TORSION AND BENDING OF SWEPT AND TAPERED WINGS WITH RIBS PARALLEL TO THE ROOT

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SUMMARY

The problem considered is that of a swept wing, either conical or cylindrical, of arbitrary section, under any system of bending and torsion loads. The wing is assumed to consist of a non-buckling outer skin, a series of booms located along generators of the tube, and a series of closely spaced ribs all parallel to the root section. The ribs are assumed to be rigid in their own planes but to offer no resistance to warping out of their planes. No restriction is placed on either the taper or the sweep of the wing.

The theory is developed in general terms, for arbitrary wing section, arbitrary variation of stress bearing area over the tube, and arbitrary applied loads. The fundamental equations are expressed in terms of a stress function which is found to satisfy a complicated integro-differential equation. Analytical solutions of this equation are obtained for certain simple types of tube and applied load by a process consisting essentially of separation of the variables.

Those solutions in some cases lead to formulae exactly analagous to those of the simple theories of bending and torsion for an unswept wing. They are slightly more complicated than the latter formulae, in that they show an interaction between bending and torsion which is characteristic of this type of wing.

The Appendix gives detailed numerical applications of the theory to a highly tapered unswept four boom tube of rectangular cross section, with a completely constrained root section, under varying bending moment and torque. It is shown that when the taper is large the root effect is of prime importance and the analysis of a delta wing would therefore be very tedious.

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INTRODUCTION :

In recent years considerable attention has been given to the use of swept back wings for high speed aeroplanes, the reason being primarily an aerodynamic one concerned with compressibility effects. Such wingsmay be divided roughly into two classes. In the first type, both the leading and trailing edges of the wing are swept through approximately the same angle so that, although the sweep may be large, the taper of the wing is only slight. In the second type, known as a delta wing, the leading edge is highly swept but the trailing edge is more or less unswept so that both the taper and the "average" sweep are large.

Consider for the moment only the first type, that is, a highly swept but slightly tapered wing. We may consider this to be derived from a corresponding unswept wing in one of two ways. Fig.1(a) shows the plan view of a conventional unswept wing and indicates the directions of the internal ribs which maintain the fore and aft section of the wing. If this wing is rotated bodily about a vertical axis through the point A it will take up a position as shown in Fig.1(b). The addition of a "root triangle" then gives a swept wing in which the ribs are more or less perpendicular to the leading edge. However, an alternative procedure would be to shear the unswept wing backwards until it takes up a position as shown in Fig.1(c). This gives a swept wing in which the ribs are parallel to the root section. There is of course an infinite number of intermediate directions for the ribs, but the configurations shown in Figs.1(b) and 1(c) illustrate the two extremes and probably the two most likely rib directions.

As far as the first type, Fig.1(b), is concerned, it is clear that, away from the root, conventional wing stressing methods, based on the concept of a flexural axis, will apply. The treatment of the root triangle is a problem in stress diffusion which may present difficulties in determining the stresses in the vicinity of the root but will have little effect on the behaviour of the wing as a whole.

However, if the ribs are all parallel to the root section as in Fig.1(c), the mode of deformation of the wing may be changed considerably, depending upon the spacing and the stiffness of the ribs. The limiting case occurs when the ribs are infinitely closely spaced and have an infinite stiffness in their own planes so that all sections of the wing parallel to the root section retain their original shape under load.

There seems to be no definite opinion as to which rib direction is the most desirable, particularly from an aeroelastic point of view. It has been pointed out by Collar (Ref.4) that with ribs perpendicular to the leading edge, as in Fig.1(b), the flexural stiffness may be just as important as the torsional stiffness in determining the critical aeroelastic speeds, such as the aileron reversal speed. Applying this idea in very general terms to Fig.1(c), it may well be that the aeroelastic properties would be improved by having the ribs parallel to the root section. However, before any rational conclusion can be reached it is necessary to derive a theory from which the stresses and deformations in a wing of this type under any given system of loads can be determined. Such is the aim of the analysis given in the subsequent pages.

To simplify the problem the limiting case of infinitely closely spaced ribs parallel to the root section, completely rigid in their own planes but offering no resistance to warping out of their planes, is taken. Apart from the ribs, the wing is assumed to be either a swept cylinder or a swept cone of arbitrary section, consisting of a non-buckling outer skin, together with a series of booms, or concentrated tension members, of the same material as the skin and located along generators of the tube. If the skin is stabilised by stringers these are considered under the general heading of booms. No restriction is placed on either the sweep or the taper of the tube which means that the theory is applicable not only to highly swept and slightly tapered wings but also to wings of the delta type with ribs parallel to the root section.

At this stage it is worth considering the problem in perspective. The tubes which are being analysed here are all part of a broad class, being defined completely in their exterior shape by the shape of the root section, which we may take in a fixed reference plane, and by the position of the apex of the cone. The structure inside this outer surface consists of skin, booms and ribs of the type previously described. If the perpendicular distance of the apex from the root section is large in comparison with the dimensions of the root section the tube may be said to be slightly tapered. A cylindrical tube can be regarded as the limiting case of a conical tube as the apex is taken to infinity in a direction which defines the sweep. Conventional wings in which both the taper and the sweep are small may be defined as those in which the angle between any generator and the perpendicular to the root section is small enough for its cosine to be taken as unity. This latter group has been exhaustively treated by Hadji-Argyris and Dunne (Refs.1,2,3) and the present analysis can be looked upon as an extension of their work to the whole class of tubes fitting into this broad picture.

In thinking of the deformations of this type of tube the conventional ideas of twist and bending must be revised. It is clear that the only sections which will retain their original shape under load are the rib sections. Consequently the displacements of any point on the surface of the tube can be defined by the rigid body movements in the plane of the rib passing through the point together with a warping displacement perpendicular to the plane of the rib. The rigid body movements will vary from rib to rib so that the "twist" of the tube may be regarded as the relative rotation of adjacent ribs about an axis perpendicular to the root section. In a similar way we can think of "bending" as being brought about by relative displacements of the ribs in a direction parallel to the root section and "curvatures" can be defined on this basis.

To fit in with these definitions of twist and bending we must regard a "torque" as a couple applied about an axis perpendicular to the root section and "bending moments" as couples applied about axes parallel to the root section.

With these definitions in mind it will be evident that if a pure torque is applied to the wing shown in Fig.1(c), the tube will not only twist but will also bend. Similarly, if a pure bending moment is applied, about a horizontal axis say, the tube will both bend and twist. This interaction between torsion and bending is characteristic of wings of this type and, in the author's opinion, constitutes the fundamental difference between a swept and an unswept wing. This fact can indeed be used as a definition of an unswept wing. The mathematical conditions to be satisfied are derived in § 3.2, the results of which are in effect an extension of the simple theory of bending and torsion and clearly show the interaction between the two types of deformation. It will also be clear from the above remarks that there is no such thing as a flexural axis for this type of wing.

A point of comparison between the two wings shown in Figs.1(b) and 1(c) which is worth emphasising is that in Fig.1(b) there is a discontinuity of structure in passing over the section AB into the root triangle so that an analysis of this type of wing must necessarily be divided into two parts. In Fig.1(c), however, no such discontinuity exists so that the analysis will be valid right up to the root section.

We now turn to a comparison of the two rib directions from the point of view of aeroelastic phenomena. Collar (Ref.4) discussed the aeroelastic effects of sweepback on the implied assumption that the ribs were perpendicular to the leading edge, as in Fig.1(b), and rigid in their own planes. This leads naturally to the idea of using the section perpendicular to the leading edge as the basic aerofoil section of the wing, since during deformation this section only changes incidence without distorting. Consequently the total air speed in the horizontal plane is resolved into a component perpendicular to, and one in the direction of, the leading edge.

In an infinitely stiff wing this latter component would have little effect on the aerodynamic forces. (This is strictly true only in the case of a yawed two-dimensional aerofoil but can be expected to be reasonably true away from the root triangle of a three dimensional wing. Since aeroelastic phenomena are concerned primarily with redistributions of aerodynamic forces near the wing tips the root effect can be ignored.) However, if the wing bends (in the conventional sense) the component in the horizontal plane parallel to the undistorted leading edge would have a component perpendicular to the distorted leading edge and this would produce an effective incidence change quite apart from that due to twist of the wing. Collar therefore concludes that the flexural stiffness of the wing plays just as important a part as torsional stiffness in determining the control characteristics at high speeds. A low flexural stiffness has an adverse effect both on the aileron or elevon reversal speed and on the longitudinal stability, the latter being probably the most important. This fact complicates considerably the stiffness criterion for a swept wing with ribs perpendicular to the leading edge.

If a similar analysis of a wing of the type shown in Fig.1(c) is made, we see first that the basic aerofoil section must be taken as that parallel to the root section. Since the total air flow is parallel to this section it is clear that bending of the wing, as defined previously, will have no effect on the aileron reversal speed or on the longitudinal stability of the aircraft. It is not possible to say, from a general argument such as this, which rib direction would give the best aeroelastic properties since weight saving would play a part in deciding this question. The analysis given in this paper does however provide a foundation on which a rational treatment of the problem can be based.

In calculating the deformations of a conventional unswept wing under load, the methods almost invariably used in practice depend on an application of simple bending theory and the Bredt-Batho theory of torsion. These theories, however, are exact in very few cases. The simplest requirements for their validity are as follows:-

(a) the tube section (both external and internal) is constant along the length;

(b) the bending moment and torque are constant,

and (c) are applied to the end sections by stresses distributed in a prescribed manner.

If these conditions are satisfied an originally plane cross section of the tube in general warps out of its plane, and all sections warp identically. If, however, the section, the bending moment or the torque varies along the tube, as happens in practice, the various sections tend to warp by different amounts with the result that, in order to have compatible displacements, the stresses are redistributed and the simple theories are no longer exact. This effect will be felt over the whole tube. Similarly, if the loads are applied to the end sections by a stress distribution other than that prescribed by the simple theory there will exist an additional self-equilibrating stress system which will have its greatest magnitude at the ends and decay towards the centre of the tube. In spite of these inexactitudes, however, the simple theories are still applied.

The analysis given in this paper takes all these effects into account but, in the process, formulae exactly analagous to those of the simple theory of bending and torsion are derived. These are given in § 3.2 for a conical tube, and in §6 for a cylindrical tube. The formulae for a conical tube are valid if the skin thickness remains constant along any generator with all boom areas and the bending moment and torque varying linearly from zero at the apex to a maximum at the root. The corresponding formulae for the cylindrical tube are true if the section, the bending moment and the torque remain constant along the length of the tube. Both sets of formulae depend upon the loads being applied to the end sections in a prescribed manner.

In view of the fact that simple bending and torsion theory is used so widely for unswept wings, even when it is not strictly applicable, it is felt that a similar use could be made of these corresponding formulae for a slightly tapered swept wing, at least as far as the calculation of deformations which determine the aeroelastic properties is concerned.

For a highly tapered tube, such as a delta wing, the position is not so simple. There the semi-span is of the same order of magnitude as the root chord and the disturbances arising from the root do not decay rapidly towards the tip. No simple formulae can be expected to give a reasonably exact knowledge of the stresses and deformations, and the calculation of the critical aeroelastic speeds would be very tedious. This effect is clearly shown by the numerical work described in the Appendix. The tube considered is highly tapered but unswept and it is shown that the self-equilibrating stress systems arising from the root constraint are of prime importance throughout the whole tube.

It might be expected from the preceding discussion that the results for a swept cylindrical tube could be obtained as the limiting case of those for a conical tube as the apex is taken to infinity. However, the coordinate system which was found most convenient for a cylindrical tube is not easily derived as the limiting case of the system for a conical tube. Consequently, in § 1, which gives the fundamental equations applicable to any load distribution, the two types of tube are considered side by side. Equations which correspond exactly in the two cases bear the same number, followed by a letter "a" if they refer to a conical tube, and by "b" if they refer to a cylindrical tube. Thus, equations (12a) and (12b) for example are corresponding equations for conical and cylindrical tubes respectively.

In succeeding sections, which deal with solutions of the fundamental equations for the two types of tube, the equations are numbered consecutively without any accompanying letters.

NOTATION

SYMBOL	CONICAL TUBE	CYLINDRICAL TUBE
Ox_o, y_o, z_o	An orthogonal R.H. system of axes with origin O at the apex of the tube. The z_o axis is perpendicular to the rib planes and x_o, y_o are any two axes forming an orthogonal R.H. system with z_o. (Fig.2)	An orthogonal R.H. system of axes with origin O at any point in the tip section. The z_o axis is perpendicular to the rib planes and y_o is parallel to the plane of sweep. (Fig.3.)
Z	The constant z_o coordinate of the root section. (Fig.2)	-----
Oz	-----	An axis parallel to the generators with origin at point O in the tip section (Fig.3).
x, y	The x_o, y_o coordinates of any point on the periphery of the root section. (Fig.2)	A pair of coordinate axes parallel to Ox_o, y_o with origin at the point of intersection C of the axis Oz with any rib plane. (Fig.3)
ρ	$= z_o/Z$ (Fig.2)	-----
s	Clockwise distance round the periphery of the root section (Fig.2)	Clockwise distance round the periphery of any rib section, measured from a fixed generator. (Fig.3)
r	$= (x^2 + y^2 + Z^2)^{\frac{1}{2}}$. Length of a generator from the apex to its point of intersection with the root section.	-----
α	-----	Angle of sweep, i.e. the angle between Oz_o and Oz. (Fig.3)
ψ	Angle between the positive directions of s and y. (Figs.2, 3).	
p	$= (x \cos\psi - y \sin\psi)$ — see Figs.2 and 3.	
Δ	$= (x \sin\psi + y \cos\psi)$ — see Figs.2 and 3.	
Γ	$= \Delta\left(1 - \frac{E}{4G}\right)^{-\frac{1}{2}}(Z^2 + p^2)^{-\frac{1}{2}}$	-----

NOTATION - Continued.

SYMBOL	CONICAL TUBE	CYLINDRICAL TUBE
A	Area enclosed by root section.	Area enclosed by any rib section.
A_{os}	Area swept by radius from the point of intersection of Oz_o with root section as its end moves over distance s.	Area swept by radius from C, the point of intersection of Oz with any rib section, as its end moves over distance s.
$\bar{x},\bar{y}$	The x,y coordinates of the centroid of the area A.	
ϕ	Angle in the tangent plane between the generator and the tangent to the rib at any point on the tube surface. (Fig.4).	
l_o, m_o, n_o	Direction cosines of a generator, referred to Ox_o, y_o, z_o. (Fig.4).	
l, m, n	Direction cosines of a line in the tangent plane perpendicular to the tangent to the rib at any point on the tube surface. (Fig.4).	
l', m', n'	Direction cosines of a line in the tangent plane perpendicular to the generator at any point on the tube surface (Fig.4).	
N	Number of booms in the tube section.	
W	Number of walls in a multi-cell tube section.	
E	Young's modulus.	
G	Modulus of rigidity.	
μ	Poisson's ratio.	
E'	$= E/(1 - \mu^2)$.	
t	Skin thickness.	
t', k	Defined by equation (48).	Defined by equation (148).
B_m	Cross sectional area of the mth boom.	

NOTATION - Continued.

SYMBOL	CONICAL TUBE	CYLINDRICAL TUBE
B'_m	Value of B_m at $\rho=1$.	Value of B_m at $z=0$.
u, v	Displacements in the x_o, y_o directions of the point of intersection of the axis Oz_o with a rigid rib.	Displacements in the x_o, y_o directions of the point of intersection C of the axis Oz with a rigid rib.
θ	Clockwise rotation of a rigid rib in its own plane.	
w	Displacement in the z_o direction of any point on the tube surface.	
v_T	Displacement in the direction of s of any point on the tube surface.	
w_P	Displacement in the tangent plane in a direction perpendicular to s of any point on the tube surface.	
w_g	Displacement in the direction of the generator of any point on the tube surface.	
w^*	That part of w corresponding to warping of the ribs out of their own planes.	
$\bar{u}, \bar{v}, \bar{\theta}$	Curvature and twist constants, defined as required in the text.	
γ	Shear strain of the skin referred to directions along and perpendicular to s (Fig.4).	
ε	Linear strain of the skin in a direction perpendicular to s (Fig.4).	
ε_g	Linear strain of the skin along the generators (Fig.4).	
q	$= Gt\gamma$. Shear flow in the skin in directions parallel and perpendicular to s (Fig.5).	
f	$= E't\varepsilon$. Direct stress flow in the skin in a direction perpendicular to s (Fig.5).	
q_g	Shear flow in the skin in directions parallel and perpendicular to the generators (Fig.5).	

SYMBOL	CONICAL TUBE	CYLINDRICAL TUBE
f_g	Direct stress flow in the skin perpendicular to the generators (Fig.5).	
P_m	$= E\,\varepsilon_g\,B_m$. Tensile force in the mth boom.	
R	Total applied force at any section in a direction normal to the ribs.	
S_x, S_y	Shear forces at any section along the x and y axes.	
M_x, M_y	Bending moments at any section about the x and y axes.	
T	Torque at any section about the axis Oz_o.	Torque at any section about an axis parallel to Oz_o and passing through C.
η, t_E	Defined by equations (95) and (96).	Defined by equations (203) and (204).
ξ, ξ_x, ξ_y	Defined by equations (104).	Defined by equations (208).
$\left.\begin{matrix}\lambda, \lambda_x, \lambda_y \\ I_{xx}, I_{yy}, I_{xy}\end{matrix}\right\}$	Defined by equations (105).	Defined by equations (209).
F	Stress function.	
h	Function of s) quantities involved in particular	
β	A constant) stress functions.	
C, D	Constants of integration.	
J_β, W_β	Functions defined by equations (131).	-----

Suffixes i,j refer to values associated with the ith and jth eigenvalues.

Suffix m refers to values associated with the mth boom.

Suffix m,m+1 refers to the value at the mth boom in the wall joining the mth and (m+1)th booms.

§1. FUNDAMENTAL EQUATIONS FOR ARBITRARY LOAD DISTRIBUTION :

§1.1. Coordinate systems :

In the analysis for both conical and cylindrical tubes it is necessary to use two different coordinate systems for locating a point on the surface of the tube. In each case the first system refers to an orthogonal set of axes, and the second system defines the generator and the rib intersecting at the point. It is convenient to consider the two cases separately.

(a) Conical tubes.

Let x_o, y_o, z_o be a right handed orthogonal set of axes, as shown in Fig.2, with origin O at the apex of the tube. The axis z_o is perpendicular to the rib planes and x_o, y_o are in any other directions perpendicular to z_o and to each other. We may define any rib section by z_o = constant, and in particular the root section corresponds to $z_o = Z$, where Z is a constant equal to the perpendicular distance from the apex to the root section.

Further let x,y be the coordinates referred to the axes x_o, y_o of any point on the root section, and let s be the distance round the periphery of the root section. Then any generator may be defined by the coordinates (x,y) or s of its point of intersection with the root section. Also, if $z_o = \rho Z$, any rib section may be defined by ρ = constant, the root section being $\rho = 1$. Hence it will be seen that any point on the surface of the tube may be located by the coordinates (x,y,ρ) or (s,ρ), defining the generator and the rib intersecting at the point.

It should be noted that the operators $\partial/\partial\rho$ and $\partial/\partial s$ in the subsequent analysis refer to variations along a generator and round the periphery of a rib section respectively.

Since all rib sections are similar in shape to the root section the two coordinate systems are related by

$$\frac{x_o}{x} = \frac{y_o}{y} = \frac{z_o}{Z} = \rho \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

(b) Cylindrical tubes.

Let x_o, y_o, z_o be a right-handed orthogonal set of axes, as shown in Fig.3, with origin at any point O in the tip section. The axis z_o is perpendicular to the rib planes and y_o is parallel to the plane of sweep, so that all the generators are parallel to the plane containing the axes y_o and z_o.

Now let z be an axis parallel to the generators of the tube and with its origin at the point O. Let this axis intersect a given rib section at the point C, so that OC = z. Then the length of any generator from the tip section to its point of intersection with the given rib section is also equal to z, and we may therefore define a rib by z = constant. To locate a point completely we need its position on the rib and for this purpose we take two new axes Cx and Cy with origin at the point of intersection C of the z axis with the rib plane and parallel to Ox_o and Oy_o respectively. Also let s be the distance round the periphery of the rib. Then we may define the position of any point on the tube surface by (x,y,z) or (s,z), the coordinate z defining the rib and x,y or s the generator intersecting at the point.

The operators $\partial/\partial z$ and $\partial/\partial s$ refer to variations along a generator and round the periphery of a rib section respectively.

The angle α between z and z_o is called the angle of sweep, and the two coordinate systems are related by the equations

$$\left.\begin{aligned} x &= x_o \\ y &= y_o - z_o\tan\alpha \\ z &= z_o\sec\alpha \end{aligned}\right) \quad \text{or} \quad \left(\begin{aligned} x_o &= x \\ y_o &= y + z\sin\alpha \\ z_o &= z\cos\alpha \end{aligned}\right. \qquad \text{.......(2)}$$

§1.2 Geometry :

In a conical tube all rib sections are similar in shape to the root section (ρ=1), the linear dimensions being in the ratio ρ. Consequently the geometry of the whole tube may be defined simply by the geometry of the root section. Let ψ in Fig.2 be the angle between the tangent to the periphery of the root section and the axis y_0, and let p be the length of the perpendicular to this tangent from the axis z_0, so that

$$\sin\psi = \frac{dx}{ds}, \qquad \cos\psi = \frac{dy}{ds} \qquad \ldots\ldots\ldots (3)$$

$$p = (x\cos\psi - y\sin\psi) \qquad \ldots\ldots\ldots (4)$$

Further let Δ be the distance between the foot of the perpendicular and the point of tangency so that

$$\Delta = (x\sin\psi + y\cos\psi) \qquad \ldots\ldots\ldots (5)$$

In the case of a cylindrical tube the angle ψ in Fig.3 is defined as the angle between the tangent to the periphery of any rib section and the axis y_0, whilst p is the length of the perpendicular to this tangent from the point C (the intersection of the z axis with the rib plane). The distance between the foot of this perpendicular and the point of tangency is Δ. It will be seen from a comparison of Figs.2 and 3 that equations (3), (4) and (5) are still applicable.

We now consider the directions of various lines lying in the tangent plane at any point A on the surface of the tube. It should be noted that this plane will be tangential to the surface at all points on the generator through A.

Referring to Fig.4, the line AT is the tangent to the rib through A, the line AG is a generator, and AP and AQ are perpendicular to AT and AG respectively.

Let (l_0, m_0, n_0) be the direction cosines with respect to the orthogonal axes (x_0, y_0, z_0) of the generator AG. In a conical tube these are given by

$$\left.\begin{aligned} l_0 = \frac{x}{r}, \quad m_0 = \frac{y}{r}, \quad n_0 = \frac{Z}{r} \\ \text{where} \quad r^2 = x^2 + y^2 + Z^2 = p^2 + \Delta^2 + Z^2 \end{aligned}\right\} \ldots\ldots\ldots (6a)$$

whilst in a cylindrical tube

$$l_0 = 0, \quad m_0 = \sin\alpha, \quad n_0 = \cos\alpha \qquad \ldots\ldots\ldots (6b)$$

In each case the direction cosines of the tangent AT are $(\sin\psi, \cos\psi, 0)$. Let (l,m,n) and (l',m',n') be the direction cosines of AP and AG respectively.

Since AP and AT are perpendicular we have

$$l\sin\psi + m\cos\psi = 0 \qquad \ldots\ldots\ldots (7)$$

Also, since AP, AT and AG are coplanar

$$\begin{vmatrix} \sin\psi & \cos\psi & 0 \\ l_0 & m_0 & n_0 \\ l & m & n \end{vmatrix} = 0 \qquad \ldots\ldots\ldots (8)$$

Equations (7) and (8) in conjunction with

$$l^2 + m^2 + n^2 = 1$$

suffice to determine (l,m,n). On solving we obtain

$$\left.\begin{aligned} l &= \cos\psi\,(l_0\cos\psi - m_0\sin\psi)\frac{n}{n_0} \\ m &= -\sin\psi\,(l_0\cos\psi - m_0\sin\psi)\frac{n}{n_0} \\ n &= n_0\left[1 - (l_0\sin\psi + m_0\cos\psi)^2\right]^{-\frac{1}{2}} \end{aligned}\right\} \qquad \ldots\ldots (9)$$

Proceeding in an exactly similar way we find that the direction cosines (l',m',n') of the line AQ are given by

$$\left.\begin{aligned} l' &= \frac{n}{n_0}\left[\sin\psi - l_0(l_0\sin\psi + m_0\cos\psi)\right] \\ m' &= \frac{n}{n_0}\left[\cos\psi - m_0(l_0\sin\psi + m_0\cos\psi)\right] \\ n' &= -n\,(l_0\sin\psi + m_0\cos\psi) \end{aligned}\right\} \ldots\ldots (10)$$

where n is given by the third of equations (9).

If ϕ is the angle between the tangent AT and the generator AG we have

$$\left.\begin{aligned} \cos\phi &= (l_0\sin\psi + m_0\cos\psi) \\ \sin\phi &= n_0/n \end{aligned}\right\} \ldots\ldots\ldots\ldots (11)$$

On substituting (6a) into equations (9), (10) and (11) it is found that for a conical tube

$$\left.\begin{aligned} l &= p\cos\psi\,(Z^2 + p^2)^{-\frac{1}{2}} \\ m &= -p\sin\psi\,(Z^2 + p^2)^{-\frac{1}{2}} \\ n &= Z(Z^2 + p^2)^{-\frac{1}{2}} \end{aligned}\right\} \ldots\ldots\ldots\ldots (12a)$$

$$\left.\begin{aligned} l' &= \frac{nr}{Z}\left[\sin\psi - \frac{x\Delta}{r^2}\right] \\ m' &= \frac{nr}{Z}\left[\cos\psi - \frac{y\Delta}{r^2}\right] \\ n' &= -\frac{n\Delta}{r} \end{aligned}\right) \quad \ldots\ldots\ldots\ldots \quad (13a)$$

$$\left.\begin{aligned} \cos\phi &= \frac{\Delta}{r} \\ \sin\phi &= \frac{1}{r}\left(Z^2 + p^2\right)^{\frac{1}{2}} \end{aligned}\right) \quad \ldots\ldots\ldots\ldots \quad (14a)$$

Similarly, substitution of (6b) into (9),(10) and (11) gives for a cylindrical tube

$$\left.\begin{aligned} l &= -n\tan\alpha\sin\psi\cos\psi \\ m &= \ \ n\tan\alpha\sin^2\psi \\ n &= \left[1 + \tan^2\alpha\sin^2\psi\right]^{-\frac{1}{2}} \end{aligned}\right) \quad \ldots\ldots\ldots\ldots \quad (12b)$$

$$\left.\begin{aligned} l' &= n\sec\alpha\sin\psi \\ m' &= n\cos\alpha\cos\psi \\ n' &= -n\sin\alpha\cos\psi \end{aligned}\right) \quad \ldots\ldots\ldots\ldots \quad (13b)$$

$$\left.\begin{aligned} \cos\phi &= \sin\alpha\cos\psi \\ \sin\phi &= \cos\alpha/n \end{aligned}\right) \quad \ldots\ldots\ldots\ldots \quad (14b)$$

Frequent reference will be made to the above equations during the subsequent analysis.

§1.3 Strains and displacements:

We now consider the tube to be loaded so that it undergoes small displacements from its unstrained position. Since it is assumed that the ribs are infinite in number and rigid in their own planes the displacement of any point on the surface of the tube may be defined by the rigid body movements of the rib passing through the point, together with a warping displacement in a direction normal to the rib.

In both conical and cylindrical tubes we let θ be the rotation of a rib about an axis parallel to z_o, and w be the component displacement in the direction of z_o of any point A on the surface of the tube. Also let v_T, w_P and w_g be the components of the displacement of A in the directions AT, AP and AG respectively.

The shear strain in the skin, referred to the directions AT and AP, is denoted by γ, and the linear strains in the directions AP and AG by ε and ε_g respectively. From the assumption of rigid ribs the linear strain in the direction of AT must be zero.

It is convenient now to consider the conical and cylindrical tubes separately.

(a) Conical tubes.

In addition to the definitions given above we let u and v be the displacements in the directions of x_o and y_o of the point of intersection of a rib with the axis z_c, so that u, v, θ define the rigid body movements of the rib.

The displacements of the point A in the x_o and y_o directions are then equal to $(u - \rho y\theta)$ and $(v + \rho x\theta)$ respectively. Resolving these in the directions AT, AP and AG we obtain the equations

$$v_T = \sin\psi\,(u - \rho y\theta) + \cos\psi\,(v + \rho x\theta) \quad \ldots\ldots \quad (15a)$$

$$w_P = l(u - \rho y\theta) + m(v + \rho x\theta) + nw \quad \ldots\ldots\ldots \quad (16a)$$

$$w_g = l_o u + m_o v + n_o w \quad \ldots\ldots\ldots\ldots\ldots \quad (17a)$$

It should be noted that no terms in θ appear in (17) because the generators all intersect the axis z_o at the apex and therefore a pure rotation θ about z_o cannot give rise to a displacement along the generators.

The shear strain γ in the skin is given by the equation

$$\gamma = \frac{\partial v_T}{\partial z_P} + \frac{\partial w_P}{\rho\partial s} \quad \ldots\ldots\ldots\ldots\ldots \quad (18a)$$

where dz_P is an element of length along AP.

It will be seen from a consideration of Fig.4 that

$$\frac{\partial v_T}{\partial z_P} = \operatorname{cosec}\phi\,\frac{\partial v_T}{r\partial\rho} - \cot\phi\,\frac{\partial v_T}{\rho\partial s}$$

Also, since the ribs are rigid in their own planes

$$\frac{\partial v_T}{\rho \partial s} = 0$$

Hence,

$$\gamma = \operatorname{cosec} \phi \frac{\partial v_T}{r \partial \rho} + \frac{\partial w_\rho}{\rho \partial s} \qquad \ldots\ldots\ldots\ldots\ldots (19a)$$

Remembering that u, v and θ are functions of ρ only, x and y functions of s only and w a function of both ρ and s, substitution of (15a) and (16a) into (19a) and simplification by means of (4) and (12a) gives

$$\gamma = \frac{n}{Z} \left[\frac{Z}{\rho} \frac{\partial w}{\partial s} + \sin \psi \frac{du}{d\rho} + \cos \psi \frac{dv}{d\rho} + p\rho \frac{d\theta}{d\rho} \right] \qquad \ldots\ldots (20a)$$

The linear strain ε_g along the generator is given by

$$\varepsilon_g = \frac{1}{r} \frac{\partial w_g}{\partial \rho}$$

Substitution of equation (17a) then gives

$$\varepsilon_g = \frac{1}{r^2} \left[Z \frac{\partial w}{\partial \rho} + x \frac{du}{d\rho} + y \frac{dv}{d\rho} \right] \qquad \ldots\ldots\ldots\ldots\ldots (21a)$$

It is well known that if the strains in three directions at a point are given, then the strain in any other direction may be determined by simple geometrical considerations. Hence, using the fact that the linear strain along AT is zero we can obtain in this way a relation between ε , ε_g and γ . This relation is found to be

$$\varepsilon_g = \varepsilon \sin^2 \phi + \gamma \sin \phi \cos \phi \qquad \ldots\ldots\ldots (22)$$

where ϕ is the angle shown in Fig.4.

By virtue of equations (11) and (14a) this becomes

$$\varepsilon_g = \frac{n_o^2}{n^2} \varepsilon + \frac{n_o \Delta}{nr} \gamma \qquad \ldots\ldots\ldots\ldots\ldots (23a)$$

If equations (20a) and (21a) are solved for $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial \rho}$ respectively we obtain

$$\frac{\partial w}{\partial s} = \frac{\rho \gamma}{n} - \frac{\rho}{Z} \left[\frac{du}{d\rho} \sin \psi + \frac{dv}{d\rho} \cos \psi + p\rho \frac{d\theta}{d\rho} \right] \qquad \ldots\ldots\ldots (24a)$$

$$\frac{\partial w}{\partial \rho} = \frac{r^2 \varepsilon_g}{Z} - \frac{1}{Z} \left[x\frac{du}{d\rho} + y\frac{dv}{d\rho} \right] \qquad \ldots\ldots\ldots\ldots (25a)$$

On differentiating these two equations with respect to ρ and s respectively and then subtracting, the displacement w is eliminated and we obtain the equation

$$\frac{Z}{n}\frac{\partial}{\partial\rho}(\rho\gamma) - \frac{\partial}{\partial s}(r^2\varepsilon_g) = \rho\left[\frac{d^2u}{d\rho^2}\sin\Psi + \frac{d^2v}{d\rho^2}\cos\Psi + p\frac{d^2}{d\rho^2}(\rho\theta)\right] \quad \ldots\ldots\ldots \quad (26a)$$

Equation (26a) will be called the equation of compatibility of strains.

(b) Cylindrical tubes.

Let u, v be the displacements in the x_o, y_o directions of the point of intersection of a rib with the z axis (not the z_o axis as in the conical tube) so that again u,v,θ define the rigid body movements of the rib.

The derivation of the equations corresponding to (15a) - (26a) above follows an exactly similar course and to save unnecessary repetition the equations will be written down without reproducing the full arguments. Thus

$$v_T = \sin\Psi\,(u - y\theta) + \cos\Psi\,(v + x\theta) \quad \ldots\ldots\ldots \quad (15b)$$

$$w_P = l\,(u - y\theta) + m(v + x\theta) + nw \quad \ldots\ldots\ldots \quad (16b)$$

$$w_g = (v + x\theta)\sin\alpha + w\cos\alpha \quad \ldots\ldots\ldots \quad (17b)$$

$$\gamma = \frac{\partial v_T}{\partial z_P} + \frac{\partial w_P}{\partial s} \quad \ldots\ldots\ldots \quad (18b)$$

$$= \operatorname{cosec}\phi\,\frac{\partial v_T}{\partial z} + \frac{\partial w_P}{\partial s} \quad \ldots\ldots\ldots \quad (19b)$$

$$= n\sec\alpha\left[\cos\alpha\,\frac{\partial w}{\partial s} + \sin\Psi\left(\frac{du}{dz} + \theta\sin\alpha\right) + \cos\Psi\frac{dv}{dz} + p\frac{d\theta}{dz}\right] \quad (20b)$$

$$\varepsilon_g = \frac{\partial w_g}{\partial z} = \cos\alpha\,\frac{\partial w}{\partial z} + \sin\alpha\,\frac{dv}{dz} + x\,\sin\alpha\frac{d\theta}{dz} \quad \ldots\ldots \quad (21b)$$

Equation (22) still applies so that

$$\varepsilon_g = \frac{\cos^2\alpha}{n^2}\,\varepsilon + \frac{1}{n}\sin\alpha\cos\alpha\cos\Psi.\gamma \quad \ldots\ldots \quad (23b)$$

Solving (20b) and (21b) for $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial z}$ respectively gives

$$\frac{\partial w}{\partial s} = \frac{\gamma}{n} - \sec\alpha\left[\sin\Psi\left(\frac{du}{dz} + \theta\sin\alpha\right) + \cos\Psi\frac{dv}{dz} + p\frac{d\theta}{dz}\right] \quad \ldots (24b)$$

$$\frac{\partial w}{\partial z} = \sec\alpha\ \varepsilon_g - \tan\alpha\frac{dv}{dz} - x\tan\alpha\frac{d\theta}{dz} \quad \ldots\ldots\ldots (25b)$$

Elimination of w between (24b) and (25b) then gives the equation of compatibility

$$\frac{\cos\alpha}{n}\frac{\partial\gamma}{\partial z} - \frac{\partial\varepsilon_g}{\partial s} = \frac{d^2u}{dz^2}\sin\psi + \frac{d^2v}{dz^2}\cos\psi + p\frac{d^2\theta}{dz^2} \quad \ldots\ldots (26b)$$

§1.4 Twist and curvatures:

It is clear from purely physical grounds that if the internal strains are known at every point in the tube then it should be possible to calculate the "twist" and "curvatures", defined here as $\frac{d\theta}{d\rho}$, $\frac{d^2u}{d\rho^2}$ and $\frac{d^2v}{d\rho^2}$ in a conical tube, and $\frac{d\theta}{dz}$, $\frac{d^2u}{dz^2}$ and $\frac{d^2v}{dz^2}$ in a cylindrical tube. It is highly important to obtain these relations since the whole basis of the subsequent treatment of the differential equations depends upon them.

(a) Conical tubes.

To obtain the expression for the twist $\frac{d\theta}{d\rho}$ take the circuit integral of equation (24a) round the periphery of the root section, thus

$$\oint\frac{\partial w}{\partial s}ds = \rho\oint\frac{\gamma}{n}ds - \frac{\rho}{Z}\left[\frac{du}{d\rho}\oint dx + \frac{dv}{d\rho}\oint dy + \rho\frac{d\theta}{d\rho}\oint p\,ds\right]$$

Since w, x and y are the same at both ends of the circuit, the L.H.S. and the second and third terms on the R.H.S. of the above equation vanish. Also we have

$$\oint p\,ds = 2A$$

where A is the area enclosed by the root section. Hence the above equation gives

$$\rho\frac{d\theta}{d\rho} = \frac{Z}{2A}\oint\frac{\gamma}{n}ds \quad \ldots\ldots\ldots\ldots (27a)$$

The corresponding equation for the curvature $\frac{d^2u}{d\rho^2}$ is obtained by multiplying equation (26a) throughout by y and again taking the circuit integral. Using equation (27a), the fact that y, r and ε_g

are the same at both ends of the circuit, and the relations

$$\oint y\,dx = -A, \qquad \oint y\,dy = 0, \qquad \oint p\,y\,ds = 3A\bar{y}$$

where $\bar{x},\bar{y}$ are the coordinates of the centroid of the area A enclosed by the root section, we finally obtain the equation

$$\rho\frac{d^2u}{d\rho^2} = -\frac{Z}{2A}\oint\frac{(2y-3\bar{y})}{n}\frac{\partial}{\partial\rho}(\rho\gamma)ds - \frac{1}{A}\oint r^2\varepsilon_g\cos\Psi\,ds. \quad \ldots\ldots(28a)$$

In an exactly similar way

$$\rho\frac{d^2v}{d\rho^2} = \frac{Z}{2A}\oint\frac{(2x-3\bar{x})}{n}\frac{\partial}{\partial\rho}(\rho\gamma)ds + \frac{1}{A}\oint r^2\varepsilon_g\sin\Psi\,ds. \quad \ldots\ldots(29a)$$

(b) Cylindrical tubes.

The equations for a cylindrical tube corresponding to those above follow in an exactly similar way and the results are

$$\frac{d\theta}{dz} = \frac{\cos\alpha}{2A}\oint\frac{\gamma}{n}\,ds \quad \ldots\ldots\ldots\ldots\ldots(27b)$$

$$\frac{d^2u}{dz^2} = -\frac{\cos\alpha}{2A}\oint\frac{(2y-3\bar{y})}{n}\frac{\partial\gamma}{\partial z}ds - \frac{1}{A}\oint\varepsilon_g\cos\Psi\,ds \quad \ldots\ldots(28b)$$

$$\frac{d^2v}{dz^2} = \frac{\cos\alpha}{2A}\oint\frac{(2x-3\bar{x})}{n}\frac{\partial\gamma}{\partial z}ds + \frac{1}{A}\oint\varepsilon_g\sin\Psi\,ds \quad \ldots\ldots(29b)$$

where in this case A is the area of any rib section and $\bar{x},\bar{y}$ are the coordinates, referred to the x and y axes, of its centroid.

It is perhaps worth emphasising that all the above equations are perfectly general in that there is no restriction on either the taper, the sweep or nature of the applied loads. They are a natural consequence of the fact that the ribs retain their shape. Previous investigators into the theory of unswept cylindrical tubes have derived an equation corresponding to (27a) and (27b) above. Indeed, if $\alpha=0$ so that $n=1$, equation (27b) becomes identical with equation (14) of Ref.1. However, in spite of the necessity for their existence on physical grounds, to the author's knowledge no relations between the curvatures and internal strains have previously been derived.

§1.5. Warping of the ribs:

We consider now the warping of a rib section out of its original plane. Again from physical considerations it is evident that if the internal strains are known this warping should be completely determinate except for arbitrary rigid body movements.

(a) Conical tubes.

If equation (24a) is integrated with respect to s we obtain

$$w = w_{so} - \frac{\rho}{Z}\frac{du}{d\rho}(x - x_{so}) - \frac{\rho}{Z}\frac{dv}{d\rho}(y - y_{so}) + \rho\left[\int_0^s \frac{\gamma}{n}ds - \frac{A_{os}}{A}\oint \frac{\gamma}{n}ds\right] \qquad \ldots\ldots\ (30a)$$

where suffix "so" refers to the value at s=0 and A_{os} is the area swept by the radius from the z_o axis as its end moves over the distance s on the periphery of the root section.

The first three terms on the R.H.S. of (30a) represent rigid body movements of the ribs, by uniform displacement in the z_o direction and by rotations about axes parallel to y_o and x_o respectively. The remaining terms must, therefore, include all displacements corresponding to warping of the ribs out of their original planes. We may now define a warping function w^* by the equation

$$w^* = \rho\left[\int_0^s \frac{\gamma}{n}ds - \frac{A_{os}}{A}\oint \frac{\gamma}{n}ds\right] \qquad \ldots\ldots\ (31a)$$

Alternatively the rate of change of warping round the section is given by

$$\frac{\partial w^*}{\partial s} = \rho\left[\frac{\gamma}{n} - \frac{p}{2A}\oint \frac{\gamma}{n}ds\right] \qquad \ldots\ldots\ (32a)$$

(b) Cylindrical tubes.

The corresponding results for a cylindrical tube are obtained in an exactly similar way from equation (24b), thus

$$w = w_{so} - \sec\alpha\left(\frac{du}{dz} + \theta\sin\alpha\right)(x - x_{so}) - \sec\alpha\frac{dv}{dz}(y - y_{so}) + \left[\int_0^s \frac{\gamma}{n}ds - \frac{A_{os}}{A}\oint \frac{\gamma}{n}ds\right] \qquad \ldots\ldots\ (30b)$$

Hence we define w^* by

$$w^* = \int_0^s \frac{\gamma}{n} ds - \frac{A_{os}}{A} \oint \frac{\gamma}{n} ds \qquad \text{.......(31b)}$$

and $$\frac{\partial w^*}{\partial s} = \frac{\gamma}{n} - \frac{p}{2A} \oint \frac{\gamma}{n} ds \qquad \text{.......(32b)}$$

It should be emphasised that the warping function w^* as defined by equations (31a) and (31b) above is not in general the complete normal displacement w, and care should be exercised in using it. It is convenient however to have such a simple warping function, for the purpose of dealing with end effects, but whenever complete displacements are required equations (30a) and (30b) must be used.

§1.6 Stress flows in the skin:

So far only displacements and strains have been considered and we turn now to the stress flows in the skin. (Following standard practice a stress flow is defined as a stress times the skin thickness, or the force per unit sectional length of skin.)

Let q be the shear flow in the skin in the directions AT and AP, and let f be the direct stress flow in the direction AP, as shown in Fig.5. Since the linear strain along AT is zero the direct stress flow in this direction must be equal to μf, where μ is Poisson's ratio. Then, if E and G are the Young's modulus and modulus of rigidity respectively and if $E' = E/(1-\mu^2)$, the stress flows q and f are related to the strains γ and ε by the equations

$$q = Gt\gamma \qquad \text{...............(33)}$$

$$f = E't\varepsilon \qquad \text{...............(34)}$$

where t is the thickness of the skin.

When the equations of equilibrium of the skin and booms are considered it will be necessary to have the shear and direct stress flows q_g and f_g over the generator AG (see Fig.5) in terms of q and f. Considering the equilibrium of the small element shown in Fig.5, we have

$$q_g = q(\sin^2\phi - \cos^2\phi) - (1-\mu)f \sin\phi \cos\phi$$
$$f_g = f(\cos^2\phi + \mu \sin^2\phi) - 2q \sin\phi \cos\phi$$

The angle ϕ is given by equations (14a) for a conical tube and (14b) for a cylindrical tube.

Hence, for a conical tube

$$q_g = \left[2\frac{n_0^2}{n^2} - 1\right]q - (1-\mu)\frac{\Delta Z}{nr^2} f$$
$$f_g = \left[1-(1-\mu)\frac{n_0^2}{n^2}\right]f - 2\frac{\Delta Z}{nr^2} q \qquad \text{.........(35a)}$$

whilst for a cylindrical tube

$$q_g = \left[2\frac{\cos^2\alpha}{n^2} - 1\right]q - \frac{(1-\mu)}{n}\sin\alpha\cos\alpha\cos\psi\, f \qquad \text{.........(35b)}$$
$$f_g = \left[1-(1-\mu)\frac{\cos^2\alpha}{n^2}\right]f - \frac{2}{n}\sin\alpha\cos\alpha\cos\psi\, q$$

§1.7 The shear stress at a built-in end:

We are now in a position to reach certain general conclusions about the distribution of shear stress round the root section if the tube is held in such a way that w, and therefore $\frac{\partial w}{\partial s}$, is zero at all points on this section.

