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BENDING AND BUCKLING OF TAPERED STEEL BEAM STRUCTURES

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ABSTRACT

This paper describes an efficient finite element method of analysing the elastic in-plane bending and out-of-plane buckling of indeterminate beam structures whose members may be tapered and of mono-symmetric I cross-section. The structure's loading includes concentrated moments and concentrated or uniformly distributed off-axis transverse and longitudinal forces, and its deformations may be prevented or resisted by concentrated or continuous rigid or elastic off-axis restraints.

Tapered finite element formulations are developed by numerical integration instead of the closed forms often used for uniform elements. Difficulties in specifying the load positions for tapered mono-symmetric members caused by the variations of the centroidal and shear centre axes are avoided by using an arbitrary axis system based on the web mid-line. Account is taken of additional Wagner torque terms arising from the inclination of the shear centre axis.

A computer program based on this method is used to analyse a number of examples of the elastic in-plane bending of tapered cantilevers and built-in beams, and very close agreement is found between its predictions and closed form solutions.

The program's predictions of the elastic out-of-plane flexural-torsional buckling of a large number of uniform and tapered doubly and mono-symmetric beams and cantilevers under various loading and restraint conditions are generally in close agreement with existing predictions and test results. The common approximation in which tapered elements are replaced by uniform elements is shown to converge slowly, and to lead to incorrect predictions for tapered mono-symmetric beams.

KEYWORDS

Beam-columns, Bending, Buckling, Mono-symmetry, Steel, Structures, Taper, Torsion

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1 INTRODUCTION

The distributions of axial forces and bending moments in steel beam structures caused by the applied loads need to be determined before their elastic flexural-torsional buckling resistances can be analysed [1]. While these may be easily found for indeterminate structures with uniform members and for determinate structures with tapered members, they are less easily found for indeterminate tapered structures. It is noteworthy that most if not all of the early and later studies of the elastic buckling of tapered structures cited in [2, 3] and more recently in [4, 5] are for statically determinate beams or cantilevers.

The finite element stiffness matrices used to analyse the elastic in-plane bending and out-of-plane buckling of a tapered beam structure depend on the variations of the section properties along each element of which a structure is composed, and in general require numerical integrations to be made, in place of the use of the common closed form solutions for uniform elements.

When the tapered elements are of mono-symmetric cross-section (Fig.1a), the centroidal and shear centre axes wander [6] along the element length (Fig. 1b). If the problem is simplified by using an arbitrary axis system [7], then modifications need to be made for the eccentricity of the centroid from the chosen axis to allow for the moments induced by eccentric longitudinal loads or restraints.

This paper describes a finite element method of analysing the elastic in-plane bending and out-of-plane buckling of indeterminate beam structures whose members may be tapered and of mono-symmetric I-section. The structure's loading includes concentrated moments and concentrated or uniformly distributed off-axis transverse and longitudinal forces, and its deformations may be prevented or resisted by concentrated or continuous rigid or elastic off-axis restraints. The method is superior to the commonly used approximate method of replacing a tapered element by a large number of uniform elements, with a significant reduction in the number of elements required to obtain an accurate solution.

While this method gives solutions for bending and out-of-plane buckling, it also allows the analysis of the inelastic buckling of uniform steel beams for which non-uniform yielding along the beam causes both the in-plane and out-of-plane properties to vary, so that the member is effectively tapered [8]. It may be noted that the combination of the anti-symmetrical stresses due to the applied loading with the symmetrical residual stresses in a doubly symmetric I-section beam causes the effective section to become mono-symmetric [9].

A computer program FTBTM based on the method described in this paper is used to analyse a number of examples of the bending and out-of-plane buckling of tapered beam structures, and its predictions are compared with previous theoretical and experimental values.

2 TAPERED MONOSYMMETRIC MEMBERS

The member cross-section shown in Fig. 1a has top and bottom flange and web widths b_t, b_b, and b_w and thicknesses t_t, t_b, and t_w, respectively. The element shown in Fig.1b may be linearly tapered in all cross-section dimensions, and while thickness tapering is very rare in practice, depth tapering is comparatively common. The member has an arbitrary but convenient [7] longitudinal axis Oz which is the locus of the web mid-heights, and an Oy axis which coincides with the web centreline. The section properties required for an in-plane analysis are

$$I_0 = A = \int_A dA = b_t t_t + b_b t_b + b_w t_w \tag{1}$$

$$I_1 = y_c A = \int_A y dA = -b_t t_t b_w / 2 + b_b t_b b_w / 2 \tag{2}$$

$$I_2 = I_x - y_c^2 A = \int_A y^2 dA = b_t t_t b_w^2 / 4 + b_b t_b b_w^2 / 4 + b_w^3 t_w / 12 + b_w t_w y_c^2 \tag{3}$$

in the coordinate y_c of the centroid is given by

$$y_c = \frac{b_w}{2} \frac{b_b t_b - b_t t_t}{A} \tag{4}$$

The coordinate of the shear centre S (Fig. 1a) is defined by

$$y_s = y_c + y_0 = \frac{b_w}{2}\frac{b_b^3 t_b - b_t^3 t_t}{b_b^3 t_b + b_t^3 t_t} \tag{5}$$

Concentrated transverse and longitudinal loads Q (at y_Q), N (at y_N), and moments M may act at a point z along the axis Oz as shown in Fig. 2a, as well as rigid or elastic restraints (at y_R). Distributed transverse and longitudinal loads q (at y_q) and n (at y_n) may act along a length L of the element as shown in Fig. 2b, as well as elastic restraints (at y_r).

3 IN-PLANE ANALYSIS AND BEHAVIOUR

3.1 FINITE ELEMENT FORMULATION

A finite element method of carrying out the in-plane (pre- flexural-torsional buckling) linear elastic analysis of uniform beam-columns is detailed in [1]. In this, the equilibrium equations are represented by

$$K_i \Delta_i = Q_{in} + Q_{ie} \tag{6}$$

in which K_i is the in-plane global stiffness matrix, Δ_i are the in-plane global nodal deflections and rotations, Q_{in} are the nodal loads, and Q_{ie} are the nodal loads equivalent to the loads distributed along the elements. The global stiffness matrix is given by

$$K_i = \sum_e T_{ie}^T k_{ie} T_{ie} \tag{7}$$

in which k_{ie} is an element stiffness matrix and T_{ie} is a matrix which transforms the element in-plane end deflections and rotations δ_{ie} into the corresponding global deflections and rotations Δ_i according to

$$\delta_{ie} = T_{ie}\Delta_i \tag{8}$$

The element stiffness matrix is given by

$$k_{ie} = C_i^{-T}\int_0^L B_i^T D_i B_i dz\, C_i^{-1} \tag{9}$$

in which C_i is a matrix related the cubic and linear functions used to approximate the element deflections v and w, B_i is a matrix related to the length L of the element and the distance z along it, and D_i is given by

$$D_i = \begin{bmatrix} EI_x & 0 \\ 0 & EA \end{bmatrix} \tag{10}$$

in which E is the Young's modulus of elasticity.

