THE COMPLEX IRREDUCIBLE CHARACTERS OF

$Sp(6, q)$, q EVEN

The University of Sydney

MEMORANDUM TO

Rare Book Librarian,
FISHER LIBRARIAN, F03

I attach copy of a letter awarding the degree of Master of Science/Doctor of Philosophy.

A. Thesis Attached.

~~No thesis (essay and course work degree).~~

B. Undertaking re immediate availability signed.

C.S. Davidson,
Graduate Assistant

Mr Swinbourne/640867

24 August 1977

Mr J.C. Looker
20 Carrodus Street
FRASER 2615

Dear Mr Looker,

I am pleased to advise you that you have been awarded the Degree of Doctor of Philosophy. This follows consideration of the reports of the examiners of your thesis entitled "The complex irreducible characters of Sp(6,q), q Even".

The next ceremony at which your degree could be conferred in person will be held on 21 January 1978. If you would prefer that the degree be conferred in absentia, which could be done in the near future, would you please let me know.

I should be pleased if you would inform me of any change in your address as the University wishes to maintain contact with its graduates to whom it sends the Gazette.

I am enclosing a copy of comments which examiners have agreed may be released to you for your information.

Yours sincerely,

for Kenneth W. Knight
Registrar,

c.c. Head of Department of Pure Mathematics
Dr. S.B. Conlon

Faculty of Science Monthly Report

Rare Book Librarian

RECORDS TO FORWARD: 1. M. Gaffney (Enrolments)
2. RECORDS

THE COMPLEX IRREDUCIBLE CHARACTERS OF

Sp(6,q), q EVEN

John Looker

Ph.D. Thesis

Department of Pure Mathematics,
University of Sdyney,
1977.

PREFACE

The topic of this thesis was suggested to me by Professor G.E. Wall. I am grateful to Professor Wall for this and for his supervision and encouragement during the course of this work.

The first two sections of the work are summaries of known results drawn from a variety of sources. In §3 these results are applied in the case of the symplectic group. Although I have not seen this done explicitly before, I believe it to be part of the folklore on the subject. The remaining sections are my own work except for those few paragraphs which obviously serve to introduce well-known concepts. Some of the results appearing in §5 on characters induced from orthogonal subgroups are known for $q = 2$ and are used in [12]. The characters of $Sp(4,q)$, q even, appear in [11] but the method used for determining them in §6 and §7 differs from the method appearing *loc. cit.*

I extend thanks to Dr. G.B. Elkington for his assistance particularly in providing material from lecture notes not readily available in this country. I extend thanks also to Mr. W.P. McLennan and Dr. H.N. Johnston of the Australian Bureau of Statistics for their encouragement and provision of time towards the preparation of the manuscript. This work would not have been possible without the understanding of my wife and children when I might have had much more free time to spend with them. Finally, I would like to thank all my friends who have assisted in many little ways and Ms. C. Kicinski for her skilled typing of the manuscript.

CONTENTS

Table 10: The Green polynomials.

Table 11: The irreducible characters $(\chi_{[\xi]}^{(i)})$ of $G = Sp(2n,q)$

References

INTRODUCTION

Amongst the factors contributing to the present interest in the problem of determining the complex irreducible characters of the finite classical groups, none has produced as much impetus as the work of J.A. Green [14]. Prior to this work, I. Schur [16] had solved the problem for the general linear groups in two dimensions and R. Steinberg [22] had extended this to three and four dimensions. Green, however, was able to solve the problem for all the general linear groups by the introduction of certain functions (now known as "Green functions") which are defined on the unipotent conjugacy classes (i.e. those whose eigenvalues are all unity), one such function corresponding to each conjugacy class of maximal tori (maximal diagonalizable subgroups). Given a symmetric function in the eigenvalues of the elements an irreducible character is obtained by using the Green functions according to a "degeneracy rule". Moreover all irreducible characters can be obtained this way.

For the general linear group in n dimensions over a Galois field of q elements, the values of the Green functions are all integral polynomials in q. Moreover, the Green functions satisfy quite striking orthogonality relations. Since the classes of maximal tori are, in this case, indexed by the partitions of n so are the Green functions, and, by an unusual coincidence, the unipotent classes are similarly indexed. It is chiefly this fact which allows Green's method to produce all the irreducible characters for the general linear groups.

In many respects the unitary groups behave like the general linear groups. For example, the unipotent conjugacy classes for the unitary group in n dimensions over a field of q^2 elements are indexed by the partitions of n. Moreover, the maximal tori can also be indexed by the partitions of n in such a way that their orders can be obtained from the orders of the corresponding maximal tori in the general linear group simply by replacing q by -q. Realizing this V. Ennola [10] conjectured that the characters of the unitary groups could be obtained in terms of "Green functions" where the values for the Green function corresponding to a given maximal torus could be obtained by substituting -q for q in the values for the Green function in the general linear group for the corresponding maximal torus. The lack of sufficient parabolic subgroups in the unitary group (i.e. subgroups containing the normalizer of a Sylow p-subgroup where p is the characteristic of the Galois field) meant that the arguments used by Green in his paper would not work in this case and as far as I am aware this conjecture is still unproved.*

In her paper [20] on the characters of the symplectic group in four dimensions over a field of q elements for q odd, B. Srinivasan obtained an analogue of the Green functions. These are similar to the Green functions for the general linear groups in the respect that they are each associated with a conjugacy class of maximal tori, take values which are integral polynomials in q and satisfy certain orthogonality relations. However, unlike the general linear group, the symplectic group has more unipotent conjugacy classes of elements than it has conjugacy classes of maximal tori. This is sufficient to prevent all the complex characters being obtained as in the former case. The situation is similar for the symplectic groups in four

---

* Recently a paper of G. Lusztig has appeared in Proc. Lond. Math. Soc. (3) 33 (1976), 443-475, which calculates the values for the Green functions associated with the Coxeter torus. This calculation establishes Ennola's conjecture for this case.

dimensions when q is even as can be easily seen from the character table for these groups (see e.g. H. Enomoto [11]).

The recent paper of P. Deligne and G. Lusztig [8] provides a beautiful theoretical framework in which "Green functions" arise naturally for all the finite classical groups from what they call "virtual representations". A virtual representation is in fact an alternating sum of representations of the group in the ℓ-adic cohomology groups of the associated algebraic group, for ℓ relatively prime to q. The "Green functions" thus obtained are again associated with conjugacy classes of maximal tori and in the case of the general linear groups and the four dimensional symplectic group are precisely the functions already mentioned. Again they satisfy important orthogonality conditions. Deligne and Lusztig show how these "Green functions" can be used to construct a large class of irreducible representations. However in the general case there are insufficient Green funtions to enable all the irreducible characters to be obtained.

This present work is motivated by a desire to find an adequate generalization of the Green functions. The end product is an explicit determination of the irreducible characters of the symplectic groups in six dimensions over a finite field of q elements where q is a power of two. The determination is in terms of a set of functions which I call "Green functions" which are associated with the conjugacy classes of "pseudo-tori". A pseudo-torus is either a maximal torus or a particular type of torus exhibiting a "rank defect". The "Green functions" corresponding to (proper) maximal tori are precisely those given by the Deligne-Lusztig theory. This wider class of "Green functions" is shown to take only values which are integral polynomials in q and to satisfy (slightly modified) orthogonality relations. The total number of "Green functions" for the six dimensional symplectic group over a finite field of characteristic two is twelve which is precisely the number of unipotent conjugacy classes.

The calculation of the (generalized) Green functions was done before the appearance of the Deligne-Lusztig paper (*loc. cit.*) but, because of the importance of their work and the simplification it provides, the text has been rewritten to incorporate many of their ideas. Thus in §1 we give a general recapitulation of some of the main results from algebraic group theory followed in §2 by a brief sketch of some of the results of the Deligne-Lusztig theory. This is applied specifically to the case of the symplectic group in §3. We then turn our attention to the task of obtaining explicit values for the "unipotent" characters from which the Green functions can be obtained. Thus in §4 we obtain properties of the orthogonal subgroups of the symplectic group which in §5 are used to determine the unipotent characters in terms of characters induced from orthogonal subgroups and parabolic subgroups. In §6 machinery is set up to facilitate computations and this is used to evaluate the unipotent characters on the unipotent conjugacy classes. Towards the end of section 6, the pseudo-tori are introduced and the Green functions are evaluated. Finally in §7, the virtual characters associated with the additional psuedo-tori introduced in §6 are obtained and linear combinations of these used to obtain all the irreducible characters of Sp(6,q), q even.

An examination of the Green functions (shown in table 10 of the appendix) reveals that by substituting -q for q and interchanging some of the unipotent conjugacy classes each Green function is transformed into another. This result suggests a generalization of Ennola's conjecture.

§1. NOTATION AND RECOLLECTIONS.

In this section we recapitulate some results from the theory of linear algebraic groups. We also take the opportunity to specify some of the notation to be employed in subsequent sections. Standard references for this material are [1] and [5] but other sources include [2], [3], [19], [23], [24] and [25].

1.1 We denote by k a finite field of q elements, where q is a power of a prime p, and by $\tilde{k}$ an algebraic closure of k. The additive group of elements of k and the multiplicative group of non-zero elements of k will be denoted by k^+ and k^* respectively. We will be considering two classes of objects, namely those associated with the finite field k and those associated with $\tilde{k}$. The notation introduced for one case will be intended to apply to the other case in an obvious way. Thus for example $\tilde{k}^+$ and $\tilde{k}^*$ will denote the additive group of elements of $\tilde{k}$ and the multiplicative group of non-zero elements of $\tilde{k}$ respectively.

We denote by $\tilde{G}$ a connected reductive linear algebraic group defined over k and by F the Frobenius endomorphism of $\tilde{G}$ induced by the endomorphism $F: \alpha \mapsto \alpha^q$ of $\tilde{k}$. We set $G = \tilde{G}^F$ where $\tilde{G}^F$ denotes the set of F-fixed points of $\tilde{G}$.

1.2 For each element g of $\tilde{G}$ we denote by adg the functor induced by the inner automorphism $x \mapsto gxg^{-1}$ of $\tilde{G}$ on the category whose objects are the (closed) subgroups and quotients of subgroups of $\tilde{G}$ and whose morphisms are algebraic homomorphisms.

Each element g of $\tilde{G}$ has a Jordan decomposition, i.e. it can be expressed as a commuting product of a semisimple element g_s and a unipotent element g_u. This decomposition is necessarily unique.

The elements g_s and g_u are referred to as the semisimple and unipotent parts of g, respectively.

1.3 For a torus $\tilde{T}$ we denote by $\tilde{\sigma}(\tilde{T})$ its $\tilde{k}$-rank and by $\sigma(\tilde{T})$ its k-rank i.e. $\sigma(\tilde{T})$ is the dimension of the unique maximal k-split subtorus of $\tilde{T}$. The k-rank $\sigma(\tilde{G})$ of $\tilde{G}$ is defined as the k-rank of a maximally k-split maximal torus in $\tilde{G}$. A maximal torus $\tilde{T}$ in G is said to be anisotropic if $\sigma(\tilde{T}) = 0$ and minisotropic if $\sigma(\tilde{T})$ assumes its minimum possible value. Note that if $\tilde{G}$ is semisimple then $\tilde{G}$ contains an anisotropic torus.

For a torus $\tilde{T}$ we denote by $X(\tilde{T})$ the (discrete) character group of $\tilde{T}$ written additively i.e. $X(\tilde{T}) = \mathrm{Hom}(\tilde{T},\tilde{k}^*)$ where the functor Hom is taken in the sense of algebraic groups. Similarly we define the group (again written additively) $Y(\tilde{T})$ of one parameter subgroups of $\tilde{T}$ as $\mathrm{Hom}(\tilde{k}^*,\tilde{T})$. Both $X(\tilde{T})$ and $Y(\tilde{T})$ are free abelian groups of rank $\tilde{\sigma}(\tilde{T})$ and can be considered as dual $\mathbb{Z}$-modules under the form defined by

(1.3.1) $$(x,y) = x_\circ y \in \mathrm{Hom}(\tilde{k}^*,\tilde{k}^*) = \mathbb{Z}$$

for all x in $X(\tilde{T})$ and y in $Y(\tilde{T})$.

1.4 If $\tilde{T}$ is a maximal torus of $\tilde{G}$ the Weyl group $\tilde{W}(=\tilde{W}_{\tilde{T}})$ relative to $\tilde{T}$ (i.e. the group $N_{\tilde{G}}(\tilde{T})/\tilde{T}$ where $N_{\tilde{G}}(\tilde{T})$ is the normalizer of $\tilde{T}$ in $\tilde{G}$) acts on $Y(\tilde{T})$ as a left $\tilde{W}$-module ($wy = \mathrm{ad}w_\circ y$ for all y in $Y(\tilde{T})$ and w in $\tilde{W}$) and on $X(\tilde{T})$ in contragredient fashion ($wx = x_\circ \mathrm{ad}w^{-1}$ for all x in $X(\tilde{T})$ and w in $\tilde{W}$).

The set $\Sigma(=\Sigma_{\tilde{T}})$ of roots relative to $\tilde{T}$ is a subset of $X(\tilde{T})$ and the corresponding coroots are elements of $Y(\tilde{T})$. The group $\tilde{W}$ (i.e. its action on $X(\tilde{T})$) is generated by the "reflections" w_r: $x \mapsto x - (x,r^*)r$

for x in $X(\tilde{T})$ where r^* is the coroot corresponding to the root r. The space E defined as $X(\tilde{T})\otimes\mathbb{R}$ possesses a positive definite symmetric bilinear form (unique up to scalar multiplication if $\tilde{G}$ is almost simple) which is invariant under the action of $\tilde{W}$. This allows $Y(\tilde{T})$ to be identified as a subset of E where in particular $r^* = \frac{2r}{(r,r)}$ for each element r of Σ.

<u>1.5</u> Corresponding to each root r in Σ is a one-parameter unipotent subgroup X_r: $\tilde{k}^+ \to \tilde{G}$ of $\tilde{G}$. A choice of a Borel subgroup $\tilde{B}$ of $\tilde{G}$ containing $\tilde{T}$ determines an ordering on Σ and thus a unique set of positive roots Σ^+ and a set Π of fundamental roots within Σ^+. For each subset J of Π there is a parabolic subgroup $\tilde{W}_J$ of $\tilde{W}$ generated by the reflections w_r where $r \in J$ and, by the Tits correspondence, a parabolic subgroup $\tilde{P}_J$ of $\tilde{G}$ generated by the subgroup $\tilde{B}$ and the $X_{-r}(\tilde{k}^+)$ for r in J.

<u>1.6</u> If $\tilde{T}$ is an F-stable maximal torus of $\tilde{G}$ then F acts on $\tilde{W}$, $Y(\tilde{T})$ and $X(\tilde{T})$ where its action on $X(\tilde{T})$ is the transpose of its action on $Y(\tilde{T})$. These induced actions of F will be denoted either as $F_{\tilde{W}}$, $F_{Y(\tilde{T})}$ and $F_{X(\tilde{T})}$ respectively or, more concisely, simply as F. F also permutes the one parameter unipotent subgroups $\chi_r(\tilde{k}^+)$ of $\tilde{G}$ thus inducing a permutation τ_F in Σ which is such that $F(w_r) = w_{\tau_F(r)}$ for each r in Σ. Considering τ_F and the elements of $\tilde{W}$ purely as transformations on Σ (or on $X(\tilde{T})$) we see that for each w in $\tilde{W}$

$$(1.6.1) \qquad F(w) = (\mathrm{ad}\tau_F)(w) = \tau_F w \tau_F^{-1} \quad .$$

An F-stable maximal torus $\tilde{T}$ of $\tilde{G}$ which is contained in F-stable Borel subgroup $\tilde{B}$ of $\tilde{G}$ will be called a <u>standard torus in $\tilde{G}$</u>. All standard tori in $\tilde{G}$ are conjugate under the action of G. For such $\tilde{T}$ and $\tilde{B}$ the corresponding system of fundamental roots Π is τ_F-stable and the F-stable parabolic subgroups of $\tilde{G}$ containing $\tilde{B}$ are precisely the $\tilde{P}_J$ for τ_F-stable subsets J of Π.

<u>1.7</u> Let $\mathcal{T}(\tilde{G})$ denote the set of maximal tori in $\tilde{G}$. $\mathcal{T}(\tilde{G})$ forms a single orbit under the action of $\tilde{G}$ ($\tilde{G}$ acting by conjugation). Suppose $\tilde{T}, \tilde{T}' \in \mathcal{T}(\tilde{G})^F$ and choose an element g in $\tilde{G}$ such that $\tilde{T}' = \mathrm{adg}\,\tilde{T}$. Set $F^* = (\mathrm{adg}^{-1})(F)$. Then

$$F^* = \mathrm{adg}^{-1} \circ F \circ \mathrm{adg} = \mathrm{ad}(g^{-1}F(g)) \circ F$$

whence $g^{-1}F(g) \in w$ for some element w in $\tilde{W}$ $(=\tilde{W}_{\tilde{T}})$. Thus $F^*|_{\tilde{T}} = \mathrm{adw} \circ F|_{\tilde{T}}$ and adg^{-1} carries T' $(=(\tilde{T}')^F)$ onto $\tilde{T}^{F^*}$. We say that $\tilde{T}'$ is <u>twisted by w relative to $\tilde{T}$</u>. Similarly $F^*_{\tilde{W}} = \mathrm{adw} \circ F_{\tilde{W}} = \mathrm{ad}(w\tau_F)$ and adg^{-1} carries $W_{\tilde{T}'}$ $(=\tilde{W}_{\tilde{T}'}{}^F)$ onto $\tilde{W}^{F^*}$.

Let $\mathcal{F}_T = \{\mathrm{adw} \circ F|_{\tilde{T}} \mid w \in \tilde{W}\}$. For w_1 in $\tilde{W}$ the functor adw_1 acts on both $\mathcal{F}_{\tilde{T}}$ and $W\tau_F$ so that $\mathcal{F}_{\tilde{T}}$ and $W\tau_F$ may be regarded as W-spaces. There is a canonical 1-1 correspondence of orbit spaces

$$(1.7.1) \qquad \mathcal{T}(\tilde{G})^F/G \leftrightarrow \mathcal{F}_{\tilde{T}}/\tilde{W} \leftrightarrow \tilde{W}\tau_F/\tilde{W} \quad .$$

Two elements w and w' in $\tilde{W}$ are <u>F-conjugate</u> if $w = w_1^{-1}w'F(w_1)$ for some element w_1 in $\tilde{W}$. The set of F-conjugacy classes in $\tilde{W}$ will be denoted by $\tilde{W}_F^{\natural}$. Thus by (1.7.1) there is a 1-1 correspondence

$$(1.7.2) \qquad \mathcal{T}(\tilde{G})^F/G \leftrightarrow \tilde{W}_F^{\natural}$$

which maps each G-orbit of maximal tori $\tilde{T}'$ in $\mathcal{T}(\tilde{G})^F$ to the set of elements in $\tilde{W}$ which twist $\tilde{T}'$ relative to $\tilde{T}$.

1.8 For each element g in $\tilde{G}$ we will denote by $Z_{\tilde{G}}(g)$ its centralizer in $\tilde{G}$ (i.e. $Z_{\tilde{G}}(g) = \tilde{G}^{adg}$) and by $Z^{\circ}_{\tilde{G}}(g)$ the connected component of the identity element in $Z_{\tilde{G}}(g)$. If $\tilde{T} \in T(\tilde{G})$, Σ is the set of roots relative to $\tilde{T}$ and $t \in \tilde{T}$ we set $\Sigma_t = \{r \in \Sigma | r(t) = 1\}$, $Z_{\tilde{W}_{\tilde{T}}}(t) = \{w \in \tilde{W}_{\tilde{T}} | adw(t) = t\}$ and $Z^{\circ}_{\tilde{W}_{\tilde{T}}}(t) = \langle w_r | r \in \Sigma_t \rangle$. For a Borel subgroup $\tilde{B}$ containing $\tilde{T}$ the subgroup $\tilde{B} \cap Z^{\circ}_{\tilde{G}}(t)$ is a Borel subgroup of $Z^{\circ}_{\tilde{G}}(t)$ and is generated by the subgroups $\tilde{T}$ and the $X_r(\tilde{k}^+)$ for r in Σ_t. Further $Z_{\tilde{G}}(t)$ is generated by $\tilde{B} \cap Z^{\circ}_{\tilde{G}}(t)$ and the w in $Z_{\tilde{W}_{\tilde{T}}}(t)$ while $Z^{\circ}_{\tilde{G}}(t)$ is generated by $\tilde{B} \cap Z^{\circ}_{\tilde{G}}(t)$ and the w in $Z^{\circ}_{\tilde{W}_{\tilde{T}}}(t)$. In particular $Z^{\circ}_{\tilde{G}}(t)$ is a reductive group and its Weyl group relative to $\tilde{T}$ is $Z^{\circ}_{\tilde{W}_{\tilde{T}}}(t)$.

1.9 Suppose $\tilde{T} \in T(\tilde{G})^F$, $\tilde{W} = \tilde{W}_{\tilde{T}}$ and s is a semisimple element of G. Choose $\tilde{T}'$ in $T(Z^{\circ}_{\tilde{G}}(s))^F$ and g in $\tilde{G}$ such that $\tilde{T}' = adg\tilde{T}$. Further set $n_w = g^{-1}F(g)$ where $w \in \tilde{W}$ and $n_w \in w$, set $F^* = adn_w \circ F$ and set $t = adg^{-1}(s)$. Then adg^{-1} carries $Z^{\circ}_{\tilde{G}}(s)$ $(=Z^{\circ}_{\tilde{G}}(s)^F)$ onto $Z^{\circ}_{\tilde{G}}(t)^{F^*}$ and the Weyl group $Z^{\circ}_{\tilde{W}_{\tilde{T}'}}(s)$ of $Z^{\circ}_{\tilde{G}}(s)$ relative to $\tilde{T}'$ onto $Z^{\circ}_{\tilde{W}}(t)$. Moreover if τ_{F^*} is the action on $\Sigma(=\Sigma_{\tilde{T}})$ induced by F^* then by (1.7.1) there is a canonical 1-1 correspondence

$$T(Z^{\circ}_{\tilde{G}}(s))^F / Z^{\circ}_G(s) \leftrightarrow Z^{\circ}_{\tilde{W}}(t)\tau_{F^*} / Z^{\circ}_{\tilde{W}}(t) \quad .$$

Thus if $\mathcal{O} \in T(\tilde{G})^F/G$ and corresponds under (1.7.2) to the element $\bar{\mathcal{O}}$ in $\tilde{W}_F^{\natural}$ then the classes in $\mathcal{O} \cap T(Z^{\circ}_{\tilde{G}}(s))^F$ under the action of $Z^{\circ}_G(s)$ are in 1-1 correspondence with the classes in $\bar{\mathcal{O}} \cap Z^{\circ}_{\tilde{W}}(t)w$ under the F-conjugacy action of $Z^{\circ}_{\tilde{W}}(t)$.

1.10 A semisimple group $\tilde{G}$ is simply connected if $X(\tilde{T})$ contains a $\mathbb{Z}$-basis dual to a set of fundamental coroots. For a connected reductive group $\tilde{G}$ whose derived group $[\tilde{G},\tilde{G}]$ is simply connected the centralizer of any semisimple element of $\tilde{G}$ is connected.

1.11 We denote by $C(\tilde{G})$ the set of conjugacy classes of $\tilde{G}$ and by $C_s(\tilde{G})$ and $C_u(\tilde{G})$ the sets of semisimple and unipotent conjugacy classes respectively. For each $\tilde{c}$ in $C(\tilde{G})$ we set $\tilde{c}_s = \{x_s | x \in \tilde{c}\}$ and $\tilde{c}_u = \{x_u | x \in \tilde{c}\}$. Then $\tilde{c}_s \in C_s(\tilde{G})$ and $\tilde{c}_u \in C_u(\tilde{G})$. Similar notation is used in respect of G. Again if $c \in C(G)$ then $c_s \in C_s(G)$ and $c_u \in C_u(G)$.

For each element g in $\tilde{G}$ we denote by $\tilde{c}(g)$ (or $\tilde{c}_{\tilde{G}}(g)$) the class in $C(\tilde{G})$ to which it belongs. Similarly for each g in G we denote by $c(g)$ (or $c_{\tilde{G}}(g)$) the G-conjugacy class of g.

If $c \in C(G)$ and $t \in c_s$ we set $c_u^{(t)} = \{x_u | x \in c,\ x_s = t\}$. Then $c_u^{(t)} \in C_u(Z_G(t))$ and the mapping $c \mapsto c_u^{(t)}$ defines a 1-1 correspondence between the set of elements c in $C(G)$ for which $t \in c_s$ and $C_u(Z_G(t))$.

1.12 If $\tilde{c} \in C(\tilde{G})^F$ then $\tilde{c}^F$ is non-void and for any element g in $\tilde{c}^F$ the G-conjugacy classes into which $\tilde{c}^F$ splits are in 1-1 correspondence with the elements of $\left(Z_{\tilde{G}}(g)/Z^{\circ}_{\tilde{G}}(g)\right)^{\natural}_F$ (for notation see 1.7). Let us assume that the derived subgroup $[\tilde{G},\tilde{G}]$ of G is simply connected. Then 1.10 implies the existence of a canonical bijection

(1.12.1) $$C_s(\tilde{G})^F \to C_s(G).$$

Suppose $\tilde{T} \in \mathcal{T}(\tilde{G})^F$ and $\tilde{W} = \tilde{W}_{\tilde{T}}$. There is an obvious canonical F-commuting bijection $\mathcal{C}_s(\tilde{G}) \to \tilde{T}/\tilde{W}$. Combining this with (1.12.1) we obtain a bijection in case $[\tilde{G},\tilde{G}]$ is simply connected

$$(\tilde{T}/\tilde{W})^F \to \mathcal{C}_s(G) \quad . \tag{1.12.2}$$

Under this bijection if $c \in (\tilde{T}/\tilde{W})^F$ and $t \in c$ then c maps to the element $\tilde{c}(t)^F$ of $\mathcal{C}_s(G)$.

<u>1.13</u> Let $\tilde{P}$ be a parabolic subgroup of $\tilde{G}$, let $\tilde{U}^+$ be the unipotent radical of $\tilde{P}$ and let $\tilde{\pi}$ be the canonical projection $\tilde{P} \to \tilde{P}/\tilde{U}^+$. Then, if $\tilde{T}$ is a maximal torus in $\tilde{P}$, $\tilde{\pi}(\tilde{T})$ is a maximal torus in $\tilde{P}/\tilde{U}^+$. Moreover $\tilde{\pi}$ induces a bijection of $\mathcal{C}_s(\tilde{P})$ onto $\mathcal{C}_s(\tilde{P}/\tilde{U}^+)$. If $\tilde{P}$ is F-stable then so is $\tilde{U}^+$. Set $P = \tilde{P}^F$, $U^+ = (\tilde{U}^+)^F$ and let π denote the canonical projection of P onto P/U^+. Then $(\tilde{P}/\tilde{U}^+)^F \cong P/U^+$ and π induces a bijection of $\mathcal{C}_s(P)$ onto $\mathcal{C}_s(P/U^+)$.

<u>1.14</u> An element g of $\tilde{G}$ is <u>regular in $\tilde{G}$</u> if the dimension of $Z_{\tilde{G}}(g)$ is minimal. Since $g \in Z^\circ_{\tilde{G}}(g_s)$ for any element g of $\tilde{G}$, it follows that g is regular in $\tilde{G}$ if and only if g_u is regular in $Z^\circ_{\tilde{G}}(g_s)$. In particular a semisimple element g of $\tilde{G}$ is regular if and only if $Z^\circ_{\tilde{G}}(g)$ is a torus (which is necessarily the unique maximal torus containing g). On the other hand, the regular unipotent elements of $\tilde{G}$ form a single conjugacy class in $\tilde{G}$.

<u>1.15</u> It will be convenient in the sequel to take certain liberties with the terminology so far introduced. For example we shall frequently refer to a maximal torus T of G where we mean that $T = \tilde{T}^F$ for some F-stable maximal torus $\tilde{T}$ of $\tilde{G}$ or to a parabolic subgroup P of G where

again we mean that $P = \tilde{P}^F$ for some F-stable parabolic subgroup $\tilde{P}$ of $\tilde{G}$. In such cases the meaning should be clear and cause no confusion.

<u>1.16</u> We turn now to the consideration of characters of G over the complex field $\mathbb{C}$. We denote by ch(G) the ring of generalized complex characters of G. The elements of ch(G) will also be regarded as defined on $C(G)$ in the obvious way. For elements ϕ and ψ in ch(G) their scalar product $(\phi,\psi)_G$ is defined as usual by

$$(\phi,\psi)_G = \frac{1}{|G|} \sum_{g\in G} \phi(g)\,\overline{\psi(g)}$$

and the norm $\|\phi\|_G$ as $(\phi,\phi)_G$. We denote by irr(G) the set of irreducible characters of G i.e. the set of elements ϕ in ch(G) for which $\|\phi\| = 1$ and deg $\phi(=\phi(1)) > 0$. A <u>proper character</u> of G is defined to be a <u>sum</u> of irreducible characters of G.

If P is a parabolic subgroup of G with unipotent radical U^+ and π is the canonical projection $P \to P/U^+$ then π induces an injection $ch(P/U^+) \to ch(P)$. For each element ϕ in $ch(P/U^+)$ we will normally denote by ϕ^* its image $\phi_\circ\pi$ under this injection and refer to ϕ^* as the <u>lift</u> of ϕ to P.

<u>1.17</u> An irreducible character χ of G is said to be in the <u>principal series</u> for G if it is a component of $\mathrm{Ind}_B^G(1)$ where B is a Borel subgroup of G. It is shown in [6] that, if W is the Weyl group of G relative to a standard torus (i.e. $\tilde{W}$ is the Weyl group of $\tilde{G}$ relative to a standard torus and $W = \tilde{W}^F$, *vide* 1.15), there is a bijective mapping $\phi \mapsto \chi_\phi$ of irr(W) onto the set of principal series

characters of G such that for each τ_F-stable subset J of Π (notation as in 1.6),

$$\left(\chi_\phi, \mathrm{Ind}_{P_J}^G(1)\right)_G = \left(\phi, \mathrm{Ind}_{W_J}^W(1)\right)_W \quad .$$

<u>1.18</u> An irreducible character χ of G is in the <u>discrete series</u> if for each proper parabolic subgroup P of G (i.e. $P \neq G$) the restriction $\chi|_P$ contains no irreducible component which can be obtained as the lift to P of an element in $\mathrm{irr}(P/U^+)$ (or, without loss of generality, in the discrete series of P/U^+). If J_1 and J_2 are τ_F-stable subsets of Π and $W(J_1,J_2)$ denotes the set of all elements in W which map J_1 onto J_2 then for each element w of $W(J_1,J_2)$ the mappings adw: $\tilde{T} \to \tilde{T}$ and adw: $X_r(\tilde{k}^+) \to X_{wr}(\tilde{k}^+)$ for roots r with support in J_1 induce an isomorphism $\hat{w}$: $P_{J_1}/U^+_{J_1} \to P_{J_2}/U^+_{J_2}$ where each $U^+_{J_i}$ is the unipotent radical of P_{J_i} for $i = 1,2$. Springer [18] has shown that if χ is a discrete series character of $P_{J_1}/U^+_{J_1}$ and ζ is a discrete series character of $P_{J_2}/U^+_{J_2}$ then

$$\left(\mathrm{Ind}_{P_{J_1}}^G(\chi^*), \mathrm{Ind}_{P_{J_2}}^G(\zeta^*)\right)_G = \sum_{w \in W(J_1,J_2)} (\chi, \zeta\circ\hat{w})_{P_{J_1}/U^+_{J_1}} \quad .$$

§2. DELIGNE-LUSZTIG THEORY

In this section we review some of the results in [8] for use in subsequent sections. The notations introduced in §1 will be used freely.

2.1 Suppose $\tilde{T} \in \mathcal{T}(\tilde{G})^F$ and $w \in \tilde{W}_{\tilde{T}}$. We write $\tilde{W}$ for $\tilde{W}_{\tilde{T}}$ and set $F^* = \mathrm{ad}w_\circ F|_{\tilde{T}}$. Because of the nature of the actions of F and w (see 1.4 and 1.6) $F^* = w_\circ F$ on $Y(\tilde{T})$ while on $X(\tilde{T})$, $F^* = F_\circ w^{-1}$.

Set $(\tilde{T}^{F^*})^\vee = \mathrm{Hom}(\tilde{T}^{F^*}, \mathbb{Q}/\mathbb{Z})$ and assume chosen a fixed embedding $k^* \to \mathbb{Q}/\mathbb{Z}$. Each element of $X(\tilde{T})$ can be regarded as an element of $\mathrm{Hom}(\tilde{T}, \mathbb{Q}/\mathbb{Z})$ which upon restriction to T^{F^*} yields an element of $(T^{F^*})^\vee$. Accordingly there is a short exact sequence

$$(2.1.1) \qquad 0 \to X(\tilde{T}) \xrightarrow{F^*-1} X(\tilde{T}) \to (\tilde{T}^{F^*})^\vee \to 0 .$$

If $\eta \in X(T)\otimes\mathbb{Q}/\mathbb{Z}$ we may choose an element $\hat{\eta}$ in $X(\tilde{T})\otimes\mathbb{Q}$ which maps under the canonical projection $X(\tilde{T})\otimes\mathbb{Q} \to X(\tilde{T})\otimes\mathbb{Q}/\mathbb{Z}$ to η. Then $(F^*-1)\eta = 0$ if and only if $(F^*-1)\hat{\eta} = \sum_i \theta_i \otimes m_i$ for elements θ_i in $X(\tilde{T})$ and integers m_i. Upon restriction to $\tilde{T}^{F^*}$ the element $\sum_i m_i\theta_i$ of $X(\tilde{T})$ yields an element of $(\tilde{T}^{F^*})^\vee$. Thus there is a short exact sequence

$$(2.1.2) \qquad 0 \to (\tilde{T}^{F^*})^\vee \to X(\tilde{T})\otimes\mathbb{Q}/\mathbb{Z} \xrightarrow{F^*-1} X(\tilde{T})\otimes\mathbb{Q}/\mathbb{Z} \to 0 .$$

2.2 Suppose $\tilde{T}' \in \mathcal{T}(\tilde{G})^F$, g is an element of $\tilde{G}$ such that $\tilde{T}' = \mathrm{ad}g\,\tilde{T}$ and $g^{-1}F(g) \in w$. Then by 1.7 there are isomorphisms

$$T' \to \tilde{T}^{F^*} : s \mapsto \mathrm{ad}g^{-1}(s)$$

and

$$(T')^\vee \to (\tilde{T}^{F^*})^\vee : \theta \mapsto \theta_\circ\, \mathrm{ad}g$$

(recall that $T' = (\tilde{T}')^F$, see 1.15).

Thus equation (2.1.2) may be written

$$(2.2.1)\quad 0 \to (T')^{\vee} \to X(\tilde{T})\otimes\mathbb{Q}/\mathbb{Z} \xrightarrow{F_\circ w^{-1}-1} X(\tilde{T})\otimes\mathbb{Q}/\mathbb{Z} \to 0.$$

It is apparent from (2.2.1) that an element η in $X(\tilde{T})\otimes\mathbb{Q}/\mathbb{Z}$ is the image of an element in $(T')^{\vee}$ for some $\tilde{T}'$ in $\mathcal{T}(\tilde{G})^F$ if and only if it lies in an F-stable $\tilde{W}$-orbit in $X(\tilde{T})\otimes\mathbb{Q}/\mathbb{Z}$. If $\tilde{T}_1, \tilde{T}_2 \in \mathcal{T}(\tilde{G})^F$, $\theta_1 \in T_1^{\vee}$ and $\theta_2 \in T_2^{\vee}$ the pairs $(\tilde{T}_1,\theta_1)$ and $(\tilde{T}_2,\theta_2)$ are <u>geometrically conjugate</u> if the images of θ_1 and θ_2 in $\tilde{X}(T)\otimes\mathbb{Q}/\mathbb{Z}$ under injections of the form (2.2.1) are in the same $\tilde{W}$-orbit. The set of geometric conjugacy classes of pairs $(\tilde{T}',\theta')$ where $\tilde{T}' \in \mathcal{T}(\tilde{G})^F$ and $\theta' \in (T')^{\vee}$ will be denoted by $\mathcal{S}$. Our discussion above sets up a bijection

$$(2.2.2)\qquad \mathcal{S} \to [(X(\tilde{T})\otimes\mathbb{Q}/\mathbb{Z})/\tilde{W}]^F \quad .$$

<u>2.3</u> Suppose $\theta' \in (T')^{\vee}$ and η is the image of θ' in $X(\tilde{T})\otimes\mathbb{Q}/\mathbb{Z}$ under (2.2.1). For $\tilde{T}_1$ in $\mathcal{T}(\tilde{G})^F$ let im $T_1^{\vee}$ denote the image of $T_1^{\vee}$ in $X(\tilde{T})\otimes\mathbb{Q}/\mathbb{Z}$ under an injection of the form (2.2.1) and suppose $\tilde{T}_1$ is twisted by w_1 relative to $\tilde{T}$. Then im $T_1^{\vee}$ contains an element in the $\tilde{W}$-orbit $\mathcal{O}_\eta$, say, of η if and only if w_1 is F-conjugate to an element in $Z_{\tilde{W}}(\eta)w$. Thus the G-conjugacy classes of F-stable maximal tori $\tilde{T}_1$ in $\tilde{G}$ for which there is an element θ_1 of $T_1^{\vee}$ such that $(\tilde{T}_1,\theta_1)$ is geometrically conjugate to $(\tilde{T}',\theta')$ are in 1-1 correspondence with the F-conjugacy classes in $\tilde{W}$ which intersect $Z_{\tilde{W}}(\eta)w$ (*cf.* 1.9).