By virtue of equations (20a) and (33), with $\frac{\partial w}{\partial s} = 0$ and $\rho = 1$, we obtain for the conical tube

$$\frac{q}{t} = \frac{Gn}{Z}\left[\sin\psi\frac{du}{d\rho} + \cos\psi\frac{dv}{d\rho} + p\frac{d\theta}{d\rho}\right] \qquad \text{.........(36a)}$$

where $\frac{du}{d\rho}$, $\frac{dv}{d\rho}$ and $\frac{d\theta}{d\rho}$ take their values at the root section (ρ=1).

In a similar way, from equations (20b) and (33) with $\frac{\partial w}{\partial s} = 0$ and $\theta = 0$, we have for the cylindrical tube

$$\frac{q}{t} = Gn\sec\alpha\left[\sin\psi\frac{du}{dz} + \cos\psi\frac{dv}{dz} + p\frac{d\theta}{dz}\right] \qquad \text{......(36b)}$$

where $\frac{du}{dz}$, $\frac{dv}{dz}$ and $\frac{d\theta}{dz}$ are again the values at the root section.

In either case it will be seen from the two above equations that if the section of the tube is a polygon, so that in each straight side ψ, p and n are constants, then the shear stress (q/t) is also a constant in each straight side of the root section. This result is useful in checking the accuracy of an approximate numerical method of dealing with this type of end constraint described later.

Hadji-Argyris and Dunne (Ref.1) showed that in an unswept cylindrical tube the shear stress (q/t) round a non-warping section is statically determinate. This result follows from the fact that the root shear forces (in the x and y directions) and the root torque (about the axis z_o) are balanced entirely by the shear flow q acting round the root section, thus leading to three simultaneous equations from which the root values of $\frac{du}{dz}$, $\frac{dv}{dz}$ and $\frac{d\theta}{dz}$ can be determined. Knowing these values the shear stress distribution is obtained from equation (36b) with n=1 and α=0. However, in the type of tube with which we are concerned here this argument does not apply, since the direct stress flow f and the boom tensions also contribute towards the root shear forces and torque, and it is no longer possible to calculate the shear stress distribution from considerations of statics.

§1.8 Equilibrium of the skin and the stress function F:

The subsequent solutions are most conveniently obtained by expressing the various stress flows in the skin in terms of a stress function F which is defined in such a way that the equation of equilibrium of the skin is automatically satisfied. To obtain this equation of equilibrium we consider a small element of the skin bounded by two generators and by two ribs. The forces acting on the sides of the element (arising from the various stress flows) are shown in Fig.6(a) for a conical tube and Fig.6(b) for a cylindrical tube. The direction cosines of these stress flows are as follows:-

q_g ....(l_o,m_o,n_o) given by equations (6a) or (6b)
f_g ....(l',m',n') given by equations (13a) or (13b)
f ....(l ,m ,n) given by equations (12a) or (12b)
q ....($\sin\psi$, $\cos\psi$, 0)

In addition to these forces the ribs will also be exerting forces on the inside face of the element. However, since the ribs are assumed to be incapable of exerting forces normal to their own planes, the stress flows on the sides of the element of the skin must have zero resultant in the direction normal to the ribs (i.e. parallel to z_o). This leads to an equation of equilibrium, and it is convenient now to consider the conical and cylindrical tubes separately.

(a) Conical tubes.

Resolving the forces on the sides of the element shown in Fig.6(a) in the z_o direction we obtain the equation

$$\frac{\partial}{\partial \rho}(n f \rho) + \frac{\partial}{\partial s}\left[r(n_o q_g + n' f_g)\right] = 0 \qquad \ldots\ldots\ldots (37a)$$

This is satisfied by a stress function F defined by

$$f = \frac{1}{n\rho}\frac{\partial F}{\partial s} \qquad \ldots\ldots\ldots (38a)$$

$$n_o q_g + n' f_g = -\frac{1}{r}\frac{\partial F}{\partial \rho} \qquad \ldots\ldots\ldots (39a)$$

But by virtue of equations (35a), (6a) and (13a)

$$n_o q_g + n' f_g = \frac{1}{r}\left[Zq - n\Delta f\right]$$

On substituting this equation into (39a), using (38a) and solving for q we obtain

$$q = \frac{1}{Z}\left[\frac{\Delta}{\rho}\frac{\partial F}{\partial s} - \frac{\partial F}{\partial \rho}\right] \qquad \ldots\ldots\ldots (40a)$$

To obtain the linear strain ε_g along the generator in terms of the stress function we have from equations (23a), (33), (34), (38a) and (40a)

$$\varepsilon_g = \frac{1}{Gntr^2}\left[\left(\Delta^2 + \frac{G}{E'}\frac{Z^2}{n^2}\right)\frac{\partial F}{\rho\partial s} - \Delta\frac{\partial F}{\partial \rho}\right]$$

But since $E = E'(1-\mu^2) = 2G(1+\mu)$ we may write

$$\frac{G}{E'} = 1 - \frac{E}{4G}$$

The expression for ε_g then reduces to

$$\varepsilon_g = \frac{1}{Gntr^2}\left[\left(r^2 - \frac{E}{4G}\frac{Z^2}{n^2}\right)\frac{\partial F}{\rho\partial s} - \Delta\frac{\partial F}{\partial \rho}\right] \qquad \ldots\ldots\ldots (41a)$$

(b) Cylindrical tubes.

The corresponding results for the cylindrical tube follow in an exactly similar way. Thus, the equation of equilibrium is

$$\frac{\partial}{\partial z}(nf) + \frac{\partial}{\partial s}\left(\cos\alpha\, q_g + n' f_g\right) = 0 \qquad \text{.........(37b)}$$

which is satisfied by a stress function F defined by

$$f = \frac{1}{n}\frac{\partial F}{\partial s} \qquad \text{.........(38b)}$$

$$\cos\alpha\, q_g + n' f_g = -\frac{\partial F}{\partial z} \qquad \text{.........(39b)}$$

The expression for q becomes

$$q = \tan\alpha\cos\psi\frac{\partial F}{\partial s} - \sec\alpha\frac{\partial F}{\partial z} \qquad \text{.........(40b)}$$

and the linear strain along the generators

$$\varepsilon_g = \frac{1}{Gnt}\left[\left(1 - \frac{E}{4G}\frac{\cos^2\alpha}{n^2}\right)\frac{\partial F}{\partial s} - \sin\alpha\cos\psi\frac{\partial F}{\partial z}\right] \qquad \text{....(41b)}$$

§1.9 The integrodifferential equation in F:

The stress function F has been defined in such a way that every element of the skin is automatically in equilibrium, but we must now ensure that the strains corresponding to F constitute a physically possible system. To do this we turn to the equations of compatibility (26a) and (26b).

(a) Conical tubes.

Substitution of (33), (40a) and (41a) into (26a) yields the equation

$$\frac{\partial}{\partial \rho}\left[\frac{1}{nt}\left(\Delta\frac{\partial F}{\partial s} - \rho\frac{\partial F}{\partial \rho}\right)\right] - \frac{\partial}{\partial s}\left[\frac{1}{nt}\left\{\left(r^2 - \frac{E}{4G}\frac{Z^2}{n^2}\right)\frac{\partial F}{\rho\,\partial s} - \Delta\frac{\partial F}{\partial \rho}\right\}\right]$$
$$= G\rho\left[\frac{d^2u}{d\rho^2}\sin\psi + \frac{d^2v}{d\rho^2}\cos\psi + \rho\frac{d^2}{d\rho^2}(\rho\theta)\right] \qquad \text{.........(42a)}$$

The three terms involving u, v and θ on the R.H.S. of (42a) may themselves be obtained in terms of F by means of equations (27a), (28a) and (29a), thus

$$\rho\frac{d^2}{d\rho^2}(\rho\theta) = \frac{d}{d\rho}\left(\rho^2\frac{d\theta}{d\rho}\right) = \frac{1}{2AG}\oint\frac{\partial}{\partial\rho}\left\{\frac{1}{nt}\left(\Delta\frac{\partial F}{\partial s} - \rho\frac{\partial F}{\partial\rho}\right)\right\}ds \qquad \text{......(43a)}$$

$$\rho \frac{d^2 u}{d\rho^2} = -\frac{1}{2AG}\oint (2y - 3\bar{y})\frac{\partial}{\partial \rho}\left\{\frac{1}{nt}\left(\Delta\frac{\partial F}{\partial s} - \rho\frac{\partial F}{\partial \rho}\right)\right\}ds$$

$$-\frac{1}{AG}\oint \frac{\cos\Psi}{nt}\left\{\left(r^2 - \frac{E}{4G}\frac{Z^2}{n^2}\right)\frac{\partial F}{\rho\,\partial s} - \Delta\frac{\partial F}{\partial \rho}\right\}ds \qquad \text{......(44a)}$$

$$\rho \frac{d^2 v}{d\rho^2} = \frac{1}{2AG}\oint (2x - 3\bar{x})\frac{\partial}{\partial \rho}\left\{\frac{1}{nt}\left(\Delta\frac{\partial F}{\partial s} - \rho\frac{\partial F}{\partial \rho}\right)\right\}ds$$

$$+\frac{1}{AG}\oint \frac{\sin\Psi}{nt}\left\{\left(r^2 - \frac{E}{4G}\frac{Z^2}{n^2}\right)\frac{\partial F}{\rho\,\partial s} - \Delta\frac{\partial F}{\partial \rho}\right\}ds \qquad \text{......(45a)}$$

If equations (43a), (44a) and (45a) are substituted into the R.H.S. of (42a) an integrodifferential equation in F is obtained. It should be pointed out however that this equation can only exist if the boundary conditions for F (considered later) ensure that ε_g is a single valued function of s, since the derivation of equations (27a) to (29a) depended on this fact.

<u>(b) Cylindrical tubes</u>.

The corresponding equations for a cylindrical tube are obtained in a similar way and we obtain

$$\frac{\partial}{\partial z}\left[\frac{1}{nt}\left(\sin\alpha\cos\Psi\frac{\partial F}{\partial s} - \frac{\partial F}{\partial z}\right)\right] - \frac{\partial}{\partial s}\left[\frac{1}{nt}\left\{\left(1 - \frac{E}{4G}\frac{\cos^2\alpha}{n^2}\right)\frac{\partial F}{\partial s} - \sin\alpha\cos\Psi\frac{\partial F}{\partial z}\right\}\right]$$

$$= G\left[\frac{d^2 u}{dz^2}\sin\Psi + \frac{d^2 v}{dz^2}\cos\Psi + p\frac{d^2\theta}{dz^2}\right] \qquad \text{. . . . .(42b)}$$

where

$$\frac{d^2\theta}{dz^2} = \frac{1}{2AG}\oint \frac{\partial}{\partial z}\left[\frac{1}{nt}\left(\sin\alpha\cos\Psi\frac{\partial F}{\partial s} - \frac{\partial F}{\partial z}\right)\right]ds \qquad \text{......(43b)}$$

$$\frac{d^2 u}{dz^2} = -\frac{1}{2AG}\oint (2y - 3\bar{y})\frac{\partial}{\partial z}\left[\frac{1}{nt}\left(\sin\alpha\cos\Psi\frac{\partial F}{\partial s} - \frac{\partial F}{\partial z}\right)\right]ds$$

$$-\frac{1}{AG}\oint \frac{\cos\Psi}{nt}\left[\left(1 - \frac{E}{4G}\frac{\cos^2\alpha}{n^2}\right)\frac{\partial F}{\partial s} - \sin\alpha\cos\Psi\frac{\partial F}{\partial z}\right]ds \qquad \text{......(44b)}$$

$$\frac{d^2v}{dz^2} = \frac{1}{2AG}\oint(2x-3\bar{x})\frac{\partial}{\partial z}\left[\frac{1}{nt}\left(\sin\alpha\cos\Psi\frac{\partial F}{\partial s} - \frac{\partial F}{\partial z}\right)\right]ds$$

$$+\frac{1}{AG}\oint\frac{\sin\Psi}{nt}\left[\left(1-\frac{E}{4G}\frac{\cos^2\alpha}{n^2}\right)\frac{\partial F}{\partial s} - \sin\alpha\cos\Psi\frac{\partial F}{\partial z}\right]ds \qquad \text{......(45b)}$$

The above integrodifferential equations in F can be readily solved for certain types of tube section and thickness variation, by a process consisting essentially of separation of the variables. By combining a number of such solutions it is theoretically possible to build up a solution for arbitrary distribution of applied load and at the same time satisfy the conditions at the ends of the tube. Although the methods of solution are fundamentally the same for both conical and cylindrical tubes, the functions involved in the solutions are quite different, and it is convenient now to consider the two types of tube separately.

§2. METHOD OF SOLUTION FOR CONICAL TUBES :

§2.1 Separation of the variables:

We now consider the possibilities of obtaining basic solutions of equations (42a) in the form

$$F = g(\rho)h(s) \qquad \ldots\ldots\ldots\ldots\ldots (46)$$

In the most general case the skin thickness t will be a function of both s and ρ but it is clear from equation (42a) that separation of the variables will be feasible only if

$$t = t'(s)t^*(\rho) \qquad \ldots\ldots\ldots\ldots\ldots (47)$$

where t' is a function of s and t^* a function of ρ.

If equations (46) and (47) are substituted into (42a), with the terms involving u, v and θ replaced by equations (43a), (44a) and (45a), the resulting equation consists of a series of terms, each of which is a function of ρ times a function of s. Each of these functions of ρ is one or other of the following :-

$$\frac{1}{t^*}\frac{g}{\rho}, \quad \frac{1}{t^*}\frac{dg}{d\rho}, \quad \frac{d}{d\rho}\left(\frac{g}{t^*}\right) \text{ and } \frac{d}{d\rho}\left(\frac{\rho}{t^*}\frac{dg}{d\rho}\right)$$

If then we divide throughout by the first of these functions it will be seen that each of the terms

$$\frac{\rho}{g}\frac{dg}{d\rho}, \quad \frac{\rho t^*}{g}\frac{d}{d\rho}\left(\frac{g}{t^*}\right) \text{ and } \frac{\rho t^*}{g}\frac{d}{d\rho}\left(\frac{\rho}{t^*}\frac{dg}{d\rho}\right)$$

must be a constant independent of ρ. This is possible only if

$$g = \rho^{\beta}, \qquad t = \rho^{k}$$

where β and k are any constants.

Consequently we now restrict the tubes to those in which the thickness variation is governed by

$$t = \rho^k t'(s) \qquad \ldots\ldots\ldots\ldots\ldots (48)$$

and take the stress function to be of the form

$$F = \rho^{\beta} h(s) \qquad \ldots\ldots\ldots\ldots\ldots (49)$$

Physically, equations (48) and (49) mean that the variation of skin thickness and of the stress flows is the same round all rib sections, the magnitudes varying from section to section according to a power of the distance from the apex.

Using equations (48) and (49) we have

$$\frac{\partial}{\partial\rho}\left[\frac{1}{nt}\left(\Delta\frac{\partial F}{\partial s} - \rho\frac{\partial F}{\partial \rho}\right)\right] = (\beta - k)\left[\frac{1}{nt'}\left(\Delta\frac{dh}{ds} - \beta h\right)\right]\rho^{\beta-k-1} \quad \text{......(50)}$$

$$\frac{1}{nt}\left[\left(r^2 - \frac{E}{4G}\frac{Z^2}{n^2}\right)\frac{\partial F}{\rho\partial s} - \Delta\frac{\partial F}{\partial\rho}\right] = \frac{1}{nt'}\left[\left(r^2 - \frac{E}{4G}\frac{Z^2}{n^2}\right)\frac{dh}{ds} - \beta\Delta h\right]\rho^{\beta-k-1} \quad \text{......(51)}$$

It will now be seen from (43a), (44a) and (45a) that each of the terms

$$\rho\frac{d^2}{d\rho^2}(\rho\theta), \quad \rho\frac{d^2u}{d\rho^2} \quad \text{and } \rho\frac{d^2v}{d\rho^2}$$

is simply a constant times $\rho^{\beta-k-1}$. In fact we may write

$$\rho\frac{d^2u}{d\rho^2} = \bar{u}\rho^{\beta-k-1} \quad \text{. . . . .(52)}$$

$$\rho\frac{d^2v}{d\rho^2} = \bar{v}\rho^{\beta-k-1} \quad \text{. . . . .(53)}$$

$$\rho\frac{d^2}{d\rho^2}(\rho\theta) = \bar{\theta}\rho^{\beta-k-1} \quad \text{. . . . .(54)}$$

where $\bar{u}$, $\bar{v}$ and $\bar{\theta}$ are constants.

If equations (50) - (54) are substituted into (42a) and the term $\rho^{\beta-k-1}$ cancelled, the following ordinary differential equation in h is obtained

$$\left(\frac{\beta-k}{nt'}\right)\left(\Delta\frac{dh}{ds} - \beta h\right) - \frac{d}{ds}\left[\frac{1}{nt'}\left\{\left(r^2 - \frac{E}{4G}\frac{Z^2}{n^2}\right)\frac{dh}{ds} - \beta\Delta h\right\}\right]$$
$$= G\left(p\bar{\theta} + \bar{u}\sin\Psi + \bar{v}\cos\Psi\right) \quad \text{. . . . . . (55)}$$

It will be seen that the above differential equation in h is linear and of second order, the coefficients and the right hand side being functions of s. Its solutions will therefore be of the form

$$h = CH_1(s) + DH_2(s) + \bar{\theta}H_3(s) + \bar{u}H_4(s) + \bar{v}H_5(s) \quad \text{......(56)}$$

where C and D are constants of integration, H_1 and H_2 represent the complementary function, and H_3, H_4, H_5, the particular integral of (55).

For each wall of the tube in which t' and ψ are continuous functions of s there will be a separate solution of the form (56), each solution containing the two unknown constants C and D, which will in general have different values in the various walls, together with the three unknown twist and curvature constants $\overline{\theta}$, $\overline{u}$, $\overline{v}$ which are the same for all walls. Thus, in a single cell tube with N booms, connected by N continuous walls, the total number of unknowns appearing in the solutions for h will in general be (2N+3) and we therefore require (2N+3) equations for their evaluation. The first 2N of these arise from conditions of continuity of strain and of equilibrium at each of the N booms and the remaining three from the values of the stress resultants.

Before deriving these equations it is convenient to summarise the expressions for the stress flows corresponding to the stress function (49). Substitution of this equation into (38a), (39a) and (40a) gives

$$f = \frac{1}{n}\frac{dh}{ds}\rho^{\beta-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (57)$$

$$q = \frac{1}{Z}\left(\Delta\frac{dh}{ds} - \beta h\right)\rho^{\beta-1} \quad \ldots\ldots\ldots\ldots\ldots (58)$$

$$n_c q_g + n' f_g = -\beta\frac{h}{r}\rho^{\beta-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (59)$$

Also, from (41a), the linear strain along the generators becomes

$$\varepsilon_g = \frac{1}{Gnr^2t'}\left[\left(r^2 - \frac{E}{4G}\frac{Z^2}{n^2}\right)\frac{dh}{ds} - \beta\Delta h\right]\rho^{\beta-k-1} \quad \ldots\ldots\ldots (60)$$

Consider now the equations arising from the continuity of strain at the mth boom (Fig.7). It is evident that the linear strain ε_g along the generator must be the same in each of the two walls meeting at the mth boom. Thus, by virtue of (60), we have

$$\left[\frac{1}{nt'}\left\{\left(r^2 - \frac{E}{4G}\frac{Z^2}{n^2}\right)\frac{dh}{ds} - \beta\Delta h\right\}\right]_{m,m-1} = \left[\frac{1}{nt'}\left\{\left(r^2 - \frac{E}{4G}\frac{Z^2}{n^2}\right)\frac{dh}{ds} - \beta\Delta h\right\}\right]_{m,m+1} \quad \ldots\ldots\ldots (61)$$

where suffix m,m-1 refers to the value at the mth boom in the wall joining the mth and (m-1)th booms, etc. The common value of the quantity on the two sides of equation (61) will now be denoted by suffix m.

Since the dimensions of a rib section are proportional to ρ and the skin thickness to ρ^k, the total cross sectional area of skin in a rib section is proportional to ρ^{k+1}. Consequently we must take the variation of boom area to obey the same law, so that

$$B_m = B'_m \rho^{k+1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (62)$$

where B_m is the cross sectional area of the mth boom and B'_m is its value at the root (ρ=1).

By virtue of (60) and (62) the tensile force P_m in the mth boom is given by

$$P_m = \frac{E}{G}\frac{B'_m}{r^2_m}\left[\frac{1}{nt'}\left\{\left(r^2 - \frac{E}{4G}\frac{Z^2}{n^2}\right)\frac{dh}{ds} - \beta\Delta h\right\}\right]_m \rho^{\beta} \quad \ldots\ldots\ldots (63)$$

Consider now the equilibrium of an elementary length of the mth boom. Each of the two walls meeting at the boom exert forces on the boom element, arising from the stress flows f_g and q_g (Figs. 4 and 5). Also the tension P_m will have different values at the two ends of the element. The ribs, to which the skin and the booms are attached, cannot exert forces in the direction normal to themselves and it is evident, therefore, that the component in the z_o direction of the forces on the sides of the element must be in equilibrium with the component in this direction of the change in boom tension over the element. This leads to the equation

$$\left(n_o q_g + n' f_g\right)_{m,m-1} - \left(n_o q_g + n' f_g\right)_{m,m+1} = \frac{n_o}{r_m}\frac{dP_m}{d\rho} \quad \ldots\ldots (64)$$

On substituting (59) and (63) this becomes

$$h_{m,m+1} - h_{m,m-1} = \frac{E}{G}\frac{ZB'_m}{r^3_m}\left[\frac{1}{nt'}\left\{\left(r^2 - \frac{E}{4G}\frac{Z^2}{n^2}\right)\frac{dh}{ds} - \beta\Delta h\right\}\right]_m \quad \ldots\ldots (65)$$

Equations (61) and (65) applied at each of the N booms in turn provide 2N linear simultaneous equations in the (2N+3) unknowns.

It should be noted that equation (65), in conjunction with (63), provides an alternative and simpler expression for the boom tension, thus

$$P_m = \frac{r_m}{Z}\left(h_{m,m+1} - h_{m,m-1}\right)\rho^{\beta} \quad \ldots\ldots\ldots\ldots (66)$$

§2.2. The stress resultants:

In order to derive the remaining three equations for determining the unknowns, we now consider the values of the stress resultants. At every rib section these will be resolved into two bending moments M_x and M_y about axes parallel to x_o and y_o, a torque T about the z_o axis, shear forces S_x and S_y along the x_o and y_o axes and a direct force R in the direction of the z_o axis. The sign convention is as follows:-

M_x,M_y	......	positive when tending to produce tensile stresses for positive y_o and x_o respectively.
T	......	positive when tending to produce positive shear flow q and positive twist $\dfrac{d\theta}{d\rho}$.
S_x,S_y	......	positive when they correspond to positive M_y and M_x respectively.
R	......	positive when tending to produce tensile stresses over the section.

Each of the above stress resultants may be expressed as an integral of the stress flows, thus

$$R = \rho \oint nfds + \sum_m (n_o P)_m \quad \ldots\ldots\ldots\ldots\ (67)$$

$$M_x = \rho^2 \oint nfyds + \rho \sum_m (n_o Py)_m \quad \ldots\ldots\ldots\ldots\ (68)$$

$$M_y = \rho^2 \oint nfxds + \rho \sum_m (n_o Px)_m \quad \ldots\ldots\ldots\ldots\ (69)$$

$$T = \rho^2 \oint \left[pq + (mx - l\,y)f\right] ds \quad \ldots\ldots\ldots\ldots\ (70)$$

$$S_x = \frac{1}{Z}\frac{dM_y}{d\rho} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ (71)$$

$$S_y = \frac{1}{Z}\frac{dM_x}{d\rho} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ (72)$$

In the above expressions the circuit integral, which is taken over all walls, gives the contribution of the skin, and the summation gives the contribution of the boom tensions. It should be noted that in equation (70) for the torque there is no contribution from the booms because all the generators intersect the axis z_o, to which the torque is referred, at the apex.

Consider first the expression for the direct force R. On substituting equations (57) and (66), equation (67) becomes

$$R = \left\{ \oint \frac{dh}{ds} ds + \sum_m \left(h_{m,m+1} - h_{m,m-1} \right) \right\} \rho^{\beta}$$

$$= \left\{ \oint h \right] + \sum_m \left(h_{m,m+1} - h_{m,m-1} \right) \Big\} \rho^{\beta} \quad \ldots\ldots\ldots \quad (73)$$

where $\oint h]$ is the sum of the differences of h between the ends of each wall. It is clear from a consideration of Fig.6 that for any single cell tube

$$\oint h] = \sum_m \left(h_{m,m-1} - h_{m,m+1} \right)$$

so that finally we have

$$R = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad (74)$$

Hence the assumption of the stress function F = gh automatically infers no force on the tube in a direction normal to the ribs. Fortunately such forces are quite insignificant in practice so that (74) is not a serious restriction.

Substitution of (57) and (66) into (68) gives

$$M_x = \rho^{\beta+1} \left\{ \oint y \frac{dh}{ds} ds + \sum_m y_m \left(h_{m,m+1} - h_{m,m-1} \right) \right\}$$

$$= \rho^{\beta+1} \Big\{ -\oint h \cos\Psi ds + \oint yh] + \sum_m y_m \left(h_{m,m+1} - h_{m,m-1} \right) \Big\}$$

Again we have, for any single cell tube,

$$\oint yh] = \sum_m y_m \left(h_{m,m-1} - h_{m,m+1} \right)$$

so that finally

$$M_x = -\rho^{\beta+1} \oint h \cos\Psi \, ds \quad \ldots\ldots\ldots\ldots\ldots \quad (75)$$

In an exactly similar way

$$M_y = -\rho^{\beta+1} \oint h \sin\Psi \, ds \quad \ldots\ldots\ldots\ldots\ldots \quad (76)$$

Substitution of (75) and (76) into (71) and (72) now gives for the shear forces

$$S_x = -\frac{(\beta+1)}{Z} \rho^{\beta} \oint h \sin\Psi \, ds \quad \ldots\ldots\ldots\ldots \quad (77)$$

$$S_y = -\frac{(\beta+1)}{Z} \rho^{\beta} \oint h \cos\Psi \, ds \quad \ldots\ldots\ldots\ldots \quad (78)$$

Finally, on substituting equations (57) and (58) into (70) and simplifying by means of (4), (6a) and (12a), the expression for the torque becomes

$$T = -\frac{\beta}{Z}\rho^{\beta+1}\oint hpds \quad \dots\dots\dots\dots\dots\dots (79)$$

It is now evident that the stress function F = gh will give a possible solution, if the applied loads give rise to bending moment and torque variations of the form

$$\left.\begin{aligned} M_x &= M'_x\rho^{\beta+1} \\ M_y &= M'_y\rho^{\beta+1} \\ T &= T'\rho^{\beta+1} \end{aligned}\right) \quad \dots\dots\dots\dots\dots\dots (80)$$

where M'_x, M'_y and T' are constants, being the values at the root section. The remaining three equations for the determination of the unknown constants then become

$$\left.\begin{aligned} \oint h\cos\psi\, ds &= -M'_x \\ \oint h\sin\psi\, ds &= -M'_y \\ \beta\oint hpds &= -ZT' \end{aligned}\right) \quad \dots\dots\dots\dots\dots (81)$$

The (2N+3) linear simultaneous equations (61), (65) and (81) will give a unique solution for the unknown constants, unless it so happens that the load variation (80) corresponds to a value of β which makes one of the equations a linear combination of the remaining ones. It is unlikely, however, that such a case would need consideration in practice and nothing more will be said about it. Values of β at which this irregularity occurs are called the eigenvalues, and their practical importance will appear in § 2.3 when self-equilibrating stress systems are considered.

Any smooth variation of bending moment or torque along the tube may be represented approximately by a few terms of a power series in ρ. Each term of the series will correspond to a stress function of the type being considered, so that the stress function gh should give a possible solution for any smooth load variation. The accuracy of the solution will depend entirely on how many terms of the power series are taken.

§ 2.3 Self equilibrating stress systems and end effects:

The method so far described gives a possible solution for the stresses and deformations due to any smooth variation of applied load, but neglects entirely the conditions at the two ends of the tube, i.e. at the root and tip. In this respect it is exactly similar to Saint Venant's classical theory of torsion of prismatical bars which gives an accurate representation of the stresses in the bar except in regions near the two ends. In this case the disturbing stresses are highly localised, and decay rapidly from the two ends towards the middle of the bar. However, in a highly tapered tube this is no longer true, and the disturbing stresses may be appreciable over the whole length of the tube, so that it is imperative to be able to derive systems of self-equilibrating stresses to superimpose on the previous solutions, in order to satisfy the end conditions.

If the stress system corresponding to F = gh is self-equilibrating it is evident from equations (81) that we must have

$$\oint h \cos\psi \, ds = \oint h \sin\psi \, ds = \oint hp\,ds = 0 \quad . \; . \; . \; . \quad (82)$$

The conditions (61), (65) and (82) again give (2N+3) linear simultaneous equations in the (2N+3) unknowns, but these equations are now all homogeneous. It is evident, therefore, that a non-zero solution can exist only if the determinant formed by the coefficients of the unknowns in the (2N+3) equations is zero. These coefficients are functions of β so that the determinantal equation is essentially an equation in β. This equation is called the "characteristic equation" and it will be seen later that it is in general transcendental with an infinite set of roots β_i. These roots are the eigenvalues already mentioned and are essentially the "decay factors" for the self-equilibrating stress systems. Corresponding to each eigenvalue are definite ratios between the unknown constants which enable them all to be expressed in terms of any one of them. The magnitude of this remaining unknown defines the "amplitude" of the stress system and is arbitrary. Thus we conclude that in general there are an infinite number of possible self-equilibrating stress systems, each one being derived from a stress function

$$F_i = h_i \rho^{\beta_i}$$

where β_i is one of an infinite set of eigenvalues or roots of a characteristic equation and h_i is its associated eigenfunction. For each β_i the eigenfunction h_i is defined completely in functional form but not in absolute magnitude. The magnitude or amplitude depends entirely upon the purpose for which the stress system is to be used.

In order to build up a given end condition it is theoretically necessary to apply the infinite set of self-equilibrating stress systems, each with a suitable amplitude. For an unswept slightly tapered tube in which $n \approx 1$, Hadji-Argyris and Dunne (Ref.1) showed that the eigenfunctions are mutually orthogonal in the sense that

$$\oint \frac{h_i h_j}{t'} ds \begin{array}{l} = 0 \text{ if } i \neq j \\ \neq 0 \text{ if } i = j \end{array}$$

This result enabled them to combine the whole set of eigenfunctions as terms of a generalised Fourier series and calculate the coefficient of each term individually, in much the same way as the coefficients of an ordinary Fourier series are determined. However, for the tubes with which we are concerned here the eigenfunctions are not orthogonal so that the possibility of obtaining an analytical solution to satisfy the conditions at all points in the end sections is ruled out.

It is possible, however, when applying the theory numerically, to get a good approximation by making an intelligent choice of only a few eigenfunctions and satisfying the conditions at a few chosen points in the end section. To fix our ideas suppose that the tube section is symmetrical about the y axis, inferring zero sweep in the x direction. Further suppose that the tube is continuous to the apex, that the warping is completely prevented at the root section and that a torque $T = T'\rho^4$ (i.e. $\beta = 3$) is being applied to the tube. It is possible to eliminate the warping at eight points on the root section (viz. the two points on the axis of symmetry, three points above the y axis and the three corresponding points below the y axis) by the application of two self-equilibrating stress systems only. These systems must correspond to positive β_i so that the self-equilibrating stresses decay towards the apex, and the natural eigenvalues to choose are those nearest to $\beta = 3$, in which case the "residual" warping over the remainder of the root section may be expected to be as small as possible. It will always be possible to calculate this residual warping, giving a check on the errors involved in the approximation. A further check is obtainable from the results of § 1.7 for the root shear flow. This method is illustrated numerically in the Appendix.

§ 2.4 The warping of the ribs:

For this type of stress function (F = gh) it is possible to obtain a very convenient expression for the warping w^*.

Substitution of equations (33), (48) and (58) into (32a) gives

$$\frac{\partial w^*}{\partial s} = \frac{\rho^{\beta-k}}{GZ}\left[\frac{1}{nt'}\left(\Delta\frac{dh}{ds}-\beta h\right) - \frac{p}{2A}\oint\frac{1}{nt'}\left(\Delta\frac{dh}{ds}-\beta h\right)ds\right] \qquad \text{......(83)}$$

Also, from equations (54) and (27a), in conjunction with (33) and (58)

$$\bar{\theta} = \frac{(\beta-k)}{2AG}\oint\frac{1}{nt'}\left(\Delta\frac{dh}{ds}-\beta h\right)ds \qquad \text{.........(84)}$$

Substitution of (84) into (83) then gives

$$\frac{\partial w^*}{\partial s} = \frac{\rho^{\beta-k}}{(\beta-k)GZ}\left[\frac{(\beta-k)}{nt'}\left(\Delta\frac{dh}{ds}-\beta h\right) - Gp\bar{\theta}\right] \qquad \text{.........(85)}$$

Now using the differential equation (55) in conjunction with (60) we obtain the equation

$$\frac{\partial w^*}{\partial s} = \frac{1}{(\beta-k)Z}\left[\rho\frac{\partial}{\partial s}(r^2\varepsilon_g) + \rho^{\beta-k}(\bar{u}\sin\psi + \bar{v}\cos\psi)\right] \qquad \text{........(86)}$$

Integration of (86) gives

$$w^* = \frac{1}{(\beta-k)Z}\left[\rho r^2 \varepsilon_g + \rho^{\beta-k}(\bar{u}x + \bar{v}y)\right] + \text{function}(\rho) \qquad \ldots\ldots(87)$$

The terms in $\bar{u}$ and $\bar{v}$ and the function of ρ correspond to rigid body movements of the ribs and may be discarded, so that the expression for the warping function w^* becomes finally

$$w^* = \frac{\rho r^2 \varepsilon_g}{(\beta-k)Z} \qquad \ldots\ldots\ldots\ldots(88)$$

Equation (88) holds, with one exception, for any stress function of the form gh. The exception is when $\beta = k$ in which case equation (84) gives $\bar{\theta} = 0$ so that (85) becomes meaningless. For this case equation (31a) must be used as it stands for calculation of the warping.

§3. CONICAL TUBE WITH SKIN THICKNESS CONSTANT ALONG THE GENERATORS:

§3.1 Introductory remarks :

So far the analysis has been concerned primarily with setting up the differential equation for the function h together with the appropriate conditions for the evaluation of the unknown constants involved in its solution. We turn now to detailed solutions for a particular type of tube, namely one in which the skin thickness does not vary along the generators so that k=0 with $t=t'(s)$ and $B_m = \rho B'_m$.

First, if $\beta = k = 0$ the differential equation (55) can be integrated directly and an analytical solution, which holds for arbitrary section and for arbitrary variation of t with s, can be obtained. Since $\beta = 0$, the load variation in this case corresponds to constant shear forces, with bending moments and torque proportional to the distance from the apex. The results obtained are remarkably similar to those arising from an application of the simple theory of bending and torsion to an unswept cylindrical tube.

Apart from this one case, however, integration of (55) can be performed only if the section and the thickness variation are specified, so that r, p, Δ, n, ψ and t are known functions of s. Even then it may not be possible to integrate (55) analytically. However, the case considered here is that of a tube whose section is polygonal with constant skin thickness in each flat wall (i.e. $t=t'=$constant). Booms, which taper linearly in area from a maximum at the root to zero at the apex, are concentrated at the intersections of the various walls. This should give a fairly good approximation to tubes occurring in practice, and, moreover, gives an integrable form for the differential equation (55).

§3.2. Solution for linear variation of bending moment and torque ($\beta = 0$)

With $\beta = 0$, equation (80) for the bending moment and torque variation may be written

$$\left.\begin{aligned} M_x &= \rho M'_x = \rho Z S_y \\ M_y &= \rho M'_y = \rho Z S_x \\ T &= \rho T' \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (89)$$

It will be seen from (79) that the stress function in the form gh with $\beta=0$ automatically infers zero torque so that F=h(s) is not the most general form to give the required load distribution. To make it quite general an extra term must be superimposed and we take

$$F = h(s) + ZC \log_e \rho \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (90)$$

where C is a constant. (The constant Z has been introduced in anticipation of later simplification.) It should be noted that at this stage the constant C may have different values in the various walls, although later considerations will show that it has the same value in all walls.

In this particular case we have $\beta=k=0$ so that, from (84), $\bar{\theta}=0$. Also, by virtue of (52) and (53)

$$\frac{d^2u}{d\rho^2} = \frac{\bar{u}}{\rho^2}, \qquad \frac{d^2v}{d\rho^2} = \frac{\bar{v}}{\rho^2} \quad \ldots\ldots\ldots\ldots \quad (91)$$

On substituting (90) and (91) into equation (42a) we obtain

$$\frac{E}{G}\frac{d}{ds}\left[\frac{1}{nt}\left\{\left(r^2-\frac{E}{4G}\frac{Z^2}{n^2}\right)\frac{dh}{ds} - ZC\Delta\right\}\right] = -E\left(\bar{u}\sin\Psi + \bar{v}\cos\Psi\right) \quad \ldots\ldots(92)$$

This equation may be integrated directly to give

$$\frac{E}{Gnt}\left\{\left(r^2-\frac{E}{4G}\frac{Z^2}{n^2}\right)\frac{dh}{ds} - ZC\Delta\right\} = E\left(D-\bar{u}x-\bar{v}y\right) \quad \ldots\ldots(93)$$

where D is a constant of integration.

On solving (93) for $\frac{dh}{ds}$ we obtain

$$\frac{dh}{ds} = C\eta + \frac{ED}{Z^2}t_E - \frac{E\bar{u}}{Z^2}xt_E - \frac{E\bar{v}}{Z^2}yt_E \quad \ldots\ldots(94)$$

where $$\eta = \Delta Z\left(r^2-\frac{E}{4G}\frac{Z^2}{n^2}\right)^{-1} \quad \ldots\ldots(95)$$

$$t_E = \frac{G}{E}Z^2nt\left(r^2-\frac{E}{4G}\frac{Z^2}{n^2}\right)^{-1} \quad \ldots\ldots(96)$$

There is no necessity to integrate equation (94) any further in this case.

On substituting equations (90) and (94) into (41a) the linear strain ε_g along the generators becomes

$$\varepsilon_g = \frac{1}{\rho r^2}\left(D - \bar{u}x - \bar{v}y\right) \quad \ldots\ldots (97)$$

Equation (97) shows that the condition of continuity of strain through the various booms is automatically satisfied if the value of the constant D is the same in all walls.

Also, since each boom area is proportional to ρ and the stress in the boom ($=E\varepsilon_g$) is inversely proportional to ρ by virtue of (97), it is clear that the tension in each boom is constant. Consequently the boom equilibrium equation (64) becomes

$$(n_o q_g + n' f_g)_{m,m+1} = (n_o q_g + n' f_g)_{m,m-1}$$

By virtue of equations (39a) and (90) this equation is automatically satisfied if the constant C has the same value in all walls.

Hence, in this case, however many booms the tube contains, we have only four unknown constants, namely C, D, $\bar{u}$ and $\bar{v}$. These may be determined from the values of the applied loads.

Substituting (90) into (38a) and (40a), the stress flows f and q become

$$f = \frac{1}{\rho n}\frac{dh}{ds} \quad \ldots\ldots\ldots\ldots\ldots\ldots (98)$$

$$q = \frac{1}{\rho}\left[\frac{\Delta}{Z}\frac{dh}{ds} - C\right] \quad \ldots\ldots\ldots\ldots (99)$$

Consider now the expression for the torque T. Substituting (98) and (99) into (70) and simplifying by means of (4), (5), (6a) and (12a) we obtain the equation

$$T = -\rho C \oint p\,ds = -2AC\rho$$

$$\text{i.e.} \quad C = -\frac{T'}{2A} \quad \ldots\ldots\ldots\ldots\ldots\ldots (100)$$

As would be expected the value of C does not depend on the magnitudes of the applied bending moments since the second term in equation (90) for F was introduced purely to take account of the applied torque.

In a similar way equations (67), (68) and (69) for the remaining stress resultants become, after using (100),

$$R = -\frac{T'}{2A}\xi + \frac{ED}{Z^2}\lambda - \frac{E\bar{u}}{Z^2}\lambda_y - \frac{E\bar{v}}{Z^2}\lambda_x \qquad \ldots\ldots (101)$$

$$M_x' = ZS_y = -\frac{T'}{2A}\xi_x + \frac{ED}{Z^2}\lambda_x - \frac{E\bar{u}}{Z^2}I_{xy} - \frac{E\bar{v}}{Z^2}I_{xx} \qquad \ldots\ldots (102)$$

$$M_y' = ZS_x = -\frac{T'}{2A}\xi_y + \frac{ED}{Z^2}\lambda_y - \frac{E\bar{u}}{Z^2}I_{yy} - \frac{E\bar{v}}{Z^2}I_{xy} \qquad \ldots\ldots (103)$$

where

$$\left.\begin{aligned} \xi &= \oint \eta\, ds = Z\oint \Delta\left(r^2 - \frac{E}{4G}\frac{Z^2}{n^2}\right)^{-1} ds \\ \xi_x &= \oint \eta y\, ds = Z\oint \Delta y\left(r^2 - \frac{E}{4G}\frac{Z^2}{n^2}\right)^{-1} ds \\ \xi_y &= \oint \eta x\, ds = Z\oint \Delta x\left(r^2 - \frac{E}{4G}\frac{Z^2}{n^2}\right)^{-1} ds \end{aligned}\right\} \qquad \ldots\ldots (104)$$

and

$$\left.\begin{aligned} \lambda &= \oint t_E\, ds + \sum_m n_o^3 B_m' \\ \lambda_x &= \oint t_E y\, ds + \sum_m n_o^3 y_m B_m' \\ \lambda_y &= \oint t_E x\, ds + \sum_m n_o^3 x_m B_m' \\ I_{xx} &= \oint t_E y^2 ds + \sum_m n_o^3 y_m^2 B_m' \\ I_{yy} &= \oint t_E x^2 ds + \sum_m n_o^3 x_m^2 B_m' \\ I_{xy} &= \oint t_E xy\, ds + \sum_m n_o^3 x_m y_m B_m' \end{aligned}\right\} \qquad \ldots\ldots (105)$$

The properties of the root section defined by equations (105) are exactly similar to those arising in the simple theory of bending and torsion for an unswept cylindrical tube. Thus the term λ is analagous to the stress bearing area of the section (t_E being regarded as an effective skin thickness), λ_x and λ_y are analagous to the moments of the stress bearing area about the x and y axes,

I_{xx} and I_{yy} to the second moments of the stress bearing area about the x and y axes and I_{xy} to the product of inertia with respect to these axes. The essential difference between equations (101) to (103) and the corresponding ones for an unswept cylindrical tube is in the presence of the three terms involving ξ, ξ_x and ξ_y. It is worthy of note that, by virtue of their definitions (104), these three terms are in no way dependent upon the distribution of stress bearing area round the section but only on the shape of the ribs and the position of the apex, which defines the taper and aerodynamic sweep.

In what follows we shall be concerned only with bending and torsion loads so that we put R = 0. Equations (101) to (103) then give

$$D = \frac{Z^2 T'}{2AE}\frac{\xi}{\lambda} + \bar{u}\frac{\lambda_y}{\lambda} + \bar{v}\frac{\lambda_x}{\lambda} \quad \ldots\ldots\ldots (106)$$

$$M_x' = ZS_y = \frac{T'}{2A}\frac{1}{\lambda}\left(\xi\lambda_x - \lambda\xi_x\right) + \frac{E\bar{u}}{Z^2}\frac{1}{\lambda}\left(\lambda_x\lambda_y - \lambda I_{xy}\right) + \frac{E\bar{v}}{Z^2}\frac{1}{\lambda}\left(\lambda_x^2 - \lambda I_{xx}\right) \quad \ldots\ldots\ldots (107)$$

$$M_y' = ZS_x = \frac{T'}{2A}\frac{1}{\lambda}\left(\xi\lambda_y - \lambda\xi_y\right) + \frac{E\bar{u}}{Z^2}\frac{1}{\lambda}\left(\lambda_y^2 - \lambda I_{yy}\right) + \frac{E\bar{v}}{Z^2}\frac{1}{\lambda}\left(\lambda_x\lambda_y - \lambda I_{xy}\right) \quad \ldots\ldots\ldots (108)$$

Knowing the values of M_x', M_y' and T', equations (107) and (108) give $\bar{u}$ and $\bar{v}$, equation (106) then gives D, and C is given by (100), so that the solution is now complete.

