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FINITE ELEMENT ANALYSIS OF THE FLEXURAL BUCKLING OF COLUMNS WITH OBLIQUE RESTRAINTS

By
N S Trahair BSc BE MEngSc PhD DEng and
K J R Rasmussen MSciEng PhD

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N.S. Trahair, BSc, BE, MEngSc, PhD, DEng
and K.J.R.Rasmussen, MSciEng, PhD

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ABSTRACT

This paper is concerned with the elastic flexural buckling of doubly symmetric columns with oblique restraints under concentric loading. Oblique restraints cause coupling between the principal axis deflections and rotations, and the flexural buckling mode involves simultaneous bending about both principal axes.

Oblique restraints may resist deflections as well as rotations, and may be rigid or elastic. They may be concentrated at point along a column, or distributed along portions of its length.

This paper discusses the nature of oblique end restraints, summarises their finite element analysis, presents examples of their effects on the elastic buckling of columns, and demonstrates the design of columns with oblique restraints.

Keywords: buckling, columns, design, elasticity, restraints, steel.

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N.Trahair@civil.usyd.edu.au,..K.Rasmussen@civil.usyd.edu.au
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1 INTRODUCTION

This paper is concerned with the elastic flexural buckling of doubly symmetric columns with oblique restraints under concentric loading. This subject may be considered to be a special case of the general problem of a compression member in a three-dimensional frame, which ignores any primary bending moments and inelastic behaviour.

The elastic buckling of a doubly symmetric column with restraints which act in a principal plane has been thoroughly researched and is well understood. Such a column buckles in one of the two principal planes xz or yz shown in Fig. 1. In this case, the buckling analysis is simplified to the consideration of the independent planar buckling modes. The effects of unequal rotational end restraints (Trahair et al, 2001) on planar buckling have been analysed, and graphs (SA, 1998 and BSI, 2000) and nomograms (AISC, 1999) for calculating the column effective length factors used to represent the buckling loads are available.

However, there appears to have been few studies of the elastic buckling of a column with oblique restraints which act in an oblique plane Xz inclined at an angle θ to the principal xz plane, as shown in Fig. 2. In this case, the oblique restraints cause coupling between the principal axis deflections and rotations, and the flexural buckling mode involves simultaneous bending about both principal axes.

Trahair (1969) studied the elastic buckling of an unequal angle section column with restraint lines (pinned about the X axis so that the moment M_X = 0 and fixed in the Xz plane so that the rotation dU/dz = 0) at each end. The elastic buckling load varied with the angle θ between the restraint lines and the column principal x axis in an approximately sinusoidal fashion. The magnitudes of the buckling loads were affected by torsional effects because of the asymmetry of the unequal angle.

Practical examples of oblique restraints include zed-section columns connected to end restraints through the web, and columns whose principal axes are rotated relative to the supporting beams or grids of beams. Depending on the stiffness of the supporting beam(s) relative to the stiffness of the column, the supporting beam(s) may be assumed to rigidly or elastically restrain rotations of the column in one or two oblique planes.

Oblique restraints may resist deflections as well as rotations, and may be rigid or elastic. They may be concentrated at point along a column, or distributed along portions of its length. The analysis of principal axis concentrated and distributed restraints on the flexural-torsional buckling of columns, beams, and beam-columns has been discussed in Trahair (1993).

In this paper, only the effects of concentrated oblique restraints on the flexural buckling of columns are considered, as the extension to the effects of continuous restraints is very straightforward. The following sections discuss the nature of oblique end restraints, summarise their analysis, present examples of their effects on the elastic buckling of columns, and demonstrate the design of columns with oblique restraints. A companion paper (Rasmussen and Trahair 2004) presents exact and approximate solutions for the buckling of columns with rigid and elastic rotational end restraints.

2 RESTRAINTS

An elastic column of length L and axial compression N is shown in Fig. 1, in which x and y are the principal axes of the column cross-section and z is the distance along the column. The column is restrained against deflections and/or rotations at one or more points along its length, as shown in Fig. 2, in which X and Y are two oblique restraint axes inclined at θ to the x, y principal axes. The restraints may exert forces F_X, F_Y which oppose deflections U, V in the X, Y directions and moments M_X, M_Y which oppose rotations $-dV/dz$, dU/dz about the X, Y axes.

When the column buckles, it deflects u, v in the x, y directions. These deflections are related to the deflections U, V in the X, Y directions by

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = [T] \begin{Bmatrix} U \\ V \end{Bmatrix} \tag{1}$$

in which [T] is given by

$$[T] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{2}$$

If the restraints are elastic, then they may be expressed as

$$\{F_R\} = [\alpha_R]\{\delta_R\} \tag{3}$$

in which

$$\{F_R\} = \{F_X \quad F_Y \quad M_Y \quad M_X\}^T \tag{4}$$

is the vector of the restraint actions,

$$\{\delta_R\} = \{U_R \quad V_R \quad dU_R/dz \quad -dV_R/dz\}^T \tag{5}$$

is the vector of the oblique plane deformations, and

$$\{\alpha_R\} = \begin{bmatrix} \alpha_{TX} & 0 & 0 & 0 \\ 0 & \alpha_{TY} & 0 & 0 \\ 0 & 0 & \alpha_{RY} & 0 \\ 0 & 0 & 0 & \alpha_{RX} \end{bmatrix} \tag{6}$$

is the matrix of restraint stiffnesses.

