TORSION EQUATIONS FOR LATERAL BUCKLING

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ABSTRACT

A torsion differential equation previously used for analysing the elastic lateral buckling of simply supported doubly symmetric beams with distributed loads acting away from the centroidal axis omits an expected term and includes an unexpected term. A different equation is derived by two different methods, either by using the calculus of variations with the second variation of the total potential, or by considering the equilibrium of the deflected and twisted beam.

Four different methods are used to find solutions for the elastic buckling of beams with uniformly distributed loads. Two of these solve the differential equations numerically, either by using a computer program based on the method of finite integrals, or by making hand calculations with a single term approximation of the buckled shape. These methods produce different solutions for the two torsion differential equations.

The two other methods used are based on the energy equation for lateral buckling. The first of these uses hand calculations and a limited series for the buckled shape, while the second uses a finite element computer program based on cubic deformation fields. Both of these produce solutions which agree closely with the finite integral and approximate solutions for the different differential equation derived in this paper, but are markedly different from the solutions for the previously used equation.

It is concluded that the previously used torsion differential equation is in error.

KEYWORDS

Beam, buckling, differential equation, distributed load, elasticity, torsion

TABLE OF CONTENTS

1. INTRODUCTION

The differential equation for the variation of the twist rotation ϕ along the z centroidal axis of a doubly symmetric beam loaded in the YZ principal plane is reported in [1] as being

$$EI_w \frac{d^4\phi}{dz^4} - GJ\frac{d^2\phi}{dz^2} - V_y y_q \frac{d\phi}{dz} - \frac{M_x^2}{EI_y}\phi = 0 \tag{1}$$

in which E, G are the elastic moduli, I_y, J, and I_w are the second moment of area about the y axis, the torsion section constant, and the warping section constant, V_y and M_x are the internal shear and moment stress resultants, and y_q is the distance below the centroidal axis at which a distributed load q acts. For a simply supported beam under uniformly distributed load q (Fig. 1)

$$\begin{aligned} V_y &= \frac{qL}{2}\left(1-\frac{2z}{L}\right) \\ M_x &= \frac{qL^2}{8}4\left(\frac{z}{L}-\frac{z^2}{L^2}\right) \end{aligned} \tag{2}$$

Equation 1 omits an expected term of the type $qy_q\phi$ and includes an unexpected term $V_y y_q\, d\phi/dz$. It is asserted that Equation 1 is incorrect. The purpose of this paper is to show how the correct torsion differential equation can be derived and to compare its predictions with those of Equation 1.

2. DERIVATION OF TORSION DIFFERENTIAL EQUATION

2.1 Calculus of Variations

The calculus of variations can be used to derive the torsion differential equation from the energy equation for lateral buckling. The energy equation for doubly symmetric sections with distributed loads only is [2]

$$\frac{1}{2}\delta^2 U_T = \frac{1}{2}\int_0^L \{EI_y u''^2 + GJ\phi'^2 + EI_w\phi''^2 + 2M_x u''\phi + qy_q\phi\}dz = 0 \tag{3}$$

in which U_T is the total potential, L is the length, u is the lateral displacement, and ' $\equiv$ d/dz. According to the calculus of variations, the functions u, ϕ which make

$$\frac{1}{2}\delta^2 U_T = \int F(z, u'', \phi, \phi', \phi'')dz \tag{4}$$

stationary satisfy the equations

$$\begin{aligned} &\frac{d^2}{dz^2}\frac{\partial F}{\partial u''} = 0 \\ &\frac{\partial F}{\partial \phi} - \frac{d}{dz}\frac{\partial F}{\partial \phi'} + \frac{d^2}{dz^2}\frac{\partial F}{\partial \phi''} = 0 \end{aligned} \tag{5}$$

This leads to [3]

$$(EI_y u'')''-(M_x\phi)''=0$$
$$(EI_w\phi'')''-(GJ\phi')'+qy_q\phi-M_x u''=0 \tag{6}$$

For beams with end twist rotation prevented, the first of Equations 6 can be integrated to

$$EI_y u''-M_x\phi=0 \tag{7}$$

Substituting this into the second of Equations 6 leads to

$$(EI_w\phi'')''-(GJ\phi')'-(M_x^2/EI_y-qy_q)\phi=0 \tag{8}$$

This torsion differential equation includes the $qy_q\phi$ term missing from Equation 1 and omits the unexpected $-V_y y_q d\phi/dz$ term. Reference 1 includes an argument for the inclusion of this unexpected term based on the assumption that the internal shear V_y may be treated as a vertical external force that displaces laterally as the beam deflects and twists.