When the typical element is of mono-symmetric cross-section it is convenient [7] to use the arbitrary axis system of Fig. 1. Thus all deflections v, and w are referred to these axes, as are the nodal loads and moments N, Q, and M and distributed loads n and q. When N or n act eccentrically to the Oz axis, then they are replaced by their static equivalents referred to the Oz axis.

The D matrix then needs to be replaced by

$$D = \begin{bmatrix} EI_2 & EI_1 \\ EI_1 & EA \end{bmatrix} \tag{11}$$

in which the values of A, I_1, and I_2 are given in Equations 1-3.

When the element is tapered so that A, I_1, and I_2 vary with z, it is difficult to develop exact formulations for the integrals required in Equation 9. Instead, it is much simpler to use the approximate but highly accurate 4-point Gaussian integration [10].

The effects of rigid nodal restraints which prevent the corresponding nodal deflections or rotations are allowed for by making appropriate modifications to the equilibrium Equations 6. When a rigid restraint against deflection parallel to the Oz axis acts eccentrically, then it is convenient to transform the global Δ_i so that they include the deflection at the eccentric restraint, as in [11]. The effects of nodal elastic restraints against the in-plane deflections and rotations may be allowed for by augmenting the global stiffness matrix K_i by the stiffnesses of the elastic restraints, as in [1]. The effects of uniformly distributed element restraints may be allowed for by augmenting the element stiffness matrix k_{ei} by the stiffnesses of the elastic restraints, as in [1].

A MATLAB [12] computer program FTBTM for the elastic flexural-torsional buckling of tapered members has been written which includes the above formulation for the in-plane (pre- flexural-torsional buckling) analysis. Examples of the application of this program are discussed in the following sub-sections.

3.2 TAPERED CANTILEVERS

A tapered cantilever of length L shown in Fig. 3a has end actions M or Q or uniformly distributed load q. The second moment of area I_x varies linearly along its length according to

$$I_x = I_0(1+\alpha\, z/L) \tag{12}$$

in which α is a constant which defines the degree of taper. The in-plane deformations of the cantilever are analysed in Appendix A. The variations of the dimensionless in-plane end (z = 0) deflections v_0/v_{0u} (in which v_{0u} is the value of v_0 for a uniform beams with α = 0) with α are shown in Fig. 4. These are in very close agreement with the values determined by the computer program FTBTM, as demonstrated in Fig. 4.

3.3 BUILT-IN BEAMS

A built-in tapered beam of length L with a uniformly distributed load q is shown in Fig. 3b. The second moment of area I_x varies along its length according to Equation 12. The redundant end actions M_0 and Q_0 are analysed in Appendix B. The variations of the dimensionless redundant actions $12M_0/qL^2$ and $2Q_0/qL$ with α are shown in Fig. 5. These are in very close agreement with the values determined by the computer program FTBTM, as demonstrated in Fig. 5.

4 FLEXURAL-TORSIONAL BUCKLING

4.1 BUCKLING ANALYSIS

4.1.1 Uniform members

A finite element method of carrying out the out-of-plane flexural-torsional buckling analysis of uniform beam-columns is detailed in [1]. In this, the energy equation at elastic buckling is represented by

$$\frac{1}{2}\Delta_o^T(K_o+\lambda G_o)\Delta_o = 0 \tag{13}$$

in which K_o and G_o are the out-of-plane global stiffness and stability matrices, Δ_o are the global out-of-plane nodal deflections and rotations, and λ is the buckling load factor.

The global stiffness matrix is given by

$$K_o = \sum_e T_{oe}^T k_{oe} T_{oe} \tag{14}$$

in which k_{oe} is an element stiffness matrix and T_{oe} is a matrix which transforms the element end deflections and rotations δ_{oe} into the corresponding global deflections and rotations Δ_o according to

$$\delta_{oe} = T_{oe}\Delta_o \tag{15}$$

The element stiffness matrix k_{oe} may be obtained from the element strain energy increase during buckling

$$\frac{1}{2}U_e = \frac{1}{2}\int_0^L \varepsilon_u{}^T D_u \varepsilon_u dz \tag{16}$$

in which

$$\varepsilon_u = \{u_s'', \phi', \phi''\}^T \tag{17}$$

and

$$D_u = \begin{bmatrix} EI_y & 0 & 0 \\ 0 & GJ & 0 \\ 0 & 0 & EI_w \end{bmatrix} \tag{18}$$

in which ' ≡ d/dz, u_s is the shear centre displacement, ϕ is the twist rotation, G is the shear modulus of elasticity, I_y is the second moment of area about the y axis of symmetry, and J and I_w are the uniform torsion and warping section constants. The element stiffness matrix can be expressed as

$$k_{oe} = C_u^{-T}\int_0^L B_u^T D_u B_u dz\, C_u^{-1} \tag{19}$$

in which C_u is a matrix related the cubic functions used to approximate the element out-of-plane deflections and rotations and B_u is a matrix related to the length L of the element and the distance z along it.

The out-of-plane global stability matrix is given by

$$G_o = \frac{1}{2}\sum_n Q(y_Q - y_s)\phi_n^2 + \sum_e T_{oe}^T g_{oe} T_{oe} \tag{20}$$

in which the first term represents the increase in potential energy during buckling of the initial nodal loads Q which result from their distance ($y_Q - y_s$) below the shear centre, and the second term contains the element stability matrix g_{oe}. It may be obtained from the increase in the element potential energy

$$\frac{1}{2}\lambda V_e = \frac{\lambda}{2}\int_0^L \varepsilon_v^T D_v \varepsilon_v dz \tag{21}$$

in which

$$\varepsilon_v = \{u_s', u_s'', \phi, \phi', \phi''\}^T \tag{22}$$

and

$$D_v = -\begin{bmatrix} N_z & 0 & 0 & N_z y_0 & 0 \\ 0 & 0 & -M_x & 0 & 0 \\ 0 & -M_x & -q\,(y_q - y_s) & 0 & 0 \\ N_z y_0 & 0 & 0 & N_z(I_p / A + y_0^2) - M_x\beta_x & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{23}$$

in which N_z and M_x, are the initial internal axial compression and moment, q is the distributed vertical load, ($y_q - y_s$) is the distance below the shear centre at which q acts, I_p is the polar moment of area, and β_x is the mono-symmetry constant [1]. The element stability matrix can be expressed as

$$g_{oe} = C_v^{-T}\int_0^L B_v^T D_v B_v dz\, C_v^{-1} \tag{24}$$

in which C_v is a matrix related the cubic functions used to approximate the element out-of-plane deflections and rotations, and B_v is a matrix related to the length L of the element and the distance z along it.