Now $\mathrm{im}(T')^{\vee} = \ker(F^*-1)$ where $\ker(F^*-1)$ is the kernel of the mapping F^*-1 in (2.2.1). Thus, using 1.7, it is easily seen that the $\tilde{W}^{F^*}$-orbits into which $\mathcal{O}_\eta \cap \ker(F^*-1)$ splits are in 1-1 correspondence with the G-classes of pairs of the form $(\tilde{T}',\theta_1)$ where $\theta_1 \in (T')^\vee$ which are geometrically conjugate to $(\tilde{T}',\theta')$.

<u>2.4</u> A torus $\tilde{T}^*$ is said to be dual to $\tilde{T}$ if $X(\tilde{T}^*) = Y(\tilde{T})$ (whence $Y(\tilde{T}^*) = X(\tilde{T})$) and the k-structure of $\tilde{T}^*$ is defined by transferring the action of F on $X(\tilde{T})$ to $Y(\tilde{T}^*)$. A group $\tilde{G}^*$ is said to be dual to $\tilde{G}$ if it is a connected reducitve k-group containing a standard torus $\tilde{T}^*$ which is provided with an isomorphism to the dual of a standard torus $\tilde{T}$ of $\tilde{G}$, the isomorphism carrying the fundamental roots of $\tilde{G}^*$ to the fundamental coroots of $\tilde{G}$.

The mapping $(y,\alpha) \to y(\alpha)$ for y in $Y(\tilde{T})$ and α in $\tilde{k}^*$ induces an isomorphism $Y(\tilde{T})\otimes\tilde{k}^* \cong \tilde{T}$ and thus an injection

(2.4.1) $$\tilde{T} \to Y(\tilde{T})\otimes\mathbb{Q}/\mathbb{Z}$$

arising from the embedding $\tilde{k}^* \to \mathbb{Q}/\mathbb{Z}$. This yields not only the short exact sequence

(2.4.2) $$0 \to \tilde{T}^{F^*} \to Y(\tilde{T})\otimes\mathbb{Q}/\mathbb{Z} \xrightarrow{F^*-1} Y(\tilde{T})\otimes\mathbb{Q}/\mathbb{Z} \to 0$$

but also a bijection

(2.4.3) $$(\tilde{T}/\tilde{W})^F \to [(Y(\tilde{T})\otimes\mathbb{Q}/\mathbb{Z})/\tilde{W}]^F$$

Thus from (2.2.2) and 1.12 there is a canonical bijection

(2.4.4) $$C_s(\tilde{G}^*)^F \to S \ .$$

(*cf.* (1.12.1)).

If s is a semisimple element in T' and η is its image in $Y(\tilde{T})\otimes\mathbb{Q}/\mathbb{Z}$ under (2.4.2) after identification of T' with $\tilde{T}^{F*}$ then the $\tilde{W}^{F*}$-orbits into which $\bar{\mathcal{O}}_\eta \cap \ker(F^*-1)$ splits, where $\bar{\mathcal{O}}_\eta$ is the $\tilde{W}$-orbit of η in $Y(\tilde{T})\otimes\mathbb{Q}/\mathbb{Z}$, are in 1-1 correspondence with the $W_{\tilde{T}'}$-classes of elements in $T' \cap c_G(s)$ (*cf.* 1.9 and 2.3).

<u>2.5</u> We now relate the pairs $(\tilde{T},\theta)$ where $\tilde{T} \in T(\tilde{G})^F$ and $\theta \in T^\vee$ to characters of G. The mapping $\alpha \mapsto e^{2\pi\alpha i}$ of $\mathbb{Q}/\mathbb{Z}$ into $\mathbb{C}^*$ induces an isomorphism $T^\vee \to \mathrm{irr}(T)$ which we use to identify $T^\vee$ with irr(T). For each element θ in $T^\vee$ we denote by $\rho_{\tilde{T}}^\theta$ the character of the virtual representation $R_{\tilde{T}}^\theta$ of G which is constructed in 1.9 of [8]. The $R_{\tilde{T}}^\theta$ are constructed *loc. cit.* over the algebraic closure $\bar{\mathbb{Q}}_\ell$ of the ℓ-adic field for ℓ a prime different from p. But $\bar{\mathbb{Q}}_\ell$ and $\mathbb{C}$ contain a common splitting field for G so the $\rho_{\tilde{T}}^\theta$ may be regarded as elements of ch(G). We note that the $\rho_{\tilde{T}}^\theta$ depend only on the G-conjugacy class of the pair $(\tilde{T},\theta)$.

<u>2.6</u> The Green function $Q_{\tilde{T},G}$ is an integer valued class function defined on G for each G class of F-stable maximal tori $\tilde{T}$ as follows. For each element g in G,

$$Q_{\tilde{T},G}(g) = \begin{cases} \rho_{\tilde{T}}^1(g) & \text{if } g \text{ is unipotent} \\ 0 & \text{otherwise} \end{cases} .$$

The importance of the Green functions is demonstrated by the following result (theorem 4.2 *loc. cit.*) If $\tilde{T} \in T(\tilde{G})^F$ and $\theta \in T^\vee$ then

$$(2.6.1) \qquad \rho_{\tilde{T}}^\theta(x) = \frac{1}{|Z_G^\circ(x_s)|} \sum_{\substack{g\in G \\ \mathrm{ad}g\tilde{T}\subseteq Z_{\tilde{G}}^\circ(x_s)}} Q_{\mathrm{ad}g\tilde{T},Z_G^\circ(x_s)}(x_u)[(\mathrm{ad}g)(\theta)](x_s)$$

for each element x in G, where of course
$[(\mathrm{ad}g)(\theta)](\dot{x}_s) = \theta((\mathrm{ad}g^{-1})x_s)$.

If $\tilde{T},\tilde{T}' \in \mathcal{T}(\tilde{G})$ we set

$$N_{\tilde{G}}(\tilde{T},\tilde{T}') = \{g\in\tilde{G} \mid \tilde{T} = \mathrm{ad}g\ \tilde{T}'\} \quad .$$

If $\tilde{T},\tilde{T}' \in \mathcal{T}(\tilde{G})^F$ we also set

$$N_G(\tilde{T},\tilde{T}') = N_{\tilde{G}}(\tilde{T},\tilde{T}')^F \quad .$$

Then,

(2.6.2) $$\frac{1}{|G|}\sum_{u\in G} Q_{\tilde{T},G}(u)\, Q_{\tilde{T}',G}(u) = \frac{|N_G(\tilde{T},\tilde{T}')|}{|T|\ |T'|}$$

<u>2.7</u> St_G will denote the Steinberg character of G. If we denote by $|G|_p$ the order of $\tilde{U}^F$ for some F-stable maximal unipotent subgroup $\tilde{U}$ of $\tilde{G}$ then according to [21] the value of St_G for any element g of G is given by

(2.7.1) $$St_G(g) = \begin{cases} (-1)^{\sigma(\tilde{G})-\sigma(Z^\circ_{\tilde{G}}(g))}\,|Z^\circ_G(g)|_p & \text{if } g \text{ is semisimple} \\ 0 & \text{otherwise} \end{cases} .$$

The following are true.

(2.7.2) $$Q_{\tilde{T},G}(1) = (-1)^{\sigma(\tilde{G})-\sigma(\tilde{T})}\ \frac{|G|}{St_G(1)\,|T|}$$

(2.7.3) $$\frac{1}{|G|}\sum_{u\in G} Q_{\tilde{T},G}(u) = \frac{1}{|T|}$$

(2.7.4) $Q_{\tilde{T},G}(u) = 1$ if u is a regular unipotent element of G.

If $\tilde{\bar{G}}$ is a connected reductive group with Frobenius endomorphism F then an isogeny $g \to \bar{g}$ of $\tilde{G}$ onto $\tilde{\bar{G}}$ commuting with F induces an isomorphism of the varieties of unipotent elements and a bijection $\tilde{T} \to \tilde{\bar{T}}$ of $\mathcal{T}(\tilde{G})^F$ onto $\mathcal{T}(\tilde{\bar{G}})^F$. Then

$$Q_{\tilde{T},G}(u) = Q_{\tilde{\bar{T}},\bar{G}}(u) \tag{2.7.5}$$

for all unipotent elements u in G.

2.8 If $\tilde{G}_1$ and $\tilde{G}_2$ are two F-stable connected reductive subgroups of $\tilde{G}$ such that $\tilde{G} = \tilde{G}_1 \times \tilde{G}_2$, any element $\tilde{T}$ in $\mathcal{T}(\tilde{G})^F$ can be written $\tilde{T}_1 \times \tilde{T}_2$ where $\tilde{T}_1 \in \mathcal{T}(\tilde{G}_1)^F$ and $\tilde{T}_2 \in \mathcal{T}(\tilde{G}_2)^F$. Moreover, each θ in $T^\vee$ can be written $\theta_1 \times \theta_2$ where $\theta_1 \in T_1^\vee$ and $\theta_2 \in T_2^\vee$. Then

$$\rho_{\tilde{T}}^{\theta} = \rho_{\tilde{T}_1}^{\theta_1} \times \rho_{\tilde{T}_2}^{\theta_2} \tag{2.8.1}$$

and

$$Q_{\tilde{T},G} = Q_{\tilde{T}_1,G_1} \times Q_{\tilde{T}_2,G_2} . \tag{2.8.2}$$

Let P be a parabolic subgroup of G with unipotent radical U^+, let $\tilde{T}$ be a member of $\mathcal{T}(\tilde{G})^F$ such that $\tilde{T} \le \tilde{P}$ and let θ be an element of $T^\vee$. Then (see 1.13) $\tilde{\pi}$ maps $\tilde{T}$ isomorphically onto $\tilde{\pi}(\tilde{T})$ and π maps T isomorphically onto $\pi(T)$ carrying θ to an element $\bar{\theta}$, say, of $\pi(T)^\vee$. Then

$$\rho_{\tilde{T}}^{\theta} = \mathrm{Ind}_P^G\left(\left(\rho_{\tilde{\pi}(\tilde{T})}^{\bar{\theta}}\right)^*\right) . \tag{2.8.3}$$

2.9 If $\tilde{T},\tilde{T}' \in \mathcal{T}(\tilde{G})^F$, $\theta \in T^\vee$ and $\theta' \in (T')^\vee$ then no element of irr(G) can occur as a component in both $\rho_{\tilde{T}}^{\theta}$ and $\rho_{\tilde{T}'}^{\theta'}$ unless the pairs $(\tilde{T},\theta)$ and $(\tilde{T}',\theta')$ are geometrically conjugate.

We define $\tilde{W}(\tilde{T},\tilde{T}') = \tilde{T}\backslash N_{\tilde{G}}(\tilde{T},\tilde{T}') = N_{\tilde{G}}(\tilde{T},\tilde{T}')/\tilde{T}'$ and $W(\tilde{T},\tilde{T}') = \tilde{W}(\tilde{T},\tilde{T}')^F$. Then $W(\tilde{T},\tilde{T}') = T\backslash N_G(\tilde{T},\tilde{T}') = N_G(\tilde{T},\tilde{T}')/T'$ and the following is true.

(2.9.1) $$(\rho_{\tilde{T}}^{\theta},\rho_{\tilde{T}'}^{\theta'})_G = \#\{w\in W(\tilde{T},\tilde{T}') \mid (\mathrm{ad}w)(\theta') = \theta\} \quad .$$

2.10 We note the following important property of the $\rho_{\tilde{T}}^{\theta}$. If $\chi \in \mathrm{irr}(G)$ then for some element $\tilde{T}$ in $\mathcal{T}(\tilde{G})^F$ and some θ in $T^\vee$

(2.10.1) $$(\chi,\rho_{\tilde{T}}^{\theta})_G \neq 0 \quad .$$

An element χ of irr(G) is said to be unipotent if $(\chi,\rho_{\tilde{T}}^{1})_G \neq 0$ for some $\tilde{T}$ in $\mathcal{T}(\tilde{G})^F$. If T is a standard torus in G then $\rho_{\tilde{T}}^{1} = \mathrm{Ind}_B^G(1)$ for a Borel subgroup B of G which implies that all principal series characters of G are unipotent.

If χ is a unipotent character of G and s is a semisimple element of G then

(2.10.2) $$\chi(s) = \frac{1}{\mathrm{St}_G(s)} \sum_{\tilde{T}\in\mathcal{T}(Z^{\circ}_{\tilde{G}}(s))^F} (-1)^{\sigma(\tilde{G})-\sigma(\tilde{T})}(\chi,\rho_{\tilde{T}}^{1})_G \quad .$$

In particular if s is regular and contained in T

(2.10.3) $$\chi(s) = (\chi,\rho_{\tilde{T}}^{1})_G \ .$$

Finally,

(2.10.4) $$\chi(1) = \frac{1}{\mathrm{St}_G(1)} \sum_{\tilde{T}\in\mathcal{T}(\tilde{G})^F} (-1)^{\sigma(\tilde{G})-\sigma(\tilde{T})}(\chi,\rho_{\tilde{T}}^{1})_G \quad .$$

2.11 The following are true.

$$(2.11.1) \qquad 1_G = \sum_{(\tilde{T})\in T(G)^F/G} \frac{1}{|\tilde{W}_{\tilde{T}}^F|} \rho_{\tilde{T}}^1$$

and

$$(2.11.2) \qquad St_G = \sum_{(\tilde{T})\in T(\tilde{G})^F/G} \frac{(-1)^{\sigma(\tilde{G})-\sigma(\tilde{T})}}{|\tilde{W}_{\tilde{T}}^F|} \rho_{\tilde{T}}^1$$

where in both cases the sum is taken over the G-orbits in $T(\tilde{G})^F$. In particular for any $\tilde{T}$ in $T(\tilde{G})^F$

$$(2.11.3) \qquad (1_G, \rho_{\tilde{T}}^1)_G = 1$$

and

$$(2.11.4) \qquad (St_G, \rho_{\tilde{T}}^1)_G = (-1)^{\sigma(\tilde{G})-\sigma(\tilde{T})} \quad .$$

These results are extended in [8] in the case where the centre $Z(\tilde{G})$ of $\tilde{G}$ is connected. As before S denotes the set of geometric conjugacy classes of pairs $(\tilde{T},\theta)$. For each element x in S we define

$$(2.11.5) \qquad \rho_x = \sum_{[(\tilde{T},\theta)]\in x/G} \frac{(-1)^{\sigma(\tilde{G})-\sigma(\tilde{T})}}{\|\rho_{\tilde{T}}^\theta\|} \rho_{\tilde{T}}^\theta$$

and

$$(2.11.6) \qquad \rho_x' = (-1)^{\sigma(\tilde{G})-\delta_x} \sum_{[(\tilde{T},\theta)]\in x/G} \frac{1}{\|\rho_{\tilde{T}}^\theta\|} \rho_{\tilde{T}}^\theta$$

where the sums are taken over the G-conjugacy classes $[(\tilde{T},\theta)]$ of pairs $(\tilde{T},\theta)$ in x and δ_x denotes the k-rank of the centralizer of a semisimple element in the dual group $\tilde{G}^*$ of $\tilde{G}$ which corresponds under the bijection (2.4.3) with the element x in S. Then ρ_x and $\rho'_x \in \mathrm{irr}(G)$ for all x in S.

Let ℓ be the semisimple rank of $\tilde{G}$ and let Δ_G be the function defined on G by

$$(2.11.7) \qquad \Delta_G(g) = \begin{cases} |Z(G)|\, q^{\ell} & \text{if g is regular unipotent} \\ 0 & \text{otherwise} \end{cases}$$

Then $\Delta_G \in \mathrm{ch}(G)$ and

$$(2.11.8) \qquad \Delta_G = \sum_{x \in S} (-1)^{\sigma(\tilde{G})-\delta_x} \rho'_x = \sum_{[(\tilde{T},\theta)]} \frac{1}{\|\rho_{\tilde{T}}^{\theta}\|} \rho_{\tilde{T}}^{\theta}$$

where the final sum is over all G-conjugacy classes $[(\tilde{T},\theta)]$ of pairs $(\tilde{T},\theta)$ with $\tilde{T}$ in $T(\tilde{G})^F$ and θ in $T^{\vee}$.

§3. THE SYMPLECTIC GROUP

Our aim is now to set down explicitly some of the structure of the symplectic group in the light of §§1 and 2. Throughout this section $\tilde{G}$ will denote the group $Sp(\tilde{V})$ defined below.

3.1 V will denote a finite dimensional vector space over k. We extend scalars to obtain the space $\tilde{V}$ $(=\tilde{k}\otimes_k V)$ and a Frobenius endomorphism F on $\tilde{V}$. We denote by $GL(\tilde{V})$, $GL(m,\tilde{k})$, $GL(V)$ and $GL(m,q)$ the appropriate groups of non-singular linear transformations on $\tilde{V}$ or V where $m = \dim V$. Clearly $GL(V) = GL(\tilde{V})^F$.

Suppose now that V is endowed with a fixed non-degenerate alternating bilinear form (,). Then $m = 2n$ for some integer n and $Sp(V)$ or $Sp(2n,q)$ will denote the group of (,)-invariant linear automorphisms of V. Extending (,) to $\tilde{V}$ allows us to define $Sp(\tilde{V})$ or $Sp(2n,\tilde{k})$ similarly and obtain $Sp(V)$ as $Sp(\tilde{V})^F$.

3.2 A hyperbolic pair in V is an ordered pair of elements e, d in V such that $(e,d) = 1$. We denote by $\binom{e}{d}$ a basis of orthogonal hyperbolic pairs in V i.e. a basis $e_1,\ldots,e_n,d_1,\ldots,d_n$ of V such that $(e_i,e_j) = (d_i,d_j) = 0$ and $(e_i,d_j) = \delta_{ij}$ for $i,j = 1,\ldots,n$ where δ_{ij} is the Kronecker delta.

The set of transformations in $\tilde{G}$ which are diagonal with respect to $\binom{e}{d}$ is a standard torus $\tilde{T}$ in $\tilde{G}$. Henceforth this definition of $\tilde{T}$ will apply unless specific reference is made to the contrary. For each t in $\tilde{T}$ we can write $te_i = x_i(t)e_i$ for $i = 1,\ldots,n$. Then $x_1,\ldots,x_n$ is a $\mathbb{Z}$-basis for $X(\tilde{T})$. If for each element α in k^* we denote by $y_i(\alpha)$ the element in $\tilde{G}$ which fixes the e_j and d_j for $j \neq i$ and maps e_i and d_i to αe_i and $\alpha^{-1}d_i$ respectively then $y_1,\ldots,y_n$ is a $\mathbb{Z}$-basis of $Y(\tilde{T})$ dual to $x_1,\ldots,x_n$.

<u>3.3</u> The Weyl group $\tilde{W}(=\tilde{W}_{\tilde{T}})$ acts faithfully on the sets $\{\pm x_1,\ldots,\pm x_n\}$ and $\{\pm y_1,\ldots,\pm y_n\}$ as the group of permutations with arbitrary sign changes. We will therefore frequently identify the elements of $\tilde{W}$ with their cycle decompositions. Note that these cylces can be of two types. A cycle will be called a <u>negative cycle</u> if it contains the same symbol twice with opposite signs and will be called a <u>positive cycle</u> otherwise.

<u>3.4</u> The isomorphism $Y(\tilde{T}) \to X(\tilde{T})$ which maps y_i to x_i for $i = 1,\ldots,n$ is clearly $\tilde{W}$ invariant. Thus the positive definite symmetric bilinear form $(\ ,\ )$ on $E(=X(\tilde{T})\otimes\mathbb{R})$ for which $x_1,\ldots,x_n$ is an orthonormal basis is $\tilde{W}$-invariant.

<u>3.5</u> Relative to the ordering $x_1 < x_2 < \ldots < x_n$ on $X(\tilde{T})$ the set $\Pi = \{p_1,\ldots,p_n\}$ where $p_i = x_{i+1} - x_i$ for $i = 1,\ldots,n-1$ and $p_n = 2x_n$ is a system of fundamental roots so $\tilde{G}$ corresponds to the Dynkin diagram of type C_n. The fundamental coroots $p_1^*,\ldots,p_n^*$ (where $p_i^* = \dfrac{2p_i}{(p_i,p_i)}$ for $i = 1,\ldots,n$) are dual to the basis $-x_1,-x_1-x_2,\ldots,-x_1-\ldots-x_{n-1},\ x_1+x_2+\ldots+x_n$ of $X(\tilde{T})$. Thus $\tilde{G}$ is a semisimple simply connected group. We note further that when p is odd $|Z(\tilde{G})| = 2$ while when $p = 2$, $|Z(\tilde{G})| = 1$.

3.6 The Frobenius endomorphism F acts on $\tilde{T}$ simply by raising each element to its qth power and so acts on $X(\tilde{T})$ and $Y(\tilde{T})$ by multiplication by q. Thus the identification of $Y(\tilde{T})$ with $X(\tilde{T})$ in 3.4 identifies $\tilde{T}$ as its own dual (see 2.4). Clearly τ_F acts trivially on Σ and F acts trivially on $\tilde{W}$. In particular $W = \tilde{W}$ and $W_F^\natural = C(W)$.

3.7 The conjugacy classes in W correspond to ordered pairs (μ,ν) of partitions such that $|\mu| + |\nu| = n$ (see e.g. Carter [4]) the cycle structure of positive and negative cycles being given by μ and ν respectively. Thus by 1.7 the G-classes in $\mathcal{T}(\tilde{G})^F$ are also indexed by such ordered pairs (μ,ν). We will denote by $\tilde{T}_{\mu,\nu}$ a representative element in the G-class in $\mathcal{T}(\tilde{G})^F$ which is indexed by (μ,ν). We wish to determine the structure of $T_{\mu,\nu}$. By 1.7, if w is an element of W with cycle structure (μ,ν) and $F^* = (\mathrm{ad}w)_\circ F|_{\tilde{T}}$ then this is equivalent to determining the structure of $\tilde{T}^{F^*}$.

3.8 We use (2.4.1) to determine the structure of $\tilde{T}^{F^*}$. Clearly the cycle decomposition of w yields an F^*-invariant direct sum decomposition of $Y(\tilde{T})\otimes\mathbb{Q}/\mathbb{Z}$ and so a direct product decomposition of $\tilde{T}^{F^*}$. We are therefore reduced to the case in which w consists of a single cycle of the form $(12\ldots n)$ or $(12\ldots n{-}1{-}2\ldots{-}n)$.

Let $\varepsilon_w = \pm 1$ according as the cycle is positive or negative. Then an element $\sum_{i=1}^{n} y_i\otimes a_i$ in $Y(\tilde{T})\otimes\mathbb{Q}/\mathbb{Z}$ is in the kernel of F^*-1 if and only if $a_i = q^{i-1}a$ for $i = 1,\ldots,n$ where a is an element of $\mathbb{Q}/\mathbb{Z}$ such that $(q^n-\varepsilon_w)a = 0$. Thus

$$\tilde{T}^{F^*} \cong (q^n-\varepsilon_w)^{-1}\mathbb{Z}/\mathbb{Z} \quad . \tag{3.8.1}$$

3.9 We continue to assume that w consists of a single positive or negative cycle of order n. As in 3.8 an element $\sum_{i=1}^{n} x_i \otimes \alpha_i$ in $X(\tilde{T})\otimes\mathbb{Q}/\mathbb{Z}$ is in the kernel of F^*-1 if and only if $\alpha_i = q^{i-1}\alpha_1$ for $i = 1,\ldots,n$ and $(q^n-\varepsilon_w)\alpha_1 = 0$. We use (2.1.2) and the preceding discussion to determine $(\tilde{T}^{F^*})^\vee$. We choose α in $\mathbb{Q}$ such that $\alpha = \alpha_1 \bmod \mathbb{Z}$ and set $\beta = (q^n-\varepsilon_w)\alpha$. Then $\beta \in \mathbb{Z}$ and $(F^*-1)(\sum_{i=1}^{n} x_i\otimes q^{i-1}\alpha) = x_1\otimes\beta$. We will denote by $\hat{\beta}$ the restriction of the element βx_1 in $X(\tilde{T})$ to $\tilde{T}^{F^*}$. Now if we use (3.8.1) to identify $\tilde{T}^{F^*}$ with $(q^n-\varepsilon_w)^{-1}\mathbb{Z}/\mathbb{Z}$ then

(3.9.1) $$\hat{\beta}(a) = \beta a \bmod \mathbb{Z}$$

for each element a in $\tilde{T}^{F^*}$. Note that since α was determiend modulo $\mathbb{Z}$, β is determined modulo $(q^n-\varepsilon_w)\mathbb{Z}$ so β may be regarded as an element of $\mathbb{Z}/(q^n-\varepsilon_w)\mathbb{Z}$. Thus the mapping $\beta \mapsto \hat{\beta}$ is an isomorphism

(3.9.2) $$\mathbb{Z}/(q^n-\varepsilon_w)\mathbb{Z} \cong (\tilde{T}^{F^*})^\vee .$$

3.10 By 1.7, the determination of the structure of $W_{\mu,\nu}(=\tilde{W}^F_{\tilde{T}_{\mu,\nu}})$ and its action on $\tilde{T}^\vee_{\mu,\nu}$ is equivalent to the determination of the structure of $\tilde{W}^{F^*}$ and its action on $(\tilde{T}^{F^*})^\vee$. Now, for w consisting of a single cycle of order n it is clear that $\tilde{W}^{F^*} = \langle w\rangle$ if $\varepsilon_w = -1$ and $\tilde{W}^{F^*} = \langle w,w_0\rangle$ when $\varepsilon_w = +1$ where w_0 is the element (1-1)(2-2)...(n-n) of W. Now since $w^{-1}(\sum_{i=1}^{n} y_i\otimes q^{i-1}a) = q(\sum_{i=1}^{n} y_i\otimes q^{i-1}a)$ when $(q^n-\varepsilon_w)a = 0$ we see that $w\hat{\beta} = q\hat{\beta}$ for all β in $\mathbb{Z}/(q^n-\varepsilon_w)\mathbb{Z}$. Similarly $w_0\hat{\beta} = -\hat{\beta}$. Elements in $(\tilde{T}^{F^*})^\vee$ in the same $\tilde{W}^{F^*}$-orbit will be called <u>conjugate</u> characters.

In table 1 of the appendix the information in 3.8, 3.9 and 3.10 is summarized for cycles where $p = 2$ and $n \leq 3$. Also contained in this table is a standard notation for the characters of $\tilde{T}^{F*}$ which will be used in later sections. The mappings $\mathbb{Z}/(q-1)\mathbb{Z} \overset{*}{\to} (q+1)\mathbb{Z}/(q^2-1)\mathbb{Z}$ etc. shown in this table are the canonical bijections obtained by multiplying a representative in $\mathbb{Z}$ by q+1 etc.

3.11 Now let w be an arbitrary element in $\tilde{W}$. The subgroup of $\tilde{W}^{F*}$ which fixes each of the subgroups of $\tilde{T}^{F*}$ corresponding to the cycle decomposition of w is a normal subgroup of $\tilde{W}^{F*}$ and is a direct product of subgroups of the type discussed in 3.10. Modulo this subgroup, $\tilde{W}^{F*}$ acts as the full group of permutations on each of the sets of subgroups of $\tilde{T}^{F*}$ which correspond to the cycles of w of a given type. In particular, if w has a cycle decomposition (μ,ν) where $\mu = (1^{m_1},2^{m_2},\ldots)$ and $\nu = (1^{n_1},2^{n_2},\ldots)$ then $|\tilde{W}^{F*}| = \prod_i (2i)^{m_i+n_i} m_i!n_i!$.

In table 2 (of the appendix) are listed for $p = 2$ and $n \leq 3$ a representative from each conjugacy class in $\tilde{W}$, its cycle type (μ,ν), the structure of $\tilde{T}^{F*}(\cong T_{\mu,\nu})$ and a representative θ of each $\tilde{W}^{F*}$-conjugacy class of elements in $(\tilde{T}^{F*})^{\vee}$ together with the subgroup $Z_{\tilde{W}^{F*}}(\theta)$ of $\tilde{W}^{F*}$ which fixes θ. The other entries appearing in this table will be explained later. By 1.7 and (2.9.1) it follows that

$$\|\rho^{\theta}_{\tilde{T}_{\mu,\nu}}\| = |Z_{\tilde{W}^{F*}}(\theta)|$$

for each ordered pair (μ,ν) of partitions and element θ in $T^{\vee}_{\mu,\nu}$. In table 2 the notation $\rho^{\theta}_{\tilde{T}_{\mu,\nu}}$ is abbreviated to $\rho^{\theta}_{\mu,\nu}$. We will also often conform to this practice in the text.

3.12 We denote by $t(a_1,\dots,a_n)$ the element of $\tilde{T}$ which maps under the injection (2.4.1) to the element $\sum_{i=1}^{n} y_i \otimes a_i$ of $Y(\tilde{T})\otimes\mathbb{Q}/\mathbb{Z}$. Further, $\tilde{c}(a_1,\dots,a_n)$ will denote the conjugacy class in $\tilde{G}$ to which $t(a_1,\dots,a_n)$ belongs. Then (from 3.3) $\tilde{c}(a_1,\dots,a_n) = \tilde{c}(b_1,\dots,b_n)$ if and only if $b_i = \pm a_i'$ for $i = 1,\dots,n$ where the a_i' are some rearrangement of the a_i.

Since $\tilde{G}$ is simply connected (3.5) it follows from (1.12.2) and (2.4.3) that the semisimple conjugacy classes of G are of the form $c(a_1,\dots,a_n)$ $(=\tilde{c}(a_1,\dots,a_n)^F)$ for all $a_1,\dots,a_n$ in $\mathbb{Q}/\mathbb{Z}$ such that $\sum_{i=1}^{n} y_i\otimes a_i$ is in some F-stable $\tilde{W}$-orbit in $Y(\tilde{T})\otimes\mathbb{Q}/\mathbb{Z}$. Thus $c(a_1,\dots,a_n)$ represents an element of $C_s(G)$ is and only if it can be written in the form

$$(3.12.1)\qquad c(a_1,\dots,a_1,qa_1,\dots,qa_1,\dots,q^{r_1}a_1,\dots,q^{r_1}a_1,a_2,\dots,a_2,\dots)$$

where there are m_1 elements $q^i a_1$ for $i = 1,\dots,r_1$, m_2 elements $q^i a_2$ for $i = 1,\dots,r_2$ etc., where r_i is the least non-negative integer such that $(q^{r_i+1}+1)a_i \in \mathbb{Z}$ or $(q^{r_i+1}-1)a_i \in \mathbb{Z}$ and if $S_i = \{\pm a_i, \pm qa_i,\dots,\pm q^{r_i}a_i\}$ for $i = 1,\dots$ then $S_i \cap S_j = \emptyset$ for $i \neq j$.

Comparison of (2.1.2) with (2.4.2) taking into account the self-duality of $\tilde{T}$ (see 3.6) shows that the determination of the geometric conjugacy classes in $\tilde{G}$ is similar to the determination of the semisimple conjugacy classes in G. Using 2.3 we can also determine the G-conjugacy classes of pairs $(\tilde{T}',\theta')$ in each geometric conjugacy class. These are shown in table 3 of the appendix for

$p = 2$ and $n \leq 3$. For example, table 3 shows that for $n = p = 2$ there is a geometric conjugacy class denoted by $[\hat{\alpha}\times\hat{\alpha}]$ (one for each class of conjugate characters $\hat{\alpha}$ shown in table 1) which contains two G-conjugacy classes with representatives $(\tilde{T}_{(1^2),(0)},\hat{\alpha}\times\hat{\alpha})$ and $(\tilde{T}_{(2),(0)},\hat{\alpha}^*)$ (appropriate identifications having been made). Note that no geometric conjugacy class in table 3 contains two pairs $(\tilde{T}',\theta')$, $(\tilde{T}',\theta'')$ where $\theta',\theta'' \in (T')^\vee$ belonging to distinct G-classes. This however is not true for $n = 4$. As for table 2 we defer the explanation of the other entries in this table.

3.13 It is easily seen that the centralizer of an element in the class (3.12.1) in $\tilde{G}$ splits into a direct product thereby reducing the determination of the centralizer to the case of an element in

$$(3.13.1) \qquad c = c(a,\ldots,a,qa,\ldots,qa,\ldots,q^r a,\ldots,q^r a)$$

where each $q^i a$ occurs m times for $i = 1,\ldots,r$ with r the least non-negative integer such that $(q^{r+1}+1)a \in \mathbb{Z}$ or $(q^{r+1}-1)a \in \mathbb{Z}$ and $mr = n$.

Set $t = t(a,\ldots,a,qa,\ldots,qa,\ldots,q^r a,\ldots,q^r a)$. If $s \in c$ we can choose an element g in $\tilde{G}$ such that $s = (\mathrm{ad}g)(t)$ in which case $g^{-1}F(g) = n_w \in w$ for some element w of W (see 1.7). By 1.9, $Z_G(s) = (\mathrm{ad}g)\, Z_{\tilde{G}}(t)^{F^*}$ where $F^* = \mathrm{ad}n_w \circ F$. We now determine $Z_{\tilde{G}}(t)^{F^*}$. There are three cases:-

Case 1. $2a \in \mathbb{Z}$. Then $r = 0$, $t \in Z(\tilde{G})$ and we may take $F^* = F$. Thus $Z_{\tilde{G}}(t)^{F^*} = G$. This disposes of the first case.

Suppose $2a \notin \mathbb{Z}$, set $\tilde{V}_i' = \langle e_{m(i-1)+1},\ldots,e_{mi}\rangle_{\tilde{k}}$ and $\tilde{V}_i'' = \langle d_{m(i-1)+1},\ldots,d_{mi}\rangle_{\tilde{k}}$ for $i = 1,\ldots,r$. Then $Z_{GL(\tilde{V})}(t) = \prod_{i=1}^{r} [GL(\tilde{V}_i') \times GL(\tilde{V}_i'')]$. For convenience we can identify $\tilde{V}_1'$ with each $\tilde{V}_i'$ and with each $\tilde{V}_i''$ by identifying e_j with e_{im+j} and d_{im+j} for $i = 0,\ldots,r-1$ and $j = 1,\ldots,m$. Now $\tilde{V}_i'$ and $\tilde{V}_i''$ are in duality under (,) for $i = 1,\ldots,r$ so each element g in $GL(\tilde{V}_i')$ has a contragredient $g^{\dagger}$, say. Moreover, it follows that $Z_{\tilde{G}}(t) = \prod_{i=1}^{r} \tilde{G}_i$ where $\tilde{G}_i = \{(g,g^{\dagger}) \mid g \in GL(\tilde{V}_i')\}$.

<u>Case 2</u>: $2a \notin \mathbb{Z}$ but $(q^{r+1}-1)a \in \mathbb{Z}$. Let n_w be the element of $\tilde{G}$ which maps $e_{(i-1)m+j}$ to e_{im+j} and $d_{(i-1)m+j}$ to d_{im+j} for $i = 1,\ldots,r-1$ and $j = 1,\ldots,m$ and maps $e_{(r-1)m+j}$ to e_j and $d_{(r-1)m+j}$ to d_j for $j = 1,\ldots,m$. If $F^* = \mathrm{ad}\,n_w \circ F$ then $F^*(t) = t$ so $Z_G(s)$ is indeed isomorphic to $Z_G(t)^{F^*}$. But the elementsof $Z_{\tilde{G}}(t)^{F^*}$ have the form $((g,g^{\dagger}),(F(g),F(g)^{\dagger}),\ldots,(F^r(g),F^r(g)^{\dagger}))$ with $F^{r+1}(g) = g$ and g in $GL(\tilde{V}_1')$. Thus $Z_G(s) \cong GL(m,q^{r+1})$.

<u>Case 3</u>: $2a \notin \mathbb{Z}$ but $(q^{r+1}+1)a \in \mathbb{Z}$. Let n_w be the element of $\tilde{G}$ which maps $e_{(i-1)m+j}$ to e_{im+j} and $d_{(i-1)m+j}$ to d_{im+j} for $i = 1,\ldots,r-1$ and $j = 1,\ldots,m$ and maps $e_{(r-1)m+j}$ to d_j and $d_{(r-1)m+j}$ to $-e_j$ for $j = 1,\ldots,m$. If $F^* = \mathrm{ad}\,n_w \circ F$ then $F^*(t) = (t)$ whence $Z_G(s)$ is isomorphic to $Z_{\tilde{G}}(t)^{F^*}$. But in this case the elements of $Z_{\tilde{G}}(t)^{F^*}$ have the same form as in case 2 except that $F^{r+1}(g) = g^{\dagger}$. Thus in this case $Z_G(s) \cong U(m,q^{2(r+1)})$.

