Vagueness, Identity, and Quantum Objects

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Abstract

While classical accounts of identity (‘classical’ here pertaining to both logics and physics) are generally well understood, the advent of quantum theory, specifically quantum statistics, has cast shadow over these conceptions. Dealing with the consequently surfacing problems is a philosophically rich and interesting enterprise. I begin this thesis by providing an exegesis of the roles played by, and features of, identity in logics, classical physics, and quantum physics. Therein I consider how under a quantal description of reality, classical notions of identity and individuality break down. In the second chapter, I address how this problem has launched an arc of thought in analytic metaphysics and formal philosophy motivating the development of non-standard formal frameworks with which philosophical sense can be made of quantal objects. Among these, I explore and critically evaluate quaset theory, quasi-set theory, and non-reflexive Schrödinger logics, identifying some significant problems with quaset theory that arise in defining cardinality and later, pointing out a problem with Schrödinger logics in their modelling of the continuity between quantal and classical treatments of the world. The queer character of identity in the quantal regime motivates a turn to vagueness which I introduce in the third chapter, providing a brief outline of vagueness and the sorites paradox. Further, I reflect on the fundamental nature of vagueness, outlining and evaluating the semantic and ontic conceptions thereof. In the final chapter, I proceed to explicate and assess notions that identity and quantal objects can be vague. I shall discuss accounts according to which the vagueness of identity and quantum objects is posited as a feature of nature emerging in quantum systems — the ontic vagueness of identity — finding that these ideas are flawed and/or rely on misinterpretations of vagueness. Finally, I present an argument which suggests how the vagueness of identity can arise as an artifact of the differing treatments of identity in the quantal and classical regimes in which the vagueness involved can be semantic rather than ontic.
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1 Identity

Monty Python’s Life of Brian

1.1 Identity in Logic/s

In logic, identity is the binary relation denoted by the symbol ‘=’. When added to our logical apparatus, the syntax of the symbol is simple; if $x$ and $y$ are variables or names, then $x = y$ is a well-formed formula (wff). As for what it means, let us begin with first-order logic. Here, the relation is taken to be a primitive concept and, as such, it cannot be defined in terms of more basic notions. That is, if we try to explain identity in first-order logic, we will inevitably end up circularly appealing to notions of identity which we already possess. For instance, in ordinary language we could say that identity is the relation any object $a$ bears to itself and nothing else, but of course, by using the term ‘nothing else’ we are already invoking identity (here, negated identity) in that ‘nothing else’ means nothing that is non-identical to $a$. A formalisation back into first-order logic clearly exposes this circularity: $\forall x \forall y ((x = x) \land (\neg y = x \rightarrow \neg y = x))$. The impossibility of defining identity in first-order logic is analogous to our inability to define, say, negation using only conjunction and disjunction in propositional logic. In this case we thus require a primitive symbol for negation.
While in first-order logic identity is built-in and implicitly understood, we can outline some of its properties. Identity is commonly thought of as an *equivalence relation* for which there is simple set of necessary and sufficient conditions. That is, we say ‘=’ is an equivalence relation iff all the following properties are satisfied:

**Reflexivity:** \( \forall x (x = x) \)

**Symmetry:** \( \forall x \forall y (x = y \leftrightarrow y = x) \)

**Transitivity:** \( \forall x \forall y \forall z ((x = y) \land (y = z) \rightarrow (x = z)) \)

These are easy to understand. If a relation is reflexive then everything bears that relation to itself and it is symmetric when insensitive to permutations of its relata. Taking the arbitrary objects \( a, b, \) and \( c \), a relation’s transitivity simply means that if \( a \) bears it to \( b \) and \( b \) bears it to \( c \), then \( a \) must bear it to \( c \).

The above description of identity is skeletal, as expected by virtue of its primitive status in first-order logic. To construct definitions identity, we require the apparatus of, at least, second-order logics. Here we extend our scope of quantification in that we can quantify over the extensions of predicates rather than just names represented by variables. For instance we can make statements of the form ‘for all properties \( P \)...’ in addition to ones such as ‘for all objects \( x \)...’. Employing second-order logic, identity can be defined as an equivalence relation satisfying *Leibniz’s law* (hereafter also referred to as ‘LL’). LL, or the *indiscernibility of identicals* states that if two objects are identical, then they must have all the same properties i.e. they are *indiscernible*/indistinguishable. If we wish to express this formally we must do so in (at least) second-order logic,
since the statement quantifies over properties as well as objects.

**Leibniz’s Law:** \( \forall x \forall y (x = y \rightarrow \forall P (P_x \leftrightarrow P_y)) \)

The above principle is uncontroversial and makes intuitive sense, for example since Superman and Clark Kent are the same person, it follows that every property possessed by Superman will be possessed by Clark Kent and vice versa. Note that LL can also pertain to predicates of arbitrary arity in that we can express such predicates as unary for we are only talking about one relatum. For instance, if Alice is taller than Bob then we express this as a property of Alice i.e. Alice has the property of being taller than Bob.

The definition of identity built so far seems tenable, though we can go a step further and construct a stronger definition by invoking the converse of LL — the **identity of indiscernibles** (IoI). This principle states that if objects are indiscernible, i.e. bearing all the same properties, then they are identical.

**Identity of Indiscernibles:** \( \forall x \forall y (\forall P(P_x \leftrightarrow P_y) \rightarrow x = y) \)

Evidently the IoI is far more metaphysically contentious than LL. For if one can conceive of two separate objects that share all the same properties (indiscernibles) then — contrary to the principle — the fact that they are not one and the same object means they cannot be identical. Identity is the relation between objects in which the relata stand for one and the same object (numerical identity), so the notion of there being two identical objects rather than two symbols referring to one and the same object is incoherent. This being said, we can attempt preserve the IoI if we so desire and I show this in the following section. For now, an important idea is that if we accept both LL and the IoI, we

---

1 There are also conceptions of LL in which the main conditional is replaced with a biconditional, i.e. the indiscernibility of identicals plus its converse. This interpretation thus equates identity with indiscernibility, which I shortly address.
are mandated to conclude that indistinguishability and identity are equivalent. Taking the conjunction of both principles gives us the biconditional:

$$\forall x \forall y (\forall P (Px \leftrightarrow Py) \leftrightarrow x = y)$$  (1)

So according to (1), whenever objects are identical they are also indiscernible and vice versa. This is quite a strong claim from a philosophical standpoint (since the IoI is involved) but I introduce it here because, as we shall see, this is often the way identity is talked about in physics — as indistinguishability.

### 1.2 Identity in Classical Physics

In classical physics, objects such as particles are taken to exist as individual entities; they are ‘well-behaved’, occupying their own, unique spatiotemporal trajectories. Here, identity between, say, \(a\) and \(b\) means that \(a\) and \(b\) are the one and the same object, that is, ordinary numerical identity. In the classical picture, we can also preserve the IoI. Suppose we have two particles of the same kind \(a\) and \(b\) as shown in Figure 1. Since they are of the same kind, they have exactly

![Figure 1: Two, classical particles \(a\) and \(b\) in three dimensional space, at some time.](image-url)
the same intrinsic properties such as their mass, charge, and so on. Yet, there
are still two individual and thus non-identical particles. But despite their prima
facie indiscernibility, the two particles are in fact discernible because classical
physics requires them to have unique spatiotemporal trajectories; they can never
occupy the same space at the same time. This is known as the impenetrability
assumption, since the particles cannot, in a sense, ‘move through’ each other.
As such, they cannot possibly share all the same properties for they must have
different properties of location in spacetime. I formalise this as particle $a$ having
the following spatiotemporal coordinates or, more rigorously, the spacetime four-
vector

$$x^\mu_a = (x_a, y_a, z_a, ct_a)$$

where $c$ is the speed of light. And similarly, particle $b$ has the spacetime four-
vector

$$x^\mu_b = (x_b, y_b, z_b, ct_b).$$

By the impenetrability assumption we require that

$$x^\mu_a \neq x^\mu_b.$$

Here the spatiotemporal locations of both particles are represented by vectors
in $\mathbb{R}^4$, capturing their $x$, $y$ and $z$ spatial coordinates along with the moments in
time at which these spatial locations are occupied. Another way of saying this
would be that the particle’s worldlines (spacetime trajectories represented by
time on a vertical axis and spatial dimensions on the others) never intersect.

A modification of Figure 1, Figure 2 shows $a$ and $b$ with their distinct
spatiotemporal properties (here, distinct spatial properties). So it is impossible
to have two, indiscernible particles because the impenetrability assumption

---

2 The superscript ‘$\mu$’ is to denote that something is a four-vector. That is, $A^\mu$ is just
shorthand for a four vector $(A_1, A_2, A_3, A_4)$. 

8
prohibits objects from holding the same properties of spatiotemporal locality. In classical physics, even if we have two of the same exact kind of particle, they remain treated as non-identical individuals (as we shall see, this is in direct contrast to the quantum case).

This treatment of classical particles is exemplified in classical statistical mechanics. Here, we wish to study the aggregate behaviour of many, microscopic particles from which we can derive the system’s macroscopic characteristics using statistical techniques (the details of which shall not be given here because they are not pertinent to this thesis). Individual particles in the system have microscopic properties such as their individual momenta or energies. We cannot practically measure each of these directly, but together they determine the system’s macroscopic properties such as entropy, pressure, and temperature. Now, we define a system’s macrostate as the specific set of macroscopic properties it bears. For instance, suppose our system consists of some gas in an enclosed box;

Figure 2: Two, classical particles $a$ and $b$ with their denoted spatiotemporal coordinates.
the system’s macrostate is defined as its volume, density, temperature, pressure, and so forth. On the other hand, a system’s microstate is the specific state of each individual particle therein at some instant in time. Consider our box of gas again. To specify its microstate we would need to specify all the properties of every gas particle in the box, so, their mass, velocity, angular momentum and such. It is important to see that the same macrostate may be achieved by many, many different microstates as there is a very large number of particle configurations that give rise to a particular set of macroscopic characteristics. The number of microstates that can give a specific macrostate is called that macrostate’s multiplicity ($\Omega$).

Multiplicity is important, for instance, because it allows us to calculate the probability of certain macrostates and thus derive macroscopic properties such as entropy, which is defined as

$$ S := k_b \ln(\Omega) \quad (5) $$

where $k_b$ is Boltzmann’s constant. In statistical mechanics, physicists need to ‘count’ up the particles/microstates of a system to determine the aforementioned quantities and in classical statistical mechanics the way that this is done sheds light on the metaphysical treatment of particles’ individuality and identity. Suppose we have a box with two qualitatively identical gas particles inside, particle $a$ and particle $b$. At an instant in time we have one particular microstate, with $a$ and $b$ in some specific locations with their respective velocities, which we shall assume are the same — shown on the left in Figure 3. Now suppose we switch the locations of $a$ and $b$ as shown on the right in Figure 3. Although an observer would not be able to tell the difference between the microstates in Figure 3, classical statistical mechanics treats them as two, distinct microstates that
would correspond the same macrostate. Suppose (grossly unrealistically) these are the only two microstates that give this macrostate. Then our multiplicity for the macrostate would be

$$\Omega = \frac{2 \text{ microstates}}{1 \text{ macrostate}} = 2$$

(6)

and thus the entropy, for example, would be

$$S = k_b \ln(2) \approx 9.57 \times 10^{-24} J K^{-1}.$$  

(7)

This macroscopic result would be different were we to treat the particles as non-individual identicals and thus have only one microstate (this will be clarified when I address the quantal situation). The above instance, however, exemplifies the treatment of the metaphysics of identity by classical physics as in line with its treatment in standard metaphysics and logics. Although we shall see that this is fundamentally different in the quantum case. 

