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APPROXIMATE ANALYSIS OF NON-UNIFORM TORSION

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ABSTRACT

It is difficult to obtain closed form exact solutions for the non-uniform torsion of beams. Advanced finite element methods easily produce very accurate solutions, but simple programs are not widely available. One simple program is validated by comparing its solutions with closed form solutions.

A simple approximate method is proposed which combines the uniform and warping torsion stiffness approximations. The determination of the warping stiffness approximation is simplified by using the moment-area method. The approximate twist rotations are compared with the accurate solutions, are shown to be of generally acceptable accuracy, and in most cases to be conservative.

Approximate solutions for the central twist rotations of simply supported beams with off-centre torques can be combined using superposition to find solutions for a wide range of torque loadings. A worked example is given.

KEYWORDS

Analysis, Bimoment, Rotation, Steel, Torque, Torsion, Warping

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1 INTRODUCTION

The accurate manual analysis of the elastic non-uniform torsion of steel members is not easy, because it requires the solution of a differential equation [1]. The alternatives are to use second-order finite element computer programs with plate or solid elements [2, 3], but these are overly sophisticated, or first-order elastic finite element programs with line elements, but these are not widely available. This paper proposes a relatively simple method of approximate manual analysis which does not require the use of a computer program, is of sufficient accuracy for design purposes, and uses techniques that are familiar to structural designers.

Non-uniform torsion in steel members is resisted by a combination of uniform torsion and warping torsion [4]. Uniform torsion dominates the behaviour of closed section members, and of angle, tee, and cruciform section members, while warping torsion dominates for members that are very thin-walled. However, both uniform and warping torsion are important for a wide range of I and channel section members.

Secondary uniform torques are often induced in the members of steel frame structures by the differential end rotations resulting from joint continuity [4], but these small actions are easily accounted for in frame analysis programs, and are often ignored.

Primary torques rarely occur in isolation. More commonly, primary torques are induced by the eccentricities of transverse loads which cause biaxial bending. Primary torsion increases the minor axis bending moments [5] and reduces the resistance to flexural-torsional buckling [6]. The results of an analysis for torsion alone can often be combined with those for bending to form a close prediction of the structural behaviour [5], and of the design actions necessary for the assessment of structural adequacy.

2 METHODS OF TORSION ANALYSIS

2.1 EXACT SOLUTION

The exact first-order twist rotations ϕ_e caused by a torque variation M_z along a member can be determined by finding the solution of the torsion differential equation [4]

$$GJ\phi_e' - EI_w\phi_e''' = M_z \tag{1}$$

which satisfies the boundary conditions. In this equation, G and E are the elastic shear and Young's moduli of elasticity, J and I_w are the torsion and warping section constants, and ' ≡ d/dz.

The general solution is

$$\phi = \phi_{PI} + A_1 e^{z/a} + A_2 e^{-z/a} + A_3 \tag{2}$$

in which ϕ_{PI} is a particular integral [1] which depends on the torque distribution M_z, A_1 - A_3 are constants of integration which depend on the boundary conditions, and

$$a^2 = EI_w/GJ \tag{3}$$

The exact mid-span twist rotation for the simply supported beam with a central concentrated torque M shown in Fig. 1a is

$$\phi_{ce} = \frac{ML}{4GJ}\left\{1 - \frac{2}{L/a}\left(\frac{1-e^{-L/a}}{1+e^{-L/a}}\right)\right\} = \frac{M}{k_e} \tag{4}$$

in which k_e is the central twist rotation stiffness. Solutions for the twist rotations ϕ_e can be used to determine the distributions of the uniform torques and bimoments

$$M_{ue} = GJ\phi_e' \tag{5}$$

$$B_e = EI_w \phi_e'' \tag{6}$$

Exact solutions for some beams and cantilevers are shown in Table 1.

2.2 FINITE ELEMENT SOLUTION

For this paper, many of the approximate solutions are compared with those obtained from a finite element program FELT derived from an elastic non-linear torsion analysis program FENLT [7], which is based on cubic twist rotation fields [8]. The exact solutions of Table 1 have been used to demonstrate the very high accuracy of the FELT solutions.

The FELT graphical output of the solutions for an example simply supported beam with two concentrated torques is shown in Fig. 2.

2.3 APPROXIMATE SOLUTION

2.3.1 Uniform Torsion Solution

Approximate uniform torsion solutions may be obtained by using formal integration. For the simply supported beam with central concentrated torque M shown in Fig. 1a, the uniform torsion approximations can be obtained by integrating

$$GJ\phi' = M/2 \tag{7}$$

so that

$$GJ\phi = Mz/2 + B_1 \tag{8}$$

and using the boundary condition ϕ = 0 at z = 0, so that B_1 =0. This allows the central twist rotation to be approximated by

$$\phi_{uc} = M/k_u \tag{9}$$

in which

$$k_u = 4GJ/L \tag{10}$$

and the end twist by

$$\phi_{uo}' = 2M/Lk_u \tag{11}$$

2.3.2 Warping Torsion Solution by Formal Integration

Approximate warping torsion solutions may also be obtained by using formal integration. For the simply supported beam with central concentrated torque M shown in Fig. 1a, the warping torsion approximations can be obtained by integrating

$$-EI_w\phi''' = M/2 \tag{12}$$

so that

$$-EI_w\phi'' = Mz/2 + C_1 \tag{13a}$$

$$-EI_w\phi' = Mz^2/4 + C_1 z + C_2 \tag{13b}$$

$$-EI_w\phi = Mz^3/12 + C_1 z^2/3 + C_2 z + C_3 \tag{13c}$$

Using the boundary conditions ϕ = 0 at z = 0, ϕ'' = 0 at z = 0, and ϕ' = 0 at z = L/2 leads to

