

# Appendix B

## Fourier Series

This appendix shows the use of the Fourier Transforms and the Discrete Fourier Transforms to approximate continuous and discrete functions. Fourier Series can be used to approximate a periodic function  $f(t)$  through a weighted sum of *sine* and *cosine* terms. For a detailed presentation on the topic of Fourier Transforms and Discrete Fourier Transforms the reader is referred to [46, 67].

### B.1 Fourier Series - real notation

A continuous function  $f(t)$  with period  $T$  can be approximated using the Fourier Series given by

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos(k\omega t) + b_k \sin(k\omega t)] \quad (\text{B.1})$$

whereas the Fourier coefficients  $a_k$  and  $b_k$  are calculated using

$$a_k = \frac{2}{T} \int_0^T f(t) \cos(k\omega t) dt \quad \forall k = 0, 1, 2, \dots, n \quad (\text{B.2})$$

$$b_k = \frac{2}{T} \int_0^T f(t) \sin(k\omega t) dt \quad \forall k = 0, 1, 2, \dots, n \quad (\text{B.3})$$

The angular frequency  $\omega$  is given by

$$\omega = \frac{2\pi}{T} \quad (\text{B.4})$$

## B.2 Fourier Series - complex notation

An alternative possibility is to describe the Fourier approximation of a continuous function with period  $T$  using complex notation.

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{ik\omega t} \quad c_k \in \mathbb{C} \quad (\text{B.5})$$

In this case the complex Fourier coefficients  $c_k$  are given by

$$c_k = \frac{1}{T} \int_0^T f(t) e^{-ik\omega t} \quad (\text{B.6})$$

The angular frequency  $\omega$  is again given by

$$\omega = \frac{2\pi}{T} \quad (\text{B.7})$$

The relationship between the real Fourier coefficients  $a_k$ ,  $b_k$  and the complex Fourier coefficients  $c_k$  is given by

$$c_k := \frac{a_k - ib_k}{2} \quad c_k \in \mathbb{C} \quad (\text{B.8})$$

$$a_k = \frac{c_n + c_{-n}}{2} \quad b_k = i \frac{c_n - c_{-n}}{2} \quad (\text{B.9})$$

The coefficients  $a_k$  correspond to the real part of  $c_k$  and the  $b_k$  correspond to the imaginary part, respectively.

## B.3 Discrete Fourier Transform

If a function  $f(x)$  consists in the interval  $0 \leq x < L$  only of a discrete system of points  $x_k$ ,

$$x_k = \frac{kL}{N} \quad k = 0, 1, \dots, N-1 \quad (\text{B.10})$$

i.e. the function is discretely sampled at a fixed sampling frequency, then this function can be approximated by a trigonometric polynomial given by

$$T_N(x) = \sum_{l=0}^{N-1} c_l \exp(2\pi i \frac{lx}{L}) \quad (\text{B.11})$$

The complex Fourier coefficients are given by

$$c_l = \frac{1}{N} \sum_{j=0}^{N-1} f(x_j) \exp(-2\pi i \frac{jx_j}{N}) \quad c_l \in \mathbb{C} \quad (\text{B.12})$$

## B.4 Nyquist theorem

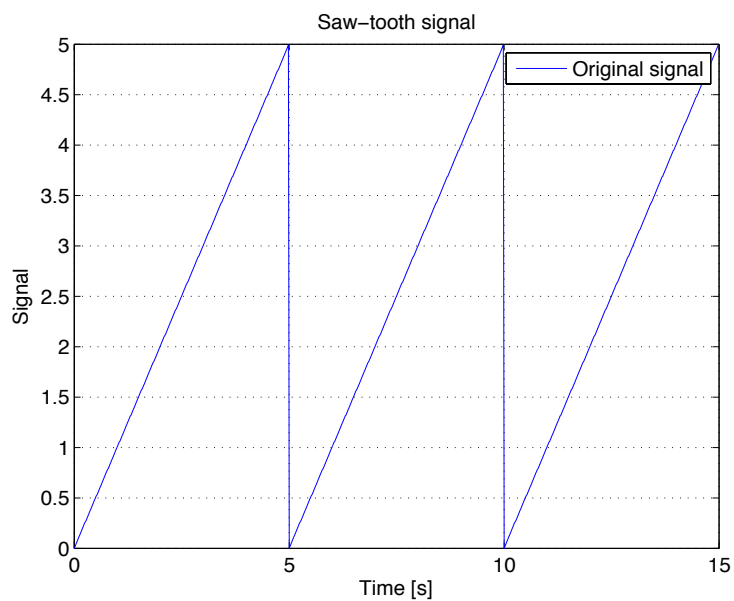
The Nyquist theorem is concerned with the reconstruction of a signal from samples. In brief it states that a band limited signal has to be sampled at twice the maximum frequency occurring in the sampled signal, or faster (equation (B.13)). This is to ensure that the signal can be reconstructed unambiguously.

