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Three-Dimensional Non-Conforming Elements

Research Report No. R808

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ABSTRACT:

The performance of three-dimensional hexahedral elements has been compared in this study. The standard quadratic 20-noded hexahedral element has shown robust performance in many applications, however, it has a high number of nodes, which necessitates relatively large computational time. The standard linear 8-noded hexahedral element has the least number of nodes in elements of this geometric type, but it is unable to model adequately a state of pure bending. The 8-noded element can be modified by including non-conforming displacements to improve its efficiency. The modified hexahedral element has been adopted in finite element analyses and its excellent performance in terms of accuracy and time efficiency is described.

Keywords: Non-conforming elements, 3-D finite element analysis.

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INTRODUCTION

The linear isoparametric elements are perhaps the simplest constant strain elements. However, the standard linear quadrilateral or hexahedral elements have shown some deficiencies in finite element analyses. For example, they are unable to represent accurately one of the most commonly occurring stress states, i.e. the state of bending stress. Therefore, a large number of these elements, in their standard form, have to be used in order to achieve an acceptable accuracy in problems where bending action is important. The use of a greater number of elements in a finite element analysis usually means a longer computational time, which becomes more profound in three-dimensional analyses. The use of higher order finite elements also leads to a greater number of unknowns for a given number of elements and therefore increases the computational cost. It is therefore desirable to seek an element with minimum number of nodes and acceptable accuracy. Such an element can be developed by removing the deficiencies associated with the linear hexahedral element.

Several methods have been introduced previously to modify the standard linear quadrilateral and hexahedral elements. In particular, Wilson *et al.* (1973) proposed a general method for modification of these elements. This method was modified subsequently by Taylor *et al.* (1976) for the quadrilateral linear element. The aim of the present study is to apply these modifications to a linear hexahedral element and evaluate the efficiency and accuracy of the modified element by performing finite element analyses of some example problems.

The well-known deficiencies associated with linear quadrilateral and hexahedral elements are discussed briefly in this article. Some of the recommendations proposed for the repair of the deficiencies are described. In particular, the method used to modify the deficiency of the standard 8-noded hexahedral linear element will be explained. The accuracy and efficiency of the modified element will be discussed through evaluating the results of three-dimensional analyses using different types of elements.

SOURCES OF ERROR

One of the main causes of the deficiencies associated with lower order finite elements is their inability to represent the state of pure bending strains. This can be illustrated by subjecting a simple rectangular planar element to a pure bending stress as shown in Fig. 1 (Wilson *et al.*, 1973). The top and bottom edges of the finite element remain straight under pure bending moment. The correct displacements increase the strain energy of the system which is generated by the normal strains, ε_x and ε_y. However, the finite element stores the strain energy generated by both the normal strain, ε_x, and the shear strain, γ_{xy}. In other words, the approximation of the state of pure bending with the finite element results in a fictitious prediction of large shear strains. The unwanted shear strain, which is often called 'parasitic shear' (Cook, 1975), causes the behaviour of the finite element to be too stiff. Cook *et al.* (1989) obtained the ratio of the internal bending resistance of the finite element with the exact bending moment as follows:

$$\frac{M_f}{M_e} = \frac{1}{1+\nu}\left(\frac{1}{1-\nu} + \frac{1}{2}\left(\frac{a}{b}\right)^2\right) \qquad (1)$$

where ν is Poisson's ratio, M_f is the finite element bending resistance, M_e is the exact bending moment, a and b are dimensions of the element as defined in Fig. 1. It can be seen from Equation (1) that the effect of the parasitic shear strain becomes significant for elements with large aspect ratio (a/b).

The three-dimensional 8-noded hexahedral element also presents the same deficiency in modelling the state of bending stresses. Application of this element in a finite element analysis of problems that include bending moment usually causes deflections to be grossly underestimated.

REMEDY

Several methods have been proposed to improve the deficiencies of linear elements. Most of the methods were used to modify the performance of the two-dimensional quadrilateral element. However, the same principle could have been used in modification of the three-dimensional linear element. These methods are briefly discussed herein.

There are some basic principles that define a robust and accurate finite element. These principles are usually used to check the validity of the element. For example, any robust finite element must pass a patch test and must not be invariant. It is generally accepted that any element which passes the patch test is a convergent element, provided that no zero energy modes are contained in a finite mesh and the mesh is restrained against rigid body motions. The element must not produce an invariant stiffness matrix which has directional properties. These principles are used in the following sections to evaluate the accuracy of the element being modified by different methods.