When the column buckles, these restraints store strain energy

$$U_R = \frac{1}{2}\{\delta_R\}^T [\alpha_R]\{\delta_R\} \tag{7}$$

which may be transformed into the principal axis system to

$$U_r = \frac{1}{2}\{\delta_r\}^T[\alpha_r]\{\delta_r\} \tag{8}$$

in which

$$\{\delta_r\} = \{u \quad v \quad du/dz \quad -dv/dz\}^T \tag{9}$$

and

$$[\alpha_r] = \begin{bmatrix}[T] & [0]\\ [0] & [T]\end{bmatrix}^T [\alpha_R] \begin{bmatrix}[T] & [0]\\ [0] & [T]\end{bmatrix} \tag{10}$$

3 FINITE ELEMENT ANALYSIS OF ELASTIC BUCKLING

A finite element computer program has been prepared which can analyse the elastic buckling of columns with varying axial force and concentrated elastic or rigid oblique restraints against deflection and rotation which can act at a number of points along the column length. This program uses the principle of conservation of energy at elastic buckling, as expressed by the energy equation

$$\sum_e \frac{1}{2}\int_0^{L_e}\left(EI_{ye}u''^2 + EI_{xe}v''^2\right)dz + \sum_r \frac{1}{2}\{\delta_r\}^T[\alpha_r]\{\delta_r\} - \sum_e \frac{\lambda}{2}\int_0^{L_e} N_e\left(u'^2 + v'^2\right)dz = 0 \tag{11}$$

in which E is the Young's modulus of elasticity, I_{xe}, I_{ye} are an element's second moments of area about the x, y principal axes, L_e is the length of and N_e is the initial axial force in an element, z is the distance along the element, $\{\delta_r\}$ is the vector of deformations and $[\alpha_r]$ is the restraint stiffness matrix for an elastic restraint point, and λ is the load factor at buckling.

In the finite element method, cubic fields (Trahair, 1993) are used to represent the displacements u, v in terms of the element nodal deformations $\{\delta_e\}$, and the components of the first and third terms of Equation 11 are transformed to

$$\frac{1}{2}\int_0^{L_e}\left(EI_{ye}u''^2 + EI_{xe}v''^2\right)dz = \frac{1}{2}\{\delta_e\}^T[k_e]\{\delta_e\} \tag{12a}$$

$$\frac{1}{2}\int_0^{L_e} N_e\left(u'^2 + v'^2\right)dz = \frac{1}{2}\{\delta_e\}^T[g_e]\{\delta_e\} \tag{12b}$$

in which $[k_e]$ and $[g_e]$ are the element stiffness and stability matrices (Trahair, 1993). These are then summed for all the elements to form the global matrices $[K]$ and $[G]$ and these are added to the restraint stiffnesses $[\alpha_r]$ to form

$$\frac{1}{2}\{\Delta\}^T\left([K] + [\alpha] - \lambda[G]\right)\{\Delta\} = 0 \tag{13}$$

Rigid restraints require some of the global deformations $\{\Delta\}$ to be zero, and so these are condensed out of Equation 13, reducing it to

$$\frac{1}{2}\{\Delta_C\}^T\left([K_C]+[\alpha_C]-\lambda[G_C]\right)\{\Delta_C\}=0 \tag{14}$$

This condensation is straightforward when the rigid restraints act in the principal planes, since the global deformations {Δ} are initially principal plane values. When the rigid restraints act in oblique planes, it is simpler to transform the corresponding principal plane deformations to the oblique planes, using

$$\begin{Bmatrix} u_r \\ v_r \\ du_r/dz \\ -dv_r/dz \end{Bmatrix} = \begin{bmatrix} [T] & [0] \\ [0] & [T] \end{bmatrix} \begin{Bmatrix} U_R \\ V_R \\ dU_R/dz \\ -dV_R/dz \end{Bmatrix} \tag{15}$$

in which $[T]$ is given by Equation 2.

The lowest buckling load factor λ and the corresponding buckling mode defined by {Δ} may be extracted from Equation 14 by using standard eigenvalue procedures. The computer program has been written using the MATLAB (Mathworks, 1995) language and functions.

4 APPLICATIONS

4.1 Rigid Rotational End Restraints

The finite element computer program has been used to analyse the elastic buckling of a uniform section column whose ends are prevented from deflecting and which has rigid oblique restraints against end rotations in a plane Xz at θ to the xz principal plane, as shown in Fig. 3. Sixteen equal length elements have been used to obtain the numerical results shown in this and subsequent sections.

When θ = 90°, the column buckles in a half sine wave $u = u_0 \sin(\pi z/L)$ out of the restraint plane at $N_0 = N_y$ in which

$$N_y = \pi^2 EI_y / L^2 \tag{16}$$

When θ = 0° and $I_x < 4I_y$, the column buckles (Timoshenko and Gere, 1961) by deflecting $v = v_0 \sin(\pi z/L)$ out of the restraint plane at $N_0 = N_x = \pi^2 EI_x / L^2$. When θ = 0° and $I_x > 4I_y$, the column buckles (Timoshenko and Gere, 1961) in the restraint plane at $N_0 = 4N_y$ by deflecting $u = u_0\{1 - \cos(2\pi z/L)\}/2$.

More generally, the elastic buckling load N_0 decreases from the lesser of N_x and 4 N_y to N_y as the angle θ increases from 0° to 90°, as shown in Fig. 3, and increases from N_y to 4 N_y as I_x / I_y increases from 1 to ∞.

Also shown in Fig. 3 are independent approximate values of the dimensionless elastic buckling load N_0/N_y obtained as set out in Rasmussen and Trahair (2004) and Appendix 3 by using the energy method of Equation 11 with

$$U = U_0\{1 - \cos(2\pi z / L)\}/2 \tag{17a}$$

$$V = V_0 \sin(\pi z / L) + \gamma U_0\{1 - \cos(2\pi z / L)\}/2 \tag{17b}$$

in which γ is a parameter used to satisfy the boundary conditions of free rotation in the Yz plane. These approximate values are very close to the more accurate finite element solutions and the exact solutions presented in Rasmussen and Trahair (2004).

4.2 Elastic Rotational End Restraints

The finite element program has also been used to analyse the buckling of a column whose ends are prevented from deflecting and which have equal elastic oblique restraints of stiffness α_{RY} against end rotations in a plane Xz at θ to the xz principal plane, as shown in Fig. 4.

The variations of the dimensionless buckling load N_0/N_y with the dimensionless restraint stiffness parameter

$$R_{RY} = \frac{\alpha_{RY} L/(2EI_y)}{1 + \alpha_{RY} L/(2EI_y)} \tag{18}$$

are shown in Fig. 4 for I_x/I_y = 4 and θ = 90°, 60°, 30°, and 0°. For θ = 0°, the dimensionless buckling load N_0/N_y increases from 1 to 4 as the dimensionless restraint parameter R_{RY} increases from 0 to 1 (so that α_{RY} increases from 0 to ∞) (Trahair et al, 2001). For higher values of θ, the dimensionless buckling load increases at a slower rate. The results shown in Fig. 4 are in close agreement with the exact values presented in Rasmussen and Trahair (2004).