4.1.2 Tapered members

When the element is tapered, it is convenient to replace the shear centre deflection u_s and its derivatives by the web mid-height values

$$\begin{aligned} u &= u_s + y_s\phi \\ u' &= u_s' + y_s'\phi + y_s\phi' \\ u'' &= u_s'' + 2y_s'\phi' + y_s\phi'' \end{aligned} \tag{25}$$

When these are used, Equation 18 becomes

$$D_{ut} = \begin{bmatrix} (EI_t + EI_b) & (EI_t - EI_b)b_w' & (EI_t - EI_b)b_w/2 \\ (EI_t - EI_b)b_w' & GJ + (EI_t + EI_b)b_w'^2 & (EI_t + EI_b)b_w b_w'/2 \\ (EI_t - EI_b)b_w/2 & (EI_t + EI_b)b_w b_w'/2 & (EI_t - EI_b)b_w^2/4 \end{bmatrix} \tag{26}$$

in which I_t and I_b are the top and bottom flange second moments of area about the y axis, and Equation 23 becomes

$$D_{vt} = -\begin{bmatrix} N_z & 0 & -N_z y_s' & -N_z y_c & 0 \\ 0 & 0 & -M_x & 0 & 0 \\ -N_z y_s' & -M_x & N_z y_s'^2 - q(y_q - y_s) & N_z y_s'(y_s + y_c) + 2.5M_x y_s' & M_x y_s \\ -N_z y_c & 0 & N_z y_s'(y_s + y_c) + 2.5M_x y_s' & N_z(I_p/A + y_c^2) - M_x\beta_x & 0 \\ 0 & 0 & M_x y_s & 0 & 0 \end{bmatrix} \tag{27}$$

This contains additional terms which arise from the additional Wagner [6, 13] torque components $(-N_z y_0 y_s'\phi)$ and $(-M_x y_s'\phi)$ in

$$T_w = -N_z\{(I_p/A + y_0^2)\phi' + y_0 y_s'\phi\} + M_x\{\beta_x\phi' - y_s'\phi\} \tag{28}$$

caused by the slope y_s' of the shear centre axis. The derivation of the Wagner torque T_w is given in Appendix C.

When a rigid restraint against deflection parallel to the Ox axis or rotation about the Oy axis acts eccentrically, then it is convenient to transform the global Δ_o so that they include the deflection or rotation at the eccentric restraint, as in [11]. The effects of nodal elastic restraints against the out-of-plane deflections and rotations may be allowed for by augmenting the global stiffness matrix K_o by the stiffnesses of the elastic restraints, as in [1]. The effects of uniformly distributed element restraints may be allowed for by augmenting the element stiffness matrix k_{eo} by the stiffnesses of the elastic restraints, as in [1].

Examples of the application of a MATLAB [12] computer program FTBTM for the elastic flexural-torsional buckling of tapered members are discussed in the following sub-sections.

4.2 MONO-SYMMETRIC UNIFORM BEAMS

Flexural-torsional buckling predictions obtained using the computer program FTBTM have demonstrated good agreement with a wide range of independent theoretical [1, 14] and test results. An example [15] is shown in Fig. 6 for a mono-symmetric uniform cantilever with an end load acting at different distances ($y_Q - y_s$) below the shear centre (it may be noted that although the theory presented in Section 4.1 above was based on a mid-web axis system, the distance below the shear centre at which transverse loads Q, q act is used in Equations 20 and 27). There is close agreement with previous theoretical and test results.

4.3 DOUBLY SYMMETRIC TAPERED BEAMS

Flexural-torsional buckling predictions obtained using FTBTM for doubly symmetric tapered beams [2] are shown in Fig. 7. There is good agreement with previous theoretical and test results.

4.4 MONO-SYMMETRIC TAPERED BEAMS AND CANTILEVERS

Flexural-torsional buckling predictions obtained using FTBTM for mono-symmetric width-tapered beams [6] are shown in Fig. 8. There is good agreement with previous theoretical and test results. The convergence of the dimensionless buckling loads Q/Q_{test} with the number of elements is demonstrated in Fig. 9. It can be seen that convergence within about 2% is obtained using 4 elements, and that the converged values are within about 3% of the test values [6].

Also shown in Fig. 9 are the corresponding values for approximations for which the tapered elements are replaced by uniform elements. Not only is the convergence much slower, but the converged values are significantly different from the test values. It is probable that this occurs because the use of uniform elements neglects the effects of the slope of the shear centre axis, which is responsible for the additional terms in the Wagner torque of Equation 28.

Buckling predictions obtained using FTBTM for mono-symmetric tapered cantilevers are shown in Fig. 10, and are a little lower than previous theoretical results [3].

5 CONCLUSIONS

This paper describes a finite element method of analysing the elastic in-plane bending and out-of-plane flexural-torsional buckling of indeterminate beam structures whose members may be tapered and of mono-symmetric I cross-section. The structure's loading includes concentrated moments and concentrated or uniformly distributed off-axis transverse and longitudinal forces, and its deformations may be prevented or resisted by concentrated or continuous rigid or elastic off-axis restraints.

Tapered finite element formulations developed by numerical integration replaced the closed forms often used for uniform elements. Difficulties in specifying the load positions for tapered mono-symmetric members caused by the variations of the centroidal and shear centre axes were avoided by using an arbitrary axis system based on the web mid-line. Account was taken of additional Wagner torque terms arising from the inclination of the shear centre axis.

A computer program based on this method was used to analyse a number of examples of the elastic in-plane bending of tapered cantilevers and built-in beams, and very close agreement was found between its predictions and closed form solutions.

The program's predictions of the elastic out-of-plane flexural-torsional buckling of a large number of uniform beams and cantilevers under various loading and restraint conditions were in close agreement with those of an existing finite element program [14]. The predictions for cantilevers with concentrated end loads were also in close agreement with test results.

The program was used to analyse the flexural-torsional buckling of a number of tapered doubly symmetric beams, and again good agreement was found with existing theoretical predictions and test results. Close agreement was also found for tapered mono-symmetric beams with central concentrated loads. Predictions for tapered mono-symmetric cantilevers with end loads were a little lower that previous theoretical predictions.

It may be concluded that the computer program developed from the finite element analysis of this paper provides an efficient method of predicting the in-plane bending and out-of-plane buckling of indeterminate beam structures composed of tapered mono-symmetric members under a wide range of loading and restraint conditions.