In order to achieve a more accurate quadrilateral linear element, the element can be subdivided into two or more triangular sub-regions (e.g., Herrmann, 1973; Felippa, 1966, Cook, 1975, Zienkiewicz and Taylor, 1989). These triangular sub-regions may join each other at an imaginary node inside the original quadrilateral element. The displacement field is then approximated within each triangular sub-region by linear interpolation polynomials. The internal imaginary node is condensed prior to assembly of elements. Application of this method to three-dimensional elements might improve their efficiencies. However, subdivision of three-dimensional hexahedral elements into tetrahedral sub-regions increases the complexity in visualization of the problem. Therefore this approach will not be considered in this study.

Doherty *et al.* (1969) noted that the shear strains predicted by the standard quadrilateral finite element under pure bending stresses are zero at the centre of the element. Therefore they proposed that the terms related to the shear strain energy to be integrated with a low order numerical scheme. In this way the fictitious shear strains near the edges of the element will be ignored. Practically, when integrating to form the element stiffness matrix, terms that include shear strains are evaluated at the centre of the element, regardless of the actual coordinates of the Gauss points at hand. It was found that the bending performance of the modified element was greatly improved. However, the element modified by this method fails the patch test unless it is of rectangular shape. Moreover, the modified element can produce an invariant stiffness matrix and therefore gives different answers when it is used in different directions in global coordinates.

Wilson *et al.* (1973) proposed another method to modify quadrilateral linear elements. In this approach four incompatible nodeless displacement modes are added to the element to improve the representation of bending in the interior of the element. The incompatible degrees of freedom are then condensed at element level. A rectangular element of this type gives exact displacement under pure bending moment, but it should be noted that non-rectangular elements of this type fail the patch test. Wilson *et al.* (1973) also outlined the formulation of the method required to modify of the three-dimensional hexahedral linear element, but did not describe its implementation.

Taylor *et al.* (1976) detected the defects in the element modified by Wilson *et al.* (1973) and repaired them. They proposed that the 'Jacobian' of the incompatible shape functions should be calculated at the centre of the element.

In the current study, the method proposed by Wilson *et al.* (1973), which was modified later by Taylor *et al.* (1976), is used to improve the accuracy of the three-dimensional linear hexahedral element under the state of bending stresses.

BASIC FORMULATION OF INCOMPATIBLE ELEMENTS

Wilson *et al.* (1973) formally introduced non-conforming elements by including additional incompatible displacement modes at the element level. The magnitudes of the modes are selected by requiring that the total strain energy of the element be a minimum. In general these extra displacement modes may violate inter-element compatibility. Therefore the elements modified by this method are termed ‘incompatible’ elements.

The non-conforming displacement modes have been added to two-dimensional quadrilateral elements (Wilson *et al.*, 1973; Taylor *et al.*, 1976). Excellent results have been obtained from planar elements. However, application of the method to axi-symmetric elements yields less convincing results (Wanji and Cheung, 1996). The modification process used for two-dimensional quadrilateral elements can, at least in principle, be applied to the three-dimensional hexahedral elements.

The standard isoparametric interpolation functions for the linear hexahedral element (Fig. 2) are given by:

$$N_i = \frac{1}{8}(1+r_i r)(1+s_i s)(1+t_i t) \quad i=1 \text{ to } 8 \tag{2}$$

Three non-conforming interpolation functions can be considered as:

$$P_1 = 1 - r^2 \ , \qquad P_2 = 1 - s^2 \ , \qquad P_3 = 1 - t^2 \tag{3}$$

In the above equations, r, s and t are the natural coordinates for the element, r_i, s_i and t_i are the values of r, s and t at node i, as shown in Fig. 2.

The local and global coordinate systems of the element are related using the standard interpolation function given by Equation (2):

$$x_j = \sum_{i=1}^{8} x_{ij} N_i \quad , \quad j=1, 3 \tag{4}$$

where x_j is the global coordinate in direction j and x_{ij} is the coordinate of node i in direction j.

The displacement field is approximated considering both standard and non-conforming interpolation functions of (2) and (3):

$$u_j = \sum_{i=1}^{8} u_{ij} N_i + \sum_{i=1}^{3} a_{ij} P_i \quad , \quad j=1, 3 \tag{5}$$

where u_j is the global displacement in direction j, u_{ij} is the displacement of node i in direction j, and a_{ij} is the nodeless non-conforming displacement of mode i in direction j. The vector of non-conforming displacement has 9 components in a three-dimensional hexahedral element.