4.3 Rigid Translational End Restraint

The finite element program has also been used to analyse the buckling of a column that is prevented from deflecting and rotating at one end (z = 0), as shown in Fig. 5. At the other end ($z = L$), the column is free to rotate, has a rigid oblique translational restraint in a plane Xz at θ to the xz principal plane, and is free to displace in the perpendicular Yz plane.

The computed values of the dimensionless buckling load N_0/N_y are shown in Fig. 5. When θ = 90°, the column buckles as a cantilever out of the restraint plane at $N_0 = N_y / 4$. When θ = 0° and $I_x < 8.18 I_y$, the column buckles (Timoshenko and Gere, 1961) as a cantilever out of the restraint plane at $N_0 = N_x$. When θ = 0° and $I_x > 8.18 I_y$, the column buckles as a propped cantilever in the restraint plane at $N_0 = 2.045 N_y$ (Timoshenko and Gere, 1961).

More generally, the patterns of the variations of the dimensionless buckling load N_0/N_y with θ and I_x/I_y are similar to those shown in Fig. 3 for columns with rigid rotational end restraints.

4.4 Rigid Translational Central Restraint

The finite element program has also been used to analyse the buckling of a column that is prevented from deflecting but free to rotate at both ends, and which has a central rigid oblique translational restraint in a plane Xz at θ to the xz principal plane, as shown in Fig. 6.

The computed values of the dimensionless buckling load N_0/N_y are shown in Fig. 6. When $\theta = 90^o$, the column buckles in a half sine wave $u = u_0 \sin(\pi z/L)$ out of the restraint plane at $N_0 = N_y$. When $\theta = 0^o$ and $I_x < 4I_y$, the column buckles in a half sine wave $v = v_0 \sin(\pi z/L)$ out of the restraint plane at $N_0 = N_x$. When $\theta = 0^o$ and $I_x > 4I_y$, the column buckles in a full sine wave $u = u_0 \sin(2\pi z/L)$ in the restraint plane at $N_0 = 4N_y$.

More generally and while $I_x < 4I_y$, the patterns of the variations of the dimensionless buckling load N_0/N_y with θ and I_x/I_y are similar to those shown in Fig. 5 for columns with rigid translational end restraints. However, when $I_x > 4I_y$, the buckling mode changes suddenly at intermediate values of θ from a planar mode $u = u_0 \sin(2\pi z/L)$ to a biplanar mode.

4.5 Elastic Translational Central Restraint

The finite element program has also been used to analyse the buckling of a column that is prevented from deflecting but free to rotate at both ends, and which has a central elastic oblique translational restraint of stiffness α_{TX} against deflection in a plane Xz at θ to the xz principal plane, as shown in Fig. 7.

The variations of the dimensionless buckling load N_0/N_y with the dimensionless restraint stiffness parameter

$$R_{TX} = \frac{\alpha_{TX} L^3 /(48EI_y)}{1 + \alpha_{TX} L^3 /(48EI_y)} \tag{19}$$

are shown in Fig. 7 for $I_x/I_y = 4$ and $\theta = 0^o$, 30^o, 60^o, and 90^o. For $\theta = 0^o$, the dimensionless buckling load N_0/N_y increases from 1 to 4 as the dimensionless restraint stiffness parameter R_{TX} increases from 0 to 0.767, when the buckling mode changes from one of symmetry to an anti-symmetrical full sine wave (Trahair et al, 2001), and then remains constant. For greater angles θ, the buckling load increases steadily to a lower maximum value at $R_{TX} = 1.0$ corresponding to infinite restraint stiffness α_{TX}.

5 DESIGN OF COLUMNS WITH OBLIQUE RESTRAINTS

5.1 Design by Buckling Analysis

The simplest method of designing steel columns with oblique restraints which is compatible with the widely used methods of designing columns with principal axis restraints is to use the method of design by buckling analysis, which is incorporated either directly or indirectly in design codes such as the AISC Specification (AISC, 1999), the Australian Standard AS4100 (SA, 1998), or the British Standard BS5950

(BSI, 2000). In this method, the results of an analysis for the elastic buckling load N_0 of the obliquely restrained column is used with the squash load

$$N_Y = Af_y \tag{19}$$

to calculate a modified slenderness

$$\lambda_c = \sqrt{(N_Y / N_0)} \tag{20}$$

which is then used to determine the nominal design capacity N_n.

For example, the AISC Specification and the Australian/New Zealand Standard (SA, 1996) for cold-formed steel structures both use

$$\frac{N_n}{N_Y} = 0.648\, e^{\lambda_c^2} \qquad (\lambda_c \le 1.5) \tag{21a}$$

$$\frac{N_n}{N_Y} = 0.877 / (\lambda_c^{\,2}) \qquad (\lambda_c > 1.5) \tag{21b}$$

as shown in Fig. 8.

5.2 Worked Example

A 2000mm long cold-formed steel zed-section column has the mid-line dimensions shown in Fig. 9a. Its section properties were determined using the computer program THIN-WALL (Papangelis and Hancock, 1997), and are shown in Fig. 9b. The column is prevented from deflecting at both ends, and is prevented from rotating in the plane of the web but is free to rotate out of the plane of the web at both ends. The design compression capacity may be determined as follows.

θ = 61.23°, I_x / I_y = 1.357E6 / 1.359E5 = 9.99

(Fig. 3) N_0/N_y = 2.3

(Equation 16) $N_y = \pi^2$ x 2E5 x 1.359E5 / 2000^2 = 67.1E3 N.

N_0 = 2.3 x 67.1E3 = 154E3 N.

(Or using the finite element program, N_0 = 152.4E3 N.)

(Equation 19) N_Y = 652.4 x 450 = 293.6E3 N.

(Equation 20) λ_c = √(293.6E3 / 152.4E3) = 1.388 < 1.5

(Equation 21a) N_n = 293.6E3 x (0.658 exp(1.388^2) = 131.1E3 N.

