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Second Order Moments in Torsion Members

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SECOND ORDER MOMENTS IN TORSION MEMBERS

by

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Abstract

This paper is concerned with the elastic flexural buckling of structural members under torsion, and with second-order moments in torsion members. Previous research is reviewed, and the energy method of predicting elastic buckling is presented. This is used to develop the differential equilibrium equations for a buckled member.

Approximate solutions based on the energy method are obtained for a range of conservative applied torque distributions and flexural boundary conditions. A comparison with the limited range of independent solutions available and with independent finite element solutions suggests that the errors in the approximate solutions may be as small as 1%.

The predicted linear elastic buckling torques may be used to approximate the second-order bending moments caused by torsion in members under more general loading. A method is developed for approximating these second-order moments. This is used as the basis of a method of estimating when these second-order moments may be significant by comparing the actual member slenderness with a reference value.

Reference values of slenderness are calculated for two examples involving an equal angle member and a circular hollow section member (both simply supported), and the importance of second-order torsion effects in an I-section member is estimated. The reference values of slenderness are found to be very high, and it is concluded that second-order moments caused by torsion in typical structural steel members with slenderness ratios L/r_y less than 300 are very small and may be neglected.

Keywords: bending, buckling, elasticity, flexure, moments, second-order effects, steel structures, torsion.

1. INTRODUCTION

Second-order moment components of applied torques are usually ignored in structural analysis, but in the most extreme cases, they may lead to flexural buckling. While it is possible that these components are very small in most practical cases and may be neglected, it is nevertheless important to be able to determine when this is not so, so that these second-order effects may be accounted for. The purpose of this paper is to present solutions for the elastic flexural buckling of torsion members, and to use these to develop a method of estimating the importance of second-order moment effects of applied torques in members under more general loading.

The flexural buckling of torsion shafts of circular cross-section is known to mechanical engineers, who prevent this from occurring in long shafts by using intermediate bearings to decrease the unbraced length of the shaft. The most common application is to a shaft of length L which is simply supported at both ends and has a concentrated torque M applied at one end and reacted at the other, as shown in Fig. 1a. It has been reported that such a shaft may buckle elastically (Timoshenko and Gere [1]; Ziegler [2]; Bazant and Cedolin [3]) under a torque

$$M_o = 2 p E I / L \quad (1)$$

in which E is the Young's modulus of elasticity and $I = I_x = I_y$ is the second moment of area of the circular shaft. The buckling mode involves lateral deflections u, v of the longitudinal axis of the shaft in the X, Y directions, which are given by

$$u = U \sin (2p z / L) \quad (2)$$

and
$$v = U (\cos (2p z / L) - 1) \quad (3)$$

in which U is an undetermined magnitude, or vice versa. These deflections correspond to a spiral-like buckled shape without any buckling twist rotations.

Early research on flexural buckling under torque has concentrated on the mechanical engineering problem of circular cross-section shafts with concentrated end torques and axial compressions, and has been summarised by Timoshenko and Gere [1], Ziegler [2], and Bazant and Cedolin [3]. A number of discussions have been made (Ziegler [2]) of the nature of concentrated end torques. In particular, it has been supposed that an axial torque may be non-conservative when it acts at a simply supported end of a shaft which is able to rotate u' or v' ($\circ d/dz$). As a result, a number of other types of end torque have been postulated which are conservative, such as "quasi-tangential" and "semi-tangential" torques (Ziegler, [2]). More recently, Teh and Clarke [4] have proposed a new type of conservative torque.

Ziegler [2] has presented solutions for a number of the flexural boundary conditions indicated in Fig. 2 for a number of different types of concentrated end torque. Barsoum and Gallagher [5] have developed a finite element method of analysis and used it to confirm Ziegler's results. Goto *et al* [6] have studied the effects of pre-buckling twist rotations on the buckling of shafts built-in at both ends, and have investigated their post-buckling behaviour.

However, much of this research has little relevance to civil engineering structures, where the torsion actions result from eccentric loads acting between the end supports, rather than from a concentrated end torque. In addition, the members of civil engineering structures are commonly of non-circular cross-section (although an extension by Grammel in 1923 to non-circular shafts is referred to by Timoshenko and Gere [1]).

In this paper, the nature of conservative end torques is first discussed, and then the energy equation for flexural buckling under conservative tangential torque is presented and used to derive the general differential equations of equilibrium. Approximate solutions are obtained for the flexural buckling of members of non-circular cross-section under concentrated end torque, uniformly distributed torque, and central concentrated torque (Fig. 3) for the flexural boundary conditions illustrated in Fig. 2. These solutions are used to develop a method of estimating the importance of second-order moments caused by torsion in members under combined bending, torsion, and compression actions.

2. CONSERVATIVE FORCES AND TORQUES

2.1 Conservative Forces

A conservative force P is often defined as a force of constant magnitude for which the work done

$$W = \int (P_x du + P_y dv + P_z dw) \quad (4)$$

when its point of application moves s from one position A to another B along any path $x(s), y(s), z(s)$ is independent of the path, in which P_x, P_y, P_z are the components of the force and du, dv, dw are the corresponding incremental displacements. In structural engineering applications, the point of application is usually a point fixed on a body, so that the force moves with the body.

When only conservative forces act on a body, its stability may be analysed by using the energy method based on the principle of conservation of energy, according to which the increase in the strain energy of the body when it moves under constant forces from an initial (pre-buckled) equilibrium position A to an adjacent buckled equilibrium position B is equal to the work done by the forces. The application of this principle is simplified because it is only necessary to consider the two positions A and B, and not the path s between them.

The work done by a general force may depend on both the path s of its point of application and its directions q as it moves along the path. For example, the direction of the end “follower” force shown in Fig. 4 remains tangential to the end slope of the cantilever column. For the first buckling path shown in Fig. 4a, the cantilever end and the force first rotate q without deflection, and then deflect v without rotation, but for the second path shown in Fig. 4b, the order is reversed. The amounts of work done by the force for these two paths differ by $-(P \cos q) v$ approximately, and so the force is non-conservative. It can be seen that the changes

in the directions q of a force P change its components P_x, P_y, P_z , and may cause changes in the work done by the force (see Equation 4).