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7 PRINCIPAL NOTATION

A	Area of cross-section
$b_{b,t,w}$	Bottom and top flange widths and web depth
$D_{i,u,v,ut}$	Generalised elasticity matrices
E	Young's modulus of elasticity
f	Stress
G	Shear modulus of elasticity
G_o	Global stability matrix
g_{oe}	Element stability matrix
$I_{x,y}$	Second moments of area about x, y axes
$I_{b,t}$	Second moments of area of bottom and top flanges
I_0	Value of I_x at $z = 0$
I_p	$= I_x + I_y$
I_w	Warping section constant
J	Uniform torsion constant
$K_{i,o}$	Global in-plane and out-of-plane stiffness matrices
$k_{ie,oe}$	Element in-plane and out-of-plane stiffness matrices
L	Length
M	External moment
M_x	Internal bending moment
N, n	Concentrated and distributed longitudinal loads
N_z	Internal compression force
Q, q	Concentrated and distributed transverse loads
$Q_{ie,in}$	Generalised equivalent distributed and concentrated nodal loads
Q_u	Value of Q for a uniform member
r_s	Distance of point (x, y) from shear centre $(0, y_s)$
$T_{ie,oe}$	In-plane and out-of-plane element to global transformation matrices
T_w	Wagner torque
$t_{t,w,b}$	Bottom and top flange and web thicknesses
U_e	Element strain energy increase
u, v, w	Displacements in x, y, z directions
u_s	Shear centre displacement parallel to x axis
V_e	Potential energy increase
x, y	Cross-section coordinates of arbitrary axis system
$y_{c,s}$	Centroid and shear centre coordinates
y_o	Distance of shear centre from centroid
$y_{N,n}$	Positions of N, n loads
$y_{Q,q}$	Positions of Q, q loads
$y_{R,r}$	Positions of concentrated and distributed restraints
z	Distance along member
$\alpha_{d,t,w}$	Taper constants
β	$= (1 + \alpha z/L)$
β_x	Mono-symmetry section constant
$\Delta_{i,o}$	Global in-plane and out-of-plane nodal deflections and rotations
$\delta_{ie,oe}$	Element in-plane and out-of-plane nodal deflections and rotations
$\varepsilon_{u,v}$	Generalised strain vectors
λ	Buckling load factor
ϕ	Twist rotation

APPENDIX A IN-PLANE DEFORMATIONS OF TAPERED CANTILEVERS

A tapered cantilever of length L is shown in Fig. 3a. The second moment of area I_x varies linearly along its length according to

$$I_x = I_0(1+\alpha\, z/L) \tag{A1}$$

The in-plane deformations of the cantilever caused by the loading M, Q, q shown in Fig. 3a may be determined from the differential equation of bending

$$-EI_x v'' = -M - Qz - qz^2/2 \tag{A2}$$

The general solution of this equation is

$$EI_0 v = \begin{Bmatrix} \beta\ln\beta - \beta \\ \beta\ln\beta - \beta - \beta^2/2 \\ \beta\ln\beta - \beta - \beta^2/2 - \beta^3/6 \end{Bmatrix}^T \begin{Bmatrix} ML^2/\alpha^2 \\ QL^3/\alpha^3 \\ qL^4/2\alpha^4 \end{Bmatrix} + A_1\beta + A_2 \tag{A3}$$

in which

$$\beta = (1+\alpha\, z/L) \tag{A4}$$

and A_1 and A_2 are constants of integration. These may be determined by using the boundary conditions

$$v_L = 0\,; \quad v_L' = 0 \tag{A5}$$

as

$$A_1 = EI_0 \begin{Bmatrix} -\ln(1+\alpha) \\ \ln(1+\alpha) - (1+\alpha) \\ -\ln(1+\alpha) + 2(1+\alpha) - (1+\alpha)^2/2 \end{Bmatrix}^T \begin{Bmatrix} ML^2/\alpha^2 \\ QL^3/\alpha^3 \\ qL^4/2\alpha^4 \end{Bmatrix} \tag{A6}$$

and

$$A_2 = \begin{Bmatrix} (1+\alpha) \\ -(1+\alpha) + (1+\alpha)^2/2 \\ (1+\alpha) - (1+\alpha)^2 + (1+\alpha)^3/3 \end{Bmatrix}^T \begin{Bmatrix} ML^2/\alpha^2 \\ QL^3/\alpha^3 \\ qL^4/2\alpha^4 \end{Bmatrix} \tag{A7}$$

The tip deformations may then be determined from

$$EI_0 v_0 = \begin{Bmatrix} -1 \\ 3/2 \\ -11/6 \end{Bmatrix}^T \begin{Bmatrix} ML^2/\alpha^2 \\ QL^3/\alpha^3 \\ qL^4/2\alpha^4 \end{Bmatrix} + A_1 + A_2 \tag{A8}$$

$$EI_0 v_0' = \begin{Bmatrix} 0 \\ 1 \\ -3/2 \end{Bmatrix}^T \begin{Bmatrix} ML^2/\alpha^2 \\ QL^3/\alpha^3 \\ qL^4/2\alpha^4 \end{Bmatrix} + A_1 \tag{A9}$$

APPENDIX B FIXED END ACTIONS OF TAPERED BEAMS

A fixed ended tapered beam of length L with a uniformly distributed load q is shown in Fig. 3b. The second moment of area I_x varies along its length according to Equation A1. The redundant end actions M_0 and Q_0 may be determined by applying the boundary conditions v_0 = v_0' = 0 to Equations A8 and A9, which can then be rearranged in the form of

$$[F]\begin{Bmatrix} M_0 \\ -Q_0 L/\alpha \end{Bmatrix} = \frac{q}{2L^2/\alpha^2}\{d\} \tag{B1}$$

whence

$$\begin{Bmatrix} M_0 \\ -Q_0 L/\alpha \end{Bmatrix} = \frac{q}{2L^2\alpha^2}[F]^{-1}\{d\} \tag{B2}$$

APPENDIX C ADDITIONAL WAGNER TORQUE COMPONENTS

When a member twists, tensile longitudinal stresses f exert restoring torque components, often described as Wagner [13] torques. The origin of these is demonstrated in Fig. 11, which shows that twist rotations ϕ about the shear centre S (0, y_s) cause a longitudinal fibre through a point (x, y) to rotate ($r_s\ \phi$)' about the line r_s joining the point to the shear centre. Thus the force fdA along the fibre develops a transverse force component fdA($r_s\ \phi$)', which exerts a torque component fdA($r_s\ \phi$)' r_s about the shear centre. The total torque is