To obtain the twist of the tube we substitute equations (33), (99) and (94) into (27a), thus

$$2AG\rho^2\frac{d\theta}{d\rho} = \frac{T'}{2A}\oint\frac{Z-\Delta\eta}{nt}ds + \frac{ED}{Z^2}\oint\frac{\Delta t_E}{nt}ds - \frac{E\bar{u}}{Z^2}\oint\frac{\Delta x t_E}{nt}ds - \frac{E\bar{v}}{Z^2}\oint\frac{\Delta y t_E}{nt}ds \quad \ldots\ldots (109)$$

Noticing that, by virtue of (95) and (96),

$$\frac{\Delta t_E}{nt} = \frac{G}{E}Z\eta \quad \ldots\ldots\ldots (110)$$

equation (109), after substitution of (106), becomes

$$\rho^2 \frac{d\theta}{d\rho} = \frac{T'}{4A^2G}\left[\oint \frac{Z-\Delta\eta}{nt} ds + \frac{G}{E}\frac{Z\xi^2}{\lambda}\right]$$

$$+ \frac{\bar{u}}{2AZ}\frac{1}{\lambda}\left(\xi\lambda_y - \lambda\xi_y\right) + \frac{\bar{v}}{2AZ}\frac{1}{\lambda}\left(\xi\lambda_x - \lambda\xi_x\right) \quad \ldots\ldots (111)$$

In the past the word "unswept" has been used rather loosely and, to the author's knowledge, no mathematical definition has been given to a "structurally unswept tapered tube". It seems logical to suppose that such a tube would be one that satisfies the following conditions:

(a) shear forces S_x and S_y applied to the tube in planes parallel to the ribs along lines intersecting the z_o axis produce bending without twist

and

(b) a torque applied about an axis perpendicular to the ribs produces twist without bending.

Considering equations (107) and (108) it will be seen that the terms in T' disappear if

$$\frac{\xi}{\lambda} = \frac{\xi_x}{\lambda_x} = \frac{\xi_y}{\lambda_y} \quad \ldots\ldots\ldots\ldots (112)$$

and the curvatures $\bar{u}$ and $\bar{v}$ then depend only on the values of the constant shear forces S_x and S_y. Further, if (112) is satisfied, the terms in $\bar{u}$ and $\bar{v}$ disappear from equation (111) and the twist depends only on the torque. Consequently, for linear bending moment and torque variation, equations (112) represent the condition that the tube should be structurally unswept according to the above definition. It does not follow, however, that (112) is also the condition for zero sweep if the bending moment and torque variation is anything but linear. Also it should be realised that in the above argument no account has been taken of the conditions at the end sections.

An investigation into the effects of symmetry of the section on the equations so far derived is useful. Consider first the case of a tube whose section is singly symmetrical about the y axis, inferring no sweep in the x direction. We then have automatically

$$\xi = \xi_x = \lambda_y = I_{xy} = 0 \quad \ldots\ldots\ (113)$$

and equations (106), (107), (108) and (111) give

$$\left.\begin{aligned} \frac{E\bar{v}}{Z^2} &= \frac{\lambda}{\lambda_x}\frac{ED}{Z^2} = -\frac{M_x'}{I_{xx} - \frac{\lambda_x^2}{\lambda}} \\ \frac{E\bar{u}}{Z^2} &= -\frac{1}{I_{yy}}\left(M_y' + \frac{T'}{2A}\xi_y\right) \\ \rho^2\frac{d\theta}{d\rho} &= \frac{T'}{4A^2G}\oint\frac{Z-\Delta\eta}{nt}\,ds - \frac{\bar{u}}{2AZ}\xi_y \end{aligned}\right\} \quad \ldots\ldots (114)$$

If the tube is doubly symmetrical about both the x and y axes, so that there is no sweep in either direction, we have in addition to (113)

$$\lambda_x = \xi_y = 0 \quad \ldots\ldots\ldots\ldots\ (115)$$

so that equations (114) reduce further to

$$\left.\begin{aligned} D &= 0 \\ \frac{E\bar{u}}{Z^2} &= -\frac{M_y'}{I_{yy}} \\ \frac{E\bar{v}}{Z^2} &= -\frac{M_x'}{I_{xx}} \\ \rho^2\frac{d\theta}{d\rho} &= \frac{T'}{4A^2G}\oint\frac{Z-\Delta\eta}{nt}\,ds \end{aligned}\right\} \quad \ldots\ldots\ldots (116)$$

Equations (116) are identical in form with the corresponding results arising from the simple theory of bending and torsion. The expressions for I_{xx} and I_{yy} however place no restriction on the amount of taper. The equation for the twist should be compared with that corresponding to the Bredt-Batho formulae, namely

$$\rho^2\frac{d\theta}{d\rho} = \frac{T'}{4A^2G}\oint\frac{Z}{t}\,ds \quad \ldots\ldots\ldots (117)$$

The two formulae become identical if the apex is taken to infinity (still maintaining the double symmetry) in which case $\frac{\Delta}{Z} \longrightarrow 0$ and $n \longrightarrow 1$.

For this doubly symmetrical tube the stress flows f and q are given by

$$f = \frac{1}{\rho}\left[-\frac{T'}{2A}\frac{\eta}{n} + \frac{M_y'}{I_{yy}}\frac{x t_E}{n} + \frac{M_x'}{I_{xx}}\frac{y t_E}{n}\right]$$
$$q = \frac{1}{\rho}\left[\frac{T'}{2A}\left(1-\frac{\Delta\eta}{Z}\right) + \frac{M_y'}{I_{yy}}\frac{\Delta x t_E}{Z} + \frac{M_x'}{I_{xx}}\frac{\Delta y t_E}{Z}\right] \qquad \ldots\ldots(118)$$

In particular, if the bending moments are zero, the stress flows due to pure torque become

$$f = -\frac{T'}{2A\rho}\frac{\eta}{n}$$
$$q = \frac{T'}{2A\rho}\left(1-\frac{\Delta\eta}{Z}\right) \qquad \ldots\ldots\ldots\ldots(119)$$

Now $\frac{\Delta\eta}{Z} > 0$ and $\to 0$ as $Z \to \infty$. Hence

$$q < \frac{T'}{2A\rho}$$

and the Bredt-Batho formula overestimates the shear flow by an amount which decreases as the taper decreases.

A further result of interest for the doubly symmetrical tube is the stress in the booms. From (97) and (116) the boom stress is given by

$$E\varepsilon_g = \frac{1}{\rho}\left(\frac{Z}{r}\right)^2\left[\frac{M_y'}{I_{yy}}.x + \frac{M_x'}{I_{xx}}.y\right] \qquad \ldots\ldots(120)$$

In particular the boom stress due to a pure torque $T=\rho T'$ is zero.

§ 3.3. Tube of polygonal section - solutions for non-zero values of β:

As mentioned previously (§ 3.1), integration of equation (55) with β other than zero is only possible if the shape of the section is known. Consequently we now confine our attention to tubes of polygonal section with the thickness constant over each flat wall.

In any side of the polygon both ψ and p are constant and, by virtue of (12a), n is also constant. Also, from equations (3), (5) and (6a)

$$\frac{d\Delta}{ds} = 1 \qquad \ldots\ldots\ldots\ldots(121)$$

$$\frac{d}{ds}(r^2) = 2\Delta \qquad \ldots\ldots\ldots\ldots(122)$$

Then, on substituting k = 0 and t' = t = constant into (55), and performing the necessary differentiation by means of (121) and (122), we obtain the equation

$$\left(r^2 - \frac{E}{4G}\frac{Z^2}{n^2}\right)\frac{d^2h}{d\Delta^2} - 2(\beta-1)\Delta\frac{dh}{d\Delta} + \beta(\beta-1)h$$
$$= -Gnt\left(p\bar{\theta} + \bar{u}\sin\Psi + \bar{v}\cos\Psi\right) \quad \ldots\ldots (123)$$

But, from (6a) and (12a), we have

$$r^2 - \frac{E}{4G}\frac{Z^2}{n^2} = \Delta^2 + \left(1 - \frac{E}{4G}\right)(Z^2 + p^2) \quad \ldots\ldots\ldots (124)$$

Hence, if we change the independent variable from Δ to Γ where

$$\Delta = \Gamma\left(1 - \frac{E}{4G}\right)^{\frac{1}{2}}(Z^2 + p^2)^{\frac{1}{2}} \quad \ldots\ldots\ldots (125)$$

the differential equation (123) becomes

$$(1 + \Gamma^2)\frac{d^2h}{d\Gamma^2} - 2(\beta-1)\Gamma\frac{dh}{d\Gamma} + \beta(\beta-1)h$$
$$= -Gnt\left(p\bar{\theta} + \bar{u}\sin\Psi + \bar{v}\cos\Psi\right) \quad \ldots\ldots\ldots (126)$$

The R.H.S. of (126) is a constant in each wall and a complete solution is obtainable in terms of elementary functions.

The solution for β = 0 has already been obtained and this can be excluded. A special case of (126) occurs when β = 1 in which case the second and third terms on the L.H.S. are zero. The general solution then becomes

$$h = C + D\Gamma - Gnt\left(p\bar{\theta} + \bar{u}\sin\Psi + \bar{v}\cos\Psi\right)\left[\Gamma\tan^{-1}\Gamma - \tfrac{1}{2}\log_e(1+\Gamma^2)\right] \quad \ldots\ldots (127)$$

For all other values of β a particular integral of equation (126) is

$$h_p = -\frac{Gnt}{\beta(\beta-1)}\left(p\bar{\theta} + \bar{u}\sin\Psi + \bar{v}\cos\Psi\right) \quad \ldots\ldots (128)$$

To obtain the complementary function of (126) we make the R.H.S. zero and then change the variables by means of the transformation*

$$\left.\begin{aligned} \Gamma &= \tan\Theta \\ h &= \Omega\cos^{-\beta}\Theta \end{aligned}\right\} \quad \ldots\ldots\ldots (129)$$

---

* This transformation is due to Mr.W.B.Smith-White of the Department of Mathematics, University of Sydney.

Equation (126) then reduces to the simple harmonic equation

$$\frac{d^2\Omega}{d\Theta^2} + \beta^2\Omega = 0$$

and its solution is

$$\Omega = C\cos\beta\Theta + D\sin\beta\Theta$$

On changing back to the variables h and Γ and adding the particular integral (128) the complete solution for h becomes

$$h = CJ_\beta(\Gamma) + DW_\beta(\Gamma) + h_P \quad \ldots\ldots\ldots \text{(130)}$$

where

$$\left.\begin{aligned} J_\beta(\Gamma) &= (1+\Gamma^2)^{\frac{\beta}{2}}\cos(\beta\tan^{-1}\Gamma) \\ W_\beta(\Gamma) &= (1+\Gamma^2)^{\frac{\beta}{2}}\sin(\beta\tan^{-1}\Gamma) \end{aligned}\right\} \quad \ldots\ldots\ldots \text{(131)}$$

The solution (131) holds for all non-zero values of β other than $\beta = 1$ in which case (127) must be used. For positive integral values of β the two functions J_β and W_β reduce to polynomials and for convenience these are tabulated below for values of β from +1 to +7.

TABLE 1.

β	$J_\beta(\Gamma)$	$W_\beta(\Gamma)$
+ 1	1	Γ
+ 2	$1 - \Gamma^2$	2Γ
+ 3	$1 - 3\Gamma^2$	$3\Gamma - \Gamma^3$
+ 4	$1 - 6\Gamma^2 + \Gamma^4$	$4\Gamma - 4\Gamma^3$
+ 5	$1 - 10\Gamma^2 + 5\Gamma^4$	$5\Gamma - 10\Gamma^3 + \Gamma^5$
+ 6	$1 - 15\Gamma^2 + 15\Gamma^4 - \Gamma^6$	$6\Gamma - 20\Gamma^3 + 6\Gamma^5$
+ 7	$1 - 21\Gamma^2 + 35\Gamma^4 - 7\Gamma^6$	$7\Gamma - 35\Gamma^3 + 21\Gamma^5 - \Gamma^7$

These polynomials could have been obtained by assuming at the outset that the complementary function of equation (126) is a power series in Γ. Proceeding in this way we find that

$$\left.\begin{aligned} J_\beta(\Gamma) &= 1 + \sum_{r=1}^{\infty}\frac{\beta(\beta-1)(\beta-2)\ldots\ldots(\beta-2r+1)}{(2r)!}(-\Gamma^2)^r \\ W_\beta(\Gamma) &= \beta\Gamma\left\{1 + \sum_{r=1}^{\infty}\frac{(\beta-1)(\beta-2)\ldots\ldots(\beta-2r)}{(2r+1)!}(-\Gamma^2)^r\right\} \end{aligned}\right\} \quad \ldots\ldots\ldots \text{(132)}$$

The two series in (132) terminate at a finite number of terms if β is a positive integer, giving the polynomials previously derived.

For negative integral values of β we use the fact that

$$\left.\begin{aligned} J_{-\beta}(\Gamma) &= (1+\Gamma^2)^{-\beta} J_\beta(\Gamma) \\ -W_{-\beta}(\Gamma) &= (1+\Gamma^2)^{-\beta} W_\beta(\Gamma) \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots (133)$$

In particular we have

$$\left.\begin{aligned} J_{-1}(\Gamma) &= \frac{1}{1+\Gamma^2} \\ W_{-1}(\Gamma) &= -\frac{\Gamma}{1+\Gamma^2} \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots (134)$$

The two functions J_β and W_β have several properties which are useful for computation. A list of formulae which have been found particularly useful in the numerical work described later are given below. For convenience in writing, the variable Γ has been omitted from the functions (e.g. $J_\beta(\Gamma)$ is written as J_β).

$$\left.\begin{aligned} J_\beta &= J_{\beta-1} - \Gamma W_{\beta-1}, & (1+\Gamma^2)J_{\beta-1} &= J_\beta + \Gamma W_\beta \\ W_\beta &= W_{\beta-1} + \Gamma J_{\beta-1}, & (1+\Gamma^2)W_{\beta-1} &= W_\beta - \Gamma J_\beta \end{aligned}\right\} \quad \ldots\ldots (135)$$

$$\left.\begin{aligned} J_{\alpha+\beta} &= J_\alpha J_\beta - W_\alpha W_\beta \\ W_{\alpha+\beta} &= W_\alpha J_\beta + J_\alpha W_\beta \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots (136)$$

$$\frac{dJ_\beta}{d\Gamma} = -\beta W_{\beta-1}, \qquad \frac{dW_\beta}{d\Gamma} = \beta J_{\beta-1} \qquad \ldots\ldots (137)$$

$$\left.\begin{aligned} \int J_\beta \, d\Gamma &= \frac{W_{\beta+1}}{\beta+1} \text{ if } \beta \neq -1 \\ &= \tan^{-1}\Gamma \text{ if } \beta = -1 \\ \int W_\beta \, d\Gamma &= -\frac{J_{\beta+1}}{\beta+1} \text{ if } \beta \neq -1 \\ &= -\tfrac{1}{2}\log_e(1+\Gamma^2) \text{ if } \beta = -1 \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots (138)$$

$$\left.\begin{aligned} (1+\Gamma^2)\frac{dJ_\beta}{d\Gamma} - \beta\Gamma J_\beta &= -\beta W_\beta \\ (1+\Gamma^2)\frac{dW_\beta}{d\Gamma} - \beta\Gamma W_\beta &= \beta J_\beta \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots (139)$$

$$\left.\begin{aligned} \Gamma\frac{dJ_\beta}{d\Gamma} - \beta J_\beta &= -\beta J_{\beta-1} \\ \Gamma\frac{dW_\beta}{d\Gamma} - \beta W_\beta &= -\beta W_{\beta-1} \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots(140)$$

All the formulae given above may be verified by direct substitution of equations (131).

Having established general solutions for h in terms of the independent variable Γ, we now reconsider the boundary conditions and the expressions for the stress resultants and stress flows. On using equations (127), (128), (130) and (131) the boundary conditions (61) and (65) become

$$\left[\frac{Z^2+p^2}{t}\left\{(1+\Gamma^2)\frac{dh}{d\Gamma} - \beta\Gamma h\right\}\right]_{m,m+1} = \left[\frac{Z^2+p^2}{t}\left\{(1+\Gamma^2)\frac{dh}{d\Gamma} - \beta\Gamma h\right\}\right]_{m,m-1}$$

$$= \frac{G}{E}\left(1-\frac{E}{4G}\right)^{-\frac{1}{2}} \frac{r_m^3}{B_m'}\left(h_{m,m+1} - h_{m,m-1}\right) \qquad \ldots\ldots(141)$$

Also, the expressions for the stress resultants (75), (76) and (79) become

$$\left.\begin{aligned} M_x' &= M_x\rho^{-(\beta+1)} = -\left(1-\frac{E}{4G}\right)^{\frac{1}{2}}\oint h\cos\Psi\,(Z^2+p^2)^{\frac{1}{2}}d\Gamma \\ M_y' &= M_y\rho^{-(\beta+1)} = -\left(1-\frac{E}{4G}\right)^{\frac{1}{2}}\oint h\sin\Psi\,(Z^2+p^2)^{\frac{1}{2}}d\Gamma \\ T' &= T\rho^{-(\beta+1)} = -\frac{\beta}{Z}\left(1-\frac{E}{4G}\right)^{\frac{1}{2}}\oint h\,p\,(Z^2+p^2)^{\frac{1}{2}}d\Gamma \end{aligned}\right\} \qquad \ldots\ldots(142)$$

Finally the equations (57), (58) and (60) for the stress flows and linear strain along the generator become

$$f = \frac{1}{Z}\left(1-\frac{E}{4G}\right)^{-\frac{1}{2}}\frac{dh}{d\Gamma}\rho^{\beta-1} \qquad \ldots\ldots\ldots(143)$$

$$q = \frac{1}{Z}\left(\Gamma\frac{dh}{d\Gamma} - \beta h\right)\rho^{\beta-1} \qquad \ldots\ldots\ldots(144)$$

$$\varepsilon_g = \frac{1}{G}\left(1-\frac{E}{4G}\right)^{\frac{1}{2}}\frac{Z^2+p^2}{Zr^2t}\left[(\Gamma^2+1)\frac{dh}{d\Gamma} - \beta\Gamma h\right]\rho^{\beta-1} \qquad \ldots\ldots(145)$$

The above expressions for h and the associated boundary conditions give all the information necessary for solving completely for a polygonal tube of this type. However, at this stage, nothing more can be said until the numerical work on a particular tube section is described in the Appendix.

§4. EXPONENTIAL SOLUTION FOR SWEPT CYLINDRICAL TUBES :

§4.1. Separation of the variables:

As in the case of the conical tube we first look for solutions of equation (42b) in the form

$$F = g(z)h(s) \quad \dots\dots\dots\dots\dots \quad (146)$$

so that the distribution of stress flow is the same round all rib sections. Again, solutions of this type will be possible only if the thickness variation over the tube is of the form

$$t = t'(s)t^{*}(z) \quad \dots\dots\dots\dots\dots \quad (147)$$

Substitution of (146) and (147) into (42b)-(45b) then yields an equation in which every term is a function of z times a function of s. It is found that the variables can be separated only if

$$g = e^{\beta z}, \qquad t^{*} = e^{kz}$$

where β and k are any constants.

Consequently we now confine our attention to tubes in which the thickness variation is governed by

$$t = e^{kz}\,t'(s) \quad \dots\dots\dots\dots\dots \quad (148)$$

and take the stress function to be of the form

$$F = e^{\beta z}\,h(s) \quad \dots\dots\dots\dots\dots \quad (149)$$

By virtue of (148) and (149) we have

$$\frac{\partial}{\partial z}\left[\frac{1}{nt}\left\{\sin\alpha\cos\psi\frac{\partial F}{\partial s}-\frac{\partial F}{\partial z}\right\}\right]=(\beta-k)\,e^{(\beta-k)z}\left[\frac{1}{nt'}\left\{\sin\alpha\cos\psi\frac{dh}{ds}-\beta h\right\}\right] \quad \dots(150)$$

$$\frac{1}{nt}\left[\left(1-\frac{E}{4G}\frac{\cos^2\alpha}{n^2}\right)\frac{\partial F}{\partial s}-\sin\alpha\cos\psi\frac{\partial F}{\partial z}\right]=e^{(\beta-k)z}\left[\frac{1}{nt'}\left\{\left(1-\frac{E}{4G}\frac{\cos^2\alpha}{n^2}\right)\frac{dh}{ds}-\beta\sin\alpha\cos\psi\, h\right\}\right]$$

$$\dots\dots\dots \quad (151)$$

Hence, by virtue of equations (43b)-(45b), we may write

$$\frac{d^2u}{dz^2} = \bar{u}\,e^{(\beta-k)z} \quad \dots\dots\dots\dots\dots \quad (152)$$

$$\frac{d^2v}{dz^2} = \bar{v}\,e^{(\beta-k)z} \quad \dots\dots\dots\dots\dots \quad (153)$$

$$\frac{d^2\theta}{dz^2} = \bar{\theta}\,e^{(\beta-k)z} \quad \dots\dots\dots\dots\dots \quad (154)$$

where $\bar{u}$, $\bar{v}$ and $\bar{\theta}$ are constants.

If now equations (150)-(154) are substituted into (42b) the following ordinary differential equation in h is obtained.

$$\frac{\beta-k}{nt'}\left\{\sin\alpha\cos\psi\frac{dh}{ds}-\beta h\right\}-\frac{d}{ds}\left[\frac{1}{nt'}\left\{\left(1-\frac{E}{4G}\frac{\cos^2\alpha}{n^2}\right)\frac{dh}{ds}-\beta\sin\alpha\cos\psi\, h\right\}\right]$$

$$=G\left(p\bar{\theta}+\bar{u}\sin\psi+\bar{v}\cos\psi\right) \qquad \ldots\ldots(155)$$

This equation is of the same general type as the corresponding equation (55) for the conical tube, being linear and of second order with coefficients and a particular integral which are functions of s. Its solution is therefore of the form given in equation (56) and in an N boom tube there will be in general (2N+3) unknown constants in the expressions for h in the various walls. The equations necessary for their evaluation are obtained in the same way, that is 2N equations from continuity of strain and equilibrium at each of the N booms and the remaining three from the stress resultants.

Substitution of the stress function (149) into (38b) to (40b) gives the expressions for the stress flows, thus

$$f = \frac{1}{n}\frac{dh}{ds}\, e^{\beta z} \qquad (156)$$

$$q = \sec\alpha\left(\sin\alpha\cos\psi\frac{dh}{ds}-\beta h\right)\cdot e^{\beta z} \qquad \ldots\ldots\ldots(157)$$

$$\cos\alpha\, q_g + n' f_g = -\beta h\cdot e^{\beta z} \qquad \ldots\ldots\ldots\ldots\ldots(158)$$

Also, from (41b), the linear strain along the generators is given by

$$\varepsilon_g = \frac{1}{Gnt'}\left[\left(1-\frac{E}{4G}\frac{\cos^2\alpha}{n^2}\right)\frac{dh}{ds}-\beta\sin\alpha\cos\psi\, h\right]e^{(\beta-k)z} \qquad \ldots\ldots(159)$$

The cross sectional area of skin in a rib section is proportional to e^{kz} so that we take the variation of the area of the mth boom to be given by

$$B_m = B'_m\, e^{kz} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(160)$$

By virtue of equation (159) the condition of continuity of strain through the mth boom is

$$\left[\frac{1}{nt'}\left\{\left(1-\frac{E}{4G}\frac{\cos^2\alpha}{n^2}\right)\frac{dh}{ds}-\beta\sin\alpha\cos\psi\, h\right\}\right]_{m,m+1}$$

$$=\left[\frac{1}{nt'}\left\{\left(1-\frac{E}{4G}\frac{\cos^2\alpha}{n^2}\right)\frac{dh}{ds}-\beta\sin\alpha\cos\psi\, h\right\}\right]_{m,m-1} \qquad \ldots\ldots\ldots(161)$$

Denoting the common value at the mth boom of the quantity on the two sides of (161) by a suffix m, the tension P_m in the mth boom is then given by

$$P_m = \frac{E}{G} B_m' \left[\frac{1}{nt'} \left\{ \left(1 - \frac{E}{4G}\frac{\cos^2\alpha}{n^2}\right)\frac{dh}{ds} - \beta \sin\alpha \cos\psi h \right\} \right]_m e^{\beta z} \quad \ldots.(162)$$

The equation of equilibrium of an elementary length of the mth boom in the z direction is

$$(\cos\alpha\, q_g + n' f_g)_{m,m-1} - (\cos\alpha\, q_g + n' f_g)_{m,m+1} = \cos\alpha \frac{dP_m}{dz} \quad (163)$$

Using equation (158) this becomes

$$h_{m,m+1} - h_{m,m-1} = \frac{E}{G} B_m' \cos\alpha \left[\frac{1}{nt'} \left\{ \left(1 - \frac{E}{4G}\frac{\cos^2\alpha}{n^2}\right)\frac{dh}{ds} - \beta \sin\alpha \cos\psi h \right\} \right]_m \quad \ldots\ldots (164)$$

Equation (164), in conjunction with (162), provides an alternative expression for the boom tension, thus

$$P_m = \sec\alpha\,(h_{m,m+1} - h_{m,m-1})\, e^{\beta z} \quad \ldots\ldots\ldots\ldots(165)$$

Equations (161) and (164) applied at each of the N booms in turn provide 2N of the necessary (2N+3) linear simultaneous equations in the unknown constants.

§ 4.2 The stress resultants:

We now consider the values of the stress resultants corresponding to the stress function gh. At every rib section these are resolved into two bending moments M_x and M_y about the x and y axes, shear forces S_x and S_y along the x and y axes, a direct force R in the z_o direction and a torque T about an axis parallel to z_o and passing through the origin C of the x,y axes. Each of these stress resultants may be expressed as an integral of the stress flows, thus

$$\left.\begin{aligned}
R &= \oint nfds + \cos\alpha \sum_m P_m \\
M_x &= \oint nfyds + \cos\alpha \sum_m P_m y_m \\
M_y &= \oint nfxds + \cos\alpha \sum_m P_m x_m \\
T &= \oint \left[pq + (mx - l\,y)f \right] ds + \sin\alpha \sum_m P_m x_m \\
S_x &= \sec\alpha \, \frac{dM_y}{dz} \\
S_y &= \sec\alpha \, \frac{dM_x}{dz} + R\tan\alpha
\end{aligned}\right\} \quad \ldots\ldots(166)$$

If equations (156), (157) and (165) are substituted into the above expressions we obtain, after some reduction,

$$\left.\begin{aligned}
R &= 0 \\
M_x &= -e^{\beta z} \oint h \cos\psi \, ds \\
M_y &= -e^{\beta z} \oint h \sin\psi \, ds \\
T &= -\beta \sec\alpha \, e^{\beta z} \oint hpds + M_y \tan\alpha \\
S_x &= -\beta \sec\alpha \, e^{\beta z} \oint h \sin\psi \, ds \\
S_y &= -\beta \sec\alpha \, e^{\beta z} \oint h \cos\psi \, ds
\end{aligned}\right\} \quad \ldots\ldots(167)$$

Hence, as in the conical tube, the assumption of the stress function gh automatically infers no load normal to the ribs. The remaining stress resultants vary exponentially along the length of the tube. If then the applied loads are such that they give rise to an exponential variation of bending moment and torque so that

$$\left.\begin{aligned}
M_x &= M_x' \, e^{\beta z} \\
M_y &= M_y' \, e^{\beta z} \\
T &= T' \, e^{\beta z}
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (168)$$

where M_x', M_y' and T' are the values at z=0, then the remaining three equations for the evaluation of the unknown constants are given by

$$\left.\begin{aligned}\oint h\cos\Psi\,ds &= -M'_x\\ \oint h\sin\Psi\,ds &= -M'_y\\ \beta\oint hpds &= (M'_y\sin\alpha - T'\cos\alpha)\end{aligned}\right\} \quad \ldots\ldots (1\text{6.})$$

In practice it is highly improbable that the applied loads will give anything approaching an exponential variation of bending moment and torque, and an arbitrary smooth distribution cannot be represented adequately by a series of exponentials. Hence, for obtaining stress systems in equilibrium with the applied loads it is unlikely that this type of stress function will be of much use. It does, however, enable self-equilibrating stress systems to be derived and these will be considered next.

§4.3. Self-equilibrating stress systems :

If the stress system corresponding to F = gh is self-equilibrating then by virtue of equations (167) we must have

$$\oint h\cos\Psi\,ds = \oint h\sin\Psi\,ds = \oint hpds = 0 \quad \ldots\ldots (170)$$

Equations (161), (164) and (170) provide (2N+3) linear simultaneous equations in the (2N+3) unknowns and all the equations are homogeneous. A non-zero solution exists only if the determinant formed by the coefficients of the (2N+3) unknowns is zero, thus leading to a characteristic equation in β . The roots β_i of this equation, or the eigenvalues, are in general infinite in number and each one is associated with an eigenfunction h_i which is defined completely in functional form but not in absolute magnitude.

It is fairly easy to show from the differential equation (155) and the "boundary conditions" (161), (164) and (170) that the eigenfunctions h_i and h_j satisfy the reciprocal relation

$$\oint \frac{h_i}{mt'}\left(\sin\alpha\cos\Psi\frac{dh_j}{ds} - \beta_j h_j\right)ds = \oint \frac{h_j}{mt'}\left(\sin\alpha\cos\Psi\frac{dh_i}{ds} - \beta_i h_i\right)ds$$

$$\ldots\ldots\ldots\ldots (171)$$

If $\alpha = 0$, in which case n = 1, this reduces to the equation

$$(\beta_i - \beta_j)\oint \frac{h_i h_j}{t'} ds$$

so that $\oint \frac{h_i h_j}{t'} ds = 0$ if $i \neq j$

This is the orthogonal relation for unswept cylindrical tubes derived by Hadji-Argyris and Dunne. It will be seen from (171), however, that in the swept tube no such orthogonality exists so that, as in the conical tube, it will be possible to satisfy the conditions at only a few chosen points in the end sections.

§4.4. The warping of the ribs :

In order to deal with the conditions at an end which is constrained against warping an expression for the warping function w^* corresponding to the stress function gh is required. The derivation of this expression is exactly similar to that of the corresponding equation (88) for the conical tube, giving finally

$$w^* = \frac{\varepsilon_g \sec\alpha}{(\beta - k)} \quad \ldots\ldots\ldots\ldots\ (172)$$

Again it should be emphasised that the warping w^* in equation (172) is not the complete displacement w but the difference between w and w^* corresponds to rigid body movements of the ribs.

§4.5. Solution of equation (155) for a polygonal tube with constant wall thickness :

To conclude this discussion of the stress function gh, the solution of equation (155) will be obtained for a tube of polygonal section with k = 0 and t = t' = constant in each flat wall. In this case ψ, p and n are also constant in each wall so that equation (155) becomes

$$\left(1 - \frac{E}{4G}\frac{\cos^2\alpha}{n^2}\right)\frac{d^2h}{ds^2} - 2\beta\sin\alpha\cos\psi\frac{dh}{ds} + \beta^2 h$$
$$= -C_i n t\left(p\bar{\theta} + \bar{u}\sin\psi + \bar{v}\cos\psi\right)$$
$$\ldots\ldots\ldots\ (173)$$

This is a second order linear differential equation with constant coefficient and a constant particular integral. Its general solution is

$$h = e^{\beta_1 s}\left[C\sin\beta_2 s + D\cos\beta_2 s\right] - \frac{G_1 n t}{\beta^2}\left(p\bar{\theta} + \bar{U}\sin\psi + \bar{V}\cos\psi\right) \quad \ldots\ldots(174)$$

where

$$\left.\begin{aligned} \beta_1 &= \beta\sin\alpha\cos\psi\left(1 - \frac{E}{4G}\frac{\cos^2\alpha}{n^2}\right)^{-1} \\ \beta_2 &= \beta\frac{\cos\alpha}{n}\left(1-\frac{E}{4G}\right)^{\frac{1}{2}}\left(1 - \frac{E}{4G}\frac{\cos^2\alpha}{n^2}\right)^{-1} \end{aligned}\right\} \quad \ldots\ldots\ldots(175)$$

The solution (174) holds for all values of β other than zero but this special case will be considered at a later stage when another type of stress function is discussed (§ 6).

§5. SINOIDAL SOLUTION FOR SWEPT CYLINDRICAL TUBE OF CONSTANT POLYGONAL SECTION :

It is evident from the preceding discussion that, unlike the conical tube, the stress function in its simplest form gh is not likely to be of much use for dealing with an arbitrary smooth variation of bending moment and torque. Such a distribution can, however, be represented adequately over the length of the tube by a few terms of a Fourier series. Apart from the constant term, each term of the series will be proportional to either sin β z or cos β z, where β is a constant, and we now look for a method of dealing with a load variation of this type. Because of the greater complexity of the stress function the method of solution is developed only for the case of a tube of polygonal section with the skin thickness constant in each flat wall and each boom area constant over its length. In this case the stress function is of the form

$$F = h_1(s)\sin\beta z + h_2(s)\cos\beta z \quad \ldots\ldots\ldots\ldots(176)$$

Using this equation and the fact that ψ, p, n and t are constants in each wall we have

$$\frac{\partial}{\partial z}\left[\frac{1}{nt}\left(\sin\alpha\cos\psi\frac{\partial F}{\partial s}-\frac{\partial F}{\partial z}\right)\right]$$

$$= \frac{\beta}{nt}\left[\sin\alpha\cos\psi\frac{dh_1}{ds}+\beta h_2\right]\cos\beta z - \frac{\beta}{nt}\left[\sin\alpha\cos\psi\frac{dh_2}{ds}-\beta h_1\right]\sin\beta z$$

$$\ldots\ldots\ldots(177)$$

and

$$\frac{1}{nt}\left[\left(1-\frac{E}{4G}\frac{\cos^2\alpha}{n^2}\right)\frac{\partial F}{\partial s}-\sin\alpha\cos\psi\frac{\partial F}{\partial z}\right]$$

$$= \frac{1}{nt}\left[\left(1-\frac{E}{4G}\frac{\cos^2\alpha}{n^2}\right)\frac{dh_1}{ds}+\beta\sin\alpha\cos\psi\, h_2\right]\sin\beta z$$

$$+ \frac{1}{nt}\left[\left(1-\frac{E}{4G}\frac{\cos^2\alpha}{n^2}\right)\frac{dh_2}{ds}-\beta\sin\alpha\cos\psi\, h_1\right]\cos\beta z \quad \ldots\ldots\ldots(178)$$

It will therefore be seen from equations (43b)-(45b) that the curvatures and twist may be written

$$\left.\begin{aligned}\frac{d^2u}{dz^2} &= \bar{u}_1 \sin\beta z + \bar{u}_2 \cos\beta z\\ \frac{d^2v}{dz^2} &= \bar{v}_1 \sin\beta z + \bar{v}_2 \cos\beta z\\ \frac{d^2\theta}{dz^2} &= \bar{\theta}_1 \sin\beta z + \bar{\theta}_2 \cos\beta z\end{aligned}\right) \quad \ldots\ldots\ldots (179)$$

where $\bar{u}_1$, $\bar{u}_2$, $\bar{v}_1$, $\bar{v}_2$, $\bar{\theta}_1$ and $\bar{\theta}_2$ are constants.

If now equations (177)-(179) are substituted into (42b) the resulting equation becomes

$$\begin{aligned}&\sin\beta z\left[\left(1-\frac{E}{4G}\frac{\cos^2\alpha}{n^2}\right)\frac{d^2h_1}{ds^2}-\beta^2h_1+2\beta\sin\alpha\cos\psi\frac{dh_2}{ds}+Gnt\left(\bar{U}_1\sin\psi+\bar{V}_1\cos\psi+p\bar{\theta}_1\right)\right]\\ &+\cos\beta z\left[\left(1-\frac{E}{4G}\frac{\cos^2\alpha}{n^2}\right)\frac{d^2h_2}{ds^2}-\beta^2h_2-2\beta\sin\alpha\cos\psi\frac{dh_1}{ds}+Gnt\left(\bar{U}_2\sin\psi+\bar{V}_2\cos\psi+p\bar{\theta}_2\right)\right]\\ &= 0 \qquad \ldots\ldots (180)\end{aligned}$$

The two terms in this equation are compatible with each other only if they are both zero, thus giving the two linear simultaneous differential equations

$$\left[\left(1-\frac{E}{4G}\frac{\cos^2\alpha}{n^2}\right)\frac{d^2}{ds^2}-\beta^2\right]h_1+2\beta\sin\alpha\cos\psi\frac{dh_2}{ds}=-Gnt\left(\bar{U}_1\sin\psi+\bar{V}_1\cos\psi+p\bar{\theta}_1\right)$$

$$\ldots.(181)$$

$$\left[\left(1-\frac{E}{4G}\frac{\cos^2\alpha}{n^2}\right)\frac{d^2}{ds^2}-\beta^2\right]h_2-2\beta\sin\alpha\cos\psi\frac{dh_1}{ds}=-Gnt\left(\bar{U}_2\sin\psi+\bar{V}_2\cos\psi+p\bar{\theta}_2\right)$$

Since in each wall ψ, p, n and t are constants these two equations have constant coefficients and particular integrals and it is a simple matter to show that the complete solution is

$$\begin{aligned}h_1 &= e^{\beta_2 s}\left(C\sin\beta_1 s+D\cos\beta_1 s\right)+e^{-\beta_2 s}\left(H\sin\beta_1 s+K\cos\beta_1 s\right)\\ &\quad+\frac{Gnt}{\beta^2}\left(\bar{U}_1\sin\psi+\bar{V}_1\cos\psi+p\bar{\theta}_1\right)\\ h_2 &= e^{\beta_2 s}\left(D\sin\beta_1 s-C\cos\beta_1 s\right)+e^{-\beta_2 s}\left(K\sin\beta_1 s-H\cos\beta_1 s\right)\\ &\quad+\frac{Gnt}{\beta^2}\left(\bar{U}_2\sin\psi+\bar{V}_2\cos\psi+p\bar{\theta}_2\right)\end{aligned} \quad \ldots(182)$$

where β_1 and β_2 are given by equations (175).

The constants of integration C, D, H and K in general take different values in the various walls, whereas the curvature and twist constants $\bar{u}_1$, $\bar{u}_2$, $\bar{v}_1$, $\bar{v}_2$, $\bar{\theta}_1$ and $\bar{\theta}_2$ are the same in all walls. Thus in an N boom tube there are in general (4N+6) unknown constants in the solutions for h_1 and h_2, so that (4N+6) equations are required for their evaluation.

Substitution of the stress function (176) into equations (38b)-(41b) gives the following expressions for the stress flows and linear strain along the generators

$$f = \frac{1}{n}\left[\frac{dh_1}{ds}\sin\beta z + \frac{dh_2}{ds}\cos\beta z\right] \qquad \ldots\ldots\ldots(183)$$

$$\cos\alpha\, q_g + n' f_g = \beta\left[h_2 \sin\beta z - h_1\cos\beta z\right] \qquad \ldots\ldots\ldots(184)$$

$$q = \sec\alpha\left[\left(\sin\alpha\cos\psi\frac{dh_1}{ds} + \beta h_2\right)\sin\beta z + \left(\sin\alpha\cos\psi\frac{dh_2}{ds} - \beta h_1\right)\cos\beta z\right] \qquad \ldots\ldots\ldots(185)$$

$$\varepsilon_g = \frac{1}{Gnt}\left(1 - \frac{E}{4G}\frac{\cos^2\alpha}{n^2}\right)\left[\left(\frac{dh_1}{ds} + \beta_1 h_2\right)\sin\beta z + \left(\frac{dh_2}{ds} - \beta_1 h_1\right)\cos\beta z\right] \qquad \ldots\ldots\ldots(186)$$

It will be seen that the linear strain ε_g consists of two parts, one proportional to sin β z and the other to cos β z. Each part must have the same value on the two sides of each boom so that the condition of continuity of strain through the mth boom becomes

$$\left[\frac{1}{nt}\left(1 - \frac{E}{4G}\frac{\cos^2\alpha}{n^2}\right)\left(\frac{dh_1}{ds} + \beta_1 h_2\right)\right]_{m,m+1} = \left[\frac{1}{nt}\left(1 - \frac{E}{4G}\frac{\cos^2\alpha}{n^2}\right)\left(\frac{dh_1}{ds} + \beta_1 h_2\right)\right]_{m,m-1} \qquad \ldots\ldots\ldots(187)$$

$$\left[\frac{1}{nt}\left(1 - \frac{E}{4G}\frac{\cos^2\alpha}{n^2}\right)\left(\frac{dh_2}{ds} - \beta_1 h_1\right)\right]_{m,m+1} = \left[\frac{1}{nt}\left(1 - \frac{E}{4G}\frac{\cos^2\alpha}{n^2}\right)\left(\frac{dh_2}{ds} - \beta_1 h_1\right)\right]_{m,m-1}$$

Denoting now by a suffix m the value at the mth boom of each of the two quantities involved in equations (187), the tension P_m in the boom becomes

$$P_m = \frac{E}{G}B_m\left\{\left[\frac{1}{nt}\left(1 - \frac{E}{4G}\frac{\cos^2\alpha}{n^2}\right)\left(\frac{dh_1}{ds} + \beta_1 h_2\right)\right]_m \sin\beta z + \left[\frac{1}{nt}\left(1 - \frac{E}{4G}\frac{\cos^2\alpha}{n^2}\right)\left(\frac{dh_2}{ds} - \beta_1 h_1\right)\right]_m \cos\beta z\right\} \qquad \ldots\ldots\ldots(188)$$

If now equations (184) and (188) are substituted into (163) it will be found that the equation of boom equilibrium also consists of a part proportional to sin β z and one proportional to cos β z. These may be separated to give the two equations

$$(h_1)_{m,m+1} - (h_1)_{m,m-1} = \frac{E}{G_1} B_m \cos\alpha \left[\frac{1}{nt}\left(1 - \frac{E}{4G_1}\frac{\cos^2\alpha}{n^2}\right)\left(\frac{dh_1}{ds} + \beta_1 h_2\right)\right]_m$$

$$\ldots\ldots(189)$$

$$(h_2)_{m,m+1} - (h_2)_{m,m-1} = \frac{E}{G_1} B_m \cos\alpha \left[\frac{1}{nt}\left(1 - \frac{E}{4G_1}\frac{\cos^2\alpha}{n^2}\right)\left(\frac{dh_2}{ds} - \beta_1 h_1\right)\right]_m$$

Equations (187) and (189) applied at each of the N booms in turn provide 4N of the necessary (4N+6) linear simultaneous equations in the unknown constants. The remaining six equations are obtained from the values of the stress resultants.

Equations (189) may be used to give an alternative expression to (188) for the boom tension P_m, thus

$$P_m = \sec\alpha\left[(h_1)_{m,m+1} - (h_1)_{m,m-1}\right]\sin\beta z + \sec\alpha\left[(h_2)_{m,m+1} - (h_2)_{m,m-1}\right]\cos\beta z \quad \ldots\ldots\ldots(190)$$

Substitution of (183), (185) and (190) into equations (166) for the stress resultants gives, after simplification,

$$\left.\begin{aligned}
R &= 0 \\
M_x &= -\sin\beta z \oint h_1 \cos\Psi\, ds - \cos\beta z \oint h_2 \cos\Psi\, ds \\
M_y &= -\sin\beta z \oint h_1 \sin\Psi\, ds - \cos\beta z \oint h_2 \sin\Psi\, ds \\
T &= \beta\sec\alpha\left[\sin\beta z \oint h_2\, p\, ds - \cos\beta z \oint h_1\, p\, ds\right] + M_y \tan\alpha \\
S_x &= \beta\sec\alpha\left[\sin\beta z \oint h_2 \sin\Psi\, ds - \cos\beta z \oint h_1 \sin\Psi\, ds\right. \\
S_y &= \beta\sec\alpha\left[\sin\beta z \oint h_2 \cos\Psi\, ds - \cos\beta z \oint h_1 \cos\Psi\, ds\right.
\end{aligned}\right\} (191)$$

Hence if the applied loads give rise to a bending moment and torque variation of the form

$$\left.\begin{aligned}
M_x &= M'_{1x} \sin\beta z + M'_{2x} \cos\beta z \\
M_y &= M'_{1y} \sin\beta z + M'_{2y} \cos\beta z \\
T &= T'_1 \sin\beta z + T'_2 \cos\beta z
\end{aligned}\right\} \ldots\ldots\ldots\ldots(192)$$

where M'_{1x}, M'_{2x}, etc. are constants, the remaining six equations necessary for the determination of the unknowns are given by

$$\left.\begin{aligned}
&\oint h_1 \cos\psi\, ds = -M'_{1x} \;, \qquad \oint h_2 \cos\psi\, ds = -M'_{2x} \\
&\oint h_1 \sin\psi\, ds = -M'_{1y} \;, \qquad \oint h_2 \sin\psi\, ds = -M'_{2y} \\
&\beta\oint h_2\, pds = \left(T'_1 \cos\alpha - M'_{1y} \sin\alpha\right), \beta\oint h_1\, pds = \left(M'_{2y} \sin\alpha - T'_2 \cos\alpha\right)
\end{aligned}\right)$$

. . . . (193)

To allow for the conditions at the ends of the tube self-equilibrating stress systems of the type described in § 4.3 must be superimposed.

§6. SWEPT CYLINDRICAL TUBE OF ARBITRARY CONSTANT SECTION UNDER CONSTANT BENDING MOMENT AND TORQUE :

If a cylindrical tube of constant section is subjected to constant bending moments and torque it is possible to obtain a complete analytical solution which holds for any shape of tube section and for any distribution of stress bearing area round the section.

The stress function is of the form

$$F = h(s) + Cz \quad \dots\dots\dots\dots\dots \quad (194)$$

where C is a constant.