2.2 Equilibrium of the Twisted Beam

The torsion differential equation can also be obtained by considering the equilibrium of the applied loads in the buckled position shown in Fig. 2. For overall equilibrium, the LH end reactions consist of a vertical force $qL/2$ and a torque $\int_0^L q(u-y_q\phi)dz/2$ about the fixed Z axis. The global reactants of these and the distributed load q at a distance z from the LH end are M_x, V_y ($=M_x$') and

$$M_Z=\int_0^L q(u-y_q\phi)dz/2-\int_0^z (q(u-y_q\phi)dz \tag{9}$$

acting about the fixed Z axis, and the torque resultant of these acting about the displaced and rotated z axis is

$$M_z=M_Z+M_x u'-V_y u \tag{10}$$

This torque is resisted by the uniform torsion and the warping rigidities, so that

$$EI_w\phi'''-GJ\phi'=M_z=\int_0^L q(u-y_q\phi)dz/2-\int_0^z q(u-y_q\phi)dz+M_x u'-M_x' u \tag{11}$$

Differentiating this equation and using $M_x'' = -q$ leads to the uniform beam version of Equation 8.

3. SOLUTIONS

3.1 Solution by Finite Integrals

Equations 1 and 8 may be solved numerically by the method of finite integrals [4, 5], as explained in the Appendix. The data used for an example are E = 2E5 N/mm^2, G = 76923 N/mm^2, I_y = 2281E5 mm^4, J = 512E4 mm^4, I_w = 64877E8 mm^6, d_w = 337.3 mm, and L = 5000 mm, in which $d_w = 2\sqrt{(I_w/I_y)}$ is the distance between flange centroids. Finite integral solutions are given in Fig. 3 for the variation of the dimensionless elastic buckling moment M_{yq}/M_0 with the dimensionless load distance y_qP_y/M_{yz} in which M_0 is the value of M_{yq} for y_q = 0 and

$$P_y = \pi^2 EI_y / L^2$$
$$M_{yz} = \sqrt{P_y (GJ + \pi^2 EI_w / L^2)} \tag{12}$$

3.2 Finite Element Solutions

Finite element solutions for the dimensionless elastic buckling moments have been obtained by using the computer program PRFELB [2, 6, 7], and are shown in Fig. 3. They are in very close agreement with the finite integral solutions of Equation 8, but differ markedly from the finite integral solutions of Equation 1.

3.3 Timoshenko's Solutions

Timoshenko [8] determined approximate solutions for simply supported beams with uniformly distributed load by using

$$\phi = \sin \pi z / L + a \sin 3\pi z / L \tag{13}$$

in the energy equation (Equation 3) and minimising with respect to the undetermined parameter a. The solutions shown in Fig. 3 are in very close agreement with the finite integral solutions of Equation 8, but differ markedly from the finite integral solutions of Equation 1.

3.4 Approximate Solutions

Approximate solutions of the torsion equations may be made by substituting $\phi = \sin \pi z/L$ and integrating each term over the beam length L. Thus for Equation 1,

$$\int_0^L \{EI_w \phi'''' - GJ\phi'' - V_y y_q \phi' - (M_x^2 / EI_y)\phi\} dz = 0 \tag{14}$$

leads to

$$1 + \frac{4 y_q P_y}{\pi^2 M_{yz}} \left(\frac{M_{yq}}{M_{yz}} \right) - \frac{32(12 - \pi^2)}{\pi^4} \left(\frac{M_{yq}}{M_{yz}} \right)^2 = 0 \tag{15}$$

which can be solved for values of (M_{yq} / M_{yz}) for given values of $y_q P_y / M_{yz}$. These solutions have been used to determine the values of M_{yq} / M_0 shown in Fig. 3. These values are quite close to the finite integral solutions of Equation 1 shown in Fig. 3 but very different from the values obtained for Equation 8. More accurate solutions could be obtained by using Equation 13 and finding the values of the parameter a which minimise the solutions, in much the same way as did Timoshenko [8] for his energy method solutions.