The normal and shear strains, ε and γ, can also be calculated as follows:

$$(\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \gamma_{12}, \gamma_{23}, \gamma_{31})^T = \sum_{i=1}^{8} \boldsymbol{B}_i (u_{1i}, u_{2i}, u_{3i})^T + \sum_{i=1}^{3} \boldsymbol{G}_i (a_{1i}, a_{2i}, a_{3i})^T \tag{6}$$

where $\boldsymbol{B_i}$ and $\boldsymbol{G_i}$ are strain matrixes as defined below.

$$\boldsymbol{B}_i = \left\{ \begin{bmatrix} N_{i,1} & 0 & 0 \\ 0 & N_{i,2} & 0 \\ 0 & 0 & N_{i,3} \\ N_{i,2} & N_{i,1} & 0 \\ 0 & N_{i,3} & N_{i,2} \\ N_{i,3} & 0 & N_{i,1} \end{bmatrix} \right\} \quad , \quad \boldsymbol{G}_i = \left\{ \begin{bmatrix} P_{i,1} & 0 & 0 \\ 0 & P_{i,2} & 0 \\ 0 & 0 & P_{i,3} \\ P_{i,2} & P_{i,1} & 0 \\ 0 & P_{i,3} & P_{i,2} \\ P_{i,3} & 0 & P_{i,1} \end{bmatrix} \right\} \tag{7}$$

Application of the principle of virtual work to a solid medium leads to the following equation:

$$\begin{bmatrix} \boldsymbol{K}_{uu} & \boldsymbol{K}_{ua} \\ \boldsymbol{K}_{au} & \boldsymbol{K}_{aa} \end{bmatrix} \begin{Bmatrix} u_o \\ a_o \end{Bmatrix} = \begin{Bmatrix} F \\ 0 \end{Bmatrix} \tag{8}$$

where u_o and a_o are the set of nodal values of u_{ij} and a_{ij} for the element, F is the vector of body forces and surface tractions and

$$(\boldsymbol{K}_{uu})_{ij} = \int \boldsymbol{B}_i^T \boldsymbol{D} \boldsymbol{B}_j dV \tag{9}$$

$$(\boldsymbol{K}_{ua})_{ij} = \int \boldsymbol{B}_i^T \boldsymbol{D} \boldsymbol{G}_j dV \tag{10}$$

$$(\boldsymbol{K}_{au})_{ij} = \int \boldsymbol{G}_i^T \boldsymbol{D} \boldsymbol{B}_j dV \tag{11}$$

$$(\boldsymbol{K}_{aa})_{ij} = \int \boldsymbol{G}_i^T \boldsymbol{D} \boldsymbol{G}_j dV \, . \tag{12}$$

In the above equations $\boldsymbol{D}$ is the matrix of material properties, relating stresses to strain components, and V denotes volume.

To eliminate the incompatible displacements from Equation (8), the lower portion of Equation (8) can be solved for a_o, i.e.

$$a_o = -\boldsymbol{K}_{aa}^{-1}\boldsymbol{K}_{au}u_o \tag{13}$$

By substituting Equation (13) into the upper portion of Equation (8), a new form of finite element equation can be obtained which does not include the incompatible displacements, i.e.,

$$\left(\boldsymbol{K}_{uu} - \boldsymbol{K}_{ua}\boldsymbol{K}_{aa}^{-1}\boldsymbol{K}_{au}\right)u_o = F \tag{14}$$

Equation (14) presents a new stiffness matrix in the form of:

$$\boldsymbol{K}_{new} = \boldsymbol{K}_{uu} - \boldsymbol{K}_{ua}\boldsymbol{K}_{aa}^{-1}\boldsymbol{K}_{au} \tag{15}$$

The element can now be treated in the standard fashion, i.e., the new stiffness matrix, $\boldsymbol{K}_{new}$, is assembled, boundary conditions are imposed, and the nodal displacements, u_o, are calculated. The incompatible displacements, a_o, can be recovered using Equation (13). These displacements are then used in conjunction with u_o to approximate the strain field by employing Equation (6).