$$C_3 = 0,\ \ C_1 = 0,\ \ C_2 = -ML/4 \tag{14}$$

This allows the central twist rotation, end twist, and central "twistature" to be approximated by

$$\phi_{wc} = M/k_w \tag{15}$$

$$\phi_{wo}' = 3M/Lk_w \tag{16}$$

$$\phi_{wc}'' = 12M/L^2k_w \tag{17}$$

in which

$$k_w = 48EI_w/L^3 \tag{18}$$

2.3.3 Warping Torsion Solution by Moment-Area

The warping torsion approximations determined by formal integration may often be found more easily (especially so for the beam loadings shown in Figs 1g, h) by using the moment-area theorems [9], which are illustrated in Fig. 3. The first moment-area theorem is that the angle between the tangents at points A and C along a beam is equal to the area of the M/EI diagram between the two points. The second is that the intercept at A between the tangents is equal to the moment of the M/EI diagram between the two points about A.

For this purpose, the simply supported beam with central concentrated torque M shown in Figs 1a and 3a is modelled as two fictitious flanges spaced h apart and having second moments of area $I_f = 2I_w/h^2$, and the torque M is replaced by equal and opposite flange forces $Q_f = M/h$, as shown in Fig. 3b. In this case, the central twist rotation is $\phi_c = 2v_{fc}/h$ in which v_{fc} is the central flange deflection. This model is sometimes described as the "twin beam analogy".

The central flange deflection v_{fc} can be found using the second moment-area theorem, by considering the tangents at the left support A and the mid-span C in Fig. 3d. Thus

$$v_{fc} = \frac{1}{2}\frac{Q_f L}{4EI_f}\frac{L}{2}\frac{L}{3} = \frac{Q_f L^3}{48EI_f} \tag{19}$$

so that

$$\phi_c = \frac{2v_{fc}}{h} = \frac{2}{h}\frac{M}{h}\frac{L^3}{48E}\frac{h^2}{2I_w} = \frac{ML^3}{48EI_w} = \frac{M}{k_w} \tag{20}$$

The end flange rotation v_{fo}' can be found by using the first moment-area theorem, by considering the tangents at the left support A and the mid-span C in Fig. 3d. Thus

$$v_{fo}' = \frac{1}{2}\frac{Q_f L}{4EI_f}\frac{L}{2} = \frac{Q_f L^2}{16EI_f} \tag{21}$$

so that

$$\phi_o' = \frac{2v_{fo}'}{h} = \frac{2}{h}\frac{M}{h}\frac{L^2}{16E}\frac{h^2}{2I_w} = \frac{ML^2}{16EI_w} = \frac{3M}{Lk_w} \tag{22}$$

2.3.4 Non-Uniform Torsion Solution

The separate uniform and warping torsion approximations may be combined [4, 5] to form approximations for the non-uniform torsion central twist rotation, end uniform torque, and central bimoment as

$$\phi_{ca} = M/(k_u + k_w) = M/k_a \tag{23}$$

$$M_{uoa} = GJ\phi_{oa}' = \frac{Mk_u/2}{(k_u + 2k_w/3)} \tag{24}$$

and

$$B_{ca} = EI_w \phi_{ca}'' = ML/4 \tag{25}$$

The accuracy of these approximations is demonstrated in Figs 4 and 5. The variations of the central twist rotation stiffness approximation of $k_a = k_u + k_w$ with the torsion parameter $k_w / k_u = 12EI_w / GJL^2$ are compared in Fig. 4 with that of the exact stiffness k_e. Low values of $12EI_w / GJL^2$ correspond to long slender members for which uniform torsion is dominant, while high values correspond to short stocky members for which warping torsion is dominant. The approximation using k_a is conservative and close to the exact k_e. Also shown in Fig. 4 are approximations using k_u and k_w. It can be seen that these are conservative, and generally very much so. The warping approximation using k_w corresponds to the "twin-beam analogy", for which it is assumed that the torque M is solely resisted by warping torsion through flange forces M/h, instead of by the sum of the warping and uniform torques. The errors of this approximation may be very high, being 55% approximately for $12EI_w / GJL^2 = 1$.

The variations of the ratios of the approximations ϕ_c, M_{uo} and B_c to the exact values are shown in Fig. 5. It can be seen that the approximations are of high accuracy for high values of k_w / k_u, but become less accurate as k_w / k_u decreases The values *for* ϕ_{ca} and M_{uoa} are of an accuracy which is acceptable for practical design, but the accuracy for B_{ca} appears to be less so. It is probable that the inaccuracies at low values of k_w / k_u are not important because the design of such slender members is likely to be governed by other considerations. If they are not, then an improved approximation of high accuracy is given by

$$B_c = \frac{B_{ca}}{(1 - 0.06 k_u / k_w)} \tag{26}$$

3 TORSION APPROXIMATIONS

3.1 CONCENTRATED TORQUES

The equation parameters ϕ_{PI}, A_1, A_2, and A_3 of Equation 2 for the exact solutions of Equation 1 for the maximum twist rotations ϕ_e of some members with concentrated torques (Fig 1a – 1c) are given in Table 1.

The parameters a_u and a_w of the equation

$$\phi_a = \frac{M}{(a_u k_u - a_w k_w)} = \frac{M}{k_a} \tag{27}$$

for the corresponding approximate twist rotations ϕ_a are given in Table 2. The variations of the ratio ϕ_a / ϕ_e of these with the torsion parameter $12EI_w / GJL^2$ are shown in Fig. 6. These variations demonstrate a maximum error in the approximations of Equation 27 of 12% on the conservative side.