Using the same method for Equation 8 leads to

$$1 + \frac{8 y_q P_y}{\pi^2 M_{yz}} \left(\frac{M_{yq}}{M_{yz}} \right) - \frac{32(12 - \pi^2)}{\pi^4} \left(\frac{M_{yq}}{M_{yz}} \right)^2 = 0 \tag{16}$$

The solutions of this have been used to determine the values of M_{yq} / M_0 shown in Fig. 3. These values are reasonably close to the finite integral solutions of Equation 8 shown in Fig. 3, but very different from the values obtained for Equation 1. The reason for this can be seen to be the change of the value of 4 in Equation15 to 8 in Equation16, which suggests that Equation 1 underestimates the significance of the load distance y_q by a factor of 2.

4. CONCLUSIONS

A torsion differential equation previously used [1] for analysing the elastic lateral buckling of simply supported doubly symmetric beams with distributed loads acting away from the centroidal axis omits an expected term and includes an unexpected term. A different equation which includes the expected term and omits the unexpected term is derived in this paper by two different methods. The first method uses the calculus of variations with the second variation of the total potential, while the second method considers the equilibrium of the deflected and twisted beam.

A number of different methods are used to find solutions for the elastic buckling of beams with uniformly distributed loads. Two of these solve the torsion differential equations numerically. The first method uses a computer program based on finite integrals [4, 5], while the second uses hand calculations with a single term approximation of the buckled shape. These methods produce different solutions for the two torsion differential equations.

Two other methods used are based on the energy equation for lateral buckling. The first of these by Timoshenko [8] uses hand calculations and a limited series for the buckled shape, while the second uses a finite element computer program [6, 7] based on cubic deformation fields [2]. Both of these produce solutions which agree closely with the finite integral and approximate solutions for the different differential equation derived in this paper, but are markedly different from the solutions for the previously used equation.

It is concluded that the previously used torsion differential equation is in error.

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6. APPENDIX – FINITE INTEGRALS

In the method of finite integrals [4, 5], a differential equation is replaced by a set of simultaneous equations which represent the differential equation at each of a number of points along a beam, as in the method of finite differences. However, the unknowns in these equations are the highest order differential operators at the points, instead of the lowest order as in the method of finite differences. This allows the use of integration which is more accurate than differentiation, leading to faster convergence or more accurate solutions. In addition, the boundary conditions are treated naturally using the constants of integration, and no fictitious points are required.

The terms of Equation 1 may be represented by using

$$\begin{aligned}
\phi''' &= \int_0^z \phi'''' \, dz + A_1 \\
\phi'' &= \int_0^z \int_0^z \phi'''' \, dzdz + A_1 z + A_2 \\
\phi' &= \int_0^z \int_0^z \int_0^z \phi'''' \, dzdzdz + A_1 z^2/2 + A_2 z + A_3 \\
\phi &= \int_0^z \int_0^z \int_0^z \int_0^z \phi'''' \, dzdzdzdz + A_1 z^3/6 + A_2 z^2/2 + A_3 z + A_4
\end{aligned} \qquad \text{(A-1)}$$

in which

$$\begin{aligned}
A_1 &= (\phi''')_0 \\
A_2 &= (\phi'')_0 \\
A_3 &= (\phi')_0 \\
A_4 &= (\phi)_0
\end{aligned} \qquad \text{(A-2)}$$

Boundary conditions of $(\phi)_0 = 0$ and $(\phi'')_0 = 0$ require $A_4 = A_2 = 0$. Boundary conditions of $(\phi)_L = 0$ and $(\phi'')_L = 0$ require

$$\begin{aligned}
A_3 L &= -\int_0^L \int_0^z \int_0^z \int_0^z \phi'''' \, dzdzdzdz - A_1 L^3/6 \\
A_1 L &= -\int_0^L \int_0^z \phi'''' \, dzdz
\end{aligned} \qquad \text{(A-3)}$$

If the beam is divided into an even number n of equal intervals by n+1 nodes, then each continuous integral may be replaced by combinations of the values of the integrand at the nodes, such as

$$\int_0^z \phi''' \, dz = (h/12)[N]\{\phi''''\} \qquad \text{(A-4)}$$

in which $h = L/n$ is the interval length and [N] is the integration matrix

$$[N] = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & . & 0 \\
5 & 8 & -1 & 0 & 0 & . & 0 \\
4 & 16 & 4 & 0 & 0 & . & 0 \\
4 & 16 & 9 & 8 & -1 & . & 0 \\
4 & 16 & 8 & 16 & 4 & . & 0 \\
. & . & . & . & . & . & . \\
4 & 16 & 8 & 16 & 8 & 16 & 4
\end{bmatrix} \qquad \text{(A-5)}$$

which is based on fitting a series of parabolas to the integrand.

