infty$ regularity estimate.

Corollary 2.1.5. *Let Ω be an open domain in $\mathbb{R}^d$, $d \geq 1$ and $T > 0$. Let $p > 1$, $0 < s < 1$, $m \geq 1$ and $1 \leq \ell < m + 1$ such that (2.1.8) holds. Suppose that $sp < d$ or Ω is bounded. Let $q_s = p_s$ if $p \neq \frac{d}{s}$ and $q_s > \max \left\{ p, 1 + \frac{1}{m}, \frac{p\ell}{m(p-1)+\ell-1} \right\}$ if $p = \frac{d}{s}$. Further, suppose that $f(\cdot, u)$ satisfies (2.0.3a)-(2.0.3b) and $g \equiv 0$. Then for every $u_0 \in L^\ell \cap L^1$ the mild solution u of (1.2.2) in L^1 satisfies*

$$\|u(t)\|_\infty \leq C e^{\omega\beta t} t^{-\alpha} \|u_0\|_\ell^\gamma \quad (2.1.13)$$

for all $t \in (0, T]$, where

$$\alpha = \frac{1}{m(p-1) - 1 + \ell(1 - \frac{p}{q_s})}, \quad \gamma = \frac{\ell(1 - \frac{p}{q_s})}{m(p-1) - 1 + \ell(1 - \frac{p}{q_s})},$$

$$\beta = \begin{cases} \frac{\frac{1}{p} - \frac{m}{m+1}}{\frac{m}{m+1} - \frac{1}{p} + \frac{\ell}{m+1}(\frac{1}{p} - \frac{1}{q_s})} & \text{if } m(p-1) < 1, \\ 0 & \text{if } m(p-1) \geq 1, \end{cases}$$

and $C > 0$ depends on m , p , s , d , q_s and ℓ (and $|\Omega|$ when $sp \geq d$).

Such regularity and decay estimates are common for these diffusion problems, see for example [104] and the references therein. In particular, with $f \equiv 0$, $g \equiv 0$ we have Corollary 2.1.5 with $\omega = 0$ and hence a decay in time. Similar estimates have been found for related problems, including for a class of local doubly nonlinear problems [104] related to a doubly nonlinear p -Laplacian evolution equation. In [40], an $L^1 - L^\infty$

estimate is found corresponding to the fractional Laplacian with $s = \frac{1}{2}$ and in [99] for the fractional porous medium equation with $s = \frac{1}{2}$. In the case of the fractional porous medium equation on $\mathbb{R}^d$ ($d \geq 1$), the same authors in [100] find such an $L^\ell - L^\infty$ regularizing effect for all $\ell \geq 1$. We can also compare this to the Barenblatt solutions found for related evolution problems in [47, 121, 126], noting the convergence in $L^1(\mathbb{R}^d)$ as $t \rightarrow \infty$. We emphasise that in all cases the exponents given by (2.1.12) in Corollary 2.1.5 agree with those found in these papers; in the case of [104], by taking $s = 1$. We note that compared to [66], we restrict to $m \geq 1$ rather than $0 < m < 1$ due to the limitation of Lemma 2.7.7. Such regularizing effects require separate consideration in this case.

For $\varphi(r) = r^m$, $r \in \mathbb{R}$, the operator $(-\Delta_p)^s \varphi$ is homogeneous and we have the following Lipschitz estimate via the regularising effect of homogeneous operators in [20]. We also obtain derivative and energy estimates for strong distributional solutions. Here we define, for $g \in L^1_{\text{loc}}(0, T; L^1)$ and $0 \leq t \leq T \leq \infty$,

$$\tilde{V}(g, t) = \limsup_{\xi \rightarrow 0^+} \int_0^{t/(1+\xi)} \frac{\|g(\tau(1+\xi)) - g(\tau)\|_1}{\xi} d\tau. \quad (2.1.14)$$

Note that $\tilde{V}(g, T) < \infty$ is equivalent to $\tau \rightarrow \tau g(\tau)$ having (essentially) finite variation on $[0, T]$. Also, in the following theorem and corollary, $\mathcal{E}$ is the energy functional given by (2.0.4). These results are proved in Section 2.8.

Theorem 2.1.6 (Derivative estimates for $\varphi(u) = u^m$). *Let Ω be an open domain in $\mathbb{R}^d$, $d \geq 1$ and $T > 0$. Let $p > 1$, $0 < s < 1$, $m > 0$ and $f(\cdot, u)$ satisfy (2.0.3a)-(2.0.3b). Then we have the following regularity estimates.*

- (1) *Suppose $m(p-1) \neq 1$, $g \in L^1(0, T; L^1)$ and $\tilde{V}(g, T) < \infty$. Then every mild solution u of (1.2.2) with $u_0 \in L^1$ is Lipschitz continuous on each compact subset of $(0, T]$, satisfying*

$$\begin{aligned} \limsup_{h \rightarrow 0^+} \frac{\|u(t+h) - u(t)\|_1}{h} &\leq \frac{C e^{2\omega t}}{t} \left(\|u_0\|_1 + \int_0^t \|g(\tau)\|_1 d\tau \right) \\ &\quad + \frac{e^{\omega t}}{t} \tilde{V}(g, t) \end{aligned} \quad (2.1.15)$$

where $C = \frac{m(p-1)+2}{|m(p-1)-1|}$. In particular, if $m > s$ then we have such a unique mild solution for all $u_0 \in L^1$.

- (2) *Let $g \in BV(0, T; L^1) \cap L^1(0, T; L^{1 \cap \infty})$. Further suppose that $m \geq 1$ satisfies $m(p-1) \neq 1$ and*

$$m(p-1) > 1 - \frac{sp}{d}. \quad (1.2.3)$$

For $\rho \geq m+1$, take $q_s > \max \left\{ p, 1 + \frac{1}{m}, \frac{p}{m(p-1)}, \frac{p\rho}{m(p-1)+\rho-1} \right\}$ if $p = \frac{d}{s}$ and $q_s = p_s$ if $p \neq \frac{d}{s}$. Suppose $sp < d$ or Ω is bounded. If $\psi > 1$ satisfies (2.1.9) and g also belongs to $L^\psi(0, T; L^\rho) \cap L^{m+1}(0, T; L^\infty)$, then for every $u_0 \in L^1$ and all $\varepsilon \geq 0$, the mild solution u to (1.2.2) is a strong distributional solution in L^1 and satisfies

$$\begin{aligned} & \int_0^t \tau^{\tilde{\alpha}} \int_\Omega \left| \frac{d}{d\tau} u^{\frac{m+1}{2}}(\tau) \right|^2 d\mu d\tau + t^{\tilde{\alpha}} [u^m(t)]_{s,p}^p \\ & \leq Ct(1+\omega t)^2 \max \left\{ (1+\omega t)^\alpha e^{\omega\beta_1 t}, t^\alpha e^{\omega\beta_2 t} \|g\|_{L^\psi(0,t;L^\rho)}^\eta \right\}^{\frac{m}{\theta}} \\ & \quad \times \left(e^{\omega t} \|u_0\|_1 + \int_0^t e^{\omega(t-\tau)} \|g(\tau)\|_1 d\tau + \varepsilon \right)^{\frac{\gamma m}{\theta(m+1)}+1} \\ & \quad \times (1+N(t)^{\gamma m}) \\ & \quad + C \int_0^t \tau^{\tilde{\alpha}+m-1} \|g(\tau)\|_{m+1}^{m+1} d\tau \end{aligned} \tag{2.1.16}$$

for all $t \in (0, T]$ with $\tilde{\alpha} = \frac{\alpha m}{\theta} + 2$, $N(t)$ given by (2.1.11) with $\ell = 1$, constants given by (2.1.12) and where $C > 0$ depends on $m, p, s, d, q_s, \psi, \rho$ (and $|\Omega|$ when $sp \geq d$).

In the case of no forcing term, $g \equiv 0$, this simplifies to the following derivative estimate.

Corollary 2.1.7. *Let Ω be an open domain in $\mathbb{R}^d$, $d \geq 1$ and $T > 0$. Let $p > 1$, $0 < s < 1$, $f(\cdot, u)$ satisfy (2.0.3a)-(2.0.3b) and $g \equiv 0$. Suppose that $sp < d$ or Ω is bounded. Let $m \geq 1$ such that $m(p-1) \neq 1$ and*

$$m(p-1) > 1 - \frac{sp}{d}. \tag{1.2.3}$$

Take $q_s = p_s$ if $p \neq \frac{d}{s}$ and $q_s > \max \left\{ p, 1 + \frac{1}{m}, \frac{p}{m(p-1)} \right\}$ if $p = \frac{d}{s}$. Then for every $u_0 \in L^1$, the mild solution u to (1.2.2) is a strong distributional solution in L^1 and satisfies

$$\begin{aligned} & \int_0^t \tau^{\tilde{\alpha}} \int_\Omega \left| \frac{d}{d\tau} u^{\frac{m+1}{2}}(\tau) \right|^2 d\mu d\tau + t^{\tilde{\alpha}} \mathcal{E}(u^m(t)) \\ & \leq Ct(1+\omega t)^{\tilde{\alpha}} e^{\omega\beta t} \|u_0\|_1^{\frac{\gamma m}{\theta(m+1)}+1} \end{aligned} \tag{2.1.17}$$

for all $t \in (0, T]$ with

$$\tilde{\alpha} = \frac{\alpha m}{\theta} + 2, \quad \beta = \frac{m\beta_1}{\theta} + \frac{\gamma m}{\theta(m+1)} + 1,$$

constants given by (2.1.12) and where $C > 0$ depends on m, p, s, d and q_s (and $|\Omega|$ when $sp \geq d$).

Applying the previous derivative estimates and the strong distributional solutions obtained on finite domains in Theorem 2.1.2, we obtain strong distributional solutions for all initial data in $L^{1 \cap \infty}$ or, when the $L^1 - L^\infty$ regularizing effect is available, for all initial data in L^1 .

Theorem 2.1.8 (Strong distributional solutions). *Let Ω be an open domain in $\mathbb{R}^d$, $d \geq 1$ and $T > 0$. Let $1 < p < \infty$, $0 < s < 1$ and $m > s$. Suppose $f(\cdot, u)$ satisfy (2.0.3a)-(2.0.3b) and $g \in BV(0, T; L^1) \cap L^1(0, T; L^{1 \cap \infty})$. Further, suppose that either $m(p-1) \neq 1$ or there exists $(v_{0,n})_{n \geq 1} \subset D((-\Delta_p)_{1 \cap \infty}^s)$ such that $v_{0,n} \rightarrow u_0^m$ in L^1 as $n \rightarrow \infty$ and $\|(-\Delta_p)_{1 \cap \infty}^s v_{0,n}\|_1$ is uniformly bounded. If either*

(1) $u_0 \in L^{1 \cap \infty}$, or

(2) $u_0 \in L^1$, $m \geq 1$ and p, s and m satisfy (1.2.3). Moreover, $g \in L^\psi(0, T; L^\rho)$ where ρ and ψ are restricted as in Theorem 2.1.3 with $\ell = 1$,

then every mild solution in L^1 of (1.2.2) is a strong distributional solution in L^1 .

In comparison with previous results in this direction [65, 66], here we consider open domains rather than bounded domains. We also obtain these solutions for all initial data in L^1 rather than restricting to L^∞ and Sobolev-type spaces.

When the domain is bounded, we obtain local Hölder regularity for mild solutions u to (1.2.2) with initial data in L^1 . Here $\tilde{V}(g, T)$ is defined as in (2.1.14).

Theorem 2.1.9 (Local Hölder continuity). *Let Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 1$ and $T > 0$. Assume $2 \leq p < \infty$, $0 < s < 1$ such that $sp \geq d$ and $m > s$ satisfying $m(p-1) \neq 1$. Suppose f satisfies (2.0.3a)-(2.0.3b) and $g \in BV(0, T; L^1) \cap L^1(0, T; L^\infty)$ such that $\tilde{V}(g, T) < \infty$. Let $u(t)$ be a mild solution to (1.2.1) with $u_0 \in L^1 \cap L^\infty$. Then $u^m(t) \in C_{\text{loc}}^\delta(\Omega)$ for every $0 < \delta < \min \left\{ \frac{sp-d}{p-1}, 1 \right\}$ and a.e. $t \in (0, T)$. In particular, for $m \geq 1$, $u(t) \in C_{\text{loc}}^\delta(\Omega)$ for every $0 < \delta < \min \left\{ \frac{sp-d}{p-1}, 1 \right\}$ and a.e. $t \in (0, T)$.*

Local Hölder regularity has been established for the fractional porous medium equation in [83] for $\frac{n-2s}{n+2s} < m < 1$ via an oscillation lemma and in [100] for the fractional porous medium equation on $\mathbb{R}^d$ for $m \geq 1$. We apply the elliptic local Hölder regularity result of [30] to prove Theorem 2.1.9. Hereby we extend the work in [31], which considers Hölder regularity in space and time for a weak formulation of the fractional p -Laplacian evolution problem.

Furthermore, for φ given by the identity, we have continuity in time and global Hölder continuity in space for a bounded domain. Here $C_0(\Omega)$ denotes the set of continuous functions $u : \Omega \rightarrow \mathbb{R}$ vanishing on the boundary $\partial\Omega$. Both Hölder regularity results are proved in Section 2.10.

Theorem 2.1.10 (Global Hölder regularity for $\varphi(r) = r$). *Let Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 2$, with a boundary $\partial\Omega$ of the class $C^{1,1}$ and $T > 0$. Let $1 < p < \infty$, $0 < s < 1 - \frac{1}{p}$, and suppose f satisfies (2.0.3a)-(2.0.3b) with F the Nemytskii operator of f on $C_0(\Omega)$. Then $-((- \Delta_p)_{|C_0}^s + F)$ generates a strongly continuous semigroup of quasi contractions on $C_0(\Omega)$. In particular, for $\varphi(r) = r$, $r \in \mathbb{R}$, $u_0 \in C_0(\Omega)$ and $g \in L^\infty(0, T; L^\infty) \cap BV(0, T; L^\infty)$, let u be the unique mild solution of the initial boundary value problem (1.2.1). Then $u \in C^{lip}([\delta, T]; C_0(\Omega))$ for every $0 < \delta < T$ and for some $\alpha \in (0, s]$, $u(t) \in C^\alpha(\overline{\Omega})$ for all $t \in (0, T)$.*

This result extends [67], wherein Giacomoni and Tiwari obtain continuity up to the boundary in space, uniform in time. Here we refine the estimate, proving that the resolvent is m -accretive in C_0 . We use the global Hölder regularity of the elliptic problem given by [73] with semigroup theory for the operator restricted to the space of continuous functions. Hölder regularity has been considered for the elliptic problem, for example, in [30].

We now introduce some general theory for accretive operators and associated evolution problems which we will use throughout this thesis.

2.2 Accretive operators

We begin by reviewing some basic definitions and important results in nonlinear semigroup theory from the standard literature [12, 34] and the monograph [50]. In particular, we introduce the relevant definitions and accretivity theory which will apply to the composed operator $(-\Delta_p)^s \varphi$. We will see in Corollary 2.4.4 that when $\varphi \in C(\mathbb{R})$ is strictly increasing with $\varphi(0) = 0$, the operator $(-\Delta_p)^s \varphi$ is m - T -accretive in L^1 with complete resolvent.

2.2.1 The general framework

Let (Σ, μ) be a measure space with a positive σ -finite measure μ , and $M(\Sigma, \mu)$ be the set of μ -a.e. equivalence classes of measurable functions $u : \Sigma \rightarrow \mathbb{R}$. For $1 \leq q \leq \infty$, we denote by L_μ^q the classical *Lebesgue space* equipped with the standard L^q -norm

$$\|u\|_q := \begin{cases} \left(\int_\Sigma |u|^q d\mu \right)^{1/q} & \text{if } q < \infty, \\ \inf \left\{ k \in [0, \infty] \mid \mu(\{|u| > k\}) = 0 \right\} & \text{if } q = \infty. \end{cases}$$

If Ω is an open subset of $\mathbb{R}^d$, $d \geq 1$, and μ is the classical d -dimensional Lebesgue measure restricted to the trace- σ -algebra $\mathcal{B}(\mathbb{R}^d) \cap \Omega$ then we write L^q instead of L_μ^q . We may also write $L^q(\Omega)$ if the domain is not clear from the setting.

Let $X \subseteq M(\Sigma, \mu)$ be a Banach space with norm $\|\cdot\|_X$. The main object in this section is the abstract Cauchy problem (in X)

$$\begin{cases} \frac{du}{dt}(t) + Au(t) \ni g(t) & \text{for a.e. } t \in (0, T), \\ u(0) = u_0, \end{cases} \quad (2.2.1)$$

for given initial value $u_0 \in \overline{D(A)}^X$ and forcing term $g \in L^1(0, T; X)$. In (2.2.1), A denotes a (possibly) multi-valued operator $A : D(A) \rightarrow 2^X$ on X with *effective domain* $D(A) := \{u \in X \mid Au \neq \emptyset\}$, the closure of $D(A)$ in X denoted by $\overline{D(A)}^X$, and *range* $\text{Rg}(A) := \bigcup_{u \in D(A)} Au$. In this setting, it is standard to view an operator A as a subset of $X \times X$, or *relation* on X , and to identify A with its graph

$$A := \left\{ (u, v) \in X \times X \mid v \in Au \right\}.$$

Then we have the following key definition.

Definition 2.2.1. *An operator A on X is called accretive (in X) if*

$$\|u - \hat{u}\|_X \leq \|u - \hat{u} + \lambda(v - \hat{v})\|_X \quad (2.2.2)$$

for every $(u, v), (\hat{u}, \hat{v}) \in A$ and every $\lambda > 0$. Further, an operator A on X is called quasi accretive if there is an $\omega \in \mathbb{R}$ such that $A + \omega I$ is accretive in X .

Clearly, if $A + \omega I$ is accretive in X for some $\omega \in \mathbb{R}$ then $A + \tilde{\omega} I$ is accretive for every $\tilde{\omega} \geq \omega$. Thus, there is no loss of generality in assuming that if A is quasi accretive then there is an $\omega \geq 0$ such that $A + \omega I$ is accretive in X .

Equivalently, A is accretive in X if and only if, for every $\lambda > 0$, the *resolvent operator* of A , defined by $J_\lambda := (I + \lambda A)^{-1}$, is a single-valued mapping from $\text{Rg}(I + \lambda A)$ to $D(A)$ which is *contractive* (also called *non-expansive*) with respect to the norm of X . That is,

$$\|J_\lambda u - J_\lambda \hat{u}\|_X \leq \|u - \hat{u}\|_X$$

for all $u, \hat{u} \in \text{Rg}(I + \lambda A)$ and $\lambda > 0$.

The following notation was introduced by Coulhon and Hauer [50] to simplify the verification of accretivity for a given operator A . Further, it may be used to introduce functional inequalities such as *Gagliardo-Nirenberg* or *Sobolev* inequalities.

Notation 2.2.2. *For $1 \leq q < \infty$, we define the q -bracket on L_μ^q to be the mapping $[\cdot, \cdot]_q : L_\mu^q \times L_\mu^q \rightarrow \mathbb{R}$ defined by*

$$[u, v]_q := \lim_{\lambda \rightarrow 0+} \frac{\frac{1}{q} \|u + \lambda v\|_q^q - \frac{1}{q} \|u\|_q^q}{\lambda}$$

for $u, v \in L_\mu^q$. We further define the bracket

$$[u, v]_+ := \lim_{\lambda \rightarrow 0+} \frac{\|[u + \lambda v]^+\|_1 - \|[u]^+\|_1}{\lambda}$$

for $u, v \in L_\mu^1$.

Then (cf. [50, pg. 13] and [12, pg. 103]) for $1 \leq q < \infty$, an operator A on L_μ^q is accretive in L_μ^q if and only if

$$[u - \hat{u}, v - \hat{v}]_q \geq 0 \quad \text{for all } (u, v), (\hat{u}, \hat{v}) \in A.$$

Furthermore, the q -bracket $[\cdot, \cdot]_q$ is upper semicontinuous (respectively, continuous if $1 < q < \infty$) and if $1 < q < \infty$,

$$[u, v]_q = \int_{\Sigma} |u|^{q-2} u v \, d\mu \quad (2.2.3)$$

for every $u, v \in L_\mu^q$. While for $q = 1$, $[\cdot, \cdot]_1$ reduces to the classical brackets $[\cdot, \cdot]$ on L_μ^1 given by

$$[u, v]_1 = \int_{\{u \neq 0\}} \text{sign}_0(u) v \, d\mu + \int_{\{u=0\}} |v| \, d\mu \quad (2.2.4)$$

for $u, v \in L_\mu^1$, where the *restricted signum* sign_0 is defined by

$$\text{sign}_0(s) = \begin{cases} 1 & \text{if } s > 0, \\ 0 & \text{if } s = 0, \\ -1 & \text{if } s < 0, \end{cases}$$

for $s \in \mathbb{R}$ (cf. [21, Section 2.2 and Example 2.8] or applying [12, Proposition 3.7]). When $q = +$, we can write $[\cdot, \cdot]_+$ as

$$[u, v]_+ = \int_{\{u \neq 0\}} \text{sign}_0^+(u) v \, d\mu + \int_{\{u=0\}} [v]^+ \, d\mu \quad (2.2.5)$$

where sign_0^+ is the sign-plus or Heaviside function,

$$\text{sign}_0^+(s) = \begin{cases} 1 & \text{if } s > 0 \\ 0 & \text{if } s \leq 0. \end{cases}$$

When applying this to functions u defined on Σ , we may also write the composition as $\mathbb{1}_{u>0} := \text{sign}_0^+(u)$.

Operators with m -accretivity

Next, we introduce the following class of operators focused on ensuring existence at each step of the discretized problem of Definition 2.2.4.

Definition 2.2.3. *An operator A on X is called m -accretive in X if A is accretive in X and satisfies the so-called range condition*

$$\text{Rg}(I + \lambda A) = X \quad \text{for some (or equivalently all) } \lambda > 0, \quad (2.2.6)$$

and an operator A on X is called quasi m -accretive in X if there is an $\omega \geq 0$ such that $A + \omega I$ is m -accretive in X .

By the classical theory of nonlinear evolution problems (cf. [21], or alternatively, [12, Corollary 4.1]), the condition ‘ A is quasi m -accretive in X ’ ensures that for given $u_0 \in \overline{D(A)}^X$ and $g \in L^1(0, T; X)$, the Cauchy problem (2.2.1) admits a unique *mild solution*, which is continuously dependent on u_0 and g .

Definition 2.2.4. For given $g \in L^1(0, T; X)$, $T > 0$ and $u_0 \in X$, a mild solution u in X of Cauchy problem (2.2.1) is a function $u \in C([0, T]; X)$ such that for every $\varepsilon > 0$ there is a partition $\sigma_\varepsilon : 0 = t_0 < \dots < t_N = T$ of the interval $[0, T]$ and a finite sequence $(g_i)_{i=1}^N \subset X$ with the following properties: $T - t_N < \varepsilon$, $t_i - t_{i-1} < \varepsilon$ for every $i = 1, \dots, N$,

$$\sum_{i=1}^N \int_{t_{i-1}}^{t_i} \|g(\tau) - g_i\|_X d\tau < \varepsilon,$$

there exists a step function $u_{\varepsilon, \sigma} : [0, T] \rightarrow X$ of the form

$$u_{\varepsilon, \sigma}(t) = u_0 \mathbb{1}_{\{t_0=0\}}(t) + \sum_{i=1}^N u_i \mathbb{1}_{(t_{i-1}, t_i]}(t), \quad (2.2.7)$$

where the coefficients u_i recursively solve the finite difference equation

$$u_i + (t_i - t_{i-1})Au_i \ni (t_i - t_{i-1})g_i + u_{i-1} \quad (2.2.8)$$

for every $i = 1, \dots, N$, and

$$\sup_{t \in [0, T]} \|u(t) - u_{\varepsilon, \sigma}(t)\|_X \leq \varepsilon.$$

Remark 2.2.5. To demonstrate the notion of mild solutions in the specific case of (1.2.1), we consider an open domain $\Omega \subseteq \mathbb{R}^d$, choose $X = L^1$ and the operator A to be $(-\Delta_p)^s \varphi + F$ where F is the Nemytskii operator of f . Then the recursion relation (2.2.8) becomes

$$u_i + (t_i - t_{i-1})((-\Delta_p)^s \varphi(u_i) + F(u_i)) \ni (t_i - t_{i-1})g_i + u_{i-1}.$$

Applying Definition 2.0.1 (see also Section 2.3.4), there exists a sequence $(\nu_k)_{k \geq 1}$ with $\varphi(\nu_k) \in D((-\Delta_p)^s) \subset W_0^{s, (2, p)}$ for $k \geq 1$ such that $\nu_k \rightarrow u_i$ in L^1 . Moreover, $u_{\varepsilon, \sigma}$ converges to the mild solution u in L^1 uniformly in time. So pointwise for $t \in [0, T]$ we have a diagonal sequence of such ν_k converging to $u(t)$ in L^1 . It is in this limit sense that mild solutions to satisfy the boundary condition of (1.2.1). In particular, since, even in the case $\Omega \subset \mathbb{R}^d$, the definition of $(-\Delta_p)^s$ depends on the data outside Ω to be zero (see also Section 2.3).

In the homogeneous case $g \equiv 0$, if A is quasi m -accretive in X , then the Crandall-Liggett theorem [51, Theorem I] (see also [12, Section 4]) states that for every element $u_0 \in \overline{D(A)}^X$, there is a unique mild solution

u of (2.2.1) in X for every $T > 0$ and this solution u can be given by the *exponential formula*

$$u(t) = \lim_{n \rightarrow \infty} \left(I + \frac{t}{n}A\right)^{-n} u_0 \quad (2.2.9)$$

uniformly in t on compact intervals. For every $u_0 \in \overline{D(A)}^X$, setting

$$T_t u_0 := u(t), \quad \text{for every } t \geq 0, \quad (2.2.10)$$

defines a (nonlinear) *strongly continuous semigroup* $\{T_t\}_{t \geq 0}$ of $(\omega-)$ quasi contractions $T_t : \overline{D(A)}^X \rightarrow \overline{D(A)}^X$ with $\omega \in \mathbb{R}$. More precisely, the family $\{T_t\}_{t \geq 0}$ satisfies the following three properties:

- *semigroup property*

$$T_{s+t} = T_t \circ T_s \quad \text{for every } s, t \geq 0,$$

- *strong continuity*

$$\lim_{t \rightarrow 0^+} \|T_t u - u\|_X = 0 \quad \text{for every } u \in \overline{D(A)}^X,$$

- *exponential growth property in X or $(\omega-)$ quasi contractivity in X*

$$\|T_t u - T_t v\|_X \leq e^{\omega t} \|u - v\|_X \quad \text{for all } u, v \in \overline{D(A)}^X, t \geq 0.$$

For the family $\{T_t\}_{t \geq 0}$ on $\overline{D(A)}^X$, the operator

$$A_o := \left\{ (u, v) \in X \times X \mid \lim_{h \rightarrow 0^+} \frac{T_h u - u}{h} = v \text{ in } X \right\}$$

is a well-defined mapping $A_o : D(A_o) \rightarrow X$ with domain

$$D(A_o) := \left\{ u \in X \mid \lim_{h \rightarrow 0^+} \frac{T_h u - u}{h} \text{ exists in } X \right\}$$

called the *infinitesimal generator* of $\{T_t\}_{t \geq 0}$. If $\{T_t\}_{t \geq 0}$ is ω -quasi contractive in X , then $-A_o$ is ω -quasi accretive in X .

Since mild solutions of Cauchy problem (2.2.1) are merely the locally uniform (in time) limit of step functions (2.2.7) with values in X , it is important to know whether they are actually *strong solutions* of (2.2.1) in X .

Definition 2.2.6. Given $u_0 \in X$ and $g \in L^1(0, T; X)$ for some $T > 0$, a function $u \in C([0, T]; X)$ is called a *strong (in time) solution in X of Cauchy problem (2.2.1)* if $u(0) = u_0$, u belongs to $W_{loc}^{1,1}((0, T); X)$ and for a.e. $0 < t < T$, one has that $u(t) \in D(A)$ and $g(t) - \frac{du}{dt}(t) \in Au(t)$.

Since we consider the closure of $(-\Delta_p)_{|L^1 \cap \infty}^s \varphi$ in $L^1 \times L^1$ for (1.2.1), we also want to consider solutions with further regularity on $\varphi(u)$, in particular with $\varphi(u) \in D((-\Delta_p)^s)$. Hence we consider *distributional (in space) solutions* as introduced in the following definition.

Definition 2.2.7. For given $T > 0$, $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying (2.0.3a)-(2.0.3b) and $g \in L^1_{loc}((0, T); L^1_{loc})$ for some $T > 0$, a function $u \in C([0, T]; L^1)$ is called a *distributional (in space) solution in L^1 of initial boundary-value problem (1.2.1)* if $\varphi(u) \in L^p_{loc}((0, T); W^{s, (2, p)})$, $u(0) = u_0$ in L^1 , and for every test function $\xi \in C_c^\infty([0, T] \times \mathbb{R}^d)$, one has that

$$\begin{aligned} & - \int_{\mathbb{R}^d} u \xi \, dx \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \int_{\mathbb{R}^d} u \xi_t \, dx \, dt + \int_{t_1}^{t_2} \int_{\mathbb{R}^d} (f(x, u) - g) \xi \, dx \, dt \\ & + \int_{t_1}^{t_2} \int_{\mathbb{R}^{2d}} \frac{(\varphi(u)(t, x) - \varphi(u)(t, y))^{p-1} (\xi(t, x) - \xi(t, y))}{|x - y|^{d+sp}} d(x, y) \, dt = 0 \end{aligned}$$

for all $0 < t_1 < t_2 < T$.

Furthermore, if u is also differentiable with $u_t(t) \in L^1$ for a.e. $t \in (0, T)$, then we call this a *strong distributional solution in L^1* .

If, for example, $X = L^q_\mu$ for $1 < q < \infty$ and $g \equiv 0$, then X is a uniformly convex Banach space and so the classical regularity theory of nonlinear semigroups (cf. [12, Theorem 4.6]) applies: let A be a quasi m -accretive operator on L^q_μ , then for every $u_0 \in D(A)$ the mild solution u of Cauchy problem (2.2.1) is a strong solution of (2.2.1) and $t \mapsto T_t u_0$ given by (2.2.10) satisfies

$$\frac{d}{dt} T_t u_0 = -A^\circ T_t u_0 \quad \text{for every } t > 0, \quad (2.2.11)$$

where A° denotes the *minimal selection* of A given by the operator

$$A^\circ := \left\{ (x, y) \in A \mid \|y\| = \min_{\hat{y} \in Ax} \|\hat{y}\| \right\}.$$

Under additional geometric conditions on the Banach space X , one has that $-A_\circ \subseteq A^\circ$. Ignoring these details on X , we nevertheless say that a strongly continuous semigroup $\{T_t\}_{t \geq 0}$ of quasi contractions on $\overline{D(A)}^X$ is *generated by $-A$* if A is quasi m -accretive in X and $\{T_t\}_{t \geq 0}$ is the family induced by (2.2.10).

2.2.2 T -accretive operators and complete resolvent

Here we introduce two stronger notions of accretivity, completely accretive operators and T -accretive operators, requiring further contractivity-type properties of the resolvent. We further extend these with the complete resolvent property.

Completely accretive operators

The notion of *completely accretive operators* was introduced in [18] by Crandall and B enilan and further developed in [50]. Following the same these two references, we introduce the following notation.

Notation 2.2.8. The set $\mathcal{J}_0$ is comprised of all convex, lower semicontinuous functions $j : \mathbb{R} \rightarrow [0, +\infty]$ satisfying $j(0) = 0$.

Definition 2.2.9. A mapping $S : D(S) \rightarrow M(\Sigma, \mu)$ with domain $D(S) \subseteq M(\Sigma, \mu)$ is called a complete contraction if

$$\int_{\Sigma} j(Su - S\hat{u}) \, d\mu \leq \int_{\Sigma} j(u - \hat{u}) \, d\mu \quad (2.2.12)$$

for all $j \in \mathcal{J}_0$ and every $u, \hat{u} \in D(S)$. An operator A on $M(\Sigma, \mu)$ is called completely accretive if for every $\lambda > 0$, the resolvent operator J_{λ} of A is a complete contraction.

As mentioned in the introduction, it is well-known (see, e.g., [50, 94]) that the fractional p -Laplacian $(-\Delta_p)^s$ equipped with Dirichlet boundary conditions is m -completely accretive in L^2 . While this property doesn't translate in full to the doubly nonlinear case, we may still obtain a weaker version of (2.2.12) in the following sense.

T -accretive operators

A more general class of operators is that of T -accretive operators. In particular, choosing $j(\cdot) = |[\cdot]^+|^q \in \mathcal{J}_0$ if $1 \leq q < \infty$ and $j(\cdot) = |[\cdot]^+ - k|^+ \in \mathcal{J}_0$ for $k \geq 0$ large enough if $q = \infty$ shows that a complete contraction S satisfies the following T -contractivity property in L_{μ}^q for every $1 \leq q \leq \infty$. Hence a completely accretive operator is T -accretive in L_{μ}^q for all $1 \leq q \leq \infty$.

Definition 2.2.10. A mapping $S : D(S) \rightarrow L_{\mu}^q$ with domain $D(S) \subseteq L_{\mu}^q$, $1 \leq q \leq \infty$, is called a T -contraction if

$$\|[Su - S\hat{u}]^+\|_q \leq \|[u - \hat{u}]^+\|_q$$

for every $u, \hat{u} \in D(S)$. We say that an operator A on L_{μ}^q is T -accretive if, for every $\lambda > 0$, the resolvent J_{λ} of A defines a T -contraction with domain $D(J_{\lambda}) = Rg(I + \lambda A)$.

A T -accretive operator in L_{μ}^q , $1 \leq q \leq \infty$, is accretive in L_{μ}^q and the resolvent is order-preserving in L_{μ}^q . That is, denoting the usual order relation on L_{μ}^q by $\leq$, if S is T -contractive in L_{μ}^q , $1 \leq q \leq \infty$, then $u \leq \hat{u}$ implies that $Su \leq S\hat{u}$ for $u, \hat{u} \in L^q$ (see [41] and [21, Section 19.4] for further properties).

We note in particular the following useful comparison property when A is an ω -quasi T -accretive operator with $\omega > 0$. We have

$$\int_{\Omega} [w_1 - w_2]^+ \, dx \leq \frac{1}{1 - \lambda\omega} \int_{\Omega} [h_1 - h_2]^+ \, dx \quad (2.2.13)$$

for all $0 < \lambda < 1/\omega$ and every $h_1, h_2 \in L^1$, where $w_1 = J_{\lambda}^A h_1$ and $w_2 = J_{\lambda}^A h_2$. This signed estimate will be of great use for the proof of

global barrier functions in Section 3.3 and the finite time of extinction in Section 3.4 as it allows a direct comparison between solutions (as well as super-solutions).

We can naturally extend these definitions to quasi m -completely accretive operators (and similarly for quasi m - T -accretive operators).

Definition 2.2.11. *An operator A on $M(\Sigma, \mu)$ is called quasi completely accretive if there is an $\omega > 0$ such that for every $\lambda > 0$, the resolvent operator J_λ of $A + \omega I$ is a complete contraction. Moreover, for $1 \leq q < \infty$, an operator A on L_μ^q is said to be quasi m -completely accretive on L_μ^q if there is an $\omega > 0$ such that $A + \omega I$ is completely accretive and the range condition (2.2.6) holds with $X = L_\mu^q$.*

Accretive operators in L^1 with complete resolvent

The class of operators which are merely T -accretive in L_μ^1 but have a so-called *complete resolvent* was introduced in [19] and further elaborated in [50]. Importantly for our problem, for a given completely accretive operator A in L^1 , the composed operator $A\varphi$ becomes T -accretive in L^1 with complete resolvent so long as φ is a strictly increasing, continuous function on $\mathbb{R}$ (see [50]). Typical examples of this class of operators include the doubly-nonlinear operators $-\Delta_p \varphi$ and $(-\Delta_p)^s \varphi$. Hence we now introduce the notion of *complete mappings*.

Definition 2.2.12. *Let $D(S)$ be a subset of $M(\Sigma, \mu)$. A mapping $S : D(S) \rightarrow M(\Sigma, \mu)$ is called complete if*

$$\int_\Sigma j(Su) \, d\mu \leq \int_\Sigma j(u) \, d\mu \quad (2.2.14)$$

for every $j \in \mathcal{J}_0$ and $u \in D(S)$.

We now introduce the class of accretive operators in L_μ^1 with complete resolvent (similarly for T -accretive operators with complete resolvent).

Definition 2.2.13. *An operator A on L_μ^1 is called (m) -accretive in L_μ^1 with complete resolvent if A is (m) -accretive in L_μ^1 and for every $\lambda > 0$, the resolvent operator $J_\lambda : Rg(I + \lambda A) \rightarrow D(A)$ of A is a complete mapping. For $\omega \in \mathbb{R}$, we call an operator A on L_μ^1 ω -quasi (m) -accretive in L_μ^1 with complete resolvent (or simply quasi (m) -accretive in L_μ^1 with complete resolvent) if $A + \omega I$ is (m) -accretive in L_μ^1 with complete resolvent.*

The condition (2.2.14) on the resolvent provides a growth estimate on $u(t)$, allowing us to estimate $u(t)$ in L_μ^p by the norms of u_0 and g in L_μ^p . In particular, we prove such an estimate in Lemma 2.7.3, extending [19, Proposition 2.4].

2.2.3 Subdifferential operators

If X is a Hilbert space H with inner product $(\cdot, \cdot)_H$, then an important class of m -accretive operators in H is given by the *subdifferential operator*

$$\partial\mathcal{E} := \left\{ (u, v) \in H \times H \mid (v, \xi - u)_H \leq \mathcal{E}(\xi) - \mathcal{E}(u) \text{ for all } \xi \in H \right\}$$

of a proper, convex, lower semicontinuous functional $\mathcal{E} : H \rightarrow (-\infty, +\infty]$. In Hilbert spaces, accretivity is equivalent to *monotonicity*; that is, an operator A is monotone if

$$(u - \hat{u}, v - \hat{v})_H \geq 0 \quad \text{for all } (u, v), (\hat{u}, \hat{v}) \in A$$

(cf. [34], and see also [6, 44]). For this class of operators $A = \partial\mathcal{E}$, the Cauchy problem (2.2.1) has the smoothing effect that every mild solution u of (2.2.1) is strong. This result is due to Brezis [32] (see also [6]).

It is well known that the *fractional p -Laplacian* $(-\Delta_p)^s$ defined in (2.0.3) equipped with homogeneous Dirichlet boundary conditions on an open set Ω can be realized as such a subdifferential operator $\partial\mathcal{E}$ in L^2 (see [94] Proposition 2.3.1). We will introduce this operator in more detail in Section 2.3.

The subdifferential operator in X

We rely on the m -accretivity of $(-\Delta_p)^s$ in L^1 to obtain mild and strong solutions to (1.2.1). We also find that it is sufficient to work with the part in $L^{1 \cap \infty}$ to establish existence of such solutions. Hence we consider operators restricted to $L^{1 \cap \infty}$ and introduce a definition of subdifferential operators from [18] which we will apply in the case $X = L^1$. We will also use this to apply the results of [22] and in particular to obtain strong solutions. We include functionals on a (possibly distinct) subspace Y for completeness.

Definition 2.2.14. *Let X , Y and Z be linear subspaces of $M(\Sigma, \mu)$. Then for an energy functional $\mathcal{E} : Y \rightarrow (-\infty, \infty]$ with effective domain $D(\mathcal{E}) := \{u \in Y \mid \mathcal{E}(u) < \infty\}$, we define the part of $\mathcal{E}$ in X by*

$$\mathcal{E}|_X(u) = \begin{cases} \mathcal{E}(u) & \text{for } u \in D(\mathcal{E}) \cap X, \\ \infty & \text{otherwise.} \end{cases}$$

Further, we define the operator $\partial_X \mathcal{E}$ in X by the graph

$$\left\{ (u, v) \in X \times X \mid \begin{array}{l} u \in D(\mathcal{E}) \text{ and } \int_{\Sigma} v(w - u) d\mu \leq \mathcal{E}(w) - \mathcal{E}(u) \\ \text{for all } w \in X \text{ with } v(w - u) \in L^1_{\mu} \end{array} \right\}$$

and the part of $\partial_X \mathcal{E}$ in Z by

$$(\partial_X \mathcal{E})|_Z := \left\{ (u, h) \in Z \times Z \mid (u, h) \in \partial_X \mathcal{E} \right\}.$$

In the case $X = L_\mu^2$ and Y a linear subspace of $M(\Sigma, \mu)$, this coincides with the previous definition of the subdifferential $\partial\mathcal{E}$ in L_μ^2 . One sees that for a functional $\mathcal{E} : Y \rightarrow (-\infty, \infty]$, we have $\partial_{L_\mu^2}\mathcal{E} = \partial(\mathcal{E}|_{L_\mu^2})$.

Remark 2.2.15. *In the case $X = L_\mu^1$, $Y = L_\mu^2$ (the relevant case for this thesis), we have the inclusion $(\partial\mathcal{E})|_{L_\mu^1 \cap \infty} \subseteq (\partial_{L_\mu^1}\mathcal{E})|_{L_\mu^1 \cap \infty}$. However, since we are primarily interested in m -accretivity, they will be largely interchangeable due to the m -accretivity properties of $(\partial\mathcal{E})|_{L_\mu^1 \cap \infty}$ proved in Theorem 2.4.1.*

For the class of operators $\partial_X\mathcal{E}$, B enilan and Crandall [18] found the following sufficient conditions implying that the closure $\overline{\partial_X\mathcal{E}^X}$ of $\partial_X\mathcal{E}$ in X is m -completely accretive. We note that if $\mathcal{E}$ is lower semicontinuous then $\overline{\partial_{L_\mu^2}\mathcal{E}^{L_\mu^2}} = \partial_{L_\mu^2}\mathcal{E}$.

Theorem 2.2.16 ([18, Lemma 7.1 & Theorem 7.4]). *Let X be either L_μ^r , $1 \leq r \leq \infty$, or $L_\mu^{1 \cap \infty}$. Then the following statements hold.*

(1) *If a functional $\mathcal{E} : X \rightarrow (-\infty, \infty]$ satisfies*

$$\begin{cases} \mathcal{E}(u + q(\hat{u} - u)) + \mathcal{E}(\hat{u} + q(\hat{u} - u)) \leq \mathcal{E}(u) + \mathcal{E}(\hat{u}) \\ \text{for all } u, \hat{u} \in X \text{ and } q \in C^\infty(\mathbb{R}) \text{ such that } q(0) = 0, \\ q' \text{ has compact support and } 0 \leq q' \leq 1 \text{ on } \mathbb{R} \end{cases} \quad (2.2.15)$$

then the operator $\partial_X\mathcal{E}$ is completely accretive in X .

(2) *If $\mathcal{E} : X \rightarrow [0, \infty]$ satisfies (2.2.15), $(0, 0) \in \partial_X\mathcal{E}$, and if $\mathcal{E}$ is lower semicontinuous for the topology of $X + L_\mu^2$, then the closure $\overline{\partial_X\mathcal{E}^X}$ of $\partial_X\mathcal{E}$ in X is m -completely accretive in X .*

Note that if $\mathcal{E}(0) = 0$ and $\mathcal{E}(u) \geq 0$ for all $u \in X$ with $\partial_X\mathcal{E}$ completely accretive, then the condition $(0, 0) \in \partial\mathcal{E}$ of Theorem 2.2.16 is satisfied.

2.3 The fractional p -Laplacian

We now introduce the primary operator of interest, the fractional p -Laplacian $(-\Delta_p)^s$, the doubly nonlinear operator $(-\Delta_p)^s\varphi$, as well as the relevant function space settings.

2.3.1 Gagliardo-Sobolev-Slobodeck i  spaces

We first provide a short summary of Gagliardo-Sobolev-Slobodeck i  spaces, which are necessary to study the initial boundary value problem (1.2.1) with functional analytical tools. For a deeper understanding of this theory, we refer the interested reader to [1, 90] or [129].

For $1 < p < \infty$, $0 < s < 1$, and an open subset Ω of $\mathbb{R}^d$, we denote by

$$W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega) \mid [u]_{W^{s,p}(\Omega)} < \infty \right\}$$

the s -Gagliardo-Sobolev-Slobodeckii space, also known as the s -fractional Sobolev space, where

$$[u]_{W^{s,p}(\Omega)} := \left(\int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{d+sp}} dy dx \right)^{1/p} \quad (2.3.1)$$

denotes the s -Gagliardo semi-norm. The space $W^{s,p}(\Omega)$ defines a Banach space if it is equipped with the norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\|u\|_{L^p(\Omega)}^p + [u]_{W^{s,p}(\Omega)}^p \right)^{1/p}$$

or, equivalently, $\|u\|_{L^p} + [u]_{W^{s,p}(\Omega)}$.