The errors in the solutions for the cantilevers of Fig. 1c can be reduced to less than 12% by changing the parameters for Equation 27 from 1/4 and 1/6 to $a_u = 0.26$ and $a_w = 0.07$ respectively, as shown in Fig. 6.

Identical equation parameters for the approximate central twist rotations ϕ_{ca} of simply supported beams with either one off-centre torque (Fig. 1g) or two equal and equally spaced torques (Fig. 1h) are given in Table 2. Accurate values ϕ_{ce} of these twist rotations were obtained using the finite element program FELT [7]. The variations of the ratio ϕ_{ca} / ϕ_{ce} of these with the dimensionless distance s/L to the torque and with the torsion parameter $12EI_w / GJL^2$ are shown in Fig. 6. The variations are identical for both torque distributions. The variations for $s/L = 0.5$ correspond to those for beams with central concentrated torque (Fig. 1a). The ratios decrease with the dimensionless torque distance s/L. The variations of the approximate dimensionless central twist rotations ϕ_{sa}/ϕ_{ca} with s/L are shown in Fig. 7 for values of $12EI_w/GJL^2$ of 0, 1, and ∞.

A wide range of approximate solutions for beams with multiple torques can be obtained using superposition of the individual solutions for beams with off-centre torques.

3.2 UNIFORMLY DISTRIBUTED TORQUES

The equation parameters ϕ_{PI}, A_1, A_2, and A_3 of Equation 2 for the exact solutions of Equation 1 for the maximum twist rotations ϕ_e of some members with uniformly distributed torques (Fig 1d – 1f) are given in Table 1.

The parameters a_u and a_w of Equation 27 (with $M = mL$) for the corresponding approximate twist rotations ϕ_a are given in Table 2. The variations of the ratio ϕ_a / ϕ_e of these with the torsion parameter $12EI_w / GJL^2$ are shown in Fig. 8.

The solutions for cantilevers indicate errors of up to 29%. These errors can be reduced to less than 12% by changing the parameters for Equation 27 from 1/2 and 1/6 to a_u = 0.55 and a_w = 0.2 respectively, as shown in Fig. 8. The errors for built-in beams are less than 12%, while those for simply supported beams are less than 1.5%.

The end torque and bimoment solutions for built-in beams of $mL/2$ and $mL^2/12$ are the fixed end actions of a finite element, which allows the element to be replaced in a finite element analysis by the negatives of these fixed end actions. They are used in this way in the finite element program FELT.

4 WORKED EXAMPLE

The data for a worked example is shown in Fig. 9. The beam has two unequal concentrated torques arranged asymmetrically. The FELT [7] solutions for this example are shown in Fig. 2. The approximate solution for this example is most easily found by using the asymmetric torque (Fig. 1g) approximations of Table 2 for each load.

Using Equations 10 and 18, k_u = 6e7 and k_w = 6e7
Using Table 2 a_{uB} = 1.250 and a_{wB} = 1.059, and
a_{uD} = 1.429 and a_{wD} = 1.138
Using Equation 27 k_{aB} = 1.386e8 and k_{aD} = 1.540e8
Using Equation 27 ϕ_{cB}= 0.0722 and ϕ_{cD} = 0.0455
Using superposition $\phi_{ca} = \phi_{cB} + \phi_{cD}$ = 0.118

This is close to the exact solution of ϕ_{ce} = 0.107 obtained using FELT.

5 CONCLUSIONS

It is difficult to obtain closed form exact solutions for the elastic non-uniform torsion of steel beams, except for a limited number of examples. Some closed form solutions are given in Table 1. Powerful finite element computer programs produce very accurate solutions, but simple programs are not widely available. A simple program used in this paper is validated by comparing its solutions with the closed form solutions.

A simple approximate method is proposed which combines the uniform and warping torsion approximations by the addition of the stiffnesses. The determination of the warping stiffness approximation is simplified by using the moment-area method. The approximate solutions for the twist rotations are compared with accurate solutions obtained using the simple finite element program. These are shown to be of generally acceptable accuracy, and in most cases to be conservative. The solutions are given in Table 2.

Approximate solutions for the central twist rotations of simply supported beams with off-centre torques can be combined using superposition to find solutions for a wide range of torque loadings. A worked example is given, and its approximate solution compares favourably with the accurate finite element solution.

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7 NOTATION

7.1 SUBSCRIPTS

a,e = approximate or exact
c = centre
u,w = uniform or warping torsion

7.2 PRINCIPAL NOTATION

A, B, C = constants of integration
a_u, a_w = uniform and warping torsion stiffness constants
a^2 $= EI_w/GJ$
E = Young's modulus of elasticity
G = shear modulus of elasticity
h = distance between flange shear centres
I_f = flange second moment of area
I_w = warping section constant
J = uniform torsion section constant
k = stiffness
L = member length
M = concentrated torque
M_z = torque variation along the member
m = uniformly distributed torque
Q_f = flange force
s = distance to concentrated torque
v_f = flange displacement
z = distance along member