Substitution of (194) into equations (38b)-(41b) gives

$$f = \frac{1}{n}\frac{dh}{ds} \quad \dots\dots\dots\dots\dots\dots \quad (195)$$

$$q = \sec\alpha\left(\sin\alpha\cos\psi\frac{dh}{ds} - C\right) \quad \dots\dots\dots \quad (196)$$

$$\cos\alpha\, q_g + n' f_g = -C \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (197)$$

$$\varepsilon_g = \frac{1}{Gnt}\left[\left(1 - \frac{E}{4G}\frac{\cos^2\alpha}{n^2}\right)\frac{dh}{ds} - C\sin\alpha\cos\psi\right] \quad (198)$$

Since both the boom area and boom stress ($=E\varepsilon_g$) are independent of z, the tension in each boom is constant so that, by virtue of (197), the equation of boom equilibrium (163) is automatically satisfied if C has the same value in all walls.

If equation (194) is substituted into (43b)-(45b) it is found that

$$\left.\begin{aligned} \frac{d^2\theta}{dz^2} &= 0 \\ \frac{d^2u}{dz^2} &= \text{constant} = \bar{u} \\ \frac{d^2v}{dz^2} &= \text{constant} = \bar{v} \end{aligned}\right\} \quad \dots\dots\dots\dots \quad (199)$$

The differential equation (42b) then becomes

$$\frac{d}{ds}\left[\frac{1}{nt}\left\{\left(1 - \frac{E}{4G}\frac{\cos^2\alpha}{n^2}\right)\frac{dh}{ds} - C\sin\alpha\cos\psi\right\}\right] = -G\left(\bar{u}\sin\psi + \bar{v}\cos\psi\right)$$

$$\dots\dots\dots (200)$$

This equation may be integrated as it stands to give

$$\left(1 - \frac{E}{4G}\frac{\cos^2\alpha}{n^2}\right)\frac{dh}{ds} - C\sin\alpha\cos\psi = Gnt\,(D - \bar{u}x - \bar{v}y) \quad \dots\dots(201)$$

where D is a constant of integration. Solving for $\frac{dh}{ds}$ then gives

$$\frac{dh}{ds} = C\eta + Et_E(D - \bar{u}x - \bar{v}y) \quad \ldots\ldots\ldots\ldots (202)$$

where
$$\eta = \sin\alpha\cos\psi\left(1 - \frac{E}{4G}\frac{\cos^2\alpha}{n^2}\right)^{-1} \quad \ldots\ldots\ldots (203)$$

$$t_E = \frac{G}{E}nt\left(1 - \frac{E}{4G}\frac{\cos^2\alpha}{n^2}\right)^{-1} \quad \ldots\ldots\ldots (204)$$

By virtue of (198) and (201) the linear strain ε_g may be written

$$\varepsilon_g = D - \bar{u}x - \bar{v}y \quad \ldots\ldots\ldots\ldots\ldots (205)$$

and the condition of continuity of strain through the various booms is therefore automatically satisfied if D has the same value in all walls. Hence, however many booms the tube contains only four unknown constants are involved in the solution, namely C, D, $\bar{u}$ and $\bar{v}$. These are determined from the magnitudes of the stress resultants.

By virtue of (205) the tension in the mth boom is given by

$$P_m = EB_m(D - \bar{u}x - \bar{v}y) \quad \ldots\ldots\ldots\ldots (206)$$

Substitution of (195), (196), (202) and (206) into equations (166) for the stress resultants then yields the expressions

$$\left.\begin{aligned}
R &= C\xi + ED\lambda - E\bar{u}\lambda_y - E\bar{v}\lambda_x \\
M_x &= C\xi_x + ED\lambda_x - E\bar{u}I_{xy} - E\bar{v}I_{xx} \\
M_y &= C\xi_y + ED\lambda_y - E\bar{u}I_{yy} - E\bar{v}I_{xy} \\
T &= -2AC\sec\alpha + M_y\tan\alpha \\
S_x &= 0 \\
S_y &= R\tan\alpha
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (207)$$

where

$$\left.\begin{aligned}
\xi &= \oint\eta\,ds = \oint\sin\alpha\cos\psi\left(1 - \frac{E}{4G}\frac{\cos^2\alpha}{n^2}\right)^{-1}ds \\
\xi_x &= \oint\eta\,y\,ds = \oint y\sin\alpha\cos\psi\left(1 - \frac{E}{4G}\frac{\cos^2\alpha}{n^2}\right)^{-1}ds \\
\xi_y &= \oint\eta\,x\,ds = \oint x\sin\alpha\cos\psi\left(1 - \frac{E}{4G}\frac{\cos^2\alpha}{n^2}\right)^{-1}ds
\end{aligned}\right\} \ldots (208)$$

and
$$\left.\begin{aligned}
\lambda &= \oint t_E\,ds + \cos\alpha\sum_m B_m \\
\lambda_x &= \oint t_E\,y\,ds + \cos\alpha\sum_m B_m y_m \\
\lambda_y &= \oint t_E\,x\,ds + \cos\alpha\sum_m B_m x_m \\
I_{xx} &= \oint t_E\,y^2\,ds + \cos\alpha\sum_m B_m y_m^2
\end{aligned}\right\} \ldots\ldots\ldots\ldots (209)$$

$$\left.\begin{aligned} I_{yy} &= \oint t_E x^2 ds + \cos\alpha \sum_m B_m x_m^2 \\ I_{xy} &= \oint t_E xyds + \cos\alpha \sum_m B_m x_m y_m \end{aligned}\right\} \ldots\ldots (209) \text{ continued}$$

Equations (207)-(209) are exactly analagous to the corresponding equations (100)-(105) for the conical tube.

The fourth of equations (207) gives the value of the constant C, thus

$$C = -\frac{1}{2A}(T\cos\alpha - M_y \sin\alpha) \quad \ldots\ldots\ldots\ldots (210)$$

As we are interested here only in the application of constant bending moment and torque we make R = 0 so that $S_y = 0$ also. The first of equations (207) then gives, after using (210),

$$D = \frac{1}{\lambda}\left[\frac{\xi}{2AE}(T\cos\alpha - M_y \sin\alpha) + \bar{u}\lambda_y + \bar{v}\lambda_x\right] \quad \ldots\ldots (211)$$

Hence, from the second and third of equations (207)

$$M_x = \frac{1}{2A\lambda}(\xi\lambda_x - \lambda\xi_x)(T\cos\alpha - M_y\sin\alpha) + \frac{E\bar{u}}{\lambda}(\lambda_x\lambda_y - \lambda I_{xy}) + \frac{E\bar{v}}{\lambda}(\lambda_x^2 - \lambda I_{xx}) \quad \ldots\ldots\ldots (212)$$

$$M_y = \frac{1}{2A\lambda}(\xi\lambda_y - \lambda\xi_y)(T\cos\alpha - M_y\sin\alpha) + \frac{E\bar{u}}{\lambda}(\lambda_y^2 - \lambda I_{yy}) + \frac{E\bar{v}}{\lambda}(\lambda_x\lambda_y - \lambda I_{xy}) \quad \ldots\ldots\ldots (213)$$

The constant C is obtained from (210), $\bar{u}$ and $\bar{v}$ from (212) and (213) and then D from (211), so that the solution is now complete.

So far the origin of the coordinates x,y has been chosen arbitrarily but a certain amount of simplification is now possible if the origin is defined as being that point in the section for which

$$\lambda_x = \lambda_y = 0.$$

(If the tube is unswept this amounts to taking the origin at the centroid of the stress bearing area of the section). Fixing the origin in this way equations (211)-(213) reduce to

$$\left.\begin{aligned} D &= \frac{\xi}{2AE\lambda}(T\cos\alpha - M_y\sin\alpha) \\ M_x &= -\frac{\xi_x}{2A}(T\cos\alpha - M_y\sin\alpha) - E\bar{u}I_{xy} - E\bar{v}I_{xx} \\ M_y &= -\frac{\xi_y}{2A}(T\cos\alpha - M_y\sin\alpha) - E\bar{u}I_{yy} - E\bar{v}I_{xy} \end{aligned}\right\} \ldots\ldots (214)$$

It is worthy of note that if the tube is unswept so that $\alpha = 0$, we have, from (208), $\xi = \xi_x = \xi_y = 0$. The two curvatures $\bar{u}$ and $\bar{v}$ are then independent of the torque T, and equations (210) and (214) become

$$\left.\begin{aligned} C &= -\frac{T}{2A} \\ D &= 0 \\ M_x &= -E\bar{u}I_{xy} - E\bar{v}I_{xx} \\ M_y &= -E\bar{u}I_{yy} - E\bar{v}I_{xy} \end{aligned}\right) \quad \ldots\ldots\ldots\ldots\ldots (215)$$

Also, in this case equation (204) gives

$$t_E = t\left[\frac{E}{G}\left(1 - \frac{E}{4G}\right)\right]^{-1} = \frac{E'}{E}t \quad \ldots\ldots\ldots\ldots (216)$$

Now for an unswept tube $\frac{E'}{E}t$ is the effective skin thickness for stress in the direction of the generators, the linear strain in the perpendicular direction being zero. Consequently the properties of the section λ, λ_x, λ_y, I_{xx}, I_{yy} and I_{xy} defined by equation (209) become respectively the effective stress bearing area of the section, the moments of this area about the x and y axes, the second moments about these axes, and the product of inertia (or cross moment) of the effective stress bearing area with respect to these axes.

The twist of the tube is obtained by substitution of (33), (196) and (202) into (27b). After a certain amount of simplification we obtain the equation

$$\frac{d\theta}{dz} = -\frac{C}{2AG}\oint\frac{(1-\eta\sin\alpha\cos\psi)}{nt}ds + \frac{1}{2A}\left(D\xi - \bar{u}\xi_y - \bar{v}\xi_x\right)$$

On using equations (210) and (211) this becomes

$$\frac{d\theta}{dz} = \frac{(T\cos\alpha - M_y\sin\alpha)}{4A^2G}\left[\oint\frac{(1-\eta\sin\alpha\cos\psi)}{nt}ds + \frac{G}{E}\frac{\xi^2}{\lambda}\right] + \frac{\bar{u}}{2A\lambda}\left(\xi\lambda_y - \lambda\xi_y\right) + \frac{\bar{v}}{2A\lambda}\left(\xi\lambda_x - \lambda\xi_x\right)$$

$$\ldots\ldots\ldots\ldots (217)$$

Note that if $\alpha = 0$ this reduces to the Bredt-Batho formula

$$\frac{d\theta}{dz} = \frac{T}{4A^2G}\oint\frac{ds}{t}$$

Suppose now that the couples M_x, M_y and T are of such magnitudes that the curvatures $\bar{u}$ and $\bar{v}$ are zero. Then, from the last two of equations

(214), we have

$$\frac{M_x}{\xi_x} = \frac{M_y}{\xi_y} = \frac{-T}{(2A\sec\alpha - \xi_y\tan\alpha)} \qquad (218)$$

This result means that the resultant couple, whose components are M_x, M_y and T, is being applied about an axis whose direction cosines (l_c, m_c, n_c) are given by

$$l_c : m_c : n_c = \xi_x : \xi_y : (2A\sec\alpha - \xi_y\tan\alpha) \quad \ldots\ldots \quad (219)$$

This axis will be called the "zero curvature axis" and its significance is that if a pure couple is applied about it the tube twists without bending. By virtue of (217) and (218) the twist is then given by

$$\frac{d\theta}{dz} = -\frac{M_y}{2AG\xi_y}\left[\oint\left(\frac{1-\sin\alpha\cos\psi.\eta}{nt}\right)ds + \frac{G}{E}\frac{\xi^2}{\lambda}\right] \quad \ldots\ldots \quad (220)$$

In the special case of a tube whose section is symmetrical about the y axis the preceding results simplify considerably by virtue of the fact that

$$\xi = \xi_x = \lambda_y = I_{xy} = 0 \quad \ldots\ldots\ldots\ldots \quad (221)$$

Equations (214) then give

$$\left.\begin{aligned} D &= 0 \\ E\bar{u} &= -\frac{1}{I_{yy}}\left[M_y + \frac{\xi_y}{2A}(T\cos\alpha - M_y\sin\alpha)\right] \\ E\bar{v} &= -\frac{M_x}{I_{xx}} \end{aligned}\right\} \quad \ldots\ldots (222)$$

and the twist (217) becomes

$$\frac{d\theta}{dz} = \frac{(T\cos\alpha - M_y\sin\alpha)}{4A^2G}\left[\oint\left(\frac{1-\eta\sin\alpha\cos\psi}{nt}\right)ds + \frac{G}{E}\frac{\xi_y^2}{I_{yy}}\right] + \frac{M_y}{2AE}\frac{\xi_y}{I_{yy}} \quad \ldots\ldots\ldots (223)$$

Also the direction cosine l_c is zero so that the zero curvature axis is parallel to the plane of sweep. Its direction in this plane is as shown in Fig.8, where

$$\tan\Omega = \frac{n_c}{m_c} = \frac{2A}{\xi_y}\sec\alpha - \tan\alpha \quad \ldots\ldots\ldots\ldots \quad (224)$$

If the section is rectangular as in Fig.9, it can be shown that

$$\xi_y = \frac{A\sin\alpha}{1 - \frac{E}{4G}\cos^2\alpha}$$

and equation (224) then becomes

$$\tan \Omega = \tan \alpha + 2\left(1 - \frac{E}{4G}\right)\cot \alpha$$

$$= \tan \alpha + (1 - \mu)\cot \alpha \quad \ldots\ldots\ldots\ldots (225)$$

The angle Ω in (225) has a minimum value $\Omega_{min} = \tan^{-1}(2\sqrt{1-\mu})$ occurring at an angle of sweep $\alpha = \tan^{-1}(\sqrt{1-\mu})$. If $\mu = \frac{1}{3}$ the minimum value of Ω is 58.5^{o}, when the angle of sweep is 39.2^{o}. Also when $\alpha = 0$ or 90^{o} equation (225) gives $\Omega = 90^{o}$ so that for the tube of symmetrical rectangular section the zero curvature axis lies somewhere in the range $58.5^{o} < \Omega < 90^{o}$.

A further result of interest for the symmetrical rectangular section is that if $\alpha = \tan^{-1}\sqrt{\mu}$ then $\Omega = \tan^{-1}\sqrt{1/\mu}$, so that the zero curvature axis is parallel to the generators. If $\mu = \frac{1}{3}$ this occurs when $\alpha = 30^{o}$.

Considering the complexity of the analysis leading up to equation (225) its simplicity is remarkable. It is interesting also that for any tube of singly symmetrical section the angle Ω, defining the direction of the zero curvature axis, depends only on the shape of the section and not upon the distribution of stress bearing area. For this statement to be true it is only necessary for the shape of the section to be symmetrical (so that $\xi = 0$) and no restriction of symmetry need be placed on the distribution of stress bearing area round the section.

§7. SWEPT CYLINDRICAL TUBE OF CONSTANT POLYGONAL SECTION UNDER LINEARLY VARYING BENDING MOMENT AND TORQUE :

For this type of load variation the stress function is of the form

$$F = h_0(s) + zh_1(s) + \tfrac{1}{2}Cz^2 \quad \ldots\ldots\ldots\ldots\ldots \quad (226)$$

where C is a constant.

This stress function is simply an extension of (194) for which the author is indebted to a suggestion by Mr. J.J. Thompson of the Department of Aeronautics, University of Sydney.

Equations (38b)-(41b) for the stress flows now take the form

$$f = \frac{1}{n}\frac{dh_0}{ds} + \frac{z}{n}\frac{dh_1}{ds} \quad \ldots\ldots\ldots\ldots\ldots \quad (227)$$

$$q = \sec\alpha\left(\sin\alpha\cos\psi\frac{dh_0}{ds} - h_1\right) + z\sec\alpha\left(\sin\alpha\cos\psi\frac{dh_1}{ds} - C\right) \ldots\ldots (228)$$

$$\cos\alpha\, q_g + n' f_g = -(h_1 + Cz) \quad \ldots\ldots\ldots\ldots\ldots \quad (229)$$

Also

$$\varepsilon_g = \frac{1}{Gnt}\left[\left(1 - \frac{E}{4G}\frac{\cos^2\alpha}{n^2}\right)\frac{dh_0}{ds} - \sin\alpha\cos\psi\, h_1\right]$$

$$+ \frac{z}{Gnt}\left[\left(1 - \frac{E}{4G}\frac{\cos^2\alpha}{n^2}\right)\frac{dh_1}{ds} - C\sin\alpha\cos\psi\right] \quad \ldots\ldots (230)$$

In developing the method of solution we shall consider only the case of a tube of constant polygonal section so that in each flat wall ψ, p, n and t are constants.

If the stress function (226) is substituted into equations (43b)-(45b) it is found that the rate of change of twist and the two curvatures may be written

$$\left.\begin{aligned} \frac{d^2\theta}{dz^2} &= \bar{\theta}_0 \\ \frac{d^2u}{dz^2} &= \bar{u}_0 + \bar{u}_1 z \\ \frac{d^2v}{dz^2} &= \bar{v}_0 + \bar{v}_1 z \end{aligned}\right) \quad \ldots\ldots\ldots\ldots\ldots \quad (231)$$

where $\bar{\theta}_0$, $\bar{u}_0$, $\bar{u}_1$, $\bar{v}_0$ and $\bar{v}_1$ are constants.

The solution now follows in an exactly similar way to that for a sinoidal load variation (§ 5) and to save unnecessary repetition only the outline will be given.

It is found that the two functions h_0 and h_1 must satisfy the two simultaneous differential equations

$$\left(1-\frac{E}{4G}\frac{\cos^2\alpha}{n^2}\right)\frac{d^2h_1}{ds^2} = -Gnt\left(\bar{u}_1\sin\psi + \bar{v}_1\cos\psi\right)$$

$$\left(1-\frac{E}{4G}\frac{\cos^2\alpha}{n^2}\right)\frac{d^2h_0}{ds^2} = 2\sin\alpha\cos\psi\,\frac{dh_1}{ds} - C - Gnt\left(\bar{u}_0\sin\psi + \bar{v}_0\cos\psi + p\bar{\theta}_0\right) \qquad (232)$$

Since in each wall ψ, p, n and t are constant and ds = d Δ, these equations may be integrated to give

$$h_1 = C\eta\Delta + Et_E\left[L + D\Delta - \tfrac{1}{2}\Delta^2(\bar{u}_1\sin\psi + \bar{v}_1\cos\psi) - p\Delta\left(\bar{u}_1\cos\psi - \bar{v}_1\sin\psi\right)\right] \qquad \ldots\ldots (233)$$

$$\frac{dh_0}{ds} = 2\eta h_1 + Et_E\left(K - \bar{u}_0x - \bar{v}_0y - 2A_{0s}\bar{\theta}_0\right) - C\Delta\left(1-\frac{E}{4G}\frac{\cos^2\alpha}{n^2}\right)^{-1} \qquad \ldots\ldots (234)$$

where K, L and D are constants of integration and η and t_E are given by equations (203) and (204) respectively.

Note that there is no necessity to integrate (234) any further since h_0 only occurs in the form of its first derivative in equations (227)-(230).

The conditions of continuity of strain and of boom equilibrium give rise to four equations at each boom, two of which are automatically satisfied if C and D have the same value in all walls. The remaining two equations to be satisfied at the mth boom are

$$\left[K + \frac{1}{Gnt}\left(\sin\alpha\cos\psi\, h_1 - C\Delta\right)\right]_{m,m+1} = \left[K + \frac{1}{Gnt}\left(\sin\alpha\cos\psi\, h_1 - C\Delta\right)\right]_{m,m-1} \qquad \ldots\ldots (235)$$

$$\left(h_1\right)_{m,m+1} - \left(h_1\right)_{m,m-1} = EB_m\left(D - \bar{u}_1x_m - \bar{v}_1y_m\right)\cos\alpha \qquad \ldots\ldots (236)$$

The total number of unknown constants in the solutions for h_0 and h_1 is in general (2N+7) in an N boom tube, composed of 2N of the type K and L, which have different values in the various walls, together with the seven absolute constants C, D, $\bar{\theta}_0$, $\bar{u}_0$, $\bar{u}_1$, $\bar{v}_0$ and $\bar{v}_1$. Equations (235) and (236) applied at each boom give 2N equations which enable the values of K and L in each wall to be expressed in terms of the other seven absolute constants.

By substitution of equations (227), (228) and (230) into (166) it is found that the stress function (226) infers stress resultants of the form

$$\left.\begin{aligned} R &= \text{constant} \\ M_x &= M'_{ox} + zM'_{1x} \\ M_y &= M'_{oy} + zM'_{1y} \\ T &= T'_o + zT'_1 \end{aligned}\right) \quad \ldots\ldots\ldots\ldots\ldots \quad (237)$$

where M'_{ox}, M'_{1x}, etc. are constants. These constants, together with R, are expressed in terms of the seven remaining unknowns C, D, $\bar{\theta}_o$, $\bar{u}_o$, $\bar{u}_1$, $\bar{v}_o$ and $\bar{v}_1$ which may therefore be determined to complete the solution.

§8. SWEPT CYLINDRICAL TUBE WITH STRINGERS REPLACED BY EQUIVALENT SHEET :

In all the solutions so far described, any stringers which the tube contains for the purpose of stabilising the skin have been considered under the general name of booms. It has been seen that if the number of booms (including stringers) is N, the number of simultaneous equations necessary to obtain a solution is in general at least (2N+3). Now if the booms consist only of four spar flanges (N=4) this would involve in general eleven simultaneous equations, although symmetry of the section and the applied loads would approximately halve this number. However if the skin depended upon stringers for its stability, the number of stringers required would probably be at the very least ten and in all probability considerably more for a wing of reasonable size. Consequently the number of simultaneous equations would then reach an impracticable level. The theory so far developed can therefore be regarded as having a practical value only if the skin is either so thick that it does not require stabilisation -- a remote contingency -- or is stabilised by a continuous medium which may be considered to be an integral part of the skin, as in a sandwich.

In conventional wing structures the difficulty described above is usually overcome by replacing the stringers for the purpose of stress analysis by an equivalent skin on the basis of equal areas. The fact that the stringers are usually fairly closely spaced and have a cross sectional area small in comparison with that of the spar flanges is sufficient justification for this approximation. This immediately suggests the possibility of adopting a similar procedure for the types of tube with which we are concerned here.

In order to represent adequately the effect of the stringers, the equivalent skin, henceforth called the "stringer sheet", must be considered to be composed of a series of elemental fibres located along the generators, which do not interact with each other and which are therefore subjected to pure tension or compression in the direction of the generators. These fibres will be "glued" to the skin so that equilibrium considerations will not be violated if the tension or compression varies along their length. With this basic conception the foregoing theory can be readily adjusted to take the stringer sheet into account. It is found, however, that in a conical tube the fact that the generators all lie in different directions makes the resulting differential equations impossible to integrate analytically (Ref.5, App.II). In the analysis that follows, therefore, consideration is given only to the case of swept cylindrical tubes.

It is clear that the direct stress flow in the stringer sheet in the direction of the generators will be $(Et_s \varepsilon_g)$, where t_s is the thickness of the stringer sheet. It is convenient to resolve this into a direct stress flow f_s and a shear flow q_s over the rib section, as shown in Fig.10. Since there is no stress flow over the generator AG, equilibrium considerations for the triangular element give

$$\left.\begin{aligned} f_s &= Et_s \varepsilon_g \sin^2 \phi = Et_s \varepsilon_g \frac{\cos^2 \alpha}{n^2} \\ q_s &= Et_s \varepsilon_g \sin \phi \cos \phi = Et_s \varepsilon_g \frac{\sin \alpha \cos \alpha \cos \Psi}{n} \end{aligned}\right\} \quad \ldots\ldots(238)$$

The total stress flows f_t and q_t over the rib section due to both skin and stringer sheet are given by

$$\left.\begin{aligned} f_t &= f + f_s \\ q_t &= q + q_s \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (239)$$

It will now be seen that the equation of equilibrium of an element of the stringer-skin combination is the same as (37b) if f is replaced by f_t. The stress function F is now defined in such a wa that

$$f_t = \frac{1}{n} \frac{\partial F}{\partial s} \quad \ldots\ldots\ldots\ldots\ldots\ldots (240)$$

$$\cos \alpha \, q_g + n' f_g = - \frac{\partial F}{\partial z} \quad \ldots\ldots\ldots\ldots\ldots\ldots (241)$$

From equations (23b), (33), (34) and (238) we have

$$f_t = \left(1 + \frac{E}{E'} \frac{t_s}{t} \frac{\cos^4 \alpha}{n^4}\right) f + \frac{E}{G} \frac{t_s}{t} \frac{\sin \alpha \cos^3 \alpha \cos \Psi}{n^3} q \quad \ldots(242)$$

Also, from equations (35b) and (13b),

$$(\cos\alpha\, q_g + n' f_g) = \cos\alpha\, q - n \sin\alpha \cos\Psi\, f \quad \ldots\ldots (243)$$

On substituting (242) and (243) into (240) and (241), and solving for f and q we obtain the expressions

$$f = Q_1 \frac{\partial F}{\partial s} + S_1 \frac{\partial F}{\partial z} \quad \ldots\ldots\ldots\ldots (244)$$

$$q = \sec\alpha \left(Q_2 \frac{\partial F}{\partial s} + S_2 \frac{\partial F}{\partial z} \right) \quad \ldots\ldots\ldots\ldots (245)$$

where

$$\left.\begin{aligned} Q_1 &= \frac{t}{n\zeta} \\ Q_2 &= \frac{t \sin\alpha \cos\Psi}{\zeta} \\ S_1 &= \frac{Et_s \sin\alpha \cos^2\alpha \cos\Psi}{Gn^3 \zeta} \\ S_2 &= -\frac{1}{\zeta}\left(t + \frac{Et_s \cos^4\alpha}{E' n^4} \right) \\ \text{and} \quad \zeta &= t + \frac{Et_s \cos^2\alpha}{Gn^2}\left(1 - \frac{E}{4G}\frac{\cos^2\alpha}{n^2} \right) \end{aligned}\right\} \quad \ldots\ldots (246)$$

The linear strain ε_g is obtained from (238) and (239), thus

$$\varepsilon_g = \frac{n^2 (f_t - f)}{Et_s \cos^2\alpha}$$

Using (240) and (244) this gives

$$\varepsilon_g = \frac{1}{Gnt}\left(Q_3 \frac{\partial F}{\partial s} + S_3 \frac{\partial F}{\partial z} \right) \quad \ldots\ldots\ldots\ldots (247)$$

where

$$\left.\begin{aligned} Q_3 &= \frac{t}{\zeta}\left(1 - \frac{E}{4G}\frac{\cos^2\alpha}{n^2} \right) \\ S_3 &= -\frac{t \sin\alpha \cos\Psi}{\zeta} = -Q_2 \end{aligned}\right\} \quad \ldots\ldots\ldots (248)$$

and ζ is given by (246).

Finally, the total shear flow q_t, obtained from equations (238), (239), (245) and (247), becomes

$$q_t = \sec\alpha \left(\sin\alpha \cos\Psi \frac{\partial F}{\partial s} - \frac{\partial F}{\partial z} \right) \quad \ldots\ldots\ldots (249)$$

The differential equation (42b) is now replaced by

$$\frac{\partial}{\partial z}\left[\frac{1}{nt}\left(Q_2 \frac{\partial F}{\partial s} + S_2 \frac{\partial F}{\partial z} \right) \right] - \frac{\partial}{\partial s}\left[\frac{1}{nt}\left(Q_3 \frac{\partial F}{\partial s} + S_3 \frac{\partial F}{\partial z} \right) \right]$$
$$= G\left(\frac{d^2 u}{dz^2} \sin\Psi + \frac{d^2 v}{dz^2} \cos\Psi + p \frac{d^2\theta}{dz^2} \right)$$
$$\ldots\ldots\ldots (250)$$

and equations (43b)-(45b) become

$$\frac{d^2\theta}{dz^2} = \frac{1}{2AG}\oint \frac{\partial}{\partial z}\left[\frac{1}{nt}\left(Q_2\frac{\partial F}{\partial s} + S_2\frac{\partial F}{\partial z}\right)\right]ds$$

$$\frac{d^2u}{dz^2} = -\frac{1}{2AG}\oint(2y-3\bar{y})\frac{\partial}{\partial z}\left[\frac{1}{nt}\left(Q_2\frac{\partial F}{\partial s} + S_2\frac{\partial F}{\partial z}\right)\right]ds$$

$$-\frac{1}{AG}\oint\frac{\cos\psi}{nt}\left(Q_3\frac{\partial F}{\partial s} + S_3\frac{\partial F}{\partial z}\right)ds \qquad \text{.......(251)}$$

$$\frac{d^2v}{dz^2} = \frac{1}{2AG}\oint(2x-3\bar{x})\frac{\partial}{\partial z}\left[\frac{1}{nt}\left(Q_2\frac{\partial F}{\partial s} + S_2\frac{\partial F}{\partial z}\right)\right]ds$$

$$+\frac{1}{AG}\oint\frac{\sin\psi}{nt}\left(Q_3\frac{\partial F}{\partial s} + S_3\frac{\partial F}{\partial z}\right)ds$$

The solutions for the various types of stress function now follow in an exactly similar way to those previously described, although two essential differences should be emphasised. First, the stringers no longer count as booms, which now include the spar flanges only. Secondly, the contribution of the stringers to the stress resultants are taken into account by replacing f and q in equations (166) by f_t and q_t respectively, the summations in these equations then referring to the spar flanges only.

Finally, it should be mentioned that if the section of the tube is constant and polygonal, and if t and t_s are constant in each flat wall, then the functions Q_1, Q_2, Q_3, S_1, S_2 and S_3 are also constants. This means that the differential equations for the h functions will be linear with constant coefficients and constant particular integrals, and analytical solutions can therefore be readily obtained.

§9. EXTENSION TO MULTI-CELL TUBES :

The preceding theory has been confined, for simplicity, to tubes containing a single cell, but the extension to multi-cell tubes of the type shown in Fig.11 presents no particular difficulty. The analysis is almost identical with that given previously, with the exception of a few small modifications. To save unnecessary repetition the main modifications will be enumerated for the conical tube analysis only, and the corresponding changes to be made in the swept cylindrical tube analysis will then be apparent.

(a) The circuit integrals in equations (27a), (28a) and (29a) for the twist and curvatures can now be taken round any closed circuit of the section. A similar interpretation must be made of the circuit integrals in equations (31a) and (32a) for the warping.

(b) The differential equation (55) for the function h is still applicable but the condition of continuity of strain (61) through the mth boom must now be extended to take account of the fact that the number of walls meeting at a boom may be more than two. The linear strain ε_g at the boom must be the same in all of them.

(c) Equations (64) and (65) for the equilibrium of an element of the boom must also be extended to include the forces exerted on the element by all the walls meeting there. On making this alteration it is found that the L.H.S. of (65) may be conveniently written as $(\nabla_m h)$, being the resultant "flow" of h out of the mth boom when h is treated as a fluid flow in each wall and is counted positive when it is flowing in the direction of s.

(d) Using this new equation of boom equilibrium it is found that equations (74)-(79) for the stress resultants remain unaltered so long as the circuit integrals are taken to mean the integrals over all walls of the section and not merely round a single closed cell.

(e) A review of the number of unknown constants involved in the solutions for h is required. If the tube contains N booms joined by W continuous walls it is easy to see that the number of cells must be (W-N+1). Now the number of unknown constants in the solutions for h in the various walls will in general be (2W+3), composed of 2W constants of integration of the type C and D in equation (56), together with $\bar{u}$, $\bar{v}$ and $\bar{\theta}$ which are the same for all walls. The conditions of continuity of strain through the various booms leads to (2W-N) equations in the unknowns, since at each boom the number of available equations is one less than the number of walls meeting there. Further, the number of equations arising from boom equilibrium considerations is N, and finally there are three more equations from the stress resultants, thus giving (2W+3) equations in all.

(f) The only other essential modification is in the solution for linear variation of bending moment and torque (§ 3.2). The constant C in equation (90) is no longer the same in all walls of the tube but is such that at the mth boom $(\nabla_m C)$ is zero, where, as above, $(\nabla_m C)$ means the "flow of C" out of the boom when it is treated as a fluid flow in each wall in the direction of s. This complicates the algebra considerably and it is no longer possible to proceed without defining the number of cells. This complication corresponds exactly to that encountered when the simple Bredt-Batho formulae for an unswept cylindrical tube are generalised to apply to a mult-cell tube.

§10. CONCLUSIONS:

The general theory has been used to obtain analytical solutions for certain fairly simple types of tube and applied load. These may be summarised as follows:-

(a) Conical tube of arbitrary section with the skin thickness constant along all generators, all boom areas tapering linearly to zero at the apex, and subjected to bending moments and torque varying linearly from zero at the apex to a maximum at the root.

(b) Swept cylindrical tube of arbitrary section with the skin thickness constant along all generators, all boom areas constant along the length of the tube, and subjected to constant bending moments and torque.

The formulae of (a) and (b) represent extensions of the simple theories of bending and torsion for an unswept tube.

(c) Conical tube of polygonal section, with the skin thickness constant over each flat wall, all boom areas tapering linearly to zero at the apex, and subjected to bending moments and torque which vary from section to section as a power of the distance from the apex.

(d) Self-equilibrating stress systems for the tube defined in (c).

(e) Swept cylindrical tubes of polygonal section, with the skin thickness constant over each flat wall, all boom areas constant along the length of the tube, and subjected to bending moments and torque which vary along the length of the tube according to either an exponential, sine or power law.

(f) Self-equilibrating stress systems for the tube defined in (e).

All the solutions exhibit an interaction between bending and torsion, which constitutes the essential difference between an unswept wing and a swept one of the type considered here.

The Appendix shows that if the tube is highly tapered, the root effect is of prime importance in determining the stresses and deformations. It is highly unlikely, therefore, that any simple formulae, similar to those of the simple bending theory for example, will give anything approaching the true stresses and deformations.

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APPENDIX.

Numerical solutions for a highly tapered unswept tube of doubly symmetrical rectangular cross section.

§ A.1. Description of tube and applied loads :

The foregoing theory will now be applied numerically to the tube shown in Fig.12. This is a highly tapered unswept conical tube with a doubly symmetrical rectangular section, the lengths of the sides of the rectangle being d and $\frac{1}{4}$d at the root. The perpendicular distance Z from the apex to the root section is also equal to d. The thickness of the top and bottom skins is constant and equal to t_0 whilst that of the two side walls is also constant and equal to $2t_0$. The tube contains four equal booms, located at the corners of the rectangle, and varying linearly in area from zero at the apex to dt_0 at the root. The root section is held in such a way that it is prevented from warping, and the tube is continuous to the apex of the cone.

The numerical work is confined to two types of applied load. The first type gives rise to a pure torque variation according to the equation

$$T = T'\rho^{\beta+1}$$

where T' is the value at the root section. The values of β considered are 0,1,2,3,4 and 5. Two self-equilibrating stress systems are derived in order to deal with the zero-warping condition at the root.

The second type of load is one giving rise to bending moments about the horizontal axis of symmetry, with variation of moment according to

$$M_y = M_y' \rho^{\beta+1}$$

where M_y' is the value at the root and the values of β considered are again 0,1,2,3,4 and 5. The root condition is again dealt with by two self-equilibrating stress systems. The methods used in obtaining the stresses and deformations are as set out in § 2 and § 3. The variation of Γ (equation 125) round the section is as shown in Fig.13

which also indicates the values of p, n and ψ in each wall.

The value of Poisson's ratio μ is taken to be $\frac{1}{3}$, so that $\frac{E}{G} = \frac{8}{3}$.

§ A.2. Stress systems in equilibrium with torque $T = T'\rho^{\beta+1}$:

We first derive stress systems to balance the applied torque, regardless of the non-warping root section.

First, if $\beta = 0$ so that $T = \rho T'$, the twist and the stress flows may be computed from equations (116) and (119). On substituting numbers these give

$$\frac{d\theta}{d\rho} = \frac{7.765\, T'}{\rho^2 G d^2 t_0}$$

$$f = -\frac{T'}{2A\rho}\,\frac{\Gamma\sqrt{3}}{1+\Gamma^2}$$

$$q = \frac{T'}{2A\rho}\cdot\frac{1}{1+\Gamma^2}$$

where $A = d^2/4$ is the area enclosed by the root section.

The warping w^*, which in this case is equal to the complete displacement w since $u = v = 0$, is obtained from equation (31a) and gives

$$w^* = \frac{T'}{Gdt_0}\left(0.722 \tan^{-1}\Gamma - 2.506\,\Gamma\right) \text{ in the side walls}$$

$$= \frac{T'}{Gdt_0}\left(1.173 \tan^{-1}\Gamma - 0.565\,\Gamma\right) \text{ in the top and bottom walls.}$$

For the other values of β considered, namely 1,2,3,4 and 5, the results of § 3.3 are applicable. However it is clear from the symmetry that the tube will twist about its central axis so that $u = v = 0$, and we look for stress systems in which q is symmetrical and f and w^* anti-symmetrical about both the horizontal and vertical centre lines of the section. Since Γ is anti-symmetrical this means that h must be symmetrical about each centre line so that in equations (127) and (130) we make $D = 0$. It is then only necessary to consider the conditions of continuity of strain and of equilibrium at one of the four booms (chosen here to be the top left hand one), and the conditions of zero bending moment about the x and y axes are automatically satisfied.

Denoting now by suffixes 1 and 2 the values of h in the left hand and top walls respectively, the expressions for h_1 and h_2 are as follows :-

$$\beta = 1 \qquad h_1 = Gdt_0\left[C_1 - 0.8944\bar{\theta}\left\{\Gamma\tan^{-1}\Gamma - \tfrac{1}{2}\log_e(1+\Gamma^2)\right\}\right]$$

$$h_2 = Gdt_0\left[C_2 - 0.1240\bar{\theta}\left\{\Gamma\tan^{-1}\Gamma - \tfrac{1}{2}\log_e(1+\Gamma^2)\right\}\right]$$

$$\left.\begin{aligned}\beta = 2,3,4,5,\; h_1 &= Gdt_0\left[C_1 J_\beta(\Gamma) - 0.8944\bar{\theta}/\beta(\beta-1)\right]\\ h_2 &= Gdt_0\left[C_2 J_\beta(\Gamma) - 0.1240\bar{\theta}/\beta(\beta-1)\right]\end{aligned}\right\} \quad \ldots\ldots(252)$$

The functions J_β reduce to the polynomials given in Table 1 for the integral values of β considered. The constants C_1, C_2 and $\bar{\theta}$ are at present unknown.

Equations (141) applied at the top left hand boom give

$$\left[(1+\Gamma^2)\frac{dh_1}{d\Gamma} - \beta\Gamma h_1\right]_{\Gamma=\cdot1936} = 1\cdot625\left[(1+\Gamma^2)\frac{dh_2}{d\Gamma} - \beta\Gamma h_2\right]_{\Gamma=-\cdot8593}$$

$$= 1\cdot4797\left[(h_2)_{\Gamma=-\cdot8593} - (h_1)_{\Gamma=\cdot1936}\right] \quad \ldots(253)$$

whilst the third of equations (142) for torque equilibrium becomes

$$2.2361\int_0^{.1936} h_1\,d\Gamma + 0.5039\int_{-.8593}^{0} h_2\,d\Gamma = -\frac{\sqrt{3}}{\beta}\,\frac{T'}{d} \quad \ldots\ldots(254)$$

If we now substitute the expressions for h_1 and h_2 given above into equations (252) and (254) we obtain three simultaneous equations for each value of β, which may be solved to give C_1, C_2 and $\bar{\theta}$. These are tabulated in Table 2.

TABLE 2.

β	$\bar{\theta}Gd^2t_0/T'$	$C_1Gd^2t_0/T'$	$C_2Gd^2t_0/T'$
1	9.402	-1.572	-2.241
2	-30.398	-18.187	0.637
3	5.275	-0.506	0.188
4	12.643	0.148	0.174
5	12.979	-0.393	-0.225

Now that the values of C_1, C_2 and $\bar{\theta}$ are known, the expressions for the stress flows f and q follow from equations (143) and (144). The boom strain is derived from (145), which then gives the tension P in the boom. Also the warping w^* follows from equation

(88). The twist of the tube, obtained by integration of (54), is given by

$$\frac{d\theta}{d\rho} = \frac{\bar{\theta}}{\beta}\,\rho^{\beta-2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (255)$$

These are all given in Tables 3, 4, 5 and 6. (Note that P is the tension in the top left hand boom.) For comparison the corresponding results for $\beta = 0$ are included.

TABLE 3. Shear Flows.

β	$2Aq\rho^{1-\beta}/T'$	
	Side Walls	Top and bottom walls
0	$(1+\Gamma^2)^{-1}$	$(1+\Gamma^2)^{-1}$
1	$0.786 - 2.102\log_e(1+\Gamma^2)$	$1.121 - 0.292\log_e(1+\Gamma^2)$
2	4.592	-2.522
3	$0.760(1-\Gamma^2) + 1.179$	$-0.283(1-\Gamma^2) + 0.164$
4	$-0.297(1-3\Gamma^2) + 1.885$	$-0.348(1-3\Gamma^2) + 0.261$
5	$0.984(1-6\Gamma^2+\Gamma^4) + 1.451$	$0.562(1-6\Gamma^2+\Gamma^4) + 0.201$

TABLE 4. Direct Stress Flows.

β	$2Af\rho^{1-\beta}/T'$	
	Side Walls	Top and bottom walls.
0	$-1.732\,\Gamma(1+\Gamma^2)^{-1}$	$-1.732\,\Gamma(1+\Gamma^2)^{-1}$
1	$-7.282\tan^{-1}\Gamma$	$-1.010\tan^{-1}\Gamma$
2	$31.500\,\Gamma$	$-1.103\,\Gamma$
3	$2.631\,\Gamma$	$-0.979\,\Gamma$
4	$-0.514(3\Gamma-\Gamma^3)$	$-0.603(3\Gamma-\Gamma^3)$
5	$6.814(\Gamma-\Gamma^3)$	$3.894(\Gamma-\Gamma^3)$

TABLE 5. Warping

β	$w^* G d t_0 \rho^{-\beta}/T'$	
	Side Walls	Top and bottom walls
0	$0.722 \tan^{-1}\Gamma - 2.506\Gamma$	$1.173 \tan^{-1}\Gamma - 0.565\Gamma$
1	$0.567\Gamma - 3.034\left[\tan^{-1}\Gamma + \frac{\Gamma}{2}\log_e(1+\Gamma^2)\right]$	$1.314\Gamma - 0.684\left[\tan^{-1}\Gamma + \frac{\Gamma}{2}\log_e(1+\Gamma^2)\right]$
2	8.220Γ	-1.852Γ
3	$0.183(3\Gamma - \Gamma^3) \quad +0.284\Gamma$	$-0.111(3\Gamma - \Gamma^3) \quad +0.064\Gamma$
4	$-0.214(\Gamma - \Gamma^3) \quad +0.340\Gamma$	$-0.408(\Gamma - \Gamma^3) \quad +0.077\Gamma$
5	$0.142(5\Gamma - 10\Gamma^3 + \Gamma^5) + 0.209\Gamma$	$0.132(5\Gamma - 10\Gamma^3 + \Gamma^5) + 0.047\Gamma$

TABLE 6. Twist and Boom Tensions.

β	0	1	2	3	4	5
$\frac{Gd^2 t_0}{T'} \frac{d\theta}{d\rho}\rho^{2-\beta}$	7.765	9.402	-15.199	1.758	3.161	2.596
$P\frac{d}{T'}\rho^{-\beta}$	0	-1.014	6.708	1.010	0.218	1.768

In interpreting these results consider first the values of the twist $\frac{d\theta}{d\rho}$ given in Table 6. For all values of β the root torque is the same, namely T', so that the value of the twist at the root ($\rho=1$) is simply equal to $T'/Gd^2 t_0$ times the appropriate figure given in Table 6. At first sight the variation of the root twist with β appears to be very erratic, in fact when $\beta = 2$ the twist is negative, i.e. in the opposite direction to the torque. It must be remembered however that the figures for the twist given in Table 6 have taken no account of the root constraint. The stress systems so far obtained are those for which the distribution of stress is the same round all sections. They can only exist alone if the appropriate stresses are applied at the root. In the case of $\beta = 2$ the warping w^* and the direct stress flow f at the root are both large and the positive work done by f in moving through the root warping more than counteracts the negative work done by the torque due to the negative twist. The apparently erratic variation of root twist with β can be explained by considering the eigenvalues, the first two of which occur at $\beta = 1.827$ and $\beta = 4.709$ as will be found later. If a torque $T = T'\rho^{2.827}$ or $T'\rho^{5.709}$ is applied to the tube and a solution sought in which the stress distribution is the same round all sections it would be found that $\bar{\dot{\theta}}$ must necessarily be infinite. Consequently the variation of root twist with β must presumably be something like that indicated in Fig.14. It will be seen later that when the root is brought back into its original plane by the application of self-equilibrating stress systems the variation of root twist with β is quite smooth so that we conclude that the effect of high taper is to make the root condition of prime importance.