We can similarly introduce the mixed Sobolev space, for $1 \leq q \leq \infty$,

$$W^{s,(q,p)}(\Omega) = \{ u \in L^q(\Omega) \mid [u]_{W^{s,p}(\Omega)} < \infty \}$$

with norm $\|u\|_{W^{s,(q,p)}(\Omega)} := \|u\|_{L^q(\Omega)} + [u]_{W^{s,p}(\Omega)}$. Importantly, this mixed setting will allow us to ensure lower semicontinuity of the subdifferential and thereby the necessary accretivity properties and related existence results such as Theorem 2.4.1.

For convenience we will use the notation $W^{s,p} := W^{s,p}(\mathbb{R}^d)$ and $[u]_{s,p} := [u]_{W^{s,p}(\mathbb{R}^d)}$ for the Gagliardo seminorm on $\mathbb{R}^d$ (and similarly for the mixed Sobolev spaces).

In this thesis we incorporate boundary conditions by considering the seminorm $[u]_{W^{s,p}(\mathbb{R}^d)}$ and instead restricting the domain in the following way. We let $W_0^{s,p}(\Omega)$, denoted by $W_0^{s,p}$, be the closure in $W^{s,p}$ of the set $C_c^\infty(\Omega)$ of test functions with norm $\|u\|_{W_0^{s,p}} := \|u\|_{L^p} + [u]_{s,p}$. Note that we must extend u to $\mathbb{R}^d$ by zero to interpret $[u]_{s,p}$. By [1, Theorem 10.1.1], the space $W_0^{s,p}$ admits, for $1 < p < \infty$, the characterization

$$W_0^{s,p}(\Omega) = \left\{ u \in W^{s,p}(\mathbb{R}^d) \mid \begin{array}{l} \exists \bar{u} : \mathbb{R}^d \rightarrow \mathbb{R} \text{ s.t. } \bar{u} = u \text{ a.e. on } \mathbb{R}^d \\ \text{and } \bar{u} = 0 \text{ quasi-everywhere on } \mathbb{R}^d \setminus \Omega \end{array} \right\},$$

where $\bar{u}$ denotes a (quasi-continuous) representative of u . Therefore, the space $W_0^{s,p}$ incorporates *homogeneous Dirichlet boundary conditions* in a weak sense.

We similarly define the corresponding mixed Sobolev space for $1 < p < \infty$, $0 < s < 1$ and $1 \leq q \leq \infty$ as the closure of $C_c^\infty(\Omega)$ in the mixed norm $\|u\|_{W_0^{s,(q,p)}} := \|u\|_{L^q} + [u]_{s,p}$. We similarly use $W_0^{s,(q,p)}$ to denote $W_0^{s,(q,p)}(\Omega)$. Note that when $q = p$, we regain the standard fractional Sobolev space. The space $W_0^{s,(q,p)}$ is a Banach space, which is reflexive if $1 < p < \infty$, $0 < s < 1$ and $1 \leq q < \infty$. Moreover, it contains the function space $\{u \in C_c^1(\mathbb{R}^d) \mid \text{supp}(u) \subseteq \Omega\}$.

The fractional p -Laplacian is a natural extension to the fractional Laplacian and p -Laplacian operators. We now introduce this operator, as well as its linear counterpart, the fractional Laplacian.

2.3.2 The fractional Laplacian

In the case of the fractional Laplacian, many equivalent definitions are available (see [86] where equivalence is shown for ten such definitions). We illustrate this and the connection to the local case with a few equivalent definitions on $\mathbb{R}^d$.

We have a direct comparison with the Laplacian via the following Fourier multiplier definition. Let $0 < s < 1$, $u \in L^q(\mathbb{R}^d)$ for $q \in [1, 2]$ and $\mathcal{F}$ be the Fourier transform. If there exists $v \in L^q(\mathbb{R}^d)$ such that

$$\mathcal{F}(v)(\xi) = -|\xi|^{2s} \mathcal{F}u(\xi) \quad (2.3.2)$$

then we define $(-\Delta)^s u = v$. Some authors equivalently consider $|\xi|^\alpha$ and $\alpha \in (0, 2)$, however (2.3.2) is more natural when generalizing to the fractional p -Laplacian as the fractional Laplacian then corresponds to the case $p = 2$ and the parameter s denoting the fractional derivative is in the natural range $(0, 1)$.

For u in an appropriate vector space such as $V = L^q(\mathbb{R}^d)$, $q \in [1, \infty)$ and sufficiently regular, we have the following singular integral definition, taking the Cauchy principal value

$$(-\Delta)^s u = \lim_{r \rightarrow 0^+} c_{d,s} \int_{\mathbb{R}^d \setminus B_r(x)} \frac{u(x) - u(y)}{|x - y|^{d+2s}} dy$$

where

$$c_{d,s} := \frac{4^s \Gamma\left(\frac{d+2s}{2}\right)}{\pi^{d/2} |\Gamma(-s)|}$$

and Γ is the standard gamma function. To roughly see the connection to the Laplacian in this definition, we can rearrange to have

$$(-\Delta)^s u = -c_{d,s} \int_{\mathbb{R}^d \setminus B_r(0)} \frac{1}{|z|^s} \left(\frac{u(x+z) - u(x)}{|z|^s} - \frac{u(x) - u(x-z)}{|z|^s} \right) \frac{dz}{|z|^d}.$$

Then we can view

$$\frac{u(x+z) - u(x)}{|z|^s} \quad (2.3.3)$$

as approximating a fractional derivative (of power s) in the direction z . Here we can again see the connection to the standard Laplacian when $s \rightarrow 1$ as this fractional derivative (2.3.3) becomes an approximation of the standard derivative at x in the direction z . We note that the nonlocal second (fractional) derivative is weighted by the term $|z|^{-d}$. This ensures integrability in an appropriate setting and provides a chain rule for linear transformations expected of such a (fractional) differential operator.

We can also view this operator as the subdifferential of a convex, lower semi-continuous functional. Define the energy functional $\mathcal{E} : L^2(\mathbb{R}^d) \rightarrow (-\infty, \infty]$ by

$$\mathcal{E}(u) = \begin{cases} \frac{c_{d,s}}{4} [u]_{s,2}^2 & \text{if } [u]_{s,2} < \infty, \\ \infty & \text{otherwise,} \end{cases} \quad (2.3.4)$$

Then we can consider the subdifferential of $\mathcal{E}$ in $L^2(\mathbb{R}^d)$, and define

$$(-\Delta)^s u = \partial_{L^2} \mathcal{E}(u)$$

for $u \in D(\partial_{L^2} \mathcal{E})$. These three definitions are equivalent on appropriate functions spaces (L^2 in particular). We note that the singular integral formulation in particular provides a very convenient form for direct calculations. In the case of an open domain $\Omega \subset \mathbb{R}^d$, in order to interpret Dirichlet boundary conditions, for example in the singular integral definition, we still integrate over all of $\mathbb{R}^d$. However, we restrict the domain of the operator to functions which are zero outside Ω and only use this formula for points in the domain. The alternative, integrating only over Ω , is also possible and is known as the *regional fractional Laplacian* (see e.g. [64, 69]).

One other definition which may be considered is the fractional power obtained via a spectral decomposition. This is distinct from the previous definitions (see [111]) and not considered in this thesis, both due to the nonlinear setting and as it is less commonly used as a fractional version of the Laplacian.

The fractional p -Laplacian

In the case of the fractional p -Laplacian we do not have the Fourier transform approach. However, the singular integral and subdifferential definitions have natural extensions. Since the constant $c_{d,s}$ does not impact the mathematical analysis, it is common to ignore it here as we will do in this thesis. In particular, on an open domain $\Omega \subseteq \mathbb{R}^d$, for $1 < p < \infty$ and $0 < s < 1$ we define the energy functional

$$\mathcal{E}(u) = \begin{cases} \frac{1}{2p} [u]_{s,p}^p & \text{if } u \in W_0^{s,(2,p)}, \\ \infty & \text{if } u \in L^2 \setminus W_0^{s,(2,p)}, \end{cases} \quad (2.0.4)$$

where $[\cdot]_{s,p}$ is the Gagliardo seminorm on $\mathbb{R}^d$ given by

$$[u]_{s,p} = \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^p}{|x - y|^{d+sp}} dy dx \right)^{1/p}. \quad (2.0.5)$$

Note that if u is defined on $\Omega \subset \mathbb{R}^d$, we extend u by zero to be defined on $\mathbb{R}^d$ in order to evaluate (2.0.5).

Then we can define the fractional p -Laplacian as an operator on L^2 by

$$(-\Delta_p)^s u := \partial \mathcal{E}(u).$$

Similarly, we can consider the associated singular integral via the Cauchy principal value,

$$(-\Delta_p)^s u(x) = \lim_{r \rightarrow 0^+} \int_{\mathbb{R}^d \setminus B_r(x)} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))}{|x - y|^{d+sp}} dy \quad (2.3.5)$$

for all $x \in \mathbb{R}^d$. Importantly, we have the following characterization of the subdifferential via a variational definition.

Proposition 2.3.1 (Characterization of $(-\Delta_p)^s$). *For $1 < p < \infty$ and $0 < s < 1$, let $\mathcal{E}$ be given by (2.0.4). Then for every $u \in W_0^{s,(2,p)}$,*

$$\partial\mathcal{E}(u) = \left\{ h \in L^2 \left| \int_{\Omega} h(x)v(x) \, dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u(x) - u(y))^{p-1}(v(x) - v(y))}{|x - y|^{d+sp}} \, dx \, dy \right. \right. \\ \left. \left. \text{for all } v \in W_0^{s,(2,p)} \right. \right\}.$$

The subdifferential condition can be written equivalently with the following variational formulation. In particular, $v \in (-\Delta_p)^s u$ in the variational formulation will satisfy

$$\int_{\Omega} v \xi \, dx := \frac{1}{2} \int_{\mathbb{R}^{2d}} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y)) (\xi(x) - \xi(y))}{|x - y|^{d+sp}} \, d(x, y) \quad (2.3.6)$$

for all $\xi \in W_0^{s,(2,p)}(\Omega)$.

From Proposition 2.3.1 it is clear that if $u \in W_0^{s,(2,p)}$ and the singular integral (2.3.5) exists and is also in L^2 then $u \in D(\partial\mathcal{E})$ and the definitions coincide. However the reverse implication is less clear. A key result in this direction is [85] which provides an equivalence between weak solutions of the elliptic problem, (s, p) -harmonic functions and (s, p) -viscosity solutions. Moreover, they obtain sufficient conditions for the singular integral to exist. In particular, they are able to estimate the integral around the blowup uniformly for $x \in \Omega$, but only away from points of zero gradient and in a restricted range of p and s (see [85, Lemma 3.7] in particular). Hence, to avoid the restrictive regularity requirements when obtaining solutions, we use the subdifferential (equivalently, variational) definitions. Nevertheless, the singular integral is still a very useful tool for calculation both when the estimates of [85] may be applied and when taking a limit within the variational double integral.

There are number of natural extensions and generalizations to the fractional p -Laplacian. First, we note that the case $p = 1$ can be characterized similarly with a formulation presented in [94], although we do not consider $p = 1$ in this work. Many authors have also considered such problems with $|x - y|^{d+sp}$ replaced by a kernel K which satisfies two sided estimates of a form similar to $|x - y|^{d+sp}$. The case with variable exponents, called the fractional $p(x)$ -Laplacian in which p in (2.3.5) is replaced by $p(x)$ has also been introduced in the fractional setting [82].

Alternative definitions for the fractional p -Laplacian have also been proposed, for example in [118]. It is also possible to interpret the restriction to a domain $\Omega \subset \mathbb{R}^d$ in the sense of a regional fractional p -Laplacian [4].

2.3.3 Key properties

The energy functional (2.0.4) for the fractional p -Laplacian satisfies the following key properties which we include as a lemma for completeness (cf. [94], [50]). Importantly, these properties provide m -accretivity of the

subdifferential which is key for obtaining existence results as we will see in Section 2.4.

Lemma 2.3.2. *Let Ω be an open domain in $\mathbb{R}^d$, $d \geq 1$. Suppose $1 < p < \infty$ and $0 < s < 1$. Then the energy functional $\mathcal{E}$ given by (2.0.4) is convex, lower semicontinuous and proper with effective domain $D(\mathcal{E}) = W_0^{s,(2,p)}$.*

Proof. It is clear that $\mathcal{E}$ is proper with $0 \in W_0^{s,(2,p)}$ and convex due to the convexity of the power function $|\cdot|^p$ for $p > 1$. To see that $\mathcal{E}$ is lower semicontinuous on L^2 , we note that $\mathcal{E}$ is lower semicontinuous if and only if, for all $\alpha \geq 0$,

$$E_\alpha := \{ u \in L^2 \mid \mathcal{E}(u) \leq \alpha \}$$

is closed in L^2 . Moreover, $\mathcal{E}$ is convex if and only if E_α is convex and so by [33, Theorem 3.7], lower semicontinuity is equivalent to requiring that E_α be weakly closed in L^2 .

Let $\alpha \geq 0$ and $(u_n)_{n \in \mathbb{N}} \subseteq E_\alpha$ such that $u_n \rightarrow u$ in L^2 . We note that u_n must be bounded in L^2 for convergence. Since $u \in E_\alpha$, $u \in D(\mathcal{E})$ and u_n is bounded in $[\cdot]_{s,p}$. Hence u_n is bounded in $W_0^{s,(2,p)}$ and so there exists a subsequence $(u_{k_n})_{n \in \mathbb{N}}$ converging weakly to some $\tilde{u} \in W_0^{s,(2,p)}$. This implies that $u_{k_n} \rightharpoonup \tilde{u}$ in L^2 as $n \rightarrow \infty$, hence $\tilde{u} = u$ (using [33, Theorem 3.10]). Defining $T : u \mapsto \nabla_{x,y} u$ where $\nabla_{x,y} u = u(x) - u(y)$, we have that $\nabla_{x,y} u_{k_n} \rightharpoonup \nabla_{x,y} u$ in $L^p(\mathbb{R}^{2d}; \frac{d(x,y)}{|x-y|^{d+sp}})$ as $n \rightarrow \infty$. Then using the weak convergence in $L^p(\mathbb{R}^{2d}; \frac{d(x,y)}{|x-y|^{d+sp}})$,

$$[u]_{s,p} \leq \liminf_{n \rightarrow \infty} [u_{k_n}]_{s,p} = \liminf_{n \rightarrow \infty} \mathcal{E}(u_{k_n}) \leq \alpha$$

(see [33, Proposition 3.5]). Hence E_α is closed in L^2 and so we have lower semicontinuity. $\square$

Translation and rotation invariance

We now mention some key properties of the fractional p -Laplacian. First, we note that the fractional p -Laplacian on $\mathbb{R}^d$ is translation and rotation invariant. This can be seen by a simple change of variable, the main point being that the Jacobian of this transformation is one and the distance function $|x - y|$ remains unchanged when this transformation is applied to both x and y . Then the transformation may be transferred to the test function ξ in (2.3.6). Suppose $u \in D((-\Delta_p)^s)$ and let $w(x) := u(\mathcal{R}x + z)$

where $\mathcal{R}$ is a rotation on $\mathbb{R}^d$ and $z \in \mathbb{R}^d$. Then for $\xi \in W^{s,(2,p)}$,

$$\begin{aligned}
& \frac{1}{2} \int_{\mathbb{R}^{2d}} \frac{(w(x) - w(y))^{p-1} (\xi(x) - \xi(y))}{|x - y|^{d+sp}} dx dy \\
&= \frac{1}{2} \int_{\mathbb{R}^{2d}} \frac{(u(x) - u(y))^{p-1} (\xi(x - z) - \xi(y - z))}{|\mathcal{R}^{-1}(x - z) - \mathcal{R}^{-1}(y - z)|^{d+sp}} dx dy \\
&= \int_{\mathbb{R}^d} (-\Delta_p)^s u \xi(\mathcal{R}^{-1}(x - z)) dx \\
&= \int_{\mathbb{R}^d} ((-\Delta_p)^s u)(\mathcal{R}x + z) \xi(x) dx,
\end{aligned}$$

so that by Proposition 2.3.1, $((-\Delta_p)^s w)(x) = ((-\Delta_p)^s u)(\mathcal{R}x + z)$.

Scaling and homogeneity

We also have the following homogeneity and scaling property for linear maps which may be viewed as a simple chain rule in this setting resulting from the difference-based fractional derivative approximation and the weighting $|x - y|^{-d}$. For $a \in \mathbb{R}$ and $k \in \mathbb{R}$, let $w(x) := au(kx)$ for all $x \in \mathbb{R}^d$. Then for all $\xi \in W^{s,(2,p)}$,

$$\begin{aligned}
& \frac{1}{2} \int_{\mathbb{R}^{2d}} \frac{(w(x) - w(y))^{p-1} (\xi(x) - \xi(y))}{|x - y|^{d+sp}} dx dy \\
&= \frac{a^{p-1}}{2|k|^{2d}} \int_{\mathbb{R}^{2d}} \frac{(u(x) - u(y))^{p-1} (\xi(x/k) - \xi(y/k))}{|x/k - y/k|^{d+sp}} dx dy \\
&= a^{p-1} |k|^{sp-d} \int_{\mathbb{R}^d} (-\Delta_p)^s u(x) \xi(x/k) dx \\
&= a^{p-1} |k|^{sp} \int_{\mathbb{R}^d} ((-\Delta_p)^s u)(kx) \xi(x) dx
\end{aligned}$$

so that $((-\Delta_p)^s w)(x) = a^{p-1} |k|^{sp} ((-\Delta_p)^s u)(kx)$ for all $x \in \mathbb{R}^d$.

2.3.4 The doubly nonlinear operator $(-\Delta_p)^s \varphi$

For $\Omega \subseteq \mathbb{R}^d$ we restrict the variational formulation with the following subset of $(-\Delta_p)^s$

$$(-\Delta_p)^s|_{L^{1\infty}} := \left\{ (u, v) \in L^{1\infty} \times L^{1\infty} \left| \begin{array}{l} u \in W_0^{s,(2,p)} \text{ and } (u, v) \\ \text{satisfy (2.3.6) for all} \\ \xi \in W_0^{s,(2,p)} \end{array} \right. \right\}. \quad (2.3.7)$$

Throughout this thesis we focus on the composition of $(-\Delta_p)^s|_{L^{1\infty}}$ and φ . Hence, we introduce the composition operator on L^1 of the form $A\varphi$ for A an operator on L_μ^1 and φ a function on $\mathbb{R}$.

Definition 2.3.3. For A an operator on L_μ^1 and φ a function on $\mathbb{R}$, we define the composed operator $A\varphi$ in L_μ^1 as a graph by

$$A\varphi = \{(u, v) \in L_\mu^1 \times L_\mu^1 \mid (\varphi(u), v) \in A\},$$

and interchangeably as a (possibly multi-valued) operator on L_μ^1 . We also define the closure in L^1 of $A\varphi$ by

$$\overline{A\varphi}^{L^1} := \left\{ (u, v) \in L_\mu^1 \times L_\mu^1 \left| \begin{array}{l} \text{there exists } ((u_k, v_k))_{k \geq 1} \text{ such that} \\ v_k \in A\varphi(u_k) \ \forall k \geq 1, \\ \lim_{k \rightarrow \infty} u_k = u \text{ in } L^1 \text{ and } \lim_{k \rightarrow \infty} v_k = v \text{ in } L^1 \end{array} \right. \right\}.$$

Notation 2.3.4. In the case $A = (-\Delta_p)^s$, for convenience we define $(-\Delta_p)^s \varphi$ to be the closure $\overline{(-\Delta_p)^s}_{L^{1 \cap \infty}} \varphi^{L^1}$.

When we want to make the dependence on the domain explicit, particularly for the approximation argument in Section 2.9, we write $(-\Delta_p)_{\Omega_n}^s$, $(-\Delta_p)_{\Omega_n, 1 \cap \infty}^s$ and A_n instead (see also these definitions in Section 2.9).

Remark 2.3.5. As in the case of the fractional p -Laplacian, we have that the composed operator $(-\Delta_p)^s \varphi$ on $\mathbb{R}^d$ is translation and rotation invariant. We also have similar scaling homogeneity properties. In particular, for $k \in \mathbb{R}$ and $w(x) := u(kx)$, we have

$$((-\Delta_p)^s \varphi(w))(x) = |k|^{sp} ((-\Delta_p)^s \varphi(u))(kx)$$

for all $x \in \mathbb{R}^d$. Moreover, when $\varphi(r) = r^m$, $r \in \mathbb{R}$ with $m > 0$, for $a \in \mathbb{R}$, we have $(-\Delta_p)^s (au)^m = a^{m(p-1)} (-\Delta_p)^s u^m$.

Existence results for the composed operator $A\varphi$

We may also extend the domain of an operator defined on $L_\mu^{1 \cap \infty}$ in the following manner introduced by [22].

Definition 2.3.6. For an operator A defined on $L_\mu^{1 \cap \infty} \times L_\mu^1$ we can extend the domain to

$$\tilde{D}(A) = \left\{ u \in L^{1 \cap \infty} \left| \begin{array}{l} \exists (u_n, h_n)_{n \geq 1} \subseteq A \text{ such that} \\ u_n \rightarrow u \text{ in } L^1 \text{ and} \\ (u_n, h_n)_{n \geq 1} \text{ is bounded in } L^\infty \times L^1 \end{array} \right. \right\}.$$

Note that $D(A) \subseteq \tilde{D}(A) \subseteq L_\mu^1$. Then for $\varphi(u_0) \in \tilde{D}((\partial_{L^1} \mathcal{E})|_{L^{1 \cap \infty}})$ we have the following theorem for existence of strong solutions from [22]. Note that here β corresponds to φ^{-1} in our setting and v to $\varphi(u)$. The statement in [22] also uses a slightly different implementation of the sub-differential, closer to $\partial_{L^\infty} \mathcal{E}$ (giving potentially a larger operator), however this does not affect the proof. A similar result for the homogeneous evolution problem can be found in [50].

Theorem 2.3.7 ([22, Theorem 4.1], Existence of strong solutions). *Let Ω be an open domain in $\mathbb{R}^d$, $d \geq 1$, with finite measure and $T > 0$. Suppose $\mathcal{E} : L^2 \rightarrow [0, \infty]$ is a lower semicontinuous function with $\mathcal{E}(0) = 0$ satisfying (2.2.15) for $X = L^2$, $\beta \in AC_{\text{loc}}(\mathbb{R})$ is nondecreasing with $\beta(\mathbb{R}) = \mathbb{R}$. For every $v_0 \in \tilde{D}((\partial_{L^1}\mathcal{E})|_{L^1 \cap \infty})$ and $f \in BV((0, T); L^1) \cap L^1(0, T; L^1) \cap L^1(0, T; L^\infty)$, there exists $v \in L^\infty((0, T) \times \Omega)$ such that $u = \beta(v) \in W^{1,\infty}(0, T; L^1)$ is the unique strong solution to*

$$\begin{cases} u'(t) + (\partial_{L^1}\mathcal{E})|_{L^1 \cap \infty} v(t) \in f(t) & \text{for a.e. } t \in (0, T), \\ u(0) = \beta(v_0). \end{cases} \quad (2.3.8)$$

We also have the following regularity result from [22] for strong solutions to subgradient problems with such a nonlinear composition φ .

Theorem 2.3.8 ([22, Theorem 1.1]). *Let (Σ, μ) be a measure space. If $w \in W^{1,1}((0, T); L_\mu^1)$, $v \in L_{\text{loc}}^1(\mathbb{R})$ and*

$$u = \int_0^w v(r) \, dr \in BV(0, T; L_\mu^1) \cap L^1((0, T); L_\mu^1)$$

then $u \in W^{1,1}((0, T); L_\mu^1)$ and for a.e. $t \in (0, T)$

$$\frac{du}{dt}(t) = v(w(t)) \frac{dw}{dt}(t)$$

for μ -a.e. $x \in \Sigma$.

2.4 Existence and uniqueness of mild solutions

We now prove accretivity and density results for composed operators arising from the subdifferential of a convex, lower semicontinuous functional. We then apply these results using the nonlinear semigroup theory summarized in Section 2.2 to the doubly nonlinear operator $(-\Delta_p)^s \varphi$ to prove Theorem 2.1.1.

In comparison with Theorem 2.3.7, the domain of initial data is not restricted to the domain of the subdifferential or the extension given in Definition 2.3.6, but rather the closure of this domain in L^1 (see, in particular, [12, Section 4]).

2.4.1 Accretivity and density results for $(\partial\mathcal{E})|_{L^1 \cap \infty} \varphi$

We first prove the following key result for subdifferential operators of the form $\overline{(\partial\mathcal{E})|_{L^1 \cap \infty} \varphi}^{L^1} + F$. In particular, we obtain m - T -accretivity with complete resolvent when F is a Lipschitz perturbation. This follows from [50, Proposition 2.17 and Proposition 2.19]. A comparable result can also be found for operators defined on finite measure spaces (Σ, μ) in [22, pg. 22].

Theorem 2.4.1. *Suppose (Σ, μ) is a σ -finite measure space, $\mathcal{E} : L_\mu^2 \rightarrow [0, \infty]$ is convex, lower semicontinuous with $\mathcal{E}(0) = 0$ and satisfying (2.2.15). Let $\varphi \in C(\mathbb{R})$ be strictly increasing with $\varphi(0) = 0$ and satisfying*

$$[\beta_\lambda(u), (\partial\mathcal{E})|_{L^{1\cap\infty}} u]_1 \geq 0 \quad (2.4.1)$$

and

$$[\beta_\lambda(u), (\partial\mathcal{E})|_{L^{1\cap\infty}} u]_2 \geq 0 \quad (2.4.2)$$

for every $\lambda > 0$ and $u \in D((\partial\mathcal{E})|_{L^{1\cap\infty}})$, where $\beta = \varphi^{-1}$ and $\beta_\lambda := \lambda^{-1}(I - (I + \lambda\beta)^{-1})$ is the Yosida approximation of β . Suppose $F : L_\mu^1 \rightarrow L_\mu^1$ satisfies the Lipschitz property

$$|F(u) - F(\hat{u})| \leq \omega|u - \hat{u}| \quad \text{on } \Sigma \quad (2.4.3)$$

for all $u, \hat{u} \in L_\mu^1$ with constant $\omega \geq 0$ and satisfies $F(0) = 0$. Then $\overline{(\partial\mathcal{E})|_{L^{1\cap\infty}} \varphi^{L^1}} + F$ is ω -quasi m - T -accretive in L_μ^1 with complete resolvent.

Proof. Since $\mathcal{E}$ is convex and attains its global minimum at 0, we have that $(0, 0) \in \partial\mathcal{E}$. Then by the complete accretivity property, $\partial\mathcal{E}$ has complete resolvent. Hence $\partial\mathcal{E}$ is m -completely accretive in L^2 with complete resolvent so that $(\partial\mathcal{E})|_{L^{1\cap\infty}}$ is also completely accretive. Since φ is injective, we have by [50, Proposition 2.17] that $(\partial\mathcal{E})|_{L^{1\cap\infty}} \varphi$ is T -accretive in L_μ^1 with complete resolvent. Then we can apply [50, Proposition 2.19] to obtain the range condition for the closure and hence that the operator $A := \overline{(\partial\mathcal{E})|_{L^{1\cap\infty}} \varphi^{L^1}} + F$ is ω -quasi m -accretive in L_μ^1 with complete resolvent. To see that this is T -accretive in L_μ^1 , it is sufficient to prove that for all $(u, v), (\hat{u}, \hat{v}) \in A$ there is a $w \in L_\mu^\infty$ such that $w(x) \in \text{sign}^+(u(x) - \hat{u}(x))$ for a.e. $x \in \Sigma$ and

$$\int_\Sigma w(v - \hat{v}) \, d\mu \geq 0. \quad (2.4.4)$$

We refer to [50, pg. 14] and [17]. Since $\partial\mathcal{E}$ is completely accretive, applying [18, Proposition 2.2] we have that for all $(u, v), (\hat{u}, \hat{v}) \in \partial\mathcal{E}$,

$$\int_\Sigma T(u - \hat{u})(v - \hat{v}) \, d\mu \geq 0$$

whenever $T \in C^\infty(\mathbb{R})$ satisfies $0 \leq T' \leq 1$, T' has compact support and $T(0) = 0$. Applying this to $\varphi(u)$ and $\varphi(\hat{u})$,

$$\int_\Sigma T(\varphi(u) - \varphi(\hat{u})) \left((\partial\mathcal{E})|_{L_\mu^{1\cap\infty}} \varphi(u) - (\partial\mathcal{E})|_{L_\mu^{1\cap\infty}} \varphi(\hat{u}) \right) \, d\mu \geq 0.$$

Approximating sign^+ by such T , applying dominated convergence and noting that φ is strictly increasing, we have

$$\int_{u > \hat{u}} \text{sign}^+(u - \hat{u}) \left((\partial\mathcal{E})|_{L_\mu^{1\cap\infty}} \varphi(u) - (\partial\mathcal{E})|_{L_\mu^{1\cap\infty}} \varphi(\hat{u}) \right) \, d\mu \geq 0.$$

By the Lipschitz condition on F , the perturbed operator A satisfies (2.4.4) and so is T -accretive in L^1 . $\square$

As mentioned in Section 2.2.1, m -accretivity in L^1 provides existence of mild solutions to the associated Cauchy problem (2.2.1) for all initial data $u_0 \in \overline{D(A)}^{L^1}$, where in this case $A = (\partial\mathcal{E})_{|_{L^1 \cap \infty}} \varphi^{L^1} + F$. Hence, in order to apply our regularity results to all initial data in L^1 , we use the following density result for the composition operator $(\partial\mathcal{E})_{|_{L^1 \cap \infty}} \varphi$. This generalizes the classic density result for subdifferential operators (cf. [34] or [12, Proposition 1.6]) and in particular generalizes an idea from [12, p.48]. Note that we define $u^+ := \max\{u, 0\}$ and $u^- := \min\{u, 0\}$.

Theorem 2.4.2 (Density of $D((\partial\mathcal{E})_{|_{L^1 \cap \infty}} \varphi)$ in L^1_μ). *Let (Σ, μ) be a σ -finite measure space, $\mathcal{E} : L^2_\mu \rightarrow (-\infty, \infty]$ a proper, lower semicontinuous, convex functional, and $\varphi \in C(\mathbb{R})$ be a strictly increasing and surjective function satisfying $\varphi(0) = 0$. Suppose that $(0, 0) \in (\partial\mathcal{E})_{|_{L^1 \cap \infty}}$, $(\partial\mathcal{E})_{|_{L^1 \cap \infty}}$ is completely accretive and the Yosida approximation of φ satisfies (2.4.1) and (2.4.2). Finally, we require that for every $u \in D(\mathcal{E}\varphi)$, one has that u^+ and $u^- \in D(\mathcal{E}\varphi)$. Then the domain $D((\partial\mathcal{E})_{|_{L^1 \cap \infty}} \varphi)$ is a dense subset of $D(\mathcal{E}\varphi)$ in L^1 .*

In particular, if the set $D(\mathcal{E}_{|_{L^1 \cap \infty}} \varphi)$ is dense in L^1_μ , then $D((\partial\mathcal{E})_{|_{L^1 \cap \infty}} \varphi)$ is also dense in L^1_μ . We note that the condition $(0, 0) \in (\partial\mathcal{E})_{|_{L^1 \cap \infty}}$ will hold if $\mathcal{E}(0) = 0$. In specific cases of A and φ , the density of $D(A\varphi)$ in L^1 has been proved: for example, Evans [60, Sect. 2, Proposition 1] for $A = -\Delta$ in L^1 and $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ increasing and φ^{-1} Lipschitz continuous, or Igbida [74, Proposition 2.1] for $A = -\operatorname{div}(|\nabla \cdot|^{p-2} \nabla \cdot)$ in $L^{1 \cap \infty}$ and $\varphi(r) = |r|^{m-1}r$ for $r \in \mathbb{R}$ and $m \geq 1$.

Proof of Theorem 2.4.2. Applying [50, Lemma A.3.1] to $\partial\mathcal{E}_{|_{L^1 \cap \infty}}$, we obtain that for every $\lambda > 0$, every $\varepsilon > 0$ sufficiently small, and every $u \in D(\mathcal{E}_{|_{L^1 \cap \infty}} \circ \varphi)$, there is a unique $u_\lambda \in D(\partial\mathcal{E}_{|_{L^1 \cap \infty}} \circ \varphi)$ satisfying

$$u_\lambda + \lambda \left(\varepsilon \varphi(u_\lambda) + \partial\mathcal{E}_{|_{L^1 \cap \infty}} \varphi(u_\lambda) \right) \ni u,$$

or equivalently, there exists $v_\lambda \in \partial\mathcal{E}_{|_{L^1 \cap \infty}} \varphi(u_\lambda)$ such that

$$u_\lambda + \lambda (\varepsilon \varphi(u_\lambda) + v_\lambda) = u. \quad (2.4.5)$$

Multiplying (2.4.5) by $\varphi(u_\lambda) - \varphi(u)$ gives

$$\begin{aligned} (u_\lambda - u, \varphi(u_\lambda) - \varphi(u))_{L^2_\mu} &= -\lambda \varepsilon (\varphi(u_\lambda), \varphi(u_\lambda) - \varphi(u))_{L^2_\mu} \\ &\quad - \lambda (v_\lambda, \varphi(u_\lambda) - \varphi(u))_{L^2_\mu}. \end{aligned}$$

The first term on the right-hand side can be estimated by

$$\begin{aligned} -(\varphi(u_\lambda), \varphi(u_\lambda) - \varphi(u))_{L^2_\mu} &= -(\varphi(u_\lambda) - \varphi(u), \varphi(u_\lambda) - \varphi(u))_{L^2_\mu} \\ &\quad - (\varphi(u), \varphi(u_\lambda) - \varphi(u))_{L^2_\mu} \\ &\leq \|\varphi(u)\|_2^2 + \|\varphi(u_\lambda)\|_\infty \|\varphi(u)\|_1 \\ &\leq \|\varphi(u)\|_2^2 + \sup_{[-\|u\|_\infty, \|u\|_\infty]} \|\varphi\|_\infty \|\varphi(u)\|_1, \end{aligned}$$

and since $v_\lambda \in \partial \mathcal{E}|_{L_\mu^1 \cap \infty} \varphi(u_\lambda)$ and noting that $\mathcal{E}(\varphi(u_\lambda)) \geq 0$, it follows that

$$\begin{aligned} -(v_\lambda, \varphi(u_\lambda) - \varphi(u))_{L_\mu^2} &= -((\partial \mathcal{E})\varphi(u_\lambda), \varphi(u_\lambda) - \varphi(u))_{L_\mu^2} \\ &\leq -(\mathcal{E}(\varphi(u_\lambda)) - \mathcal{E}(\varphi(u))) \\ &\leq \mathcal{E}(\varphi(u)) \end{aligned}$$

for all $\lambda > 0$. Thus, and since φ is increasing, we have shown that

$$\begin{aligned} 0 \leq (u_\lambda - u, \varphi(u_\lambda) - \varphi(u))_{L_\mu^2} &\leq \lambda \mathcal{E}(\varphi(u)) + \lambda \varepsilon \|\varphi(u)\|_2^2 \\ &\quad + \lambda \varepsilon \sup_{[-\|u\|_\infty, \|u\|_\infty]} \|\varphi\|_\infty \|\varphi(u)\|_1 \end{aligned}$$

for all $\lambda > 0$, from where we can conclude that

$$\lim_{\lambda \rightarrow 0^+} \int_{\Sigma} (u_\lambda - u)(\varphi(u_\lambda) - \varphi(u)) d\mu = 0.$$

Since

$$f_\lambda(x) := (u_\lambda(x) - u(x))(\varphi(u_\lambda(x)) - \varphi(u(x))) \geq 0 \quad \mu\text{-a.e. on } \Sigma,$$

the latter limit means that $f_\lambda \rightarrow 0$ in L_μ^1 . After possibly passing to a subsequence, which we denote by $(\lambda_n)_{n \in \mathbb{N}}$, we know that

$$\lim_{n \rightarrow \infty} f_{\lambda_n}(x) = 0 \quad \text{for } \mu\text{-a.e. } x \in \Sigma,$$

which due to the strict monotonicity of φ implies that

$$\lim_{n \rightarrow \infty} u_{\lambda_n}(x) = u(x) \quad \text{for } \mu\text{-a.e. } x \in \Sigma. \quad (2.4.6)$$

It remains to find a dominating function for a subsequence of $(u_{\lambda_n})_{n \in \mathbb{N}}$ in L_μ^1 so that we would have convergence in L_μ^1 and thereby density. For this we estimate u_λ pointwise above and below by replacing u with u^+ and u^- in (2.4.5).

We recall from [50, Lemma 2.2.1] that if $u_\lambda \geq 0$ for all $\lambda > 0$, then the μ -pointwise limit (2.4.6) together with the fact that u_λ satisfies

$$\|u_\lambda\|_1 \leq \|u\|_1 \quad \text{for all } \lambda > 0, \quad (2.4.7)$$

implies that $u_\lambda \rightarrow u$ in L_μ^1 .

We now denote $A := \partial \mathcal{E}|_{L_\mu^1 \cap \infty}$. Let $u_{+, \lambda} := J_\lambda^{\varepsilon \varphi_1 + A \varphi}(u^+)$, where $u^+ = u \vee 0$ is the positive part of u . Since $\varepsilon \varphi_1 + A \varphi$ is T -accretive in L_μ^1 (cf. [50, Proposition 2.3.6]), one has that $u_{+, \lambda} \geq 0$ for all $\lambda > 0$. Moreover, by the above argument, $u_{+, \lambda}$ satisfies (2.4.6) and (2.4.7). Therefore one has that $u_{+, \lambda} \rightarrow u^+$ in L_μ^1 . Next, let $u^- = (-u) \vee 0$ be the negative part of u and set $u_{-, \lambda} := J_\lambda^{\varepsilon \varphi_1 + A \varphi}(-u^-)$. Then, one also has that $u_{-, \lambda}$ satisfies (2.4.6) and so, in particular, $-u_{-, \lambda}$ satisfies (2.4.6). Since $-u_{-, \lambda}$ is positive

and satisfies (2.4.7), it follows that $-u_{-, \lambda} \rightarrow -u^-$ in L_μ^1 . Moreover, for $u_\lambda := J_\lambda^{\varepsilon\varphi_1 + A\varphi} u$, one has that

$$-u_{-, \lambda} \leq u_\lambda \leq u_{+, \lambda} \quad \text{for every } \lambda > 0.$$

From this sandwich inequality and since $u_{+, \lambda} \rightarrow u^+$ in L_μ^1 and $-u_{-, \lambda} \rightarrow -u^-$ in L_μ^1 , one can extract from every zero sequence $(\lambda_n)_{n \geq 0}$ a subsequence $(\lambda_{k_n})_{n \geq 1}$ and finds a positive function $g \in L_\mu^1$ such that $|u_{\lambda_{k_n}}| \leq g$ μ -a.e. on Σ for all $n \geq 1$. Thus, by Lebesgue's dominated convergence theorem, it follows that $u_{\lambda_{k_n}} \rightarrow u$ in L_μ^1 as $n \rightarrow \infty$ and, thereby we have shown that the domain $D(\partial \mathcal{E}|_{L_\mu^{1 \cap \infty}} \circ \varphi)$ lies dense in the closure $\overline{D(\mathcal{E}|_{L_\mu^{1 \cap \infty}} \circ \varphi)}^{L_\mu^1}$ of $D(\mathcal{E}|_{L_\mu^{1 \cap \infty}} \circ \varphi)$ with respect to the L_μ^1 -norm topology. $\square$

We check conditions (2.2.15), (2.4.1) and (2.4.2) in the case of the fractional p -Laplacian, where $\mathcal{E}$ is given by (2.0.4) in order to apply Theorem 2.4.1 and Theorem 2.4.2.

Lemma 2.4.3. *Let Ω be an open domain in $\mathbb{R}^d$, $d \geq 1$, $0 < s < 1$ and $p > 1$. Then the energy functional $\mathcal{E}$ given by (2.0.4) satisfies (2.2.15). Moreover, for $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ such that φ is bijective, increasing and satisfies $\varphi(0) = 0$, $\mathcal{E}$ satisfies (2.4.1) and (2.4.2) for every $\lambda > 0$ and $u \in D((-\Delta_p)_s|_{L^{1 \cap \infty}})$.*

Proof. It is sufficient to show that $\mathcal{E}$ satisfies (2.2.15) for every v and $\hat{v} \in D(\mathcal{E})$. For given $x, y \in \mathbb{R}^d$, set $a = v(x) - v(y)$ and $b = \hat{v}(x) - \hat{v}(y)$, and let $q \in C^\infty(\mathbb{R})$ satisfying $q(0) = 0$ and $0 \leq q' \leq 1$. Since $0 \leq q' \leq 1$, there is a $k \in [0, 1]$ such that $q(b - a) = k(b - a)$. Then, by the convexity of $|\cdot|^p$, one has that

$$|ka + (1 - k)b|^p + |(1 - k)a + kb|^p \leq |a|^p + |b|^p$$

holds, and so we have

$$|b - q(b - a)|^p + |a + q(b - a)|^p \leq |a|^p + |b|^p.$$

Therefore (2.2.15) follows by integrating over $\mathbb{R}^{2d}$ with respect to $|x - y|^{-d-sp} dx dy$.

Since φ is increasing, φ^{-1} is also increasing and so for $\lambda \geq 0$,

$$|a - b| \leq |a - b + \lambda(\varphi^{-1}(a) - \varphi^{-1}(b))|$$

for all $a, b \in \mathbb{R}$. Further, $\varphi(0) = 0$ implies $\beta_\lambda(0) = 0$ and so β_λ is Lipschitz continuous. Hence $u \in D((-\Delta_p)_s|_{L^{1 \cap \infty}})$ implies that $\beta_\lambda(u) \in W_0^{s,p} \cap L^2$. It then follows from the characterization of $(-\Delta_p)_s$ given in Proposition 2.3.1 that the part $(-\Delta_p)_s|_{L^{1 \cap \infty}}$ of $(-\Delta_p)_s$ in $L^{1 \cap \infty} \times L^{1 \cap \infty}$ satisfies (2.4.2).

For (2.4.1) we approximate sign_0 in the following way as done in [50] for the p -Laplacian. We define the sequence $(\gamma_\varepsilon)_{\varepsilon>0} \subset C(\mathbb{R})$ by

$$\gamma_\varepsilon(r) = \begin{cases} 1 & \text{if } r > \varepsilon, \\ \frac{r}{\varepsilon} & \text{if } -\varepsilon \leq r \leq \varepsilon, \\ -1 & \text{if } r < -\varepsilon, \end{cases}$$

for $r \in \mathbb{R}$. Then for all $\varepsilon > 0$ and $\lambda > 0$, $\gamma_\varepsilon(\beta_\lambda)$ is bounded and locally Lipschitz continuous so that $\gamma_\varepsilon(\beta_\lambda(u)) \in W_0^{s,p} \cap L^2$ and using the characterization of $(-\Delta_p)^s$,

$$\int_{\Omega} \left((-\Delta_p)_{|L^{1 \cap \infty}}^s u \right) \gamma_\varepsilon(\beta_\lambda(u)) \, dx \geq 0 \quad (2.4.8)$$

for all $u \in D((-\Delta_p)_{|L^{1 \cap \infty}}^s)$. Moreover we have the bound $\gamma_\varepsilon(\beta_\lambda(\cdot)) \leq 1$ on $\mathbb{R}$,

$$\lim_{\varepsilon \rightarrow 0+} \gamma_\varepsilon(\beta_\lambda(u(x))) = \text{sign}_0(\beta_\lambda(u(x))) \quad \text{for a.e. } x \in \Omega,$$

and $(-\Delta_p)_{|L^{1 \cap \infty}}^s u \in L^1$, so we can apply dominated convergence to (2.4.8) to obtain (2.4.1) when taking $\varepsilon \rightarrow 0+$. $\square$

Hence we may apply Theorem 2.4.1 to the doubly nonlinear operator $(-\Delta_p)^s \varphi$ to obtain the following corollary.

Corollary 2.4.4. *Let Ω be an open domain in $\mathbb{R}^d$, $d \geq 1$. Suppose $F : L^1 \rightarrow L^1$ satisfies the Lipschitz property (2.4.3) for all $u, \hat{u} \in L^1$ with constant $\omega \geq 0$ and satisfies $F(0) = 0$. Suppose $\varphi \in C(\mathbb{R})$ such that φ is bijective, increasing and satisfies $\varphi(0) = 0$. Then the operator $(-\Delta_p)^s \varphi + F$ is ω -quasi m - T -accretive in L^1 with complete resolvent.*

We can now apply this accretivity result for $(-\Delta_p)^s \varphi$ and the previous density result to obtain existence of unique mild solutions.