The directions of a force may be defined in a number of ways, and two examples are shown in Fig. 5. In Fig. 5a, the force direction remains fixed (“gravity” loading), and the elastic buckling load P_o is equal to $p^2 EI / 4L^2$ (Timoshenko and Gere [1]). In Fig. 5b, the force direction remains radial (this problem is often referred to as the “flagpole” problem), and the elastic buckling load is equal to $p^2 EI / L^2$ (Simitses [7]). Both of these are examples of conservative forces.

On the other hand, the follower force shown in Fig. 4 is non-conservative, and the final position is not one of static equilibrium (because the moments exerted by the force are of opposite sign to those which correspond to the column curvatures). A dynamic analysis of this problem (Ziegler [2]) shows that dynamic deflections become possible at a load equal to $2.03p^2 EI / L^2$ approximately.

2.2 Conservative Moments and Torques

Whether a constant moment (or torque) M is conservative or not is often discussed by using an analogy between moments and forces. Thus a constant moment is considered to move s along a path $x(s), y(s), z(s)$ from one position A to another position B, during which its directions q are defined. The work done by the moment is given by

$$W = \int (M_x dq_x + M_y dq_y + M_z dq_z) \quad (5)$$

in which M_x, M_y, M_z are the components of the moment M and dq_x, dq_y, dq_z are the corresponding incremental rotations. Thus the moment is conservative if the work done is independent of the path. The analogy is not perfect, however, since while the work done by a force is affected by both its incremental deflections and rotations as it moves along the path, the work done by a moment is affected only by its incremental rotations.

Concerns about the nature of an external torque are affected by considerations of how it may arise in practice and of how it may rotate during rotations of the body to which it is attached. A torque whose directions do not change with the rotations q_x, q_y of the longitudinal axis of the member to which it is attached may be described as axial. However, it seems likely that such a torque will rarely occur in practice.

A type of torque which is more likely to occur is one whose directions change with the rotations q_x, q_y of the longitudinal axis of the member. This type of torque is often referred to as a “configuration dependent” torque, and has characteristics somewhat analogous to those of a follower force, whose directions also change with the member rotations q_x, q_y . Because the follower force is known to be non-conservative, there may be concern that a configuration dependent torque may also be non-conservative.

Ziegler's [2] argument that an axial torque may be non-conservative is related to the torque vector M acting about the Z axis shown in Fig. 6, and uses the definition of a conservative action as one for which the work done during any motion depends only on the initial and final positions of the action and not on the path followed. If the torque M in Fig. 6 moves through a first path by rotating $q_X (= -v') = p$ about the X axis and then by $q_Y (= u') = p$ about the Y axis, then no work is done. If however the torque moves to the same final position by rotating $q_Z = p$ about the Z axis, then the work done Mp is non-zero. Thus the work done is path-dependent, and so the torque must be non-conservative. It may be noted that some researchers [1, 8, 9] show that this conclusion is also valid if the torque M does not rotate with the body, although the physical reality of such a torque is doubtful [10].

On the other hand, a configuration dependent torque is conservative if it can be generated by a set of conservative forces. Ziegler [2] discussed three types of conservative configuration dependent torques which can be generated by sets of conservative forces. The first of these is the quasi-tangential torque, an example of which is shown in Fig. 7, where universal joints are used to transmit a torque through a simply supported member.

The behaviour of a universal joint is illustrated in Fig. 8. This joint has a rigid central "spider" in the shape of a cruciform, the four ends of which are connected through frictionless bearings to two rigid yokes, which themselves are rigidly connected to two shafts. This joint allows relative rotations $q_X (= -v')$, $q_Y (= u')$ between the shafts to take place about the X , Y axes while transmitting a torque M from one shaft to the other.

For the simply supported member shown in Fig. 7, a conservative torque M (generated by a pair of equal and opposite conservative forces) is applied to a rigid drive shaft which can only rotate about the fixed Z axis. This torque is transmitted through a universal joint to the simply supported member, and then through a second universal joint to a reaction point. If the arm of the universal joint spider connected to the drive shaft is parallel to the fixed Y axis, then the rigid drive shaft may supply a reaction moment M_{XL} to the member. This moment can be determined as $M_{XL} = M q_{YL}$ by satisfying the local member end condition $M_{XL} = 0$ associated with the freedom of the end to rotate about the arm of the spider in the XZ plane. The resultant of the moments M and M_{XL} is equivalent to two equal and opposite horizontal forces acting at the ends of the spider arm parallel to the Y axis. The torque exerted by these forces on the member is a quasi-tangential torque (Ziegler [2]).

The elastic buckling resistance of a simply supported member under equal and opposite quasi-tangential torques applied through universal joints depends on the angle between the arms of the spiders connected to the drive shaft at $z = L$ and the reaction at $z = 0$ (Koiter [10]). When these arms are parallel, the linear elastic buckling torque is given by

$$M_o = p EI / L \quad (6)$$

which is half the value given by Equation 1.

Another type of conservative configuration dependent torque is the semi-tangential torque (Ziegler [2]). This is similar to the quasi-tangential torque, except that the resultant end torque M corresponds to two equal end torques $0.5 M$, which are equivalent to two pairs of equal and opposite conservative transverse forces at 90° to each other. In this case, the member end conditions are $M_{XL} = 0.5 M q_{YL}$ and $M_{YL} = -0.5 M q_{XL}$. The linear elastic buckling torque is reported by Ziegler [2] as being given by

$$M_o = 1.564 p E I / L \quad (7)$$

Ziegler [2] also discussed a third type of conservative configuration dependent torque, referred to as a pseudo-tangential torque, but because the value of this torque decreases towards zero as the end twist rotation increases towards 90° , it does not lead to instability.

Recently, Teh and Clarke [4] proposed a fourth type of conservative configuration dependent torque, and demonstrated a physical mechanism which produces this as an external torque. This mechanism is shown in Fig. 9. The torque exerted on the member $M = P a$ is generated by two equal conservative forces P , each of which is transmitted along a cable, over a fixed pulley, to a pulley mounted on a moveable slide, and finally to the end of a rigid arm of length a which is rigidly connected to the end of the member.

