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INTERACTION BUCKLING OF TAPERED BEAMS
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ABSTRACT

There have been comparatively few studies of the elastic lateral buckling of braced or continuous tapered beams, and these are limited in their application. Lateral buckling is affected by the separate effects of moment distribution, taper, and restraints between adjacent segments.

Moment distribution effects are commonly allowed for in design codes by using C_{bm} factors to multiply the classic lateral buckling moments M_u of simply supported uniform beam segments in uniform bending.

Taper effects for linearly web tapered beam segments may conveniently be allowed for by multiplying the segment lateral buckling moments C_{bm} M_u computed using the mid-segment section properties by taper factors C_{bt}. Values of C_{bt} for a number of different segment moment distributions have been determined using a finite element computer program for the buckling of tapered beam structures.

Lateral buckling of a braced or continuous beam is also affected the interaction between the segments into which it is divided by its braces and supports. One segment will be more critical than its neighbours, which will restrain the critical segment and increase its buckling resistance. The effects of restraints on buckling are commonly allowed for by using effective length factors to multiply the segment length used in the formulation of the elastic lateral buckling moment M_u of a uniform segment in uniform bending. Methods of determining the critical segment and of approximating its increased resistance developed for uniform beams have been adapted for web-tapered braced and continuous beams.

This paper shows how these effects can be allowed for separately to develop good approximations for the elastic lateral buckling resistances of tapered braced and continuous beams. The accuracy of the approximations is demonstrated by comparisons with the predictions of the finite element computer program for the buckling of tapered beam structures.

KEYWORDS

Beams, Braces, Buckling, Continuity, Interaction, Steel, Structures, Taper

TABLE OF CONTENTS

1 INTRODUCTION

There have been comparatively few studies of the elastic lateral buckling of braced or continuous tapered beams [1-6], and these are limited in their application. On the other hand, the buckling of uniform braced and continuous beams has been thoroughly investigated [7-12].

Early studies [1, 2] of the elastic buckling of simply supported I-beams with constant web taper (Fig. 1) under constant moment gradient have been used to develop predictions for beams with equally spaced interior braces which prevent lateral displacement and twist rotation [2]. The motive for these studies has been to apply them to the design of the tapered rafters of portal frames with negligible axial forces. A subsequent investigation [3] has extended these early studies.

The buckling of simply supported tapered beams is affected by the moment distribution as well as by the taper. These effects have been combined and represented [1] by dual factors to be used with the classic elastic buckling moment resistance M_u of uniform beams in uniform bending [6, 11-13] given by

$$M_u = \sqrt{\left\{\left(\frac{\pi^2 EI_y}{L^2}\right)\left(GJ + \frac{\pi^2 EI_w}{L^2}\right)\right\}} \tag{1}$$

in which E and G are the Young's and shear moduli of elasticity, L is the beam length, and I_y, J and I_w are the minor axis second moment of area, the uniform torsion constant, and the warping constant of the beam cross-section, respectively.

The buckling of a braced or continuous beam is also affected the interaction between the segments into which the beam is divided by its braces and supports. This effect has been described as interaction buckling [7, 8]. This interaction effect has been combined [2] with those of taper and moment distribution [1]. The results of these studies have been adopted in [5] and described in [6], although not in current editions. A separation of the effects of interaction from the combined effect of taper and moment distribution has been investigated [3].

It can be said that these studies of the elastic buckling of web-tapered braced beams are limited in their application, and are probably of somewhat doubtful accuracy because of the combining together of the separate effects of taper, moment distribution, and interaction between adjacent segments. The purpose of this paper is to show how these effects can be allowed for separately to develop good approximations for the elastic buckling resistance. This is done by extending to tapered beams an approximate method [9-12] of allowing for interaction buckling in uniform braced and continuous beams. The accuracy of this extension is demonstrated by comparisons with the predictions of a general finite element computer program FTBTM for tapered beam structures [14].

2 INTERACTION BUCKLING OF UNIFORM BEAMS

2.1 INTERACTION BUCKLING

When a braced or continuous beam buckles, there are interactions between the segments into which the beam is divided by its braces and supports. One segment will be more critical than its neighbours, which will restrain the critical segment and increase its buckling resistance. For example, three buckling modes of a braced beam are shown in Fig. 2.

(1) When the end segments are comparatively long, they are restrained during buckling by the short centre segment, and the buckled shape has inflection points in the end segments, as shown in Fig 2b.

(2) When the centre segment is comparatively long, it is restrained by the short end segments, and the buckled shape has inflection points in the centre segment, as shown in Fig. 2c.

(3) Between these two situations lies the zero interaction brace arrangement for which the buckled shape has inflection points at the brace points, as shown in Fig. 2d, and each segment buckles as if unrestrained.

2.2 METHODS OF ANALYSIS

The effects of interactions on the elastic lateral buckling of a braced or continuous beam may be determined by using a computer program such as PRFELB [15] or FTBTM [14] to analyse the complete beam. Alternatively, approximate methods may be used.

A very simple approximate method of buckling analysis [16] is often used, in which the interactions between adjacent segments are ignored and each segment is analysed as if simply supported laterally. The so-determined elastic buckling moment of each segment is then used to evaluate a corresponding beam load set, and the lowest of these (which defines the critical segment) is taken as the elastic buckling load set. This method produces a lower bound estimate which is sometimes remarkably close to the true buckling load set.

A much more accurate but still reasonably simple method of buckling analysis has been developed [9-12]. In this method, the accuracy of the lower bound estimate (obtained as described above) is improved by allowing for the interactions between the critical segment and the adjacent segments at buckling. This is done by using a simple approximation for the destabilising effects of the in-plane bending moments on the stiffnesses of the adjacent segments, and by approximating the restraining effects of these segments on the critical segment by using the effective length chart of Fig. 3 (for the buckling of braced compression members) to estimate the effective length factor of the critical beam segment. A step by step summary is as follows:

(1) Determine the properties EI_y, GJ, EI_w, L of each segment.

(2) Analyse the in-plane bending moment distribution through the beam, and determine the moment distribution factors C_{bm} [5, 11, 12] for each segment.

(3) Assume all effective length factors k are equal to unity.

(4) Calculate the maximum moment M in each segment at elastic buckling from

$$M = C_{bm} \sqrt{\left\{ \left(\frac{\pi^2 EI_y}{(kL)^2} \right) \left(GJ + \frac{\pi^2 EI_w}{(kL)^2} \right) \right\}} \tag{2}$$

and the corresponding beam buckling load set Q.

(5) Determine a lower bound estimate Q_m of the beam buckling load set as the lowest value of the loads Q, and identify the segment associated with this as the critical segment AB (this is the simple approximate lower bound method [16] described above).

(6) If a more accurate estimate of the beam buckling load is required, use the values Q_m and Q_{RA}, Q_{RB} calculated in Step 5 together with Fig. 4 to approximate the stiffnesses α_{RA}, α_{RB} of the segments adjacent to the critical segment AB.

(7) Calculate the stiffness of the critical segment AB from $2EI_{yAB}/L_{AB}$.

(8) Calculate the stiffness ratios G_A, G_B from

$$G_{A,B} = \frac{2EI_{yAB} / L_{AB}}{0.5\alpha_{RA,RB} + 2EI_{yAB} / L_{AB}} \,. \tag{3}$$

(9) Determine the effective length factor k for the critical segment AB from Fig. 3.