Proof of Theorem 2.1.1. Let $\mathcal{E}$ be given by (2.0.4) and φ, f satisfy (2.0.1) and (2.0.3a)-(2.0.3b), respectively. Then, it follows from Corollary 2.4.4 that $(-\Delta_p)^s$ is m -completely accretive and $(-\Delta_p)_{|L^{1 \cap \infty}}^s$ satisfies (2.4.1) and (2.4.2) with respect to the Yosida approximation β_λ of φ^{-1} . Further, under the hypothesis $\varphi \in W_{loc}^{1,\infty}(\mathbb{R})$, one has that

$$[\varphi(\xi)]_{s,p} \leq \|\varphi'\|_{L^\infty(-\|\xi\|_\infty, \|\xi\|_\infty)} [\xi]_{s,p}$$

for every $\xi \in C_c^\infty(\Omega)$, and if $\varphi \in W_{loc}^{1,q}(\mathbb{R})$ for $q > 1/(1-s)$, then Hölder's

inequality yields that

$$\begin{aligned}
& \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\varphi(\xi(x)) - \varphi(\xi(y))|^p}{|x - y|^{d+sp}} dy dx \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\left| \int_{\xi(y)}^{\xi(x)} \varphi'(r) dr \right|^p}{|x - y|^{d+sp}} dy dx \\
&\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\left| \left(\int_{\xi(y)}^{\xi(x)} |\varphi'(r)|^q dr \right)^{\frac{p}{q}} \right|^{\frac{p}{q}}}{|x - y|^{d+sp}} |\xi(x) - \xi(y)|^{\frac{p}{q'}} dy dx \\
&\leq \|\varphi'\|_{L^q(-\|\xi\|_\infty, \|\xi\|_\infty)}^p \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\xi(x) - \xi(y)|^{\frac{p}{q'}}}{|x - y|^{d+sp}} dy dx \\
&= \|\varphi'\|_{L^q(-\|\xi\|_\infty, \|\xi\|_\infty)}^p [\xi]_{sq', \frac{p}{q'}}
\end{aligned}$$

for every $\xi \in C_c^\infty(\Omega)$. In the last estimate, we note that $q > 1/(1-s)$ is equivalent to $0 < sq' < 1$ and hence the seminorm, $[\xi]_{sq', \frac{p}{q'}}$ is finite. Thus, under either condition $\varphi \in W_{loc}^{1,\infty}(\mathbb{R})$ or $\varphi \in W_{loc}^{1,q}(\mathbb{R})$ for $q > 1/(1-s)$, one has that the set $C_c^\infty(\Omega)$ is contained in $D(\mathcal{E}|_{L^1 \cap \infty} \varphi)$ and dense in L^1 . Thus Theorem 2.4.2 implies that under those conditions on φ , the domain $D((-\Delta_p)^s|_{L^1 \cap \infty} \varphi)$ is dense in L^1 .

Since by Corollary 2.4.4, the operator $(-\Delta_p)^s \varphi + F$ is m - T accretive in L^1 with complete resolvent, it follows from standard semigroup theory (e.g. [12, Corollary 4.2]) that for every $u_0 \in \overline{D(\mathcal{E}|_{L^1 \cap \infty} \circ \varphi)}^{L^1}$, there exists a unique mild solution u to problem (1.2.1). Moreover this mild solution satisfies growth estimate (2.1.2) (see also Lemma 2.7.3) and (2.1.3) for $\nu = 1$ (e.g. [12, Theorem 4.1]). The case $\nu = +$ follows in the same way, applying the T -contractivity condition of the resolvent. This completes the proof of this theorem. $\square$

2.5 Strong solutions on domains with finite volume

We now consider strong solutions on domains with finite Lebesgue measure, applying Theorem 2.3.7 to show that for φ strictly increasing with $\varphi(0) = 0$ and $\varphi^{-1} \in AC_{loc}(\mathbb{R})$, mild solutions to (1.2.1) are in fact strong distributional solutions. Moreover, we obtain general derivative and energy estimates for such φ when Ω is a general open domain in $\mathbb{R}^d$. We later refine these in the case $\varphi(r) = r^m$ to obtain the estimates of Theorem 2.1.6.

We first introduce a Lipschitz continuity result for mild solutions of (1.2.1) with initial data $u_0 \in L^1$ satisfying $\varphi(u_0) \in D((-\Delta_p)^s|_{L^1 \cap \infty})$. For this, we define

$$V(g, t+) := \limsup_{h \rightarrow 0^+} \int_0^{t-h} \frac{\|g(\tau+h) - g(\tau)\|_1}{h} d\tau$$

and for $t \in [0, T)$, $g(t+)$ is the essential limit of g from the right.

This result is presented in [21, Lemma 7.8] in the setting of T -accretive operators. However, considering the potential availability of [21], we include the short proof for completeness.

Lemma 2.5.1 ([21, Lemma 7.8]). *Let Ω be an open domain in $\mathbb{R}^d$, $d \geq 1$ and $T > 0$. Let $1 < p < \infty$, $0 < s < 1$, $\varphi \in C(\mathbb{R})$ satisfy (2.0.1) and $\varphi^{-1} \in AC_{\text{loc}}(\mathbb{R})$. Suppose $f(\cdot, u)$ satisfies (2.0.3a)-(2.0.3b) and let F be the Nemystkii operator of f . Further suppose that $g \in BV(0, T; L^1) \cap L^1(0, T; L^{1 \cap \infty})$, $u_0 \in L^1$ and $\varphi(u_0) \in D((-\Delta_p)^s|_{L^{1 \cap \infty}})$. Then the mild solution u of (1.2.1) is Lipschitz continuous in L^1 for all $t \in (0, T)$ with Lipschitz constant*

$$\begin{aligned} L := & e^{\omega T} \|g(0+) - (-\Delta_p)^s \varphi(u_0)\|_1 + V(g, T+) \\ & + \omega \int_0^T e^{\omega(T-\tau)} V(g, \tau+) \, d\tau. \end{aligned} \quad (2.5.1)$$

Proof. Let $h \in (0, T)$ and $t \in [0, T - h]$. From the growth estimate (2.1.3), we obtain

$$\begin{aligned} \|u(t+h) - u(t)\|_1 & \leq e^{\omega t} \|u(h) - u_0\|_1 \\ & \quad + \int_0^t e^{\omega(t-\tau)} \|g(\tau+h) - g(\tau)\|_1 \, d\tau \end{aligned} \quad (2.5.2)$$

noting the definition of $[\cdot, \cdot]_1$ given by (2.2.4) for the term involving g . Using (2.1.3) again, now with the constant solution obtained by setting $g_2(t) = u_0$ for $t \in [0, h]$,

$$\|u(h) - u_0\|_1 \leq \int_0^h e^{\omega(h-\tau)} \|g(\tau) - (-\Delta_p)^s \varphi(u_0)\|_1 \, d\tau.$$

Dividing (2.5.2) by h and taking the limit supremum as $h \rightarrow 0^+$, we then have the Lipschitz estimate (2.5.1). $\square$

We now prove the existence of strong distributional solutions.

Proof of Theorem 2.1.2. By Theorem 2.1.1, for every $u_0 \in L^1$, there is a unique mild solution to Cauchy problem (1.2.1). Now, let $u_0 \in \hat{D}((-\Delta_p)^s|_{L^{1 \cap \infty}} \varphi)$. Then by the definition of $\hat{D}$ given in (2.1.4), there exists a sequence $(v_n, w_n) \in (-\Delta_p)^s|_{L^{1 \cap \infty}}$ for $n \in \mathbb{N}$ such that $v_n \rightarrow \varphi(u_0)$ in L^1 as $n \rightarrow \infty$ and $(w_n)_{n \in \mathbb{N}}$ is bounded in L^1 . Since $(v_n)_{n \in \mathbb{N}}$ is bounded in L^∞ and Ω has finite measure, $\varphi^{-1}(v_n) \in L^{1 \cap \infty}$ uniformly for $n \in \mathbb{N}$. Let $(u_n)_{n \in \mathbb{N}}$ be the mild solutions with initial data $\varphi^{-1}(v_n)$. By Lemma 2.5.1, each u_n is Lipschitz continuous on $[0, T]$. Hence estimating the Lipschitz continuity of u ,

$$\begin{aligned} \|u(t+h) - u(t)\|_1 & \leq \|u_n(t+h) - u_n(t)\|_1 + (1 + e^{\omega h}) \|u(t) - u_n(t)\|_1 \\ & \leq L_n h + 2e^{\omega(t+h)} \|u_0 - u_n(0)\|_1 \end{aligned}$$

for all $n \in \mathbb{N}$. We note that since w_n is bounded in L^1 , L_n is bounded uniformly for all $n \in \mathbb{N}$. Since v_n converges to $\varphi(u_0)$ in L^1 , we have pointwise convergence of $u_n(0)$ to u_0 almost everywhere in Ω . Hence, using the uniform bound for v_n in L^∞ , we can take the limit supremum as $n \rightarrow \infty$ and apply Fatou's lemma to obtain Lipschitz continuity for u . Then by the Lipschitz continuity of F we have that $F(u) \in BV((0, T); L^1)$. Moreover by the standard growth estimate (2.1.2), since $\tilde{D}((-\Delta_p)_{|L^1 \cap \infty}^s) \subset L^\infty$, $u \in L^\infty([0, T]; L^\infty)$ and so $F(u) \in L^1(0, T; L^\infty)$. Applying Theorem 2.3.7 with forcing term $\tilde{g} = -F(u) + g$ we have that $u \in W^{1,\infty}(0, T; L^1)$ is a strong distributional solution to (1.2.1).

The chain rule (2.1.5) follows from the proof of [22, Theorem 4.1] and we have $u \in C([0, T]; L^q)$ for $1 \leq q < \infty$ due to the regularity of mild solutions in L^1 . Multiply the doubly nonlinear problem (1.2.1) by $\frac{d}{dt}\varphi(u)$ to obtain

$$\varphi'(u) \left| \frac{du}{dt} \right|^2 + \frac{d}{dt} \mathcal{E}(\varphi(u(t))) + (F(u(t)) - g(t)) \frac{du}{dt}(t) \varphi'(u(t)) = 0$$

giving (2.1.6). $\square$

We now introduce a lemma to obtain similar derivative and energy estimates to Theorem 2.1.6 for a more general class of subdifferential operators and nonlinearities φ . In particular, we consider the problem

$$\begin{cases} \frac{\partial u}{\partial t} + \overline{A\varphi^{L^1}} u + f(x, u) = g(x, t) & \text{in } \Omega \times (0, T), \\ u = 0 & \text{in } \mathbb{R}^d \setminus \Omega \times (0, T), \\ u(x, 0) = u_0(x) & \text{on } \Omega, \end{cases} \quad (2.5.3)$$

where A is the subdifferential in L^1 of a proper, lower semicontinuous convex functional $\mathcal{E} : L^2 \rightarrow (-\infty, \infty]$. Here we use the notation $\Phi(r) := \int_0^r \varphi(s) ds$ for $r \in \mathbb{R}$ and $\varphi \in C(\mathbb{R})$.

We first prove an intermediate result which we will further estimate differently depending on the regularity of φ to obtain estimates with differing dependence on the forcing term g .

Lemma 2.5.2. *Let Ω be an open domain in $\mathbb{R}^d$, $d \geq 1$ and $T > 0$. Let $A := (\partial_{L^1} \mathcal{E})_{|L^1 \cap \infty}$ where $\mathcal{E} : L^2 \rightarrow (-\infty, \infty]$ is a proper, lower semicontinuous, convex functional satisfying (2.2.15) and $\mathcal{E}(0) = 0$. Let $\varphi \in C(\mathbb{R})$ be a strictly increasing function such that $\varphi^{-1} \in AC_{\text{loc}}(\mathbb{R})$ and $\varphi(0) = 0$. Suppose that $f(\cdot, u)$ satisfies (2.0.3a)-(2.0.3b) and $g \in L^1(0, T; L^{1 \cap \infty})$. Then every strong distributional solution u of (2.5.3) in*

L^1 satisfies

$$\begin{aligned}
& \frac{1}{2} \int_0^t s^{k+2} \int_{\Omega} \varphi'(u) \left| \frac{du}{ds} \right|^2 dx ds + t^{k+2} \mathcal{E}(\varphi(u(t))) \\
& \leq (k+2) \int_0^t (k+1 + \omega s) s^k \|u\varphi(u)\|_1 ds \\
& \quad + \int_0^t ((k+2)^2 + \omega^2 s^2) s^k \|u^2 \varphi'(u)\|_1 ds \\
& \quad + (k+2) \int_0^t s^{k+1} \int_{\Omega} g \varphi(u) dx ds \\
& \quad + \int_0^t s^{k+2} \int_{\Omega} g \varphi'(u) \frac{du}{ds} dx ds
\end{aligned} \tag{2.5.4}$$

for all $t \in (0, T]$ and $k > -1$.

Proof. We write $v := \varphi(u)$. Multiplying (2.5.3) by $s^{k+2} \frac{dv}{ds}$ for $k > -1$, we can estimate u in $W_{\text{loc}}^{1,2}((0, T]; L^2)$, as in [22], by

$$\begin{aligned}
& \int_0^t s^{k+2} \int_{\Omega} \varphi'(u(s)) \left| \frac{du}{ds} \right|^2 dx ds + t^{k+2} \mathcal{E}(\varphi(u(t))) \\
& = (k+2) \int_0^t s^{k+1} \mathcal{E}(\varphi(u(s))) ds \\
& \quad + \int_0^t s^{k+2} \int_{\Omega} (g(s) - F(u(s))) \frac{dv}{ds}(s) dx ds.
\end{aligned} \tag{2.5.5}$$

Estimating $\int_0^t s^{k+1} \mathcal{E}(v(s)) ds$, we note that A is the subdifferential of $\mathcal{E}$, so

$$\langle A(v(s)) - A(0), 0 - v(s) \rangle \leq \mathcal{E}(0) - \mathcal{E}(v(s))$$

for all $0 < s \leq T$. Then we estimate $\mathcal{E}(v(s))$ by

$$\mathcal{E}(v) \leq - \int_{\Omega} \frac{du}{ds} v dx - \int_{\Omega} F(u) v dx + \int_{\Omega} g v dx \tag{2.5.6}$$

for $0 < s \leq T$. Since φ is increasing,

$$\Phi(r) \leq \varphi(r)r \quad \text{for all } r \in \mathbb{R}.$$

We multiply (2.5.6) by s^{k+1} and integrate over $(0, t)$ to obtain

$$\begin{aligned}
& \int_0^t s^{k+1} \mathcal{E}(v(s)) ds + t^{k+1} \int_{\Omega} \Phi(u(t)) dx \\
& \leq (k+1) \int_0^t s^k \int_{\Omega} \Phi(u(s)) dx ds \\
& \quad + \int_0^t s^{k+1} \int_{\Omega} u(s) \frac{dv}{ds}(s) ds \\
& \quad + \int_0^t s^{k+1} \int_{\Omega} (g(s) - F(u(s))) v dx ds.
\end{aligned} \tag{2.5.7}$$

Since $\varphi(0) = 0$ and φ is non-decreasing, $\Phi(r) \geq 0$ for all $r \in \mathbb{R}$ and $\varepsilon > 0$. Hence

$$\int_{\Omega} \Phi(u(t)) \, dx \geq 0$$

for all $0 < t \leq T$. Then returning to (2.5.5) we can estimate $\int_0^t s^{k+1} \mathcal{E}(v(s)) \, ds$, giving

$$\begin{aligned} & \int_0^t s^{k+2} \int_{\Omega} \varphi'(u) \left| \frac{du}{ds} \right|^2 \, dx \, ds + t^{k+2} \mathcal{E}(\varphi(u(t))) \\ &= (k+2)(k+1) \int_0^t s^k \int_{\Omega} \Phi(u(s)) \, dx \, ds + (k+2) \int_0^t s^{k+1} \int_{\Omega} u \frac{dv}{ds} \, ds \\ &+ (k+2) \int_0^t s^{k+1} \int_{\Omega} (g(s) - F(u)) v \, dx \, ds \\ &+ \int_0^t s^{k+2} \int_{\Omega} (g(s) - F(u)) \frac{dv}{ds} \, dx \, ds, \end{aligned}$$

where we leave the dependence $u = u(s)$ implicit in these integrals. Further, estimating $\Phi(u)$ and applying the Lipschitz property of F ,

$$\begin{aligned} & \int_0^t s^{k+2} \int_{\Omega} \varphi'(u) \left| \frac{du}{ds} \right|^2 \, dx \, ds + t^{k+2} \mathcal{E}(\varphi(u(t))) \\ & \leq (k+2) \int_0^t (k+1 + \omega s) s^k \|u\varphi(u)\|_1 \, ds \\ & + \int_0^t (k+2 + \omega s) s^{k+1} \int_{\Omega} |u| \varphi'(u) \left| \frac{du}{ds} \right| \, dx \, ds \quad (2.5.8) \\ & + (k+2) \int_0^t s^{k+1} \int_{\Omega} g \varphi(u) \, dx \, ds \\ & + \int_0^t s^{k+2} \int_{\Omega} g \frac{dv}{ds} \, dx \, ds. \end{aligned}$$

Applying Young's inequality, we combine $\left| \frac{du}{ds} \right|$ terms to obtain (2.5.4). $\square$

We now extend (2.5.4) when g has bounded variation in L^1 .

Lemma 2.5.3. *Suppose Ω is an open domain in $\mathbb{R}^d$, $d \geq 1$ and $T > 0$. Let $A := (\partial_{L^1} \mathcal{E})|_{L^1 \cap \infty}$ where $\mathcal{E} : L^2 \rightarrow (-\infty, \infty]$ is a proper, lower semi-continuous, convex functional satisfying (2.2.15) and $\mathcal{E}(0) = 0$. Let φ satisfy (2.0.1) and $\varphi^{-1} \in AC_{\text{loc}}(\mathbb{R})$. Suppose that $f(\cdot, u)$ satisfies (2.0.3a)-(2.0.3b) and $g \in BV(0, T; L^1) \cap L^1(0, T; L^{1 \cap \infty})$. Then every strong dis-*

tributonal solution u of (2.5.3) in L^1 with $u_0 \in L^\infty$ satisfies

$$\begin{aligned}
& \frac{1}{2} \int_0^t s^{k+2} \int_{\Omega} \varphi'(u(s)) \left| \frac{du}{ds} \right|^2 dx ds + t^{k+2} \mathcal{E}(\varphi(u(t))) \\
& \leq (k+2) \int_0^t (k+1 + \omega s) s^k \|u\varphi(u)\|_1 ds \\
& \quad + \int_0^t ((k+2)^2 + \omega^2 s^2) s^k \|u^2 \varphi'(u)\|_1 ds \\
& \quad + \lim_{h \rightarrow 0^+} \frac{1}{h} \int_0^{t-h} s^{k+2} \|g(s+h) - g(s)\|_1 ds \|\varphi(u)\|_{L^\infty(0,t;L^\infty)} \\
& \quad + 2(k+2) \int_0^t s^{k+1} \|g(s)\|_1 ds \|\varphi(u)\|_{L^\infty(0,t;L^\infty)}
\end{aligned} \tag{2.5.9}$$

for all $t \in (0, T]$ and $k > -1$.

Proof. We integrate the last term of (2.5.4) by parts, giving

$$\begin{aligned}
& \int_0^t s^{k+2} \int_{\Omega} g \frac{dv}{ds} dx ds \\
& \leq \lim_{h \rightarrow 0^+} \frac{1}{h} \int_0^{t-h} \|(s+h)^{k+2} g(t+s) - s^{k+2} g(s)\|_1 ds \|v\|_{L^\infty(0,t;L^\infty)} \\
& \leq \lim_{h \rightarrow 0^+} \frac{1}{h} \int_0^{t-h} s^{k+2} \|g(s+h) - g(s)\|_1 ds \|\varphi(u)\|_{L^\infty(0,t;L^\infty)} \\
& \quad + (k+2) \int_0^t s^{k+1} \|g(s)\|_1 ds \|\varphi(u)\|_{L^\infty(0,t;L^\infty)}.
\end{aligned}$$

□

Before extending the estimates of this section further in the case $\varphi(r) = r^m$, $r \in \mathbb{R}$ with $m > 0$, we introduce the following refined version of the dissipative estimate for T -accretive operators and an $L^1 - L^\infty$ regularizing effect.

2.6 Refined dissipation in L^1

Due to the L^1 setting that we have introduced for $(-\Delta_p)^s \varphi$ and, in particular, the T -accretivity properties, we find dissipation of differences $(u_1 - u_2)^+$ in L^1 for solutions to (1.2.1). This follows the case of the evolution fractional p -Laplacian given by [121]. We note that this is a refinement of (2.1.3), the difference being that here we keep the full $(-\Delta_p)^s \varphi(u)$ terms when estimating rather than applying T -accretivity to estimate these by zero. Due to this we cannot use the T -accretivity directly.

We note that in the case $f \equiv 0$ and $g_1 \leq g_2$ this provides a strict dissipative effect for $(u_1 - u_2)^+$ whenever we have a non-zero set of times t with $|\{u_1 > u_2\}| > 0$ and $|\{u_1 \leq u_2\}| > 0$.

Theorem 2.6.1. *Let Ω be an open domain in $\mathbb{R}^d$, $d \geq 1$ and $T > 0$. Suppose φ satisfies (2.0.1), f satisfies (2.0.3a)-(2.0.3b) and $g_1, g_2 \in L^1(0, T; L^1)$. We consider u_1, u_2 to be two strong distributional solutions to (1.2.1) in L^1 . Then for all $0 < t_1 < t_2 < T$,*

$$\begin{aligned} \|[u_1 - u_2]^+(t_2)\|_1 &\leq e^{\omega(t_2-t_1)} \|[u_1 - u_2]^+(t_1)\|_1 - \frac{1}{2} \int_{t_1}^{t_2} I(\tau) d\tau \\ &\quad + e^{\omega(t_2-t_1)} \int_{t_1}^{t_2} \int_{\{u_1 > u_2\}} (g_1 - g_2)^+ dx d\tau \\ &\quad - \int_{t_1}^{t_2} \int_{\{u_1 > u_2\}} (g_2 - g_1)^+ dx d\tau \end{aligned} \quad (2.6.1)$$

where for all $\tau \in [0, T]$, omitting τ in $u_1(x, \tau)$ etc., $I(\tau)$ is given by

$$\int_{D(\tau)} \frac{|(\varphi(u_1(x)) - \varphi(u_1(y)))^{p-1} - (\varphi(u_2(x)) - \varphi(u_2(y)))^{p-1}|}{|x - y|^{d+sp}} d(x, y) \quad (2.6.2)$$

with the domain of integration

$$\begin{aligned} D(\tau) &= \{u_1(x, \tau) > u_2(x, \tau), u_1(y, \tau) \leq u_2(y, \tau)\} \\ &\quad \cup \{u_1(x, \tau) \leq u_2(x, \tau), u_1(y, \tau) > u_2(y, \tau)\}. \end{aligned}$$

Proof. We multiply (1.2.1) by $\text{sign}_0^+(u_1 - u_2)$ and integrate over $\Omega \times (t_1, t_2)$. Then

$$\begin{aligned} &\|[u_1 - u_2]^+(t_1)\|_1 - \|[u_1 - u_2]^+(t_2)\|_1 \\ &= \int_{t_1}^{t_2} \int_{\Omega} ((-\Delta_p)^s \varphi(u_1) - (-\Delta_p)^s \varphi(u_2)) \mathbb{1}_{\{u_1 > u_2\}} dx d\tau \\ &\quad + \int_{t_1}^{t_2} \int_{\Omega} (F(u_1) - F(u_2)) \mathbb{1}_{\{u_1 > u_2\}} dx d\tau \\ &\quad - \int_{t_1}^{t_2} \int_{\Omega} (g_1 - g_2) \mathbb{1}_{\{u_1 > u_2\}} dx d\tau \end{aligned}$$

We evaluate the fractional p -Laplacian term as in [121, Section 5], noting that φ strictly increasing implies that $\mathbb{1}_{\{u_1 > u_2\}} = \mathbb{1}_{\{\phi(u_1) > \phi(u_2)\}}$. Here we omit the time dependence of $u_1(x, \tau)$ and $u_2(x, \tau)$ for brevity of notation. We have, for $\tau \in (0, T)$,

$$\begin{aligned} &\int_{\Omega} ((-\Delta_p)^s \varphi(u_1) - (-\Delta_p)^s \varphi(u_2)) \mathbb{1}_{\phi(u_1) > \phi(u_2)} dx \\ &= \frac{1}{2} \int_{\mathbb{R}^{2d}} \frac{(\varphi(u_1(x)) - \varphi(u_1(y)))^{p-1} - (\varphi(u_2(x)) - \varphi(u_2(y)))^{p-1}}{|x - y|^{d+sp}} \\ &\quad \times (\mathbb{1}_{\{\phi(u_1(x)) > \phi(u_2(x))\}} - \mathbb{1}_{\{\phi(u_1(y)) > \phi(u_2(y))\}}) d(x, y). \end{aligned} \quad (2.6.3)$$

Looking at the integrand, we notice that

$$\mathbb{1}_{\{\phi(u_1(x)) > \phi(u_2(x))\}} - \mathbb{1}_{\{\phi(u_1(y)) > \phi(u_2(y))\}} \quad (2.6.4)$$

is 1 if and only if both $u_1(x) > u_2(x)$ and $u_1(y) \leq u_2(y)$. Similarly, (2.6.4) is -1 if and only if $u_1(x) \leq u_2(x)$ and $u_1(y) > u_2(y)$. Otherwise, this difference term is zero. So in both cases where (2.6.4) is non-zero, the integrand of (2.6.3) will be positive and in fact we can write

$$\int_{\Omega} ((-\Delta_p)^s \varphi(u_1) - (-\Delta_p)^s \varphi(u_2)) \mathbb{1}_{\{u_1 > u_2\}} dx = I(\tau)$$

with I given by (2.6.2).

Importantly, this provides a dissipative effect for this difference $u_1 - u_2$ whenever $|\{u_1 > u_2\}| > 0$ and $|\{u_1 \leq u_2\}| > 0$.

For the remaining terms, we have

$$\int_{t_1}^{t_2} \int_{\Omega} (F(u_1) - F(u_2)) \mathbb{1}_{\{u_1 > u_2\}} dx d\tau \geq -\omega \int_{t_1}^{t_2} \| [u_1 - u_2]^+ \|_1 d\tau$$

and we split $g_1 - g_2$ according to its sign. Then we can group together terms with the same sign and apply a Grönwall inequality to obtain (2.6.1). $\square$

2.7 A general $L^\ell - L^\infty$ regularizing effect

As in Section 2.2, we state these results for Lebesgue spaces L_μ^q with $1 \leq q \leq \infty$. We note that the following theorem generalizes [104, Theorem 2.1] and [40, Theorem 1]. While these two theorems in [40, 104] are restricted to derive $L_\mu^q - L_\mu^\infty$ regularity estimates of solutions of parabolic diffusion problems with homogeneous forcing terms $g \equiv 0$ and without Lipschitz perturbations (see also [103]), the following results can also treat evolution problems involving Lipschitz continuous nonlinearities and L_μ^∞ -bounded forcing terms. In particular we consider mild solutions u to

$$\begin{cases} u'(t) + Au(t) = g(t) & \text{for a.e. } t \in (0, T), \\ u(0) = u_0, \end{cases} \quad (2.7.1)$$

where A is quasi m -accretive in $L_\mu^{q_0}$ with complete resolvent for some $1 \leq q_0 < \infty$.

For $\lambda \geq 0$, we define the signed truncator $G_\lambda(s) := [|s| - \lambda]^+ \text{sign}(s)$ for every $s \in \mathbb{R}$ and we set $0 < T \leq \infty$. Note that $q - 1 \in [0, \infty)$ so we are using the notation $\|\cdot\|_{q-1}$ to denote the usual Lebesgue integral even when this is not a norm. When $q - 1 = 0$ this is given by

$$\|u\|_0 = \int_{\Sigma} \text{sign}_0(|u|) d\mu.$$

The condition (2.7.2) for the forcing term can always be satisfied by choosing $\rho = \infty$ and $\psi = \infty$.

Theorem 2.7.1. *Let $T > 0$. For $1 \leq q < r \leq \infty$, $1 \leq \sigma < r$, $q \leq \rho \leq \infty$ and $1 < \psi \leq \infty$ satisfying*

$$\begin{cases} \frac{1}{\rho} < \left(1 - \frac{1}{\psi}\right) \left(1 - \frac{\sigma}{r}\right) & \text{if } \sigma \geq q, \\ \frac{1}{\rho} \leq \left(1 - \frac{1}{\psi}\right) \left(\frac{\sigma}{q} - \frac{\sigma}{r}\right) & \text{if } \sigma < q, \end{cases} \quad (2.7.2)$$

and

$$\frac{1}{\rho} \leq \frac{1}{q} - \frac{\sigma}{r} \left(1 - \frac{1}{\psi}\right), \quad (2.7.3)$$

let $g \in L^\psi(0, T; L_\mu^\rho) \cap L^1(0, T; L_\mu^q)$ and $u_0 \in L_\mu^q$. Suppose $u \in C([0, T]; L_\mu^q)$ satisfies $u(0) = u_0$ and for some $L > 0$, $\omega \geq 0$ and every $\lambda \geq 0$, the “level set energy inequality”

$$\begin{aligned} & \|G_\lambda(e^{-\omega t_2} u(t_2))\|_q^q + L \int_{t_1}^{t_2} e^{\omega(\sigma-q)s} \|G_\lambda(e^{-\omega s} u(s))\|_r^\sigma ds \\ & \leq \|G_\lambda(e^{-\omega t_1} u(t_1))\|_q^q + q\lambda\omega \int_{t_1}^{t_2} \|G_\lambda(e^{-\omega s} u(s))\|_{q-1}^{q-1} ds \\ & \quad + q \int_{t_1}^{t_2} e^{-\omega s} |[G_\lambda(e^{-\omega s} u), g(s)]_q| ds \end{aligned} \quad (2.7.4)$$

holds for all $0 \leq t_1 < t_2 \leq T$. Further, assume $t \mapsto G_\lambda(e^{-\omega t} u(t))$ satisfies the following growth estimate in the L^q -norm,

$$\begin{aligned} \|G_\lambda(e^{-\omega t} u(t))\|_q & \leq \|G_\lambda(e^{-\omega s} u(s))\|_q \\ & \quad + \int_s^t e^{-\omega \tau} \|g(\tau) \mathbf{1}_{\{e^{-\omega \tau} |u(\tau)| > \lambda\}}\|_q d\tau \end{aligned} \quad (2.7.5)$$

for all $0 \leq s < t \leq T$. Then there exists $C > 0$ depending on σ , q , r , ρ , ψ and L such that

$$\begin{aligned} \|u(t)\|_\infty & \leq C \max \left\{ (e^{\omega \beta_1 t} \left(\frac{1}{t} + \omega\right)^{\frac{1}{\sigma(1-\frac{q}{r})}} (\|u_0\|_q + \|g\|_{L^1(0,t;L^q)})^\gamma, \right. \\ & \quad \left. e^{\omega \beta_2 t} \|g\|_{L^\psi(0,T;L^\rho)}^\eta (\|u_0\|_q + \|g\|_{L^1(0,t;L^q)})^{\gamma_\psi} \right\} \end{aligned} \quad (2.7.6)$$

for all $t \in (0, T]$ with the exponents

$$\begin{aligned} \gamma &= \frac{\frac{1}{\sigma} - \frac{1}{r}}{\frac{1}{q} - \frac{1}{r}}, \quad \gamma_\psi = \frac{(1-\frac{1}{\psi})(\frac{1}{\sigma} - \frac{1}{r}) - \frac{1}{\rho\sigma}}{(1-\frac{1}{\psi})(\frac{1}{q} - \frac{1}{r}) - \frac{1}{\rho\sigma} + \frac{1}{q\sigma}}, \\ \beta_1 &= \begin{cases} \frac{\frac{1}{\sigma} - \frac{1}{q}}{\frac{1}{q} - \frac{1}{r}} & \text{if } \sigma < q, \\ 0 & \text{if } \sigma \geq q, \end{cases} \quad \beta_2 = \begin{cases} \eta(q - \sigma)(1 - \frac{1}{\psi}) & \text{if } \sigma < q, \\ 0 & \text{if } \sigma \geq q, \end{cases} \\ \eta &= \frac{\frac{1}{q\sigma}}{(1-\frac{1}{\psi})(\frac{1}{q} - \frac{1}{r}) - \frac{1}{\rho\sigma} + \frac{1}{q\sigma}}. \end{aligned} \quad (2.7.7)$$

Our proof of Theorem 2.7.1 is based on a De Giorgi iteration inspired by [104] and [40]. For this, we modify [87, Chapter 2.5, Lemma 5.6] to prove convergence of the following recurrence relation.

Lemma 2.7.2. *Let $b \geq 1$, $0 < f < 1$ and $M \in \mathbb{N} \setminus \{0\}$. Suppose a sequence $(y_k)_{k \geq 0}$ in $[0, \infty)$ satisfies the recursion relation*

$$y_{k+1} \leq b^k \sum_{i=1}^M c_i y_k^{1+\delta_i} \quad \text{for all } k \in \mathbb{N}$$

where c_i and δ_i are positive constants for all $i \in \{1, \dots, M\}$. Choose

$$C = \min_{i \in \{1, \dots, M\}} \left(c_i^{-\frac{1}{\delta_i}} \right)$$

and $\delta_m = \min_{i \in \{1, \dots, M\}} \delta_i$. If

$$y_0 \leq \frac{C}{M} b^{-\frac{1}{\delta_m^2}},$$

then

$$y_k \leq \frac{C}{M} b^{-\frac{1}{\delta_m^2}} b^{-\frac{k}{\delta_m}} \quad \text{for all } k \in \mathbb{N}. \quad (2.7.8)$$

In particular, if $b > 1$ then $y_k \rightarrow 0$ as $k \rightarrow \infty$.

Proof. Estimate (2.7.8) follows via induction with

$$\begin{aligned} y_{k+1} &\leq b^k \sum_{i=1}^M c_i y_k^{1+\delta_i} \\ &\leq \frac{C b^k}{M} \left(b^{-\frac{1}{\delta_m^2}} b^{-\frac{k}{\delta_m}} \right)^{1+\delta_m} \sum_{i=1}^M c_i \left(\frac{C}{M} \right)^{\delta_i} \\ &\leq \frac{C}{M} b^{-\frac{1}{\delta_m^2}} b^{-\frac{k+1}{\delta_m}}. \end{aligned}$$

□

Proof of Theorem 2.7.1. By (2.7.4), one sees that $G_\lambda(u) \in L^\sigma(0, T; L_\mu^r)$ for every $\lambda \geq 0$. Let $\lambda \geq 0$ and $t \in (0, T]$ and for every integer $k \geq 0$, set

$$t_k = t(1 - 2^{-k}), \quad \lambda_k = \lambda(1 - 2^{-k}), \quad G_k(\cdot) = G_{\lambda_k}(\cdot),$$

and

$$U_k = \sup_{\hat{s} \in [t_k, t]} \|G_k(e^{-\omega \hat{s}} u(\hat{s}))\|_q^q + L \int_{t_k}^t e^{\omega(\sigma-q)s} \|G_k(e^{-\omega s} u(s))\|_r^\sigma ds.$$

Then the aim is to choose $\lambda \geq 0$ such that $U_k \rightarrow 0$ as $k \rightarrow \infty$. By the continuity of $t \mapsto \|G_k(e^{-\omega t} u(t))\|_q^q$, there is an $s_k \in (t_{k-1}, t_k)$ satisfying

$$\|G_k(e^{-\omega s_k} u(s_k))\|_q^q = \frac{2^k}{t} \int_{t_{k-1}}^{t_k} \|G_k(e^{-\omega s} u(s))\|_q^q ds. \quad (2.7.9)$$

Further, note that

$$\mathbb{1}_{\{|e^{-\omega s}u|>\lambda_k\}} \leq \mathbb{1}_{\{|e^{-\omega s}u|>\lambda_{k-1}\}} \left(\frac{2^k [|e^{-\omega s}u| - \lambda_{k-1}]^+}{\lambda} \right)^\ell$$

for every $\ell \geq 0$. We can then estimate

$$|G_k(e^{-\omega s}u(s))|^q \leq \left(\frac{2^k}{\lambda} \right)^\ell |G_{k-1}(e^{-\omega s}u(s))|^{q+\ell} \quad (2.7.10)$$

on $[t_{k-1}, t]$ for $q \geq 0$ and $\ell \geq 0$. We now aim to obtain a recurrence relation for U_k of the form in Lemma 2.7.2. Taking a supremum over $[t_k, t]$ in (2.7.4) we can bound U_k by

$$\begin{aligned} U_k &\leq 2\|G_k(e^{-\omega t_k}u(t_k))\|_q^q + 2q\lambda_k\omega \int_{t_k}^t \|G_k(e^{-\omega s}u(s))\|_{q-1}^{q-1} ds \\ &\quad + 2q \int_{t_k}^t e^{-\omega s} \|G_k(e^{-\omega s}u)g(s)\|_q ds. \end{aligned} \quad (2.7.11)$$

Estimating the first term by Lemma 2.7.3 and choosing s_k according to (2.7.9),

$$\begin{aligned} &\|G_k(e^{-\omega t_k}u(t_k))\|_q^q \\ &\leq \left(\|G_k(e^{-\omega s_k}u(s_k))\|_q + \int_{s_k}^{t_k} e^{-\omega \tau} \|g(\tau)\mathbb{1}_{\{|e^{-\omega \tau}u(\tau)|>\lambda_k\}}\|_q d\tau \right)^q \\ &\leq \frac{2^{k+q}}{t} \int_{t_{k-1}}^{t_k} \|G_k(e^{-\omega s}u(s))\|_q^q ds \\ &\quad + 2^q \left(\int_{s_k}^{t_k} e^{-\omega \tau} \|g(\tau)\mathbb{1}_{\{|e^{-\omega \tau}u(\tau)|>\lambda_k\}}\|_q d\tau \right)^q. \end{aligned}$$

Separating the g in the second term here by Hölder's inequality, we have

$$\begin{aligned} &\int_{s_k}^{t_k} e^{-\omega \tau} \|g(\tau)\mathbb{1}_{\{|e^{-\omega \tau}u(\tau)|>\lambda_k\}}\|_q d\tau \\ &\leq \left(\int_{s_k}^{t_k} \|e^{-\omega \tau}g(\tau)\|_\rho^\psi d\tau \right)^{\frac{1}{\psi}} \left(\int_{s_k}^{t_k} \|\mathbb{1}_{\{|e^{-\omega \tau}u(\tau)|>\lambda_k\}}\|_{\rho'_q}^{\psi'} d\tau \right)^{\frac{1}{\psi'}} \end{aligned}$$

where we choose ρ'_q and ψ' such that $\frac{1}{\rho} + \frac{1}{\rho'_q} = \frac{1}{q}$ and $\frac{1}{\psi} + \frac{1}{\psi'} = 1$, respectively. We can then estimate U_k , extending the time integrals

(t_{k-1}, t) , with

$$\begin{aligned}
U_k &\leq \frac{2^{k+q+1}}{t} \int_{t_{k-1}}^t \|G_k(e^{-\omega s} u(s))\|_q^q ds \\
&\quad + 2^{q+1} \|g\|_{L^\psi(t_{k-1}, t; L_\mu^\rho)}^q \left(\int_{t_{k-1}}^t \|\mathbb{1}_{\{e^{-\omega \tau} |u(\tau)| > \lambda_k\}}\|_{\rho'_q}^{\psi'} d\tau \right)^{\frac{q}{\psi'}} \\
&\quad + 2q\lambda_k \omega \int_{t_{k-1}}^t \|G_k(e^{-\omega s} u(s))\|_{q-1}^{q-1} ds \\
&\quad + 2q \|g\|_{L^\psi(t_{k-1}, t; L_\mu^\rho)} \left(\int_{t_{k-1}}^t \|G_k(e^{-\omega s} u(s))\|_{(q-1)\rho'}^{(q-1)\psi'} ds \right)^{\frac{1}{\psi'}},
\end{aligned}$$

where we choose $\frac{1}{\rho} + \frac{1}{\rho'} = 1$.

We apply (2.7.10) to each G_k term, as well as $\mathbb{1}_{\{e^{-\omega \tau} |u(\tau)| > \lambda_k\}}$, with $\ell = \varepsilon_1, 1 + \varepsilon_1, q + \varepsilon_2 - \rho'_q$ and $q + \varepsilon_3 - (q-1)\rho'$. The positive constants $\varepsilon_1, \varepsilon_2$ and ε_3 will later be chosen such that $\ell \geq 0$ in each case and an appropriate recurrence relation may be obtained. Note that the requirement $\ell \geq 0$ will be satisfied as a result of assumption (2.7.2). Then noting that $\lambda_k < \lambda$, there exists $C > 0$ depending on q such that

$$\begin{aligned}
\frac{U_k}{C} &\leq \frac{2^{k(1+\varepsilon_1)}}{\lambda^{\varepsilon_1}} \left(\frac{1}{t} + \omega \right) \int_{t_{k-1}}^t \|G_{k-1}(e^{-\omega s} u(s))\|_{q+\varepsilon_1}^{q+\varepsilon_1} ds \\
&\quad + \left(\frac{2^k}{\lambda} \right)^{\frac{q(q+\varepsilon_2)}{\rho'_q}} \|g\|_{L^\psi(0, t; L_\mu^{\rho_q})}^q \left(\int_{t_{k-1}}^t \|G_{k-1}(e^{-\omega s} u(s))\|_{q+\varepsilon_2}^{(q+\varepsilon_2)\frac{\psi'}{\rho'_q}} d\tau \right)^{\frac{q}{\psi'}} \\
&\quad + \|g\|_{L^\psi(0, t; L_\mu^\rho)} \left(\frac{2^k}{\lambda} \right)^{\frac{q+\varepsilon_3}{\rho'} - (q-1)} \left(\int_{t_{k-1}}^t \|G_{k-1}(e^{-\omega s} u(s))\|_{q+\varepsilon_3}^{(q+\varepsilon_3)\frac{\psi'}{\rho'}} ds \right)^{\frac{1}{\psi'}}.
\end{aligned} \tag{2.7.12}$$

Now it remains to recover U_{k-1} from integrals of the form

$$\int_{t_{k-1}}^t \|G_{k-1}(e^{-\omega s} u(s))\|_{q+\varepsilon}^{(q+\varepsilon)M} ds \tag{2.7.13}$$

where $\varepsilon > 0$ and $M > 0$. In particular, we set $q_\varepsilon := q + \varepsilon$ and choose ε as follows. To obtain U_{k-1} from (2.7.13) we will apply Hölder's inequality, so choose $\varepsilon > 0$ and $\theta \in [0, 1]$ such that

$$\frac{1}{q_\varepsilon} = \frac{\theta}{q} + \frac{1-\theta}{r} \quad \text{and} \quad (1-\theta)q_\varepsilon M = \sigma.$$

In particular, we choose

$$\varepsilon = \begin{cases} \frac{\sigma}{M} \left(1 - \frac{q}{r}\right) & \text{if } r < \infty, \\ \frac{\sigma}{M} & \text{if } r = \infty, \end{cases} \tag{2.7.14}$$

and

$$\theta = \begin{cases} 1 - \frac{1}{1+q(\frac{1}{\sigma} - \frac{1}{r})} & \text{if } r < \infty, \\ \frac{Mq}{\sigma + Mq} & \text{if } r = \infty, \end{cases} \quad (2.7.15)$$

satisfying $\theta < 1$ and $\varepsilon > 0$ given that $M > 0$. The condition $\theta \geq 0$ requires that $M \geq \frac{\sigma}{r}$. Since we take $M = 1$, $\frac{\psi'}{\rho'_q}$ and $\frac{\psi'}{\rho'}$ in the case of (2.7.12), this is satisfied by assumptions (2.7.2) and (2.7.3). Then applying standard L^p interpolation with θ ,

$$\begin{aligned} & \int_{t_{k-1}}^t \|G_{k-1}(e^{-\omega s}u(s))\|_{q_\varepsilon}^{q_\varepsilon M} ds \\ & \leq \int_{t_{k-1}}^t e^{-\omega(\sigma-q)s} (\|G_{k-1}(e^{-\omega s}u(s))\|_q^q)^{\frac{\theta q_\varepsilon M}{q}} \times \\ & \quad e^{\omega(\sigma-q)s} \|G_{k-1}(e^{-\omega s}u(s))\|_r^{(1-\theta)q_\varepsilon M} ds \\ & \leq \frac{1}{L} \sup_{\hat{s} \in [t_{k-1}, t]} e^{-\omega(\sigma-q)\hat{s}} (\|G_{k-1}(e^{-\omega \hat{s}}u(\hat{s}))\|_q^q)^{\frac{\theta q_\varepsilon M}{q}} \times \\ & \quad L \int_{t_{k-1}}^t e^{\omega(\sigma-q)s} \|G_{k-1}(e^{-\omega s}u(s))\|_r^\sigma ds. \end{aligned}$$

We estimate $e^{-\omega(\sigma-q)s}$ on $[t_{k-1}, t]$ according to the sign of $\sigma - q$ so that

$$\sup_{s \in [t_{k-1}, t]} e^{-\omega(\sigma-q)s} = \begin{cases} e^{-\omega(\sigma-q)t} & \text{if } \sigma \leq q, \\ 1 & \text{if } \sigma > q. \end{cases}$$

Hence, applying Young's inequality such that both terms have the same exponent and evaluating, we have

$$\int_{t_{k-1}}^t \|G_{k-1}(e^{-\omega s}u(s))\|_{q_\varepsilon}^{q_\varepsilon M} ds \leq \frac{1}{L} e^{\omega t(q-\sigma)^+} U_{k-1}^{M - \frac{\sigma}{r} + 1}$$

where $(q - \sigma)^+ = \max\{0, q - \sigma\}$.