ϕ = twist rotation

Figure	Support and Restraint	Torque	ϕ_{PI}	A_1	A_2
1a	Beam Free - Free	M	$\frac{Mz}{2GJ}$	$-\frac{Ma}{2GJ}\left(\frac{1}{e^{L/2a}+e^{-L/2a}}\right)$	$-A_1$
1b	Beam Fix - Fix	M	$\frac{Mz}{2GJ}$	$\frac{Ma}{2GJ}\left(\frac{e^{-L/2a}-1}{e^{L/2a}-e^{-L/2a}}\right)$	$A_1+\frac{Ma}{2GJ}$
1c	Cantilever Fix - Free	M	$\frac{Mz}{GJ}$	$-\frac{Ma}{GJ}\left(\frac{1}{e^{2L/a}+1}\right)$	$A_1+\frac{Ma}{GJ}$
1d	Beam Free - Free	$mL=M$	$\frac{mL^2}{2GJ}\left(\frac{z}{L}-\frac{z^2}{L^2}\right)$	$\frac{ma^2}{GJ}\left(\frac{1}{e^{L/a}+1}\right)$	$-A_1+\frac{ma^2}{GJ}$
1e	Beam Fix - Fix	$mL=M$	$\frac{mL^2}{2GJ}\left(\frac{z}{L}-\frac{z^2}{L^2}\right)$	$-\frac{mLa}{2GJ}\left(\frac{1+e^{L/a}}{1-e^{2L/a}}\right)$	$A_1+\frac{mLa}{2GJ}$
1f	Cantilever Fix - Free	$mL=M$	$\frac{mL^2}{GJ}\left(\frac{z}{L}-\frac{z^2}{2L^2}\right)$	$\frac{ma^2}{GJ}\left(\frac{e^{L/a}-L/a}{e^{2L/a}+1}\right)$	$A_1+\frac{mLa}{GJ}$

$$\phi_e = \phi_{PI} + A_1 e^{z/a} + A_2 e^{-z/a} - (A_1 + A_2)$$

Table 1. Parameters for Exact Solutions

Figure	Support	End Restraint	Torque	a_u	a_w
1a	Beam	Free - Free	Central Concentrated	1	1
1b	Beam	Fixed - Fixed	Central Concentrated	1	4
1c	Cantilever	Fixed - Free	Concentrated End	$\frac{1}{4}$ (0.26)	$\frac{1}{16}$ (0.07)
1d	Beam	Free - Free	Uniformly Distributed	2	$\frac{8}{5}$
1e	Beam	Fixed - Fixed	Uniformly Distributed	2	8
1f	Cantilever	Fixed - Free	Uniformly Distributed	0.5 (0.55)	$\frac{1}{6}$ (0.2)
1g	Beam	Free - Free	Symmetrical Concentrated	$\frac{1}{2(s/L)}$	$\frac{1}{(s/L)(3-4s^2/L^2)}$
1h	Beam	Free - Free	Concentrated Off Centre	$\frac{1}{2(s/L)}$	$\frac{1}{(s/L)(3-4s^2/L^2)}$