<u>3.14</u> We require a parametrization of $T(Z_{\tilde{G}}(s))^F/Z_G(s)$. For case 1 of 3.13 this has been dealt with in 3.7. We now consider cases 2 and 3. Since $\tilde{G}_i \cong GL(\tilde{V}_i')$ for each i, its Weyl group is precisely S_m,

the symmetric group on m symbols. Thus the Weyl group of $Z_{\tilde{G}}(t)$ is S_m^r. Now the injection $w \mapsto w \times 1 \times \ldots \times 1$ of S_m into S_m^r induces a bijection $(S_m)^{\natural}_{(F^*)^r} \to (S_m^r)^{\natural}_{F^*}$ so by (1.7.1) and (1.7.2) there are bijections

$$T(Z_{\tilde{G}}(s))^F/Z_G(s) \to (S_m^r)^{\natural}_{F^*} \to (S_m)^{\natural}_{(F^*)^r} \to S_m\,\tau_{(F^*)^r}/S_m \quad .$$

Now for case 2, $\tau_{(F^*)^r} = 1$ while for case 3, $\tau_{(F^*)^r} = (1-1)(2-2)\ldots(m-m)$. In either case $S_m\tau_{(F^*)^r}/S_m$ is in 1-1 correspondence with $C(S_m)$. Thus the orbits in $T(Z_{\tilde{G}}(s))^F/Z_G(s)$ are indexed by a single partition μ of m corresponding to the cycle type of the corresponding conjugacy class of S_m.

In table 4 of the appendix is a list for $p = 2$ and $n \leq 3$ of the elements $\tilde{c}$ in $C_s(\tilde{G})^F$ together with $Z_{\tilde{G}}(t)$ and $Z_G(s)$ for representative elements t in $\tilde{T} \cap \tilde{c}$ and s in c. Moreover, for each $Z_{\tilde{G}}(s)$, parameters corresponding to the orbits in $T(Z_{\tilde{G}}(s))^F/Z_G(s)$ are shown together with the parameters for the orbit in $T(\tilde{G})^F/G$ in which each such orbit is contained. From this table or from 1.9 we find that for $p = 2$ and $n \leq 3$, each orbit in $T(\tilde{G})^F/G$ contains at most one orbit in $T(Z_{\tilde{G}}(s))^F/Z_G(s)$ for each semisimple element s of G. This result is not true for $n = 4$ (*cf.* remarks on table 3 in 3.12).

<u>3.15</u> If $V_0 \leq V$ we denote its (,)-orthogonal space by $V_0^{\perp}$. V_0 is said to be totally isotropic if $V_0 \leq V_0^{\perp}$. A totally isotropic chain $\underline{\underline{c}}$: $(0) < V_1 < \ldots < V_r$ in V is a sequence of nested totally isotropic subspaces of V. We let $\underline{\underline{C}}(V)$ denote the set of totally isotropic chains in V. There is an obvious identification $\underline{\underline{C}}(V) = \underline{\underline{C}}(\tilde{V})^F$. We define the rank $r_{\underline{\underline{c}}}$ of the chain $\underline{\underline{c}}$: $(0) < V_1 < \ldots < V_r$ to be r and the type $\nu_{\underline{\underline{c}}}$ of the chain to be the

sequence of numbers ($\dim V_1$, $\dim V_2/V_1,\ldots,\dim V_r/V_{r-1}$). The groups G and $\tilde{G}$ act as permutation groups on $\underline{\underline{C}}(V)$ and $\underline{\underline{C}}(\tilde{V})$ respectively. Under these actions the totally isotropic chains fall into orbits according to their type.

3.16 If $\underline{\underline{c}}$: $(0) < \tilde{V}_1 < \ldots < \tilde{V}_r$ is in $\underline{\underline{C}}(\tilde{V})$ we denote by $\tilde{P}_{\underline{\underline{c}}}$ the set of elements in $\tilde{G}$ fixing $\underline{\underline{c}}$. The action on $\tilde{V}$ of an element in $\tilde{P}_{\underline{\underline{c}}}$ induces linear transformations on the spaces $\tilde{V}_1, \tilde{V}_2/\tilde{V}_1, \ldots, \tilde{V}_r/\tilde{V}_{r-1}$ and $\tilde{V}_r^{\perp}/\tilde{V}_r^{\perp}$. Since $\tilde{V}_r$ is totally isotropic (,) induces a non-degenerate form (,) on $\tilde{V}_r^{\perp}/\tilde{V}_r$ and we denote by $\widetilde{Sp}(\underline{\underline{c}})$ the group $Sp(\tilde{V}_r^{\perp}/\tilde{V}_r)$ relative to this form. Further we set $\widetilde{GL}(\underline{\underline{c}}) = GL(\tilde{V}_1) \times GL(\tilde{V}_2/\tilde{V}_1) \times \ldots \times GL(\tilde{V}_r/\tilde{V}_{r-1})$. There is a well defined projection $\tilde{\pi}_{\underline{\underline{c}}}: \tilde{P}_{\underline{\underline{c}}} \to \widetilde{GL}(\underline{\underline{c}}) \times \widetilde{Sp}(\underline{\underline{c}})$.

3.17 If $\Pi = \{p_1,\ldots,p_n\}$ is the system of fundamental roots in 3.5 and if ν is the sequence of numbers $(m_1,\ldots,m_r)$ where $\sum_{i=1}^{r} m_i \leq n$ we define $J_\nu = \Pi \sim \{p_{m_1}, p_{m_1+m_2}, \ldots, p_{m_1+\ldots+m_r}\}$. Then $\tilde{P}_{\underline{\underline{c}}}$ is a parabolic subgroup of $\tilde{G}$ conjugate to $\tilde{P}_{J_{\nu_{\underline{\underline{c}}}}}$ with unipotent radical $\ker \tilde{\pi}_{\underline{\underline{c}}}$. In cases where $\nu = \nu_{\underline{\underline{c}}}$ but the particular chain $\underline{\underline{c}}$ in question is unimportant we will simply write $\tilde{P}_\nu$ for $\tilde{P}_{\underline{\underline{c}}}$. Similarly we write W_ν for the parabolic subgroup W_{J_ν} of W. If $\underline{\underline{c}} \in \underline{\underline{C}}(V)$ then $\tilde{P}_{\underline{\underline{c}}}$ is F-stable and we set $P_{\underline{\underline{c}}} = \tilde{P}_{\underline{\underline{c}}}^F$. We will denote by $\pi_{\underline{\underline{c}}}$ the projection $P_{\underline{\underline{c}}} \to GL(\underline{\underline{c}}) \times Sp(\underline{\underline{c}})$ obtained by restriction of $\tilde{\pi}_{\underline{\underline{c}}}$ to $P_{\underline{\underline{c}}}$. Then $\ker \pi_{\underline{\underline{c}}} = (\ker \tilde{\pi}_{\underline{\underline{c}}})^F$.

§4. THE ORTHOGONAL SUBGROUPS IN CHARACTERISTIC 2.

Henceforth we will assume that $p = 2$. To assist with the determination of the unipotent characters of G ($=Sp(2n,q)$) we establish in this section some properties of the orthogonal subgroups of G. As a reference to some of the basic properties of these groups we cite Dieudonné [9].

4.1 We denote by $\mathcal{Q}$ the set of quadratic forms on V with polar form $(\ ,\ )$, i.e. $\mathcal{Q}$ consists of those functions $Q\colon V \to k$ satisfying the conditions

$$(4.1.1) \qquad Q(v_1+v_2) = Q(v_1) + Q(v_2) + (v_1,v_2)$$

and

$$(4.1.2) \qquad Q(\alpha v_1) = \alpha^2 Q(v_1)$$

for all α in k and v_1,v_2 in V.

Let $\binom{e}{d}$ be a fixed basis of orthogonal hyperbolic pairs in V. We write $k_\circ = \{\gamma+\gamma^2 \mid \gamma\in k\}$. If for each element Q in $\mathcal{Q}$ we define its pseudo-discriminant

$$(4.1.3) \qquad \Delta(Q) = \sum_{i=1}^{n} Q(e_i)Q(d_i) \quad \text{modulo } k_\circ ,$$

then Δ is a well defined function $\mathcal{Q} \to k^+/k_\circ$ independent of particular basis $\binom{e}{d}$ of orthogonal hyperbolic pairs used in its definition. This can be restated another way. If we define for each Q in $\mathcal{Q}$ and g in G the function Q^g by setting $Q^g(v) = Q(gv)$ for all v in V then G permutes the elements of $\mathcal{Q}$ in two orbits (for $n \geq 1$) distinguished by the values of Δ on $\mathcal{Q}$.

4.2 For each Q in $\mathcal{Q}$ we define O(Q), the orthogonal group of Q, as the set of linear automorphisms of V which leave Q invariant. The subgroups O(Q) fall into two conjugacy classes under the action of G. We write $O_+ = O_+(2n,q)$ and $O_- = O_-(2n,q)$ for typical groups corresponding to these two cases i.e. corresponding to O(Q) for $\Delta(Q) = 0$ and $\Delta(Q) \neq 0$ respectively.

4.3 For each Q in $\mathcal{Q}$ the Dickson invariant of Q is a homomorphism $D_Q\colon O(Q) \to k^+$ defined by

(4.3.1)

$$D_Q(g) = \sum_{i,j=1}^{n} \Big[(ge_i,d_j)(gd_i,d_j)Q(e_j) + (ge_i,e_j)(gd_i,e_j)Q(d_j) + (ge_i,e_j)(gd_i,d_j)\Big]$$

for all elements g in O(Q). This definition is independent of the chosen basis $\binom{e}{d}$ of orthogonal hyperbolic pairs used (see [9]). We note that for $n \geq 1$ im $D_Q = \{0,1\}$.

4.4 Lemma. If $Q \in \mathcal{Q}$ and $g \in G$ then

$$D_Q^g = D_{Q^g} \text{ (as functions on } O(Q)^g = O(Q^g)) \quad .$$

Proof. Applying g to the elements of $\binom{e}{d}$ yields another basis of orthogonal hyperbolic pairs for V. We evaluate D_Q with respect to this basis using (4.3.1). Thus if $g' \in O(Q)^g$

$$D_Q^g(g') = D_Q(gg'g^{-1})$$

$$= \sum_{i,j=1}^{n} \Big[(gg'e_i, gd_j)(gg'd_i, gd_j)Q(ge_j)$$

$$+ (gg'e_i, ge_j)(gg'd_i, ge_j)Q(gd_j) + (gg'e_i, ge_j)(gg'd_i, gd_j) \Big]$$

$$= D_{Q^g}(g')$$

since (,) is g-invariant.

4.5 Lemma. For each Q in $\mathcal{Q}$ and v in V let Q_v be the function $V \to k$ defined by

$$Q_v(v') = Q(v') + (v, v')^2$$

for all v' in V. Then the mapping $Q_\circ : v \mapsto Q_v$ defines a bijection $V \to \mathcal{Q}$ such that

(4.5.1) $$Q_\circ(gv) = Q_\circ(v)^{g^{-1}}$$

for all g in O(Q). Moreover

(4.5.2) $$\Delta(Q_\circ(v)) = \Delta(Q) + Q(v) \quad \text{modulo } k_\circ$$

for each v in V.

Proof. A simple verification shows that Q_v is indeed a quadratic form on V with polar form (,) whence $Q_\circ$ is well defined. The injectivity of $Q_\circ$ is a result of the non-degeneracy of (,). Now if $Q' \in \mathcal{Q}$ then $\sqrt{Q + Q'}$ is a linear form on V and so can be expressed as the mapping $v' \mapsto (v, v')$ of V into k for a suitable element v of V. Then $Q' = Q_\circ(v)$ and the bijectivity of $Q_\circ$ follows.

Suppose $g \in O(Q)$. Then for each v' in V

$$Q_{gv}(v') = Q(v') + (gv,v')^2 = Q(g^{-1}v') + (v,g^{-1}v')^2$$
$$= Q_v(g^{-1}v') = Q_v^{g^{-1}}(v') \ .$$

This implies that $Q_\circ(gv) = Q_\circ(v)^{g^{-1}}$ as claimed.

The final assertion follows immediately if $v = 0$. Thus we assume that $v \neq 0$ and suppose that the basis $\binom{e}{d}$ of orthogonal hyperbolic pairs is chosen so that $v = e_1$. Then working modulo $k_\circ$ we have from (4.1.3)

$$\begin{aligned} \Delta(Q_v) &= \sum_{i=1}^{n} Q_v(e_i)Q_v(d_i) \\ &= \sum_{i=1}^{n} \left[Q(e_i) + (v,e_i)^2\right]\left[Q(d_i) + (v,d_i)^2\right] \\ &= \sum_{i=1}^{n} Q(e_i)Q(d_i) + Q(e_1) \\ &= \Delta(Q) + Q(v). \end{aligned}$$

This completes the proof.

<u>4.6</u> <u>Lemma</u>. Suppose $Q,Q' \in \mathcal{Q}$ and let $S = O(Q) \cap O(Q')$. Then $D_Q|_S = D_{Q'}|_S$.

<u>Proof</u>. Using the previous lemma we set $v = Q_\circ^{-1}(Q')$. If $v = 0$ i.e. if $Q = Q'$ then the desired result follows immediately. We now suppose $v \neq 0$ and choose a basis $\binom{e}{d}$ of orthogonal hyperbolic pairs in V such that $v = e_1$. Since $Q' = Q_v$ a simple evaluation yields the

equations $Q(e_i) + Q'(e_i) = 0$ for $i = 1,\dots,n$, $Q(d_i) + Q'(d_i) = 0$ for $i = 2,\dots,n$ and $Q(d_1) + Q'(d_1) = 1$. Thus for g in S we find from (4.5.1) that $ge_1 = e_1$ whence substituting in (4.3.1) we obtain

$$D_Q(g) + D_{Q'}(g) = \sum_{i=1}^{n} (ge_i,e_1)(gd_i,e_1) = 0$$

This completes the proof.

<u>4.7</u> <u>Lemma</u>. Suppose $Q,Q' \in \mathcal{Q}$, let $g_1,\dots,g_s$ be elements of G such that

$$G = O(Q)g_1O(Q') \cup \dots \cup O(Q)g_sO(Q')$$

is the $(O(Q),O(Q'))$-double coset decomposition of G and set $S_i = O(Q)^{g_i} \cap O(Q')$ for $i = 1,\dots,s$.

(i) If $\Delta(Q) \neq \Delta(Q')$ then $s = q/2$.

(ii) If $\Delta(Q) = \Delta(Q')$ and $n \geq 2$ then $s = q/2 + 1$.

(iii) Under the conditions of either (i) or (ii) $D_{Q'}|_{S_i} \neq 0$ for $i = 1,\dots,s$.

(iv) If $\Delta(Q) = \Delta(Q')$ and $n = 1$ then

(a) in case $\Delta(Q) = 0$, $s = q/2 + 1$ and $D_{Q'}|_{S_i} = 0$ for precisely one value of i,

and (b) in case $\Delta(Q) \neq 0$, $s = q/2$ and $D_{Q'}|_{S_i} \neq 0$ for $i = 1,\dots,s$.

<u>Proof</u>. Let Q^G denote the orbit of Q in $\mathcal{Q}$ under the action of G. There is a canonical 1-1 correspondence of the $(O(Q),O(Q'))$-double cosets in G with the $O(Q')$-orbits in Q^G. Moreover if Q_i is an

element in the orbit in Q^G which corresponds to the double coset $O(Q)g_iO(Q')$ then S_i is the stabilizer in $O(Q')$ of Q_i. Now by lemma 4.5 if we set $v = [Q_o']^{-1}(Q)$ then Q_o' is an $O(Q')$-orbit preserving bijection of $\{v' \in V | Q'(v') = Q(v) \bmod k_o\}$ onto Q^G. Thus the $(O(Q),O(Q'))$-double cosets of G are in 1-1 correspondence with the $O(Q')$-orbits in $\{v' \in V | Q'(v') \notin k_o\}$ when $\Delta(Q) \neq \Delta(Q')$ and with the $O(Q')$-orbits in $\{v' \in V | Q'(v') \in k_o\}$ when $\Delta(Q) = \Delta(Q')$ such that in either case if v_i is an element in the orbit corresponding to the double coset $O(Q)g_iO(Q')$ then S_i is the stabilizer in $O(Q')$ of the point v_i.

Now in $\{v' \in V | Q'(v') \notin k_o\}$ there are $q/2$ $O(Q')$-orbits corresponding to the $q/2$ values outside k_o which Q' can assume on this set. $\{0\}$ is an $O(Q')$-orbit in $\{v' \in V | Q'(v') \in k_o\}$ and there are $q/2-1$ others corresponding to the non-zero values in k_o. Finally $\{v' \in V | Q'(v') \in k_o\}$ contains an orbit of Q'-isotropic points except when $n = 1$ and $\Delta(Q') \neq 0$. This completes the evaluation of s.

Let S_i be the stabilizer in $O(Q')$ of a point v_i in V. If v_i' is an element of $\langle v_i \rangle^{\perp}$ such that $Q'(v_i') \neq 0$ then the transvection $g: V \to V$, defined by $gv'' = v'' + Q'(v_i')^{-1}(v'', v_i')v_i'$ for each v'' in V, fixes Q' and v_i and so is an element of S_i. Moreover $D_{Q'}(g) = 1$. This shows that $D_{Q'}|_{S_i} \neq 0$ in this case. The remaining case is when all elements of $\langle v_i \rangle^{\perp}$ are Q'-isotropic i.e. when $n = 1$ and $Q'(v_i) = 0$. But this situation can only occur when $\Delta(Q') = 0$ and then only for v_i in a single orbit in $\{v' \in V | Q'(v') \in k_o\}$. In this case we find that $S_i = 1$. This completes the proof.

4.8 We return now to the consideration of totally isotropic chains in V. For $\underline{\underline{c}}$: $(0) < V_1 < \ldots < V_r$ in $\underline{\underline{C}}(V)$ and Q in $\mathcal{Q}$ we define $\eta^Q_{\underline{\underline{c}}}$ to be the integer i in the range 0 to r such that $Q|_{V_i} = 0$ but $Q|_{V_{i+1}} \neq 0$. For an ordered sequence ν of numbers $(m_1,\ldots,m_r)$ we define the order $|\nu|$ of ν as $m_1 +\ldots+ m_r$ and the rank $r(\nu)$ of ν as r. Clearly $r(\nu_{\underline{\underline{c}}}) = r_{\underline{\underline{c}}}$ for all $\underline{\underline{c}}$ in $\underline{\underline{C}}(V)$.

4.9 Lemma. The O(Q)-orbits of isotropic chains $\underline{\underline{c}}$ in V are distinguished by the values $\nu_{\underline{\underline{c}}}$ and $\eta^Q_{\underline{\underline{c}}}$. Moreover the values $\nu_{\underline{\underline{c}}}$ and $\eta_{\underline{\underline{c}}}$ are subject only to the restrictions $|\nu_{\underline{\underline{c}}}| \leq n$ and $\eta^Q_{\underline{\underline{c}}} \leq r_{\underline{\underline{c}}}$ except in the case when $\Delta(Q) \neq 0$ and $|\nu_{\underline{\underline{c}}}| = n$ when the further restriction $\eta^Q_{\underline{\underline{c}}} < r_{\underline{\underline{c}}}$ applies .

Proof. Clearly two totally isotropic chains $\underline{\underline{c}}$ and $\underline{\underline{c}}'$ in V cannot be in the same O(Q)-orbit unless $\nu_{\underline{\underline{c}}} = \nu_{\underline{\underline{c}}'}$ and $\eta^Q_{\underline{\underline{c}}} = \eta^Q_{\underline{\underline{c}}'}$. If $\underline{\underline{c}}$ is the chain $(0) < V_1 < \ldots < V_r$ then $Q|_{V_r}$ is a semilinear transformation $V_r \to k$. Now if $Q|_{V_r} \neq 0$ we put $K = \ker Q|_{V_r}$ and choose a basis $e_1,\ldots,e_s$, where $s = |\nu_{\underline{\underline{c}}}| - 1$, of K compatible with the sequence $(0) \leq V_1 \cap K \leq \ldots \leq V_r \cap K = K$. Further, we choose an element u from $V_{\eta^Q_{\underline{\underline{c}}}+1}$ such that $Q(u) = 1$. If we follow the same procedure for $\underline{\underline{c}}'$ to obtain $e_1',\ldots,e_s'$ and u' then the mapping $e_i \mapsto e_i'$ for $i = 1,\ldots,s$ and $u \mapsto u'$ extends by Cahit Arf's theorem (see e.g. [26]) to an element of O(Q) which carries $\underline{\underline{c}}$ to $\underline{\underline{c}}'$. The case $Q|_{V_r} = 0$ is similar. Clearly for any totally isotropic chain $\underline{\underline{c}}$ in V the conditions

$|\nu_{\underline{\underline{c}}}| \leq n$ and $n^Q_{\underline{\underline{c}}} \leq r_{\underline{\underline{c}}} = r(\nu_{\underline{\underline{c}}})$ must apply. Now suppose we are given a sequence ν such that $|\nu| \leq n$ and a non-negative integer η such that $\eta \leq r(\nu)$. We exclude for the moment the case where $|\nu| = n$, $\eta = r(\nu)$ and $\Delta(Q) \neq 0$. We can choose a basis $\binom{e}{d}$ of orthogonal hyperbolic pairs in V such that $Q(e_i) = Q(d_i) = 0$ for $i = 1,\ldots,n-1$ and, when $\Delta(Q) = 0$, such that $Q(e_n) = Q(d_n) = 0$ also. Thus there is an obvious totally isotropic chain $\underline{\underline{c}}$ in V such that $\nu_{\underline{\underline{c}}} = \nu$ and $n^Q_{\underline{\underline{c}}} = \eta$. Finally, the case excluded is impossible for if $|\nu_{\underline{\underline{c}}}| = n$ and $n^Q_{\underline{\underline{c}}} = r_{\underline{\underline{c}}}$ then $\dim V_r = n$ and $Q|_{V_r} = 0$ whence $\Delta(Q) = 0$.

This completes the proof.

<u>4.10</u> <u>Lemma</u>. Suppose $Q \in \mathcal{Q}$, $\underline{\underline{c}}$ is the totally isotropic chain $(0) < V_1 < \ldots < V_r$ in V and $S_{\underline{\underline{c}}}$ is the stabilizer of $\underline{\underline{c}}$ in $O(Q)$.

(i) If $n^Q_{\underline{\underline{c}}} < r_{\underline{\underline{c}}}$, $D_Q|_{\ker\pi_{\underline{\underline{c}}} \cap S_{\underline{\underline{c}}}} \neq 0$.

(ii) If $n^Q_{\underline{\underline{c}}} = r_{\underline{\underline{c}}}$, $\pi_{\underline{\underline{c}}}$ maps $S_{\underline{\underline{c}}}$ onto $GL(\underline{\underline{c}}) \times O(\bar{Q})$ where $\bar{Q}$ is the quadratic form induced by Q on $V_r^{\perp}/V_r$. Further, if $\bar{D}_{\bar{Q}}$ is the Dickson invariant on $O(\bar{Q})$ regarded as a function on $GL(\underline{\underline{c}}) \times O(\bar{Q})$ by lifting via the projection $GL(\underline{\underline{c}}) \times O(\bar{Q}) \to O(\bar{Q})$ then $D_Q|_{S_{\underline{\underline{c}}}} = \bar{D}_{\bar{Q}} \circ \pi_{\underline{\underline{c}}}$.

<u>Proof</u>. (i) Choose an element v in V_r such that $Q(v) \neq 0$ and let g be the transvection $v' \mapsto v' + Q(v)^{-1}(v',v)v$ for all v' in V. An easy computation then yields the result that $g \in S_{\underline{\underline{c}}}$, $\pi_{\underline{\underline{c}}}(g) = 1$ and $D_Q(g) = 1$ which establishes the first case.

(ii) That $\pi_{\underline{\underline{c}}}$ maps $S_{\underline{\underline{c}}}$ onto $GL(\underline{\underline{c}}) \times O(\bar{Q})$ can be obtained by an application of Cahit Arf's theorem. Now let $\binom{e}{d}$ be a basis of orthogonal hyperbolic pairs in V chosen so that $V_r = \langle e_1,\ldots,e_s \rangle$ where $s = |\nu_{\underline{\underline{c}}}|$. Then for any g in $S_{\underline{\underline{c}}}$ we have $(ge_i,e_j) = 0$ for $i,j = 1,\ldots,s$ and since $\eta_{\underline{\underline{c}}} = r_{\underline{\underline{c}}}$ we have further that $Q(e_j) = 0$ for $j = 1,\ldots,s$. This yields by substitution into (4.3.1)

$$D_Q(g) = \sum_{i,j=m+1}^{n} \Big[(ge_i,d_j)(gd_i,d_j)Q(e_j) + (ge_i,e_j)(gd_i,e_j)Q(d_j) + (ge_i,e_j)(gd_i,d_j)\Big]$$

and the final statement of the lemma follows.

§5. UNIPOTENT CHARACTERS

Again we assume that $G = Sp(2n,q)$ where q is even. Induction from subgroups of G allows us to identify all the unipotent characters of G for $n \leq 3$. Further, each of the virtual characters $\rho_{\tilde{T}}^{1}$ is identified as a combination of these unipotent characters. This lays the groundwork for the explicit determination of the Green functions in the next section.

5.1 According to 3.3 the Weyl group W of G acts as a set of permutations on n objects with sign changes. If we denote by N the subgroup of W which fixes each object apart from sign changes then N is a normal subgroup of W and $W/N \cong S_n$, the symmetric group on n symbols. The irreducible characters of S_n were determined by Frobenius [13] and are indexed by partitions as are the conjugacy classes of S_n. For each partition ν of n we will denote by ψ_ν the corresponding element in $irr(S_n)$. To avoid possible confusion between the character associated with a partition and that associated with its dual we will specify that $\psi_{(n)}$ denotes the principal character on S_n while $\psi_{(1^n)}$ is the alternating character. The isomorphism $W/N \cong S_n$ allows us to lift each character ψ_ν of S_n to a character ψ_ν^* of W.

5.2 We consider now the representation of W on the Euclidean space E of 3.4. For $m \leq n$ we denote by W_m the group of permutations with sign changes acting on the subspace $\langle x_1,\ldots,x_m\rangle$ of E and by W_m' the group of permutations with sign changes acting on

$<x_{n-m+1},\ldots,x_n>$. Then $W_m \times W'_{n-m}$ is a subgroup of W with $W_m/W_m \cap N \cong S_m$ and $W'_{n-m}/W'_{n-m} \cap N \cong S_{n-m}$.

Let δ'_m be the determinant character of the representation of W'_m on $<x_{n-m+1},\ldots,x_n>$ and set $\varepsilon'_m = \delta'_m.\psi'_{(1^n)}$ where ψ'_ν denotes the lift of ψ_ν from S_m to W'_m. Then ε'_m takes the value -1 on any single sign change in W'_m and the value +1 on any transposition in W'_m. Finally for each ordered pair (μ,ν) of partitions such that $|\mu| + |\nu| = n$ we define

$$\phi_{\mu,\nu} = \operatorname{Ind}_{W_{|\mu|}\times W'_{|\nu|}}^{W} (\psi^*_\mu \times \psi'_\nu.\varepsilon'_{|\nu|})$$

where ψ^*_μ denotes the lift of ψ_μ from $S_{|\mu|}$ to $W_{|\mu|}$. Then the $\phi_{\mu,\nu}$ are uniquely determined by the (μ,ν) and form a complete set of irreducible characters of W. Since

$$\phi_{\mu,\nu}(1) = \frac{n!}{|\mu|!|\nu|!}\psi_\mu(1)\psi_\nu(1)$$

it follows immediately that the linear characters of W are precisely $\phi_{(n),(0)}$ $(=1_W)$, $\phi_{(1^n),(0)}$ $(=\psi^*_{(1^n)})$, $\phi_{(0),(n)}$ $(=\delta'_n\psi^*_{(1^n)})$ and $\phi_{(0),(1^n)}$ $(=\delta'_n)$.

The characters of W for $n \leq 3$ are written out explicitly in table 5 of the appendix.

5.3 By 1.17 there is a 1-1 correspondence between the ordered pairs of partitions (μ,ν) such that $|\mu| + |\nu| = n$ and the characters $\chi_{\mu,\nu}$ in the principal series of G, this correspondence having the property that for each subset J of a system of fundamental roots Π for G as in 3.5

$$(\chi_{\mu,\nu}, \mathrm{Ind}_{P_J}^{G}(1))_G = (\phi_{\mu,\nu}, \mathrm{Ind}_{W_J}^{W}(1))_W .$$

<u>5.4</u> <u>Lemma</u>. If $J = \{p_2, \ldots, p_n\}$ then

$$\mathrm{Ind}_{P_J}^{G}(1) = \chi_{(n),(0)} + \chi_{(n-1),(1)} + \chi_{(1,n-1),(0)}$$

<u>Proof</u>. $W_J N = W_{n-1} \times W_1'$ so that

$$\mathrm{Ind}_{W_J}^{W_J N}(1) = 1_{W_J N} + 1_{W_{n-1}} \times \varepsilon_1' .$$

Now $1_{W_{n-1}} \times \varepsilon_1' = \psi^*_{(n-1)} \times \psi'_{(1)} \cdot \varepsilon_1'$ which induces to $\phi_{(n-1),(1)}$.

Further

$$\mathrm{Ind}_{W_J N}^{W}(1) = \left[\mathrm{Ind}_{S_{n-1}}^{S_n}(1)\right]^* = \left[\psi_{(n)} + \psi_{(1,n-1)}\right]^*$$

$$= \phi_{(n),(0)} + \phi_{(1,n-1),(0)} .$$

The result now follows from 5.3.

<u>5.5</u> <u>Lemma</u>. If $\chi \in \mathrm{irr}(G)$ and $(\chi, \mathrm{Ind}_B^G(1))_G = 1$ then $\chi = \chi_{(n),(0)}$, $\chi_{(1^n),(0)}$, $\chi_{(0),(n)}$ or $\chi_{(0),(1^n)}$.

<u>Proof</u>. From 1.17, χ corresponds to an element ϕ in irr(W) such that $(\phi, \mathrm{Ind}_1^W(1))_W = 1$ so that ϕ is a linear character of W. The result now follows from the remarks in 5.2 and 5.3.

<u>5.6</u> <u>Lemma</u>. Suppose $J = \{p_1, \ldots, p_{n-1}\}$. Then

$$\mathrm{Ind}_{P_J}^{G}(1) = \sum_{i=0}^{n} \chi_{(i),(n-i)} .$$

Proof. Since, for each i, $N \leq W_i \times W'_{n-i}$ and $W_J N = W$ there is precisely one $(W_J, W_i \times W'_{n-i})$-double coset in W. Moreover $W_J \cap W_i \times W'_{n-i} = S_i \times S_{n-i}$. Thus by Mackey's theorem

$$(\mathrm{Ind}_{W_J}^{W}(1), \phi_{(i),(n-i)})_W = (1, \psi^*_{(i)} \times \psi'_{(n-i)}\varepsilon'_{n-i})_{S_i \times S_{n-i}}$$

$$= (1,1)_{S_i \times S_{n-i}} = 1 .$$

But $W = W_J \cup W_J(1-1)W_J \cup \ldots \cup W_J(1-1)(2-2)\ldots(n-n)W_J$ so again by Mackey's theorem $\|\mathrm{Ind}_{W_J}^{W}(1)\|_W = n + 1$. Thus $\mathrm{Ind}_{W_J}^{W}(1) = \sum_{i=0}^{n} \phi_{(i),(n-i)}$ and the result follows from 5.3.

5.7 We now turn to the consideration of some characters induced from orthogonal subgroups of G. As shown in 4.2 there are two distinct conjugacy classes of such subgroups in G for which we denote representatives by $O_+(=O_+(2n,q))$ and $O_-(=O_-(2n,q))$. For any Q in $\mathcal{Q}$ we will denote by $\varepsilon_{O(Q)}$ the complex character of O(Q) obtained in the obvious way from the Dickson invariant D_Q: $O(Q) \to k^+$ (see 4.3).

5.8 Lemma.

(i) $\|\mathrm{Ind}_{O_+}^{G}(1)\| = \|\mathrm{Ind}_{O_+}^{G}(\varepsilon)\| = q/2 + 1.$

(ii) $\|\mathrm{Ind}_{O_-}^{G}(1)\| = \|\mathrm{Ind}_{O_-}^{G}(\varepsilon)\| = q/2 + 1$ unless n=1 in which case $\|\mathrm{Ind}_{O_-}^{G}(1)\| = \|\mathrm{Ind}_{O_-}^{G}(\varepsilon)\| = q/2$.

(iii) $(\mathrm{Ind}_{O_+}^{G}(1), \mathrm{Ind}_{O_-}^{G}(1)) = (\mathrm{Ind}_{O_+}^{G}(\varepsilon), \mathrm{Ind}_{O_-}^{G}(\varepsilon)) = q/2$.

(iv) $(\mathrm{Ind}_{O_+}^G(1), \mathrm{Ind}_{O_-}^G(\varepsilon)) = (\mathrm{Ind}_{O_-}^G(1), \mathrm{Ind}_{O_+}^G(\varepsilon))$

$$= (\mathrm{Ind}_{O_-}^G(1), \mathrm{Ind}_{O_-}^G(\varepsilon)) = 0 .$$

(v) $$(\mathrm{Ind}_{O_+}^G(1), \mathrm{Ind}_{O_+}^G(\varepsilon)) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1 . \end{cases}$$

<u>Proof</u>. Let $\tau = 1_{O(Q)}$ or $\varepsilon_{O(Q)}$ and $\tau' = 1_{O(Q')}$ or $\varepsilon_{O(Q')}$. We may apply Mackey's theorem and lemma 4.4 to obtain

$$(\mathrm{Ind}_{O(Q)}^G(\tau), \mathrm{Ind}_{O(Q')}^G(\tau'))_G = \sum_{i=1}^{s} (\tau^{g_i}, \tau')_{S_i}$$

$$= \sum_{i=1}^{s} (\tau'', \tau')_{S_i}$$

where S_i is as in 4.7 and where $\tau'' = 1_{O(Q')}$ if $\tau = 1_{O(Q)}$ and $\tau'' = \varepsilon_{O(Q')}$ if $\tau = \varepsilon_{O(Q)}$. The results now follow immediately from 4.7.

<u>5.9</u> <u>Corollary</u>: If $n \geq 2$ the characters $\mathrm{Ind}_{O_+}^G(1) - \mathrm{Ind}_{O_-}^G(1)$ and $\mathrm{Ind}_{O_+}^G(\varepsilon) - \mathrm{Ind}_{O_-}^G(\varepsilon)$ are each the difference of two distinct irreducible characters of G and have no common irreducible component. When $n = 1$,
$\mathrm{Ind}_{O_+}^G(1) - \mathrm{Ind}_{O_-}^G(1) = \mathrm{Ind}_{O_+}^G(\varepsilon) - \mathrm{Ind}_{O_-}^G(\varepsilon) \in \mathrm{irr}(G)$.

<u>5.10</u> <u>Lemma</u>. Let $\underline{\underline{c}}$ be a totally isotropic chain in V and let $P_{\underline{\underline{c}}}$ be its stabilizer in G. Further, let O be the subgroup O_+ or O_- of G and let $\bar{O}$ be the orthogonal subgroup of $Sp(\underline{\underline{c}})$ corresponding to a quadratic form $\bar{Q}$ for which $\Delta(\bar{Q})$ is zero or non-zero according as $O = O_+$ or O_- respectively. Then if $\zeta \in ch(GL(\underline{\underline{c}}))$ and $\chi \in ch(Sp(\underline{\underline{c}}))$

$$(\mathrm{Ind}_O^G(\varepsilon),\mathrm{Ind}_{P_{\underline{c}}}^G((\zeta\times\chi)^*))_G = \begin{cases} (1,\zeta)_{GL(\underline{c})}(\mathrm{Ind}_O^{Sp(\underline{c})}(\varepsilon),\chi)_{Sp(\underline{c})} \\ \quad \text{if } |\nu_{\underline{c}}| < n \\ (1,\zeta)_{GL(\underline{c})}\chi(1) \text{ if } |\nu_{\underline{c}}|=n,\ O=O_+ \\ 0 \quad \text{if } |\nu_{\underline{c}}|=n,\ O=O_- \end{cases}$$

Proof. Let Q be the quadratic form in $\mathcal{Q}$ such that $O = O(Q)$ and let $G(\underline{c})$ be the G-orbit of totally isotropic chains in V containing $\underline{c}$. We set $\ell = r_{\underline{c}}$ except when $\Delta(Q) \neq 0$ and $|\nu_{\underline{c}}| = n$ in which case we set $\ell = r_{\underline{c}} - 1$ and for $i = 1,\dots,\ell$ we choose an element $\underline{c}_i$ in $G(\underline{c})$ such that $n_{\underline{c}_i}^Q = i$. By lemma 4.9 such chains $\underline{c}_i$ exist and form a complete set of representatives of the $O(Q)$-orbits in $G(\underline{c})$. Now choose elements g_i in G for $i = 1,\dots,\ell$ such that $\underline{c}_i = g_i\underline{c}$. Then the g_i form a complete set of $(O,P_{\underline{c}})$-double coset representatives in G so by Mackey's theorem and lemma 4.4 we obtain

$$(\mathrm{Ind}_O^G(\varepsilon),\mathrm{Ind}_{P_{\underline{c}}}^G((\zeta\times\chi)^*))_G = \sum_{i=1}^{\ell} (\varepsilon_O^{g_i},(\zeta\times\chi)^*)_{O^{g_i}\cap P_{\underline{c}}}$$

$$= \sum_{i=1}^{\ell} (\varepsilon_{O(Q_i)},(\zeta\times\chi)^*)_{O(Q_i)\cap P_{\underline{c}}}$$

where $Q_i = Q^{g_i}$ for $i = 1,\dots,\ell$. Now clearly $n_{\underline{c}}^{Q_i} = n_{\underline{c}_i}^Q$ so that for $i < r_{\underline{c}}$ lemma 4.10 (i) implies that $\varepsilon_{O(Q_i)}$ is a non-trivial linear character on $\ker \pi_{\underline{c}} \cap O(Q_i) \cap P_{\underline{c}}$. Now by definition $(\zeta\times\chi)^*$ is trivial on $\ker \pi_{\underline{c}}$ whence $(\varepsilon_{O(Q_i)},(\zeta\times\chi)^*)_{O(Q_i)\cap P_{\underline{c}}} = 0$. When $i = r_{\underline{c}}$ lemma 4.10 (ii) implies that $\varepsilon_{O(Q_i)} = (1_{GL(\underline{c})} \times \varepsilon_{O(\bar{Q}_i)})^*$ where $\bar{Q}_i$

is the quadratic form induced on $V_r^{\perp}/V_r$ by Q_i and the lifting takes place via the projection

$$\pi_{\underline{\underline{c}}}: O(Q_i) \cap P_{\underline{\underline{c}}} \rightarrow GL(\underline{\underline{c}}) \times O(\bar{Q}_i) \quad .$$

Thus in this case

$$(\varepsilon_{O(Q_i)}, (\zeta\times\chi)^*)_{O(Q_i)\cap P_{\underline{\underline{c}}}} = (1,\zeta)_{GL(\underline{\underline{c}})} \quad (\varepsilon_{O(\bar{Q}_i)},\chi)_{O(\bar{Q}_i)} \, .$$

The result now follows since $\Delta(\bar{Q}_i) = \Delta(Q)$.