The hexahedral element developed in this way passes the patch test only if the element is a parallelogram. A similar deficiency can be detected in the performance of the two-dimensional quadrilateral linear element modified by the same method, i.e., by the method proposed by Wilson *et al.* (1973). Taylor *et al.* (1976) proposed a remedy for the modified two-dimensional element. The same remedy is used here to correct the defective terms, and consists of calculating the Jacobian at the centre of the element whenever it is going to be used in conjunction with incompatible terms, i.e., with matrix $\boldsymbol{G}$. Application of this remedy to the modified hexahedral linear element improves the performance of the element.

To demonstrate the accuracy and efficiency of the modified element, finite element solutions of some example problems using different types of elements are compared in the next section.

EXAMPLE PROBLEMS

Two example problems are considered in this section to verify the efficiency and accuracy of the newly developed element. A cantilever beam subjected to a concentrated load is considered as the first example. The problem of the bearing capacity of a shallow circular foundation is also considered. In these problems, the performance of the modified element is compared with those of the standard linear and quadratic hexahedral elements. All elements are integrated numerically by Gauss quadrature rules using a full integration scheme, i.e., using 27 Gauss points for the quadratic 20-noded elements and 8 Gauss points for the standard linear elements and the non-conforming elements.

Cantilever Beam

A cantilever beam subjected to bending moment under a concentrated transverse load is considered here. The length of the beam was divided into *5* elements. The dimension of the beam and the configuration of the finite element mesh used in the analyses are shown in Fig. 3.

The standard linear element, the non-conforming element, and the standard quadratic element were used in the analyses of the beam. The performance of different elements is compared in Fig. 4. In this figure δ is the deflection of the beam, x is the distance from the fixed support, L is the length of the beam, P is the concentrated end load, E is the Young's modulus of the beam material, and I is the moment of inertia of the beam cross section. The results of the analysis obtained using five non-conforming elements are very close to those obtained using five standard quadratic 20-noded elements, and both agree very well with the exact theoretical solution from the engineering theory of bending. However, application of five standard linear 8-noded elements results in significant inaccuracies, e.g., 32% error in prediction of the tip deflection of the beam.

Theoretically accurate results may be obtained using a sufficiently large number of standard linear elements in a fine grid. Therefore two more analyses have been performed using finer grids of the standard linear 8-noded elements. The grids were generated by bisecting the elements in the web of the beam, producing grids of 20 elements and 80 elements. As the number of elements in a mesh increases, the predicted displacement of the beam moves toward the more accurate solutions (Fig. 4). The results of the analysis using 80 linear elements are comparable with the results obtained using five non-conforming elements.

It is worth noting that application of non-parallelogram isoparametric elements in a finite element analysis usually decreases the accuracy of the predicted nodal variables. This is also the case for the non-conforming element, i.e., if non-parallelogram, non-conforming elements are used in a finite element analysis, the analysis will result in a less accurate prediction of displacements. For example, if the mesh configuration shown in Fig. 3 by dashed-lines is used in the analysis of the cantilever beam, the predicted deflection of the beam is underestimated by 4%.

Bearing Capacity of A Shallow Circular Foundation

Finite element analyses of a shallow circular foundation were performed using the newly developed non-conforming element and the standard 8-noded hexahedral elements. The accuracy and efficiency of the different elements are compared here.

The foundation was assumed to be rigid and smooth with a diameter D, and subjected to vertical loading. The soil was assumed to obey the Tresca failure criterion deforming under undrained (constant volume) conditions. The soil was assumed to have an undrained shear strength of s_u, a Poisson's ratio of $\nu = 0.49$

($\cong 0.5$), and a rigidity index of $G/s_u=1000$, where G is the elastic shear modulus. The finite element mesh used in the analysis is shown in Fig. 5. Only a quarter of a cylinder of the soil was modelled. The external diameter and the depth of the soil cylinder were both assumed to be $5D$.

The analyses of the problem have been performed with three different types of elements, i.e., the standard linear element, the standard quadratic element, and the modified non-conforming element. All finite element meshes consisted of an equal number of elements. The load-deflection curves predicted from the analyses are compared in Fig. 6. The finite element predictions of the ultimate bearing capacity of the foundation using both standard quadratic elements and non-conforming elements are very close to the exact theoretical value ($5.69s_u$). However, the finite element analysis using the standard linear hexahedral elements cannot predict the ultimate bearing capacity of the foundation. The predicted load-deflection curve obtained using the linear elements is grossly different from those obtained using the non-conforming and the standard quadratic elements.