$$k_a = a_u k_u + a_w k_w$$

Table 2. Parameters for Twist Rotation Stiffnesses

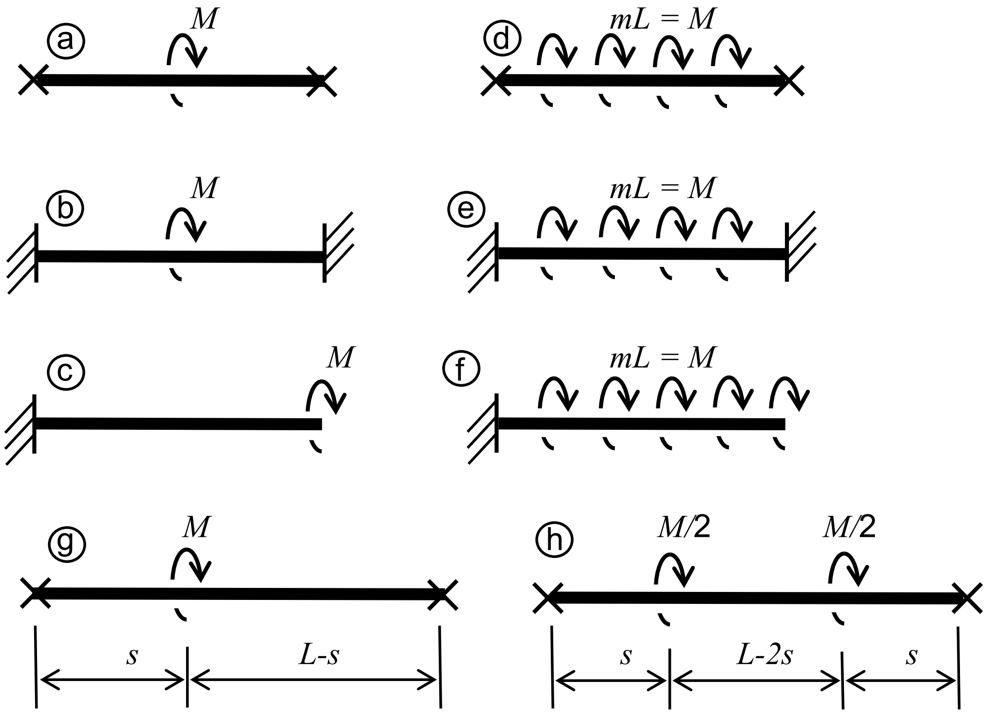

Fig. 1   Beams and Torques

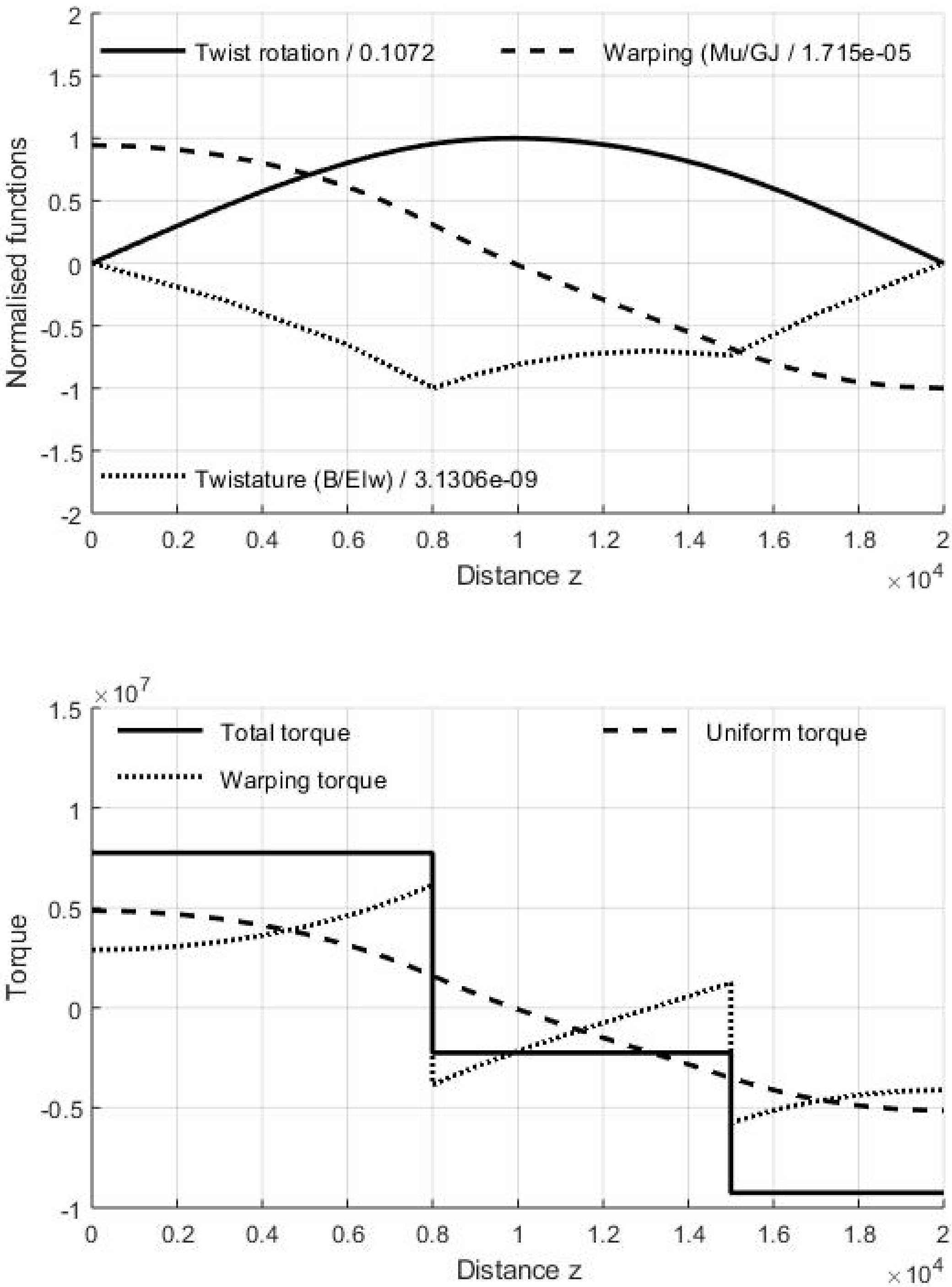

Fig. 2 Non-Uniform Torsion Solutions