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\[3\] This is a significant simplification of how the idea is presented in physics but it illustrates my point just as well. For a more in-depth information see [Schroeder, 1999, Ch. 2].

\[4\] One should bear in mind that this classical treatment of what are in fact quantal objects is incorrect, failing to yield results that are consistent with experiment. Consequently, such treatment is only valid when a system’s temperature is sufficiently high and particle density sufficiently low for quantal effects to be considered negligible.
1.3 Identity in Quantum Physics

In our classically conceived system, each particle had a unique spatiotemporal trajectory and thus all particles were distinguishable. This makes sense if our system is sufficiently rarefied but the model breaks down when the system becomes dense and we consider the true, quantal nature of the particles in question. Why is this? We must first understand that a particle is not accurately described as a ‘tiny sphere’ or a single point, as implied §1.2. Rather, quantal particles do not have well-defined positions/momenta/etc., they are only described by what we call their wavefunctions. These are complex-valued functions, often denoted \( \Psi \), that extend through space (this could be position space, momentum space, or others) contrasting with the classical picture in which particles are thought to occupy a well-defined, specific set of points in space and time. Figure 4 illustrates a simple, one-dimensional wavefunction of position.

![Wavefunction Graph](image)

Figure 4: An example of what position wavefunction in one-dimension might look like. The real part is shown in black and the imaginary part in blue.

\(^5\)Here ‘dense’ and ‘rarefied’ are technical terms which I will define.
\(^6\)In this section I begin by presenting the wavefunction as a function of position (\( \Psi(x) \)), however we can also have wavefunctions of momentum (\( \Psi(p) \)) and of various other domains.
The wavefunction is everything we know about a particle, for it contains all its measurable properties and describes the superposition of possible states in which it could be. For instance, if we have the wavefunction of a particle in three spatial dimensions $\Psi(x, y, z, t)$ we can then perform operations on the wavefunction in order to find out where the particle is likely to be found (its spatial position). This does not necessarily return one, specific location; it gives us a probability density function which, unlike the original wavefunction, is real-valued and positive. The probability density function $\rho(x, y, z, t)$ can be obtained by taking the wavefunction’s modulus squared (multiplying it by its complex conjugate $\Psi(x, y, z, t)^*$) as such:

$$
\rho(x, y, z, t) = |\Psi(x, y, z, t)|^2 = \Psi(x, y, z, t)\Psi(x, y, z, t)^* \quad (8)
$$

The probability distribution needs to integrate to unity over its domain — this is called the normalisation condition. It is important because integrating $\rho(x, y, z, t)$ over a volume corresponds to the probability $P$ of finding the particle therein, so over all space at some time the total probability must sum up to 1:

$$
P(x, y, z \in [-\infty, +\infty]) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Psi(x, y, z, t)\Psi(x, y, z, t)^* dxdydz = 1 \quad (9)
$$

This means that our particle will definitely need to be somewhere. According to the standard interpretation of quantum mechanics (the Copenhagen Interpretation\footnote{As the standard canonical view, the CI is the most commonly taught and understood interpretation of quantum mechanics at the undergraduate and early graduate level although it is not entirely well-received by modern, working theoreticians. Moreover, many non-standard interpretations of quantum mechanics would require independent and significantly different philosophical treatment along with their own explanations. As a consequence thereof, other interpretations fall beyond the scope of this thesis and I restrict discussion of quantum mechanics and its philosophy to the CI alone.}) if we are to observe or interact with the particle its wavefunction will collapse to give us one, specific location. The probability distribution just tells us exactly how likely it is that the particle will ‘pick’ out each possible location.
It is very important to note that the fact that we don’t know exactly where the particle is prior to observation is *not* merely due to an epistemic deficiency. It is a fundamental, ontological property of quantal particles according to the standard interpretation that they have no single position prior to the collapse of the wavefunction. That is, the question ‘where was the particle before it was observed?’ is meaningless — there was no single location, only a probability distribution.

Now, consider a system of very rarefied\(^8\) particles of the same kind. Since the system is rarefied, the particles are far away from each other, this means they are non-interacting and all occupy their own single-particle states and can be modelled as essentially classical objects (quantal effects are negligible). However, as we increase the density of our system the wavefunctions become crowded together until they begin to interact and overlap. Consequently, at high densities it is no longer possible for us to determinately distinguish any particles of the same kind as we could in the classical case, since we can no longer track their spatiotemporal trajectories. Note that in the rarefied case the particles are sufficiently spread out (with no interacting wavefunctions) thus allowing us to ignore quantal effects, and thereby allowing us to track, or at least approximate, their trajectories as if they were classical. Figure 5 illustrates this transition from rarefied to dense with the wavefunction of two particles in one dimension. In the top left of Figure 5, we see two particle’s wavefunctions as they would appear in a rarefied system. As the system increases in density, the two particles move closer together as shown in the top right; the wavefunctions are beginning to overlap slightly though we can still discern both of them. Further increasing the density gives the situation shown in the bottom plot; the wavefunctions are severely overlapping, in which case the particles are rendered indistinguishable.

\(^8\)‘Rarefied’ here means spread out i.e. few objects in a given volume. This is the opposite of dense (many objects in a given volume).
When wavefunction-overlapping occurs, there is no distinct spatiotemporal trajectory that each particle occupies since quantal effects are no longer negligible, and as such, there is no property we can appeal to in order to assert any kind of distinguishability given that the particles are of the same kind. If we invoke the IoI, which we showed holds in the classical case, it follows then that the particles are identical. In quantum physics, such particles are said to be \textit{exactly identical} which is taken to mean that they are indistinguishable \cite{Feynman1965, Townsend2000, Flammia2019}.

The exact identity of quantal particles that I have described has important physical and philosophical consequences. Physically, the statistical mechanics of systems containing identical particles must be different to classical statistical mechanics with non-identical particles. This is primarily because the

Figure 5: Wavefunction-overlapping in one dimension.
particle-counting is different. Since particles are exactly identical, swapping their positions to achieve a new microstate like we did in the classical case is no longer possible. Rather, the particles' identity is strongly reflected in the counting in that swapping their states results in one and the same microstate — not just two separate microstates corresponding to a single macrostate. This leads to the theories of quantum statistics, the quantal analogues of classical statistical mechanics.

Philosophically, our ordinary conception of identity as described in §1.2 seems to break down. Since a dense system of particles of the same kind entails that the multiple particles are exactly identical to each other, we have entities that are identical without being one and the same thing. Here the case of multiple, identical objects goes against the notion that the identity of, say, objects $a$ and $b$ meaning that they are in fact the same object. This looks like an incoherent idea given our understanding of numerical identity, so perhaps I ought to address the language used by physicists. When ‘identical particles’ are talked about, we could say that identity is not used in the traditional numerical sense. Rather, it could actually just mean qualitative identity in the strongest possible sense whereby the particles share exactly the same properties. In this case, we no longer have the aforementioned paradoxical situation which violates our definition of numerical identity and further, we must reject the IoI since we have indiscernible objects that are not actually (numerically) identical. However, there is still a problem here because the particles remain absolutely indistinguishable. Given a group of such particles, we should not be able to tell how many there are in total or even tell that there is a group of particles in the first place for they are all totally indistinguishable. Despite the impossibility of counting/ordering them, the number of particles in such a group can still be stated in physics. In the following sections I show how we might be able to make sense of this philosophically; here I am only grazing the surface of a
philosophical problem with which I intend to engage. This being said, I will not eliminate the idea that physics may in fact be referring to numerical identity, for there is no definitive answer here [Krause and Arenhart, 2020, p. 6ff.].

To illustrate which particles exhibit this ‘identicality’ and further motivate quantum statistics, we can define the exchange operator \( \hat{P} \). This operator acts on a wavefunction of two exactly identical particles and swaps their labels. Suppose subscripts ‘1’ and ‘2’ are fictional labels for the two particles and \( r \) is the position vector \((x, y, z)\). The exchange operator then acts as such:

\[
\hat{P}\Psi(r_1, r_2) = \Psi(r_2, r_1)
\]  

(10)

Here, the ‘swapped’ wavefunction’s probability distribution \(|\Psi(r_2, r_1)|^2\) must be the same as the original wavefunction’s probability distribution for indistinguishable particles. This means that the wavefunctions themselves can only differ by an overall phase factor \(e^{i\varphi}\) (where \(\varphi\) is some phase angle). In quantum physics, an overall phase factor on a wavefunction has no observable effects because it reduces to unity once we take its modulus squared: \(|e^{i\varphi}|^2 = e^{i\varphi}e^{-i\varphi} = 1\). We have:

\[
|\Psi(r_1, r_2)|^2 = |\Psi(r_2, r_1)|^2 \rightarrow \Psi(r_1, r_2) = e^{i\varphi}\Psi(r_2, r_1)
\]  

(11)

So our indistinguishable particles are physically invariant under exchange as they may only differ by an inconsequential, overall phase factor. If we apply the exchange operator to the original wavefunction twice, i.e. \(\hat{P}^2\Psi(r_1, r_2)\), we should get the wavefunction back again. As such, we can say that \(\hat{P}^2 = 1\). Given that

\[
\Psi(r_1, r_2) = e^{i\varphi}\Psi(r_2, r_1) = \hat{P}\Psi(r_2, r_1)
\]  

(12)

Interestingly, there are even some theories according to which certain particles can be considered instances of one and the same thing such as John Wheeler’s famous albeit very much unlikely ‘One-Electron Universe’ [Feynman, 1965].
and that $\hat{P}^2 = 1$ we know that $e^{2i\varphi} = 1$ also. This means that $e^{i\varphi}$ itself is either equal to -1 or +1 (that is, $\varphi$ must be an integer multiple of $\pi$). Therefore, identical particles come in two kinds; those for which $e^{i\varphi} = +1$ and those for which $e^{i\varphi} = -1$. The former are called *bosons* and the latter *fermions*. We can express this with the exchange operator as follows.

\[
\text{Bosons: } \Psi(r_1, r_2) = \hat{P}\Psi(r_2, r_1) = +\Psi(r_2, r_1)
\]

\[
\text{Fermions: } \Psi(r_1, r_2) = \hat{P}\Psi(r_2, r_1) = -\Psi(r_2, r_1)
\]

Fermions and bosons are elementary particles that happen to have half-integer and integer spin respectively. They are the two types of exactly identical particles and so there are two quantum-statistical models that describe their aggregate behaviour. Systems of fermions are governed by *Fermi-Dirac* statistics and systems of bosons by *Bose-Einstein* statistics. Due to the particles' exact identicality these models are very different from the model of classical statistical mechanics (*Maxwell-Boltzmann* statistics) corresponding to the treatment outlined in §1.2.\footnote{At higher energies, the distributions given by quantum and Maxwell-Boltzmann statistics converge. This corresponds to my point in\textsuperscript{4} about being able to ignore quantal effects at sufficiently high temperatures.}

\footnotetext[4]{At higher energies, the distributions given by quantum and Maxwell-Boltzmann statistics converge. This corresponds to my point in\textsuperscript{4} about being able to ignore quantal effects at sufficiently high temperatures.}
2 Formal Treatment in the Quantal Context

2.1 Quaset Theory (QST)

With an appreciation for the appropriate treatment of quantal particles, we can begin to explore some philosophical formalisms attempting to make sense of them. One attempt at providing a formal framework with which we can model the metaphysics of groups of quantal objects was quaset theory (QST). As I alluded to in §1.3, if particles of some quantum system can all be absolutely indistinguishable how then can we make sense of there being a group of particles? For we should not be able to tell that there are multiple things in the first place, let alone count them, if they are all totally indistinguishable (what the physicist calls ‘exact identity’) because we cannot appeal to any distinct properties of the particles in order to enumerate them. Thus, standard set-theoretic tools and classical logics cannot be deployed to accurately understand the metaphysics of such systems. Modelling the particles as an ordinary set requires that we have unique, distinguishable elements otherwise we cannot verify that the set would satisfy, say, the Zermelo-Fraenkel (ZF) axioms [Dalla Chiara and Di Francia, 1993, p. 268]. Originally, Toraldo Di Francia outlined the direction we would need to take, positing that we either develop a specialized intensional semantics designed to capture such systems, or use the existing framework of fuzzy set theory to do so. Ultimately, this led to the development of QST by Di Francia and Maria Luisa Dalla Chiara [Dalla Chiara and Di Francia, 1993].