The variations of the non-dimensional stress flows $\frac{2Aq}{T'}$ and $\frac{2Af}{T'}$ round the top left hand quarter of the root section ($\rho=1$), obtained from Tables 3 and 4, are shown in Figs.15 to 20 for values of $\beta = 0, 1, 2, 3, 4$ and 5 respectively. It will be seen that in some cases the shear flow q is negative, i.e. opposes the torque, over parts of the section, but this also is due to the fact that no account has been taken of root constraint.

Finally the variation of the non-dimensional warping $\frac{Gdtw^*}{T'}$ round the top left hand quarter of the root section, obtained from Table 5, is shown in Fig.21.

§ A.3. Self-equilibrating stress systems for elimination of warping due to torque :

In order to eliminate the root warping due to the stress systems just derived, self-equilibrating stress systems giving a warping distribution anti-symmetrical about both centre lines of the cross section are required. These are obtained from the symmetrical h functions given in equations (252) and the problem now is to find possible values for β , i.e. the eigenvalues.

Substitution of (252) into (253) and (254), with T'=0, gives the following three equations:

$$\left[W_\beta(.1936)\right] C_1 + \left[1.625W_\beta(.8593)\right] C_2 - .3464\bar{\theta}/\beta(\beta-1) = 0$$

$$\left[J_\beta(.1936)\right] C_1 + \left[1.0983\beta W_\beta(.8593) - J_\beta(.8593)\right] C_2 - \left[.7704 + .1171\beta\right] \bar{\theta}/\beta(\beta-1) = 0$$

$$\left[2.2361W_\beta(.1936) + .4330J_\beta(.1936)\right] C_1 + \left[.5039W_\beta(.8593) + .4330J_\beta(.8593)\right] C_2 - .4410(\beta+1)\bar{\theta}/\beta(\beta-1) = 0$$

. . . . (256)

These equations are homogeneous and a non-zero solution exists only if the determinant formed by the coefficients of C_1, C_2 and $\bar{\theta}$ is zero. On expanding this third order determinant and substituting for J_β and W_β from (131), the following characteristic equation in β is finally derived:

$$(5.5978 + \beta^2) \cos(.5186\beta) - (4.3589 + \beta^2) \cos(.9012\beta) - .6345\beta\left[4.2 \sin(.9012\beta) + \sin(.5186\beta)\right] = 0 \quad \text{....(257)}$$

The periodic nature of this equation indicates that it has an infinite number of real roots. Since the L.H.S. is an even function of β the roots occur in equal positive and negative pairs. It was found that the first three pairs of roots occur at $\beta = \pm 1$, ± 1.827 and ± 4.709. The root + 1 may be discounted since the solution (130) of the differential equation in h breaks down for this value of β. The root - 1 may also be discounted because of the singularity which occurs at that value of β in the integral of the function J_β (equations 138). Consequently the first two pairs of eigenvalues are $\beta = \pm 1.827$ and $\beta = \pm 4.709$.

The fact that the roots occur in pairs is interesting from the point of view of the relative importance of the root and tip effects. The self-equilibrating stresses corresponding to the first pair of eigenvalues vary along the length of the tube as $\rho^{0.827}$ and $\rho^{-2.827}$ respectively. Consequently the system corresponding to the positive eigenvalue decays from the root towards the tip only very slowly, but that corresponding to the negative eigenvalue decays much more rapidly from the tip to the root. This means that, whereas the root effect is likely to be appreciable over the whole tube, the tip effect will be fairly localised. In seeking the main trends of high taper we are therefore justified in taking the tube under consideration to be continuous to the apex, and are interested only in stress systems decaying towards the apex, i.e. with positive eigenvalues.

Substitution of $\beta = +\,1.827$ into any two of equations (256) enables the constants C_1 and C_2 to be determined in terms of $\bar{\theta}$, thus

$$C_1 = 0.6846\bar{\theta}\ , \qquad C_2 = -0.0051\bar{\theta}$$

The expressions for h_1 and h_2 (252) are then known and the stress flows q and f and the warping w^* then follow. The magnitude of $\bar{\theta}$ remains arbitrary. This stress system will, for convenience, be called the first torque-eigenload, and the stress flows and warping are given in Table 7.

TABLE 7. 1st Torque-Eigenload.

	Side walls	Top and bottom walls
$\frac{q}{Gt_o\bar{\theta}}\rho^{-.827}$	$-1.2508J_{.827}(\Gamma) + 1.0815$	$.0092J_{.827}(\Gamma) + .1500$
$\frac{f}{Gt_o\bar{\theta}}\rho^{-.827}$	$-2.1665W_{.827}(\Gamma)$	$.0160W_{.827}(\Gamma)$
$\frac{w^*}{d\bar{\theta}}\rho^{-1.827}$	$-.2470W_{1.827}(\Gamma) + .2136\Gamma$	$.0030W_{1.827}(\Gamma) + .0481\Gamma$

The twist of the tube, from equation (255), is given by

$$\frac{d\theta}{d\rho} = 0.5473\bar{\theta}\rho^{-0.173} \qquad \text{. . . . . (258)}$$

whilst the tension P in the top L.H. boom becomes

$$P = -0.1774Gdt_0\bar{\theta}\rho^{1.827} \qquad \text{. . . . . (259)}$$

An exactly similar procedure enables the second torque-eigenload, corresponding to $\beta = +\ 4.709$, to be derived. The stress flows and warping are given in Table 8.

<u>TABLE 8.</u> <u>2nd Torque-Eigenload.</u>

	Side walls	Top and bottom walls
$\frac{q}{Gt_0\bar{\theta}}\rho^{-3.709}$	$-.6374J_{3.709}(\Gamma) + .2411$	$-.3777J_{3.709}(\Gamma) + .0334$
$\frac{f}{Gt_0\bar{\theta}}\rho^{-3.709}$	$-1.1039W_{3.709}(\Gamma)$	$-.6541W_{3.709}(\Gamma)$
$\frac{w^*}{d\bar{\theta}}\rho^{-4.709}$	$-.0488W_{4.709}(\Gamma) + .0185\Gamma$	$-.0470W_{4.709}(\Gamma) + .0042\Gamma$

The twist and the tension in the top left hand boom are given by

$$\frac{d\theta}{d\rho} = 0.2124\bar{\theta}\rho^{2.709} \qquad \text{. . . . . (260)}$$

$$P = -0.3784Gdt_0\bar{\theta}\rho^{4.709} \qquad \text{. . . . . (261)}$$

The variation of the stress flows and the warping round the root section (ρ=1) due to the first and second torque-eigenloads is shown in Figs.22 and 23 respectively. The marked difference between them is quite obvious.

§A.4. Elimination of warping due to torque :

So far, the amplitudes of the first and second torque-eigenloads derived in § A.3 are arbitrary. However, by giving suitable values to $\overline{\theta}$ and then superimposing them on the stress systems due to the torque $T=T'\rho^{\beta+1}$, derived in §A.2, it is possible to eliminate the warping at any two chosen points in each quarter of the root section. The points chosen for this purpose are the booms and the quarter points of the top and bottom walls. The necessary values of $\overline{\theta}$, denoted by $\overline{\theta}_{E1}$ and $\overline{\theta}_{E2}$, are given in Table 9.

TABLE 9. Amplitudes of Torque-Eigenloads.

β	0	1	2	3	4	5
$Gd^2 t_0 \overline{\theta}_{E1}/T'$	-8.128	-10.662	34.545	3.729	1.527	0.816
$Gd^2 t_0 \overline{\theta}_{E2}/T'$	0.718	0.262	-0.010	-0.317	-1.167	3.414

The warping at points in the root section other than those at which it has been made zero can now be determined by substituting $\overline{\theta} = \overline{\theta}_{E1}$ and $\overline{\theta}_{E2}$ into Tables 7 and 8 respectively and adding the warping due to the torque (Table 5). Proceeding in this way it was found that in no case was the maximum residual warping in the root section more than $1\frac{1}{2}\%$ of the maximum value before the eigenloads were superimposed. It is therefore considered that, for $0 \leq \beta \leq 5$, the first two eigenloads give a sufficiently accurate representation of the root condition.

The variation of twist, due to both torque and eigenloads, is shown in Table 10, which gives also the value at the root ($\rho=1$). These root values of the twist should be compared with those in Table 6 before the two eigenloads were added. It will be seen that the apparently erratic variation of root twist with β has now been smoothed out, thus confirming the conclusion that the root effect is highly important.

TABLE 10. Final Twists.

β	$Gd^2 t_o \frac{d\theta}{d\rho} \Big/ T'$			
	Variation with ρ			Root Value
	$x\ \rho^{\beta-2}$	$x\rho^{-.173}$	$x\rho^{2.709}$	
0	7.765	-4.448	0.152	3.469
1	9.402	-5.835	0.056	3.623
2	-15.199	18.907	-0.002	3.706
3	1.758	2.041	-0.067	3.732
4	3.161	0.836	-0.248	3.749
5	2.596	0.447	0.725	3.768

The total stress flows, q and f, can also be obtained by adding those due to the eigenloads on to those due to the torque. The variation of shear flow q round the root section (ρ=1) and also round the section mid-way between the root and the apex ($\rho=\frac{1}{2}$) is shown graphically in Figs. 24(a) to 29(a) for values of β = 0 to 5 respectively. The variation of direct stress flow f round the same two sections is shown similarly in Figs. 24(b) to 29(b). These should be compared with Figs.15 to 20 which show the variations of q and f before the application of the eigenloads.

The theory of § 1.7 proves that in the exact solution the shear flow q should be constant in each side of the root section. An inspection of Figs. 24(a) to 29(a) shows that the approximate solution, using two eigenloads only, gives a remarkably good representation of this condition.

The marked difference between the stress distributions round the sections ρ=1 and $\rho=\frac{1}{2}$ further emphasises the importance of the root condition.

§ A.5. Stress systems in equilibrium with bending moment $M_y = M_y' \rho^{\beta+1}$:

We now consider solutions for an applied bending moment $M_y = M_y' \rho^{\beta+1}$. This also implies a shear force $S_x = (\beta+1) M_y' \rho^{\beta}/Z$.

First, if $\beta = 0$, equations (116) give

$$\bar{v} = d\theta/d\rho = 0$$

$$\bar{u} = -\frac{d^2 M_y'}{EI}$$

It is found that $I_{yy} = .07638 d^3 t_o$ so that

$$\bar{u} = -4.910 M_y'/G d t_o$$

The stress flows are given by (118), with $M_x' = T' = 0$, thus

$$\frac{\rho d^2 f}{M_y'} = \frac{15.212 \Gamma}{1 + \Gamma^2} \quad \text{in the L.H.Wall}$$

$$= \frac{1.813}{1 + \Gamma^2} \quad \text{in the top wall}$$

and

$$\frac{\rho d^2 q}{M_y'} = \frac{8.783 \Gamma^2}{1 + \Gamma^2} \quad \text{in the L.H.wall}$$

$$= \frac{1.047 \Gamma}{1 + \Gamma^2} \quad \text{in the top wall}$$

For $\beta = 1,2,3,4$ and 5 we use equations (127) and (130) for the stress function h. From symmetry, however, it is clear that the tube will neither twist nor bend in the y-direction so that we take $\bar{\theta} = \bar{v} = 0$. Also the shear flow q must be symmetrical about the y-axis and anti-symmetrical about the x-axis, and vice-versa for the direct stress flow f and the warping w^*. This means that h must be symmetrical about the y-axis and anti-symmetrical about the x-axis. The functions h_1 and h_2 in the L.H. and top walls respectively, are therefore as shown below, the constants C, D and $\bar{u}$ being at present unknown.

$\underline{\beta = 1}$

$$h_1 = G t_o \left[C - 1.7889 \bar{u} \left\{ \Gamma \tan^{-1} \Gamma - \tfrac{1}{2} \log_e (1 + \Gamma^2) \right\} \right]$$

$$h_2 = G t_o D \Gamma$$

$\underline{\beta = 2,3,4,5}$

$$h_1 = G t_o \left[C J_\beta(\Gamma) - 1.7889 \bar{u}/\beta(\beta - 1) \right]$$

$$h_2 = G t_o D W_\beta(\Gamma) \qquad \text{.......(262)}$$

The condition $M_x = T = 0$ is automatically satisfied by the symmetry but the second of equations (142) becomes

$$\int_0^{.1936} h_1 \, d\Gamma = -.3873 M_y'/d \quad \ldots\ldots\ldots\ldots\ldots \quad (263)$$

This, together with equations (253), enables the unknown constants C, D and $\bar{u}$ to be determined, and the results are given in Table 11.

TABLE 11.

β	$Gdt_0\bar{u}/M_y'$	Gdt_0C/M_y'	Gdt_0D/M_y'
1	-3.448	-2.038	0.983
2	2.327	0.083	0.873
3	-4.044	-3.331	-0.849
4	9.358	-0.654	-0.162
5	19.253	-0.317	-0.104

The stress flows f and q can now be obtained and are given in Tables 12 and 13. The corresponding expressions for $\beta = 0$ are also included.

TABLE 12. Direct Stress Flows.

β	$fd^2\rho^{1-\beta}/M_y'$	
	L.H.wall	Top wall
0	$15.232\,\Gamma(1+\Gamma^2)^{-1}$	$1.813(1+\Gamma^2)^{-1}$
1	$10.685 \tan^{-1}\Gamma$	1.702
2	$-0.287\,\Gamma$	3.024
3	$34.614\,\Gamma$	$-4.413(1-\Gamma^2)$
4	$4.530(3\Gamma - \Gamma^3)$	$-1.119(1-3\Gamma^2)$
5	$10.985(\Gamma - \Gamma^3)$	$-0.900(1-6\Gamma^2+\Gamma^4)$

TABLE 13. Shear Flows.

β	$qd^2\rho^{1-\beta}/M'_y$	
	L.H.wall	Top wall
0	$8.783\Gamma^2(1+\Gamma^2)^{-1}$	$1.047\Gamma(1+\Gamma^2)^{-1}$
1	$2.038 + 3.085\log_e(1+\Gamma^2)$	0
2	3.998	-1.746Γ
3	$9.992(1-\Gamma^2) - 3.617$	5.096Γ
4	$2.615(1-3\Gamma^2) + 5.580$	$0.646(3\Gamma-\Gamma^3)$
5	$1.586(1-6\Gamma^2+\Gamma^4) + 8.611$	$2.078(\Gamma-\Gamma^3)$

The constant $\bar{u}$ defines the magnitude of the curvature of the tube (equation (253) and its apparently erratic variation with β (Table 11) can be explained in a similar way to the variation of the twist with β in §A.2. It is due to the fact that $\bar{u}$ must become infinite at the eigenvalues, the first two of which will be found later to be β = 2.642 and 6.808. When the root warping has been eliminated by the superposition of self-equilibrating stress systems the variation of the root value of the curvature with β becomes quite smooth.

The variation of the stress flows f and q round the top L.H. quarter of the root section are shown graphically in Figs.30 to 35 for values of β = 0,1,2,3,4 and 5 respectively. In comparing these graphs it should be noticed that Figs.33-35 are drawn to a smaller vertical scale than Figs.30-32.

We turn now to the warping of the section. Since, by symmetry, the twist is zero, equation (27a) gives

$$\oint \frac{\gamma}{n}\,ds = 0$$

Hence, from (31a),

$$w^* = \frac{\rho}{G}\int_0^s \frac{q}{nt}\,ds \quad \dots\dots\dots\dots\dots \quad (264)$$

Taking the point $s = 0$ to be at the middle of the L.H.wall, substituting the expressions for q from Table 13 and performing the integrations, the expressions for w^* are as shown in Table 14. The values at the top booms are also given.

TABLE 14.

β	$Gdt_o w^* \rho^{-\beta}/M_y'$		
	L.H.wall.	Top booms	Top wall.
0	$3.169(\Gamma - \tan^{-1}\Gamma)$	.0075	$-0.162+0.307 \log_e(1+\Gamma^2)$
1	$-1.491\Gamma+2.226\left[\tan^{-1}\Gamma + \frac{1}{2}\Gamma\log_e(1+\Gamma^2)\right]$	.145	0.145
2	1.443Γ	.279	$0.512(1-\Gamma^2) + 0.146$
3	$1.202(3\Gamma-\Gamma^3) - 1.305\Gamma$	.437	$-0.498(1-3\Gamma^2) - 0.169$
4	$0.944(\Gamma-\Gamma^3) + 2.014\Gamma$	.566	$-0.095(1-6\Gamma^2+\Gamma^4) + 0.293$
5	$0.114(5\Gamma-10\Gamma^3+\Gamma^5)+3.107\Gamma$	.704	$-0.061(1-10\Gamma^2+5\Gamma^4) + 0.481$

The warping w^* given in Table 14 is not necessarily the complete displacement w. From equation (30a), with $w_{so} = x_{so} = v = 0$ and $\rho = 1$, the value of w at the root is given by

$$w = w^* - \frac{x}{d}\left(\frac{du}{d\rho}\right)_{\rho=1} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (265)$$

Hence, by giving a suitable value to the root slope $(du/d\rho)_{\rho=1}$ the complete displacement w can be made zero at all four booms. This amounts to giving the whole tube a rigid body rotation about the y-axis in the root section.

Since at the top booms $x = d/8$, the root slope necessary to eliminate w at the booms is simply equal to eight times the value of w^* at each of the top booms in the root section, as obtained from Table 14. This leads to Table 15 which shows the root slope necessary to eliminate warping at the booms in the root section, and also the expressions for the displacement w at other points in this section.

TABLE 15.

β	$\left(\frac{du}{d\rho}\right)_{\rho=1}$	Gdt_0w/M'_y at the root.	
		L.H. wall	Top wall
0	.060	$3.130\Gamma - 3.169\tan^{-1}\Gamma$	$-0.170 + 0.307\log_e(1+\Gamma^2)$
1	1.162	$-2.242\Gamma + 2.226\left[\tan^{-1}\Gamma + \frac{1}{2}\Gamma\log_e(1+\Gamma^2)\right]$	0
2	2.235	0	$0.512(1-\Gamma^2) - 0.133$
3	3.494	$1.202(3\Gamma-\Gamma^3) - 3.563\Gamma$	$-0.498(1-3\Gamma^2) - 0.606$
4	4.526	$0.944(\Gamma-\Gamma^3) - 0.910\Gamma$	$-0.095(1-6\Gamma^2+\Gamma^4) - 0.273$
5	5.634	$0.114(5\Gamma-10\Gamma^3+\Gamma^5)-0.533\Gamma$	$-0.061(1-10\Gamma^2+5\Gamma^4) - 0.223$

The distribution of w round the top L.H. quarter of the root section, computed from Table 15, is shown plotted in Fig.36. In no case does the maximum value of Gdt_0w/M'_y in the L.H. wall exceed .003 which, in comparison with the values in the top wall, may be assumed to be zero.

Finally, the tension P in each of the two top booms is given in Table 16.

TABLE 16.

β	0	1	2	3	4	5
$Pd\rho^{-\beta}/M'_y$	1.293	1.214	0.564	3.826	2.304	2.348

§ A.6. Self-equilibrating stress systems for elimination of warping due to bending moment :

To eliminate the warping due to the bending moment M_y, self-equilibrating stress systems giving a warping distribution symmetrical about the x-axis and anti-symmetrical about the y-axis are required. These are derived from the h functions (262) and will be called bending eigenloads.

The procedure for determing the eigenvalues is exactly the same as for the torque eigenloads (§ A.3) and leads to the following characteristic equation :

$$(7.8174+\beta^2)\sin(.5186\beta) - (7.4648+\beta^2)\sin(.9012\beta) + .7628\beta\left[3.3876\cos(.9012\beta)+\cos(.5186\beta)\right] = 0$$

The L.H.S. of this equation is an odd function of β so that once again the roots occur in equal positive and negative pairs. The first two positive eigenvalues were found to be $\beta = 2.642$ and $\beta = 6.808$.

For each of these values of β, the constants C and D in equations (262) can be obtained in terms of $\bar{u}$, and the stress flows f and q then follow. These are given in Tables 17 and 18.

TABLE 17. 1st Bending Eigenload.

	L.H.wall	Top wall
$\frac{fd}{Gt_0\bar{u}}\rho^{-1.642}$	$-1.9386W(\Gamma_{1.642})$	$0.6107J(\Gamma_{1.642})$
$\frac{qd}{Gt_0\bar{u}}\rho^{-1.642}$	$-1.1192J(\Gamma_{1.642}) + 1.0894$	$-0.3526W(\Gamma_{1.642})$

TABLE 18. 2nd Bending Eigenload

	L.H.wall	Top Wall
$\frac{fd}{Gt_0\bar{u}}\rho^{-5.808}$	$-0.7006W(\Gamma_{5.808})$	$-0.5165J(\Gamma_{5.808})$
$\frac{qd}{Gt_0\bar{u}}\rho^{-5.808}$	$-0.4045J(\Gamma_{5.808}) + 0.3080$	$0.2982W(\Gamma_{5.808})$

The warping w at the root can be adjusted so that it is zero at all four booms by a judicious choice of the root slope. The necessary value of $(du/d\rho)_{\rho=1}$ and the consequent expressions for the warping are shown in Table 19 for both eigenloads.

TABLE 19.

		1st Bending-Eigenload	2nd Bending-Eigenload
$\frac{w}{\bar{u}}$	L.H.wall	$-.1529W_{2.642}(\Gamma) + .4008\,\Gamma$	$-.0214W_{6.808}(\Gamma) + .1210\,\Gamma$
	Top wall	$.0783J_{2.642}(\Gamma) + .0488$	$-.0257J_{6.808}(\Gamma) + .0203$
$\left(\frac{du}{d\rho}\right)_{\rho=1}$		$-.0118\bar{u}$	$-.0153\bar{u}$

Finally the tension P in each of the top booms is given by

$$\frac{P}{Gt_o\bar{u}} = -.2715\rho^{2.642} \quad \text{for the 1st bending-eigenload}$$

$$= -.2908\rho^{6.808} \quad \text{for the 2nd bending-eigenload.}$$

The variation of f, q and w round the top L.H. quarter of the root section due to each of the two eigenloads is shown graphically in Figs.37 and 38 respectively.

§A.7. Elimination of warping due to bending moment :

By giving suitable values to $\bar{u}$ in the 1st and 2nd bending-eigenloads and then superimposing them on the stresses due to the bending moment (§A.5) it is possible to eliminate the warping at the middle and the quarter points of the top and bottom walls in the root section. The necessary values of $\bar{u}$, denoted by $\bar{u}_{E1}$ and $\bar{u}_{E2}$ respectively, are given in Table 20.

TABLE 20.

β	0	1	2	3	4	5
$Gdt_o\bar{u}_{E1}/M'_y$	1.329	0	-2.975	8.683	2.891	2.214
$Gdt_o\bar{u}_{E1}/M'_y$	-0.150	0	0.016	-0.045	-0.161	-0.487

After applying the two eigenloads with these amplitudes the residual warping in the root section is in no case greater than 1½% of the maximum value before they are applied. This is considered to be sufficiently accurate.

The variation of curvature along the tube, its value at the root and the value of the root slope, due to both the bending moment and the eigenloads, are given in Table 21.

TABLE 21.

β	$Gdt_o \dfrac{d^2u}{d\rho^2} \Big/ M'_y$				Root Slope
	Variation with ρ			Root Value	$\dfrac{Gdt_o}{M'_y}\left(\dfrac{du}{d\rho}\right)_{\rho=1}$
	$x\rho^{\beta-2}$	$x\rho^{0.642}$	$x\rho^{4.808}$		
0	-4.910	1.329	-0.150	-3.731	0.047
1	-3.449	0	0	-3.449	1.162
2	2.327	-2.975	0.016	-0.632	2.270
3	-4.045	8.683	-0.045	4.594	3.392
4	9.359	2.891	-0.161	12.088	4.495
5	19.253	2.214	-0.478	20.989	5.615

It will be seen from Table 21 that the curvatures at the root section become positive as β increases. The explanation of this may be found in the presence of the shear force S_x which is equal to $(\beta+1)M_y'\rho^{\beta}/d$. This gives rise to shear stresses in the side walls which produce positive shear slopes $(du/d\rho)$. For a given value of M_y', S_x increases more and more rapidly towards the root as β increases so that the resulting positive curvatures also increase. These curvatures due to shear more than counteract the negative curvatures due to bending moment. This explanation is confirmed by the increase of root slope with β, shown in the last column of Table 21.

The total stress flows f and q, due to both bending moment and eigenloads, are shown plotted in Figs.39 to 44 round the top left hand quarter of the root section ($\rho=1$) and the section midway between the root and the tip ($\rho=\frac{1}{2}$).

The shear flow should actually be constant in each straight wall of the root section, as shown in §1.7. The fact that q is anti-symmetrical about the vertical centre line shows further that in the top and bottom walls of the root section the shear flow should be zero. An inspection of Figs.39 to 44 shows that these conditions are very nearly satisfied by the application of two eigenloads only. In all cases the shear flow in the top wall of the root section is too small to plot.

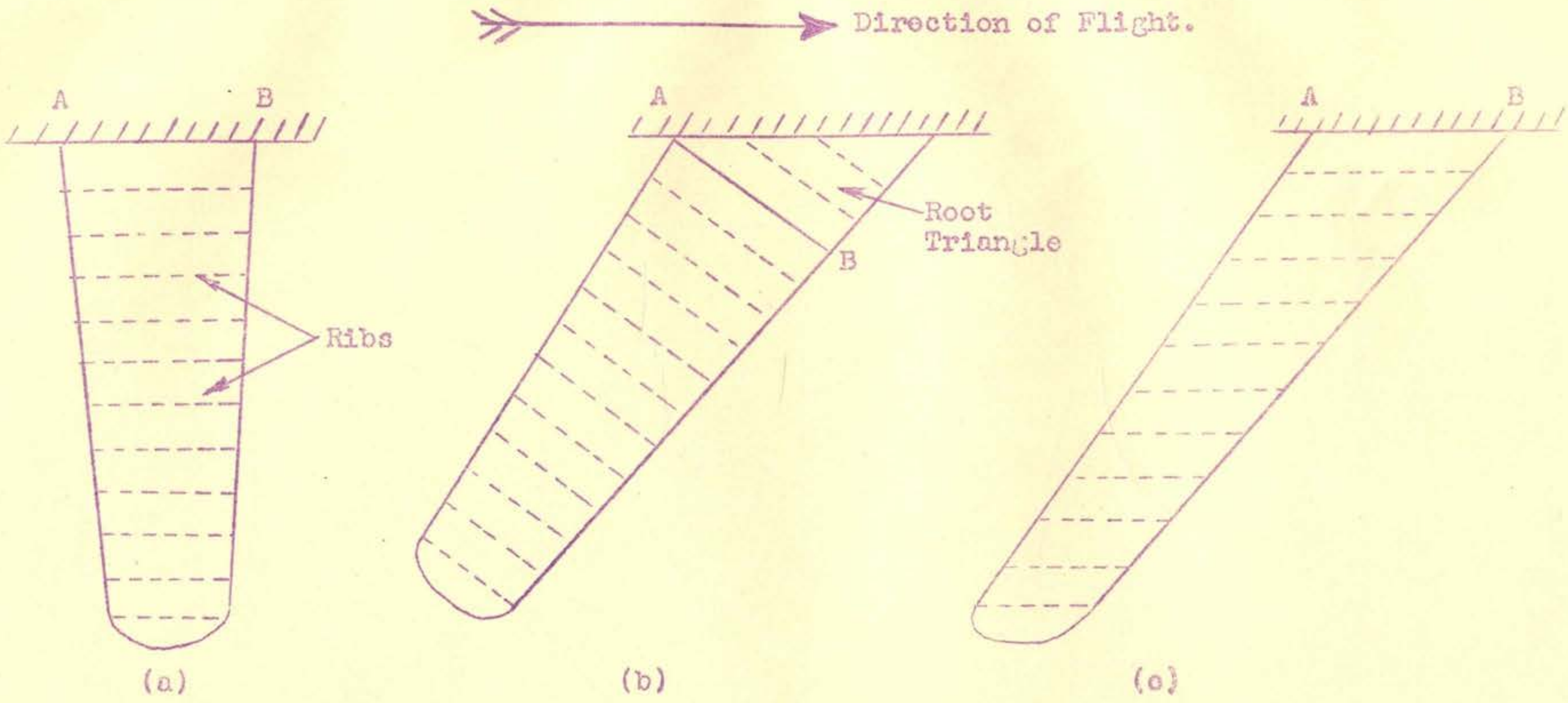


FIG.1. DERIVATION OF A SWEPT WING FROM AN UNSWEPT ONE.

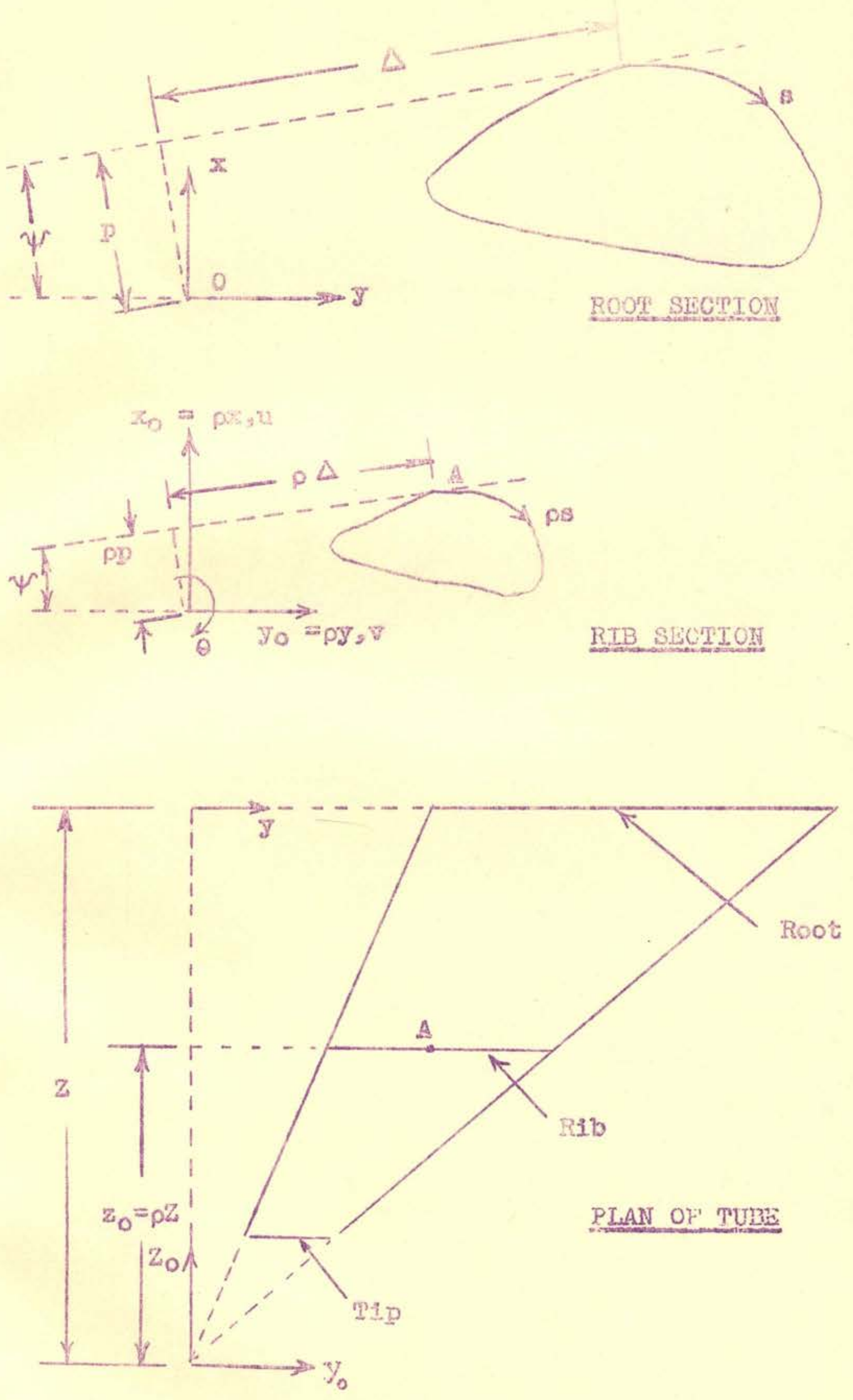


FIG. 2. GEOMETRY OF CONICAL TUBE.

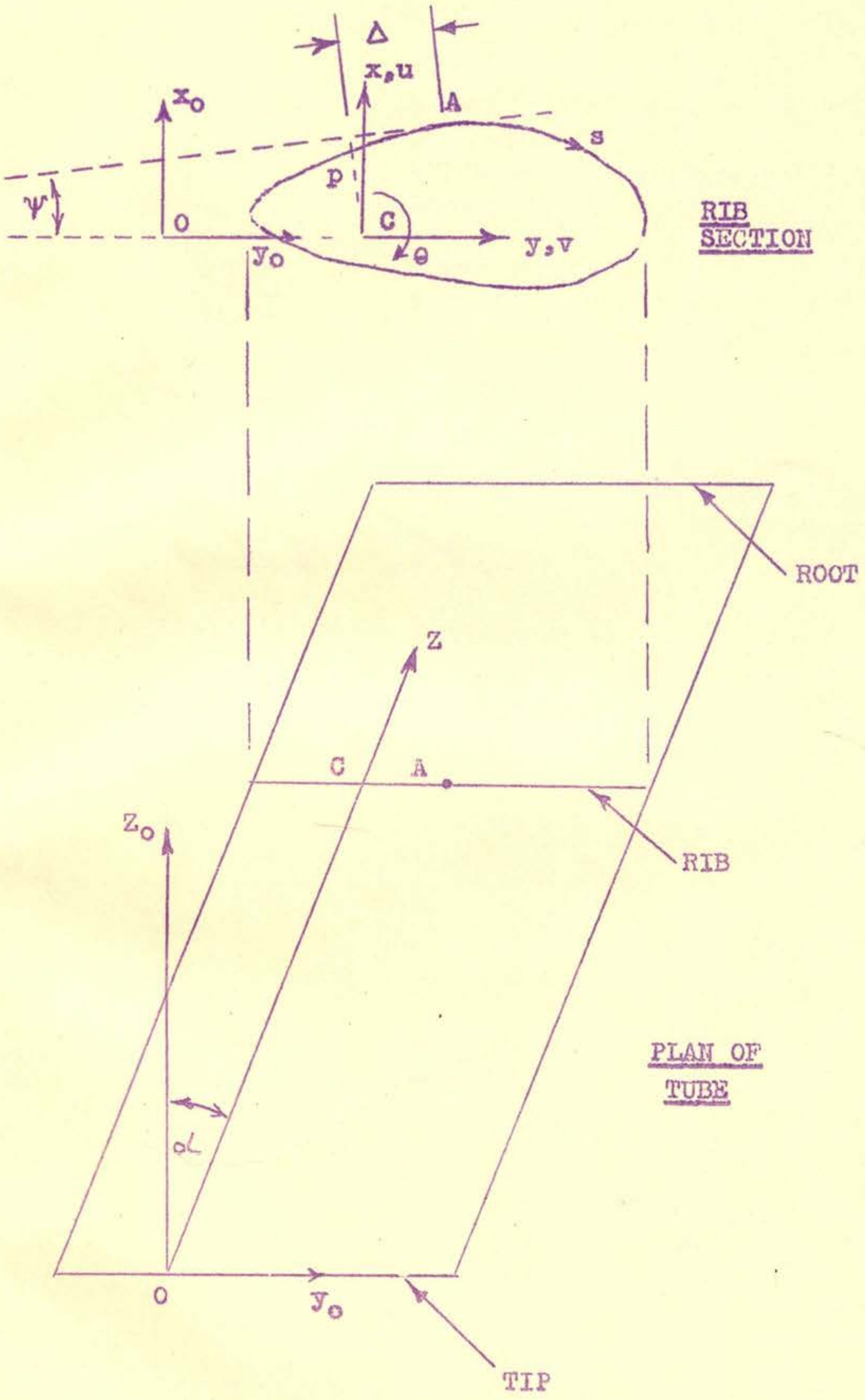


FIG.3. GEOMETRY OF CYLINDRICAL TUBE.

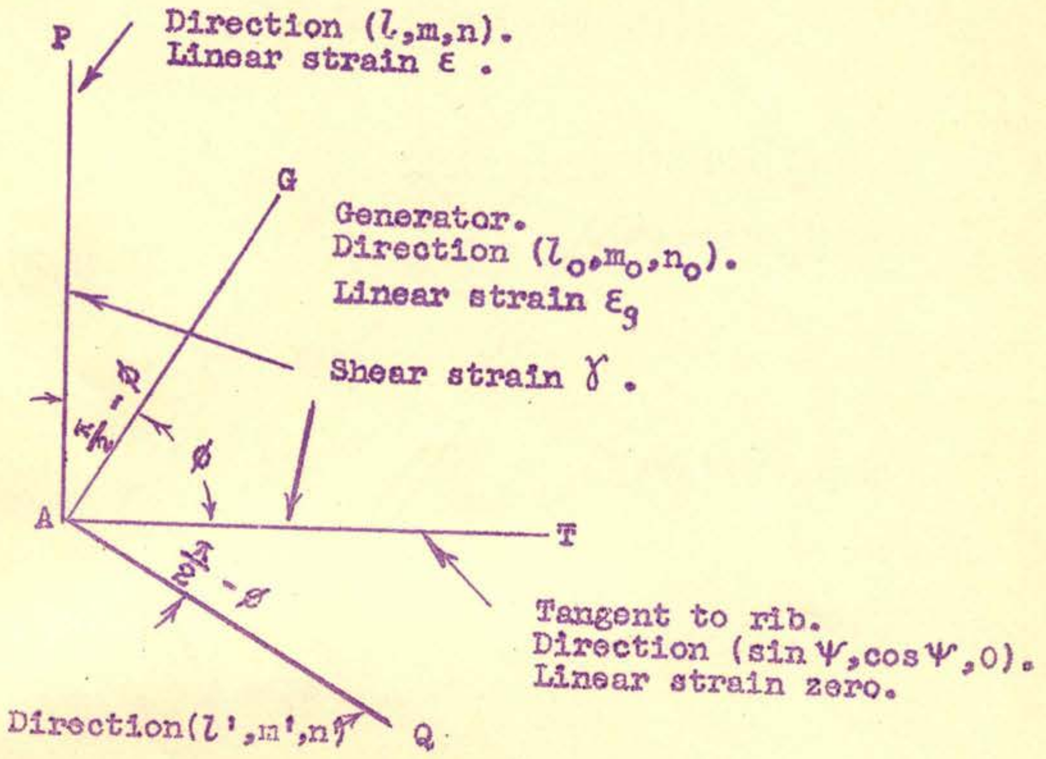


FIG.4. GEOMETRY OF TANGENT PLANE.

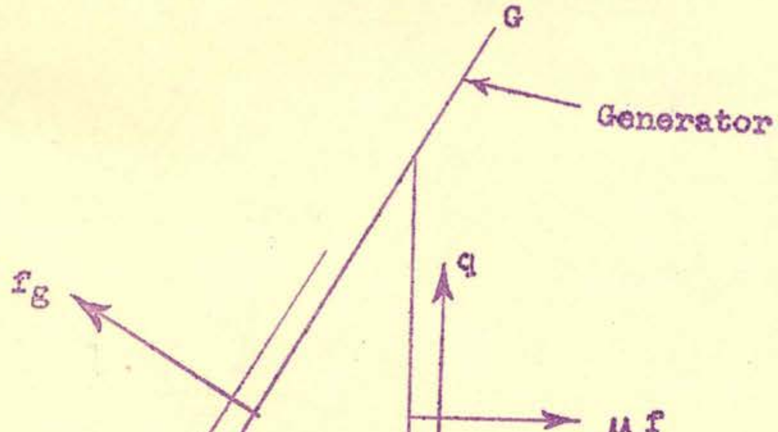


FIG.5. STRESS FLOWS IN THE SKIN.

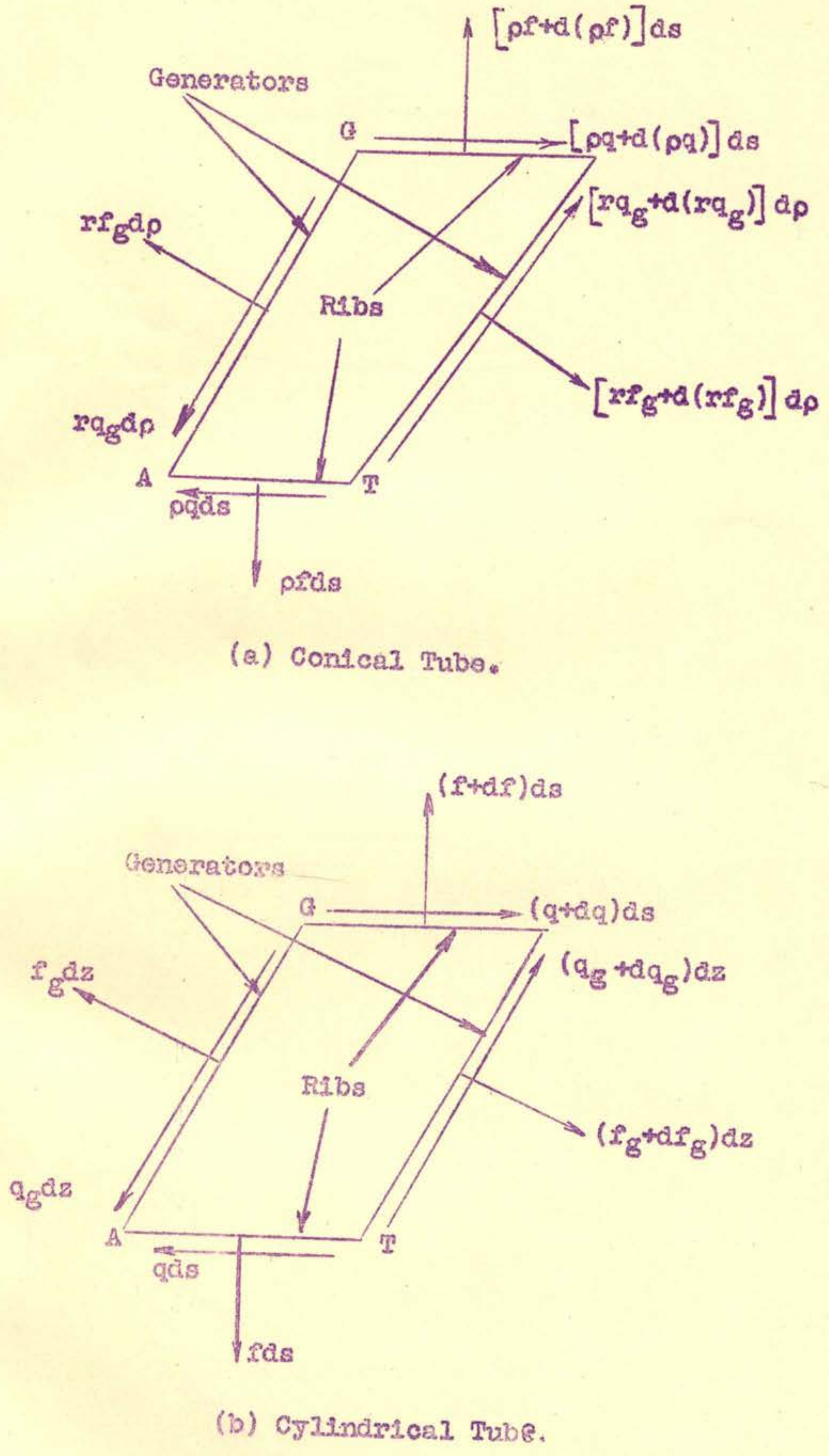


FIG.6. FORCES ON AN ELEMENT OF SKIN.