$$T_w = \int_A f r_s (r_s \phi)' dA \tag{C1}$$

The tensile stress is given by

$$f = -N_z / A + M_x (y - y_c) / I_x \tag{C2}$$

and the distance r_s by

$$r_s = \sqrt{\{x^2 + (y - y_s)^2\}} \tag{C3}$$

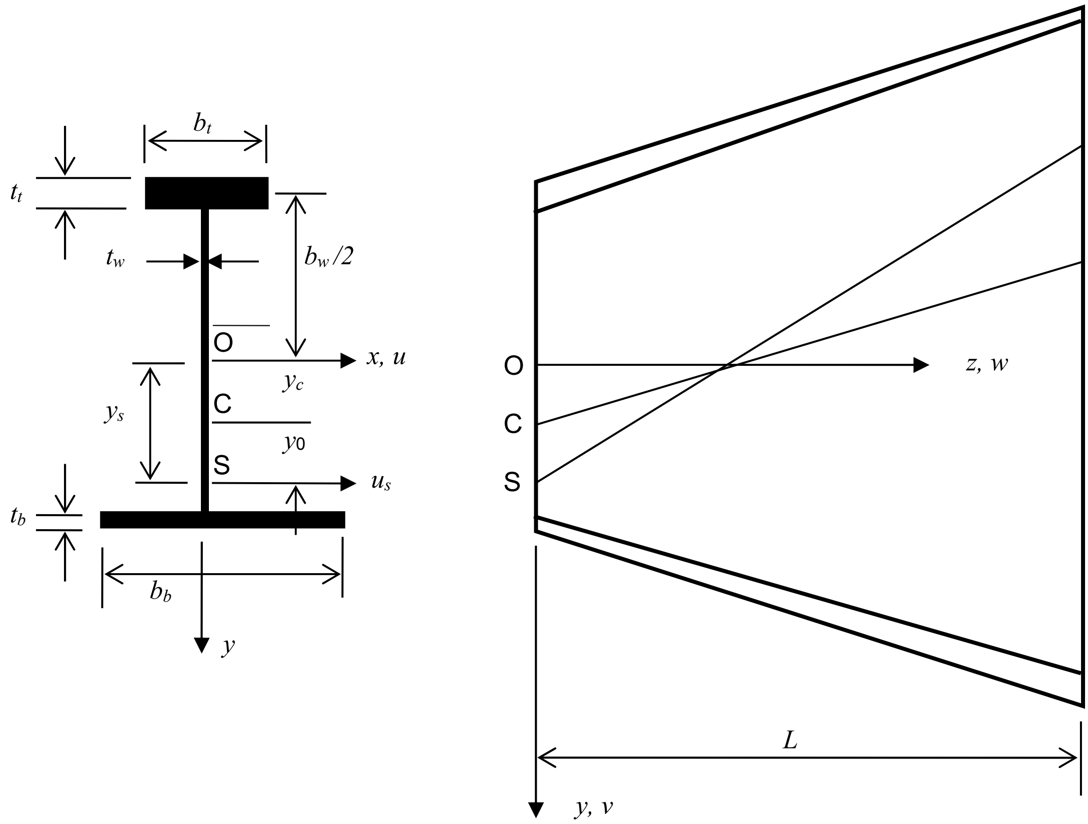
It can be shown that

$$\begin{aligned} &\int_A r_s^2 dA = I_p + A y_0^2 \\ &\int_A (y - y_c) r_s^2 dA = I_x \beta_x \\ &\int_A r_s r_s' dA = y_0 y_s' A \\ &\int_A (y - y_c) r_s r_s' dA = -y_s' I_x \end{aligned} \tag{C4}$$

so that

$$T_w = -N_z \{[I_p / A + y_0^2]\phi' + y_0 y_s' \phi\} + M_x \{\beta_x \phi' - y_s' \phi\} \tag{C5}$$

FIGURES



(a) Section (b) Element

Fig. 1 Geometry of a Tapered Element

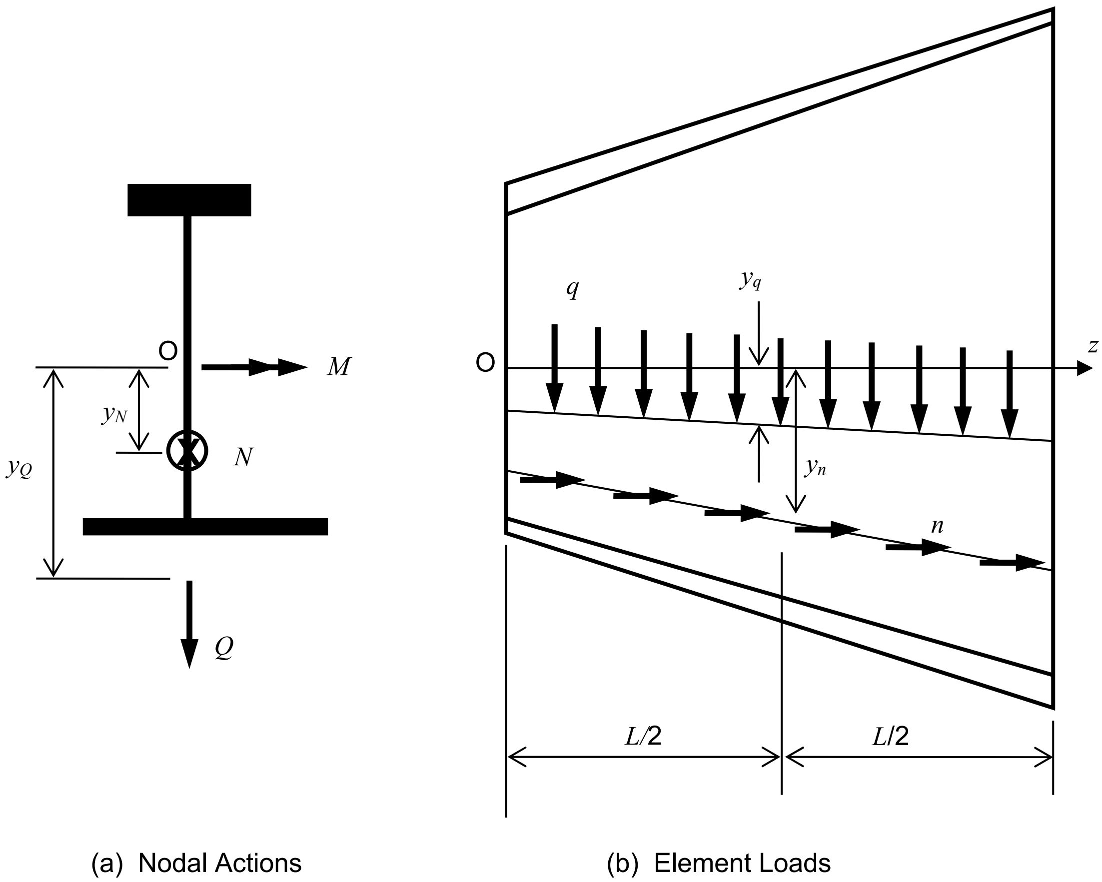
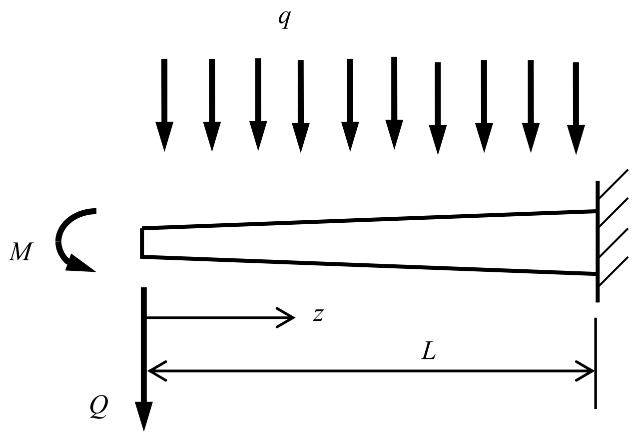
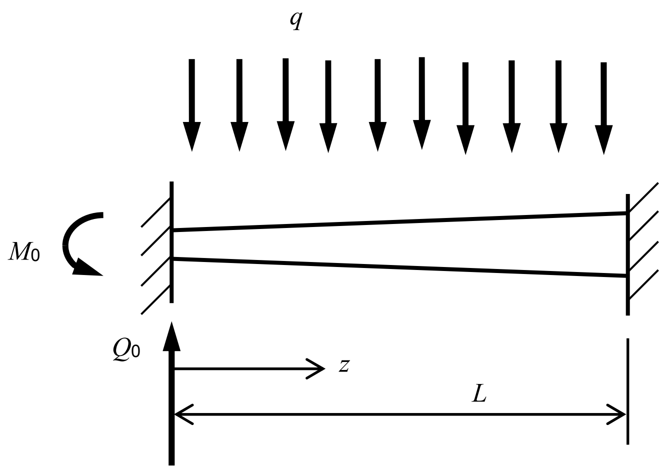