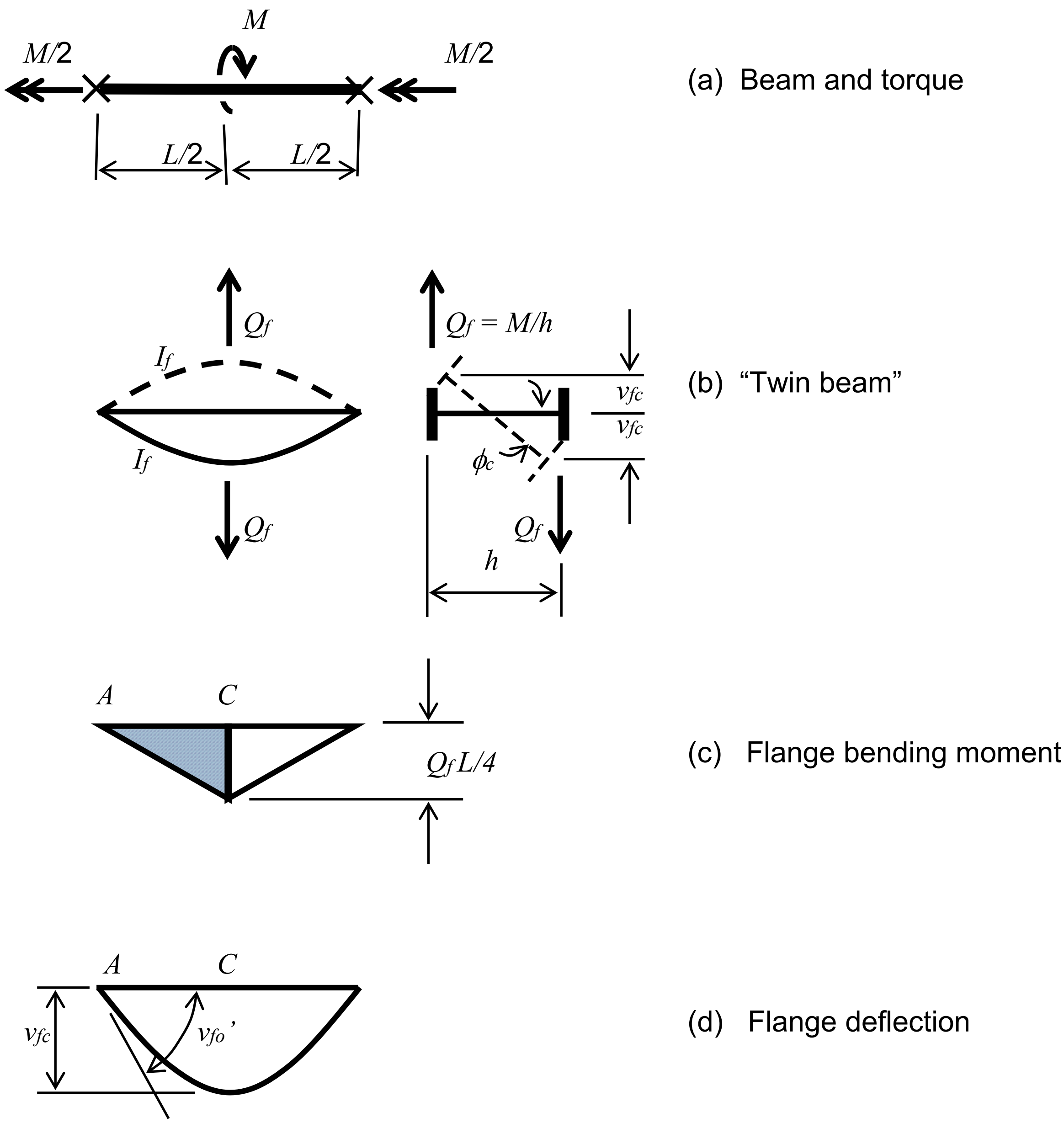

Fig. 3 "Twin Beam" Model of Warping Torsion

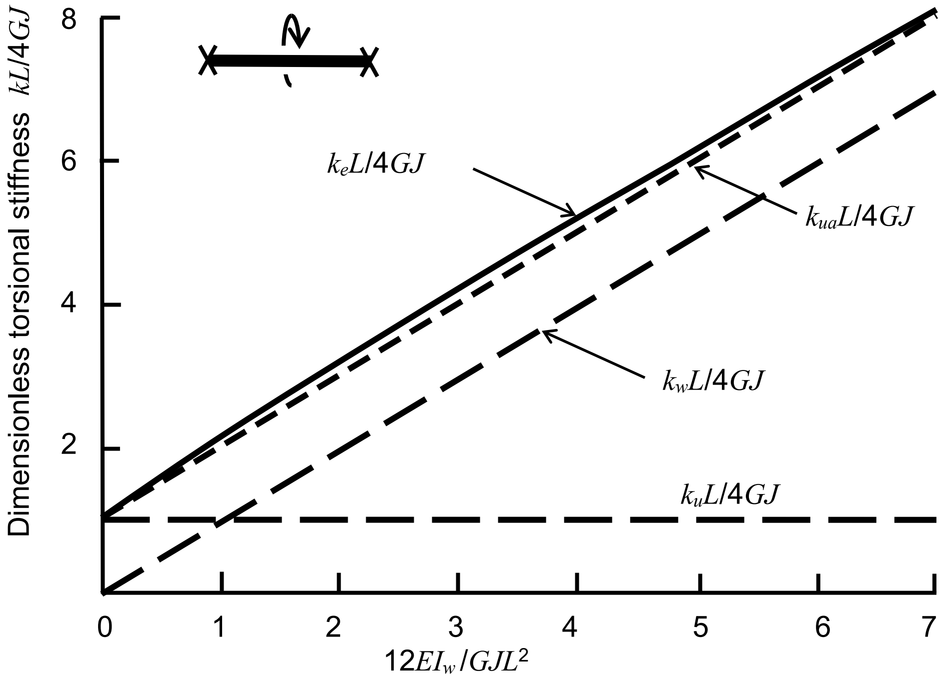

Fig. 4 Torsional Stiffness Approximations

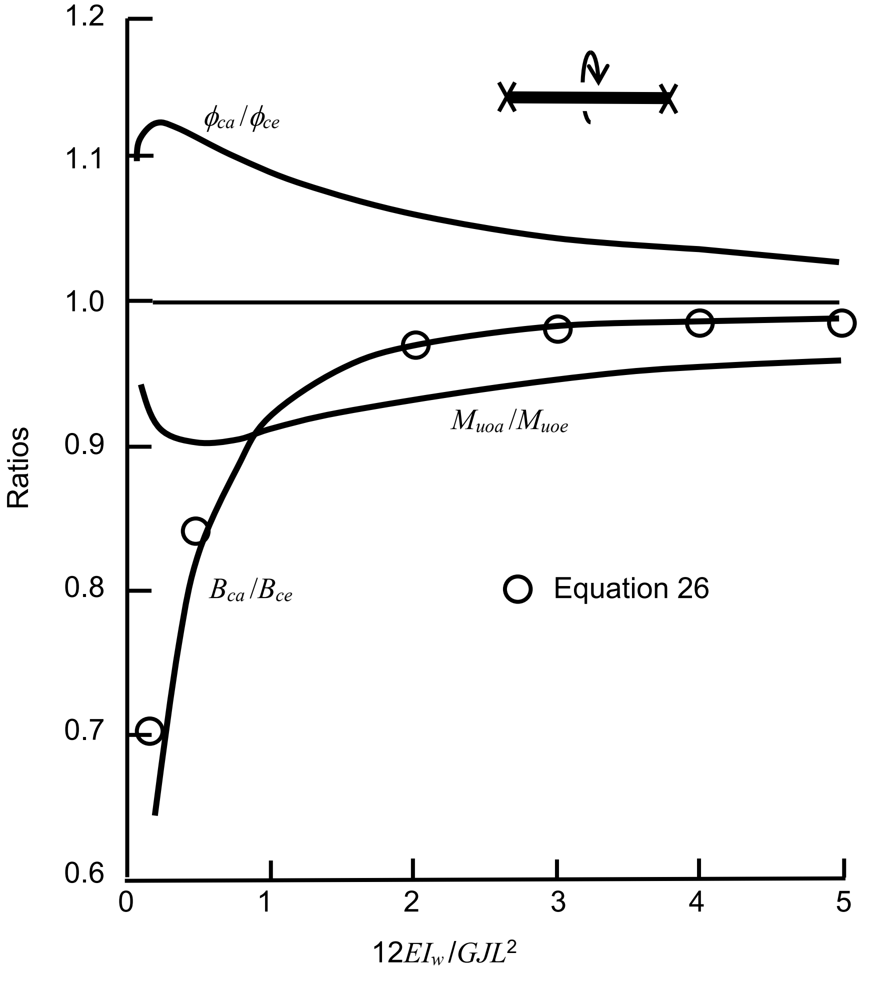