According to QST, a quantal particle’s name has a determined intension but is not a precise designator. Let us unpack this. If a name is a precise designator then there is a unique entity to which it determinately refers. Since particle names are imprecise designators, there is indeterminacy as to which objects are their referents. For instance, suppose I have a pair of identical twins
as neighbours but only ever see one at a time. Suppose also that I know the person I see as ‘Alice’ and am oblivious to the fact that there are actually two people living next door. It follows that whenever I refer to ‘Alice next door’, it is indeterminate as to which of the twins I am picking out. Here, ‘Alice’ is an imprecise designator because there is no entity to which it determinately refers.

On the other hand, a particle’s name’s intension, unlike its extension (referent), is entirely determinate. In this context, a name’s intension may be thought of as the meaning it carries rather than the object it designates. For example, the terms ‘the largest country’ and ‘Russia’ both have the same extensions but the meaning invoked by the former is distinct from that of the latter; the intensions are different. Now, under QST, a particle’s, say, an electron’s name’s intension is said to be precisely the bundle of properties bestowed upon it by physical law. These are properties like rest mass, charge, spin, and so forth, and they are given by physical law in the sense that physics dictates an electron’s combination of rest mass, charge, spin and the rest of its intrinsic and state-independent properties. Nomologically, all electrons are defined by and must have these essential properties. This is an intensional definition as it appeals to the necessary and sufficient conditions that must be satisfied by the object, as opposed to an extensional definition which would define the name by looking at everything in its extension. For instance, something is an electron iff it has mass \( m_e \approx 9.109 \times 10^{-31} \text{kg} \), charge \( q_e \approx -1.602 \times 10^{-19} \text{C} \), spin \( \pm \hbar/2 \) etc.; the particle is defined through its fundamental, nomological properties. So we can say that the intension of some particle’s name is the precise and determinate group of properties that nomologically define the object. Suppose we have a two-particle quantum system forming the entangled state \( |\psi\rangle \) where \( |+\rangle \)
and $|\uparrow\rangle$ represent spin-up and spin-down states respectively.\textsuperscript{11}

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_1 |\downarrow\rangle_2 + |\downarrow\rangle_1 |\uparrow\rangle_2)$$  \hspace{1cm} (13)

Here, subscripts ‘1’ and ‘2’ are completely fictional, physically useless labels (exactly how they are used in equation (1) of [Morris et al., 2019, p. 1]). Think of each as denoting a possible extension of an imprecise designator. To emphasise this, the following state would be the exactly same state as that in (13):

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_2 |\downarrow\rangle_1 + |\downarrow\rangle_2 |\uparrow\rangle_1)$$  \hspace{1cm} (14)

In $|\psi\rangle$, it is indeterminate as to which particle is electron 1 and which is electron 2 but there are definitely two electrons in total (as we shall soon see, $q\text{card}(|\psi\rangle) = 2$) and that whatever electron 1 and electron 2 could refer to, we know will have an electron’s spin, charge, and so on.

Now we get to the main thrust of QST. If we claim that a group of particles forms a quaset, the key feature is that the quaset has some cardinal but no ordinal. Cardinals, in this case finite cardinals, are a generalisation of the naturals $\mathbb{N}$ used to measure the ‘sizes’ of sets. If a set $S$ has $n$ elements its cardinality is $\text{card}(S) = n$. Formally, a set’s cardinality is defined though one-to-one correspondence functions or bijective functions. A bijection between two sets is a pairing of each element from one set with exactly one element in the other set so that nothing is left out. Two sets have the same cardinal iff there exists a bijection between them. For example, let

$S = \{1, 2, 3, 4\} \ni T = \{-1, -2, -3, -4\}$.

\textsuperscript{11}If unfamiliar with the notation here, see chapters one and four of McIntyre et al., 2012 for an explanation of Dirac notation and entangled particles respectively.
We can construct a bijective function

\[ f : S \rightarrow T \text{ where } f = \{(1,-1), (2,-2), (3,-3), (4,-4)\}. \]

Since \( f \) is bijective, we know that

\[ \text{card}(S) = \text{card}(T). \]

Ordinals are another generalisation of the natural numbers whose purpose is to represent ways in which elements of a set may be ordered. They are used to describe the position of elements in well-ordered sets, for instance ‘first’ or ‘second’. A well-ordered set is some set endowed with a transitive relation \( > \) such that for elements \( a \) and \( b \) either \( a > b \), or \( b > a \), or \( a = b \) (exactly one of the three is true), and for which there is a smallest element i.e. an element \( a \) for which \( \neg \exists x a > x \). For instance, \( \mathbb{N} \) under \( > \) is well-ordered since \( > \) is transitive, 0 is the smallest element (that is, \( \neg \exists x 0 > x \)), and for any two naturals, e.g. 0 and 1 only one of the following is true: \( 0 > 1, 1 > 0, 1 = 0 \). The ordinal, or order type of \( \mathbb{N} \) is denoted \( \omega \) which is the first transfinite ordinal, though for our purposes we will only need to consider finite ordinals. Now, we can determine a quaset’s cardinality, for instance we can have a total of \( n \) electrons, but there will be no well-ordering for them. We have no way of arranging the electrons one after another like we could for the elements of \( \mathbb{N} \) under \( > \). Recall our entangled state \(|\psi\rangle \). Modelling \(|\psi\rangle \) as a quaset accounts for our ability to say that there are a total of two electrons in \(|\psi\rangle \) corresponding to its cardinality — which in QST is called its quasi-cardinality. This modelling is also claimed to account for the fictional labels ‘electron 1’ and ‘electron 2’ being physically useless as there is no way to say which electron is the first or second due to their absolute indistinguishability, reflected by the absence of a well-ordering for the quaset.
QST can be formalised, as presented by Steven French and Décio Krause, in classical, first-order logic with identity as a logical constant [French et al., 2006, p. 234]. French and Krause provide this with three definitions and 12 axioms which are unnecessary to explicate entirely in this thesis as I intend to give an outline of only the salient points relevant to the systems of quantal particles at hand. As an extension of ordinary ZF set theory, QST incorporates ordinary ZF sets while imposing quasets ‘over the top’ in that we retain ZF set theory, but quasets generalise the apparatus to model quantal objects as well as classical ones. This is clear as per the first axiom and definition which state that anything with elements or anything that isn’t an urobject (an object that isn’t a set but can be an element thereof) counts as a quaset — so all ordinary sets are also quasets. Therefore in addition to modelling quantum systems, since ZF set theory is embedded in QST we can use the same modelling apparatus for the classical world. As alluded to above, a key concept here is the idea of a quaset’s quasi-cardinality (qcard) which is the size of the quaset. Just as the quaset is a generalisation of the ZF set, quasi-cardinality is the generalisation of cardinality thereby allowing us to talk about the size of quasets as well as ZF sets. Accordingly, quasi-cardinality is the total number of elements in a quaset, where for an ordinary set Z (which is, of course, also a quaset) qcard(Z) = card(Z). The addition of quasi-cardinality allows us to express the number of particles in some quantum system as the quasi-cardinal of the quaset representing said system.

Since QST models systems containing an exact number of absolutely indistinguishable particles as a quaset with a quasi-cardinal but no ordinal, we are supposedly absolved of the problem whereby we can determine the number of such particles despite their indistinguishability. How exactly this is the case is somewhat unclear though I think there are two ways in which it could work. Firstly, the particles’ names’ indeterminate reference accounts for our inability
to pick them out and count them one by one, and secondly, the quaset having a cardinal without an ordinal models the fact that we can say how many of these indistinguishable particles comprise a given system even though we cannot enumerate them. Whether it does properly solve the problem or not, I will soon identify what I think is a fundamental flaw of QST.

As a somewhat esoteric formalism, QST has a very limited philosophical literature. I thus propose two main interpretations thereof in the context of quantal metaphysics. The first interpretation would be that we ought to take QST as a rigorous model for the metaphysics of groups of quantal and classical objects. As a consequence of the detail in which it is presented, it seems to me that this is likely what Di Francia and Dalla Chiara intended. On accepting this rigid interpretation I claim that there is a major flaw with the theory’s idea of ‘cardinals without ordinals’ due to the concept of an ordinal being necessary for the standard definition of cardinality.

As discussed earlier in this section, cardinality is defined through the existence of a bijective function between sets, where such a function is a one-to-one pairing of every element from either set. The problem surfaces when we consider that to talk of cardinality requires that we can pick out individual elements of sets such that we may construct the bijection. Yet, for a quaset of quantal particles, we cannot pick out individual elements determinately because according to QST the particle names are imprecise designators, nor can we properly enumerate the particles to find the cardinality because this task requires the concept of ordinals. Ordinals are required if we wish to enumerate anything because the task of enumeration means to construct an ordered list, for instance we can enumerate \( \mathbb{N} \) and \( \mathbb{Z} \) by writing them out in the forms \( \{1, 2, 3, 4, 5, \ldots\} \) and \( \{0, -1, 1, -2, 2, \ldots\} \) respectively. Therefore we have a systematic way of listing all the elements of \( \mathbb{N} \) and \( \mathbb{Z} \) such that we can pick out the \( n^{th} \) element of one set and map it to, say, the \( n^{th} \) element of the other set thereby forming a bijection.
which proves $\text{card}(\mathbb{N}) = \text{card}(\mathbb{Z}) = \aleph_0$. To construct such a bijection, each element must be precisely designated and determinately distinct from or identical to the others thereby allowing us to know whether or not some element has already been mapped. Furthermore, cardinals are standardly thought of as certain ordinals such as in von Neumann cardinal assignment [Moschovakis, 1994, §12.25] and thus we require ordinals if we are to define cardinals in the standard sense. It follows that we should not be able to talk about the quasi-cardinality of quasets because in doing so we are implicitly relying on ordinals and the ability to determinately tell whether or not we have already mapped some element in the construction of a bijection. Keep in mind that quasi-cardinality is not said to be defined differently from ordinary cardinality for it is just the extension of the idea from ZF set theory to QST. As such, QST has a serious issue it must deal with if interpreted in this manner. Although, I see a possible solution to this problem if the quaset theorist defines quasi-cardinality in a non-standard way which does not rely on ordinals. Alas this is a rather drastic measure because QST is an extension of the ZF set theory which is embedded therein. That is, everything from ZF set theory is still present in QST, so to use non-ZF-theoretic definitions of cardinality such as those in other foundations of mathematics (type theory, category theory, and the like) may elicit further complications or outright incompatibility.[12]

The alternative interpretation I suggest is motivated by the obstacles with which the first contends. Rather than taking it as a strict and rigorous model, QST may be interpreted as a formal framework designed to provide an intuitive conception of groups of indistinguishable objects — a sort of mathematical metaphor suggesting how we ought to think of such situations. Along these lines, groups of identical objects, which are paradoxical from a classical perspective (using ZF set theories and classical logic), can be conceived with

[12]Such complications or incompatibility problems are only speculated as the details of competing foundations of mathematics are beyond the scope of this thesis.
Formal Treatment in the Quantal Context

QST as quasets — having quasi-cardinals but no ordinals. However, what distinguishes this interpretation is that we should not take QST ‘too seriously’ as a logico-mathematical foundation of quantal metaphysics, rather we should appreciate its utility as a guide to making intuitive sense of groups of identical objects. This interpretation absolves QST from the issues which arise when subject to the first interpretation because it is not necessary to rigorously define quasi-cardinality in such a way as to avoid its reliance on ordinals. QST thus amounts to a metaphorical, intuition-building tool designed to make sense of groups of absolutely indistinguishable objects. While it does not provide a rigorous, formal foundation for such systems, the fact that particle names are not precise designators and that quasets have some sense of orderless cardinality aids their conceivability (such systems may not be totally incoherent on grounds that identity between terms means their precise co-reference, or that we should not be able to tell there is a group of objects provided they are absolutely indistinguishable).