Using a capacity factor of 0.85, the design capacity is N_d = 0.85 x 131.1E3 = 111.6 N.

6 CONCLUSIONS

Column restraints which act in planes oblique to the principal planes cause coupling between the principal axis deflections, so that the buckling mode is bi-planar, rather than planar. Restraints may resist both rotations and deflections, may be rigid or elastic, and may be concentrated at a restraint point or distributed along a portion of the column length.

This paper describes a finite element program for analyzing the elastic buckling of columns with oblique, concentrated, rigid or elastic restraints and varying axial compression. The program was validated by comparisons with well-known results for columns with principal axis rotational and translational restraints, and with independently obtained solutions for columns which are prevented from deflecting at both ends and from rotating in oblique planes.

The program was used to determine the variations of the dimensionless elastic buckling loads N_0/N_y with the second moment of area ratio I_x/I_y and the inclination θ of the restraint plane to the principal xz plane. The buckling loads generally decrease with θ and increase with I_x/I_y, although the effect of I_x/I_y decreases with increase in θ.

A worked example is presented which demonstrates, by using the method of design by buckling analysis, that obliquely restrained steel columns can be designed in a way which is consistent with the traditional methods of designing columns with principal axis restraints.

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APPENDIX 2 NOTATION

A	area of cross-section
$a_{0,1,2}$	see Equations 31
$b_{0,1,2}$	see Equations 28
$c_{0,1,2}$	see Equations 29
E	Young's modulus of elasticity
$\{F_R\}$	vector of restraint actions
F_X, F_Y	restraint forces
f_y	yield stress
$[G]$, $[K]$	global stiffness and stability matrices
$[G_C]$, $[K_C]$	condensed global stiffness and stability matrices
$[g_e]$, $[k_e]$	element stiffness and stability matrices

I_x, I_y	column second moments of area about the x, y principal axes
I_{xe}, I_{ye}	element second moments of area about the x, y principal axes
L	column length
L_e	element length
M_X, M_Y	restraint moments
$(M_X)_0$	value of M_X at z = 0
N	concentric axial load
N_d	design compression capacity
N_e	element axial compression
N_n	nominal compression capacity
N_x, N_y	Euler flexural buckling loads
N_Y	squash load
N_0	elastic buckling load
R_{RY}, R_{TX}	restraint parameters
$[T]$	transformation matrix
u, v	shear centre deflections parallel to the x, y principal axes
u_r, v_r	shear centre deflections at restraint point
u_0, v_0	maximum values of shear centre deflections
U, V	shear centre deflections parallel to the X, Y restraint axes
U_0, V_0	maximum values of oblique shear centre deflections
x, y	principal axes
X, Y	oblique directions parallel and perpendicular to restraint plane
z	distance along element
$[\alpha]$	restraint stiffness matrix
$[\alpha_C]$	condensed restraint stiffness matrix
$[\alpha_R]$	oblique restraint stiffness matrix
$[\alpha_r]$	principal axis restraint stiffness matrix
α_{RX}, α_{RY}	stiffnesses of rotational restraints
α_{TX}, α_{TY}	stiffnesses of translational restraints
γ	constant used to satisfy end condition
$\{\delta_R\}$	vector of oblique plane deformations at a restraint point
$\{\delta_r\}$	vector of principal axis deformations at a restraint point
$\{\Delta\}$	vector of global nodal deformations
$\{\Delta_C\}$	condensed vector of global nodal deformations
θ	inclination of Xz restraint plane to the xz principal plane
λ	buckling load factor
λ_c	modified slenderness

APPENDIX 3 APPROXIMATE ANALYSIS OF THE ELASTIC BUCKLING OF A COLUMN WITH OBLIQUE END RESTRAINTS

A doubly symmetric column of length L and axial load N is prevented from deflecting at both ends. Each end of the column is prevented from rotating in a plane Xz at an angle θ to the xz principal plane, but is free to rotate out of this plane. It is assumed that $I_x > I_y$.

It may be assumed that the column buckles by deflecting

$$U = U_0\{1-\cos(2\pi z/L)\}/2 \tag{22}$$

$$V = V_0\sin(\pi z/L) + \gamma\, U_0\{1-\cos(2\pi z/L)\}/2 \tag{23}$$

in the X and Y directions (in and out of the restraint plane). The buckled shape U satisfies the restraint plane boundary condition $(dU/dz)_0 = 0$, and the second term in the buckled shape V ensures that the restraint plane boundary condition $(M_X)_0 = 0$ is also satisfied. Using Equations 1 and 2,

$$(M_X)_0 = -EI_x v''\cos\theta + EI_y u''\sin\theta = 0 \tag{24}$$

whence

$$\gamma = \frac{\sin\theta\cos\theta\,(I_x/I_y - 1)}{(I_x/I_y)\cos^2\theta + \sin^2\theta} \tag{25}$$

The energy equation for the elastic buckling load N_0 of the column is

$$N_0 = \frac{EI_y\int_0^L u''^2\,dz + EI_x\int_0^L v''^2\,dz}{\int_0^L (u'^2 + v'^2)\,dz} \tag{26}$$

which can be written as

$$\frac{N_0}{\pi^2 EI_y/L^2} = \frac{b_0 + b_1(V_0/U_0) + b_2(V_0/U_0)^2}{c_0 + c_1(V_0/U_0) + c_2(V_0/U_0)^2} \tag{27}$$

in which

$$b_0 = 4\{\cos\theta - \gamma\sin\theta)^2 + (I_x/I_y)(\sin\theta + \gamma\cos\theta)^2\} \tag{28a}$$

$$b_1 = -(16/3\pi)\{\sin\theta(\cos\theta - \gamma\sin\theta) - (I_x/I_y)\cos\theta(\sin\theta + \gamma\cos\theta)\} \tag{28b}$$

$$b_2 = \sin^2\theta + (I_x/I_y)\cos^2\theta \tag{28c}$$

$$c_0 = 1 + \gamma^2 \tag{29a}$$

$$c_1 = (16/3\pi)\,\gamma \tag{29b}$$

$$c_2 = 1 \tag{29c}$$

The dimensionless buckling load N_0 / N_y has a minimum value when

$$\frac{V_0}{U_0} = \frac{-a_1 - \sqrt{a_1^{\,2} - 4a_0 a_2})}{2a_2} \tag{30}$$

in which

$$a_0 = b_1 c_0 - b_0 c_1 \tag{31a}$$

$$a_1 = 2\,(b_2 c_0 - b_0 c_2) \tag{31b}$$

$$a_2 = b_2 c_1 - b_1 c_2 \tag{31c}$$

The variations of the minimum value of N_0 / N_y with θ for given values of I_x / I_y are shown in Fig. 3.