$$f_{\text{sampling}} \geq 2f_{\text{signal,max}} \quad (\text{B.13})$$

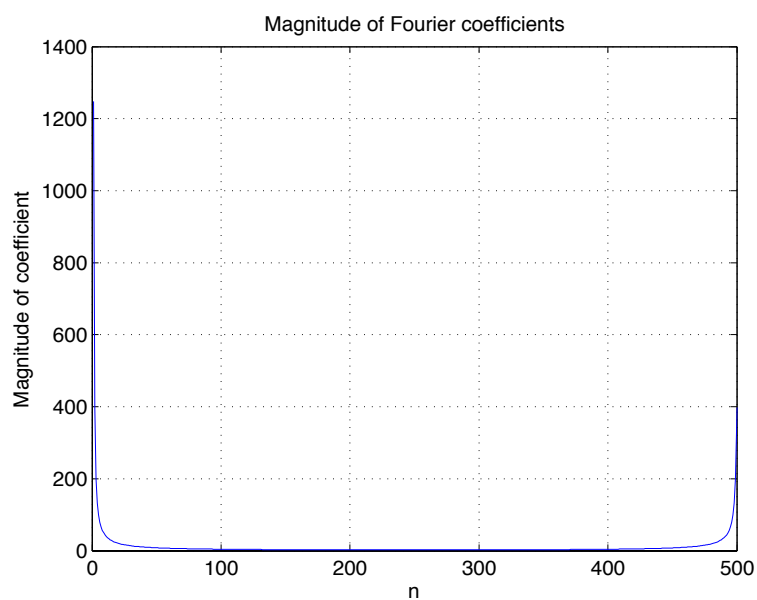
### Example B.1

*This example shows how a sampled signal can be described using complex Fourier coefficients and how the original signal can be reconstructed. Let's assume we have a saw-tooth signal with a period of 5s sampled at 100Hz. This signal is shown in figure B.1. Applying a Fast Fourier Transformation (FFT) to one period of this signal we obtain the complex Fourier coefficients. The magnitude of the coefficients is shown in figure B.2. As the sampling rate was 100Hz and the period length was 5s, we obtain  $5 \cdot 100 = 500$  complex Fourier coefficients, one per sampling point, from the FFT.*

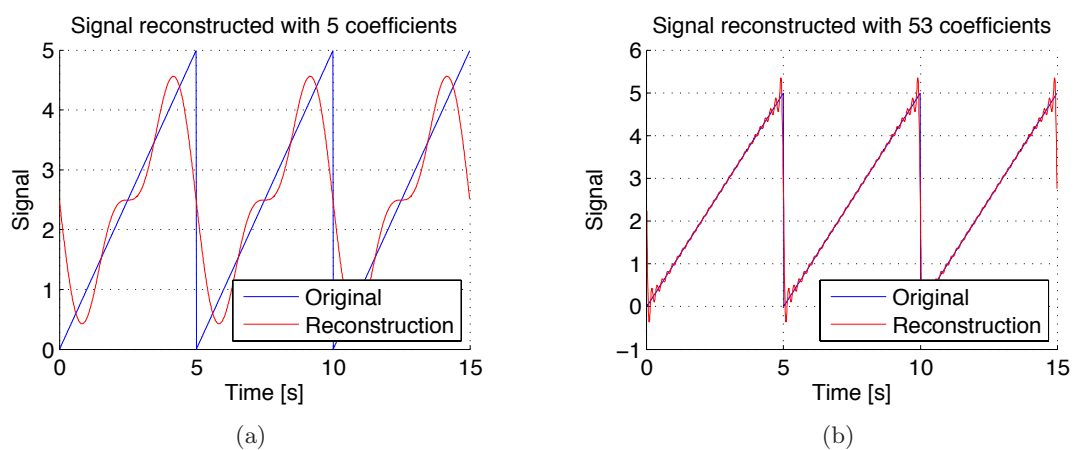
*When reconstructing the original signal we choose only the dominant coefficients, i.e. the coefficients whose magnitude is highest. Figure B.3 shows two examples of the signal reconstruction with two different numbers of dominant coefficients used, the number in (a) being smaller than in (b). It can be seen how the approximation gets better the more coefficients are used.*



**Figure B.1:** Saw-tooth signal — this signal has a period of 5s and was sampled at 100Hz.



**Figure B.2:** Magnitude of the Fourier coefficients of the saw-tooth signal which were obtained using a FFT. The first coefficient corresponds to the DC bias. One should also note the symmetry of the magnitude plot.



**Figure B.3:** Reconstruction of the original signal with two different numbers of Fourier coefficients used for the reconstruction — (a) shows the result when using only the 5 dominant coefficients, while (b) shows the reconstruction using the 53 dominant coefficients.