To apply Lemma 2.7.2, the exponents of U_{k-1} corresponding to (2.7.12) must be of the form $1 + \delta$ with $\delta > 0$. Hence we require

$$\frac{q}{\rho'_q} + \frac{q}{\psi'} \left(1 - \frac{\sigma}{r}\right) > 1 \quad \text{and} \quad \frac{1}{\rho'} + \frac{1}{\psi'} \left(1 - \frac{\sigma}{r}\right) > 1$$

which follow from (2.7.2). Rewriting (2.7.12) as a recurrence relation for U_{k+1} , we introduce the following constants

$$\begin{aligned} c_1 &= \left(\frac{1}{\lambda}\right)^{\varepsilon_1} \left(\frac{1}{t} + \omega\right) e^{\omega t(q-\sigma)^+}, \quad c_2 = \left(\frac{1}{\lambda}\right)^{\frac{q(q+\varepsilon_2)}{\rho'_q}} \|g\|_{L^\psi(0,t;L_\mu^{\rho_q})}^q e^{\frac{q\omega t}{\psi'}(q-\sigma)^+}, \\ c_3 &= \left(\frac{1}{\lambda}\right)^{\frac{q+\varepsilon_3}{\rho'} - q + 1} \|g\|_{L^\psi(0,t;L_\mu^{\rho_q})}^{\frac{\omega t}{\psi'}(q-\sigma)^+}, \\ b &= \max \left\{ 2^{1+\varepsilon_1}, 2^{\frac{q(q+\varepsilon_2)}{\rho'_q}}, 2^{\frac{q+\varepsilon_3}{\rho'} - q + 1} \right\}, \end{aligned}$$

and exponents

$$\delta_1 = 1 - \frac{\sigma}{r}, \quad \delta_2 = \frac{q}{\psi'} \left(1 - \frac{\sigma}{r}\right) + \frac{q}{\rho'_q} - 1, \quad \delta_3 = \frac{1}{\psi'} \left(1 - \frac{\sigma}{r}\right) + \frac{1}{\rho'} - 1.$$

Then we obtain

$$U_{k+1} \leq b^{k+1} \sum_{i=1}^3 C_{C_i} U_k^{1+\delta_i}$$

for some $C > 0$ depending on q , L and ψ . Then setting

$$\delta_m := \min \{\delta_1, \delta_2, \delta_3\} = \delta_3,$$

in order to apply Lemma 2.7.2, we require that

$$U_0 \leq \frac{1}{3 b^{\frac{1}{\delta_m^2}}} \min_{i \in \{1,2,3\}} \frac{1}{(C_{C_i})^{\frac{1}{\delta_i}}}. \quad (2.7.16)$$

We estimate U_0 by (2.7.11) and (2.7.5), so that

$$\begin{aligned} U_0 &\leq 2 \left(\|u_0\|_q^q + q \int_0^t \|e^{-\omega s} u(s)\|_q^{q-1} e^{-\omega s} \|g(s)\|_q \, ds \right) \\ &\leq 2 \left(\|u_0\|_q^q + q \left(\|u_0\|_q + \int_0^t e^{-\omega r} \|g(r)\|_q \, dr \right)^{q-1} \int_0^t e^{-\omega s} \|g(s)\|_q \, ds \right) \\ &\leq 2(1+q) \left(\|u_0\|_q + \int_0^t e^{-\omega s} \|g(s)\|_q \, ds \right)^q. \end{aligned}$$

As the previous estimates were for arbitrary $\lambda \geq 0$, relabelling $C > 0$ to include b , we want to find λ such that

$$c_i \leq \frac{C}{\left(\|u_0\|_q + \|g\|_{L^1(0,t;L^q)} \right)^{q\delta_i}} \quad (2.7.17)$$

for $i \in \{1, 2, 3\}$. Set

$$\begin{aligned} \beta_1 &= \begin{cases} \frac{q-\sigma}{\varepsilon_1} & \text{if } \sigma \leq q, \\ 0 & \text{if } \sigma > q, \end{cases} \\ \kappa_1 &= \begin{cases} \frac{(q-\sigma)\rho'_q}{\psi' q(q+\varepsilon_2)} & \text{if } \sigma \leq q, \\ 0 & \text{if } \sigma > q, \end{cases} \\ \kappa_2 &= \begin{cases} \frac{q-\sigma}{\psi' \left(\frac{q+\varepsilon_3}{\rho'} - q + 1 \right)} & \text{if } \sigma \leq q, \\ 0 & \text{if } \sigma > q. \end{cases} \end{aligned}$$

Then (2.7.17) holds if

$$\begin{aligned}\lambda &\geq Ce^{\omega\beta_1 t} \left(\left(\frac{2^q}{t} + q\omega \right) \left(\|u_0\|_q + \int_0^t e^{-\omega s} \|g(s)\|_q ds \right)^{q\delta_1} \right)^{\frac{1}{\varepsilon_1}}, \\ \lambda &\geq Ce^{\omega\kappa_1 t} \left(\|g\|_{L^\psi(0,t;L_\mu^{\rho_q})}^q \left(\|u_0\|_q + \int_0^t e^{-\omega s} \|g(s)\|_q ds \right)^{q\delta_2} \right)^{\frac{\rho'_q}{q(q+\varepsilon_2)}}, \text{ and} \\ \lambda &\geq Ce^{\omega\kappa_2 t} \left(\|g\|_{L^\psi(0,t;L_\mu^\rho)} \left(\|u_0\|_q + \int_0^t e^{-\omega s} \|g(s)\|_q ds \right)^{q\delta_3} \right)^{\frac{1}{\frac{q+\varepsilon_3}{\rho'} - (q-1)}},\end{aligned}$$

for some $C > 0$ depending on q, σ, r, ρ, ψ and L . So taking λ as the maximum of these estimates, we have by Fatou's Lemma,

$$0 = \liminf_{k \rightarrow \infty} U_k \geq \sup_{\hat{s} \in [t, t]} \|G_\lambda(e^{-\omega\hat{s}}u(\hat{s}))\|_q^q + \int_t^t e^{\omega(\sigma-q)t} \|G_\lambda(e^{-\omega s}u(s))\|_r^\sigma ds.$$

Noting that t was chosen arbitrarily, this implies that

$$\|u(t)\|_\infty \leq \lambda \quad \text{for all } t \in (0, T].$$

Evaluating constants and simplifying, we obtain (2.7.6). $\square$

The following lemma shows that the growth condition on $G_\lambda(e^{-\omega t}u(t))$ given by (2.7.5) holds for operators with complete resolvent. In the case $\lambda = 0$ this reduces to the standard growth estimate for accretive operators with complete resolvent.

Lemma 2.7.3. *Let $1 \leq q_0 < \infty$ and suppose A is ω -quasi m -accretive in $L_\mu^{q_0}$ with complete resolvent for some $\omega \geq 0$. Let $T > 0$, $1 \leq q \leq \infty$ such that $g \in L^1(0, T; L_\mu^q \cap L_\mu^{q+\varepsilon})$ for some $\varepsilon > 0$ and $u_0 \in \overline{D(A)}^{L_\mu^{q_0}} \cap L_\mu^q$. Denote by $u(t)$ the mild solution to (2.7.1). Then we have the growth estimate*

$$\begin{aligned}\|G_\lambda(e^{-\omega t}u(t))\|_q &\leq \|G_\lambda(e^{-\omega s}u(s))\|_q \\ &\quad + \int_s^t e^{-\omega\tau} \|g(\tau)\mathbf{1}_{\{e^{-\omega\tau}|u(\tau)| > \lambda\}}\|_q d\tau\end{aligned}\tag{2.7.18}$$

for all $0 \leq s \leq t \leq T$ and $\lambda \geq 0$.

Proof. For $u \in D(J_h^A)$, we can rewrite the resolvent operator in the following way,

$$J_{\frac{h}{1-h\omega}}^{A+\omega I} u = (1 - h\omega) J_h^A u.$$

Then for $A + \omega I$ having complete resolvent, consider $\alpha \in \mathbb{R}$ and take $j(\cdot) = |G_\lambda(\alpha \cdot)|^q$ in the complete resolvent property (2.2.14) with the resolvent operator $J_{\frac{h}{1-h\omega}}^{A+\omega I}$ to obtain the estimate

$$\begin{aligned}\int_\Sigma |G_\lambda(\alpha v)|^q d\mu &\geq \int_\Sigma \left| G_\lambda \left(\alpha J_{\frac{h}{1-h\omega}}^{A+\omega I} v \right) \right|^q d\mu \\ &= \|G_\lambda(\alpha(1 - h\omega)J_h^A v)\|_q^q.\end{aligned}\tag{2.7.19}$$

Given $s < t$, take a partition $(s_n)_{n \in \{0,1,\dots,N\}}$ of $[s, t]$ given by $s_n = s + \frac{n(t-s)}{N}$. Let

$$g_n := \frac{N}{t-s} \int_{s_n}^{s_{n+1}} g(\tau) d\tau. \quad (2.7.20)$$

Then let $(v_n)_{n \in \{0,1,\dots,N\}}$ be the solution to the discrete problem

$$\begin{cases} v_n + \frac{t-s}{N} A v_n = v_{n-1} + \frac{t-s}{N} g_{n-1} & \text{for } n = 1, \dots, N, \\ v_0 = u(s). \end{cases}$$

We can apply the resolvent estimate (2.7.19) to v_n , taking $h = \frac{t-s}{N}$. Further, let $S_n = \{x \in \Sigma : \frac{e^{-\omega t}}{(1-h\omega)^{N-n}} |v_n + h g_n| > \lambda\}$ so that we may separate terms.

$$\begin{aligned} \left\| G_\lambda \left(\frac{e^{-\omega t}}{(1-h\omega)^{N-n}} v_n \right) \right\|_q &\leq \left\| G_\lambda \left(\frac{e^{-\omega t}}{(1-h\omega)^{N-n+1}} (v_{n-1} + h g_{n-1}) \right) \right\|_q \\ &\leq \left\| G_\lambda \left(\frac{e^{-\omega t}}{(1-h\omega)^{N-n+1}} v_{n-1} \right) \right\|_q + \frac{e^{-\omega t} h}{(1-h\omega)^{N-n+1}} \|g_{n-1} \mathbf{1}_{S_{n-1}}\|_q. \end{aligned}$$

Repeating this, we have

$$\begin{aligned} \|G_\lambda(e^{-\omega t} v_N)\|_q &\leq \left\| G_\lambda \left(\frac{e^{-\omega t}}{\left(1 - \frac{\omega(t-s)}{N}\right)^N} v_0 \right) \right\|_q \\ &\quad + \frac{t-s}{N} \sum_{n=0}^{N-1} \frac{e^{-\omega t}}{\left(1 - \frac{\omega(t-s)}{N}\right)^{N-n}} \|g_n \mathbf{1}_{S_n}\|_q \end{aligned}$$

which converges to (2.7.18) as $N \rightarrow \infty$ by the definition of mild solution and the projection (2.7.20). $\square$

The following proposition introduces the pointwise estimate (2.7.21) for operators with complete resolvent which we will use as the condition for applying Theorem 2.7.1 to the doubly nonlinear problem (1.2.1) (see Section 2.7.1). In particular, this provides (2.7.4).

Proposition 2.7.4. *For $1 \leq q_0 < \infty$ and $\omega \geq 0$, let A be an ω -quasi m -accretive operator on $L_\mu^{q_0}$ with complete resolvent. Suppose there are $q_0 \leq q < r \leq \infty$, $1 \leq \sigma < r$ and $C > 0$ such that A satisfies the one-parameter Sobolev type inequality*

$$\|G_\lambda(u)\|_r^\sigma \leq C [G_\lambda(u), v + \omega(G_\lambda(u) + \lambda \mathbf{1})]_q \quad (2.7.21)$$

for every $(u, v) \in A$ and $\lambda \geq 0$. Let $T > 0$, $g \in L^1(0, T; L_\mu^{q_0}) \cap L^1(0, T; L_\mu^{q+\varepsilon})$ for some $\varepsilon > 0$ and $u \in C([0, T]; L_\mu^{q_0}) \cap L^1(0, T; L_\mu^{1 \wedge \infty})$

be the mild solution to (2.7.1) where $u_0 \in \overline{D(A)}^{L_\mu^{q_0}} \cap L_\mu^{1 \cap \infty}$. Then for every $\lambda \geq 0$, u satisfies the “level set energy inequality”

$$\begin{aligned} & \|G_\lambda(e^{-\omega t_2} u(t_2))\|_q^q + \frac{q}{C} \int_{t_1}^{t_2} e^{\omega(\sigma-q)s} \|G_\lambda(e^{-\omega s} u(s))\|_r^\sigma ds \\ & \leq \|G_\lambda(e^{-\omega t_1} u(t_1))\|_q^q + \lambda \omega \int_{t_1}^{t_2} \|G_\lambda(e^{-\omega s} u(s))\|_{q-1}^{q-1} ds \\ & \quad + q \int_{t_1}^{t_2} e^{-\omega s} |[G_\lambda(e^{-\omega s} u(s)), g]_q| ds \end{aligned} \quad (2.7.22)$$

for all $0 \leq t_1 < t_2 \leq T$.

Proof. Let $\{s_n\}_{n \in \{0,1,\dots,N\}}$ be the discretization of the interval $[t_1, t_2]$ given by $s_n := t_1 + \frac{n(t_2-t_1)}{N}$. Then set

$$g_N(s) := \frac{N}{t_2 - t_1} \int_{s_n}^{s_{n+1}} g(\tau) d\tau \quad \text{for } s \in [s_n, s_{n+1})$$

for all $n \in \{0, \dots, N-1\}$, which will converge to g in $L^1(0, T; L_\mu^{q_0}) \cap L^1(0, T; L_\mu^q)$ as $N \rightarrow \infty$. To see this, first check for continuous g then argue by density. Let $\{v_n\}_{n \in \{0,\dots,N\}}$ be the associated family of solutions to the time discretized Cauchy problem satisfying

$$v_{n+1} = J_{\frac{t_2-t_1}{N}}^A \left(v_n + \frac{t_2-t_1}{N} g_N(s_n) \right) \quad (2.7.23)$$

for all $n \in \{0, \dots, N-1\}$ with $v_0 = u(t_1)$. Note that by the complete resolvent property of A with $u_0 \in L_\mu^q$ and $g \in L^1(0, T; L_\mu^q)$, $v_n \in L_\mu^q$ for all $n \in \{0, \dots, N\}$. We first obtain a discrete version of the integral estimate (2.7.22) by discretizing with a telescoping sum and applying a product rule. For $q \geq 1$ we use the following property of q -brackets,

$$[u, v]_q \leq \frac{1}{q} \|u + v\|_q^q - \frac{1}{q} \|u\|_q^q \quad (2.7.24)$$

for every $u, v \in L_\mu^q$. Here we apply (2.7.24) to the following telescoping sum, taking $u = G_\lambda(e^{-\omega s_n} v_n)$ and $v = G_\lambda(e^{-\omega s_{n-1}} v_{n-1}) - G_\lambda(e^{-\omega s_n} v_n)$.

$$\begin{aligned} & \|G_\lambda(e^{-\omega t_2} v_N)\|_q^q - \|G_\lambda(e^{-\omega t_1} u(t_1))\|_q^q \\ & = \sum_{n=1}^N \|G_\lambda(e^{-\omega s_n} v_n)\|_q^q - \|G_\lambda(e^{-\omega s_{n-1}} v_{n-1})\|_q^q \\ & \leq \sum_{n=1}^N q [G_\lambda(e^{-\omega s_n} v_n), G_\lambda(e^{-\omega s_n} v_n) - G_\lambda(e^{-\omega s_{n-1}} v_{n-1})]_q. \end{aligned}$$

Noting that G_λ is a Lipschitz continuous function, we can differentiate almost everywhere on $\mathbb{R}$. Here we define

$$G'_\lambda(s) = \begin{cases} 1 & \text{if } |s| > \lambda, \\ 0 & \text{if } |s| \leq \lambda, \end{cases}$$

and

$$c_n = \int_0^1 G'_\lambda(\theta e^{-\omega s_n} v_n + (1-\theta)e^{-\omega s_{n-1}} v_{n-1}) d\theta,$$

so that we can rewrite this difference as an integral of the derivative with

$$G_\lambda(e^{-\omega s_n} v_n) - G_\lambda(e^{-\omega s_{n-1}} v_{n-1}) = c_n (e^{-\omega s_n} v_n - e^{-\omega s_{n-1}} v_{n-1}).$$

Then returning to the estimate in the discrete setting,

$$\begin{aligned} & \|G_\lambda(e^{-\omega t_2} v_N)\|_q^q - \|G_\lambda(e^{-\omega t_1} u(t_1))\|_q^q \\ & \leq q \sum_{n=1}^N [G_\lambda(e^{-\omega s_n} v_n), c_n (e^{-\omega s_n} v_n - e^{-\omega s_{n-1}} v_{n-1})]_q \\ & \leq q \sum_{n=1}^N [G_\lambda(e^{-\omega s_n} v_n), c_n e^{-\omega s_{n-1}} (v_n - v_{n-1})]_q \\ & \quad + q \sum_{n=1}^N [G_\lambda(e^{-\omega s_n} v_n), c_n (e^{-\omega s_n} - e^{-\omega s_{n-1}}) v_n]_q. \end{aligned}$$

Defining

$$R_n = [G_\lambda(e^{-\omega s_n} v_n), (c_n - 1)(v_n - v_{n-1})]_q,$$

we can rewrite the previous estimate as

$$\begin{aligned} & \|G_\lambda(e^{-\omega t_2} v_N)\|_q^q - \|G_\lambda(e^{-\omega t_1} u(t_1))\|_q^q \\ & \leq \frac{q(t_2 - t_1)}{N} \sum_{n=1}^N e^{-\omega s_{n-1}} [G_\lambda(e^{-\omega s_n} v_n), -A v_n + g_N(s_{n-1})]_q \\ & \quad + q \sum_{n=1}^N e^{-\omega s_{n-1}} R_n + q \sum_{n=1}^N [G_\lambda(e^{-\omega s_n} v_n), e^{-\omega s_n} v_n c_n]_q \left(1 - e^{\frac{\omega(t_2 - t_1)}{N}}\right). \end{aligned}$$

We now consider the values of v_n and v_{n-1} for almost every $x \in \Sigma$ to show that R_n is non-positive. Note that $c_n \leq 1$ and in particular,

$$c_n(x) \in \begin{cases} \{0\} & \text{if } |\theta e^{-\omega s_n} v_n + (1-\theta)e^{-\omega s_{n-1}} v_{n-1}| \leq \lambda \text{ for a.e. } \theta \in (0, 1), \\ \{1\} & \text{if } |\theta e^{-\omega s_n} v_n + (1-\theta)e^{-\omega s_{n-1}} v_{n-1}| > \lambda \text{ for a.e. } \theta \in (0, 1), \\ (0, 1) & \text{otherwise.} \end{cases}$$

Since $|e^{-\omega s_n} v_n(x)| \leq \lambda$ implies that $G_\lambda(e^{-\omega s_n} v_n(x)) = 0$ and so does not contribute to R_n , we consider only $x \in \Sigma$ such that $|e^{-\omega s_n} v_n(x)| > \lambda$. Then there will be some subinterval of $\theta \in (0, 1)$ such that

$$|\theta e^{-\omega s_n} v_n(x) + (1-\theta)e^{-\omega s_{n-1}} v_{n-1}(x)| > \lambda$$

implying that $c_n(x) > 0$. If $c_n = 1$ then $c_n(x) - 1 = 0$ and so this will not contribute to R_n . Hence we only consider x such that $c_n(x) \in (0, 1)$. Since $|e^{-\omega s_n} v_n(x)| > \lambda$, this implies that either

$$|e^{-\omega s_{n-1}} v_{n-1}(x)| \leq \lambda$$

or

$$\text{sign}(v_{n-1}(x)) = -\text{sign}(v_n(x)).$$

For the first case, $G_\lambda(e^{-\omega s_{n-1}} v_{n-1}(x)) = 0$ so

$$\begin{aligned} & (G_\lambda(e^{-\omega s_n} v_n(x)))^{q-1} (v_n(x) - v_{n-1}(x)) \\ &= ((G_\lambda(e^{-\omega s_n} v_n(x)))^{q-1} - (G_\lambda(e^{-\omega s_{n-1}} v_{n-1}(x)))^{q-1}) (v_n(x) - v_{n-1}(x)) \\ &\geq 0. \end{aligned}$$

Note that for $q = 1$, $(G_\lambda(v_n(x)))^{q-1} = \text{sign}(G_\lambda(v_n(x)))$. For the second case, $\text{sign}(v_n(x) - v_{n-1}(x)) = \text{sign}(v_n(x))$ so

$$(G_\lambda(e^{-\omega s_n} v_n(x)))^{q-1} (v_n(x) - v_{n-1}(x)) \geq 0.$$

Putting this together we have that $R_n \leq 0$. Returning to the discrete estimate,

$$\begin{aligned} & \|G_\lambda(e^{-\omega t_2} v_N)\|_q^q - \|G_\lambda(e^{-\omega t_1} u(t_1))\|_q^q \\ &\leq \frac{q(t_2 - t_1)}{N} \sum_{n=1}^N e^{-\omega s_{n-1}} [G_\lambda(e^{-\omega s_n} v_n), -Av_n + g_N(s_n)]_q \\ &\quad + \sum_{n=1}^N \|G_\lambda(e^{-\omega s_n} v_n)\|_q^q \left(1 - e^{\frac{q\omega(t_2-t_1)}{N}}\right). \end{aligned}$$

Note that by (2.7.21),

$$\begin{aligned} & e^{-\omega s_{n-1}} [G_\lambda(e^{-\omega s_n} v_n), Av_n]_q = e^{-\omega s_{n-1}} e^{-(q-1)\omega s_n} [G_{\lambda e^{\omega s_n}}(v_n), Av_n]_q \\ &= e^{\frac{\omega(t_2-t_1)}{N}} e^{-q\omega s_n} [G_{\lambda e^{\omega s_n}}(v_n), Av_n]_q \\ &\geq e^{\frac{\omega(t_2-t_1)}{N}} e^{-q\omega s_n} \frac{1}{C} \|G_{\lambda e^{\omega s_n}}(v_n)\|_r^\sigma \\ &\quad - e^{\frac{\omega(t_2-t_1)}{N}} e^{-q\omega s_n} \omega (\|G_{\lambda e^{\omega s_n}}(v_n)\|_q^q + \lambda e^{\omega s_n} \|G_{\lambda e^{\omega s_n}}(v_n)\|_{q-1}^{q-1}) \\ &= e^{\frac{\omega(t_2-t_1)}{N}} \frac{e^{\omega s_n(\sigma-q)}}{C} \|G_\lambda(e^{-\omega s_n} v_n)\|_r^\sigma \\ &\quad - e^{\frac{\omega(t_2-t_1)}{N}} \omega (\|G_\lambda(e^{-\omega s_n} v_n)\|_q^q + \lambda \|G_\lambda(e^{-\omega s_n} v_n)\|_{q-1}^{q-1}). \end{aligned}$$

Hence we have

$$\begin{aligned} & e^{-\omega s_{n-1}} [G_\lambda(e^{-\omega s_n} v_n), Av_n]_q \\ &\geq \frac{e^{\omega s_n(\sigma-q)}}{C} \|G_\lambda(e^{-\omega s_n} v_n)\|_r^\sigma \\ &\quad - e^{\frac{\omega(t_2-t_1)}{N}} \omega (\|G_\lambda(e^{-\omega s_n} v_n)\|_q^q + \lambda \|G_\lambda(e^{-\omega s_n} v_n)\|_{q-1}^{q-1}). \end{aligned}$$

We now aim to take $N \rightarrow \infty$, first converting the discrete sums to integrals. Let U_N be a stepwise solution to (2.7.23) such that

$$U_N(s) = v_0 \mathbf{1}_{\{0\}}(s) + \sum_{n=1}^N v_n \mathbf{1}_{(s_{n-1}, s_n]}(s)$$

for every $s \in [t_1, t_2]$. We have

$$\begin{aligned}
& \|G_\lambda(e^{-\omega t_2} v_N)\|_q^q - \|G_\lambda(e^{-\omega t_1} u(t_1))\|_q^q \\
& \leq -\frac{q(t_2 - t_1)}{N} \sum_{n=1}^N \left(\frac{e^{\omega s_n(\sigma-q)}}{C} \|G_\lambda(e^{-\omega s_n} v_n)\|_r^\sigma \right. \\
& \quad + e^{\frac{\omega(t_2-t_1)}{N}} \omega [G_\lambda(e^{-\omega s_n} v_n), G_\lambda(e^{-\omega s_n} v_n) - \lambda \operatorname{sign}(G_\lambda(e^{-\omega s_n} v_n))]_q \\
& \quad \left. + e^{-\omega s_{n-1}} [G_\lambda(e^{-\omega s_n} v_n), g_N(s_n)]_q \right) \\
& \quad + \sum_{n=1}^N \|G_\lambda(e^{-\omega s_n} v_n)\|_q^q \left(1 - e^{\frac{q\omega(t_2-t_1)}{N}} \right) \\
& \leq -\frac{q}{C} \int_{t_1}^{t_2} e^{\omega(\sigma-q)(s-\operatorname{sign}(\sigma-q)\frac{t_2-t_1}{N})} \left\| G_\lambda(e^{-\omega(s+\frac{t_2-t_1}{N})} U_N) \right\|_r^\sigma ds \\
& \quad + \left| \frac{e^{\frac{q\omega(t_2-t_1)}{N}} - 1}{\frac{t_2-t_1}{N}} - q\omega e^{\frac{\omega(t_2-t_1)}{N}} \right| \int_{t_1}^{t_2} \|G_\lambda(e^{-\omega s} U_N)\|_q^q ds \\
& \quad + q\omega \lambda e^{\frac{\omega(t_2-t_1)}{N}} \int_{t_1}^{t_2} \|G_\lambda(e^{-\omega s} U_N)\|_{q-1}^{q-1} ds \\
& \quad + q \int_{t_1}^{t_2} \int_{\Sigma} |G_\lambda(e^{-\omega s} U_N)|^{q-1} |g_N| d\mu ds.
\end{aligned}$$

We now prove convergence of each term in the estimate in order to obtain the continuous version (2.7.22). Noting that $U_N(s) \rightarrow u(s)$ in $C([0, T]; L_\mu^{q_0})$ and since $\|\cdot\|_q^q$ and $\|\cdot\|_r^\sigma$ are lower semicontinuous on $L_\mu^{q_0}$, we have that

$$\liminf_{N \rightarrow \infty} \|G_\lambda(e^{-\omega t_2} v_N)\|_q^q \geq \|G_\lambda(e^{-\omega t_2} u(t_2))\|_q^q$$

and applying Fatou's lemma,

$$\begin{aligned}
& \liminf_{N \rightarrow \infty} \int_{t_1}^{t_2} e^{\omega(\sigma-q)s} \left\| G_\lambda(e^{-\omega(s+\frac{t_2-t_1}{N})} U_N(s)) \right\|_r^\sigma ds \\
& \geq \int_{t_1}^{t_2} e^{\omega(\sigma-q)s} \|G_\lambda(e^{-\omega s} u(s))\|_r^\sigma ds.
\end{aligned}$$

Next, note that by the complete resolvent property of A and Lemma 2.7.3 we can estimate U_N and u in L_μ^q uniformly on $[0, T]$. Hence let M bound both $\|U_N\|_q$ and $\|u\|_q$.

$$\begin{aligned}
& \left| \frac{1 - e^{-\frac{q\omega t}{N}}}{\frac{t}{N}} - q\omega \right| \int_{t_1}^{t_2} \|G_\lambda(e^{-\omega s} U_N(s))\|_q^q ds \leq \left| \frac{1 - e^{-\frac{q\omega t}{N}}}{\frac{t}{N}} - q\omega \right| \int_{t_1}^{t_2} M^q ds \\
& \rightarrow 0
\end{aligned}$$

as $N \rightarrow \infty$. For the next term we prove uniform convergence of $G_\lambda(e^{-\omega s} U_N)$ to $G_\lambda(e^{-\omega s} u)$ in $C([0, T]; L_\mu^{q-1})$ when $\lambda > 0$. For this fix $s \in [0, T]$ and let

$$f_N = G_\lambda(e^{-\omega s} U_N(s)) - G_\lambda(e^{-\omega s} u(s))$$

so that $f_N \rightarrow 0$ in L^{q_0} . We note that

$$\|f_N\|_q \leq 2M$$

and by Chebyshev's inequality

$$\begin{aligned} \mu(\{x \in \Sigma : |f_N| > 0\}) &\leq \mu(\{x \in \Sigma : e^{-\omega s}|U_N| \geq \lambda \text{ or } e^{-\omega s}|u| \geq \lambda\}) \\ &\leq \frac{1}{\lambda^q} (\|e^{-\omega s}U_N(s)\|_q + \|e^{-\omega s}u(s)\|_q) \\ &\leq \frac{2M}{\lambda^q}. \end{aligned}$$

Here we consider cases for q . For $q - 1 \geq q_0$, apply Hölder's inequality with θ chosen to satisfy

$$\frac{\theta}{q_0} + \frac{1 - \theta}{q} = \frac{1}{q - 1}$$

to obtain

$$\lim_{N \rightarrow \infty} \|f_N\|_{q-1} \leq \lim_{N \rightarrow \infty} (\|f_N\|_{q_0}^\theta \|f_N\|_q^{1-\theta}) = 0.$$

For $q - 1 < q_0$, we apply Jensen's inequality noting that $|\cdot|^{\frac{q_0}{q-1}}$ is convex. Let

$$\Sigma_N = \{x \in \Sigma : |f_N| > 0\}$$

then we can again estimate f_N with

$$\|f_N\|_{q-1} \leq \mu(\Sigma_N)^{\frac{q_0}{q-1}-1} \|f_N\|_{q_0}.$$

Note that for $q - 1 < 1$, we have

$$\begin{aligned} & \left| \|G_\lambda(e^{-\omega s}U_N)\|_{q-1}^{q-1} - \|G_\lambda(e^{-\omega s}u)\|_{q-1}^{q-1} \right| \\ & \leq \int_\Sigma \left| |G_\lambda(e^{-\omega s}U_N)|^{q-1} - |G_\lambda(e^{-\omega s}u)|^{q-1} \right| d\mu \\ & \leq \int_\Sigma |f_N|^{q-1} d\mu \end{aligned}$$

so that

$$\lim_{N \rightarrow \infty} \|G_\lambda(e^{-\omega s}U_N(s))\|_{q-1}^{q-1} = \|G_\lambda(e^{-\omega s}u(s))\|_{q-1}^{q-1}.$$

For the last term, note that $g_N \rightarrow g$ in $L^1(0, T; L_\mu^q)$ as $N \rightarrow \infty$. So by a corollary of Riesz-Fischer, there exists a subsequence N_k and a function $h \in L^1(0, T; L_\mu^q)$ such that $|g_{N_k}(x)| \leq h(x)$ for all k and a.e. $x \in \Sigma$. Similarly, interpolating between q_0 and $q + \varepsilon$,

$$\begin{aligned} & \|G_\lambda(e^{-\omega s}U_N(s)) - G_\lambda(e^{-\omega s}u(s))\|_q \\ & \leq \|G_\lambda(e^{-\omega s}U_N(s)) - G_\lambda(e^{-\omega s}u(s))\|_{q_0}^\theta \times \\ & \quad \|G_\lambda(e^{-\omega s}U_N(s)) - G_\lambda(e^{-\omega s}u(s))\|_{q+\varepsilon}^{1-\theta} \end{aligned}$$

for some $\theta \in (0, 1]$. Then with U_N and u bounded in $L_\mu^{q+\varepsilon}$,

$$G_\lambda(e^{-\omega s} U_N(s)) \rightarrow G_\lambda(e^{-\omega s} u(s)) \quad \text{in } L_\mu^{q_0} \text{ as } N \rightarrow \infty$$

uniformly for all $t \in [0, T]$. Hence taking another subsequence we have a dominant $H_\lambda \in L^\infty(0, T; L_\mu^q)$. So we can estimate the integrand pointwise

$$|G_\lambda(e^{-\omega s} U_{N_k}(s))^{q-1} g_N| \leq H_\lambda^{q-1} h \quad \text{a.e. on } \Sigma \times [0, T].$$

Moreover this dominant is in $L^1(0, T; L_\mu^1)$ with

$$\int_{t_1}^{t_2} \int_\Sigma H_\lambda^{q-1} h \, d\mu \, ds \leq \|H_\lambda\|_{L^\infty(0, T; L_\mu^q)} \|h\|_{L^1(0, T; L_\mu^q)}.$$

Hence we apply dominated convergence to obtain the continuous estimate (2.7.22). $\square$

We now show that the pointwise estimate (2.7.21) from Proposition 2.7.4 implies a similar estimate when adding a Lipschitz perturbation.

Lemma 2.7.5. *For $1 \leq q < \infty$, let A be an operator on L_μ^q and suppose there are $1 \leq r \leq \infty$, $\sigma > 0$, $\omega \in \mathbb{R}$, $\lambda \geq 0$ and $C > 0$ such that (2.7.21) is satisfied for all $(u, v) \in A$. Let $F : L_\mu^q \rightarrow L_\mu^q$ be Lipschitz continuous with Lipschitz constant $\omega' \geq 0$ and satisfying $F(0) = 0$. Then, the operator $A + F$ in L_μ^q satisfies*

$$\|G_\lambda(u)\|_r^\sigma \leq C [G_\lambda(u), v + (\omega + \omega')(G_\lambda(u) + \lambda \operatorname{sign}(u))]_q. \quad (2.7.25)$$

Proof. Let $\hat{v} = v + F(u)$. Then, since $[\cdot, \cdot]_q$ is linear in the second term,

$$\begin{aligned} [G_\lambda(u), \hat{v} + (\omega + \omega')(G_\lambda(u) + \lambda \operatorname{sign}(u))]_q \\ = [G_\lambda(u), v + \omega(G_\lambda(u) + \lambda \operatorname{sign}(u))]_q \\ + [G_\lambda(u), F(u) + \omega'(G_\lambda(u) + \lambda \operatorname{sign}(u))]_q. \end{aligned} \quad (2.7.26)$$

By the Lipschitz condition,

$$-\omega'|u| \leq F(u) \leq \omega'|u|.$$

Hence

$$\begin{aligned} [G_\lambda(u), F(u)]_q &\geq -\omega'[G_\lambda(u), u]_q \\ &= -\omega'[G_\lambda(u), G_\lambda(u) + \lambda \operatorname{sign}(u)]_q. \end{aligned}$$

So applying this and the initial assumption to (2.7.26), we have (2.7.25). $\square$

We now extend the $L_\mu^q - L_\mu^\infty$ regularity of Theorem 2.7.1 to obtain $L_\mu^\ell - L_\mu^\infty$ regularity as in Theorem 2.1.3. For this we consider (2.7.6) applied to $\tilde{u}_0 = u(\frac{t}{2})$ and $\tilde{g}(s) = g(s + \frac{t}{2})$. The following theorem is an extension of [50, Chapter 4] and in particular with $r = \infty$ gives the desired regularity result.

Theorem 2.7.6. For $1 \leq \ell < q < r \leq \infty$ and $T > 0$, let $g \in L^1(0, T; L_\mu^\ell \cap L_\mu^q)$ and $u : [0, T] \rightarrow L_\mu^\ell$ such that $u \in L^\infty(0, T; L_\mu^\ell \cap L_\mu^r)$ satisfying the exponential growth property (2.7.18) for all $0 < s \leq t \leq T$ and some $\omega \geq 0$. Suppose there exist increasing functions $c_1(t)$, $c_2(t)$ with $c_1(t) > 0$ and $c_2(t) \geq 0$ for all $t \in (0, T]$ and exponents $\alpha \geq 0$, $0 < \gamma^* \leq \gamma < \infty$ such that

$$\|u(t)\|_r \leq \max \left\{ c_1\left(\frac{t}{2}\right) \left(\frac{2}{t} + \omega\right)^\alpha \left(\|u\|_{L^1(\frac{t}{2}, t; L_\mu^q)} + \|g\|_{L^1(\frac{t}{2}, t; L_\mu^q)}\right)^\gamma, \right. \\ \left. c_2\left(\frac{t}{2}\right) \left(\|u\|_{L^1(\frac{t}{2}, t; L_\mu^q)} + \|g\|_{L^1(\frac{t}{2}, t; L_\mu^q)}\right)^{\gamma^*} \right\} \quad (2.7.27)$$

for every $t \in (0, T]$. Define

$$\theta := 1 - \gamma \left(\frac{\frac{1}{\ell} - \frac{1}{q}}{\frac{1}{\ell} - \frac{1}{r}} \right)$$

and suppose that $\theta > 0$. Then for all $\varepsilon \geq 0$ one has the $L_\mu^\ell - L_\mu^r$ estimate

$$\|u(t)\|_r \leq 2^\gamma \left(2^{\frac{\alpha}{\gamma\theta}} + \sup_{s \in (0, t]} N(s)^\theta \right)^{\frac{\gamma}{\theta}} \max \left\{ c_1(t)^{\frac{1}{\theta}} \left(\frac{1}{t} + \omega\right)^{\frac{\alpha}{\theta}}, c_2(t)^{\frac{1}{\theta}} \right\} \times \\ \left(e^{\omega t} \|u(0)\|_\ell + \int_0^t e^{\omega(t-\tau)} \|g(\tau)\|_\ell d\tau + \varepsilon \right)^{\frac{\theta_\ell \gamma}{\theta}} \quad (2.7.28)$$

for all $t \in (0, T]$ where

$$N(t) := \sup_{s \in (0, t]} \frac{M\left(\frac{s}{2}\right) \|g\|_{L^1(\frac{s}{2}, s; L_\mu^q)} + c_2\left(\frac{s}{2}\right)^{\frac{1}{\gamma}}}{M(s)^{\frac{1}{\theta}} \left(e^{\omega s} \|u(0)\|_\ell + \int_0^s e^{\omega(s-\tau)} \|g(\tau)\|_\ell d\tau + \varepsilon \right)^{\frac{\theta_\ell}{\theta}}}, \\ M(t) := \max \left\{ c_1(t)^{\frac{1}{\gamma}} \left(\frac{1}{t} + \omega\right)^{\frac{\alpha}{\gamma}}, c_2(t)^{\frac{1}{\gamma}} \right\},$$

and

$$\theta_\ell := \frac{\frac{1}{q} - \frac{1}{r}}{\frac{1}{\ell} - \frac{1}{r}}.$$

Note that (2.7.28) may be infinite when $\varepsilon = 0$ and $\|u(0)\|_\ell = 0$.

Proof. We first note that since $\gamma^* \leq \gamma$, we can estimate

$$\left(\|u\|_{L^1(\frac{t}{2}, t; L_\mu^q)} + \|g\|_{L^1(\frac{t}{2}, t; L_\mu^q)} \right)^{\gamma^*} \leq \max \left\{ 1, \left(\|u\|_{L^1(\frac{t}{2}, t; L_\mu^q)} + \|g\|_{L^1(\frac{t}{2}, t; L_\mu^q)} \right)^\gamma \right\}$$

for $t \in (0, T]$. Hence, taking (2.7.27) to the power $\frac{1}{\gamma}$, we have

$$\|u(t)\|_r^{\frac{1}{\gamma}} \leq \max \left\{ c_1\left(\frac{t}{2}\right)^{\frac{1}{\gamma}} \left(\frac{2}{t} + \omega\right)^{\frac{\alpha}{\gamma}} \left(\|u\|_{L^1(\frac{t}{2}, t; L_\mu^q)} + \|g\|_{L^1(\frac{t}{2}, t; L_\mu^q)} \right), \right. \\ \left. c_2\left(\frac{t}{2}\right)^{\frac{1}{\gamma}} \left(\|u\|_{L^1(\frac{t}{2}, t; L_\mu^q)} + \|g\|_{L^1(\frac{t}{2}, t; L_\mu^q)} \right), c_2\left(\frac{t}{2}\right)^{\frac{1}{\gamma}} \right\}.$$

We then apply standard interpolation on the L_μ^q norm with exponent θ_ℓ and the growth estimate (2.7.18) on $\|u(\frac{t}{2})\|_\ell$ to obtain

$$\|u(t)\|_q \leq \left(e^{\omega t} \|u(0)\|_\ell + \int_0^t e^{\omega(t-\tau)} \|g(\tau)\|_\ell d\tau \right)^{\theta_\ell} \|u(t)\|_r^{1-\theta_\ell}$$

for all $t \in (0, T]$. By choice of θ , θ_ℓ we have the relation

$$\gamma(1 - \theta_\ell) = 1 - \theta.$$

Then for all $t \in (0, T]$,

$$\begin{aligned} \|u(t)\|_r^{\frac{1}{\gamma}} &\leq M(\tfrac{t}{2}) \left(e^{\frac{\omega t}{2}} \|u(0)\|_\ell + \int_0^{\frac{t}{2}} e^{\omega(\frac{t}{2}-\tau)} \|g(\tau)\|_\ell d\tau \right)^{\theta_\ell} \|u(\tfrac{t}{2})\|_r^{\frac{1-\theta}{\gamma}} \\ &\quad + M(\tfrac{t}{2}) \|g\|_{L^1(\frac{t}{2}, t; L_\mu^q)} + c_2(\tfrac{t}{2})^{\frac{1}{\gamma}}. \end{aligned} \quad (2.7.29)$$

We aim to produce comparable terms on either side of this equation. Since c_1 and c_2 are increasing, $M(\frac{t}{2}) \leq 2^{\frac{\alpha}{\gamma}} M(t)$. Furthermore,

$$\begin{aligned} e^{\frac{\omega t}{2}} \|u(0)\|_\ell + \int_0^{\frac{t}{2}} e^{\omega(\frac{t}{2}-\tau)} \|g(\tau)\|_\ell d\tau \\ \leq e^{-\frac{\omega t}{2}} \left(e^{\omega t} \|u(0)\|_\ell + \int_0^t e^{\omega(t-\tau)} \|g(\tau)\|_\ell d\tau \right). \end{aligned}$$

Hence we define

$$K_u(t) := \frac{M(t)^{-\frac{1}{\theta}} \|u(t)\|_r^{\frac{1}{\gamma}}}{\left(e^{\omega t} \|u(0)\|_\ell + \int_0^t e^{\omega(t-\tau)} \|g(\tau)\|_\ell d\tau + \varepsilon \right)^{\frac{\theta_\ell}{\theta}}} \quad \text{for } t \in (0, T] \quad (2.7.30)$$

in order to estimate and rearrange (2.7.29) into a relation involving K_u and N . Now we fix $t \in (0, T]$ so that after rearranging we may take a supremum over $(0, t]$ to obtain

$$\sup_{s \in (0, t]} K_u(s) \leq 2^{\frac{\alpha}{\gamma\theta}} \sup_{s \in (0, \frac{t}{2}]} K_u(s)^{1-\theta} + N(t).$$

We now aim to split the forcing term $N(t)$ so as to incorporate this into each K_u term. For this we define $D(t) \geq (2^{\frac{\alpha}{\gamma\theta}})^{\frac{1}{\theta}}$ for $t \in (0, T]$ such that

$$D(t) - 2^{\frac{\alpha}{\gamma\theta}} D(t)^{1-\theta} = N(t). \quad (2.7.31)$$

Noting that $\theta \in (0, 1)$, this is possible as the function $f(x) = x - cx^\alpha$ for $c \geq 0$ and $\alpha \in (0, 1)$ is continuous, satisfies $f(c^{\frac{1}{1-\alpha}}) = 0$ and is strictly

increasing for $x > (\alpha c)^{\frac{1}{1-\alpha}}$ (in particular for $x \geq c^{\frac{1}{1-\alpha}}$). Further, we can estimate (2.7.31) by

$$\begin{aligned} N(t) &= (D(t)^\theta)^{\frac{1-\theta}{\theta}} \left(D(t)^\theta - 2^{\frac{\alpha}{\gamma\theta}} \right) \\ &\geq \left(D(t)^\theta - 2^{\frac{\alpha}{\gamma\theta}} \right)^{\frac{1}{\theta}} \end{aligned}$$

so that

$$D(t) \leq \left(2^{\frac{\alpha}{\gamma\theta}} + N(t)^\theta \right)^{\frac{1}{\theta}}. \quad (2.7.32)$$

Then,

$$\sup_{s \in (0, t]} K_u(s) - D(t) \leq 2^{\frac{\alpha}{\gamma\theta}} \left(\left(\sup_{s \in (0, \frac{t}{2}]} K_u(s) \right)^{1-\theta} - D(t)^{1-\theta} \right).$$

Noting that $K_u(s)$ is bounded for all $s \in (0, T]$, either

$$\sup_{s \in (0, t]} K_u(s) \leq D(t)$$

or we can extend to a supremum over $(0, t]$ and combine terms, obtaining

$$\begin{aligned} \sup_{s \in (0, t]} K_u(s) - D(t) &\leq 2^{\frac{\alpha}{\gamma\theta}} \left(\sup_{s \in (0, t]} K_u(s) - D(t) \right)^{1-\theta} \\ \left(\sup_{s \in (0, t]} K_u(s) - D(t) \right)^\theta &\leq 2^{\frac{\alpha}{\gamma\theta}}. \end{aligned}$$

In either case, we have the uniform bound,

$$K_u(s) \leq \left(2^{\frac{\alpha}{\gamma\theta}} \right)^{\frac{1}{\theta}} + D(t)$$

for all $s \in (0, t]$. Applying (2.7.32), we have

$$\begin{aligned} K_u(t) &\leq \left(2^{\frac{\alpha}{\gamma\theta}} \right)^{\frac{1}{\theta}} + \left(2^{\frac{\alpha}{\gamma\theta}} + N(t)^\theta \right)^{\frac{1}{\theta}} \\ &\leq 2 \left(2^{\frac{\alpha}{\gamma\theta}} + N(t)^\theta \right)^{\frac{1}{\theta}}. \end{aligned}$$

Rewriting this as an estimate on $\|u(t)\|_r$ we obtain (2.7.28). $\square$

2.7.1 Application to the doubly nonlinear fractional diffusive equation

By Theorem 2.4.1 and the proof of Theorem 2.1.1 we know that the operator $(-\Delta_p)^s \varphi + F$ is m-accretive in L^1 with complete resolvent where F is the Nemytskii operator of $f(\cdot, u)$ satisfying (2.0.3a)-(2.0.3b). Hence

to apply Theorem 2.7.1 we first prove the pointwise estimate (2.7.21) for the operator $(-\Delta_p)^s \varphi$ in L^1 , giving Proposition 2.7.4 and thereby outlining the proof of the $L^{m+1} - L^\infty$ estimate of Theorem 2.7.9. We then apply Theorem 2.7.6, proving the $L^\ell - L^\infty$ estimate of Theorem 2.1.3.