Consequently, without an alternative definition of cardinality I tend towards accepting the second interpretation since it allows us to dismiss the aforementioned problem of cardinality; investigating it would be akin to taking the metaphor too ‘literally’ thereby missing the point. Although, if this is the correct interpretation, it warrants clearer articulation by Di Francia, Dalla Chiara, French, and Krause because, as it stands, the manner in which we ought to interpret QST is indefinite. Going forward, QST beset with these difficulties is overtaken by a promising alternative/modification which has garnered far more attention. As such, I put QST to the side in favour of the systems to be discussed in §2.2.
2.2 Quasi-Set Theory ($\mathcal{Q}$) and Non-Reflexive Schrödinger Logics

So far, QST seems to be a somewhat useful alternative to standard set theory if we want to make sense of objects in the microphysical regime. It attempts to address the problems outlined in §1.3 which surface as a consequence of ascribing classical conceptions of identity and indistinguishability to quantal objects, for it allows us to model such objects through a more suitable mathematical framework in which such problems do not seem to arise. However, QST itself mentions nothing of the indistinguishability and identity relations borne between the objects in the quasets themselves. Rather, it purports to solve the problem only in that it admits the conception of a precise number of quantal particles despite their absolute indistinguishability.

Motivated by the inadequacy of existing systems in making sense of the microphysical regime, mathematician Newton Da Costa developed a new first-order logic $S$ [Da Costa and Krause, 1994] in which both classical and quantal objects along with their identity and indistinguishably relations, or lack thereof, could be modelled. Da Costa does not however, try to construct a new ‘quantum identity’ relation, rather the logic contains two distinct sorts or ‘species’ of objects, one for which identity is not applicable i.e. if $x$ or $y$ are of a particular sort, then $x = y$ is not a wff. Here, the species for which identity claims are forbidden represents quantal objects, and the other species corresponds to ordinary, classical objects. As such, we get to preserve our well-founded, classical notions of identity, whilst also having a logic in which we can speak of quantal objects. The idea of prohibiting identity statements about such objects was originally articulated by Erwin Schrödinger in the earlier days of quantum physics, claiming that quantal particles have no individuality and that the concept of identity makes no sense in this context. Or as he put it:
I beg to emphasize this and I beg you to believe it: It is not a question of being able to ascertain the identity [of particles] in some instances and not being able to do so in others. It is beyond doubt that the question of ‘sameness’, of identity, really and truly has no meaning. [Schrödinger, 1951, p. 16]

Consequently, the first-order system $S$ is known as a *Schrödinger logic*. Let us now lay out the syntax for $S$:

(a) **Connectives:**
   - Unary: $\neg$;
   - Binary: $\lor, \land, \rightarrow, \leftrightarrow$;

(b) **Quantifiers**: $\forall, \exists$;

(c) **Punctuation**: $(, )$ and $;$

(d) **Terms of the First Species** (*m-terms)*:
   - The set of variables $\{x_1, x_2, \ldots\}$ with cardinality $\aleph_0$;
   - The set of constants $\{a_1, a_2, \ldots\}$ with cardinality $\aleph_0$;

(e) **Terms of the Second Species** (*M-terms)*:
   - The set of variables $\{X_1, X_2, \ldots\}$ with cardinality $\aleph_0$;
   - The set of constants $\{A_1, A_2, \ldots\}$ with cardinality $\aleph_0$;

(f) **Symbol of Equality**: $=$;

(g) **Predicates**: For each $n \in \mathbb{Z}^+$ at least one predicate $P^n$ of arity $n$.

Note that here a ‘term’ means any constant or variable of the first or second species. Da Costa does not explicate what constitutes a formula of $S$, instead proclaiming that the wffs of $S$ are defined according to the standard syntax for many-sorted logics (in general) given by Hao Wang [Wang, 1952]. To complete the syntax, I explain what I think Wang’s definition of many-sorted logic wffs means for the two-sorted system $S$. I use a recursive definition for wffs in $S$ as follows:
1. An atomic formula is an expression of the form $P^n t_1 t_2 \ldots t_n$ where $P^n$ is an $n$-ary predicate and $t_1, t_2, \ldots, t_n$ are terms.

2. Every atomic formula is a wff.

3. If $\varphi$ and $\psi$ are wffs then so are $\neg \varphi$, $(\varphi \lor \psi)$, $(\varphi \land \psi)$, $(\varphi \rightarrow \psi)$, and $(\varphi \leftrightarrow \psi)$.

4. If $\varphi$ is a wff and $\alpha$ a variable of any sort then $\exists \alpha \varphi$ and $\forall \alpha \varphi$ are wffs.

5. If $\alpha_M$ and $\beta_M$ are $M$-terms then $\alpha_M = \beta_M$ is a wff.

6. Nothing else is a wff.

This means that, as required, $x = y$ is not a wff if either $x$ or $y$ are $m$-terms. As such, under $S$ we are forbidden from making identity claims involving $m$-terms which are taken to represent quantal objects. Note that this does not mean we cannot speak of the identity of $m$-terms themselves, for instance we can say that ‘$a_1$’ and ‘$a_2$’ are different terms, but we cannot use such terms in identity claims.

A semantics for system $S$ can be generated, although Da Costa and Krause claim that doing so using standard ZF set theories is problematic since identity claims cannot be made with $m$-terms \textsuperscript{[Da Costa and Krause, 1994, p. 539]. For instance, we could say that the domain for $S$, call it $D$, is the union of the set of objects represented by $M$-constants, $D_M$, and the set of objects represented by $m$-constants, $D_m$ with $D_M \cap D_m = \emptyset$, as there are no terms that are of both sorts. So we have $D = D_M \cup D_m$, and then for every $n$-place predicate we can have the subset $D^n$ of $D$ containing $n$-tuples of the objects under each predicate. Now, we can understand equality as the diagonal of $D_M$, denoted $\Delta_{D_M}$ which is the set containing all the pairs of elements of $D_M$ that
are paired with themselves. That is $\Delta_{D_M} = \{(x, x) : x \in D_M\}$. This leaves us with a notion of equality exclusively for all the $M$-terms. However, the problem here is that we cannot consider $D_m$ to be an ordinary set of ZF set theory or similar standard set theories (according to Da Costa and Krause) because we are syntactically prohibited from using the identity predicate with its elements. Thus Da Costa suggested that an alternative to standard set theories should be developed in which a semantics for system $S$ could be constructed — he named this quasi-set theory [Da Costa, 1980] although the theory was only properly developed decades thereafter.

At this point one may fail to see how an inability to make identity claims about the elements of $D_m$ removes its status as a ZF set especially since Da Costa and Krause do not explain why this is the case. We could ostensibly have a set and just lack a means of making identity claims about its members. Upon closer investigation, I realise that an inability to make identity statements about a set’s elements conflicts with at least one of the ZF axioms. Looking at only the first ZF axiom (extensionality) which is common amongst all standard set theories, it is easy to see that it would not apply to $D_m$. The axiom of extensionality sates that if two sets have the same elements, then they are the same. If the sets we are concerned with contain anything represented by an $m$-term, we cannot apply this axiom because there is no notion of sameness between elements if they are represented by $m$-terms. As such, we cannot strictly call $D_m$ a set of ZF set theory since it does not obey at least one axiom — as Da Costa suggested, an alternative is in fact required.

In 1992 Krause developed quasi-set theory (following Krause, we shall henceforth call this $\mathcal{Q}$) for the first time [Krause et al., 1992] and subsequently a semantics for $S$ was developed by both Krause and Da Costa [Da Costa et al., 1997]. $\mathcal{Q}$ is very similar to although much more detailed than QST (with 24 axioms and 10 definitions) and unlike QST, it is very clearly designed as an
alternative mathematical foundation for quantum physics as this interpretation is made explicit by Krause. To that end, $\mathcal{Q}$ has indeed been applied in the derivation of certain principles of quantum statistics from the ground up [Krause et al., 1999].

Similarly to QST, $\mathcal{Q}$ is an extension of ordinary ZFU (ZF set theory with urelements i.e. elements that cannot themselves be sets) and has a similar concept of quasi-cardinality, although unlike QST it does not suffer from the requirement of ordinals for the definition of quasi-cardinality, which French and Krause indeed explicate. Rather, in $\mathcal{Q}$ quasi-cardinally is taken to be a primitive concept [Da Costa and Krause, 2007, p. 1] and as such need not be defined with reliance on ordinals/enumeration and bijective functions. The reason $\mathcal{Q}$ is an extension of ZFU specifically is of course to accommodate the two types of urobjects in the ontology, namely $m/M$-objects which cannot themselves be sets. Further, $\mathcal{Q}$ treats the separate class of $m$-objects as indistinguishable non-individuals (objects which lack self-identity e.g. $a$ is a non-individual iff $a = a$ is false or not a wff), as is required for the theory if it is to faithfully capture the nature of quantal objects — at least in Schrödinger’s view. While they cannot bear identity relations, the indistinguishability of $m$-objects is represented by a separate, primitive indistinguishability relation ‘$\equiv$’. Importantly, $a \equiv b$ does not necessarily entail $a = b$ since the latter would fail to count as a wff had $a$ or $b$ been $m$-terms. For the $m$-objects $a_m$ and $b_m$, $a_m \equiv b_m$ can hold true, thereby formally accounting for their indistinguishability whilst keeping the identity relation in the classical domain of $M$-objects. With a syntax and semantics for $S$, a higher-order extension thereof, called $S_\omega$ was also developed, allowing the IoI to be articulated. $S_\omega$ thus provided a well-founded formal language with which concepts lacking an adequate formal treatment in quantal contexts could be made sense of rigorously [French et al., 2006]. Note that since $m$-objects may be indistinguishable but never identical, we are mandated to reject the IoI.
for \( m \)-objects in \( \mathfrak{Q} \) and \( S/\omega \). This being said, \( \mathfrak{Q} \) and \( S/\omega \) are not specifically designed to account for identity in the quantal context (really they don’t at all) nor do they use identity in the physicist’s sense outlined in §1.3, rather they provide a language in which we can talk about quantal objects as ones that lack individuality and for which identity makes no sense — as Schrödinger thought.