Fig. 2 Nodal Actions and Element Loads



(a) Cantilever



(b) Built-in beam

Fig. 3 Tapered Members

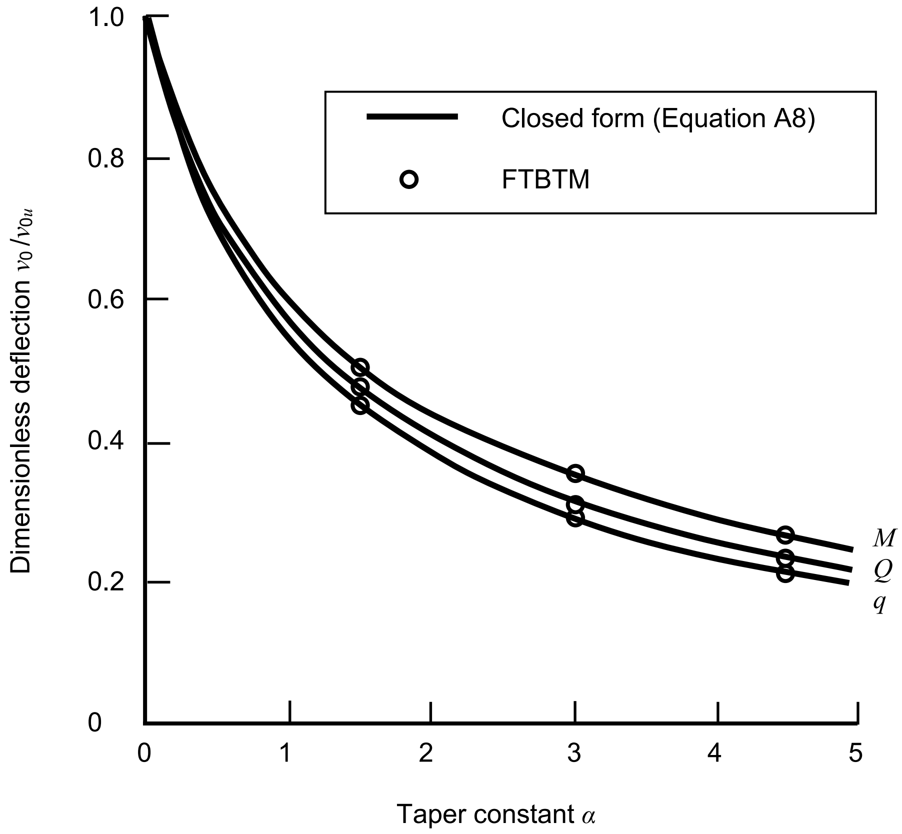


Fig. 4 In-Plane Deflections of Tapered Cantilevers

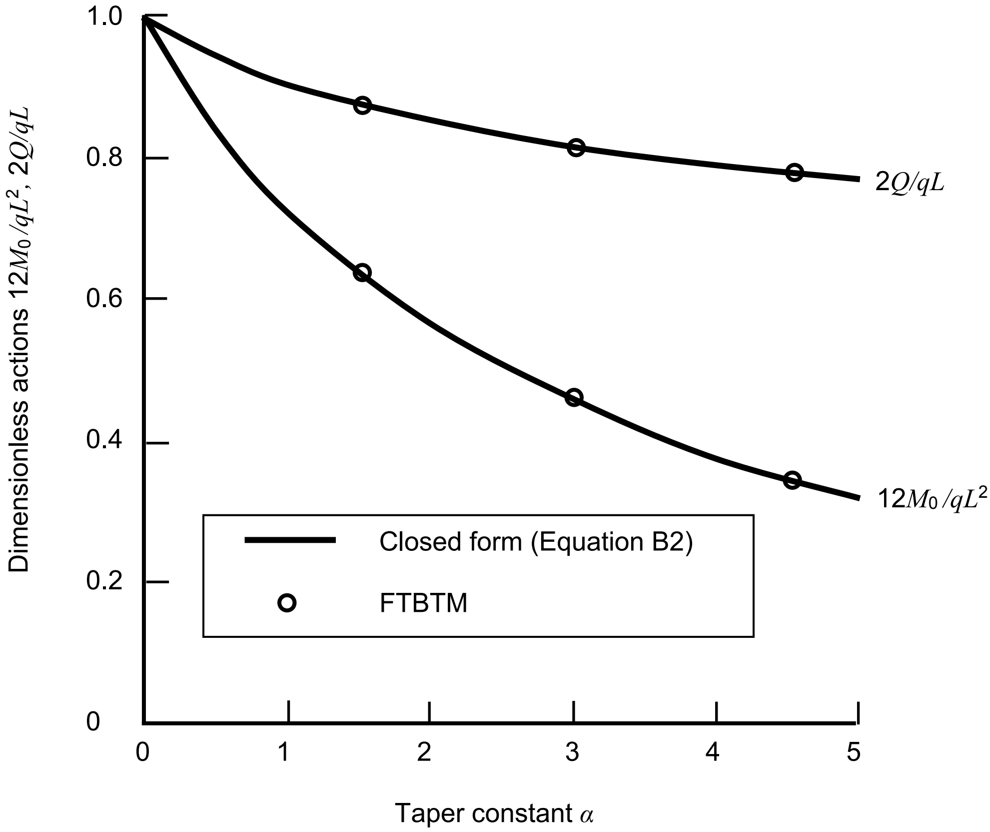


Fig. 5 Fixed End Actions of Tapered Beams