The CPU time required for each analysis can be compared with the time spent for the analysis using the standard linear elements, as follows:

$$\frac{T_{non-conforming}}{T_{standard\ \ linear}} = 1.4$$

$$\frac{T_{standard\ \ quadratic}}{T_{standard\ \ linear}} = 36.8$$

The CPU time required for the analysis using the non-conforming element is less than 4% of the time required for the analysis using the standard quadratic element. This makes application of the non-conforming 8-node elements in a finite element analysis of plasticity problems more attractive than the standard 20-node elements.

COMPUTATIONAL EFFICIENCY

In order to demonstrate further the efficiency of the non-conforming hexahedral element, a block of 5×5×5 elastic elements was analysed using different types of elements. The mesh used in the analyses is presented in Fig. 7. A comparison of the accuracy of different elements and the CPU time required for each analysis is indicated in Table 1. All analyses consisted of calculating the stiffness matrix and force vector, solving equations, and calculating strains and stresses. The analysis using the non-conforming elements predicts a displacement within 0.4% of the displacement predicted by the analysis using the standard quadratic elements. However, the real advantage of the non-conforming elements is its time efficiency. The time required for the analysis using the non-conforming elements was about 10% of the time needed for the analysis using the standard hexahedral 20-noded elements. The results of the analysis with 5×5×5 linear elements show about 9% error in calculation of the displacement. To achieve a more accurate prediction of displacements with the standard linear elements, the number of elements in the finite element mesh was increased to 10×10×10 and 15×15×15 elements. As the number of elements in the mesh increases, the predicted displacement moves toward the more accurate solution, i.e. the prediction based on the standard quadratic elements. However, as may be seen from Table 1, the time required for the analysis of a grid of 15×15×15 standard linear elements is about 760 times greater than the time required for the analysis using the non-conforming elements.

CONCLUSIONS

The standard linear hexahedral element exhibits some well-known deficiencies in finite element analyses. This element was modified by including non-conforming displacement modes. The accuracy and efficiency of the modified element were demonstrated by analysing some example problems. The performance of the non-conforming element is very similar to the higher order element, i.e., the quadratic hexahedral element. It was also found that the modified element is extremely efficient and effective in the analysis of three-dimensional problems.

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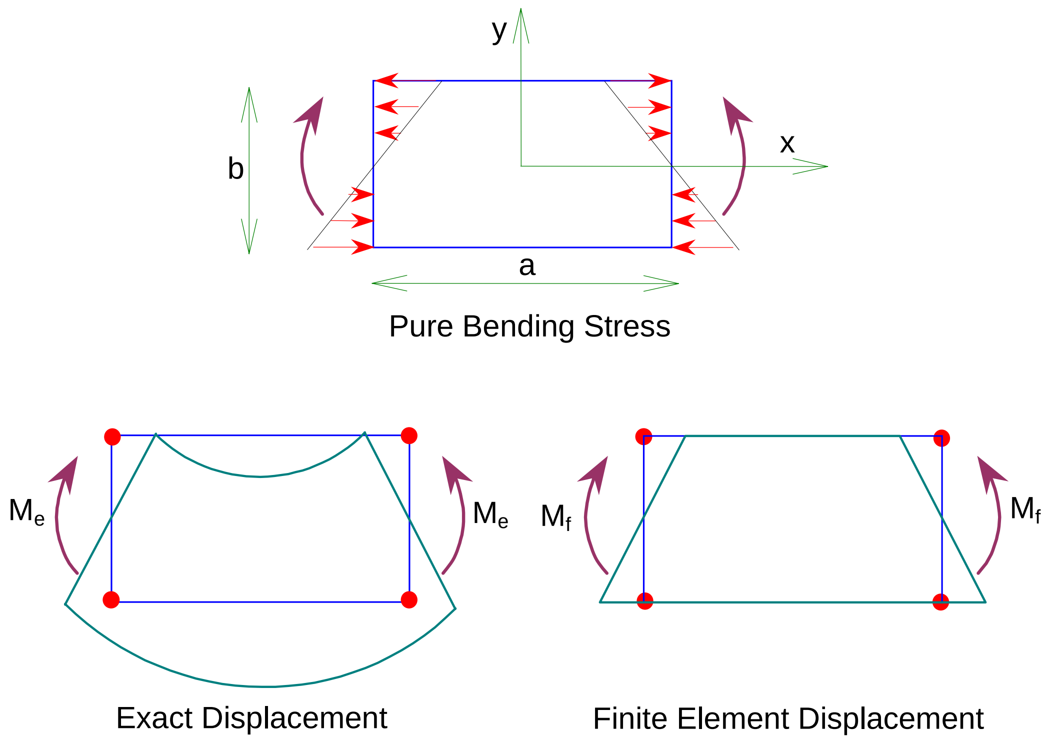