If the end of the member rotates q_{YL} about an axis parallel to the Y axis, then the corresponding rotation of the rigid arm a causes relative displacements of the ends of the arm of $a q_{YL}$ parallel to the Z direction, so that the end forces P on the arm exert a moment

$$M_{XL} = P a q_{YL} = M q_{YL} \quad (8)$$

If the end of the member rotates q_{XL} about an axis parallel to the X axis, then the winders (of radius r) at the ends of the rigid arm rotate, and wind up and unwind cables attached to them by distances $r q_{XL}$, which cause horizontal displacements $r q_{XL}$ of the pulleys on the moveable slides. These displacements cause the cables transmitting the forces P to the rigid arm to rotate $r q_{XL} / s$, so that horizontal components $P r q_{XL} / s$ act parallel to the Z direction at the ends of the rigid arm. These components exert a moment about the Y axis which is given by

$$M_{YL} = - P (r q_{XL} / s) a = - M r q_{XL} / s \quad (9)$$

When $r = 0$, then $M_{YL} = 0$, and the torque M is quasi-tangential. When $r = s$, then

$$M_{YL} = - M q_{XL} \quad (10)$$

In this case, the resultant of M , M_{YL} , M_{XL} is aligned with the rotated member axis, and so the conservative torque of Teh and Clarke [4], is a follower torque. By analogy with the use of semi-tangential to describe a conservative torque M with components $M_{XL} = 0.5 M q_{YL}$ and $M_{YL} = -0.5 M q_{XL}$, the conservative follower torque of Teh and Clarke with components given by Equations 8 and 10 may be described as a

“tangential” torque. For this paper, it is assumed that all applied torques are of this tangential type.

3. THEORY

3.1 Energy Equation

The energy equation for flexural buckling of a member of length L under conservative tangential torques (Teh and Clarke [4]) is

$$\frac{1}{2} \int_0^L \{ EI_x (v'')^2 + EI_y (u'')^2 - M_z (u'v'' - v'u'') \} dz = 0 \quad (11)$$

in which I_x and I_y are the second moments of area about the x and y principal axes, M_z is the internal torque at a distance z along the member, and (') is equivalent to (d/dz) .

The first two terms in this equation equal the increase in the strain energy of the member during buckling and the third term equals the work done by the torsional loading. The term $M_z u'v'' dz$ can be thought of as representing the work done by the moment component about the x axis $M_z u'$ (see Fig. 10a) of the torque M_z when there is an angle change $v'' dz$ about the x axis between the ends of an element of length dz (see Fig. 10b). The term $-M_z v'u'' dz$ is similar.

Equation 11 may be transformed to

$$\frac{1}{2} \begin{Bmatrix} U \\ V \end{Bmatrix}^T \begin{bmatrix} \int_0^L EI_y (f_u'')^2 dz & -\frac{1}{2} M_o \int_0^L f_m (f_u' f_v'' - f_v' f_u'') dz \\ -\frac{1}{2} M_o \int_0^L f_m (f_u' f_v'' - f_v' f_u'') dz & \int_0^L EI_x (f_v'')^2 dz \end{bmatrix} \begin{Bmatrix} U \\ V \end{Bmatrix} = 0 \quad (12)$$

in which

$$u = U f_u(z) \quad (13)$$

$$v = V f_v(z) \quad (14)$$

$$M_z = M_o f_m(z) \quad (15)$$

in which U, V, M_o are constants and $f_u(z), f_v(z)$, and $f_m(z)$ are shapes. Equation 12 may be solved as

$$M_o^2 = \frac{4 EI_y EI_x \left\{ \int_0^L (f_u'')^2 dz \right\} \left\{ \int_0^L (f_v'')^2 dz \right\}}{\left\{ \int_0^L f_m (f_u' f_v'' - f_v' f_u'') dz \right\}^2} \quad (16)$$

Approximate values of the magnitude of the buckling torque M_o may be obtained by substituting guessed functions for the buckled shapes f_u, f_v into Equation 16. If these shapes satisfy the kinematic boundary conditions, the approximate solution will be

greater than or equal to the true value of M_o (Trahair [11]). The accuracy of the approximate solution may therefore be improved by including arbitrary parameters a in the buckled shapes in the form of

$$f_u = u_1 + a_u u_2 \quad (17)$$

$$f_v = v_1 + a_v v_2 \quad (18)$$

and then minimising M_o with respect to these parameters.

A convenient method of obtaining the improved approximation is to use a computer program such as Mathcad (MathSoft, [12]) to define the initial buckling shapes of Equations 17 and 18, to evaluate the differentials and integrals of Equation 16, and to solve it for the buckling torque M_o . Repeated trials using different parameters a can then be made to minimise the solution and improve the accuracy.

3.2 Differential Equations of Equilibrium.

The differential equilibrium equations for flexural buckling under torque can be obtained from the energy equation (Equation 11) by using the calculus of variations, according to which a function

$$\int_0^L F(z, u', u'', v', v'') dz = 0 \quad (19)$$

is stationary when

$$-\frac{d}{dz} \left(\frac{\partial F}{\partial v'} \right) + \frac{d^2}{dz^2} \left(\frac{\partial F}{\partial v''} \right) = 0 \quad (20)$$

and

$$-\frac{d}{dz} \left(\frac{\partial F}{\partial u'} \right) + \frac{d^2}{dz^2} \left(\frac{\partial F}{\partial u''} \right) = 0 \quad (21)$$

These conditions allow the differential equilibrium equations to be obtained as

$$-(EI_x v'')'' = -\frac{1}{2}(M_z u'')' - \frac{1}{2}(M_z u')'' \quad (22)$$

$$(EI_y u'')'' = -\frac{1}{2}(M_z v'')' - \frac{1}{2}(M_z v')'' \quad (23)$$

For the special case of simply supported beams with constant $M_z = M_o$, these become

$$-EI_x v'' = -M_o \{u' - (u_L' - u_o')z/L - u_o'\} \quad (24)$$

and

$$EI_y u'' = -M_o \{v' - (v_L' - v_o')z/L - v_o'\} \quad (25)$$

The first terms on the right hand sides of these equations are the second-order moment components about the local x and y axes of the torque M_o (see Fig. 10a, for example).