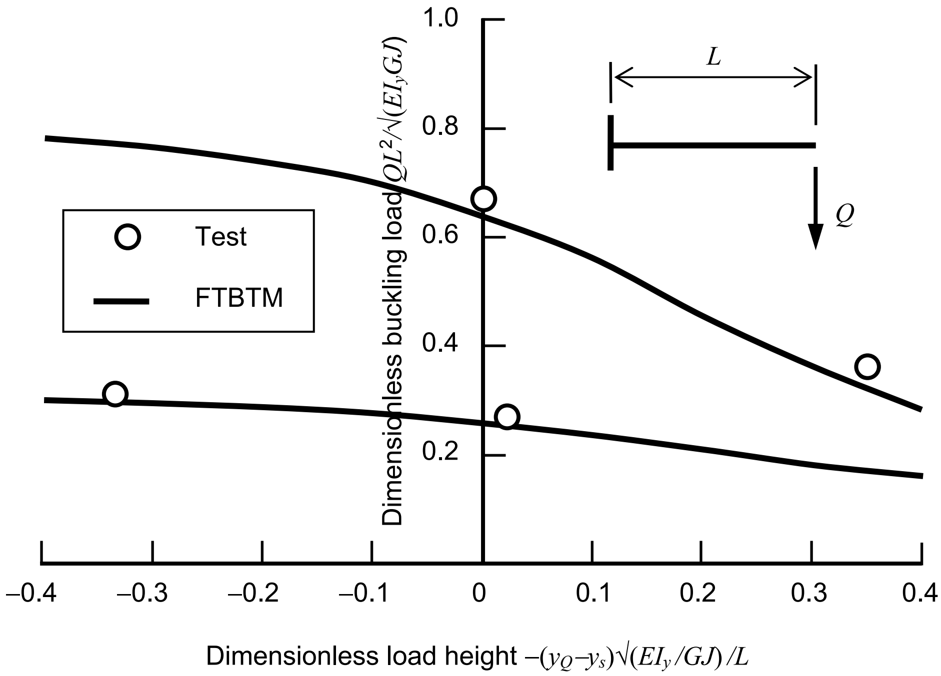


Fig. 6 Buckling of Uniform Mono-Symmetric Cantilevers

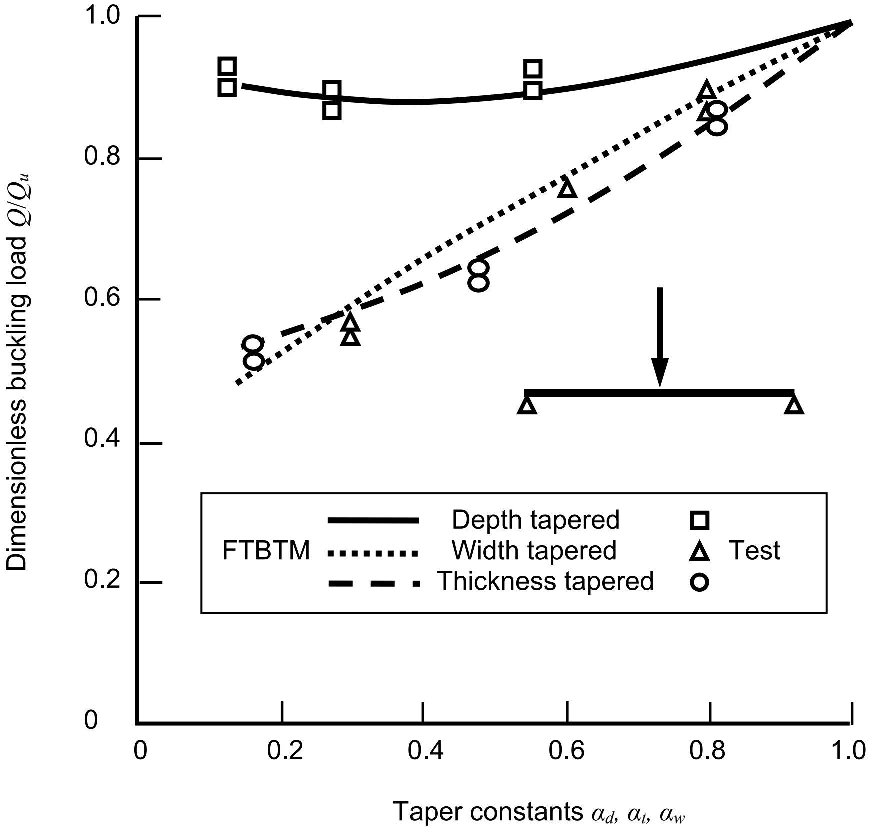


Fig. 7 Buckling of Doubly Symmetric Tapered Beams

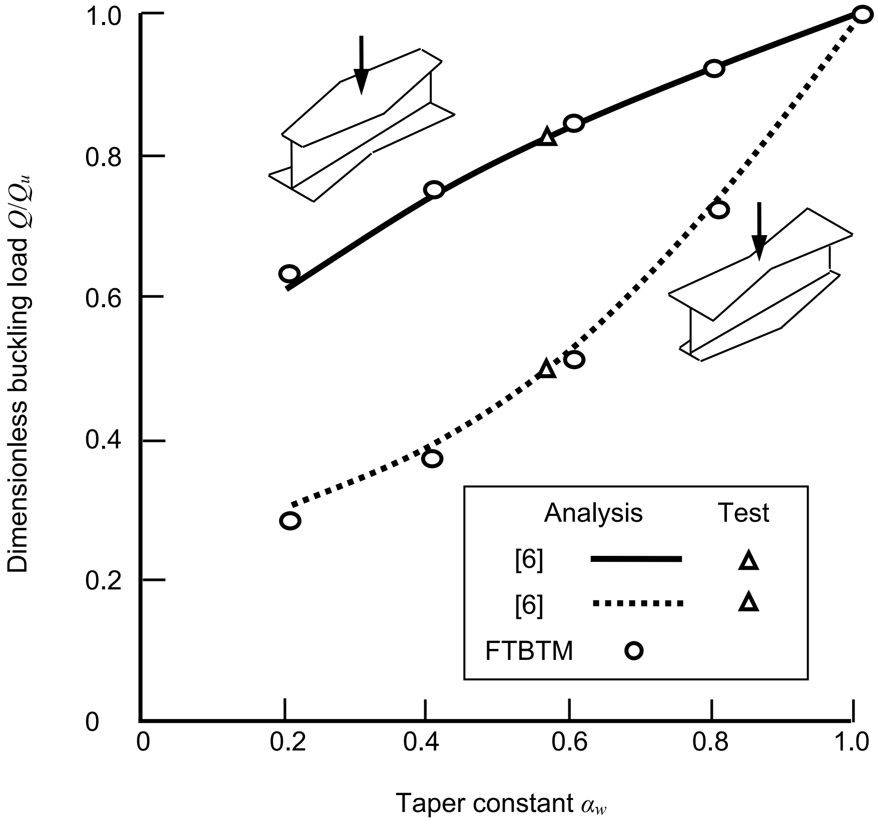