(10) Calculate the elastic buckling moment M of the critical segment AB using k in Equation 2, and from this the corresponding improved approximation of the elastic buckling load set of the beam.

It should be noted that while the calculations for the lower bound (the first five steps) are made for all segments, those for the improved estimate are only made for the critical segment, and so comparatively little extra effort is involved. The development and application of this method is further described in [9-10].

2.3 EXAMPLE

The buckling moments of the simply supported braced uniform beam 1234 shown in Fig. 5a have been analysed using the lower bound zero interaction approximation of [16], more accurately using the approximate interaction method, and most accurately using the computer program PRFELB [15], and are shown by the dimensionless moments M / M_{120} (in which $M = M_2$ at buckling and M_{120} is the value of M_2 for a simply supported segment 12 with k_{12} = 1.0 and α_{m12} =1.844) in Fig. 6. Also shown in Fig. 6 are upper bounds obtained by assuming that the effective length factors are $k_{12,\ 23}$ = 0.7, 0.5. It can be seen that the values for the approximate interaction method are close to the accurate values, and significantly higher than those for the zero interaction method, except for span ratios L_{23}/L_{12} close to the zero interaction value of 0.64.

3 INTERACTION BUCKLING OF TAPERED BEAMS

3.1 GENERAL

The maximum moment M_t at the elastic lateral buckling of an I-section beam segment of constant web taper (Fig. 1) for which end lateral deflections and twist rotations are prevented may be approximated by using

$$M_t = C_{bm} C_{bt} \sqrt{\left\{ \left(\frac{\pi^2 EI_{yL/2}}{(kL)^2} \right) \left(GJ_{L/2} + \frac{\pi^2 EI_{wL/2}}{(kL)^2} \right) \right\}} \tag{4}$$

in which C_{bm} and C_{bt} are factors which allow for the effects of the moment distribution and taper, k is an effective length factor which allows for the effects of end restraints against end minor axis rotation and warping, and the section properties $I_{yL/2}$, $J_{L/2}$ and $I_{wL/2}$ are the values at the middle of the segment length L. This is somewhat different from the approximations used in [1, 2] which use the properties at the smaller end cross-section, and dual factors for the combined effects of moment distribution, taper and end restraint. The formulation of Equation 4 allows these effects to be separated, and has the advantage of conforming with the widely accepted methods of allowing for moment distribution and end restraints in uniform beam segments.

3.2 EFFECTS OF MOMENT DISTRIBUTION

The effects of moment distribution on the lateral buckling of a uniform beam segment may be determined from

$$C_{bm} = M_t / M_u \tag{5}$$

in which M_t is the maximum moment in a simply supported uniform (C_{bt} =1) segment without end restraints (k = 1) with the actual moment distribution. Values of M_t and M_u have been determined using the computer program FTBTM [14]. The variations of C_{bm} with the end moment ratio β_m for segments with either end moments M and $\beta_m M$, or with central concentrated load Q at the shear centre and restraining moments $3\beta_m QL/16$ at one end, or with central concentrated load Q at the shear centre and restraining moments $\beta_m QL/8$ at both ends are shown in Fig. 7. These accurate values are somewhat different from the approximations derived from [5].

3.3 EFFECTS OF TAPER

The effects of taper on the lateral buckling of a web-tapered beam segment without end restraints (k = 1) may be determined from

$$C_{bt} = M_t / (C_{bm} M_u) \tag{6}$$

in which M_t is the maximum moment in a simply supported tapered segment with the actual moment distribution. Values of M_t and M_u have been determined using the computer program FTBTM [14] and values of C_{bm} from Fig. 7. The variations of C_{bt} with the web depth ratio b_{wL} / b_{w0} for segments with either end moments M and βM which cause single curvature bending, or with central concentrated load Q at the shear centre and restraining moments $3\beta_m QL/16$ at one end, are shown in Fig. 8. The variations of C_{bt} with the web depth ratio b_{wL} / b_{w0} for segments with central concentrated load Q at the shear centre and restraining moments $3\beta_m QL/16$ at one end are very small, because the use of the mid-segment values of the section properties leads to very close approximations for tapered segments. This is in contrast with the significant corrections for taper which must be used when the properties of the smaller end section are used [1]. The values of C_{bt} for segments with end moments M and βM which cause single curvature bending are more variable, but not greatly so.

3.4 EFFECTS OF END RESTRAINTS

The effects of end restraints on the lateral buckling of a web-tapered beam segment may be determined by first using the computer program FTBTM [14] to determine the value of M_t. These effects can be expressed as effective length factors k obtained by substituting the values of M_t, C_{bm}, and C_{bt} in the rearrangement of Equation 4 as

$$\left\{ \frac{(M_t / C_{bm} C_{bt})^2 L^2}{\pi^2 EI_{yL/2} GJ_{L/2}} \right\} k^4 - k^2 - \left\{ \frac{\pi^2 EI_{wL/2}}{GJ_{L/2} L^2} \right\} = 0 \tag{7}$$

and solving. A series of web-tapered beams with end moments 0 (at the smaller end) and M (at the larger end) has been analysed for the effects of equal stiffness restraints acting on both flanges at the loaded (larger) end. These restraints are equivalent to minor axis rotational and warping restraints of stiffness α_{Ry} and $\alpha_{Ry} b_w^2/4$ respectively. The variations of the effective length factors k for these beams with the dimensionless restraint stiffness parameters $(\alpha_{Ry} / 2EI_{yL/2}) / (1 + \alpha_{Ry} / 2EI_{yL/2})$ and the web depth ratio bw_L / bw_0 are shown in Fig. 9. Also shown in Fig. 9 are the corresponding variations for uniform beams in uniform bending with restraints at one or both ends. The values for the tapered beams vary little with the web depth ratio bw_L / bw_0, and generally lie between these latter variations.

3.5 INTERACTION BUCKLING OF TAPERED BRACED BEAMS

A symmetrical three segment tapered braced beam is shown in Fig. 5b. Lateral deflections and twist rotations are prevented at the segment ends by the supports and braces. The interaction buckling of a number of these beams with different segment length ratios L_{23}/L_{12} has been analysed using the computer program FTBTM [14]. The dimensionless elastic buckling moments M/M_{120} (in which M_{120} is the elastic buckling moment of an unrestrained end segment 12) are compared in Fig. 10 with the lower bound solutions [16] which ignore interactions between the adjacent segments. Also shown in Fig. 10 are upper bound solutions obtained by assuming that minor axis rotations and warping are prevented at the brace points. The dimensionless elastic buckling moments M/M_{120} lie between the upper and lower bounds. They decrease from the upper bound values towards the lower bound values as the length ratio L_{23}/L_{12} increases towards the zero interaction value L_{23}/L_{12} = 0.65 approximately, and then approach asymptotically towards the upper bound values.

Approximate interaction buckling moments may be determined by adapting the procedure for uniform beams discussed in Section 2.2 above. For this adaptation, the mid-segment values are used for the section properties in Steps 1, 6, 7 and 8, and Equation 2 is replaced by Equation 4 in Step 4 (with k = 1) and Step 10. These approximations are also shown in Fig. 10, and are either conservative (for $0 < L_{23}/L_{12} < 0.65$) or in close agreement with the FTBTM values.