The following lemma allows us to estimate the key term for (2.7.4), namely the q -bracket $[G_\lambda(u), (-\Delta_p)^s(u^m)]_{m+1}$. In particular, the restriction $m \geq 1$ in this lemma results in the same restriction in Theorem 2.1.3. Recall that we use the notation $r^m = |r|^{m-1}r$ for powers and that $G_\lambda(r) = [|r| - \lambda]^+ \text{sign}(r)$ for $r \in \mathbb{R}$.

Lemma 2.7.7. *Let $1 < p < \infty$, $m \geq 1$ and $\lambda \geq 0$. Given $a, b \in \mathbb{R}$ define $a_\lambda = G_\lambda(a)$, $b_\lambda = G_\lambda(b)$. Then,*

$$(a^m - b^m)^{p-1}(a_\lambda^m - b_\lambda^m) \geq |a_\lambda^m - b_\lambda^m|^p.$$

Proof. If $a_\lambda - b_\lambda = 0$, both sides are 0. For $a_\lambda - b_\lambda \neq 0$,

$$\begin{aligned} \text{sign}(a_\lambda^m - b_\lambda^m) &= \text{sign}(a^m - b^m) \\ |a^m - b^m|^{p-2}(a^m - b^m)(a_\lambda^m - b_\lambda^m) &= |a^m - b^m|^{p-1}|a_\lambda^m - b_\lambda^m| \end{aligned}$$

so we only need to prove that $|a_\lambda^m - b_\lambda^m| \leq |a^m - b^m|$. We take cases, assuming without loss of generality that $|a| \geq |b|$. First suppose that $|a| \leq \lambda$ or $|b| \leq \lambda$ so that $a_\lambda = b_\lambda = 0$ and the inequality is clear. Next, if $|b| \leq \lambda$ and $|a| > \lambda$, then

$$|a_\lambda^m - b_\lambda^m| = (|a| - \lambda)^m \leq (|a| - |b|)^m \leq |a|^m - |b|^m \leq |a^m - b^m|.$$

Hence we consider cases for a, b corresponding to $|a| > \lambda$ and $|b| > \lambda$. Suppose that $a > \lambda$ and $b > \lambda$. Then noting that for $m \geq 1$, $|x|^{m-1}$ is non-decreasing on the set $[0, \infty)$,

$$\begin{aligned} a^m - b^m &= \int_b^a \frac{d}{dx} x^m dx \\ &\geq \int_b^a m|x - \lambda|^{m-1} dx \\ &= (a - \lambda)^m - (b - \lambda)^m. \end{aligned}$$

Similarly for $a < -\lambda$, $b < -\lambda$, noting that $|x|^{m-1}$ is non-increasing on $(-\infty, 0]$,

$$a^m - b^m \geq (a + \lambda)^m - (b + \lambda)^m.$$

Finally, suppose that $a > \lambda$ and $b < -\lambda$ (similarly for $a < -\lambda$ and $b > \lambda$). Then

$$\begin{aligned} |a_\lambda^m - b_\lambda^m| &= (a - \lambda)^m - (b + \lambda)^m \\ &\leq a^m - b^m \\ &= |a^m - b^m|. \end{aligned}$$

□

We can now derive the pointwise estimate for Proposition 2.7.4, giving the Sobolev-type inequality required for Theorem 2.7.1 in the case of (1.2.1). Recall the notation of q -brackets from Section 2.2. In this case with $q = m + 1$ for $m \geq 1$, the q -bracket is given by

$$[u, v]_{m+1} = \int_{\Omega} |u|^{m-1} u v \, d\mu$$

for every $u, v \in L^{m+1}$.

Lemma 2.7.8. *Let Ω be an open domain in $\mathbb{R}^d$, $d \geq 1$. For $1 < p < \infty$, $0 < s < 1$, define p_s according to the Sobolev embedding (2.1.7). Suppose that $sp < d$ or Ω is bounded. Let $\varphi(r) = r^m$ for $r \in \mathbb{R}$ where $m \geq 1$. Then $(-\Delta_p)^s \varphi$ satisfies the one-parameter Sobolev type inequality*

$$\|G_\lambda(u)\|_{mp_s}^{mp} \leq C_d [G_\lambda(u), (-\Delta_p)^s \varphi(u)]_{m+1} \quad (2.7.33)$$

for all $\varphi(u) \in W_0^{s,(2,p)}$ and $\lambda \geq 0$. In particular, this is (2.7.21) with $q = m + 1$, $\omega = 0$, $\sigma = mp$ and $r = mp_s$. If $sp < d$, C_d depends only on d , otherwise it depends on s, p, d and $|\Omega|$.

Proof. Let $u^m \in W_0^{s,(2,p)}$. By Lemma 2.7.7 one sees that

$$\begin{aligned} & [G_\lambda u, (-\Delta_p)^s(u^m)]_{m+1} \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|((u(t, x))^m - (u(t, y))^m)^{p-2} ((u(t, x))^m - (u(t, y))^m)}{|x - y|^{d+sp}} \times \\ & \quad ((G_\lambda u(t, x))^m - (G_\lambda u(t, y))^m) \, dx \, dy \\ &\geq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|(G_\lambda u(t, x))^m - (G_\lambda u(t, y))^m|^p}{|x - y|^{d+sp}} \, dx \, dy \\ &= [(G_\lambda u(t, x))^m]_{s,p}^p. \end{aligned}$$

By the classical Sobolev inequality for Gagliardo semi-norms (cf. [89]), when $sp < d$

$$\|u\|_{p_s} \leq C_d [u]_{s,p}$$

where p_s is given by (2.1.7). Otherwise we use the embedding,

$$\|u\|_{p_s} \leq C_d (\|u\|_p + [u]_{s,p})$$

and the general estimate (see [91])

$$\|u\|_p \leq C [u]_{s,p}$$

for $u \in W_0^{s,p}$ where C depends on s, p, d and $|\Omega|$. Applying these to $G_\lambda(u)$ we obtain (2.7.33). $\square$

With these preliminaries we can now apply Proposition 2.7.4 and Theorem 2.7.1 to prove the $L^{m+1} - L^\infty$ regularisation effect for the doubly nonlinear nonlocal problem (1.2.1).

Theorem 2.7.9. *Let Ω be an open domain in $\mathbb{R}^d$, $d \geq 1$. Suppose $p > 1$, $0 < s < 1$, and $m \geq 1$ such that*

$$m(p-1) > 1 - (m+1)\frac{sp}{d}. \quad (2.7.34)$$

Suppose that $sp < d$ or Ω is bounded. Let $q_s = p_s$ if $p \neq \frac{d}{s}$ and $q_s > \max\{p, 1 + \frac{1}{m}\}$ if $p = \frac{d}{s}$. Let $T > 0$ and $g \in L^1(0, T; L^1) \cap L^{\psi_s}(0, T; L^\rho) \cap L^1(0, T; L^{m+1+\varepsilon})$ for some $\varepsilon > 0$ where $\rho \geq m+1$ and $\psi > 1$ satisfy

$$\begin{cases} \frac{1}{\rho} < \left(1 - \frac{1}{\psi}\right) \left(1 - \frac{p}{q_s}\right) & \text{if } m(p-1) \geq 1, \\ \frac{1}{\rho} \leq \left(1 - \frac{1}{\psi}\right) p \left(\frac{m}{m+1} - \frac{1}{q_s}\right) & \text{if } m(p-1) < 1. \end{cases} \quad (2.1.9)$$

Let $u(t)$ be the mild solution to (1.2.1) for $u_0 \in L^1 \cap L^{m+1}$. Then one has that

$$\begin{aligned} \|u(t)\|_\infty \leq C \max & \left\{ e^{\beta_1 \omega t} \left(\frac{1}{t} + \omega\right)^\alpha \left(\|u_0\|_{m+1} + \|g\|_{L^1(0,t;L^{m+1})}\right)^\gamma, \right. \\ & \left. e^{\beta_2 \omega t} \|g\|_{L^\psi(0,t;L^\rho)}^\eta \left(\|u_0\|_{m+1} + \|g\|_{L^1(0,t;L^{m+1})}\right)^{\gamma_\psi} \right\} \end{aligned} \quad (2.7.35)$$

for all $0 < t \leq T$, where we have the exponents

$$\begin{aligned} \gamma &= \frac{\frac{1}{p} - \frac{1}{q_s}}{\frac{m}{m+1} - \frac{1}{q_s}}, \quad \gamma_\psi = \frac{\left(1 - \frac{1}{\psi}\right) \left(\frac{1}{p} - \frac{1}{q_s}\right) - \frac{1}{\rho p}}{\left(1 - \frac{1}{\psi}\right) \left(\frac{m}{m+1} - \frac{1}{q_s}\right) - \frac{1}{\rho p} + \frac{1}{(m+1)p}}, \\ \alpha &= \frac{1}{mp(1 - \frac{m+1}{mq_s})}, \quad \eta = \frac{1}{mp \left(1 - \frac{1}{\psi}\right) \left(1 - \frac{m+1}{mq_s}\right) + 1 - \frac{m+1}{\rho}}, \end{aligned} \quad (2.7.36)$$

and

$$\begin{aligned} \beta_1 &= \begin{cases} \frac{\frac{1}{mp} - \frac{1}{m+1}}{\frac{1}{m+1} - \frac{1}{mq_s}} & \text{if } m(p-1) < 1, \\ 0 & \text{if } m(p-1) \geq 1, \end{cases} \\ \beta_2 &= \begin{cases} \eta(m+1 - mp) \left(1 - \frac{1}{\psi}\right) & \text{if } m(p-1) < 1, \\ 0 & \text{if } m(p-1) \geq 1. \end{cases} \end{aligned} \quad (2.7.37)$$

Proof. Let F be the Nemytskii operator of $f(\cdot, u)$ and $\mathcal{E}$ the energy functional (2.0.4). Then by Theorem 2.1.1 and the proof thereof, $(-\Delta_p)^s \varphi + F$ is ω -quasi m -accretive in L^1 and a mild solution to (1.2.1) exists in L^1 for all $u_0 \in L^1$. By Lemma 2.7.8 and Proposition 2.7.5 we have that $(-\Delta_p)^s \varphi + F$ satisfies the one-parameter Sobolev inequality (2.7.21) with $q = m+1$, $\sigma = mp$, $r = mq_s$, $C = C_d$ and ω the Lipschitz constant of f . Note that for q_s we have chosen $\tilde{p}$ in the Sobolev embedding (2.1.7). Then we can apply Proposition 2.7.4 with $q_0 = 1$ to obtain an estimate of the form (2.7.4) with L given by $\frac{m+1}{C_d}$. To satisfy the conditions on q , σ , r we first note that all are in $[1, \infty]$ with q and σ finite given that $p < \infty$.

Then $\sigma < r$ is equivalent to $p < q_s$. This is clear when $p \neq \frac{d}{s}$. However in the case $p = \frac{d}{s}$ we must choose $q_s > p$. For $q < r$, we require that $m+1 < mq_s$. When $p < \frac{d}{s}$, this implies that $1 + \frac{1}{m} < (\frac{1}{p} - \frac{s}{d})^{-1}$, equivalent to (2.1.8) with $\ell = m+1$ and so is satisfied given that $\ell < m+1$. When $p = \frac{d}{s}$, we choose $q_s > 1 + \frac{1}{m}$. In the case $p > \frac{d}{s}$ the inequality is clear. We now apply Theorem 2.7.1 to obtain (2.7.35). For this we also need $m+1 \leq \rho \leq \infty$ and $1 < \psi \leq \infty$ to satisfy (2.7.2), hence requiring (2.1.9). $\square$

We now extend this to the $L^\ell - L^\infty$ regularisation estimate of Theorem 2.1.3 for the doubly nonlinear nonlocal problem (1.2.1) by applying Theorem 2.7.6.

Proof of Theorem 2.1.3. Note that for $\ell < m+1$, (2.1.8) implies (2.7.34). We apply Theorem 2.7.9 to $\tilde{u}(s) = u(s + \frac{t}{2})$ and $\tilde{g}(s) = g(s + \frac{t}{2})$ to obtain (2.7.27) for all $t \in (0, T]$ with

$$c_1(t) = Ce^{\beta_1 \omega t}, \quad c_2(t) = Ce^{\beta_2 \omega t} \|g\|_{L^\psi(0,t;L^\rho)}^\eta, \quad \gamma = \frac{\frac{1}{p} - \frac{1}{q_s}}{\frac{m}{m+1} - \frac{1}{q_s}}.$$

We now aim to apply Theorem 2.7.6, taking q and r in this theorem to be $m+1$ and ∞ , respectively. By Corollary 2.4.4 we have that the operator $(-\Delta_p)^s \varphi + F$ is ω -quasi m -accretive with complete resolvent. Hence by Lemma 2.7.3, u satisfies the exponential growth property (2.7.18). We also require that $\gamma_\psi \leq \gamma$, equivalent to

$$\rho \geq \frac{1 - m(p-1)}{1 - \frac{p}{q_s}}.$$

This is satisfied by (2.1.8) when $p < \frac{d}{s}$ since $\rho \geq m+1 > \ell$, by the choice of q_s when $p = \frac{d}{s}$ and since $\rho \geq 1$ when $p > \frac{d}{s}$. For the condition $\theta > 0$ of Theorem 2.7.6, we require that

$$1 - \gamma \left(1 - \frac{\ell}{m+1}\right) > 0.$$

In particular, this holds by (2.1.8) in the case $p < \frac{d}{s}$, by the choice of q_s when $p = \frac{d}{s}$ and since $\ell \geq 1$ when $p > \frac{d}{s}$. Hence we may apply Theorem 2.7.6 to obtain (2.1.10). $\square$

2.8 Derivative estimates

We now consider the case $\varphi(r) = r^m$ with $m \geq 1$ satisfying (1.2.3), refining the estimates of Section 2.5 in the case of the fractional p -Laplacian. Importantly in this case we have the $L^\ell - L^\infty$ regularizing effect of Section 2.7 and the function $\varphi'(r)$ does not blow up as $r \rightarrow 0$. This allows us to remove the L^∞ condition on the initial data under the assumptions of Theorem 2.1.3.

We first introduce a Lipschitz continuity result for $t > 0$ analogous to Lemma 2.5.1, but with less restrictive initial data. This is due to the homogeneity of the composed operator for such power-like φ . Recall that $\tilde{V}(g, t)$ is defined by

$$\tilde{V}(g, t) = \limsup_{\xi \rightarrow 0^+} \int_0^{t/(1+\xi)} \frac{\|g(\tau(1+\xi)) - g(\tau)\|_1}{\xi} d\tau \quad (2.1.14)$$

for $0 \leq t \leq T \leq \infty$.

Lemma 2.8.1. *Let Ω be an open domain in $\mathbb{R}^d$, $d \geq 1$ and $T > 0$. Let $p > 1$, $0 < s < 1$ and $f(\cdot, u)$ satisfy (2.0.3a)-(2.0.3b). Suppose $\varphi(r) = r^m$, $r \in \mathbb{R}$ for $m > 0$ such that $m(p-1) \neq 1$. Suppose $g \in L^1(0, T; L^1)$ and $\tilde{V}(g, T) < \infty$. Then every mild solution u to (1.2.1) with $u_0 \in L^1$ is Lipschitz continuous on each compact subset of $(0, T]$, satisfying*

$$\begin{aligned} \limsup_{h \rightarrow 0^+} \frac{\|u(t+h) - u(t)\|_1}{h} &\leq \frac{Ce^{2\omega t}}{t} \left(\|u_0\|_1 + \int_0^t \|g(\tau)\|_1 d\tau \right) \\ &\quad + \frac{e^{\omega t}}{t} \tilde{V}(g, t) \end{aligned} \quad (2.1.15)$$

where $C = \frac{m(p-1)+2}{|m(p-1)-1|}$.

Proof. We note that $(-\Delta_p)^{s,m}$ is homogeneous of order $\alpha = m(p-1)$, so we apply [20, Theorem 4] with forcing term $\tilde{f}(t) = -F(u(t)) + g(t)$. Using the Lipschitz property of F and supposing that u is a mild solution to (1.2.1), we have for all $t \in (0, T)$,

$$\begin{aligned} \|u(t(1+\xi)) - u(t)\|_1 &\leq |1 - (1+\xi)^{\frac{1}{1-\alpha}}| \left(\int_0^t \omega \|u(\tau)\|_1 + \|g(\tau)\|_1 d\tau \right) \\ &\quad + \frac{|1+\xi - (1+\xi)^{\frac{1}{1-\alpha}}|}{1+\xi} \left(\int_0^{(1+\xi)t} \omega \|u(\tau)\|_1 + \|g(\tau)\|_1 d\tau \right) \\ &\quad + (1+\xi)^{\frac{1}{1-\alpha}} \omega \int_0^t \|u(\tau(1+\xi)) - u(\tau)\|_1 d\tau \\ &\quad + (1+\xi)^{\frac{1}{1-\alpha}} \int_0^t \|g(\tau(1+\xi)) - g(\tau)\|_1 d\tau \\ &\quad + 2|1 - (1+\xi)^{\frac{1}{1-\alpha}}| \|u_0\|_1. \end{aligned}$$

Applying Grönwall's inequality,

$$\begin{aligned} &\|u(t(1+\xi)) - u(t)\|_1 \\ &\leq \left(|1 - (1+\xi)^{\frac{1}{1-\alpha}}| \left(2\|u_0\|_1 + \int_0^t \omega \|u(\tau)\|_1 + \|g(\tau)\|_1 d\tau \right) \right. \\ &\quad + |1 - (1+\xi)^{\frac{\alpha}{1-\alpha}}| \left(\int_0^{(1+\xi)t} \omega \|u(\tau)\|_1 + \|g(\tau)\|_1 d\tau \right) \\ &\quad \left. + (1+\xi)^{\frac{1}{1-\alpha}} \int_0^t \|g(\tau(1+\xi)) - g(\tau)\|_1 d\tau \right) e^{(1+\xi)^{\frac{1}{1-\alpha}} \omega t}. \end{aligned}$$

Letting $\alpha = m(p-1)$ and dividing through by ξ , we can take the lim sup as $\xi \rightarrow 0^+$ to obtain the estimate

$$\begin{aligned} \limsup_{\xi \rightarrow 0^+} \frac{\|u(t(1+\xi)) - u(t)\|_1}{\xi} &\leq \frac{e^{\omega t}(1+\alpha)}{|1-\alpha|} \left(\int_0^t \omega \|u(\tau)\|_1 + \|g(\tau)\|_1 \, d\tau \right) \\ &\quad + e^{\omega t} \left(\frac{2\|u_0\|_1}{|1-\alpha|} + \tilde{V}(g, t) \right). \end{aligned}$$

So we apply the growth estimate on u in L^1 given by Theorem 2.1.1 and divide through again by t to obtain (2.1.15). $\square$

We have the following corollary to Lemma 2.5.3, applied when $\varphi(r) = r^m$, $r \in \mathbb{R}$ with $m > 0$ and $\mathcal{E}$ is given by (2.0.4).

Corollary 2.8.2. *Let $\Omega \subseteq \mathbb{R}^d$, $d \geq 1$, and $m > 0$. Suppose that $f(\cdot, u)$ satisfies (2.0.3a)-(2.0.3b) and $g \in BV(0, T; L^1) \cap L^1(0, T; L^\infty)$. Then every strong distributional solution u of (1.2.2) in L^1 with initial data $u_0 \in L^\infty$ satisfies*

$$\begin{aligned} &\frac{2m}{(m+1)^2} \int_0^t s^{k+2} \int_\Omega \left| \frac{d}{ds} u^{\frac{m+1}{2}} \right|^2 dx \, ds + \frac{t^{k+2}}{2p} [u^m(t)]_{s,p}^p \\ &\leq (k+2) \int_0^t (k+1+\omega s) s^k \|u(s)\|_{m+1}^{m+1} \, ds \\ &\quad + m \int_0^t ((k+2)^2 + \omega^2 s^2) s^k \|u(s)\|_{m+1}^{m+1} \, ds \\ &\quad + \lim_{h \rightarrow 0^+} \frac{1}{h} \int_0^{t-h} s^{k+2} \|g(s+h) - g(s)\|_1 \, ds \|u\|_{L^\infty(0,t;L^\infty)}^m \\ &\quad + 2(k+2) \int_0^t s^{k+1} \|g(s)\|_1 \, ds \|u\|_{L^\infty(0,t;L^\infty)}^m \end{aligned} \tag{2.8.1}$$

for all $t \in (0, T]$ and $k > -1$.

We now apply and extend Lemma 2.5.3 using the $L^1 - L^\infty$ regularity of Theorem 2.1.3.

Lemma 2.8.3. *Let Ω be an open domain in $\mathbb{R}^d$, $d \geq 1$, $T > 0$, $p > 1$ and $0 < s < 1$. Suppose that $sp < d$ or Ω is bounded. Let $m \geq 1$ satisfy*

$$m(p-1) > 1 - \frac{sp}{d}. \tag{1.2.3}$$

Further suppose that $f(\cdot, u)$ satisfies conditions (2.0.3a)-(2.0.3b) and $g \in BV(0, T; L^1) \cap L^1(0, T; L^{1 \cap \infty})$. Then every strong distributional solution

u of (1.2.2) in L^1 with $u_0 \in L^1$ satisfies, for all $\varepsilon \geq 0$,

$$\begin{aligned}
& \int_0^t \tau^{\tilde{\alpha}} \int_{\Omega} \left| \frac{d}{d\tau} u^{\frac{m+1}{2}}(\tau) \right|^2 d\mu d\tau + t^{\tilde{\alpha}} [u^m(t)]_{s,p}^p \\
& \leq Ct(1+\omega t)^2 \max \left\{ (1+\omega t)^{\alpha} e^{\omega\beta_1 t}, t^{\alpha} e^{\omega\beta_2 t} \|g\|_{L^{\psi}(0,t;L^{\rho})}^{\eta} \right\}^{\frac{m}{\theta}} \\
& \quad \times \left(e^{\omega t} \|u_0\|_1 + \int_0^t e^{\omega(t-\tau)} \|g(\tau)\|_1 d\tau + \varepsilon \right)^{\frac{\gamma m}{\theta(m+1)} + 1} (1 + N(t)^{\gamma m}) \\
& \quad + C \int_0^t \tau^{\tilde{\alpha}+m-1} \|g(\tau)\|_{m+1}^{m+1} d\tau
\end{aligned} \tag{2.1.16}$$

for all $t \in (0, T]$, where $\tilde{\alpha} = \frac{\alpha m}{\theta} + 2$, $N(t)$ is given by (2.1.11), exponents are given by (2.1.12) and the estimate is possibly infinite when $\varepsilon = 0$. Here C depends on m, p, s, d, q_s, ρ and ψ (and $|\Omega|$ when $sp \geq d$).

Proof. We estimate (2.5.4) from Lemma 2.5.2 further by also applying Young's inequality to the last term. This gives

$$\begin{aligned}
& \frac{m}{(m+1)^2} \int_0^t \tau^{k+2} \int_{\Omega} \left| \frac{d}{d\tau} u^{\frac{m+1}{2}} \right|^2 d\mu d\tau + \frac{t^{k+2}}{2p} [u^m(t)]_{s,p}^p \\
& \leq \int_0^t ((k+2)^2(m+1) + (k+2)\omega\tau + \omega^2 m\tau^2) \tau^k \|u\|_{m+1}^{m+1} d\tau \\
& \quad + (k+2) \int_0^t \tau^{k+1} \int_{\Omega} g u^m d\mu d\tau \\
& \quad + m \int_0^t \tau^{k+2} \int_{\Omega} |g|^2 u^{m-1} d\mu d\tau.
\end{aligned}$$

Furthermore, we have

$$\tau^{k+1} \int_{\Omega} |g| |u|^m d\mu \leq \frac{1}{m+1} \tau^{k+m+1} \|g\|_{m+1}^{m+1} + \frac{m}{m+1} \tau^k \|u\|_{m+1}^{m+1}$$

and

$$\tau^{k+2} \int_{\Omega} |g|^2 |u|^{m-1} d\mu \leq \frac{2}{m+1} \tau^{k+m+1} \|g\|_{m+1}^{m+1} + \frac{m-1}{m+1} \tau^k \|u\|_{m+1}^{m+1}.$$

So we can estimate

$$\begin{aligned}
& \frac{m}{(m+1)^2} \int_0^t \tau^{k+2} \int_{\Omega} \left| \frac{d}{d\tau} u^{\frac{m+1}{2}} \right|^2 d\mu d\tau + \frac{t^{k+2}}{2p} [u^m(t)]_{s,p}^p \\
& \leq 2 \int_0^t ((k+2)^2(m+2) + \omega^2(m+1)\tau^2) \tau^k \|u\|_{m+1}^{m+1} d\tau \\
& \quad + \left(2 + \frac{k}{m+1} \right) \int_0^t \tau^{k+m+1} \|g\|_{m+1}^{m+1} d\tau.
\end{aligned}$$

We estimate $\|u\|_{m+1}$ by $\|u\|_1$ by applying Theorem 2.1.3 to $\|u\|_\infty$ and the standard growth estimate to $\|u\|_1$. We have,

$$\begin{aligned} \|u(t)\|_{m+1}^{m+1} &\leq \|u(t)\|_1 \|u(t)\|_\infty^m \\ &\leq C \max \left\{ e^{\omega\beta_1 t} \left(\frac{1}{t} + \omega \right)^\alpha, e^{\omega\beta_2 t} \|g\|_{L^\psi(0,t;L^\rho)}^\eta \right\}^{\frac{m}{\theta}} (1 + N(t)^{\gamma m}) \times \\ &\quad \left(e^{\omega t} \|u_0\|_1 + \int_0^t e^{\omega(t-\tau)} \|g(\tau)\|_1 d\tau + \varepsilon \right)^{\frac{\gamma m}{(m+1)\theta} + 1} \end{aligned}$$

where variables are given by (2.1.11) and (2.1.12) with $\ell = 1$. Note that we require (1.2.3) to satisfy condition (2.1.8) of Theorem 2.1.3. Then for $k = \frac{\alpha m}{\theta}$ we define $\tilde{\alpha} = \frac{\alpha m}{\theta} + 2$, giving (2.1.16). $\square$

For Theorem 2.1.6, it remains to prove that under the given assumptions, for all $u_0 \in L^1$ the mild solution u to (1.2.1) is a strong distributional solution in L^1 . We prove this similarly to [22], noting that the $L^1 - L^\infty$ regularizing effect and the Lipschitz continuity in L^1 given by Lemma 2.8.1 allow us to obtain similar key estimates with less regularity of the initial data. However, we use [75, Lemme 2.1] rather than [22, Theorem 1.1] to apply this to general open domains.

Proof of Theorem 2.1.6. Fix $u_0 \in L^1$ and let u be the mild solution in L^1 to (1.2.1) with initial data u_0 . Consider a sequence $(u_{0,n})_{n \in \mathbb{N}} \subset \hat{D}((-\Delta_p)_{|L^{1 \cap \infty}^s}^s \varphi)$ such that $u_{0,n} \rightarrow u_0$ in L^1 as $n \rightarrow \infty$. Such a sequence exists because $D((-\Delta_p)_{|L^{1 \cap \infty}^s}^s \varphi) \subseteq \hat{D}((-\Delta_p)_{|L^{1 \cap \infty}^s}^s \varphi)$ and

$$\overline{D((-\Delta_p)_{|L^{1 \cap \infty}^s}^s \varphi)}^{L^1} = L^1$$

by Theorem 2.1.1. Applying Theorem 2.1.2, $(u_{0,n})_{n \in \mathbb{N}}$ generate strong distributional solutions $(u_n)_{n \in \mathbb{N}}$ in L^1 satisfying $u_n(0) = u_{0,n}$ for each $n \in \mathbb{N}$. Moreover,

$$u_n \rightarrow u \quad \text{in } C([0, T]; L^1)$$

as $n \rightarrow \infty$.

Consider $0 < t_1 < t_2 < T$. Then by Theorem 2.1.3 and the standard growth estimate in L^1 , $u_n(t) \in L^1 \cap L^\infty$ uniformly for all $t \in (t_1, t_2)$ and $n \in \mathbb{N}$. Let $v_n = \varphi(u_n)$ for $n \in \mathbb{N}$. Then since $(u_n)_{n \in \mathbb{N}}$ is uniformly bounded in $L^\infty(t_1, t_2; L^\infty)$, we have that $(v_n)_{n \geq 1}$ is bounded in $L^\infty(t_1, t_2; L^\infty)$ and so taking a subsequence, we can relabel such that

$$v_n \xrightarrow{*} v \quad \text{weakly-* in } L^\infty(t_1, t_2; L^\infty)$$

for some $v \in L^\infty(t_1, t_2; L^\infty)$. Since $\varphi(r) = r^m$ with $m \geq 1$ is increasing, one has that $v = \varphi(u)$.

Define $\beta = \varphi^{-1}$,

$$\beta_{1/2}(r) = \int_0^r ((\beta)'(\tau))^{1/2} d\tau,$$

and $w_n = \beta_{1/2}(\varphi(u_n))$ for $n \in \mathbb{N}$ so that

$$w'_n(t) = \sqrt{\varphi'(u_n(t))} \frac{du_n}{dt}(t) \quad \text{for almost every } t \in (0, T).$$

By (2.1.16), we have that $(w_n)_{n \in \mathbb{N}}$ is uniformly bounded in $W^{1,2}(t_1, t_2; L^2)$ uniformly for all $n \in N$. So taking a further subsequence $(n_k)_{k \geq 1}$ with $n_k \rightarrow \infty$ for weak convergences and relabelling, we have

$$\begin{aligned} w_k &\rightarrow w \quad \text{in } C([t_1, t_2]; L^2) \quad \text{and} \\ w'_k &\rightharpoonup w' \quad \text{weakly in } L^2(t_1, t_2; L^2). \end{aligned}$$

We have

$$\begin{aligned} |\beta_{1/2}(v_k) - \beta_{1/2}(v)| &= \left| \int_v^{v_k} ((\beta)')^{\frac{1}{2}}(r) dr \right| \\ &\leq |v_k - v|^{\frac{1}{2}} |\beta(v_k) - \beta(v)|^{\frac{1}{2}}. \end{aligned}$$

Hence $w = \beta_{1/2}(\varphi(u))$. Define $\Psi : R(\beta_{1/2}) \rightarrow \mathbb{R}$ such that

$$\Psi \circ \beta_{1/2} = \beta.$$

Then $\Psi(0) = 0$ and by [22, Lemma 4.3], Ψ is locally absolutely continuous on $R(\beta_{1/2})$. In particular, $\Psi \in W^{1,1}(R(\beta_{1/2}))$. So $u = \Psi(w)$ where $w \in W^{1,2}(t_1, t_2; L^2)$ and $\Psi \in W^{1,1}(R(\beta_{1/2}))$. Moreover by Lemma 2.8.1, $u \in BV(t_1, t_2; L^1)$. Then by [75, Lemme], $u \in W^{1,1}(t_1, t_2; L^1)$ and for a.e. $t \in (t_1, t_2)$,

$$\frac{du}{dt} = \Psi'(w(t)) \frac{dw}{dt}.$$

Since t_1, t_2 were arbitrary, we have that $u \in W_{\text{loc}}^{1,1}((0, T); L^1)$ and hence is a strong solution to (1.2.1). Then by a standard localisation argument and the continuity of $\sqrt{\varphi'}$, we can apply lower semicontinuity of the L^2 norm and $\mathcal{E}$ to obtain (2.1.16) for $u_0 \in L^1$.

To prove the distributional property we apply the method in [22, Lemma 4.2] given that $u \in W^{1,1}(t_1, t_2; L^1)$. Then for $t \in (0, T)$, $\phi(u(t)) \in D((-\Delta_p)^s_{|L^1 \cap \infty})$. $\square$

2.9 Strong solutions on open domains

In this section we prove Theorem 2.1.8. That is, we establish the existence of strong distributional solutions of (1.2.2) in L^1 for an open domain $\Omega \subseteq \mathbb{R}^d$. To do this, we approximate the unique mild solution u given by Theorem 2.1.1 by strong distributional solutions u_n in $L^1(\Omega_n)$. For this we cover Ω by $(\Omega_n)_{n \geq 1}$, an increasing sequence of open and bounded subsets of Ω . By Theorem 2.1.2, we have strong distributional solutions $(u_n)_{n \geq 1}$ of the associated initial boundary-value problem

$$\begin{cases} \frac{\partial u_n}{\partial t} + (-\Delta_p)^s u_n^m + f(x, u) = g(x, t) & \text{in } \Omega_n \times (0, T), \\ u_n = 0 & \text{in } \mathbb{R}^d \setminus \Omega_n \times (0, T), \\ u_n(x, 0) = u_{0,n} & \text{on } \Omega_n, \end{cases} \quad (2.9.1)$$

where for each $n \geq 1$, $u_{0,n}$ is a function in $\hat{D}((-\Delta_p)_{|L^{1 \cap \infty}}^s \cdot^m)$ and $u_{0,n}$ converges to the u_0 of (1.2.2) in L^1 .

We note that the existence of mild solutions to (1.2.2) for initial data in L^1 and the density for initial data which we use for this approximation both rely on the density result (2.1.1) of Theorem 2.1.1 for which we have the sufficient condition $\varphi \in W_{\text{loc}}^{1,q}(\mathbb{R})$ for $q \in (\frac{1}{1-s}, \infty]$. This results in the restriction $m > s$ in the statement of Theorem 2.1.8.

Due to our subdifferential setting, $(-\Delta_p)^s$ depends on the domain Ω_n moreso than just extending by 0. In particular, this is due to the space for test functions, $W_0^{s,(2,p)}$ being larger for larger Ω_n (see Proposition 2.3.1). To provide the rigorous functional analytical setting of this approximation argument and the dependence on Ω_n explicit, we introduce notation for the doubly nonlinear operator on $L^1(\Omega_n)$ for $n \geq 1$.

We define the energy functional $\mathcal{E}_n : L^2(\Omega_n) \rightarrow (-\infty, +\infty]$ by

$$\mathcal{E}_n(u) = \begin{cases} \frac{1}{2p} [u]_{s,p}^p & \text{if } u \in W_0^{s,(2,p)}(\Omega_n), \\ +\infty & \text{otherwise.} \end{cases} \quad (2.9.2)$$

We then define the fractional p -Laplacian on Ω_n as

$$(-\Delta_p)_{\Omega_n}^s u := \partial_{L^2(\Omega_n)} \mathcal{E}_n(u)$$

for every $u \in D(\partial_{L^2(\Omega_n)} \mathcal{E}_n)$. We denote the restriction to $L^{1 \cap \infty}(\Omega_n)$ by

$$(-\Delta_p)_{\Omega_n, 1 \cap \infty}^s := \left\{ (u, v) \in L^{1 \cap \infty}(\Omega_n) \times L^{1 \cap \infty}(\Omega_n) \mid v = (-\Delta_p)_{\Omega_n}^s u \right\}.$$

Finally, we have the associated doubly nonlinear operator on $L^1(\Omega_n)$ given by $A_n := \overline{(-\Delta_p)_{\Omega_n, 1 \cap \infty}^s \cdot^m L^1(\Omega_n)}$, defined as in Definition 2.0.1.

For $f(x, u)$ satisfying (2.0.1) and (2.0.3a) and F the Nemytskii operator of f on L^1 as usual, we denote the associated Nemytskii operator on $L^1(\Omega_n)$ by F_n for $n \geq 1$. We can compare such F_n much more simply by extending functions by zero since here we only have dependence on the domain of the function itself (and not any test functions which could restrict the domain of F_n).

With this setting, we now introduce the following Cauchy problem in $L^1(\Omega_n)$, corresponding to (2.9.1),

$$\begin{cases} \frac{du_n}{dt}(t) + A_n u_n(t) + F_n(u_n(t)) = g(t) & \text{in } (0, T), \\ u_n(0) = u_{0,n}. \end{cases} \quad (2.9.3)$$

In order to compare solutions to (2.9.1) with solutions to (1.2.2), we want to view both as problems in L^1 . Hence we introduce the following extension for A_n to an operator on L^1 . That is, on $L^1(\Omega)$ rather than $L^1(\Omega_n)$. Let

$$\tilde{A}_n u(x) = \begin{cases} [A_n(u \mathbb{1}_{\Omega_n})](x) & \text{for } x \in \Omega_n, \\ 0 & \text{otherwise,} \end{cases} \quad (2.9.4)$$

for $n \geq 1$. This simply restricts functions on Ω to Ω_n and extends functions in the range of the operator by zero. So solutions u_n to the Cauchy problem (2.9.3) can be extended by zero to be defined on $\Omega \times (0, T)$ and thereby satisfy

$$\begin{cases} \frac{du_n}{dt}(t) + \tilde{A}_n u_n(t) + F(u_n(t)) = g(t) & \text{in } (0, T), \\ u_n(0) = u_{0,n}. \end{cases} \quad (2.9.5)$$

To prove the convergence of $(u_n)_{n \geq 1}$ as Ω_n grows to cover Ω , we require the following two results. The first lemma gives us a comparison principle for the fractional p -Laplacian on subdomains which we will then apply in order to approximate solutions to (1.2.2) by solutions to (2.9.1).

Lemma 2.9.1. *Let $1 < p < \infty$, $0 < s < 1$, $m > s$, $h \in L^1(\mathbb{R}^d)$ and Ω_1, Ω_2 be open subsets of $\mathbb{R}^d$ satisfying $\Omega_1 \subseteq \Omega_2$. Define*

$$\begin{aligned} w_1 &:= (I + \lambda(\tilde{A}_1 + F))^{-1}(h \mathbf{1}_{\Omega_1}) \quad \text{and} \\ w_2 &:= (I + \lambda(\tilde{A}_2 + F))^{-1}(h \mathbf{1}_{\Omega_2}) \end{aligned}$$

for $0 < \lambda < 1/\omega$. Suppose $h(x) \geq 0$ a.e. on $\mathbb{R}^d$. Then $w_1 \leq w_2$ a.e. on $\mathbb{R}^d$.

Proof. By a standard density argument it is sufficient to consider $h \in L^1 \cap L^\infty$ with $h \geq 0$ so that $w_1, w_2 \in L^1 \cap L^\infty$. In particular, then $w_1 \in D((-\Delta_p)_{\Omega_1, 1 \cap \infty}^s)$ and $w_2 \in D((-\Delta_p)_{\Omega_2, 1 \cap \infty}^s)$ when viewed as functions on Ω_1 and Ω_2 respectively, since they are both zero outside their respective domains. Moreover, by (2.2.13), since $h \geq 0$ we have that $w_1 \geq 0$ and $w_2 \geq 0$. We aim to estimate $(w_1 - w_2)^+$, hence we introduce a sequence of $q \in C^1(\mathbb{R})$ to approximate $\text{sign}_0^+(w_1 - w_2) := \mathbf{1}_{\{w_1 > w_2\}}$. In particular, we require that $0 \leq q \leq 1$, $q(r) = 0$ for $r \leq 0$ and that there exists $M > 0$ such that $0 < q'(r) < M$ for all $r > 0$. For convenience, we define $Q := q(w_1 - w_2)$. Note that $Q = 0$ in $\mathbb{R}^d \setminus \Omega_1$ and

$$\begin{aligned} [Q]_{s,p} &= \frac{1}{p} \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|q(w_1(x) - w_2(x)) - q(w_1(y) - w_2(y))|^p}{|x - y|^{d+sp}} dy dx \right)^{\frac{1}{p}} \\ &\leq \frac{M}{p} \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|w_1(x) - w_2(x) - (w_1(y) - w_2(y))|^p}{|x - y|^{d+sp}} dy dx \right)^{\frac{1}{p}} \\ &\leq M([w_1]_{s,p} + [w_2]_{s,p}) \end{aligned}$$

so that $Q \in W_0^{s,(2,p)}(\Omega_1)$ and hence also in $W_0^{s,(2,p)}(\Omega_2)$. We have that

$$\begin{aligned} \int_{\mathbb{R}^d} (w_1 - w_2) q(w_1 - w_2) dx &= -\lambda \int_{\mathbb{R}^d} (\tilde{A}_1 w_1 - \tilde{A}_2 w_2) Q dx \\ &\quad - \lambda \int_{\mathbb{R}^d} (F(w_1) - F(w_2)) Q dx. \end{aligned}$$

Estimating the first term,

$$\begin{aligned} & -\lambda \int_{\Omega_1} (-\Delta_p)^s_{\Omega_1, 1 \cap \infty}(w_1^m) Q \, dx + \lambda \int_{\Omega_2} (-\Delta_p)^s_{\Omega_2, 1 \cap \infty}(w_2^m) Q \, dx \\ & = -\lambda \int_{\mathbb{R}^{2d}} \frac{I(x, y)(Q(x) - Q(y))}{|x - y|^{d+sp}} d(x, y) \end{aligned}$$

where

$$I(x, y) := (w_1^m(x) - w_1^m(y))^{p-1} - (w_2^m(x) - w_2^m(y))^{p-1}$$

and

$$\text{sign}_0(I(x, y)) = \text{sign}_0(Q(x) - Q(y))$$

for all $x, y \in \mathbb{R}^d$. In particular,

$$-\lambda \int_{\mathbb{R}^d} (\tilde{A}_1 w_1 - \tilde{A}_2 w_2) Q \, dx \leq 0.$$

For the second term we apply the Lipschitz continuity of F ,

$$-\lambda \int_{\mathbb{R}^d} (F(w_1) - F(w_2)) Q \, dx \leq \lambda \omega \int_{\mathbb{R}^d} (w_1 - w_2)(q(w_1) - q(w_2)) \, dx.$$

By a standard approximation for sign_0^+ , we then have that

$$(1 - \lambda \omega) \int_{\mathbb{R}^d} (w_1 - w_2)^+ \, dx \leq 0$$

so that $w_1 \leq w_2$ a.e. in $\mathbb{R}^d$ given that $\lambda < 1/\omega$. $\square$

We can now prove the following approximation result for (1.2.2).

Proposition 2.9.2. *Let Ω be an open subset of $\mathbb{R}^d$, $d \geq 1$, $p > 1$, $0 < s < 1$, $m > s$ and suppose $(\Omega_n)_{n \geq 1}$ is the sequence of open subsets of Ω defined by $\Omega_n = \Omega \cap B_n$ where B_n is the ball centered at the origin with radius n . Then, for every $u_0 \in L^1$, there is a sequence of functions $(u_{0,n})_{n \geq 1}$ with $u_{0,n} \in L^1(\Omega_n)$ and $u_{0,n}^m \in D((-\Delta_p)^s_{\Omega_n, 1 \cap \infty})$ for each $n \geq 1$ such that*

$$\lim_{n \rightarrow \infty} u_{0,n} \mathbf{1}_{\Omega_n} = u_0 \quad \text{in } L^1.$$

Moreover, for each $n \geq 1$, there is a unique strong distributional solution u_n of the Cauchy problem (2.9.3) such that extending to Ω by zero,

$$\lim_{n \rightarrow \infty} u_n \mathbf{1}_{\Omega_n} = u \quad \text{in } C([0, T]; L^1) \quad (2.9.6)$$

for every $T > 0$, where u is the unique mild solution of problem (1.2.2).