In the final chapter I briefly present what I claim to be a problem with Schrödinger logics and \( \mathfrak{Q} \). This arises as an artifact of the continuity between classical and quantal modelling of the world in contrast to the discrete, two-sorted systems presented above. Further, I do not wish to fully assess these systems here although I’d like to point out that taking quasi-cardinality as a primitive concept seems like an easy evasion of the requirement for definition (which I claimed would be a difficult task) — perhaps a rather shallow victory. Ideally, there ought to be a rigorous definition of quasi-cardinality as there is for cardinality in set theories of which \( \mathfrak{Q} \) is an extension. Notwithstanding, this an is entirely valid albeit suboptimal maneuver.
3 An Outline of Vagueness

3.1 Vagueness and the Sorites

The unusual behaviour of the identity relation in quantal contexts has motivated the idea that the identity predicate or the objects themselves may exhibit vagueness. In order to investigate these claims and ultimately present my own, an initial outline of vagueness is required. Predicates in classical among other logics enforce clear delineations between what does and what does not fall into their extensions. That is, every object either possesses a certain property or it doesn’t. However, significant challenges arise if predicates that do not have such well-defined extensions are introduced. In natural and even formal/scientific language we often find ourselves using predicates which permit borderline cases. These are vague. For instance, the predicate ‘tall’ is vague because it is indeterminate at which height something ceases/begins to be tall. We could say that the Taipei 101 is tall, though the Eiffel Tower and/or a basketball player may also be deemed to be tall in certain contexts. One may argue that such predicates are vague because they are context-sensitive, that a specification of the context under which the predicate applies shall remove its vagueness; we would say the Eiffel Tower is tall for a 19th century tower but not a modern tower. Alas, context specification for a vague predicate fails to eliminate its vagueness, the remains vague even within the specified context, regardless of its specificity. For instance, stating that ‘the Taipei 101 is tall for a modern tower’ does not settle the vague nature of tallness for even in the domain of modern towers we have not defined which are tall and which are not. This vague predicate is markedly distinct from one such as ‘≥400m tall’ which gives us a clear demarcation of its extension. We must also mark the distinction between vagueness and ambiguity as they are, at times, conflated. Ambiguity is a property of a term indicating that it has more than one meaning, whereas vague terms have
fuzzily defined meanings, but not necessarily more than one.¹³

A well-known paradox that surfaces when working with vagueness in classical among other logics is the sorites. A canonical way of generating the sorites in classical logic relies on modus ponens and the principle of tolerance, an intuitive logical mechanism that applies to vague predicates. Tolerance shows how such predicates are permissive to borderline cases by simply stating that if an object is within the extension of a vague predicate, then an object which is slightly different will also fall into that extension. For instance, if one with \( n \) hairs is bald, one with \( n + 1 \) hairs is also bald, as is one with \( n - 1 \) hairs. I formalise tolerance for an arbitrary vague predicate \( P \) and arbitrary objects \( a_i \) as such (excuse the abuse of notation):¹⁴

\[
\forall a_i (Pa_i \rightarrow (Pa_{i+1} \land Pa_{i-1})) \text{ where } i \in \mathbb{Z}
\]  

Here, the arbitrary objects \( a_{i+1} \) and \( a_{i-1} \) are different from \( a_i \) by a minimal, discrete amount. Accepting this, the sorites paradox may be generated by iterative applications of modus ponens, universal instantiation, and conjunction elimination. For example, consider the set \( \{x \in \mathbb{Z} | 0 \leq x \leq 1000\} \), and suppose that the integer 1 is small and that 1000 is not. Starting at 1, we apply tolerance and conclude, by modus ponens, universal instantiation, and conjunction elimination that 2 is also small. We then repeat this reasoning for 2 and all subsequent resulting integers until we conclude that 1000 is small. At this point we have reached a contradiction, that 1000 is small and not small — an example of the sorites paradox. I have formalised the sorites series over

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¹³ At a surface level, vagueness and ambiguity are different although some theories of vagueness, e.g. certain supervaluationist theories, end up regrading vagueness as a multiplicity of meaning and in that sense akin to ambiguity.

¹⁴ Normally, tolerance as a premise of the sorites goes in one direction (the other being trivial); if one with 10 hairs is bald, then obviously one with 9 hairs is bald. However, applying tolerance to say that one with > 10 hairs is bald will actually generate the paradox. In my formalisation, I aim to capture tolerance at a higher level of generality.

¹⁵ There are also notions of the ‘continuous sorites’ and even the ‘topological sorites’ [Weber and Colyvan, 2010] but in this canonical formulation tolerance works in discrete steps.
\{i, j \in \mathbb{Z} | i \ll j \vee i \gg j\} as follows:

(1) \(P_a_i\) Assumption
(2) \(\neg P_a_j\) Assumption
(3) \(\forall a_i (P_a_i \rightarrow (P_a_{i+1} \land P_a_{i-1}))\) Tolerance Principle
(4) \(P_a_{i\pm1}\) By (1) and (3)
(5) \(P_a_{i\pm2}\) By (4) and (3)

\ldots

(\lfloor i - j \rfloor + 1) \(P_a_{j\mp2}\) By (\lfloor i - j \rfloor) and (3)
(\lfloor i - j \rfloor + 2) \(P_a_{j\mp1}\) By (\lfloor i - j \rfloor + 1) and (3)
(\lfloor i - j \rfloor + 3) \(P_a_j\) By (\lfloor i - j \rfloor + 2) and (3)
(\lfloor i - j \rfloor + 4) \(P_a_j \land \neg P_a_j\) By (\lfloor i - j \rfloor + 3) and (2)

As we can see, by iterating the three inference rules we have arrived at a contradiction, that \(\neg P_a_j \land P_a_j\). Note that if \(P\) was a precise predicate, premise (3) would be clearly false which would remove the paradoxical character of the argument. For example, tolerance would not work for \(\geq 400\text{m}\) for if something goes from 400m to 399m, then it immediately escapes the predicate’s extension. When our predicate is tolerant, the sorites makes sense ostensibly because the premises seem reasonable and all the inference steps are valid — hence the paradox. But, if a predicate were not tolerant then the sorites would be clearly incorrect since premise (3) would be false. Therefore we can say that a susceptibility to the sorites paradox is characteristic of vagueness and so constructing the paradox serves as a way to demonstrate that a predicate is vague, which I attempt with identity in §4.3. This being said, there are various other methods that can be used to construct different kinds of soritic paradoxes such as the continuous or topological sorites. Likewise, there are strong cases for tolerance as a condition for vagueness to be replaced in favour of alternative
principles such as closeness [Smith, 2005]. The above explanation showcases a basic, canonical approach to the subject which is sufficient for our purposes.

3.2 Sources of Vagueness

Now that we have outlined vagueness itself, we can address the question of its source. The dominant view among philosophers today (myself included) is that vagueness is a fundamentally linguistic phenomenon, arising as an artifact of semantic imprecision. For instance, the vague nature of a concept such as ‘redness’ is a consequence of the linguistic apparatus not being used to articulate exactly which wavelengths of light constitute red (probably because such an ability was unnecessary for humans as natural language evolved). The usual view of semantic vagueness is that we are talking about a multitude of perfectly precise things in the world, such as light of specific wavelengths or towers of specific heights, but we have not fixed on exactly which ones we are talking about. For example we have not decided on which out of all the precise towers are the tall ones hence the vagueness of the predicate ‘tall’. I partly agree here though I take issue with the notion of describing an object as ‘precise’. On my view this looks like a category-error because precision pertains to descriptions of things or actions and the like but not to objects themselves. I can describe a tree precisely or measure a the length of a desk precisely but I cannot make sense of a ‘precise tree’ or a ‘precise desk’. Even though I can make sense of something like a ‘precise thermometer’ I do not actually mean that the thermometer itself is precise, I really mean that it measures temperature precisely; the precision concerns how the thermometer functions. Thus, I claim that vagueness is still linguistic but that this need not commit us to belief in precise objects and that it is erroneous to describe objects themselves as vague or precise — similarly to how it is erroneous to describe them as, say, necessary (□a is not a wff of...
ordinary modal logics). In any case, the view in common is that vagueness has its roots in language alone rather than there being any other underlying sources such as vagueness in the real world (which I shortly discuss).

Often vagueness induced by semantic imprecision is framed as an undesirable linguistic bug — which, unsurprisingly, was Frege’s view (who implied it was a byproduct of ‘laziness’ [Sorensen, 2018 §8]) — although it can also be thought of as a useful semantic feature, for instance in that it allows language users to interpret meanings broadly and gradually precisify as new information is acquired. It seems to me that this normative status of linguistic vagueness depends on the goals of the language users. For example, it would be undesirable in the case of language users unknowingly employing vague terms who intend to be precise, whereas it could be useful in artistic endeavours where it is desirable for a work to have a plurality of interpretations or in situations where excessive precision is undesirable for epistemic reasons. For the artist, this could be that musicians interpret intentionally vague concepts given by composers such as ‘forte’ or ‘fortissimo’ so as to allow the performer room for creative expression/interpretation.

An alternate, less well-received view about where vagueness comes from is that it exists in the world itself — it is an ontological phenomenon. The assertion that certain objects are vague seems puzzling to friends of the semantic view since we would think of vagueness as a property of predicates rather than of the referents of names. However this ontic conception would have it that the reason we think predicates are vague is actually because the objects they represent are vague. Things such as clouds or mountains are often mentioned as examples of vague objects in this sense as they have no precisely demarcated boundaries. In the case of the cloud, there are certain water molecules that are definitely part of it as well as ones that definitely are not, however there are also molecules that are not determinately part of or excluded from it. As such,
those who support this view claim that the cloud is vague. I disagree because I fail to see how the cloud as an object is anything other than an artificially-imposed demarcation of spacetime made by language users, i.e. the world does not have any sense of what is in the cloud and what isn’t because the cloud is an entirely human-made concept. Here I invoke mereological universalism; that all compositions of material objects constitute other objects (we can delineate the cloud however we like). Universalism contrasts with mereological nihilism according to which only simples exist i.e. objects with no proper parts (parts of objects that are not the objects themselves) [Van Inwagen, 1995]. Philosophers often question universalism because it entails that there are single, whole objects such as the one made up of myself, Russia, and Kripke which seems like an absurd collection of parts to call a whole [Rea, 1998]. As a universalist, I would say that while it is perfectly valid to speak of such objects, we are at liberty to choose groupings of parts based on their usefulness — myself, Russia, and Kripke is not a useful object to talk about. It is not forbidden to delineate such an object, merely pointless. Analogously, the string ‘Tasty flavourless numbers are cats.’ is a perfectly valid, grammatical sentence albeit an essentially useless and nonsensical way to put words together (or Chomsky’s ‘Colourless green ideas sleep furiously.’ [Chomsky and Lightfoot, 2002 p. 15]) — just as myself, Russia, and Kripke is a valid, yet purposeless object to acknowledge.

The idea of mereological universalism makes it clearer that when we talk about an object, we are just referring to a region of the universe that happens to be useful to talk about. Along these lines, when a certain molecule’s membership in the cloud is questioned what is really being asked is ‘does this water molecule belong to the bit of space I am calling ‘the cloud’?’ That is, in trying to determine what is part of the cloud we are doing nothing more than deciding which bits of the universe we think are most useful to consider as a whole object. If one agrees with this, the term’s vagueness becomes apparent as
a semantic phenomenon since what we are really talking about is what we mean by ‘the cloud’. In response, those in favour of the ontic view may claim that even if ‘the cloud’ is just whatever we mean by ‘the cloud’, we could still just be referring to a single, ontically vague object such that it is indeterminate whether a certain molecule is in it. However, this assumes that there is in fact some fundamental, ontological notion of ‘the cloud’ as something existing outside of language whereas my view relies on the assumption that only we decide that a certain group of molecules is a cloud. As such, my argument here only works if one believes that only language users assign names to bits of the universe and that these bits of the universe otherwise have no sense of being particular objects such as clouds.