Fig. 1: Bending moment on linear elements

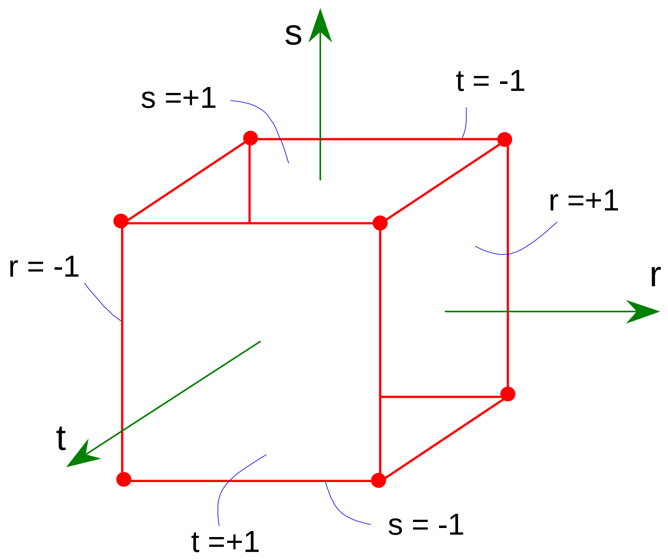

Fig. 2: Linear hexahedral element

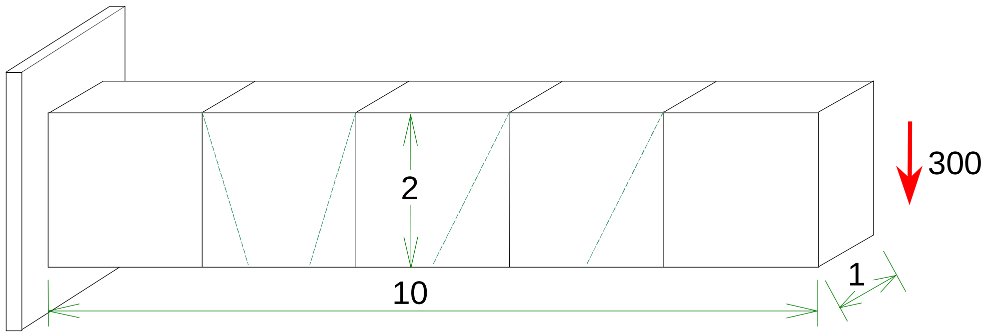

Fig. 3: Geometry of cantilever beam

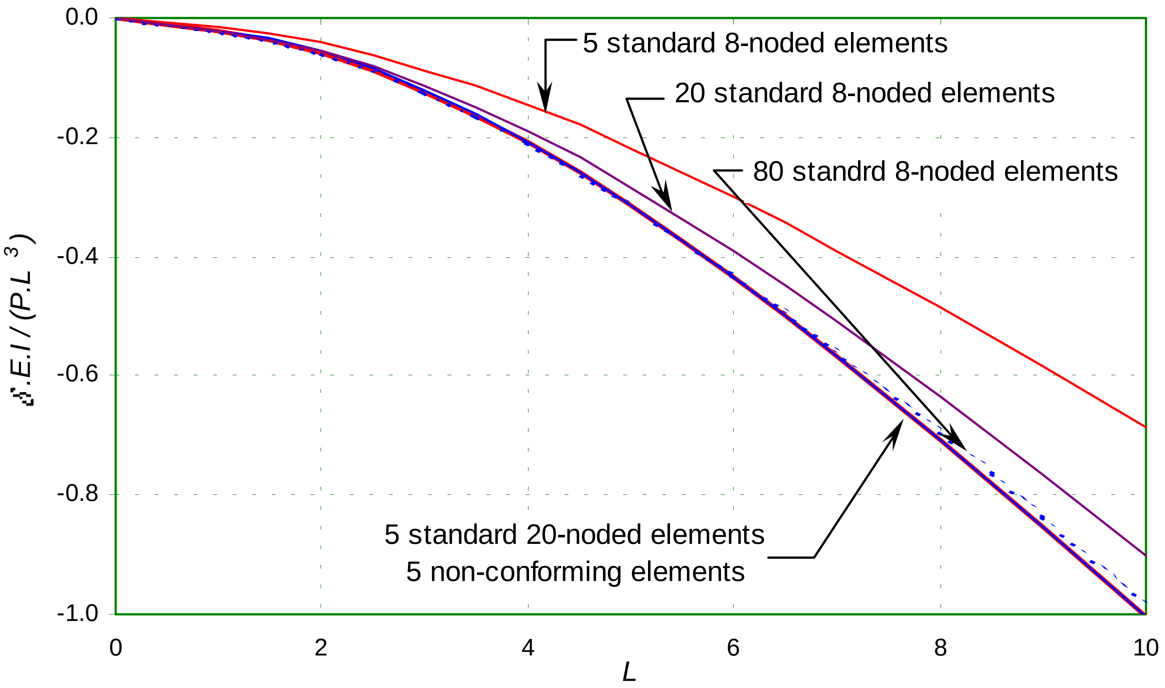

Fig. 4: Deflection of cantilever beam

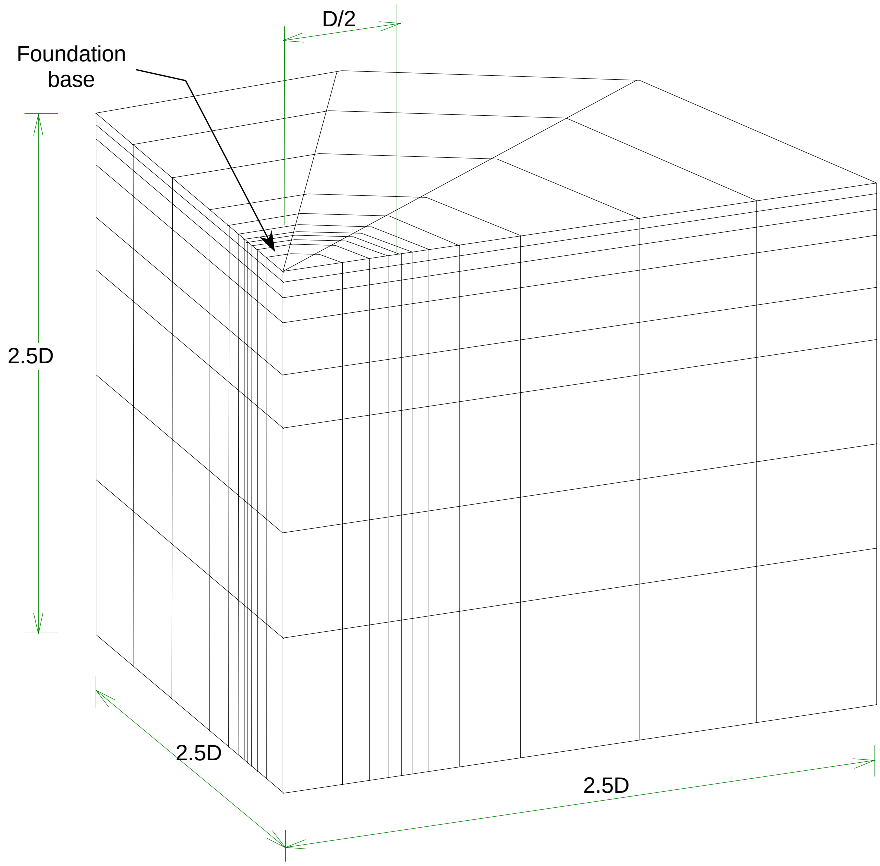