Proof. By Theorem 2.1.1, for each $n \geq 1$, we can approximate $u_0 \mathbf{1}_{\Omega_n} \in L^1(\Omega_n)$ by a sequence $(u_{0,k})_{k \in \mathbb{N}}$ with $u_{0,k}^m \in D((-\Delta_p)^s_{\Omega_n, 1 \cap \infty})$. Taking a diagonal subsequence and relabelling, we obtain the sequence $(u_{0,n})_{n \geq 1}$ of the proposition. Moreover, by Theorem 2.1.2, for each $n \geq 1$ there

is a unique strong distributional solution u_n of the initial value problem (2.9.3) with initial data $u_{0,n}$. Then, when extended by zero to be defined on Ω , u_n is also a strong distributional solution of (2.9.1) and therefore a strong solution of (2.9.5).

Next, by Corollary 2.4.4 (see also [50, Section 2]), for every $\lambda > 0$, the resolvent operators $J_\lambda^{\tilde{A}_n+F} := (I + \lambda(\tilde{A}_n + F))^{-1}$ and $J_\lambda^{A+F} := (I + \lambda(A + F))^{-1}$ are contractions on L^1 . Furthermore, by the m -accretivity of $A_n + F_n$ on $L^1(\Omega_n)$, $\tilde{A}_n + F$ is also m -accretive on L^1 . Then since $u_{n,0} \rightarrow u_0$ in L^1 , applying [12, Proposition 4.4 and Theorem 4.14], (2.9.6) is equivalent to showing that for every $h \in L^{1 \cap \infty}$ and every $0 < \lambda < 1/\omega$, for

$$w := J_\lambda^{A+F} h \quad \text{and} \quad w_n := J_\lambda^{\tilde{A}_n+F} h,$$

one has that

$$\lim_{n \rightarrow \infty} w_n = w \quad \text{in } L^1. \quad (2.9.7)$$

Hence we now aim to prove (2.9.7). By definition of $\tilde{A}_n$, we can write

$$w_n(x) = \begin{cases} [J_\lambda^{A_n+F_n}(h \mathbf{1}_{\Omega_n})](x) & \text{if } x \in \Omega_n, \\ 0 & \text{otherwise.} \end{cases}$$

By [50, Proposition 2.19],

$$w^m \in D((-\Delta_p)_{1 \cap \infty}^s) \quad \text{and} \quad w_n^m \in D((-\Delta_p)_{\Omega_n, 1 \cap \infty}^s)$$

for every $n \geq 1$. Moreover,

$$\|w\|_{1 \cap \infty} \leq \frac{1}{1-\lambda\omega} \|h\|_{1 \cap \infty}$$

and

$$\|w_n\|_{1 \cap \infty} \leq \frac{1}{1-\lambda\omega} \|h\|_{1 \cap \infty} \quad \text{for every } n \geq 1. \quad (2.9.8)$$

Thus, multiplying

$$w_n \mathbf{1}_{\Omega_n} + \lambda \left((-\Delta_p)_{\Omega_n, 1 \cap \infty}^s (w_n \mathbf{1}_{\Omega_n})^m + F_n(w_n \mathbf{1}_{\Omega_n}) \right) = h \mathbf{1}_{\Omega_n} \quad (2.9.9)$$

by w_n^m and subsequently applying (2.9.8), one sees that

$$(1 - \lambda\omega) \|w_n\|_{m+1}^{m+1} + \lambda [w_n^m]_{s,p}^p \leq \frac{1}{(1-\lambda\omega)^m} \|h\|_1 \|h\|_\infty^m.$$

From this and by (2.9.8), we can conclude that $(w_n^m)_{n \geq 1}$ is bounded in $W_0^{s,(2,p)}$. Thus, there is a $v \in W_0^{s,(2,p)}$ such that, after possibly passing to a subsequence, and by setting $\tilde{w}^m := v$,

$$\lim_{n \rightarrow \infty} w_n^m = \tilde{w}^m \quad \text{weakly in } W_0^{s,(2,p)}. \quad (2.9.10)$$

Moreover, the fractional Rellich-Kondrachov theorem (see [117, Theorem 2.1]) yields that after possibly passing to a subsequence, $w_n^m \rightarrow \tilde{w}^m$ in $L^q(K)$ for every compact subset K of $\mathbb{R}^d$, where $q \in [1, \infty]$ depends on

s , p and d . In particular, again possibly passing to a subsequence, one has that $w_n(x) \rightarrow \tilde{w}(x)$ for a.e. $x \in \mathbb{R}^d$.

By Lemma 2.9.1, supposing first that $h \geq 0$, we have that $w_n \leq w_{n+1}$ a.e. on $\mathbb{R}^d$. Thus and by (2.9.8), Beppo-Levi's theorem of monotone convergence yields that

$$\lim_{n \rightarrow \infty} w_n = \tilde{w} \quad \text{in } L^1 \quad (2.9.11)$$

holds. Similarly, if $h \leq 0$ then for every integer $n \geq 1$, $0 \leq w_1 - w_n \leq w_1 - w_{n+1}$ a.e. on $\mathbb{R}^d$ and by (2.9.8), it follows again from Beppo-Levi's theorem of monotone convergence that (2.9.11) holds. Now, let $h \in L^{1 \cap \infty}$ be general. Then we decompose $h = h^+ - h^-$ into the positive part $h^+ := h \vee 0$ and negative part $h^- := (-h) \vee 0$ of h , and set for every integer $n \geq 1$,

$$w_n^+(x) := \begin{cases} [J_\lambda^{A_n+F_n}[h^+ \mathbf{1}_{\Omega_n}]](x) & \text{if } x \in \Omega_n, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$w_n^-(x) := \begin{cases} [J_\lambda^{A_n+F_n}[-h^- \mathbf{1}_{\Omega_n}]](x) & \text{if } x \in \Omega_n, \\ 0 & \text{otherwise.} \end{cases}$$

Since $-h^- \leq h \leq h^+$, it follows from (2.2.13) that

$$w_n^- \leq w_n \leq w_n^+ \quad \text{a.e. on } \mathbb{R}^d \text{ for all } n \geq 1.$$

Moreover, by the previous monotone convergence arguments with $h^+ \geq 0$ and $-h^+ \leq 0$, $w_n^+ \uparrow \tilde{w}^+$ and $w_n^- \downarrow \tilde{w}^-$ in L^1 for some limits $\tilde{w}^-$, $\tilde{w}^+ \in L^{1 \cap \infty}$. Hence, $\tilde{w}^+ + |\tilde{w}^-| \in L^{1 \cap \infty}$ and one has that $|w_n| \leq \tilde{w}^+ + |\tilde{w}^-|$ a.e. in Ω for all $n \geq 1$. Therefore, Lebesgue's dominated convergence theorem implies that (2.9.11) holds for every $h \in L^{1 \cap \infty}$.

Since $v^{\frac{1}{m}} = \tilde{w}$, it remains to show that $v^{\frac{1}{m}} = w := J_\lambda^A h$. To see this, we first note that by (2.9.8) and (2.9.11), one has that for every $1 \leq q < \infty$,

$$\lim_{n \rightarrow \infty} w_n = \tilde{w} \quad \text{in } L^q \quad (2.9.12)$$

and so, multiplying (2.9.9) by w_n^m and subsequently, letting $n \rightarrow \infty$ gives that

$$\lim_{n \rightarrow \infty} [w_n^m]_{s,p}^p = \frac{\langle h - \lambda F(\tilde{w}), \tilde{w}^m \rangle_{W_0^{-s,(2,p)'}, W_0^{s,(2,p)}} - \|\tilde{w}\|_{m+1}^{m+1}}{\lambda} \quad (2.9.13)$$

where we also apply the Lipschitz continuity of F and (2.9.10). Further, since $(w_n^m)_{n \geq 1}$ is bounded in $W_0^{s,(2,p)}$, also the sequence $((-\Delta_p)^s(w_n^m))_{n \geq 1}$ in $W_0^{-s,(2,p)'}$ given by

$$\begin{aligned} & \langle (-\Delta_p)^s(w_n^m), \xi \rangle_{W_0^{-s,(2,p)'}, W_0^{s,(2,p)}} \\ &= \int_{\mathbb{R}^{2d}} \frac{|w_n^m(x) - w_n^m(y)|^{p-2} (w_n^m(x) - w_n^m(y)) (\xi(x) - \xi(y))}{|x - y|^{d+sp}} dy dx \end{aligned}$$

for every $\xi \in W_0^{s,(2,p)}$, is bounded. Therefore, there is a $\chi \in W_0^{-s,(2,p)'}$ such that after possibly passing to a subsequence, one has that

$$\lim_{n \rightarrow \infty} (-\Delta_p)^s(w_n^m) = \chi \quad \text{weakly}^* \text{ in } W_0^{-s,(2,p)'}. \quad (2.9.14)$$

By the two limits (2.9.12) and (2.9.14), multiplying the equation (2.9.9) by $\xi \in C_c^\infty$ and subsequently, sending $n \rightarrow \infty$ yields that

$$\langle \tilde{w} + \lambda (\chi + F(\tilde{w})) - h, \xi \rangle_{W_0^{-s,(2,p)'}, W_0^{s,(2,p)}} = 0$$

for all $\xi \in C_c^\infty$, from where a standard density argument yields that

$$\tilde{w} + \lambda (\chi + F(\tilde{w})) = h \quad \text{in } W_0^{-s,(2,p)'}. \quad (2.9.15)$$

Therefore, it remains to show that

$$\chi = (-\Delta_p)^s_{1 \cap \infty} \tilde{w}^m. \quad (2.9.16)$$

To prove this, we use the classical *monotonicity trick* (see, e.g., [92, p. 172]). We begin by multiplying (2.9.15) by w_n^m . Due to (2.9.10), (2.9.12) and (2.9.13), sending $n \rightarrow \infty$ in the resulting equation yields that

$$\begin{aligned} \langle \chi, \tilde{w}^m \rangle_{W_0^{-s,(2,p)'}, W_0^{s,(2,p)}} &= \frac{\langle h - \lambda F(\tilde{w}), \tilde{w}^m \rangle_{W_0^{-s,(2,p)'}, W_0^{s,(2,p)}} - \|\tilde{w}\|_{m+1}^{m+1}}{\lambda} \\ &= \lim_{n \rightarrow \infty} [w_n^m]_{s,p}^p. \end{aligned} \quad (2.9.17)$$

Next, let $\xi \in W_0^{s,(2,p)}$. Since the variational fractional p -Laplace operator is a monotone operator $(-\Delta_p)^s : W_0^{s,(2,p)} \rightarrow W_0^{-s,(2,p)'}$ in the sense that

$$\langle (-\Delta_p)^s(v) - (-\Delta_p)^s(\xi), v - \xi \rangle_{W_0^{-s,(2,p)'}, W_0^{s,(2,p)}} \geq 0$$

for every $v, \xi \in W_0^{s,(2,p)}$, one has that

$$\begin{aligned} 0 &\leq \langle (-\Delta_p)^s(w_n^m) - (-\Delta_p)^s(\xi), w_n^m - \xi \rangle_{W_0^{-s,(2,p)'}, W_0^{s,(2,p)}} \\ &= [w_n^m]_{s,p}^p - \langle (-\Delta_p)^s(w_n^m), \xi \rangle_{W_0^{-s,(2,p)'}, W_0^{s,(2,p)}} - \langle (-\Delta_p)^s(\xi), w_n^m - \xi \rangle_{W_0^{-s,(2,p)'}, W_0^{s,(2,p)}} \end{aligned}$$

for every $n \geq 1$. Hence, sending $n \rightarrow \infty$ in the previous inequality and using (2.9.10), (2.9.14) and (2.9.17) gives that

$$0 \leq \langle \chi - (-\Delta_p)^s(\xi), \tilde{w}^m - \xi \rangle_{W_0^{-s,(2,p)'}, W_0^{s,(2,p)}}.$$

Since in the last inequality $\xi \in W_0^{s,(2,p)}$ was arbitrary, we can choose $\xi = \tilde{w}^m - \mu \zeta$ for any $\zeta \in W_0^{s,(2,p)}$ and $\mu > 0$. It follows that

$$0 \leq \langle \chi - (-\Delta_p)^s(\tilde{w}^m - \mu \zeta), \zeta \rangle_{W_0^{-s,(2,p)'}, W_0^{s,(2,p)}}.$$

In this last inequality, we can send $\mu \rightarrow 0+$ and obtain that

$$0 \leq \langle \chi - (-\Delta_p)^s(\tilde{w}^m), \zeta \rangle_{W_0^{-s,(2,p)'}, W_0^{s,(2,p)}} \quad \text{for every } \zeta \in W_0^{s,(2,p)},$$

implying that $\chi = (-\Delta_p)^s(\tilde{w}^m)$. Since $\tilde{w}$ and $h \in L^{1 \cap \infty}$, it follows from (2.9.15) that (2.9.16) holds and so we have shown that $\tilde{w} + \lambda(A\tilde{w} + F(\tilde{w})) = h$, or, equivalently, $\tilde{w} = J_\lambda^{A+F} h$. Since the resolvent J_λ^{A+F} is a contraction on L^1 , it follows that $\tilde{w} = w$ and thereby we have shown that (2.9.7) holds. This completes the proof of this lemma. $\square$

With the preceding lemma in mind, we can now give the proof of Theorem 2.1.8. Note, below we write $L^p(W_0^{s,(2,p)})$ instead of $L^p(0, T; W_0^{s,(2,p)})$ for $T > 0$, and denote by $L^{p'}(W_0^{-s,(2,p)'})$ the dual space of $L^p(W_0^{s,(2,p)})$.

Proof of Theorem 2.1.8. We only provide the proof for initial values $u_0 \in L^{1 \cap \infty}$ since by Corollary 2.1.5 the same conclusion holds for general $u_0 \in L^1$ under the given restrictions on p, s , and m . So let $u_0 \in L^{1 \cap \infty}$, take $(\Omega_n)_{n \geq 1}$ to be the sequence $(\Omega \cap B_n)_{n \geq 1}$ of intersections with balls of radius n and $(u_{0,n})_{n \geq 1}$ the sequence of functions converging to u_0 in L^1 given by Proposition 2.9.2. In particular, $u_{0,n}^m \in D((-\Delta_p)_{\Omega_n, 1 \cap \infty}^s)$. Then define u_n to be the associated sequence of strong distributional solutions to (2.9.3). We now prove the strong distributional properties of the mild solution u of Cauchy problem (1.2.2) by considering the boundedness and convergence of u_n and $\frac{du_n}{dt}$ in appropriate Banach spaces.

Multiplying the differential equation in (2.9.1) by u_n^m and subsequently, for every $T > 0$, integrating over $(0, T)$ yields that

$$\begin{aligned} & \frac{1}{m+1} \|u_n(t) \mathbf{1}_{\Omega_n}\|_{m+1}^{m+1} \Big|_0^T + \int_0^T [u_n^m(t)]_{s,p}^p dt \\ &= \int_0^T \int_{\Omega_n} (-F_n(u_n) + g_n) u_n^m dx dt \end{aligned} \quad (2.9.18)$$

which we estimate by

$$\begin{aligned} & \frac{1}{m+1} \|u_n(t) \mathbf{1}_{\Omega_n}\|_{m+1}^{m+1} \Big|_0^T + \int_0^T [u_n^m(t)]_{s,p}^p dt \\ & \leq \int_0^T (1 + \omega) \|u_n(t) \mathbf{1}_{\Omega_n}\|_{m+1}^{m+1} + \|g(t)\|_{m+1}^{m+1} dt. \end{aligned} \quad (2.9.19)$$

Since $u_0 \in L^{1 \cap \infty}$, Proposition 2.9.2 and the standard growth estimate of Theorem 2.1.1 imply that for every $1 \leq q < \infty$,

$$\lim_{n \rightarrow \infty} u_n \mathbf{1}_{\Omega_n} = u \quad \text{in } C([0, T]; L^q) \quad (2.9.20)$$

for every $T > 0$. Thus (2.9.19) implies that $(u_n^m)_{n \geq 1}$ is bounded in $L^p(W_0^{s,(2,p)})$. Since the space $L^p(W_0^{s,(2,p)})$ is reflexive and by (2.9.20),

we can conclude that $u^m \in L^p(W_0^{s,(2,p)})$ and after possibly passing to a subsequence,

$$\lim_{n \rightarrow \infty} u_n^m \mathbb{1}_{\Omega_n} = u^m \quad \text{weakly in } L^p(W_0^{s,(2,p)}). \quad (2.9.21)$$

In particular, one has that the sequence $(\mathcal{A}_{s,p}(u_n^m))_{n \geq 1}$ of linear bounded functionals given by

$$\begin{aligned} & \langle \mathcal{A}_{s,p}(u_n^m), \xi \rangle_{L^{p'}(W_0^{-s,(2,p)'}), L^p(W_0^{s,(2,p)})} \\ &= \frac{1}{2} \int_0^T \int_{\mathbb{R}^{2d}} \frac{(u_n^m(x) - u_n^m(y))^{p-1} (\xi(x) - \xi(y))}{|x - y|^{d+sp}} d(x, y) dt \end{aligned}$$

for every $\xi \in L^p(W_0^{s,(2,p)})$, is bounded in $L^{p'}(W_0^{-s,(2,p)'})$. Therefore, there is a $\chi \in L^{p'}(W_0^{-s,(2,p)'})$ such that after possibly passing to a subsequence, one has that

$$\lim_{n \rightarrow \infty} \mathcal{A}_{s,p}(u_n^m) = \chi \quad \text{weakly}^* \text{ in } L^{p'}(W_0^{-s,(2,p)'}). \quad (2.9.22)$$

We now consider the following equation in $L^1(\Omega_n)$,

$$\frac{du_n}{dt}(t) + A_n u_n(t) + F_n(u_n)(t) = g_n(t) \quad \text{for } t \in (0, T). \quad (2.9.23)$$

We first note that $(F_n(u_n) \mathbb{1}_{\Omega_n})_{n \geq 1}$ and $(g_n \mathbb{1}_{\Omega_n})_{n \geq 1}$ are bounded in the dual space $L^{p'}(W_0^{-s,(2,p)'})$ since, by the Lipschitz property of F and the growth estimate (2.1.2), $F(u_n)$ and g_n are bounded in the dual space $L^{p'}(0, T; L^{p'})$. Hence, given (2.9.22), we have that $(\frac{du_n}{dt} \mathbb{1}_{\Omega_n})_{n \geq 1}$ is also bounded in $L^{p'}(W_0^{-s,(2,p)'})$. Thus, taking another subsequence, $(\frac{du_n}{dt} \mathbb{1}_{\Omega_n})_{n \geq 1}$ converges weak-* in $L^{p'}(W_0^{-s,(2,p)'})$. Applying test functions and (2.9.20), we can conclude that $\frac{du}{dt} \in L^{p'}(W_0^{-s,(2,p)'})$ and we have the weak-* convergence,

$$\left(\frac{du_n}{dt} + F_n(u_n) - g_n \right) \mathbb{1}_{\Omega_n} \xrightarrow{*} \frac{du}{dt} + F(u) - g \quad \text{in } L^{p'}(W_0^{-s,(2,p)'}) \quad (2.9.24)$$

as $n \rightarrow \infty$.

Hence, multiplying (2.9.23) by a test function $\xi \in C_c^\infty((0, T) \times \mathbb{R}^d)$ and sending $n \rightarrow \infty$ leads to

$$\left\langle \frac{du}{dt}(t) + \chi + F(u) - g, \xi \right\rangle_{L^{p'}(W_0^{-s,(2,p)'}), L^p(W_0^{s,(2,p)})} = 0.$$

Therefore, we have shown that

$$\frac{du}{dt}(t) + \chi + F(u) = g \quad \text{in } W_0^{-s,(2,p)'} \text{ for a.e. } t \in (0, T). \quad (2.9.25)$$

Now we can show that u is a distributional solution of (1.2.1) by proving that $\chi = \mathcal{A}_{s,p}(u^m)$. Here, $\mathcal{A}_{s,p}$ is the lifted operator $\mathcal{A}_{s,p} : L^p(W_0^{s,(2,p)}) \rightarrow L^{p'}(W_0^{-s,(2,p)'})$ given by

$$\begin{aligned} & \langle \mathcal{A}_p^s(v), \xi \rangle_{L^{p'}(W_0^{-s,(2,p)'}), L^p(W_0^{s,(2,p)})} \\ &= \frac{1}{2} \int_{t_1}^{t_2} \int_{\mathbb{R}^{2d}} \frac{|v(x) - v(y)|^{p-2} (v(x) - v(y)) (\xi(x) - \xi(y))}{|x - y|^{d+sp}} d(x, y) dt \end{aligned}$$

for every $v, \xi \in L^p(W_0^{s,(2,p)})$. To prove this, we again use the monotonicity trick (cf, the proof of Proposition 2.9.2), but to the lifted operator $\mathcal{A}_p^s$. First, we note that multiplying (2.9.25) by u^m yields that

$$\frac{1}{m+1} \|u\|_{m+1}^{m+1} \Big|_0^T + \langle \chi + F(u) - g, u^m \rangle_{L^{p'}(W_0^{-s,(2,p)'}), L^p(W_0^{s,(2,p)})} = 0.$$

Thus (2.9.18) and (2.9.20) imply that

$$\lim_{n \rightarrow \infty} \int_0^T [u_n^m(t)]_{p,s}^s dt = \langle \chi, u^m \rangle_{L^{p'}(W_0^{-s,(2,p)'}), L^p(W_0^{s,(2,p)})}. \quad (2.9.26)$$

By the monotonicity of $\mathcal{A}_{s,p}$, one has that

$$\begin{aligned} 0 &\leq \langle \mathcal{A}_p^s(u_n^m) - \mathcal{A}_{s,p}(\xi), u_n^m - \xi \rangle_{L^{p'}(W_0^{-s,(2,p)'}), L^p(W_0^{s,(2,p)})} \\ &= \int_0^T [u_n^m(t)]_{s,p}^p dt - \langle \mathcal{A}_p^s(u_n^m), \xi \rangle_{L^{p'}(W_0^{-s,(2,p)'}), L^p(W_0^{s,(2,p)})} \\ &\quad - \langle \mathcal{A}_{s,p}(\xi), u_n^m - \xi \rangle_{L^{p'}(W_0^{-s,(2,p)'}), L^p(W_0^{s,(2,p)})} \end{aligned}$$

for all $n \geq 1$. Thus, sending $n \rightarrow \infty$ in the last inequality and by using (2.9.21), (2.9.22), and (2.9.26), one finds that

$$0 \leq \langle \chi - \mathcal{A}_{s,p}(\xi), u^m - \xi \rangle_{L^{p'}(W_0^{-s,(2,p)'}), L^p(W_0^{s,(2,p)})}$$

for every $\xi \in L^p(W_0^{s,(2,p)})$. Now, by proceeding as in the proof of Proposition 2.9.2, we can choose $\xi = u^m - \mu \zeta$, taking $\mu \rightarrow 0+$, to conclude that $\chi = \mathcal{A}_{s,p}(u^m)$.

Applying Corollary 2.8.2 and the standard growth estimate (2.1.2), we can deduce that the sequence $(u_n^m(t))_{n \geq 1}$ is bounded in $W_0^{s,(2,p)}$ for every $t > 0$, and that $(\frac{d}{dt} u_n^{\frac{m+1}{2}} \mathbf{1}_{\Omega_n})_{n \geq 1}$ is bounded in $L^2(t_1, t_2; L^2)$ for every $0 < t_1 < t_2 < \infty$. Then by (2.9.20) and Fatou's lemma, we can conclude that $\frac{d}{dt} u^{\frac{m+1}{2}} \in L_{loc}^2((0, \infty); L^2)$ and $u^m(t) \in W_0^{s,(2,p)}$ for every $t > 0$. In particular, one has that $w := u^{\frac{m+1}{2}} \in W_{loc}^{1,1}((0, \infty); L_{loc}^1)$.

If $m(p-1) \neq 1$, then by Theorem 2.1.6, u is locally Lipschitz continuous on $(0, T)$ with values in L^1 . In the case $m(p-1) = 1$, by assumption we have that $\|(-\Delta_p)_{1 \cap \infty}^s u_{0,n}^m\|_1$ is uniformly bounded for $n \geq 1$. Then applying the Lipschitz estimate for u_n given by Lemma 2.5.1, together with (1.2.1), we have that u is Lipschitz continuous on $(0, T)$ with values in L^1 . Therefore, and since $p(r) := r^{\frac{2}{m+1}-1} r \in L_{loc}^1(\mathbb{R})$, it follows from [22, Theorem 1.1] that $u \in W_{loc}^{1,1}((0, T); L_{loc}^1)$.

Now in the case $m(p-1) \neq 1$, we apply the Lipschitz estimate on u (both in the case $m(p-1) \neq 1$ and $m(p-1) = 1$), to obtain

$$\begin{aligned} \|\partial_t u(t)\|_{L^1(\Omega_n)} &= \limsup_{h \rightarrow 0+} \frac{\|u(t+h) - u(t)\|_{L^1(\Omega_n)}}{h} \\ &\leq \limsup_{h \rightarrow 0+} \frac{\|u(t+h) - u(t)\|_{L^1}}{h} \\ &\leq L(t) \end{aligned}$$

for every $t > 0$ and $n \geq 1$ where $L(t)$ is obtained from (2.1.15) when $m(p-1) \neq 1$ and (2.5.1) when $m(p-1) = 1$. Sending $n \rightarrow \infty$ in this inequality shows that $u \in W_{loc}^{1,1}((0,T); L^1)$, and in particular a strong solution of (1.2.1). $\square$

Thus, we also prove the regularity of solutions stated in Theorem 2.1.6.

2.10 Hölder regularity

This section is dedicated to the parabolic Hölder regularity of mild solutions to the initial boundary value problem

$$\begin{cases} u_t(t) + (-\Delta_p)^s u^m(t) + f(\cdot, u(t)) = g(\cdot, t) & \text{in } \Omega \times (0, T), \\ u(t) = 0 & \text{in } \mathbb{R}^d \setminus \Omega \times (0, T), \\ u(0) = u_0 & \text{on } \Omega, \end{cases} \quad (2.10.1)$$

for $1 < p < \infty$ with $p \neq 1 + \frac{1}{m}$, $0 < s < 1$ and where Ω is an open, bounded domain in $\mathbb{R}^d$, $d \geq 2$. For global Hölder regularity we consider only the case $m = 1$. For the local Hölder result, we apply the following local elliptic Hölder regularity result from [30].

Theorem 2.10.1 ([30, Theorem 1.4]). *Let $\Omega \subset \mathbb{R}^d$ be a bounded and open set. Assume $2 \leq p < \infty$, $0 < s < 1$ and $q \geq 1$ satisfies $q > \frac{d}{sp}$. We define the exponent*

$$\Theta(d, s, p, q) := \min \left\{ \frac{1}{p-1} \left(sp - \frac{d}{q} \right), 1 \right\}. \quad (2.10.2)$$

Let $u \in W_{loc}^{s,p}(\Omega) \cap L_{loc}^\infty(\Omega) \cap L_{sp}^{p-1}(\mathbb{R}^d)$ be a local weak solution of

$$(-\Delta_p)^s u = h \quad \text{in } \Omega,$$

where $h \in L_{loc}^q(\Omega)$. Then $u \in C_{loc}^\delta(\Omega)$ for every $0 < \delta < \Theta$.

Proof of Theorem 2.1.9. In order to apply Theorem 2.10.1, we require a distributional solution. Hence, first consider u to be a strong distributional solution to (1.2.1) with $u_0 \in \hat{D}((-\Delta_p)^s_{|L^1 \cap \infty} \varphi)$. By the standard growth estimate (2.1.2), $u(t) \in L^1 \cap L^\infty$ for $t \in [0, T]$ and hence by the Lipschitz condition of f , for F the Nemytskii operator of f , $F(u(t)) \in L^1$ for all $t \in [0, T]$. We now apply Theorem 2.10.1 with

$$h(t) := g(t) - F(u(t)) - u_t(t) \in L^1$$

for almost every $t \in (0, T)$. Then by [30, Theorem 1.4], for almost every $t \in (0, T)$,

$$[u^m(t)]_{C^\delta(B_{R/8}(x_0))} \leq C \left[\|u_0\|_{L^\infty(B_R(x_0))} + \|u(t)\|_{p-1} + \|h(t)\|_1^{\frac{1}{p-1}} \right]$$

for some $C > 0$ depending on R, s, p, d and δ . Applying the homogeneous regularizing effects of Theorem 2.1.6, we estimate $u_t \in L^1$ so that

$$\begin{aligned} \|h(t)\|_1 &\leq \|g(t)\|_1 + \omega e^{\omega t} \left(\|u_0\|_1 + \int_0^t \|g(r)\|_1 dr \right) \\ &\quad + \frac{\tilde{C} e^{2\omega t}}{t} \left(\|u_0\|_1 + \int_0^t \|g(r)\|_1 dr + V(t, g) \right) \end{aligned}$$

for $t \in (0, T)$ where $\tilde{C} > 0$ depends on m and p .

We now approximate mild solutions u with $u_0 \in L^1$ using the density result of Theorem 2.1.1, and applying Fatou's lemma, we have $u^m(t) \in C_{\text{loc}}^\delta(\Omega)$ for almost every $t \in (0, T)$.

Since $r^m, r \in \mathbb{R}$, is locally Lipschitz continuous for $m \geq 1$, we obtain Hölder continuity of $u(t)$ in this case by composition. $\square$

Our proof of global Hölder regularity employs the following elliptic Hölder regularity result for the fractional p -Laplacian from [73].

Theorem 2.10.2 ([73, Theorem 1.1]). *Let Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 2$, with a boundary $\partial\Omega$ of the class $C^{1,1}$, $p \in (1, \infty)$ and $0 < s < 1$. There exists $\alpha \in (0, s]$ and $C_\Omega > 0$ depending on d, p, s, Ω such that if $h \in L^\infty$, then the weak solution $u \in W_0^{s,p}$ of*

$$\begin{cases} (-\Delta_p)^s u = h & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^d \setminus \Omega, \end{cases} \quad (2.10.3)$$

belongs to $C^\alpha(\overline{\Omega})$ and satisfies

$$\|u\|_{C^\alpha(\overline{\Omega})} \leq C_\Omega \|h\|_{L^\infty}^{\frac{1}{p-1}}. \quad (2.10.4)$$

For this we use a restriction to the set of continuous functions $u : \Omega \rightarrow \mathbb{R}$ vanishing on the boundary $\partial\Omega$, which we denote by $C_0(\Omega)$. Following the notation in Definition 2.2.14, we denote by $(-\Delta_p)_{|C_0}^s$ the restriction of $(-\Delta_p)^s$ to $C_0(\Omega) \times C_0(\Omega)$. We first prove accretivity and density results for this operator on $C_0(\Omega)$. The proof of density follows the idea in [98, Proposition 5.4].

Proposition 2.10.3 (Density of $D((-\Delta_p)_{|C_0}^s)$ in $C_0(\Omega)$). *Let Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 2$, with a boundary $\partial\Omega$ of the class $C^{1,1}$, F the Nemytskii operator of f on $C_0(\Omega)$ satisfying (2.0.3a)-(2.0.3b), $p \in (1, \infty)$ and $0 < s < 1$. Then $(-\Delta_p)_{|C_0}^s + F$ is m -completely accretive in $C_0(\Omega)$. Furthermore, if $s < 1 - \frac{1}{p}$ then the set $D((-\Delta_p)_{|C_0}^s)$ is dense in $C_0(\Omega)$.*

Proof. Since Ω has finite Lebesgue measure, one has that the subdifferential satisfies $(-\Delta_p)_{|C_0}^s \subseteq (-\Delta_p)^s$. Then since $(-\Delta_p)^s + F$ is ω -quasi m -completely accretive in L^2 , for every $g \in C_0(\Omega)$ and $\lambda > 0$, there is a unique $u_\lambda \in L^2$ satisfying

$$(1 + \lambda\omega)u_\lambda + \lambda(\partial\mathcal{E}(u_\lambda) + F(u_\lambda)) = g \quad \text{in } L^2. \quad (2.10.5)$$

Moreover, by the complete accretivity condition, $u_\lambda \in L^\infty$. Hence u_λ is a weak solution of the non-local Poisson problem (2.10.3) with

$$\begin{aligned} h &:= -F(u_\lambda) + \frac{g - (1 + \lambda\omega)u_\lambda}{\lambda} \\ &\in L^\infty, \end{aligned}$$

and so Theorem 2.10.2 yields that $u_\lambda \in C_0^\alpha(\Omega)$, satisfying

$$(1 + \lambda\omega)u_\lambda + \lambda((-\Delta_p)_{|C_0}^s u_\lambda + F(u_\lambda)) = g \quad \text{in } L^2.$$

As $g \in C_0(\Omega)$ and $\lambda > 0$ were arbitrary, we have thereby shown that the shifted operator $(-\Delta_p)_{|C_0}^s u_\lambda + F(u_\lambda) + \omega I_{C_0}$ satisfies the range condition (2.2.6).

To prove the density result, fix $u \in C_c^\infty(\Omega)$. We first prove that $u + (-\Delta_p)^s u \in L^\infty$ by splitting the domain to deal with local and nonlocal estimates. For $\varepsilon > 0$,

$$(-\Delta_p)^s u \leq \int_{B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-1}}{|x - y|^{d+sp}} dy + \int_{\mathbb{R}^d \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-1}}{|x - y|^{d+sp}} dy.$$

Estimating $|u(x) - u(y)|$ by the derivative $\sup_{s \in \Omega} |u'(s)| |x - y|$ for the first term,

$$\begin{aligned} \int_{B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-1}}{|x - y|^{d+sp}} dy &\leq \int_{B_\varepsilon(x)} \frac{\sup_{s \in \Omega} |u'(s)|^{p-1} |x - y|^{p-1}}{|x - y|^{d+sp}} dy \\ &\leq C_B \sup_{s \in \Omega} |u'(s)|^{p-1} \int_0^\varepsilon \frac{1}{r^{d+1-(1-s)p}} r^{d-1} dr \end{aligned}$$

where C_B is a constant for integration over a d -dimensional ball. Then we have

$$\int_0^\varepsilon \frac{1}{r^{2-(1-s)p}} dr = C [r^{(1-s)p-1}]_0^\varepsilon$$

which is bounded for $s < 1 - \frac{1}{p}$. For the nonlocal term,

$$\begin{aligned} \int_{\mathbb{R}^d \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-1}}{|x - y|^{d+sp}} dy &\leq C_B \|u\|_\infty \int_\varepsilon^\infty \frac{1}{r^{d+sp}} r^{d-1} dr \\ &= C \|u\|_\infty [r^{-sp}]_\infty^\varepsilon \end{aligned}$$

for some $C > 0$. Since $sp > 0$, this is bounded and $(-\Delta_p)^s u \in L^\infty$. We now define $f := A_1 u = u + (-\Delta_p)^s u \in L^\infty$ and approximate by $f_n \in C_c^\infty(\Omega) \subset C_0(\Omega)$ such that $f_n \rightarrow f$ in $L^{p_s^*}$ as $n \rightarrow \infty$ with $\|f_n\|_\infty \leq \|f\|_\infty$ and where $\frac{1}{p_s^*} + \frac{1}{p_s} = 1$ and p_s is given by (2.1.7).

We now solve $A_1 u_n = f_n \in C_0(\Omega)$ using the m -accretivity of $(-\Delta_p)_{|C_0}^s$, finding that $u_n \in C_0(\Omega)$ and so $u_n \in D((-\Delta_p)_{|C_0}^s)$. Moreover, by the elliptic Hölder estimate (2.10.4) with $(f_n)_{n \in \mathbb{N}}$ bounded in L^∞ , $(u_n)_{n \in \mathbb{N}}$ is

bounded in $C^\alpha(\Omega)$. Hence taking a subsequence and relabelling, we have convergence to a function in $C(\Omega)$.

Next, we prove that this limit is u by considering convergence in $W^{s,p}$. For $w \in D((-\Delta_p)_{|C_0}^s)$ we can define $\varphi_w(v) := (A_1 w, v)_{L^2}$ for all $v \in W^{s,p}$. Estimating by Hölder's inequality,

$$\begin{aligned} |\varphi_{u_n}(v) - \varphi_u(v)| &\leq \int_{\Omega} |(f_n - f)v| \, d\mu \\ &\leq \|f_n - f\|_{p_s^*} \|v\|_{p_s} \\ &\leq C_d \|f_n - f\|_{p_s^*} \|v\|_{W^{s,p}} \end{aligned}$$

by the standard Sobolev embedding (2.1.7). Hence $\varphi_{u_n} \rightarrow \varphi_u$ in $(W^{s,p})'$ and so $(\varphi_{u_n})_{n \geq 1}$ is bounded in $(W^{s,p})'$. Then we also have that

$$\begin{aligned} (A_1 u_n, u_n)_{L^2} &= \|u_n\|_2^2 + [u_n]_{s,p}^p \\ &\leq C_u (1 + \|u_n\|_{W^{s,p}}). \end{aligned}$$

Now

$$\begin{aligned} \liminf_{\|v\|_{W^{s,p}} \rightarrow \infty} \frac{(A_1 v, v)_{L^2}}{\|v\|_{W^{s,p}}} &= \frac{\|v\|_2 + [v]_{s,p}^p}{\|u\|_{W^{s,p}}} \\ &\geq \|v\|_{W^{s,p}}^{p-1} \\ &\rightarrow \infty \end{aligned}$$

for $p \in (1, \infty)$. Hence $(u_n)_{n \geq 1}$ is bounded in $W^{s,p}$ and passing to a subsequence we have that $u_n \rightharpoonup \tilde{u}$ in $W^{s,p}$. We extend $\varphi_w(v)$ to $w \in W^{s,p}$ by defining $\varphi_w(v) = (w + (-\Delta_p)^s w, v)_{L^2}$ for $w \in W^{s,p} \setminus D((-\Delta_p)_{|C_0}^s)$. By Minty's theorem, [112, Proposition II.2.2],

$$\langle \varphi_v - \varphi_{u_n}, v - u_n \rangle \geq 0$$

for all $v \in W^{s,p}$. We have the convergences $\varphi_{u_n} \rightarrow \varphi_u$ in $(W^{s,p})'$ and $u_n \rightharpoonup \tilde{u}$ in $W^{s,p}$ so that

$$\langle \varphi_v - \varphi_{\tilde{u}}, v - u \rangle \geq 0$$

for all $v \in W^{s,p}$. Applying Minty's theorem again, $\varphi_{\tilde{u}} = \varphi_u$ so that by the uniqueness provided by the accretivity of $(-\Delta_p)^s$ in L^2 , $\tilde{u} = u$. Also, since $u_n \rightharpoonup u$ in $W^{s,p}$, we have that $u_n \rightarrow u$ in $C_0(\Omega)$, relabelling by appropriate subsequences. Then the density of $C_c^\infty(\Omega)$ in $C_0(\Omega)$ gives us the desired density result. $\square$

We now prove Hölder continuity in the identity case, $m = 1$. In this theorem, the case $p = 2$ is well-known. We note that this proof of parabolic regularity relies on the global Hölder regularity estimate of the elliptic problem which we believe is not optimal, in particular with the L^∞ norm required in (2.10.4), and that a stronger elliptic result would also improve this parabolic result.

Proof of Theorem 2.1.10. By Proposition 2.10.3, we have that $(-\Delta_p)_{|_{C_0}}^s + F$ is ω -quasi m -completely accretivity in $C_0(\Omega)$ and $\overline{D((-\Delta_p)_{|_{C_0}}^s)}^{C_0} = C_0(\Omega)$. The Crandall-Liggett theorem (see [51], [11]) says that $-(\partial_{C_0}\mathcal{E} + F)$ generates a strong continuous semigroup of ω -quasi contractions on $C_0(\Omega)$ where $\mathcal{E}$ is given by (2.0.4). Further, since $(-\Delta_p)_{|_{C_0}}^s$ is homogeneous of order $p - 1$, and since

$$(-\Delta_p)_{|_{C_0}}^s \subseteq \partial_{L^q}\mathcal{E}$$

for all $1 \leq q \leq \infty$, it follows from [71] that for initial data $u_0 \in C_0(\Omega)$ and $g \in C((0, T); C_0(\Omega)) \cap BV(0, T; C_0(\Omega))$, the mild solution $u \in C([0, T]; C_0(\Omega))$ of the initial boundary value problem (2.10.1) is a strong solution with $u \in W^{1,\infty}(\delta, T; C_0(\Omega))$. Moreover, $u \in C^{lip}([\delta, T]; C_0(\Omega))$ for every $0 < \delta < T$. In particular, u is a weak solution of the non-local Poisson problem (2.10.3) with

$$h := g(t) - F(u) - u_t(t) \in C((0, T); C_0(\Omega)).$$

Hence, by the elliptic regularity result Theorem 2.10.2, we obtain that $u(t) \in C^\alpha(\overline{\Omega})$ for all $t \in (0, T)$ for some $\alpha \in (0, s]$. $\square$

Chapter 3

Barenblatt solutions

Our focus in this chapter is to investigate the self-similar properties of (1.1.3) and establish an existence result for *Barenblatt solutions*. Such self-similar solutions with Dirac delta as initial data have a long history in the context of linear elliptic and parabolic equations in particular, where they are *fundamental solutions* which provide a rich theory for the generation of general solutions [61]. Despite lacking this property in the nonlinear setting, Barenblatt solutions are nevertheless of great interest as they give insight into the existence and regularity theory. Moreover, they provide insight into the asymptotic behaviour of solutions. We refer to [15, 47, 54, 120] and the references therein. Barenblatt solutions are often otherwise referred to as *fundamental* or *source-type* solutions due to their role in the theory of linear equations.

Such Barenblatt solutions in the nonlinear setting have been introduced first in the case of the porous medium equation (1.1.2) by Zel'dovich and Kompaneets [131], by Barenblatt in [13, 14] and by Pattle in [102]. These techniques have been extended to a wide range of nonlinear problems, both local and nonlocal. Here the connection to the asymptotic behaviour of the porous medium equation has been introduced in [23, 63, 78, 79]. Naturally, they have been considered for p -Laplacian evolution equations, studied by Kamin and Vázquez [80] and the doubly nonlinear diffusive equation by Saá [110]. We also note [47] where the anisotropic case of the doubly nonlinear diffusive equation is considered. In the nonlocal case, the fractional diffusive problem and the fractional p -Laplacian evolution problem has been considered by Vázquez [121, 122, 126].

The general idea for obtaining such self-similar solutions for these evolution equations is to use self-similarity properties to reduce the equation to an ODE via a change of coordinates to ‘self-similar variables’. The solution of this ODE then forms the self-similar profile of the Barenblatt solution. Importantly, this relationship between Barenblatt solutions and inherent scaling properties provides a connection to the asymptotic behaviour of solutions as we will see when obtaining global barrier functions and comparison estimates in Section 3.3. Moreover, this places Baren-

blatt solutions into the wider context of scaling techniques which have given rise to a wide array of key results in the analysis of elliptic and parabolic problems, particularly for physical processes. We refer to [15, 16] and [120].

In the case of $m = 1$, Barenblatt solutions for (1.1.3) have been studied by Vázquez in [121, 126]. Our results and proofs extend those to the case $m \geq 1$, and $p > p_c$ where

$$p_c := \frac{1 + m}{\frac{s}{d} + m}. \quad (3.0.1)$$

We note that $p > p_c$ corresponds to (1.2.3) and is a standard condition in this context due to the self-similar scaling properties of (1.1.3). See, for example, the case $p = 2$ [122] with the condition $m > (\frac{d-2s}{d})^+$ and $m = 1$ [126] requiring $p > \frac{2d}{d+s}$.

In particular, these are self-similar solutions with a Dirac delta as initial data in a limiting sense as $t \rightarrow 0$. We introduce self-similar solutions to (1.1.3) in Section 3.1, in particular the appropriate scaling transformation under which these solutions are invariant. Furthermore, we prove an Aleksandrov symmetry principle in Section 3.2 and a global barrier for solutions to (1.1.3) in Section 3.3 which we use in the proof of our main result.

Our main result is the following theorem.

Theorem 3.0.1. *Suppose $m \geq 1$, $p > p_c$, $0 < s < 1$, $d \geq 1$ are such that $m(p-1) \neq 1$ and $sp < d$. Then for all $M > 0$ there exists a unique Barenblatt solution Γ to (1.1.3) of the form*

$$\Gamma(x, t; M) = M^{sp\beta} t^{-d\beta} F(M^{-(m(p-1)-1)\beta} x t^{-\beta}) \quad (3.0.2)$$

where

$$\beta = \frac{1}{d(m(p-1)-1) + sp} \quad (3.0.3)$$

and the profile function $F : \mathbb{R}^d \rightarrow \mathbb{R}$ is positive, radially symmetric and is decreasing such that $F(r)$ has decay at infinity with order depending on p and given by (3.3.2). Moreover, Γ decays in time uniformly with respect to x .

In particular, Γ is a strong distributional solution in L^1 of the doubly nonlinear nonlocal diffusion equation (1.1.3) having $M\delta_0$ as initial datum as $t \rightarrow 0+$, where δ_0 is the Dirac δ -function with mass $\int_{\mathbb{R}^d} \delta_0 dx = 1$.