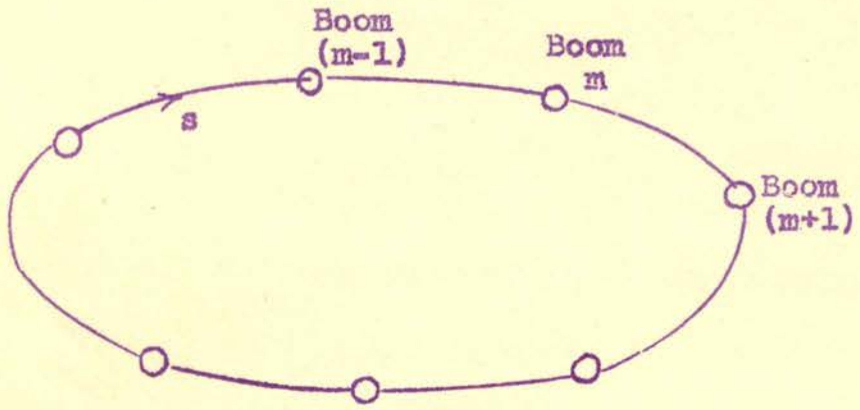


FIG.7. NOTATION FOR BOOMS

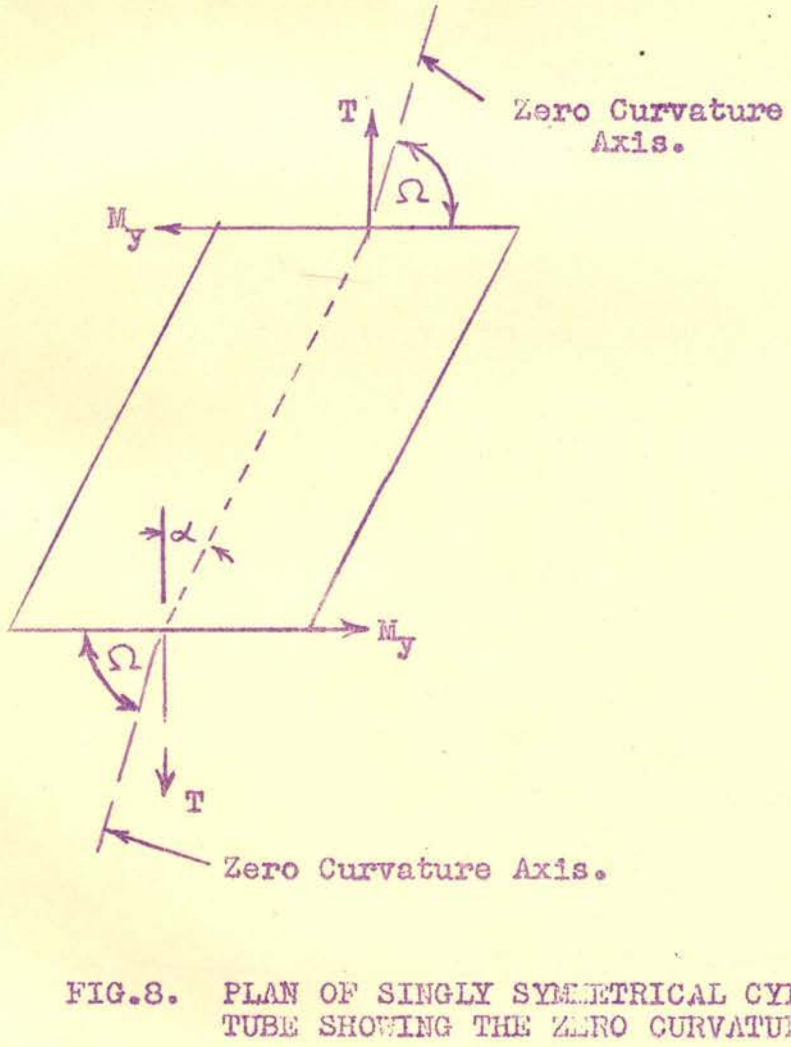


FIG.8. PLAN OF SINGLY SYMMETRICAL CYLINDRICAL
TUBE SHOWING THE ZERO CURVATURE AXIS.

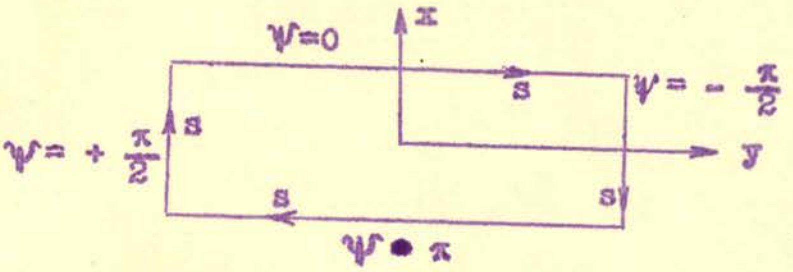


FIG.9. RECTANGULAR SECTION OF SWEPT CYLINDRICAL TUBE.

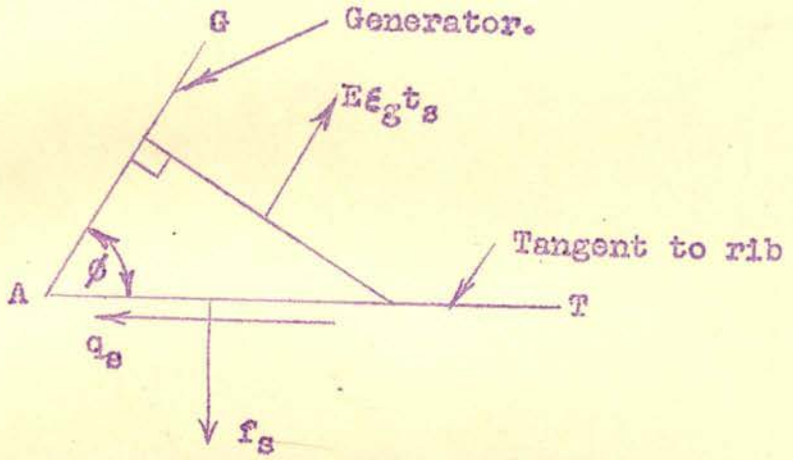


FIG.10. STRESS FLOWS IN THE STRINGER SHEET.

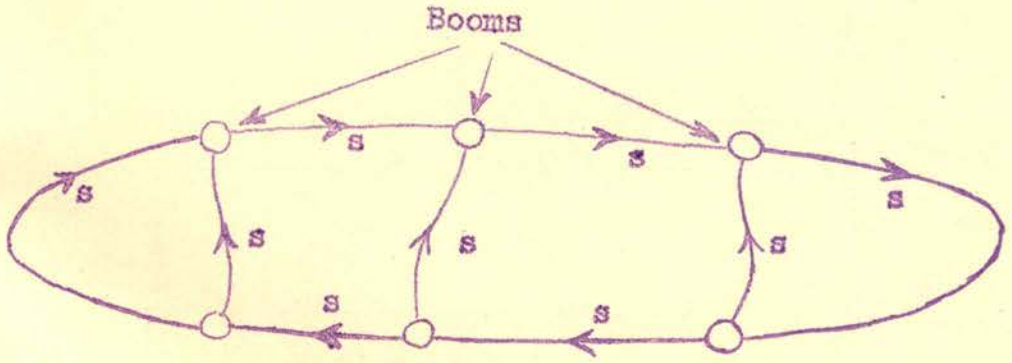


FIG.11. EXAMPLE OF MULTI-CELL TUBE SECTION.

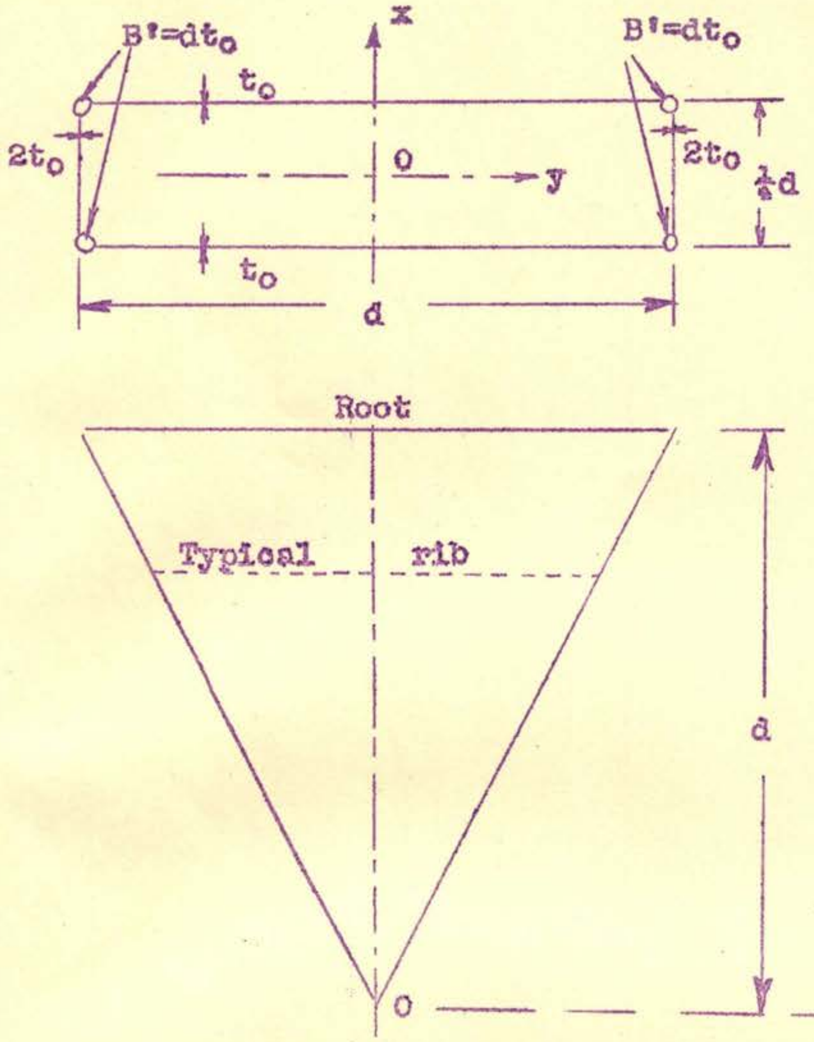


FIG.12. DETAILS OF TUBE CONSIDERED IN THE APPENDIX.
(Note: all subsequent Figures refer to this tube.)

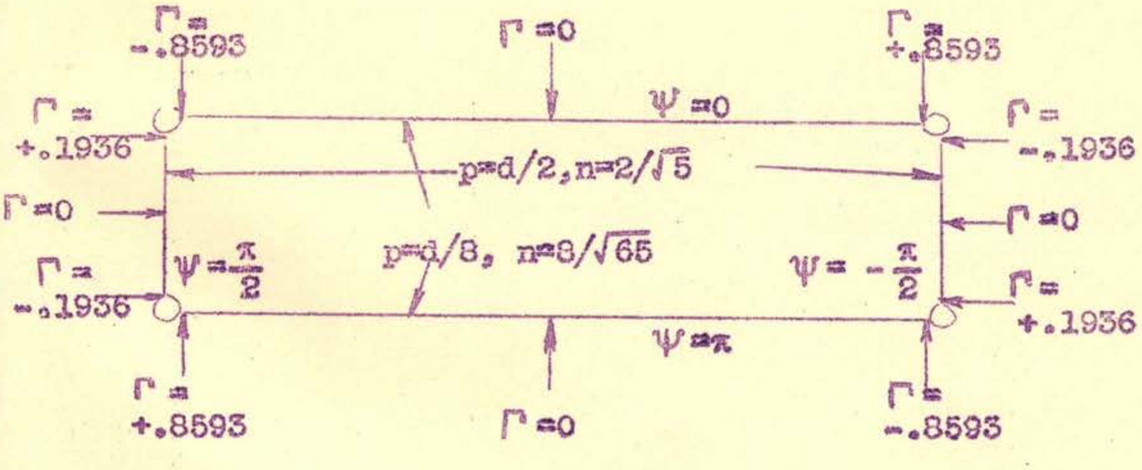


FIG.13. VARIATION OF Γ, p, n AND Ψ ROUND THE SECTION.
(Note: the variation of Γ in each wall is linear.)

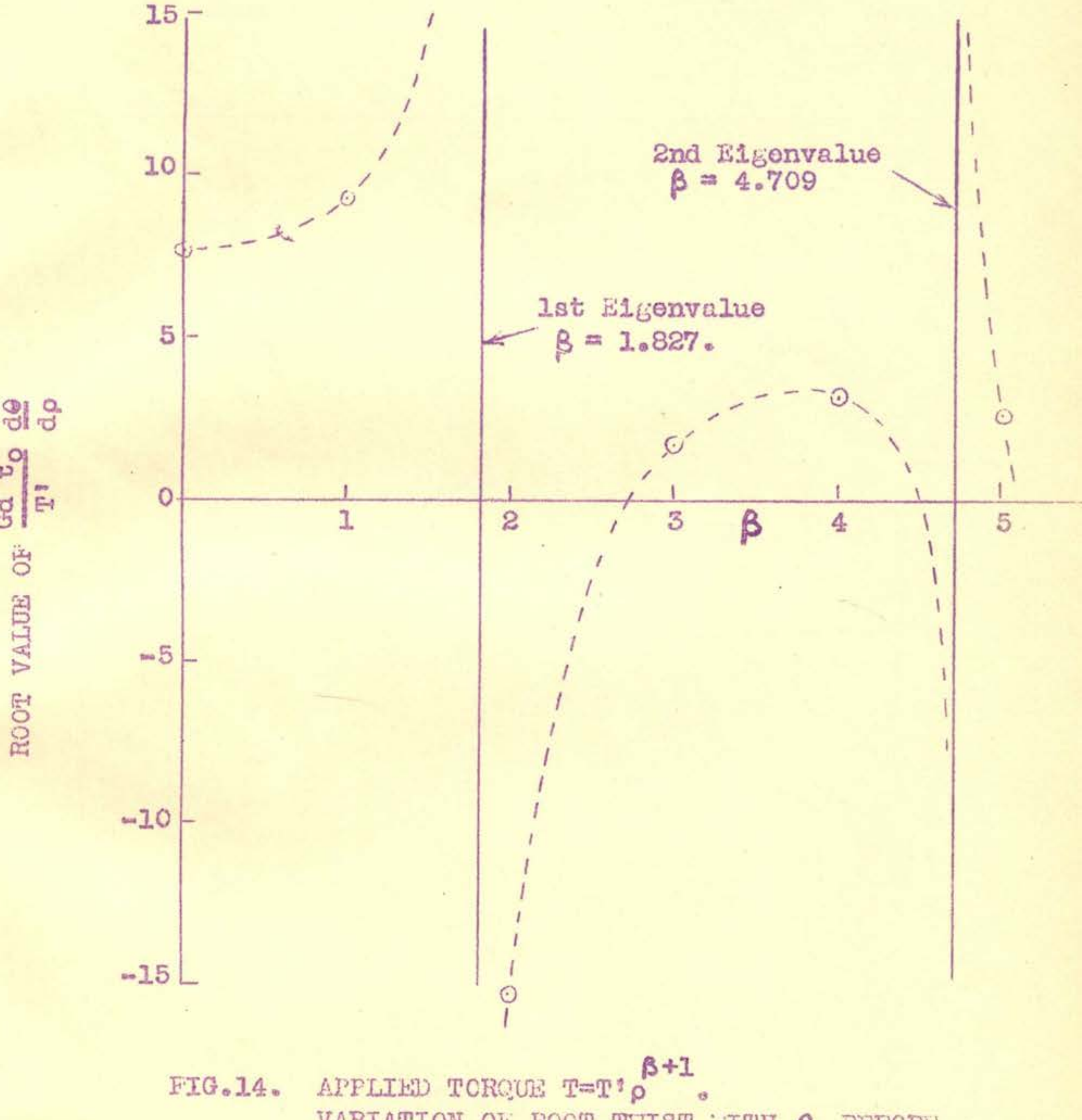


FIG.14. APPLIED TORQUE $T=T'\rho^{\beta+1}$.
VARIATION OF ROOT TWIST WITH β BEFORE ROOT WARPING IS ELIMINATED.
(Note : points ⊙ obtained from Table 6. dotted lines are guessed.)

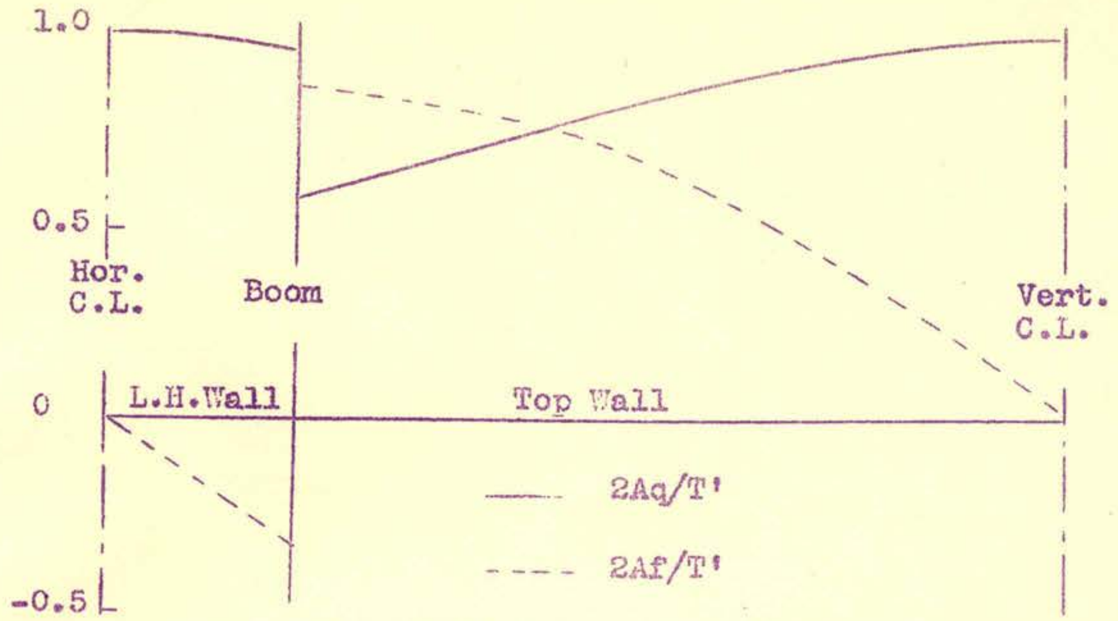


FIG.15. APPLIED TORQUE T=ρT'. ROOT STRESS FLOWS BEFORE WARPING IS ELIMINATED.

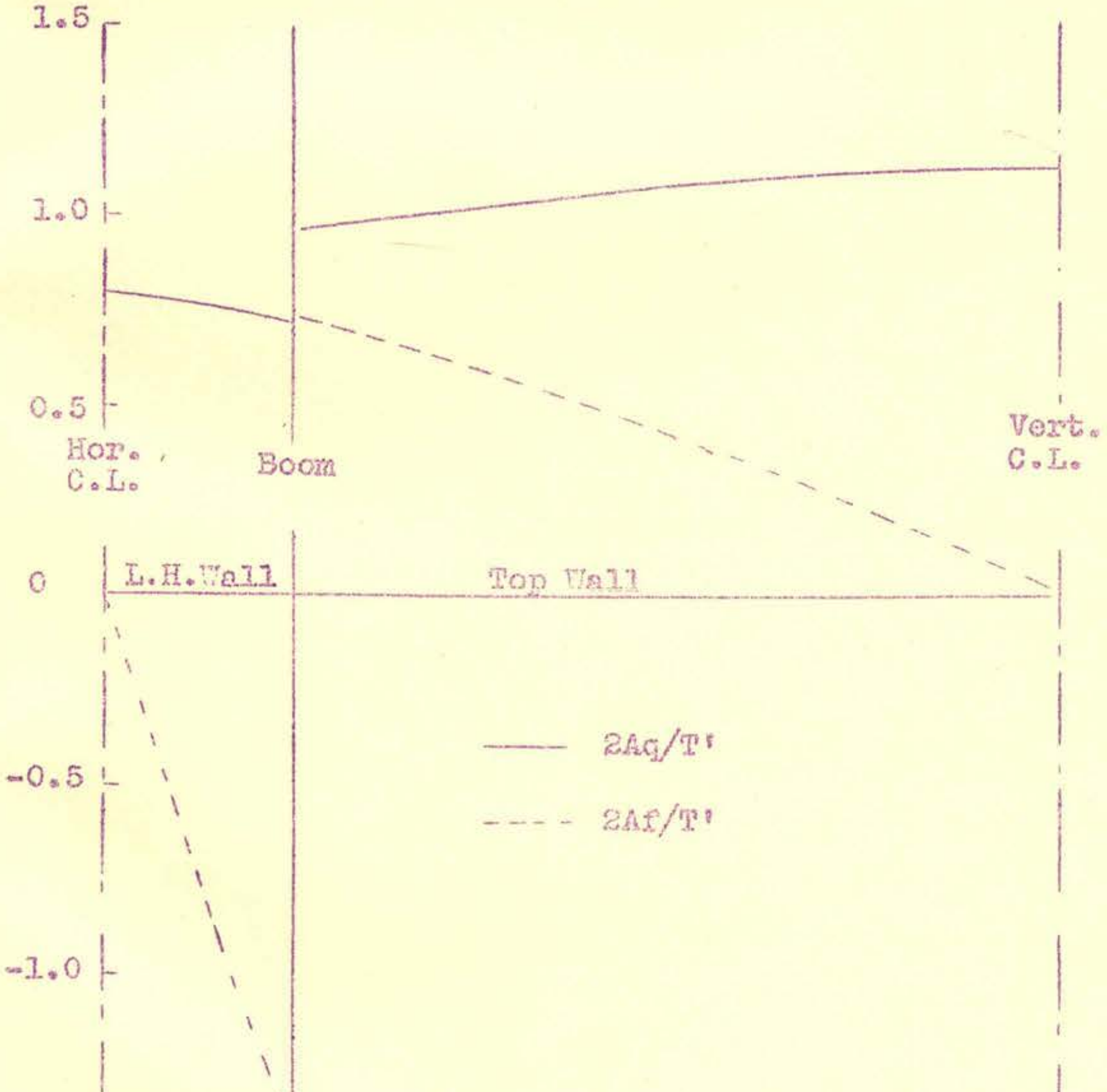


FIG.16. APPLIED TORQUE $T=\rho^2 T'$. ROOT STRESS FLOWS BEFORE WARPING IS ELIMINATED.

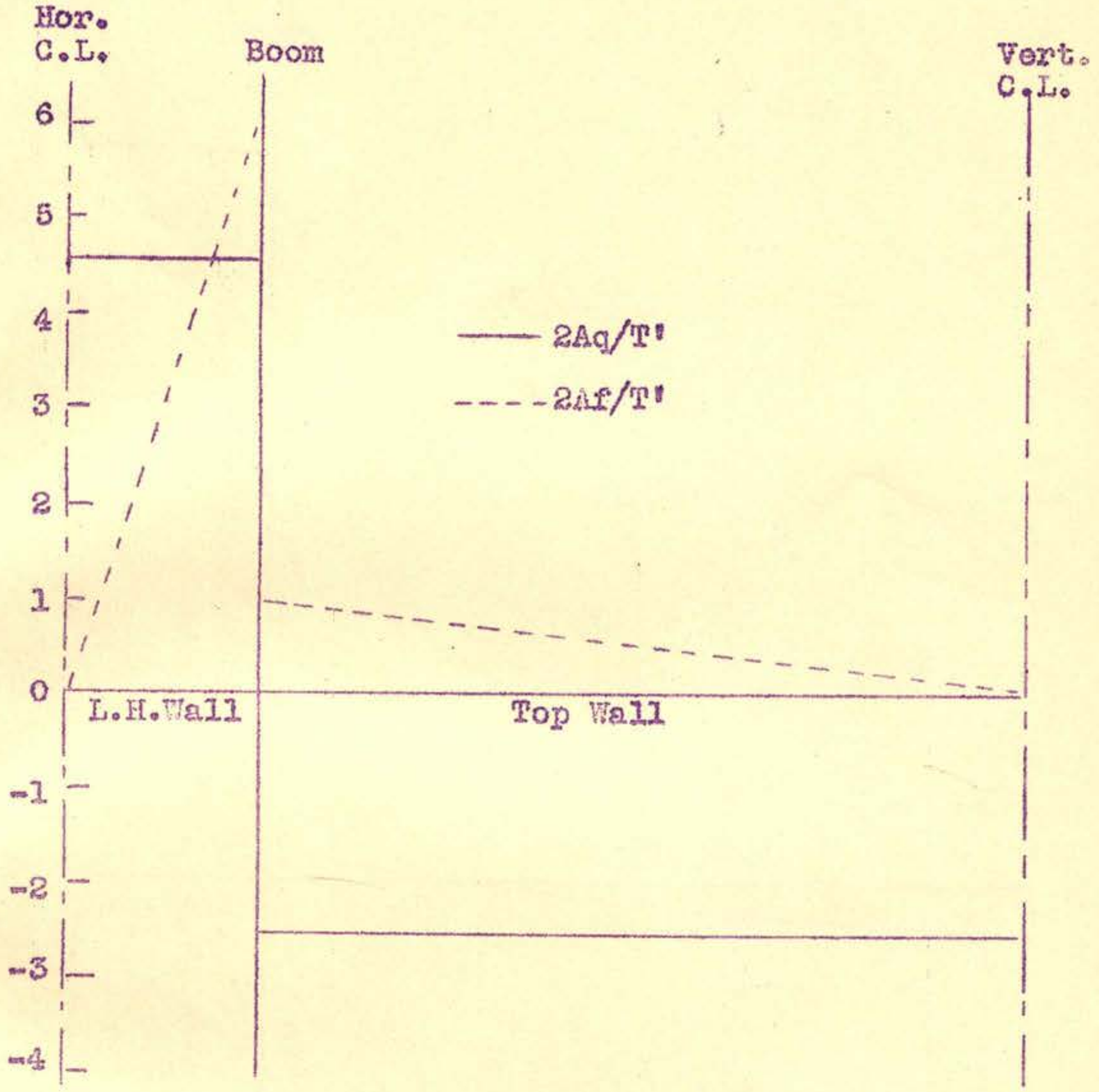


FIG.17. APPLIED TORQUE $T = \rho^3 T'$. ROOT STRESS FLOWS BEFORE WARPING IS ELIMINATED.
(Note the reduced scale.)

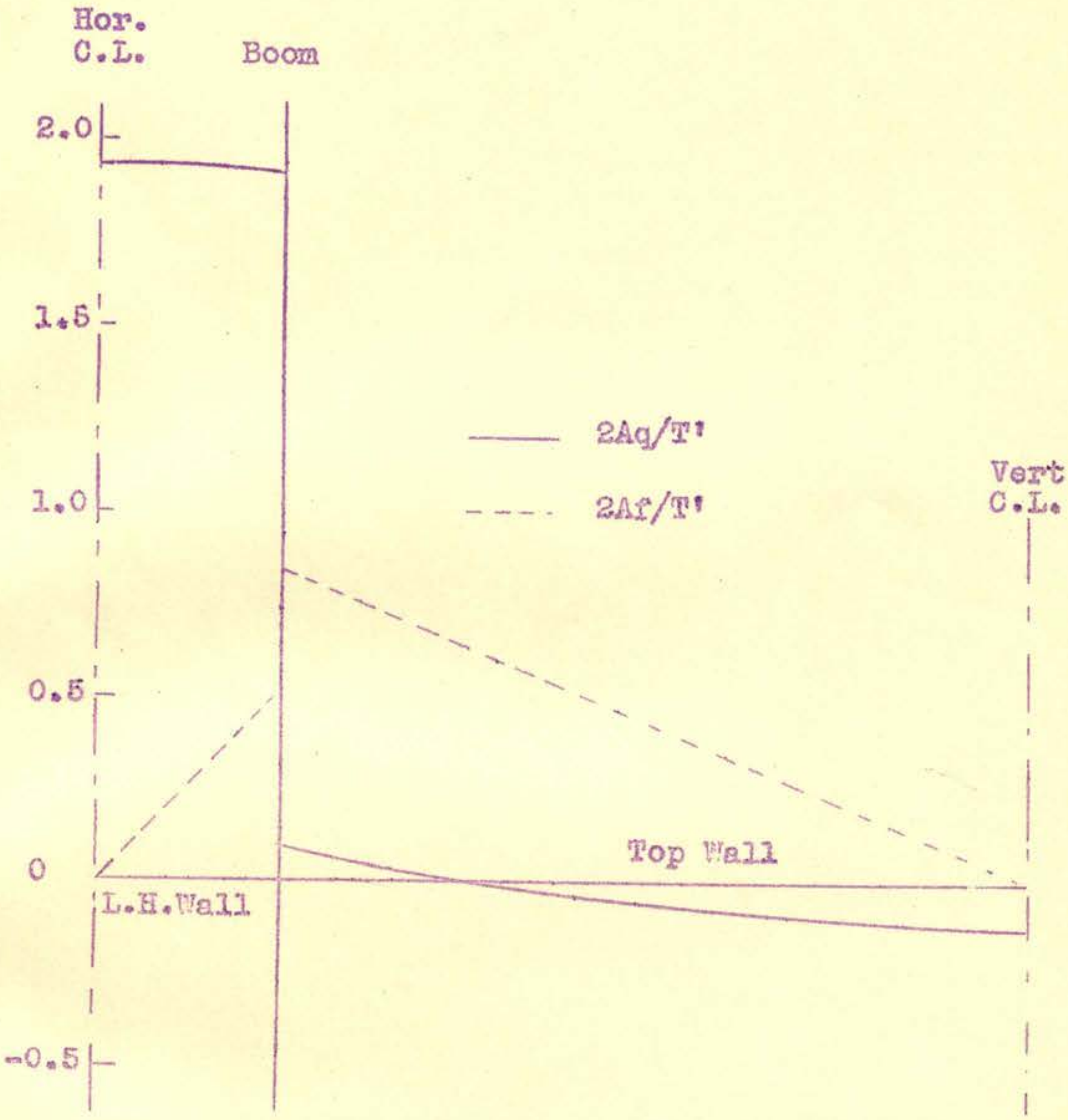


FIG.18. APPLIED TORQUE $T=\rho^4 T'$. ROOT STRESS FLOWS BEFORE WARPING IS ELIMINATED.

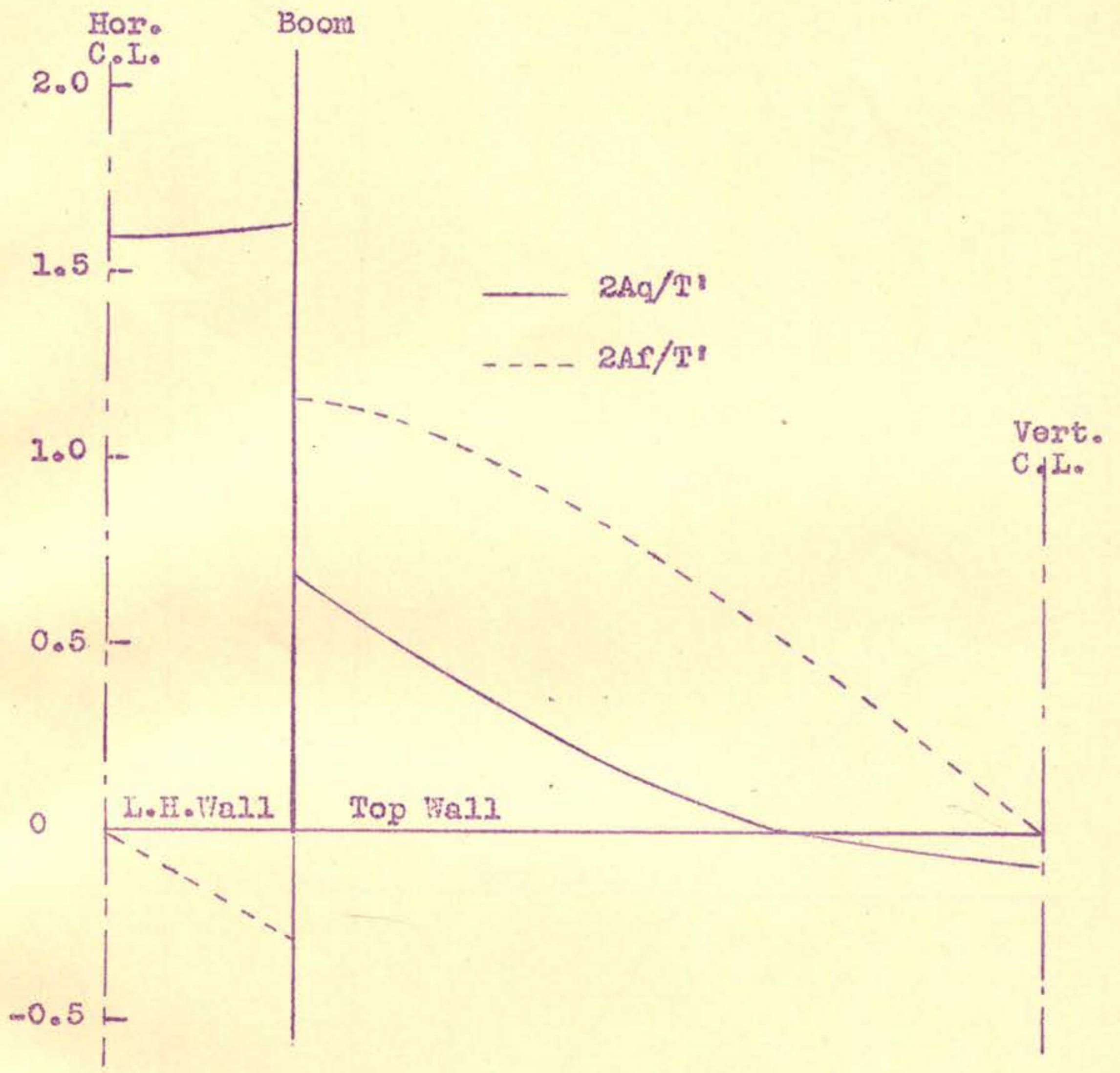


FIG.19. APPLIED TORQUE $T = \rho^5 T'$. ROOT STRESS FLOWS BEFORE WARPING IS ELIMINATED.

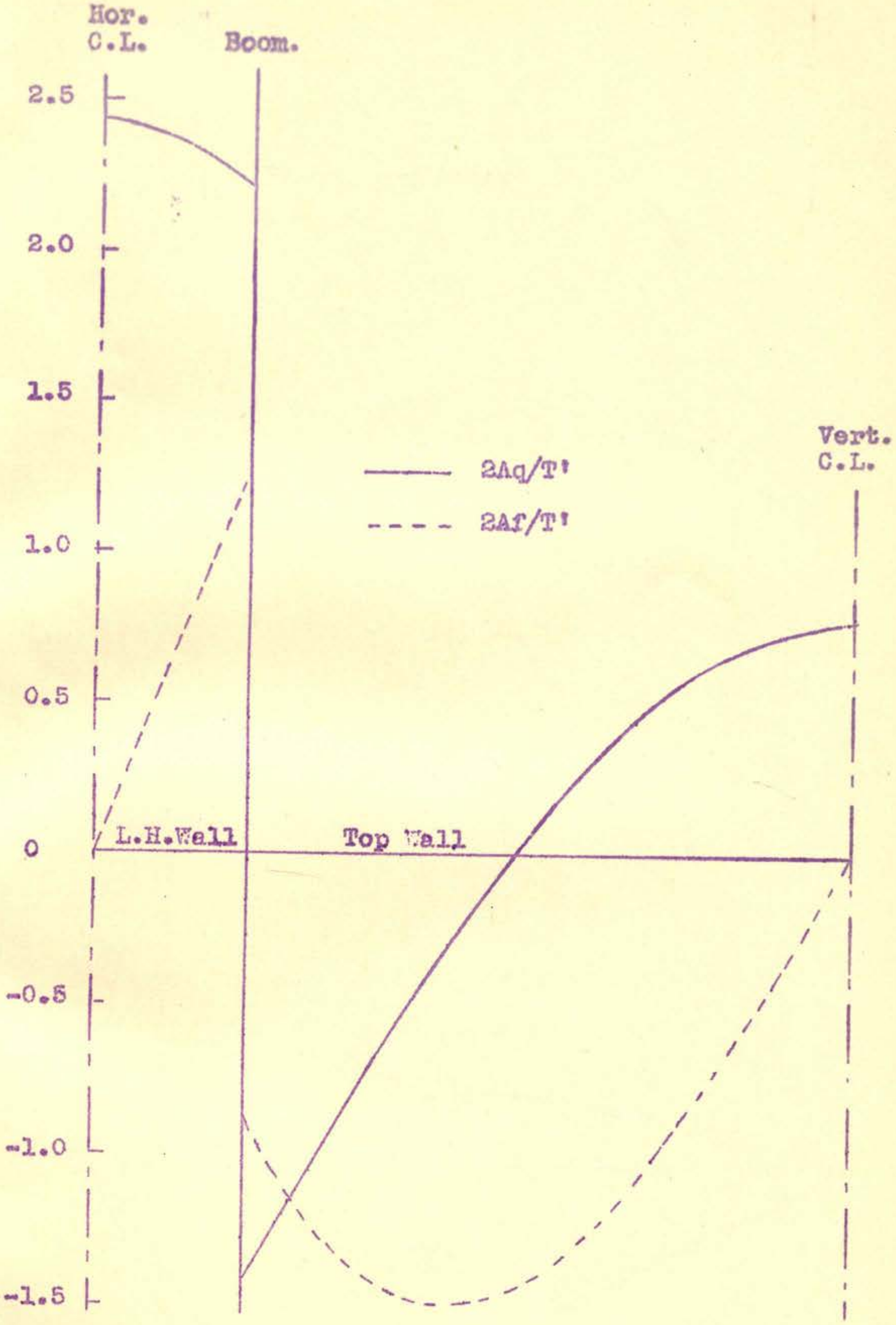


FIG.20. APPLIED TORQUE $T = \rho^6 T'$. ROOT STRESS FLOWS BEFORE WARPING IS ELIMINATED.

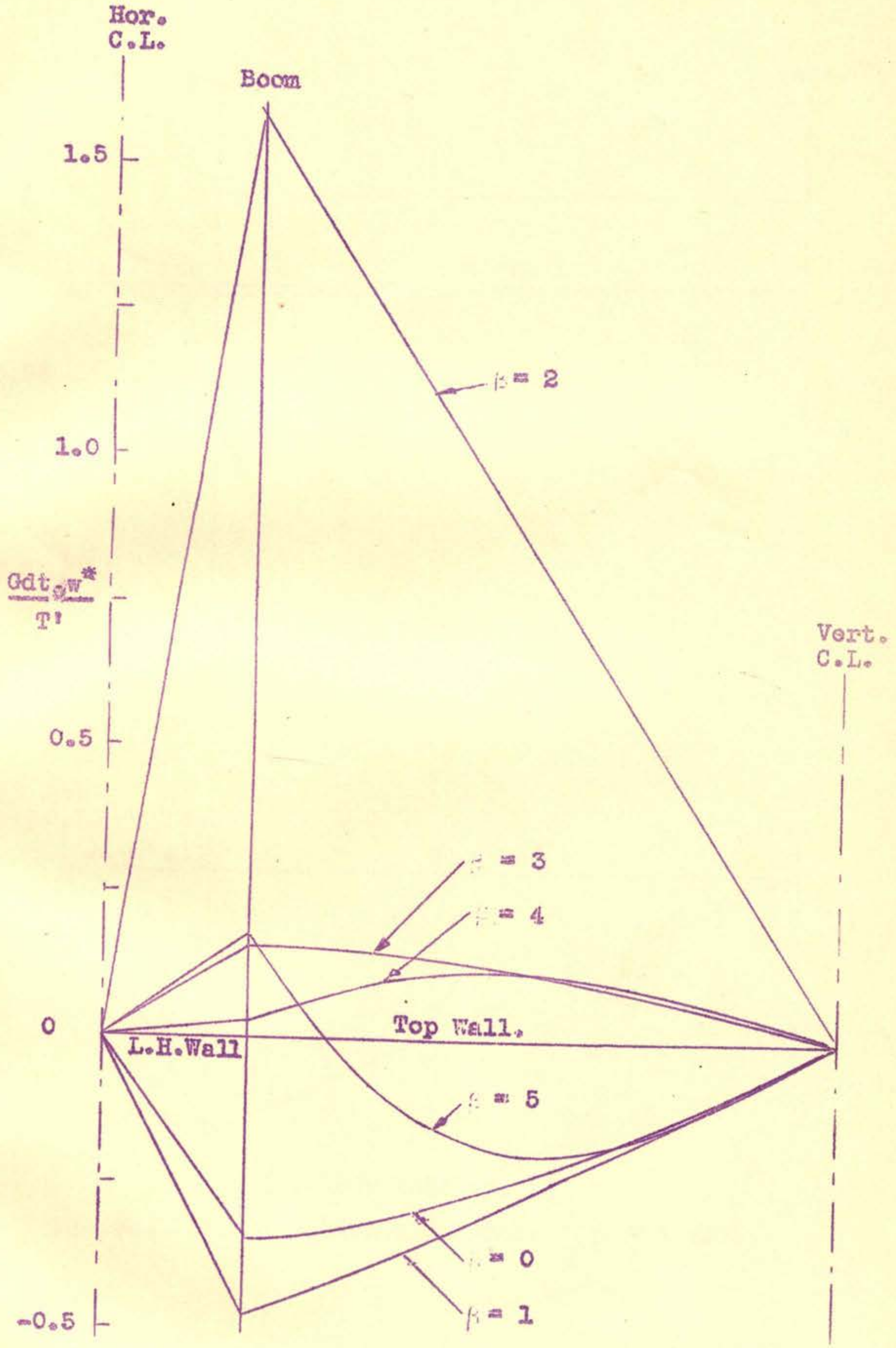
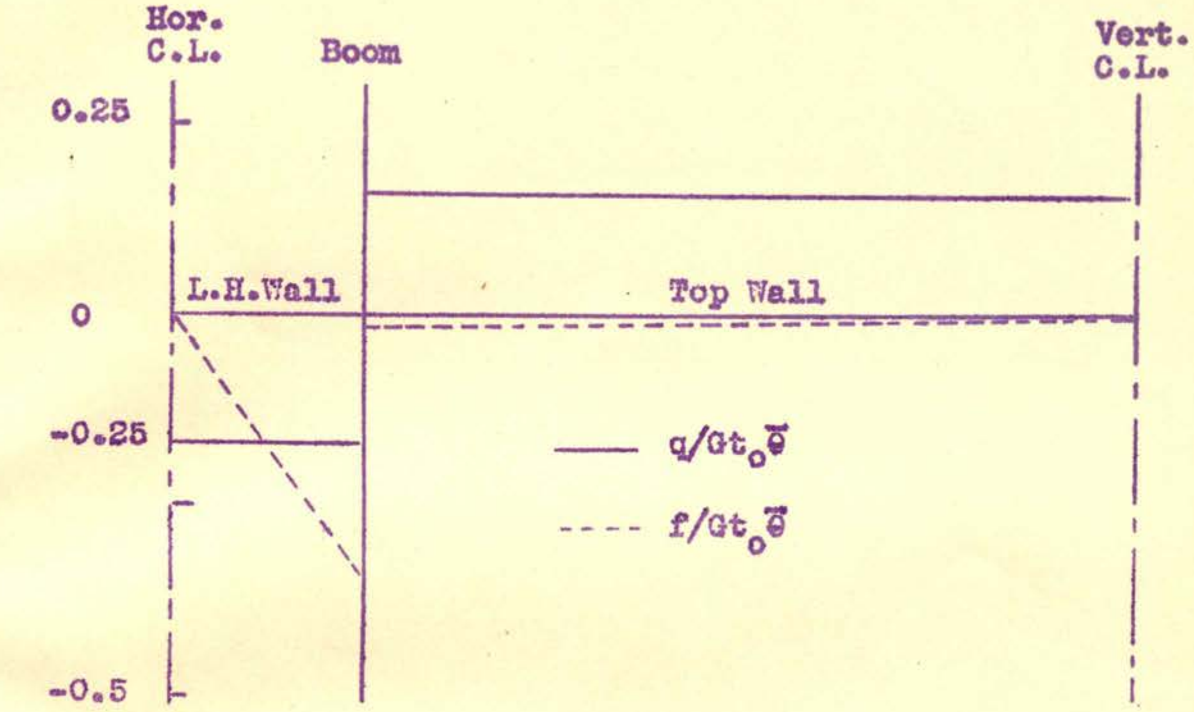
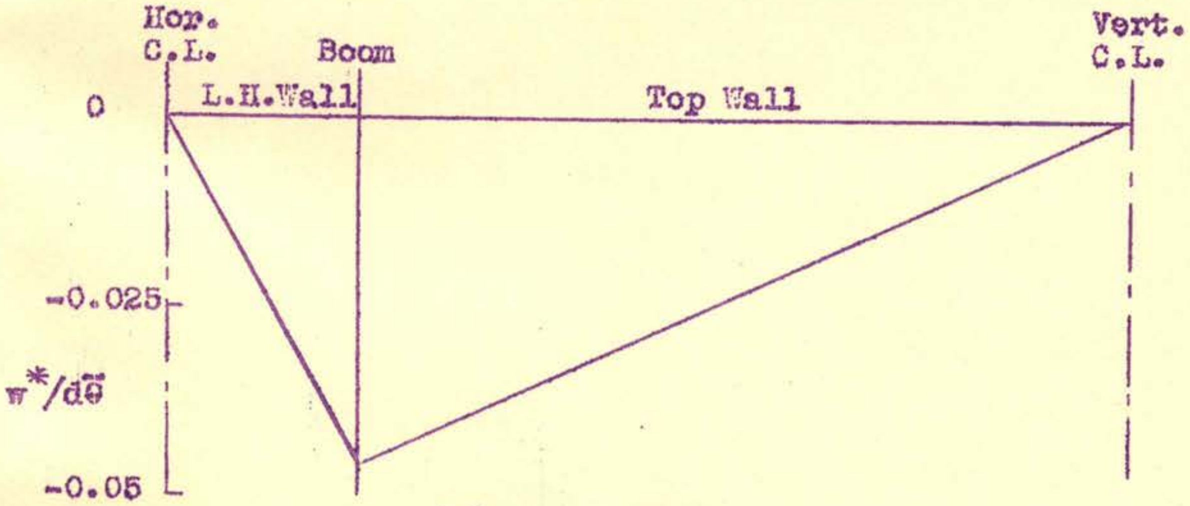


FIG.21. APPLIED TORQUE $T = T'\rho^{\beta+1}$.
WARPING OF ROOT SECTION BEFORE BEING ELIMINATED.