5.11 Lemma. If $\underline{\underline{c}} \in \underline{\underline{C}}(V)$,

$$(\mathrm{Ind}_O^G(1), \mathrm{Ind}_{P_{\underline{\underline{c}}}}^G(1)) = \begin{cases} r_{\underline{\underline{c}}} & \text{if } O = O_- \text{ and } |\nu_{\underline{\underline{c}}}| = n \\ r_{\underline{\underline{c}}}+1 & \text{otherwise} \end{cases}$$

Proof. The $(O,P_{\underline{\underline{c}}})$-double cosets of G are in 1-1 correspondence with the O-orbits in $G(\underline{\underline{c}})$. Since $\nu_{\underline{\underline{c}}'} = \nu_{\underline{\underline{c}}}$ for all $\underline{\underline{c}}'$ in $G(\underline{\underline{c}})$ the result is immediate from lemma 4.9.

5.12 Theorem. Suppose $n \geq 2$. Then

(i) $$\mathrm{Ind}_{O_+}^G(1) - \mathrm{Ind}_{O_-}^G(1) = \chi_{(n-1),(1)} - \chi_{(1,n-1),(0)}$$

and (ii) $$\mathrm{Ind}_{O_+}^G(\varepsilon) - \mathrm{Ind}_{O_-}^G(\varepsilon) = \chi_{(0),(n)} - \zeta$$

where ζ is an irreducible character of G not in the principal series.

(iii) In case $n = 1$

$$\mathrm{Ind}_{O_+}^G(1) - \mathrm{Ind}_{O_-}^G(1) = \mathrm{Ind}_{O_+}^G(\varepsilon) - \mathrm{Ind}_{O_-}^G(\varepsilon) = \chi_{(0),(1)} \; (=\mathrm{St}_G) \quad .$$

Proof. Applying 5.11 we obtain, for $n \geq 2$,
$(\mathrm{Ind}_{O_+}^G(1), \mathrm{Ind}_{P_{(1)}}^G(1))_G = (\mathrm{Ind}_{O_-}^G(1), \mathrm{Ind}_{P_{(1)}}^G(1))_G = (\mathrm{Ind}_{O_+}^G(1), \mathrm{Ind}_{P_{(n)}}^G(1))_G = 2$ and $(\mathrm{Ind}_{O_-}^G(1), \mathrm{Ind}_{P_{(n)}}^G(1))_G = 1$. Now since $\chi_{(n),(0)} (=1_G)$ occurs as an irreducible component in both $\mathrm{Ind}_{O_+}^G(1)$ and $\mathrm{Ind}_{O_-}^G(1)$, part (i) follows easily from 5.4, 5.5 and 5.9. Observing that $P_{(1^n)}$ is a Borel subgroup B of G we apply lemma 5.10 to obtain

$$(\mathrm{Ind}_{O_+}^G(\varepsilon), \mathrm{Ind}_B^G(1))_G = (\mathrm{Ind}_{O_+}^G(\varepsilon), \mathrm{Ind}_{P_{(n)}}^G(1))_G = 1$$

and $$(\mathrm{Ind}_{O_-}^G(\varepsilon), \mathrm{Ind}_B^G(1))_G = 0 .$$

Now $\chi_{(n),(0)}$ is not a component of $\mathrm{Ind}_{O_+}^G(\varepsilon)$ so, by lemmas 5.5 and 5.6, $\chi_{(0),(n)}$ occurs in $\mathrm{Ind}_{O_+}^G(\varepsilon)$ with multiplicity 1 and is the only principal series character to do so. Cases (ii) and (iii) now follow easily from 5.9.

5.13 Corollary: $\chi_{(n-1),(1)} = \frac{1}{2}\left\{\mathrm{Ind}_{O_+}^G(1) - \mathrm{Ind}_{O_-}^G(1) + \mathrm{Ind}_{P_{J(1)}}^G(1) - 1_G\right\}$
and $\chi_{(n-1,1),(0)} = \frac{1}{2}\left\{\mathrm{Ind}_{P_{J(1)}}^G(1) - 1_G - \mathrm{Ind}_{O_+}^G(1) + \mathrm{Ind}_{O_-}^G(1)\right\}$.

Proof. Use lemmas 5.4 and 5.12(i).

5.14 Aside: By using a more powerful version of the result of Springer stated in 1.18 or by considering the components of $\mathrm{Ind}_B^G((\hat{\alpha}\times 1\times\ldots\times 1)^*)$ (character notations as in tables 1 and 2) we find that $\mathrm{Ind}_{P_{(1)}}^G((\hat{\alpha}\times 1)^*)$ where 1 denotes the principal character on $Sp(2(n-1),q)$ is an irreducible character of G for each α in

$\mathbb{Z}/(q-1)\mathbb{Z}$, $\alpha \neq 0$. Further by extending lemma 5.11 we find that

$$\text{(i)} \quad \mathrm{Ind}_{O_+}^{G}(1) = \chi_{(n-1),(1)} + \sum_{\{\alpha,-\alpha\}} \mathrm{Ind}_{P_{(1)}}^{G}((\hat{\alpha}\times 1)^*)$$

and

$$\text{(ii)} \quad \mathrm{Ind}_{O_-}^{G}(1) = \chi_{(1,n-1),(0)} + \sum_{\{\alpha,-\alpha\}} \mathrm{Ind}_{P_{(1)}}^{G}((\hat{\alpha}\times 1)^*)$$

where $\sum_{\{\alpha,-\alpha\}}$ denotes the sum over all the (unordered) pairs $\{\alpha,-\alpha\}$

where $\alpha \in \mathbb{Z}/(q-1)\mathbb{Z}$, $\alpha \neq 0$. Then from lemma 5.4 we find

$$1_G = \mathrm{Ind}_{P_{(1)}}^{G}(1) - \mathrm{Ind}_{O_+}^{G}(1) - \mathrm{Ind}_{O_-}^{G}(1) - \sum_{\substack{\alpha\in\mathbb{Z}/(q-1)\mathbb{Z} \\ \alpha\neq 0}} \mathrm{Ind}_{P_{(1)}}^{G}((\hat{\alpha}\times 1)^*).$$

This yields the following interesting characterization of characters of G.

(5.14.1) A complex class function χ on G is a character of G if and only if its restrictions to O_+, O_- and $P_{(1)}$ are all characters.

5.15 We now proceed to determine the unipotent characters for some special groups. For the first such exercise (5.16) we interrupt our present development and draw only on earlier results. The discussion carried out in §3 for Sp(2n,q) can be carried out analogously for GL(n,q) as is well known. As mentioned in 3.14 the Weyl group of $GL(n,\tilde{k})$ is S_n, the symmetric group on n symbols, and its conjugacy classes are indexed by partitions corresponding to the cycle type. Thus the GL(n,q)-conjugacy classes in $T(GL(n,\tilde{k}))^F$ are indexed by partitions and we will denote by $\tilde{T}_\nu$ a representative of the class corresponding to the partition ν of n. For each $\tilde{T}'$ in $T(GL(n,\tilde{k}))^F$

and each θ' in $(T')^\vee$ the character $\rho^{\theta'}_{\tilde{T}'}$ of $GL(n,q)$ has been defined in 2.5. As usual we abbreviate the notation $\rho^{\theta}_{\tilde{T}_\nu}$ to ρ^{θ}_{ν} for each partition ν of n and θ in $T^\vee_\nu$.

The irreducible characters ψ_ν of S_n were mentioned in 5.1. We denote by χ_ν the principal series character of $GL(n,q)$ corresponding under the bijection in 1.17 to ψ_ν. The χ_ν can all be obtained as integral linear combinations of characters $\mathrm{Ind}_P^{GL(n,q)}(1)$ for parabolic subgroups P of $GL(n,q)$.

<u>5.16</u> <u>Theorem</u>. The virtual character $\rho^1_{\tilde{T}}$ of $GL(n,q)$ corresponding to a minisotropic torus $\tilde{T}$ is given by

$$\rho^1_{(n)} = \sum_\nu \psi_\nu((n))\chi_\nu \quad . \tag{5.16.1}$$

<u>Proof</u>. We use induction on n the result being clear for $n = 1$. We define a character $\xi_{(n)}$ of $W = S_n$ by $\xi_{(n)} = \sum_\nu \psi_\nu((n))\psi_\nu$. Then by an application of the orthogonality relations for characters (see e.g. [7]) we obtain for each element w in S_n

$$\xi_{(n)}(w) = \begin{cases} n & \text{if } w \text{ has cycle type } (n) \\ 0 & \text{otherwise} \end{cases} \quad .$$

We now suppose inductively that for each $m < n$ each virtual character ρ^1_μ of $GL(m,q)$ is a combination of principal series charactersof $GL(m,q)$ and that under the bijection of 1.17 $\rho^1_{(m)}$ corresponds to $\xi_{(m)}$. Now let μ be a partition of n other than (n). Let us suppose $\mu = (1^{m_1}, 2^{m_2}, \ldots)$. Then by (2.8.3)

$\rho^1_\mu = \mathrm{Ind}_{P_\mu}^{GL(n,q)}(((\rho^1_{(1)})^{m_1} \times (\rho^1_{(2)})^{m_2} \times \ldots)^*)$ where P_μ is a parabolic subgroup of $GL(n,q)$ with unipotent radical U_μ^+ such that $P_\mu/U_\mu^+ \cong GL(1,q)^{m_1} \times GL(2,q)^{m_2} \times \ldots$. In particular ρ_μ^1 is a combination of principal series characters of $GL(n,q)$ whence by (2.11.1) $\rho_{(n)}^1$ is also. Now setting $\xi_\mu = \mathrm{Ind}_{W_\mu}^W((\xi_{(1)})^{m_1} \times (\xi_{(2)})^{m_2} \times \ldots)$ where W_μ is a parabolic subgroup of W isomorphic to $S_1^{m_1} \times S_2^{m_2} \times \ldots$ we see from 1.17 that under the bijection therein ρ_μ^1 corresponds to ξ_μ. Thus if $\rho_{(n)}^1$ corresponds to the element ξ in ch(W), 1.17 together with the orthogonality relations implicit in (2.9.1) imply that $(\xi,\xi_\mu)_W = 0$ for each μ. By a simple evaluation we find that for each w in W

$$\xi_\mu(w) \neq 0 \quad \text{if } w \text{ has cycle type } \mu \text{ and}$$

$$\xi_\mu(w) = 0 \quad \text{otherwise} \quad .$$

We are therefore reduced to the conclusion that ξ is a multiple of $\xi_{(n)}$. But by (2.11.3) we have $(1,\rho_{(n)}^1) = 1$ and by definition $(1,\xi_{(n)}) = 1$. The result now follows from 1.17.

5.17 Remarks. The coefficients $\psi_\nu((n))$ in (5.16.1) vanish unless ψ_ν is the character of the ith exterior power of the reflection representation (see e.g. [6]) of W for some $i = 0,1,\ldots,n-1$. In this case $\psi_\nu((n)) = (-1)^i$. I conjecture that, in general, if $\tilde{G}$ a connected reductive k-group, $\tilde{W}$ its Weyl group, χ a principal series character of G corresponding to the irreducible character ψ, say, of W as in 1.17 and $\tilde{T}'$ is a Coxeter torus of $\tilde{G}$ then $(\rho_{T'}^1,\chi) = (-1)^i$ if ψ is the character of the ith exterior power of the reflection representation of W and $(\rho_{\tilde{T}}^1,\chi) = 0$ otherwise.

5.18 Theorem. If $n = 2$ there are six unipotent characters of G namely $\chi_{(2),(0)}$, $\chi_{(0),(1^2)}$, $\chi_{(1),(1)}$, $\chi_{(1^2),(0)}$, $\chi_{(0),(2)}$ and $\zeta_{(0),(0)}$ where $\zeta_{(0),(0)}$ denotes the character ζ in 5.12(ii). They can be obtained as follows:-

$$\chi_{(2),(0)} = 1_G$$

$$\chi_{(0),(1^2)} = St_G$$

$$\chi_{(1),(1)} = \tfrac{1}{2}\{Ind^G_{P_{(1)}}(1) - 1_G + Ind^G_{O_+}(1) - Ind^G_{O_-}(1)\}$$

$$\chi_{(1^2),(0)} = Ind^G_{P_{(1)}}(1) - 1_G - \chi_{(1),(1)}$$

$$\chi_{(0),(2)} = Ind^G_{P_{(2)}}(1) - 1_G - \chi_{(1),(1)}$$

$$\zeta_{(0),(0)} = Ind^G_{O_-}(\varepsilon) - Ind^G_{O_+}(\varepsilon) + \chi_{(0),(2)} \; .$$

Finally, the virtual characters $\rho^{1}_{\tilde{T}}$ can be obtained as linear combinations of the unipotent characters with the coefficients as shown in table 6 of the appendix.

Proof. As remarked in 2.10 all principal series characters are unipotent. That they along with $\zeta_{(0),(0)}$ can be obtained as shown is a direct application of 5.6 and 5.12. Let $c = c(d,qd)$ where $d \in (q^2+1)^{-1}\mathbb{Z}/\mathbb{Z}$ and $d \neq 0$ (see 3.13 and table 4). Then c is a class of regular semisimple elements of G from which a representative s, say, can be chosen in $T_{(0),(2)}$. Thus c avoids all proper parabolic subgroups of G. Now using table 4 we obtain

$$[Ind^G_{O_+}(1)](s) = [Ind^G_{O_+}(\varepsilon)](s) = 0$$

and

$$[Ind^G_{O_-}(1)](s) = [Ind^G_{O_-}(\varepsilon)](s) = 1 \; .$$

Thus $\chi_{(1),(1)}(s) = -1$, $\chi_{(1^2),(0)}(s) = 0$, $\chi_{(0),(2)}(s) = 0$ and $\zeta_{(0),(0)}(s) = 1$. Now from table 2 $\|\rho^1_{(0),(2)}\| = 4$ so applying (2.10.3), (2.11.3) and (2.11.4) we obtain $\rho^1_{(0),(2)} = 1_G + St_G - \chi_{(1),(1)} + \zeta_{(0),(0)}$. In particular, $\zeta_{(0),(0)}$ is unipotent.

We now apply (2.8.3), 5.3 and 5.16 (in conjunction with (2.7.5) and (2.8.1))

$$\rho^1_{(1^2),(0)} = \mathrm{Ind}_B^G(1) = 1_G + St_G + 2\chi_{(1),(1)} + \chi_{(1^2),(0)} + \chi_{(0),(2)}.$$

$$\begin{aligned}\rho^1_{(1),(1)} &= \mathrm{Ind}^G_{P_{(1)}}((1\times\rho^1_{(0),(1)})^*) = \mathrm{Ind}^G_{P_{(1)}}((1\times(1_S - St_S))^*)\\ &= 2\ \mathrm{Ind}^G_{P_{(1)}}(1) - \mathrm{Ind}^G_B(1)\\ &= 1_G - St_G + \chi_{(1^2),(0)} - \chi_{(0),(2)}\end{aligned}$$

where $S = Sp(2,q)$.

$$\begin{aligned}\rho^1_{(2),(0)} &= \mathrm{Ind}^G_{P_{(2)}}((\rho^1_{(2)})^*) = \mathrm{Ind}^G_{P_{(2)}}((1_{G_1} - St_{G_1})^*)\\ &= 2\ \mathrm{Ind}^G_{P_{(2)}}(1) - \mathrm{Ind}^G_B(1)\\ &= 1_G - St_G - \chi_{(1^2),(0)} + \chi_{(0),(2)}\end{aligned}$$

where $G_1 = GL(2,q)$.

Finally, we obtain from (2.11.1) that

$$\rho^1_{(0),(1^2)} = 1_G + St_G - \chi_{(1^2),(0)} - \chi_{(0),(2)} - 2\zeta_{(0),(0)}.$$

This completes the verification of the entries in table 6 and establishes that the unipotent characters listed indeed form a complete set.

<u>5.19</u> <u>Lemma</u>. $\zeta_{(0),(0)}$ is in the discrete series for Sp(4,q).

<u>Proof</u>. From 5.12(ii) $\zeta_{(0),(0)}$ is a component of $\mathrm{Ind}_{0_-}^{G}(\varepsilon)$ where $G = Sp(4,q)$. But from 5.10 $(\mathrm{Ind}_{0_-}^{G}(\varepsilon),\mathrm{Ind}_{P}^{G}(\chi^*))_G = 0$ if $P = P_{(2)}$ or $P_{(1^2)}$ for any character χ in $ch(P/U^+)$. Further applying 5.10 and 5.12 parts (ii) and (iii) we obtain for any discrete series characters ζ in ch(GL(1,q)) and χ in ch(Sp(2,q))

$$(\chi_{(0),(2)} - \zeta_{(0),(0)},\mathrm{Ind}_{P_{(1)}}^{G}((\zeta\times\chi)^*))_G$$

$$= (\mathrm{Ind}_{0_-}^{G}(\varepsilon) - \mathrm{Ind}_{0_-}^{G}(\varepsilon),\mathrm{Ind}_{P_{(1)}}^{G}((\zeta\times\chi)^*))_G$$

$$= (1,\zeta)_{GL(1,q)}(\mathrm{Ind}_{0_+(2,q)}^{Sp(2,q)}(\varepsilon) - \mathrm{Ind}_{0_-(2,q)}^{Sp(2,q)}(\varepsilon),\chi)_{Sp(2,q)}$$

$$= (1,\zeta)_{GL(1,q)}(\chi_{(0),(1)},\chi)_{Sp(2,q)}$$

$$= 0$$

where the last step follows because the Steinberg character $\chi_{(0),(1)}$ is not in the discrete series of Sp(2,q). But since $\chi_{(0),(2)}$ is in the principal series for G it cannot be a component of $\mathrm{Ind}_{P_{(1)}}^{G}((\zeta\times\chi)^*)$ so neither can $\zeta_{(0),(0)}$. This completes the proof.

5.20 Lemma. For $n > 2$ the character ζ in theorem 5.12 (ii) is unipotent and is a component of $\mathrm{Ind}^G_{P_{(1^{n-2})}}((1\times\zeta_{(0),(0)})^*)$ where 1 denotes the principal character on $GL(1,q)^{n-2}$ and $\zeta_{(0),(0)}$ is as defined in 5.18.

Proof. Applying 5.10 and 5.12 (ii) we obtain

$$(\chi_{(0),(n)} - \zeta, \mathrm{Ind}^G_{P_{(1^{n-2})}}((1\times\zeta_{(0),(0)})^*))_G$$

$$= (\mathrm{Ind}^G_{O_+}(\varepsilon) - \mathrm{Ind}^G_{O_-}(\varepsilon), \mathrm{Ind}^G_{P_{(1^{n-2})}}((1\times\zeta_{(0),(0)})^*))_G$$

$$= (\mathrm{Ind}^{Sp(4,q)}_{O_+(4,q)}(\varepsilon) - \mathrm{Ind}^{Sp(4,q)}_{O_-(4,q)}(\varepsilon), \zeta_{(0),(0)})_{Sp(4,q)}$$

$$= (\chi_{(0),(2)} - \zeta_{(0),(0)}, \zeta_{(0),(0)})_{Sp(4,q)}$$

$$= -1$$

Thus ζ is a component of $\mathrm{Ind}^G_{P_{(1^{n-2})}}((1\times\zeta_{(0),(0)})^*)$. Now from table 6, $\rho^1_{(0),(2)} = 1_G + St_G - \chi_{(1),(1)} + \zeta_{(0),(0)}$. Thus from (2.8.1) and (2.8.3) $\rho^1_{(1^{n-2}),(2)}$ is a combination of principal series charactersplus $\mathrm{Ind}^G_{P_{(1^{n-2})}}((1\times\zeta_{(0),(0)})^*)$ and since ζ is not in the principal series it follows that $(\zeta, \rho^1_{(1^{n-2}),(2)})_G \neq 0$. This completes the proof.

5.21 **Lemma.** If n = 3, then

$$\chi_{(3),(0)} = 1_G$$

$$\chi_{(0),(1^3)} = St_G$$

$$\chi_{(2),(1)} = \tfrac{1}{2}\{\mathrm{Ind}^G_{O_+}(1)-\mathrm{Ind}^G_{O_-}(1)+\mathrm{Ind}^G_{P_{(1)}}(1)-1_G\}$$

$$\chi_{(1,2),(0)} = \mathrm{Ind}^G_{P_{(1)}}(1)-\chi_{(2),(1)}-1_G$$

$$\chi_{(1),(1^2)} = \mathrm{Ind}^G_{P_{(1)}}(\chi^*_{(1),(1)})-\mathrm{Ind}^G_{P_{(2)}}(1)+\chi_{(1,2)(0)}+1_G$$

$$\chi_{(0),(1,2)} = \mathrm{Ind}^G_{P_{(2,1)}}(1)-\mathrm{Ind}^G_{P_{(3)}}(1)-\mathrm{Ind}^G_{P_{(2)}}(1)-\chi_{(1),(1^2)}+1_G$$

$$\chi_{(1^2),(1)}+\chi_{(1),(2)} = \mathrm{Ind}^G_{P_{(2)}}(1)-\mathrm{Ind}^G_{P_{(3)}}(1)$$

$$\chi_{(0),(3)}+\chi_{(1),(2)} = \mathrm{Ind}^G_{P_{(1)}}(1)-\chi_{(2),(1)}-1_G$$

$$\chi_{(1^3),(0)}+\chi_{(1^2),(1)} = \mathrm{Ind}^G_{P_{(1,1)}}(1)-\mathrm{Ind}^G_{P_{(2)}}(1)-\mathrm{Ind}^G_{P_{(3)}}(1)-\chi_{(1),(1^2)}+1_G$$

$$\zeta_{(1),(0)}+\zeta_{(0),(1)} = \mathrm{Ind}^G_{P_{(1)}}((1\times\zeta_{(0),(0)})^*)$$

$$\chi_{(0),(3)}-\zeta_{(1),(0)} = \mathrm{Ind}^G_{O_+}(\varepsilon)-\mathrm{Ind}^G_{O_-}(\varepsilon)$$

where $\zeta_{(1),(0)}$ denotes the character ζ of 5.12(ii) and $\zeta_{(0),(1)}$ is an irreducible character of G not in the principal series.

Proof. An examination of the character table (table 5) of the Weyl group W yields by Frobenius reciprocity:-

$$\mathrm{Ind}^W_{W_{(1,1)}}(1) = 1_W+\phi_{(1^3),(0)}+2\phi_{(2),(1)}+\phi_{(1),(1^2)}+2\phi_{(1^2),(1)}+\phi_{(1),(2)} + 2\phi_{(1,2),(0)}$$

$$\mathrm{Ind}^W_{W_{(2,1)}}(1) = 1_W+\phi_{(0),(3)}+2\phi_{(2),(1)}+\phi_{(1),(1^2)}+\phi_{(1^2),(1)}+2\phi_{(1),(2)} + \phi_{(1,2),(0)}+\phi_{(0),(1,2)}$$

$$\mathrm{Ind}_{W_{(2)}}^{W}(1) = 1_W + \phi_{(2),(1)} + \phi_{(1^2),(1)} + \phi_{(1),(2)} + \phi_{(1,2),(0)}$$

and

$$\mathrm{Ind}_{W_{(1)}}^{W}(\phi_{(1),(1)}) = \phi_{(2),(1)} + \phi_{(1),(1^2)} + \phi_{(1^2),(1)} + \phi_{(1),(2)}$$

Using 5.3 we convert these expressions into ones involving principal series characters of G. Then using 5.4, 5.6 and 5.13 we can verify the first 9 equations directly.

Now by 5.19, $1\times\zeta_{(0),(0)}$ is a discrete series character of $GL(1,q) \times Sp(4,q)$ so we may apply 1.18 to obtain $\|\mathrm{Ind}_{P_{(1)}}^{G}((1\times\zeta_{(0),(0)})^*)\|$. For $J = J_{(1)}$ (see 3.17 and 3.4) the only elements w of W stabilizing J are the elements 1 and (1-1). Thus in both cases the mapping $\hat{w}$ of $P_{(1)}/U_{(1)}^{+}$ (=$GL(1,q)\times Sp(4,q)$) onto itself leaves the direct factor $Sp(4,q)$ fixed pointwise. It therefore follows that $\|\mathrm{Ind}_{P_{(1)}}^{G}((1\times\zeta_{(0),(0)})^*)\| = 2$. Again applying 5.19 we find that $(\mathrm{Ind}_{P_{(1)}}^{G}((1\times\zeta_{(0),(0)})^*), \mathrm{Ind}_{B}^{G}(1)) = 0$ so that $\mathrm{Ind}_{P_{(1)}}^{G}((1\times\zeta_{(0),(0)})^*)$ has no components in the principal series. By 5.20 $\zeta_{(1),(0)}$ is one component of $\mathrm{Ind}_{P_{(1)}}^{G}((1\times\zeta_{(0),(0)})^*)$ so that the tenth equation follows. The final equation follows immediately from 5.12(ii).

<u>5.22</u> <u>Theorem</u>. If n = 3 the principal series characters of G together with $\zeta_{(1),(0)}$ and $\zeta_{(0),(1)}$ form the complete set of unipotent characters of G. Moreover the virtual characters $\rho_{\tilde{T}}^{1}$ can be obtained as linear combinations of the unipotent characters with coefficients as shown in table 6 of the appendix.

Proof. Using (2.8.3) it is straightforward to verify the coefficients in table 6 for the first 7 virtual characters of G listed. Now suppose $s \in c(f,qf,q^2f)$ where $f \in (q^3+1)^{-1}\mathbb{Z}/\mathbb{Z}$ but $f \notin (q+1)^{-1}\mathbb{Z}/\mathbb{Z}$. Then s is a regular semisimple element of G which is not conjugate to any element of any proper parabolic subgroup of G. Moreover from table 4 we obtain

$$[\mathrm{Ind}_{O_+}^G(1)](s) = [\mathrm{Ind}_{O_+}^G(\varepsilon)](s) = 0$$

and

$$[\mathrm{Ind}_{O_-}^G(1)](s) = [\mathrm{Ind}_{O_-}^G(\varepsilon)](s) = 1 \ .$$

Thus from 5.21,

$$\chi_{(2),(1)}(s)=-1, \quad \chi_{(1,2),(0)}(s)=0, \quad \chi_{(1),(1^2)}(s)=1,$$

$$\chi_{(0),(1,2)}(s)=0, \quad \chi_{(1^2),(1)}(s)+\chi_{(1),(2)}(s)=0,$$

$$\chi_{(0),(3)}(s)+\chi_{(1),(2)}(s)=0, \quad \chi_{(1^3),(0)}(s)+\chi_{(1^2),(1)}(s)=0,$$

$$\zeta_{(1),(0)}(s)+\zeta_{(0),(1)}(s)=0 \quad \text{and} \quad \chi_{(0),(3)}(s)-\zeta_{(1),(0)}(s)=-1 \ .$$

Now if $\chi_{(0),(3)}(s) \neq 0$ then $\chi_{(1),(2)}(s) \neq 0$ and $\chi_{(1^2),(1)}(s) \neq 0$ which implies from (2.10.3), (2.11.3) and (2.11.4) that $\|\rho^1_{(0),(3)}\| \geq 7$ contrary to table 2. Thus $\chi_{(0),(3)}(s) = 0$ and the entry for $\rho^1_{(0),(3)}$ in table 6 can be obtained from (2.10.3).

Next we consider an element s in $c(b,d,qd)$ where $b \in (q+1)^{-1}\mathbb{Z}/\mathbb{Z}$, $b \neq 0$ and $d \in (q^2+1)^{-1}\mathbb{Z}/\mathbb{Z}$, $d \neq 0$. Then again s is a regular semisimple element of G and we obtain as before

$$[\mathrm{Ind}_{0_+}^G(1)](s) = [\mathrm{Ind}_{0_+}^G(\varepsilon)](s) = 1$$

and

$$[\mathrm{Ind}_{0_-}^G(1)](s) = [\mathrm{Ind}_{0_-}^G(\varepsilon)](s) = 0\ ,$$

Thus from 5.21,

$$\chi_{(2),(1)}(s)=0, \quad \chi_{(1,2),(0)}(s)=-1, \quad \chi_{(1),(1^2)}(s)=0,$$

$$\chi_{(0),(1,2)}(s)=1, \quad \chi_{(1^2),(1)}(s)+\chi_{(1),(2)}(s)=0$$

$$\chi_{(0),(3)}(s)+\chi_{(1),(2)}(s)=-1, \quad \chi_{(1^3),(0)}(s)+\chi_{(1^2),(1)}(s)=1,$$

$$\zeta_{(1),(0)}(s)+\zeta_{(0),(1)}(s)=0 \quad \text{and } \chi_{(0),(3)}(s)-\zeta_{(1),(0)}(s)=1.$$

Now set $\alpha = \chi_{(0),(3)}(s)$. Then

$$\chi_{(1),(2)}(s)=-1-\alpha, \quad \chi_{(1^2),(1)}(s)=1+\alpha, \quad \chi_{(1^3),(0)}(s)=-\alpha$$

$$\zeta_{(1),(0)}(s)=-1+\alpha \quad \text{and } \zeta_{(0),(1)}(s)=1-\alpha$$

so by (2.10.3), (2.11.3) and (2.11.4)

$$\rho^1_{(0),(1,2)} = 1_G-St_G-\chi_{(1,2),(0)}+\chi_{(0),(2,1)}+\alpha\chi_{(0),(3)}-(1+\alpha)\chi_{(1),(2)}$$

$$+ (1+\alpha)\chi_{(1^2),(1)}-\alpha\chi_{(1^3),(0)}-(1-\alpha)\zeta_{(1),(0)}+(1-\alpha)\zeta_{(0),(1)}$$

so that $\|\rho^1_{(0),(1,2)}\| = 6\alpha^2 + 8$. Comparing this with the entry in table 2 implies that $\alpha = 0$. This verifies the entry in table 6 for $\rho^1_{(0),(1,2)}$. The entry for $\rho^1_{(0),(1^3)}$ is now obtained by an application of (2.11.1).

§6. DETERMINATION OF THE GREEN FUNCTIONS

We begin by introducing the pseudo-tori of Sp(4,q) and Sp(6,q) and define the Green functions associated with them. An explicit determination of the Green functions requires the determination of the unipotent conjugacy classes in G and this is discussed. Next we determine the values of the functions $g_P^G(c;d)$ associated with the induction of characters from parabolic subgroups of G. Finally the Green functions are written out explicitly for Sp(6,q) and its centralizers of semisimple elements.

6.1 Let G = Sp(4,q). We will denote the trivial subgroup 1 of G by $T_{(0),(0),(2)}$ where we refer to the number in brackets in the last subscript as the rank defect of $T_{(0),(0),(2)}$. The maximal tori of G together with $T_{(0),(0),(2)}$ will be called collectively the pseudo-tori of G. The maximal tori of G will sometimes be referred to as the proper tori of G and are said to have rank defect zero. The proper torus $T_{\mu,\nu}$ of G will also be denoted by $T_{\mu,\nu,(0)}$ although we will usually omit the subscript (0).

Let G = Sp(6,q) and let H be a hyperbolic plane in V i.e. H is a subspace of V generated by a hyperbolic pair in V (for notations see 3.1 and 3.2). Then Sp(H) × $T_{(0),(0),(2)}$ is a subgroup of G and the tori which are maximal in Sp(H) × $T_{(0),(0),(2)}$ fall into 2 distinct conjugacy classes in G corresponding to the classes of maximal tori in Sp(H). We will denote the torus $T_{(1),(0)} \times T_{(0),(0),(2)}$ by $T_{(1),(0),(2)}$ and $T_{(0),(1)} \times T_{(0),(0),(2)}$ by $T_{(0),(1),(2)}$. The (proper) maximal tori of G together with the conjugates under G of $T_{(1),(0),(2)}$ and $T_{(0),(1),(2)}$ will be called the pseudo-tori of G in this case. Again each pseudo-torus of G can be denoted by a symbol of the form $T_{\mu,\nu,\lambda}$ indicating which conjugacy class in G it belongs to where (μ,ν) is an ordered pair of partitions, λ(=(0) or (2)) is the rank defect and $|\mu| + |\nu| + |\lambda| = n$. Clearly the discussion we have carried out for the finite groups G could as easily have been carried out for the algebraic group $\tilde{G}$. That being done we can define $P(\tilde{G})$ as the set of pseudo-tori in $\tilde{G}$ and set $P(G) = P(\tilde{G})^F$. Then of course the pseudo-tori in the finite case are just the $T = \tilde{T}^F$ for elements $\tilde{T}$ in $P(G)$.

6.2 We now define

(6.2.1) $$\rho^1_{(0),(0),(2)} = -\chi_{(1^2),(0)} - \chi_{(0),(2)} + \chi_{(1),(1)} + \zeta_{(0),(0)}.$$

This gives rise to the appropriate entries in tables 2 and 6. We also define the extended geometric conjugacy classes of Sp(4,q) by making the single entry shown in table 3 under the column headed "(0),(0),(2)". Then making use of 2.9 we easily see that 2.9 continues to hold if we replace the words "geometrically conjugate" by the words "in the same extended geometric conjugacy class".

We use (2.8.3) to define $\rho^1_{(1),(0),(2)}$ i.e. we define $\rho^1_{(1),(0),(2)} = \mathrm{Ind}^G_{P_{(1)}}((1\times\rho^1_{(0),(0),(2)})^*)$ where of course G = Sp(6,q). Then

(6.2.2) $$\rho^1_{(1),(0),(2)} = -\chi_{(1^3),(0)} - \chi_{(0),(3)} - \chi_{(1,2),(0)} - \chi_{(0),(1,2)} + \chi_{(2),(1)} + \chi_{(1),(1^2)} + \zeta_{(1),(0)} + \zeta_{(0),(1)}.$$

Finally we define

$$(6.2.3)\qquad \rho^{1}_{(0),(1),(2)} = -\chi_{(1^3),(0)}+\chi_{(0),(3)}+\chi_{(1,2),(0)}-\chi_{(0),(1,2)}$$

$$-\ \chi_{(2),(1)}+\chi_{(1),(1^2)}-\zeta_{(1),(0)}+\zeta_{(0),(1)}\ .$$

Corresponding to these definitions we have made appropriate entries in the tables. The extended geometric conjugacy classes are as shown in table 3 but we defer the definition of the $\rho^{\hat{\alpha}}_{(1),(0),(2)}$ and $\rho^{\hat{\beta}}_{(0),(1),(2)}$ until §7.

<u>6.3</u> We extend the definition of the Green functions in 2.6 to cover all pseudo-tori. Thus if $\tilde{T}$ is a pseudo-torus of $\tilde{G}$ (=Sp(4,$\tilde{k}$) or Sp(6,$\tilde{k}$)) we define for each element g in G

$$(6.3.1)\qquad Q_{\tilde{T},G}(g) = \begin{cases} \rho^{1}_{\tilde{T}}(g) & \text{if g is unipotent} \\ 0 & \text{otherwise} \end{cases}$$

We discuss the properties of these functions at the end of the section. We turn now to the conjugacy classes of G.

<u>6.4</u> The conjugacy classes of the finite Symplectic, Orthogonal and Unitary groups for characteristic 2 have been determined by Wall [26]. Each of these groups is obtained as the group of non-singular linear transformations on a finite dimensional vector space V over the finite field k which leave invariant a non-degenerate symmetric bilinear form (,) on V and in the orthogonal case a quadratic form as well. If for an element g in G (where here G is one of the groups referred to above) we write T = 1 + g where 1 denotes the unit element of G then T(V) is called the <u>Cayley space</u> of g. The

Cayley form of g is the bilinear form [,] defined on T(V) by setting [Tu, Tv] = (u,Tv) for all u,v in V. It is useful to observe in this context that [,] is a non-degenerate form on T(V) and that $T(V) = (\ker T)^{\perp}$. Moreover [Tu, Tv] + [Tv, Tu] = (Tu,Tv) for all u,v in V. The conjugacy class in G to which g belongs is determined by its elementary divisors and by certain invariants associated with its Cayley space and form. We will refer to the collection of elementary divisors and other distingusihing invariants as the Wall invariants of the element or of the conjugacy class in G to which it belongs.

The unipotent conjugacy classes together with their orders are shown for selected groups in table 7 of the appendix. They have been determined by the methods of Wall (loc.cit.). In table 7 only those invariants necessary to distinguish the conjugacy class are shown. This practice is also adopted in the text. The semisimple classes of selected orthogonal groups together with their orders have been determined by the same methods and are shown in table 4. The latter have already been used in §5.