Here M is called the mass. We note that $p > p_c$ ensures that we have self-similar transformations of the form presented in Section 3.1 and, in particular, that we obtain the global bound stated in Section 3.3.

3.1 Self-similar solutions

We are interested in scaling transformations of time and space under which (1.1.3) is invariant. Moreover, we consider self-similar solutions,

that is, classes of solutions which are invariant under such a one-parameter group of scaling symmetries. Such solutions play a key role in the asymptotic behaviour as well as the existence and regularity of such evolution equations.

We first introduce two important scaling transformations for (1.1.3). Scaling time, space and amplitude, we have that (1.1.3) is invariant under the transformation

$$T_k u(x, t) = k^d u(kx, k^{dm(p-1)-d+sp} t) \quad (3.1.1)$$

for all $k > 0$. By the amplitude scaling, this transformation also conserves mass. We can scale only the time and amplitude with

$$\hat{T}_M u(x, t) = M u(x, M^{m(p-1)-1} t) \quad (3.1.2)$$

which will scale the mass of u by the factor $M > 0$.

Since (1.1.3) is invariant under the scaling transformation (3.1.1), we look for self-similar solutions of the form

$$U(x, t; M) = t^{-d\beta} F(xt^{-\beta}; M) \quad (3.1.3)$$

for some $\beta \in \mathbb{R}$. Here $F(z; M)$ is chosen to have mass $M > 0$ and the factor $t^{-d\beta}$ conserves the mass of U in time. Substituting (3.1.3) into (1.1.3), we have that setting

$$\beta = \frac{1}{dm(p-1) - d + sp} \quad (3.0.3)$$

we obtain such a self-similar solution where the profile F satisfies

$$-d\beta F(z) - \beta \nabla F(z) \cdot z + (-\Delta_p)^s (F(z))^m = 0.$$

Equivalently,

$$(-\Delta_p)^s (F(z))^m = \beta \nabla \cdot (zF) \quad (3.1.4)$$

which we refer to as the self-similar profile equation. We will also use the radial form with $r = |z|$, given by

$$(-\Delta_p)^s (F(z))^m = \beta r^{1-d} (r^d F(z))_r. \quad (3.1.5)$$

Note that we can scale the profile function with $M = 1$, $F(z; 1)$, to have unit mass. Then by the rescaling (3.1.2), we can deduce the profile of mass M to be

$$F(z; M) = M^{sp\beta} F(M^{-(m(p-1)-1)\beta} z; 1).$$

3.2 An Aleksandrov symmetry principle

Using the radial symmetry of the fractional p -Laplacian, we prove an Aleksandrov symmetry principle [2] for

$$\begin{cases} \frac{du}{dt}(t) + (-\Delta_p)^s u^m(t) = 0 & \text{in } (0, \infty), \\ u(0) = u_0, \end{cases} \quad (3.2.1)$$

in $L^1(\mathbb{R}^d)$, extending the fractional p -Laplacian with $m = 1$ from [121]. We consider a reflection around a hyperplane in $\mathbb{R}^d$, denoting the reflection map by $\Pi : \mathbb{R}^d \rightarrow \mathbb{R}^d$. Then the hyperplane splits $\mathbb{R}^d$ into two disjoint open sets. The idea of this method is to obtain a typical difference comparison in the sense of T -accretive operators, such as the estimate (2.1.3), but for a solution and its reflection on one half-space. By the translation and rotational invariance of the operator $(-\Delta)_s^m$, we can consider the specific case where the hyperplane is $\{x \in \mathbb{R}^d \mid x_1 = 0\}$ and the reflection is given by $\Pi(x) := (-x_1, x_2, \dots, x_d)$.

Here we modify a standard comparison principle idea, used for example in [127], obtaining estimates on the whole domain.

Theorem 3.2.1. *Suppose $0 < s < 1$, $p > 1$, $d \geq 1$ and $m > s$. Let u be the mild solution of (3.2.1) with initial data $u_0 \in L^1$. Let $\Pi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ denote reflection around a hyperplane H , splitting $\mathbb{R}^d$ into Ω_1 and Ω_2 . If*

$$u_0(x) \leq u_0(\Pi(x)) \quad \text{in } \Omega_1,$$

then

$$u(x, t) \leq u(\Pi(x), t) \quad \text{in } \Omega_1 \times [0, T].$$

Of important consequence are the following two corollaries giving us radially decreasing solutions with bounded initial data and radial symmetry of solutions with radially symmetric initial data, respectively.

Corollary 3.2.2. *Suppose $0 < s < 1$, $p > 1$, $d \geq 1$ and $m > s$. Then mild solutions u of (3.2.1) with non-negative, compactly supported initial data u_0 in a ball $B_R(0)$ are radially decreasing in space for all $|x| \geq R$ and $t \geq 0$.*

Proof. This follows from Theorem 3.2.1 by taking hyperplanes perpendicular to the radial direction and with distance at least R from the origin. Then data closer to the origin is non-negative and the data away from the origin zero. $\square$

The next corollary follows similarly.

Corollary 3.2.3. *Suppose $0 < s < 1$, $p > 1$, $d \geq 1$ and $m > s$. Then mild solutions u of (3.2.1) with non-negative, radially symmetric and radially decreasing initial data are radially symmetric and radially decreasing in space for all $x \in \mathbb{R}^d$ and $t \geq 0$.*

Before proving Theorem 3.2.1, we prove two lemmas. The first allows us to use the variational formulation of the fractional p -Laplacian when restricting to the half-space in Theorem 3.2.1.

Lemma 3.2.4. *Let $\Pi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be the reflection around a hyperplane H in $\mathbb{R}^d$ and Ω_1 be a half-space associated with H . If a measurable function u satisfies $[u]_{s,p} < \infty$ and*

$$|u(x)| \leq |u(x) - u(\Pi(x))| \quad (3.2.2)$$

for all $x \in \Omega_1$ then $[u\mathbb{1}_{\Omega_1}]_{s,p} \leq 2[u]_{s,p}$.

Proof. We let Ω_2 be the other half-space given by reflecting Ω_1 around H . To estimate the seminorm

$$[u\mathbb{1}_{\Omega_1}]_{s,p}^p = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x)\mathbb{1}_{\Omega_1}(x) - u(y)\mathbb{1}_{\Omega_1}(y)|^p}{|x - y|^{d+sp}} dx dy, \quad (3.2.3)$$

we consider the integrand for $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$, taking x and y in each half-space given by the hyperplane H .

In the region $\Omega_1 \times \Omega_1$, the integrand of (3.2.3) will correspond to the integrand of $[u]_{s,p}$ in the same region. Moreover, on $\Omega_2 \times \Omega_2$ the integrand of (3.2.3) is zero. Hence we consider the case where x and y are in opposite regions, assuming without loss of generality that $x \in \Omega_1$ and $y \in \Omega_2$. Then we can apply (3.2.2) to the integrand of (3.2.3), with

$$\begin{aligned} \int_{\Omega_2} \int_{\Omega_1} \frac{|u(x)\mathbb{1}_{\Omega_1}(x) - u(y)\mathbb{1}_{\Omega_1}(y)|^p}{|x - y|^{d+sp}} dx dy &= \int_{\Omega_2} \int_{\Omega_1} \frac{|u(x)|^p}{|x - y|^{d+sp}} dx dy \\ &\leq \int_{\Omega_2} \int_{\Omega_1} \frac{|u(x) - u(y) - (u(\Pi(x)) - u(y))|^p}{|x - y|^{d+sp}} dx dy \\ &\leq 2^{p-1} \int_{\Omega_2} \int_{\Omega_1} \frac{|u(x) - u(y)|^p + |u(\Pi(x)) - u(y)|^p}{|x - y|^{d+sp}} dx dy. \end{aligned}$$

We can then estimate the second term by a change of variable,

$$\begin{aligned} \int_{\Omega_2} \int_{\Omega_1} \frac{|u(\Pi(x)) - u(y)|^p}{|x - y|^{d+sp}} dx dy &= \int_{\Omega_2} \int_{\Omega_2} \frac{|u(x) - u(y)|^p}{|\Pi(x) - y|^{d+sp}} dx dy \\ &\leq \int_{\Omega_2} \int_{\Omega_2} \frac{|u(x) - u(y)|^p}{|x - y|^{d+sp}} dx dy. \end{aligned}$$

Note that for the last inequality, choosing coordinates such that the hyperplane H is given by the set $\{x \in \mathbb{R}^d : x_1 = 0\}$, we have

$$|\Pi(x) - y|^2 = |x_1 + y_1|^2 + \sum_{i=2}^d |x_i - y_i|^2 \geq |x - y|^2 \quad (3.2.4)$$

when x_1 and y_1 have the same sign and hence when x and y are in the same half-space. Combining the integrals on each region, we have the desired estimate. $\square$

The next lemma is a typical key estimate for obtaining comparison principles as we will be able to transfer this estimate to the time derivative of two solutions. However, notably in this case, we restrict to the half-space Ω_1 which requires additional considerations compared to the full domain $\mathbb{R}^d$. For this lemma we note that we can approximate the positive part of the sign function sign_0^+ by a sequence $(q_M)_{M>0}$ where

$$q_M(s) = \begin{cases} 1 & \text{if } s \geq \frac{1}{M}, \\ Ms & \text{if } 0 < s < \frac{1}{M}, \\ 0 & \text{if } s \leq 0. \end{cases}$$

Lemma 3.2.5. *Let $0 < s < 1$, $p > 1$, $d \geq 1$ and $m > 0$. Let $\Pi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ denote reflection around a hyperplane H , splitting $\mathbb{R}^d$ into open subsets Ω_1 and Ω_2 . Suppose $u \in L^{1 \cap \infty}$ such that $u^m \in D((-\Delta_p)^s_{|L^{1 \cap \infty}})$ and let $\hat{u}(x) := u(\Pi(x))$ for all $x \in \mathbb{R}^d$. We consider $q \in C^1(\mathbb{R})$ satisfying $0 \leq q \leq 1$, $q(s) = 0$ for $s \leq 0$ and $0 < q'(s) < M$ for all $s > 0$ given $M > 0$. Then*

$$\int_{\Omega_1} ((-\Delta_p)^s u^m - (-\Delta_p)^s \hat{u}^m) q(u^m - \hat{u}^m) dx \geq 0. \quad (3.2.5)$$

In particular, one has that

$$\int_{\Omega_1} ((-\Delta_p)^s u^m - (-\Delta_p)^s \hat{u}^m) \mathbb{1}_{\{u > \hat{u}\}} dx \geq 0. \quad (3.2.6)$$

Proof. We let $v = u^m$, $\hat{v} = \hat{u}^m$. Note that

$$\begin{aligned} [q(v - \hat{v})]_{s,p} &= \frac{1}{p} \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|q(v(x) - \hat{v}(x)) - q(v(y) - \hat{v}(y))|^p}{|x - y|^{d+sp}} dx dy \right)^{\frac{1}{p}} \\ &\leq M \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|v(x) - \hat{v}(x) - (v(y) - \hat{v}(y))|^p}{|x - y|^{d+sp}} dx dy \right)^{\frac{1}{p}} \\ &\leq 2M[v]_{s,p}. \end{aligned}$$

Let $Q(x)$ denote $q(v(x) - \hat{v}(x))$ for $x \in \mathbb{R}^d$. Since $q(s) = 0$ for $s \leq 0$, we have either $q(v(x) - v(\Pi(x))) = 0$ or $q(v(\Pi(x)) - v(x)) = 0$, and so Q satisfies (3.2.2). Then by Lemma 3.2.4, $[Q \mathbb{1}_{\Omega_1}]_{s,p} < \infty$ and so we apply the variational formulation,

$$\begin{aligned} &\int_{\mathbb{R}^d} ((-\Delta_p)^s v(x) - (-\Delta_p)^s \hat{v}(x)) Q(x) \mathbb{1}_{\Omega_1} dx \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(v(x) - v(y))^{p-1} - (\hat{v}(x) - \hat{v}(y))^{p-1}}{|x - y|^{d+sp}} \times \\ &\quad (Q(x) \mathbb{1}_{\Omega_1}(x) - Q(y) \mathbb{1}_{\Omega_1}(y)) dy dx. \end{aligned}$$

To show that this integral is non-negative, consider x and y in the two regions Ω_1 and Ω_2 on either side of the hyperplane H . First note that in

$\Omega_1 \times \Omega_1$, we have

$$\int_{\Omega_1} \int_{\Omega_1} \frac{((v(x) - v(y))^{p-1} - (\hat{v}(x) - \hat{v}(y))^{p-1}) (Q(x) - Q(y))}{|x - y|^{d+sp}} dy dx. \quad (3.2.7)$$

Considering the sign of the integrand in (3.2.7) and noting that q is non-decreasing, we have that whenever $Q(x) \neq Q(y)$,

$$\begin{aligned} \text{sign}((v(x) - v(y))^{p-1} - (\hat{v}(x) - \hat{v}(y))^{p-1}) \\ &= \text{sign}(v(x) - \hat{v}(x) - (v(y) - \hat{v}(y))) \\ &= \text{sign}(Q(x) - Q(y)). \end{aligned}$$

So (3.2.7) is non-negative, while in the case of $\Omega_2 \times \Omega_2$, both indicator functions are zero. For the cross terms $\Omega_1 \times \Omega_2$ and $\Omega_2 \times \Omega_1$, we apply a change of variables, mapping y to $\Pi(y)$ and x to $\Pi(x)$, respectively. Note that $|x - \Pi(y)| = |\Pi(x) - y|$. For the sum of these terms, we have

$$\begin{aligned} &\int_{\Omega_1} \int_{\Omega_2} \frac{((v(x) - v(y))^{p-1} - (\hat{v}(x) - \hat{v}(y))^{p-1}) Q(x)}{|x - y|^{d+sp}} dy dx \\ &\quad - \int_{\Omega_2} \int_{\Omega_1} \frac{((v(x) - v(y))^{p-1} - (\hat{v}(x) - \hat{v}(y))^{p-1}) Q(y)}{|x - y|^{d+sp}} dy dx \\ &= \int_{\Omega_1} \int_{\Omega_1} \frac{((v(x) - \hat{v}(y))^{p-1} - (\hat{v}(x) - v(y))^{p-1}) (Q(x) + Q(y))}{|x - \Pi(y)|^{d+sp}} dy dx. \end{aligned}$$

If this is non-negative we are done, so consider the domain on which this integrand may be negative. In particular, since Q is only non-zero when $v - \hat{v} > 0$, this can only occur when $v(x) > \hat{v}(x)$ and $v(y) < \hat{v}(y)$ or when $v(y) > \hat{v}(y)$ and $v(x) < \hat{v}(x)$. By symmetry of x and y , we may consider only the first case. Let $\Omega_x = \Omega_1 \cap \{v(x) > \hat{v}(x)\}$ and $\Omega_y = \Omega_1 \cap \{v(y) < \hat{v}(y)\}$. Then for $(x, y) \in \Omega_x \times \Omega_y$, we have

$$\begin{aligned} &\int_{\Omega_x} \int_{\Omega_y} \frac{((v(x) - \hat{v}(y))^{p-1} - (\hat{v}(x) - v(y))^{p-1}) (Q(x) + Q(y))}{|x - \Pi(y)|^{d+sp}} dy dx \\ &\geq - \int_{\Omega_x} \int_{\Omega_y} \frac{((v(x) - v(y))^{p-1} - (\hat{v}(x) - \hat{v}(y))^{p-1}) (Q(x) - Q(y))}{|x - y|^{d+sp}} dy dx \end{aligned}$$

where we also use that $|x - \Pi(y)| \geq |x - y|$ for $x, y \in \Omega_1$ as in (3.2.4). Hence the sum of both cases are bounded by the original $\Omega_1 \times \Omega_1$ term, (3.2.7). So we have the desired non-negativity,

$$\int_{\mathbb{R}^d} ((-\Delta_p)^s v(x) - (-\Delta_p)^s \hat{v}(x)) q(v(x) - \hat{v}(x)) \mathbf{1}_{\Omega_1} dx \geq 0.$$

Letting q converge to sign_0^+ and noting that r^m is strictly increasing for $r \in \mathbb{R}$, we can replace v by u^m , giving (3.2.6). $\square$

We finally apply this half-space estimate to (1.2.1) and prove Theorem 3.2.1.

Proof of Theorem 3.2.1. Using the translation and rotational invariance of the operator $(-\Delta_p)^{s,m}$, we may assume that $\Omega_1 = \{x \in \mathbb{R}^d : x_1 > 0\}$. Denote $\hat{u}(x) := u(\Pi(x))$ for all $x \in \mathbb{R}^d$. First suppose that u is a strong distributional solution to (1.2.1) with initial data $u_0 \in L^1 \cap L^\infty$ and that $(q_M)_{M>0}$ converges to the positive indicator function $[\text{sign}_0]^+$. Then $\hat{u}(x, t) := u(\Pi(x), t)$ is also a strong distributional solution with initial data $\hat{u}_0$. By the chain rule and applying Lemma 3.2.5,

$$\begin{aligned} \frac{d}{dt} \int_{\Omega_1} (u(t) - \hat{u}(t))^+ dx &= \int_{\Omega_1} (u'(t) - \hat{u}'(t)) \mathbb{1}_{\{u > \hat{u}\}} dx \\ &= - \int_{\Omega_1} ((-\Delta_p)^s u^m(t) - (-\Delta_p)^s \hat{u}^m(t)) \mathbb{1}_{\{u > \hat{u}\}} dx \\ &\leq 0 \end{aligned}$$

noting that the composition $(-\Delta_p)^s u^m(t) \in L^1$. Then we have,

$$\int_{\Omega_1} (u(t) - \hat{u}(t))^+ dx \leq \int_{\Omega_1} (u_0 - \hat{u}_0)^+ dx. \quad (3.2.8)$$

We approximate L^1 initial data by functions $u_{0,k} \in L^1 \cap L^\infty$ such that $u_{0,k}^m \in D((-\Delta_p)^s_{1 \cap \infty})$. Then since these are strong distributional solutions, taking $k \rightarrow \infty$, we obtain (3.2.8) for mild solutions u with initial data $u_0 \in L^1$. Applying (3.2.8) when $u_0(x) \leq u_0(\Pi(x))$ then gives the result. $\square$

3.3 Barrier construction

In this section, we produce global barriers for solutions u to (1.2.1) with initial data which is bounded with compact support. These barrier functions are radially symmetric, decreasing in x and have sufficient decay at infinity to be integrable in space for all $t \geq 0$. Moreover, we will use these barrier functions to construct Barenblatt solutions to (1.2.1). Such barrier functions have been proven for the fractional p -Laplacian case with $m = 1$ in [121] and [126]. We apply the same methods for more general $m > s$.

We show the existence of such global barriers to (1.2.1) in the range $0 < s < 1$, $m > s$ and $p > p_c$, defined by (3.0.1), corresponding to the homogeneity condition

$$m(p-1) > 1 - \frac{sp}{d}.$$

In particular, this matches the range of the self-similar scaling transformation (3.1.1) and associated self-similar solutions (3.0.2) with $\beta > 0$ defined by (3.0.3).

When considering these barriers, we split the range of p into three regions. Define p_1 to be the positive solution to

$$\frac{msp_1(p_1-1)}{1-m(p_1-1)} = d,$$

corresponding to the homogeneity condition

$$m(p_1 - 1) = \frac{d}{d + sp_1}. \quad (3.3.1)$$

Note that $1 < p_c < p_1 < 1 + \frac{1}{m}$.

We separately consider the upper region where $p > p_1$, the critical case $p = p_1$ and the lower sublinear region $p_c < p < p_1$. We find barriers with different rates of decay at infinity in each region. Hence the critical case p_1 is the transition point between the decay regimes of $|z|^{-\frac{sp}{1-m(p-1)}}$ (for $p < p_1$) and $|z|^{-(d+sp)}$ (for $p > p_1$) since

$$d + sp_1 = \frac{sp_1}{1 - m(p_1 - 1)}.$$

We first define the decay function g by

$$g(r) := \begin{cases} r^{-d-sp} & \text{if } p > p_1, \\ r^{-d-sp} \log(r) & \text{if } p = p_1, \\ r^{-\frac{sp}{1-m(p-1)}} & \text{if } p_c < p < p_1. \end{cases} \quad (3.3.2)$$

for all $r > 0$. Then for positive constants A, C_1, C_2, R_1 and R_2 , we introduce the barrier function $H : \mathbb{R}^d \times [0, \infty) \rightarrow [0, \infty)$ in the following way. For $p \geq p_1$,

$$H(x, t) = \begin{cases} A(t+1)^{-d\beta} & \text{if } |x(t+1)^{-\beta}| \leq R_1, \\ C_1|x|^{-d} & \text{if } R_1 < |x(t+1)^{-\beta}| \leq R_2, \\ C_2(t+1)^{-d\beta} g(|x|(t+1)^{-\beta}) & \text{if } |x(t+1)^{-\beta}| > R_2. \end{cases} \quad (3.3.3)$$

For $p_c < p < p_1$,

$$H(x, t) = \begin{cases} A(t+1)^{-d\beta} & \text{if } |x(t+1)^{-\beta}| \leq R_2, \\ C_2(t+1)^{-d\beta} g(|x|(t+1)^{-\beta}) & \text{if } |x(t+1)^{-\beta}| > R_2. \end{cases} \quad (3.3.4)$$

We then have existence of barrier functions with the following theorem.

Theorem 3.3.1 (Global barrier). *Let $d \geq 1$, $0 < s < 1$, $m > s$ and $p > 1$. Suppose u is a mild solution to (3.2.1) with initial data $u_0 \in L^1(\mathbb{R}^d)$ which is bounded and has compact support. Define $H : \mathbb{R}^d \times [0, \infty) \rightarrow [0, \infty)$ by (3.3.3) or (3.3.4) as appropriate. Then there exist positive constants A, C_1, C_2, R_1 and R_2 such that $|u(t)| \leq H(t)$ a.e. in $\mathbb{R}^d$ for $t \geq 0$.*

To prove this we apply the methods presented in [121] in the superlinear case and [126] in the sublinear case. The proofs follow essentially by taking $\hat{p} := m(p-1) + 1$, $\hat{s} := \frac{sp}{\hat{p}}$ and applying the relevant calculations to $(-\Delta_{\hat{p}})^{\hat{s}}$ in place of $(-\Delta_p)^{s \cdot m}$. We can apply such a replacement despite not being able to directly combine the exponent m with $p-1$

due to the nature of the estimates required for these barrier functions. In particular, focusing on the spatial coordinates, for a barrier function H and a point $y_0 \in \mathbb{R}^d$, we will only need to estimate $(-\Delta_p)^s H(y_0)$ from below. So we can typically ignore the first term of the difference, given that H is non-negative, i.e.

$$(H(y_0)^m - H(y)^m)^{p-1} \geq -H(y)^{m(p-1)}.$$

Meanwhile the issue of $\hat{s} \geq 1$ can be resolved by instead applying the smoothness of H^m and using the associated boundedness properties of $(-\Delta_p)^s H^m$, in particular those given by [85].

The outline in each case will be to find super-solutions to the self-similar profile equation (3.1.4), rescale these to obtain super-solutions to (1.2.1) and then apply a comparison principle on $\mathbb{R}^d \times [0, \infty)$.

3.3.1 Super-solutions to the profile equation

To obtain pointwise estimates of (3.1.4), we use the Cauchy principal value of $(-\Delta_p)^s$ which will agree with Definition 2.0.1 when both exist but will not necessarily have the same domain. In particular, we introduce the following approximation for $\varepsilon > 0$,

$$(-\Delta_p)_\varepsilon^s u(z) := \int_{\mathbb{R}^d \setminus B_\varepsilon(z)} \frac{(u(z) - u(y))^{p-1}}{|z - y|^{d+sp}} dy \quad (3.3.5)$$

for all $z \in \mathbb{R}^d$. Then $(-\Delta_p)_\varepsilon^s$ converges to the principal value as $\varepsilon \rightarrow 0$.

The following lemma provides a global bound with decay of order $|x|^{-d}$. The idea is to refine this estimate for large $|x|$ such that the global barrier is integrable in space.

Lemma 3.3.2. *Let $d \geq 1$, $p > 1$, $0 < s < 1$ and $m > s$. Suppose u is a mild solution to (3.2.1) with initial data $u_0 \in L^1(\mathbb{R}^d)$ which is bounded with compact support. Then for $C > 0$ sufficiently large, depending on d and u_0 , we have*

$$|u(x, t)| \leq C|x|^{-d} \quad (3.3.6)$$

for all $x \in \mathbb{R}^d$ and $t > 0$.

Proof. Given a ball B centered at the origin and containing the support of u_0 , we bound $|u_0|$ pointwise a.e. by a smooth, radially symmetric and radially decreasing function $\hat{u}_0$ with compact support defined such that

$$\hat{u}_0(x) = \|u_0\|_\infty$$

for all $x \in B$. We let $\hat{u}(x, t)$ be the associated mild solution to (3.2.1) in L^1 . Since $\hat{u}_0$ is radially symmetric and radially decreasing, by Corollary 3.2.3, $\hat{u}(x, t)$ is radially decreasing in $\mathbb{R}^d$ for all $t \geq 0$. Since $\hat{u}(x, t)$ is

monotone decreasing in the radial variable, there exists $M > 0$, depending only on d , such that

$$\hat{u}(x, t)|x|^d \leq M \int_{\{y \in \mathbb{R}^d: |y| < |x|\}} \hat{u}(y, t) dy \leq M \|\hat{u}_0\|_1$$

for a.e. $x \in \mathbb{R}^d$, where we have used the standard accretive growth estimate in $L^1(\mathbb{R}^d)$. Hence $\hat{u}(x, t) \leq C|x|^{-d}$ in $\mathbb{R}^d \times [0, \infty)$ for some $C > 0$ depending on d and u_0 . Similarly, we have that $-\hat{u} \geq -C|x|^{-d}$. So comparing u , $-u$ with $\hat{u}$ and $-\hat{u}$ by (2.1.3), we have (3.3.6). $\square$

For $p \geq p_1$ we choose a profile barrier function such that the intermediate region will be bounded by Lemma 3.3.2. We choose

$$G(z) = \begin{cases} A & \text{if } |z| \leq R_1, \\ C_1|z|^{-d} & \text{if } R_1 < |z| \leq R_2, \\ C_2g(|z|) & \text{if } |z| > R_2, \end{cases} \quad (3.3.7)$$

where $g(r)$ is defined by (3.3.2). In particular, g is smooth, decreasing and such that $G(z)$ is integrable at infinity. We glue the separate regions by matching constants. In particular, we require that

$$A = C_1 R_1^{-d} \quad \text{and} \quad C_2 = \frac{C_1}{R_2^d g(R_2)}. \quad (3.3.8)$$

We then require the super-solution condition in the remaining regions. We first consider the near region since this is independent of g .

Lemma 3.3.3 (Near region for $p > 1$). *Let $d \geq 1$, $p > 1$, $0 < s < 1$ and $m > 0$. Define $G : \mathbb{R}^d \rightarrow [0, \infty)$ by (3.3.7) where the positive constants C_1 and R_1 are chosen to satisfy (3.3.11), $R_2 \geq R_1$ and A, C_2 are given by (3.3.8). Then*

$$(-\Delta_p)_\varepsilon^s G(z)^m - \beta \nabla \cdot (zG(z)) \geq 0 \quad (3.3.9)$$

for all $\varepsilon > 0$ sufficiently small and all $z \in \mathbb{R}^d$ with $|z| \leq R_1$.

Proof. We have that

$$-\beta \nabla \cdot (zG(z)) = -\beta dA \quad (3.3.10)$$

for $|z| \leq R_1$. Since the integrand is non-negative,

$$(-\Delta_p)_\varepsilon^s G(z)^m \geq \int_{\{|z-y| \geq 3R_1\} \cap \{|y| \geq 2R_1\}} \frac{(A^m - G(y)^m)^{p-1}}{|z-y|^{d+sp}} dy$$

for all $0 < \varepsilon < R_1$ and all $z \in \mathbb{R}^d$ satisfying $|z| \leq R_1$. Then using (3.3.8),

$$(A^m - G(y)^m)^{p-1} \geq A^{m(p-1)}(1 - 2^{-dm})^{p-1}.$$

Integrating and relabelling $C > 0$ as needed,

$$\begin{aligned} (-\Delta_p)_\varepsilon^s G(z)^m &\geq CA^{m(p-1)} \int_{3R_1}^\infty r^{-sp-1} dr \\ &= CA^{m(p-1)} R_1^{-sp}. \end{aligned}$$

So to satisfy the super-solution equation (3.3.9) in this region, comparing with (3.3.10), relabelling C and rearranging, we want to satisfy

$$A^{m(p-1)-1} \geq CR_1^{sp}$$

where $C > 0$ depends on s, p, d and m . Applying (3.3.8) gives

$$C_1^{m(p-1)-1} \geq CR_1^{d(m(p-1)-1)+sp}. \quad (3.3.11)$$

□

We now prove the super-solution condition in the most involved region, the far region. For this we first consider the case $p > p_1$, defining

$$G(z) = \begin{cases} A & \text{if } |z| \leq R_1, \\ C_1|z|^{-d} & \text{if } R_1 < |z| \leq R_2, \\ C_2|z|^{-(d+sp)} & \text{if } |z| > R_2, \end{cases} \quad (3.3.12)$$

with matching conditions

$$A = C_1 R_1^{-d} \quad \text{and} \quad C_2 = C_1 R_2^{sp}. \quad (3.3.13)$$

Here we will make room around the singularity in (3.3.5) at $y = z$ by considering $|z| \geq 2R_2$. The intermediate region where $G(z) = C_1|z|^{-d}$ is important to be able to do this.

Lemma 3.3.4 (The far region for $p > p_1$). *Let $d \geq 1$, $0 < s < 1$, $m > 0$ and $p > p_1$. Define $G : \mathbb{R}^d \rightarrow [0, \infty)$ by (3.3.12) where positive constants C_1, R_1, R_2 satisfy (3.3.15) and A, C_2 are given by (3.3.13). Then*

$$(-\Delta_p)_\varepsilon^s G(z)^m - \beta \nabla \cdot (zG(z)) \geq 0 \quad (3.3.9)$$

for all $\varepsilon > 0$ sufficiently small and all $z \in \mathbb{R}^d$ with $|z| \geq 2R_2$.

Proof. Fix $z \in \mathbb{R}^d$ with $|z| \geq 2R_2$. Applying (3.3.13), we have that

$$-\beta \nabla \cdot (zG(z)) = \beta sp C_1 R_2^{sp} |z|^{-d-sp} > 0 \quad (3.3.14)$$

which we will use to control the fractional p -Laplacian term. Hence we estimate $(-\Delta_p)_\varepsilon^s G(z)^m$ by separating (3.3.5) into regions corresponding to G .

In the region $\{y \in \mathbb{R}^d : |y| \leq R_1\}$, noting that $G \geq 0$ on $\mathbb{R}^d$ and $|z| \leq 2|z - y|$, we have

$$\begin{aligned} \int_{\{y \in \mathbb{R}^d : |y| \leq R_1\}} \frac{(G(z)^m - G(y)^m)^{p-1}}{|z - y|^{d+sp}} dy &\geq -C R_1^d A^{m(p-1)} |z|^{-d-sp} \\ &= -C C_1^{m(p-1)} R_1^{-d(m(p-1)-1)} |z|^{-d-sp} \end{aligned}$$

for some $C > 0$ depending on d .

In the region $\{y \in \mathbb{R}^d : R_1 < |y| \leq R_2\}$, relabelling the constant C as needed, we similarly have

$$\begin{aligned} \int_{\{R_1 < |y| \leq R_2\}} \frac{(G(z)^m - G(y)^m)^{p-1}}{|z - y|^{d+sp}} dy &\geq -C \int_{R_1}^{R_2} \frac{(C_1 r^{-d})^{m(p-1)}}{(|z|/2)^{d+sp}} r^{d-1} dr \\ &= -C C_1^{m(p-1)} \int_{R_1}^{R_2} r^{-dm(p-1)+d-1} dr |z|^{-d-sp} \end{aligned}$$

where $C > 0$ depends on d, s, p and m .

In the region $\{y \in \mathbb{R}^d : |y| > |z|\}$, the integrand will be positive so this can be ignored. The intermediate region $\{y \in \mathbb{R}^d : R_2 \leq |y| \leq |z|\}$ contains the singularity at $y = z$. Hence we further split this region by integrating the ball $B_{|z|/2}(z)$ centered at z separately.

We treat the annulus region $D = \{y \in \mathbb{R}^d : R_2 \leq |y| \leq |z|\} \setminus B_{|z|/2}(z)$ as in the previous cases, noting that $|z - y| \geq |z|/2$. Then again relabelling C ,

$$\begin{aligned} \int_D \frac{(G(z)^m - G(y)^m)^{p-1}}{|z - y|^{d+sp}} dy &\geq - \int_D \frac{(C_2 |y|^{-(d+sp)})^{m(p-1)}}{(|z|/2)^{d+sp}} dy \\ &\geq -C C_1^{m(p-1)} R_2^{-d(m(p-1)-1)} |z|^{-d-sp} \end{aligned}$$

since $(d + sp)m(p - 1) > d$ by assumption (3.3.1), where $C > 0$ depends on d, s, p and m . In particular, (3.3.1) ensures that this and the following estimate on $B_{|z|/2}(z)$ do not grow relative to (3.3.14) as $|z| \rightarrow \infty$.

Finally, we consider the ball centered around z , $B_{|z|/2}(z)$. By [85, Lemma 3.6], since G^m is C^2 without critical points in $B_{|z|/2}(z)$, the integral

$$\int_{B_{|z|/2}(z) \setminus B_\varepsilon(z)} \frac{(G(z)^m - G(y)^m)^{p-1}}{|z - y|^{d+sp}} dy$$

is bounded independently of $\varepsilon > 0$. To determine the dependence on $|z|$, we apply a rescaling, finding that

$$\begin{aligned} \left| \int_{B_{|z|/2}(z) \setminus B_\varepsilon(z)} \frac{(G(z)^m - G(y)^m)^{p-1}}{|z - y|^{d+sp}} dy \right| &\leq C C_2^{m(p-1)} |z|^{-(d+sp)m(p-1)-sp} \\ &\leq C C_1^{m(p-1)} R_2^{-d(m(p-1)-1)} |z|^{-(d+sp)}. \end{aligned}$$

where $C > 0$ depends on d, s, p and m , again using that $(d + sp)m(p - 1) > d$ by assumption. Note that this matches the previous estimate.

For $G(z)$ to be a super-solution we require that (3.3.14) bounds all these terms together for all $|z| \geq 2R_2$. We then multiply the estimate in each region by $|z|^{d+sp}$ and apply (3.3.8) to R_2 to reduce variables. Then the super-solution condition (3.3.9) holds given that

$$\begin{aligned} C_1^{1-m(p-1)} &\geq C R_1^{-d(m(p-1)-1)} R_2^{-sp}, \\ C_1^{1-m(p-1)} &\geq C \int_{R_1}^{R_2} r^{-d(m(p-1)-1)-1} dr R_2^{-sp}, \\ C_1^{1-m(p-1)} &\geq C R_2^{-d(m(p-1)-1)-sp} \end{aligned} \quad (3.3.15)$$

for $C > 0$ depending on d, s, p and m . $\square$

We prove the critical case $p = p_1$ very similarly to the case $p > p_1$, but now with a correction term. Define $G : \mathbb{R}^d \rightarrow [0, \infty)$ by

$$G(z) = \begin{cases} A & \text{if } |z| \leq R_1, \\ C_1 |z|^{-d} & \text{if } R_1 < |z| \leq R_2, \\ C_2 |z|^{-(d+sp_1)} \log(|z|)^\gamma & \text{if } |z| > R_2, \end{cases} \quad (3.3.16)$$

for all $z \in \mathbb{R}^d$ where $\gamma = \frac{1}{1-m(p-1)}$. We then have the matching conditions,

$$A = C_1 R_1^{-d} \quad \text{and} \quad C_2 = \frac{C_1 R_2^{sp_1}}{\log(R_2)^\gamma}. \quad (3.3.17)$$

Lemma 3.3.5 (The far region for $p = p_1$). *Let $d \geq 1$, $0 < s < 1$, $m > 0$ and $p = p_1$. Define $G : \mathbb{R}^d \rightarrow [0, \infty)$ by (3.3.16) where positive constants C_1, R_1, R_2 satisfy (3.3.18) and A, C_2 are given by (3.3.17). Then*

$$(-\Delta_p)_\varepsilon^s G(z)^m - \beta \nabla \cdot (z G(z)) \geq 0 \quad (3.3.9)$$

for all $\varepsilon > 0$ sufficiently small and all $z \in \mathbb{R}^d$ with $|z| > \min \{ 2R_2, 2e^{\frac{\gamma}{sp_1}} \}$.

Proof. In radial coordinates with $r = |z|$,

$$-\beta r^{1-d} (r^d G(y))_r = C_2 \beta \left(sp_1 - \frac{\gamma}{\log(r)} \right) r^{-(d+sp_1)} \log(r)^\gamma.$$

For $r > 2e^{\frac{\gamma}{sp_1}}$, we have

$$-\beta r^{1-d} (r^d G(z))_r \geq \frac{C C_1 R_2^{sp_1}}{\log(R_2)^\gamma} |z|^{-(d+sp_1)} \log(|z|)^\gamma$$

for some $C > 0$ depending on m, s and d . As usual we now estimate the fractional p -Laplacian term $(-\Delta_{p_1})_\varepsilon^s G(z)^m$ for $z \in \mathbb{R}^d$ with $|z| > \min \{ 2R_2, 2e^{\frac{\gamma}{sp_1}} \}$. These estimates are done similarly to those in Lemma 3.3.4. For $\{y \in \mathbb{R}^d : |y| \leq R_2 \text{ or } |y| \geq |z|\}$, the calculations are the same.

However, in the intermediate region $\{y \in \mathbb{R}^d : R_2 \leq |y| \leq |z|\}$ we must account for the new decay. For the annulus region

$$D := \{y \in \mathbb{R}^d : R_2 \leq |y| \leq |z|\} \setminus B_{|z|/2}(z),$$

we integrate by parts to obtain,

$$\int_D \frac{(G(z)^m - G(y)^m)^{p_1-1}}{|z-y|^{d+sp_1}} dy \geq -CC_2^{m(p_1-1)} |z|^{-d-sp} \log(|z|)^\gamma$$

since $(d+sp_1)m(p_1-1) = d$, where $C > 0$ depends on d, s and m .

For the ball $B_{|z|/2}(z)$, we again apply [85, Lemma 3.6] and a rescaling, finding that

$$\left| \int_{B_{|z|/2}(z) \setminus B_\varepsilon(z)} \frac{(G(z)^m - G(y)^m)^{p-1}}{|z-y|^{d+sp}} dy \right| \leq CC_2^{m(p-1)} |z|^{-d-sp} \log(|z|)^{\gamma-1}$$

again noting that $(d+sp)m(p_1-1) = d$.

For $G(z)$ to be a super-solution we require that (3.3.14) bounds all these terms together for all $|z| \geq 2R_2$. Then, relabelling $C > 0$ as needed, we can multiply each condition by $|z|^{d+sp_1} \log(|z|)^{-\gamma}$ and rearrange to obtain the conditions

$$C_1 \geq CR_2^{d-sp_1\gamma}, \quad C_1 \geq C \log(R_2)^\gamma R_2^{-sp_1}, \quad C_1 \geq CR_2^{-sp_1}. \quad (3.3.18)$$

□

We now consider the lower sublinear case $p_c < p < p_1$. Here we use a barrier function of the form

$$G(z) = \begin{cases} A & \text{if } |z| \leq R_2, \\ C_2 |z|^{-\frac{sp}{1-m(p-1)}} & \text{if } |z| > R_2, \end{cases} \quad (3.3.19)$$

for $z \in \mathbb{R}^d$ with the matching condition

$$A = C_2 R_2^{-\frac{sp}{1-m(p-1)}}. \quad (3.3.20)$$

Note that $p > p_c$ ensures that G is integrable at infinity.

Lemma 3.3.6 (The far region for $p_c < p < p_1$). *Let $d \geq 1$, $0 < s < 1$, $m > 0$ and $p_c < p < p_1$. Define $G : \mathbb{R}^d \rightarrow [0, \infty)$ by (3.3.19) where $C_2 > 0$ satisfies (3.3.21), $R_2 > 0$ and A is given by (3.3.20). Then*

$$(-\Delta_p)_\varepsilon^s G(z)^m - \beta \nabla \cdot (zG(z)) \geq 0 \quad (3.3.9)$$

for all $\varepsilon > 0$ sufficiently small and all $z \in \mathbb{R}^d$ with $|z| > R_2$.

Proof. In radial coordinates with $r = |z|$,

$$-\beta r^{1-d}(r^d G(y))_r = \frac{C_2}{1-m(p-1)} r^{-\frac{sp}{1-m(p-1)}} > 0.$$

We use this to compensate for the (possibly negative) fractional p -Laplacian term. Evaluating $(-\Delta_p)_\varepsilon^s G^m$, first consider $y_0 \in \mathbb{R}^d$ such that $|y_0| = 1$. Then,

$$(-\Delta_p)_\varepsilon^s G(y_0)^m = C_2^{m(p-1)} \int_{\mathbb{R}^d \setminus B_\varepsilon(y_0)} \frac{\left(1 - r^{-\frac{spm}{1-m(p-1)}}\right)^{p-1}}{|y_0 - y|^{d+sp}} dy.$$

The integrand is positive for $r > 1$, so this region can be ignored. For $p < p_1$, $G^{m(p-1)} \in L^1(B_1(0))$ so the integrand is also bounded in L^1 near $y = 0$. Hence it remains to estimate near $y = y_0$. Since $G(y)^m$ is a C^2 function without critical points, we can also apply [85] so that the principal value is bounded in a small ball around y_0 with bound independent of y_0 . So as in [126], there is a finite constant $k(d, s, p, m)$ such that

$$(-\Delta_p)^s G(y_0)^m = -k C_2^{m(p-1)}.$$

We evaluate for $|y_0| \neq 1$ by the spatial scaling transformation

$$v_h(y) := h^\gamma v(hy)$$

for $h > 0$ with $\gamma = \frac{sp}{1-m(p-1)}$. This leaves the profile bound (3.3.19) invariant and scales the profile equation (3.1.5) by $r^{-\gamma}$ such that

$$(-\Delta_p)^s G(y)^m - \beta r^{1-d}(r^d G(y))_r = \left(\frac{C_2}{1-m(p-1)} - k C_2^{m(p-1)} \right) r^{-\gamma}.$$

Then we satisfy the super-solution condition (3.3.9) for

$$C_2 \geq (k(1-m(p-1)))^{1/(1-m(p-1))}. \quad (3.3.21)$$

□

3.3.2 A comparison principle

We now convert profile functions back to the standard coordinates of (1.2.1) and apply a comparison principle.

Lemma 3.3.7 (Comparison principle). *Let $0 < s < 1$, $p > 1$, $m > s$ and $d \geq 1$. Let u be a mild solution to (1.2.1) with initial data $u_0 \in L^1$. Suppose $H : \mathbb{R}^d \times [0, \infty) \rightarrow \mathbb{R}$ satisfies the super-solution condition*

$$(H'(t) + (-\Delta_p)_\varepsilon^s H^m(t)) \mathbf{1}_{\{u > H\}} \geq 0 \quad (3.3.22)$$

a.e. in $\mathbb{R}^d$ for all $t > 0$ and all $\varepsilon > 0$ sufficiently small. Suppose $|u_0| \leq H(0)$ a.e. in $\mathbb{R}^d$. Then for all $t \geq 0$, $u(t) \leq H(t)$ almost everywhere in $\mathbb{R}^d$.

Proof. By Theorem 2.1.1, we can approximate u_0 by smooth data to get strong distributional solutions converging to u in $L^1(\mathbb{R}^d)$. Hence first suppose that u is a strong distributional solution of (1.2.1).