One closed form solution of Equations 24 and 25 is given by Equations 1-3 for $I_x = I_y = I$. More generally, however, closed form solutions are not available for Equations 22 and 21, and numerical methods must be used, such as the finite element method of Teh and Clarke [4]. Alternatively, approximate solutions may be obtained by using the energy method presented in Section 3.1 above.

4 SOLUTIONS

4.1 End Torque

An elastic buckling solution for the concentrated end torque M_o acting on the simply supported member shown in Fig. 1a is given by Equations 1-3 with EI replaced by $\ddot{O}(EI_x EI_y)$. These equations also satisfy the kinematic boundary conditions $u_0 = u_L = v_0 = v_L = 0$ and the differential equations of equilibrium (Equations 24 and 25). It can be shown that the buckled shapes defined by Equations 2 and 3 can be rotated into perpendicular planes inclined at any angle to the XZ , YZ planes without any change from the elastic buckling torque of Equation 1.

A lower elastic buckling solution for the concentrated tangential end torque M_o acting on the simply supported member shown in Fig. 1a is given by Equation 7 with EI replaced by $\ddot{O}(EI_x EI_y)$. The corresponding buckled shapes are shown in Fig. 1b. The buckled shape u is symmetric with non-zero end slopes, so that the tangential end torques M have equal out-of-balance components $Mq_{y0,L}$ acting about the Y axis. These are balanced by equal and opposite end reactions Mq_{y0}/L , which cause anti-symmetric bending about the X axis and deflection v , as shown in Fig. 1b. The combination of the symmetric torque distribution with the symmetric u and the anti-symmetric v allows the value of $\partial M_z(u'v'' - v'u'')dz$ in Equation 11 to be non-zero (a zero value would lead to an infinite solution for M_o being obtained from Equation 16).

The solutions obtained using the finite element method of Teh and Clarke [13] for the flexural boundary conditions illustrated in Fig. 2 are given in Table 1 by the values of the buckling factor k in

$$M_o = k p \ddot{O}(EI_x EI_y)/L \quad (26)$$

in which M_o is the maximum torque in the member at elastic flexural buckling (see Fig. 3). Also given in Table 1 (and shown in brackets) are a number of approximate solutions obtained by the energy method of Section 3.1. The approximate energy method values in Table 1 of $k = 2.891$, 2.188 , and 2.019 for built-in, propped, and sway members are about 1% higher than the values of $k = 2.861$, 2.168 , and 2.0 quoted by Ziegler [2], and the values of $k = 1.658$ and 1.053 are about 6% higher than the values of $k = 1.564$ and 1.0 quoted for simply supported and cantilevered

members. The values obtained by the finite element method are in agreement with those quoted by Ziegler.

4.2 Other Torque Loadings

The finite element solutions for the buckling factor k for the flexural boundary conditions illustrated in Fig. 2 for central concentrated or uniformly distributed torque are also given in Table 1.

The distributions of u and v of a simply supported member with central concentrated torque (see Fig. 11) are both symmetric. The change from the anti-symmetric v shown in Fig. 1b for a concentrated end torque to the symmetric v shown in Fig. 11b is caused by the change in the torque distribution from symmetric to symmetric, as shown in Fig. 3a and c. The combination of the anti-symmetric torque distribution with the symmetric u and v allows the value of $\delta M_z(u'v''-v'u'')dz$ in Equation 11 to be non-zero

The buckling torque coefficients k for members with central concentrated torque are generally greater than those for concentrated end torque (except for sway members), and the coefficients for uniformly distributed torque are even greater.

Table 1. Finite Element (and approximate) Elastic Buckling Coefficients k

Torque Loading	Concentrated End	Uniformly Distributed	Uniformly Distributed	Central Concentrated
Torque Reactions	One End	One End	Both Ends	Both Ends
Built-In	2.861 (2.891)	-	6.719	4.000
Propped	2.168 (2.188)	-	3.725	2.663
Simply Supported	1.564 (1.658)	-	2.567	2.000
Sway	2.000 (2.019)	-	4.842	2.000
Cantilevered	1.000 (1.053)	1.215 (???)	-	-

5 REFERENCE SLENDERNESS

5.1 Second-Order Compression Effects

The designers of civil engineering steel structures are used to allowing approximately for any small second-order effects Pv (or Pu) of axial compression on the bending moment distribution. This is commonly done (SA [14]; Trahair and Bradford [15]) by amplifying the first-order bending moments M_x (or M_y) by multiplying them by approximate amplification factors of the form of

$$d_p = \frac{1}{1 - P/P_o} \tag{27}$$

in which P is the axial compression in the member and P_o is the value of P which causes elastic buckling. This approximation is generally conservative, especially when the shape of the first-order bending deflections is different from that of the lowest buckling mode associated with P_o .

Equation 27 shows that the relative importance of these second-order effects depends on the ratio P/P_o , which has a maximum value of P_p/P_o , in which P_p is the value of P at plastic collapse. Second-order effects are always small when P_p/P_o is small because when the load P reaches the plastic collapse load P_p , the load P is still small compared with the elastic buckling load P_o . Second-order effects become more significant when P_p/P_o approaches 1, when failure by plastic collapse and by elastic buckling are equally likely. The value of $P_p/P_o = 1$ may therefore be regarded as a reference condition.

In the specific case of simply supported compression members,

$$\frac{P_p}{P_o} = \frac{Af_y}{\mathbf{p}^2 EI / L^2} \quad (28)$$

in which A is the cross-sectional area and f_y is the yield stress. The value of $P_p/P_o = 1$ corresponds to a reference geometrical slenderness

$$\left(\frac{L}{r} \right)_r = \sqrt{\mathbf{p}^2 E / f_y} \quad (29)$$

For steel members with $E = 200,000$ MPa and $f_y = 300$ MPa, this reference slenderness is $(L/r)_r = 81.1$. Thus for low slenderness members (say $L/r \gg 20$), second-order effects are small ($d_p < 1 / (1 - (20 / 81.1)^2) = 1.065$) and may be neglected.