6.5 Henceforth G will denote the group Sp(V) for a finite dimensional vector space V over the field k of q elements where q is a power of 2 and dim V = 2n. All other notations as in §3 will also be assumed.

It will be useful to have the following result.

6.6 Lemma. Suppose $g \in G$, $T = 1 + g$ and let [,] be the Cayley form of g. We define and $\pi(x^g) \in d$. It is easily seen that this definition is independent of the choice of x in c.

6.9 Suppose P_1 and P_2 are parabolic subgroups of G with unipotent radicals U_1^+ and U_2^+ respectively such that $P_2 \leq P_1$. Then P_2/U_1^+ is a parabolic subgroup of P_1/U_1^+ with unipotent radical U_2^+/U_1^+ and if $c \in C(G)$ and $d \in C(P_2/U_2^+)$ then identifying P_2/U_2^+ with $(P_2/U_1^+)/(U_2^+/U_1^+)$ we have

$$g_{P_2}^G(c;d) = \sum_{e \in C(P_1/U_1^+)} g_{P_1}^G(c;e) g_{P_2/U_1^+}^{P_1/U_1^+}(e;d) .$$

6.10 The importance of the g(c;d) stems from the following.

Suppose P is a parabolic subgroup of G with unipotent radical U^+ and suppose $\phi \in ch(P/U^+)$. Then (Green loc.cit.)

$$[\mathrm{Ind}_P^G(\phi^*)](c) = \sum_{d \in C(P/U^+)} g(c;d)\phi(d) .$$

The following lemma reduces the determination of the g(c;d) to the unipotent case.

6.11 Lemma. Let P be a parabolic subgroup of G and let c be an element of C(G). Then $g_P^G(c;d) = 0$ for all d in $C(P/U^+)$ unless $\pi(t) \in d_s$ for some element t in $c_s \cap P$. In this case (see 1.11 for notation) $g_P^G(c;d) = g_{Z_P(t)}^{Z_G(t)}(c_u^{(t)}; d_u^{(\pi(t))})$.

Proof. The first assertion is clear. Suppose $d \in C(P/U^+)$, $t \in c_s \cap P$ and $\pi(t) \in d_s$, Choose an element x in c such that

$x_s = t$. If $x^g \in P$ and $\pi(x^g) \in d$ then $\pi(t^g) \in d_s$ so that $t^{gy} = t$ for some element y of P. Thus $g \in Z_G(t)P$. Since the left cosets gP in $Z_G(t)P$ are in 1-1 correspondence with the cosets $gZ_P(t)$ in $Z_G(t)$ it follows from the definition of $g_P^G(c;d)$ that $g_P^G(c;d)$ is the number of cosets $gZ_P(t)$ in $Z_G(t)$ such that $x_u^g \in P$ and $\pi(x_u^g) \in d_u^{(\pi(t))}$. The result now follows.

<u>6.12</u> <u>Lemma</u>. For each element d in $\mathcal{C}_s(P/U^+)$ let $\bar{t}_d$ be a representative element in P/U^+ and let t_d be an element of P such that $\pi(t_d) = \bar{t}_d$. Moreover for each e in $\mathcal{C}_u(Z_{P/U^+}(\bar{t}_d))$ let u_e be a representative element in $Z_{P/U^+}(t_d)$. Then for all ϕ in $ch(P/U^+)$ and c in $\mathcal{C}(G)$,

(6.12.1)

$$[\mathrm{Ind}_P^G(\phi^*)](c) = \sum_{\substack{d \in \mathcal{C}_s(P/U^+) \\ d \subseteq \pi(c_s \cap P)}} \sum_{e \in \mathcal{C}_u(Z_{P/U^+}(\bar{t}_d))} g_{Z_P(t_d)}^{Z_G(t_d)}(c_u^{(t_d)};e)\phi(\bar{t}_d u_e).$$

<u>Proof</u>. We apply 6.10, 1.11 and 6.11. Thus

$$[\mathrm{Ind}_P^G(\phi^*)](c) = \sum_{e \in \mathcal{C}(P/U^+)} g_P^G(c;e)\phi(e)$$

$$= \sum_{d \in \mathcal{C}_s(P/U^+)} \sum_{e \in \mathcal{C}_u(Z_{P/U^+}(\bar{t}_d))} g_P^G(c;c_{P/U^+}(\bar{t}_d u_e))\phi(\bar{t}_d u_e)$$

$$= \sum_{\substack{d \in \mathcal{C}_s(P/U^+) \\ d \subseteq (c_s \cap P)}} \sum_{e \in \mathcal{C}_u(Z_{P/U^+}(\bar{t}_d))} g_{Z_P(t_d)}^{Z_G(t_d)}(c_u^{(t_d)};e)\phi(\bar{t}_d u_e) .$$

This completes the proof.

We will make use of the following lemma which is similar to a result in [14] .

6.13 Lemma. Suppose $\underline{\underline{c}} \in \underline{\underline{C}}(V)$, $c \in C(G)$, $d \in C(GL(\underline{\underline{c}}) \times Sp(\underline{\underline{c}}))$ and d has Wall invariants $\mathcal{W}_d$. Then if $x \in c$, $g^G_{P_{\underline{\underline{c}}}}(c;d)$ is the number of chains $\underline{\underline{c}}'$ in $\underline{\underline{C}}(V)$ of type $\nu_{\underline{\underline{c}}}$ fixed by x and such that $\pi_{\underline{\underline{c}}'}(x)$ has Wall invariants $\mathcal{W}_d$.

Proof. There is a 1-1 correspondence between the set of cosets $G/P_{\underline{\underline{c}}}$ and the set of chains $\underline{\underline{c}}'$ of type $\nu_{\underline{\underline{c}}}$ given by $gP_{\underline{\underline{c}}} \to g\underline{\underline{c}}$. Moreover $x^g \in P_{\underline{\underline{c}}}$ if and only if $x \in \text{adg}\, P_{\underline{\underline{c}}} = P_{g\underline{\underline{c}}}$. Further, $\pi_{\underline{\underline{c}}}(x^g) \in d$ if and only if $\pi_{\underline{\underline{c}}}(x^g)$ has invariants $\mathcal{W}_d$. Now adg maps $GL(\underline{\underline{c}}) \times Sp(\underline{\underline{c}})$ onto $GL(g\underline{\underline{c}}) \times Sp(g\underline{\underline{c}})$ and g maps the factor spaces associated with $\underline{\underline{c}}$ isometrically. Thus adg maps the Cayley space and form isometrically whence $(\text{adg})\pi_{\underline{\underline{c}}}(x^g)$ and $\pi_{\underline{\underline{c}}}(x^g)$ have the same Wall invariants. But $(\text{adg})\pi_{\underline{\underline{c}}}(x^g) = \pi_{g\underline{\underline{c}}}(x)$ so that $\pi_{\underline{\underline{c}}}(x^g) \in d$ if and only if $\pi_{g\underline{\underline{c}}}(x)$ has Wall invariants $\mathcal{W}_d$.

This completes the proof.

6.14 Lemma. The $g^G_P(c;d)$ where $G = Sp(6,q)$ and P is a maximal parabolic subgroup of G are as shown in table 8.

Proof. We merely illustrate the method used to determine the entries in this table in several cases. The other cases are similar.

(i) Let c have Wall invariants $2\times(x+1)^3$, $\delta_1 = 0$ and let $P = P_{(1)}$. Let $\binom{e}{d}$ be a basis of orthogonal hyperbolic pairs for V and let g be the element of Sp(V) defined by:-

$$g: \begin{cases} d_1 \mapsto d_1 + d_2 + d_3 \\ d_2 \mapsto d_2 + d_3 \\ d_3 \mapsto d_3 \\ e_3 \mapsto e_3 + e_2 \\ e_2 \mapsto e_2 + e_1 \\ e_1 \mapsto e_1 \end{cases}$$

Then g leaves invariant an element Q in $\mathcal{Q}$ if and only if $Q(e_1) = Q(e_2) = Q(d_3) = Q(d_2) = 0$. Thus g leaves invariant an element Q in $\mathcal{Q}$ for which $\Delta(Q) = 0$. This is sufficient to show that $g \in c$.

We apply lemma 6.13. A chain of type (1) corresponds to a 1-dimensional subspace, $\langle v \rangle$ say, of V. Moreover $\langle v \rangle$ is fixed by g if and only if $\langle v \rangle \leq \ker T$ where $T = 1 + g$. Now $\ker T = \langle e_1, d_3 \rangle$ so $v = \alpha e_1 + \beta d_3$ for some α, β in k not both zero, $\langle v \rangle^\perp = \langle e_1, d_3, e_2, d_2, \beta d_1 + \alpha e_3 \rangle$ and $T(\langle v \rangle^\perp / \langle v \rangle) = \langle e_1, d_3, \beta d_2 + \beta d_3 + \alpha e_2 \rangle / \langle v \rangle$. If $\beta = 0$ then $\langle v \rangle = \langle e_1 \rangle$, d_3, e_2 is a basis mod $\langle v \rangle$ of $T(\langle v \rangle^\perp / \langle v \rangle)$ and the form corresponding to the Cayley space $T(\langle v \rangle^\perp / \langle v \rangle)$ is represented by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ i.e. $\pi_{\underline{\underline{c}}}(g)$, where $\underline{\underline{c}}$ is the chain $(0) < \langle v \rangle$, has Wall invariants $2\times(x+1), (x+1)^2, s_1 = 0$.

If $\beta \neq 0$ we may choose $\beta = 1$. Thus e_1, $d_2 + d_3 + \alpha e_2$ is a basis mod $\langle v \rangle$ of the Cayley space $T(\langle v \rangle^\perp / \langle v \rangle)$ and the corresponding Cayley form is represented by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & \alpha \end{pmatrix}$. Since this is equivalent to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ if $\alpha \neq 0$ this yields q–1 possible choices for $\underline{\underline{c}}$ such that $\pi_{\underline{\underline{c}}}(g)$ has Wall invariants $2\times(x+1), (x+1)^2, s_1 = 1$. When $\alpha = 0$ the invariants are $2\times(x+1), (x+1)^2, s_1 = 0$.

(ii) Let g have Wall invariants $(x+1)^6$, δ_1 and let $P = P_{(1)}$. Choose $\binom{e}{d}$ so that ker $T = \langle e_1 \rangle$ where $T = 1 + g$. Then there is precisely one one-dimensional subspace fixed by g namely $\langle e_1 \rangle$. Let Q be an element of $\mathcal{Q}$ fixed by g. Then we observe from table 7 that $\Delta(Q) = \delta_1$. Since T is nilpotent and $T^2 \neq 0$ we can choose an element v in T(V) such that $Tv = e_1$. By lemma 6.6,
$Q(e_1) = Q_g(e_1) = [e_1,e_1] = (v,e_1) = 0$ since $T(V) = \langle e_1 \rangle^\perp$ (see 6.4). Thus Q induces a quadratic form on $\langle e_1 \rangle^\perp / \langle e_1 \rangle$ with polar form (,) which is invariant under the action of g on $\langle e_1 \rangle^\perp / \langle e_1 \rangle$ and has pseudo discriminant δ_1. This shows that the transformation induced by g on $\langle e_1 \rangle^\perp / \langle e_1 \rangle$ has Wall invariants $(x+1)^4$, δ_1.

(iii) Let c have Wall invariants $2\times(x+1)$, $2\times(x+1)^2$, $s_1 = 0$ and suppose $P = P_{(1)}$. It is easily shown that if we define g by $Te_2 = e_1$, $Td_1 = d_2$, $Te_1 = Te_3 = Td_2 = Td_3 = 0$ where $T = 1 + g$ then $g \in c$. Let K = ker T. Then $K = \langle e_1,d_2,e_3,d_3 \rangle$. We again choose 1-dimensional subspaces $\langle v \rangle$ of K.

(a) Suppose $v \in K \cap K^\perp$. If $u \in \langle v \rangle^\perp$ then $(Tu,u) = Q_g(Tu) = 0$ since $s_1 = 0$ and $Tu \in T(V) = K^\perp$. But $Tu \in K$ since $T^2 = 0$. Thus $Tu \in K \cap K^\perp \cap \langle v \rangle^\perp = \langle v \rangle$ whence $T(\langle v \rangle^\perp / \langle v \rangle) = 0$. This implies that, for the q + 1 possible $\langle v \rangle$ in $K \cap K^\perp$, g acts trivially on $\langle v \rangle^\perp / \langle v \rangle$.

(b) $v \in K$ but $v \notin K^\perp$. Then $K \not\leq \langle v \rangle^\perp$ so $V = K + \langle v \rangle^\perp$. Thus $T(V) = T(\langle v \rangle^\perp)$ and since $T(V) = K^\perp$ we find that

$$T(\langle v \rangle^\perp / \langle v \rangle) = \frac{T(\langle v \rangle^\perp) + \langle v \rangle}{\langle v \rangle} \cong \frac{T(\langle v \rangle^\perp)}{T(\langle v \rangle^\perp) \cap \langle v \rangle} = T(\langle v \rangle^\perp) = T(V) .$$

Since this isomorphism is compatible with the action of T and hence of g we see that the Wall invariants of the action of g on

$\langle v\rangle^{i}/\langle v\rangle$ are $2x(x+1)^2$, $s_1 = 0$. There are $\frac{q^4-1}{q-1} - (q+1)$ such $\langle v\rangle$.

This completes our illustration of the method used to determine the $g_P^G(c;d)$ for maximal parabolic subgroups P of G. If P is not maximal the $g_P^G(c;d)$ are obtained by applying 6.9.

<u>6.15</u> <u>Lemma</u>. Let U be a maximal unipotent subgroup of G. Then

$\Delta_G\big|_U = \sum_{(\tilde{T})\in T(\tilde{G})^F/G} \frac{|T|}{|W_{\tilde{T}}|} \rho_{\tilde{T}}^{1}\big|_U$, where the sum is over all G-conjugacy classes $(\tilde{T})$ of elements of $T(\tilde{G})^F$.

<u>Proof</u>. By (2.9.1) $\|\rho_{\tilde{T}}^{\theta}\| = |Z_{W_{\tilde{T}}}(\theta)|$ for all $\tilde{T}$ in $T(\tilde{G})^F$ and θ in $T^{\vee}$. But for θ in $T^{\vee}$ the number of elements θ' in $T^{\vee}$ such that $[(\tilde{T},\theta')] = [(\tilde{T},\theta)]$, where $[(\tilde{T},\theta)]$ denotes the G-conjugacy class of the pair $(\tilde{T},\theta)$, is $\frac{|W_{\tilde{T}}|}{|Z_{W_{\tilde{T}}}(\theta)|}$ and from (2.6.1) $\rho_{\tilde{T}}^{\theta}\big|_U = Q_{\tilde{T},G} = \rho_{\tilde{T}}^{1}\big|_U$. Thus from 3.5 and (2.11.8)

$$\begin{aligned}
\Delta_G\big|_U &= \sum_{[(\tilde{T},\theta)]} \frac{1}{\|\rho_{\tilde{T}}^{\theta}\|} \rho_{\tilde{T}}^{\theta}\big|_U \\
&= \sum_{[(\tilde{T},\theta)]} \frac{1}{|Z_{W_{\tilde{T}}}(\theta)|} \rho_{\tilde{T}}^{1}\big|_U \\
&= \sum_{(\tilde{T})\in T(G)/G} \sum_{\theta\in T^{\vee}} \frac{1}{|W_{\tilde{T}}|} \rho_{\tilde{T}}^{1}\big|_U \\
&= \sum_{(\tilde{T})\in T(G)/G} \frac{|T|}{|W_{\tilde{T}}|} \rho_{\tilde{T}}^{1}\big|_U \quad .
\end{aligned}$$

<u>6.16</u> <u>Lemma</u>. Δ_G and $q^3 1_G + St_G + \frac{1}{2}q(\chi_{(1^3),(0)} + \chi_{(0),(2,1)} + \chi_{(1),(1^2)} + \zeta_{(0),(1)}) - \frac{1}{2}q^2(\chi_{(0),(3)} + \chi_{(2,1),(0)} + \chi_{(2),(1)} + \zeta_{(1),(0)})$ take the same values on the unipotent elements of G.

Proof. We simply apply lemma 6.15 using the expressions for $\rho^1_{\tilde{T}}$ as given in table 6 and the values of $|T|$ and $|W_{\tilde{T}}|$ given in table 2.

6.17 Theorem. The values of the unipotent characters of G on the unipotent classes of G and the Green functions for G are as shown in tables 9 and 10 respectively.

Proof. Combining lemmas 5.12 and 6.16 allows us to calculate using 6.10 and tables 4 and 8 the values of the unipotent characters of G on the unipotent classes of G. The values of the Green functions then can be computed for G using table 6. The values of the Green functions for the unitary groups can be written down immediately from the results in 2.7.

6.18 Remarks. For each partition ν we will denote the dual partition by ν^*. Then in accordance with a theorem of Kawanaka [15] we have

$$(6.18.1) \qquad \sum_{\substack{u\in G \\ u \text{ unipotent}}} \chi_{\mu,\nu}(u) = |G|_p \, \chi_{\nu^*,\mu^*}(1)$$

for all ordered pairs (μ,ν) of partitions such that $|\mu| + |\nu| = n \leq 3$. But further we have

$$(6.18.2) \qquad \sum_{\substack{u\in G \\ u \text{ unipotent}}} \zeta_{\mu,\nu}(u) = |G|_p \, \zeta_{\nu^*,\mu^*}(1)$$

for all ordered pairs (μ,ν) of partitions such that $|\mu| + |\nu| = n-2$ where $n \leq 3$.

Using (2.9.1) we may rewrite (2.6.2) as follows:-

(6.18.3)

$$\frac{1}{|G|}\sum_{u\in G} Q_{\tilde{T},G}(u)Q_{\tilde{T}',G}(u) = \begin{cases} \dfrac{\|\rho^{1}_{\tilde{T}}\|}{|T|} & \text{if } \tilde{T},\tilde{T}' \text{ conjugate under } G \\ 0 & \text{otherwise.} \end{cases}$$

Then (6.18.3) can be checked to hold when $\tilde{T},\tilde{T}'$ are pseudo-tori of G for $n \leq 3$.

§7. THE IRREDUCIBLE CHARACTERS OF Sp(6,q), q EVEN

In this final section we define and establish some properties of virtual representations associated with the pseudo-tori of Sp(6,q) with positive rank defect. It is then possible to write down all the complex irreducible characters of Sp(6,q) for q even.

7.1 We rewrite (2.6.1) in the following form. If $\tilde{G}$ is a connected reductive k-group, $\tilde{T} \in \mathcal{T}(\tilde{G})^F$ and $\theta \in T^\vee$ then

(7.1.1)

$$\rho_{\tilde{T}}^{\theta}(x) = \frac{1}{|Z_G^\circ(x_s)|} \sum_{\substack{g\in G \\ \mathrm{ad}g\ Z_{\tilde{G}}^\circ(\tilde{T})\subseteq Z_{\tilde{G}}^\circ(x_s)}} Q_{\mathrm{ad}g\ \tilde{T},Z_G^\circ(x_s)}(x_u)[(\mathrm{ad}g)\theta](x_s)$$

for all elements x in G.

Our aim will be to show that for G = Sp(4,q) or Sp(6,q) if we define $\rho_{\tilde{T}}^{\theta}$ in accordance with (7.1.1.) for pseudo-tori $\tilde{T}$ of positive rank defect then such $\rho_{\tilde{T}}^{\theta}$ are indeed elements of ch(G) and satisfy certain properties.

We define the Weyl group of a pseudo-torus $\tilde{T}$ of $\tilde{G}$ as $\tilde{W}_{\tilde{T}} = N_{\tilde{G}}(\tilde{T})/Z_{\tilde{G}}^\circ(\tilde{T})$ where $Z_{\tilde{G}}(\tilde{T})$ denotes the set of elements of $\tilde{G}$ which commute with all elements of $\tilde{T}$ and $Z_{\tilde{G}}^\circ(\tilde{T})$ is the connected component of the identity in $Z_{\tilde{G}}(\tilde{T})$. As before, we define $W_{\tilde{T}} = \tilde{W}_{\tilde{T}}^F$. Further for two pseudo-tori $\tilde{T},\tilde{T}'$ in G we define

$$N_{\tilde{G}}(\tilde{T},\tilde{T}') = \{g\in\tilde{G} \mid \tilde{T} = \mathrm{ad}g\ \tilde{T}'\}$$

and $N_G(\tilde{T},\tilde{T}') = N_{\tilde{G}}(\tilde{T},\tilde{T}')^F$ which is as for proper tori (see 2.6).

Further we define $\tilde{W}(\tilde{T},\tilde{T}') = Z^{\circ}_{\tilde{G}}(\tilde{T})\backslash N_{\tilde{G}}(\tilde{T},\tilde{T}') = N_{\tilde{G}}(\tilde{T},\tilde{T}')/Z^{\circ}_{\tilde{G}}(\tilde{T}')$ and $W(\tilde{T},\tilde{T}') = \tilde{W}(\tilde{T},\tilde{T}')^F$ which is compatible with the definitions in 2.9 for proper tori.

For the purposes of computation we may rewrite (7.1.1) using the fact that $\tilde{G}$ is a semisimple simply connected group (3.4) (which allows us to apply 1.10) and the discussion in 3.14. If $G = Sp(2n,q)$ with q even and $n \leq 3$, $\tilde{T}$ is a pseudo-torus of G and $\theta \in T^{\vee}$ then for each element g of G we choose an element of G with semisimple part t in T and unipotent part u which is G-conjugate to g if this is possible. In this case

$$(7.1.2) \qquad \rho^{\theta}_{\tilde{T}}(g) = Q_{\tilde{T},Z_G(t)}(u)\zeta_\theta(t)$$

where

$$(7.1.3) \qquad \zeta_\theta(t) = \frac{1}{|Z_{W_{\tilde{T}}}(t)|} \sum_{w \in W_{\tilde{T}}} [(\mathrm{ad}w)\theta](t) = \sum_{t' \in c_{W_{\tilde{T}}}(t)} \theta(t')$$

where $c_{W_{\tilde{T}}}(t)$ denotes the set of conjugates of t under the action of $W_{\tilde{T}}$.

If g_s is not conjugate to an element of T

$$(7.1.4) \qquad \rho^{\theta}_{\tilde{T}}(g) = 0 \quad .$$

<u>7.2</u> <u>Lemma</u>. Suppose $G = Sp(4,q)$ or $Sp(6,q)$, $\tilde{T},\tilde{T}' \in P(G)$, $\theta \in T^{\vee}$ and $\theta' \in (T')^{\vee}$. Then if $\rho^{\theta}_{\tilde{T}}$ and $\rho^{\theta'}_{\tilde{T}'}$ are defined as in (7.1.1)

$$(\rho^{\theta}_{\tilde{T}},\rho^{\theta'}_{\tilde{T}'})_G = \#\{w \in W(\tilde{T},\tilde{T}') \mid (\mathrm{ad}w)\theta' = \theta\} \frac{\|\rho^{1}_{\tilde{T}}\|_G}{|W_{\tilde{T}}|} \quad .$$

<u>Proof</u>. The proof is omitted being identical with the proof of lemma 6.10 in [8] except that we use the modified orthogonality relations (6.18.3).

7.3 Lemma. If $G = Sp(4,q)$, $\tilde{T} = \tilde{T}_{(0),(0),(2)}$, $\theta = 1_T$ and ρ is defined in accordance with (7.1.1) then $\rho = \rho^1_{(0),(0),(2)}$ where $\rho^1_{(0),(0),(2)}$ as defined in 6.2.

Proof. Clearly $\rho = \rho^1_{(0),(0),(2)}$ on unipotent elements of G and ρ vanishes on all other elements of G. Thus by an easy computation $\frac{1}{|G|}\sum_{g\in G} |\rho(g)|^2 = 4 = \|\rho^1_{(0),(0),(2)}\|$ which implies that $\rho^1_{(0),(0),(2)}$ also vanishes on the non-unipotent elements of G.

7.4 Lemma. If $G = Sp(6,q)$, $\tilde{T} = \tilde{T}_{(1),(0),(2)}$, $\theta \in T^{\vee}$ and ρ is defined in accordance with (7.1.1) then

(7.4.1) $$\rho = \mathrm{Ind}^G_{P_{(1)}}(\theta\times\rho^1_{(0),(0),(2)})^*)$$

where in (7.4.1) θ is the character of $GL(1,q)$ given by the canonical isomorphism $T \cong GL(1,q)$.

Proof. Now $Q_{\tilde{T},G} = \rho^1_{(1),(0),(2)}$ on unipotent elements of G by definition where $\rho^1_{(1),(0),(2)}$ is as defined in 6.2. Thus from 6.2 it follows that (7.4.1) holds for unipotent elements of G. If c is a non-unipotent conjugacy class of G we apply 6.12 with $\phi = \theta\times\rho^1_{(0),(0),(2)}$. Since $\rho^1_{(0),(0),(2)}$ vanishes on all elements of $Sp(4,q)$ except for those with elementary divisors $(x+1)^4$ the right hand side of equation (6.12.1) vanishes unless $\bar{t}_d = t \times 1$ for some (non-identity) element t in $GL(1,q)$ and $u_e = 1 \times u$ for some element u in $Sp(4,q)$ with elementary divisors $(x+1)^4$.

Thus $[\mathrm{Ind}^G_{P_{(1)}}(\phi^*)](c) = 0$ unless $c_s = c(a,0,0)$ for some (non-zero) element a in $(q-1)^{-1}\mathbb{Z}/\mathbb{Z}$ and c_u has elementary divisors $2\times(x+1),(x+1)^4$. In this case equation (6.12.1) reduces to $[\mathrm{Ind}_{P_{(1)}}^G(\phi^*)](c) = \rho^1_{(0),(0),(2)}(u)\ (\theta(c(a))+\theta(c(-a)))$ since

$Z_{P_{(1)}}(t_d) = Z_G(t_d) = GL(1,q) \times Sp(4,q)$ when $\bar{t}_d = t \times 1$ with t in

$GL(1,q)$ and $c_s \cap P_{(1)}$ splits into two classes in $P_{(1)}$ corresponding

to the elements $t(a) \times 1$ and $t(-a) \times 1$ in $GL(1,\tilde{k}) \times Sp(4,\tilde{k})$. Comparing this with (7.1.2),(7.1.3) and (7.1.4) we find that (7.4.1) holds for all elements of G. This completes the proof.

7.5 Lemma. Let $G = Sp(6,q)$, let $Z_1 = Z_G(t_1)(=GL(1,q) \times Sp(4,q))$ where $t_1 \in c(a,0,0)$ for some non-zero a in $(q-1)^{-1}\mathbb{Z}/\mathbb{Z}$ and let $Z_2 = Z_G(t_2)(=U(1,q^2) \times Sp(4,q))$ where $t_2 \in c(b,0,0)$ for some non-zero b in $(q+1)^{-1}\mathbb{Z}/\mathbb{Z}$. For any element β in $\mathbb{Z}/(q+1)\mathbb{Z}$ we define ρ in accordance with (7.1.1) where $\tilde{T} = \tilde{T}_{(0),(1),(2)}$ and $\theta = \hat{\beta} \in T^\vee$. Then

$$\rho\big|_{Z_1} = q\phi_1 \times \rho^1_{(0),(0),(2)}$$

and

$$\rho\big|_{Z_2} = [(q-2)\phi_2 + \hat{\beta} + (\widehat{-\beta})] \times \rho^1_{(0),(0),(2)}$$

where ϕ_1 is the character of $GL(1,q)$ defined by

$$\phi_1(t) = \begin{cases} q-1 & \text{if } t = 1 \\ 0 & \text{otherwise} \end{cases}$$

and ϕ_2 is the character of $U(1,q^2)$ defined by

$$\phi_2(t) = \begin{cases} q + 1 & \text{if } t = 1 \\ 0 & \text{otherwise} \end{cases} .$$

Proof. This is by direct verification using (7.1.2), (7.1.3) and (7.1.4.).

7.6 Lemma. Let $G = Sp(6,q)$ and let ρ be defined as in lemma 7.5. Then $\rho \in ch(G)$.

Proof. We use Brauer's characterization of characters. Let E be an elementary subgroup of G, i.e. $E = \langle g\rangle \times A$ where $\langle g\rangle$ denotes the cyclic subgroup of G generated by an element g and A is a subgroup of prime power order. If E consists entirely of unipotent elements then $\rho|_E$ is by definition of ρ the restriction of a Green function which is itself the restriction of a character of G so $\rho|_E \in ch(E)$. Otherwise $\langle g\rangle$ contains a non-trivial semisimple element or A contains a non-trivial semisimple element in its centre so that in either case $E \leq Z_G(s)$ for some non-trivial semisimple element s of G. Thus the lemma will follow if we show that $\rho|_{Z_G(s)}$ is a character for each non-trivial semisimple element s of G. An examination of table 4 shows that $\rho|_{Z_G(s)} = 0$ for each non-trivial semisimple element s of G unless $s \in c(a,0,0)$ for some non-zero element a in $(q-1)^{-1}\mathbb{Z}/\mathbb{Z}$ or $s \in c(b,0,0)$ for some non-zero element b in $(q+1)^{-1}\mathbb{Z}/\mathbb{Z}$. But by 7.5 $\rho|_{Z_G(s)}$ is a character in each of these cases. The result now follows.

7.7 Lemma. Let $G = Sp(6,q)$ and let ρ be as defined in 7.5 when $\beta = 0$. Then $\rho = \rho^1_{(0),(1),(2)}$ where $\rho^1_{(0),(1),(2)}$ is the character defined in 6.2.

Proof. By 7.2, $(\rho,\rho^1_{\tilde{T}})_G = 0$ for any proper torus T of G or for $T = T_{(1),(0),(2)}$. Using this in conjunction with the definitions of such $\rho^1_{\tilde{T}}$ as appearing in table 6 we find immediately that

$$(7.7.1) \qquad \rho = m\rho^1_{(0),(1),(2)} + \chi$$

for some number m and some class function χ which is orthogonal to all unipotent characters of G. Now by direct computation we can show that $(\rho,\mathrm{Ind}^G_{O_+}(1))_G = (\rho|_{O_+},1)_{O_+} = -1$ and $(\rho,\mathrm{Ind}^G_{O_-}(1))_G = (\rho|_{O_-},1)_{O_-} = +1$ whence by 5.12(i) $(\rho,\chi_{(2),(1)} - \chi_{(1,2),(0)}) = -2$. But $\rho \in ch(G)$ by 7.6 so comparing this relation with (7.7.1) and (6.2.3) it follows that $m = 1$. Finally since by 7.2 $\|\rho\|_G = 8$ and by (6.2.3) $\|\rho^1_{(0),(1),(2)}\|_G = 8$ it follows that $\chi = 0$ as required.

7.8 We are now in a position to construct the irreducible characters of G where $G = Sp(4,q)$ or $Sp(6,q)$. For each extended geometric conjugacy class $[\xi]$ of G shown in table 3 and for each i as shown in column 2 of table 11 we define the class function $\chi^{(i)}_{[\xi]}$ on G by

$$(7.8.1) \qquad \chi^{(i)}_{[\xi]} = \sum_{[(\tilde{T},\theta)]\in[\xi]/G} \frac{c^{(i)}_{\tilde{T},\theta}}{\|\rho^{\theta}_{\tilde{T}}\|} \rho^{\theta}_{\tilde{T}}$$

where the sum is over all G-conjugacy classes $[(\tilde{T},\theta)]$ of pairs $(\tilde{T},\theta)$ in $[\xi]$ and the values $c^{(i)}_{\tilde{T},\theta}$ are as shown in table 11.

7.9 Lemma. The functions $\chi^{(i)}_{[\xi]}$ have positive degree and

$$(\chi^{(i)}_{[\xi]},\chi^{(i')}_{[\xi']})_G = \begin{cases} 1 & \text{if } [\xi] = [\xi'],\ i = i' \\ 0 & \text{otherwise} . \end{cases}$$

Proof. The first claim can be checked by direct computation. If $[\xi] \neq [\xi']$ it follows immediately from (7.8.1) and 7.2 that $(\chi^{(i)}_{[\xi]},\chi^{(i')}_{[\xi']})_G = 0$ for any i and i'. Now applying the orthogonality property implicit in 7.2 to (7.8.1) we obtain

$$(7.9.1) \qquad \chi(^{(i)}_{[\xi]},\ ^{(i')}_{[\xi]})_G = \sum_{[(\tilde{T},\theta)]\in[\xi]/G} \frac{c^{(i)}_{\tilde{T},\theta}\ c^{(i')}_{\tilde{T},\theta}}{\|\rho^{\theta}_{\tilde{T}}\|}$$

for all i and i'. The result now follows from an examination of the $c^{(i)}_{\tilde{T},\theta}$ and $\|\rho^{\theta}_{\tilde{T}}\|$ in tables 11 and 2 respectively.

7.10 Lemma. The functions $\chi^{(i)}_{[1\times 1]}$, $i = 1,\ldots,6$ are the unipotent characters of Sp(4,q). The functions $\chi^{(i)}_{[1\times 1\times 1]}$, $i = 1,\ldots,12$ are the unipotent characters of Sp(6,q).

Proof. Lemmas 7.3, 7.4 and 7.7 show the equivalence of (7.1.1) when $\theta = 1$ and the definitions in table 6. The lemma can thus be verified directly from table 6.

7.11 Lemma. If G = Sp(4,q) or Sp(6,q) and $[\xi]$ is an extended geometric conjugacy class of G then $\chi^{(i)}_{[\xi]} \in \text{irr}(G)$ for i = 1 or 2.

Proof. In each of these cases $\chi^{(i)}_{[\xi]}$ is in one of the forms (2.11.5) or (2.11.6) and is therefore known to be in irr(G).

<u>7.12</u> <u>Theorem</u>. If $G = Sp(4,q)$ the $\chi^{(i)}_{[\xi]}$ defined in table 9 form a complete set of irreducible characters of G.

<u>Proof</u>. All such characters are irreducible by 7.10 and 7.11. But since the $\chi^{(i)}_{[\xi]}$ exhaust the irreducible components in the $\rho^{\theta}_{\tilde{T}}$ for $\tilde{T}$ in $T(\tilde{G})^F$ and θ in $T^{\vee}$ the set is complete by (2.10.1).

<u>7.13</u> <u>Lemma</u>. If $G = Sp(6,q)$ the functions $\chi^{(i)}_{[\xi]}$ are irreducible characters of G for the following values of $[\xi]$ and i

(a) $[\xi] = [\hat{\alpha}\times\hat{\alpha}\times 1]$ or $[\hat{\beta}\times\hat{\beta}\times 1]$, $i = 3,4$

(b) $[\xi] = [\hat{\alpha}\times\hat{\alpha}\times\hat{\alpha}]$ or $[\hat{\beta}\times\hat{\beta}\times\hat{\beta}]$, $i = 3$.

<u>Proof</u>. We will denote by $\zeta^{(1)}_{\hat{\alpha}\times\hat{\alpha}}$ and $\zeta^{(2)}_{\hat{\alpha}\times\hat{\alpha}}$ the characters of $GL(2,q)$ defined by

$$\zeta^{(1)}_{\hat{\alpha}\times\hat{\alpha}} = \tfrac{1}{2}\,\rho^{\hat{\alpha}\times\hat{\alpha}}_{(1^2)} + \tfrac{1}{2}\,\rho^{\hat{\alpha}*}_{(2)} \qquad \text{and}$$

$$\zeta^{(2)}_{\hat{\alpha}\times\hat{\alpha}} = \tfrac{1}{2}\,\rho^{\hat{\alpha}\times\hat{\alpha}}_{(1^2)} - \tfrac{1}{2}\rho^{\hat{\alpha}*}_{(2)} \qquad \text{respectively,}$$

where ρ^{θ}_{μ} denotes the virtual character $\rho^{\theta}_{\tilde{T}_\mu}$ for θ in $T^{\vee}_{\mu}$ with $\tilde{T}_\mu$ a maximal torus of $GL(2,\tilde{k})$ twisted by an element of cycle type μ. That $\zeta^{(1)}_{\hat{\alpha}\times\hat{\alpha}}$ and $\zeta^{(2)}_{\hat{\alpha}\times\hat{\alpha}}$ are proper (in fact irreducible) characters of $GL(2,q)$ is evident from [14] or from methods similar to those employed in 7.10 and 7.11. Now from table 11

$$\chi^{(1)}_{[1]} = \tfrac{1}{2}\,\rho^{1}_{(1),(0)} + \tfrac{1}{2}\,\rho^{1}_{(0),(1)} \quad .$$

Thus from (2.8.3) ,

$$\mathrm{Ind}_{P_{(2)}}^{G}((\zeta_{\alpha\times\alpha}^{(2)}\times\chi_{[1]}^{(1)})^{*}) = \tfrac{1}{4}\rho_{(1^3),(0)}^{\hat{\alpha}\times\hat{\alpha}\times 1} + \tfrac{1}{4}\rho_{(1^2),(1)}^{\hat{\alpha}\times\hat{\alpha}\times 1} - \tfrac{1}{4}\rho_{(1,2),(0)}^{1\times\hat{\alpha}*}$$

$$- \tfrac{1}{4}\rho_{(2),(1)}^{\hat{\alpha}*\times 1} = \chi_{[\hat{\alpha}\times\hat{\alpha}\times 1]}^{(3)}$$

Similarly $\mathrm{Ind}_{P_{(2)}}^{G}((\zeta_{\hat{\alpha}\times\hat{\alpha}}^{(1)} \times \chi_{[1]}^{(2)})*) = \chi_{[\hat{\alpha}\times\hat{\alpha}\times 1]}^{(4)}$. Thus $\chi_{[\hat{\alpha}\times\hat{\alpha}\times 1]}^{(i)} \in \mathrm{ch}(G)$ for i = 3.4.