3.6 INTERACTION BUCKLING OF TAPERED CONTINUOUS BEAMS

A symmetrical two span tapered continuous beam is shown in Fig. 5c. Lateral deflections and twist rotations are prevented at the supports and load points. The interaction buckling of a number of these beams with different span length ratios L_{23}/L_{12} has been analysed using the computer program FTBTM [14]. The dimensionless elastic buckling moments M/M_{120} (in which M_{120} is the elastic buckling moment of an unrestrained end segment 12) are compared in Fig. 11 with the lower bound solutions [16] which ignore interactions between the adjacent segments. Also shown are upper bound solutions obtained by assuming that minor axis rotations and warping are prevented at the load points. The dimensionless elastic buckling moments M/M_{120} lie between the upper and lower bounds. They decrease from the upper bound values towards the lower bound values as the length ratio L_{23}/L_{12} increases towards the zero interaction value L_{23}/L_{12} = 1.21 approximately, and then approach asymptotically towards the upper bound values.

Approximate interaction buckling moments may be determined by using the adaptation discussed in Section 3.5 above of the procedure for uniform beams discussed in Section 2.2. These approximations are also shown in Fig. 11, and are either conservative (for $0 < L_{23}/L_{12} < 1.21$) or in close agreement with the FTBTM values.

It can be seen from Fig. 11 that the accurate buckling moment for the continuous beam at the zero interaction length ratio of L_{23}/L_{12} = 1.21 is approximately 10% higher than those for the separate segments 12 and 23. The reason for this is illustrated in Fig. 12, which shows the buckled shapes for the top and bottom flange deflections u_t and u_b. The continuity of these deflections at the segment junction 2 displayed for the half continuous beam 123 (Fig. 12b) is not achieved between the separate segments 12 and 23 (Fig. 12 c and d). The achievement of continuity in the continuous beam is due to non-zero internal restraints between the segments, which are assumed to be zero for the zero interaction approximations. These non-zero restraints increase the buckling load of the continuous beam 123 above the zero interaction approximations.

4 CONCLUSIONS

The elastic lateral buckling of braced and continuous beams is affected by the separate effects of moment distribution, taper, and restraints between adjacent segments. Segment moment distribution effects are commonly allowed for in design codes [5] by using C_{bm} factors to multiply the classic lateral buckling moments M_u (Equation 1) of simply supported uniform beam segments in uniform bending. Approximate methods for determining C_{bm} are available [5, 11, 12], or finite element computer buckling programs [15] may be used.

Taper effects for linearly web tapered segments may conveniently be allowed for by multiplying the classic lateral buckling moments $C_{bm} M_u$ computed using the mid-segment section properties by taper factors C_{bt}. Values of C_{bt} for a number of different segment moment distributions have been determined using the computer program FTBTM [14]. The variations of these with the degree of taper is comparatively small, in contrast to those based on the use of the properties of the smallest section [1].

The effects of end restraints on buckling are commonly allowed for by using effective length factors k to multiply the segment length L used in the formulation (Equation 1) of the elastic lateral buckling of uniform segments in uniform bending. Lower bound solutions [16] may be obtained by setting the effective length factors of the segments equal to 1.0 so that the effects of end restraints are ignored. Improved solutions can be obtained by adapting a previously developed procedure [9-12] for uniform beams. In this procedure, the critical segment is identified as the one which would buckle first in the absence of restraints between segments. The increased buckling resistance of the critical segment is found by approximating the restraining stiffnesses of the adjacent segments, and by using these in an established method of determining the effective length factors for elastically restrained compression members.

The adapted procedure has been used to approximate the elastic buckling resistances of series of tapered braced and continuous beams, and has been shown to produce solutions which are conservative or close to the accurate solutions.

5 REFERENCES

[1] Lee, GC, Morrell, ML, and Ketter, RL, Design of tapered members, *Welding Research Council Bulletin,* 1972; 173: 1-32.

[2] Morrell, ML and Lee, GC, Allowable stress for web tapered beams with lateral restraints, *Welding Research Council Bulletin,* 1974; 192: 1-12.

[3] Polyzois, D and Raftoyiannis, IG, Lateral-torsional stability of steel web-tapered !-beams, *Journal of Structural Engineering*, ASCE, 1998; 124(10): 1208-15.

[4] Andrade, A, Providencia, P, and Camotim, D, *Computers and Structures*, 2010; 88: 1179-96.

[5] AISC, *LRFD specification for structural steel buildings*, American Institute of Steel Construction, 1993.

[6] Stability Research Council, *Guide to stability design criteria for metal structures,* John Wiley and Sons, New York, 1988.

[7] Trahair NS, Interaction buckling of narrow rectangular continuous beams, *Civ. Engg Trans*, Inst. Engrs, Aust., 1968; CE10 (2): 167-72.

[8] Trahair NS, Elastic stability of continuous beams, *Journ. Struct. Dvn*, ASCE, 1969; 95 (ST6): 1295-312.

[9] Nethercot DA and Trahair NS, Lateral buckling approximations for elastic beams, *The Structural Engineer*, 1976; 54 (6): 197-204.

[10] Nethercot DA and Trahair NS, Lateral buckling calculations for braced beams, *Civ. Engg Trans*, Inst. Engrs, Aust., 1977; CE19 (2): 211-4.

[11] Trahair, NS, *Flexural-torsional buckling of structures*, E & FN Spon, London, 1993.

[12] Trahair, NS, Bradford, MA, Nethercot, DA, and Gardner, L, *The behaviour and design of steel structures to EC3*, Taylor and Francis, London, 2008.

[13] Timoshenko, SP and Gere, JM, *Theory of elastic stability*, McGraw-Hill, N.Y.,1961.

[14] Trahair, NS, Bending and buckling of tapered steel beam structures, *Engineering Structures, in press.*