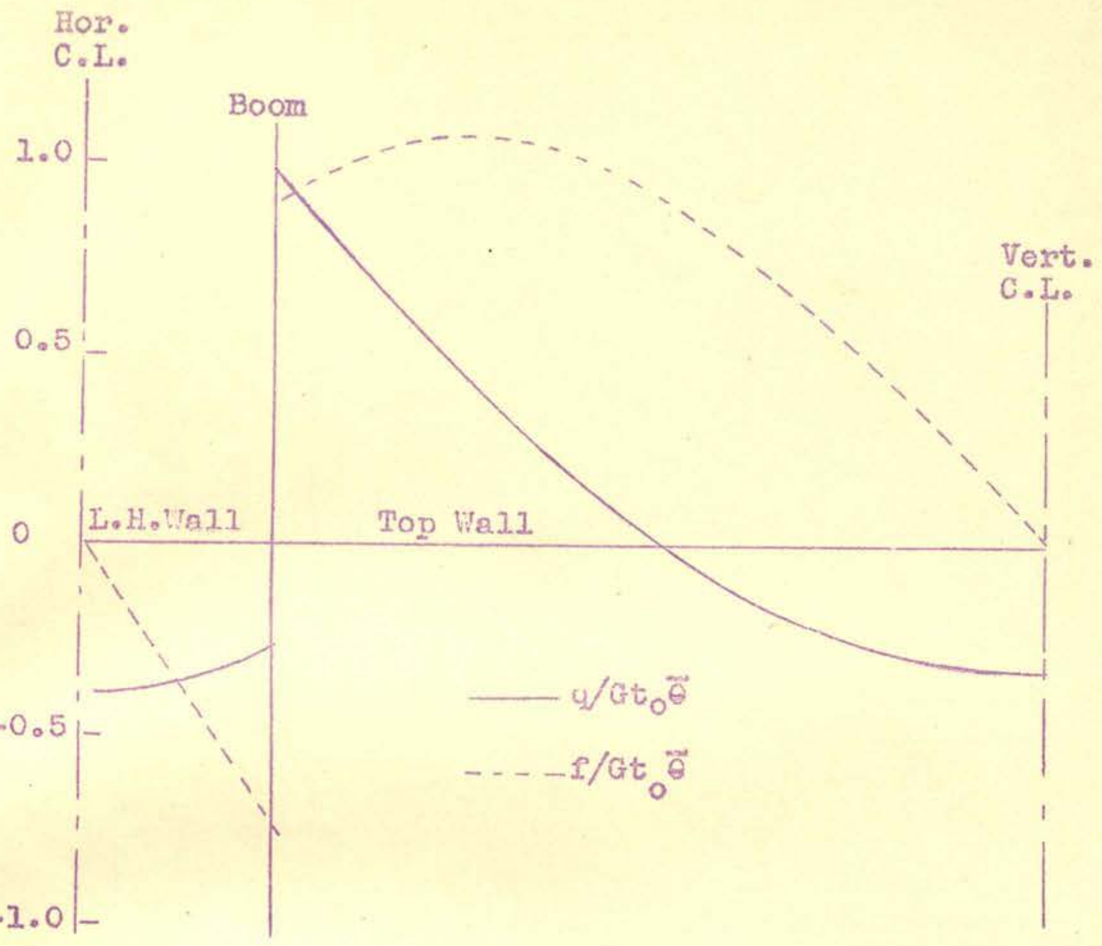


(a) ROOT STRESS FLOWS.

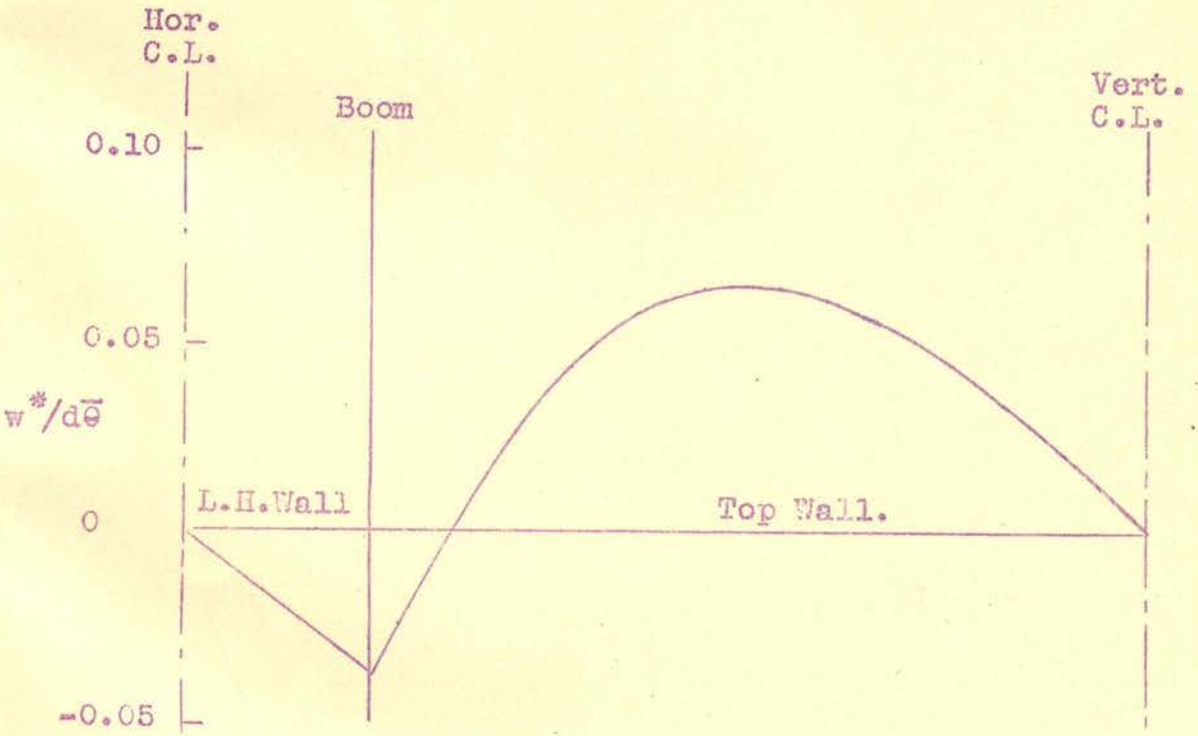


(b) ROOT WARPING.

FIG.22. FIRST TORQUE-EIGENLOAD. ($\beta = 1.827$)



(a) ROOT STRESS FLOWS.



(b) ROOT WARPING.

FIG.23. SECOND TORQUE-EIGENLOAD. (β =4.709)

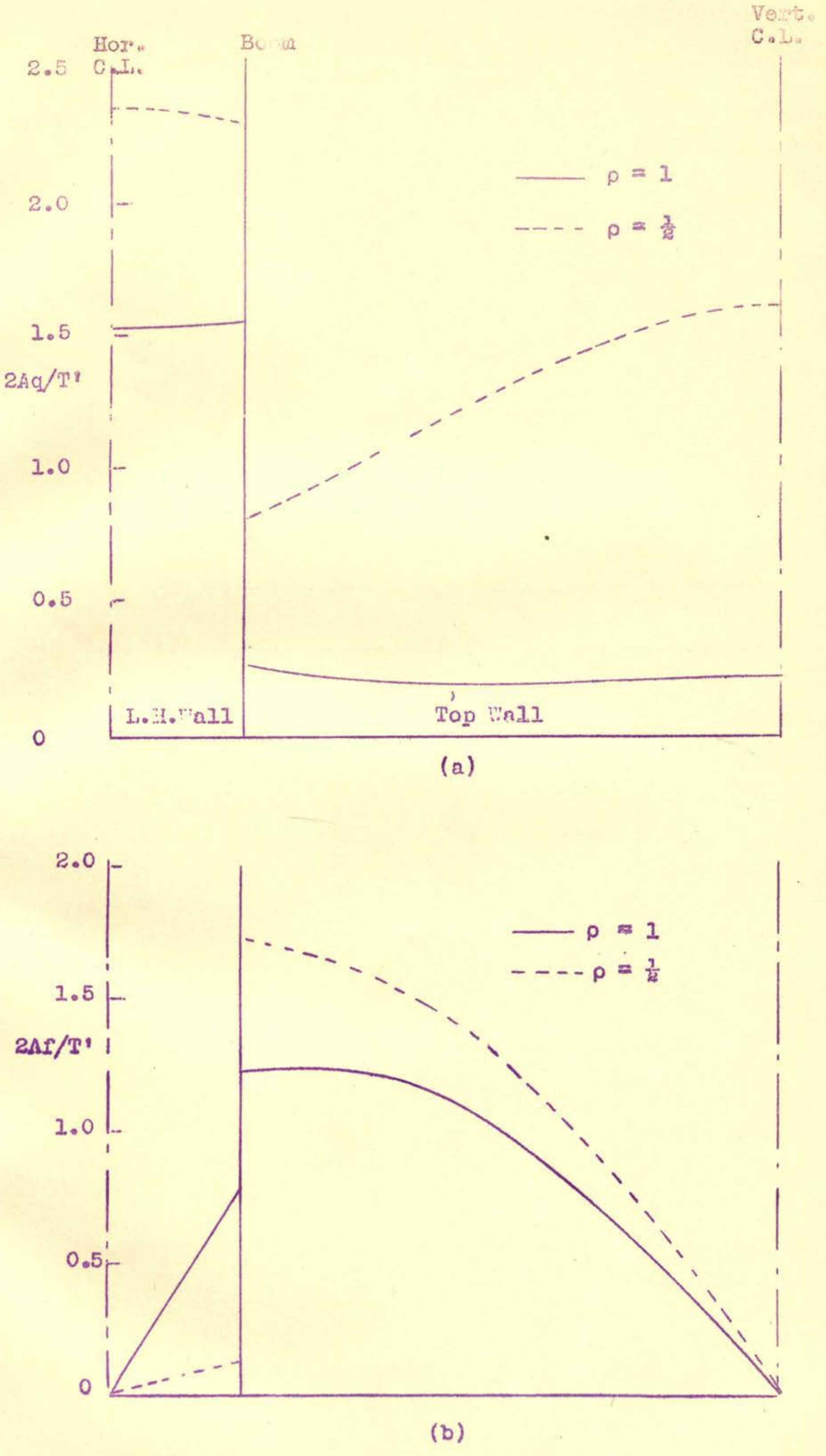


FIG.24. APPLIED TORQUE $T = \rho T'$ WITH ROOT CONSTRAINED. STRESS FLOWS ROUND SECTIONS $\rho = 1$ AND $\rho = \frac{1}{2}$.

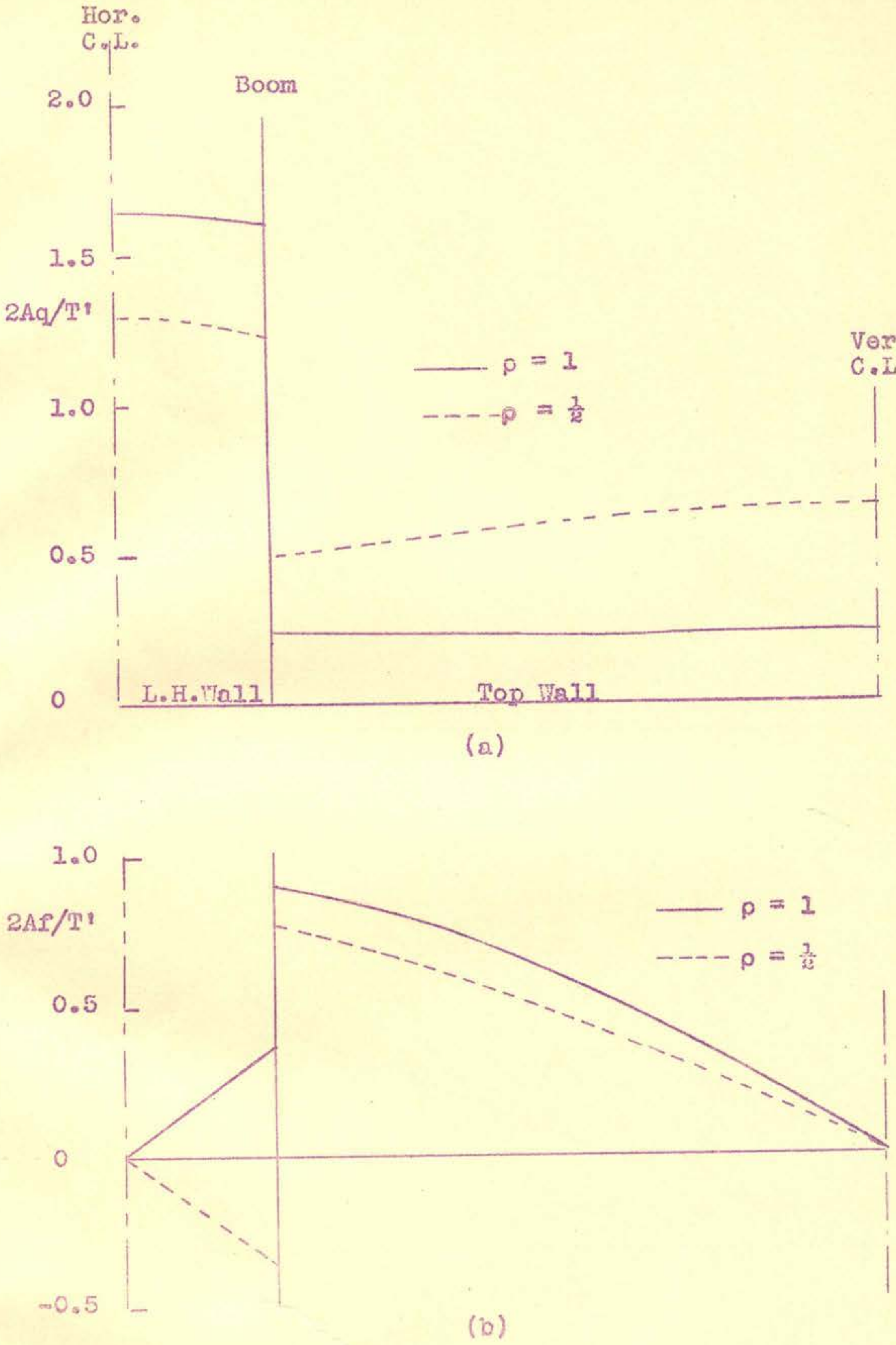


FIG.25. APPLIED TORQUE $T = \rho^2 T'$ WITH ROOT CONSTRAINED.
STRESS FLOWS ROUND SECTIONS $\rho=1$ AND $\rho=\frac{1}{2}$.

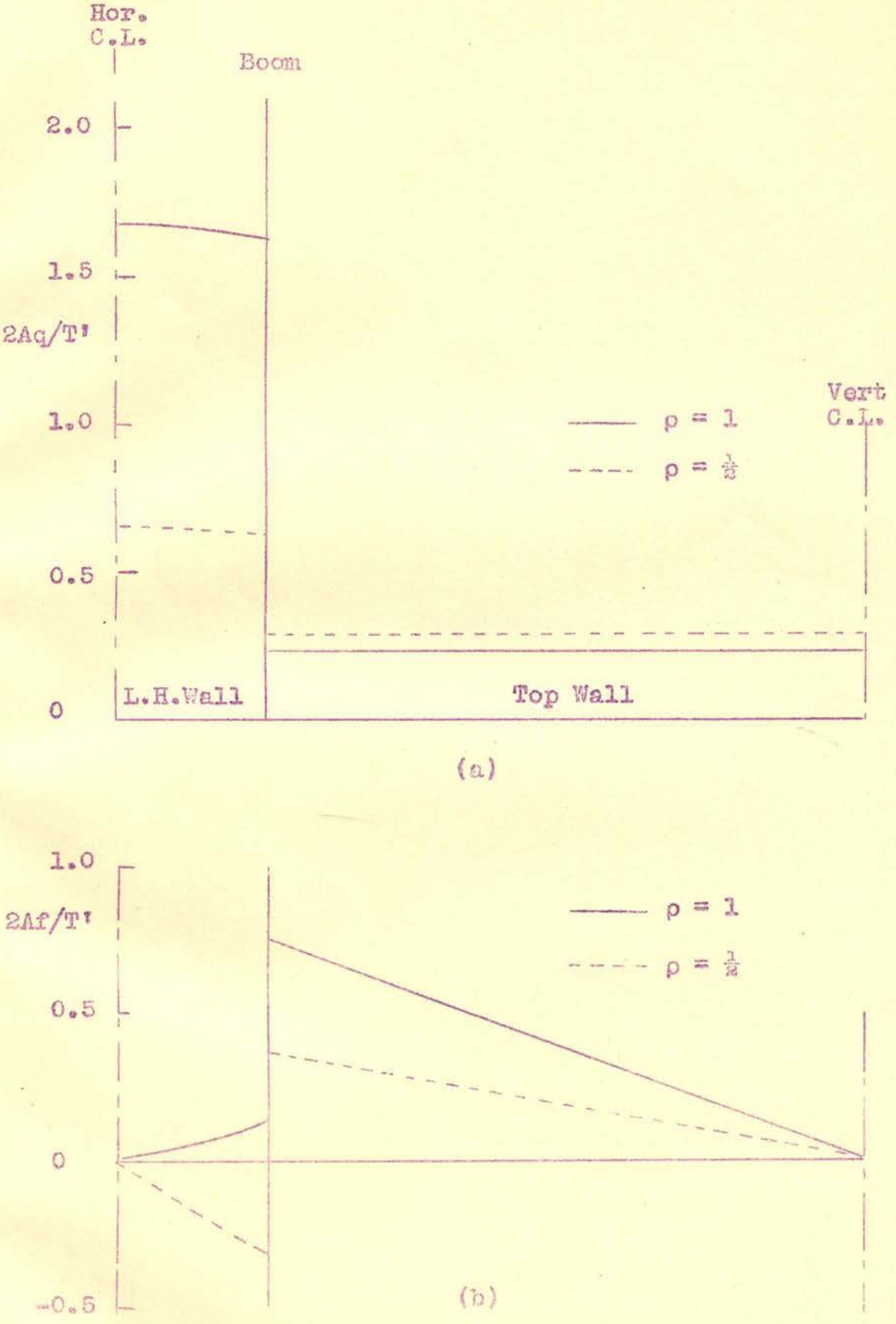


FIG.26. APPLIED TORQUE $T = \rho^3 T'$ WITH ROOT CONSTRAINED.
STRESS FLOWS ROUND SECTIONS $\rho=1$ AND $\rho=\frac{1}{2}$.

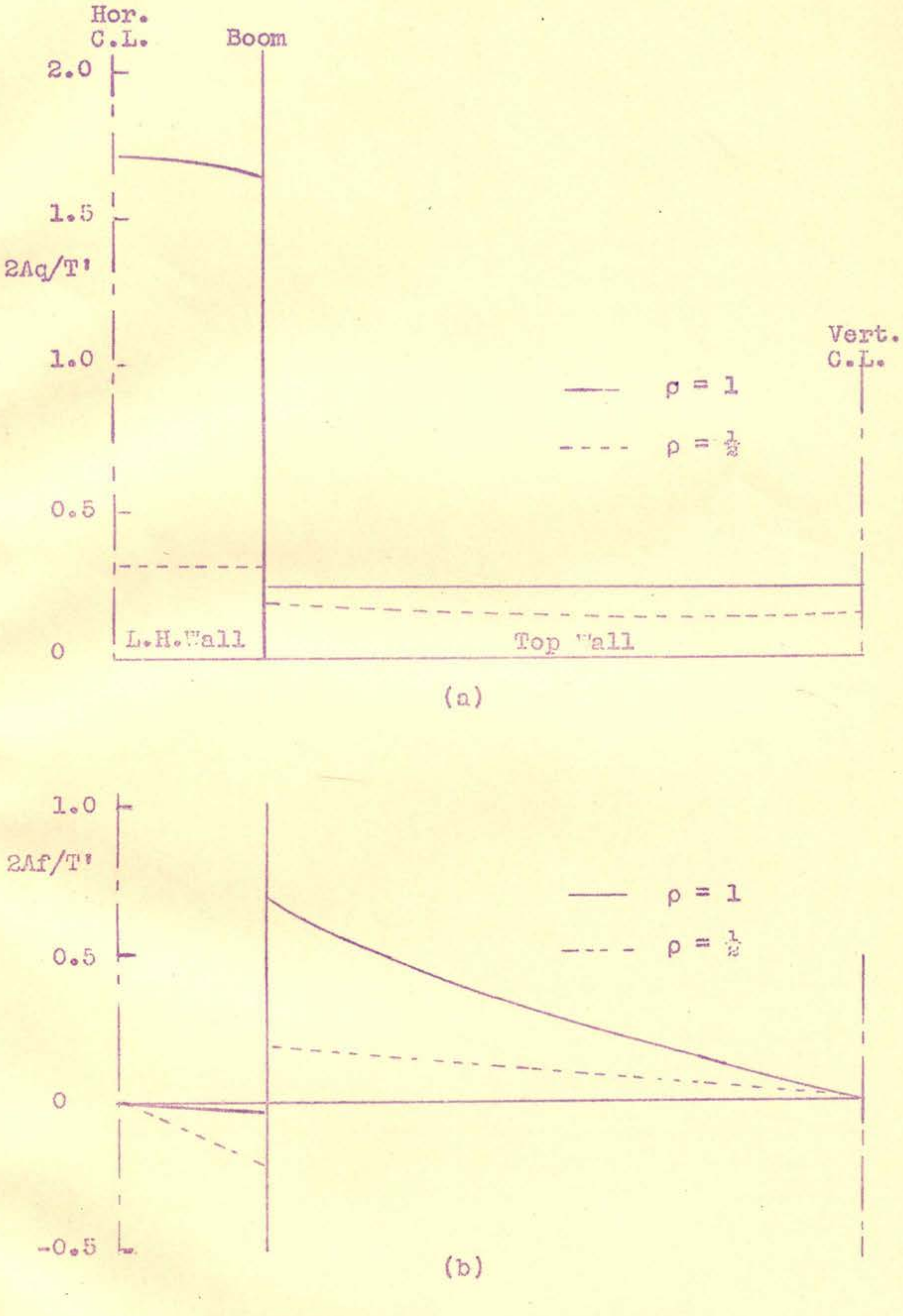


FIG.27. APPLIED TORQUE T = ρ^4T' WITH ROOT CONSTRAINED. STRESS FLOWS ROUND SECTIONS ρ=1 AND ρ=½.

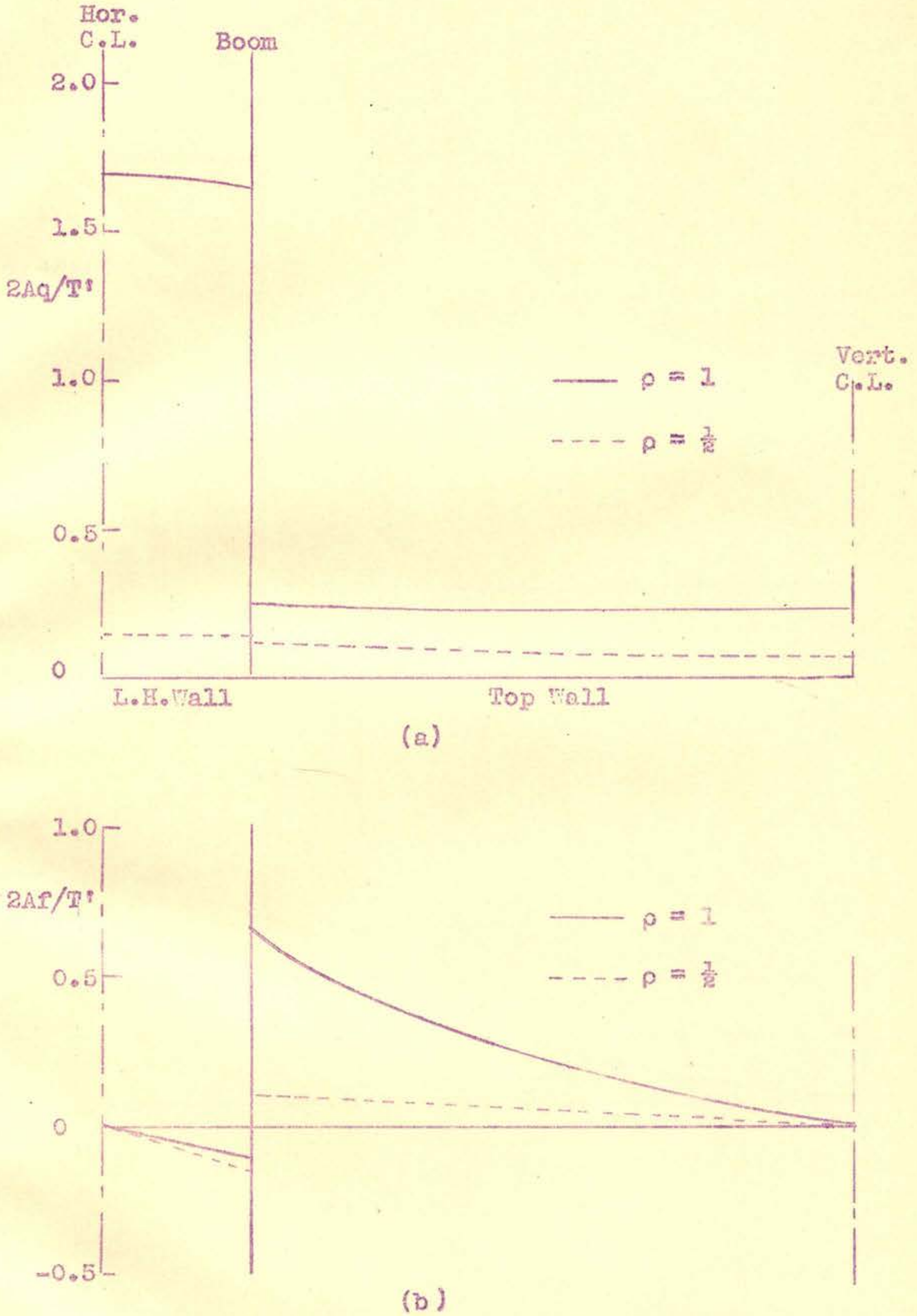


FIG.28. APPLIED TORQUE $T = \rho^5 T'$ WITH ROOT CONSTRAINED.
STRESS FLOWS ROUND SECTIONS $\rho=1$ AND $\rho=\frac{1}{2}$.

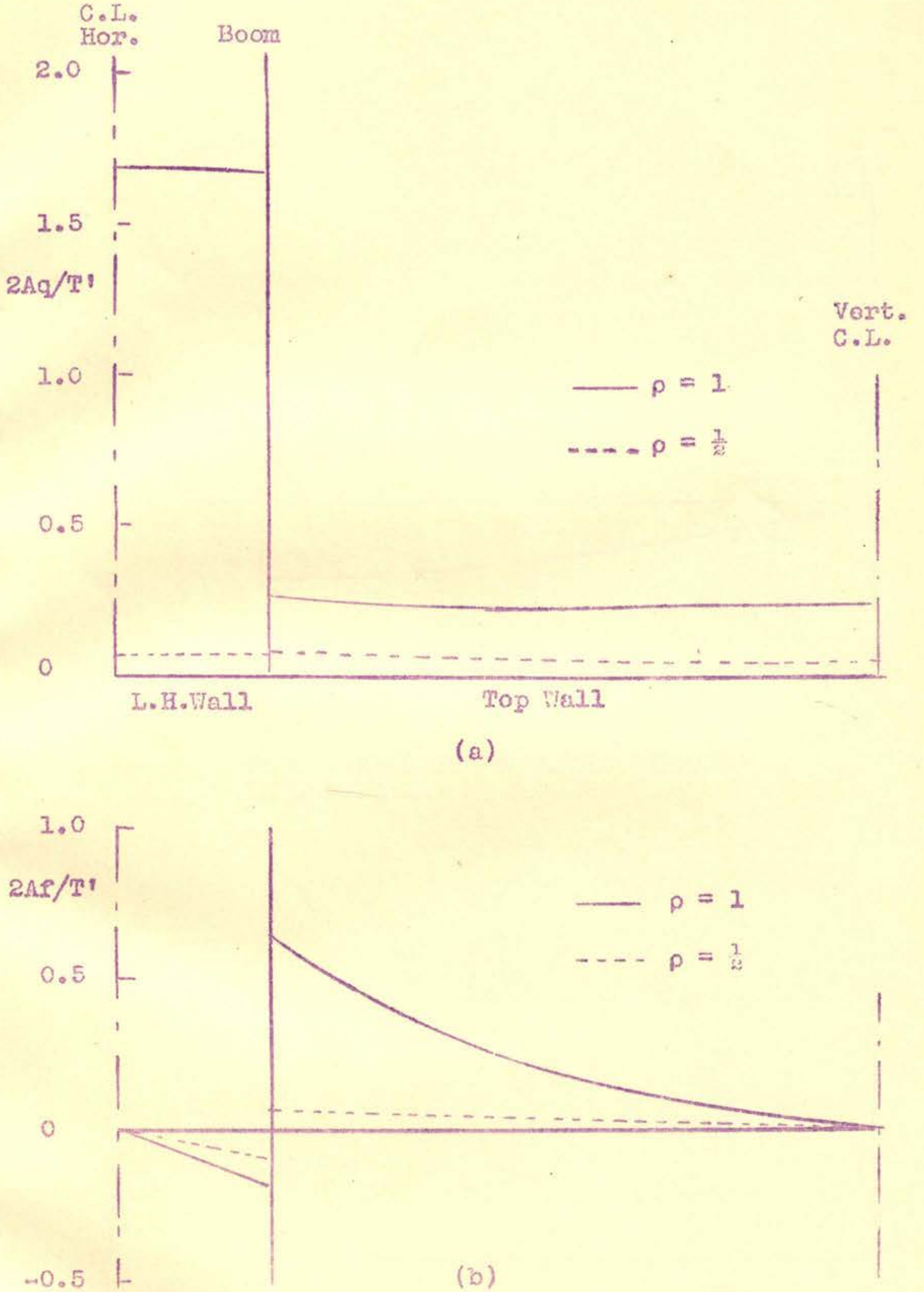


FIG.29. APPLIED TORQUE $T = \rho^{6}T'$ WITH ROOT CONSTRAINED.
STRESS FLOWS ROUND SECTIONS $\rho=1$ AND $\rho=\frac{1}{2}$.

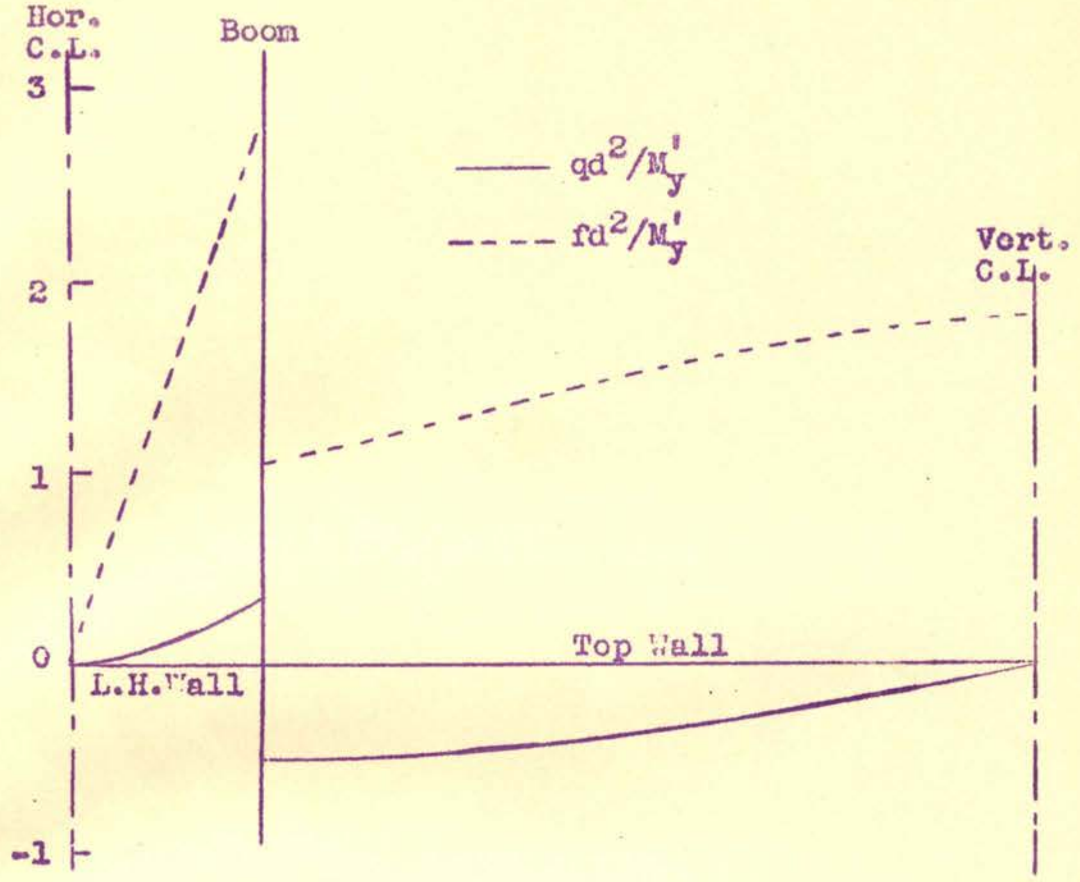


FIG.30. APPLIED BENDING MOMENT $M_y = \rho M_y'$.
ROOT STRESS FLOWS BEFORE WARPING IS ELIMINATED.

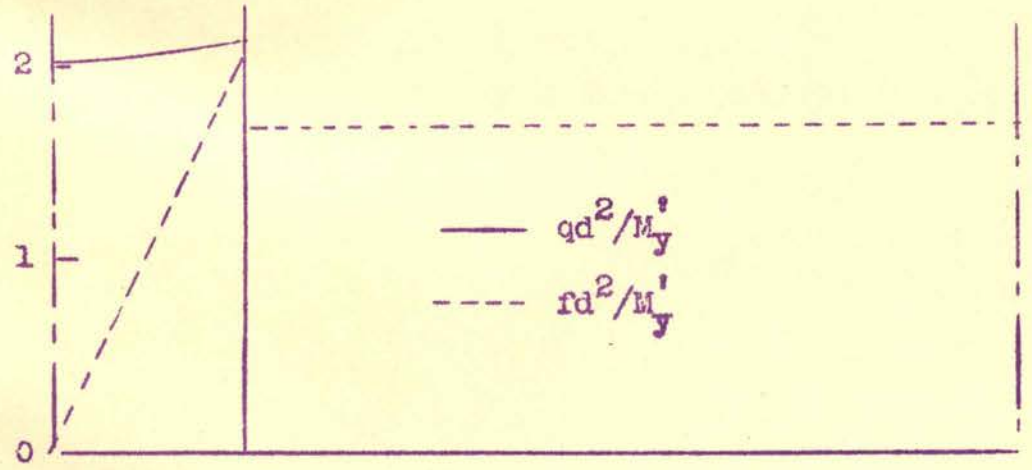


FIG.31. APPLIED BENDING MOMENT $M_y = \rho^2 M_y'$.
ROOT STRESS FLOWS BEFORE WARPING IS ELIMINATED.
(Note: q=0 in the top wall.

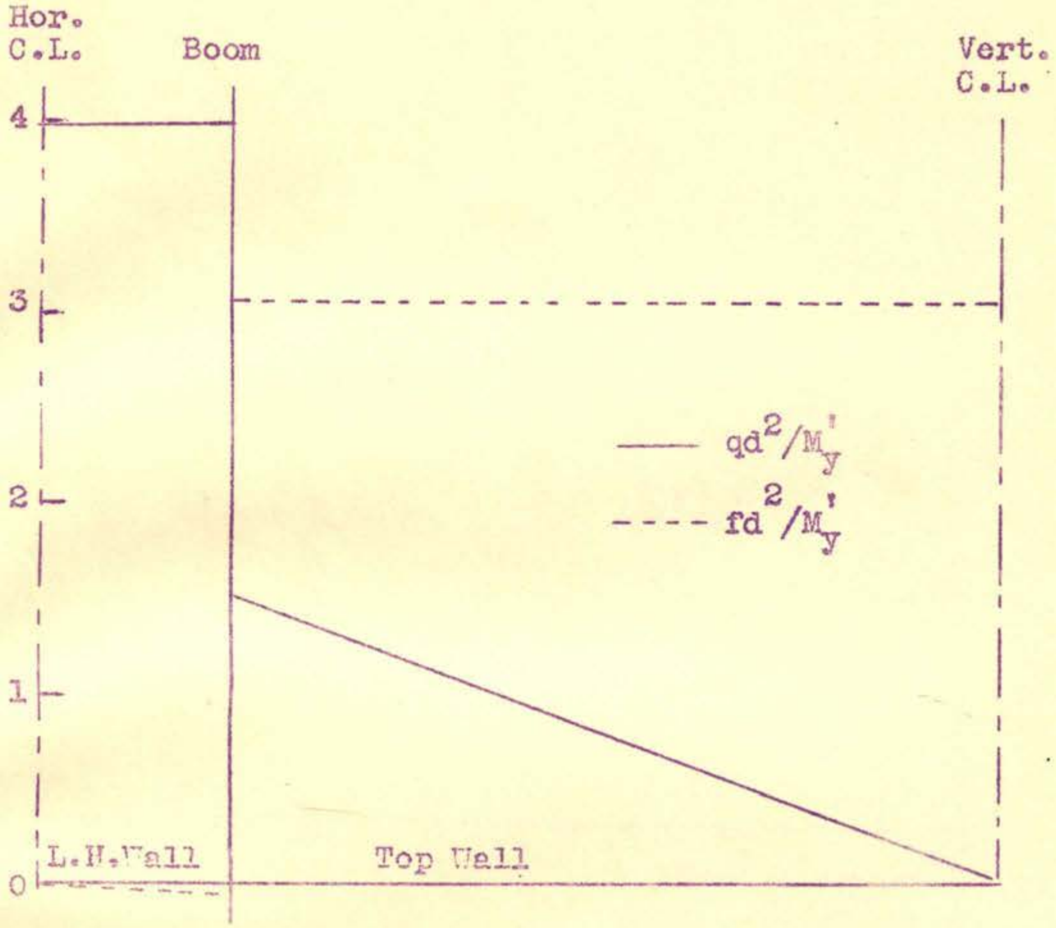


FIG.32. APPLIED BENDING MOMENT $M_y = \rho^3 M_y'$.
ROOT STRESS FLOWS BEFORE WARPING IS ELIMINATED.

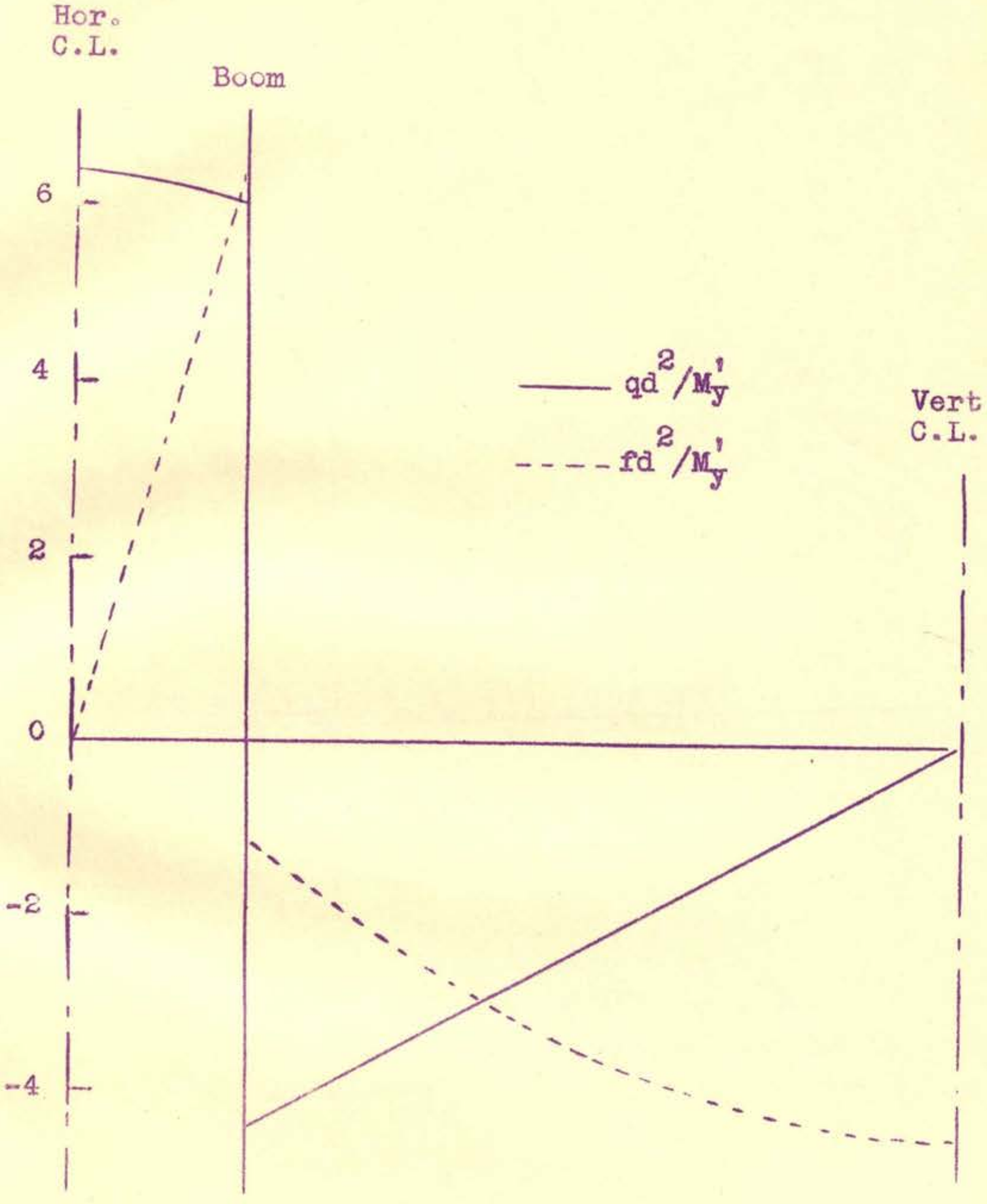


FIG.33. APPLIED BENDING MOMENT $M_y = \rho^4 M_y'$.
ROOT STRESS FLOWS BEFORE WARPING IS ELIMINATED.

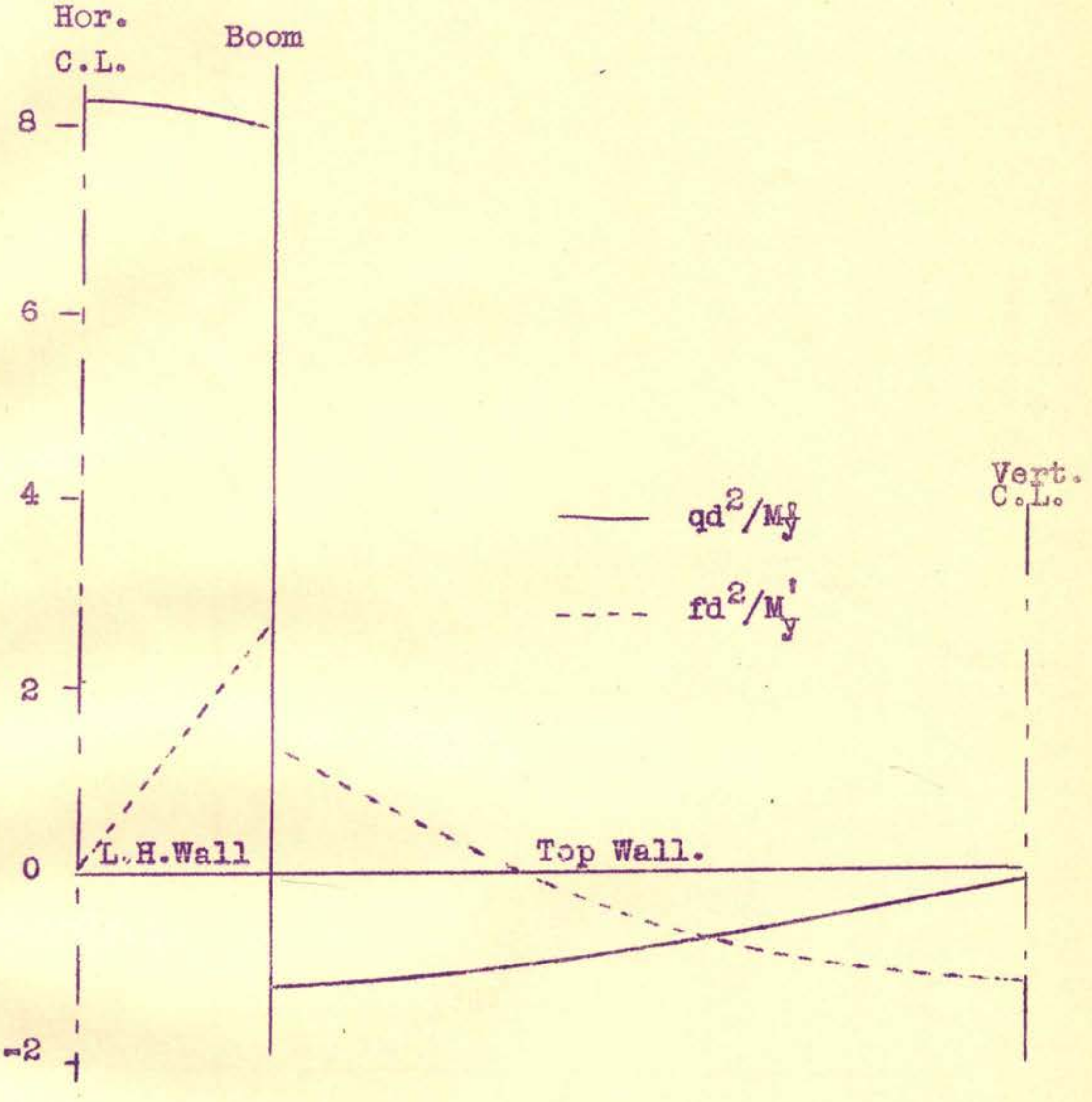


FIG.34. APPLIED BENDING MOMENT $M_y = \rho^5 M_y'$.
ROOT STRESS FLOWS BEFORE WARPING IS ELIMINATED.