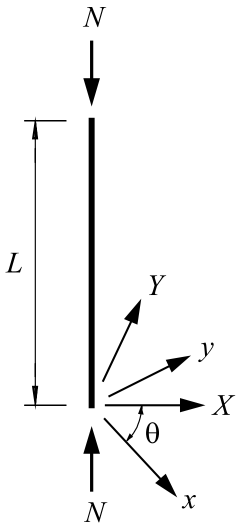


Fig.1 Elastic Column

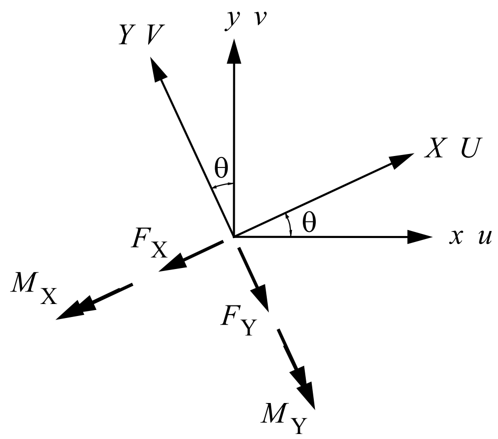


Fig.2 Oblique Axes and Restraint Actions

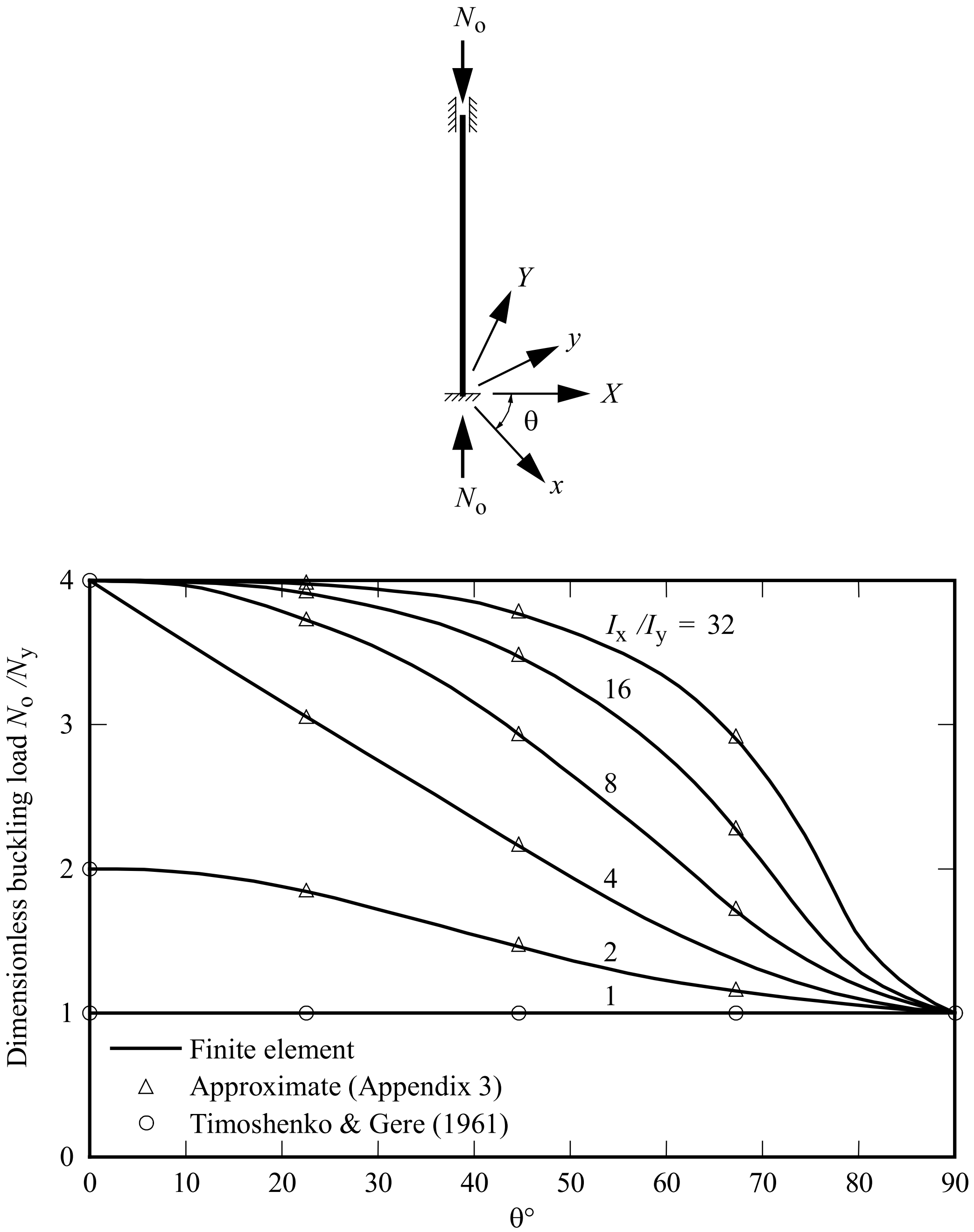


Fig.3 Buckling of Columns with Rigid Rotational End Restraints