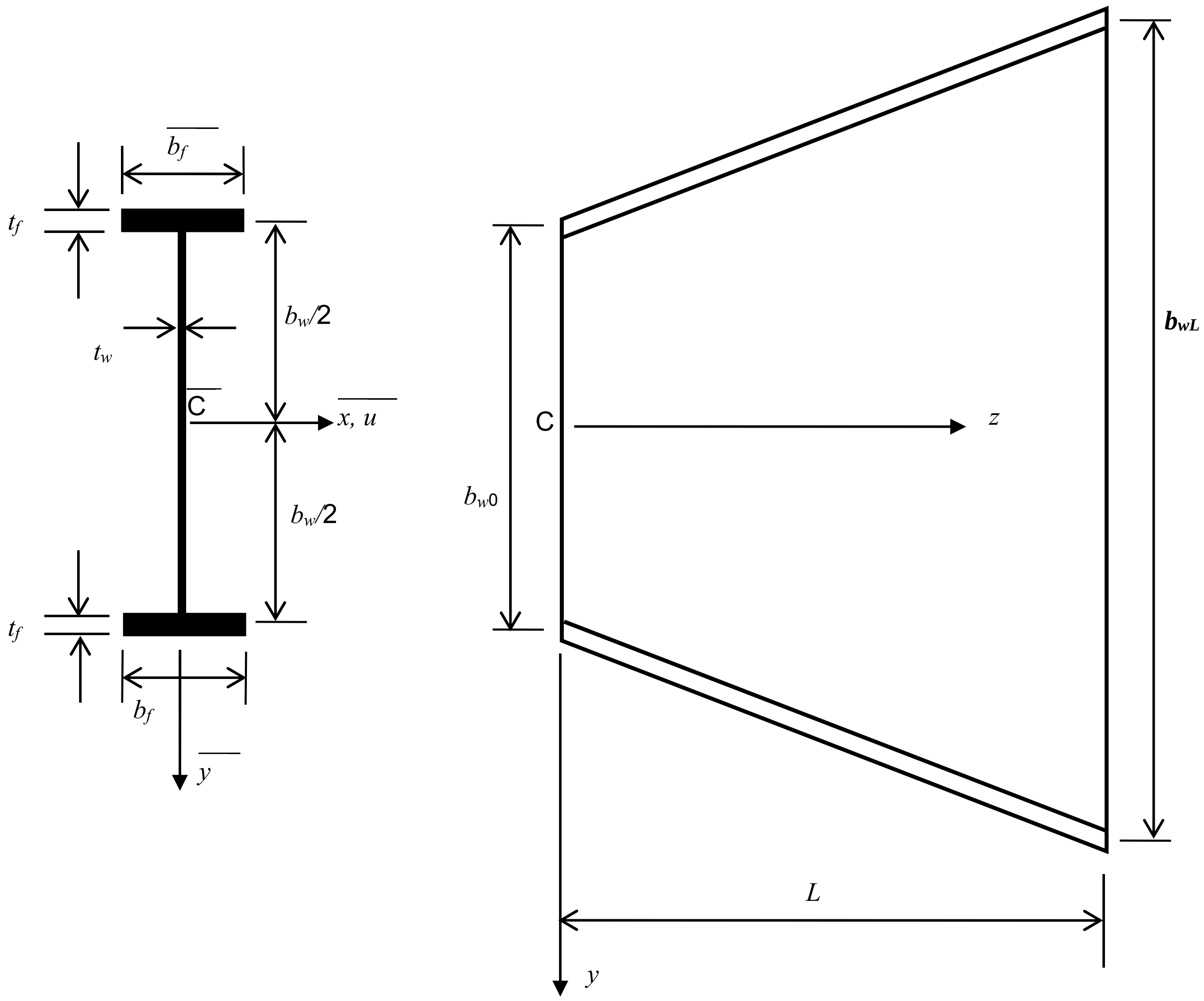
[15] Papangelis, JP, Trahair, NS, and Hancock, GJ, Elastic flexural-torsional buckling of structures by computer, *Computers and Structures*, 1998; 68: 125-137.

[16] Salvadori, MG, Lateral buckling of beams of rectangular cross-section under bending and shear, *Proceedings of the First US National Congress of Applied Mechanics*, 1951: 403-5.

6 NOTATION

$b_{f,w}$	Flange width and web depth
$b_{w0,L}$	Values of b_w at segment ends
$b_{wL/2}$	Value of b_w at mid-segment
C_{bm}	Moment distribution factor
C_{bt}	Taper factor
E	Young's modulus of elasticity
G	Shear modulus of elasticity
$G_{A,B}$	End stiffness ratios
I_y	Second moment of area about y axis
$I_{ym,R}$	Values of I_y for critical and restraining segments
I_w	Warping section constant
J	Uniform torsion constant
k	Effective length factor
L	Segment length
L_R	Length of restraining segment
M	Moment
M_t	Elastic buckling moment of a tapered segment
M_u	Elastic buckling moment of a uniform simply supported segment in uniform bending
M_{120}	Elastic buckling moment of an unrestrained segment 12
Q	Concentrated load
Q_m	Value of Q for critical segment
$Q_{RA,B}$	Values of Q for adjacent segments
$t_{f,w}$	Flange and web thicknesses
$u_{t,b}$	Buckling displacements of top and bottom flanges
x, y	Principal axis coordinates
z	Distance along segment
$\alpha_{RA,B}$	Stiffnesses of adjacent segments
$\alpha_{Ry,w}$	Flexural and warping restraint stiffnesses
β, β_m	Moment factors