5.2 Second-Order Torsion Effects

Small second-order moments caused by torque may be allowed for approximately by adapting the method described above for the second-order moments caused by axial compression. Thus the first-order bending moments M_x and M_y may be amplified by multiplying them by approximate amplification factors

$$\mathbf{d}_m = \frac{1}{1 - M / M_o} \quad (30)$$

in which M is the maximum torque in the member and M_o is the value of M at elastic flexural buckling. This approximation will generally be conservative also, because the first-order deflected shapes will usually be different from the buckling shapes.

Equation 30 shows that the relative importance of these second-order effects depends on the ratio M/M_o , which has a maximum value of M_p/M_o , in which M_p is the value of M at plastic collapse. Second-order effects are small when M_p/M_o is small, but become more significant as M_p/M_o approaches 1. Thus the value of $M_p/M_o = 1$ may also be regarded as a reference condition.

5.3 Plastic Torsion Collapse

Conservative approximate methods have been developed for determining the value of the maximum torque M_p at plastic collapse (Pi and Trahair [16]; Trahair and Bradford [15]; and Trahair [17]). These methods can be applied to doubly and mono-symmetric open sections which resist torsion by a combination of uniform and warping torsion, as well as to hollow sections and general sections which resist only uniform torsion.

5.4 Reference Slenderness Examples

5.4.1 Angle section member

For this first example, a thin-walled angle section member is assumed to be simply supported in flexure, and loaded by uniformly distributed torque which is reacted equally at both ends. For this case, Table 1 gives the maximum torque at elastic flexural buckling as

$$M_o = 2.567 p \sqrt{(EI_x EI_y)} / L \quad (31)$$

At plastic collapse, the maximum torque at the reaction M_p is equal to the uniform torsion plastic torque approximated by (Trahair and Bradford [15])

$$M_{up} = (b_1 + b_2)(t^2 / 2)t_y \quad (32)$$

in which b_1 and b_2 are the leg lengths of the angle, t is the thickness, and the shear yield stress is given by

$$t_y = f_y / \sqrt{3} \quad (33)$$

For a steel ($E = 200,000$ MPa, $f_y = 300$ MPa) thin-walled equal leg angle ($b_1 = b_2 = b$),

$$I_x = 4 I_y = b^3 t / 3 \quad (34)$$

and so the reference length defined by $M_p = M_o$ is

$$L_r = \frac{2.567 p \times 200,000 \times (b^3 t / 3) \sqrt{1/4}}{(bt^2) \times 300 / \sqrt{3}} \quad (35)$$

and if $r_y = b / \sqrt{24}$ is substituted, then this corresponds to

$$\left(\frac{L}{r_y} \right)_r = 7603 \frac{b}{t} \quad (36)$$

for the reference slenderness. Most practical angles have $b / t > 6$, and so $(L / r_y)_r > 45,600$ approximately. But practical members usually have $L / r_y < 300 \gg (L / r_y)_r / 150$. It can therefore be concluded that second-order moment effects due to torsion in practical angles will be very small, and can be neglected.

It may be noted that the torsional rigidities GJ of thin-walled angle section members are much lower than their flexural rigidities EI_x, EI_y . A comparatively low value of GJ will cause large twist rotations to occur long before the elastic buckling torque M_o , which depends on $\sqrt[3]{(EI_x EI_y)}$, can be reached.

5.4.2 Circular hollow section member

For this second example, a thin-walled circular hollow section member is assumed to be simply supported in flexure, and loaded by a central concentrated torque which is reacted equally at both ends. For this case, Table 1 gives the maximum torque at elastic flexural buckling as

$$M_o = 2.000 p \sqrt[3]{(EI_x EI_y)} / L \quad (37)$$

At plastic collapse, the maximum torque at the reaction M_p is equal to the uniform torsion plastic torque approximated by

$$M_{up} = p (d^2 t / 2) t_y \quad (38)$$

in which d is the diameter of the section and t is the thickness.

For a steel ($E = 200,000$ MPa, $f_y = 300$ MPa) thin-walled circular hollow section,

$$I_x = I_y = p d^3 t / 8 \quad (39)$$

and so the reference slenderness defined by $M_p = M_o$ corresponds to

$$L_r = \frac{2.000 p \times 200,000 \times (p d^3 t / 8)}{(p d^2 t / 2) \times 300 / \sqrt{3}} \quad (40)$$

and if $r = d / \sqrt{2}$ is substituted, then this becomes

$$\left(\frac{L}{r} \right)_r = 5126 \quad (41)$$

Thus second-order moment effects due to torsion will always be small in practical circular hollow section members with $L/r < 300$.

It may be noted that circular hollow section members have torsional rigidities GJ which are comparable with their flexural rigidities EI_x, EI_y . Despite this, large twist rotations will generally occur before the elastic buckling torque M_o can be reached.

5.4.3 I-section member

For this third example, an I-section member is assumed to be simply supported in flexure, and loaded by a central concentrated torque which is reacted equally at both ends. For this case, Table 1 gives the maximum torque at elastic flexural buckling as

$$M_o = 2.000 p \ddot{O}(EI_x EI_y) / L \quad (42)$$

At plastic collapse, the maximum torque at the reaction M_p is approximated by (Trahair and Bradford [15])

$$M_p = M_{up} + 2 M_{fp} d_f / L \quad (43)$$

in which the uniform torsion plastic torque is given by (Trahair and Bradford [13])

$$M_{up} = (b_f t_f^2 - t_f^3 / 3 + b_w t_w^2 / 2 + t_w^3 / 6) t_y \quad (44)$$

in which b_f and b_w are the flange width and web depth, and t_f and t_w are the flange and web thickness, the flange fully plastic moment is given by

$$M_{fp} = (b_f^2 t_f / 4) f_y \quad (45)$$

and $d_f (= b_w + t_f)$ is the distance between flange centroids.