The motivation for my discussion of arguments against ontic vagueness is that I shall propose my own theory of quantal, non-ontic, soritic vague identity in §4.3 (as opposed to all other theories of quantal vague identity to be discussed which require commitment to ontic vagueness). This is because given the attitudes towards ontic vagueness shared by myself among most philosophers, the fact that my theory does not require a commitment to ontic vagueness is an advantage as the notion of vague identity is easier to accept when the vagueness involved is only semantic.
4 Vague Identity

4.1 Ontic Vagueness of Identity in the Quantal Context

While the notion of ontic vagueness is already considered to be problematic, recent decades have seen arguments for ontic vagueness existing in the identity relations between quantal objects. The most notable proponent of this idea is Edward Lowe, suggesting that the indeterminate identity of entangled particles is an instance of ontic vagueness of identity [Lowe, 1994]. When we claim that entangled particles are indeterminately identical, the indeterminacy we speak of is stronger than its analogue in ordinary classical contexts in that there is no underlying fact of the matter to which we merely lack epistemic access. Rather, the indeterminacy here is ontic (note that here I am talking about indeterminacy and not necessarily vagueness). To illustrate these differing notions of indeterminacy in classical and quantal contexts, let us imagine a coin flip at the beginning of a football game and compare it to the measurement of an electron’s spin. The outcome of the coin flip is ‘indeterminate’ to the referee until she observes the flipped coin, yet we know that the outcome is fixed even before this, say, when the coin hits the ground. The indeterminacy we are referring to here is entirely epistemic because the coin is already either heads or tails, it has chosen its state, the referee just doesn’t know which. On the other hand, the outcome of the electron’s spin measurement is indeterminate in a stronger sense — not only do we not know the outcome before measurement, but unlike the coin, there is no prior fact of the matter as to in which spin state the electron will be. The idea that such quantum events lack any unknown underlying factors is most notably supported by Bell’s theorem [Bell, 1964]. While left unmentioned by Lowe, I emphasise that Bell’s result is crucial for a proper understanding of Lowe’s argument, underpinning his claims regarding ontic (rather than epistemic) indeterminacy.
Stewart Bell provided an argument suggesting that so-called hidden-variable theories attempting to explain the Einstein-Podolsky-Rosen (EPR) paradox were incorrect. An instance of the EPR paradox would be that when one member of an entangled pair’s spin is measured, we can immediately know what the spin of the other will be, no matter how far away it is. Evidently this is paradoxical because something seems to have instantaneously travelled from one particle to the other upon execution of the measurement, and as such, it has motivated the idea that there may be some epistemically inaccessible hidden variable which predetermines the spins. According to such theories the spins would have determined values all along that we just don’t know about, thus solving the EPR paradox — akin to the coin flip outcome being predetermined but unknown to the referee. However, Bell proposed an experiment whose outcome struck hidden-variable theories severely. He showed that independently measured particles of an entangled pair would yield outcomes constrained by certain inequalities (known as Bell’s inequalities) were they to hold hidden variables. As it turned out, these inequalities were violated by the predictions of quantum mechanics, thus providing evidence against hidden-variable theories. To philosophically interpret Bell’s result, quantum indeterminacy such as that seen before we measure the first particle’s spin in the EPR paradox is in fact not epistemic but entirely ontic; there is no hidden variable so the measured spin state does not supervene on any such factors. At this point I wish to emphasise that Bell’s result is not a knockdown of all hidden-variable theories (as it is often mistaken to be), rather it provides very significant evidence against such theories. Therefore we should be aware that it is not out of the question to posit alternatives to Bell’s and therefore Lowe’s conclusions, but as I mentioned in §1.3, such non-standard interpretations of quantum mechanics fall beyond our scope.

In his argument for the ontic vagueness of identity, Lowe imagines some
atom capturing an electron \( a \) and then releasing an electron \( b \) at a later time. Since, upon capture the electron becomes entangled with other electrons in the atom, their identity is indeterminate which Lowe takes to be synonymous with vague. To generalise Lowe’s argument, if we have particles of the same kind \( a \) and \( b \), proceed to entangle them, and then pick one out \( c \), then both \( a = c \) and \( b = c \) are indeterminate not due to our ignorance of some underlying fact, but because the indeterminacy is a purely ontological, non-supervenient feature. Since we have an ontological indeterminacy of identity claims, Lowe concludes that we have an ontic vagueness of identity between the particles. How exactly this indeterminacy leads to vagueness is not explained so I assume Lowe is equating the two — I address this later.

As well as dealing with objections to Lowe’s claims\(^\footnote{For some objections to Lowe see \cite{Noonan,1995}.}\), French and Krause attempt to extend his contention that identical particles exhibit vagueness by employing elements of \( Q \) and Schrödinger logics \cite{French and Krause,2003}. On Lowe’s view, the quantal particles are taken to be individuals in the sense that we can ascribe identity relations between them, such as the identity relations they each bear to themselves. On the other hand, Schrödinger logics and \( Q \) do not allow identity claims to be made involving \( m \)-terms (representing quantal particles). As such, \( m \)-objects are said to lack individuality for they cannot have self-identity — \( a_m = a_m \) is not a wff therefore we cannot say that \( a_m \) is identical to itself. French and Krause thus posit that if we to deploy Schrödinger logics and \( Q \), our particles shall exhibit vagueness on an even deeper level than Lowe’s ontic indeterminacy suggests. In their own words:

... we can have a determinate number of quantum objects in a given state without these objects possessing definite identity conditions. And it is because of this lack of self-identity that the objects can be described as vague, in perhaps the most fundamental sense one can
We ought to note that the vagueness French and Krause posit must pertain to the particles themselves whereas Lowe is talking about the ontic vagueness of identity relations between them. This is of course because French and Krause cannot assert that identity relations are vague since identity claims are not even wffs in this context. If we accept $\mathcal{Q}$ with Schrödinger logics, vague identity of quantal objects must be rejected on the grounds that identity is relegated to the domain of classical objects — behaving as it does in classical logic (a primitive, precise concept).

While Lowe’s quantal ontic vagueness may exist provided indeterminacy implies vagueness (and ignoring my claims against ontic vagueness), I object to French and Krause’s notion that lacking self-identity results in vagueness on a more fundamental level — and to that end I present the following counterexample. Suppose we have an ordinary language with predicates but no names for these predicates i.e. we cannot mention the predicates with quotation marks. Using one of these predicates $P$, we can form the expression $P = P$, which is not a wff. Note that we lack a way of talking about the predicate itself for we have no predicate names, thus we cannot claim that ‘$P$’=‘$P$’. Even though $P$ lacks self-identity, it is not necessarily vague. For a specific example here, suppose $P$ is the precise predicate: > 400m tall. According to the syntax of most logics including $S/S_\omega$, $P = P$ does not constitute a wff and therefore French and Krause would say that $P$ lacks self-identity. Despite this, $P$ is definitely not vague and hence a lack of self-identity alone does not provide adequate grounds for the assertion of vagueness. Moreover, since identity relations pertain strictly to $M$-terms alone, if we accept that a lack of self-identity implies vagueness it follows that literally anything in our language that isn’t an $M$-term is vague — all quantifiers, punctuation, connectives, the symbol of equality, and so forth —
which is a philosophically unpalatable contention. Consequently, I reject French and Krause’s argument for the vagueness of m-objects let alone their vagueness in any more ‘fundamental’ sense.

All this considered, I identify three of the most tenable positions. Firstly we could accept Lowe’s ontic vagueness of identity if we are prepared to permit identity relations between quantal objects, equate indeterminacy with vagueness, and accept ontic vagueness — a view I reject on the grounds that it equates vagueness with indeterminacy (which I shortly address) and requires an ontic conception thereof. Alternatively, and as I have just argued, we could accept Q and Schrödinger logics in which case we must reject both the quantal vagueness of identity and the vagueness of quantal objects themselves. And finally, since ontic vagueness is such a minority view amongst philosophers, Lowe’s assumption that indeterminacy implies vagueness could be rejected in favour of a criterion for vagueness such as sorites susceptibility or tolerance to borderline cases (I argue for this in §4.3). On the first and last views, Lowe’s conception of vagueness as indeterminacy seems to have been overshadowed by more recent literature on the subject. The concept of vagueness was crystallised a few years after the publication of Lowe’s paper, for instance by Peter Smith and Rosanna Keefe who identify that vague predicates lack well-defined extensions, are susceptible to the sorties, and have borderline cases [Keefe and Smith, 1996, pp. 2-3]. So, the more rigid ideas of sorites susceptibility and tolerance to borderline cases with which we are familiar today should not be used to claim that Lowe was ‘wrong’ about vagueness as mere indeterminacy. Rather, an appreciation for the different understandings of vagueness in both periods is required. As such, this is not exactly a philosophically interesting issue per se. Although, the same does not apply to French and Krause (making their argument in 2003) who either have an unsound argument or misunderstand vagueness by interpreting it in far too broad a sense according to which precise terms (among many other
symbols) would be considered vague.

Before continuing, I wish to address some potential confusion regarding \( \Omega \) and Schrödinger logics. One may be tempted to argue that this language is inadequate if we want to formally describe quantal particles because it prohibits identity claims about such particles while they are indeed identical (as the physicists say). The error here is that the identity banned by \( \Omega \) and Schrödinger logics is not the same as the physicist’s identity in the sense of ‘identical particles’. Recall that ‘identical particles’ in physics could be taken to mean the particles are absolutely indistinguishable — identity and indistinguishability being equivalent here. However, the formal system preserves primitive, separate notions of indistinguishability and identity, where \( m \)-objects are indistinguishable but identity is kept only in the classical world and does not equate with indistinguishability.

4.2 Evans’s Argument

A significant objection to the ontic vagueness of identity, and one with which Lowe contends, is Evans’s Argument. In 1978 Gareth Evans published a remarkably short paper providing a formal argument against the ontic indeterminacy of identity statements [Evans, 1978] (as with Lowe and for the same reason, Evans uses vagueness and indeterminacy interchangeably). I interpret the argument informally, with arbitrary objects \( a \) and \( b \), as follows:

1. Assume for reductio that it is indeterminate whether \( a \) is identical with \( b \).
2. It follows from (1) that \( b \) has the property of indeterminate identity with \( a \).
3. By the reflexivity of identity, it is not indeterminate whether \( a \) is identical with \( a \).
(4) It follows from (3) that \( a \) does not have the property of indeterminate identity with \( a \).

(5) Since \( b \) has a property \( a \) lacks, namely the indeterminate identity with \( a \), this means \( a \) is not identical with \( b \) through failure to satisfy LL. Consequently, it is not indeterminate whether \( a \) is identical with \( b \). We have thus reduced (1) to contradiction.

The ensuing contradiction renders (1) false, thereby refuting the objects' indeterminate identity.