FIGURES



(a) Section (b) Segment

Fig. 1 Geometry of a Web-Tapered Segment

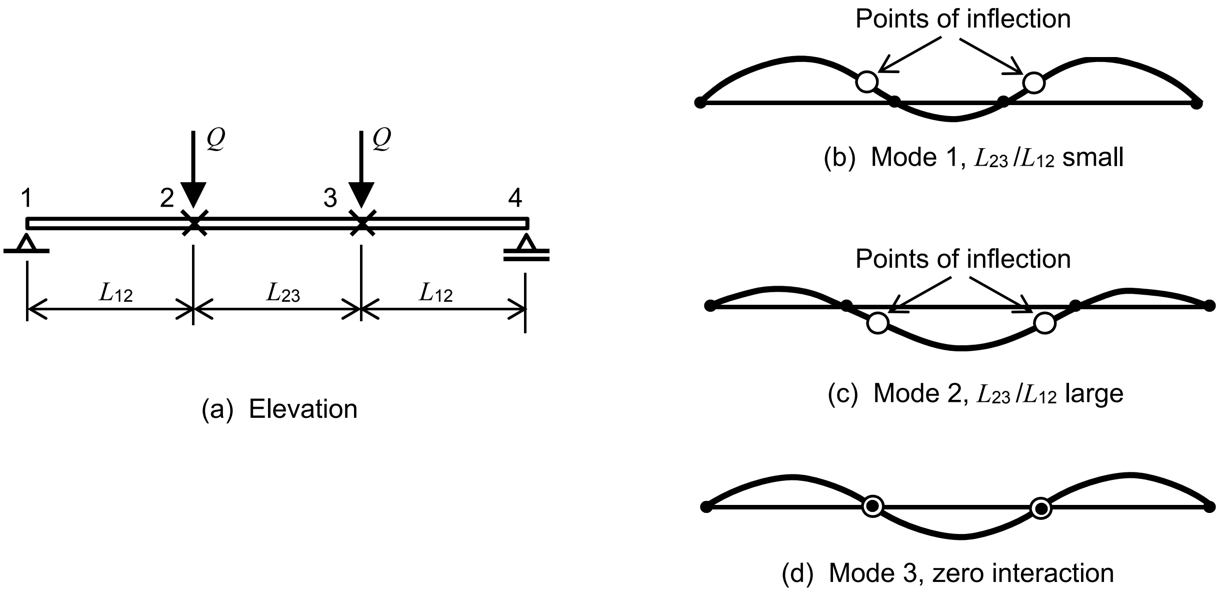


Fig. 2 Buckling Modes of Braced Beams

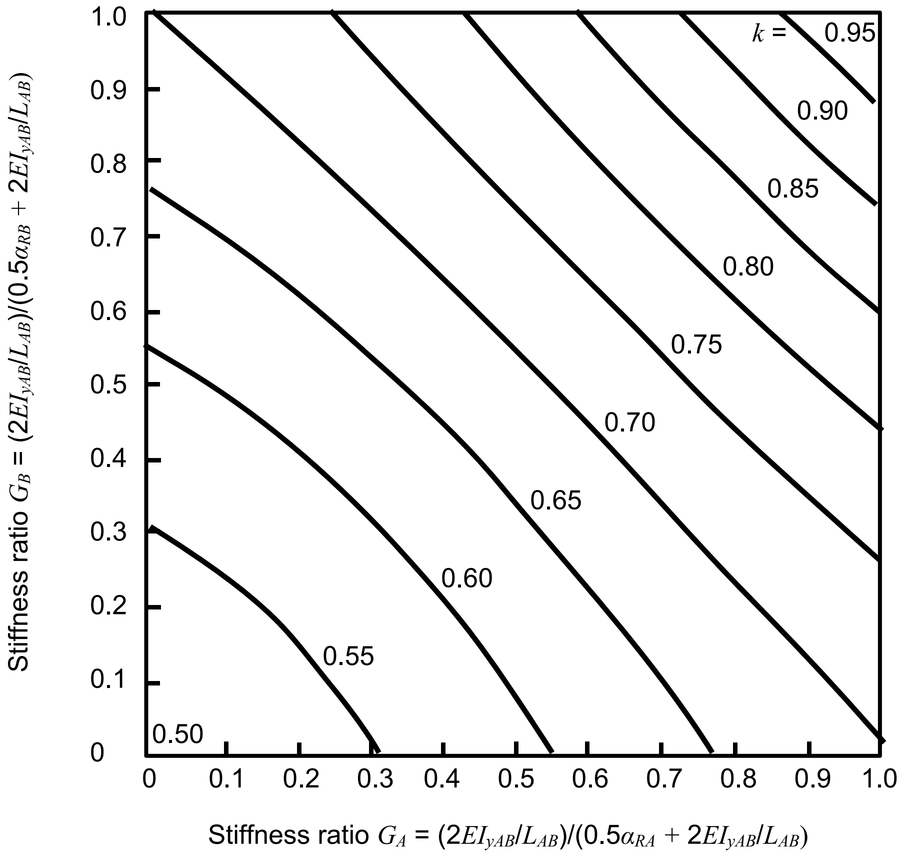
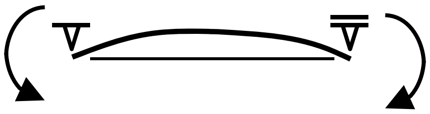
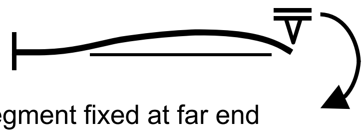


Fig. 3 Effective Length Factors k



(a) Segment providing equal end restraints $\alpha_R = \frac{2EI_{yR}}{L_R}\left(1 - \frac{Q_m}{Q_R}\right)$

(b) Segment hinged at far end $\alpha_R = \frac{3EI_{yR}}{L_R}\left(1 - \frac{Q_m}{Q_R}\right)$



(c) Segment fixed at far end $\alpha_R = \frac{4EI_{yR}}{L_R}\left(1 - \frac{Q_m}{Q_R}\right)$