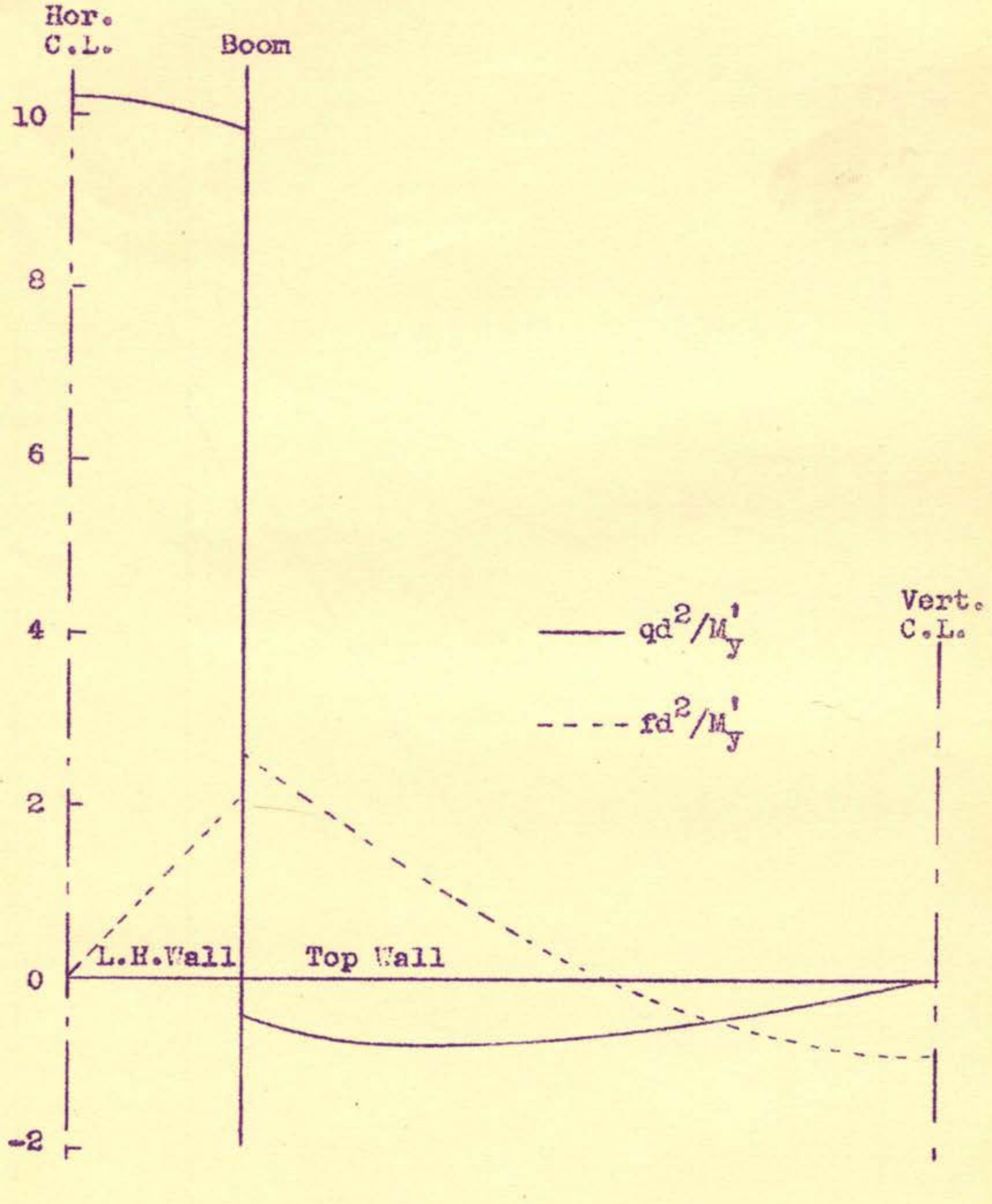


FIG.35. APPLIED BENDING MOMENT $M_y = \rho^6 M_y'$.
ROOT STRESS FLOWS BEFORE WARPING
IS ELIMINATED.

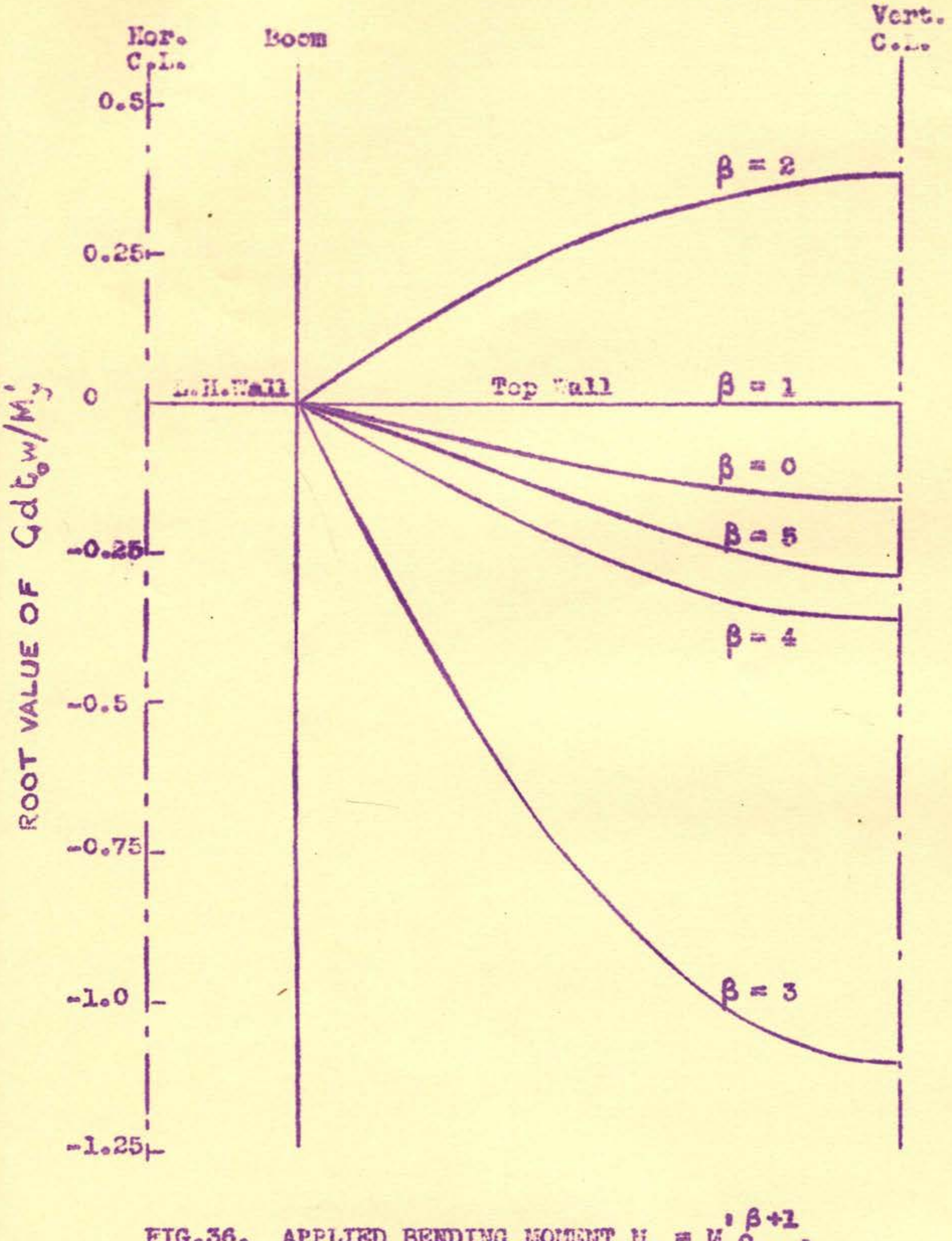
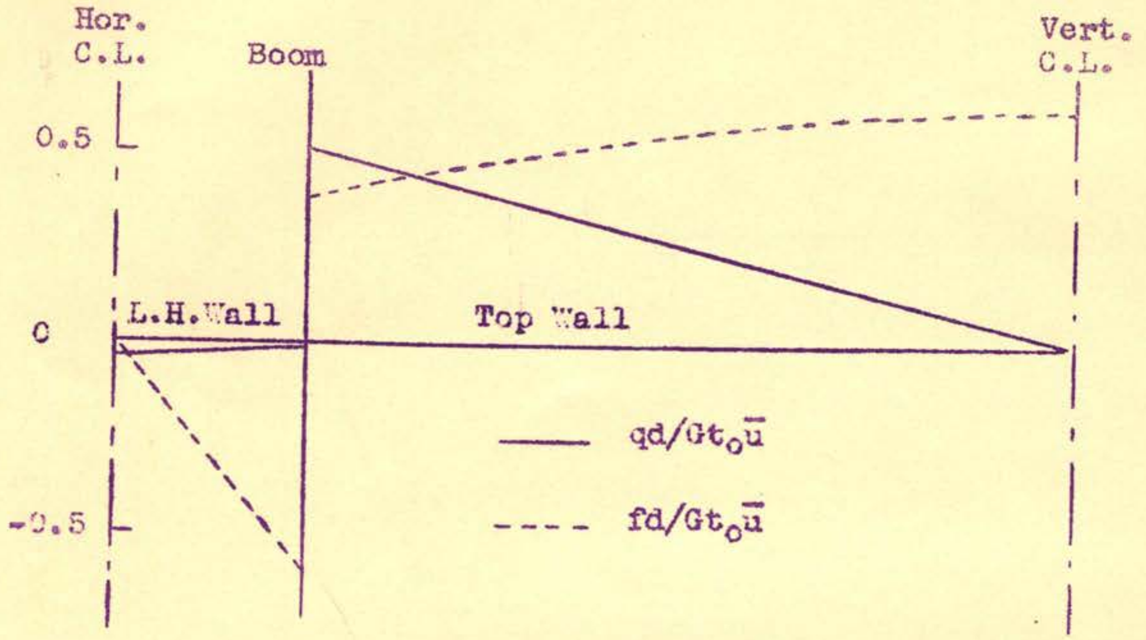
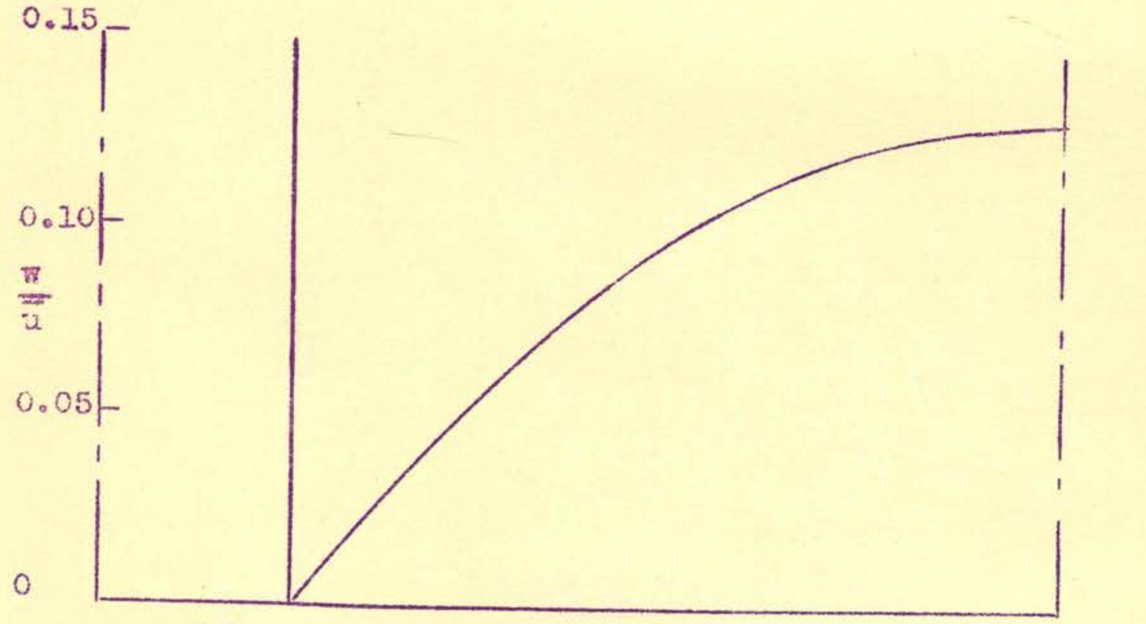


FIG.36. APPLIED BENDING MOMENT $M_y = M_y'\rho^{\beta+1}$.
ROOT WARPING w, ADJUSTED SO THAT w=0 AT EACH BOOM.
Note: in the left hand wall w is so small that it cannot be plotted. In the top wall w=0 when β =1.



(a) ROOT STRESS FLOWS



(b) ROOT WARPING, ADJUSTED SO THAT w=0 AT EACH BOOM.
(Note: in the left hand wall w is too small to plot.)

FIG.37. FIRST BENDING-EIGENLOAD. (β = 2.642)

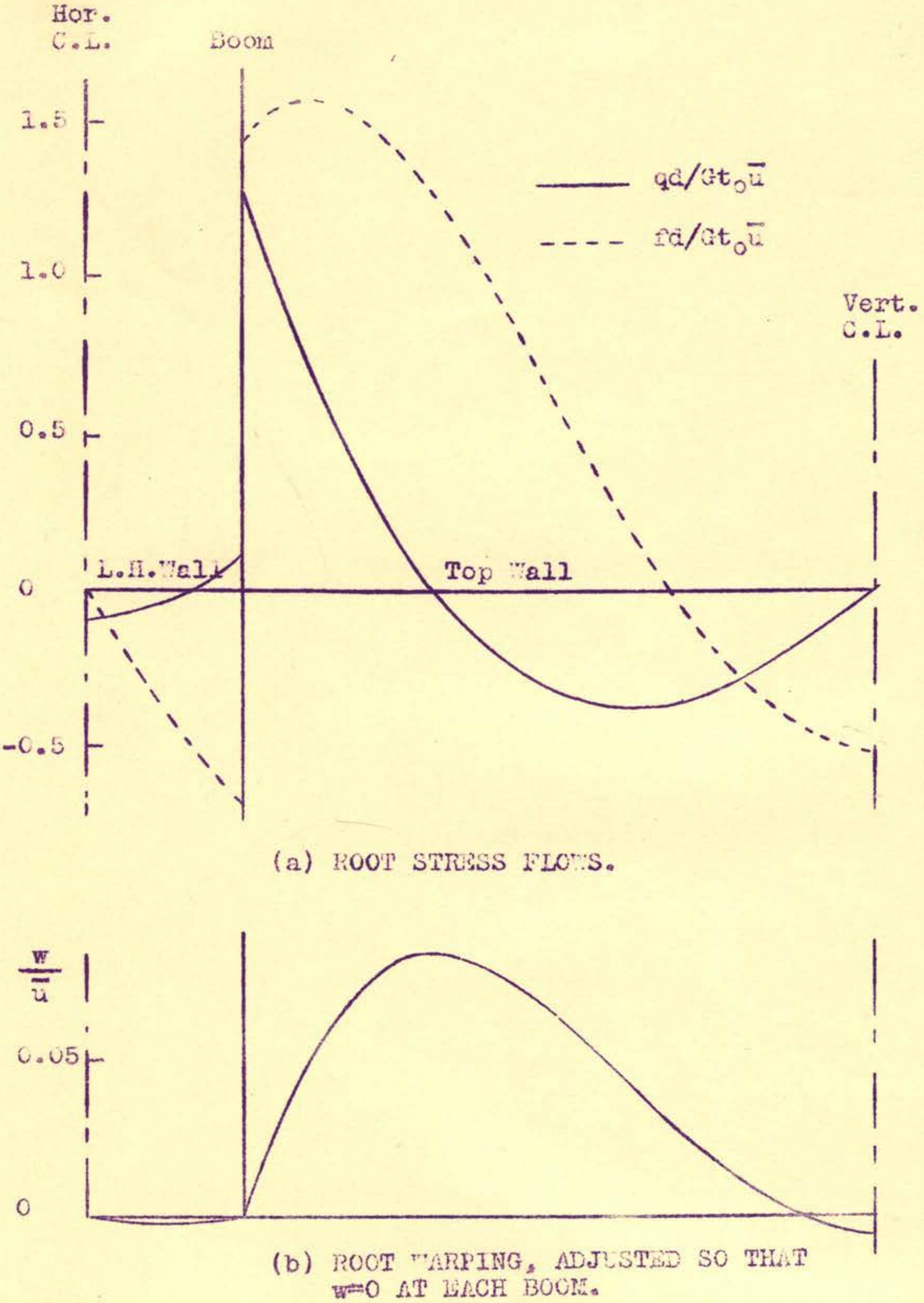


FIG.38. SECOND BENDING-EIGENLOAD. (β = 6.808).

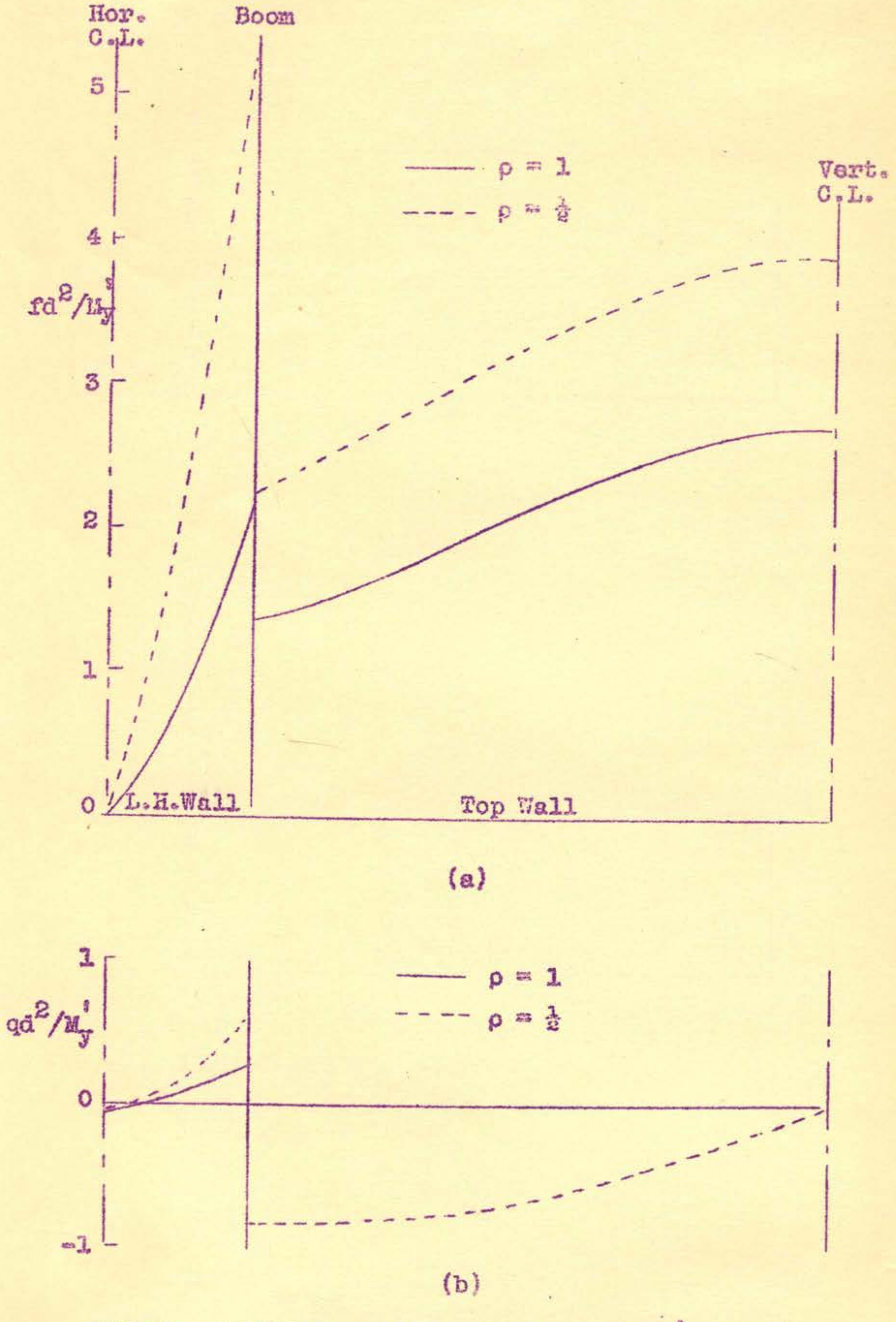
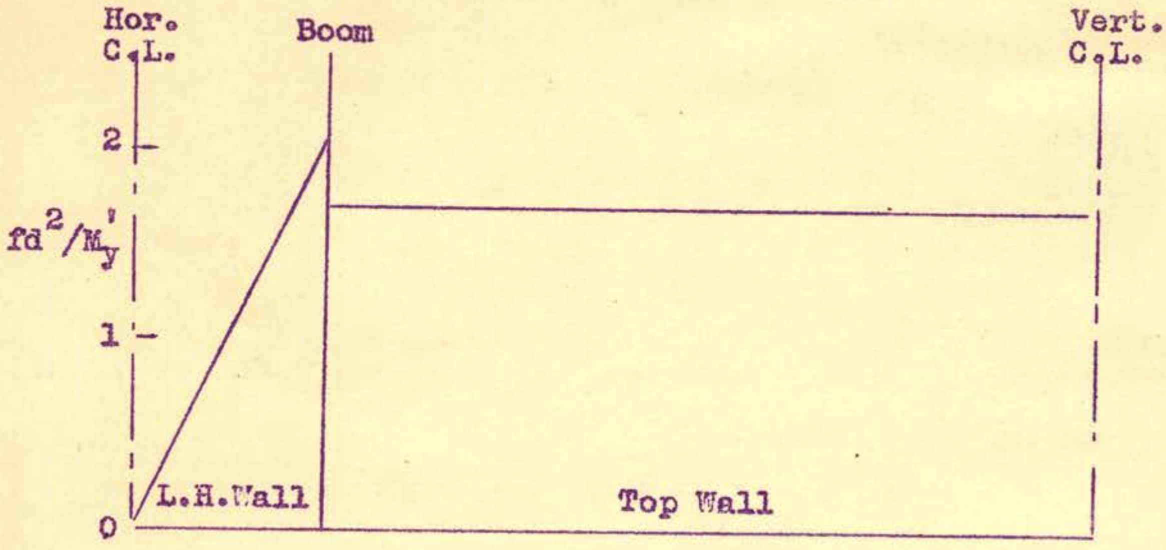
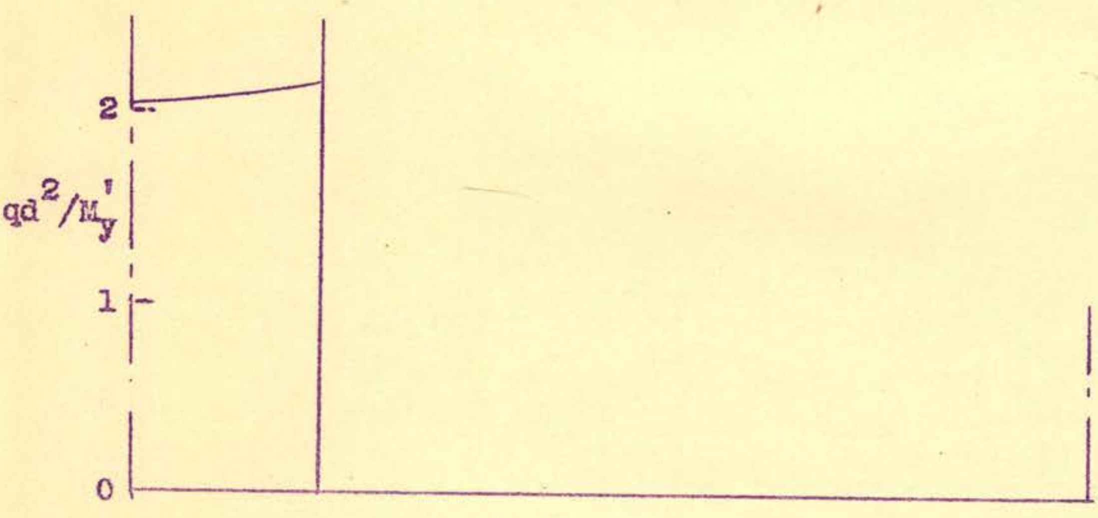


FIG. 39. APPLIED BENDING MOMENT $M_y = \rho M_y'$; WITH ROOT CONSTRAINED. STRESS FLOWS ROUND SECTIONS $\rho=1$ AND $\rho=\frac{1}{2}$.



(a)



(b)

FIG.40. APPLIED BENDING MOMENT $M_y = \rho^2 M_y'$, WITH CONSTRAINED ROOT. STRESS FLOWS ROUND ALL SECTIONS.

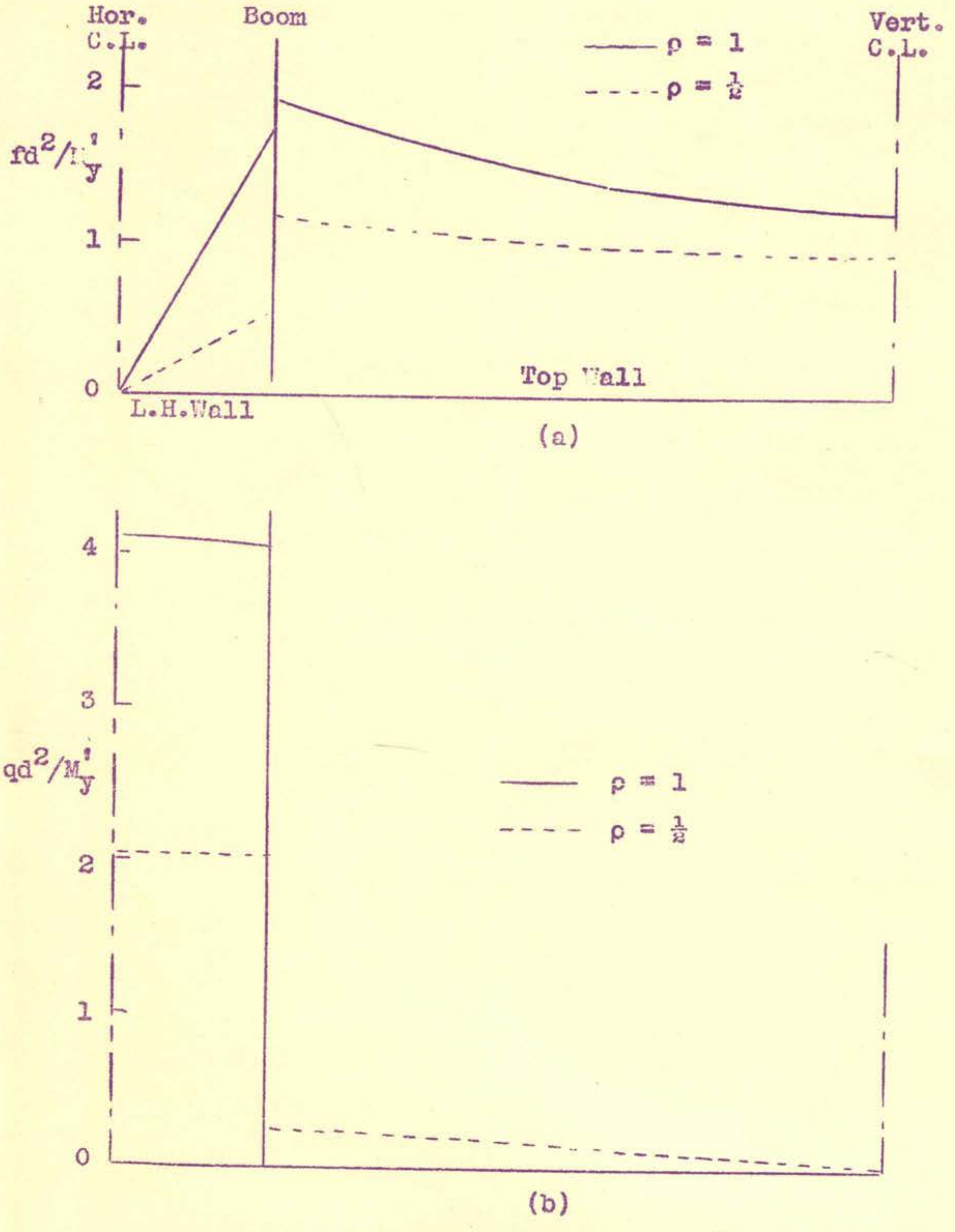
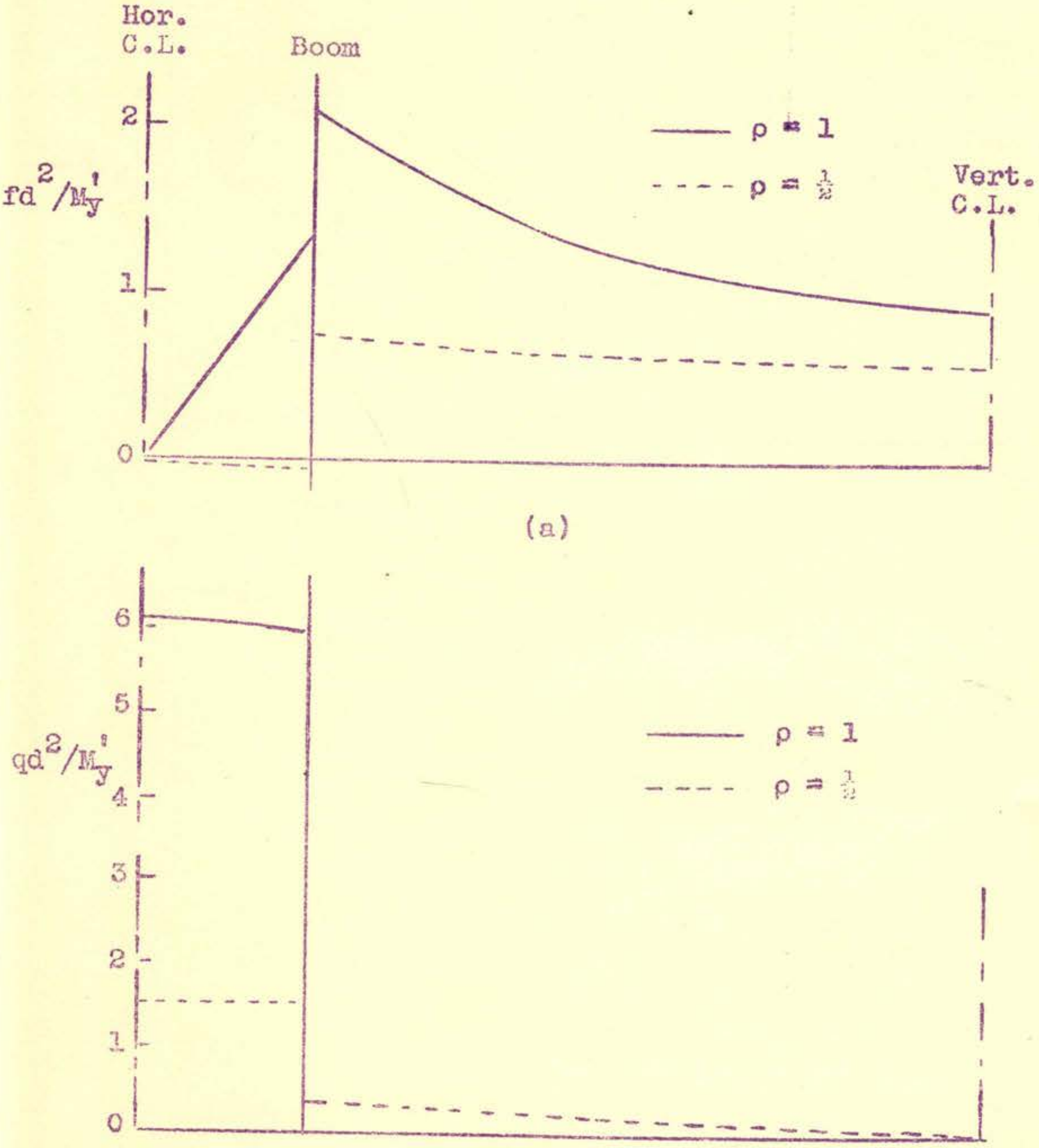
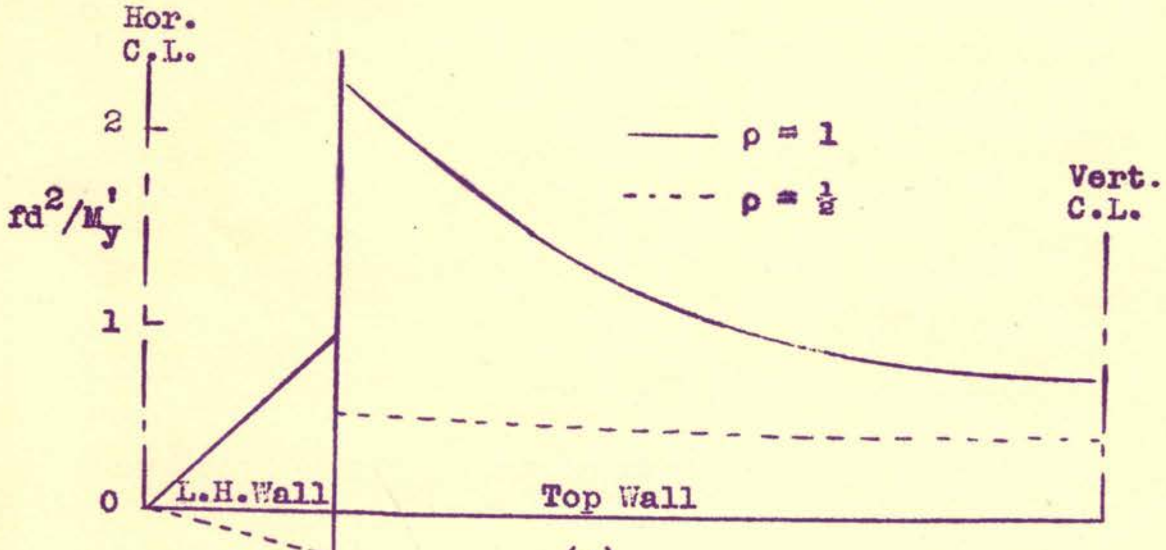


FIG.41. APPLIED BENDING MOMENT $M_y = \rho^3 M_y'$, WITH ROOT CONSTRAINED. STRESS FLOWS ROUND SECTIONS $\rho=1$ AND $\rho=\frac{1}{2}$.

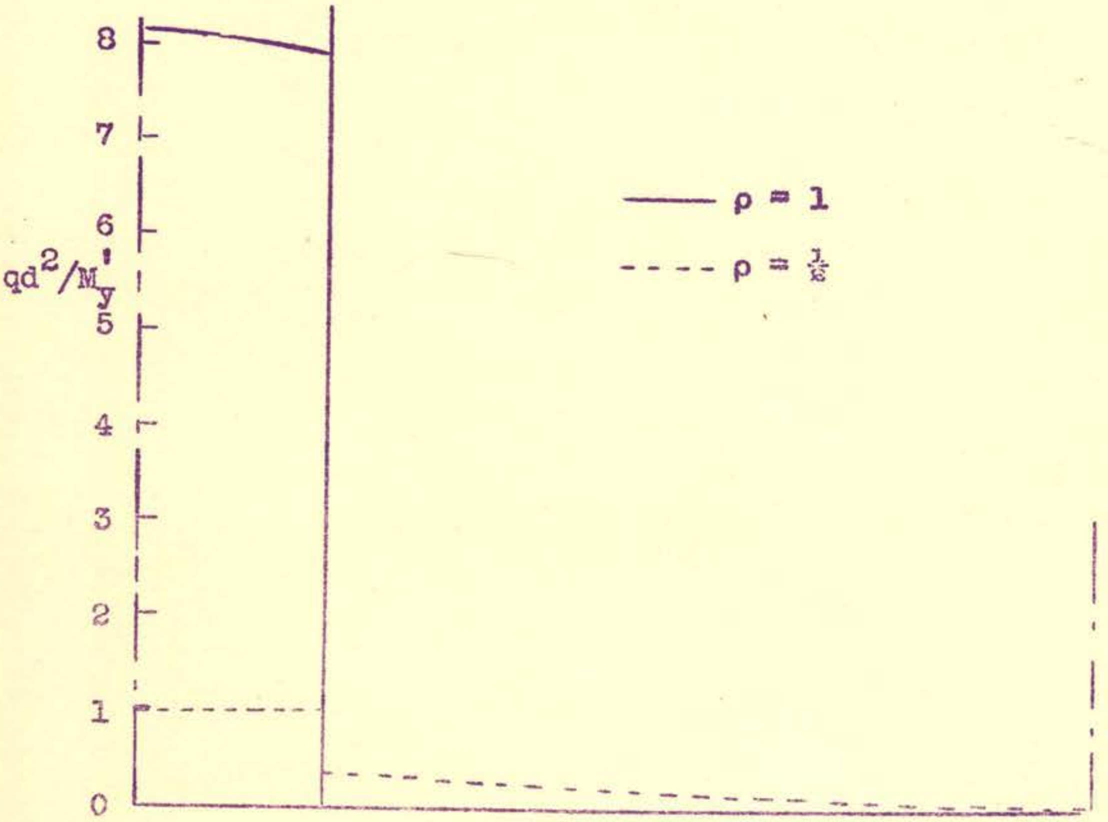


(b) Note the reduced scale.

FIG. 42. APPLIED BENDING MOMENT $M_y = \rho^4 M_y'$, WITH ROOT CONSTRAINED. STRESS FLOWS ROUND SECTIONS $\rho=1$ AND $\rho=\frac{1}{2}$.

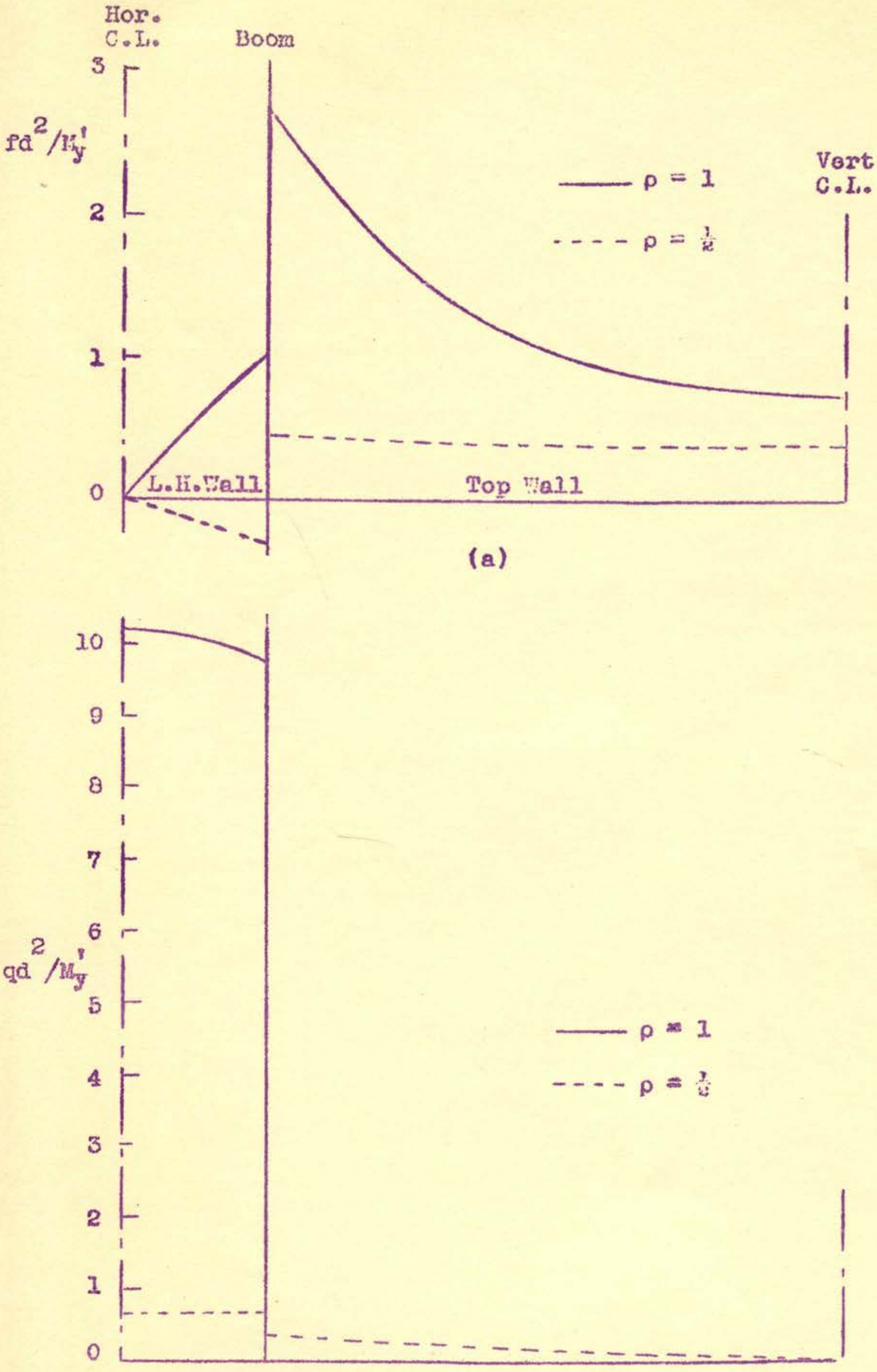


(a)



(b) Note the reduced scale.

FIG.43. APPLIED BENDING MOMENT $M_y = \rho^5 M_y'$, WITH ROOT CONSTRAINED. STRESS FLOWS ROUND SECTIONS $\rho=1$ AND $\rho=\frac{1}{2}$.



(b) Note the reduced scale.

FIG.44. APPLIED BENDING MOMENT $M_y = \rho^6 M_y'$, WITH ROOT CONSTRAINED. STRESS FLOWS ROUND SECTIONS $\rho=1$ and $\rho=\frac{1}{2}$.

TELEPHONE ; MW 2541

The University of Sydney

DEPARTMENT OF AERONAUTICAL ENGINEERING

Your Reference
JDBu.VN.

8th September, 1949.

The Registrar,
THE UNIVERSITY OF SYDNEY.

Dear Sir,

I submit herewith four copies of a thesis entitled "The Torsion and Bending of Swept and Tapered Wings with Ribs Parallel to the Root", for examination for the degree of Doctor of Philosophy. I enclose also four copies of a summary of the thesis.

The thesis embodies the results of research carried out in this University.

Yours faithfully,

WHW/MSB.

TORSION AND BENDING OF SWEPT AND TAPERED WINGS WITH RIBS PARALLEL TO THE ROOT

by

W. H. Wittrick, M.A.

Department of Aeronautical Engineering, University of Sydney.

A thesis presented for the degree of Doctor of Philosophy in the University of Sydney.

SUMMARY

The problem considered is that of a swept wing, either conical or cylindrical, of arbitrary section, under any system of bending and torsion loads. The wing is assumed to consist of a non-buckling outer skin, a series of booms located along generators of the tube, and a series of closely spaced ribs all parallel to the root section. The ribs are assumed to be rigid in their own planes but to offer no resistance to warping out of their planes. No restriction is placed on either the taper or the sweep of the wing.

The theory is developed in general terms, for arbitrary wing section, arbitrary variation of stress bearing area over the tube, and arbitrary applied loads. The fundamental equations are expressed in terms of a stress function which is found to satisfy a complicated integro-differential equation. Analytical solutions of this equation are obtained for certain simple types of tube and applied load by a process consisting essentially of separation of the variables.

Those solutions in some cases lead to formulae exactly analagous to those of the simple theories of bending and torsion for an unswept wing. They are slightly more complicated than the latter formulae, in that they show an interaction between bending and torsion which is characteristic of this type of wing.

The Appendix gives detailed numerical applications of the theory to a highly tapered unswept four boom tube of rectangular cross section, with a completely constrained root section, under varying bending moment and torque. It is shown that when the taper is large the root effect is of prime importance and the analysis of a delta wing would therefore be very tedious.

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