Thus Equation 1 is replaced by

$$[T]\{\phi''''\} = \{0\} \tag{A-6}$$

in which

$$[T] = EI_w[I_0] - GJ[I_2] - V_y y_q[I_3] - (M_x^2 / EI_y)[I_4] \tag{A-7}$$

In this equation, $[I_0]$ is a unit matrix and

$$\begin{aligned} [I_2] &= (h/12)^2[N][N] + A_1[z_1] \\ [I_3] &= (h/12)^3[N][N][N] + A_1[z_2]/2 + A_3[I_0] \\ [I_4] &= (h/12)^4[N][N][N][N] + A_1[z_3]/6 + A_3[z_1] \end{aligned} \tag{A-8}$$

in which $[z_1]$, $[z_2]$, and $[z_3]$ are diagonal matrices with the appropriate values of z, z^2, and z^3, and

$$\begin{aligned} A_1 &= -(h/12)^2\{NN_L\}^T\{\phi''''\}/L \\ A_3 &= -(h/12)^4\{NNNN_L\}^T\{\phi''''\}/L - A_1L^2/6 \end{aligned} \tag{A-9}$$

in which $\{NN_L\}^T$ and $\{NNNN_L\}^T$ are the last lines of $[N][N]$ and $[N][N][N][N]$ respectively.

Similar replacements are made for Equation 8.

The elastic buckling loads may be determined by finding the values which satisfy

$$|T| = 0 \tag{A-10}$$

Because Equation A-6 is quadratic in the load, an iterative process will be required for this.

7. NOTATION

a	Parameter in buckled shape
$A_{1\text{-}4}$	Constants of integration
d_w	Web depth
E	Young's modulus of elasticity
F	Function
G	Shear modulus of elasticity
h	Interval = L / n
I_y	Minor axis second moment of area
I_w	Warping section constant
$[I_{1\text{-}4}]$	Finite integral matrices
J	Uniform torsion section constant
L	Length
M_x	Bending moment
M_{yq}	Maximum moment at elastic buckling
M_{yz}	Uniform bending elastic buckling moment
M_z	Torque about displaced z axis
M_Z	Moment about fixed Z axis
M_0	Value of M_{yq} for $y_q = 0$
n	Number of intervals
$[N]$	Integrating matrix
$\{NNL\}^T$	Last line of $[N][N]$
$\{NNNNL\}^T$	Last line of $[N][N][N][N]$
P_y	Column elastic buckling load
q	Intensity of distributed load
$[T]$	Total torsional stiffness matrix
u	Lateral deflection
U_T	Total potential
V_y	Shear
x, y	Principal axes
y_q	Distance of load below centroidal axis
z	Buckled centroidal axis
Z	Fixed centroidal axis
$[z_{1,2,3}]$	Diagonal matrices of values of z, z^2, z^3
ϕ	Angle of twist rotation