Fig. 4 Stiffness Approximations

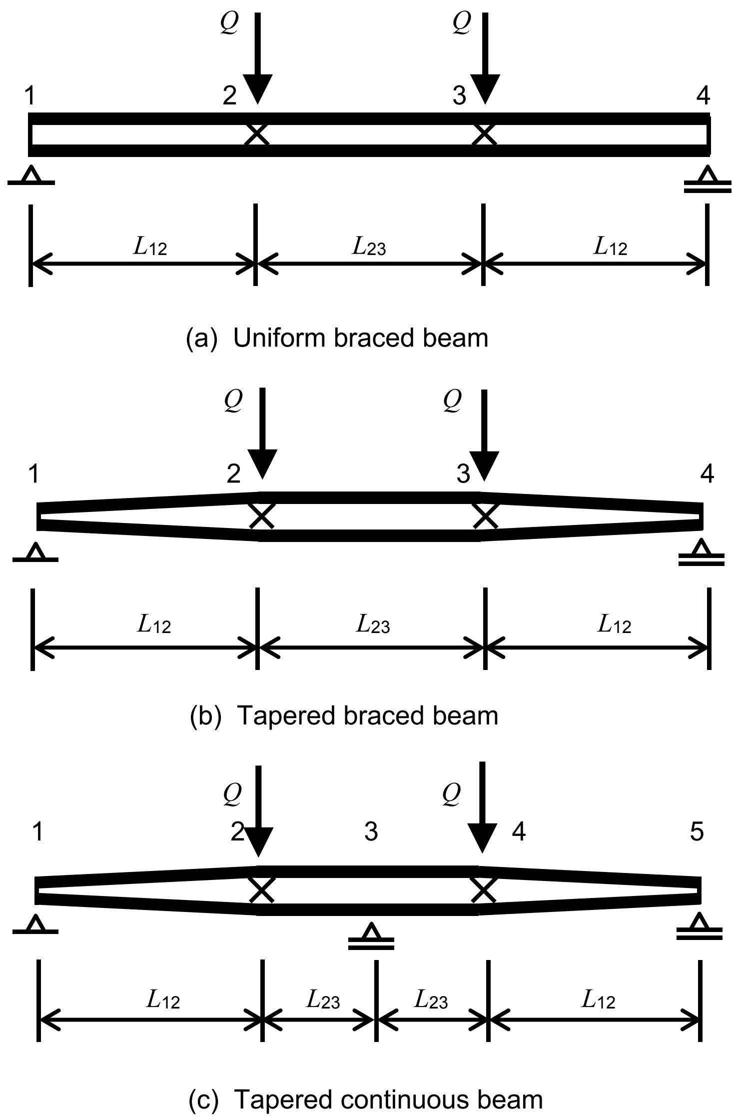


Fig. 5 Braced and Continuous Beams

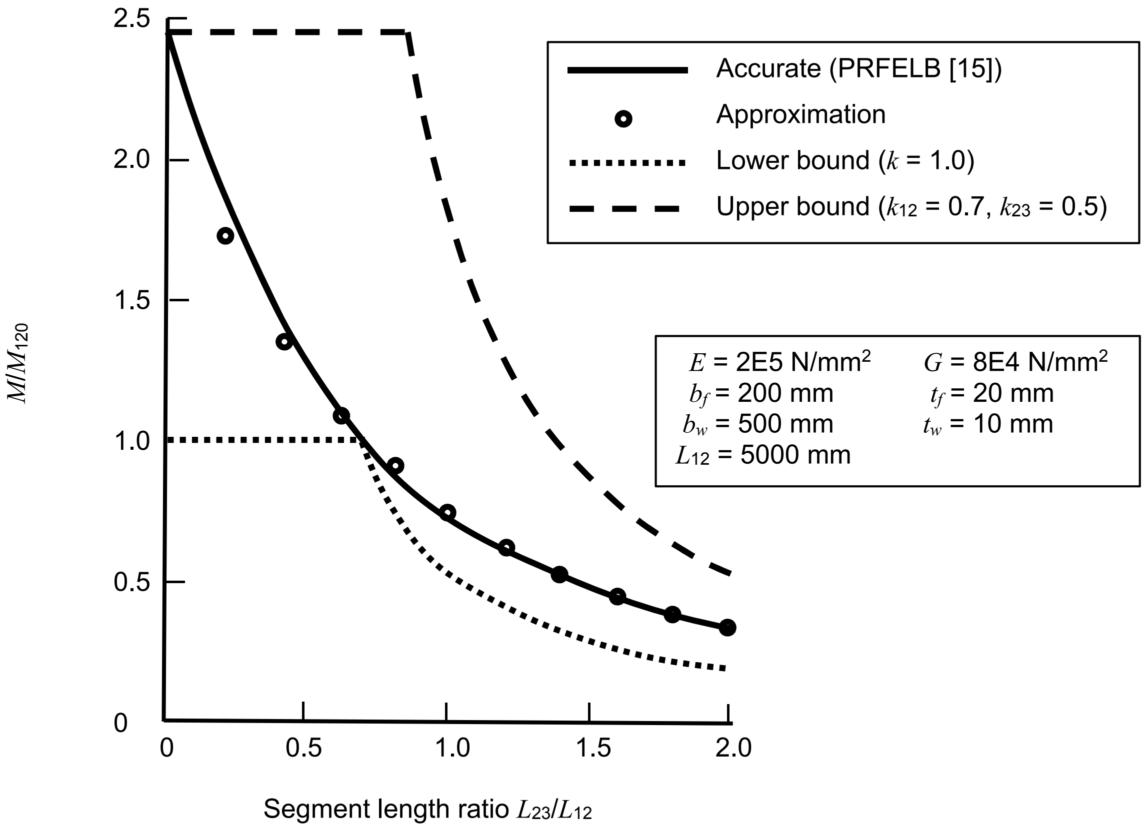


Fig. 6 Buckling of Uniform Braced Beams (Fig. 5a)