Now

$$\chi_{[\hat{\beta}\times\hat{\beta}]}^{(1)} = -\tfrac{1}{2}\rho_{(2),(0)}^{\hat{\beta}*} - \tfrac{1}{2}\rho_{(0),(1^2)}^{\hat{\beta}\times\hat{\beta}}$$

so that

$$\mathrm{Ind}_{P_{(1)}}^{G}((1\ \chi_{[\hat{\beta}\times\hat{\beta}]}^{(1)})*) = -\tfrac{1}{2}\rho_{(1,2),(0)}^{1\times\hat{\beta}*} - \tfrac{1}{2}\rho_{(1),(1^2)}^{1\times\hat{\beta}\times\hat{\beta}}$$

$$= \chi_{[1\times\hat{\beta}\times\hat{\beta}]}^{(1)} + \chi_{[1\times\hat{\beta}\times\hat{\beta}]}^{(3)} .$$

Thus since $\chi_{[1\times\hat{\beta}\times\hat{\beta}]}^{(1)}$ is in ch(G) by 7.11 so is $\chi_{[1\times\hat{\beta}\times\hat{\beta}]}^{(3)}$.
Similarly $\chi_{[1\times\hat{\beta}\times\hat{\beta}]}^{(4)} \in \mathrm{ch}(G)$.

(b) From table 11,

$$\chi_{[\hat{\alpha}\times\hat{\alpha}\times\hat{\alpha}]}^{(3)} = -\rho_{(3),(0)}^{\hat{\alpha}**} + \chi_{[\hat{\alpha}\times\hat{\alpha}\times\hat{\alpha}]}^{(1)} + \chi_{[\hat{\alpha}\times\hat{\alpha}\times\hat{\alpha}]}^{(2)} \in \mathrm{ch}(G) .$$

Similarly $\chi_{[\hat{\beta}\times\hat{\beta}\times\hat{\beta}]}^{(3)} = \rho_{(0),(3)}^{\hat{\beta}**} - \chi_{[\hat{\beta}\times\hat{\beta}\times\hat{\beta}]}^{(1)} + \chi_{[\hat{\beta}\times\hat{\beta}\times\hat{\beta}]}^{(2)} \in \mathrm{ch}(G)$.

All these characters are irreducible by 7.9. This completes the proof.

7.14 Lemma. Let $G = Sp(6,q)$. Then $\chi^{(i)}_{[\hat{\alpha}\times 1\times 1]} \in ch(G)$ for $i = 3,\dots,6$.

Proof. From table 11 and (2.8.3)

$$\mathrm{Ind}^G_{P_{(1)}}((\hat{\alpha}\times\chi^{(3)}_{[1\times 1]})^*) = \mathrm{Ind}^G_{P_{(1)}}((\hat{\alpha}\times(\tfrac{1}{8}\rho^1_{(1^2),(1)} + \tfrac{1}{4}\rho^1_{(1),(1)} - \tfrac{1}{4}\rho^1_{(2),(0)} - \tfrac{1}{8}\rho^1_{(0),(1^2)} - \tfrac{1}{4}\rho^1_{(0),(0),(2)}))^*)$$

$$= \tfrac{1}{8}\rho^{\hat{\alpha}\times 1\times 1}_{(1^3),(0)} + \tfrac{1}{4}\rho^{\hat{\alpha}\times 1\times 1}_{(1^2),(1)} - \tfrac{1}{4}\rho^{\hat{\alpha}\times 1}_{(1,2),(0)} - \tfrac{1}{8}\rho^{\hat{\alpha}\times 1\times 1}_{(1),(1^2)} - \tfrac{1}{4}\rho^{\hat{\alpha}}_{(1),(0),(2)}$$

$$= \chi^{(3)}_{[\hat{\alpha}\times 1\times 1]} .$$

Thus $\chi^{(3)}_{[\hat{\alpha}\times 1\times 1]} \in ch(G)$. The rest of the proof is similar.

7.15 Lemma. Let $G = Sp(6,q)$. Then $\chi^{(i)}_{[\hat{\beta}\times 1\times 1]} \in ch(G)$ for $i = 3,\dots,6$.

Proof. From table 11 and (2.8.3),

$$\mathrm{Ind}^G_{P_{(1)}}((1\times\chi^{(2)}_{[1\times\hat{\beta}]})^*) = -\tfrac{1}{2}\rho^{1\times 1\times\hat{\beta}}_{(1^2),(1)} + \tfrac{1}{2}\rho^{1\times\hat{\beta}\times 1}_{(1),(1^2)} .$$

Thus $-\tfrac{1}{2}\rho^{1\times 1\times\hat{\beta}}_{(1^2),(1)} + \tfrac{1}{2}\rho^{1\times\hat{\beta}\times 1}_{(1),(1^2)}$ is a proper character of G which, by table 2 and (2.9.1), has norm 3 and by 7.11 and 7.8 contains $\chi^{(2)}_{[1\times 1\times\hat{\beta}]}$ as an irreducible component occurring with multiplicity 1. We may therefore find two distinct elements ζ_β and θ_β of irr(G) such that

$$(7.15.1) \qquad \zeta_\beta + \theta_\beta = -\tfrac{1}{2}\rho^{1\times 1\times\hat\beta}_{(1^2),(1)} + \tfrac{1}{2}\rho^{1\times\hat\beta\times 1}_{(1),(1^2)} - \chi^{(2)}_{[1\times 1\times\hat\beta]}\,.$$

Similarly $-\tfrac{1}{2}\rho^{1\times 1\times\hat\beta}_{(1^2),(1)} + \tfrac{1}{2}\rho^{1\times\hat\beta}_{(2),(1)} - \chi^{(2)}_{[1\times 1\times\hat\beta]}$ is a proper character of G with norm 2 which has one irreducible component, ζ_β say, in common with $\zeta_\beta + \theta_\beta$. Thus there is an irreducible character η_β of G distinct from ζ_β and θ_β such that

$$(7.15.2) \qquad \zeta_\beta + \eta_\beta = -\tfrac{1}{2}\rho^{1\times 1\times\hat\beta}_{(1^2),(1)} + \tfrac{1}{2}\rho^{1\times\hat\beta}_{(2),(1)} - \chi^{(2)}_{[1\times 1\times\hat\beta]}\,.$$

Now define

$$(7.15.3) \qquad \phi_\beta = \rho^{\hat\beta\times 1}_{(0),(1,2)} + \chi^{(1)}_{[1\times 1\times\hat\beta]} + \chi^{(2)}_{[1\times 1\times\hat\beta]}$$

and

$$(7.15.4) \qquad \psi_\beta = \rho^{\hat\beta\times 1\times 1}_{(0),(1^3)} + \chi^{(1)}_{[1\times 1\times\hat\beta]} + \chi^{(2)}_{[1\times 1\times\hat\beta]}\,.$$

Then (by (7.9.1), table 2 and table 9) the following relations hold:-

$$(7.15.5) \qquad \|\phi_\beta\| = 2$$

$$(7.15.6) \qquad \|\psi_\beta\| = 6$$

$$(7.15.7) \qquad (\phi_\beta, \psi_\beta) = -2$$

$$(7.15.8) \qquad (\phi_\beta, \zeta_\beta + \phi_\beta) = (\phi_\beta, \zeta_\beta + \eta_\beta) = 1$$

$$(7.15.9) \qquad (\psi_\beta, \zeta_\beta + \theta_\beta) = (\psi_\beta, \zeta_\beta + \eta_\beta) = 1.$$

From (7.15.5) and (7.15.8) it follows that either $\phi_\beta = \theta_\beta + \eta_\beta$ or $\phi_\beta = \zeta_\beta + \lambda_\beta$ where $\lambda_\beta \in \mathrm{ch}(G)$, $\|\lambda_\beta\| = 1$ and $\lambda_\beta \neq \pm\chi^{(i)}_{[1\times 1\times\hat\beta]}(i=1,2)$, $\pm\theta_\beta$, $\pm\eta_\beta$, $\pm\zeta_\beta$. But in both cases (7.15.6), (7.15.7) and (7.15.9) allow us to evaluate ψ_β. We obtain the following alternatives:-

(7.15.10) $\phi_\beta = \theta_\beta + \eta_\beta$ and $\psi_\beta = -\theta_\beta - \eta_\beta + 2\zeta_\beta$

or

(7.15.11) $\phi_\beta = \zeta_\beta + \lambda_\beta$ and $\psi_\beta = \theta_\beta + \eta_\beta - 2\lambda_\beta$.

Suppose now that $\beta' \in \mathbb{Z}/(q+1)\mathbb{Z}$, $\beta' \neq 0$ and write ρ for $\rho^{\hat{\beta}'}_{(0),(1),(2)}$. Then, by 7.2,

(7.15.12) $\|\rho\| = 4$,

(7.15.13) $(\rho, \chi^{(1)}_{[1\times1\times\hat{\beta}]}) = (\rho, \chi^{(2)}_{[1\times1\times\hat{\beta}]}) = 0$

(7.15.14) $(\rho, \zeta_\beta + \theta_\beta) = (\rho, \zeta_\beta + \eta_\beta) = 0$

and

(7.15.15) $(\rho, \phi_\beta) = (\rho, \psi_\beta) = 0$.

If we assume that (7.15.10) holds then (7.15.14) and (7.15.15) imply that

(7.15.16) $(\rho, \theta_\beta) = (\rho, \eta_\beta) = (\rho, \zeta_\beta) = 0$. Moreover,

equations (7.15.1) - (7.15.4) together with (7.15.10) imply that:-

$$\rho^{1\times1\times\hat{\beta}}_{(1^2),(1)} = -\chi^{(1)}_{[1\times1\times\hat{\beta}]} - \chi^{(2)}_{[1\times1\times\hat{\beta}]} - \theta_\beta - \eta_\beta - 2\zeta_\beta ,$$

$$\rho^{1\times\hat{\beta}\times1}_{(1),(1^2)} = -\chi^{(1)}_{[1\times1\times\hat{\beta}]} + \chi^{(2)}_{[1\times1\times\hat{\beta}]} + \theta_\beta - \eta_\beta ,$$

$$\rho^{1\times\hat{\beta}}_{(2),(1)} = -\chi^{(1)}_{[1\times1\times\hat{\beta}]} + \chi^{(2)}_{[1\times1\times\hat{\beta}]} - \theta_\beta + \eta_\beta ,$$

$$\rho^{\hat\beta\times 1}_{(0),(1,2)} = -\chi^{(1)}_{[1\times 1\times\hat\beta]} - \chi^{(2)}_{[1\times 1\times\hat\beta]} + \theta_\beta + \eta_\beta,$$

$$\rho^{\hat\beta\times 1\times 1}_{(0),(1^3)} = -\chi^{(1)}_{[1\times 1\times\hat\beta]} - \chi^{(2)}_{[1\times 1\times\hat\beta]} - \theta_\beta - \eta_\beta + 2\zeta_\beta \quad .$$

Thus under the condition that (7.15.10) holds ρ has no irreducible components in common with any virtual character corresponding to the geometric conjugacy class $[1\times 1\times\hat\beta]$.

In lemmas 7.10, 7.11, 7.13 and 7.14 we identified all irreducible components of the virtual characters of G corresponding to geometric conjugacy classes of G other than those of the form $[1\times 1\times\hat\beta]$. By 7.2, none of these irreducible components appear as components of ρ . Since each irreducible component of ρ must appear in some virtual character of G (see (2.10.1)), it is now clear that we can choose a non-zero element β in $\mathbb{Z}/(q+1)\mathbb{Z}$ such that ρ has an irreducible component in common with a virtual character corresponding to the geometric conjugacy class $[1\times 1\times\hat\beta]$ and that for this β equation (7.15.11) holds. Again it is easily shown that $\chi^{(1)}_{[1\times 1\times\hat\beta]}$, $\chi^{(2)}_{[1\times 1\times\hat\beta]}$, θ_β, η_β, ζ_β and $\pm\lambda_\beta$ are the only irreducible components corresponding to $[1\times 1\times\hat\beta]$. An examination of equations (7.15.12) - (7.15.15) together with (7.15.11) now reveals that

(7.15.17) $$\rho = \varepsilon(\theta_\beta + \eta_\beta - \zeta_\beta + \lambda_\beta)$$

where $\varepsilon = \pm 1$. Now from (7.15.1), (7.15.2), (7.15.3), (7.15.11) and (7.15.17) we obtain

(7.15.18) $$\zeta_\beta = -\tfrac{1}{4}\rho^{1\times 1\times\hat\beta}_{(1^2),(1)} + \tfrac{1}{4}\rho^{\hat\beta\times 1}_{(0),(1,2)} - \frac{\varepsilon}{4}\rho^{\hat\beta'}_{(0),(1),(2)}.$$

If we let Z_2 be as in lemma 7.5 and compute directly using the values of the Green polynomials we find

$$(7.15.19) \qquad (\rho^{1\times 1\times\hat{\beta}}_{(1^2),(1)},\ \hat{\beta}\times\chi^{(5)}_{[1\times 1]})_{Z_2} = -2(q-1)$$

$$(7.15.20) \qquad (\rho^{\hat{\beta}\times 1}_{(0),(1,2)},\ \hat{\beta}\times\chi^{(5)}_{[1\times 1]})_{Z_2} = -(q-1)$$

and

$$(7.15.21) \qquad (\rho^{\hat{\beta}'}_{(0),(1),(2)},\ \hat{\beta}\times\chi^{(5)}_{[1\times 1]})_{Z_2} = \begin{cases} q-1 & \text{if } \beta' = \pm\beta \\ q-2 & \text{otherwise} \end{cases}$$

This implies

$$(7.15.22) \quad (\zeta_\beta,\ \hat{\beta}\times\chi^{(5)}_{[1\times 1]})_{Z_2} = \begin{cases} \tfrac{1}{4}(q-1)\ (1-\varepsilon) & \text{if } \beta' = \pm\beta \\ \tfrac{1}{4}[\,q(1-\varepsilon)-1+2\varepsilon\,] & \text{otherwise} \end{cases}$$

Since $(\zeta_\beta,\hat{\beta}\times\chi^{(5)}_{[1\times 1]})_{Z_2}$ is an integer and q is even $\varepsilon = 1$ and $\beta' = \pm\beta$.

The remaining details are now easily established.

<u>7.16</u> <u>Theorem</u>. For $G = Sp(6,q)$, q even, the characters $\chi^{(i)}_{[\xi]}$ as defined in table 11 form a complete set of irreducible characters for G.

<u>Proof</u>. The proof follows from 7.9, 7.10, 7.11, 7.13, 7.14 and 7.15.

APPENDIX

For each of the tables shown it is assumed that $p = \text{char}\ \tilde{k} = 2$ and $n \leq 3$. Notations and observations not appearing in the footnotes may be found in the text. These tables provide, among other things, sufficient information for the determination of the value of any irreducible character of $Sp(6,q)$, $q = p^{\alpha}$, on any conjugacy class.

TABLE 1 : Some Notations

(i) Notation for elements in $\mathbb{Q}/\mathbb{Z}$ of or $q^n \pm 1$.

Notation for element in $\mathbb{Q}/\mathbb{Z}$	Definition and Conditions	Conjugates	Number of distinct classes of conjugates
a	$(q-1)a=0, a\neq 0$	$a,-a$	$\frac{1}{2}(q-2)$
b	$(q+1)b=0, b\neq 0$	$b,-b$	$\frac{1}{2}q$
c	$(q^2-1)c=0, (q-1)c\neq 0, (q+1)c\neq 0$	$c,-c,qc,-qc$	$\frac{1}{4}q(q-2)$
d	$(q^2+1)d=0, d\neq 0$	$d,-d,qd,-qd$	$\frac{1}{4}q^2$
e	$(q^3-1)e=0, (q-1)e\neq 0$	$e,-e,qe,-qe,q^2e,-q^2e$	$\frac{1}{6}q(q^2-1)$
f	$(q^3+1)f=0, (q+1)f\neq 0$	$f,-f,qf,-qf,q^2f,-q^2f$	$\frac{1}{6}q(q^2-1)$

(ii) Notation for elements in $(\tilde{T}^{F*})^{\vee}$ where $\tilde{T} \in T(\tilde{G})^F$ and $F^* = \mathrm{ad}\,w \circ F|_{\tilde{T}}$ for w a signed cycle in $\tilde{W}_{\tilde{T}}$ of order $\tilde{\sigma}(\tilde{T}) \leq 3$.

$w \in W_{\tilde{T}}$	$\mathrm{Im}(\tilde{T}^{F*} \to \mathbb{Q}/\mathbb{Z})$ (see (3.8.1))	Character notation	Definition and conditions	Conjugates	Number of distinct classes of conjugates
(1)	$(q-1)^{-1}\mathbb{Z}/\mathbb{Z}$	$\hat{\alpha}$	$\alpha \in \frac{\mathbb{Z}}{(q-1)\mathbb{Z}}, \alpha \neq 0$	$\hat{\alpha}, -\hat{\alpha}$	$\frac{1}{2}(q-2)$
(1-1)	$(q+1)^{-1}\mathbb{Z}/\mathbb{Z}$	$\hat{\beta}$	$\beta \in \frac{\mathbb{Z}}{(q+1)\mathbb{Z}}, \beta \neq 0$	$\hat{\beta}, -\hat{\beta}$	$\frac{1}{2}q$
(12)	$(q^2-1)^{-1}\mathbb{Z}/\mathbb{Z}$	$\hat{\gamma}$	$\gamma \in \frac{\mathbb{Z}}{(q^2-1)\mathbb{Z}}, (q-1) \nmid \gamma, (q+1) \nmid \gamma$	$\hat{\gamma}, -\hat{\gamma}, q\hat{\gamma}, -q\hat{\gamma}$	$\frac{1}{4}q(q-2)$
		$\hat{\alpha}^*$	$\alpha \in \frac{\mathbb{Z}}{(q-1)\mathbb{Z}} \xrightarrow{\sim *} \frac{(q+1)\mathbb{Z}}{(q^2-1)\mathbb{Z}}, \alpha \neq 0$	$\hat{\alpha}^*, -\hat{\alpha}^*$	$\frac{1}{2}(q-2)$
		$\hat{\beta}^*$	$\beta \in \frac{\mathbb{Z}}{(q+1)\mathbb{Z}} \xrightarrow{\sim *} \frac{(q-1)\mathbb{Z}}{(q^2-1)\mathbb{Z}}, \beta \neq 0$	$\hat{\beta}^*, -\hat{\beta}^*$	$\frac{1}{2}q$
(12-1-2)	$(q^2+1)^{-1}\mathbb{Z}/\mathbb{Z}$	$\hat{\delta}$	$\delta \in \frac{\mathbb{Z}}{(q^2+1)\mathbb{Z}}, \delta \neq 0$	$\hat{\delta}, -\hat{\delta}, q\hat{\delta}, -q\hat{\delta}$	$\frac{1}{4}q^2$
(123)	$(q^3-1)^{-1}\mathbb{Z}/\mathbb{Z}$	$\hat{\varepsilon}$	$\varepsilon \in \frac{\mathbb{Z}}{(q^3-1)\mathbb{Z}}, (q^2+q+1) \nmid \varepsilon$	$\hat{\varepsilon}, -\hat{\varepsilon}, q\hat{\varepsilon}, -q\hat{\varepsilon}, q^2\hat{\varepsilon}, -q^2\hat{\varepsilon}$	$\frac{1}{6}q(q^2-1)$
		$\hat{\alpha}^{**}$	$\alpha \in \frac{\mathbb{Z}}{(q-1)\mathbb{Z}} \xrightarrow{\sim **} \frac{(q^2+q+1)\mathbb{Z}}{(q^3-1)\mathbb{Z}}, \alpha \neq 0$	$\hat{\alpha}^{**}, -\hat{\alpha}^{**}$	$\frac{1}{2}q(q-2)$
(123-1-2-3)	$(q^3+1)^{-1}\mathbb{Z}/\mathbb{Z}$	$\hat{\eta}$	$\eta \in \frac{\mathbb{Z}}{(q^3+1)\mathbb{Z}}, (q^2-q+1) \nmid \eta$	$\hat{\eta}, -\hat{\eta}, q\hat{\eta}, -q\hat{\eta}, q^2\hat{\eta}, -q^2\hat{\eta}$	$\frac{1}{6}q(q^2-1)$
		$\hat{\beta}^{**}$	$\beta \in \frac{\mathbb{Z}}{(q+1)\mathbb{Z}} \xrightarrow{\sim **} \frac{(q^2-q+1)\mathbb{Z}}{(q^3-1)\mathbb{Z}}, \beta \neq 0$	$\hat{\beta}^{**}, -\hat{\beta}^{**}$	$\frac{1}{2}q$

TABLE 2*: $\tilde{T}^{F*}$, $(\tilde{T}^{F*})^{\vee}$ and $\|\rho^{\theta}_{\mu,\nu}\|$ for $G = Sp(2n,q)$

(i) n=1

Cycle type μ,ν	Class representative w in $W_{\tilde{T}}$	Structure of $\tilde{T}^{F*}$	Character $\theta\epsilon(\tilde{T}^{F*})^{\vee}$	$Z_{\tilde{W}_{\tilde{T}}^{F*}}(\theta)$	$\|\rho^{\theta}_{\mu,\nu}\|$	Number of distinct classes of conjugate θ's.
(1),(0)	1	C_{q-1}	1	W	2	1
			$\hat{\alpha}$	1	1	$\frac{1}{2}(q-2)$
(0),(1)	(1-1)	C_{q+1}	1	W	2	1
			$\hat{\beta}$	1	1	$\frac{1}{2}q$

* Subscripts on characters are used to distinguish non-conjugate characters having the same notation in table 1 (ii).

(ii) $\underline{n = 2}$

Cycle type μ,ν or μ,ν,λ	Class representative w in $\tilde{W}_{\tilde{T}}$	Structure of $\tilde{T}^{F^*}$	Character $\theta\in(\tilde{T}^{F^*})^\vee$	$Z_{\tilde{W}_{\tilde{T}}^{F^*}}(\theta)$	$\Vert\rho^{\theta}_{\mu,\nu}\Vert$ or $\Vert\rho^{\theta}_{\mu,\nu,\lambda}\Vert$	Number of distinct classes or conjugates θ's.
$(1^2),(0)$	1	C^2_{q-1}	1	W	8	1
			$\hat{\alpha}\times 1$	$\langle(2\text{-}2)\rangle$	2	$\frac{1}{2}(q-2)$
			$\hat{\alpha}\times\hat{\alpha}$	$\langle(12)\rangle$	2	$\frac{1}{2}(q-2)$
			$\hat{\alpha}_1\times\hat{\alpha}_2$	1	1	$\frac{1}{8}(q-2)(q-4)$
$(1),(1)$	$(2\text{-}2)$	$C_{q-1}\times C_{q+1}$	1	$\langle(1\text{-}1),(2\text{-}2)\rangle$	4	1
			$\hat{\alpha}\times 1$	$\langle(2\text{-}2)\rangle$	2	$\frac{1}{2}(q-2)$
			$1\times\hat{\beta}$	$\langle(1\text{-}1)\rangle$	2	$\frac{1}{2}q$
			$\hat{\alpha}\times\hat{\beta}$	1	1	$\frac{1}{4}q(q-2)$
$(2),(0)$	(12)	C_{q^2-1}	1	$\langle(12),(1\text{-}1)(2\text{-}2)\rangle$	4	1
			$\hat{\alpha}^*$	$\langle(12)\rangle$	2	$\frac{1}{2}(q-2)$
			$\hat{\beta}^*$	$\langle(1\text{-}2)\rangle$	2	$\frac{1}{2}q$
			$\hat{\gamma}$	1	1	$\frac{1}{4}q(q-2)$
$(0),(2)$	$(12\text{-}1\text{-}2)$	C_{q^2+1}	1	$\langle(12\text{-}1\text{-}2)\rangle$	4	1
			$\hat{\delta}$	1	1	$\frac{1}{4}q^2$
$(0),(1^2)$	$(1\text{-}1)(2\text{-}2)$	C^2_{q+1}	1	W	8	1
			$\hat{\beta}\times 1$	$\langle(2\text{-}2)\rangle$	2	$\frac{1}{2}q$
			$\hat{\beta}\times\hat{\beta}$	$\langle(12)\rangle$	2	$\frac{1}{2}q$
			$\hat{\beta}_1\times\hat{\beta}_2$	1	1	$\frac{1}{4}q(q-2)$
$(0),(0),(2)$	—	1	1	—	4	1

(iii) $\underline{n = 3}$

Cycle type μ,ν or μ,ν,λ	Class representative w in $\tilde{W}_{\tilde{T}}$	Structure of $\tilde{T}^{F*}$	Character $\sigma\epsilon(\tilde{T}^{F*})$	$Z_{\tilde{W}_{\tilde{T}}^{F*}}(\theta)$	$\lVert\rho^{\theta}_{\mu,\nu}\rVert$ or $\lVert\rho^{\theta}_{\mu,\nu,\lambda}\rVert$	Number of distinct classes of conjugate θ's.
$(1^3),(0)$	1	C^3_{q-1}	1	W	48	1
			$\hat{\alpha}\times 1\times 1$	$\langle(2-2),(23)\rangle$	8	$\frac{1}{2}(q-2)$
			$\hat{\alpha}\times\hat{\alpha}\times 1$	$\langle(12),(3-3)\rangle$	4	$\frac{1}{2}(q-2)$
			$\hat{\alpha}\times\hat{\alpha}\times\hat{\alpha}$	$\langle(12),(23)\rangle$	6	$\frac{1}{2}(q-2)$
			$\hat{\alpha}_1\times\hat{\alpha}_2\times 1$	$\langle(3-3)\rangle$	2	$\frac{1}{8}(q-2)(q-4)$
			$\hat{\alpha}_1\times\hat{\alpha}_1\times\hat{\alpha}_2$	$\langle(12)\rangle$	2	$\frac{1}{4}(q-2)(q-4)$
			$\hat{\alpha}_1\times\hat{\alpha}_2\times\hat{\alpha}_3$	1	1	$\frac{1}{48}(q-2)(q-4)(q-6)$
$(1^2),(1)$	(3-3)	$C^2_{q-1}\times C_{q+1}$	1	$\langle(1-1),(12),(3-3)\rangle$	16	1
			$\hat{\alpha}\times 1\times 1$	$\langle(2-2),(3-3)\rangle$	4	$\frac{1}{2}(q-2)$
			$\hat{\alpha}\times\hat{\alpha}\times 1$	$\langle(12),(3-3)\rangle$	4	$\frac{1}{2}(q-2)$
			$\hat{\alpha}_1\times\hat{\alpha}_2\times 1$	$\langle(3-3)\rangle$	2	$\frac{1}{8}(q-2)(q-4)$
			$1\times 1\times\hat{\beta}$	$\langle(1-1),(12)\rangle$	8	$\frac{1}{2}q$
			$\hat{\alpha}\times 1\times\hat{\beta}$	$\langle(2-2)\rangle$	2	$\frac{1}{4}(q-2)$
			$\hat{\alpha}\times\hat{\alpha}\times\hat{\beta}$	$\langle(12)\rangle$	2	$\frac{1}{4}q(q-2)$
			$\hat{\alpha}_1\times\hat{\alpha}_2\times\hat{\beta}$	1	1	$\frac{1}{16}q(q-2)(q-4)$
$(1,2),(0)$	(2-3)	$C_{q-1}\times C_{q^2-1}$	1	$\langle(1-1),(23),(2-2)(3-3)\rangle$	8	1
			$\hat{\alpha}\times 1$	$\langle(23),(2-2)(3-3)\rangle$	4	$\frac{1}{2}(q-2)$
			$1\times\hat{\alpha}^*$	$\langle(1-1),(23)\rangle$	4	$\frac{1}{2}(q-2)$
			$\hat{\alpha}\times\hat{\alpha}^*$	$\langle(23)\rangle$	2	$\frac{1}{2}(q-2)$
			$\hat{\alpha}_1\times\hat{\alpha}_2^*$	$\langle(23)\rangle$	2	$\frac{1}{8}(q-2)(q-4)$
			$1\times\hat{\beta}^*$	$\langle(1-1),(2-3)\rangle$	4	$\frac{1}{2}q$
			$\hat{\alpha}\times\hat{\beta}^*$	$\langle(2-3)\rangle$	2	$\frac{1}{4}q(q-2)$
			$1\times\hat{\gamma}$	$\langle(1-1)\rangle$	2	$\frac{1}{4}q(q-2)$
			$\hat{\alpha}\times\hat{\gamma}$	1	1	$\frac{1}{8}q(q-2)^2$

Cycle type μ,ν or μ,ν,λ	Class representative w in $\tilde{W}_{\tilde{T}}$	Structure of $\tilde{T}^{F*}$	Character $\theta\in(\tilde{T}^{F*})^{\vee}$	$Z_{\tilde{W}^{F*}_{\tilde{T}}}(\theta)$	$\Vert\rho^{\theta}_{\mu,\nu}\Vert$ or $\Vert\rho^{\theta}_{\mu,\nu,\lambda}\Vert$	Number of distinct classes of conjugate θ's.
(1),(2)	(23-2-3)	$C_{q-1}\times C_{q^2+1}$	1	<(1-1),(23-2-3)>	8	1
			$\hat{\alpha}\times 1$	<(23-2-3)>	4	$\frac{1}{2}(q-2)$
			$1\times\hat{\delta}$	<(1-1)>	2	$\frac{1}{4}q^2$
			$\hat{\alpha}\times\hat{\delta}$	1	1	$\frac{1}{8}q^2(q-2)$
$(1),(1^2)$	(2-2)(3-3)	$C_{q-1}\times C^2_{q+1}$	1	<(1-1),(2-2),(23)>	16	1
			$\hat{\alpha}\times 1\times 1$	<(2-2),(23)>	8	$\frac{1}{2}(q-2)$
			$1\times\hat{\beta}\times 1$	<(1-1),(3-3)>	4	$\frac{1}{2}q$
			$\hat{\alpha}\times\hat{\beta}\times 1$	<(3-3)>	2	$\frac{1}{2}q(q-2)$
			$1\times\hat{\beta}\times\hat{\beta}$	<(1-1),(23)>	4	$\frac{1}{2}q$
			$\hat{\alpha}\times\hat{\beta}\times\hat{\beta}$	<(23)>	2	$\frac{1}{4}q(q-2)$
			$1\times\hat{\beta}_1\times\hat{\beta}_2$	<(1-1)>	2	$\frac{1}{8}q(q-2)$
			$\alpha\times\hat{\beta}_1\times\hat{\beta}_2$	1	1	$\frac{1}{16}q(q-2)^2$
(3),(0)	(123)	C_{q^3-1}	1	<(123),-1>	6	1
			$\hat{\alpha}^{**}$	<(123)>	3	$\frac{1}{2}(q-2)$
			$\hat{\varepsilon}$	1	1	$\frac{1}{6}q(q^2-1)$
(2),(1)	(12)(3-3)	$C_{q^2-1}\times C_{q+1}$	1	<(12),(1-1)(2-2),(3-3)>	8	1
			$\hat{\alpha}^{*}\times 1$	<(12),(3-3)>	4	$\frac{1}{2}(q-2)$
			$\hat{\beta}^{*}\times 1$	<(1-2),(3-3)>	4	$\frac{1}{2}q$
			$\hat{\gamma}\times 1$	<(3-3)>	2	$\frac{1}{4}q(q-2)$
			$1\times\hat{\beta}$	<(12),(1-1)(2-2)>	4	$\frac{1}{2}q$
			$\hat{\alpha}^{*}\times\hat{\beta}$	<(12)>	2	$\frac{1}{4}q(q-2)$
			$\hat{\beta}^{*}\times\hat{\beta}$	<(1-2)>	2	$\frac{1}{2}q$
			$\hat{\beta}^{*}_1\times\hat{\beta}_2$	<(1-2)>	2	$\frac{1}{4}q^2$
			$\hat{\gamma}\times\hat{\beta}$	1	1	$\frac{1}{8}q^2(q-2)$

Cycle type μ,ν or μ,ν,λ	Class representative w in $\tilde{W}_{\tilde{T}}$	Structure of $\tilde{T}^{F*}$	Character $\theta_\epsilon(T^{F*})^\vee$	$Z_{\dot{W}_{\tilde{T}}^{F*}}(\theta)$	$\Vert\rho^\theta_{\mu,\nu}\Vert$ or $\Vert\rho^\theta_{\mu,\nu,\lambda}\Vert$	Number of distinct classes of conjugate θ's.
(0),(3)	(123-1-2-3)	C_{q^3+1}	1	<(123-1-2-3)>	6	1
			$\hat{\beta}**$	<(13-2)>	3	$\frac{1}{2}q$
			$\hat{\eta}$	1	1	$\frac{1}{6}q(q^2-1)$
(0),(1,2)	(1-1)(23-2-3)	$C_{q+1}\times C_{q^2+1}$	1	<(1-1),(23-2-3)>	8	1
			$\hat{\beta}\times 1$	<(23-2-3)>	4	$\frac{1}{2}q$
			$1\times\hat{\delta}$	<(1-1)>	2	$\frac{1}{4}q^2$
			$\hat{\beta}\times\hat{\delta}$	1	1	$\frac{1}{8}q^3$
$(0),(1^3)$	-1	C^3_{q+1}	1	W	48	1
			$\hat{\beta}\times 1\times 1$	<(2-2),(23)>	8	$\frac{1}{2}q$
			$\hat{\beta}\times\hat{\beta}\times 1$	<(12),(3-3)>	4	$\frac{1}{2}q$
			$\hat{\beta}\times\hat{\beta}\times\hat{\beta}$	<(12),(23)>	6	$\frac{1}{2}q$
			$\hat{\beta}_1\times\hat{\beta}_2\times 1$	<(3-3)>	2	$\frac{1}{8}q(q-2)$
			$\hat{\beta}_1\times\hat{\beta}_1\times\hat{\beta}_2$	<(12)>	2	$\frac{1}{4}q(q-2)$
			$\hat{\beta}_1\times\hat{\beta}_2\times\hat{\beta}_3$	1	1	$\frac{1}{48}q(q-2)(q-4)$
(1),(0),(2)	—	C_{q-1}	1	—	8	1
			$\hat{\alpha}$	—	4	$\frac{1}{2}(q-2)$
(0),(1),(2)	—	C_{q+1}	1	—	8	1
			$\hat{\beta}$	—	4	$\frac{1}{2}q$

TABLE 3 : Geometric conjugacy classes for $G = Sp(2n,q)$

(i) $n = 1$

Geometric conjugacy class symbol \ Torus index	(1),(0)	(0),(1)
[1]	1	1
[$\hat{\alpha}$]	$\hat{\alpha}$	
[$\hat{\beta}$]		$\hat{\beta}$

(ii) <u>n = 2</u>

Geometric conjugacy class symbol \ Torus index	$(1^2),(0)$	$(1),(1)$	$(2),(0)$	$(0),(2)$	$(0),(1^2)$	$(0),(0),(2)$
$[1\times 1]$	1	1	1	1	1	1
$[\hat{\alpha}\times 1]$	$\hat{\alpha}\times 1$	$\hat{\alpha}\times 1$				
$[\hat{\alpha}\times\hat{\alpha}]$	$\hat{\alpha}\times\hat{\alpha}$		$\hat{\alpha}^*$			
$[\hat{\alpha}_1\times\hat{\alpha}_2]$	$\hat{\alpha}_1\times\hat{\alpha}_2$					
$[1\times\hat{\beta}]$		$1\times\hat{\beta}$			$\hat{\beta}\times 1$	
$[\hat{\alpha}\times\hat{\beta}]$		$\hat{\alpha}\times\hat{\beta}$				
$[\hat{\beta}\times\hat{\beta}]$			$\hat{\beta}^*$		$\hat{\beta}\times\hat{\beta}$	
$[\hat{\beta}_1\times\hat{\beta}_2]$					$\hat{\beta}_1\times\hat{\beta}_2$	
$[\hat{\gamma}]$			$\hat{\gamma}$			
$[\hat{\delta}]$				$\hat{\delta}$		

(iii) <u>n = 3</u>

Geometric conjugacy class \ Torus index	$(1^3),(0)$	$(1^2),(1)$	$(1,2),(0)$	$(1),(2)$	$(1),(1^2)$	$(3),(0)$	$(2),(1)$	$(0),(3)$	$(0),(1,2)$	$(0),(1^3)$	$(1),(0)$	$(0),(1)$
$[1\times 1\times 1]$	1	1	1	1	1	1	1	1	1	1	1	1
$[\hat\alpha\times 1\times 1]$	$\hat\alpha\times 1\times 1$	$\hat\alpha\times 1\times 1$	$\hat\alpha\times 1$	$\hat\alpha\times 1$	$\hat\alpha\times 1\times 1$						$\hat\alpha$	
$[\hat\alpha\times\hat\alpha\times 1]$	$\hat\alpha\times\hat\alpha\times 1$	$\hat\alpha\times\hat\alpha\times 1$	$1\times\hat\alpha^*$				$\hat\alpha^*\times 1$					
$[\hat\alpha\times\hat\alpha\times\hat\alpha]$	$\hat\alpha\times\hat\alpha\times\hat\alpha$		$\hat\alpha\times\hat\alpha^*$			$\hat\alpha^{**}$						
$[\hat\alpha_1\times\hat\alpha_2\times 1]$	$\hat\alpha_1\times\hat\alpha_2\times 1$	$\hat\alpha_1\times\hat\alpha_2\times 1$										
$[\hat\alpha_1\times\hat\alpha_1\times\hat\alpha_2]$	$\hat\alpha_1\times\hat\alpha_1\times\hat\alpha_2$		$\hat\alpha_2\times\hat\alpha_1^*$									
$[\hat\alpha_1\times\hat\alpha_2\times\hat\alpha_3]$	$\hat\alpha_1\times\hat\alpha_2\times\hat\alpha_3$											
$[1\times 1\times\hat\beta]$		$1\times 1\times\hat\beta$			$1\times\hat\beta\times 1$		$1\times\hat\beta$		$\hat\beta\times 1$	$\hat\beta\times 1\times 1$		$\hat\beta$
$[\hat\alpha\times 1\times\hat\beta]$		$\hat\alpha\times 1\times\hat\beta$			$\hat\alpha\times\hat\beta\times 1$							
$[\hat\alpha\times\hat\alpha\times\hat\beta]$		$\hat\alpha\times\hat\alpha\times\hat\beta$					$\hat\alpha^*\times\hat\beta$					
$[\hat\alpha_1\times\hat\alpha_2\times\hat\beta]$		$\hat\alpha_1\times\hat\alpha_2\times\hat\beta$										
$[1\times\hat\beta\times\hat\beta]$			$1\times\hat\beta^*$		$1\times\hat\beta\times\hat\beta$		$\hat\beta^*\times 1$			$\hat\beta\times\hat\beta\times 1$		
$[\hat\alpha\times\hat\beta\times\hat\beta]$			$\hat\alpha\times\hat\beta^*$		$\hat\alpha\times\hat\beta\times\hat\beta$							
$[\hat\beta\times\hat\beta\times\hat\beta]$							$\hat\beta^*\times\hat\beta$	$\hat\beta^{**}$		$\hat\beta\times\hat\beta\times\hat\beta$		
$[1\times\hat\beta_1\times\hat\beta_2]$					$1\times\hat\beta_1\times\hat\beta_2$					$\hat\beta_1\times\hat\beta_2\times 1$		
$[\hat\alpha\times\hat\beta_1\times\hat\beta_2]$					$\hat\alpha\times\hat\beta_1\times\hat\beta_2$							
$[\hat\beta_1\times\hat\beta_1\times\hat\beta_2]$							$\hat\beta_1^*\times\hat\beta_2$			$\hat\beta_1\times\hat\beta_1\times\hat\beta_2$		
$[\hat\beta_1\times\hat\beta_2\times\hat\beta_3]$										$\hat\beta_1\times\hat\beta_2\times\hat\beta_3$		
$[1\times\hat\gamma]$			$1\times\hat\gamma$				$\hat\gamma\times 1$					
$[\hat\alpha\times\hat\gamma]$			$\hat\alpha\times\hat\gamma$									
$[\hat\gamma\times\hat\beta]$							$\hat\gamma\times\hat\beta$					
$[1\times\hat\delta]$				$1\times\hat\delta$					$1\times\hat\delta$			
$[\hat\alpha\times\hat\delta]$				$\hat\alpha\times\hat\delta$								
$[\hat\beta\times\hat\delta]$									$\hat\beta\times\hat\delta$			
$[\hat\varepsilon]$						$\hat\varepsilon$						
$[\hat\eta]$								$\hat\eta$				

TABLE 4 : Centralizers of semisimple elements

(A) Semisimple conjugacy classes, centralizers and pseudo-tori for $G = Sp(2n,q)$*.