To express this argument rigorously and understand Evans's original formalisation, some specialized additions to our logical apparatus are required. Firstly, indeterminacy is not a predicate for it pertains to the truth value of whole propositions, not just to objects. In this sense it is somewhat similar to modality. As such, a unary, sentential operator ‘\( \nabla \)’ is introduced, acting on propositions such that we may interpret the expression ‘\( \nabla \varphi \)’ as the new proposition ‘it is indeterminate whether \( \varphi \)’. Moreover, we require a method of formal expression for properties such as the ones ascribed to \( b \) and denied to \( a \) in premises (2) and (3) respectively. These are properties of indeterminacy of their argument’s identity with some fixed object. So to be sufficiently explicit, property abstraction from Church’s lambda calculus, a formal system conventionally used to express computations, is employed to express the properties at hand. Under this system we denote propositions in which a property is ascribed to an object, such as ‘\( Pa \)’, as ‘\( \lambda x[Px]a \)’ or (as Evans does) ‘\( \hat{x}[Px]a \)’. We can now express, for instance, premise (2) as ‘\( \hat{x}[\nabla(x = a)]b \)’, thereby accurately reflecting the property’s content in the syntax. Now we can present and understand Evans’s argument in its full formality. I have reconstructed the argument below and added premise (4.1) to make the violation of LL and jump to (5) explicit:
Note that (5) follows from (4.1) because if a proposition is determinate, like (4.1), then we can negate its indeterminacy. That is, for any determinate proposition \( \varphi \) we can say that \( \varphi \) implies \( \neg \nabla \varphi \).

If Evans is correct, assuming the indeterminacy of an identity claim inevitably leads to its determinacy. As such, indeterminate identity statements are prohibited on pain of contradiction. Now, if we are to apply Evans’s argument to quantal objects while taking the position of Schrödinger and the quantum logics such as the system \( S \) which he inspired, Evans’s argument becomes impossible to formulate. Not a single proposition in the argument counts as a wff, for if \( x, a, \) and \( b \) are representative of quantal objects or variables ranging over them then they would need to be \( m \)-terms. And under such an interpretation, identity claims between \( m \)-terms are not even indeterminate, they are not wffs. Note this does not necessarily make the argument wrong, but unsayable by fiat. As such it is an easy dismissal of albeit not an objection to Evans’s argument. Although, if we do not adopt the aforementioned banning of identity statements
involving $m$-terms, several notable objections remain.

In Lowe’s original paper on quantal ontic vagueness (discussed in §4.1) he rejects Evans’s argument, making the positive case for vague identity. To that end Lowe identifies five possible problems which I henceforth call $\mathcal{L}_1$...$\mathcal{L}_5$ \cite{Lowe}. I now proceed to interpret these ideas and assess them all afterwards.

$\mathcal{L}_1$: Indeterminacy cannot be expressed as a sentential operator ‘$\nabla$’.

The idea behind $\mathcal{L}_1$ is that we should not be able to apply some logical operator such as $\nabla$ to a formula of indeterminate truth value in order to obtain a new formula of determinate truth value. In this case, $\mathcal{L}_1$ says it is illegal to turn an indeterminate formula into a determinate one through logical operation.

$\mathcal{L}_2$: The predicate symbol ‘$\hat{x}[\nabla(x = a)]$’ does not express a property.

$\mathcal{L}_2$ rejects the status of $\hat{x}[\nabla(x = a)]$ as a legitimate property of any object on the grounds that being indeterminately identical to something cannot really be a property of an object because the predicate itself involves indeterminacy. Interestingly, this bears some similarity to $\mathcal{L}_1$ as the legitimacy of the indeterminacy operator, in this case as part of a property, is called into question again.

$\mathcal{L}_3$: $\nabla(a = b)$ does not imply $\hat{x}[\nabla(x = a)]b$

and

$\nabla(a = a)$ does not imply $\neg\hat{x}[\nabla(x = a)]a$.

$L_3$ claims that premise (1) does not entail (2) and that (3) does not entail (4). That is, the indeterminate identity of $b$ with $a$ does not follow from the assumption that $a$ and $b$ are indeterminately identical. And the negation of $a$’s
indeterminate self-identity does not follow from the non-indeterminate identity of \(a\) with itself.

\[ L_4 : \nabla(a = a) \]

This is equivalent to the negation of (3). It is indeterminate whether \(a\) is identical to itself. Evidently we are thus required to reject the determinacy of the reflexivity of identity.

\[ L_5 : \text{LL requires some kind of restriction.} \]

\(L_5\) claims that the application of LL is invalid in this context, i.e. perhaps it does not work for indeterminate properties and needs to be restricted. Lowe does not elaborate on how or why we might need such restriction and dismisses \(L_5\) as a last resort. I suspect that those in favour of \(L_5\) might suggest that LL must be relegated to the domain of ordinary properties in extensional contexts, failing to apply to properties involving indeterminacy because these could be intensional contexts (I explain this shortly).

Out of the above objections, \(L_3\) and \(L_5\) seem to me to be the most tenable. \(L_4\) is a rather drastic measure as it requires the rejection of the determinate reflexivity of identity. It is quite extreme to reject determinate reflexivity in this case because in Evans’s argument we are operating under ordinary logic (not some non-reflexive logic like \(S\)) in which identity is, as outlined in §1.1, characteristically reflexive. To deny that \(x = x\) is always true a priori denies identity one of its core elements in standard logics which would have drastic consequences e.g. ‘bread is bread’ would not necessarily be a tautology. This is unless we reject determinate reflexivity via a logic such as \(S\) — although these logics are already completely incompatible with Evans’s argument as we have established. Moreover, there is no good reason to reject determinate reflexivity
here other than to render Evans’s argument invalid. The same seems true of \( L_1 \) and \( L_2 \). It is again a high-cost and inadequately motivated maneuver to claim that indeterminacy cannot be expressed as a sentential operator or that ‘\( \hat{x}[\nabla(x = a)] \)’ does not express a genuine property.

On the other hand, I think that \( L_5 \) may work as an objection because indeterminacy might actually be an intensional context, and as such LL would not always apply therein. In intensional (or opaque) contexts, propositions do not necessarily preserve their truth value under the substitution of co-referring terms and thus we cannot necessarily apply LL. This idea of different intensions changing the truth of a proposition was famously articulated by Frege [Frege, 1948, p. 215] who called intension ‘sinn’ or ‘sense’ (as opposed to extension which he called ‘bedeutung’ or ‘reference’). Subsequently, Carnap established the terms ‘intension’ and ‘extension’ [Carnap, 1947] which we use now. Common occurrences of intensional contexts are propositions of the form ‘I believe that \( x \)’ or ‘I know that \( x \)’ since one may believe/know something if referred to by some name and not believe/know it if referred to by another name.

Possibility and necessity can also give rise to intensional contexts and we already know that these are modal operators, as is indeterminacy. To use an example harking back to Quine [Quine, 1943], consider the statements ‘necessarily, 8 is \( > 7 \)’ and ‘necessarily, the number of planets is \( > 7 \)’. Both use the property ‘\( > 7 \)’ in different intensional contexts such that the former is true and the latter false\(^\text{17}\). Therefore we cannot always abstract the property within the scope of the necessity operator salva veritate, ergo LL fails here because the number of planets and the number 8 do not both possess the property of being necessarily \( > 7 \) despite being identical. Since indeterminacy is a modal operator just like necessity is, I make an argument by analogy to suggest that indeterminacy can also give intensional contexts which violate LL in the same way necessity does.

\(^{17}\) Quine used 9 instead of 8 for obvious reasons.
As such, we are required to restrict LL to extensional contexts only and thus cannot assume that its application to (2) and (4) (which involve indeterminacy) entails (4.1). Consequently, we can say that Evans’s argument may not be valid. Note that further consideration in favour of this objection comes from $\mathcal{L}_3$ as the fallacious property abstraction of indeterminate identicality with $a$ permits the very inferences $\mathcal{L}_3$ denies.

To me, $\mathcal{L}_3$ itself seems to be a stronger objection. If we claim that $\hat{x}[\nabla (x = a)]b$ (premise (2)), then by symmetry we can also claim that $\hat{x}[\nabla (x = b)]a$.\footnote{By the symmetry of indeterminate identity, which seems undeniable.} This is a formidable problem because since $\nabla (a = b)$, the propositions $\hat{x}[\nabla (x = a)]b$ and $\hat{x}[\nabla (x = b)]a$ are not actually determinately distinct, for they only differ by a swapping of the indeterminately identical $a$ and $b$. Lowe proposes this as a variation of $\mathcal{L}_3$ since it denies that (3) entails (4) on grounds that we cannot infer $\neg \hat{x}[\nabla (x = a)]a$ because of the indeterminate distinctness of properties $\hat{x}[\nabla (x = a)]$ and $\hat{x}[\nabla (x = b)]$. I have interpreted and expressed this counterargument to Evans with my own formalisation in which I introduce what I call indeterminacy propagation. Indeterminacy propagation is an intuitive principle according to which if two terms $x$ and $y$ are indeterminately identical, then a determinate wff involving one term $\varphi(x)$ becomes indeterminate under substitution of this term with the other i.e. $\nabla \varphi(y)$.

**Indeterminacy Propagation:** $\forall x \forall y ((\varphi(x) \land \nabla (x = y)) \rightarrow \nabla \varphi(y))$

This allows the counterargument to Evans to be formulated as follows:
As we can see, premise (4) above conflicts with premise (4) of Evans’s argument, for Evans’s (4) determinately states that $x$ is not indeterminately identical with $a$, whilst my (4) states that it is indeterminate whether $x$ is identical with $a$. In other words, the propositions $\neg \varphi$ and $\downarrow \varphi$ cannot both be true.

By $\mathcal{L}_3$ and $\mathcal{L}_5$, I thus reject Evans’s argument against the ontic vagueness of identity. $\mathcal{L}_5$ suggests how the argument could be invalid in that it applies LL within the potentially intensional context of indeterminacy. And moreover, by $\mathcal{L}_3$ and through my own formalisation thereof we see that Evans fails to recognise the indeterminate distinctness of the properties $\hat{x}[\nabla(x = a)]$ and $\hat{x}[\nabla(x = b)]$ — allowing us to generate the aforementioned incompatibility with Evans’s premise (4).

4.3 A Theory of Quantal Non-Ontic Vague Identity

Recall that in §1.3 we made a qualitative distinction between dense and rarefied systems. We said that a rarefied system is characterised by large distances between particles such that they do not interact and all occupy their own single-particle states. On the other hand, we said that a dense system has sufficiently short distances between particles as to elicit their interaction; the overlapping
of wavefunctions. Recall also that in rarefied systems our classical notions make sense, whereas in the case of dense systems such notions break down along with classical statistical mechanics. Consequently, we are required to turn from Maxwell-Boltzmann statistics to the quantal Bose-Einstein/Fermi-Dirac statistics in this regime, where particles of the same kind are taken to be exactly identical. But at what point does the transition occur?

In this section, I improve our qualitative understanding of dense and rarefied systems by explaining the more rigorous, quantitative approach to the distinction. Subsequently, I argue that even this distinction between rarefied and dense systems is not an entirely determinate one. And that its indeterminacy serves as a substrate on which we can construct a theory of non-ontic vagueness of identity in the physicist’s sense.