As usual we estimate the signed difference, noting that H is piecewise smooth,

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^d} (u(t) - H(t))^+ dx &= \int_{\mathbb{R}^d} (u'(t) - H'(t)) \mathbb{1}_{\{u>H\}} dx \\ &\leq - \int_{\mathbb{R}^d} ((-\Delta_p)^s u(t) + (-\Delta_p)_\varepsilon^s H^m(t)) \mathbb{1}_{\{u>H\}} dx. \end{aligned}$$

By the accretivity of $(-\Delta_p)^{s,m}$ in L^1 and using that $\cdot^m$ is strictly increasing so that $\mathbb{1}_{\{u^m>H^m\}} = \mathbb{1}_{\{u>H\}}$, we have the monotonicity estimate

$$\int_{\mathbb{R}^d} ((-\Delta_p)^s u^m(t) - (-\Delta_p)^s H^m(t)) \mathbb{1}_{\{u>H\}} dx \geq 0.$$

Hence it remains to prove the convergence of $(-\Delta_p)_\varepsilon^s H^m(t)$. We can approximate $\mathbb{1}_{u>H}$ by $q(u - H)$ for $q \in C^1(\mathbb{R})$ as usual with $0 \leq q \leq 1$, $q(r) = 0$ for $r \leq 0$, $0 < q'(r) < M$ for some $M > 0$ for all $r > 0$. Then letting $Q(t) = q(u(t) - H(t))$ for $0 < t < \infty$ and noting that $H \in W^{s,(2,p)}$ and $Q \in W^{s,(2,p)}$, we can use the symmetry of the fractional p-Laplacian, fixing $t > 0$, to obtain

$$\begin{aligned} \int_{\mathbb{R}^d} ((-\Delta_p)_\varepsilon^s H^m) Q dx &= \int_{\mathbb{R}^{2d} \setminus \{|z-y|<\varepsilon\}} \frac{(H^m(z) - H^m(y))^{p-1} Q(z)}{|z-y|^{d+sp}} d(y, z) \\ &= \frac{1}{2} \int_{\mathbb{R}^{2d} \setminus \{|z-y|<\varepsilon\}} \frac{(H^m(z) - H^m(y))^{p-1} (Q(z) - Q(y))}{|z-y|^{d+sp}} d(y, z) \end{aligned}$$

which converges to $((-\Delta_p)_\varepsilon^s H^m) Q$ as $\varepsilon \rightarrow 0$. Taking q to approximate sign_0^+ and integrating the previous difference estimate, we obtain,

$$\int_{\mathbb{R}^d} (u(t) - H(t))^+ dx \leq \int_{\mathbb{R}^d} (u(0) - H(0))^+ dx.$$

Similarly, comparing $-u$ and $-H$ and noting that $-H$ will be a subsolution to (1.2.1), we have that $|u(t)| \leq H(t)$ a.e. on $\mathbb{R}^d$ for $t \geq 0$ so long as $|u_0| \leq H(0)$. $\square$

Proof of Theorem 3.3.1. Converting the barriers G given by (3.3.12) for $p > p_1$, (3.3.16) for $p = p_1$ and (3.3.19) for $p_c < p < p_1$ back to (x, t) coordinates gives us the barrier function

$$H(x, t) = (t + 1)^{-d\beta} G(x(t + 1)^{-\beta})$$

defined by (3.3.3). Note that $H(x, 0) = G(x)$.

Hence in light of Lemma 3.3.7 and the previous pointwise estimates, we need only prove that such constants can be chosen for G in each case

so that H bounds $|u|$ at $t = 0$ and the super-solution condition (3.3.9) is satisfied whenever $u > H$.

First, in the case $p > p_1$, we want to satisfy the conditions of Lemma 3.3.3 and Lemma 3.3.4 while also ensuring that H bounds u elsewhere. In particular, this requires (3.3.11), (3.3.15) and $C_1 \geq C$ for some $C > 0$ with this last inequality coming from the pointwise bound in the intermediate region $R_1 \leq |z| \leq 2R_2$. For this we consider cases on the homogeneity of the operator $(-\Delta_p)^{s,m}$.

If $m(p-1) > 1$, we require

$$C_1 \geq CR_1^{d+\frac{sp}{m(p-1)-1}}, \quad C_1 \leq CR_1^d R_2^{\frac{sp}{m(p-1)-1}}, \quad C_1 \geq C$$

and so we can take R_1 large enough to contain the support of u_0 , C_1 large enough to satisfy the first and last inequalities while also ensuring that $A \geq \|u_0\|_\infty$ and R_2 large enough for the remaining inequality.

In the case $m(p-1) = 1$, we require

$$R_1^{sp} \leq C, \quad R_2^{sp} \geq C, \quad R_2^{sp} \geq C \log(R_2), \quad C_1 \geq C.$$

So we can take R_1 sufficiently small and R_2 sufficiently large to satisfy the first three inequalities. Then by taking C_1 large enough, we can ensure that $H(0)$ bounds $|u_0|$.

In the case $m(p-1) < 1$, we require

$$C_1 R_1^{\frac{sp}{1-m(p-1)}-d} \leq C, \quad C_1 R_2^{\frac{sp}{1-m(p-1)}-d} \geq C, \quad C_1 \geq C.$$

Note that $p_c < p < 1 + \frac{1}{m}$ implies that $\frac{sp}{1-m(p-1)} > d$. So fix $R_2 > 0$ such that B_{R_2} contains the support of u_0 . Then take C_1 large enough to satisfy the last two inequalities and ensure that H bounds u_0 at $t = 0$. Finally take R_1 small enough to satisfy the first inequality.

In the case $p = p_1$, we require (3.3.11), (3.3.18) and $C_1 \geq C$ for some $C > 0$. As in the previous case, we can take R_2 large, C_1 large and R_1 small to satisfy all conditions.

Finally, in the case $p_c < p < p_1$, we require (3.3.21) and, by applying Lemma 3.3.3 with $R_1 = R_2$,

$$C_2 R_2^{\frac{sp}{1-m(p-1)}-d} \leq C,$$

so we can satisfy all conditions by taking R_2 small and C_2 large. $\square$

3.4 Finite time of extinction

In the porous medium case, $\varphi(u) = u^m$ with $0 < m < 1$, we can obtain extinction in finite time following the method presented for the fractional Laplacian case in [83]. We first prove a comparison principle for the doubly nonlinear problem (1.2.1), extending [22, Theorem 4.1] to inhomogeneous boundary data on $\mathbb{R}^d$. By constructing an explicit supersolution and subsolution on $\mathbb{R}^N$ we can then prove extinction in finite time.

Theorem 3.4.1 (Finite time of extinction). *Let Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 1$, $p > 1$ and $0 < s < 1$. Let u be a strong distributional solution to (1.2.1) with $u_0 \in L^\infty$ where φ is strictly increasing, $\varphi(\mathbb{R}) = \mathbb{R}$, $\varphi(0) = 0$ and $\frac{1}{\varphi^{p-1}} \in L^1(0, \|u_0\|_\infty)$, $f \equiv 0$ and $g \equiv 0$. Then $u(t, \cdot) = 0$ for all $t \geq t^*$ where t^* is given by*

$$t^* = \frac{1}{C} \int_0^{\|u_0\|_\infty} \frac{1}{(\varphi(\tau))^{p-1}} d\tau \quad (3.4.1)$$

and C , given by (3.4.7), depends on Ω , p , s and d .

We have the following corollary in the case $\varphi(r) = r^m$, $r \in \mathbb{R}$.

Corollary 3.4.2. *Let Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 1$, $p > 1$ and $0 < s < 1$. Let u be a strong distributional solution to (1.2.2) with $f \equiv 0$, $g \equiv 0$, $u_0 \in L^\infty$ and $m > 0$ such that $m(p-1) < 1$. Then $u(t, \cdot) = 0$ for all $t \geq t^*$ where t^* is given by*

$$t^* = \frac{1}{C(1-m(p-1))} \|u_0\|_\infty^{1-m(p-1)}$$

and C , given by (3.4.7), depends on Ω , p , s and d .

In the case $m > s$ we may extend this result to mild solutions in L^1 with initial data in $L^{1 \cap \infty}$ by approximation due to the density result (2.1.1) and the standard growth estimate (2.1.3).

Finite time of extinction of solutions to (1.2.1) was also proved for the fractional porous medium equation ($p = 2$) in [83] in the Dirichlet case and [100] for the Cauchy problem. See also [123, 126] for discussion of extinction for the fractional p -Laplacian evolution problem on $\mathbb{R}^d$.

3.4.1 A parabolic comparison principle

We first introduce the inhomogeneous fractional Sobolev space for Ω an open domain in $\mathbb{R}^d$, $d \geq 1$, and $b \in L^1_{\text{loc}}(\mathbb{R}^d \setminus \Omega)$,

$$W_b^{s,(2,p)}(\Omega) = \left\{ u \in W^{s,(2,p)}(\mathbb{R}^d) \mid u = b \text{ a.e. on } \mathbb{R}^d \setminus \Omega \right\}$$

and the fractional p -Laplacian for $u \in W_b^{s,(2,p)}(\Omega)$. In this setting we have the energy functional $\mathcal{E} : L^2(\Omega) \rightarrow (-\infty, \infty]$ defined by

$$\mathcal{E}(u) = \begin{cases} \frac{1}{2p} [u]_{s,p}^p & \text{if } u \in W_b^{s,(2,p)}(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

so that the fractional p -Laplacian is given by the subdifferential operator of $\mathcal{E}$ in L^2 . In particular, we use the variational equation,

$$\frac{1}{2} \int_{\mathbb{R}^{2d}} \frac{(u(x) - u(y))^{p-1} (v(x) - v(y))}{|x - y|^{d+sp}} dy dx = \int_{\Omega} h(x) v(x) dx \quad (3.4.2)$$

so that we have the characterization,

$$\partial\mathcal{E}(u) = \{h \in L^2 : u, h \text{ satisfy (3.4.2) for all } v \in L^2 \text{ with } [v]_{s,p} < \infty\}.$$

For every $u \in W_b^{s,(2,p)}(\Omega)$ this is then unique, so we write $(-\Delta_p)^s u = \partial\mathcal{E}(u)$.

In this setting we consider the following inhomogeneous evolution equation,

$$\begin{cases} u_t + (-\Delta_p)^s \varphi(u) + f(\cdot, u) = g & \text{on } \Omega \times [0, T], \\ u(t) = h(t) & \text{on } \mathbb{R}^d \setminus \Omega \times [0, T], \\ u(0) = u_0 & \text{on } \Omega. \end{cases} \quad (3.4.3)$$

In particular, we have the comparison principle.

Theorem 3.4.3 (Comparison principle for inhomogeneous boundary data). *Let Ω be an open domain in $\mathbb{R}^d$, $d \geq 1$, $0 < s < 1$, $p > 1$, $T > 0$, f satisfy (2.0.3a)-(2.0.3b) and $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be strictly increasing and satisfy $\varphi(0) = 0$. Suppose u and $\hat{u} \in W^{1,1}((0, T); L^1)$ are two strong distributional solutions in L^1 to the inhomogeneous Dirichlet problem (3.4.3) with initial data $u_0, \hat{u}_0 \in L^1$, forcing terms $g, \hat{g} \in L^1((0, T); L^1)$ and boundary data $h, \hat{h} \in L^1_{\text{loc}}(\mathbb{R}^d \setminus \Omega)$, respectively. If $h(t) \leq \hat{h}(t)$ a.e. on $\mathbb{R}^d$ for a.e. $t \in (0, T)$, then*

$$\begin{aligned} \int_{\Omega} (u(t) - \hat{u}(t))^+ d\mu &\leq e^{\omega t} \int_{\Omega} (u_0 - \hat{u}_0)^+ d\mu \\ &+ \int_0^t e^{\omega(t-s)} \int_{\Omega} (g(s) - \hat{g}(s)) \mathbb{1}_{\{u > \hat{u}\}} d\mu ds \end{aligned} \quad (3.4.4)$$

for all $0 \leq t \leq T$.

Proof. Since u and $\hat{u}$ are distributional solutions, we have that $\varphi(u(t)) \in W_{\varphi(h(t))}^{s,(2,p)}(\Omega)$ and $\varphi(\hat{u}(t)) \in W_{\varphi(\hat{h}(t))}^{s,(2,p)}(\Omega)$ for a.e. $t \in (0, T)$. We first prove an estimate on the sign of $(-\Delta_p)^s u - (-\Delta_p)^s \hat{u}$ given that $u \leq \hat{u}$ on $\mathbb{R}^d \setminus \Omega$. We approximate the sign function by considering all $q \in C^1(\mathbb{R})$ satisfying $0 \leq q \leq 1$, $q(s) = 0$ for $s \leq 0$ and $q'(s) > 0$ for $s > 0$ and prove that

$$\int_{\Omega} ((-\Delta_p)^s u - (-\Delta_p)^s \hat{u}) q(u - \hat{u}) dx \geq 0.$$

By assumption, $q(u - \hat{u}) = 0$ on $\mathbb{R}^d \setminus \Omega$. So we have

$$\begin{aligned} &\int_{\Omega} ((-\Delta_p)^s u - (-\Delta_p)^s \hat{u}) q(u - \hat{u}) dx \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u(x) - u(y))^{p-1} - (\hat{u}(x) - \hat{u}(y))^{p-1}}{|x - y|^{d+ps}} \times \\ &\quad (q(u(x) - \hat{u}(x)) - q(u(y) - \hat{u}(y))) dy dx. \end{aligned}$$

We split these integrals into $\{u \geq \hat{u}\}$ and $\{u < \hat{u}\}$ terms noting that $q(u(x) - \hat{u}(x)) = 0$ on $\{u < \hat{u}\}$. On $\{u \geq \hat{u}\} \times \{u \geq \hat{u}\}$, using the monotonicity of $q(s)$, we have

$$\iint_{\{u \geq \hat{u}\}^2} \frac{(u(x) - u(y))^{p-1} - (\hat{u}(x) - \hat{u}(y))^{p-1}}{|x - y|^{d+ps}} \times \\ (q(u(x) - \hat{u}(x)) - q(u(y) - \hat{u}(y))) \, dy \, dx \geq 0.$$

On $\{u < \hat{u}\} \times \{u < \hat{u}\}$ we have

$$q(u(x) - \hat{u}(x)) - q(u(y) - \hat{u}(y)) = 0.$$

Then applying the symmetry of x and y to the remaining two terms, we have that

$$\begin{aligned} & \int_{\Omega} ((-\Delta_p)^s u - (-\Delta_p)^s \hat{u}) q(u - \hat{u}) \, dx \\ & \geq 2 \int_{\{u \geq \hat{u}\}} \int_{\{u < \hat{u}\}} \frac{(u(x) - u(y))^{p-1} - (\hat{u}(x) - \hat{u}(y))^{p-1}}{|x - y|^{d+ps}} \times \\ & \quad q(u(x) - \hat{u}(x)) \, dy \, dx \\ & \geq 2 \int_{\{u \geq \hat{u}\}} \int_{\{u < \hat{u}\}} \frac{(\hat{u}(x) - \hat{u}(y))^{p-1} - (u(x) - u(y))^{p-1}}{|x - y|^{d+ps}} \times \\ & \quad q(u(x) - \hat{u}(x)) \, dy \, dx \\ & = 0. \end{aligned}$$

Letting q converge to $[\text{sign}_0]^+$ and noting that $\varphi(u) > \varphi(\hat{u})$ if and only if $u > \hat{u}$,

$$\int_{\Omega} ((-\Delta_p)^s \varphi(u) - (-\Delta_p)^s \varphi(\hat{u})) \mathbf{1}_{\{u > \hat{u}\}} \, d\mu \geq 0. \quad (3.4.5)$$

By the chain rule, (3.4.3) and (3.4.5),

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} (u(t) - \hat{u}(t))^+ \, d\mu = \int_{\Omega} (u'(t) - \hat{u}'(t)) \mathbf{1}_{\{u > \hat{u}\}} \, d\mu \\ & = - \int_{\Omega} ((-\Delta_p)^s \varphi(u)(t) - (-\Delta_p)^s \varphi(\hat{u})(t)) \mathbf{1}_{\{u > \hat{u}\}} \, d\mu \\ & \quad + \int_{\Omega} (F(u) - F(\hat{u})) \mathbf{1}_{\{u > \hat{u}\}} \, d\mu + \int_{\Omega} (g(t) - \hat{g}(t)) \mathbf{1}_{\{u > \hat{u}\}} \, d\mu \\ & \leq \omega \int_{\Omega} (u(t) - \hat{u}(t))^+ \, d\mu + \int_{\Omega} (g(t) - \hat{g}(t)) \mathbf{1}_{\{u > \hat{u}\}} \, d\mu. \end{aligned}$$

Applying a Grönwall inequality,

$$\begin{aligned} \int_{\Omega} (u(t) - \hat{u}(t))^+ \, d\mu & \leq e^{\omega t} \int_{\Omega} (u_0 - \hat{u}_0)^+ \, d\mu \\ & \quad + \int_0^t e^{\omega(t-s)} \int_{\Omega} (g(s) - \hat{g}(s)) \mathbf{1}_{\{u > \hat{u}\}} \, d\mu \, ds. \end{aligned}$$

□

3.4.2 Proof of finite time of extinction

We now suppose that Ω is bounded in order to construct a super-solution and a sub-solution which are truncated within a ball containing Ω .

Proof of Theorem 3.4.1. We let u be a strong distributional solution to (1.2.1). We will construct a super-solution and sub-solution on $\mathbb{R}^d \times \mathbb{R}_+$ to bound u via separation of variables of the form $\mu(x)T(t)$ using the fundamental solution. In particular, we choose T to be a decreasing function such that $T(t^*) = 0$ for some $t^* > 0$.

Choose $R > 0$ such that $\Omega \subset B_R(0)$. In order to apply Theorem 3.4.3 we require that this super-solution $V(x, t)$ satisfies

$$V_t + (-\Delta_p)^s \varphi(V) \geq 0. \quad (3.4.6)$$

Letting $\beta = \varphi^{-1}$ we set $V(x, t) = \beta(W(x, t))$ with $W(x, t) = \mu(x)T(t)$, where we choose

$$\mu(x) = \begin{cases} R^{-d-ps} & \text{for } |x| \leq R, \\ |x|^{-d-ps} & \text{for } |x| \in (R, 3R), \\ 0 & \text{for } |x| \geq 3R. \end{cases}$$

Defining

$$C_R := \frac{\omega_{d-1}}{4^{d+ps}d} (3^d - 2^d) (1 - 2^{-d-ps})^{p-1} R^{-sp} \quad (3.4.7)$$

with ω_d denoting the volume of a d -dimensional unit ball, and

$$t^* = \frac{1}{C_R} \int_0^{\|u_0\|_\infty} \frac{1}{(\varphi(\tau))^{p-1}} d\tau,$$

we set

$$T(t) = \begin{cases} R^{d+ps} \varphi(\sigma(t^* - t)) & \text{if } t < t^* \\ 0 & \text{if } t \geq t^* \end{cases}$$

where $\sigma(t)$ satisfies

$$\int_0^{\sigma(t)} \frac{1}{(\varphi(\tau))^{p-1}} d\tau = C_R t$$

for $t \in [0, t^*]$. Note that $V(x, 0) = \|u_0\|_\infty$ and $V(x, t^*) = 0$ for $x \in \Omega$. Moreover, $V_t = -C_R (\varphi(\sigma(t^* - t)))^{p-1}$.

For $x \in \Omega$, we have

$$(-\Delta_p)^s \mu(x) \leq \int_{\mathbb{R}^d \setminus B_R(0)} \frac{R^{-(d+ps)(p-1)}}{|x - y|^{d+ps}} dy$$

which is bounded since $ps > 0$ so that $(-\Delta_p)^s \mu(x) \in L^\infty$ and so we can apply the singular integral form of the fractional p -Laplacian. Let

$g = V_t + (-\Delta_p)^s \varphi(V)$ on $\Omega \times (0, \infty)$. Note that $V \in W_{\text{loc}}^{1,1}(0, \infty; L^1(\mathbb{R}^d))$. Then V is a strong distributional solution to

$$\begin{cases} v_t + (-\Delta_p)^s \varphi(v) = g & \text{on } \Omega \times (0, \infty), \\ v(t) = V_t(t) & \text{on } \mathbb{R}^d \setminus \Omega \times (0, \infty), \\ v(0) = v_0 & \text{in } \mathbb{R}^d, \end{cases} \quad (3.4.8)$$

where $V(t) \geq u(t)$ on $\mathbb{R}^d \setminus \Omega$ and $V(0) \geq u_0$. Applying Theorem 3.4.3, we have that

$$\int_{\Omega} (u(t) - V(t))^+ d\mu \leq - \int_0^t \int_{\Omega} g(s) \mathbf{1}_{\{u > V\}} d\mu ds.$$

To obtain that $u(t) \leq V(t)$ almost everywhere on $\Omega \times (0, \infty)$, it therefore remains to prove that the right-hand side is bounded by zero.

For $x \in \Omega$, using the singular integral formulation,

$$\begin{aligned} (-\Delta_p)^s \mu(x) &\geq \int_{B_{3R}(0) \setminus B_{2R}(0)} \frac{(R^{-d-ps} - |y|^{-d-ps})^{p-1}}{|x - y|^{d+ps}} dy \\ &\geq (R^{-d-ps}(1 - 2^{-d-ps}))^{p-1} \int_{B_{3R}(0) \setminus B_{2R}(0)} \frac{1}{(4R)^{d+sp}} dy \\ &\geq \frac{\omega_{d-1}}{4^{d+ps}d} (3^d - 2^d) R^{-sp} (R^{-d-ps}(1 - 2^{-d-ps}))^{p-1}. \end{aligned}$$

Rewriting (3.4.6) in terms of the separated variables $\mu(x)$ and $T(t)$, and applying the estimate on $(-\Delta_p)^s \mu$, it is sufficient for $T(t)$ to satisfy

$$\frac{d}{dt} \beta(R^{-d-ps} T(t)) + C_R (R^{-d-ps} T(t))^{p-1} \geq 0 \quad \text{on } (0, t^*).$$

In particular, we require that

$$\frac{d\sigma}{dt} = C_R (\varphi(\sigma))^{p-1} \quad \text{on } (0, t^*),$$

which holds by definition of σ . Hence

$$V_t + (-\Delta_p)^s \varphi(V) \geq 0$$

for all $x \in \Omega$.

Similarly for $-u$ and $-V$ we have, with respect to the Lebesgue measure,

$$\begin{cases} (-V)_t + (-\Delta_p)^s \varphi(-V) \leq 0 & \text{on } \Omega \times (0, \infty), \\ -V(0, \cdot) \leq u_0 & \text{on } \mathbb{R}^d, \\ -V \leq u & \text{on } \mathbb{R}^d \setminus \Omega \times (0, \infty), \end{cases}$$

hence we have that for almost every $x \in \Omega$ and all $t \geq 0$, $-V(t) \leq u(t) \leq V(t)$ and so $u(t) = 0$ for $t \geq t^*$. $\square$

3.5 Existence of a Barenblatt solution

In this section, we aim to prove the existence of a Barenblatt solution which tends to a Dirac delta as $t \rightarrow 0+$. Here we use a rescaling technique from [127], see also [121] where this is proven for the fractional p -Laplacian.

Theorem 3.5.1. *Let $d \geq 1$, $0 < s < 1$, $m \geq 1$ and $p > p_c$ such that $m(p-1) \neq 1$ and $sp < d$. Then for any mass $M > 0$ there exists a strong distributional solution Γ to (1.2.1) in L^1 which is nonnegative, radially symmetric, decreasing radially in space with decay given by $g(|x|)$ and decays in time with order $t^{-\frac{1}{m(p-1)-\frac{p}{qs}}}$ uniformly in x . Moreover, Γ tends to a Dirac delta as $t \rightarrow 0+$.*

Proof. For a given smooth, positive, radially symmetric initial datum $u_0 \in L^{1 \cap \infty}$ with compact support contained in the open unit ball $B_1(0)$ and having mass $\int_{\mathbb{R}^d} u_0 \, dx = 1$, let u be the corresponding positive strong distributional solution of (1.2.1) provided by Theorem 2.1.8. By applying the scaling transformation (3.1.1) to u , one obtains that for every integer $k \geq 1$,

$$u_k(x, t) := k^d u(kx, k^{d(m(p-1)-1)+sp}t), \quad x \in \mathbb{R}^d, \, t \geq 0, \quad (3.5.1)$$

is a strong distributional solution of the doubly nonlinear nonlocal diffusion equation (1.1.3) with initial datum $u_{k,0} := k^d u_0(k \cdot)$ representing a nascent δ -function in the sense that $u_{k,0}$ converges to the Dirac delta δ_0 function in the sense of distributions as $k \rightarrow \infty$. Therefore, it remains to study the existence and properties of the limit function

$$\lim_{k \rightarrow \infty} u_k(x, t) \quad \text{for every } x \in \mathbb{R}^d \text{ and } t > 0.$$

In particular, we will take the limit of a subsequence of $(u_k)_{k \geq 1}$ to define the Barenblatt solution $\Gamma(x, t)$. By (2.1.2),

$$\|u_k(t)\|_1 \leq \|u_{k,0}\|_1 = 1 \quad (3.5.2)$$

for every $t \geq 0$ and every $k \geq 1$. From the $L^1 - L^\infty$ regularization estimate (2.1.13), we have that

$$\sup_{k \geq 1} \|u_k(t)\|_\infty \leq C t^{-\alpha} \quad \text{for every } t > 0, \quad (3.5.3)$$

where $\alpha > 0$ is defined as in Corollary 2.1.5 and so (3.5.2) yields that

$$\sup_{k \geq 1} \|u_k^m(t)\|_p \leq C^{(m-1)} t^{-(m-1)\alpha} \quad \text{for every } t > 0. \quad (3.5.4)$$

Further, applying (2.1.15) and Corollary 2.1.7 to $(u_k)_{k \geq 1}$ gives that

$$\sup_{k \geq 1} \|\partial_t u_k(t)\|_1 \leq \frac{2}{|m(p-1)-1|} \frac{1}{t} \quad \text{for every } t > 0, \quad (3.5.5)$$

and

$$\sup_{k \geq 1} [u_k^m(t)]_{s,p}^p \leq C t^{-(1+m\alpha)} \quad \text{for every } t > 0. \quad (3.5.6)$$

Therefore, for every $\delta > 0$, the sequence $(\frac{\partial u_k}{\partial t})_{k \geq 1}$ is bounded in the space $L^\infty(\delta, \infty; L^1)$. In particular, (3.5.4) and (3.5.6) yield that the sequence $(u_k^m)_{k \geq 1}$ is bounded in $L^\infty(\delta, \infty; W^{s,p})$. Since the previous estimates (3.5.2)-(3.5.6) remain valid on any compact subset K of $\mathbb{R}^d$, and $W^{s,p}(K)$ is compactly embedded into $L^1(K)$ by the Rellich-Kondrachov theorem for fractional Sobolev spaces (see [117, Theorem 2.1]), it follows from Simon's compactness result [114, Theorem 1] that $(u_k^m)_{k \geq 1}$ is relatively compact in $C([t_1, t_2]; L^1(K))$ for every $0 < t_1 < t_2$.

Now, for every integer $n \geq 1$, let $I_n = [1/n, n]$ and $K_n = \{x \in \mathbb{R}^d : |x| \leq n\}$. Then $(I_n)_{n \geq 1}$ is an increasing sequence of compact intervals, approximating the positive open real line $(0, \infty)$, and $(K_n)_{n \geq 1}$ is an increasing sequence of compact subsets of $\mathbb{R}^d$ approximating $\mathbb{R}^d$. Since both sequences $(I_n)_{n \geq 1}$ and $(K_n)_{n \geq 1}$ are countable, a standard diagonal argument yields the existence of a subsequence $(u_{\varphi(k)})_{k \geq 1}$ of $(u_k)_{k \geq 1}$ and an element $\Gamma \in C((0, \infty); L_{loc}^1)$ such that

$$\lim_{k \rightarrow \infty} u_{\varphi(k)} = \Gamma \quad \text{in } C([t_1, t_2]; L^1(K)) \quad (3.5.7)$$

for every $0 < t_1 < t_2$ and every compact subset K of $\mathbb{R}^d$. Further, since each $u_{\varphi(k)}$ is given by (3.5.1), we can apply the global bound for u given by (3.3.3) and (3.3.4), so that for all $R > R_2$ and $t \geq 0$, we have

$$\begin{aligned} & \|u_{\varphi(k)}(t)\|_{L^1(\{|x| \geq R\})} \\ & \leq C_2 (t + \varphi(k)^{-\frac{1}{\beta}})^{-d\beta} \int_{\{|x| \geq R\}} g\left(|x|(t + \varphi(k)^{-\frac{1}{\beta}})^{-\beta}\right) dx \\ & = C_2 \int_{\{|z| \geq R(t + \varphi(k)^{-\frac{1}{\beta}})^{-\beta}\}} g(|z|) dz. \end{aligned}$$

Since $g(|z|)$ is integrable in $\mathbb{R}^d$ in each case, for any given $\varepsilon > 0$, there is an $R > 0$ such that for every $0 < t_1 < t_2$,

$$\sup_{k \geq 1} \sup_{t \in [t_1, t_2]} \|u_{\varphi(k)}(t)\|_{L^1(\{|x| \geq R\})} < \varepsilon.$$

Combining this with (3.5.7) gives

$$\lim_{k \rightarrow \infty} u_{\varphi(k)} = \Gamma \quad \text{in } C([t_1, t_2]; L^1)$$

for every $0 < t_1 < t_2$, and by (2.1.13),

$$\lim_{k \rightarrow \infty} u_{\varphi(k)} = \Gamma \quad \text{in } C([t_1, t_2]; L^q) \quad (3.5.8)$$

for every $1 \leq q < \infty$. Next, let $0 < t_1 < t_2$ and K be a compact subset of $\mathbb{R}^d$. Then by (3.5.5) and (3.5.8), one sees that

$$\begin{aligned} \|\Gamma(t_2) - \Gamma(t_1)\|_{L^1(K)} &\leq \lim_{k \rightarrow \infty} \|u_k(t_2) - u_k(t_1)\|_{L^1(K)} \\ &\leq \lim_{k \rightarrow \infty} \int_{t_1}^{t_2} \|\partial_t u_k(t)\|_{L^1(K)} dt \\ &\leq \lim_{k \rightarrow \infty} \int_{t_1}^{t_2} \|\partial_t u_k(t)\|_1 dt \\ &\leq 2 \frac{\log t_2 - \log t_1}{|m(p-1) - 1|}. \end{aligned}$$

Hence, applying an increasing sequence $(K_n)_{n \geq 1}$ of compact subsets of $\mathbb{R}^d$ to the preceding inequality and subsequently sending $n \rightarrow \infty$ yields that

$$\|\Gamma(t_2) - \Gamma(t_1)\|_1 \leq 2 \frac{\log t_2 - \log t_1}{|m(p-1) - 1|} \quad (3.5.9)$$

for every $0 < t_1 < t_2$. Therefore $\Gamma \in C((0, \infty); L^1)$ and is locally absolutely continuous with values in L^1 . Moreover, by (3.5.9), Γ is in $W^{1,\infty}(\delta, T; L^1)$ satisfying

$$\lim_{h \rightarrow 0+} \left\| \frac{\Gamma(t+h) - \Gamma(t)}{h} \right\|_1 \leq 2 \lim_{h \rightarrow 0+} \frac{\log(t+h) - \log t}{h|m(p-1) - 1|} = \frac{2}{t|m(p-1) - 1|}.$$

Next, since $u_{\varphi(k)}$ is a strong distributional solution of (1.1.3), we may multiply (1.1.3) by $u_{\varphi(k)}^m$ and subsequently integrate over (t_1, t_2) for given $0 < t_1 < t_2$. Then, one obtains that

$$\frac{1}{m+1} \|u_{\varphi(k)}(t_2)\|_{m+1}^{m+1} + \int_{t_1}^{t_2} [u_{\varphi(k)}^m(t)]_{s,p}^p dt = \frac{1}{m+1} \|u_{\varphi(k)}(t_1)\|_{m+1}^{m+1}. \quad (3.5.10)$$

Integrating (3.5.6) over (t_1, t_2) , we also have that

$$\int_{t_1}^{t_2} [u_{\varphi(k)}^m(t)]_{s,p}^p dt \leq \frac{C}{m\alpha} t_1^{-m\alpha}. \quad (3.5.11)$$

Thanks to the two estimates (3.5.4) and (3.5.11), we have that for every $0 < t_1 < t_2$, $(u_{\varphi(k)}^m)_{k \geq 1}$ is bounded in $L^p(t_1, t_2; W^{s,p})$. From here, we proceed similarly to the proof of Theorem 2.1.8 in order to show that Γ is a strong distributional solution to (1.2.1). Letting $0 < t_1 < t_2$, the space $L^p(t_1, t_2; W^{s,p})$ is reflexive, so by (3.5.8) we can conclude that $\Gamma^m \in L^p(t_1, t_2; W^{s,p})$. After possibly passing to another subsequence of $(u_{\varphi(k)}^m)_{k \geq 1}$, we have that

$$\lim_{k \rightarrow \infty} u_{\varphi(k)}^m = \Gamma^m \quad \text{weakly in } L^p(t_1, t_2; W^{s,p}). \quad (3.5.12)$$

In particular, one has that the sequence $(\mathcal{A}_{s,p}(u_{\varphi(k)}^m))_{k \geq 1}$ of linear bounded functionals on $L^p(t_1, t_2; W^{s,p})$ given by

$$\begin{aligned} &\langle \mathcal{A}_{s,p}(u_{\varphi(k)}^m), \xi \rangle_{L^{p'}(t_1, t_2; W^{-s,p'}), L^p(t_1, t_2; W^{s,p})} \\ &= \frac{1}{2} \int_{t_1}^{t_2} \int_{\mathbb{R}^{2d}} \frac{(u_{\varphi(k)}^m(x) - u_{\varphi(k)}^m(y))^{p-1} (\xi(x) - \xi(y))}{|x - y|^{d+sp}} dx dy dt \end{aligned}$$

for every $\xi \in L^p(t_1, t_2; W^{s,p})$, is bounded in $L^{p'}(t_1, t_2; W^{-s,p'})$. Therefore, there is an $\chi \in L^{p'}(t_1, t_2; W^{-s,p'})$ such that after possibly passing to a subsequence, one has that

$$\lim_{n \rightarrow \infty} \mathcal{A}_{s,p}(u_{\varphi(k)}^m) = \chi \quad \text{weakly}^* \text{ in } L^{p'}(t_1, t_2; W^{-s,p'}). \quad (3.5.13)$$

Further, since $u_{\varphi(k)}$ is a strong distributional solution of (1.1.3), it follows from

$$\partial_t u_{\varphi(k)} + \mathcal{A}_{s,p}(u_{\varphi(k)}^m) = 0 \quad (3.5.14)$$

that $(\partial_t u_{\varphi(k)})_{k \geq 1}$ is bounded in $L^{p'}(t_1, t_2; W^{-s,p'})$. Thus and since $\Gamma \in L^1$, it follows from (3.5.8) that $\partial_t \Gamma \in L^{p'}(t_1, t_2; W^{-s,p'})$ and, after possibly passing to another subsequence of $(u_{\varphi(k)})_{k \geq 1}$, that

$$\lim_{k \rightarrow \infty} \partial_t u_{\varphi(k)} = \partial_t \Gamma \quad \text{weakly}^* \text{ in } L^{p'}(t_1, t_2; W^{-s,p'}). \quad (3.5.15)$$

Hence, multiplying (3.5.14) by a test function $\xi \in C_c^\infty((t_1, t_2) \times \mathbb{R}^d)$ and sending $k \rightarrow \infty$ yields that

$$\langle \partial_t \Gamma(t) + X^*, \xi \rangle_{L^{p'}(t_1, t_2; W^{-s,p'}), L^p(t_1, t_2; W^{s,p})} = 0,$$

which yields that

$$\partial_t \Gamma(t) + \chi = 0 \quad \text{in } W^{-s,p'} \text{ for a.e. } t \in (t_1, t_2). \quad (3.5.16)$$

Now, we are ready to prove that Γ is a distributional solution of (1.2.1) on (t_1, t_2) . To do this, it remains to show that $\chi = \mathcal{A}_{s,p}(\Gamma^m)$, where $\mathcal{A}_{s,p}$ is the lifted operator $\mathcal{A}_{s,p} : L^p(t_1, t_2; W^{s,p}) \rightarrow L^{p'}(t_1, t_2; W^{-s,p'})$ given by

$$\begin{aligned} & \langle \mathcal{A}_p^s(v), \xi \rangle_{L^{p'}(t_1, t_2; W^{-s,p'}), L^p(t_1, t_2; W^{s,p})} \\ &= \frac{1}{2} \int_{t_1}^{t_2} \int_{\mathbb{R}^{2d}} \frac{|v(x) - v(y)|^{p-2} (v(x) - v(y)) (\xi(x) - \xi(y))}{|x - y|^{d+sp}} dx dy dt \end{aligned}$$

for every $v, \xi \in L^p(t_1, t_2; W^{s,p})$. For this, note that multiplying (3.5.16) by Γ^m yields that

$$\frac{1}{m+1} \|\Gamma\|_{m+1}^{m+1} \Big|_{t_1}^{t_2} + \langle \chi, \Gamma^m \rangle_{L^{p'}(t_1, t_2; W^{-s,p'}), L^p(t_1, t_2; W^{s,p})} = 0.$$

Thus (3.5.8) and (3.5.10) yield that

$$\lim_{n \rightarrow \infty} \int_{t_1}^{t_2} [u_{\varphi(k)}^m(t)]_{p,s}^s dt = \langle \chi, \Gamma^m \rangle_{L^{p'}(t_1, t_2; W^{-s,p'}), L^p(t_1, t_2; W^{s,p})}. \quad (3.5.17)$$

By the monotonicity of $\mathcal{A}_{s,p}$, one has that

$$\begin{aligned} 0 & \leq \langle \mathcal{A}_p^s(u_{\varphi(k)}^m) - \mathcal{A}_{s,p}(\xi), u_{\varphi(k)}^m - \xi \rangle_{L^{p'}(t_1, t_2; W^{-s,p'}), L^p(t_1, t_2; W^{s,p})} \\ &= \int_{t_1}^{t_2} [u_{\varphi(k)}^m(t)]_{s,p}^p dt - \langle \mathcal{A}_p^s(u_{\varphi(k)}^m), \xi \rangle_{L^{p'}(t_1, t_2; W^{-s,p'}), L^p(t_1, t_2; W^{s,p})} \\ & \quad - \langle \mathcal{A}_{s,p}(\xi), u_{\varphi(k)}^m - \xi \rangle_{L^{p'}(t_1, t_2; W^{-s,p'}), L^p(t_1, t_2; W^{s,p})} \end{aligned}$$

for every $k \geq 1$. Thus, sending $k \rightarrow \infty$ in the last inequality and by using (3.5.12), (3.5.13), and (3.5.17), one obtains that

$$0 \leq \langle \chi - \mathcal{A}_{s,p}(\xi), u^m - \xi \rangle_{L^{p'}(t_1, t_2; W^{-s, p'}), L^p(t_1, t_2; W^{s, p})}$$

for every $\xi \in L^p(t_1, t_2; W^{s, p})$. Now, by proceeding as in the proof of Proposition 2.9.2, we can choose $\xi = u^m - \mu \zeta$, taking $\mu \rightarrow 0+$, to conclude that $\chi = \mathcal{A}_{s,p}(\Gamma^m)$.

Thus Γ is a distributional solution of (1.1.3) and it remains to show that Γ is differentiable with values in L^1_{loc} at almost everywhere $t \in (0, \infty)$. Since the argument is exactly the same as the one given in the proof of Theorem 2.1.1, we omit this part.

We can estimate Γ by applying the global bounds (3.3.3) and (3.3.4) to u so that there exists $C_2 > 0$ and $R_2 > 0$ such that

$$u(x, t) \leq C_2(t+1)^{-d\beta} g(|x|(t+1)^{-\beta}),$$

for a.e. $x \in \mathbb{R}^d$ and $t \geq 0$ satisfying $|x| > (t+1)^\beta R_2$. Then for $k \geq 1$,

$$u_k(x, t) \leq C_2(t + k^{-\frac{1}{\beta}})^{-d\beta} g\left(|x|(t + k^{-\frac{1}{\beta}})^{-\beta}\right)$$

for all $|x| > (t + k^{-\frac{1}{\beta}})^\beta R_2$. Hence there exists $C > 0$ such that

$$\Gamma(x, t) \leq Ct^{-d\beta} g(|x|t^{-\beta}) \quad (3.5.18)$$

for all $|x| > Ct^\beta$. So Γ tends to a Dirac delta as $t \rightarrow 0+$.

It remains to prove the radial symmetry properties of Γ . Since $u_{\phi(k)}^m$ converged to Γ^m weakly in $L^p(t_1, t_2; W^{s, p})$, we have a subsequence, relabelled as u_k , which converges pointwise almost everywhere in $[t_1, t_2] \times \mathbb{R}^d$ for $0 < t_1 < t_2$. Then applying Corollary 3.2.3, u_k is radially symmetric and radially decreasing so that the same holds for Γ almost everywhere in $(0, \infty) \times \mathbb{R}^d$. Moreover, (3.5.18) gives the decay of Γ in space and (3.5.3) gives the decay in time. $\square$

Proposition 3.5.2. *The strong distributional solution Γ given by Theorem 3.5.1 is self-similar with the form (3.0.2) where $F : \mathbb{R}^d \rightarrow \mathbb{R}$ satisfies the properties of Theorem 3.0.1.*

Proof. We apply a rescaling to self-similar variables,

$$V(y, \tau) = t^{d\beta} \Gamma(x, t) \quad \text{for } t > 0 \text{ and } x \in \mathbb{R}^d \quad (3.5.19)$$

where $y = xt^{-\beta}$ and $\tau = \log t$. Then since $\Gamma \in W^{1, \infty}(\delta, \infty; L^1) \cap L^\infty(\delta, \infty; W^{s, p})$ for all $\delta > 0$, we can apply those estimates such that $V \in W^{1, \infty}(\delta, \infty; L^1) \cap L^\infty(\delta, \infty; W^{s, p})$ for all $\delta > 0$. Moreover, since Γ is a strong distributional solution, V is a strong distributional solution to

$$\partial_\tau V - \beta \nabla \cdot (yV) + (-\Delta_p)^s V = 0$$

for $\tau \in \mathbb{R}$ and $y \in \mathbb{R}^d$. In particular, since $\partial_t \Gamma \in L^\infty(\delta, \infty; L^1)$, $\partial_\tau V \in L^\infty(\delta, \infty; L^1)$ and

$$\partial_t \Gamma = t^{-d\beta-1} (\partial_\tau V - \beta \nabla \cdot (yV))$$

by the rescaling (3.5.19), we have that $\nabla \cdot (yV) \in L^\infty(\delta, \infty; L^1)$ for all $\delta > 0$.

We have the following lemma from [121] wherein u_k is defined by (3.5.1) and u_1 is the positive strong distributional solution corresponding to a given radially symmetric initial datum $u_0 \in L^{1 \cap \infty}$ with compact support contained in the unit ball and having mass 1 as in the proof of Theorem 3.5.1.

Lemma 3.5.3 ([121, Lemma 6.1]). *If v_1 is the rescaled function from u_1 and v_k from u_k according to (3.5.19), then*

$$v_k(y, \tau) = v_1(y, \tau + \log(k)).$$

Since (relabelling by an appropriate subsequence) u_k converges pointwise almost every to Γ in $\mathbb{R}^d \times (0, \infty)$, $v_k(y, \tau)$ also converges pointwise almost everywhere to $V(y, \tau)$ in $\mathbb{R}^d \times (-\infty, \infty)$. By Lemma 3.5.3, we know that $v_k(y, \tau) = v_1(y, \tau + \log(k))$ so that, for $h > 0$, pointwise we have that

$$\begin{aligned} V(y, \tau + h) &= \lim_{k \rightarrow \infty} v_k(y, \tau + h) \\ &= \lim_{k \rightarrow \infty} v_{k+e^h}(y, \tau) \\ &= V(y, \tau) \end{aligned}$$

almost everywhere in $\mathbb{R}^d \times (-\infty, \infty)$. Hence V is independent of τ and $V_\tau = 0$ almost everywhere. Hence $F = V$ is an appropriate profile function. Moreover, we carry the radial symmetry properties of U to F . $\square$

3.6 Uniqueness of the Barenblatt solution

We obtain uniqueness of the self-similar profile using a method of mass difference analysis, applied to the evolution fractional p -Laplacian in [121] (see also [127]). This completes the proof of Theorem 3.0.1.

Theorem 3.6.1. *The Barenblatt solution given by Theorem 3.5.1 is unique for each $M > 0$.*

Proof. Let Γ_1 and Γ_2 be two Barenblatt solutions to (1.1.3) with profile functions F_1 and F_2 , respectively as defined in (3.0.2). Suppose F_1 and F_2 have the same mass M . Since F_1 and F_2 are both positive and

$$\int_{\mathbb{R}^d} F_1(z) dz = \int_{\mathbb{R}^d} F_2(z) dz = M,$$

one either has that $F_1 = F_2$ a.e. on $\mathbb{R}^d$, or

$$\int_{\mathbb{R}^d} (F_1(z) - F_2(z))^+ dz > 0$$

and

$$\int_{\mathbb{R}^d} (F_2(z) - F_1(z))^+ dz > 0.$$

Then (3.0.2) implies that for $t > 0$, the difference $\Gamma_1(t) - \Gamma_2(t)$ of the corresponding Barenblatt solutions Γ_1 and Γ_2 is also sign-changing. In particular, Γ_1 and Γ_2 satisfy the L^1 dissipation inequality for differences, Theorem 2.6.1, with strict inequality. Hence using the form (3.0.2) for Γ_1 and Γ_2 ,

$$t_2^{-d\beta} \|[F_1(xt_2^{-\beta}) - F_2(xt_2^{-\beta})]^+\|_1 < t_1^{-d\beta} \|[F_1(xt_1^{-\beta}) - F_2(xt_1^{-\beta})]^+\|_1$$

for $0 < t_1 < t_2$. Now a change of variable implies that

$$\|[F_1 - F_2]^+\|_1 < \|[F_1 - F_2]^+\|_1$$

giving a contradiction. □

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