For a steel ($E = 200,000$ MPa, $f_y = 300$ MPa) I-section with $b_f = 305$ mm, $b_w = 288.2$ mm, $t_f = 15.4$ mm, $t_w = 9.9$ mm, $I_x = 223 \times 10^6$ mm⁴, $I_y = 72.9 \times 10^6$ mm⁴, $r_y = 76.7$ mm, a slenderness ratio of $L/r_y = 300$ corresponds to $L = 23010$ mm. For this member $M_o = 6964$ kNm and $M_p = 17.4$ kNm. Thus the equivalent reference ratio of $M_p / M_o = 17.4 / 6964 \gg 0.0025$ is very small, and so the second-order moments caused by torsion will also be very small.

It may be noted that the torsional stiffnesses of I-section members are significantly lower than their flexural stiffnesses, and that large twist rotations will generally occur before elastic buckling can take place.

6 CONCLUSIONS

In this paper the second-order moment components of applied torques, which are usually ignored in structural analysis, are studied. Previous research on the elastic flexural buckling of structural members under torsion is first reviewed, and the conservative nature of applied torques is discussed. The energy method of predicting elastic buckling is then presented and used to develop the differential equilibrium equations for a buckled member.

Finite element solutions are obtained for a range of applied torque systems and flexural boundary conditions, together with a number of approximate energy method solutions. A comparison of these approximate solutions with the limited range of

independent solutions available suggests that the errors in the approximate solutions may be as low as 1%.

The predicted linear elastic buckling torques may be used to approximate the second-order bending moments caused by torsion in members under more general loading. A method is developed for approximating these second-order moments. This is used as the basis of a method of estimating when these second-order moments may be significant by comparing the actual member slenderness with a reference value. The reference value is obtained from a strength reference condition for which the torque M_p at plastic collapse is equal to the torque M_o at elastic buckling. A member whose slenderness is much lower than the reference value has a value of M_p which is much lower than M_o , and will fail by plastic collapse long before second-order effects can become significant.

Reference values of slenderness are calculated for two examples involving an equal angle member and a circular hollow section member (both simply supported), and the importance of second-order torsion effects in an I-section member is estimated. The reference values of slenderness are found to be very high, and it is concluded that second-order moments caused by torsion in typical structural steel members with slenderness ratios L/r_y less than 300 are very small and can be neglected.

More generally, it can be concluded that second-order moments caused by torsion will be small in structural steel members unless their plastic torsional strengths are higher than their elastic buckling torques. Further, the comparatively low torsional stiffnesses of many structural steel members will cause large twist rotations to occur long before their elastic buckling torques can be reached.

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8 NOTATION

a	length of rigid arm
A	cross-sectional area
b, b_1, b_2	angle section leg lengths
b_f, b_w	flange width and web depth
d	diameter of circular hollow section
d_f	distance between flange centroids
E	Young’s modulus of elasticity
F	function in energy equation
f_m	shape of torque distribution
f_u, f_v	shapes of buckling deflections
f_y	yield stress
I, I_x, I_y	second moments of area
k	buckling factor
L	member length

L_r	reference length
$(L/r_y)_r$	reference slenderness
M	concentrated moment or torque
M_{fp}	flange fully plastic moment
M_o	maximum torque at elastic buckling
M_p	value of M at plastic collapse
M_{up}	fully plastic uniform torque
M_x, M_y	bending moments
M_z	torque at z along a member
P	force
P_o	value of P at elastic buckling
P_p	value of P at plastic collapse
P_x, P_y, P_z	components of P
r	radius of winder
r, r_y	radii of gyration
s	distance along a path AB, or distance of slide from rigid arm
t, t_f, t_w	thicknesses
u, v, w	deflections in the x, y, z directions
U, V	constants defining magnitudes of u, v
u_1, u_2	shape components of u
v_1, v_2	shape components of v
W	work done
x, y	local principal axes of cross-section
X, Y, Z	global axes
z	distance along centroidal axis
a, a_u, a_v	arbitrary parameters
d_m	amplification factor for torque
d_p	amplification factor for axial compression
du, dv, dw	incremental deflections
dq_x, dq_y, dq_z	incremental rotations
q_{XL}, q_{YL}	end rotations
t_y	shear yield stress

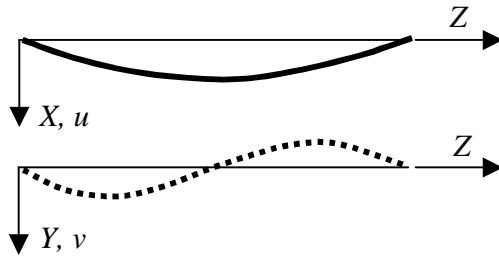
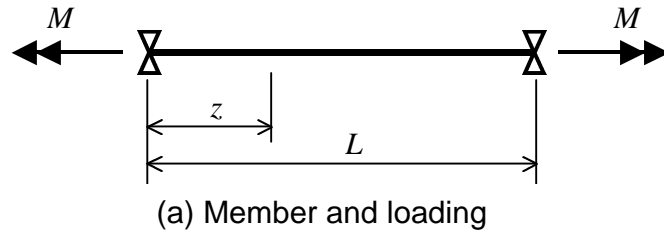


Fig. 1. Simply Supported Member with End Torque

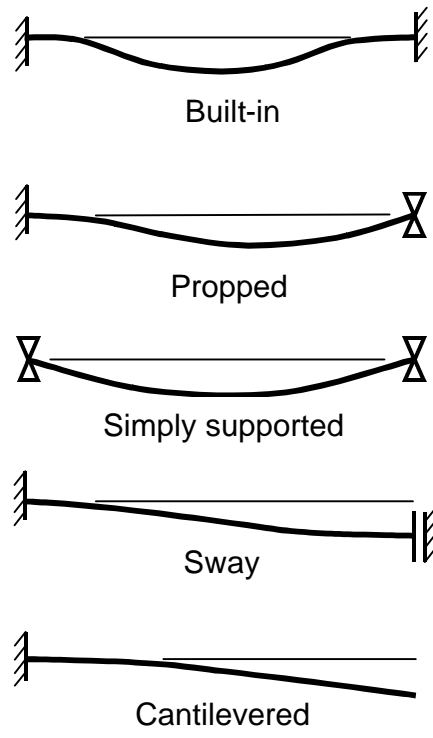
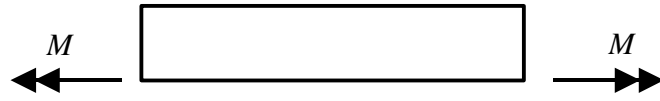
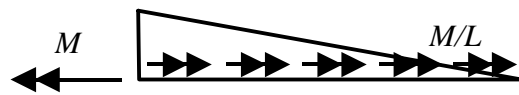