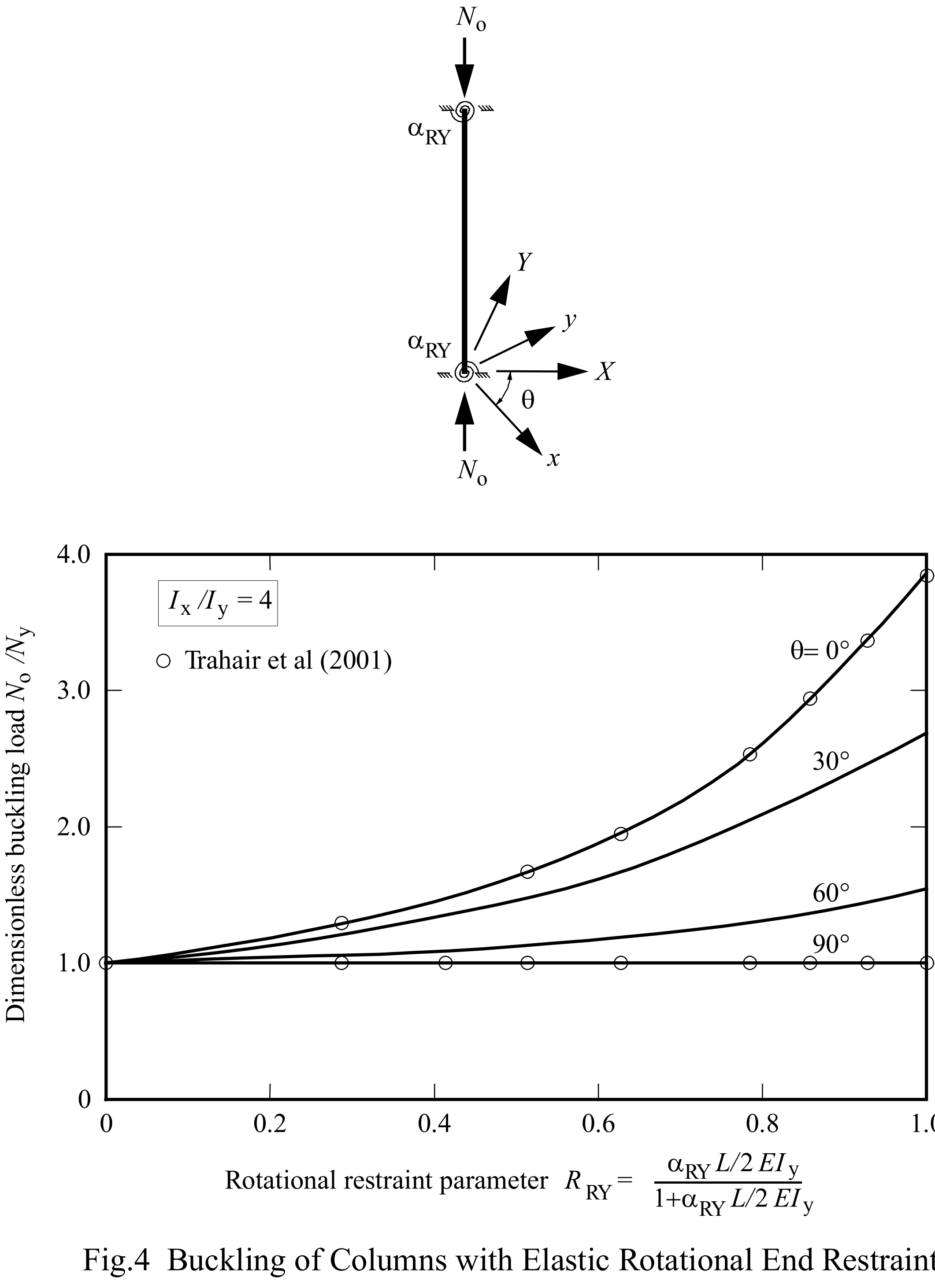


Fig.4 Buckling of Columns with Elastic Rotational End Restraints

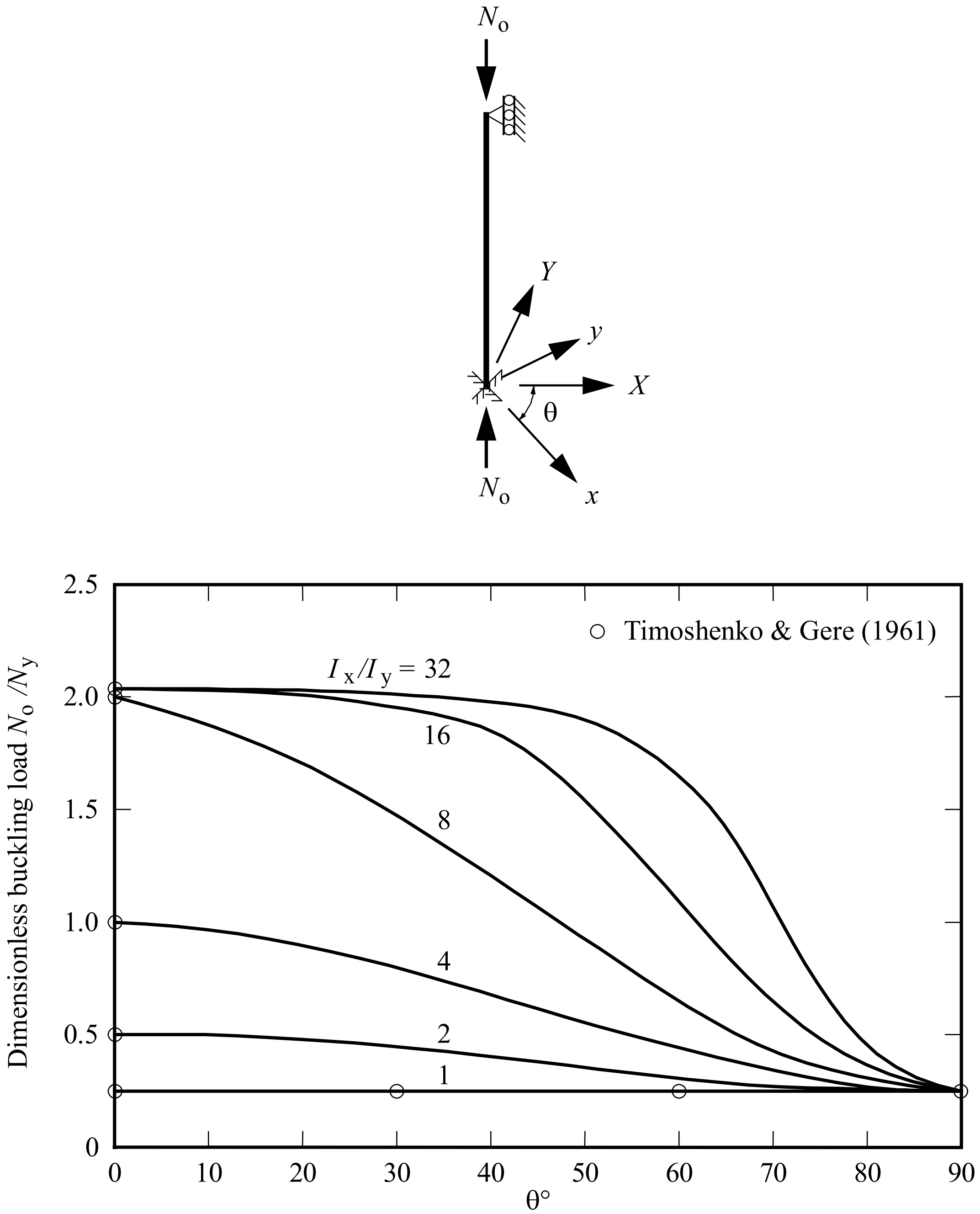


Fig.5 Buckling of a Cantilever with Rigid Translational End Restraint

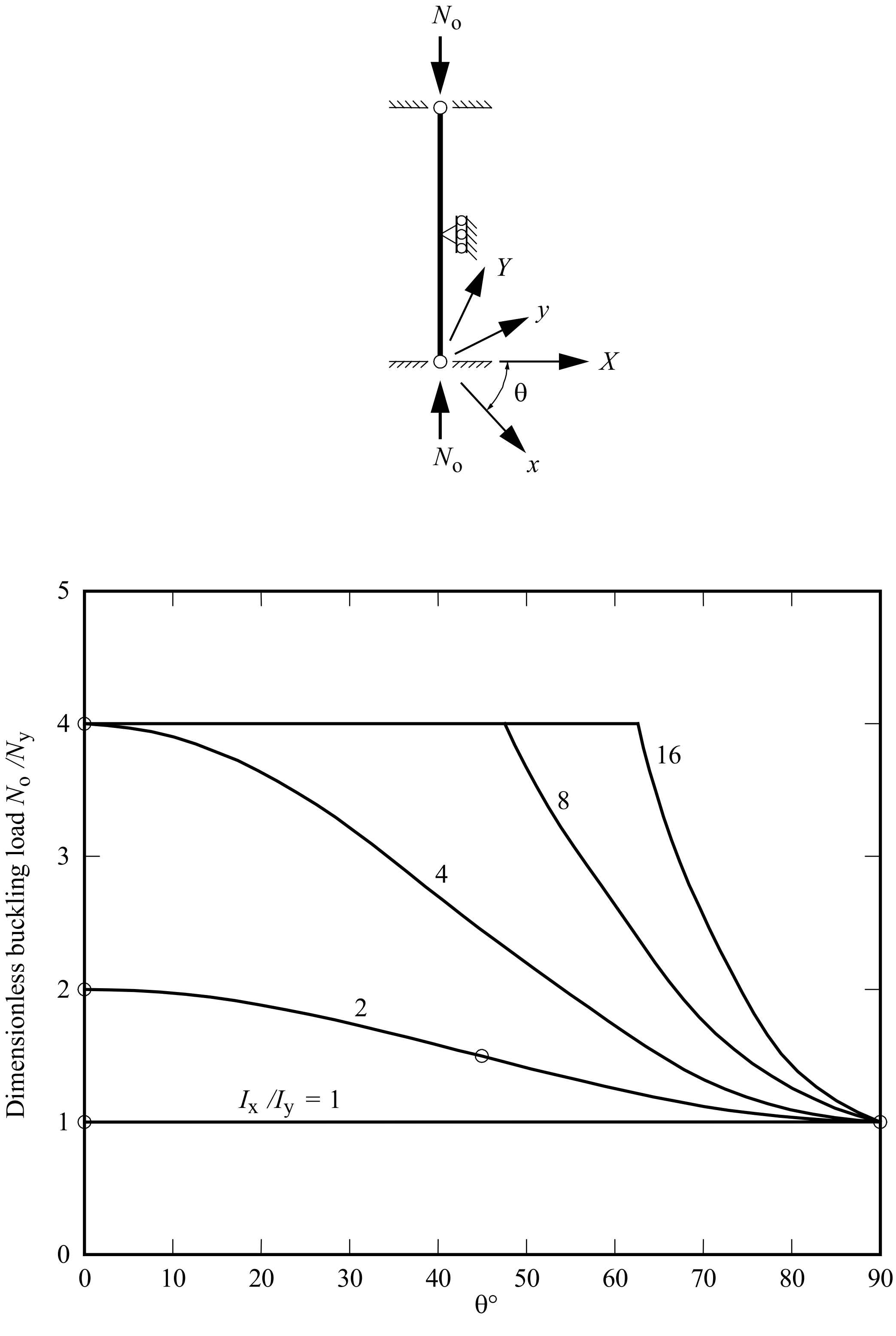