Fig. 5 Accuracy of ϕ_c, M_{uo}, B_c

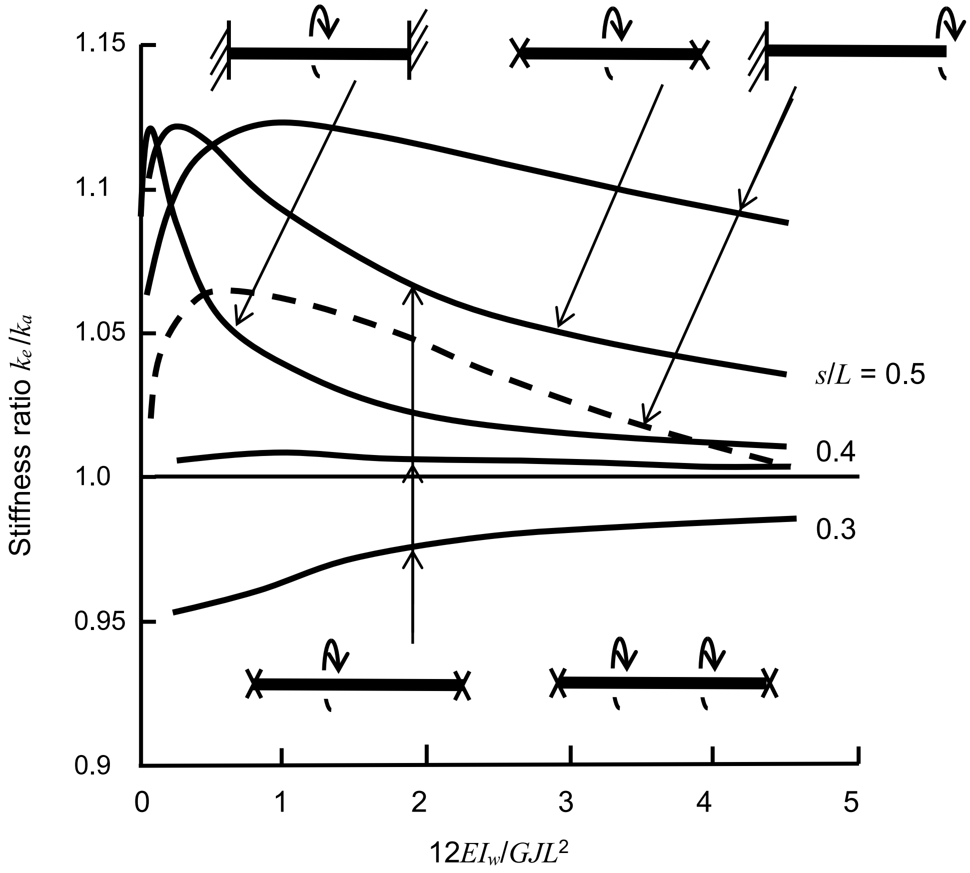

Fig. 6 Approximations for Concentrated Torques

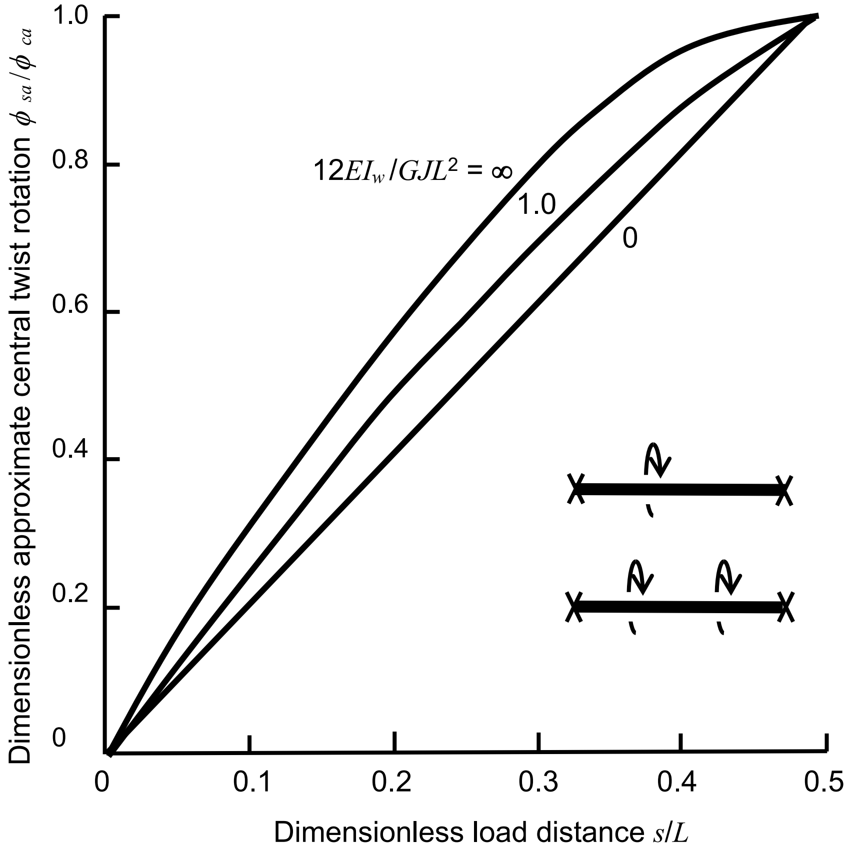