(i) $n = 1$.

Conjugacy class $\tilde{c} \in \mathcal{C}_s(\tilde{G})^F$	Structure of $Z_{\tilde{G}}(t), t \in \tilde{c}$	Structure of $Z_G(s), s \in \tilde{c}^F$	Index of orbit $\mathcal{O} \in \mathcal{T}(Z_{\tilde{G}}(s))^F/Z_G(s)$	Index of $\mathcal{O}^G$ in $\mathcal{T}(\tilde{G})^F/G$.
$\tilde{c}(0)$	$Sp(2,\tilde{k})$	$Sp(2,q)$	(1),(0)	(1),(0)
			(0),(1)	(0),(1)
$\tilde{c}(a)$	$GL(1,\tilde{k})$	$GL(1,q)$	(1)	(1),(0)
$\tilde{c}(b)$	$GL(1,\tilde{k})$	$U(1,q^2)$	(1)	(0),(1)

* (a) $\mathcal{O}^G = \{ad g(\tilde{T}') \mid \tilde{T}' \in \mathcal{O},\ g \in G\}$

(b) For $Z_{\tilde{G}}(t) = Sp(2,\tilde{k}) \times GL(2,\tilde{k})$, say, (1),(0) : (1^2) denotes the maximal torus in $Z_{\tilde{G}}(t)$ whose component in $Sp(2,\tilde{k})$ is parametrized by (1),(0) and whose component in $GL(2,\tilde{k})$ is parametrized by (1^2) (see 3.14).

(c) Note that $\mathcal{T}(\tilde{G}) \subseteq \mathcal{P}(\tilde{G})$ and $\mathcal{T}(\tilde{G})^F/G \subseteq \mathcal{P}(\tilde{G})^F/G$. See 6.1 for definition of $\mathcal{P}(\tilde{G})$.

(ii) <u>n = 2</u>

Conjugacy class $\tilde{c}\in C_s(\tilde{G})^F$	Structure of $Z_{\tilde{G}}(t), t\in\tilde{c}$	Structure of $Z_G(s), s\in\tilde{c}^F$	Index of orbit $\mathcal{O}\in P(Z_{\tilde{G}}(s))^F/Z_G(s)$	Index of $\mathcal{O}^G$ in $P(\tilde{G})^F/G$.
$\tilde{c}(0,0)$	$Sp(4,\tilde{k})$	$Sp(4,q)$	$(1^2),(0)$ $(1),(1)$ $(2),(0)$ $(0),(2)$ $(0),(1^2)$ $(0),(0),(2)$	$(1^2),(0)$ $(1),(1)$ $(2),(0)$ $(0),(2)$ $(0),(1^2)$ $(0),(0),(2)$
$\tilde{c}(a,0)$	$GL(1,\tilde{k})\times Sp(2,\tilde{k})$	$GL(1,q)\times Sp(2,q)$	$(1) : (1),(0)$ $(1) : (0),(1)$	$(1^2),(0)$ $(1),(1)$
$\tilde{c}(a,a)$	$GL(2,\tilde{k})$	$GL(2,q)$	(1^2) (2)	$(1^2),(0)$ $(2),(0)$
$\tilde{c}(a_1,a_2)$	$GL(1,\tilde{k})\times GL(1,\tilde{k})$	$GL(1,q)^2$	$(1) : (1)$	$(1^2),(0)$
$\tilde{c}(0,b)$	$Sp(2,\tilde{k})\times GL(1,\tilde{k})$	$Sp(2,q)\times U(1,q^2)$	$(1),(0) : (1)$ $(0),(1) : (1)$	$(1),(1)$ $(0),(1^2)$
$\tilde{c}(a,b)$	$GL(1,\tilde{k})\times GL(1,\tilde{k})$	$GL(1,q)\times U(1,q^2)$	$(1) : (1)$	$(1),(1)$
$\tilde{c}(b,b)$	$GL(2,\tilde{k})$	$U(2,q^2)$	(1^2) (2)	$(0),(1^2)$ $(2),(0)$
$\tilde{c}(b_1,b_2)$	$GL(1,\tilde{k})\times GL(1,\tilde{k})$	$U(1,q^2)^2$	$(1) : (1)$	$(0),(1^2)$
$\tilde{c}(c,qc)$	$GL(1,\tilde{k})^2$	$GL(1,q^2)$	(1)	$(2),(0)$
$\tilde{c}(d,qd)$	$GL(1,\tilde{k})^2$	$U(1,q^4)$	(1)	$(0),(2)$

(iii) <u>n = 3</u>

Conjugacy class $\tilde{c}\in C_s(\tilde{G})^F$	Structure of $Z_{\tilde{G}}(t), t\in\tilde{c}$	Structure of $Z_G(s), s\in\tilde{c}^F$	Index of orbit $\mathcal{O}\in P(Z_{\tilde{G}}(s))^F/Z_G(s)$	Index of $\mathcal{O}^G$ in $P(\tilde{G})^F/G$
$\tilde{c}(0,0,0)$	$Sp(6,\tilde{k})$	$Sp(6,q)$	$(1^3),(0)$	$(1^3),(0)$
			$(1^2),(1)$	$(1^2),(1)$
			$(1,2),(0)$	$(1,2),(0)$
			$(1),(2)$	$(1),(2)$
			$(1),(1^2)$	$(1),(1^2)$
			$(3),(0)$	$(3),(0)$
			$(2),(1)$	$(2),(1)$
			$(0),(3)$	$(0),(3)$
			$(0),(1,2)$	$(0),(1,2)$
			$(0),(1^3)$	$(0),(1^3)$
			$(1),(0),(2)$	$(1),(0),(2)$
			$(0),(1),(2)$	$(0),(1),(2)$
$\tilde{c}(a,0,0)$	$GL(1,k)\times Sp(4,\tilde{k})$	$GL(1,q)\times Sp(4,q)$	$(1) : (1^2),(0)$	$(1^3),(0)$
			$(1) : (1),(1)$	$(1^2),(1)$
			$(1) : (2),(0)$	$(1,2),(0)$
			$(1) : (0),(2)$	$(1),(2)$
			$(1) : (0),(1^2)$	$(1),(1^2)$
			$(1) : (0),(0),(2)$	$(1),(0),(2)$
$\tilde{c}(a,a,0)$	$GL(2,\tilde{k})\times Sp(2,\tilde{k})$	$GL(2,q)\times Sp(2,q)$	$(1^2) : (1),(0)$	$(1^3),(0)$
			$(1^2) : (0),(1)$	$(1^2),(1)$
			$(2) : (1),(0)$	$(1,2),(0)$
			$(2) : (0),(1)$	$(2),(1)$
$\tilde{c}(a,a,a)$	$GL(3,\tilde{k})$	$GL(3,q)$	(1^3)	$(1^3),(0)$
			$(1,2)$	$(1,2),(0)$
			(3)	$(3),(0)$

Conjugacy class $\tilde{c}\in C_s(\tilde{G})^F$	Structure of $Z_{\tilde{G}}(t)$, $t\in\tilde{c}$	Structure of $Z_G(s)$, $s\in\tilde{c}^F$	Index of orbit $O\in P(Z_{\tilde{G}}(s)^F/Z_G(s)$	Index of O^G in $P(\tilde{G})^F/G$
$\tilde{c}(a_1,a_2,0)$	$GL(1,\tilde{k})\times GL(1,\tilde{k})\times Sp(2,\tilde{k})$	$GL(1,q)^2\times Sp(2,q)$	$(1) : (1) : (1),(0)$	$(1^3),(0)$
			$(1) : (1) : (0),(1)$	$(1^2),(1)$
$\tilde{c}(a_1,a_1,a_2)$	$GL(2,\tilde{k})\times GL(1,\tilde{k})$	$GL(2,q)\times GL(1,q)$	$(1^2) : (1)$	$(1^3),(0)$
			$(2) : (1)$	$(1,2),(0)$
$\tilde{c}(a_1,a_2,a_3)$	$GL(1,\tilde{k})\times GL(1,\tilde{k})\times GL(1,\tilde{k})$	$GL(1,q)^3$	$(1) : (1) : (1)$	$(1^3),(0)$
$\tilde{c}(0,0,b)$	$Sp(4,\tilde{k})\times GL(1,\tilde{k})$	$Sp(4,q)\times U(1,q^2)$	$(1^2),(0) : (1)$	$(1^2)(1)$
			$(1),(1) : (1)$	$(1),(1^2)$
			$(2),(0) : (1)$	$(2),(1)$
			$(0),(2) : (1)$	$(0),(1,2)$
			$(0),(1^2) : (1)$	$(0),(1^3)$
			$(0),(0),(2) : (1)$	$(0),(1),(2)$
$\tilde{c}(a,0,b)$	$GL(1,\tilde{k})\times Sp(2,\tilde{k})\times GL(1,\tilde{k})$	$GL(1,q)\times Sp(2,q)\times U(1,q^2)$	$(1) : (1),(0) : (1)$	$(1^2),(1)$
			$(1) : (0),(1) : (1)$	$(1),(1^2)$
$\tilde{c}(a,a,b)$	$GL(2,\tilde{k})\times GL(1,\tilde{k})$	$GL(2,q)\times U(1,q^2)$	$(1^2) : (1)$	$(1^2),(1)$
			$(2) : (1)$	$(2),(1)$
$\tilde{c}(a_1,a_2,b)$	$GL(1,\tilde{k})\times GL(1,\tilde{k})\times GL(1,\tilde{k})$	$GL(1,q)^2\times U(1,q^2)$	$(1) : (1) : (1)$	$(1^2),(1)$
$\tilde{c}(0,b,b)$	$Sp(2,\tilde{k})\times GL(2,\tilde{k})$	$Sp(2,q)\times U(2,q^2)$	$(1),(0) : (1^2)$	$(1),(1^2)$
			$(1),(0) : (2)$	$(1,2),(0)$
			$(0),(1) : (1^2)$	$(0),(1^3)$
			$(0),(1) : (2)$	$(2),(1)$

Conjugacy class $\tilde{c} \in \mathcal{C}_s(\tilde{G})^F$	Strucutre of $Z_{\tilde{G}}(t), t\in\tilde{c}$	Structure of $Z_G(s), s\in\tilde{c}^F$	Index of orbit $\mathcal{O}\in\mathcal{P}(Z_{\tilde{G}}(s))^F/Z_G(s)$	Index of $\mathcal{O}^G$ in $\mathcal{P}(\tilde{G})^F/G$
$\tilde{c}(a,b,b)$	$GL(1,\tilde{k})\times GL(2,\tilde{k})$	$G(1,q)\times U(2,q^2)$	$(1) : (1^2)$ $(1) : (2)$	$(1),(1^2)$ $(1,2)(0)$
$\tilde{c}(b,b,b)$	$GL(3,\tilde{k})$	$U(3,q^2)$	(1^3) $(1,2)$ (3)	$(0),(1^3)$ $(2),(1)$ $(0),(3)$
$\tilde{c}(0,b_1,b_2)$	$Sp(2,\tilde{k})\times GL(1,\tilde{k})\times GL(1,\tilde{k})$	$Sp(2,q)\times U(1,q^2)^2$	$(1),(0) : (1) : (1)$ $(0),(1) : (1) : (1)$	$(1),(1^2)$ $(0),(1^3)$
$\tilde{c}(a,b_1,b_2)$	$GL(1,\tilde{k})\times GL(1,\tilde{k})\times GL(1,\tilde{k})$	$GL(1,q)\times U(1,q^2)^2$	$(1) : (1) : (1)$	$(1),(1^2)$
$\tilde{c}(b_1,b_1,b_2)$	$GL(2,\tilde{k})\times GL(1,\tilde{k})$	$U(2,q^2)\times U(1,q^2)$	$(1^2) : (1)$ $(2) : (1)$	$(0),(1^3)$ $(2),(1)$
$\tilde{c}(b_1,b_2,b_3)$	$GL(1,\tilde{k})\times GL(1,\tilde{k})\times GL(1,\tilde{k})$	$U(1,q^2)^3$	$(1) : (1) : (1)$	$(0),(1^3)$
$\tilde{c}(0,c,qc)$	$Sp(2,\tilde{k})\times GL(1,\tilde{k})^2$	$Sp(2,q)\times GL(1,q^2)$	$(1),(0) : (1)$ $(0),(1) : (1)$	$(1,2),(0)$ $(2),(1)$
$\tilde{c}(a,c,qc)$	$GL(1,\tilde{k})\times GL(1,\tilde{k})^2$	$GL(1,q)\times GL(1,q^2)$	$(1) : (1)$	$(1,2),(0)$
$\tilde{c}(c,qc,b)$	$GL(1,\tilde{k})^2\times GL(1,\tilde{k})$	$GL(1,q^2)\times U(1,q^2)$	$(1) : (1)$	$(2),(1)$
$\tilde{c}(0,d,qd)$	$Sp(2,\tilde{k})\times GL(1,\tilde{k})^2$	$Sp(2,q)\times U(1,q^4)$	$(1),(0) : (1)$ $(0),(1) : (1)$	$(1),(2)$ $(0),(1,2)$
$\tilde{c}(a,d,qd)$	$GL(1,\tilde{k})\times GL(1,\tilde{k})^2$	$GL(1,q)\times U(1,q^4)$	$(1) : (1)$	$(1),(2)$

Conjugacy class $\tilde{c} \in C_s(\tilde{G})^F$	Structure of $Z_{\tilde{G}}(t), t \in \tilde{c}$	Structure of $Z_G(s), s \in \tilde{c}^F$	Index of orbit $\mathcal{O} \in \mathcal{P}(Z_{\tilde{G}}(s))^F / Z_G(s)$	Index of $\mathcal{O}^G$ in $\mathcal{P}(\tilde{G})^F/G$.
$\tilde{c}(b,d,qd)$	$GL(1,\tilde{k}) \times GL(1,\tilde{k})^2$	$U(1,q^2) \times U(1,q^4)$	(1) : (1)	(0),(1,2)
$\tilde{c}(e,qe,q^2e)$	$GL(1,\tilde{k})^3$	$GL(1,q^3)$	(1)	(3),(0)
$\tilde{c}(f,qf,q^2f)$	$GL(1,\tilde{k})^3$	$U(1,q^6)$	(1)	(0),(3)

(B) Orders of centralizers of semisimple elements in $Sp(2n,q)$, $O_+(2n,q)$ and $O_-(2n,q)$. *

(i) n = 1

Conjugacy class $\tilde{c}$ in $C_s(\tilde{G})^F$	Number of such classes	$\lvert Z_G(s)\rvert, s\in\tilde{c}^F$	$\lvert Z_{O_+}(s)\rvert, s\in\tilde{c}\cap O_+$	$\lvert Z_{O_-}(s)\rvert, s\in\tilde{c}\cap O_-$
$\tilde{c}(0)$	1	$q(q^2-1)$	$2(q-1)$	$2(q+1)$
$\tilde{c}(a)$	$\frac{1}{2}q(-2)$	$q-1$	$q-1$	
$\tilde{c}(b)$	$\frac{1}{2}q$	$q+1$		$q+1$

* (a) $O_+ = O_+(2n,q)$ and $O_- = O_-(2n,q)$.

(b) $\tilde{c} \cap O_+$ is either void or a conjugacy class in O_+. Similarly for $\tilde{c} \cap O_-$.

(ii) <u>$n = 2$</u>

Conjugacy class $\tilde{c}$ in $C_s(\tilde{G})^F$	Number of such classes	$\lvert Z_G(s)\rvert, s\in\tilde{c}^F$	$\lvert Z_{O_+}(s)\rvert, s\in\tilde{c}\cap O_+$	$\lvert Z_{O_-}(s)\rvert, s\in\tilde{c}\cap O_-$
$\tilde{c}(0,0)$	1	$q^4(q^4-1)(q^2-1)$	$2q^2(q^2-1)^2$	$2q^2(q^4-1)$
$\tilde{c}(a,0)$	$\frac{1}{2}(q-2)$	$q(q^2-1)(q-1)$	$2(q-1)^2$	$2(q^2-1)$
$\tilde{c}(a,a)$	$\frac{1}{2}(q-2)$	$q(q^2-1)(q-1)$	$q(q^2-1)(q-1)$	
$\tilde{c}(a_1,a_2)$	$\frac{1}{8}(q-2)(q-4)$	$(q-1)^2$	$(q-1)^2$	
$\tilde{c}(0,b)$	$\frac{1}{2}q$	$q(q^2-1)(q+1)$	$2(q+1)^2$	$2(q^2-1)$
$\tilde{c}(a,b)$	$\frac{1}{4}q(q-2)$	q^2-1		q^2-1
$\tilde{c}(b,b)$	$\frac{1}{2}q$	$q(q^2-1)(q+1)$	$q(q^2-1)(q+1)$	
$\tilde{c}(b_1,b_2)$	$\frac{1}{8}q(q-2)$	$(q+1)^2$	$(q+1)^2$	
$\tilde{c}(c,qc)$	$\frac{1}{4}q(q-2)$	q^2-1	q^2-1	
$\tilde{c}(d,qd)$	$\frac{1}{4}q^2$	q^2+1		q^2+1

(iii) $n = 3$

Conjugacy class c in $\mathcal{C}_s(\tilde{G})^F$	Number of such classes	$\lvert Z_G(s)\rvert$, $s\in\tilde{c}^F$	$\lvert Z_{O_+}(s)\rvert$, $s\in\tilde{c}\cap O_+$	$\lvert Z_{O_-}(s)\rvert$, $s\in\tilde{c}\cap O_-$
$\tilde{c}(0,0,0)$	1	$q^9(q^6-1)(q^4-1)(q^2-1)$	$2q^6(q^3-1)(q^4-1)(q^2-1)$	$2q^6(q^3+1)(q^4-1)(q^2-1)$
$\tilde{c}(a,0,0)$	$\frac{1}{2}(q-2)$	$q^4(q^4-1)(q^2-1)(q-1)$	$2q^2(q^2-1)^2(q-1)$	$2q^2(q^4-1)(q-1)$
$\tilde{c}(a,a,0)$	$\frac{1}{2}(q-2)$	$q^2(q^2-1)^2(q-1)$	$2q(q^2-1)(q-1)^2$	$2q(q^2-1)^2$
$\tilde{c}(a,a,a)$	$\frac{1}{2}(q-2)$	$q^3(q^3-1)(q^2-1)(q-1)$	$q^3(q^3-1)(q^2-1)(q-1)$	
$\tilde{c}(a_1,a_2,0)$	$\frac{1}{8}(q-2)(q-4)$	$q(q^2-1)(q-1)^2$	$2(q-1)^3$	$2(q^2-1)(q-1)$
$\tilde{c}(a_1,a_1,a_2)$	$\frac{1}{4}(q-2)(q-4)$	$q(q^2-1)(q-1)^2$	$q(q^2-1)(q-1)^2$	
$\tilde{c}(a_1,a_2,a_3)$	$\frac{1}{48}(q-2)(q-4)(q-6)$	$(q-1)^3$	$(q-1)^3$	
$\tilde{c}(0,0,b)$	$\frac{1}{2}q$	$q^4(q^4-1)(q^2-1)(q+1)$	$2q^2(q^4-1)(q+1)$	$2q^2(q^2-1)^2(q+1)$
$\tilde{c}(a,0,b)$	$\frac{1}{4}q(q-2)$	$q(q^2-1)^2$	$2(q^2-1)(q+1)$	$2(q^2-1)(q-1)$
$\tilde{c}(a,a,b)$	$\frac{1}{4}q(q-2)$	$q(q^2-1)^2$		$q(q^2-1)^2$
$\tilde{c}(a_1,a_2,b)$	$\frac{1}{16}q(q-2)(q-4)$	$(q^2-1)(q-1)$		$(q^2-1)(q-1)$
$\tilde{c}(0,b,b)$	$\frac{1}{2}q$	$q^2(q^2-1)^2(q+1)$	$2q(q^2-1)^2$	$2q(q^2-1)(q+1)^2$
$\tilde{c}(a,b,b)$	$\frac{1}{4}q(q-2)$	$q(q^2-1)^2$	$q(q^2-1)^2$	
$\tilde{c}(b,b,b)$	$\frac{1}{2}q$	$q^3(q^3+1)(q^2-1)(q+1)$		$q^3(q^3+1)(q^2-1)(q+1)$
$\tilde{c}(0,b_1,b_2)$	$\frac{1}{8}q(q-2)$	$q(q^2-1)(q+1)^2$	$2(q^2-1)(q+1)$	$2(q+1)^3$
$\tilde{c}(a,b_1,b_2)$	$\frac{1}{16}q(q-2)^2$	$(q^2-1)(q+1)$	$(q^2-1)(q+1)$	
$\tilde{c}(b_1,b_1,b_2)$	$\frac{1}{4}q(q-2)$	$q(q^2-1)(q+1)^2$		$q(q^2-1)(q+1)^2$
$\tilde{c}(b_1,b_2,b_3)$	$\frac{1}{48}q(q-2)(q-4)$	$(q+1)^3$		$(q+1)^3$
$\tilde{c}(0,c,qc)$	$\frac{1}{4}q(q-2)$	$q(q^2-1)^2$	$2(q^2-1)(q-1)$	$2(q^2-1)(q+1)$
$\tilde{c}(a,c,qc)$	$\frac{1}{8}q(q-2)^2$	$(q^2-1)(q-1)$	$(q^2-1)(q-1)$	
$\tilde{c}(c,qc,b)$	$\frac{1}{8}q^2(q-2)$	$(q^2-1)(q+1)$		$(q^2-1)(q+1)$
$\tilde{c}(0,d,qd)$	$\frac{1}{4}q^2$	$q(q^4-1)$	$2(q^2+1)(q+1)$	$2(q^2+1)(q-1)$
$\tilde{c}(a,d,qd)$	$\frac{1}{8}q^2(q-2)$	$(q^2+1)(q-1)$		$(q^2+1)(q-1)$
$\tilde{c}(b,d,qd)$	$\frac{1}{8}q^3$	$(q^2+1)(q+1)$	$(q^2+1)(q+1)$	
$\tilde{c}(e,qe,q^2e)$	$\frac{1}{6}q(q^2-1)$	q^3-1	q^3-1	
$\tilde{c}(f,qf,q^2f)$	$\frac{1}{6}q(q^2-1)$	q^3+1		q^3+1

TABLE 5 : Irreducible characters for Weyl groups with Dynkin diagram of type C_n.

n = 2

Character\class	$(1^2),(0)$	$(1),(1)$	$(2),(0)$	$(0),(2)$	$(0),(1^2)$
$\phi_{(2),(0)}$	1	1	1	1	1
$\phi_{(0),(1^2)}$	1	-1	-1	1	1
$\phi_{(1^2),(0)}$	1	1	-1	-1	1
$\phi_{(0),(2)}$	1	-1	1	-1	1
$\phi_{(1),(1)}$	2	0	0	0	-2

n = 3

Character\class	$(1^3),(0)$	$(1^2),(1)$	$(1,2)(0)$	$(1),(2)$	$(1),(1^2)$	$(3),(0)$	$(2),(1)$	$(0),(3)$	$(0)(1,2)$	$(0),(1^3)$
$\phi_{(3),(0)}$	1	1	1	1	1	1	1	1	1	1
$\phi_{(0),(1^3)}$	1	-1	-1	1	1	1	1	-1	-1	-1
$\phi_{(1^3),(0)}$	1	1	-1	-1	1	1	-1	1	-1	1
$\phi_{(0),(3)}$	1	-1	1	-1	1	1	-1	-1	1	-1
$\phi_{(1,2),(0)}$	2	2	0	0	2	-1	0	-1	0	2
$\phi_{(0),(1,2)}$	2	-2	0	0	2	-1	0	1	0	-2
$\phi_{(2),(1)}$	3	1	1	1	-1	0	-1	0	-1	-3
$\phi_{(1),(1^2)}$	3	-1	-1	1	-1	0	-1	0	1	3
$\phi_{(1^2),(1)}$	3	1	-1	-1	-1	0	1	0	1	-3
$\phi_{(1),(2)}$	3	-1	1	-1	-1	0	1	0	-1	3

TABLE 6: Decomposition of $\rho^1_{\mu,\nu}$ into irreducible components for $G = Sp(2n,q)$.

(i) $n = 1$

Virtual character	1_G	St_G
$\rho^1_{(1),(0)}$	1	1
$\rho^1_{(0),(1)}$	1	-1

(ii) n = 2

Virtual character	1_G	St_G	$\chi_{(1^2),(0)}$	$\chi_{(0),(2)}$	$\chi_{(1),(1)}$	$\zeta_{(0),(0)}$
$\rho^1_{(1^2),(0)}$	1	1	1	1	2	
$\rho^1_{(1),(1)}$	1	-1	1	-1		
$\rho^1_{(2),(0)}$	1	-1	-1	1		
$\rho^1_{(0),(2)}$	1	1			-1	1
$\rho^1_{(0),(1^2)}$	1	1	-1	-1		-2
$\rho^1_{(0),(0),(2)}$			-1	-1	1	1

(iii) $n = 3$

Virtual character	1_G	St_G	$\chi_{(1^3),(0)}$	$\chi_{(0),(3)}$	$\chi_{(1,2),(0)}$	$\chi_{(0),(1,2)}$	$\chi_{(2),(1)}$	$\chi_{(1),(1^2)}$	$\chi_{(1^2),(1)}$	$\chi_{(1),(2)}$	$\zeta_{(1),(0)}$	$\zeta_{(0),(1)}$
$\rho^1_{(1^3),(0)}$	1	1	1	1	2	2	3	3	3	3		
$\rho^1_{(1^2),(1)}$	1	-1	1	-1	2	-2	1	-1	1	-1		
$\rho^1_{(1,2),(0)}$	1	-1	-1	1			1	-1	-1	1		
$\rho^1_{(1),(2)}$	1	1			1	1			-1	-1	1	1
$\rho^1_{(1),(1^2)}$	1	1	-1	-1			1	1	-1	-1	-2	-2
$\rho^1_{(3),(0)}$	1	1	1	1	-1	-1						
$\rho^1_{(2),(1)}$	1	1	-1	-1			-1	-1	1	1		
$\rho^1_{(0),(3)}$	1	-1					-1	1			1	-1
$\rho^1_{(0),(1,2)}$	1	-1			-1	1			1	-1	-1	1
$\rho^1_{(0),(1^3)}$	1	-1	3	-3			-1	1	-3	3	-2	2
$\rho^1_{(1),(0),(2)}$			-1	-1	-1	-1	1	1			1	1
$\rho^1_{(0),(1),(2)}$			-1	1	1	-1	-1	1			-1	1

TABLE 7 : Unipotent conjugacy classes.

(i) $G = GL(2,q)$, $Sp(2,q)$ or $U(2,q^2)$

Wall invariants	Order of Class
$2\times(x+1)$	1
$(x+1)^2$	q^2-1

(ii) $G = O_+(2,q)$

Wall invariants	Order of Class
$2\times(x+1)$	1
$(x+1)^2$	$q-1$

(iii) $G = O_-(2,q)$

Wall invariants	Order of Class
$2\times(x+1)$	1
$(x+1)^2$	$q+1$

(iv) <u>$G = GL(3,q)$</u>

Wall invariants	Order of Class
$3\times(x+1)$	1
$(x+1),(x+1)^2$	$(q^3-1)(q-1)$
$(x+1)^3$	$q(q^3-1)(q^2-1)$

(v) <u>$U = U(3,q^2)$</u>

Wall invariants	Order of Class
$3\times(x+1)$	1
$(x+1),(x+1)^2$	$(q^3+1)(q-1)$
$(x+1)^3$	$q(q^3+1)(q^2-1)$

(vi) <u>$G = Sp(4,q)$</u>

Wall invariants	Order of Class
$4\times(x+1)$	1
$2\times(x+1),(x+1)^2$	q^4-1
$2\times(x+1)^2, s_1 = 0$	q^4-1
$2\times(x+1)^2, s_1 = 1$	$(q^4-1)(q^2-1)$
$(x+1)^4, \delta_1 = 0$	$\frac{1}{2}q^2(q^4-1)(q^2-1)$
$(x+1)^4, \delta_1 \neq 0$	$\frac{1}{2}q^2(q^4-1)(q^2-1)$

(vii) $\underline{G = O_+(4,q)}$

Wall invariants	Order of Class
$4\times(x+1)$	1
$2\times(x+1),(x+1)^2$	$q(q^2-1)$
$2\times(x+1)^2, s_1 = 0$	$2(q^2-1)$
$2\times(x+1)^2, s_1 = 1$	$(q^2-1)^2$
$(x+1)^4,\ \delta_1 = 0$	$q(q^2-1)^2$

(viii) $\underline{G = O_-(4,q)}$

Wall invariants	Order of Class
$4\times(x+1)$	1
$2\times(x+1),(x+1)^2$	$q(q^2+1)$
$2\times(x+1)^2, s_1 = 1$	q^4-1
$(x+1)^4, \delta_1 \neq 0$	$q(q^4-1)$

(ix) $G = Sp(6,q)$

Wall invariants	Order of Class
$6\times(x+1)$	1
$4\times(x+1),(x+1)^2$	q^6-1
$2\times(x+1),2\times(x+1)^2,s_1=0$	$(q^6-1)(q^2+1)$
$2\times(x+1),2\times(x+1)^2,s_1=1$	$(q^6-1)(q^4-1)$
$2\times(x+1),(x+1)^4,\delta_2=0$	$\frac{1}{2}q^4(q^6-1)(q^4-1)$
$2\times(x+1),(x+1)^4,\delta_2\neq 0$	$\frac{1}{2}q^4(q^6-1)(q^4-1)$
$3\times(x+1)^2$	$q^2(q^6-1)(q^4-1)$
$(x+1)^2,(x+1)^4$	$q^4(q^6-1)(q^4-1)(q^2-1)$
$2\times(x+1)^3,\delta_1=0$	$\frac{1}{2}q^3(q^6-1)(q^4-1)(q+1)$
$2\times(x+1)^3,\delta_1\neq 0$	$\frac{1}{2}q^3(q^6-1)(q^4-1)(q-1)$
$(x+1)^6,\delta_1=0$	$\frac{1}{2}q^6(q^6-1)(q^4-1)(q^2-1)$
$(x+1)^6,\delta_1\neq 0$	$\frac{1}{2}q^6(q^6-1)(q^4-1)(q^2-1)$

(x) $G = O_+(6,q)$

Wall invariants	Order of Class
$6\times(x+1)$	1
$4\times(x+1),(x+1)^2$	$q^2(q^3-1)$
$2\times(x+1),2\times(x+1)^2,s_1 = 0$	$(q^4-1)(q^2+q+1)$
$2\times(x+1),2\times(x+1)^2,s_1 = 1$	$q(q^4-1)(q^3-1)$
$2\times(x+1),(x+1)^4,\delta_2 = 0$	$\frac{1}{2}q^3(q^4-1)(q^3-1)(q+1)$
$2\times(x+1),(x+1)^4,\delta_2 \neq 0$	$\frac{1}{2}q^3(q^4-1)(q^3-1)(q-1)$
$3\times(x+1)^2$	$q^2(q^4-1)(q^3-1)$
$(x+1)^2,(x+1)^4$	$q^3(q^4-1)(q^3-1)(q^2-1)$
$2\times(x+1)^3,\delta_1 = 0$	$q^2(q^4-1)(q^3-1)(q+1)$
$(x+1)^6,\delta_1 = 0$	$q^4(q^4-1)(q^3-1)(q^2-1)$

(xi) $G = O_-(6,q)$

Wall invariants	Order of Class
$6\times(x+1)$	1
$4\times(x+1),(x+1)^2$	$q^2(q^3+1)$
$2\times(x+1),2\times(x+1)^2,s_1 = 0$	$(q^4-1)(q^2-q+1)$
$2\times(x+1),2\times(x+1)^2,s_1 = 1$	$q(q^4-1)(q^3+1)$
$2\times(x+1),(x+1)^4,\delta_2 = 0$	$\frac{1}{2}q^3(q^4-1)(q^3+1)(q-1)$
$2\times(x+1),(x+1)^4,\delta_2 \neq 0$	$\frac{1}{2}q^3(q^4-1)(q^3+1)(q+1)$
$3\times(x+1)^2$	$q^2(q^4-1)(q^3+1)$
$(x+1)^2, (x+1)^4$	$q^3(q^4-1)(q^3+1)(q^2-1)$
$2\times(x+1)^3,\delta_1 \neq 0$	$q^2(q^4-1)(q^3+1)(q-1)$
$(x+1)^6,\delta_1 \neq 0$	$q^4(q^4-1)(q^3+1)(q2-1)$

TABLE 8 : Values of $g_P(c;d)$

(i) $G = GL(2,q)$ or $Sp(2,q)$, $P = B$ i.e. $B/U \cong GL(1,q)^2$ or $GL(1,q)$ respectively.

Wall invariants for c	$g_B^G(c;\{1\})$
$2\times(x+1)$	$q+1$
$(x+1)^2$	1

(ii) $G = GL(3,q)$, $P = P_{(1)}$ i.e. $P/U^+ \cong GL(1,q) \times GL(2,q)$.

Wall invariants for c	Wall invarinats for $d \in C_u(GL(2,q))$	$g_P^G(c;\{1\}\times d)$
$3\times(x+1)$	$2\times(x+1)$	q^2+q+1
$(x+1),(x+1)^2$	$2\times(x+1)$	1
	$(x+1)^2$	q
$(x+1)^3$	$(x+1)^2$	1

(iii) $G = Sp(4,q)$, $P = P_{(1)}$ i.e. $P/U^+ \cong GL(1,q) \times Sp(2,q)$.

Wall invariants for c	Wall invariants for $d \in C_u(Sp(2,q))$	$g_P^G(c;\{1\}\times d)$
$4\times(x+1)$	$2\times(x+1)$	$(q+1)(q^2+1)$
$2\times(x+1),(x+1)^2$	$2\times(x+1)$	1
	$(x+1)^2$	$q(q+1)$
$2\times(x+1)2, s_1 = 0$	$2\times(x+1)$	$q+1$
$2\times(x+1)^2, s_1 = 1$	$2\times(x+1)$	1
	$(x+1)^2$	q
$(x+1)^4, \delta_1 = 0$	$(x+1)^2$	1
$(x+1)^4, \delta_1 \neq 0$	$(x+1)^2$	1

(iv) $G = Sp(4,q)$, $P = P_{(2)}$ i.e. $P/U^{+} \cong GL(2,q)$.