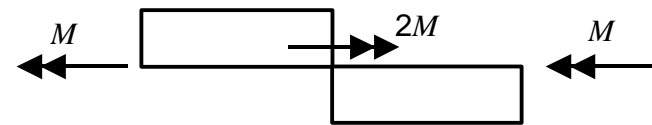
Fig. 2. Flexural Boundary Conditions



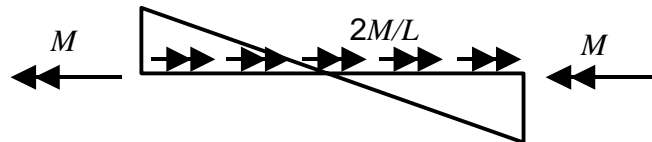
(a) Concentrated end torque



(b) Uniformly distributed torque



(c) Central concentrated torque



(d) Uniformly distributed torque

Fig. 3. Torque Distributions

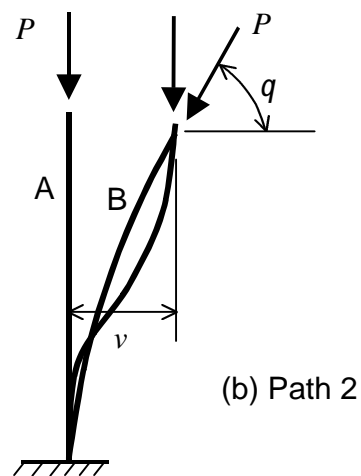
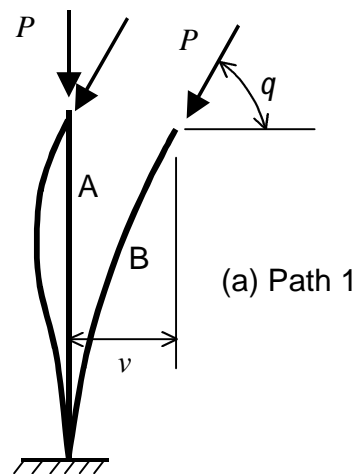


Fig. 4. "Follower" Loading

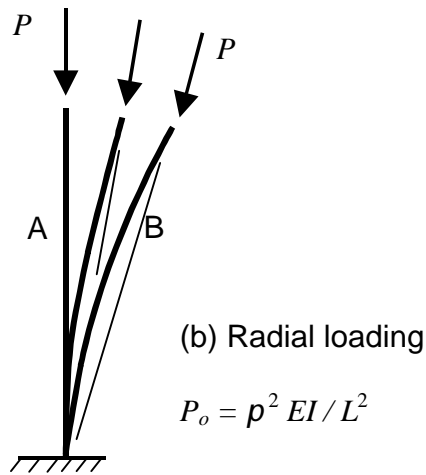
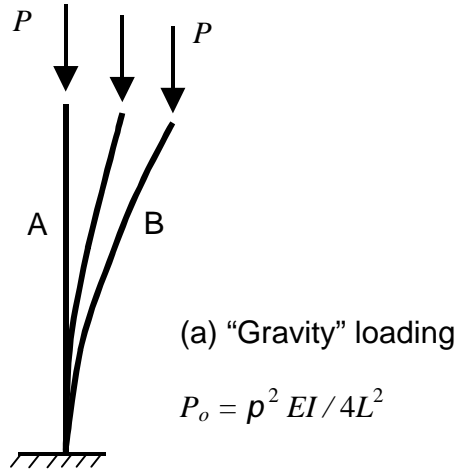
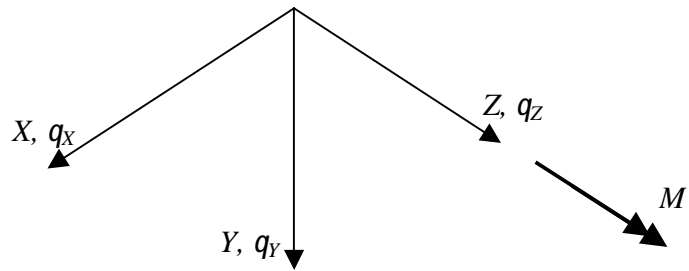


Fig. 5. Conservative Force Examples



Path 1, $(q_X, q_Y, q_Z) = (p, p, 0)$, Work = 0
Path 2, $(q_X, q_Y, q_Z) = (0, 0, p)$, Work = $M p$