Fig.6 Buckling of a Column with a Rigid Central Tranlational Restraint

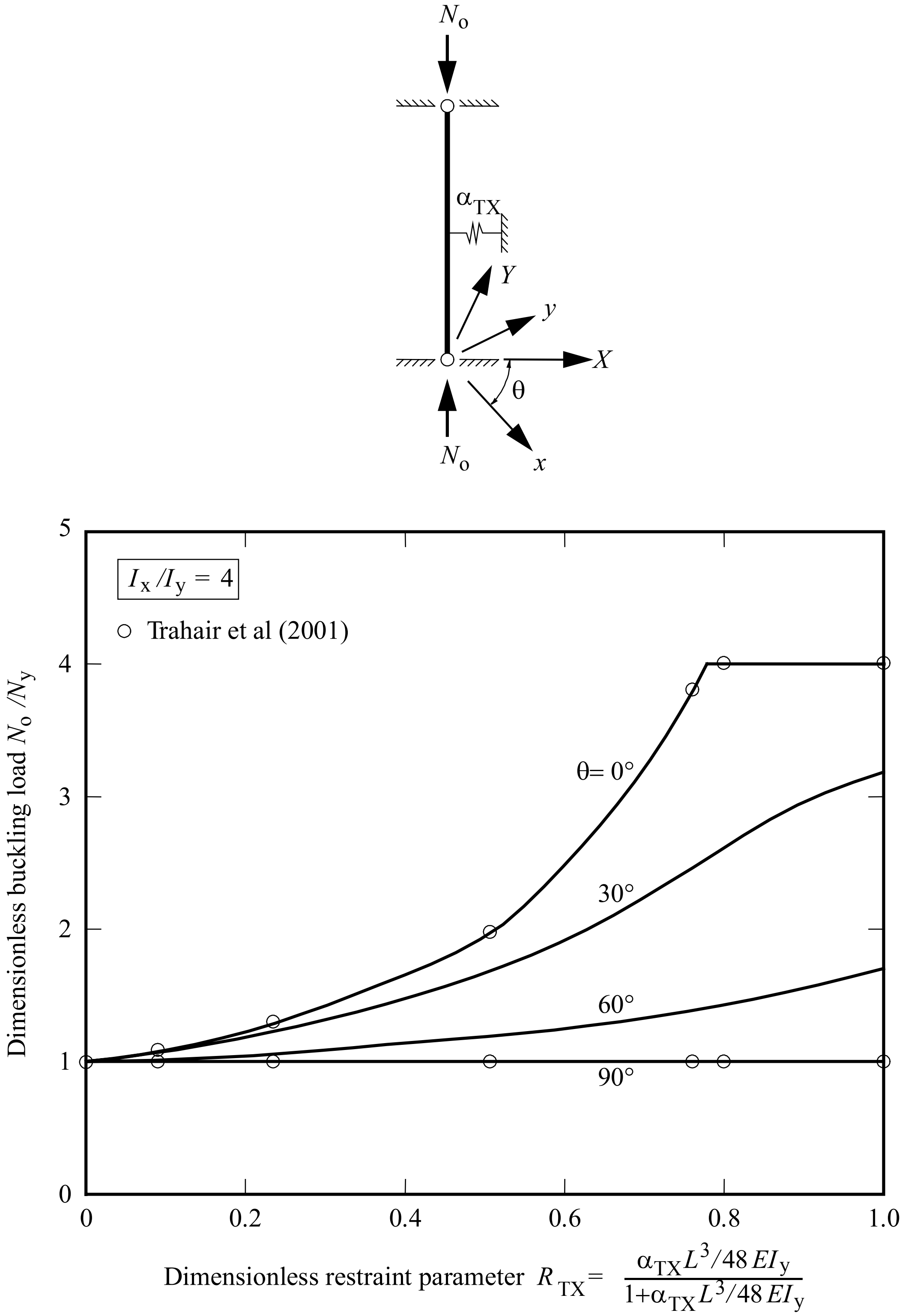


Fig.7 Buckling of a Column with Elastic Translational Central Restraint

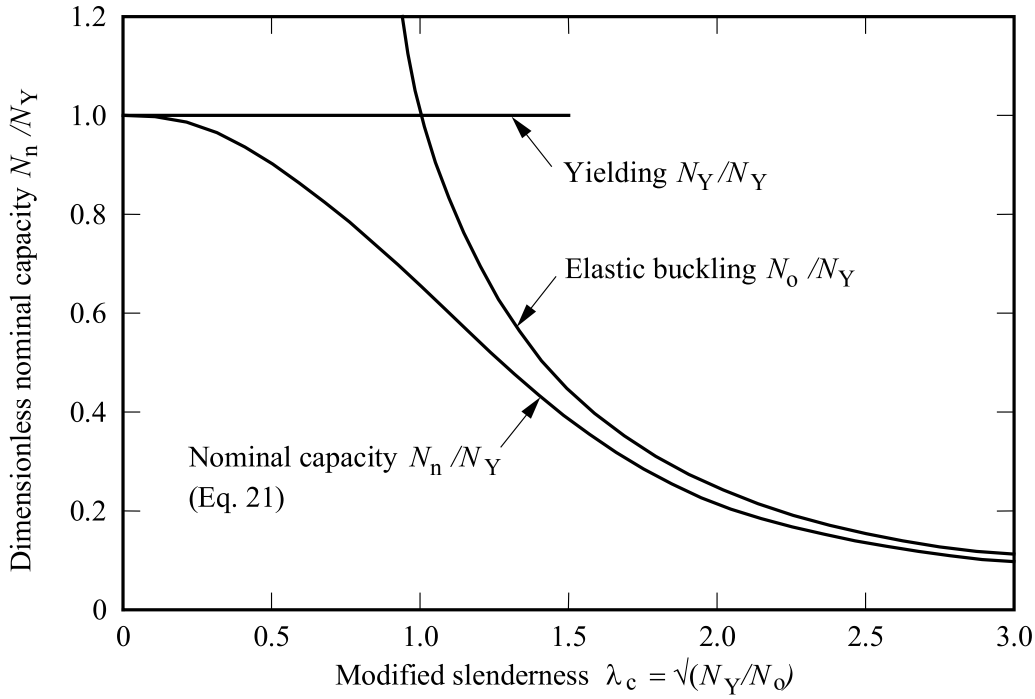
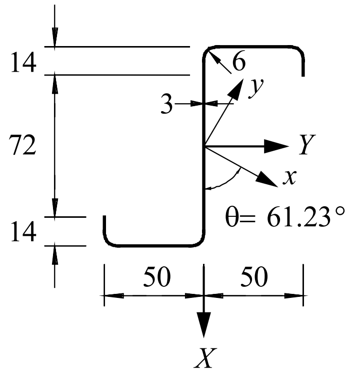


Fig.8 Design by Buckling Analysis



(a) Cross-Section

$E = 2 \times 10^5$ MPa
$f_y = 450$ MPa
$A = 652.4$ mm^2
$I_x = 1.357 \times 10^6$ mm^4
$I_y = 1.359 \times 10^5$ mm^4

(b) Properties

Fig.9 Worked Example