Wall invariants for c	Wall invariants for d	$g_P^G(c;d)$
$4\times(x+1)$	$2\times(x+1)$	$(q+1)(q^2+1)$
$2\times(x+1),(x+1)^2$	$2\times(x+1)$	$q+1$
$2\times(x+1)^2, s_1 = 0$	$2\times(x+1)$	1
	$(x+1)^2$	$q(q+1)$
$2\times(x+1)^2, s_1 = 1$	$2\times(x+1)$	1
	$(x+1)^2$	q
$(x+1)^4, \delta_1 = 0$	$(x+1)^2$	1
$(x+1)^4, \delta_1 \neq 0$	$(x+1)^2$	1

(v) $G = Sp(4,q)$, $P = B$ i.e. $B/U^{+} \cong GL(1,q)^2$

Wall invariants for c	$g_B^G(c;\{1\})$
$4\times(x+1)$	$(q+1)^2(q^2+1)$
$2\times(x+1),(x+1)^2$	$(q+1)^2$
$2\times(x+1)^2, s_1 = 0$	$(q+1)^2$
$2\times(x+1)^2, s_1 = 1$	$2q+1$
$(x+1)^4, \delta_1 = 0$	1
$(x+1)^4, \delta_1 \neq 0$	1

(vi) $G = Sp(6,q)$, $P = P_{(1)}$ i.e. $P/U^{+} \cong GL(1,q) \times Sp(4,q)$.

Wall invariants for c	Wall invariants for $d \in C_u(Sp(4,q))$	$g_P^G(c;\{1\}\times d)$
$6\times(x+1)$	$4\times(x+1)$	$(q^3+1)(q^2+q+1)$
$4\times(x+1),(x+1)^2$	$4\times(x+1)$	1
	$2\times(x+1),(x+1)^2$	$q(q+1)(q^2+1)$
$2\times(x+1),2\times(x+1)^2,s_1 = 0$	$4\times(x+1)$	$q+1$
	$2\times(x+1)^2,s_1 = 0$	$q^2(q+1)$
$2\times(x+1),2\times(x+1)^2,s_1 = 1$	$4\times(x+1)$	1
	$2\times(x+1),(x+1)^2$	q
	$2\times(x+1)^2,s_1 = 1$	$q^2(q+1)$
$2\times(x+1),(x+1)^4,\delta_1 = 0$	$2\times(x+1),(x+1)^2$	1
	$(x+1)^4,\delta_1 = 0$	$q(q+1)$
$2\times(x+1),(x+1)^4,\delta_1 \neq 0$	$2\times(x+1),(x+1)^2$	1
	$(x+1)^4,\delta_1 \neq 0$	$q(q+1)$
$3\times(x+1)^2$	$2\times(x+1),(x+1)^2$	$q+1$
	$2\times(x+1)^2,s_1 = 0$	1
	$2\times(x+1)^2,s_1 = 1$	q^2-1
$(x+1)^2,(x+1)^4$	$2\times(x+1)^2,s_1 = 1$	1
	$(x+1)^4,\delta_1 = 0$	$\frac{1}{2}q$
	$(x+1)^4,\delta_1 \neq 0$	$\frac{1}{2}q$
$2\times(x+1)^3,\delta_1 = 0$	$2\times(x+1)^2,s_1 = 0$	2
	$2\times(x+1)^2,s_1 = 0$	$q-1$
$2\times(x+1)^3,\delta_1 \neq 0$	$2\times(x+1)^2,s_1 = 1$	$q+1$
$(x+1)^6,\delta_1 = 0$	$(x+1)^4,\delta_1 = 0$	1
$(x+1)^6,\delta_1 \neq 0$	$(x+1)^4,\delta_1 \neq 0$	1

(vii) $G = Sp(6,q)$, $P = P_{(2)}$ i.e. $P/U^+ \cong GL(2,q) \times Sp(2,q)$.

Wall invariants for c	Wall invariants for $d_1 \in C_u(GL(2,q))$	Wall invariants for $d_2 \in C_u(Sp(2,q))$	$g_P^G(c;d_1 \times d_2)$
$6\times(x+1)$	$2\times(x+1)$	$2\times(x+1)$	$(q^2+1)(q^3+1)(q^2+q+1)$
$4\times(x+1),(x+1)^2$	$2\times(x+1)$	$2\times(x+1)$	$(q+1)(q^2+1)$
	$2\times(x+1)$	$(x+1)^2$	$q^2(q+1)(q^2+1)$
$2\times(x+1),2\times(x+1)^2,s_1 = 0$	$2\times(x+1)$	$2\times(x+1)$	$q(q+1)^2+1$
	$(x+1)^2$	$2\times(x+1)$	$q^3(q+1)$
$2\times(x+1),2\times(x+1)^2,s_1 = 1$	$2\times(x+1)$	$2\times(x+1)$	q^2+q+1
	$2\times(x+1)$	$(x+1)^2$	$q^2(q+1)$
	$(x+1)^2$	$2\times(x+1)$	q^3
$2\times(x+1),(x+1)^4,\delta_1$ *	$2\times(x+1)$	$(x+1)^2$	$q+1$
	$(x+1)^2$	$2\times(x+1)$	1
	$(x+1)^2$	$(x+1)^2$	q^2-1
$3\times(x+1)^2$	$2\times(x+1)$	$2\times(x+1)$	$q+1$
	$2\times(x+1)$	$(x+1)^2$	q^2
	$(x+1)^2$	$(x+1)^2$	$q^2(q+1)$
$(x+1)^2,(x+1)^4$	$2\times(x+1)$	$(x+1)^2$	1
	$(x+1)^2$	$2\times(x+1)$	1
	$(x+1)^2$	$(x+1)^2$	$q-1$
$2\times(x+1)^3,\delta_1 = 0$	$2\times(x+1)$	$2\times(x+1)$	1
	$(x+1)^2$	$2\times(x+1)$	$2q$
	$(x+1)^2$	$(x+1)^2$	$q(q-1)$
$2\times(x+1)^3,\delta_1 \neq 0$	$2\times(x+1)$	$2\times(x+1)$	1
	$(x+1)^2$	$(x+1)^2$	$q(q+1)$
$(x+1)^6,\delta_1$	$(x+1)^2$	$(x+1)^2$	1

* Value of δ_1 can be zero or non-zero modulo $k_* = \{\gamma+\gamma^2 | \gamma \in k\}$.

(viii) $G = Sp(6,q)$, $P = P_{(3)}$ i.e. $P/U^{+} \cong GL(3,q)$.

Wall invariants for c	Wall invariants for d	$g_P^G(c;d)$
$6\times(x+1)$	$3\times(x+1)$	$(q+1)(q^2+1)(q^3+1)$
$4\times(x+1),(x+1)^2$	$3\times(x+1)$	$(q+1)(q^2+1)$
$2\times(x+1),2\times(x+1)^2,s_1 = 0$	$3\times(x+1)$	$q+1$
	$(x+1),(x+1)^2$	$q^2(q+1)^2$
$2\times(x+1),2\times(x+1)^2,s_1 = 1$	$3\times(x+1)$	$q+1$
	$(x+1),(x+1)^2$	$q^2(q+1)$
$2\times(x+1),(x+1)^4,\delta_1$	$(x+1),(x+1)^2$	$q+1$
$3\times(x+1)^2$	$3\times(x+1)$	1
	$(x+1),(x+1)^2$	$q(q+1)$
$(x+1)^2,(x+1)^4$	$(x+1),(x+1)^2$	1
	$(x+1)^3$	q
$2\times(x+1)^3,\delta_1 = 0$	$(x+1),(x+1)^2$	$q+1$
	$(x+1)^3$	$2q^2$
$2\times(x+1)^3,\delta_1 \neq 0$	$(x+1),(x+1)^2$	$q+1$
$(x+1)^6,\delta_1$	$(x+1)^3$	1

(ix) $G = Sp(6,q)$, $P = P_{(1,2)}$ i.e. $P/U^+ \cong GL(1,q)^2 \times Sp(2,q)$.

Wall invariants for c	Wall invariants for $d \in C_u(Sp(2,q))$	$g_P^G(c;\{1\}^2 \times d)$
$6\times(x+1)$	$2\times(x+1)$	$(q+1)(q^2+1)(q^3+1)(q^2+q+1)$
$4\times(x+1),(x+1)^2$	$2\times(x+1)$	$(q+1)^2(q^2+1)$
	$(x+1)^2$	$q^2(q+1)^2(q^2+1)$
$2\times(x+1),2\times(x+1)^2,s_1 = 0$	$2\times(x+1)$	$(q+1)^2(2q^2+1)$
$2\times(x+1),2\times(x+1)^2,s_1 = 1$	$2\times(x+1)$	$2q^3+2q^2+2q+1$
	$(x+1)^2$	$q^2(q+1)^2$
$2\times(x+1),(x+1)^4,\delta_1$	$2\times(x+1)$	1
	$(x+1)^2$	$2q(q+1)$
$3\times(x+1)^2$	$2\times(x+1)$	$(q+1)^2$
	$(x+1)^2$	$2q^2(q+1)$
$(x+1)^2,(x+1)^4$	$2\times(x+1)$	1
	$(x+1)^2$	$2q$
$2\times(x+1)^3,\delta_1 = 0$	$2\times(x+1)$	$3q+1$
	$(x+1)^2$	$q(q-1)$
$2\times(x+1)^3,\delta_1 \neq 0$	$2\times(x+1)$	$q+1$
	$(x+1)^2$	$q(q+1)$
$(x+1)^6$, δ_1	$(x+1)^2$	1

(x) $G = Sp(6,q)$, $P = P_{(2,3)}$ i.e. $P/U^{+} \cong GL(2,q) \times GL(1,q)$

Wall invariants for c	Wall invariants for $d \in C_u(GL(2,q))$	$g_P^G(c;d\times\{1\})$
$6\times(x+1)$	$2\times(x+1)$	$(q+1)(q^2+1)(q^3+1)(q^2+q+1)$
$4\times(x+1),(x+1)^2$	$2\times(x+1)$	$(q+1)(q^2+1)(q^2+q+1)$
$2\times(x+1),2\times(x+1)^2,s_1 = 0$	$2\times(x+1)$	$(q+1)(q^3+2q^2+q+1)$
	$(x+1)^2$	$q^3(q+1)^2$
$2\times(x+1),2\times(x+1)^2,s_1 = 1$	$2\times(x+1)$	$(q+1)(2q^2+q+1)$
	$(x+1)^2$	$q^3(q+1)$
$2\times(x+1),(x+1)^4,\delta_1$	$2\times(x+1)$	$q+1$
	$(x+1)^2$	$q(q+1)$
$3\times(x+1)^2$	$2\times(x+1)$	$2q^2+2q+1$
	$(x+1)^2$	$q^2(q+1)$
$(x+1)^2,(x+1)^4$	$2\times(x+1)$	1
	$(x+1)^2$	$2q$
$2\times(x+1)^3,\delta_1 = 0$	$2\times(x+1)$	$q+1$
	$(x+1)^2$	$q(3q+1)$
$2\times(x+1)^3,\delta_1 \neq 0$	$2\times(x+1)$	$q+1$
	$(x+1)^2$	$q(q+1)$
$(x+1)^6,\delta_1$	$(x+1)^2$	1

(xi) $G = Sp(6,q)$, $P = B$ i.e. $B/U^+ \cong GL(1,q)^3$.

Wall invariants for c	$g_B^G(c;\{1\})$
$6\times(x+1)$	$(q+1)^2(q^2+1)(q^3+1)(q^2+q+1)$
$4\times(x+1),(x+1)^2$	$(q+1)^2(q^2+1)(q^2+q+1)$
$2\times(x+1),2\times(x+1)^2,s_1 = 0$	$(q+1)^3(2q^2+1)$
$2\times(x+1),2\times(x+1)^2,s_1 = 1$	$(q+1)(3q^3+3q^2+2q+1)$
$2\times(x+1),(x+1)^4,\delta_1$	$(q+1)(2q+1)$
$3\times(x+1)^2$	$(q+1)(3q^2+2q+1)$
$(x+1)^2,(x+1)^4$	$3q+1$
$2\times(x+1)^3,\delta_1 = 0$	$4q^2+3q+1$
$2\times(x+1)^3,\delta_1 \neq 0$	$(q+1)(2q+1)$
$(x+1)^6,\delta_1$	1

TABLE 9 : Values of unipotent characters on unipotent classes.

(i) $G = Sp(4,q)$

Wall invariants for conjugacy class \ Character	$\chi_{(2),(0)}$	$\chi_{(0),(1^2)}$	$\chi_{(1^2),(0)}$	$\chi_{(0),(2)}$	$\chi_{(1),(1)}$	$\zeta_{(0),(0)}$
$4\times(x+1)$	1	q^4	$\frac{1}{2}q(q^2+1)$	$\frac{1}{2}q(q^2+1)$	$\frac{1}{2}q(q+1)^2$	$\frac{1}{2}q(q-1)^2$
$2\times(x+1),(x+1)^2$	1		$\frac{1}{2}q(q+1)$	$-\frac{1}{2}q(q-1)$	$\frac{1}{2}q(q+1)$	$-\frac{1}{2}q(q-1)$
$2\times(x+1)^2, s_1 = 0$	1		$-\frac{1}{2}q(q-1)$	$\frac{1}{2}q(q+1)$	$\frac{1}{2}q(q+1)$	$-\frac{1}{2}q(q-1)$
$2\times(x+1)^2, s_1 = 1$	1		$\frac{1}{2}q$	$\frac{1}{2}q$	$\frac{1}{2}q$	$\frac{1}{2}q$
$(x+1)^4, \delta_1 = 0$	1		$-\frac{1}{2}q$	$-\frac{1}{2}q$	$\frac{1}{2}q$	$\frac{1}{2}q$
$(x+1)^4, \delta_1 \neq 0$	1		$\frac{1}{2}q$	$\frac{1}{2}q$	$-\frac{1}{2}q$	$-\frac{1}{2}q$

(ii) $G = Sp(6,q)$

Wall invariants for conjugacy class \ Character	$\chi_{(3),(0)}$	$\chi_{(0),(1^3)}$	$\chi_{(1^3),(0)}$	$\chi_{(0),(3)}$	$\chi_{(1,2)(0)}$	$\chi_{(1),(1,2)}$
$6\times(x+1)$	1	q^9	$\frac{1}{2}q^4(q^2+1)(q^2-q+1)$	$\frac{1}{2}q(q^2+1)(q^2-q+1)$	$\frac{1}{2}q(q^3+1)(q+1)$	$\frac{1}{2}q^4(q^3+1)(q+1)$
$4\times(x+1),(x+1)^2$	1		$\frac{1}{2}q^4(q^2+1)$	$-\frac{1}{2}q(q-1)(q^2+1)$	$\frac{1}{2}q(q^2+1)(q+1)$	$-\frac{1}{2}q^4(q^2-1)$
$2\times(x+1),2\times(x+1)^2,s_1 = 0$	1		$-\frac{1}{2}q^4(q-1)$	$\frac{1}{2}q(2q^2-q+1)$	$\frac{1}{2}q(q+1)$	$\frac{1}{2}q^4(q+1)$
$2\times(z+1),2\times(x+1)^2,s_1 = 1$	1		$\frac{1}{2}q^4$	$\frac{1}{2}q(q^2-q+1)$	$\frac{1}{2}q(q^2+q+1)$	$\frac{1}{2}q^4$
$2\times(x+1),(x+1)^4,\delta_2 = 0$	1		$-\frac{1}{2}q^3$	$-\frac{1}{2}q(2q-1)$	$\frac{1}{2}q$	$-\frac{1}{2}q^3$
$2\times(x+1),(x+1)^4,\delta_2 \neq 0$	1		$\frac{1}{2}q^3$	$\frac{1}{2}q$	$\frac{1}{2}q(2q+1)$	$\frac{1}{2}q^3$
$3\times(x+1)^2$	1			$-\frac{1}{2}q(q-1)$	$\frac{1}{2}q(q+1)$	
$(x+1)^2,(x+1)^4$	1			$\frac{1}{2}q$	$\frac{1}{2}q$	
$2\times(x+1)^3,\delta_1 = 0$	1			$\frac{1}{2}q(q+1)$	$-\frac{1}{2}q(q-1)$	
$2\times(x+1)^3,\delta_1 \neq 0$	1			$-\frac{1}{2}q(q-1)$	$\frac{1}{2}q(q+1)$	
$(x+1)^6,\delta_1 = 0$	1			$-\frac{1}{2}q$	$-\frac{1}{2}q$	
$(x+1)^6,\delta_1 \neq 0$				$\frac{1}{2}q$	$\frac{1}{2}q$	

Wall invariants for conjugacy class \ Character	$\chi_{(2),(1)}$	$\chi_{(1),(1^2)}$	$\chi_{(1^2),(1)}$	$\chi_{(1),(2)}$	$\zeta_{(1),(0)}$	$\zeta_{(0),(1)}$
$6\times(x+1)$	$\frac{1}{2}q(q^2+1)(q^2+q+1)$	$\frac{1}{2}q^4(q^2+1)(q^2+q+1)$	$q^3(q^4+q^2+1)$	$q^2(q^4+q^2+1)$	$\frac{1}{2}q(q-1)(q^3-1)$	$\frac{1}{2}q^4(q-1)(q^3-1)$
$4\times(x+1),(x+1)^2$	$\frac{1}{2}q(q^2+1)(q+1)$	$\frac{1}{2}q^4(q^2+1)$	$q^3(q^2+1)$	q^2	$-\frac{1}{2}q(q-1)(q^2+1)$	$-\frac{1}{2}q^4(q^2-1)$
$2\times(x+1),2\times(x+1)^2,s_1=0$	$\frac{1}{2}q(2q^2+q+1)$	$\frac{1}{2}q^4(q+1)$	q^3	$q^2(q^2+1)$	$-\frac{1}{2}q(q-1)$	$-\frac{1}{2}q^4(q-1)$
$2\times(x+1),2\times(x+1)^2,s_1=1$	$\frac{1}{2}q(q^2+q+1)$	$\frac{1}{2}q^4$	q^3	q^2	$\frac{1}{2}q(q^2-q+1)$	$\frac{1}{2}q^4$
$2\times(x+1),(x+1)^4,\delta_2=0$	$\frac{1}{2}q(2q+1)$	$\frac{1}{2}q^3$			$\frac{1}{2}q$	$\frac{1}{2}q^3$
$2\times(x+1),(x+1)^4,\delta_2\neq 0$	$\frac{1}{2}q$	$-\frac{1}{2}q^3$			$-\frac{1}{2}q(2q-1)$	$-\frac{1}{2}q^3$
$3(x+1)^2$	$\frac{1}{2}q(q+1)$		q^3	q^2	$-\frac{1}{2}q(q-1)$	
$(x+1)^2,(x+1)^4$	$\frac{1}{2}q$				$\frac{1}{2}q$	
$2\times(x+1)^3,\delta_1=0$	$\frac{1}{2}q(q+1)$			q^2	$-\frac{1}{2}q(q-1)$	
$2\times(x+1)^3,\delta_1\neq 0$	$-\frac{1}{2}q(q-1)$			q^2	$\frac{1}{2}q(q+1)$	
$(x+1)^6,\delta_1=0$	$\frac{1}{2}q$				$\frac{1}{2}q$	
$(x+1)^6,\delta_1\neq 0$	$-\frac{1}{2}q$				$-\frac{1}{2}q$	

TABLE 10 : The Green polynomials.

(i) GL(2,q)

Class	$Q_{(1^2)}$	$Q_{(2)}$
$2\times(x+1)$	$1+q$	$1-q$
$(x+1)^2$	1	1

(ii) $U(2,q^2)$

Class	$Q_{(1^2)}$	$Q_{(2)}$
$2\times(x+1)$	$1-q$	$1+q$
$(x+1)^2$	1	1

(iii) Sp(2,q)

Class	$Q_{(1),(0)}$	$Q_{(0),(1)}$
$2\times(x+1)$	$1+q$	$1-q$
$(x+1)^2$	1	1

(iv) GL(3,q)

	$Q_{(1^3)}$	$Q_{(1,2)}$	$Q_{(3)}$
$3\times(x+1)$	$(1+q)(1+q+q^2)$	$1-q^3$	$(1-q)(1-q^2)$
$(x+1),(x+1)^2$	$1+2q$	1	$1-q$
$(x+1)^3$	1	1	1

(v) $U(3,q^2)$

Class	$Q_{(1^3)}$	$Q_{(1,2)}$	$Q_{(3)}$
$3\times(x+1)$	$(1-q)(1-q+q^2)$	$1+q^3$	$(1+q)(1-q^2)$
$(x+1),(x+1)^2$	$1-2q$	1	$1-q$
$(x+1)^3$	1	1	1

(vi) $Sp(4,q)$

Class	$Q_{(1^2),(0)}$	$Q_{(1),(1)}$	$Q_{(2),(0)}$	$Q_{(0),(2)}$	$Q_{(0),(1^2)}$	$Q_{(0),(0),(2)}$
$4\times(x+1)$	$(1+q)^2(1+q^2)$	$1-q^4$	$1-q^4$	$(1-q^2)^2$	$(1-q)^2(1+q^2)$	
$2\times(x+1),(x+1)^2$	$(1+q)^2$	$1+q^2$	$1-q^2$	$1-q^2$	$(1-q)^2$	
$2\times(x+1)^2, s_1 = 0$	$(1+q)^2$	$1-q^2$	$1+q^2$	$1-q^2$	$(1-q)^2$	
$s_1 = 1$	$1+2q$	1	1	1	$1-2q$	
$(x+1)^4, \delta_1 = 0$	1	1	1	1	1	$2q$
$\delta_1 \neq 0$	1	1	1	1	1	$-2q$

(vii) Sp(6,q)

Class	$Q_{(1^3),(0)}$	$Q_{(1^2),(1)}$	$Q_{(1,2),(0)}$	$Q_{(1),(2)}$
$6\times(x+1)$	$(1+q)^2(1+q^2)(1+q^3)(1+q+q^2)$	$(1+q)(1+q^2)(1-q^6)$	$(1+q)(1+q^2)(1-q^6)$	$(1+q)(1-q^2)(1-q^6)$
$4\times(x+1),(x+1)^2$	$(1+q)^2(1+q^2)(1+q+q^2)$	$(1+q)(1+q^2)(1+q^3)$	$(1+q)(1+q^2)(1-q^3)$	$(1+q)(1-q^2)(1+q^3)$
$2\times(x+1),2\times(x+1)^2,s_1=0$	$(1+q)^3(1+2q^2)$	$(1+q)(1-q^2)(1+2q^2)$	$(1+q)(1+q^2)$	$(1+q)(1-q^2)$
$s_1=1$	$(1+q)(1+2q+3q^2+3q^3)$	$1+q+q^2+2q^3-q^4$	$(1+q)(1+q^2-q^3)$	$(1+q)(1-q^2+q^3)$
$2\times(x+1),(x+1)^4,\delta_2=0$	$(1+q)(1+2q)$	$1+q+2q^2$	$1+q$	$1+q$
$\delta_2\neq0$	$(1+q)(1+2q)$	$1+q+2q^2$	$1+q$	$1+q$
$3\times(x+1)^2$	$(1+q)(1+2q+3q^2)$	$(1+q)(1+q^2)$	$1+q+q^2-q^3$	$(1+q)(1-q^2)$
$(x+1)^2,(x+1)^4$	$1+3q$	$1+q$	$1+q$	$1+q$
$2\times(x+1)^3,\delta_1=0$	$1+3q+4q^2$	$(1-q)(1+2q)$	$1+q+2q^2$	$(1-q)(1+2q)$
$\delta_1\neq0$	$(1+q)(1+2q)$	$1+q$	$1+q$	$1+q$
$(x+1)^6,\delta_1=0$	1	1	1	1
$\delta_1\neq0$	1	1	1	1

Class	$Q_{(1),(1^2)}$	$Q_{(3),(0)}$	$Q_{(2),(1)}$	$Q_{(0),(3)}$	$Q_{(0),(1,2)}$
$6\times(x+1)$	$(1-q)(1+q^2)(1-q^6)$	$(1-q^2)(1-q^4)(1+q^3)$	$(1-q)(1+q^2)(1-q^6)$	$(1-q^2)(1-q^4)(1-q^3)$	$(1-q)(1-q^2)(1-q^6)$
$4\times(x+1),(x+1)^2$	$(1-q)(1+q^2)(1-q^3)$	$(1-q^2)(1-q^4)$	$(1-q)(1+q^2)(1+q^3)$	$(1-q^2)(1-q^4)$	$(1-q)(1-q^2)(1-q^3)$
$2\times(x+1),2\times(x+1)^2,s_1=0$	$(1-q)(1-q^2)(1+2q^2)$	$(1-q^2)(1+q^3)$	$(1-q)(1+q^2)$	$(1-q^2)(1-q^3)$	$(1-q)(1-q^2)$
$s_1=1$	$1-q+q^2-2q^3-q^4$	$1-q^2$	$(1-q)(1+q^2+q^3)$	$1-q^2$	$(1-q)(1-q^2-q^3)$
$2\times(x+1),(x+1)^4,\delta_2=0$	$1-q+2q^2$	$1-q^2$	$1-q$	$1-q^2$	$1-q$
$\delta_2\neq0$	$1-q+2q^2$	$1-q^2$	$1-q$	$1-q^2$	$1-q$
$3\times(x+1)^2$	$(1-q)(1+q^2)$	$1-q^2$	$1-q+q^2+q^3$	$1-q^2$	$(1-q)(1-q^2)$
$(x+1)^2,(x+1)^4$	$1-q$	1	$1-q$	1	$1-q$
$2\times(x+1)^3,\delta_1=0$	$1-q$	$1+q^2$	$1-q$	$1-q^2$	$1-q$
$\delta_1\neq0$	$(1+q)(1-2q)$	$1-q^2$	$1-q+2q^2$	$1+q^2$	$(1+q)(1-2q)$
$(x+1)^6,\delta_1=0$	1	1	1	1	1
$\delta_1\neq0$	1	1	1	1	1

Class	$Q_{(0),(1^3)}$	$Q_{(1),(0),(2)}$	$Q_{(0),(1),(2)}$
$6\times(x+1)$	$(1-q)^2(1+q^2)(1-q^3)(1-q+q^2)$		
$4\times(x+1),(x+1)^2$	$(1-q)^2(1+q^2)(1-q+q^2)$		
$2\times(x+1),2\times(x+1)^2,s_1=0$	$(1-q)^3(1+2q^2)$		
$s_1=1$	$(1-q)(1-2q+3q^2-3q^3)$		
$2\times(x+1),(x+1)^4,\delta_2=0$	$(1-q)(1-2q)$	$2q^2(1+q)$	$-2q^2(1-q)$
$\delta_2\neq 0$	$(1-q)(1-2q)$	$-2q^2(1+q)$	$2q^2(1-q)$
$3\times(x+1)^2$	$(1-q)(1-2q+3q^2)$		
$(x+1)^2,(x+1)^4$	$1-3q$		
$2\times(x+1)^3,\delta_1=0$	$(1-q)(1-2q)$		
$\delta_1\neq 0$	$1-3q+4q^2$		
$(x+1)^6,\delta_1=0$	1	$2q$	$-2q$
$\delta_1\neq 0$	1	$-2q$	$2q$

TABLE 11 : The irreducible characters $(\chi^{(i)}_{[\xi]})$ of $G = Sp(2n,q)$. *

(i) $n = 1$

Geometric conjugacy class $[\xi]$	Index i \ Torus Index	(1),(0)	(0),(1)
1	1	1	1
	2	1	-1
$[\hat{\alpha}]$	1	1	
$[\hat{\beta}]$	1		-1

* The entries in the table are the values of $C^{(i)}_{\tilde{T},\theta}$. The values of $\chi^{(i)}_{[\xi]}$ are given by (7.8.1).

(ii) $n = 2$

Geometric conjugacy class $[\xi]$	Index i \ Torus index	$(1^2),(0)$	$(1),(1)$	$(2),(0)$	$(0),(2)$	$(0),(1^2)$	$(0),(0),(2)$
$[1\times 1]$	1	1	1	1	1	1	
	2	1	-1	-1	1	1	
	3	1	1	-1		-1	-1
	4	1	-1	1		-1	-1
	5	2			-1		1
	6				1	-2	1
$[\hat\alpha\times 1]$	1	1	1				
	2	1	-1				
$[\hat\alpha\times\hat\alpha]$	1	1		1			
	2	1		-1			
$[\hat\alpha_1\times\hat\alpha_2]$	1	1					
$[1\times\hat\beta]$	1		-1			-1	
	2		-1			1	
$[\hat\alpha\times\hat\beta]$	1		-1				
$[\hat\beta\times\hat\beta]$	1			-1		-1	
	2			-1		1	
$[\hat\beta_1\times\hat\beta_2]$	1					1	
$[\hat\gamma]$	1			-1			
$[\hat\delta]$	1				1		

(iii) <u>n = 3</u>

Geometric conjugacy class [ξ]	Index i \ Torus index	$(1^3),(0)$	$(1^2),(1)$	$(1,2),(0)$	$(1),(2)$	$(1),(1^2)$	$(3),(0)$	$(2),(1)$	$(0),(3)$	$(0),(1,2)$	$(0),(1^3)$	$(1),(0),(2)$	$(0),(1),(2)$
[1×1×1]	1	1	1	1	1	1	1	1	1	1	1		
	2	1	-1	-1	1	1	1	1	-1	-1	-1		
	3	1	1	-1		-1	1	-1			3	-1	-1
	4	1	-1	1		-1	1	-1			-3	-1	1
	5	2	2		1		-1			-1		-1	1
	6	2	-2		1		-1			1		-1	-1
	7	3	1	1		1		-1	-1		-1	1	-1
	8	3	-1	-1		1		-1	1		1	1	1
	9	3	1	-1	-1	-1		1		1	-3		
	10	3	-1	1	-1	-1		1		-1	3		
	11				1	-2			1	-1	-2	1	-1
	12				1	-2			-1	1	2	1	1
[α̂×1×1]	1	1	1	1	1	1							
	2	1	-1	-1	1	1							
	3	1	1	-1		-1						-1	
	4	1	-1	1		-1						-1	
	5	2			-1							1	
	6				1	-2						1	
[α̂×α̂×1]	1	1	1	1				1					
	2	1	-1	-1				1					
	3	1	1	-1				-1					
	4	1	-1	1				-1					
[α̂×α̂×α̂]	1	1		1			1						
	2	1		-1			1						
	3	2					-1						

Geometric conjugacy class [ξ]	Index \ Torus index 1	$(1^3),(0)$	$(1^2),(1)$	$(1,2),(0)$	$(1),(2)$	$(1),(1^2)$	$(3),(0)$	$(2),(1)$	$(0),(3)$	$(0),(1,2)$	$(0),(1^3)$	$(1),(0),(2)$	$(0),(1),(2)$
$[\hat{\alpha}_1\times\hat{\alpha}_2\times 1]$	1	1	1										
	2	1	-1										
$[\hat{\alpha}_1\times\hat{\alpha}_1\times\hat{\alpha}_2]$	1	1		1									
	2	1		-1									
$[\hat{\alpha}_1\times\hat{\alpha}_2\times\hat{\alpha}_3]$	1	1											
$[1\times 1\times\hat{\beta}]$	1		-1			-1		-1		-1	-1		
	2		-1			1		1		-1	-1		
	3		-1			-1		1			1		1
	4		-1			1		-1			1		1
	5		-2							1			-1
	6									-1	2		-1
$[\hat{\alpha}\times 1\times\hat{\beta}]$	1		-1			-1							
	2		-1			1							
$[\hat{\alpha}\times\hat{\alpha}\times\hat{\beta}]$	1		-1					-1					
	2		-1					1					
$[\hat{\alpha}_1\times\hat{\alpha}_2\times\hat{\beta}]$	1		-1										
$[1\times\hat{\beta}\times\hat{\beta}]$	1			-1		-1		-1			-1		
	2			-1		1		1			-1		
	3			-1		-1		1			1		
	4			-1		1		-1			1		

Geometric conjugacy class[ξ]	Torus index / Index i	$(1^3),(0)$	$(1^2),(1)$	$(1,2),(0)$	$(1),(2)$	$(1),(1^2)$	$(3),(0)$	$(2),(1)$	$(0),(3)$	$(0),(1,2)$	$(0),(1^3)$	$(1),(0),(2)$	$(0),(1),(2)$
$[\hat{\alpha}\times\hat{\beta}\times\hat{\beta}]$	1			-1		-1							
	2			-1		1							
$[\hat{\beta}\times\hat{\beta}\times\hat{\beta}]$	1							1	1		1		
	2							1	-1		-1		
	3								1		-2		
$[1\times\hat{\beta}_1\times\hat{\beta}_2]$	1					1					1		
	2					1					-1		
$[\hat{\alpha}\times\hat{\beta}_1\times\hat{\beta}_2]$	1					1							
$[\hat{\beta}_1\times\hat{\beta}_1\times\hat{\beta}_2]$	1							1			1		
	2							1			-1		
$[\hat{\beta}_1\times\hat{\beta}_2\times\hat{\beta}_3]$	1										-1		
$[1\times\hat{\gamma}]$	1			-1				-1					
	2			-1				1					
$[\hat{\alpha}\times\hat{\gamma}]$	1			-1									
$[\hat{\gamma}\times\hat{\beta}]$	1							1					
$[1\times\hat{\delta}]$	1				1					1			
	2				1					-1			
$[\hat{\alpha}\times\hat{\delta}]$	1				1								
$[\hat{\beta}\times\hat{\delta}]$	1									-1			
$[\hat{\varepsilon}]$	1						1						
$[\hat{\eta}]$	1								-1				

REFERENCES

[1] Borel, A., Linear algebraic groups, Benjamin, New York (1969).

[2] Borel, A. and J. Tits, Groupes réductifs, Publ. Math. I.H.E.S. 27 (1965), 55-151.

[3] Carter, R.W., Simple groups and simple Lie algebras, J. Lond. Math. Soc. 40 (1965), 193-240.

[4] Carter, R., Conjugacy classes in the Weyl group, Part G in Seminar on algebraic groups and related finite groups, by A. Borel *et al.*, Lecture notes in mathematics 131, Springer, Berlin (1970).

[5] Chevalley, Séminaire C., Classification des groupes de Lie algébrique (two volumes), Paris (1956-8).

[6] Curtis, C.W., N. Iwahori and R. Kilmoyer, Hecke algebras and characters of parabolic type of fintie groups with (B,N)-pairs, Publ. Math. I.H.E.S. 40 (1971), 81-116.

[7] Curtis, C.W. and I. Reiner, Representation theory of finite groups and associative algebras, Wiley (Interscience), New York (1962).

[8] Deligne, P. and G. Lusztig, Representations of reductive groups over finite fields, Ann. Math. 103 (1976), 103-161.

[9] Dieudonné, J., La géometrie des groupes classiques, Springer, Berlin (1955).

[10] Ennola, V., On the characters of the finite unitary groups, Ann. Acad. Sci. Fenn. Ser. AI323 (1963), 1-23.

[11] Enomoto, H., The characters of the finite symplectic group Sp(4,q), $q = 2^f$, Osaka J. Math. 9 (1972), 75-94.

[12] Frame, J.S. and A. Rudvalis, Characters of symplectic groups over F_2, Proc. Gainsville Conference on Finite Groups; Math. Studies 7, North-Holland, Amsterdam (1972).

[13] Frobenius, G., Über die Charaktere der Symmetrischen Gruppe, Sitz. Berlin Akad. (1900), 516-534.

[14] Green, J.A., The characters of the finite general linear groups, Trans. A.M.S. 80 (1955), 402-447.

[15] Kawanaka, N., Unipotent elements and characters of finite Chevalley groups, Osaka J. Math. 12 (1975), 523-554.

[16] Schur, I., Untersuchungen über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen, J. Reine Angew. Math. 132 (1907), 85-137.

[17] Springer, T.A., Characters of special groups, Part D in Seminar on algebraic groups and related finite groups, by A. Borel *et al.*, Lecture notes in mathematics 131, Springer, Berlin (1970).

[18] Springer, T.A., Cusp forms for finite groups, Part C in Seminar on algebraic groups and related finite groups, by A. Borel *et al.*, Lecture notes in mathematics 131, Springer, Berlin (1970).

[19] Springer, T.A. and R. Steinberg, Conjugacy classes, Part E in Seminar on algebraic groups and related finite groups, by Al Borel *et al.*, Lecture notes in mathematics 131, Springer, Berlin (1970).

[20] Srinivasan, B., The characters of the finite symplectic group Sp(4,q), Trans. A.M.S. 131 (1968), 488-525.

[21] Srinivasan, B., On the Steinberg character of a finite simple group of Lie type, J. Aust. Math. Soc. 12 (1971), 1-14.

[22] Steinberg, R., The representations of GL(3,q), GL(4,q), PGL(3,q) and PGL(4,q), Canadian J. of Math. 3 (1951), 225-235.

[23] Steinberg, R., Regular elements of semisimple algebraic groups, Publ. Math. I.H.E.S. 25 (1965), 49-80.

[24] Steinberg, R., Lectures on Chevalley groups, Yale Univ. Lecture Notes (1967-8).

[25] Steinberg, R., Endomorphisms of linear algebraic groups, A.M.S. Memoirs 80 (1968), 1-107.

[26] Wall, G.E., On the conjugacy classes in the unitary, symplectic and orthogonal groups, J. Aust. Math. Soc. 3 (1963), 1-62.