Fig. 6. Non-conservative Torque

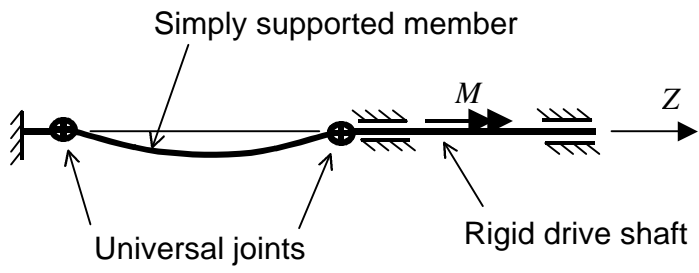


Fig. 7. Conservative Axial Torque

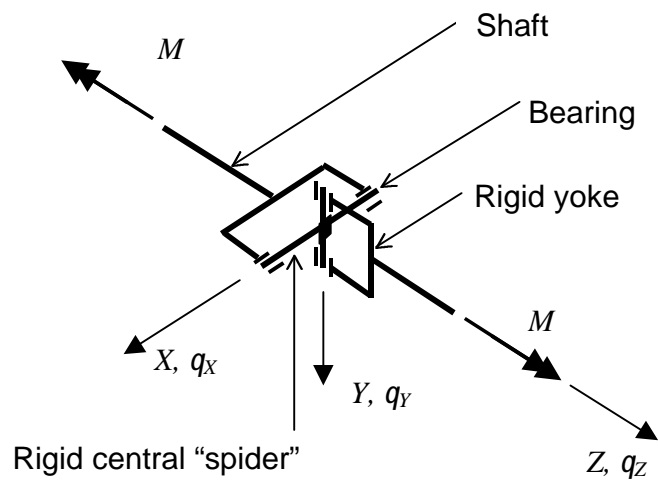


Fig. 8. Diagrammatic View of Universal Joint

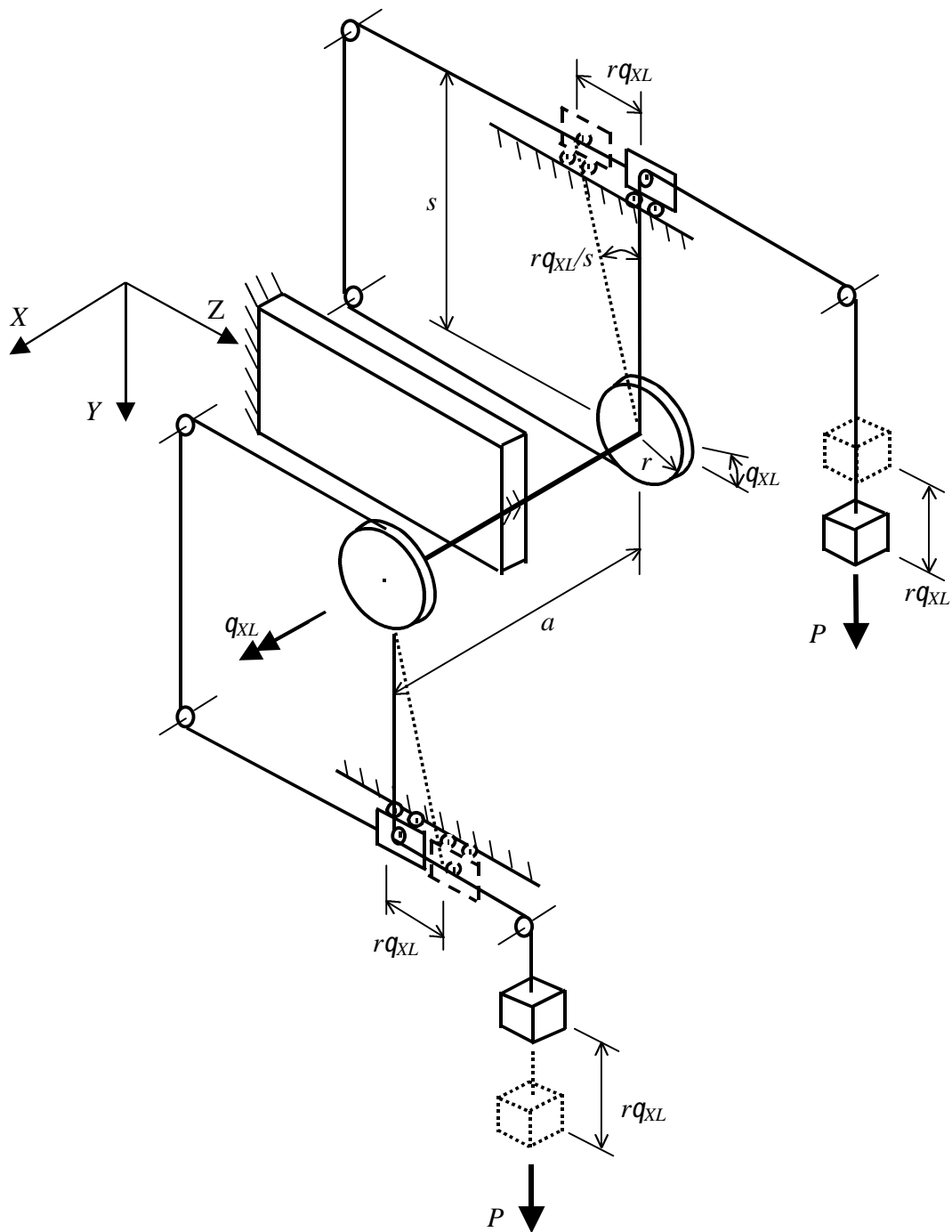
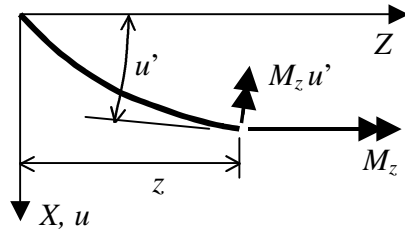
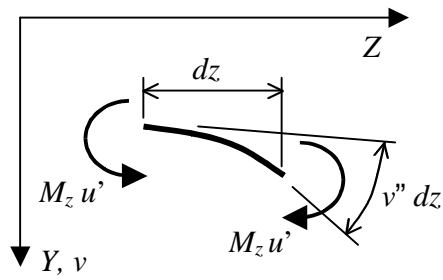


Fig. 9. Mechanism of Teh and Clarke [4]

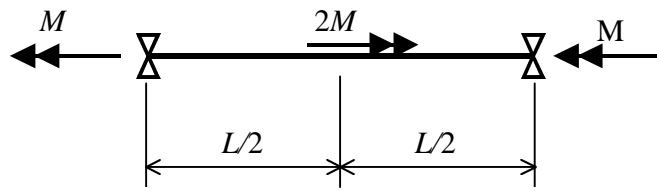


(a) Plan view of component $M_z u'$

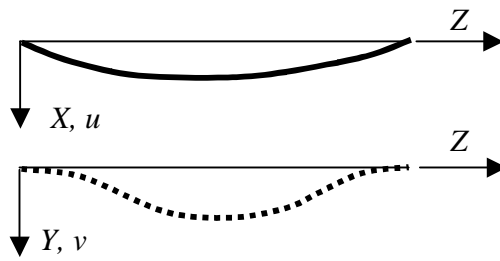


(b) Elevation of element bending

Fig. 10. Work Done by Torque Component



(a) Member and loading



(b) Buckled shapes

Fig. 11. Simply Supported Member with Central Torque