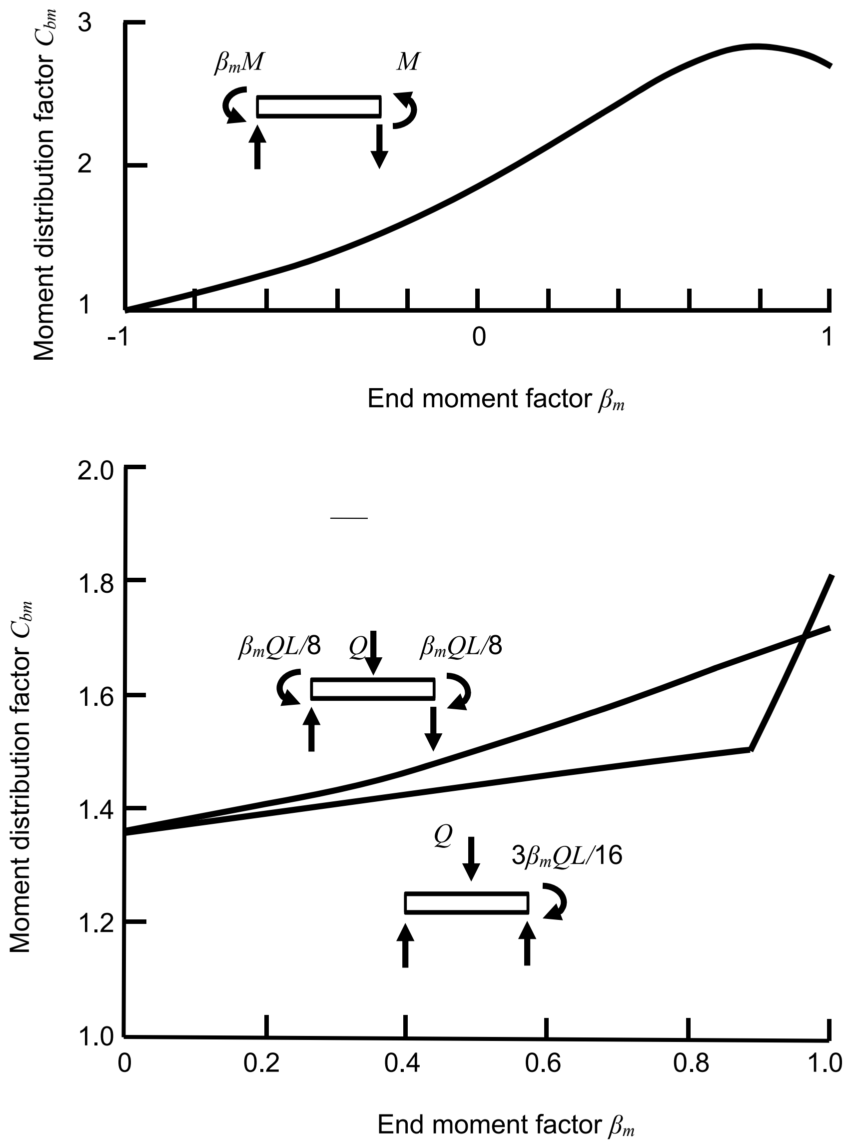


Fig. 7 Effect of Moment Distribution

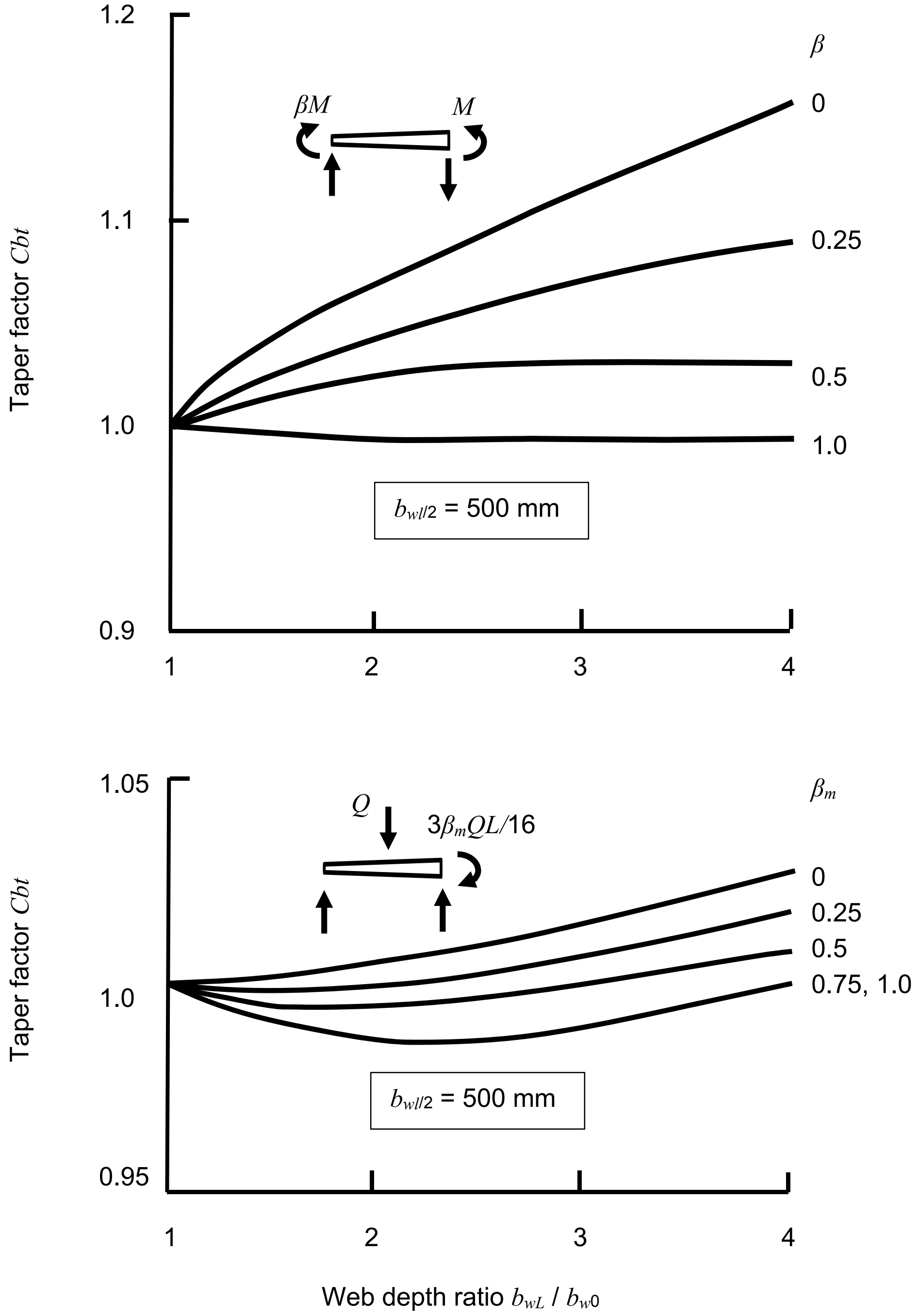
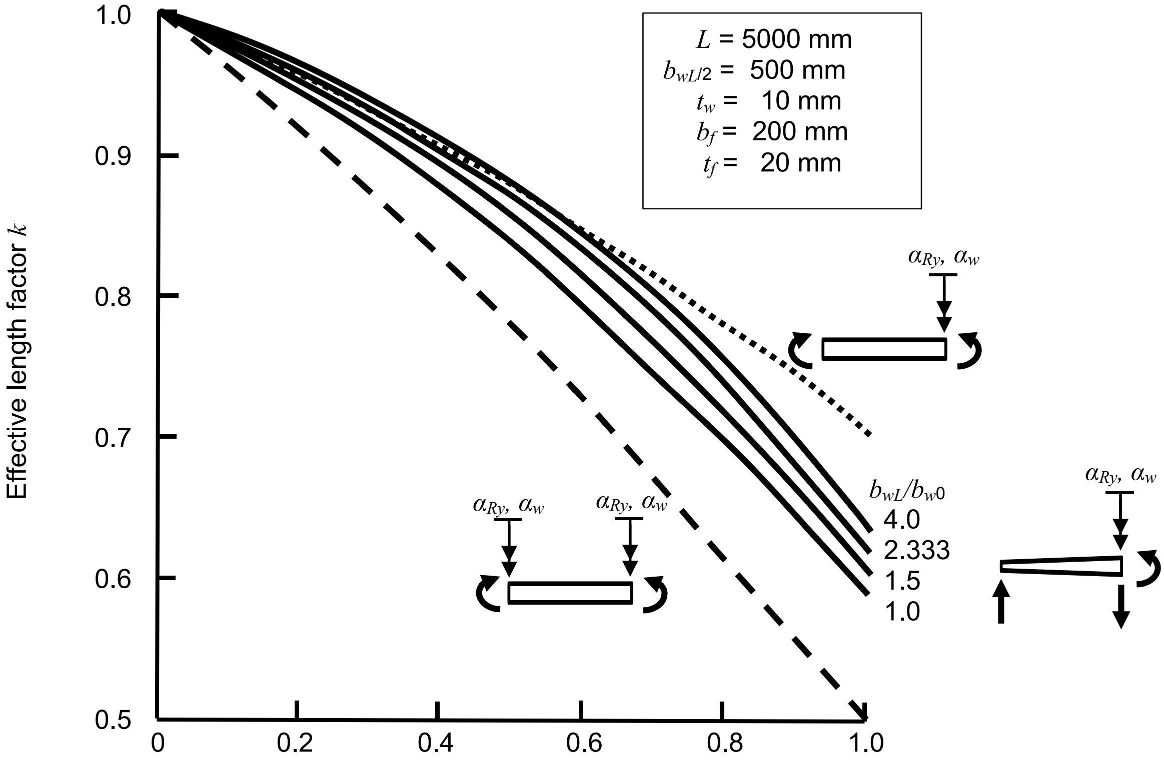


Fig. 8 Effect of Taper



$$\frac{\alpha_{Ry} L / 2EI_{yL/2}}{1 + \alpha_{Ry} L / 2EI_{yL/2}}$$

Fig. 9 Effective Length Factors k of Tapered Segments

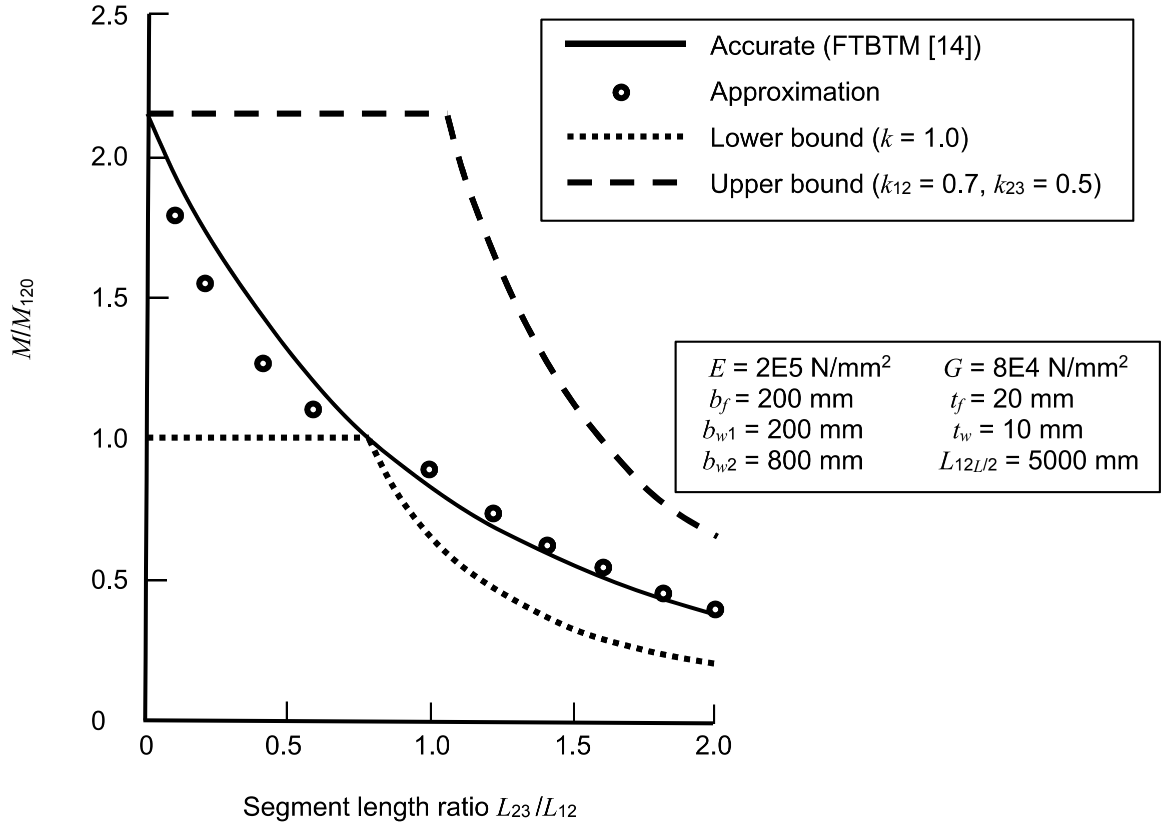


Fig. 10 Buckling of Tapered Braced Beams (Fig. 5b)