Fig. 8 Buckling of Mono-Symmetric Tapered Beams

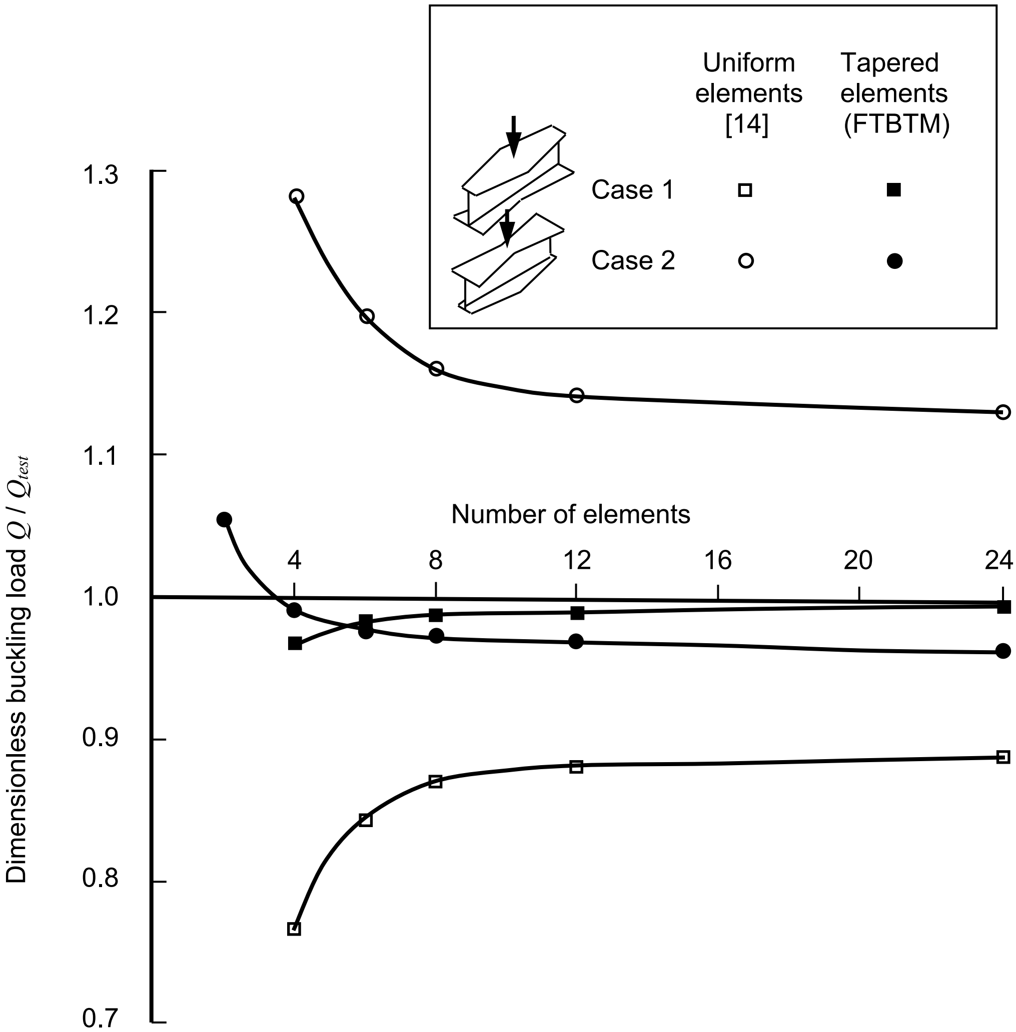


Fig. 9 Convergence

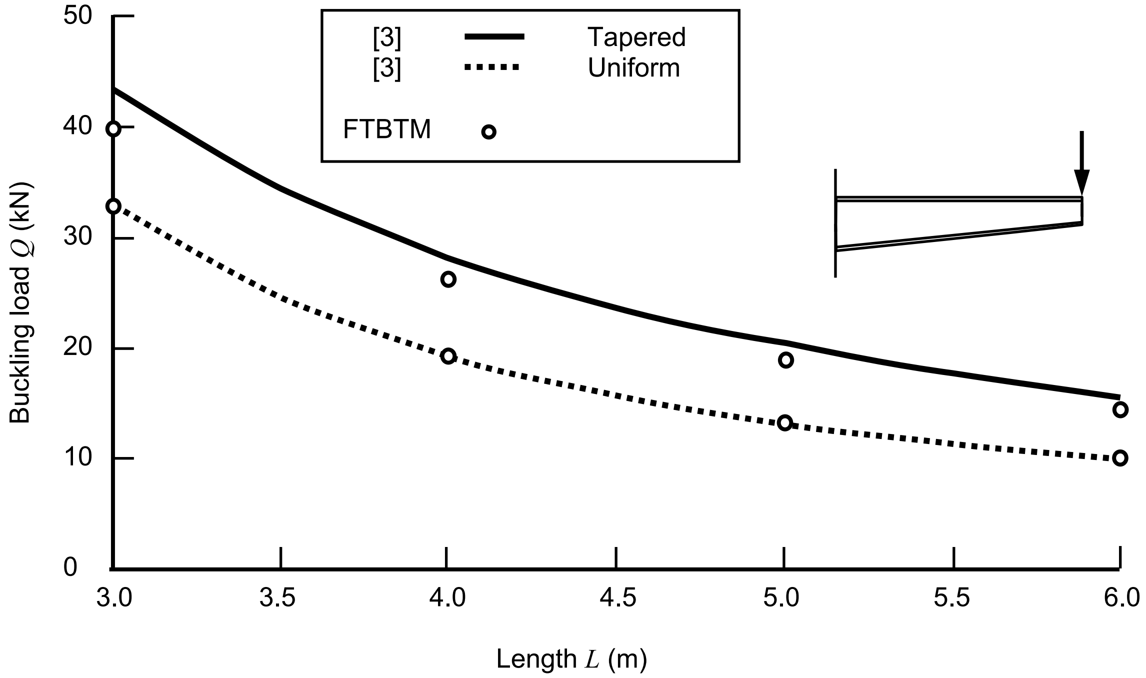


Fig. 10 Buckling of Mono-Symmetric Tapered Cantilevers

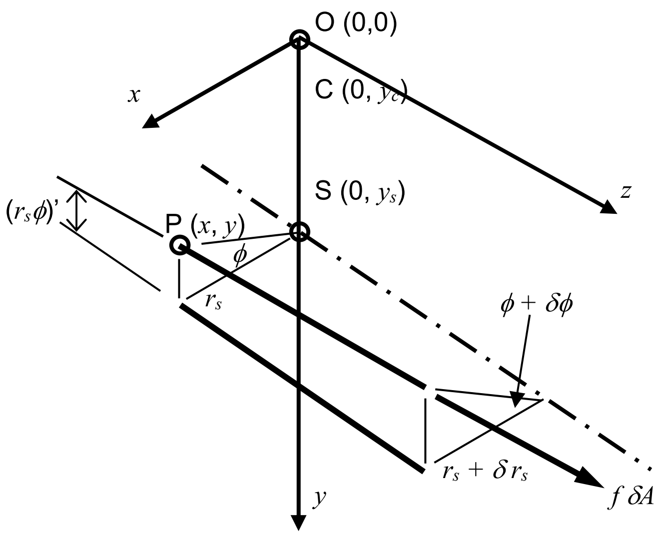


Fig. 11 Wagner Torque