In physics, there is a formal, artificially imposed distinction between dense and rarefied systems achieved through the comparison of a system’s number density $N$ to the quantum volume $v_Q$. Number density is just like ordinary density but with the number of particles (instead of mass) over volume. On the other hand, the quantum volume serves as a crude approximation of the space taken up by the bulk of a particle’s spatial wavefunction\footnote{This is rough. Really, it is presented as the cube of the particle’s De Broglie wavelength [Schroeder, 1999, p. 264] (a more primitive notion of what was to become the wavefunction) but the general concept I am getting at is the same.} Thus the physicist can use these quantities to determine at which number densities the wavefunctions will begin to non-negligibly overlap and thereby demarcate dense from rarefied systems. If the system’s number density is such that $n$ quantum volumes cannot fit in it without being forced to overlap, then it makes sense that the system is dense rather than rarefied. More precisely, a system is said to be rarefied when the reciprocal of the number density is much greater than the quantum volume, and dense when the reciprocal of the number density is...
less than or equal to the quantum volume.

**Rarefied:** \( \frac{1}{N} \gg v_Q \)

\( \Rightarrow \) **Non-identical particles; apply Maxwell-Boltzmann statistics.**

**Dense:** \( \frac{1}{N} \leq v_Q \)

\( \Rightarrow \) **Identical particles; apply Fermi-Dirac/Bose-Einstein statistics.**

This is a useful distinction for the physicist to make as it allows her to deploy the necessary statistical mechanics for each case. Further, the considered and rigorous justification behind this criterion makes physical sense. But, however useful and justified the distinction, we must remember that there is semantic vagueness here because the distinction is artificially imposed. It is not the case that when the quantum volume is equal to the reciprocal number density the system suddenly switches between dense and rarefied. Likewise, it is not the case that at some particular height one switches between tall and short.\(^{20}\)

The quantum volume is a rough guide to the space taken up by a wavefunction, really the spatial wavefunction may extend far beyond the quantum volume, having very small values everywhere except for a ‘bulge’ around its particle’s location. The values of \( N \) and \( v_Q \) at which we consider the wavefunctions to be overlapping is semantically vague ergo the distinction between rarefied and dense is also semantically vague. Since the particles of a rarefied system are non-identical and the particles of a dense system are exactly identical, the vagueness of ‘rarefied’ and ‘dense’ runs parallel to, and thus implies a vagueness of identity for the particles in question. Note that here the kind of identity we are speaking of is the physicist’s notion of identity as indistinguishability — with LL and the IoI, or just as qualitative identity.

\(^{20}\)Assuming standard interpretations of vagueness.
We can construct a soritic series to illustrate my point. Suppose we have a system of particles of the same kind where \( v_Q = \frac{10^{20}}{N} \). Let us also take any two particles and name them \( a \) and \( b \). Since the system is rarefied we can make the statement \( a \neq b \). We also know that if, say \( v_Q = \frac{1}{N} \) then the negation of our original identity statement, \( \neg(a \neq b) \) (which is the same as \( a = b \)), will be true because the system is dense i.e. \( a \) and \( b \) are exactly identical. Now, increasing the number density by a small amount (say, by decreasing the volume) such that \( v_Q = \frac{10^{19}}{N} \) will not affect whether or not the system is rarefied and thus will not change the truth value of our identity statement \( a \neq b \) — the identity relation is tolerant. If we continue to increase the number density in small steps, for each of which \( a \neq b \) holds true by tolerance, we shall find that \( a \neq b \) is true when \( v_Q = \frac{1}{N} \), contradicting what we have previously stated. I have formalised this sorites paradox for identity below with \( N \) propositions where \( i \) is a very large number such that \( i \gg 1 \) and \( \delta i \) is an arbitrarily small increment such that \( \delta i \ll i \).
As we can see, if we have $v_Q = \frac{1}{N}$ then we have both $a = b$ and $a \neq b$; (N) and (2) give us the contradiction of the sorites paradox. By demonstrating the sorites susceptibility of what we are calling identity, we reinforce the idea that it is vague. Further, I emphasise that the vagueness here is semantic and need not be ontic as it pertains to the point at which we call the particles identical or non-identical, rather than to some inherent ontological notion of their vague
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identity. For the aforementioned reasons, I consider this to be an advantage.

In light of the treatment of identity by sortal Schrödinger logics such as system $S$ from §2.2, an objection to my argument for vague identity comes to mind. Recall that these logics distinguish between two sorts of objects, $m$-objects and $M$-objects where the former are representative of quantal objects between which identity statements are not wffs, and the latter of classical ones between which identity statements are wffs. If we are to employ such a system, it follows that when we have $v_Q \geq \frac{1}{N}$, we cannot say that $a = b$ for it is not a wff since $a$ and $b$ have become $m$-terms. Rather, as a rarefied system approaches $v_Q = \frac{1}{N}$, $a \neq b$ gradually moves from a true wff to something that is not a wff at all — as $a$ and $b$ shift from $M$-terms to $m$-terms. This looks like a threat to our theory of vague identity because it does not allow us to generate a sorites paradox; we cannot make identity statements for $m$-terms and so we have nothing to contradict proposition (1). At best, we instead get a sort of vagueness of ‘wff-ness’ — if such a concept is even coherent — because we now have vagueness between $m$-terms and $M$-terms (I address this shortly).

However, I claim that this is ill suited as an objection to my argument because we need to recognise that Schrödinger logics treat indistinguishability and identity as separate concepts; recall from §2.2 that a separate indistinguishably relation ‘≡’ is defined in $Q$, as a semantics for $S_{S\omega}$, which can hold between $m$-terms. As such, we are not using the same notion of identity. This being said, an analogue of my argument for vague identity could easily be made for vague indistinguishability were we to accept Schrödinger logics, but a vagueness of identity in the sense of such logics is off the table. As we have established, Schrödinger logics and $\Omega$ confine identity to the domain of $M$-objects in which it behaves classically and precisely.

This leads me to a problem that I see with Schrödinger logics and $\Omega$. As per my argument above, the distinction between non-identical and exactly
identical particles is vague. In the same manner, the distinction between the two sorts of objects in Schrödinger logics and $\mathcal{Q}$ is vague and therefore, the status of expressions as wffs or not wffs is also vague. Interpreting identity in the sense of $S$ and $\mathcal{Q}$, we can claim that premise (1) of my soritic series is true, for $a$ and $b$ would be $M$-objects. However, as the density of the system increases $a$ and $b$ must begin to be treated as $m$-objects. Consequently, it is vague at what point $a \neq b$ ceases to be a wff. This is significant because a logic must clearly determine what does and does not constitute a wff; when Schrödinger logics fail in this regard we have a serious problem. We can see here that the continuity between classical and quantal treatments of the world leads to both the vagueness of identity (in the physicist’s sense) and the problematic vagueness of wff-status for Schrödinger logics.

The soundness of my argument for vague identity now hinges upon how we decide to interpret identity itself. Here, the various interpretations of identity correspond to different bundles of postulates about identity we accept e.g. LL and the IoI. If we interpret identity in the context of first-order logic, it seems impossible for the relation to be vague because precise, classical identity is already built into our set-theoretic modelling apparatus, which is itself a minimum requirement for sense to be made of any such concept (having a model means logical consistency). Since identity is fixed into the set-theoretic background (perfectly precisely), we cannot make sense of vague identity because we cannot model it set-theoretically. This is Nicholas Smith’s argument against vague identity [Smith, 2008] which notably also holds up against $\mathcal{Q}$ if we try to take it as an alternate modelling apparatus, for the identity in $\mathcal{Q}$ is of the same kind that Smith addresses — namely, identity as a primitive concept in first-order logic. However, interpreting identity as the physicist seems to — accepting the IoI plus LL — then the idea that this relation is vague during the transition from classical to quantal regimes, is tenable. Alternatively, if we
assume that physicists actually just mean qualitative identity and therefore absolute indistinguishability without the IoI, my argument is also tenable as this would just be the vague indistinguishability version alluded to previously (or vague qualitative identity).

Harking back to the later-Wittgenstein, the discussion I present of vague identity now resembles a case of philosophical problems arising as artefacts of linguistic confusion. In *Philosophical Investigations*, Wittgenstein introduces the notion of *language-games* [Wittgenstein, 1953]. Although he does not rigorously define it, activities in which language is employed in different ways are characterised as language-games, and the ways in which terms are used in certain language-games account for how they acquire their meanings therein. Since the meanings of terms are accounted for by their use in a particular language-game, these meanings can differ dramatically over different contexts. Instances of language-games would include issuing orders, making hypotheses and performing experiments to test them, expressing feelings, thinking about a concept, and so forth. Here, the metaphor of a game is supposed to reflect the rule-governed character of language in these different activities. Among these games I would count the language-game of physics and that of analytical philosophical inquiry. In physics, we could say that language is used in order to effectively carry out empirical and theoretical investigation of the universe whereas in philosophy it is largely used to accurately conceive, engineer, and express concepts. The term ‘identity’ is used differently in the language-game of physics, to how it is used the language-game of philosophical logic. The physicist may use the term in order to express that, say, two particles are absolutely indistinguishable or to denote a kind of mathematical object (e.g. identity matrices). In first-order logics however, identity is used as a binary, reflexive, symmetric, and transitive relation associated to a primitive concept. Either way, the term’s use differs across each discipline and thus can have different meanings in each language-
game. Consequently, interpreting identity in the physics language-game allows us to construct a notion of vague identity because it is not built-in as a precise concept of our modelling apparatus. On the other hand, we cannot make sense of identity as vague in the context of first-order logics for identity is itself embedded in the apparatus by which we make sense of things — as Smith argued.

Unlike other conceptions of vague identity I discussed, the one presented here does not commit us to a notion of ontic vagueness. As I have mentioned previously, the distinction between dense and rarefied systems is one imposed artificially (albeit justifiably) by language-users (physicists). Recall the position of mine outlined in §3.2. Density and ‘rarefiedness’ are not fundamentally ontological, they are merely our way of describing the world, and as such, their vagueness is again an artifact of the imperfections of language. For instance, this is similar to the vague predicates ‘large’ and ‘small’; these concepts are vague because of the looseness of natural language. There is no underlying ontological largeness or smallness of things in the same way that there is no underlying ontological identity or non-identity of objects. I reiterate this because unlike Lowe, French, and Krause, my theory of quantal vagueness does not require the adoption of ontological vagueness and is consequently a more palatable view in the current philosophical landscape.

4.4 Conclusions on Vague Identity

In this chapter I have shown that an adoption of the physicist’s notion of identity entails a non-ontic vagueness of the relation resultant from the penumbral region between our modelling of classical and quantum behaviours. On the other hand, identity as understood in first-order logic is not capable of being vague since it is embedded in the modelling apparatus itself. Consequently, we ought to be
careful about exactly what we mean by identity in order to avoid susceptibility to
disagreement via linguistic confusion since the term is used differently in physics
and first-order logics (including $S$). I therefore accept the idea of vague identity
insofar as we mean identity in the physicist’s sense. Even without talking of
identity as a physicist, accepting the principle of the IoI leads to the vagueness of
identity in the quantal context, for the absolute indistinguishability of quantal
particles would imply their identity — where, as per my argument in §4.3, this
can be semantically vague. In addition, I emphasise that Lowe held a rather
loose definition of vagueness due to the time at which he wrote about quantal
vague identity, and therefore to claim that he was mistaken on grounds that
indeterminacy does not necessarily imply vagueness would be another instance
of linguistic confusion. In contrast, I have provided what I believe to be strong
counterarguments to French and Krause’s assertion that quantal objects are
vague since such objects lack self-identity under Schrödinger logics and $Q$. As
such, while I can accept quantal vague identity in some senses of the word
‘identity’ and the standard, current sense of the word ‘vague’, I reject that the
relation or the quantal objects to which it does or does not pertain can be vague
given $S$, $S_\omega$, or $Q$. 
References