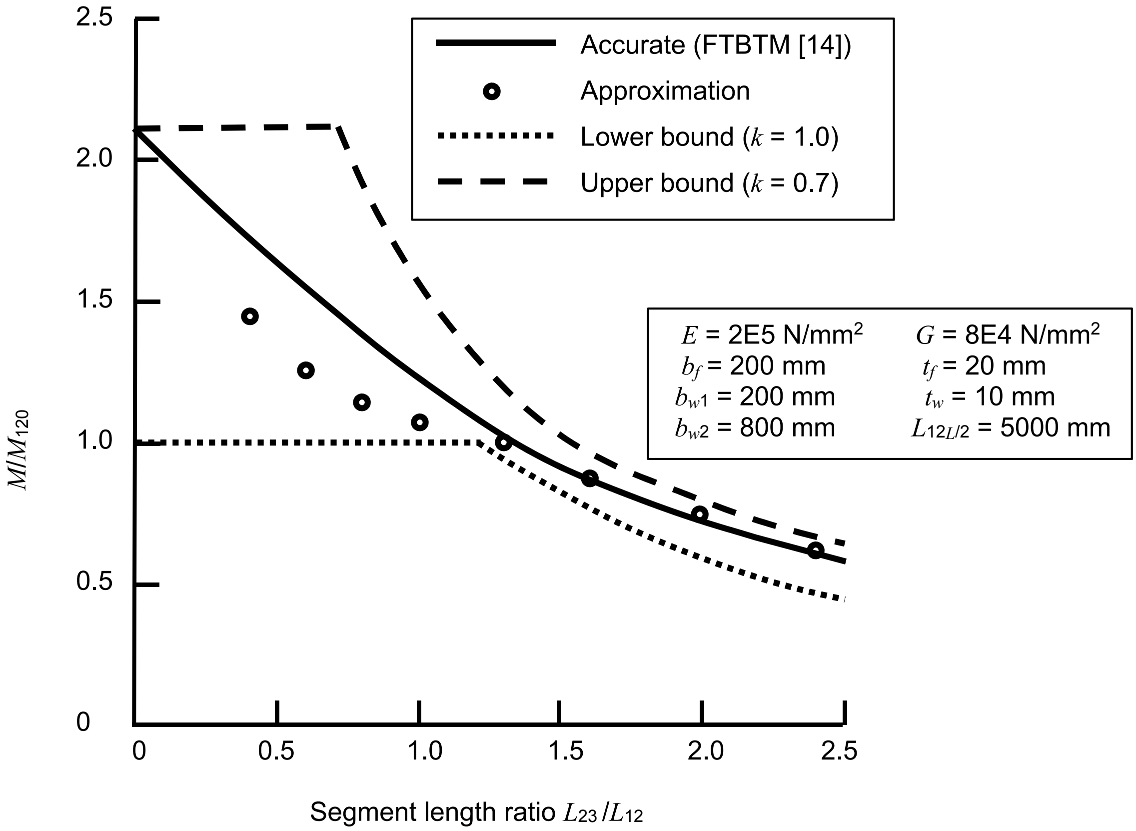


Fig. 11 Buckling of Tapered Continuous Beams (Fig. 5c)

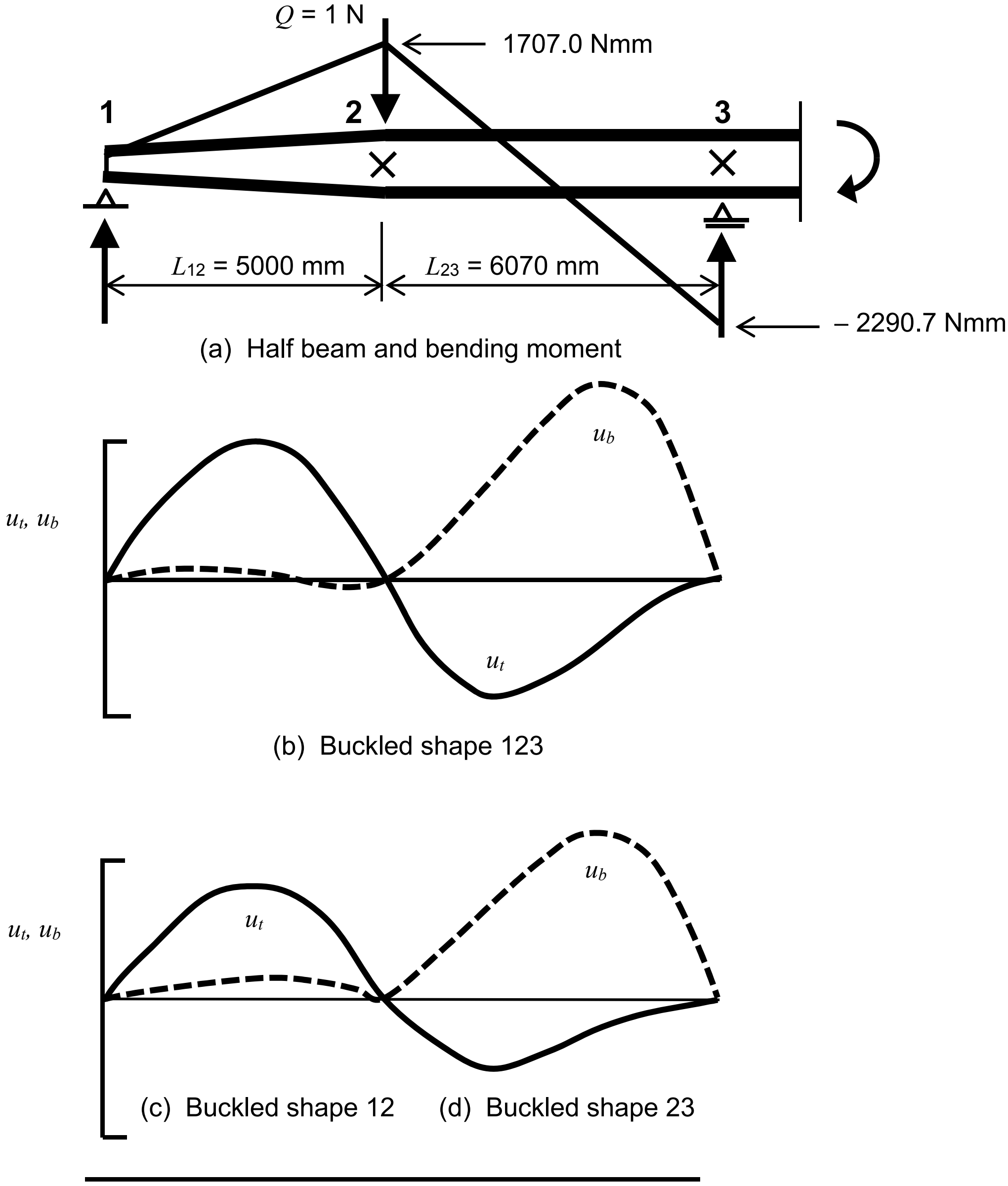


Fig. 12 "Zero Interaction" Buckled Shapes