Fig. 7 Twist Rotation Variation with Load Distance

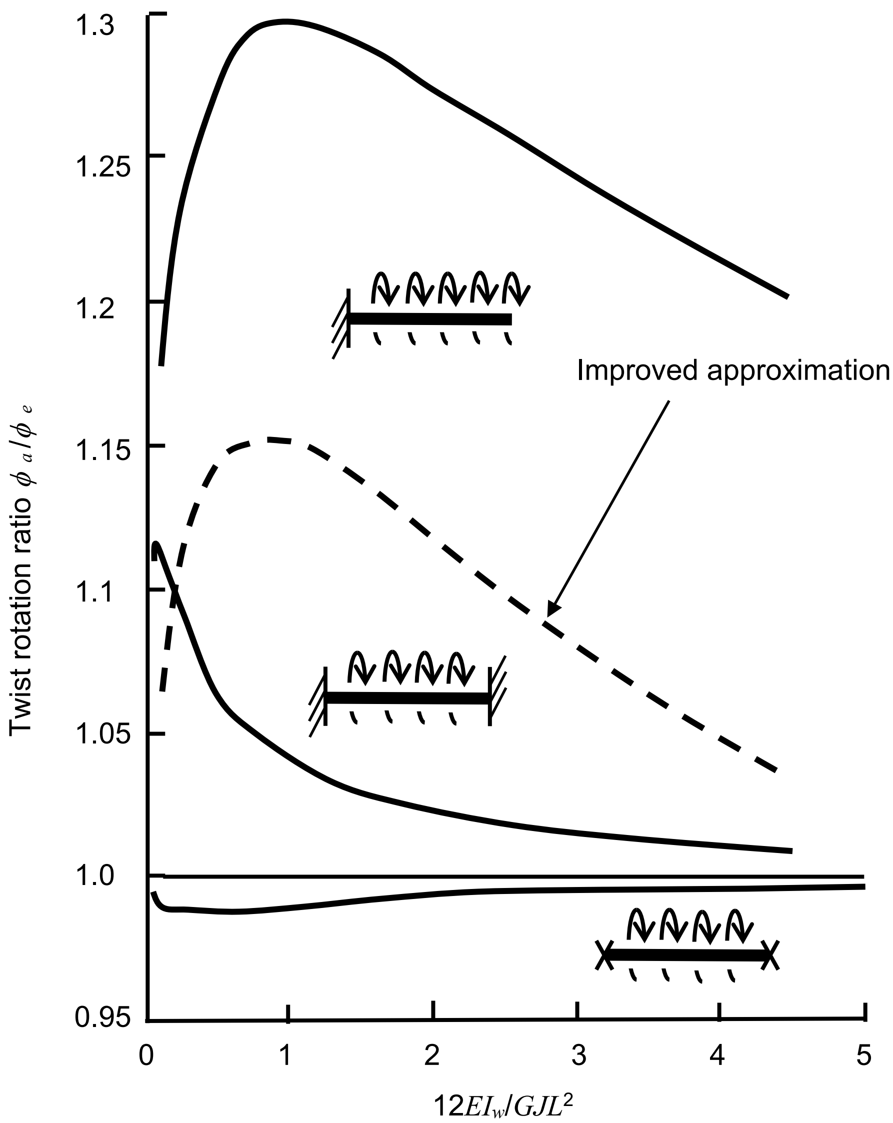

Fig. 8 Approximations for Distributed Torques

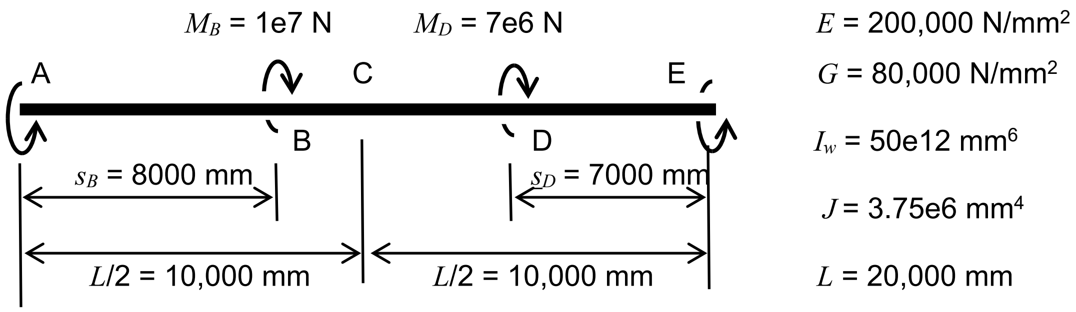

Fig. 9 Worked Example (FELT solution shown in Fig. 2)