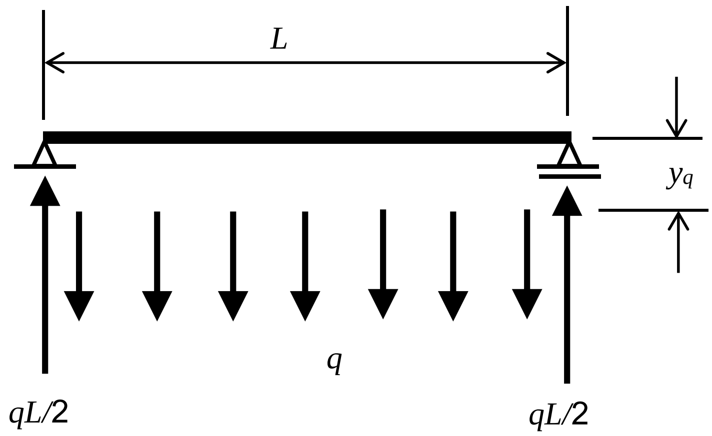

Fig. 1 Simply Supported Beam

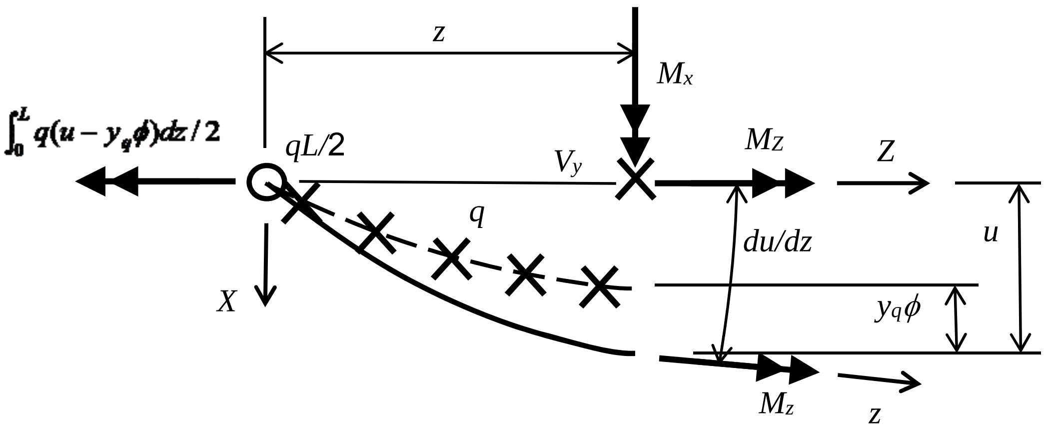

Fig. 2 Equilibrium of Buckled Beam

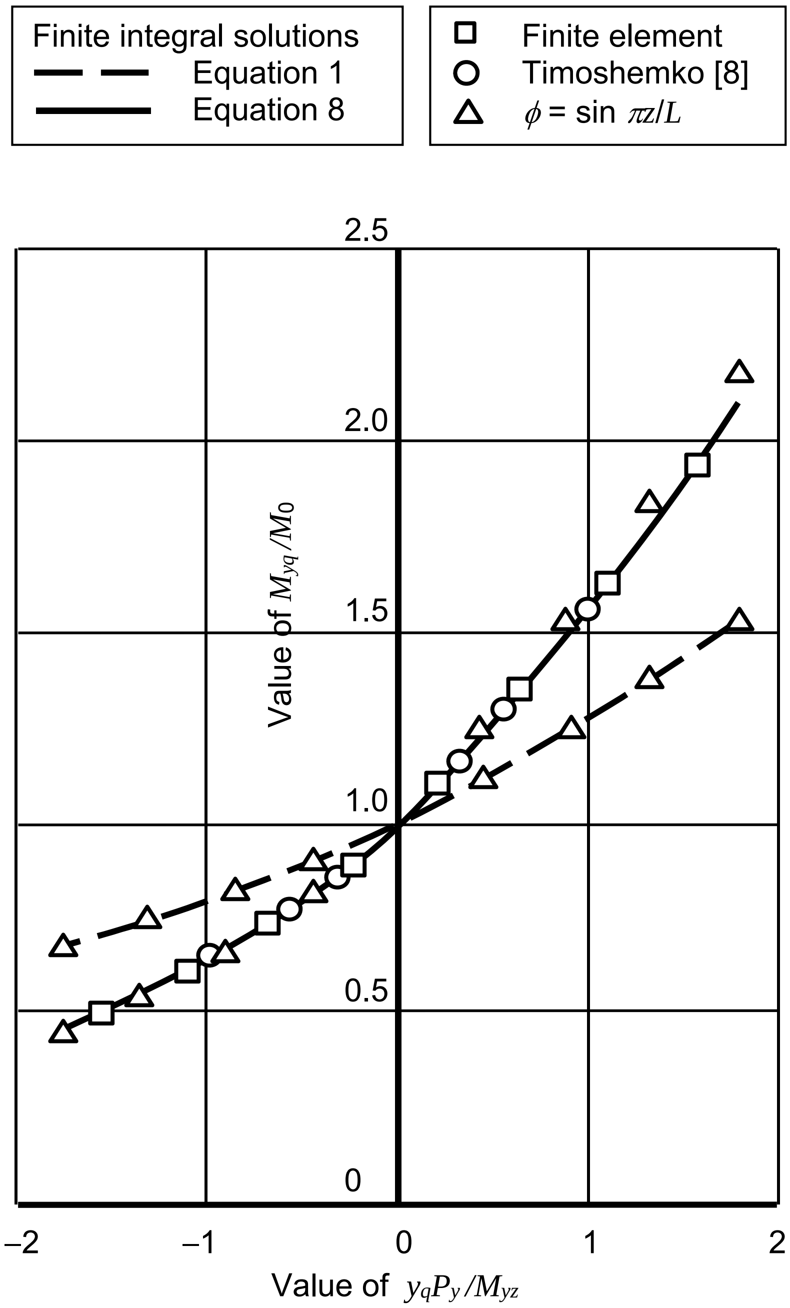

Fig. 3 Solutions of Differential Equations