Fig. 5: Finite element idealisation of the bearing capacity problem

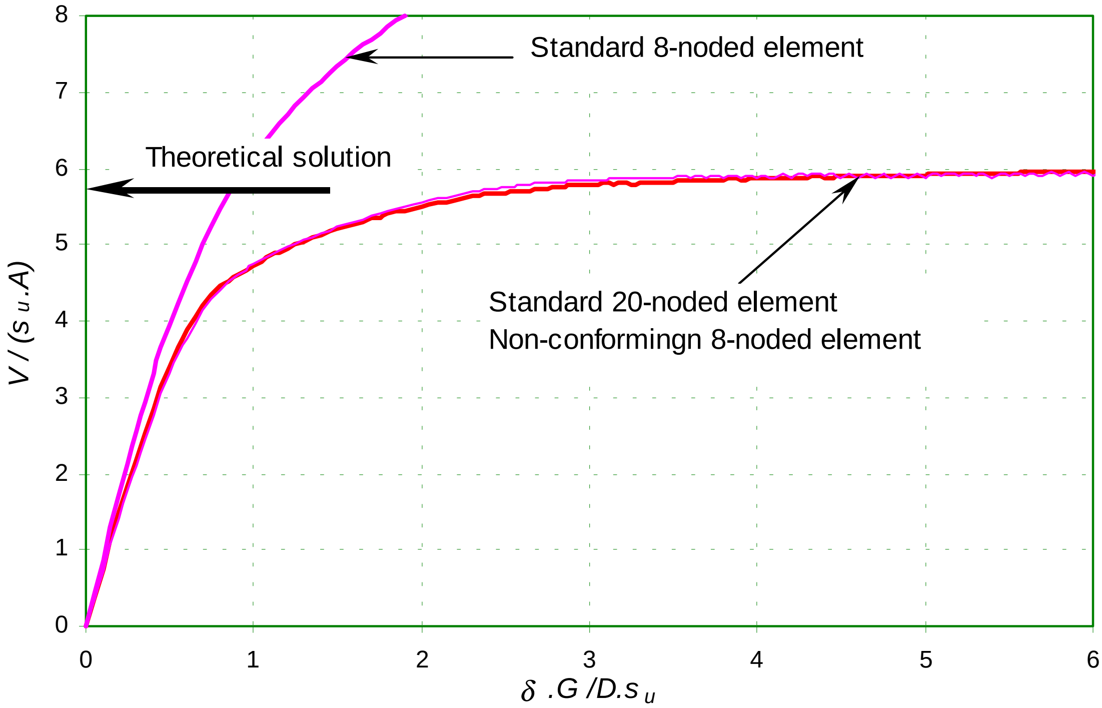

Fig. 6: Comparison of the results of analyses using different elements

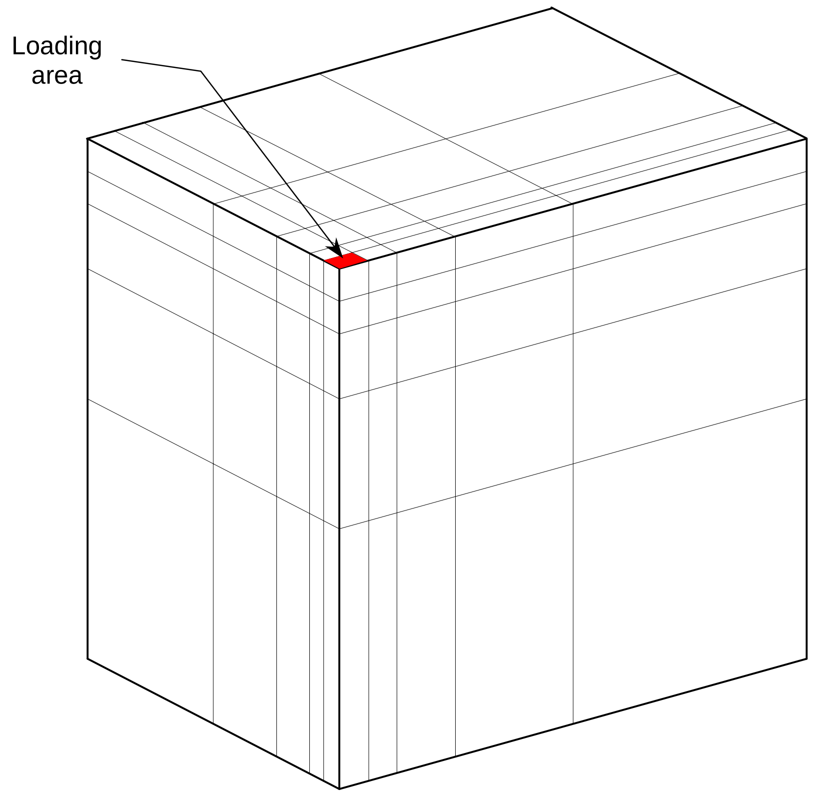

Fig. 7: Finite element mesh of 5×5×5 elements

Table 1: Comparison of the time required for analyses using different elements

Element type and number of elements	Analysis time	Displacement	Accuracy
Standard quadratic, 5×5×5	*t*	*15.64*	*100%*
Non-conforming, 5×5×5	*0.095 t*	*15.70*	*99.6%*
Standard linear, 5×5×5	*0.06 t*	*14.22*	*90.9%*
Standard linear, 10×10×10	*1.91 t*	*15.14*	*96.8%*
Standard linear, 15×15×15	*72 t*	*15.54*	*99.4%*